Proficiency Exam 5 - Vector Geometry

You will have 30 minutes to complete the exam. You may use a calculator, but you must show all steps done to get full credit for completing the problem. This means that if you use your calculator for anything other than arithmetic, you must indicate on your test paper what you did on the calculator.

1. Compute the projection of the vector $(1,5,7)$ onto the vector $(1,-1,1)$ in $\mathbb{R}^3$.

   Answer: Let $x=(1,5,7)$ and $v=(1,-1,1)$. Then

$$\mathrm{proj}_v(x)=\frac{\langle x,v\rangle}{\langle v,v\rangle}v=\frac{1\cdot 1+5\cdot(-1)+7\cdot 1}{1^2+(-1)^2+1^2}(1,-1,1)=\frac{3}{3}(1,-1,1)=(1,-1,1).$$

2. For what value of t will the set of vectors below be orthogonal?

$$\left\{\begin{bmatrix}1\\-1\\0\\-1\end{bmatrix},\begin{bmatrix}3t\\t+3\\t+3\\2t-3\end{bmatrix},\begin{bmatrix}t+1\\t-1\\-1\\2\end{bmatrix}\right\}$$

   Answer: v_1 is orthogonal to v_2 and v_3 no matter the value of t. For this to be an orthogonal set, $\langle v_2,v_3\rangle$ must be 0.

$$0=\langle v_2,v_3\rangle=3t(t+1)+(t+3)(t-1)-(t+3)+2(2t-3)\Rightarrow 0=4(t^2+2t-3)\Rightarrow t=1,-3$$

3. (TRUE or FALSE) Consider the statement and decide if it is true or false. If true, provide reasoning. If false, provide a counterexample.

   " Let $W=\mathrm{span}(w_1,w_2)$. If $\mathcal{B}=\{w_1,w_2\}$ is an orthogoanl basis for W, then $\mathrm{proj}_W(x)=\mathrm{proj}_{w_1}(x)+\mathrm{proj}_{w_2}(x)$. "

   Answer: True. Since $\mathrm{proj}_W(x)\in W$ and $\mathcal{B}$ is orthogonal,

$$\mathrm{proj}_W(x)=\frac{\langle x,w_1\rangle}{\langle w_1,w_1\rangle}w_1+\frac{\langle x,w_2\rangle}{\langle w_2,w_2\rangle}w_2=\mathrm{proj}_{w_1}(x)+\mathrm{proj}_{w_2}(x).$$

4. Find an orthonormal basis for

$$W=\mathrm{span}\left\{\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix},\begin{bmatrix}0\\4\\7\\9\end{bmatrix}\right\}.$$

   Answer: Set $b_1=\frac{v_1}{||v_1||}=(0.5,-0.5,-0.5,0.5)$. Then find

$$\tilde{b}_2=v_2-\mathrm{proj}_{b_1}(v_2)=v_2-\langle v_2,b_1\rangle b_1=(0,4,7,9)-(-1)\cdot(0.5,-0.5,-0.5,0.5)=(0.5,3.5,6.5,9.5)$$

   Finally, set

$$b_2=\frac{\tilde{b}_2}{||\tilde{b}_2||}=\frac{1}{12.04}(0.5,3.5,6.5,9.5)=(0.04,0.29,0.54,0.79).$$