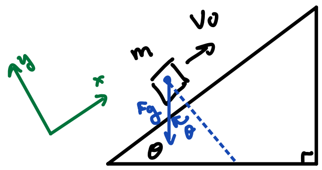
Activity 4

1. Time to review! Draw free body diagrams on top of the following figures, and write down Newton's second law (or conservation of momentum) in terms of components of your chosen coordinate system. Write them as differential equations that describe the equation of motion. In each case, outline your method of solving the differential equation and obtaining the kinematic equations.



$\vec{F} = mg\sin\theta\,\hat{x} + (F_N - mg\cos\theta)\hat{y}$

$m\ddot{x} = -mg\sin\theta$

Multiply both sides by dt & integrate for $v(t) = \dot{x}(t)$

Multiply both sides by dt & integrate for $x(t)$

Figure 1: A block of mass m slides up a hill that makes angle θ with the horizontal. Gravitational acceleration is g and gravity points downwards.

$\vec{F}_{ext} = \dot{\vec{P}}$

$-mg\,y = \frac{dP}{dt}$

$-mg\,dt = P + dP - P$

$P = mv$

$P + dP = (m+dm)(v+dv) + (-dm)(v - v_{ex})$

$= mv + mdv + \cancel{v}dm + \mathcal{O}(\varepsilon^2) - \cancel{v}dm + v_{ex}dm$

$-mg\,dt = \cancel{mv} + m\,dv + v_{ex}\,dm - \cancel{mv}$

$-mg\,dt = m\,dv + v_{ex}\,dm \rightarrow$ If you got this far, that's fine

If you want to go farther, divide by m

$-g\,dt = dv + v_{ex}\frac{dm}{m}$

Integrate $\int_{t_0}^{t} -g\,dt' = \int_{v_0}^{v} dv' + v_{ex}\int_{m_0}^{m}\frac{dm'}{m'}$

$-gt = v - v_0 + v_{ex}\ln\left(\frac{m}{m_0}\right)$

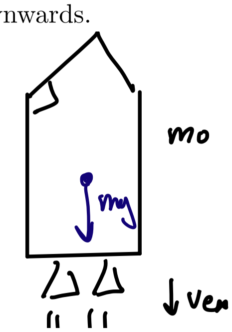
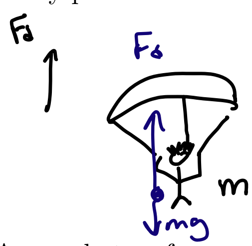


Figure 2: A rocket shoots upwards. It starts with initial mass m_0 and expels fuel at a speed v_{ex}. Gravitational acceleration is g and gravity points downwards.



$m\vec{a} = -mg\hat{y} + cv^2\hat{y}$

$m\ddot{y} = -mg + cv^2$

$m\dot{v} = -mg + cv^2$

Separate variables into dv, dt and integrate.

Rearrange to get $v(t) = \dot{y}(t)$

Multiply both sides by dt & integrate to get $y(t)$

Figure 3: A parachuter of mass m falls with drag force $\vec{F} = -cv^2\hat{v}$. Gravitational acceleration is g and gravity points downwards.

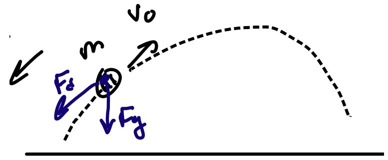


Figure 4: A projectile of mass m is shot with speed v_0 and experiences a drag force $\vec{F}_d = -b\vec{v}$ and gravity pointing downwards.

$$m\vec{a} = -b v_x \hat{x} + (-b v_y - mg)\hat{y}$$

$$m\ddot{x} = -b v_x \qquad m\ddot{y} = -b v_y - mg$$

$$\dot{v}_x = -\frac{b}{m} v_x \qquad \dot{v}_y = -\frac{b}{m} v_y - g$$

separate variables (dv_x and dt; dv_y and dt) and integrate to find $v_x(t)$, $v_y(t)$

Multiply both sides by dt & integrate to find $x(t)$, $y(t)$

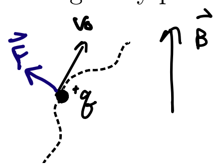


Figure 5: A charge $+q$ moves in a magnetic field $\vec{B} = B\hat{z}$ with an initial velocity $\vec{v}_0$. There is no gravity.

$$m\vec{a} = q(\vec{v}\times\vec{B}) = q v_y B\hat{x} - q v_x B\hat{y}$$

$$m\ddot{x} = q v_y B \qquad m\ddot{y} = -q v_x B$$

$$\dot{v}_x = qB/m\, v_y \qquad \dot{v}_y = -qB/m\, v_x$$

- Define $\eta = v_x + i v_y$ & derive $\dot{\eta} = f(\eta)$
- separate variables ($d\eta$, dt) and integrate
- Find $\eta(t)$
- use Euler's eqn to extract v_y, v_x
- Multiply both sides by dt & integrate to find $y(t)$, $x(t)$

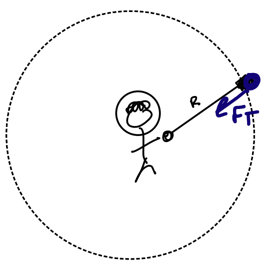


Figure 6: An astronaut swings a mass in a circle. There is no gravity.

$$\vec{F} = -F_T\hat{r}$$

$$m\vec{a} = -F_T\hat{r}$$

$$m(\ddot{r} - r\dot{\phi}^2) = -F_T$$

$$m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = 0$$

we know $r = R = \text{const.}$

$$\Rightarrow \begin{cases} -mr\dot{\phi}^2 = -F_T \\ mr\ddot{\phi} = 0 \end{cases} \Rightarrow \dot{\phi} = \omega_0$$

$$\phi = \omega_0 t + \phi_0$$

$$-mR\omega_0^2 = F_T$$

$$-mR\omega_0^2 = F_T$$

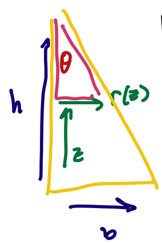
$$\vec{R}_{CM} = \frac{1}{M}\int \vec{r}\,dm \qquad I = \int \rho^2\,dm \Rightarrow I = \int \rho^2 \mu\, dV \Rightarrow I = \mu \int \rho^3\, d\rho\, d\phi\, dz$$

2. The position of the center of mass, $\vec{R}_{\rm CM}$ and the moment of inertia I of a system of N particles are defined as:

$$\vec{R}_{\rm CM} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \qquad\qquad I = \sum_{\alpha}^{N} m_\alpha \rho_\alpha^2\,, \tag{1}$$

where ρ is the distance of particle α from the axis of rotation. Find the moment of inertia of the uniform cone shown in Fig. 7.

$$I = \mu \int \rho^3 d\rho\, dz\, d\phi = \mu \int_0^h \int_0^{r(z)} \rho^3 d\rho\, dz \int_0^{2\pi} d\phi$$



we know $\tan\theta = \frac{b}{h}$ (yellow Δ)

and $\tan\theta = \frac{r(z)}{h-z}$ (pink Δ)

$\Rightarrow \frac{b}{h} = \frac{r(z)}{h-z}$

$b - \frac{bz}{h} = r(z)$

$$I = \mu \int_0^h \int_0^{bz/h} \rho^3 d\rho\, dz\ (2\pi)$$

$$I = 2\pi\mu \int_0^h \left[\frac{1}{4}\rho^4\right]_0^{bz/h} dz$$

$$I = 2\pi\mu \int_0^h \frac{b^4 z^4}{4h^4}\, dz = \frac{2\pi\mu b^4}{4h^4}\left(\frac{z^5}{5}\right)\Big|_0^h$$

$$I = \frac{\pi\mu b^4}{2h^4}\left(\frac{h^5}{5}\right) = \frac{\pi\mu b^4 h}{10}\ , \quad \mu = \frac{M}{V_{cone}} = \frac{M}{\left(\frac{\pi b^2 h}{3}\right)}$$

$$\mu = \frac{3M}{\pi b^2 h}$$

$$I = \frac{\pi b^4 h}{10}\left(\frac{3M}{\pi b^2 h}\right) = \boxed{\frac{3Mb^2}{10}}$$

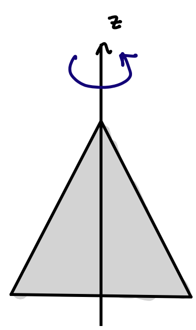


Figure 7: The cone rotating about the z-axis. It has height h and radius b at the base

3. Find the work done from point A to point B along the three paths shown in Fig. 8. Use:

$$W(A \to B) = \int_A^B \vec{F} \cdot d\vec{r} \tag{2}$$

$$\vec{F} = xy^2\hat{x} + y^3\hat{y} \tag{3}$$

Useful integral:

$$\int xe^x dx = (x-1)e^x + C \qquad \vec{F} \cdot d\vec{r} = xy^2\,dx + y^3\,dy \tag{4}$$

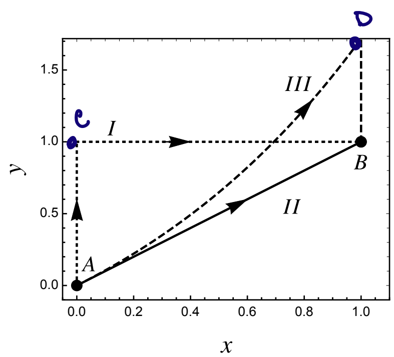


Figure 8: The paths the force acts along. Path II is defined by $y = x$ and path III is defined by $y = e^x - 1$

$$W_I = \int_A^C y^3 dy + \int_C^B xy^2 dx = \int_0^1 y^3 dy + \int_0^1 x(1)^2 dx = \frac{y^4}{4}\Big|_0^1 + \int_0^1 x\,dx$$

$$= \frac{1}{4} + \frac{x^2}{2}\Big|_0^1 = \frac{1}{4} + \frac{1}{2} = \boxed{3/4}$$

Along II, $x = y \Rightarrow dx = dy$

$$\vec{F} \cdot d\vec{r} = x^3 dx + x^3 dx = 2x^3 dx$$

$$W_{II} = \int_A^B 2x^3 dx = \int_0^1 2x^3 dx = \frac{2x^4}{4}\Big|_0^1 = \boxed{\frac{1}{2}}$$

Along III up to point D:

$y = e^x - 1$

$dy = e^x dx$

$$\vec{F} \cdot d\vec{r} = x(e^x-1)^2 dx + (e^x-1)^3 e^x dx$$

$$= (xe^{2x} - 2xe^x + x)dx + (e^{3x} - 3e^{2x} + 3e^x - 1)e^x dx$$

From point D → B:

$x = 1$, $dx = 0$ — dy only

at point D, $x = 1$, $y = e - 1$

$$W_{III} = \int_A^D \left[xe^{2x} - 2xe^x + x + e^{4x} - 3e^{3x} + 3e^{2x} - e^x\right]dx + \int_D^B y^3 dy$$

$$= \frac{1}{4}\int_0^1 2xe^{2x}(2dx) + \Big[-2(x-1)e^x + \tfrac{1}{2}x^2 + \tfrac{1}{4}e^{4x} - \tfrac{3}{3}e^{3x} + \tfrac{3}{2}e^{2x} - e^x\Big]_0^1 + \int_{e-1}^1 y^3 dy$$

$$= \frac{1}{4}\left[(2x-1)e^{2x}\right]_0^1 + \Big[0 + \tfrac{1}{2} + \tfrac{1}{4}e^4 - e^3 + \tfrac{3}{2}e^2 - e + \cancel{2}(-1)(1) - 0 - \tfrac{1}{4} + \cancel{1} - 3/2 + \cancel{1}\Big] + \frac{1}{4}y^4\Big|_{e-1}^1$$

$$= \underbrace{\frac{1}{4}\left[(1)e^2 - (-1)(1)\right]}_{1/4e^2 + 1/4} + \frac{1}{4}e^4 - e^3 + 3/2e^2 - e - 1/4 - 1 + \frac{1}{4} - \frac{1}{4}(e-1)^4$$

$$= \cancel{\tfrac{1}{4}e^4} - \cancel{e^3} + 7/4e^2 - \cancel{e} - 1 + \cancel{1/4} - \frac{1}{4}(\cancel{e^4} - \cancel{4e^3} + 6e^2 - \cancel{4e} + \cancel{1})$$

$$= (7/2 - 6/4)e^2 - 1 = \boxed{2e^2 - 1 = W_{III}}$$