Bayesian Learning

Lab 2

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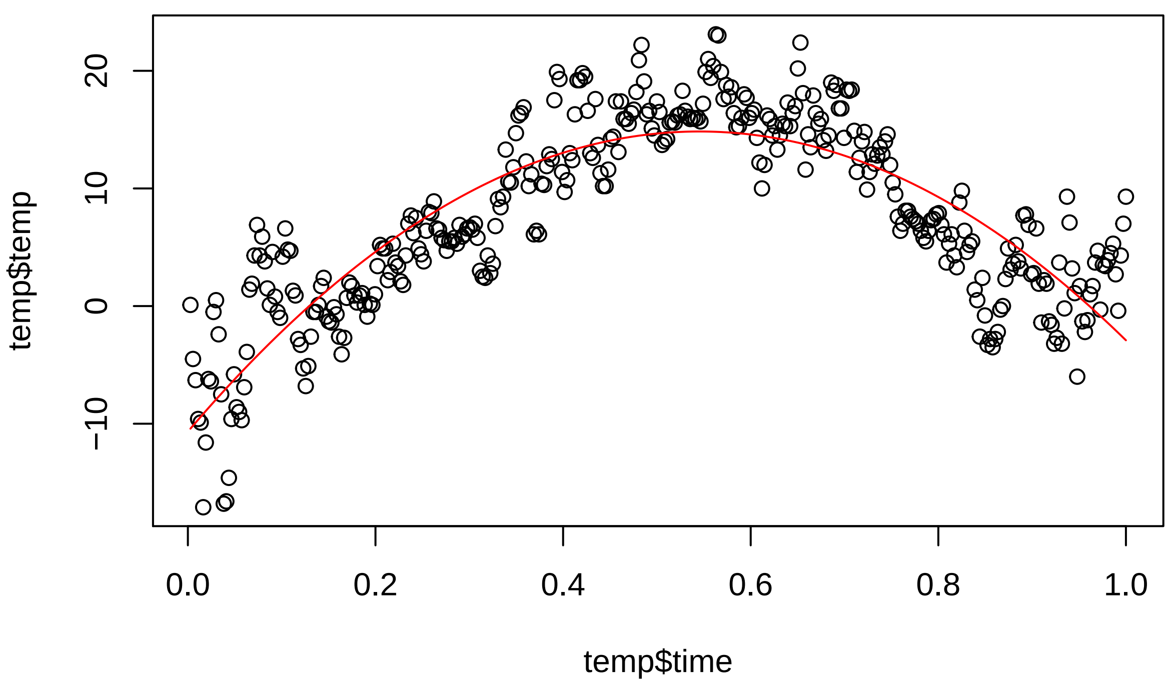
Question 1

```
temp <- read.table("../data/TempLinkoping2016.txt", header=T)

mod <- lm(temp ~ time + I(time^2), data=temp)
```

```
idx <- order(temp$time)
x <- temp$time[idx]
y <- fitted(mod)[idx]

plot(temp$time, temp$temp)
lines(x, y, col='red', type='l')
```



Prior

$$
\begin{aligned}
\sigma^2 &\sim \mathrm{Inv}-\chi^2(\nu_0, \sigma_0^2) \\
\beta|\sigma^2 &\sim \mathrm{N}(\mu_0, \sigma^2\Omega_0^{-1})
\end{aligned}
$$

Likelihood

$$
\mathbf{y}|\beta, \sigma^2, \mathbf{X} \sim \mathrm{N}(\mathbf{X}\beta, \sigma^2 I_n)
$$

Posterior

$$
\begin{aligned}
\sigma^2|\mathbf{y} &\sim \mathrm{Inv}-\chi^2(\nu_n, \sigma_n^2) \\
\beta|\sigma^2, \mathbf{y} &\sim \mathrm{N}(\mu_n, \sigma^2\Omega_n^{-1})
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_n &= (\mathbf{X}^\intercal\mathbf{X} + \Omega_0)^{-1}(\mathbf{X}^\intercal\mathbf{X}\hat{\beta} + \Omega_0\mu_0) \\
\Omega_n &= \mathbf{X}^\intercal\mathbf{X} + \Omega_0 \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (\mathbf{y}^\intercal\mathbf{y} + \mu_0^\intercal\Omega_0\mu_0 - \mu_n^\intercal\Omega_n\mu_n)
\end{aligned}
$$

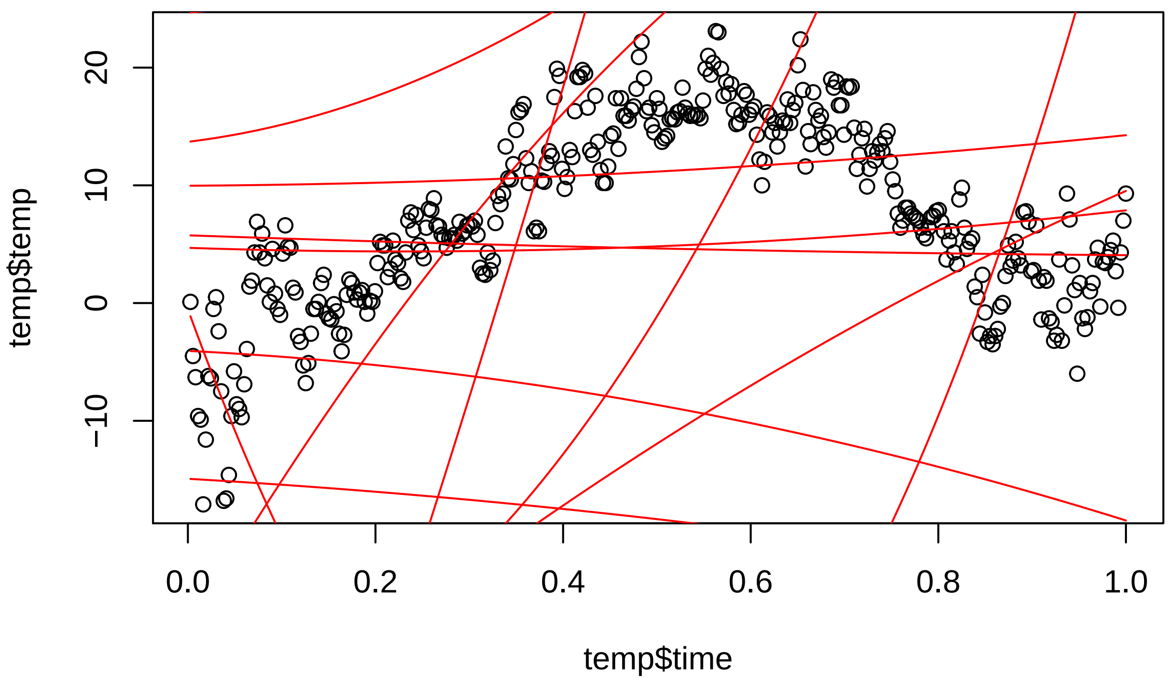
a)

```
mu0 <- c(0, 0, 0)
omega0 <- diag(3) * 0.05
nu0 <- 1
sigmasq0 <- 20

hyperparams <- list(mu=mu0, omega=omega0, nu=nu0, sigmasq=sigmasq0)
```

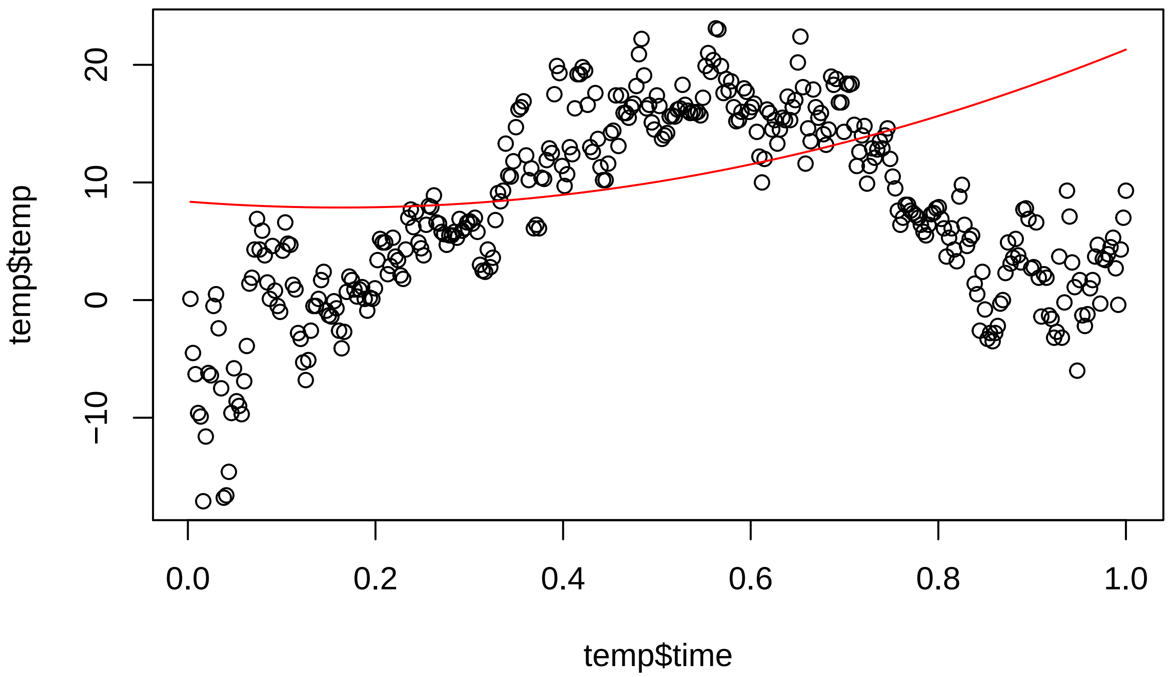
b)

```
library(geoR)
library(MASS)

time <- data.frame(rep(1,nrow(temp)), temp$time, temp$time^2)
mtime <- as.matrix(time)
mtemp <- matrix(temp$temp, ncol = 1)

prior_estimate <- function(data, params) {
    sigmasq  <- rinvchisq( n = 1, df = params$nu, scale = params$sigmasq)
    betacoef <- mvrnorm(n = 1, mu = params$mu, Sigma = sigmasq * solve(params$omega) )

    data %*% betacoef
}
```

```r
plot(temp$time, temp$temp)

x <- sort(temp$time)

set.seed(12345)

for (i in 1:20){
    y <- prior_estimate(mtime, hyperparams)[order(temp$time)]
    lines(x, y, col='red', type='l')
}
```




```r
set.seed(12345)

x <- sort(temp$time)
y <- rowMeans(sapply(1:1000, function(x) prior_estimate(mtime, hyperparams)[order(temp$time)]))

plot(temp$time, temp$temp)
lines(x, y, col='red', type='l')
```

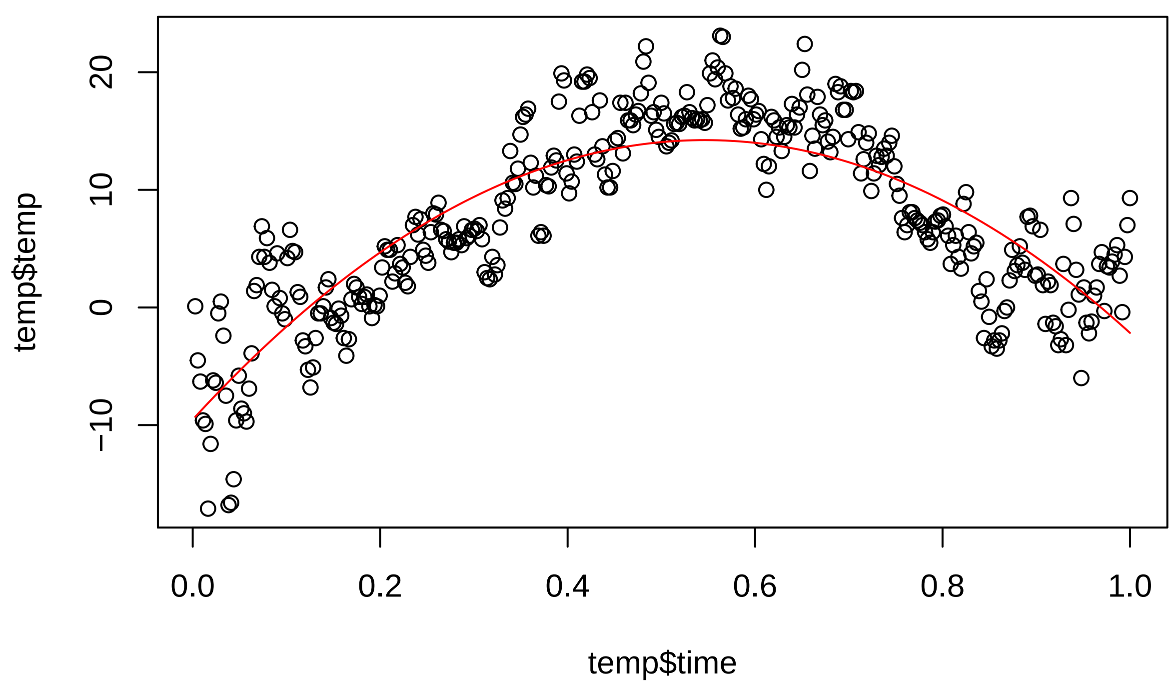


c)

```
posterior_param_sample <- function(X, y, hyperparams){
    XX <- t(X) %*% X

    betahat <- solve(XX) %*% t(X) %*% y

    mun <- solve(XX + hyperparams$omega) %*%
        (XX %*% betahat + hyperparams$omega %*% hyperparams$mu)

    omegan <- XX + hyperparams$omega

    nun <- hyperparams$nu + nrow(X)

    nunsigmasqn <- hyperparams$nu * hyperparams$sigmasq +
        (t(y) %*% y +
         t(hyperparams$mu) %*% hyperparams$omega %*% hyperparams$mu -
         t(mun) %*% omegan %*% mun )

    sigmasqn <- nunsigmasqn / nun

    sigmasq  <- rinvchisq(n = 1, df=nun, scale=sigmasqn)
    beta <- mvrnorm(n = 1, mu = mun, Sigma = as.numeric(sigmasq) * solve(omegan))
```

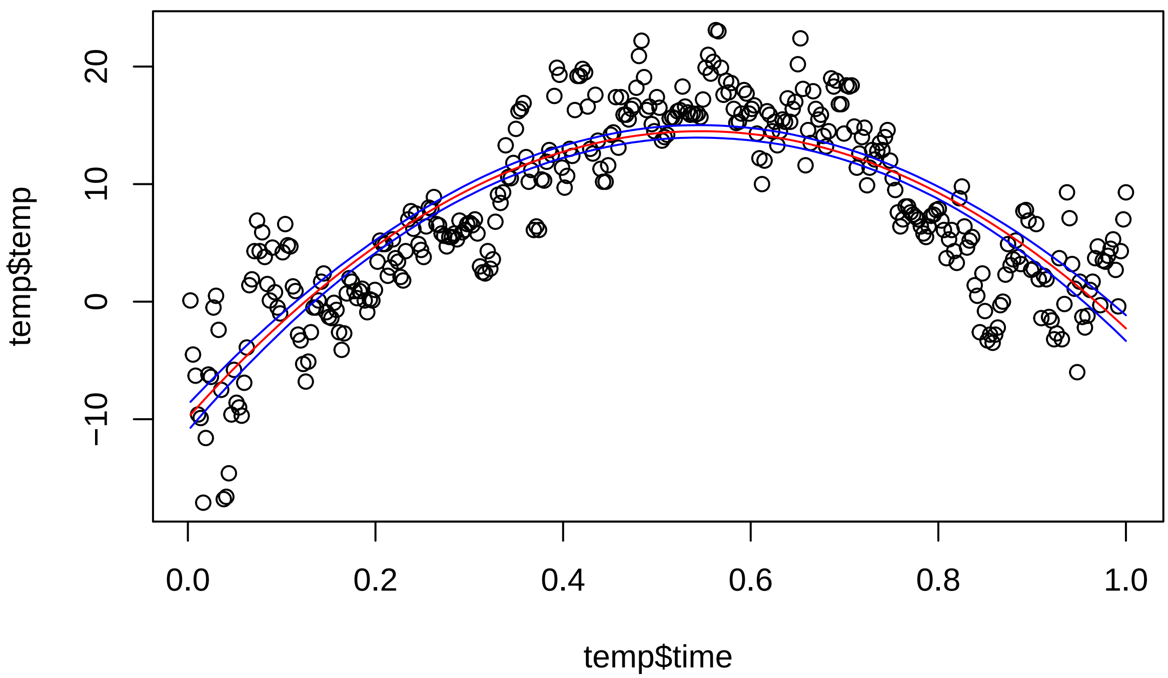
```r
    list(beta = beta, sigmasq = sigmasq)
}

posterior_estimate <- function(X, y, hyperparams){
    sample <- posterior_param_sample(X, y, hyperparams)
    X %*% sample$beta
}
```

```r
plot(temp$time, temp$temp)

set.seed(12345)

idx <- order(temp$time)
x <- temp$time[idx]
y <- posterior_estimate(mtime, mtemp, hyperparams)[idx]

lines(x, y, col='red', type='l')
```




```r
set.seed(12345)

ests <- sapply(1:1000, FUN = function(x) posterior_estimate(mtime, mtemp, hyperparams))
cred_interval <- apply(ests, MARGIN = 1, quantile, probs = c(0.05, 0.95))

idx <- order(temp$time)
x <- temp$time[idx]
y1 <- rowMeans(ests)[idx]
```

```r
y2 <- cred_interval[1,][idx]
y3 <- cred_interval[2,][idx]

plot(temp$time, temp$temp)
lines(x, y1, col='red', type='l')
lines(x, y2, col='blue', type='l')
lines(x, y3, col='blue', type='l')
```



d)

```r
set.seed(12345)

betas <- sapply(1:1000, FUN = function(x) posterior_param_sample(mtime, mtemp, hyperparams)$beta)
hot <- mean(-betas[2,] / (2 * betas[3,]))
hot * 366 # July 27, 2016 (Wed)
```

```
## [1] 200.0608
```

e)

Set μ_0 to zeros and a high Ω_0 that expresses a high degree of certainty in our prior.

Question 2

```r
women <- read.table("../data/WomenWorks.txt", header = TRUE)
```

a)

```r
glmModel <- glm(Work ~ 0 + ., data = women, family = binomial)
summary(glmModel)
```

```
##
## Call:
## glm(formula = Work ~ 0 + ., family = binomial, data = women)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.1662  -0.9299   0.4391   0.9494   2.0582
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## Constant     0.64430    1.52307   0.423 0.672274
## HusbandInc  -0.01977    0.01590  -1.243 0.213752
## EducYears    0.17988    0.07914   2.273 0.023024 *
## ExpYears     0.16751    0.06600   2.538 0.011144 *
## ExpYears2   -0.14436    0.23585  -0.612 0.540489
## Age         -0.08234    0.02699  -3.050 0.002285 **
## NSmallChild -1.36250    0.38996  -3.494 0.000476 ***
## NBigChild   -0.02543    0.14172  -0.179 0.857592
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 277.26  on 200  degrees of freedom
## Residual deviance: 222.73  on 192  degrees of freedom
## AIC: 238.73
##
## Number of Fisher Scoring iterations: 4
```

b)

```r
library(mvtnorm)

logprior <- function(beta, mean, sigma){
    dmvnorm(beta, mean = mean, sigma = sigma, log = TRUE)
}

loglikelihood <- function(beta, X, Y){
    linear_prediction <- t(X) %*% beta

    probabilities <- (Y * linear_prediction) - log(1 + exp(linear_prediction))
```

```r
    loglike <- sum(probabilities)

    ## if (abs(loglike) == Inf)
    ##     loglike = -20000

    loglike
}

logposterior <- function(beta, X, Y, mean, sigma){
   loglikelihood(beta, X, Y) + logprior(beta, mean, sigma)
}

tau <- 10
mu <- rep(0,8)
sigma <- tau^2 * diag(8)

womenX <- as.matrix(women[,2:ncol(women)])
womenY <- as.matrix(women[,1])

optpost <- optim(par = matrix(rep(0, 8), ncol = 1),
                 fn = logposterior, method = "BFGS", hessian = TRUE,
                 X = t(womenX), Y = womenY,
                 mean = mu, sigma = sigma,
                 control=list(fnscale=-1))
```

```r
posterior_beta_sample <- function(n, mu, sigma){
    rmvnorm(n, mean = mu, sigma = sigma)
}

mu <- optpost$par
sigma <- -solve(optpost$hessian)

set.seed(12345)
beta_samples <- posterior_beta_sample(n=1000, mu=mu, sigma=sigma)
cred_intervals <- apply(beta_samples, 2, quantile, prob=c(0.025, 0.975))
colnames(cred_intervals) <- colnames(women)[-1]
```

The 95% credibility intervals below show that the intercept, husband income, transformed years in work experience, and the number of children older than 6 years are statistical insignificant, i.e. 0 (no impact) is contained in the intervals. This indicates that it may be better to remove those variables from the model.

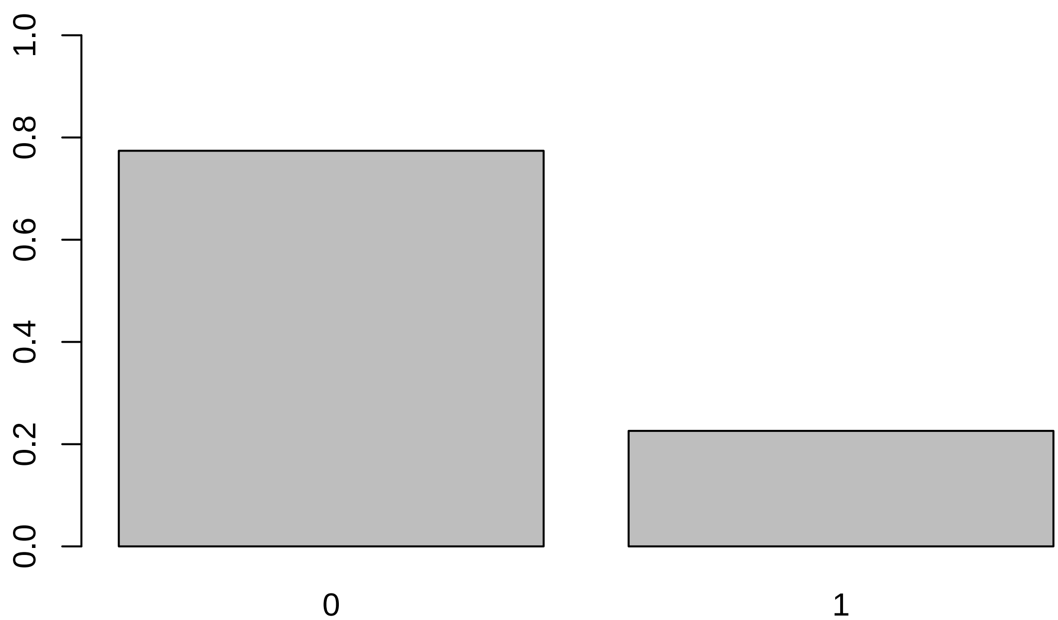
```r
cred_intervals
```

```
##        Constant  HusbandInc  EducYears    ExpYears   ExpYears2         Age
## 2.5%  -2.313434 -0.05091541 0.03095675 0.04723847 -0.6143409 -0.13476231
## 97.5%  3.533276  0.00780642 0.32739377 0.29656573  0.2943841 -0.03095853
##       NSmallChild  NBigChild
## 2.5%   -2.1296887 -0.3023168
## 97.5%  -0.5759195  0.2692414
```

c)

```
posterior_predictive_sample <- function(X, beta){
    linear_prediction <- t(X) %*% beta
    probability <- exp(linear_prediction) / (1 + exp(linear_prediction))
    rbinom(n=1, size=1, prob=probability)
}

x <- matrix(c(1, 10, 8, 10, (10 / 10)^2, 40, 1, 1), ncol=1)
prediction_samples <- apply(beta_samples, 1,  function(beta) posterior_predictive_sample(x, beta))
```

```
counts <- table(prediction_samples)
barplot(counts / sum(counts), ylim=c(0, 1))
```



The barplot above indicates that it is much more probable that given the data the woman is not working under our model which is surprising.