# Lecture 3 1/26/26

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## Review of last week:

**Wednesday:** vectors
- notation
- basic vector operations
- vector differentiation
- Newton's $1^{st}$ & $2^{nd}$ laws

**Friday:** Practiced using Newton's $2^{nd}$ law:

For a constant force, derived Kinematic equations:

$$\vec{F} = m\vec{a}, \quad F_0\hat{x} = ma\hat{x}$$

$$F_0 = ma = m\frac{d}{dt}v$$

$$\int_0^t F_0\, dt' = m\int_0^t \frac{d}{dt'}v\, dt'$$

$$\frac{F_0}{m} t' \Big|_0^t = v(t')\Big|_0^t$$

$$\frac{F_0}{m} t = v(t) - \overbrace{v(0)}^{v_0} = \frac{d}{dt}x - v_0$$

$$\int_0^t \left(\frac{F_0}{m}t' + v_0\right) dt' = \int_0^t \frac{d}{dt'}x\, dt'$$

$$\frac{F_0}{m}\frac{1}{2}(t')^2\Big|_0^t + v_0 t'\Big|_0^t = x(t')\Big|_0^t$$

$$\frac{1}{2}\frac{F_0}{m}t^2 + v_0 t = x(t) - \overbrace{x(0)}^{x_0}$$

$$\boxed{\frac{F_0}{2m}t^2 + v_0 t + x_0 = x(t)}$$

These steps are very common in kinematics problems

Newton's $3^{rd}$ law $\quad \vec{F}_{12} = -\vec{F}_{21}$

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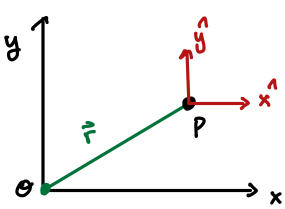
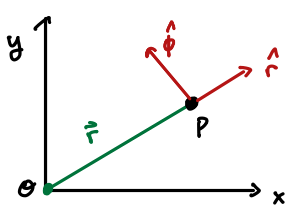
## Polar Coordinates

$$x = r\cos\phi \qquad \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\phi \quad \Longleftrightarrow \quad \phi = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\vec{r} = r\hat{r}$$




- $\hat{x}$ & $\hat{y}$ are always pointing the same direction
- $\hat{r}$ & $\hat{\phi}$ depend on where point P is located wrt the origin.

Since $\hat{r}$ & $\hat{\phi}$ span the 2D plane, we can write:

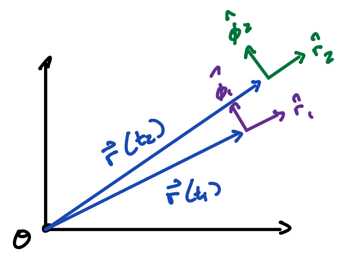
$$\vec{F} = F_r \hat{r} + F_\phi \hat{\phi}$$

$\vec{F} = m\vec{a} = m\ddot{\vec{r}}$ To find $F_r$ and $F_\phi$, we need $\ddot{\vec{r}}$

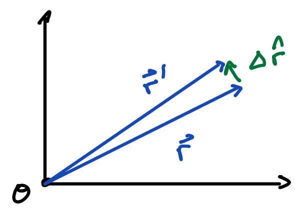
$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\underbrace{\frac{d\hat{r}}{dt}}_{\text{we need to find this}}$$

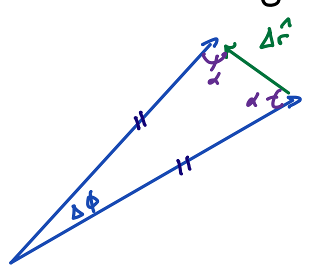
$$\frac{d\hat{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \hat{r}}{\Delta t}$$



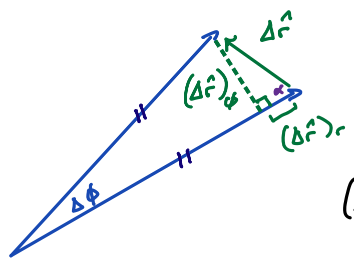

Since $\hat{\phi}$ & $\hat{r}$ don't change when $\vec{r}(t_2)$ changes length, we can simplify $|\vec{r}(t_2)| = |\vec{r}(t_1)| \equiv r = 1$ and will get the same answer




Zoom in on triangle:




$\Rightarrow$




$$\sin\alpha = \frac{(\Delta\hat{r})_\phi}{|\Delta\hat{r}|} \qquad \cos\alpha = \frac{-(\Delta\hat{r})_r}{|\Delta\hat{r}|}$$

$$(\Delta\hat{r})_\phi = |\Delta\hat{r}|\sin\alpha = |\Delta\hat{r}|\underbrace{\cos\left(\tfrac{1}{2}\Delta\phi\right)}_{\approx 1} \approx |\Delta\hat{r}|$$

$$(\Delta\hat{r})_r = -|\Delta\hat{r}|\sin\left(\tfrac{1}{2}\Delta\phi\right) \approx -|\Delta\hat{r}|\frac{\Delta\phi}{2}$$

$$\Delta\phi + \alpha + \alpha = 180^\circ$$
$$\alpha = 90^\circ - \tfrac{1}{2}\Delta\phi$$

$$\Rightarrow \sin\alpha = \cos\left(\tfrac{1}{2}\Delta\phi\right)$$
$$\cos\alpha = \sin\left(\tfrac{1}{2}\Delta\phi\right)$$

$$\sin\Delta\phi = \frac{|\Delta\hat{r}|}{|\hat{r}|} \Rightarrow |\Delta\hat{r}| = \sin\Delta\phi \approx \Delta\phi$$

$$\Rightarrow (\Delta\hat{r})_\phi = \Delta\phi$$
$$(\Delta\hat{r})_r = -\Delta\phi\frac{\Delta\phi}{2} \approx 0$$

$$\boxed{\Delta\hat{r} = \Delta\phi\,\hat{\phi}}$$

$$\lim_{\Delta t \to 0}\frac{\Delta\hat{r}}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta\phi}{\Delta t}\hat{\phi} = \dot{\phi}\hat{\phi}$$

$$\boxed{\frac{d\hat{r}}{dt} = \dot{\phi}\hat{\phi}}$$

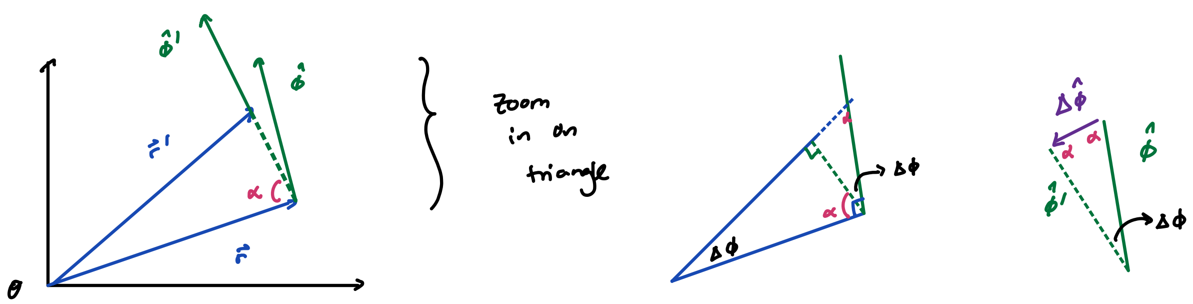
$\Rightarrow \dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\frac{d\hat{r}}{dt}$

$\boxed{\vec{v} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}}$ Can be written as $\vec{v} = v_r\,\hat{r} + v_\phi\,\hat{\phi}$

$v_\phi = r\dot{\phi} = r\omega$ (familiar)

Next, find $\ddot{\vec{r}}$ :

$$\ddot{\vec{r}} = \ddot{r}\,\hat{r} + \dot{r}\,\frac{d\hat{r}}{dt} + (\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\phi} + r\dot{\phi}\,\frac{d\hat{\phi}}{dt}$$




As $\Delta\phi \to 0$, $\alpha + \alpha + \Delta\phi = 180^\circ$, $\alpha \to 90^\circ$

$\Delta\hat{\phi} \perp \hat{\phi} \Rightarrow \Delta\hat{\phi} \parallel \hat{r}$ & $\Delta\hat{\phi}$ points towards origin

$\Rightarrow \Delta\hat{\phi} = -|\Delta\hat{\phi}|\,\hat{r}$

$$\sin\Delta\phi = \frac{|\Delta\hat{\phi}|}{|\hat{\phi}|} = |\Delta\hat{\phi}|$$

$|\Delta\hat{\phi}| = \sin\Delta\phi \approx \Delta\phi \Rightarrow \Delta\hat{\phi} = -\Delta\phi\,\hat{r}$ & $\frac{\Delta\hat{\phi}}{\Delta t} = \frac{-\Delta\phi}{\Delta t}\,\hat{r}$

$$\Rightarrow \frac{d\hat{\phi}}{dt} = -\frac{d\phi}{dt}\,\hat{r} = -\dot{\phi}\,\hat{r}$$

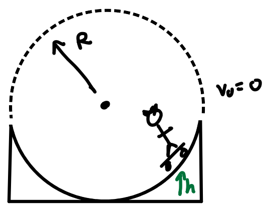
$$\vec{a} = \ddot{r}\,\hat{r} + \dot{r}\,\overbrace{\frac{d\hat{r}}{dt}}^{=\dot{\phi}\hat{\phi}} + (\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\phi} + r\dot{\phi}\,(-\dot{\phi}\,\hat{r})$$

$$= (\ddot{r} - r\dot{\phi}^2)\,\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\phi}$$

$\vec{F} = m\vec{a} \Rightarrow F_r = m\ddot{r} - mr\dot{\phi}^2$

$F_\phi = 2m\dot{r}\dot{\phi} + mr\ddot{\phi}$

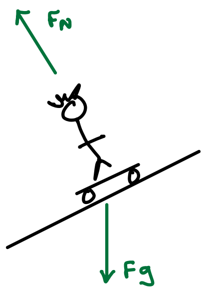
## Example:

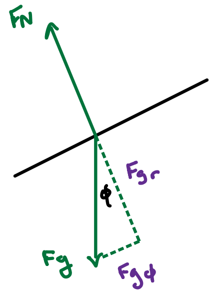



Assume $h \ll R$

Describe $\hat{r}(t)$.

$$\vec{F} = m\ddot{\vec{r}}$$




$$F_{g\phi} = F_g \sin\phi$$
$$F_{gr} = F_g \cos\phi$$

$$\Rightarrow \vec{F} = (-N + mg\cos\phi)\hat{r} - mg\sin\phi\,\hat{\phi}$$

In the $\hat{\phi}$-direction:

$$-mg\sin\phi = m a_\phi = m(2\dot{r}\dot{\phi} + r\ddot{\phi})$$

We know $r = R = \text{constant}$: $\Rightarrow$ $-mg\sin\phi = mR\ddot{\phi}$

$\phi \ll 1$ $\Rightarrow$ $-mg\phi = mR\ddot{\phi}$

$$\boxed{\ddot{\phi} = -\frac{g}{R}\phi}$$

Use a trial fn: $\phi = A\cos(kt) + B\sin(kt)$

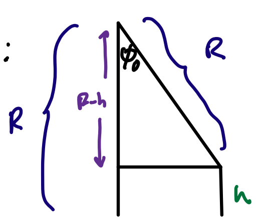
$$\dot{\phi} = -Ak\sin(kt) + Bk\cos(kt)$$

$$\ddot{\phi} = -Ak^2\cos(k) - Bk^2\sin(kt)$$

$$\ddot{\phi} = -k^2\phi \quad \Rightarrow \quad k^2 = \frac{g}{R}$$

$$\Rightarrow \phi(t) = A\sin\left(\sqrt{\frac{R}{g}}\,t\right) + B\cos\left(\sqrt{\frac{R}{g}}\,t\right)$$

Initially:




$$\cos\phi_0 = \frac{R-h}{R}$$

$$\phi_0 = A\sin(0) + B\cos(0) = B$$

$$\phi_0 = \boxed{B = \cos^{-1}\left(\frac{R-h}{R}\right)}$$

$V_0 = 0$

$$\dot{\phi}_0 = -\sqrt{\frac{R}{g}}A\cos(0) + \sqrt{\frac{R}{g}}B\sin(0) = -\sqrt{\frac{R}{g}}A = 0 \Rightarrow \boxed{A = 0}$$

$$\phi(t) = \cos^{-1}\left(\frac{R-h}{h}\right) \cos\left(\sqrt{\frac{R}{g}}\, t\right)$$

$$\boxed{\vec{r}(t) = R\hat{r} + \cos^{-1}\left(\frac{R-h}{h}\right) \cos\left(\sqrt{R/g}\; t\right) \hat{\phi}}$$

<u>Air Resistance</u>