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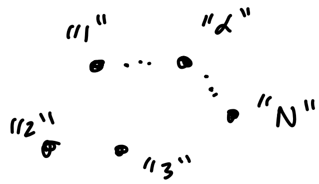
Lecture 10, 2/11/26

Angular Momentum of a System of N particles

Consider a system of N particles $\alpha = 1, 2, \dots, N$ with $\vec{l}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$



The total $\vec{L}$ is $\quad \vec{L} = \sum_{\alpha=1}^{N} l_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$

How does $\vec{L}$ change with time?

$\dot{\vec{L}} = \sum_\alpha \dot{\vec{l}}_\alpha = \vec{r}_\alpha \times \vec{F}_\alpha \qquad$ Remember, $\vec{F}_\alpha = \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext}$

$$\dot{\vec{L}} = \vec{r}_\alpha \times \left[\sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{ext} \right] = \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\alpha \times \vec{F}_\alpha^{ext}$$

We can use the same trick as before: (reorganize the sum)

$$\sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \sum_\alpha \left(\sum_{\beta < \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_{\beta > \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} \right) \qquad \vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$$

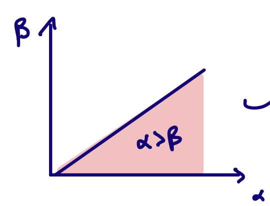
$$= \sum_\alpha \sum_{\beta < \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} - \sum_\alpha \sum_{\beta > \alpha} \vec{r}_\alpha \times \vec{F}_{\beta\alpha}$$

Next, reindex:
$\alpha \to \beta$
$\beta \to \alpha$

$$= \sum_\alpha \sum_{\beta < \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} - \sum_\beta \underbrace{\sum_{\alpha > \beta}}\, \vec{r}_\beta \times \vec{F}_{\alpha\beta}$$

Note: $\quad \sum_\beta \sum_{\alpha > \beta} f(\alpha, \beta) = \sum_\alpha \sum_{\beta < \alpha} f(\alpha, \beta)$

These sums are the same



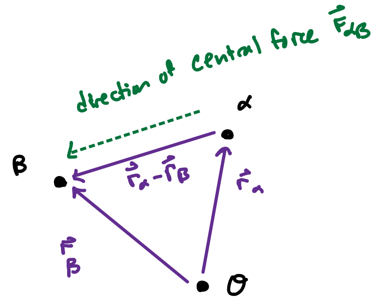
This region ($\alpha > \beta$) is identical to the $\beta < \alpha$ region

$$= \sum_\alpha \sum_{\beta < \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} - \sum_\alpha \sum_{\beta < \alpha} \vec{r}_\beta \times \vec{F}_{\alpha\beta}$$

$$= \sum_\alpha \sum_{\beta < \alpha} \left(\vec{r}_\alpha \times \vec{F}_{\alpha\beta} - \vec{r}_\beta \times \vec{F}_{\alpha\beta} \right)$$

$$= \sum_\alpha \sum_{\beta < \alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} \quad \Rightarrow \quad \boxed{\dot{\vec{L}} = \sum_{\beta < \alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} + \vec{r}_\alpha \times \vec{F}_\alpha^{ext}}$$

$$\boxed{\dot{\vec{L}} = \sum_{\beta<\alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} + \vec{r}_\alpha \times \vec{F}_\alpha^{ext}}$$



If all $\vec{F}_{\alpha\beta}$ are <u>central forces</u>, then $\vec{F}_{\alpha\beta} \parallel \vec{r}_\alpha - \vec{r}_\beta$ $\Rightarrow$

$$\Rightarrow (\vec{r}_\alpha \times \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0$$

$$\Rightarrow \dot{\vec{L}} = \vec{r}_\alpha \times \vec{F}_\alpha^{ext} = \vec{\Gamma}^{ext}$$

Conservation of angular momentum:

If a net external torque on an N-particle system $= 0$, then the total $\vec{L} = \vec{r}_\alpha \times \vec{p}_\alpha =$ constant (Assuming all internal forces are central forces, which is almost always true)

Moment of Inertia:

When a rigid body rotates on an axis (let's call it the z-axis), we can calculate L_z using the <u>moment of inertia</u>:

$$L_z = I\omega \qquad I = m_\alpha r_\alpha^2 \qquad \text{This is derived from } \vec{L} = \vec{r} \times \vec{p} \text{ in Ch.10}$$

$\omega = \dot{\phi}$

$$\hookrightarrow \int r^2\, dm = \int r^2 \rho\, dV \quad \text{for solid objects}$$

Angular momentum about CM:

In an inertial frame, $\dot{\vec{L}} = \vec{\Gamma}^{ext}$

In <u>any</u> frame, $\dot{\vec{L}}(\text{about CM}) = \vec{\Gamma}^{ext}(\text{about CM})$, even when CM is accelerating! (Proof in Ch.10)

Chapter 4: Kinetic Energy and Work

Kinetic energy: $T \equiv \frac{1}{2}mv^2$ $\Rightarrow$ useful! But why?

Let's take its derivative: $\dot{T} = \frac{1}{2}m \frac{d}{dt}(\vec{v} \cdot \vec{v})$

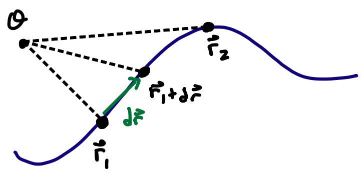
$$= \frac{1}{2}m(\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}}) = m\dot{\vec{v}} \cdot \vec{v} = m\vec{a} \cdot \vec{v}$$

$$\dot{T} = \vec{F} \cdot \vec{v} = \vec{F} \cdot \dot{\vec{r}}$$

$$\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$dT = \vec{F} \cdot d\vec{r}$$

Change in the system : $d T = \vec{F}\cdot d\vec{r} = dW$: Work done by $\vec{F}$ over a distance $d\vec{r}$

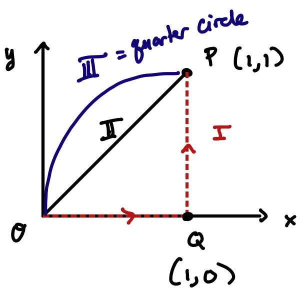


$$\Delta T \equiv T_2 - T_1 = \sum \vec{F}\cdot d\vec{r} \longrightarrow \int_1^2 \vec{F}\cdot d\vec{r}$$ (line integral)

$$\Delta T = T_2 - T_2 = \int_1^2 \vec{F}\cdot d\vec{r} \equiv W(1\to 2)$$ The work- KE theorem

This is the total force

Practice with line integrals:



$$\vec{F} = y\hat{x} + 2x\hat{y}$$

$$W_I = \int_I \vec{F}\cdot d\vec{r} = \int_O^Q \vec{F}\cdot d\vec{r} + \int_Q^P \vec{F}\cdot d\vec{r}$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

From $O \to Q$, $dy = 0 \Rightarrow d\vec{r} = dx\,\hat{x} + \cancel{dy}\,\hat{y} = dx\,\hat{x}$

From $Q \to P$, $dx = 0 \Rightarrow d\vec{r} = \cancel{dx}\,\hat{x} + dy\,\hat{y} = dy\,\hat{y}$

$$W_I = \int_{O_x}^{Q_x} \underbrace{\vec{F}\cdot\hat{x}}_{=F_x}\,dx + \int_{Q_y}^{P_y} \underbrace{\vec{F}\cdot\hat{y}}_{F_y}\,dy$$

$$W_I = \int_0^1 y\,dx + \int_0^1 2x\,dy = \int_0^1 \cancel{(0)\,dx} + \int_0^1 2(1)\,dy$$

y=0 between O & Q

x=1 between Q & P

$$W_I = \int_0^1 2\,dy = 2y\Big|_0^1 = \boxed{2 = W_I}$$

Next, calculate W_{II}:

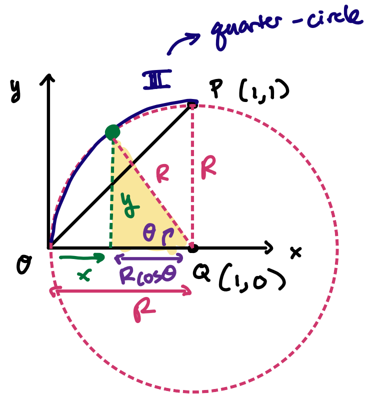
$$W_{II} = \int_{II} \vec{F}\cdot d\vec{r}$$ on path II: $x = y \Rightarrow dx = dy$

$$W_{II} = \int_{II} \vec{F}\cdot\left(dx\,\hat{x} + dy\,\hat{y}\right) = \int_{II}\left(y\,dx + 2x\,dy\right) = \int_{O_x}^{P_x}\left(x\,dx + 2x\,dx\right) = \int_0^1 3x\,dx = \frac{3}{2}x^2\Big|_0^1 = \boxed{\frac{3}{2} = W_{II}}$$

use $x=y$

$dx=dy$

W_{III}: On path III,



$\Rightarrow\ y = R\sin\theta \qquad R = 1$

$$\boxed{\begin{aligned} y &= \sin\theta \\ dy &= \cos\theta\, d\theta \end{aligned}}$$

$\Rightarrow\ x + R\cos\theta = R \qquad R = 1$

$$\boxed{x + \cos\theta = 1}$$

$dx - \sin\theta\, d\theta = 0$

$$\boxed{dx = \sin\theta\, d\theta}$$

At point P, $\theta = \pi/2$

At O, $\theta = 0$

$$W_{III} = \int_{III}\vec{F}\cdot d\vec{r}$$

$$= \int_{III} F_x\,dx + F_y\,dy$$

$$= \int_{III} y\,dx + 2x\,dy$$

$$= \int_{O_\theta}^{P_\theta}\left[(\sin\theta)(\sin\theta\,d\theta) + 2(1-\cos\theta)(\cos\theta\,d\theta)\right]$$

$$= \int_0^{\pi/2}\left(\sin^2\theta - 2\cos\theta - 2\cos^2\theta\right)d\theta = 2\int_0^{\pi/2}\cos\theta\,d\theta + \int_0^{\pi/2}\left[\left(\sin^2\theta - \cos^2\theta\right) - \cos^2\theta\right]d\theta$$

$$= 2\sin\theta\Big|_0^{\pi/2} + \int_0^{\pi/2}\left[\left(-\cos 2\theta\right) - \frac{1}{2}\left(\cos 2\theta + 1\right)\right]d\theta$$

$$= 2\sin\pi/2 - 0 + \int_0^{\pi/2}\left(-3/2\cos 2\theta - 1/2\right)d\theta$$

$$= 2(1) - \frac{3}{2}\,\frac{1}{2}\sin 2\theta\Big|_0^{\pi/2} - \frac{1}{2}\theta\Big|_0^{\pi/2}$$

$$= 2 - 3/4\left(\overset{0}{\cancel{\sin\pi}} - 0\right) - \pi/4 + 0$$

$$\boxed{W_{III} = 2 - \pi/4}$$