3.10 Rocket has initial m_0 mass and $v_0 = 0$. For what m is $p = \max$?

$$0 = \dot{p} = \frac{dp}{dt} \Rightarrow 0 = dP = [P + dP] - P = [(m+dm)(v+dv) + (-dm)(v - v_{ex})] - mv$$

$$0 = \cancel{mv} + m\,dv + \cancel{v\,dm} + \mathcal{O}(\epsilon^2) - \cancel{v\,dm} + v_{ex}\,dm - \cancel{mv}$$

$$0 = m\,dv + v_{ex}\,dm$$

$$\int_{m_0}^{m} -v_{ex}\frac{dm'}{m'} = \int_0^v dv'$$

$$-v_{ex}\ln\left(\frac{m}{m_0}\right) = v$$

$$p = mv = -mv_{ex}\ln\left(\frac{m}{m_0}\right)$$

$$\frac{dp}{dm} = -v_{ex}\ln\left(\frac{m}{m_0}\right) - mv_{ex}\frac{1}{m} = -v_{ex}\left(\ln\frac{m}{m_0} + 1\right) = 0$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = -1$$

$$\boxed{m = m_0 e^{-1}}$$

3.11

a) Show $m\dot{v} = -mv_{ex} + F^{ext}$ for rocket travelling in a straight line.

$$\frac{dP}{dt} = F^{ext}$$

$$dP = P(t+dt) - P(t)$$

$$= \underbrace{(m+dm)}_{m_{rocket}\text{ at }t+dt}\underbrace{(v+dv)}_{v_{rocket}\text{ at }t+dt} + \underbrace{(-dm)}_{m_{fuel}}\underbrace{(v+dv-v_{ex})}_{v_{fuel}} - \underbrace{m}_{m_{rocket}\text{ at }t}\underbrace{v}_{v_{rocket}\text{ at }t}$$

$$= \cancel{mv} + m\,dv + \cancel{v\,dm} + \cancel{dm\,dv} - \cancel{v\,dm} - \cancel{dv\,dm} + v_{ex}\,dm - \cancel{mv}$$

$$dP = m\,dv + v_{ex}\,dm$$

$$\frac{dP}{dt} = m\frac{dv}{dt} + v_{ex}\frac{dm}{dt}$$

$$F_{ext} = m\dot{v} + v_{ex}\dot{m}$$

$$\Rightarrow \boxed{m\dot{v} = F_{ext} - \dot{m}v_{ex} \quad ☑}$$

(b) Let $F_{ext} = -mg$ and $\dot{m} = -k$ s.t. $m(t) = m_0 - kt$. Find $v(t)$.

$$m\dot{v} = -mg - \dot{m} v_{ex}$$

$$(m_0 - kt)\frac{dv}{dt} = -(m_0 - kt)g + kv_{ex}$$

$$dv = \frac{-(m_0 - kt)g + kv_{ex}}{m_0 - kt}\, dt$$

$$\int_{v_0}^{v} dv' = \int_0^t \frac{-(m_0 - kt')g + kv_{ex}}{m_0 - kt'}\, dt'$$

$$v'\Big|_{v_0}^{v} = \int_0^t -g\, dt' + \int_0^t \frac{kv_{ex}\, dt'}{m_0 - kt'}$$

$$v - v_0 = -gt'\Big|_0^t - v_{ex}\int_0^t \frac{-k\, dt'}{m_0 - kt'}$$

$$v - v_0 = -gt - v_{ex} \ln(m_0 - kt')\Big|_0^t$$

$$\boxed{v = v_0 - gt - v_{ex}\ln\left(\frac{m_0 - kt}{m_0}\right)}$$

(c) $\left.\begin{aligned} m_0 &= 2\times10^6 \text{ kg} \\ m(t_f) &= 10^6 \text{ kg} \\ v_{ex} &= 3000 \text{ m/s} \\ v_0 &= 0 \end{aligned}\right\}$ Find $v(t_f)$ and compare to case without gravity

$$t_f = (2 \text{ min})\left(\frac{60 \text{ s}}{\text{min}}\right) = 120 \text{ s}$$

$$v(t_f) = 0 - (9.8 \text{ m/s}^2)(120 \text{ s}) - (3000 \text{ m/s}) \ln\left(\frac{10^6 \text{ kg}}{2\times10^6 \text{ kg}}\right)$$

$$\boxed{v(t_f) = 900 \text{ m/s}}$$

If $g = 0$, $v = v_0 - v_{ex}\ln\left(\frac{m}{m_0}\right)$

$$v(t_f) = 0 - (3000 \text{ m/s}) \ln\left(\frac{10^6 \text{ kg}}{2\times10^6 \text{ kg}}\right) = \boxed{2100 \text{ m/s} = v(t_f)^{[\text{no gravity}]}}$$

d) What happens when $|\dot{m} v_{ex}| \ll |m_0 g|$?

Eq. 3.30: $m\dot{v} = -\dot{m} v_{ex} - mg$

We know: $\dot{m} < 0$, $v_{ex} > 0$ $\Rightarrow$ $\dot{m} v_{ex} < 0$

$\Rightarrow$ $-\dot{m} v_{ex} > 0$

$m > 0$, $g > 0$ $\Rightarrow$ $-mg < 0$

If $|\dot{m} v_{ex}| \ll |mg|$

$-\dot{m} v_{ex} \ll mg$

$\Rightarrow$ $-\dot{m} v_{ex} - mg \ll 0$

$\Rightarrow$ $m\dot{v} < 0$

$\Rightarrow$ $\dot{v} < 0$ and $v_0 = 0$

If the rocket were suspended in space, it would fall vertically downward
If the rocket were on the ground, the normal force would keep it at $y=0$, but it wouldn't be able to take off.

3.14 Rocket feels $\vec{f} = -b\vec{v}$. Show that if $\dot{m} = -k$, $v = \frac{k}{b} v_{ex}\left[1 - \left(\frac{m}{m_0}\right)^{b/k}\right]$ $(v_0 = 0)$

$F_{ext} = \frac{dP}{dt}$

$dP = [P + dP] - P = [(m+dm)(v+dv) + (-dm)(v - v_{ex})] - mv = m\,dv + v_{ex}\,dm$

$-bv = \frac{dP}{dt}$

$\frac{dP}{dt} = m\frac{dv}{dt} + v_{ex}\frac{dm}{dt}$ $\dot{m} = -k$

$\frac{dP}{dt} = m\frac{dv}{dt} - v_{ex}k$

$-bv = m\frac{dv}{dt} - v_{ex}k$

$-bv = m\frac{dv}{dm}\cdot\frac{dm}{dt} - v_{ex}k$

$-bv = -km\frac{dv}{dm} - v_{ex}k$

$bv - kv_{ex} = km\frac{dv}{dm}$

$\frac{dm}{m} = \frac{k\,dv}{bv - kv_{ex}}$ $\Rightarrow$ $\int_{m_0}^{m}\frac{dm'}{m'} = \frac{k}{b}\int_0^v \frac{dv'}{v - \frac{kv_{ex}}{b}}$

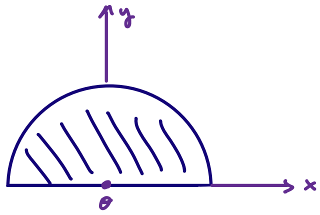
$\ln\left(\frac{m}{m_0}\right) = \frac{k}{b}\ln\left(\frac{v - kv_{ex}/b}{-kv_{ex}/b}\right)$

$\frac{b}{k}\ln\left(\frac{m}{m_0}\right) = \ln\left(\frac{v - kv_{ex}/b}{-kv_{ex}/b}\right)$

$\left(\frac{m}{m_0}\right)^{b/k} = \frac{v - kv_{ex}/b}{-kv_{ex}/b}$

$\boxed{\frac{kv_{ex}}{b}\left[-\left(\frac{m}{m_0}\right)^{b/k} + 1\right] = v}$ ☑

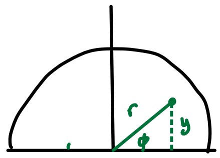
3.21 Find CM of:



$\vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$ Define $\sigma = \frac{M}{A} \Rightarrow \sigma = \frac{dm}{dA}$

$\vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,\sigma\,dA = \frac{1}{M}\int \vec{r}\,\sigma\,r\,dr\,d\phi$ we know $\vec{R}_{cm}$ lies on y-axis by symmetry $\Rightarrow R_x = 0 \Rightarrow \vec{r} = y\hat{y}$

$\vec{R}_{cm} = \frac{1}{M}\hat{y}\int y\,\sigma\,r\,dr\,d\phi$



$r\sin\phi = y$

$\vec{R}_{cm} = \frac{\sigma}{M}\hat{y}\int r\sin\phi\,r\,dr\,d\phi$

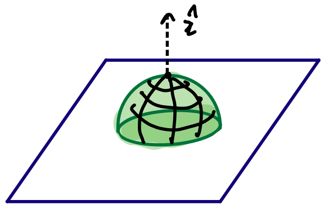
$\vec{R}_{cm} = \frac{\sigma}{M}\hat{y}\int_0^R r^2\,dr\int_0^\pi \sin\phi\,d\phi$

$\Rightarrow \vec{R}_{cm} = \frac{\sigma}{M}\hat{y}\left(\frac{r^3}{3}\right)_0^R\left(-\cos\phi\right)_0^\pi$

$\vec{R}_{cm} = \frac{R^3\sigma\hat{y}}{3M}(1+1) = \frac{2R^3\sigma\hat{y}}{3M}$ $\sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi R^2}$

$\vec{R}_{cm} = \frac{2R^3}{3M}\frac{M}{\frac{1}{2}\pi R^2}\hat{y} = \boxed{\frac{4R}{3\pi}\hat{y} = \vec{R}_{cm}}$

3.22 Find the CM of solid hemisphere, R



$\vec R_{cm} = \frac{1}{M}\int \vec r\, dm = \frac{1}{M}\int \vec r \rho\, dV$ We know by symmetry that $\vec R_{cm} \parallel \hat z$

$\Rightarrow \vec r = z\hat z$

← ½-circle

$$= \frac{1}{M}\int z\hat z \rho\, dV = \frac{\rho}{M}\hat z \int_0^{\pi/2}\int_0^{R}\int_0^{2\pi} z\, r^2 \sin\theta\, dr\, d\theta\, d\phi \Bigg|_{z=r\cos\theta}$$

$$= \frac{\rho \hat z}{M}(2\pi)\int_0^{\pi/2}\int_0^{R} r\cos\theta\, r^2\sin\theta\, dr\, d\theta = \frac{2\pi\rho}{M}\hat z \int_0^{\pi/2} \underbrace{\sin\theta}_{u}\, \underbrace{\cos\theta\, d\theta}_{du} \int_0^{R} r^3 dr$$

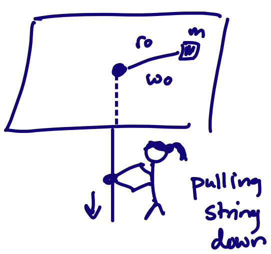
$$= \frac{2\pi\rho}{M}\hat z \; \frac{\sin^2\theta}{2}\Bigg|_0^{\pi/2} \frac{r^4}{4}\Bigg|_0^{R} = \frac{2\pi\rho}{M}\hat z \left[\frac{1^2}{2} - 0\right]\frac{R^4}{4} = \frac{2\pi\rho}{M}\frac{R^4}{8}\hat z$$

$$\rho = \frac{M}{V} = \frac{M}{\frac{1}{2}(\frac{4}{3}\pi R^3)} = \frac{M}{\frac{2}{3}\pi R^3}$$

$$= \frac{\cancel{2\pi}}{\cancel{M}} \frac{\cancel{M}}{\frac{2}{3}\cancel{\pi} R^3} \frac{R^4}{8}\hat z$$

$$= \boxed{\frac{3R}{8}\hat z}$$

3.25



Angular momentum = conserved

$L_i = L_f$

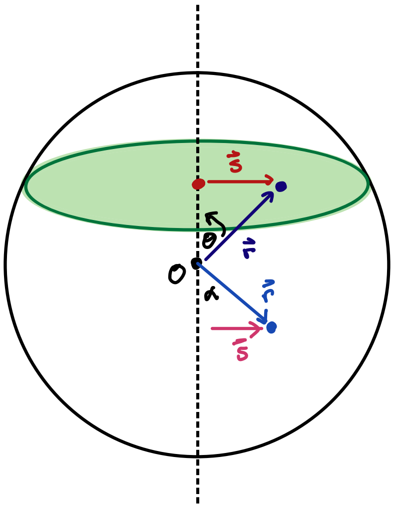
$I_i \omega_0 = I_f \omega_f$

$I = \frac{1}{2}mr^2$ $I_i = \frac{1}{2}m r_0^2$ $I_f = \frac{1}{2}mr^2$

$\frac{1}{2}m r_0^2 \omega_0 = \frac{1}{2}m r^2 \omega$

$$\boxed{\frac{r_0^2}{r^2}\omega_0 = \omega}$$

3.32 Show $I_{sphere} = \frac{2}{5} MR^2$. Use $dV = r^2 dr \sin\theta \, d\theta \, d\phi$



$$I = \sum_i m_i s_i^2 \rightarrow \int s^2 dm = \int s^2 \rho \, dV = \rho \int s^2 r^2 dr \sin\theta \, d\theta \, d\phi$$

where $\rho = \frac{M}{V}$

We know: $\sin\theta = \frac{s}{r}$ (see picture)

$$\Rightarrow s^2 = r^2 \sin^2\theta$$

$$I = \rho \int r^4 \sin^2\theta \, dr \sin\theta \, d\theta \, d\phi = \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

Do each integral one at a time

$$\int_0^R r^4 dr = \frac{r^5}{5}\Big|_0^R = \frac{1}{5}R^5$$

$$\int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi \sin\theta \left[1 - \cos^2\theta\right] d\theta = \int_0^\pi \sin\theta \, d\theta - \int_0^\pi \underbrace{\cos^2\theta}_{u^2} \underbrace{\sin\theta \, d\theta}_{-du} = -\cos\theta\Big|_0^\pi + \underbrace{\frac{\cos^3\theta}{3}}_{\frac{u^3}{3}}\Big|_0^\pi$$

$$= -\cos\pi + \cos(0) + \frac{1}{3}(\cos\pi)^3 - \frac{1}{3}\cos(0)^3$$

$$= -(-1) + 1 - 1/3 - 1/3 = 2 - 2/3 = 4/3$$

$$\int_0^{2\pi} d\phi = \phi\Big|_0^{2\pi} = 2\pi$$

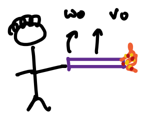
$$\Rightarrow I = \rho \left[\frac{R^5}{5}\right]\left[\frac{4}{3}\right]\left[2\pi\right] = \frac{8\pi\rho R^5}{15} = \frac{8\cancel{\pi} R^5}{15}\left[\frac{M}{4/3 \cancel{\pi} R^3}\right] = \boxed{\frac{2MR^2}{5}}$$

3.34 Juggler throws a rod. When released: rod ——— = horizontal

CM velocity = v_0

angular velocity = ω_0

He wants to catch the rod after exacty n rotations. What should v_0 be?



$ma = -mg \Rightarrow v = -gt + v_0$

$0 = -g t_{max} + v_0$

$t_{max} = v_0/g$

$t_{catch} = 2 t_{max}$ (travels up & back down to be caught)

$= \dfrac{2v_0}{g}$

ω = constant (gravity acts at CM, which is also the point the rod rotates about $\Rightarrow$ does not exert torque, $L = I\omega$ = const.)

$\omega_0 = \dfrac{d\theta}{dt} \Rightarrow \theta = \omega_0 t + \theta_0$

$(\theta - \theta_0) = 2\pi n = \omega_0 t$

$\dfrac{2\pi n}{\omega_0} = t = t_{catch} \Rightarrow \dfrac{2v_0}{g} = \dfrac{2\pi n}{\omega_0}$

$$\boxed{v_0 = \frac{n\pi g}{\omega_0}}$$