

3.10 Rocket has initial m_0 mass and $v_0 = 0$. For what m is $p = \max$?



$$0 = \dot{P} = \frac{dP}{dt} \Rightarrow 0 = dP = [P + dP] - P = [(m + dm)(v + dv) + (-dm)(v - v_{ex})] - mv$$

$$0 = mv + mdv + v_{ex}dm + O(\epsilon^2) - vdm + v_{ex}dm - vdv$$

$$0 = mdv + v_{ex}dm$$

$$\int_{m_0}^m \frac{dm'}{m'} = \int_{-v_{ex}}^v dv'$$

$$-v_{ex} \ln\left(\frac{m}{m_0}\right) = v$$

$$P = mv = -mv_{ex} \ln\left(\frac{m}{m_0}\right)$$

$$\frac{dp}{dm} = -v_{ex} \ln\left(\frac{m}{m_0}\right) - mv_{ex} \frac{1}{m} = -v_{ex} \left(\ln \frac{m}{m_0} + 1 \right) = 0$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = -1$$

$$m = m_0 e^{-1}$$

3.11

a) Show $\dot{mv} = -mv_{ex} + F_{ext}$ for rocket travelling in a straight line.

$$\frac{dP}{dt} = F_{ext}$$

$$dP = P(t+dt) - P(t)$$

$$= \underbrace{(m+dm)(v+dv)}_{m_{rocket} \text{ at } t+dt} + \underbrace{(-dm)(v+dv-v_{ex})}_{m_{fuel} \text{ at } t+dt} - mv$$

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 $v_{rocket \text{ at } t+dt}$

$$= \cancel{mv} + \cancel{mdv} + v_{ex}dm + \cancel{vdm} - \cancel{vdv} - \cancel{dm} + v_{ex}dm - \cancel{mv}$$

$$dP = mdv + v_{ex}dm$$

$$\frac{dP}{dt} = m \frac{dv}{dt} + v_{ex} \frac{dm}{dt}$$

$$F_{ext} = m \dot{v} + v_{ex} \dot{m}$$

$$\Rightarrow \dot{mv} = F_{ext} - mv_{ex} \quad \checkmark$$

(b) Let $F_{ext} = -mg$ and $\dot{m} = -k$ s.t. $m(t) = m_0 - kt$. Find $v(t)$.

$$mv = -mg - \dot{m}v_{ex}$$

$$(m_0 - kt) \frac{dv}{dt} = -(m_0 - kt)g + kv_{ex}$$

$$dv = \frac{-(m_0 - kt)g + kv_{ex}}{m_0 - kt} dt$$

$$\int_{v_0}^v dv' = \int_0^t \frac{-(m_0 - kt')g + kv_{ex}}{m_0 - kt'} dt'$$

$$v'|_{v_0}^v = \int_0^t -g dt' + \int_0^t \frac{k v_{ex} dt'}{m_0 - kt'}$$

$$v - v_0 = -g t'|_0^t - v_{ex} \int_0^t \frac{-k dt'}{m_0 - kt'}$$

$$v - v_0 = -gt - v_{ex} \ln(m_0 - kt)|_0^t$$

$$v = v_0 - gt - v_{ex} \ln\left(\frac{m_0 - kt}{m_0}\right)$$

(c) $m_0 = 2 \times 10^6 \text{ kg}$
 $m(t_f) = 10^6 \text{ kg}$
 $v_{ex} = 3000 \text{ m/s}$
 $v_0 = 0$

Find $v(t_f)$ and compare to case without gravity

$$t_f = (2 \text{ min}) \left(\frac{60 \text{ s}}{\text{min}} \right) = 120 \text{ s}$$

$$v(t_f) = 0 - (9.8 \text{ m/s}^2)(120 \text{ s}) - (3000 \text{ m/s}) \ln\left(\frac{10^6 \text{ kg}}{2 \times 10^6 \text{ kg}}\right)$$

$$v(t_f) = 900 \text{ m/s}$$

$$\text{If } g=0, \quad v = v_0 - v_{ex} \ln\left(\frac{m}{m_0}\right)$$

$$v(t_f) = 0 - (3000 \text{ m/s}) \ln\left(\frac{10^6 \text{ kg}}{2 \times 10^6 \text{ kg}}\right) = 2100 \text{ m/s} = v(t_f) [\text{no gravity}]$$

d) What happens when $|\dot{m}v_{ex}| \ll |m_0g|$?

$$\text{Eq. 3.30: } \dot{m}\ddot{v} = -\dot{m}v_{ex} - mg$$

$$\begin{aligned} \text{We know: } \dot{m} < 0, v_{ex} > 0 &\Rightarrow \dot{m}v_{ex} < 0 \\ &\Rightarrow -\dot{m}v_{ex} > 0 \end{aligned}$$

$$m > 0, g > 0 \Rightarrow -mg < 0$$

$$\text{If } |\dot{m}v_{ex}| \ll |mg|$$

$$-\dot{m}v_{ex} \ll mg$$

$$\Rightarrow -\dot{m}v_{ex} - mg \ll 0$$

$$\Rightarrow \dot{m}\ddot{v} < 0$$

$$\Rightarrow \ddot{v} < 0 \quad \text{and} \quad v_0 = 0$$

If the rocket were suspended in space, it would fall vertically downward

If the rocket were on the ground, the normal force would keep it at $y=0$, but it wouldn't be able to take off.

(3.14) Rocket feels $\vec{f} = -b\vec{v}$. Show that if $\dot{m} = -k$, $v = \frac{k}{b} v_{ex} \left[1 - \left(\frac{m}{m_0} \right)^{b/k} \right]$ ($v_0 = 0$)

$$F_{ext} = \frac{dP}{dt} \quad dP = [P + dP] - P = [(m + dm)(v + dv) + (-dm)(v - v_{ex})] - mv = mdv + v_{ex}dm$$

$$-bv = \frac{dP}{dt} \quad \frac{dP}{dt} = m \frac{dv}{dt} + v_{ex} \frac{dm}{dt} \quad \dot{m} = -k$$

$$\frac{dP}{dt} = m \frac{dv}{dt} - v_{ex} k$$

$$-bv = m \frac{dv}{dt} - v_{ex} k$$

$$-bv = m \frac{dv}{dt} + \frac{dm}{dt} - v_{ex} k$$

$$-bv = -km \frac{dv}{dm} - v_{ex} k$$

$$bv - kv_{ex} = km \frac{dv}{dm}$$

$$\frac{dm}{m} = \frac{kdv}{bv - kv_{ex}} \Rightarrow \int_{m_0}^m \frac{dm'}{m'} = \frac{k}{b} \int_0^v \frac{dv'}{v - \frac{kv_{ex}}{b}}$$

$$\ln \left(\frac{m}{m_0} \right) = \frac{k}{b} \ln \left(\frac{v - \frac{kv_{ex}}{b}}{-kv_{ex}/b} \right)$$

$$\frac{k}{b} \ln \left(\frac{m}{m_0} \right) = \ln \left(\frac{v - \frac{kv_{ex}}{b}}{-kv_{ex}/b} \right)$$

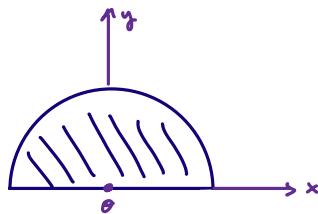
$$\left(\frac{m}{m_0} \right)^{b/k} = \frac{v - \frac{kv_{ex}}{b}}{-\frac{kv_{ex}}{b}}$$

$$\boxed{\frac{kv_{ex}}{b} \left[- \left(\frac{m}{m_0} \right)^{b/k} + 1 \right] = v}$$

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3.21

Find CM of:

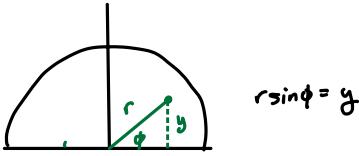


$$\vec{R}_{CM} = \frac{1}{M} \int \vec{r} dm \quad \text{Define } \sigma = \frac{M}{A} \Rightarrow \sigma = \frac{dm}{dA}$$

$$\vec{R}_{CM} = \frac{1}{M} \int \vec{r} \sigma dA = \frac{1}{M} \int \vec{r} \sigma r dr d\phi$$

we know \vec{R}_{CM} lies on y-axis by symmetry
 $\Rightarrow R_x = 0 \Rightarrow \vec{r} = y\hat{y}$

$$\vec{R}_{CM} = \frac{\sigma}{M} \hat{y} \int y \sigma r dr d\phi$$



$$\vec{R}_{CM} = \frac{\sigma}{M} \hat{y} \int r \sin \phi r dr d\phi$$

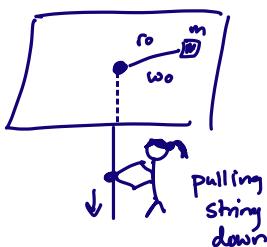
$$\vec{R}_{CM} = \frac{\sigma}{M} \hat{y} \int_0^R r^2 dr \int_0^\pi \sin \phi d\phi$$

$$\Rightarrow \vec{R}_{CM} = \frac{\sigma}{M} \hat{y} \left(\frac{r^3}{3} \right)_0^R (-\cos \phi)_0^\pi$$

$$\vec{R}_{CM} = \frac{R^3 \sigma \hat{y}}{3M} (1+1) = \frac{2R^3 \sigma \hat{y}}{3M} \quad \sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi R^2}$$

$$\vec{R}_{CM} = \frac{2R^3}{3M} \frac{M}{\frac{1}{2}\pi R^2} \hat{y} = \boxed{\frac{4R}{3\pi} \hat{y} = \vec{R}_{CM}}$$

3.25



Angular momentum = conserved

$$L_i = L_f$$

$$I_i \omega_0 = I_f \omega_f$$

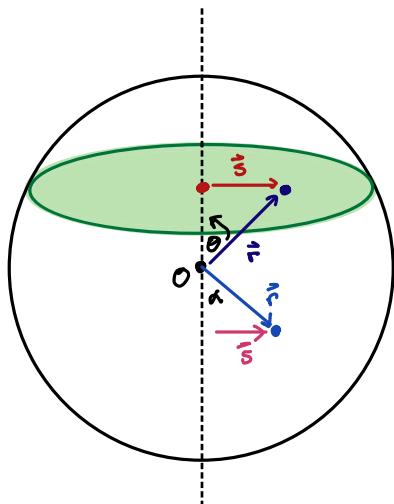
$$I = \frac{1}{2}mr^2 \quad I_i = \frac{1}{2}mr_0^2 \quad I_f = \frac{1}{2}mr^2$$

$$\frac{1}{2}mr_0^2 \omega_0 = \frac{1}{2}mr^2 \omega$$

$$\boxed{\frac{r^2}{r_0^2} \omega_0 = \omega}$$

3.32

Show $I_{\text{spec}} = \frac{2}{5} MR^2$. Use $dV = r^2 dr \sin\theta d\theta d\phi$



$$I = \sum_i m_i s_i^2 \rightarrow \int s^2 dm = \int s^2 \rho dV = \rho \int s^2 r^2 dr \sin\theta d\theta d\phi$$

We know: $\sin\theta = \frac{s}{r}$ (see picture)

where $\rho = \frac{M}{V}$

$$\Rightarrow s^2 = r^2 \sin^2\theta$$

$$I = \rho \int_0^R r^4 \sin^2\theta dr \sin\theta d\theta d\phi = \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi$$

Do each integral one at a time

$$\int_0^R r^4 dr = \frac{r^5}{5} \Big|_0^R = \frac{1}{5} R^5$$

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin\theta [1 - \cos^2\theta] d\theta = \int_0^\pi \sin\theta d\theta - \int_0^\pi \underbrace{\cos^2\theta \sin\theta}_{u^2} d\theta = -\cos\theta \Big|_0^\pi + \frac{\cos^3\theta}{3} \Big|_0^\pi$$

$$= -(\cos\pi + \cos(0)) + \frac{1}{3}(\cos\pi)^3 - \frac{1}{3}(\cos(0))^3$$

$$= -(-1) + 1 - \frac{1}{3} - \frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$\int_0^{2\pi} d\phi = \phi \Big|_0^{2\pi} = 2\pi$$

$$\Rightarrow I = \rho \left[\frac{R^5}{5} \right] \left[\frac{4}{3} \right] \left[2\pi \right] = \frac{8\pi \rho R^5}{15} = \frac{8\cancel{\pi} R^5}{15} \left[\frac{M}{\cancel{4/3} \cancel{\pi} R^3} \right] = \boxed{\frac{2MR^2}{5}}$$

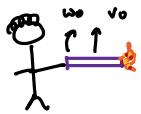
3.34 Juggler throws a rod. When released:

 = horizontal

CM velocity = v_0

angular velocity = ω_0

He wants to catch the rod after exactly n rotations. What should v_0 be?



$$ma = -mg \Rightarrow v = -gt + v_0$$

$$0 = -gt_{\max} + v_0$$

$$t_{\max} = \frac{v_0}{g}$$

$t_{\text{catch}} = 2t_{\max}$ (travels up & back down to be caught)

$$= \frac{2v_0}{g}$$

$\omega = \text{constant}$ (gravity acts at CM, which is also the point the rod rotates about \Rightarrow does not exert torque, $L = I\omega = \text{const.}$)

$$\omega_0 = \frac{d\theta}{dt} \Rightarrow \theta = \omega_0 t + \theta_0$$

$$(\theta - \theta_0) = 2\pi n = \omega_0 t$$

$$\frac{2\pi n}{\omega_0} = t = t_{\text{catch}} \Rightarrow \frac{2v_0}{g} = \frac{2\pi n}{\omega_0}$$

$$v_0 = \frac{n\pi g}{\omega_0}$$