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Lecture 6 2/2/26

Last week:

Monday: Newton's 2nd law in polar coordinates:

Cartesian	Polar
$x = r\cos\phi$	$r = \sqrt{x^2+y^2}$
$y = r\sin\phi$	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$\vec{r} = x\hat{x} + y\hat{y}$

$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y}$

$\vec{a} = \ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y}$

$\vec{F} = m(\ddot{x}\hat{x} + \ddot{y}\hat{y})$

$\vec{r} = r\hat{r}$

From taking $\frac{d\hat{r}}{dt}$

$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$

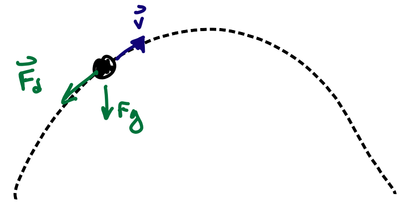
$\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$

$\vec{F} = m(\ddot{r} - r\dot{\phi}^2)\hat{r} + m(2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$

Wednesday & Friday:

Introduced Drag Force

opposes $\vec{v}$:



Linear drag: $\vec{F}_d = -bv\hat{v}$

Quadratic drag: $\vec{F}_d = -cv^2\hat{v}$ (From the Activity on Friday)

$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = -b\vec{v}$ (for example)

First order differential equation, use separation of variables

Example: vertical quadratic drag for falling object:

$$\vec{F} = m\vec{a}$$

$$-cv^2\hat{v} - mg\hat{y} = m\dot{\vec{v}}$$

$$-cv^2(-\hat{y}) - mg\hat{y} = -m\dot{v}\hat{y}$$

$$+cv^2 - mg = -m\frac{dv}{dt}$$

$$\int dt = \int \frac{-m\,dv}{cv^2 - mg}$$

$\Rightarrow$ see Activity 2

Back to Linear Drag: $\vec{F} = -b\vec{v}$

Last wednesday we found that, for linear drag:

$$x(t) = +\tau v_{x_0}\left(1 - e^{-t/\tau}\right) + x_0$$

$$y(t) = y_0 - v_{term}\, t + \tau\left(v_{oy} + v_{term}\right)\left(1 - \underbrace{e^{-t/\tau}}\right)$$

Where $\tau = 1/k$, $k = b/m$, $v_{term} = g/k$

Let $x_0 = 0$

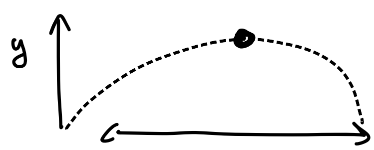
isolate t: $\frac{x(t)}{v_{ox}\tau} = 1 - e^{-t/\tau} \Rightarrow -\frac{t}{\tau} = \ln\left(1 - \frac{x(t)}{v_{ox}\tau}\right)$

$$\Rightarrow y(t) = y_0 - v_{term}\, t + \tau\left(v_{oy} + v_{term}\right)\left[1 - \overbrace{\left[1 - \frac{x(t)}{v_{x_0}\tau}\right]}\right]$$

$$\boxed{y(t) = y_0 + v_{term}\tau \ln\left(1 - \frac{x(t)}{v_{ox}\tau}\right) + \frac{v_{oy} + v_{term}}{v_{ox}}\, x(t)}$$

Find range R of the projectile. (Let $x_0 = y_0 = 0$)

In a vacuum, $R = \frac{2 v_{ox} v_{oy}}{g}$



With drag,

$$y(t) = 0 = y_0 + v_{term}\tau \ln\left(1 - \frac{R}{v_{ox}\tau}\right) + \frac{v_{oy} + v_{term}}{v_{ox}} R$$

If $R \ll v_{ox}\tau$ use $\ln(1-\varepsilon) \simeq -\left(\varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \dots\right)$

$$\Rightarrow 0 = 0 + v_{term}\tau\left(\frac{-R}{v_{ox}\tau} - \frac{R^2}{2v_{ox}^2\tau^2} - \frac{R^3}{3v_{ox}^3\tau^3} - \dots\right) + \frac{v_{oy} + v_{term}}{v_{ox}} R$$

$$0 = 0 - \cancel{\frac{v_{term} R}{v_{ox}}} - \frac{v_{term} R^2}{2v_{ox}^2\tau} - \frac{v_{term} R^3}{3v_{ox}^3\tau^2} + \frac{v_{oy} R}{v_{ox}} + \cancel{\frac{v_{term} R}{v_{ox}}}$$

$$0 = \cancel{\frac{R}{v_{ox}}}\left[-\frac{v_{term} R}{2 v_{ox}\tau} - \frac{v_{term} R^2}{3 v_{ox}^2 \tau^2} + v_{oy}\right]$$

$$0 = \cancel{\frac{v_{term}}{2 v_{ox}\tau}}\left[-R - \frac{2R^2}{3 v_{ox}\tau} + \frac{2 v_{oy} v_{ox} \tau}{v_{term}}\right]$$

$$R = \frac{2 v_{oy} v_{ox} \tau}{v_{term}} - \frac{2R^2}{3 v_{ox}\tau}$$

Remember $\tau = \frac{1}{k} = \frac{m}{b}$

$v_{term} = \frac{mg}{b}$

$\Rightarrow v_{term} = \tau g$

$$R = \underbrace{\frac{2 v_{oy} v_{ox}}{g}}_{R_{vac}} - \frac{2}{3}\frac{R^2}{v_{ox}\tau}$$

and $R \approx R_{vac}$

$$\Rightarrow \quad R = R_{vac}\left(1 - \frac{2}{3}\frac{R_{vac}}{v_{ox}\tau}\right)$$

$$R = R_{vac}\left(1 - \frac{4 v_{oy}}{3 g \tau}\right) = R\left(1 - \frac{4}{3}\frac{v_{oy}}{v_{term}}\right) \ \boxed{\checkmark}$$

You practice quadratic force in the Friday Activity.

$$\vec{F} = -cv^2\hat{v} = -c|v|\underbrace{|v|\hat{v}}_{\vec{v}} = -c|v|\,\vec{v}$$

$$m\dot{\vec{v}} = -c|v|\left(v_x\hat{x} + v_y\hat{y}\right) \qquad |v| = \sqrt{v_x^2 + v_y^2}$$

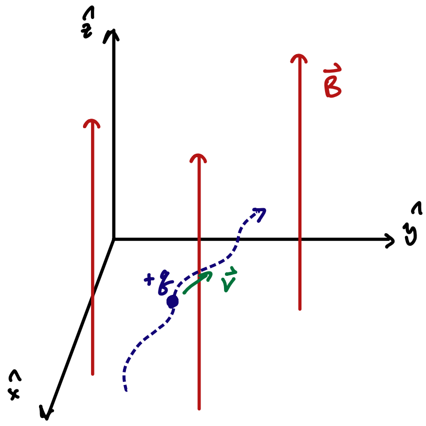
$$m\dot{v}_x\hat{x} + m\dot{v}_y\hat{y} = -cv_x\sqrt{v_x^2+v_y^2}\,\hat{x} - cv_y\sqrt{v_x^2+v_y^2}\,\hat{y}$$

$$\Rightarrow \quad \left.\begin{aligned} m\dot{v}_x &= -cv_x\sqrt{v_x^2+v_y^2} \\ m\dot{v}_y &= -cv_y\sqrt{v_x^2+v_y^2} \end{aligned}\right\}$$ Coupled! Very difficult to solve

Next, we'll consider a system with coupled dif eq's that we can solve!

Motion of a Charge in a uniform $\vec{B}$ field

Lorentz force: $\quad \vec{F} = q\vec{v}\times\vec{B}$



$\vec{B} = B\hat{z} \qquad\qquad \vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

To find $\vec{a}$ using $\vec{F}=m\vec{a}$, we first need $\vec{v}\times\vec{B}$

$\vec{F} = \vec{v}\times\vec{B}$

$(\vec{v}\times\vec{B})_i = \varepsilon_{ijk}\, v_j B_k \qquad \vec{B}=B\hat{z} \Rightarrow B_1 = B_x = 0,\ B_2 = B_y = 0,\ B_3 = B_z = B$

$(\vec{v}\times\vec{B})_x = \varepsilon_{1j3}\, v_j B_3 = \varepsilon_{123} v_2 B_3 = +v_y B$

$(\vec{v}\times\vec{B})_y = \varepsilon_{2j3}\, v_j B_3 = \varepsilon_{213} v_1 B_3 = -v_x B$

$(\vec{v}\times\vec{B})_z = \varepsilon_{3j3}\, v_j B_3 = 0$

$\vec{F} = q(\vec{v}\times\vec{B}) = q(v_y B\hat{x} - v_x B\hat{y})$

$\vec{F} = q(\vec{v}\times\vec{B}) = q(v_y B\hat{x} - v_x B\hat{y})$

$\Rightarrow \quad m\ddot{x} = q v_y B \quad\Rightarrow\quad m\dot{v}_x = q v_y B$

$m\ddot{y} = -q v_x B \quad\Rightarrow\quad m\dot{v}_y = -q v_x B$

1^{st} order coupled dif eq's! These can be solved though!

$m\ddot{z} = 0 \quad\Rightarrow\quad v_z = \text{constant}$

Rewrite: $\quad \dot{v}_x = \dfrac{qB}{m} v_y, \quad \dot{v}_y = -\dfrac{qB}{m} v_x \quad\Rightarrow\quad \boxed{\begin{aligned}\dot{v}_x &= \omega v_y \\ \dot{v}_y &= -\omega v_x\end{aligned}}$

where $\omega \equiv \dfrac{qB}{m}$ "cyclotron frequency"

Let $\boxed{\eta = v_x + i v_y}$

(This is weird, I know, but it will work out)

$\dot{\eta} = \dot{v}_x + i\dot{v}_y = (\omega v_y) + i(-\omega v_x) = \omega v_y - i\omega v_x = -i\omega\underbrace{(i v_y + v_x)}_{\eta} = -i\omega\eta$

plug in dif-eq's

$\dot{\eta} = -i\omega\eta \rightarrow$ we can solve this!

$\dfrac{d\eta}{dt} = -i\omega\eta \qquad\qquad \eta = \underbrace{A}_{\text{Let's take a closer look}}e^{-i\omega t}$