Activity 5

1. Consider the potential energy of a particle in a 1D system shown in Fig. 1.

   (a) Draw the equilibrium points on the graph. Label each as stable or unstable.

   (b) For points A, B, and C shown on the plot, draw the direction the force $\vec{F}$ points when the particle is at the corresponding position.

   (c) Rank the forces the particle experiences at point A, B, and C from largest to smallest in magnitude.

   $F_B > F_A > F_c$

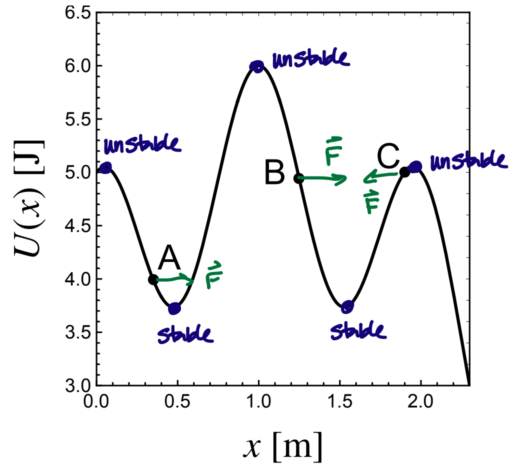


Figure 1: The path between Point A and Point B.

2. Consider the potential energy of a pair of atoms in a molecular bond as shown in Fig. 2. The x-axis displays r, the separation between the atoms. This plot is normalized to ϵ, the binding energy, and σ, the average bond length, so that the axes are dimensionless.

   (a) Draw any/all equilibrium points on the graph. Label each as stable or unstable.

   (b) If the total mechanical energy $E = -0.5\epsilon$, at what values of r does the kinetic energy, T, equal 0? These are called "turning points."

   $E_T = T + u(r)$

   When $T=0$, $E_T = u(r)$ $\Rightarrow$ $r = 1.03\ \sigma$

   $r = 1.37\ \sigma$

   (c) If the total mechanical energy $E = +0.5\epsilon$, what are the turning points?

   $r = 0.97\ \sigma$

   (d) At what value of E does the radius of separation between the atoms become unbounded?

   $E_T = 0$

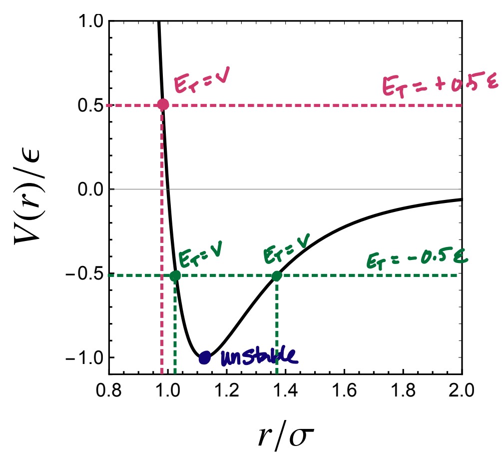


Figure 2: The path between Point A and Point B.

Activity 5

3. For 1D systems only subject to conservative forces, show that the following equation follows from conservation of mechanical energy:

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)} \tag{1}$$

$$T + U(x) = E$$

$$\frac{1}{2}m\dot{x}^2 = E - U(x)$$

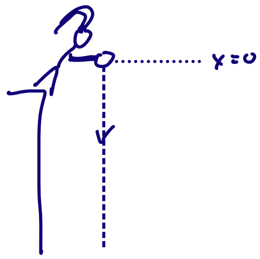
$$\dot{x}^2 = \frac{2}{m}(E - U(x))$$

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

4. Consider a ball dropped off a cliff. Let x be the vertical axis, and $x = 0$ the position of the ball when it is dropped. Using Eq. (1) to find $x(t)$ for the ball.

5. Find $\vec{\nabla} \times \vec{F}$ for the following forces. Are they conservative? If so, find $U(\vec{r})$. Draw the path you used to evaluate $U(\vec{r})$, and check that $\vec{\nabla}U = -\vec{F}$. [Note: Useful integral: $\int xe^x dx = (x-1)e^x + C$]

   (a) $F = x^2\hat{x} + 3y\hat{y}$
   (b) $F = y^2x\hat{x} + x^2y\hat{y}$
   (c) $F = xe^{xy}\hat{x} - ye^{xy}\hat{y}$
   (d) $F = \cancel{xe^{xy}\hat{x} + ye^{xy}\hat{y}}$ typo: $F = ye^{xy}\hat{x} + xe^{xy}\hat{y}$



$U = mgx$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - mgx}$$

ball falls $\downarrow$

$\frac{dx}{dt} < 0 \Rightarrow$ choose $-$

$$\frac{dx}{dt} = -\sqrt{\frac{2}{m}}\sqrt{E - mgx}$$

$$\int_0^x \frac{dx'}{\sqrt{E - mgx'}} = \int_0^t -\sqrt{\frac{2}{m}}\, dt'$$

$u = E - mgx$

$du = -mg\,dx$

$$\frac{1}{-mg}\int_{u_1}^{u_2} \frac{du}{\sqrt{u}} = -\frac{1}{mg}\, 2u^{1/2}\Big|_{u_1}^{u_2} = \frac{-2}{mg}\sqrt{E - mgx'}\Big|_0^x = -\sqrt{\frac{2}{m}}\, t'\Big|_0^t$$

$$-\frac{2}{mg}\sqrt{E - mgx} + \frac{2}{mg}\sqrt{E} = \sqrt{\frac{2}{m}}\, t$$

At $t=0$, $x=0$ & $v=0$ $\Rightarrow$ $E = T + U = 0 + mg(0)$
$= 0$

$E = 0$

$$-\frac{2}{\cancel{m}g}\sqrt{-mgx} = \sqrt{\frac{2}{\cancel{m}}}\, t$$

$$-4gx = 2t^2$$

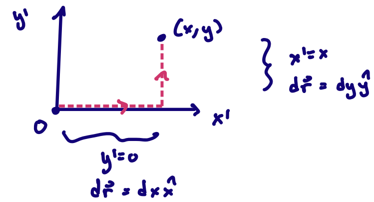
$$\boxed{x = -\frac{1}{2}gt^2}$$

(typo: a more interesting problem is $\vec{F} = y e^{xy}\hat{x} + x e^{xy}\hat{y}$)

a) $\vec{\nabla} \times \vec{F} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k \hat{x}_i = \varepsilon_{312} \frac{\partial}{\partial x} F_y \hat{z} + \varepsilon_{321} \frac{\partial}{\partial y} F_x \hat{z} = (+1)(0) + (-1)(0) = 0 \Rightarrow$ Conservative

$U(\vec{r}) = -\int_0^{\vec{r}} \vec{F} \cdot d\vec{r} = -\int_0^{\vec{r}} x^2 dx + 3y\,dy$

$U(x,y,z) = -\int_0^{(x,y,z)} (x')^2 dx' + 3y'dy'$ $\Rightarrow$ Choose a path to evaluate this



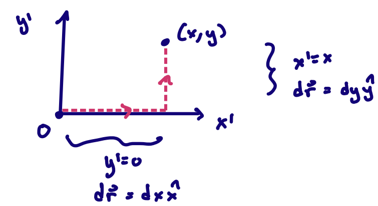
$U(x,y,z) = -\int_0^x (x')^2 dx' - \int_0^y 3y'dy' = -\frac{x^3}{3} - \frac{3y^2}{2}$

$\vec{\nabla} U = -x^2 \hat{x} - 3y\hat{y} = -\vec{F}$ ☑

b) $\vec{\nabla} \times \vec{F} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k \hat{x}_i = \varepsilon_{312} \frac{\partial}{\partial x} F_y \hat{z} + \varepsilon_{321} \frac{\partial}{\partial y} F_x \hat{z} = (+1) \frac{\partial}{\partial x} x^2 y\,\hat{z} + (-1) \frac{\partial}{\partial y} y^2 x\,\hat{z} = 2xy\,\hat{z} - 2yx\,\hat{z} = 0 \Rightarrow$ conservative

$U(\vec{r}) = -\int_0^{\vec{r}} \vec{F} \cdot d\vec{r} = -\int_0^{\vec{r}} y^2 x\,dx + x^2 y\,dy$

$U(x,y,z) = -\int_0^{(x,y,z)} (y')^2 x' dx' + (x')^2 y' dy'$ $\Rightarrow$ Choose a path to evaluate this



$U(x,y,z) = -\int_0^x \cancel{(0)\,x'dx'}^{\,0} - \int_0^y x^2 y' dy'$

constant, pull out of integral

$= -x^2 \left. \frac{(y')^2}{2} \right|_0^y = -\frac{x^2 y^2}{2}$

$\vec{\nabla} U = -xy^2 \hat{x} - x^2 y\,\hat{y} = -\vec{F}$ ☑

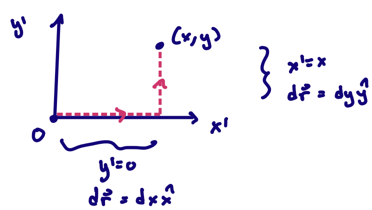
c) $\vec{\nabla} \times \vec{F} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k \hat{x}_i = \varepsilon_{312} \frac{\partial}{\partial x} F_y \hat{z} + \varepsilon_{321} \frac{\partial}{\partial y} F_x \hat{z} = (+1)(-y^2 e^{xy})\hat{z} + (-1) x^2 e^{xy} \hat{z} = -(x^2+y^2) e^{xy} \hat{z} \neq 0$

not conservative

d) With typo fixed: $\vec{F} = y e^{xy} \hat{x} + x e^{xy} \hat{y}$

$\vec{\nabla} \times \vec{F} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k \hat{x}_i = \varepsilon_{312} \frac{\partial}{\partial x} F_y \hat{z} + \varepsilon_{321} \frac{\partial}{\partial y} F_x \hat{z} = (+1)(xy\,e^{xy})\hat{z} + (-1) xy\,e^{xy} \hat{z} = 0 \Rightarrow$ conservative

$U(\vec{r}) = -\int_0^{\vec{r}} \vec{F} \cdot d\vec{r} = -\int_0^{\vec{r}} y e^{xy} dx + x e^{xy} dy$



$U(x,y) = -\int_0^x (0)\,e^{(0)} dx' - \int_0^y x\,e^{xy'} dy'$

$= 0 - x \int_0^y e^{xy'} dy' = -x \left. \frac{1}{x} e^{xy'} \right|_0^y$

$= -e^{xy} + 1 = 1 - e^{xy}$

$\vec{\nabla} U = -y e^{xy} \hat{x} - x e^{xy} \hat{y} = -\vec{F}$ ☑