Proficiency Exam 6 - Eigenvalues and Eigenvectors

You will have 30 minutes to complete the exam. You may use a calculator, but you must show all steps done to get full credit for completing the problem. This means that if you use your calculator for anything other than arithmetic, you must indicate on your test paper what you did on the calculator.

1. Compute the determinant of the following matrix.

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 5 \\ -3 & 3 & -9 \end{bmatrix}$$

Answer: (Via Cofactor Expansion) I have worked out below what I expect most students to do using the first row. Some student may expand using a different row or column. I would be good to double check my math.

$$\det(A) = 2 \cdot \begin{vmatrix} 8 & 5 \\ 3 & -9 \end{vmatrix} - 4 \cdot \begin{vmatrix} 4 & 5 \\ -3 & -9 \end{vmatrix} + 6 \cdot \begin{vmatrix} 4 & 8 \\ -3 & 3 \end{vmatrix} = 2(-87) - 4(-21) + 6(36) = 126$$

(Via Row Reduction)

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 5 \\ -3 & 3 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -7 \\ 0 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

The determinant of the end result is -63, however, during this process we divided a row by 2 and swapped two rows, so we must multiply the computed determinant by -2 to get 126.

2. The matrix A given below has an eigenvalue of 3. Find a basis for the eigenspace (the subspace of all eigenvectors for the eigenvalue of 3).

$$A = \begin{bmatrix} 43 & -15 & -10 \\ -120 & 48 & 30 \\ 360 & -135 & -87 \end{bmatrix}$$

Answer:

$$A - 3I = \begin{bmatrix} 40 & -15 & -10 \\ -120 & 45 & 30 \\ 360 & -135 & -90 \end{bmatrix} \to \begin{bmatrix} 40 & -15 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So if $(A - 2I)v = 0$,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 4v_1 - 1.5v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ -1.5 \end{bmatrix}$$

3. (TRUE or FALSE) Consider the statement and decide if it is true or false. If true, provide reasoning. If false, provide a counterexample.

“ For any 2×2 matrix, $\det(A)$ is equal to the product of the eigenvalues. ”

Answer: TRUE. Consider the characteristic polynomial in two way.

$$\lambda^2 - (a+d)\lambda + (ad - bc) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

4. Find the general solution to the following system of LODEs.

$$\frac{dx}{dt} = 10x(t) + 3y(t)$$
$$\frac{dy}{dt} = -3x(t) - 5y(t)$$

Answer: Find e-vals:

$$\det(A - \lambda I) = \begin{vmatrix} 10 - \lambda & 3 \\ -3 & -5 - \lambda \end{vmatrix} = -41 + 5\lambda + \lambda^2$$

Using quadratic formula, the e-vals are $\lambda_1 \approx 4.37$ and $\lambda_2 \approx -9.37$.

Find e-vecs: For $\lambda_1 = 4.37$

$$(A - 4.37I)v = 0 \Rightarrow \begin{bmatrix} 5.63 & 3 \\ -3 & -9.37 \end{bmatrix} v = 0 \Rightarrow 5.63v_1 + 3v_2 = 0 \Rightarrow v_2 = -1.88v_1 \Rightarrow v = \begin{bmatrix} 1 \\ -1.88 \end{bmatrix}.$$

For $\lambda_1 = -9.37$

$$(A + 9.37I)v = 0 \Rightarrow \begin{bmatrix} 19.37 & 3 \\ -3 & 4.37 \end{bmatrix} v = 0 \Rightarrow 19.37v_1 - 3v_2 = 0 \Rightarrow v_2 = -6.46v_1 \Rightarrow v = \begin{bmatrix} 1 \\ -6.46 \end{bmatrix}.$$

General solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \alpha e^{4.37t} \begin{bmatrix} 1 \\ -1.88 \end{bmatrix} + \beta e^{-9.37t} \begin{bmatrix} 1 \\ -6.46 \end{bmatrix}.$$