Proficiency Exam 3 - Linear Transformations

You will have 30 minutes to complete the exam. You may use a calculator, but you must show all steps done to get full credit for completing the problem. This means that if you use your calculator for anything other than arithmetic, you must indicate on your test paper what you did on the calculator.

1. Is the function $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$F(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 - 3x_2 \end{bmatrix}$$

a linear transformation? If it is, prove it. If it isn't, provide a counterexample that shows it is not.

Answer: Yes. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be arbitrary, then

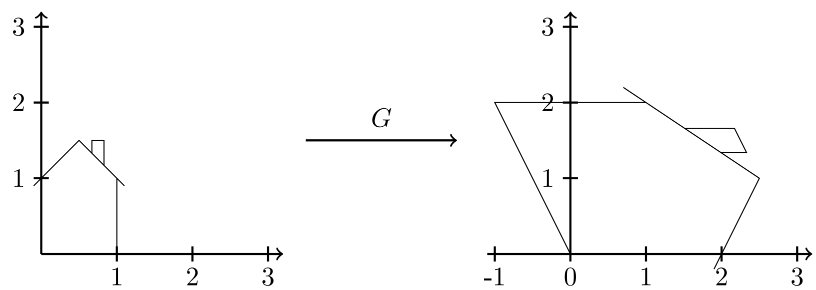
$$F(x+y) = \begin{bmatrix} (x_1+y_1) + (x_2+y_2) \\ (x_1+y_1) - (x_2+y_2) \\ 2(x_1+y_1) - 3(x_2+y_2) \end{bmatrix} = \begin{bmatrix} (x_1+x_2) + (y_1+y_2) \\ (x_1-x_2) + (y_1-y_2) \\ (2x_1-3x_2) + (2y_1-3y_2) \end{bmatrix} = F(x) + F(y).$$

So F respects addition. If $\alpha \in \mathbb{R}$, then

$$F(\alpha x) = \begin{bmatrix} \alpha x_1 + \alpha x_2 \\ \alpha x_1 - \alpha x_2 \\ 2\alpha x_1 - 3\alpha x_2 \end{bmatrix} = \begin{bmatrix} \alpha(x_1+x_2) \\ \alpha(x_1-x_2) \\ \alpha(2x_1-3x_2) \end{bmatrix} = \alpha F(x).$$

So F respects scalar multiplication, and thus is a linear transformation.

2. Find the matrix representation of the linear transformation pictured below.



Answer:

$$\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$$

3. (TRUE or FALSE) Consider the statement and decide if it is true or false. If true, provide reasoning. If false, provide a counterexample.

"If $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is onto, then $m \leq n$."

Answer: The statement is FALSE. Since the transformation to be onto, its RREF must have a pivot in every row. This can only happen if the number of rows, n is less than or equal to the number of columns, m.

4. Is the following linear transformation 1-1?

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 4 \\ -1 & 3 & 7 \end{bmatrix} \mathbf{x}$$

Answer: Yes, T is 1-1. The rref of the matrix has a pivot in every column.