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Lecture 13 2/18/26

Force as the Gradient of PE

Consider a particle acted on by $\vec{F}(\vec{r})$ (conservative) with PE $U(\vec{r})$. Let's look at how much work is done by $\vec{F}$ in a small displacement: $\vec{r} \to \vec{r} + d\vec{r}$

$$W(\vec{r} \to \vec{r}+d\vec{r}) = \vec{F}(\vec{r}) \cdot d\vec{r} \qquad d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$
$$= F_x\,dx + F_y\,dy + F_z\,dz$$

Also:

$$W(\vec{r} \to \vec{r}+d\vec{r}) = -\left[U(\vec{r}+d\vec{r}) - U(\vec{r})\right] \equiv dU$$
$$= -\left[U(x+dx,\ y+dy,\ z+dz) - U(x,y,z)\right]$$

Note: for $f(x)$ of one variable, one can write

$$df = f(x+dx) - f(x) = \frac{df}{dx}\,dx$$

For a fn of 3 variables, we have instead:

$$dU = U(x+dx,\ y+dy,\ z+dz) - U(x,y,z)$$
$$= \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz$$

partial derivatives

$$W(\vec{r} \to \vec{r}+d\vec{r}) = dU$$

$$W(\vec{r} \to \vec{r}+d\vec{r}) = -\left[\frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz\right]$$
$$= F_x\,dx + F_y\,dy + F_z\,dz$$

$$\Rightarrow F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} = -\frac{\partial U}{\partial x}\,\hat{x} - \frac{\partial U}{\partial y}\,\hat{y} - \frac{\partial U}{\partial z}\,\hat{z}$$

Remember: $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{x} + \frac{\partial f}{\partial y}\,\hat{y} + \frac{\partial f}{\partial z}\,\hat{z}$ $\Rightarrow$ $\boxed{\vec{F} = -\vec{\nabla} U}$

One can derive $\vec{F}$ from U!!

You can thing of $\vec{\nabla}$ as an operator:

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

[then $\vec{\nabla} f = \hat{x}\frac{\partial}{\partial x} f + \hat{y}\frac{\partial}{\partial y} f + \hat{z}\frac{\partial}{\partial z} f$ as usual]

We can see that $du = \underbrace{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz}_{= \vec{\nabla}u \cdot d\vec{r}}$

Thus for any function, $df = \vec{\nabla} \cdot d\vec{r}$ & in 1D, this reduces to $df = \frac{df}{dx}dx$

The 2nd condition that $\vec{F}$ is conservative:

The work done by a force $\vec{F}$ is path-independent iff:

$$\underbrace{\vec{\nabla} \times}_{\text{"the curl of } \vec{F}\text{"}} \vec{F} = 0$$

If $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ and $\vec{F} = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$, then $\vec{\nabla}\times\vec{F}$ is just a cross product

$$= \hat{x}_1\frac{\partial}{\partial x_1} + \hat{x}_2\frac{\partial}{\partial x_2} + \hat{x}_3\frac{\partial}{\partial x_3} = \hat{x}_i\frac{\partial}{\partial x_i} \qquad = F_1\hat{x}_1 + F_2\hat{x}_2 + F_3\hat{x}_3 = F_i\hat{x}_i$$

$$(\vec{\nabla}\times\vec{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}F_k \qquad \text{just like} \qquad (\vec{A}\times\vec{B})_i = \varepsilon_{ijk}A_jB_k$$

$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

Example: Is $F_g = mg\hat{z}$ conservative?

$$\vec{\nabla}\times\vec{F} = \frac{\partial}{\partial y}F_z\hat{x} - \frac{\partial}{\partial x}F_z\hat{y} \quad , \ F_z = mg$$

$$= 0 - 0 = 0 \quad \Rightarrow \text{yes} \ ☑$$

Is $F_G = \frac{GmM}{r^2}\hat{r}$ conservative?

$$= \frac{GmM\,(r\hat{r})}{r^2 r} = \frac{GmM\,\vec{r}}{r^3} = \frac{GmM\,(x\hat{x}+y\hat{y}+z\hat{z})}{(x^2+y^2+z^2)^{3/2}} = \frac{GmM\,x_a\hat{x}_a}{(x_\ell x_\ell)^{3/2}}$$

$$(\vec{\nabla}\times\vec{F})_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}F_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{GmM\,x_k}{(x_\ell x_\ell)^{3/2}}\right]$$

$$= GmM\,\varepsilon_{ijk}\left[\frac{\partial x_k}{\partial x_j}\frac{1}{(x_\ell x_\ell)^{3/2}} + x_k\frac{\partial}{\partial x_j}(x_\ell x_\ell)^{-3/2}\right]$$

$$= GmM\,\underbrace{\varepsilon_{ijk}}_{\substack{=0\\ \text{if } j=k}}\left[\underbrace{\delta_{jk}}_{\substack{=0\\ \text{if } j\neq k}}\frac{1}{(x_\ell x_\ell)^{3/2}} + x_k(-3/2)(x_{\ell'}x_{\ell'})^{-5/2}\underbrace{\left(\frac{\partial x_\ell}{\partial x_j}x_\ell + x_\ell\frac{\partial x_\ell}{\partial x_j}\right)}_{=2x_\ell\frac{\partial x_\ell}{\partial x_j}}\right]$$

$$= GmM\,\varepsilon_{ijk}\left[0 - \frac{3}{2}\frac{x_k}{(x_{\ell'}x_{\ell'})^{5/2}}\left(2x_\ell\,\delta_{\ell j}\right)\right]$$

$$= -3GmM\,\varepsilon_{ijk}\frac{x_k}{(x_{\ell'}x_{\ell'})^{5/2}}(x_j)$$

$$= -3GmM\frac{\varepsilon_{ijk}x_jx_k}{(x_\ell x_\ell)^{5/2}}$$

We know $\varepsilon_{ijk}x_jx_k = 0$

Why? $\varepsilon_{ijk}x_jx_k = \varepsilon_{ikj}x_kx_j = \varepsilon_{ikj}x_jx_k$

rename
$j \to k$
$k \to j$

$\varepsilon_{ijk}x_jx_k = \varepsilon_{ikj}x_jx_k$

switch j & k $\quad \varepsilon_{ijk} = -\varepsilon_{ikj}$

$\varepsilon_{ijk}x_jx_k = -\varepsilon_{ijk}x_jx_k \Rightarrow$ must $=0$

(only # s.t. $n=-n$ is $n=0$)

$$\Rightarrow (\vec{\nabla}\times\vec{F}_G)_i = -3mM\frac{\cancel{\varepsilon_{ijk}x_jx_k}^{\,0}}{(x_\ell x_\ell)^{5/2}} = 0 \Rightarrow \text{conservative!}$$