Activity 2: Quadratic Drag Force

1. Consider a body moving horizontally (like a railcar on a track) subject to a quadratic drag force:

$$\vec{F}_d = -cv^2\hat{v}$$

   (a) Write Newton's second law $\vec{F} = m\vec{a}$ in the x-direction: $m\ddot{x} = -cv_x^2$

   (b) Write the answer to part (a) as a differential equation for v_x $\quad m\dot{v}_x = -c\,v_x^2$

   (c) Solve for $v_x(t)$

$$\frac{dv_x}{dt} = \frac{-c}{m} v_x^2$$

$$\int_{v_{x0}}^{v_x} \frac{dv_x'}{(v_x')^2} = -\frac{c}{m}\int_0^t dt'$$

$$\left(-\frac{1}{v_x'}\right)\Big|_{v_{x0}}^{v_x} = -c/m\ t$$

$$-\frac{1}{v_x} + \frac{1}{v_{x0}} = \frac{-c}{m}t$$

$$v_x = \frac{1}{\frac{1}{v_{x0}} + \frac{c}{m}t} = \frac{v_{x0}}{1 + \frac{cv_{x0}}{m}t}$$

Let $\tau \equiv \dfrac{m}{cv_{x0}}$

$$\Rightarrow \boxed{v_x = \frac{v_{x0}}{1 + t/\tau}}$$

2. Now consider a ball that starts from rest, then falls only vertically through the air. The ball is subject to both gravity and a quadratic drag force:

$$\vec{F}_d = -cv^2\hat{v} \qquad \vec{F}_g = -mg\hat{y}$$

$\vec{F}_d = -cv^2\hat{v} = -cv^2(-\hat{y}) = cv^2\hat{y}$

(a) Write Newton's second law $\vec{F} = m\vec{a}$ in the y-direction: $m\ddot{y} = cv_y^2 - mg$

(b) Find the terminal velocity

$\vec{v}_{term} = \frac{-mg}{c}\hat{y}$, $|v_{term}| = \frac{mg}{c}$

(c) Write the answer to part (a) as a differential equation for v_y $m\dot{v}_y = cv_y^2 - mg$

(d) Solve for $v_y(t)$

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3. Now consider a projectile moving through the air along a path that's both vertical and horizontal. Write Newton's second law in both directions:

(a) Write $\vec{F}_d$ in terms of $\hat{x}$ and $\hat{y}$. (Hint: rewrite this to get a factor of $\vec{v}$, and then use $\vec{v} = v_x\hat{x} + v_y\hat{y}$)

$\vec{F}_d = -cv^2\hat{v} = -cv^2\frac{\vec{v}}{v} = -cv(v_x\hat{x} + v_y\hat{y})$, $v = \sqrt{v_x^2 + v_y^2}$

(b) $m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\, v_x$

(c) $m\dot{v}_y = -c\sqrt{v_x^2 + v_y^2}\, v_y$

(d) Based on the above, why are the equations of motion for quadratic air resistance more difficult to solve than for linear air resistance?

They are coupled!

$$\frac{dv_y}{dt} = \frac{c}{m} v_y^2 - g$$

$$\int_{v_{y0}}^{v_y} \frac{dv_y'}{\frac{c}{m}(v_y')^2 - g} = \int_0^t dt'$$ use $$\int \frac{du}{1-a^2u^2} = \frac{1}{a}\tanh^{-1}(au) + c$$

$$= -\frac{1}{g} \int_{v_{y0}}^{v_y} \frac{dv_y'}{1 - \frac{c}{mg}(v_y')^2} = t'\Big|_0^t$$

then $u = \sqrt{\frac{c}{mg}}\, v_y'$ and $a = 1$

$du = \sqrt{\frac{c}{mg}}\, dv_y'$

$$-\frac{1}{g}\sqrt{\frac{mg}{c}} \int_{v_{y0}}^{v_y} \frac{\sqrt{\frac{c}{mg}}\, dv_y'}{1 - \frac{c}{mg}(v_y')^2} = t$$

$$-\sqrt{\frac{m}{gc}} \tanh^{-1}\left(\sqrt{\frac{c}{mg}}\, v_y'\right)\Bigg|_{v_{y0}}^{v_y} = t$$

$$\tanh^{-1}\left(\sqrt{\frac{c}{mg}}\, v_y\right) - \tanh^{-1}\left(\sqrt{\frac{c}{mg}}\, v_{y0}\right) = -\sqrt{\frac{gc}{m}}\, t$$

$$\sqrt{\frac{c}{mg}}\, v_y = \tanh\left[\tanh^{-1}\left(\sqrt{\frac{c}{mg}}\, v_{y0}\right) - \sqrt{\frac{gc}{m}}\, t\right]$$

$$v_y = \sqrt{\frac{mg}{c}} \tanh\left[\tanh^{-1}\left(\sqrt{\frac{c}{mg}}\, v_{y0}\right) - \sqrt{\frac{gc}{m}}\, t\right]$$

If $v_{y0} = 0$,

$$\boxed{v_y = \sqrt{\frac{mg}{c}} \tanh\left(-\sqrt{\frac{gc}{m}}\, t\right)}$$