Proficiency Exam 2 - Vector Spaces

You will have 30 minutes to complete the exam. You may use a calculator, but you must show all steps done to get full credit for completing the problem. This means that if you use your calculator for anything other than arithmetic, you must indicate on your test paper what you did on the calculator.

1. Consider the subset $W = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0 \text{ and } y - z = 0\}$. Is W a subspace? Be sure to provide support for your answer.

   Answer:
   W is a subspace. Consider two vectors $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$ in W with sum $w_1 + w_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$. Notice that

   $$(x_1+x_2)-(y_1+y_2) = (x_1-y_1)+(x_2-y_2) = 0+0 = 0 \text{and} (y_1+y_2)-(z_1+z_2) = (y_1-z_1)+(y_2-z_2) = 0+0 = 0,$$

   so $w_1 + w_2$ belongs to W.

   Now consider $w = (x, y, z)$ in W and $\alpha \in \mathbb{R}$. Notice that

   $$\alpha x - \alpha y = \alpha(x-y) = \alpha \cdot 0 = 0 \text{and} \alpha y - \alpha z = \alpha(y-z) = \alpha \cdot 0 = 0,$$

   so αw belongs to W.

2. Find a basis for the null space of the matrix

   $$\begin{bmatrix} -3 & 1 & 0 & -2 \\ 3 & -1 & 1 & 4 \\ -6 & 2 & 3 & 2 \end{bmatrix}.$$

   Answer:

   $$RREF\left(\begin{bmatrix} -3 & 1 & 0 & -2 \\ 3 & -1 & 1 & 4 \\ -6 & 2 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} -3 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   So

   $$-3x_1 + x_2 - 2x_4 = 0 \Rightarrow x_2 = 3x_1 + 2x_4 \text{and} x_3 + 2x_4 = 0 \Rightarrow x_3 = -2x_4.$$

   Thus any solution $(x_1.x_2, x_3, x_4)$ looks like

   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 + 2x_4 \\ -2x_4 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}.$$

   So the basis of the null space is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

3. (TRUE or FALSE) Consider the statement and decide if it is true or false. If true, provide reasoning. If false, provide a counterexample.

   "Any set of 3 or more vectors in $\mathbb{R}^3$ spans the whole space."

   Answer:
   FALSE. Take the columns of the matrix in the previous problem as a counterexample. Since there are only two pivots, there are some vectors that cannot be obtained by linear combination.

4. Is the following set a basis of $\mathbb{R}^3$?

   $$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \right\}$$

   Answer:
   NO. A basis must be linearly independent, and the third vector is the negative of the sum of the first two vectors, so the set is dependent.

   $$RREF\left(\begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & -3 \\ 0 & 2 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

   Since there are only two pivots, the original vectors are linearly dependent.