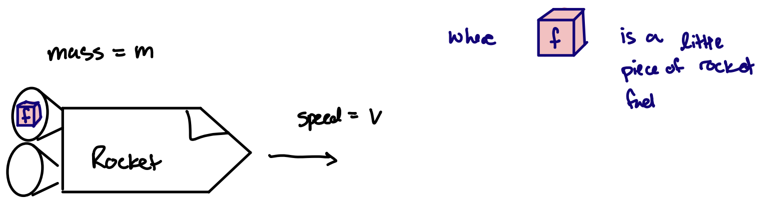
Activity 3

1. Consider a rocket and a fuel system that starts off with mass m and speed v. The rocket ejects fuel with a speed v_{ex} relative to the rocket. **There are no external forces acting on the rocket.**



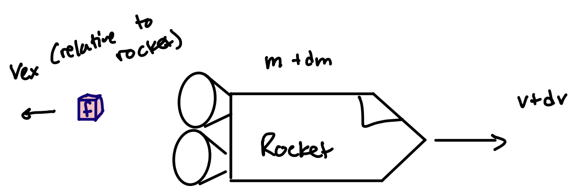
   (a) What is the change in total momentum of the system?

   $\Delta \vec{P}_{total} = 0$

   (b) At time t_i, the rocket begins ejecting fuel. Find $\vec{P}_i$ for the full rocket and fuel system right before any fuel is ejected.

   $\vec{P}_i = m\vec{v}$ let $\vec{v} = v\hat{x}$ $\Rightarrow P_i = mv$

2. Now consider how the system evolves after an infinitesimally small increment of time to $t_f = t_i + dt$.



   The rocket's new velocity is $v + dv$, and its mass is $m + dm$.

   (a) We know $m_{\text{fuel}} + m_{\text{rocket}} = m$. Find m_{fuel}.

   $m_{fuel} + (\cancel{m} + dm) = \cancel{m}$

   $\boxed{m_{fuel} = -dm}$

   (b) Find $\vec{P}_f$ and time t_f for the whole system (fuel + rocket)

$$P_f = m_{fuel} V_{fuel} + M_{rocket} V_{rocket} = (-dm)(v - v_{ex}) + (m+dm)(v+dv) = -v\cancel{dm} + v_{ex}dm + mv + mdv + v\cancel{dm} + O(dt^2)$$

$$\boxed{P_f = mv + mdv + v_{ex}dm}$$

   (c) Use conservation of momentum to find a relationship between m, dm, dv, and v_{ex}.

$$P_i = P_f$$

$$\cancel{m}v = \cancel{m}v + mdv + v_{ex}dm$$

$$\boxed{mdv = -v_{ex}dm}$$

   (d) Integrate this relationship to find v as a function of m. Let $m(t = 0) = m_0$ and $v(t = 0) = v_0$.

$$dv = -v_{ex}\frac{dm}{m}$$

$$\int_{v_0}^{v} dv' = -v_{ex}\int_{m_0}^{m} \frac{dm'}{m'}$$

$$v'\Big|_{v_0}^{v} = -v_{ex}\ln(m')\Big|_{m_0}^{m}$$

$$v - v_0 = -v_{ex}\ln\left(\frac{m}{m_0}\right)$$

$$\boxed{v = v_0 - v_{ex}\ln(m/m_0)}$$

3. The position of the center of mass (CM) of a system of N particles is defined as:

$$\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \tag{1}$$

(a) Find the position of the center of mass of three particles with positions $\vec{r}_1 = (1,1,0)$, $\vec{r}_2 = (1,-1,0)$, $\vec{r}_3 = (0,0,3)$ and masses $m_1 = m_2 = m$ and $m_3 = 10m$.

$$\vec{r} = \frac{m(\hat{x}+\hat{y}) + m(\hat{x}-\hat{y}) + 10m(3\hat{z})}{m+m+10m} = \frac{2m\hat{x} + 30m\hat{z}}{12m} = \frac{1}{6}\hat{x} + \frac{5}{2}\hat{z}$$

(b) Sometimes one needs to find the CM for an extended massive object, rather than a collection of particles. Rewrite Eq. (1) above as an integral formula. Imagine each individual particle with mass m_α gets shrunk down to an infinitesimally small piece of the object with mass dm.

$$\vec{R} = \frac{1}{M}\int \vec{r}\, dm$$

(c) Now imagine the object has constant density $\rho = m/V$. Rewrite the above integral in terms of dV instead of dm. Write an expression for just the z-component of $\vec{R}$ in Cartesian coordinates.

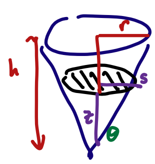
$$\rho = \frac{M}{V} = \frac{dm}{dV} \Rightarrow \boxed{\vec{R} = \frac{1}{M}\int \vec{r}\rho\, dV} = \frac{1}{M}\left[\int x\hat{x}\rho\, dV + \int y\hat{y}\rho\, dV + \int z\hat{z}\rho\, dV\right] \quad \text{and } dV = dx\,dy\,dz$$

$\rho\, dV = dm$

$$\boxed{R_z = \frac{1}{M}\int z\rho\, dx\,dy\,dz}$$

(d) Find the position of the CM of uniform solid cone. The cone lines up with the z-axis, the cone has a height h, radius r, the tip of the cone is positioned at the origin.

$\vec{R} = \frac{1}{M}\int \vec{r}\rho\, dV$ we know $R_x = R_y = 0$



$R_z = \frac{1}{M}\int z\rho\, dx\,dy\,dz$

The $\int dx\,dy$ integral gives the area of the circle at z. $A = \pi s^2$ we know $\tan\theta = \frac{s}{z}$ and $\tan\theta = \frac{r}{h} \Rightarrow \frac{s}{z} = \frac{r}{h}$, $s = \frac{zr}{h}$

$\Rightarrow A = \frac{\pi z^2 r^2}{h^2}$

$R_z = \frac{\rho}{M}\int_0^h \left(\pi \frac{z^2 r^2}{h^2}\right) z\, dz$

$R_z = \frac{1}{V}\left(\frac{\pi r^2}{h^2}\right)\int_0^h z^3\, dz$

$R_z = \frac{1}{V}\left(\frac{\pi r^2}{h^2}\right)\frac{1}{4}h^4 = \frac{1}{V}\frac{\pi r^2 h^2}{4}$

$V = \frac{1}{3}\pi r^2 h$ for a cone

$\Rightarrow R_z = \frac{3}{\pi r^2 h}\,\frac{\pi r^2 h^2}{4} = \boxed{\frac{3}{4}h}$