Activity 6

1. In your reading, you saw that

$$\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)\,, \qquad \text{where} \qquad \vec{\nabla}_i f = \hat{x}\frac{\partial f}{\partial x_i} + \hat{y}\frac{\partial f}{\partial y_i} + \hat{z}\frac{\partial f}{\partial z_i} \tag{1}$$

   (a) Show $\frac{\partial}{\partial x_1} f(x_1 - x_2) = -\frac{\partial}{\partial x_2} f(x_1 - x_2)$

$$\frac{\partial}{\partial x_1} f(x_1 - x_2) = f'(x_1-x_2)\frac{\partial}{\partial x_1}(x_1-x_2) = f'(x_1-x_2)(1-0) = f'(x_1-x_2)$$

$$\frac{\partial}{\partial x_2} f(x_1 - x_2) = f'(x_1-x_2)\frac{\partial}{\partial x_2}(x_1-x_2) = f'(x_1-x_2)(0-1) = -f'(x_1-x_2) = -\frac{\partial}{\partial x_1} f(x_1-x_2) \quad \boxtimes$$

   (b) Generalize this to argue that Eq. (1) is true.

$$[\vec{\nabla}_1 U(\vec{r}_1-\vec{r}_2)]_x = \frac{\partial}{\partial x_1}\underbrace{U(\vec{r}_1-\vec{r}_2)}_{\text{rewrite this}} = \frac{\partial}{\partial x_1} U(x_1-x_2, y_1-y_2, z_1-z_2) = U'(x_1-x_2, y_1-y_2, z_1-z_2)\frac{\partial}{\partial x_1}(x_1-x_2)$$

$$= U'(\vec{r}_1-\vec{r}_2)(1-0) = U'(\vec{r}_1-\vec{r}_2)$$

$$[\vec{\nabla}_2 U(\vec{r}_1-\vec{r}_2)]_x = \frac{\partial}{\partial x_2} U(\vec{r}_1-\vec{r}_2) = \frac{\partial}{\partial x_1} U(x_1-x_2, y_1-y_2, z_1-z_2) = U'(x_1-x_2, y_1-y_2, z_1-z_2)\frac{\partial}{\partial x_2}(x_1-x_2)$$

$$= U'(\vec{r}_1-\vec{r}_2)(0-1) = -U'(\vec{r}_1-\vec{r}_2)$$

So $(\vec{\nabla}_1 U)_x = -(\vec{\nabla}_2 U)_x$ and the same is true for $(\vec{\nabla} U)_y, (\vec{\nabla} U)_z \Rightarrow \vec{\nabla}_1 U = -\vec{\nabla}_2 U$ ☑

2. **Energy of Interaction of Two Particles**. Consider the gravitational force between two isolated masses:

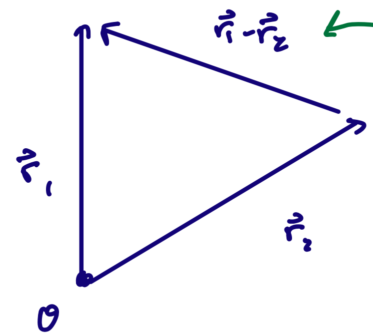
$$\vec{F}_{12} = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2) \tag{2}$$

   (a) Does $\vec{F}_{12}$ depend on the position of the origin from which $\vec{r}_1$ and $\vec{r}_2$ are measured? Explain why or why not.

No.

$$\vec{r}_1 - \vec{r}_2 \to (\vec{r}_1 + \cancel{\vec{a}}) - (\vec{r}_2 + \cancel{\vec{a}})$$

$$\to \vec{r}_1 - \vec{r}_2$$



This vector doesn't care where origin is

(b) The total work on the system is given by:

$$W = d\vec{r}_1 \cdot \vec{F}_{12} + d\vec{r}_2 \cdot \vec{F}_{21} \,. \tag{3}$$

Show that the total work on the system can be written as

$$W = -dU \,, \tag{4}$$

where recall the chain rule, $dU = d\vec{r} \cdot \vec{\nabla} U(\vec{r})$

$\vec{F}_{12} = -\nabla_1 U(\vec{r}_1)$ If particle 2 is at origin (where $\vec{u}(\vec{r})=0$)

$= -\nabla_1 U(\vec{r}_1 - \vec{r}_2)$ If particle 2 is not at origin

Note that $\vec{\nabla}_1 = \vec{\nabla}_{(1-2)} = \vec{\nabla}$

$W = -d\vec{r}_1 \cdot \vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2) + d\vec{r}_2 \cdot (-\vec{F}_{12})$

$= -d\vec{r}_1 \cdot \nabla_1 U(\vec{r}_1 - \vec{r}_2) + d\vec{r}_2 \cdot \vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2) = -d(\vec{r}_1 - \vec{r}_2) \cdot \vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)$

$= -d\vec{r} \cdot \vec{\nabla} U(\vec{r}) = -dU$

(c) Show that for an elastic collision, $\Delta T = 0$. Use the fact that an elastic collision assumes the particles interact only via a conservative force, and that their potential energy of interaction $U(\vec{r}_1 - \vec{r}_2) \to 0$ as their separation goes to infinity.

$W = W$

$\Delta T = -\Delta U$

$\Delta T = -U_f + U_i = 0 + 0$

$\Delta T = 0$ $\square$

3. As you saw in the reading, for an N-particle system,

$$T = \sum_{\alpha} T_\alpha = \sum_{\alpha} \frac{1}{2} m_\alpha v_\alpha^2 \quad \text{and} \quad U = U^{\text{int}} + U^{\text{ext}}, \quad \text{where} \quad U^{\text{int}} = \sum_{\alpha} \sum_{\beta > \alpha} U_{\alpha\beta}(|\vec{r}_\alpha - \vec{r}_\beta|)$$

(a) If these N particles form a rigid body, how does $|\vec{r}_\alpha - \vec{r}_\beta|$ change with time? In other words, from an initial to a final state, what is $\Delta(|\vec{r}_\alpha - \vec{r}_\beta|)$?

$$\Delta(|\vec{r}_\alpha - \vec{r}_\beta|) = 0$$

(b) Given this, what is ΔU_{int} for a rigid body?

$$\Delta U_{int} = 0$$

(c) Write down the change in total mechanical energy ΔE for a rigid body.

$$\Delta E = \Delta T + \Delta U = \Delta T + \cancel{\Delta U^{int}}^{\,0} + \Delta U^{ext}$$

$$\boxed{\Delta E = \Delta T + \Delta U_{ext}}$$