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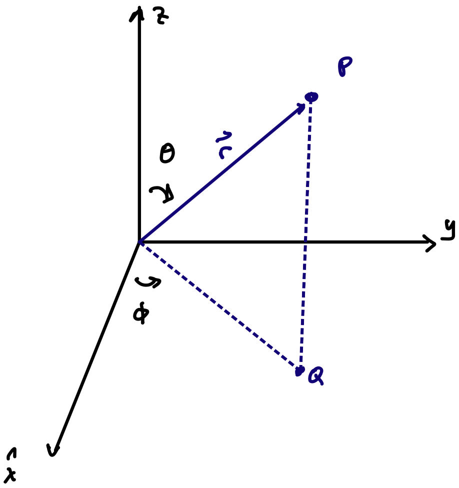
Lecture 16 2/25/26

Claim: Central forces that are spherically symmetric are conservative.

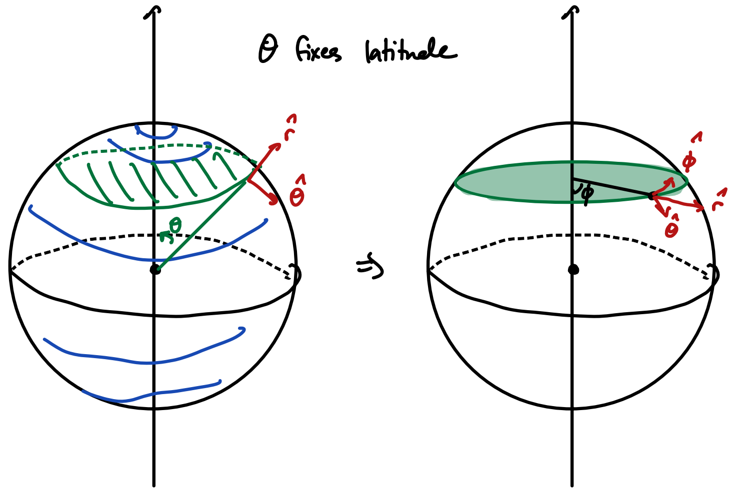
Spherical coordinates:

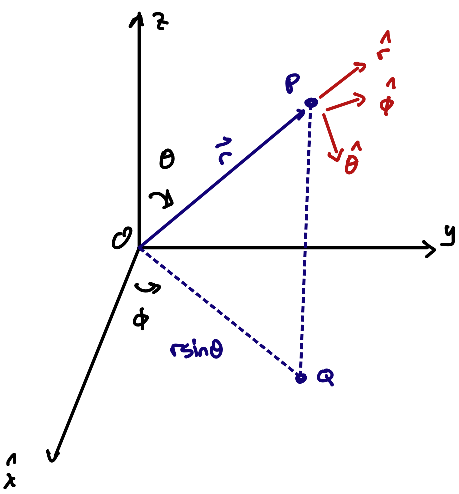


$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

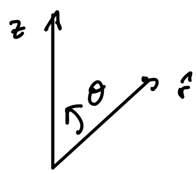
$z = r\cos\theta$





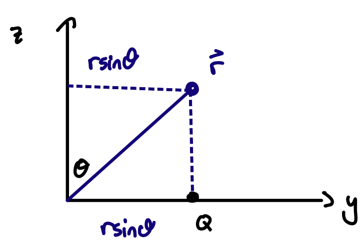
ϕ tells you where on that circle you are (longitude, if you like)

The $\measuredangle$ between $\hat{r}$ and $\hat{z}$ is θ by definition

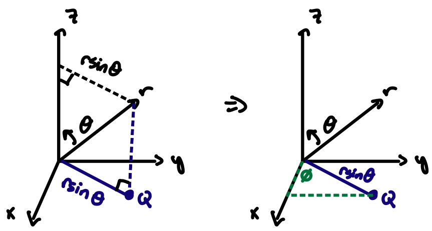


$\Rightarrow$ The projection of $\hat{r}$ onto the x-y plane is given by $r\sin\theta$

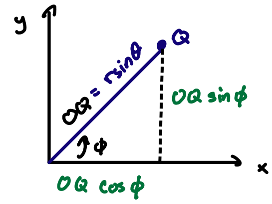
$\Rightarrow OQ = r\sin\theta$ Imagine Q sits on the y-axis:



- Once we're in the xy-plane, ϕ tells you exactly where:



Looking down at xy-plane!



$x = OQ\cos\phi = r\sin\theta\cos\phi$

$y = OQ\sin\phi = r\sin\theta\sin\phi$

- unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ are all $\perp$ to each other

$$\hat{r} \perp \hat{\theta} \perp \hat{\phi}$$

and

$$\vec{a}\cdot\vec{b} = a_r b_r + a_\theta b_\theta + a_\phi b_\phi$$

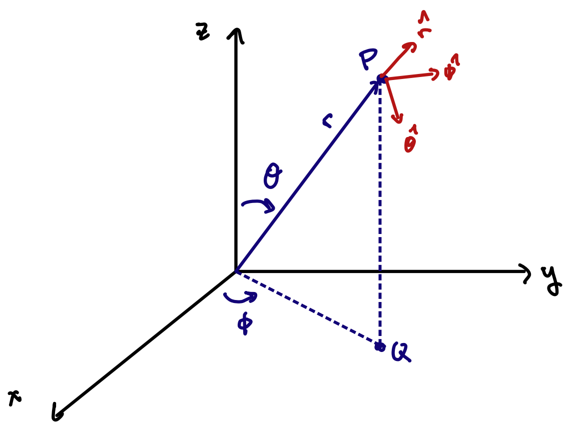
- the gradient in spherical coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} \quad \text{in Cartesian}$$

Recall: $df = \frac{\partial f}{\partial x_i} dx_i$

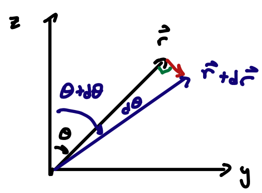
$$df = \vec{\nabla} f \cdot d\vec{r}$$

$\uparrow$ we need $d\vec{r}$ in spherical coordinates $\quad d\vec{r} = (d\vec{r})_r \hat{r} + (d\vec{r})_\theta \hat{\theta} + (d\vec{r})_\phi \hat{\phi}$



- small displacement in the $\hat{r}$-direction $\Rightarrow dr$
  (just changing the length of $\vec{r}$) $\Rightarrow (d\vec{r})_r = dr$

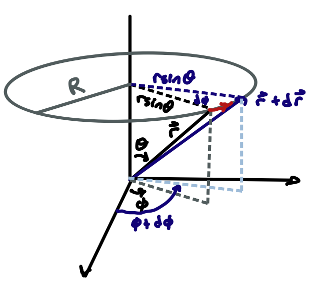
- small displacement in $\hat{\theta}$-direction



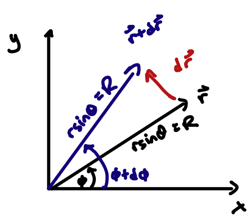
$|d\vec{r}| \approx r\sin d\theta \approx r\,d\theta = (d\vec{r})_\theta = r\,d\theta$

We did this same analysis when we first discussed polar coordinates

Finally, small displacement in $\hat{\phi}$-direction:



view looking at xy-plane



$(d\vec{r})_\phi \approx R\sin(d\phi) \approx R\,d\phi = r\sin\theta\,d\phi$

Putting this all together:

$$d\vec{r} = (d\vec{r})_r \, \hat{r} + (d\vec{r})_\theta \, \hat{\theta} + (d\vec{r})_\phi \, \hat{\phi}$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta \, d\phi \, \hat{\phi}$$ ⇒ useful for integration in spherical coordinates as well

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi$$ (chain rule)

set these equal

$$df = \vec{\nabla} f \cdot d\vec{r} = (\vec{\nabla} f)_r \, dr + (\vec{\nabla} f)_\theta \, r\,d\theta + (\vec{\nabla} f)_\phi \, r\sin\theta \, d\phi$$

$$\frac{\partial f}{\partial r} dr = (\vec{\nabla} f)_r \, dr \quad \Rightarrow \quad (\vec{\nabla} f)_r = \frac{\partial f}{\partial r}$$

$$\frac{\partial f}{\partial \theta} d\theta = (\vec{\nabla} f)_\theta \, r\,d\theta \quad \Rightarrow \quad (\vec{\nabla} f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial \phi} d\phi = (\vec{\nabla} f)_\phi \, r\sin\theta \, d\phi \quad \Rightarrow \quad (\vec{\nabla} f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\Rightarrow \quad \boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

Claim: spherically symmetric central forces are conservative

and vice versa: Central forces that are conservative must be spherically symmetric.

We'll prove the "vice versa" statement

- Let $\vec{F}$ be a conservative, central force $\vec{F} = f(\vec{r})\,\hat{r}$ ⇒ $F_r = f(\vec{r}) = f(r,\theta,\phi)$, $F_\theta = 0$, $F_\phi = 0$
- $\exists\, u(\vec{r})$ s.t. $\vec{F} = -\vec{\nabla} u$ (if $\vec{F}$ is cnstv)
- $$\vec{\nabla} u = \frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\hat{\phi}$$

$$F_r = -\frac{\partial u}{\partial r} \qquad F_\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad F_\phi = -\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = 0 \qquad \Rightarrow \frac{\partial u}{\partial \phi} = 0$$

Conclude: $u(r,\theta,\phi) = u(r)$ only

- $\vec{F} = -\frac{\partial u}{\partial r}\hat{r}$ where $u(r)$ only

  $\Rightarrow \frac{\partial u}{\partial r}$ depends on r only

  $\Rightarrow \vec{F}(r)$ only

$\Rightarrow \vec{F}(r,\theta,\phi) = \vec{F}(r) \Rightarrow$ spherically symmetric ☑

To get the other "not" vice versa direction, just run these steps backwards.

Energy of Interaction of Two Particles

- Consider two particles interacting via $\vec{F}_{12}$ & $\vec{F}_{21}$ only (no external force)

  $\uparrow$

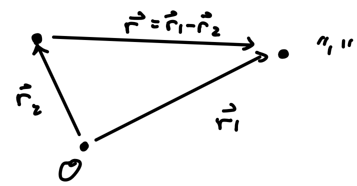
  $\vec{F}$ on "1" by "2"

$\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1, \vec{r}_2)$ depends on position of both particles

$\vec{F}_{12} = -\vec{F}_{21}$ (Newton's 3rd Law)

<u>Example</u>: Gravity: $\vec{F}_{12} = \frac{-Gm_1m_2\hat{r}}{r^2} = \frac{-Gm_1m_2\vec{r}}{r^3}$ where $\vec{r} = \vec{r}_1 - \vec{r}_2$

$= \frac{-Gm_1m_2(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$



The quantity $\vec{r}_1 - \vec{r}_2$ appears because it's <u>translationally invariant</u> (independent of position of origin)

$\Rightarrow$ For translationally invariant systems,

$$\vec{F}_{12} = \vec{F}_{12}(\vec{r}_1 - \vec{r}_2)$$

Let's fix the origin at position "2" s.t. $\vec{r}_2 = 0$:

- Focus on forces acting on particle "1"

For $\vec{F}_{12}$ to be conservative, we require $\vec{\nabla}_1 \times \vec{F}_{12} = 0$ where $\vec{\nabla}_1 = \hat{x}\frac{\partial}{\partial x_1} + \hat{y}\frac{\partial}{\partial y_1} + \hat{z}\frac{\partial}{\partial z_1}$

Define PE: $\vec{F}_{12} = -\vec{\nabla}_1 u(\vec{r}_1)$ if particle 2 sits at origin