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Lecture 15 2/23/26

Review of Last Week:

Monday:
- Practiced calculating work using line integrals.
- Introduced conservative forces. $\vec{F}$ is conservative iff
  - (i) $\vec{F}$ depends on $\vec{r}$ only
  - (ii) Work done by $\vec{F}$ is path-independent

$$W(1 \to 2) = \int_1^2 \vec{F} \cdot d\vec{r}$$

$$U(\vec{r}) \equiv -W(\vec{r}_0 \to \vec{r}) \equiv -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$ If $\vec{F}$ is conservative

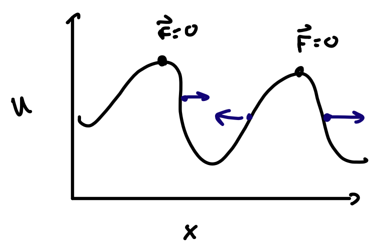
$W = \Delta T \qquad W = -\Delta U \implies \Delta T = -\Delta U \implies \Delta E = 0$ if all forces are conservative

$\Delta T + \Delta U = W_{NC}$ when some are non-conservative

Wednesday:
- $\vec{F} = -\vec{\nabla} U = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)U$
- $\vec{\nabla} \times \vec{F} = 0 \iff \vec{F}$ is conservative
- showed $\vec{F}_g$ & $\vec{F}_G$ are conservative

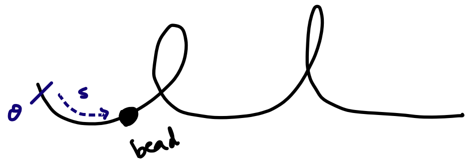
Friday:
- 1D systems & energy conservation.

$\vec{F} = -\vec{\nabla} U \implies F_x = -\frac{\partial U}{\partial x}$



- Usually you know $U(x)$. To find E_T, you need T somewhere. Then if E = conserved, E_T is constant
- Practiced $\vec{\nabla} \times \vec{F}$, $\vec{\nabla} U$, $\int \vec{F} \cdot d\vec{r}$

Curvilinear 1D Systems



- The position of the bead can be specified by s = distance along wire from origin

$$T = \tfrac{1}{2} m \dot{s}^2$$

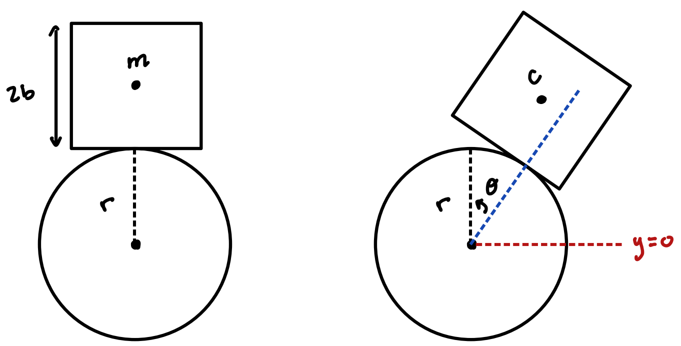
- F on bead is complicated $\vec{F}_N$ changes as bead moves.

- $\vec{F}_N$ does no work $\vec{F}_N \perp d\vec{r} \Rightarrow \vec{F}_N \cdot d\vec{r} = 0$

- We care about $F_{tang} = m\ddot{s}$ (survives dot product with $d\vec{r}$)

- We can define $F_{tang} = -\dfrac{dU}{ds}$ iff F_{tang} is conservative $\Longleftrightarrow$ $E = T + U(s)$ = constant

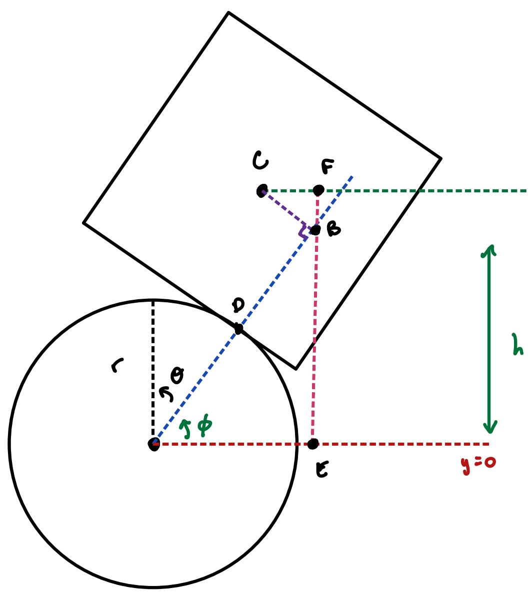
Example: Stability of Cube balanced on a cylinder:



Show $\theta = 0$ is equilibrium & is it stable/unstable?

This system can be described by θ completely
$\Rightarrow$ 1D system!

First, find $U(\theta)$.



$U = mgh = mgy$

A few useful distances:

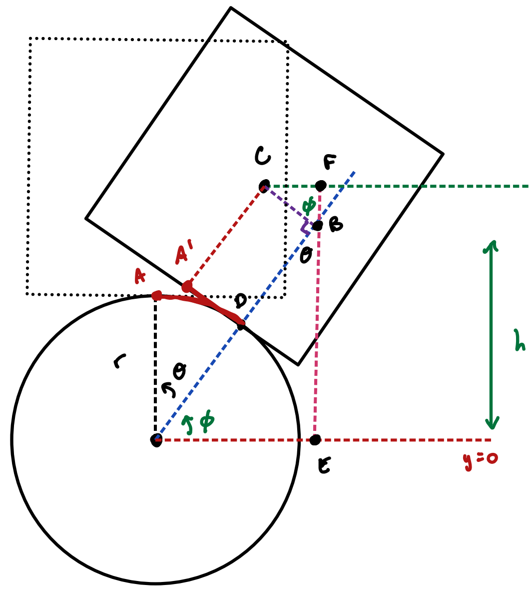
$$OD = r$$
$$DB = b$$
$$\Rightarrow OB = r + b$$

$$BE = (OB)\sin\phi = (OB)\cos\theta$$

$$BE = (r+b)\cos\theta$$

$h = BE + BF \Rightarrow$ need to find BF

• Let A be the original point of contact btw block & cylinder when $\theta=0$.



$A \to A'$

We know $CB = A'D$

and $AD = A'D = r\theta$

$\Rightarrow CB = r\theta$

$BF = CB\cos\phi = CB\sin\theta$

$BF = r\theta\sin\theta$

$\Rightarrow h = BE + BF = (r+b)\cos\theta + r\theta\sin\theta$

$U = mgh = mg(r+b)\cos\theta + mgr\theta\sin\theta$

$$\frac{\partial U}{\partial\theta} = -mg(\cancel{r}+b)\sin\theta \cancel{+mgr\sin\theta} + mgr\theta\cos\theta$$

$$= -mgb\sin\theta + mgr\theta\cos\theta$$

$\frac{dU}{d\theta} = 0 \Rightarrow b\sin\theta = r\theta\cos\theta$ true for $\theta = 0 \Rightarrow$ equilibrium point.

Stable / unstable?

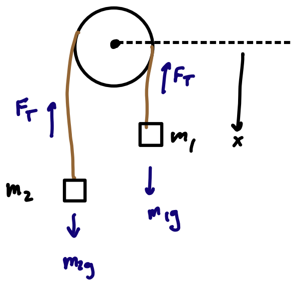
$$\frac{d^2U}{d\theta^2} = -mgb\cos\theta + mgr\cos\theta - mgr\theta\sin\theta$$

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = -mgb + mgr - 0 = mg(r-b)$$

stable if $r > b$
unstable if $r < b$

Systems of multiple connected parts can be 1D:

Example:



The length of the string is fixed
$\Rightarrow$ the whole system can be described by height of one mass

$\Delta T_1 + \Delta U_1 = W_1^{ten}$ $\qquad$ F_T = same along whole string

$\Delta T_2 + \Delta U_2 = W_2^{ten}$

$$\left.\begin{array}{l} đW_1 = F_T dx_1 \\ đW_2 = F_T dx_2 = -F_T dx_1 \end{array}\right\} \quad đW_1 = -đW_2$$

$$W_1^{ten} = -W_2^{ten}$$

$$\Delta T_1 + \Delta U_1 + \Delta T_2 + \Delta U_2 = W_1^{ten} + W_2^{ten} = 0$$

$$E = T_1 + U_1 + T_2 + U_2 = \text{constant}$$

$E = \sum_{\alpha=1}^{N} (T_\alpha + U_\alpha)$ if work is done by constraining forces between particles

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Central Forces

A central force is always directed towards or away from a fixed center

$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r}$ $\Rightarrow$ central force

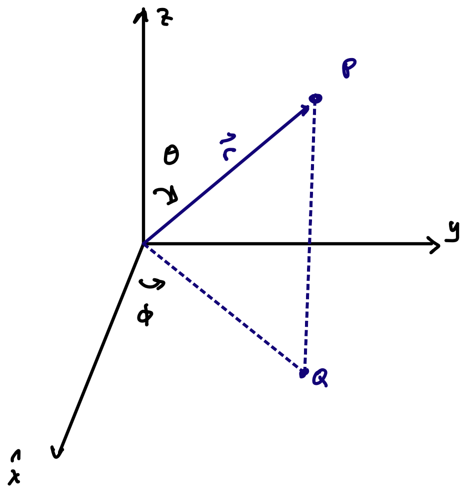
A <u>spherically symmetric</u> central force is independent of direction of $\vec{r}$:

$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r} = f(r)\,\hat{r}$ $\qquad$ (also called "rotationally invariant")

only depends on distance from origin

Claim: Spherically symmetric central forces are conservative (and vice versa)

Before we proceed, we must discuss spherical coordinates:

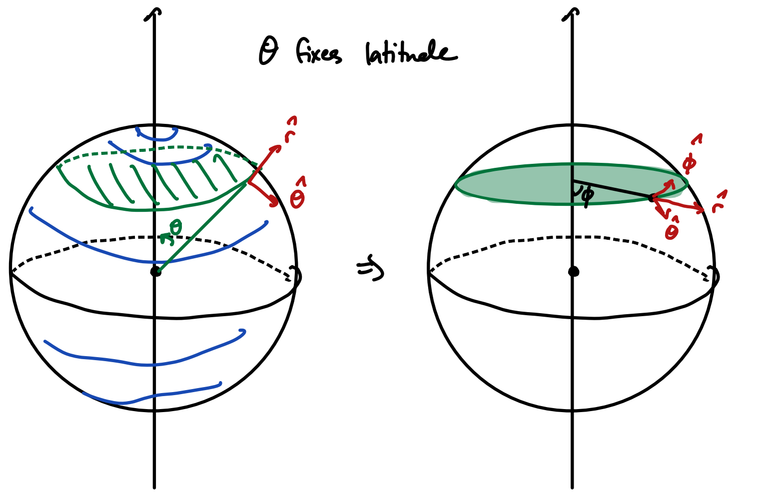


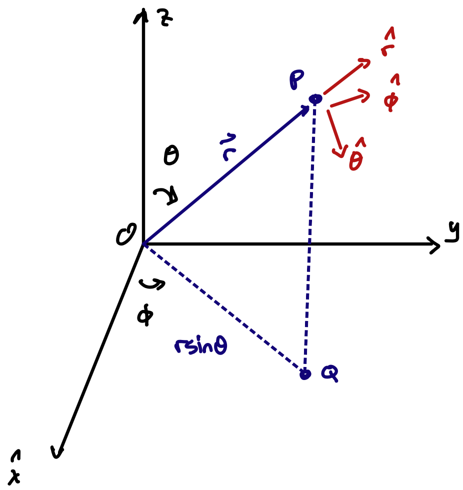
$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

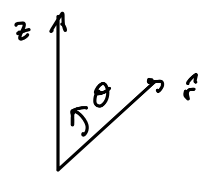
θ fixes latitude





ϕ tells you where on that circle you are (longitude, if you like)

The $\measuredangle$ between $\vec{r}$ and $\hat{z}$ is θ by definition



$\Rightarrow$ The projection of $\vec{r}$ onto the x-y plane is given by $r\sin\theta$

$\Rightarrow$ $OQ = r\sin\theta$

Imagine Q sits on the y-axis: