Lecture 4 1/28/26

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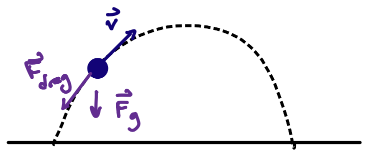
Linear Air Resistance (the thing you used to ignore)



Air resistance $\Rightarrow$ different kinds of forces (drag, lift, etc.)
Here we focus only on drag force

$$\vec{F}_{drag} = -f(v)\,\hat{v}$$

At low speeds, one can model $f(v)$ as:

$$f(v) = \underbrace{b}_{\uparrow}v + \underbrace{c}_{\uparrow}v^2 = f_{lin} + f_{quad}$$

depends on object's shape.

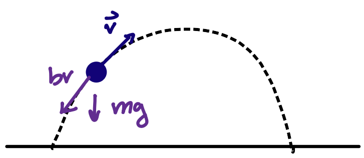
Newton's 2^{nd} law: $\vec{F} = m\vec{a}$

$$\vec{F}_{drag} + \dots = m\ddot{\vec{r}}$$

$\Rightarrow$ differential equation. Difficult to solve if $f(v) = bv + cv^2$. Luckily, usually one term dominates & you can ignore the other

Consider $\dfrac{f_{quad}}{f_{lin}} \ll 1 \Rightarrow \vec{f}(v) \approx -bv\,\hat{v} \Rightarrow$ linear air resistance

Linear Air Resistance



$$\vec{F} = \vec{F}_{drag} + \vec{F}_g = -b\vec{v} + m\vec{g}$$

$$m\ddot{\vec{r}} = -b\vec{v} + m\vec{g} \quad \Rightarrow \text{Nothing depends on position}$$

$$m\dot{\vec{v}} = -b\vec{v} - m\vec{g} \quad \text{First Order dif. eq.}$$

$$m\dot{\vec{v}} = (-bv_x)\,\hat{x} + (-\underbrace{bv_y}_{} - mg)\,\hat{y}$$

Note that v_y could be ≤ 0
or ≥ 0

<u>Horizontal drag</u>: $m\dot{v}_x = -bv_x$

$$\dot{v}_x = \frac{-b}{m} v_x \qquad \text{let } k \equiv b/m$$

$$\dot{v}_x = -k v_x$$

Just by looking at this, we can see

(derivative of v_x) = (constant) (v_x)

So you can guess $v_x \sim e^{(\text{constant})\, t}$

You can also find this directly using <u>separation of variables</u>:

$$\frac{dv_x}{dt} = -k v_x$$

$$\frac{dv_x}{v_x} = -k\, dt$$

$$\int_{\underbrace{v_x(t=0)}_{v_{x0}}}^{v_x(t)} \frac{1}{v_x'}\, dv_x' = \int_0^t -k\, dt'$$

$$\ln(v_x') \Big|_{v_{x0}}^{v_x} = -kt' \Big|_0^t \Rightarrow \ln(v_x) - \ln(v_{x0}) = -kt$$

$$\ln\left(\frac{v_x}{v_{x0}}\right) = -kt$$

$$\frac{v_x}{v_{x0}} = e^{-kt}$$

$$\boxed{v_x = v_{x0}\, e^{-kt}}$$

Often, it's convenient to define $\tau = 1/k$

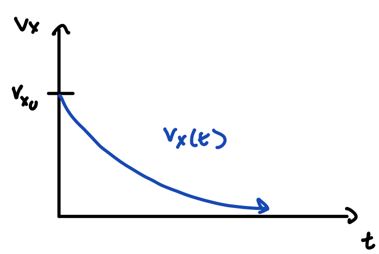
$$\boxed{v_x = v_{x0}\, e^{-t/\tau}} \quad \text{where } \tau = m/b$$

Because $[t] = s$

$[-kt] = 1$

$[k] = 1/s$

As $t \to \infty$, $e^{-t/\tau} \to 0$



Next, find $x(t)$: $v_x = v_{x0}\, e^{-t/\tau}$

$$\frac{dx}{dt} = v_{x0}\, e^{-t/\tau}$$

$$v_x = v_{x_0} e^{-t/\tau}$$

$$\frac{dx}{dt} = v_{x_0} e^{-t/\tau}$$

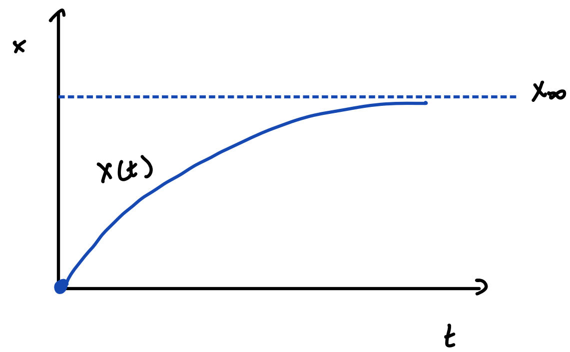
$$\int_{x_0}^{x} dx' = \int_0^t v_{x_0} e^{-t'/\tau} dt'$$

$$x' \Big|_{x_0}^{x} = v_{x_0} (-\tau) e^{-t'/\tau} \Big|_0^t$$

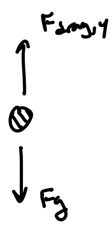
$$x - x_0 = -\tau v_{x_0} \left(e^{-t/\tau} - 1 \right)$$

$$\boxed{x(t) = \tau v_{x_0} \left(1 - e^{-t/\tau}\right) + x_0}$$

let $x_0 = 0$. As $t \to \infty$, $x(t) \to \tau v_{x_0} \equiv x_\infty$



vertical drag:



$$F_y = -b\underbrace{\dot{y}}_{=v_y} - mg$$

If $\dot{y} < 0$, then $F_{drag} \uparrow$

What is the terminal velocity?

$$F_y = 0 \Rightarrow 0 = -b\dot{y} - mg$$

$$\boxed{\dot{y} = \frac{-mg}{b}} = v_y^{term}$$

Let's find $v_y(t)$ in general:

$$m\dot{v}_y = -bv_y - mg$$

$$\frac{dv_y}{dt} = -kv_y - g$$

$$\int_{v_{y_0}}^{v_y} \frac{dv_y'}{kv_y' + g} = \int_0^t -dt'$$

$$\frac{1}{k} \int_{v_{y_0}}^{v_y} \frac{k\,dv_y'}{kv_y' + g} = -\int_0^t dt'$$

let $u = kv_y' + g$
$\Rightarrow du = k\,dv_y'$

$$\int \frac{du}{u} = \ln(u)$$

$$\frac{1}{k} \ln\left(kv_y' + g\right)\Big|_{v_{y_0}}^{v_y} = -t'\Big|_0^t$$

$$\frac{1}{k} \ln\left(\frac{kv_y + g}{kv_{y_0} + g}\right) = -t$$

$$\ln\left(\frac{kv_y+g}{kv_{y_0}+g}\right) = -kt$$

$$\frac{kv_y+g}{kv_{y_0}+g} = e^{-kt}$$

$$kv_y + g = (kv_{y_0}+g)e^{-kt}$$

$$\boxed{v_y = \left(v_{y_0} + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}}$$

let $\tau = 1/k \Rightarrow v_y(t) = (v_{y_0} + g\tau)\, e^{-t/\tau} - g\tau$

$$\lim_{t\to\infty} v_y(t) = -g\tau = \frac{-g}{k} = \frac{-g}{(b/m)} = -\frac{mg}{b}$$

let $v_{term} = \left| \frac{-mg}{b} \right|$

$$v_y = v_{y_0}e^{-kt} - v_{term}\left(1 - e^{-kt}\right)$$

Next, find $y(t)$:

$$\frac{dy}{dt} = \left(v_{y_0} + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}$$

$$\int_{y_0}^{y} dy' = \int_0^t \left[\left(v_{y_0} + \frac{g}{k}\right)e^{-kt'} - g/k\right] dt'$$

$$y'\Big|_{y_0}^{y} = -\frac{1}{k}\left(v_{y_0} + \frac{g}{k}\right)e^{-kt'} - \frac{g}{k}t' \Big|_0^t$$

$$y - y_0 = -\frac{1}{k}\left(v_{y_0} + \frac{g}{k}\right)e^{-kt} + \frac{1}{k}\left(v_{y_0} + \frac{g}{k}\right) - g/k\, t$$

remember $v_{term} = g/k$

$$\boxed{y(t) = y_0 - v_{term}t + \tau(v_{0y} + v_{term})(1 - e^{-t/\tau})}$$

Also, $x(t) = v_{0x}\tau\left(1 - e^{-t/\tau}\right) \Rightarrow \frac{x(t)}{v_{0x}\tau} = 1 - e^{-t/\tau} \Rightarrow -\frac{t}{\tau} = \ln\left(1 - \frac{x(t)}{v_{0x}\tau}\right)$

$$\Rightarrow y(t) = y_0 - v_{term}t + \cancel{\tau}(v_{0y} + v_{term})\frac{x(t)}{v_{0x}\cancel{\tau}}$$

$$\boxed{y(t) = y_0 + v_{term}\tau \ln\left(1 - \frac{x(t)}{v_{0x}\tau}\right) + \frac{v_{0y} + v_{term}}{v_{0x}}\, x(t)}$$