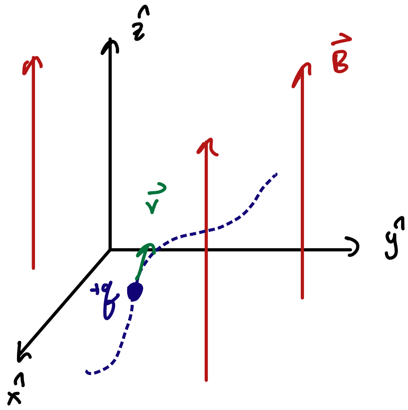
Lecture 7



$$\vec{F} = q\,\vec{v}\times\vec{B} \qquad \vec{B} = B\hat{z}$$

$$\Rightarrow \quad \vec{F} = q v_y B\,\hat{x} - q v_x B\,\hat{y}$$

$$m\dot{v}_x\,\hat{x} + m\dot{v}_y\,\hat{y} = q v_y B\,\hat{x} - q v_x B\,\hat{y}$$

$$\Rightarrow \quad m\dot{v}_x = q v_y B \qquad m\dot{v}_y = -q v_x B$$

$$\underline{\dot{v}_x = \omega v_y}\ , \quad \underline{\dot{v}_y = -\omega v_x} \qquad \omega = \frac{qB}{m} \quad \text{"cyclotron frequency"}$$

Let $\eta = v_x + i v_y$

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y$$

$$= (\omega v_y) + i(-\omega v_x) = -i\omega\,(i v_y + v_x) = -i\omega\eta$$

$\dot{\eta} = -i\omega\eta$ → we can solve this!

$$\frac{d\eta}{dt} = -i\omega\eta$$

$$\Rightarrow \quad \eta = \underbrace{Ae^{-i\omega t}}_{\text{Let's take a closer look}}$$

Complex exponentials

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots \qquad \text{let } z = i\theta$$

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \frac{i^8\theta^8}{8!} + \dots$$

$i \equiv \sqrt{-1} \quad \Rightarrow$

$i^0 = 1$	
$i^1 = i$	$i^5 = i^4 i = i$
$i^2 = -1$	$i^6 = i^4 i^2 = -1$
$i^3 = i^2 i = -i$	$i^7 = i^4 i^3 = -i$
$i^4 = i^2 i^2 = (-1)(-1) = +1$	$i^8 = i^4 i^4 = i$

} The pattern repeats

$$e^{i\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \dots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)}_{\sin\theta}$$

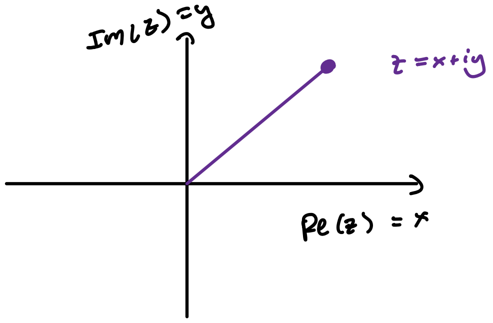
$$\Rightarrow \quad e^{i\theta} = \cos\theta + i\sin\theta$$

Euler's Formula

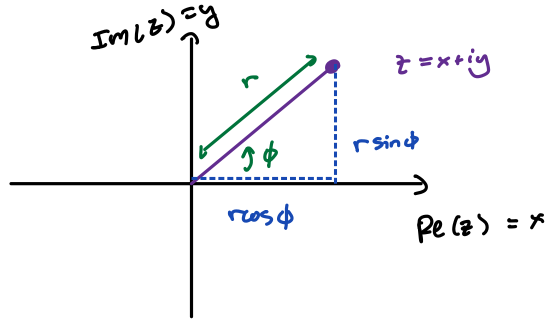
What does it mean? In general: $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$ let $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$

$\Rightarrow z = x + iy$ where x, y are any real # $\in (-\infty, \infty)$

We can represent this visually:



We can also use polar coordinates



Back to Euler's formula.

If $z = e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow x = \cos\theta \quad y = \sin\theta$

$$r = \sqrt{x^2 + y^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$r = 1$

Also $x = r\cos\phi$ & $y = r\sin\phi$

$$\left.\begin{aligned} x &= \cos\theta = r\cos\phi = \cos\phi \\ y &= \sin\theta = r\sin\phi = \sin\phi \end{aligned}\right\}$$

the θ in Euler's formula <u>is</u> the ϕ in polar coordinates!

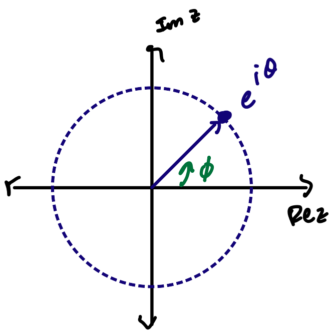
In general, any complex # can be written in two ways:

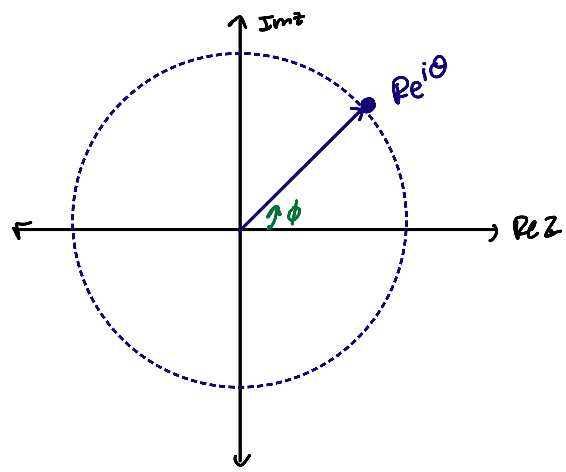
$z = x + iy$

"Cartesian"

$z = Re^{i\theta}$

"polar"

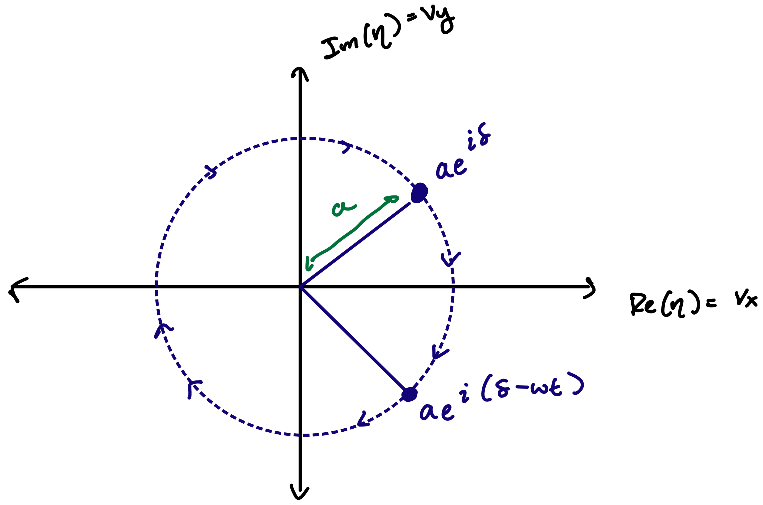




Back to our solution of a charge moving in a $\vec{B}$ field:

$$\eta = Ae^{-i\omega t} = (ae^{i\delta})\, e^{-i\omega t} = a e^{i(\delta-\omega t)} \qquad \& \qquad \eta = v_x + i v_y$$

A can be complex!



$$\eta = ae^{i(\delta-\omega t)} \qquad \eta = v_x + iv_y$$

$$\eta = a\left[\cos(\delta-\omega t) + i\sin(\delta-\omega t)\right]$$

$$\Rightarrow \quad v_x = a\cos(\delta-\omega t) \qquad v_y = a\sin(\delta-\omega t)$$

To find $x(t)$ & $y(t)$, easier to go back to polar coordinates:

$$\xi = x + iy \qquad \dot{\xi} = \dot{x} + i\dot{y} = v_x + iv_y = \eta$$

$$\frac{d\xi}{dt} = \eta$$

$$\int_{\xi_0}^{\xi} d\xi' = \int_0^t \eta\, dt' = \int_0^t ae^{i(\delta-\omega t')}\, dt'$$

$$\xi'\Big|_{\xi_0}^{\xi} = \frac{a}{-i\omega}\, e^{i(\delta-\omega t')}\Big|_0^t$$

$$\xi - \xi_0 = \frac{ai}{\omega}\left[e^{i(\delta-\omega t)} - e^{i\delta}\right]$$

$$\xi = \xi_0 + \frac{ia}{\omega}e^{i\delta} + \frac{ia}{\omega}e^{i(\delta-\omega t)}$$

$$\xi = x_0 + i y_0 - \frac{ia}{\omega}\cos\delta + \frac{a}{\omega}\sin\delta + \frac{ia}{\omega} e^{i(\delta - \omega t)}$$

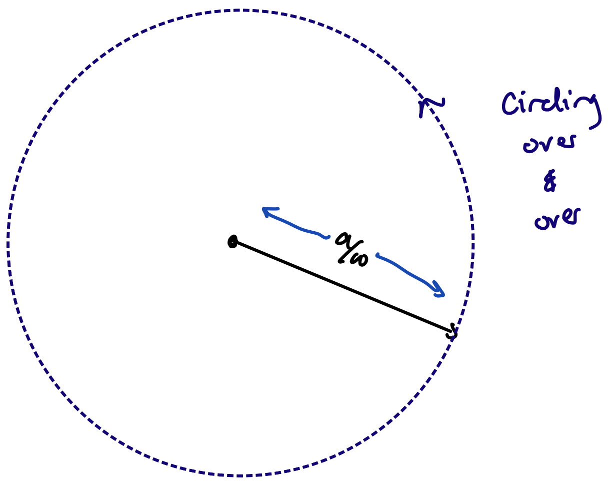
Let $x_0 = -\frac{a}{\omega}\sin\delta$ and $y_0 = \frac{a}{\omega}\cos\delta$ (change origin)

$$\xi = \frac{ia}{\omega} e^{i(\delta-\omega t)} = \frac{ia}{\omega}\cos(\delta - \omega t) - \frac{a}{\omega}\sin(\delta - \omega t) = x(t) + i y(t)$$

$$\boxed{\begin{aligned} x(t) &= -\frac{a}{\omega}\sin(\delta - \omega t) \\ y(t) &= \frac{a}{\omega}\cos(\delta - \omega t) \end{aligned}}$$

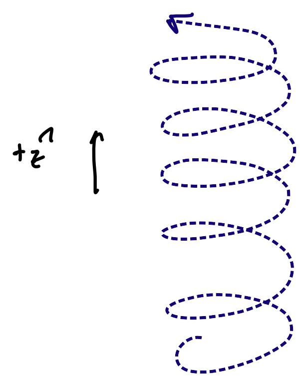
$$\xi(t) = \frac{ia}{\omega} e^{i(\delta - \omega t)}$$

In the x-y plane, this motion is



Remember $\ddot{z} = 0$

$\Rightarrow z(t) = z_0 + v_{0z} t$



$$|\vec{v}_{\text{transverse}}| = |\eta| = \left| a e^{i(\delta - \omega t)} \right| = a = v$$

$$|R| = |\xi| = \left| \frac{ia}{\omega} e^{i(\delta-\omega t)} \right| = \frac{a}{\omega} = \frac{v}{\omega} = \frac{mv}{qB}$$

End of Chapter 2!