Grade:

Physics 3221 Spring 2026: Midterm 1

1. A Falling Rocket (20pts)

Consider a rocket suspended in air, see Fig. 1. At time $t = 0$, the rocket has an initial speed $v_0 = 0$ and initial mass m_0. The rocket begins ejecting fuel from its tank at a constant speed v_{ex} relative to the rocket's speed, decreasing the mass $m(t)$ of the rocket over time with a rate $\dot{m} = -k$, where k is constant. The rocket is subject to the gravitational force $\vec{F}_g = -mg\hat{y}$.

(a) *(2 pts)* Find $m(t)$.

(b) *(4 pts)* Find $\frac{dP}{dt}$ for the rocket, including the rocket's fuel.

Hint: $dP = (P + dP) - P$ *and* $P = p_{rocket} + p_{fuel}$.

(c) *(5 pts)* Using $\frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}}$, write a differential equation for the rocket's velocity in the y-direction. Put your final differential equation in terms of v, t, v_{ex}, g, m_0, and k.

(Note: you can use the variable $v \equiv v_y$ *as all motion occurs only in the* y*-direction.)*

(d) *(5 pts)* Find $v(t)$ in terms of v_{ex}, m_0, k, and g.

(e) *(2 pts)* Assume v_{ex} is small enough that the rocket is falling right after it is released from rest. Find the t at which the rocket's acceleration becomes zero.

(f) *(2 pts)* Find the maximum v_{ex} such that the rocket is falling right after it is released from rest. *(Hint: Find* v_{ex} *such that the rocket's acceleration is never zero for* $t > 0$*).*

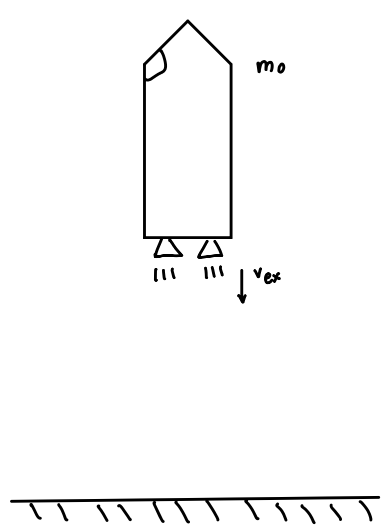

Figure 1: The falling rocket

① a) $\dot{m} = -k$

$\frac{dm}{dt} = -k$

$dm = -k\,dt$

$\int_{m_0}^{m} dm' = \int_0^t -k\,dt'$

$m'\Big|_{m_0}^{m} = -kt'\Big|_0^t$

$m - m_0 = -kt$

$\boxed{m = m_0 - kt}$

b) $P = mv$

$P + dP = \underbrace{(m+dm)(v+dv)}_{\text{rocket}} + \underbrace{(-dm)(v+dv-v_{ex})}_{\text{fuel}}$

$= mv + mdv + \cancel{vdm} + \cancel{dmdv} - \cancel{vdm} - \cancel{dvdm} + v_{ex}\,dm$

$P + dP = mv + mdv + v_{ex}\,dm$

$dP = \underbrace{(\cancel{mv} + m\,dv + v_{ex}\,dm)}_{P+dP} - \underbrace{(\cancel{mv})}_{P}$

$dP = m\,dv + v_{ex}\,dm$

$\boxed{\frac{dP}{dt} = m\frac{dv}{dt} + v_{ex}\frac{dm}{dt}}$

c) $\frac{d\vec{P}}{dt} = \vec{F}_{ext} = -mg\hat{y}$ Both point in y-direction

$\Rightarrow \frac{dP}{dt} = -mg$

$m\frac{dv}{dt} + v_{ex}\frac{dm}{dt} = -mg$

$\frac{dv}{dt} = -g - \frac{v_{ex}}{m}\frac{dm}{dt}$ use $\frac{dm}{dt} = -k$, $m = m_0 - kt$

$\frac{dv}{dt} = -g - \frac{v_{ex}}{m_0 - kt}(-k)$

$\boxed{\frac{dv}{dt} = -g + \frac{v_{ex}k}{m_0 - kt}}$

d) $\int_{v_0}^{v} dv' = \int_0^t -g\,dt' + \int_0^t \frac{v_{ex}k\,dt'}{m_0 - kt'}$

$v - v_0 = -gt'\Big|_0^t - v_{ex}\int_0^t \frac{(-k\,dt') \equiv du}{m_0 - kt' \equiv u} = -gt - v_{ex}\ln(m_0 - kt')\Big|_0^t = -gt - v_{ex}\ln\left(\frac{m_0 - kt}{m_0}\right)$

$\boxed{v(t) = v_0 - gt - v_{ex}\ln\left(1 - \frac{k}{m_0}t\right)}$

e)

$$\frac{dv}{dt} = -g + \frac{v_{ex} k}{m_0 - kt}$$

$$0 = -g + \frac{v_{ex} k}{m_0 - kt}$$

$$g(m_0 - kt) = v_{ex} k$$

$$m_0 - kt = \frac{v_{ex} k}{g}$$

$$m_0 - \frac{v_{ex} k}{g} = kt$$

$$\boxed{\frac{m_0}{k} - \frac{v_{ex}}{g} = t}$$ when $a=0$

f)

$$\frac{m_0}{k} - \frac{v_{ex}}{g} = t(a=0) \geq 0$$

$$\frac{m_0}{k} \geq \frac{v_{ex}}{g}$$

$$\boxed{\frac{m_0 g}{k} \geq v_{ex}}$$

$$\frac{m_0 g}{k} = v_{ex}^{max}$$

Grade:

2. Drag on a Train (20pts)

Consider a train on frictionless tracks, but subject to a drag force due to air resistance, see Fig. 2. The train has a mass m and an initial speed v_0. The drag force is given by $\vec{F} = -b\vec{v}$, where b is the drag coefficient and has units kg/s. There are no other forces acting on the train.

(a) *(6 points)* Find the velocity $v(t)$ in terms of v_0, m, b, and t.

(b) *(6 points)* Find the position $x(t)$ in terms of v_0, m, b, and t. Assume $x(0) = 0$.

(c) *(4 points)* Find the work done on the train when it travels a distance d from the starting position. Leave your answer in terms of v_0, m, b, and d.

(d) *(4 points)* Find the change in kinetic energy of the train from time $t = 0$ to time t_d, where $x(t_d) = d$. Leave your answer in terms of v_0, m, b, and d.

(e) *(4 points)* The train car is filled with various materials, such that the train's density is given by

$$\rho(x, y, z) = \alpha x(h - z)\,,$$

where the x-axis points in the direction of the initial velocity, and the z-axis points upwards away from the ground. The point $x = y = 0$ is the bottom, left, front corner of the train car. The train has a length l in the x-direction, width w in the y-direction pointing into the page, and height h in the z-direction. Find the center of mass of the train. Remember that $dm = \rho(x, y, z)dV$, which in Cartesian coordinates is $dm = \rho(x, y, z)dxdydz$.

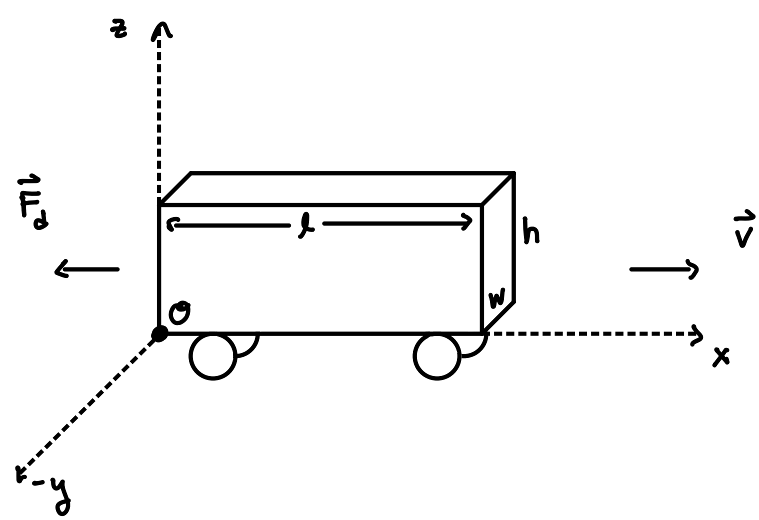

Figure 2: A train subject to a drag force

② a) $\vec{F} = m\vec{a}$

$-b\vec{v} = m\frac{d\vec{v}}{dt}$ all in $\hat{x}$-direction

$-bv = m\frac{dv}{dt}$

$-\frac{b}{m}\,dt = \frac{dv}{v}$ let $b/m \equiv k$

$\int_0^t -k\,dt' = \int_{v_0}^{v} \frac{dv'}{v'}$

$-kt'\Big|_0^t = \ln(v')\Big|_{v_0}^{v}$

$-kt = \ln\left(\frac{v}{v_0}\right)$

$e^{-kt} = \frac{v}{v_0}$

$v = v_0 e^{-kt}$

$$\boxed{v = v_0 e^{-\frac{b}{m}t}}$$

b) $\frac{dx}{dt} = v_0 e^{-kt}$

$\int_{x_0=0}^{x} dx' = \int_0^t v_0 e^{-kt'}\,dt'$

$x'\Big|_0^x = v_0\left(-\frac{1}{k}e^{-kt'}\right)\Big|_0^t$

$x = -\frac{v_0}{k}\left(e^{-kt}-1\right)$

$x = \frac{v_0}{k}\left(1-e^{-kt}\right)$

$$\boxed{x(t) = \frac{mv_0}{b}\left(1-e^{-\frac{bt}{m}}\right)}$$

c) $W = \int_{\text{initial}}^{\text{final}} \vec{F}\cdot d\vec{x} = \int_0^d -bv\,dx = -b\int_0^d v_0 e^{-kt}\,dx$

we need t in terms of x:

$\frac{kx}{v_0} = 1-e^{-kt}$

$e^{-kt} = 1-\frac{kx}{v_0}$

$W = -b\int_0^d v_0\left(1-\frac{kx}{v_0}\right)dx$

$= -bv_0\left[x - \frac{1}{2}\frac{k}{v_0}x^2\right]_0^d$

$W = -bv_0\left[d - \frac{kd^2}{2v_0}\right] = bd\left[\frac{1}{2}kd - v_0\right]$

$$\boxed{W = bd\left[\frac{bd}{2m} - v_0\right]}$$

d) $W = \Delta KE = bd\left(\frac{bd}{2m} - v_0\right)$ Also you can use:

$\Delta KE = KE_f - KE_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$

$= \frac{1}{2}mv_0^2 e^{-2kt} - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2\left(e^{-2kt}-1\right)$

$= \frac{1}{2}mv_0^2\left[\left(1-\frac{kx}{v_0}\right)^2 - 1\right]_{x=d} = \frac{1}{2}mv_0^2\left[\cancel{1} - \frac{2kd}{v_0} + \frac{k^2d^2}{v_0^2} - \cancel{1}\right] = \frac{1}{2}mkd\left[-2v_0 + kd\right]$

$= bd\left[-v_0 + \frac{1}{2}\frac{bd}{m}\right] = \boxed{bd\left[\frac{bd}{2m} - v_0\right] = \Delta KE}$

e) $\vec{R}_{cm} = \frac{1}{m}\int \vec{r}\,\rho\, dx\,dy\,dz \qquad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$= \frac{1}{m}\int_0^h\int_0^w\int_0^l (x\hat{x} + y\hat{y} + z\hat{z})\,\alpha x\,(h-z)\,dx\,dy\,dz$$

$$= \frac{\alpha}{m}\int_0^h\int_0^w\int_0^l \left[x^2(h-z)\hat{x} + xy(h-z)\hat{y} + xz(h-z)\hat{z} \right] dx\,dy\,dz$$

$$= \frac{\alpha}{m}\int_0^h\int_0^w \left[\frac{x^3}{3}(h-z)\hat{x} + \frac{x^2}{2}y(h-z)\hat{y} + \frac{x^2}{2}z(h-z)\hat{z} \right]_0^l dy\,dz$$

$$= \frac{\alpha}{m}\int_0^h\int_0^w \left[\frac{l^3}{3}(h-z)\hat{x} + \frac{l^2}{2}y(h-z)\hat{y} + \frac{l^2}{2}z(h-z)\hat{z} \right] dy\,dz$$

$$= \frac{\alpha l^2}{m}\int_0^h \left[\frac{l}{3}(h-z)y\hat{x} + \frac{1}{2}(h-z)\frac{y^2}{2}\hat{y} + \frac{1}{2}z(h-z)y\hat{z} \right]_0^w dz$$

$$= \frac{\alpha l^2}{m}\int_0^h \left[\frac{lw}{3}(h-z)\hat{x} + \frac{w^2}{4}(h-z)\hat{y} + \frac{w}{2}(hz-z^2)\hat{z} \right] dz$$

$$= \frac{\alpha l^2 w}{m}\left[\frac{l}{3}\left(hz - \frac{1}{2}z^2\right)\hat{x} + \frac{w}{4}\left(hz - \frac{1}{2}z^2\right)\hat{y} + \frac{1}{2}\left(\frac{1}{2}hz^2 - \frac{1}{3}z^3\right)\hat{z} \right]_0^h$$

$$= \frac{\alpha l^2 w}{m}\left[\frac{l}{3}\left(h^2 - \frac{1}{2}h^2\right)\hat{x} + \frac{w}{4}\left(h^2 - \frac{1}{2}h^2\right)\hat{y} + \frac{1}{2}\left(\frac{1}{2}h^3 - \frac{1}{3}h^3\right)\hat{z} \right]$$

$$= \frac{\alpha l^2 w h^2}{m}\left[\frac{l}{6}\hat{x} + \frac{w}{8}\hat{y} + \frac{1}{2}\left(\frac{1}{6}h\right)\hat{z} \right]$$

$$\boxed{\vec{R}_{cm} = \frac{\alpha l^2 w h^2}{2m}\left(\frac{l}{3}\hat{x} + \frac{w}{4}\hat{y} + \frac{h}{6}\hat{z} \right)}$$

Grade:

3. Charged Puck (20pts)

A charged puck of mass m and charge $+q$ is confined to move between two concentric cylinders, so that its center of mass is always a distance R away from the axis of the cylinder, see Fig. 3. The charge experiences a magnetic field $\vec{B}$. Ignore gravity.

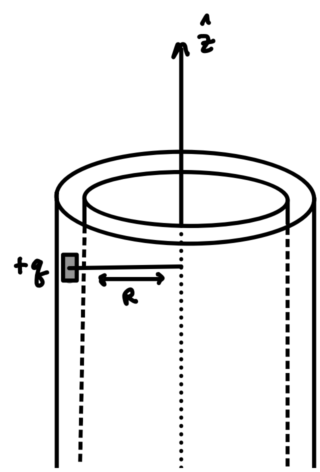

Figure 3: A puck confined to move between two concentric cylinders such that its distance from the z-axis is fixed to be $\rho = R$.

(a) *(6 points)* Using the Lorentz Force law $\vec{F} = q\vec{v} \times \vec{B}$, write down a set of differential equations that describe the motion of the puck in cylindrical polar coordinates. In this coordinate system:

$$\vec{v} = v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{z} = v_\rho \hat{\rho} + \rho \dot{\phi} \hat{\phi} + v_z \hat{z}$$
$$\vec{B} = B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z}$$

Note that ρ in cylindrical coordinates is analogous to r in polar coordinates.

(b) *(6 points)* Simplify the differential equations found in part (a) using the fact that $\rho = R =$ constant in cylindrical coordinates. Write these as a system of coupled first order differential equations for $v_z = \dot{z}$ and $v_\phi = R\dot{\phi}$.

(c) *(3 points)* Which component(s) of $\vec{B}$ impact the motion of the puck?

(d) *(5 points)* What is the torque on the puck about the axis of the cylinder? Leave your answer in terms of B_ρ, R, q, v_z, and v_ϕ. Is the angular momentum of the puck conserved?

③ a) $\vec{F} = q\vec{v}\times\vec{B}$

$$\vec{v}\times\vec{B} = \left(v_\rho \hat{\rho} + v_\phi \hat{\phi} + v_z \hat{z}\right)\times\left(B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z}\right)$$

$$= B_\phi v_\rho \underbrace{(\hat{\rho}\times\hat{\phi})}_{\hat{z}} + B_z v_\rho \underbrace{(\hat{\rho}\times\hat{z})}_{-\hat{\phi}} + B_\rho v_\phi \underbrace{(\hat{\phi}\times\hat{\rho})}_{-\hat{z}} + B_z v_\phi \underbrace{(\hat{\phi}\times\hat{z})}_{\hat{\rho}} + B_\rho v_z \underbrace{(\hat{z}\times\hat{\rho})}_{\hat{\phi}} + B_\phi v_z \underbrace{(\hat{z}\times\hat{\phi})}_{-\hat{\rho}}$$

$$= (B_\phi v_\rho - B_\rho v_\phi)\hat{z} + (B_\rho v_z - B_z v_\rho)\hat{\phi} + (B_z v_\phi - B_\phi v_z)\hat{\rho}$$

$$= (B_\phi v_\rho - B_\rho \rho\dot{\phi})\hat{z} + (B_\rho v_z - B_z v_\rho)\hat{\phi} + (B_z \rho\dot{\phi} - B_\phi v_z)\hat{\rho}$$

$\vec{F} = m\vec{a} \Rightarrow \quad q(\vec{v}\times\vec{B}) + \vec{F}_N = m\vec{a} \qquad \vec{F}_N = F_N\hat{\rho}$

$$m\vec{a} = m(\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + m\ddot{z}\hat{z} = m(\dot{v}_\rho - \rho\dot{\phi}^2)\hat{\rho} + m(\rho\ddot{\phi} + 2v_\rho\dot{\phi})\hat{\phi} + m\dot{v}_z\hat{z}$$

$$\boxed{\begin{aligned} \Rightarrow\ & q(B_\phi v_\rho - B_\rho \rho\dot{\phi}) = m\ddot{z} \\ & q(B_\rho v_z - B_z v_\rho) = m(\rho\ddot{\phi} + 2v_\rho\dot{\phi}) \\ & q(B_z\rho\dot{\phi} - B_\phi v_z) + F_N = m(\dot{v}_\rho - \rho\dot{\phi}^2) \end{aligned}}$$

b) Use $\rho = R = \text{const.}$ & $v_\phi = R\dot{\phi}$

$\Rightarrow \quad v_\rho = \dot{\rho} = 0$

$$q(B_\phi \cancel{v_\rho}^{\,0} - B_\rho\rho\dot{\phi}) = -qB_\rho R\dot{\phi} = \boxed{-qB_\rho v_\phi = m\dot{v}_z}$$

$$q(B_\rho v_z - B_z \cancel{v_\rho}^{\,0}) = qB_\rho v_z = m(R\ddot{\phi} + 2\cancel{v_\rho}^{\,0}\dot{\phi}) = mR\ddot{\phi} = m\dot{v}_\phi$$

$$\Rightarrow \boxed{qB_\rho v_z = m\dot{v}_\phi}$$

$$q(B_z R\dot{\phi} - B_\rho v_z) + F_N = m(\cancel{\dot{v}_\rho}^{\,0} - R\dot{\phi}^2) = mR\dot{\phi}^2$$

$$q(B_z v_\phi - B_\phi v_z) + F_N = \frac{m v_\phi^2}{R}$$

$\Rightarrow$ extra equation w/ extra unknown (F_N)
we know $\rho = R = \text{const}$, don't need
kinematics in $\hat{\rho}$-direction.

c) Only B_ρ affects the motion of the puck $\rightsquigarrow$ $\left\{ \begin{array}{l} \boxed{-qB_\rho v_\phi = m\dot{v}_z} \\ \boxed{qB_\rho v_z = m\dot{v}_\phi} \end{array} \right.$

d) $\vec{\Gamma} = \vec{r}\times\vec{F}$ about axis of cylinder, $\vec{r} = \rho\hat{\rho} = R\hat{\rho}$

$$\vec{F} = q(\vec{v}\times\vec{B}) + \vec{F}_N$$

$$\vec{F} = q(B_\phi v_\rho - B_\rho \rho\dot{\phi})\hat{z} + q(B_\rho v_z - B_z v_\rho)\hat{\phi} + q(B_z \rho\dot{\phi} - B_\phi v_z)\hat{\rho} + F_N\hat{\rho}$$

$$\vec{F} = F_z\hat{z} + F_\phi\hat{\phi} + F_\rho\hat{\rho}$$

$$\vec{\Gamma} = \vec{r}\times\vec{F} = R\hat{\rho}\times(F_z\hat{z} + F_\phi\hat{\phi} + F_\rho\hat{\rho}) = RF_z(-\hat{\phi}) + RF_\phi(\hat{z}) + 0$$

$$= -RF_z\hat{\phi} + RF_\phi\hat{z}$$

$$\vec{\Gamma} = Rq(B_\phi \cancelto{0}{v_\rho} - B_\rho R\dot{\phi})\hat{\phi} + Rq(B_\rho v_z - B_z\cancelto{0}{v_\rho})\hat{z}$$

$$= -RqB_\rho v_\phi\hat{\phi} + RqB_\rho v_z\hat{z}$$

$$\boxed{\vec{\Gamma} = qRB_\rho(v_z\hat{z} - v_\phi\hat{\phi})}$$

$\boxed{\vec{\ell} \text{ is not conserved } (\vec{\Gamma}\neq 0)}$

Formulas (just for the Midterm)

Newton's 2nd Law

$$\vec{F} = m\ddot{\vec{r}}$$

$$\text{(Cartesian)} \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$$

$$\text{(Polar)} \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases}$$

$$\text{(Cylindrical Polar)} \begin{cases} F_\rho = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{z} \end{cases}$$

Position vector

$$\text{(Cartesian) } \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\text{(Polar) } \vec{r} = r\hat{r}$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Momentum and Force

$$\vec{p} = m\vec{v}$$

$$\frac{d\vec{P}}{dt} = \vec{F}^{\text{ext}}$$

Angular Momentum and Torque

$$\vec{l} = \vec{r} \times \vec{p}$$

$$\dot{\vec{L}} = \vec{\Gamma}^{\text{ext}}$$

$$\vec{\Gamma} = \vec{r} \times \vec{F}$$

$$L = I\omega$$

$$I = \sum_{\alpha}^{N} m_\alpha r_\alpha^2$$

Center of Mass

$$\vec{R}_{\text{CM}} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha$$

$$\vec{R}_{\text{CM}} = \frac{1}{M}\int \vec{r}\,dm$$

Useful Integrals

$$\int \frac{du}{u} = \ln(u) + c$$

Work

$$W = \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$