(4.2) Evaluate $W = \int_0^P \vec{F}\cdot d\vec{r} = \int_0^P (F_x dx + F_y dy)$ where $\vec{F} = x^2\hat{x} + 2xy\,\hat{y}$ & $P = (1,1)$

Use the following paths:

(a) The path from $(0,0)$ to $(0,1)$ along x-axis, then from $(0,1)$ to $(1,1)$ straight up.

$$W = \int_0^P \vec{F}\cdot d\vec{r} = \int_0^{(0,1)} \vec{F}\cdot dx\,\hat{x} + \int_{(0,1)}^P \vec{F}\cdot dy\,\hat{y} = \int_0^1 x^2 dx \Big|_{y=0} + \int_0^1 2xy\,dy \Big|_{x=1} = \frac{x^3}{3}\Big|_0^1 + \int_0^1 2y\,dy = \frac{1}{3} + y^2\Big|_0^1$$
$$= \frac{1}{3} + 1 = \boxed{4/3}$$

(b) The path defined by $y = x^2$

$y = x^2$ & $\vec{F}\cdot d\vec{r} = F_x dx + F_y dy$

$dy = 2x\,dx$

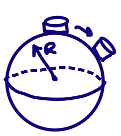
$$W = \int_0^P \vec{F}\cdot d\vec{r} = \int_0^P (F_x dx + F_y dy) = \int_0^P (x^2 dx + 2xy\,dy) = \int_0^1 \left[x^2 dx + 2x(x^2)(2x\,dx)\right] = \int_0^1 (x^2 + 4x^4)\,dx$$
$$= \left[\frac{1}{3}x^3 + \frac{4}{5}x^5\right]_0^1 = \frac{1}{3} + 4/5 = \boxed{\frac{17}{15}}$$

(c) The path defined by $x = t^3$, $y = t^2$

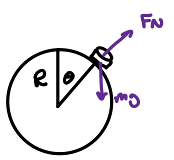
$x = t^3$ $\quad$ $y = t^2$ $\quad$ $x=0 \Rightarrow t=0$

$dx = 3t^2 dt$ $\quad$ $dy = 2t\,dt$ $\quad$ $x=1 \Rightarrow t=1$ $\quad$] bounds of integration

$$W = \int_0^P [F_x dx + F_y dy] = \int_0^P (x^2 dx + 2xy\,dy) = \int_0^P \left[(t^6)(3t^2 dt) + 2(t^3)(t^2)(2t\,dt)\right]$$
$$= \int_0^1 (3t^8 + 4t^6)\,dt = \left[\frac{3}{9}t^9 + \frac{4}{7}t^7\right]_0^1 = \frac{1}{3} + 4/7 = \boxed{\frac{19}{21}}$$

(4.8) 

Puck on sphere, no friction. If the puck slides off, how far vertically does it descend before falling off the sphere?



$E = T + U = \frac{1}{2}mv^2 + mgy$

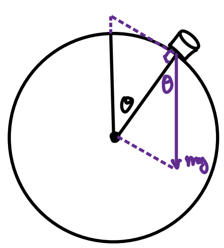
Let $y = 0$ be the center of the sphere

Also assume the small nudge gives the puck negligible initial velocity $v_0 \approx 0$

$$E\Big|_{t=0} = \frac{1}{2}mv_0^2 + mgR$$
$$= mgR = E = \text{constant}$$

$$\Rightarrow mgR = \frac{1}{2}mv^2 + mgy$$
$$g(R-y) = \frac{1}{2}v^2$$
$$\sqrt{2g(R-y)} = v$$

$y = R\cos\theta \Rightarrow v = \sqrt{2gR(1-\cos\theta)}$



$\vec{F} = m\vec{a}$

$$(F_N - mg\cos\theta)\,\hat{r} + mg\sin\theta\,\hat{\theta} = m(\ddot{r} - r\dot{\theta}^2)\,\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

We know $r = R$ = constant (until puck falls off)

$$\Rightarrow F_N - mg\cos\theta = -mR\dot{\theta}^2 \qquad mg\sin\theta = mR\ddot{\theta}$$

Also $V_\theta = R\dot{\theta} = v$ (no motion in $\hat{r}$-direction)

$$\Rightarrow F_N - mg\cos\theta = -\frac{m}{R}v^2$$

$$F_N = -\frac{m}{R}v^2 + mg\cos\theta = -2mg(1-\cos\theta) + mg\cos\theta = mg(3\cos\theta - 2)$$

$$F_N = 0 \Rightarrow 3\cos\theta = 2 \Rightarrow \cos\theta = 2/3$$

The puck's height is $R\cos\theta \Rightarrow \Delta y = R\cos\theta - R\cos(0) = 2R/3 - R$

$$\boxed{\Delta y = -R/3}$$

4.13 Calculate $\vec{\nabla} f$ for :

(a) $f = \ln(r)$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$$

$$\vec{\nabla} f = \frac{1}{r}\frac{\partial r}{\partial x}\hat{x} + \frac{1}{r}\frac{\partial r}{\partial y}\hat{y} + \frac{1}{r}\frac{\partial r}{\partial z}\hat{z}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\cancel{2}x = \frac{x}{r}, \quad \underbrace{\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}}_{\text{by the same method}}$$

$$\boxed{\vec{\nabla} f = \frac{x}{r^2}\hat{x} + \frac{y}{r^2}\hat{y} + \frac{z}{r^2}\hat{z} = \frac{\vec{r}}{r^2} = \frac{\hat{r}}{r}}$$

(b) $f = r^n$

$$\vec{\nabla} f = nr^{n-1}\frac{\partial r}{\partial x}\hat{x} + nr^{n-1}\frac{\partial r}{\partial y}\hat{y} + nr^{n-1}\frac{\partial r}{\partial z}\hat{z}$$

$$\boxed{\vec{\nabla} f = nr^{n-2}x\hat{x} + nr^{n-2}y\hat{y} + nr^{n-2}z\hat{z} = nr^{n-2}\vec{r} = nr^{n-1}\hat{r}}$$

(c) $f = g(r)$

$$\vec{\nabla} f = g'(r)\frac{\partial r}{\partial x}\hat{x} + g'(r)\frac{\partial r}{\partial y}\hat{y} + g'(r)\frac{\partial r}{\partial z}\hat{z}$$

$$\boxed{\vec{\nabla} f = \frac{g'(r)x}{r}\hat{x} + \frac{g'(r)y}{r}\hat{y} + \frac{g'(r)z}{r}\hat{z} = g'(r)\frac{\vec{r}}{r} = g'(r)\hat{r}}$$

4.17 $\vec{F} = q\vec{E}_0$, $\vec{E}_0$ = uniform

(a) Show $\vec{F}$ is conservative & show $U(\vec{r}) = -q\vec{E}_0\cdot\vec{r}$

$$\vec{\nabla}\times\vec{F} = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}\hat{x}_i = q\,\varepsilon_{ijk}\frac{\partial}{\partial x_j}(E_0)_k\,\hat{x}_i$$

E_0 = uniform $\Rightarrow$ no $x, y,$ or z-dependence $\Rightarrow \frac{\partial E_0}{\partial x_i} = 0 \quad \forall i$

$\Rightarrow \vec{\nabla}\times\vec{F} = 0$, conservative ☑

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}}\vec{F}\cdot d\vec{r} = -\int_{\vec{r}_0}^{\vec{r}} \underset{\substack{\uparrow\\ \text{doesn't depend on } \vec{r}}}{q\vec{E}_0}\cdot d\vec{r} = -q\vec{E}_0\cdot\left[\int_{\vec{r}_0}^{\vec{r}}d\vec{r}\right] = -q\vec{E}_0\cdot(\vec{r}-\vec{r}_0) = -q\vec{E}_0\cdot\vec{r} + q\vec{E}_0\cdot\vec{r}_0$$

Choose $\vec{r}_0 = 0 \Rightarrow U(\vec{r}) = -q\vec{E}_0\cdot\vec{r}$ ☑

(b) Check that $\vec{F} = -\vec{\nabla}U$

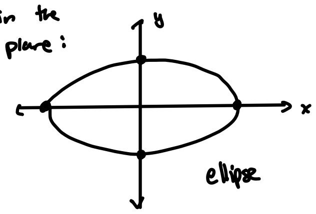
$$\vec{\nabla}U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z} \qquad U = -q\vec{E}_0\cdot(x\hat{x}+y\hat{y}+z\hat{z}) = -qE_{0x}\,x\hat{x} - qE_{0y}\,y\hat{y} - qE_{0z}\,z\hat{z}$$

$$\vec{\nabla}U = -qE_{0x}\hat{x} - qE_{0y}\hat{y} - qE_{0z}\hat{z} = -q\vec{E}_0$$

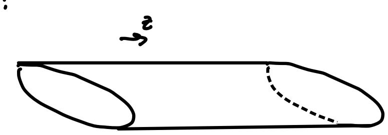
$\vec{F} = -\vec{\nabla}U = +qE_0$ ☑

4.19 a) Describe $f = x^2 + 4y^2$ = const.

A slice in the x-y plane:



In 3D:



b) Find $\hat{n}$ for $f=5$ at $(1,1,1)$

From 4.18, $\vec{\nabla}f\big|_{\vec{r}} \perp$ surface f through $\vec{r}$

$\Rightarrow \hat{n} \parallel \vec{\nabla}f$

$$\vec{\nabla}f = 2x\hat{x} + 8y\hat{y}$$

$$\vec{\nabla}f\Big|_{(1,1,1)} = 2\hat{x} + 8\hat{y} \parallel \hat{n} \qquad |\hat{n}| = 1 \quad \text{(unit normal)}$$

$$\text{so } \hat{n} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{2\hat{x}+8\hat{y}}{\sqrt{4+64}} = \frac{2\hat{x}}{2\sqrt{17}} + \frac{8\hat{y}}{2\sqrt{17}} = \boxed{\frac{1}{\sqrt{17}}\hat{x} + \frac{4}{\sqrt{17}}\hat{y} = \hat{n}}$$

4.23 Is $\vec{F}$ conservative? If so, find U and check $\vec{F} = -\vec{\nabla} U$

(a) $\vec{F} = k(x, 2y, 3z)$

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_x}{\partial z} = 0 \qquad \frac{\partial F_y}{\partial x} = \frac{\partial F_y}{\partial z} = 0 \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_z}{\partial y} = 0 \quad \Rightarrow \vec{\nabla}\times\vec{F} = 0 \Rightarrow \text{conservative}$$

$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}'$ Choose the path that goes from $(0,0,0) \xrightarrow{\mathrm{I}} (0,0,x) \xrightarrow{\mathrm{II}} (0,y,x) \xrightarrow{\mathrm{III}} (z,y,x)$ in straight lines

$$= -\int_0^{\vec{r}} k(x'dx' + 2y'dy' + 3z'dz') = \underbrace{-k\int_0^x x'dx'}_{\text{path I}} \underbrace{-k\int_0^y 2y'dy'}_{\text{path II}} \underbrace{-k\int_0^z 3z'dz'}_{\text{path III}} = \left(-\tfrac{1}{2}x^2 - y^2 - \tfrac{3}{2}z^2\right)k$$

$$-\vec{\nabla} U = \left[+x\hat{x} + 2y\hat{y} + 3z\hat{z}\right]k = \vec{F} \quad ☑$$

(b) $\vec{F} = k(y, x, 0)$

$$\vec{\nabla}\times\vec{F} = \hat{x}_i \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k = \hat{z}\,\varepsilon_{321}\frac{\partial}{\partial y}F_x + \hat{z}\,\varepsilon_{312}\frac{\partial}{\partial x}F_y = \hat{z}(-1)\frac{\partial}{\partial y}(y) + \hat{z}(+1)\frac{\partial}{\partial x}(x) = -\hat{z} + \hat{z} = 0 \Rightarrow \text{conservative}$$

Use the same path as (a)

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}' = -k\int_0^{\vec{r}} y'dx' + x'dy' = -k\int_0^x (0)\,dx' - k\int_0^y (x)\,dy' = 0 - kxy = -kxy$$

$$\vec{\nabla} U = -ky\hat{x} - kx\hat{y} = -\vec{F} \quad ☑$$

(c) $\vec{F} = k(-y, x, 0)$

$$\vec{\nabla}\times\vec{F} = \varepsilon_{ijk}\frac{\partial}{\partial x_j}F_k \hat{x}_i = \varepsilon_{312}\frac{\partial}{\partial x}F_y\hat{z} + \varepsilon_{321}\frac{\partial}{\partial y}F_x\hat{z} = (+1)\frac{\partial}{\partial x}x\,\hat{z} + (-1)\frac{\partial}{\partial y}(-y)\hat{z} = (+1)(1)\hat{z} + (-1)(-1)\hat{z} = 2\hat{z} \neq 0$$

Not conservative

4.28 mass m on spring w/ constant k. $PE = \frac{1}{2}kx^2$, $x_{max} = A$

a) Use E conservation to find $\dot{x}(x, E)$

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2$$

$$\tfrac{1}{2}m\dot{x}^2 = E - \tfrac{1}{2}kx^2$$

$$\dot{x}^2 = 2E/m - k/m\,x^2$$

$$\dot{x} = \pm\sqrt{2E/m - k/m\,x^2}$$

b) Show $E = \frac{1}{2}kA^2$ and find $\dot{x}(x, A)$. Then find $t(x)$

When $x = x_{max}$, $\dot{x} = 0$

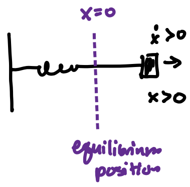
$$\Rightarrow E = \tfrac{1}{2}m(0)^2 + \tfrac{1}{2}kx_{max}^2 = 0 + \tfrac{1}{2}kA^2$$

$$E = \tfrac{1}{2}kA^2 \quad ☑$$

$$\Rightarrow \dot{x} = \pm\sqrt{\tfrac{2}{m}\tfrac{1}{2}kA^2 - k/m\,x^2}$$

$$\dot{x} = \pm\sqrt{k/m}\sqrt{A^2 - x^2} \quad ☑$$

Consider $x>0$, $\dot{x}>0$



$$\Rightarrow \quad \frac{dx}{dt} = +\sqrt{\frac{k}{m}}\sqrt{A^2-x^2}$$

$$\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{A^2-x^2}} = dt$$

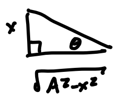
$$\int_0^x \sqrt{\frac{m}{k}}\frac{dx'}{\sqrt{A^2-(x')^2}} = \int_0^t dt'$$

$$\sqrt{\frac{m}{k}}\tan^{-1}\left(\frac{x'}{\sqrt{A^2-(x')^2}}\right)\Bigg|_0^x = t-0$$

$$\sqrt{\frac{m}{k}}\left[\tan^{-1}\left(\frac{x}{\sqrt{A^2-x^2}}\right)-0\right] = t$$

$$t = \sqrt{\frac{m}{k}}\tan^{-1}\left(\frac{x}{\sqrt{A^2-x^2}}\right)$$

$\tan\theta = \dfrac{x}{\sqrt{A^2-x^2}} \Rightarrow$



$\Rightarrow$ hypotenuse $= A$ by Pythagorean theorem

$$\Rightarrow \sin\theta = \frac{x}{A}$$

$$\theta = \tan^{-1}\left(\frac{x}{\sqrt{A^2-x^2}}\right) = \sin^{-1}\left(\frac{x}{A}\right)$$

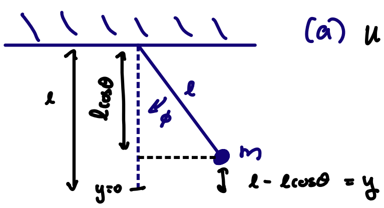
$$t = \sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{x}{A}\right)$$

c) Find $x(t)$ & show this is SHM w/ $T = 2\pi\sqrt{m/k}$

$$\sin\left(\sqrt{\frac{k}{m}}\,t\right) = \frac{x}{A} \Rightarrow x = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$\Rightarrow$ SHM ☑ The period is $T = \dfrac{2\pi}{\sqrt{k/m}} = \boxed{2\pi\sqrt{\frac{m}{k}} = T}$

4.34



(a) $U = mgy = mg(\ell - \ell\cos\phi) = mg\ell(1-\cos\phi)$ ☑

$$E = \tfrac{1}{2}mv^2 + U$$

$$v = \ell\dot{\phi}$$

$$E = \tfrac{1}{2}m\ell^2\dot{\phi}^2 + mg\ell(1-\cos\phi)$$

(b) Find EOM

$$\frac{dE}{dt} = 0 = \tfrac{1}{2}m\ell^2\, 2\dot{\phi}\ddot{\phi} + mg\ell(\sin\phi\,\dot{\phi})$$

$$-m\ell^2\dot{\phi}\ddot{\phi} = mg\ell\sin\phi\,\dot{\phi}$$

$$\underbrace{m\ell^2}_{I}\,\underbrace{\ddot{\phi}}_{\omega} = \underbrace{-mg\ell\sin\phi}_{F\ell\sin\theta = \Gamma}$$

$$I\omega = \Gamma \quad ☑$$

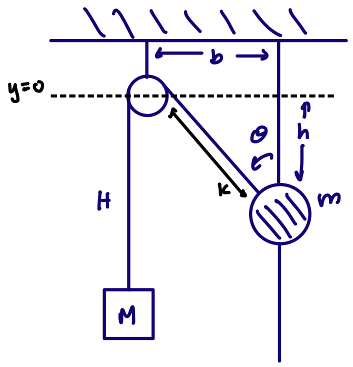
(c) Find $\phi(t)$ assuming $\phi \ll 1$

$$\ddot{\phi} = -g/\ell\,\sin\phi \approx -g/\ell\,\phi$$

$$\ddot{\phi} = -g/\ell\,\phi \Rightarrow \phi(t) = A\cos\left(\sqrt{\frac{g}{\ell}}\,t\right) + B\sin\left(\sqrt{\frac{g}{\ell}}\,t\right)$$

periodic where $\dfrac{2\pi}{T} = \sqrt{g/\ell} \Rightarrow T = 2\pi\sqrt{\ell/g}$ ☑

4.36



(a) Find $u(\theta)$

$$u = -MgH - mgh$$

$\ell = H + K$ $\quad \sin\theta = b/K$ $\quad h = K\cos\theta$

$H = \ell - K$ $\quad K = \dfrac{b}{\sin\theta}$

$$\Rightarrow h = \frac{b}{\tan\theta}$$

$$H = \ell - b/\sin\theta$$

$$u = -Mg\left(\ell - \frac{b}{\sin\theta}\right) - mg\, b/\tan\theta = -Mg\ell + gb\left(\frac{M}{\sin\theta} - \frac{m}{\tan\theta}\right)$$

(b) Find equilibria

$$u(\theta) = -Mg\ell + gb\left(\frac{M}{\sin\theta} - \frac{m}{\tan\theta}\right)$$

$$u'(\theta) = -gb\left[M\sin^{-2}\theta\cos\theta - m\tan^{-2}\theta\sec^2\theta\right]$$
$$= -gb\left[M\frac{\cos\theta}{\sin^2\theta} - m\frac{1}{\sin^2\theta}\right]$$

$$u'(\theta) = -\frac{gb}{\sin^2\theta}\left[M\cos\theta - m\right]$$

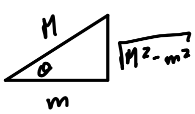
$$0 = u'(\theta) \Rightarrow M\cos\theta = m$$

$$\cos\theta = m/M = \text{equilibrium} \quad (\text{for } m < M)$$

$$u''(\theta) = 2gb\sin^{-3}\theta\cos\theta\left[M\cos\theta - m\right] - \frac{gb}{\sin^2\theta}\left[-M\sin\theta\right]$$

$$u''(\theta) = \frac{2gbM\cos^2\theta}{\sin^3\theta} - \frac{2gbm\cos\theta}{\sin^3\theta} + \frac{gbM}{\sin\theta} = -\frac{gb}{\sin\theta}\left[-\frac{2M\cos^2\theta}{\sin^2\theta} + \frac{2m\cos\theta}{\sin^2\theta} - M\right]$$

$$\theta_0 = \cos^{-1}(m/M)$$
$$= \sin^{-1}\left(\frac{\sqrt{M^2-m^2}}{M}\right) \Rightarrow u''(\theta_0) = \frac{-gbM}{\sqrt{M^2m^2}}\left[\frac{-2M(m/\cancel{M})^2}{(M^2-m^2)/\cancel{M^2}} + \frac{2m\, m/\cancel{M}}{(M^2-m^2)/M^{\cancel{2}}} - M\right]$$



$$= \frac{-gbM}{\sqrt{M^2m^2}}\left[\frac{-2Mm^2}{M^2-m^2} + \frac{2m^2M}{M^2-m^2} - M\right]$$

$$= \frac{-gbM}{\sqrt{M^2m^2}}\left[-M\right] = \frac{gbM^2}{\sqrt{M^2-m^2}} = u''(\theta_0)$$

$g>0$, $b>0$, $M^2>0$, $\sqrt{M^2-m^2}>0$ $\quad \Rightarrow u''(\theta_0) > 0$

$\Rightarrow$ <u>stable equilibrium</u>