MTH 309 - Activity 7
Eigenvalues and Eigenvectors

1. Consider an isolated ecosystem with two animal species living without competition for resources: an predator and a prey. At any point in time, the change in population size for the predator will depend positively on the population size for the predator as well as the prey. The change in the prey population will depend positively on its own population size, but negatively on the predator population size. This scenario can be modeled by a couple simple linear ordinary differential equations (LODEs):

$$\begin{aligned} \frac{dx}{dt} &= \alpha x(t) + \beta y(t) \\ \frac{dy}{dt} &= -\beta x(t) + \delta y(t). \end{aligned}$$

   Write this system in matrix form.

2. A population that grows without constraint will exhibit exponential growth. Assume that $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, plug this into the system of LODEs, and determine the values of the scalar λ and the constant vector $\mathbf{v}$.

3. Due to the nature of derivatives, the set of solutions of a LODE forms a subspace of functions. That means that if $x_1(t)$ and $x_2(t)$ are both solutions to the same LODE, then so is any linear combination. Use this fact to derive a general formula for the solutions to our system of LODEs.

4. Consider the ecosystem with the following predator prey model:

$$\begin{aligned} \frac{dx}{dt} &= -0.04x(t) + 0.2y(t) \\ \frac{dy}{dt} &= -0.2x(t) + 0.4y(t). \end{aligned}$$

   Find and analyze the general solution.

5. The parameter α is the growth rate of the predator population (birth rate - death rate) and is typically a little less than 0, whereas the growth rate for the prey population δ is typically much higher than 0. The parameter β is the predation rate and captures how successful the predators are at catching their prey. If you keep α and δ fixed, what happens to the solutions if β is varied?

6. What happens when we get imaginary numbers in the exponent?

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \theta^{(2k+1}}{(2k+1)!} \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

   This would yield a periodic solution with the predator and prey populations oscillating over time. When would this occur?