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Lecture 12, 2/16/26

Last week:

Monday:
- reviewed the rocket problem
- calculated the CM of a cone using $\vec{R} = \frac{1}{M}\int \vec{r}\, dm$, $\rho = \frac{M}{V}$

$$\Rightarrow \vec{R} = \frac{1}{M}\int \vec{r} \rho\, dV = \frac{1}{M}\int (x\hat{x} + y\hat{y} + z\hat{z})\rho\, dx\, dy\, dz$$

in Cartesian coordinates

- Introduced angular momentum $\vec{\ell} = \vec{r} \times \vec{p}$

  & showed $\dot{\vec{\ell}} = \vec{r} \times \dot{\vec{p}} = \vec{\Gamma}$ = torque

- Only central forces $\Rightarrow \dot{\vec{\ell}} = 0$
- Showed Kepler's 2nd law results from $\vec{\ell}$ = constant

Wednesday:
- Showed for a system of particles $\dot{\vec{L}} = \sum_{\beta<\alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} + \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$

  If internal forces are all central, $\dot{\vec{L}} = \vec{r}_\alpha \times \vec{F}_\alpha^{ext} = \vec{\Gamma}^{ext}$

- Moment of inertial $L_z = I\omega$, $I = m_\alpha r_\alpha^2 = \int r^2 dm = \int r^2 \rho\, dV$ for solid object

  where r is distance from $\hat{z}$-axis

- $\dot{\vec{L}} = \vec{\Gamma}^{ext}$ about CM, even when CM is accelerating!

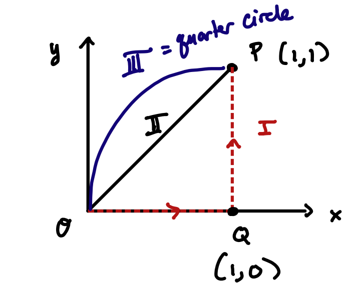
- Started Ch. 4: Energy

  Work-KE theorem: $\Delta T = T_2 - T_1 = \int_1^2 \vec{F} \cdot d\vec{r} \equiv W(1 \to 2)$

- practiced line integrals

Friday: Review Activity for M1

Practice with line integrals: (continued)



$\vec{F} = y\hat{x} + 2x\hat{y}$

$$W_I = \int_I \vec{F} \cdot d\vec{r} = \int_O^Q \vec{F} \cdot d\vec{r} + \int_Q^P \vec{F} \cdot d\vec{r}$$

$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$

We did path I last week. ($W_I = 2$)

Next, calculate W_{II}:

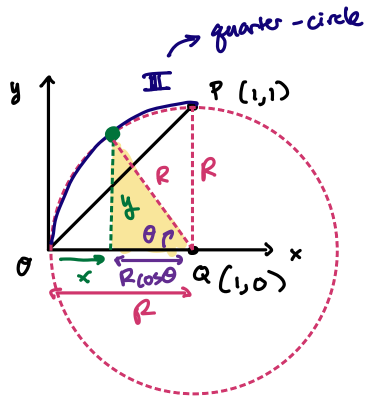
$W_{II} = \int_{II} \vec{F}\cdot d\vec{r}$ on path II: $x=y \Rightarrow dx=dy$

$$W_{II} = \int_{I} \vec{F}\cdot(dx\hat{x} + dy\hat{y}) = \int_{II}(y\,dx + 2x\,dy) = \int_{O_x}^{P_x}(x\,dx + 2x\,dx) = \int_0^1 3x\,dx = \frac{3}{2}x^2\Big|_0^1 = \boxed{\frac{3}{2} = W_{II}}$$

use $x=y$

$dx=dy$

W_{III}: On path III,



$$W_{III} = \int_{III} \vec{F}\cdot d\vec{r}$$

$$= \int_{III} F_x\,dx + F_y\,dy$$

$$= \int_{III} y\,dx + 2x\,dy$$

$\Rightarrow y = R\sin\theta$ $R=1$

$\boxed{\begin{aligned} y &= \sin\theta \\ dy &= \cos\theta\,d\theta \end{aligned}}$

$\Rightarrow x + R\cos\theta = R$ $R=1$

$\boxed{x + \cos\theta = 1}$

$dx - \sin\theta\,d\theta = 0$

$\boxed{dx = \sin\theta\,d\theta}$

At point P, $\theta = \pi/2$

At θ, $\theta = 0$

$$= \int_{\theta_O}^{P_\theta}\left[(\sin\theta)(\sin\theta\,d\theta) + 2(1-\cos\theta)(\cos\theta\,d\theta)\right]$$

$$= \int_0^{\pi/2}(\sin^2\theta - 2\cos\theta - 2\cos^2\theta)\,d\theta = 2\int_0^{\pi/2}\cos\theta\,d\theta + \int_0^{\pi/2}\left[(\sin^2\theta - \cos^2\theta) - \cos^2\theta\right]d\theta$$

$$= 2\sin\theta\Big|_0^{\pi/2} + \int_0^{\pi/2}\left[(-\cos 2\theta) - \tfrac{1}{2}(\cos 2\theta + 1)\right]d\theta$$

$$= 2\sin\pi/2 - 0 + \int_0^{\pi/2}\left(-\tfrac{3}{2}\cos 2\theta - \tfrac{1}{2}\right)d\theta$$

$$= 2(1) - \frac{3}{2}\cdot\frac{1}{2}\sin 2\theta\Big|_0^{\pi/2} - \frac{1}{2}\theta\Big|_0^{\pi/2}$$

$$= 2 - 3/4\left(\cancel{\sin\pi}^{\,0} - 0\right) - \pi/4 + 0$$

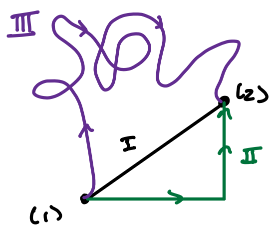
$$\boxed{W_{III} = 2 - \pi/4}$$

Potential Energy and Conservative forces

Conservative forces: $\vec{F}$ is conservative iff:

(i) $\vec{F}$ depends on the particle's position $\vec{r}$ only

(ii) For any points $\vec{r}_1$, and $\vec{r}_2$, $W(\vec{r}_1 \to \vec{r}_2)$ done by $\vec{F}$ is the same for all paths between $\vec{r}_1$ and $\vec{r}_2$



$W_I = W_{II} = W_{III} = \ldots$

Example: charge $+q$ in an electric field $\vec{E} = E\hat{x}$

Let's find the work between any two points P_1 & P_2.

$$W(1\to 2) = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_2} q\,\vec{E}\cdot d\vec{r} = \int_{P_1} q E\,\hat{x}\cdot(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= qE\int_{P_1}^{P_2} dx = qE(x_2 - x_1)$$ $\Rightarrow$ only depends on end points! Satisfies (ii)

Forces we've seen so far:

Conservative	non conservative
$\vec{F}_G$, $\vec{F}_g$, $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$	$\vec{F}_d$, $\vec{F}_f$
spring force $\vec{F} = -k\vec{x}$	

Why conservative forces?

Mechanical Energy: $E = KE + PE = T + \underbrace{U(\vec{r})}_{\text{Let's define potential energy}}$

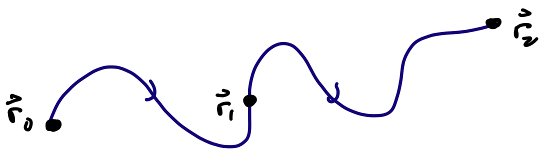
Choose a point $\vec{r}_0$ s.t. $U(\vec{r}_0) = 0$. Then:

$$\underbrace{U(\vec{r}) = -W(\vec{r}_0 \to \vec{r})}_{\text{definition}} \equiv -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

$U(\vec{r})$ is only well-defined for conservative forces

One can go backwards and find $W(\vec{r}_1 \to \vec{r}_2)$ using $U(\vec{r})$:

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2)$$



$$W(\vec{r}_1 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_2) - W(\vec{r}_0 \to \vec{r}_1)$$

$$= -U(\vec{r}_2) + U(\vec{r}_1) = -\Delta U$$

$$\boxed{W(\vec{r}_1 \to \vec{r}_2) = -\Delta U}$$

Remember: $\Delta T = W(\vec{r}_1 \to \vec{r}_2) \Rightarrow \Delta T = -\Delta U$

$\Delta(T+U) = 0$

$\Delta E = 0 \Rightarrow$ energy is conserved!

If $\exists$ several forces the total work just adds:

$$W = \int (\vec{F}_1 + \vec{F}_2 + \dots) \cdot d\vec{r} = \int \vec{F}_1 \cdot d\vec{r} + \int \vec{F}_2 \cdot d\vec{r} + \dots = W_1 + W_2 + \dots = -\Delta U_1 - \Delta U_2 - \dots$$

<u>Principle of conservation of Energy for one particle:</u>

If all N forces acting on a particle are conservative, each with corresponding $U_i(\vec{r})$, then the total <u>mechanical energy</u> is:

$$E = T + U = T + U_1(\vec{r}) + U_2(\vec{r}) + \dots + U_N(\vec{r})$$ is conserved (E is constant in time)

What about nonconservative forces (friction, drag, etc)?

$$\Delta T = W = W_{cons} + W_{NC}$$

$$= -\Delta U + W_{NC}$$

$$\Rightarrow \underbrace{\Delta T + \Delta U}_{\text{change in mechanical energy of the system}} = \underbrace{W_{NC}}_{\text{energy flowing in or out of the system}}$$ = work done by non-conservative (NC) forces

Example: You apply a constant force $\vec{F}$ on a box and move it a distance D across the carpet with constant speed v.

- How much work did you do on the block?
- How much work did the carpet's friction do on the block?

$$\Delta E = \Delta T + \Delta U = W = W_{\text{you-on-block}} + W_{\text{friction}}$$

$$W_{\text{you-on-block}} = \int \vec{F}\cdot d\vec{r} = \int F\,dr = F\int dr = FD$$

$$\Delta T = \tfrac{1}{2}m\,\Delta(v^2) = \tfrac{1}{2}m v_f^2 - \tfrac{1}{2}m v_i^2 = 0 \qquad (v = \text{constant})$$

$\Delta U = 0$ (no conservative forces did any work)

$$\Delta E = \Delta T + \Delta U = 0 = W_{\text{you-on-block}} + W_f$$

$$0 = FD + W_f$$

$$\boxed{-FD = W_f}$$

This makes sense. $\vec{F}_f = -\vec{F}$ since the block is not accelerating $\left(\sum\vec{F} = \vec{F} + \vec{F}_f = 0\right)$

$$W_{\text{total}} = \int (\vec{F} + \vec{F}_f)\cdot d\vec{r} = 0$$

<u>Force as the Gradient of PE</u>

Consider a particle acted on by $\vec{F}(\vec{r})$ (conservative) with PE $U(\vec{r})$. Let's look at how much work is done by $\vec{F}$ in a small displacement: $\vec{r} \to \vec{r} + d\vec{r}$

$$W(\vec{r} \to \vec{r} + d\vec{r}) = \vec{F}(\vec{r})\cdot d\vec{r}$$

$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$$= F_x\,dx + F_y\,dy + F_z\,dz$$

Also:

$$W(\vec{r} \to \vec{r} + d\vec{r}) = -\left[U(\vec{r} + d\vec{r}) - U(\vec{r})\right] \equiv dU$$

$$= -\left[U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)\right]$$

Note: for $f(x)$ of one variable, one can write

$$df = f(x+dx) - f(x) = \frac{df}{dx}\,dx$$

For a fn of 3 variables, we have instead:

$$dU = U(x+dx,\, y+dy,\, z+dz) - U(x,y,z)$$

$$= \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz$$

partial derivatives