Solutions
Rachel Houtz

Physics 3221 Spring 2026: Practice Midterm 1 Problem:

Grade:

1. You Found Me!

When exploring the wilderness, you find a suspiciously placed boulder at the bottom of an icy hill. On the other side of the hill, there's a hole, see Fig. 1. You hit the boulder so that it has an initial speed v_0 sliding up the incline.

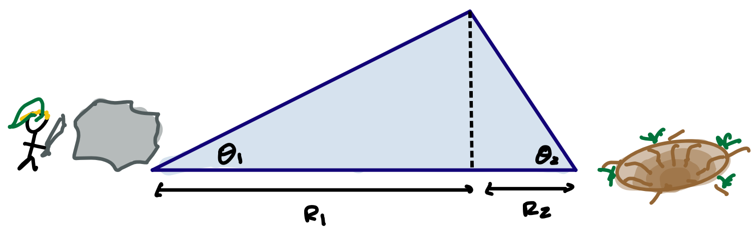
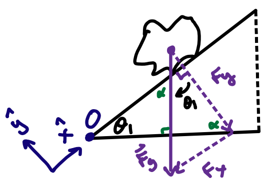


Figure 1: The icy, frictionless hill. The boulder has mass m, and the gravitational acceleration in this wilderness is g.

(a) Find the time t_* it takes for the boulder to get to the top of the hill in terms of v_0, θ_1, R_1, and g.



Choose point O as $x=y=0$ @ $t=0$

so $x_0=y_0=0$

$\vec{F}_g = mg\sin\theta(-\hat{x}) + mg\cos\theta(-\hat{y})$

$\Rightarrow \; m\ddot{x} = -mg\sin\theta_1$

$\ddot{x} = -g\sin\theta_1$

$\int_0^t \frac{d}{dt'}\dot{x}(t')\,dt' = \int_0^t -g\sin\theta_1\,dt'$

$\dot{x}(t')\Big|_0^t = -g\sin\theta_1\, t'\Big|_0^t$

$\dot{x}(t) - \dot{x}(0) = -g\sin\theta_1(t-0)$

$\dot{x} = v_0 - g\sin\theta_1 t$

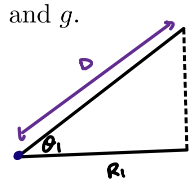
$\int_0^t \frac{dx}{dt'}\,dt' = \int_0^t (v_0 - g\sin\theta_1 t')\,dt'$

$x(t')\Big|_0^t = \left[v_0 t' - \frac{1}{2}g\sin\theta_1 (t')^2\right]_0^t$

$x(t) = \cancel{x_0}\,0 + v_0 t - \frac{1}{2}g\sin\theta_1 t^2$

$x(t) = v_0 t - \frac{1}{2}g\sin\theta_1 t$

The top of the hill is reached when the boulder travels a distance D:



$\cos\theta_1 = \frac{R_1}{D}$

$D = R_1/\cos\theta_1$

$D = \frac{R_1}{\cos\theta_1} = v_0 t_* - \frac{1}{2}g\sin\theta_1 t_*^2$

$0 = -\frac{1}{2}g\sin\theta_1 t_*^2 + v_0 t_* - R_1/\cos\theta_1$

$t_* = \dfrac{-v_0 \pm \sqrt{v_0^2 - 4(-\frac{1}{2}g\sin\theta_1)R_1/\cos\theta_1}}{-g\sin\theta_1}$

$t_* = \frac{v_0}{g\sin\theta_1} \mp \frac{1}{g\sin\theta_1}\sqrt{v_0^2 - 2gR_1\tan\theta_1}$

$t_* = \frac{v_0}{g\sin\theta}\left(1 \mp \sqrt{1 - \frac{2gR_1}{v_0^2}\tan\theta_1}\right)$

$\boxed{t_* = \frac{v_0}{g\sin\theta}\left(1 - \sqrt{1 - \frac{2gR_1\tan\theta_1}{v_0^2}}\right)}$

The first solution occurs when the boulder is on its way up (the second would be if the incline were longer and the boulder went higher, then slid back down through $x=D$)

(b) What is the minimum initial velocity, $v_0^{\min}$, required for the boulder to make it to the top of the hill? Put your answer in terms of g, θ_1 and R_1.

We know that at maximum $x(t)$, $\dot{x}(t)=0$

$\dot{x}(t) = 0 = v_0 - g\sin\theta_1 t$

$v_0 = g\sin\theta_1 t_{max}$

$\frac{v_0}{g\sin\theta_1} = t_{max}$

$x(t_{max}) = v_0 t_{max} - \frac{1}{2}g\sin\theta_1 t_{max}^2$

$x(t_{max}) = \frac{v_0^2}{g\sin\theta_1} - \frac{1}{2}g\sin\theta_1\frac{v_0^2}{g^2\sin^2\theta_1} = \frac{v_0^2}{2g\sin\theta_1}$

We want $x_{max} \geq D = R_1/\cos\theta_1$

$\frac{v_0^2}{2g\sin\theta_1} \geq \frac{R_1}{\cos\theta_1}$

$v_0^2 \geq 2gR_1\tan\theta_1$

$\boxed{v_0 \geq \sqrt{2gR_1\tan\theta_1}}$

Note that you get the same answer by requiring $t_* \in \mathbb{R} \Rightarrow \left(1 - \frac{2gR_1\tan\theta}{v_0^2}\right) \geq 0$

Physics 3221 Spring 2026: Practice Midterm 1 Problem:

Grade:

(c) Assuming $v_0 = v_0^{\min}$, how much time does it take for the boulder to make it to the bottom of the other side of the hill? Put your answer in terms of $v_0, \theta_1, R_1, g, \theta_2$, and R_2.

see next page

(d) Assuming $v_0 < v_0^{\min}$, how much time does it take for the boulder to return back to its original spot? Put your answer in terms of v_0, θ_1, R_1, and g.

First the boulder goes up until $v=0$:

$\dot{x} = v_0 - g\sin\theta_1 t$ [from part (a)]

$0 = v_0 - g\sin\theta_1 t_1$

$t_1 = \frac{v_0}{g\sin\theta_1}$

You can probably already figure out that $t_{total} = 2t_1$ by the fact that the way down = the way up (with time reversed)

If you're not convinced, though, you can also find the roots of:

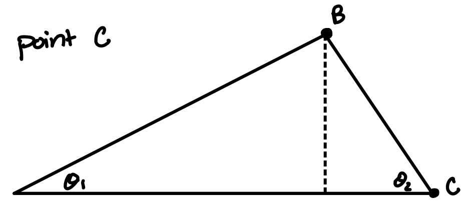
$x = v_0 t - \frac{1}{2} g\sin\theta_1 t^2$

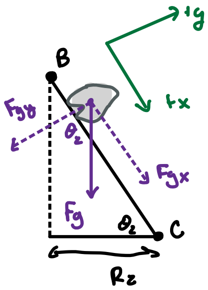
$0 = v_0 t - \frac{1}{2} g\sin\theta_1 t^2$

$0 = (t)(v_0 - \frac{1}{2} g\sin\theta\, t)$

$\boxed{t=0 \quad \text{and} \quad t = \frac{2v_0}{g\sin\theta}}$

① (c) Let's first find the time it takes to go from point B to point C





$m\ddot{x} = +mg\sin\theta_2$ (Note that $F_{gx} > 0$)

$\Rightarrow \ddot{x} = +g\sin\theta_2$

$\Rightarrow \dot{x} = gt\sin\theta_2 - v_B$

we're now starting $t=0$ at point B. Let $v(0) = v_B$

$\Rightarrow x = x_B + v_B t + \frac{1}{2}g\sin\theta_2 t^2$

using the same method as part (a)

The time it takes to go from B→C is given by:

$$x_C - x_B = v_B t + \frac{1}{2}g\sin\theta_2 t_{CB}^2$$

and

$$\cos\theta_2 = \frac{R_2}{x_C - x_B}$$

$$\frac{R_2}{\cos\theta_2} = \overset{0}{\cancel{v_B t}} + \frac{1}{2}g\sin\theta_2 t_{CB}^2$$

We know $v_B = 0$ if $v_0 = v_{min}$ (the boulder just barely makes it to the top)

$$t_{CB}^2 = \frac{2R_2}{g\sin\theta_2\cos\theta_2}$$

To get to total time it takes to go from $x=0$ to point C, just add

$$t_C = t_* + t_{CB} = \boxed{\frac{v_0}{g\sin\theta}\left(1-\sqrt{1-\frac{2gR_1\tan\theta_1}{v_0^2}}\right) + \sqrt{\frac{2R_2}{g\sin\theta_2\cos\theta_2}} = t_C}$$

Physics 3221 Spring 2026: Practice Midterm 1 Problem: Grade:

2. Air resistance

Consider the same setup shown in Fig. 1. You hit the boulder so that it has an initial speed v_0 sliding up the incline. This time, the boulder experiences air resistance:

$$\vec{F}_{\text{drag}} = -f(v)\hat{v}$$

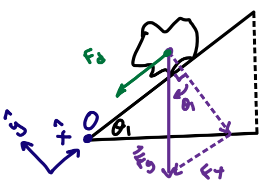
$$f(v) = bv + cv^2$$

(a) Assuming $b \gg cv$, find an implicit expression for the time t_* it takes for the boulder to get to the top of the hill in terms of v_0, θ_1, R_1, b, m, and g.

If $b \gg cv$ for all v during the boulder's trajectory, we can simplify

$$\vec F_{drag} = -bv\hat v = -b\vec v$$

On the way up, $\vec F_{drag}$ points down the incline:



$$m\vec a = -bv\hat x - mg\sin\theta_1\hat x$$

$$m\ddot x = -bv_x - mg\sin\theta_1$$

Call $v_x \equiv v$, $k \equiv b/m$

$$\dot v = -kv - g\sin\theta_1$$

Call $\alpha = \frac{g\sin\theta_1}{k} = \frac{mg\sin\theta_1}{b}$

$$\frac{-dv}{kv + g\sin\theta_1} = dt$$

$$\frac{1}{k}\int_{v_0}^{v} \frac{-dv'}{v' + \frac{g}{k}\sin\theta_1} = \int_0^t dt'$$

$$-\frac{1}{k}\ln(v'+\alpha)\Big|_{v_0}^{v} = t'\Big|_0^t$$

$$-\frac{1}{k}\ln\left(\frac{v+\alpha}{v_0+\alpha}\right) = t$$

$$v = (v_0+\alpha)e^{-kt} - \alpha$$

$$\int_0^t \dot x(t')\,dt' = \int_0^t \left[(v_0+\alpha)e^{-kt'} - \alpha\right]dt'$$

$$x(t) - \cancel{x(0)}^{\,0} = \left[-\frac{1}{k}(v_0+\alpha)e^{-kt'} - \alpha t'\right]_0^t$$

$$x(t) = \frac{v_0+\alpha}{-k}\left(e^{-kt} - 1\right) - \alpha t$$

From problem 1 (a), we know the top of the hill is $x = \frac{R_1}{\cos\theta_1}$

$$\frac{R_1}{\cos\theta_1} = \frac{v_0+\alpha}{k}\left(1 - e^{-kt_*}\right) - \alpha t^*$$

$$\boxed{\frac{R_1}{\cos\theta_1} = \left(\frac{mv_0}{b} + \frac{m^2 g\sin\theta_1}{b^2}\right)\left(1 - e^{-\frac{mt_*}{b}}\right) - \frac{mg\sin\theta_1\, t_*}{b}}$$

(b) Assuming $b \gg cv$, find an implicit expression for the minimum initial velocity, v_0^{min}, required for the boulder to make it to the top of the hill? Put your answer in terms of g, θ_1, R_1, b, m, and v_0^{min}.

We know that at maximum $x(t)$, $\dot x(t) = 0$

$$\dot x(t) = v(t) = 0 = (v_0+\alpha)e^{-kt} - \alpha$$

$$\alpha = (v_0+\alpha)e^{-kt_{max}}$$

$$-\frac{1}{k}\ln\left(\frac{\alpha}{v_0+\alpha}\right) = t_{max}$$

$$x(t_{max}) = \frac{v_0+\alpha}{k}\left(1 - e^{-kt_{max}}\right) - \alpha t_{max}$$

$$x(t_{max}) = \frac{v_0+\alpha}{k}\left[1 - e^{\ln\left(\frac{\alpha}{v_0+\alpha}\right)}\right] + \frac{\alpha}{k}\ln\left(\frac{\alpha}{v_0+\alpha}\right)$$

$$x(t_{max}) = \frac{v_0+\alpha}{k}\left[1 - \frac{\alpha}{v_0+\alpha}\right] + \frac{\alpha}{k}\ln\left(\frac{\alpha}{v_0+\alpha}\right)$$

$$x(t_{max}) = \frac{1}{k}\left[v_0+\alpha-\alpha\right] + \frac{\alpha}{k}\ln\left(\frac{\alpha}{v_0+\alpha}\right)$$

$$x(t_{max}) = \frac{v_0}{k} + \frac{\alpha}{k}\ln\left(\frac{\alpha}{v_0+\alpha}\right)$$

We want $x_{max} \geq \frac{R_1}{\cos\theta_1}$

$$\frac{v_0}{k} + \frac{\alpha}{k}\ln\left(\frac{\alpha}{v_0+\alpha}\right) \geq \frac{R_1}{\cos\theta_1}$$

$$\left(\frac{v_0}{k} - \frac{R_1}{\cos\theta_1}\right) \geq -\frac{\alpha}{k}\left[\ln\alpha - \ln(v_0+\alpha)\right]$$

$$\frac{v_0}{\alpha} - \frac{R_1 k}{\alpha\cos\theta_1} - \ln\alpha \geq \ln(v_0+\alpha)$$

$$e^{v_0/\alpha}\exp\left(\frac{-R_1 k}{\alpha\cos\theta_1}\right)(-\alpha) \geq v_0+\alpha$$

$$-\alpha - \alpha\exp\left(\frac{-R_1 k}{\alpha\cos\theta_1}\right)e^{v_0/\alpha} \geq v_0$$

$$-\frac{mg\sin\theta_1}{b}\left[1 + e^{-\frac{R_1 b^2}{m^2 g\sin\theta_1\cos\theta_1}}\, e^{\frac{v_0 b}{mg\sin\theta_1}}\right] \geq v_0$$

$$\boxed{-\frac{mg\sin\theta_1}{b}\left[1 + \exp\left(b\,\frac{mv_0\cos\theta_1 - R_1 b}{m^2 g\sin\theta_1\cos\theta_1}\right)\right] \geq v_0}$$

Grade:

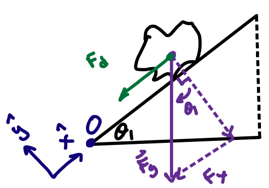
Physics 3221 Spring 2026: Practice Midterm 1 Problem:

(c) Assuming $b \ll cv$, find an implicit expression for the time t_* it takes for the boulder to get to the top of the hill in terms of v_0, θ_1, R_1, c, m, and g.

If $b >> cv$ for all v during the boulder's trajectory, we can simplify

$\vec{F}_{drag} = -cv^2\hat{v} = -\frac{c\vec{v}}{v}$

On the way up, $\vec{F}_{drag}$ points <u>down</u> the incline:



$m\vec{a} = -cv_x^2\hat{x} - mg\sin\theta_1\hat{x}$

$m\ddot{x} = -cv_x^2 - mg\sin\theta_1$ Call $v_x \equiv v$,

$\dot{v} = -\frac{c}{m}v^2 - g\sin\theta_1$ Call $\beta^2 = \frac{mg\sin\theta_1}{c}$

$$\frac{-dv}{\frac{c}{m}v^2 + g\sin\theta_1} = dt$$

$$\frac{-1}{g\sin\theta_1}\int_{v_0}^{v}\frac{dv'}{(v'/\beta)^2+1} = \int_0^t dt'$$

$$\frac{-1}{g\sin\theta_1}\left[\beta\tan^{-1}(v'/\beta)\right]_{v_0}^{v} = t'\Big|_0^t$$

$$-\frac{\beta}{g\sin\theta_1}\left[\tan^{-1}\left(\frac{v}{\beta}\right) - \tan^{-1}(v_0/\beta)\right] = t$$

$$\tan^{-1}(v/\beta) = \tan^{-1}(v_0/\beta) - \frac{tg\sin\theta_1}{\beta}$$

$$v(t) = \beta\tan\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t\right]$$

$$x(t) = \beta\int_0^t \tan\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t'\right]dt'$$

let $u = \tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t$

$du = -\frac{g\sin\theta_1}{\beta}dt$

$$x(t) = \frac{-\beta^2}{g\sin\theta_1}\int \tan(u)\,du = \frac{-1}{g\sin\theta_1}\left(-\ln\cos u\right)_{u_1}^{u_2}$$

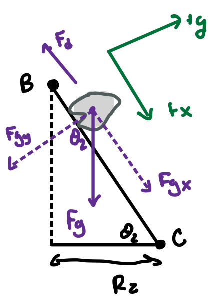
$$x(t) = \frac{\beta^2}{g\sin\theta_1}\ln\cos\left(\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t'\right)_0^t$$

(continued on next page)

(d) Assuming $b \gg cv$ and $v_0 = v_0^{\min}$, how much time does it take for the boulder to fall from the top of the hill down to the bottom of the other side of the hill? Find an implicit expression in terms of v_0, m, b, g, θ_2, and R_2.

If $v_0 = v_{min}$, the boulder just barely makes it to the top of the hill, so $v(t_*) = 0$.

On the way down,



$m\vec{a} = mg\sin\theta_2\hat{x} - b\vec{v}$

$m\ddot{x} = mg\sin\theta_2 - bv_x$

$m\dot{v} = mg\sin\theta_2 - bv$ Call $v_x \equiv v$, $k \equiv b/m$

$\frac{dv}{dt} = g\sin\theta_2 - kv$

$$-\frac{1}{k}\int_0^v \frac{dv'}{v' - \frac{g}{k}\sin\theta_2} = \int_{t_B}^{t_C} dt'$$

Call $\alpha = \frac{g\sin\theta_1}{k} = \frac{mg\sin\theta_1}{b}$

$$-\frac{1}{k}\ln(v'-\alpha)\Big|_0^v = t'\Big|_{t_B}^{t}$$

$$-\frac{1}{k}\ln\left(\frac{v-\alpha}{\alpha}\right) = t - t_B$$

$$\frac{v-\alpha}{\alpha} = e^{-k(t-t_B)}$$

$$v = \alpha e^{-k(t-t_B)} - \alpha$$

$$\int_{t_B}^{t}\dot{x}(t') = \int_{t_B}^{t}\left[\alpha e^{-k(t'-t_B)} - \alpha\right]dt'$$

$$x(t')\Big|_{t_B}^{t} = \left[-\frac{\alpha}{k}e^{-k(t'-t_B)} - \alpha t'\right]_{t_B}^{t}$$

$$x - x_B = -\alpha/k\, e^{-k(t-t_B)} + \frac{\alpha}{k}e^0 - \alpha(t-t_B)$$

$$(x - x_B) = \alpha/k\left[1 - e^{-k(t-t_B)}\right] - \alpha(t-t_B)$$

continued on next page

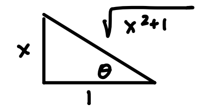
(e) Assuming $b \gg cv$ and $v_0 < v_0^{\min}$, how much time does it take for the boulder to return back to its original spot? Put your answer as an implicit expression for t in terms of v_0, θ_1, R_1, m, b, and g.

See the next pages

② (c) (continued)

$$x(t) = \frac{\beta^2}{g\sin\theta_1}\ln\cos\left(\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t'\right)\Bigg|_0^t$$

$$x(t) = \frac{\beta^2}{g\sin\theta_1}\ln\left\{\frac{\cos\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t\right]}{\cos\left[\tan^{-1}(v_0/\beta)\right]}\right\}$$



$\cos(\tan^{-1}(x))$

$\tan\theta = x$

$\tan^{-1}(x) = \theta$

$\cos(\tan^{-1}(x)) = \cos\theta$

$= \frac{1}{\sqrt{1+x^2}}$

$\cos\tan^{-1}\frac{v_0}{\beta} = \frac{\beta}{\sqrt{\beta^2+v_0^2}}$

$$x(t) = \frac{\beta^2}{g\sin\theta_1}\ln\left(\frac{\beta\cos\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t\right]}{\sqrt{\beta^2+v_0^2}}\right)$$

At the top of the hill, $x = \frac{R_1}{\cos\theta_1}$

$$\frac{R_1}{\cos\theta_1} = \frac{\beta^2}{g\sin\theta_1}\ln\left(\frac{\beta\cos\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t_*\right]}{\sqrt{\beta^2+v_0^2}}\right)$$

$$\frac{R_1 g\tan\theta_1}{\beta^2} = \ln\left(\frac{\beta\cos\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t_*\right]}{\sqrt{\beta^2+v_0^2}}\right)$$

$$e^{R_1 g\tan\theta_1/\beta^2} = \frac{\beta\cos\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t_*\right]}{\sqrt{\beta^2+v_0^2}}$$

$$\frac{1}{\beta}\sqrt{\beta^2+v_0^2}\,e^{R_1 g\tan\theta_1/\beta^2} = \cos\left[\tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t_*\right]$$

$$\cos^{-1}\left(\frac{1}{\beta}\sqrt{\beta^2+v_0^2}\,e^{\frac{R_1 g\tan\theta_1}{\beta^2}}\right) = \tan^{-1}(v_0/\beta) - \frac{g\sin\theta_1}{\beta}t_*$$

$$t_* = \frac{\beta\tan^{-1}(v_0/\beta)}{g\sin\theta_1} - \frac{\beta}{g\sin\theta_1}\cos^{-1}\left(\frac{1}{\beta}\sqrt{\beta^2+v_0^2}\,e^{R_1 g\tan\theta_1/\beta^2}\right)$$

$$\boxed{t_* = \sqrt{\frac{m}{cg\sin\theta_1}}\left[\tan^{-1}\left(\sqrt{\frac{v_0^2 c}{mg\sin\theta_1}}\right) - \cos^{-1}\left(\sqrt{1+\frac{v_0^2 c}{mg\sin\theta_1}}\;e^{cR_1/m\cos\theta_1}\right)\right]}$$

② (d) (continued)

$$(x-x_B) = \alpha/k\left[1-e^{-k(t-t_B)}\right] - \alpha(t-t_B)$$

$$x_C - x_B = \frac{R_2}{\cos\theta_2} \quad \text{[see problem ① (c)]}$$

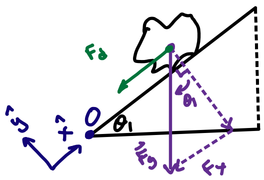
$$\boxed{\frac{R_2}{\cos\theta_2} = \alpha/k\left[1-e^{-kt_{CB}}\right] - \alpha t_{CB}}$$

where $t_{CB} \equiv t_C - t_B$

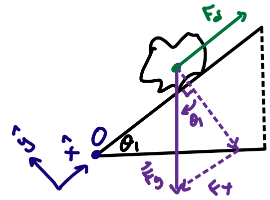
t_{CB} is the time it takes to go from the top of the hill (point B) to the bottom of the hill (point C)

② (e)

On the way up, F_d points <u>down</u> the incline



On the way down, F_d points <u>up</u> the incline



So the time for the whole trajectory won't just be $2\times$ (time it takes to get to the top)

Still, we can write $m\ddot{x} = \underbrace{-bv}_{\text{this term will change signs when } v \text{ changes signs}} - mg\sin\theta_1$

From 2(a), we have: $x(t) = \frac{v_0+\alpha}{-k}\left(e^{-kt}-1\right) - \alpha t$

The boulder returns when $x(t)=0$:

$$0 = \frac{v_0+\alpha}{-k}\left(e^{-kt}-1\right) - \alpha t$$

$$\alpha t = \frac{v_0+\alpha}{k}\left(1-e^{-kt}\right)$$

We know $k = b/m$

$\alpha = mg\sin\theta_1/b$

$$t = \frac{1}{k}\left(\frac{v_0}{\alpha}+1\right)\left(1-e^{-kt}\right)$$

$$t = \frac{m}{b}\left(\frac{v_0 b}{mg\sin\theta_1}+1\right)\left(1-e^{-bt/m}\right)$$

$$\boxed{t = \left(\frac{v_0}{g\sin\theta_1}+\frac{m}{b}\right)\left(1-e^{-bt/m}\right)}$$

Physics 3221 Spring 2026: Practice Midterm 1 Problem:

Grade:

3. Rolling Ball

Now consider a uniform solid disk of mass M and radius R rolling **without** slipping down an incline, as shown in Fig. 2.

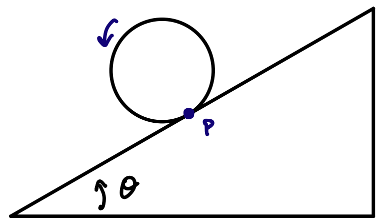
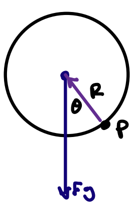


Figure 2: The disk rolls without slipping with initial angular velocity ω. This means the initial translational velocity of the CM is $v = R\omega$.

(a) Find the total torque about point P. The moment of inertia for a disk about a point on its circumference is $\frac{3}{2}MR^2$.



$\vec{\Gamma} = \vec{r} \times \vec{F}$

$\boxed{\Gamma = MgR\sin\theta}$

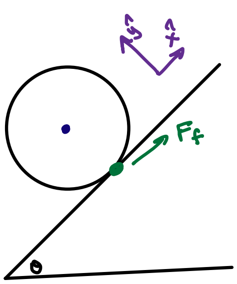
For part (c): $\vec{\Gamma} = \dot{\vec{L}}$

$$\left|MgR\sin\theta\right| = \left|\frac{dL}{dt}\right| = \left|\frac{d}{dt}(I\omega)\right| = \left|\frac{3}{2}MR^2\dot{\omega}\right| = \left|\frac{3}{2}MR\dot{v}\right|$$

$$\Rightarrow g\sin\theta = \left|\frac{3}{2}\dot{v}\right|$$

$\frac{2}{3}g\sin\theta = |\dot{v}|$ $\dot{v} = -\frac{2}{3}g\sin\theta$ (accel. down the hill)

(b) Find the total torque about the CM. The moment of inertia for a disk about its CM is $\frac{1}{2}MR^2$.



$\tau = RF_f$, must find F_f

$M\ddot{x} = -Mg\sin\theta + F_f$

$M\dot{v} = -Mg\sin\theta + F_f$

$M(-\frac{2}{3}g\sin\theta) = -Mg\sin\theta + F_f$

$\frac{1}{3}Mg\sin\theta = F_f$

$\Rightarrow \boxed{\tau = \frac{1}{3}MgR\sin\theta}$

(c) Find the linear acceleration of the disk.

$\boxed{\dot{v} = -\frac{2}{3}g\sin\theta}$ (see part a)

Physics 3221 Spring 2026: Practice Midterm 1 Problem: Grade:

4. Zonai Device

You come across a device that hovers in place. Upon examining it, you realize that it's a rocket expelling fuel vertically downwards to counteract the force of gravity. The rocket has initial mass m_0 and expels fuel at a velocity v_{ex}. The rocket can only afford to burn 90% of its mass as fuel.

(a) How long can the rocket hover?

$$\frac{dP}{dt} = F_{ext} = -mg$$

$$dP = [P+dP] - P = \Big[\underbrace{(m+dm)(\cancel{v+dv})}_{(P+dP)_{rocket}} + \underbrace{(-dm)(\cancel{v}-v_{ex})}_{(P+dP)_{fuel}}\Big] - m\cancel{v}$$

0 0 0

$$dP = v_{ex}\,dm$$

$$\frac{dP}{dt} = v_{ex}\frac{dm}{dt} = -mg$$

$$\int_{m_0}^{m} \frac{v_{ex}\,dm'}{m'} = \int_0^t -g\,dt'$$

$$v_{ex}\ln(m')\Big|_{m_0}^{m} = -gt'\Big|_0^t$$

$$v_{ex}\ln\left(\frac{m}{m_0}\right) = -gt$$

$$-\frac{v_{ex}}{g}\ln\left(\frac{m}{m_0}\right) = t$$

let $f = 0.9$

$$-\frac{v_{ex}}{g}\ln\left(\frac{f m_0}{m_0}\right) = t$$

$$\boxed{t = -\frac{v_{ex}}{g}\ln(f)} = \frac{v_{ex}}{g}\ln\left(\frac{1}{f}\right)$$

(log of # > 1 is > 0)

(b) If you increase v_{ex} but you require that the rocket must continue to hover in place, what happens to $\frac{dm}{dt}$?

$$-\frac{v_{ex}}{g}\ln\left(\frac{m}{m_0}\right) = t$$

$$m = m_0 e^{-gt/v_{ex}}$$

$$\frac{dm}{dt} = -\frac{g m_0}{v_{ex}} e^{-gt/v_{ex}}$$

Increasing v_{ex} makes the argument of the exponent less negative, longer decay time

The prefactor in front decreases as well $\Rightarrow$ $\frac{dm}{dt}$ starts off less negative (smaller in magnitude)

(c) If you double v_{ex} but you require that the rocket must continue to hover in place, how long can the rocket hover compared to your answer in part (a)?

$$t = \frac{2v_{ex}}{g}\ln(f) = 2(t_a)$$

$\Rightarrow$ doubles

(d) What happens to the rocket if you double $\frac{dm}{dt}$ but keep v_{ex} the same? Why?

$$\frac{dm}{dt} = -2\frac{g m_0}{v_{ex}} e^{-gt/v_{ex}}$$

$\Rightarrow$ must have doubled m_0

Grade:

5. Doing Work on a Charged Ball

A charged ball of mass M slides on a table, constrained to follow the path from point A to point B, as depicted in Fig. 3. This ball has charge $+q$ and is subject to an electric field (the magnetic field is zero):

$$\vec{E} = \frac{1}{2}\kappa y^2 \hat{x} + \kappa x y \hat{y}\,,$$

where κ is a constant with units of $(\mathrm{m}\cdot\mathrm{kg})/(\mathrm{C}\cdot\mathrm{s}^2)$. The ball only experiences the Lorentz force from the $\vec{E}$ field and possibly friction while sliding on the horizontal table (plus gravity and the normal force, which cancel in the vertical direction, while the normal force does no work in the horizontal plane of the table).

(a) The work done by the Lorentz force is independent of the pat taken. Knowing this, find the work done by the Lorentz force along the path shown in Fig. 3.

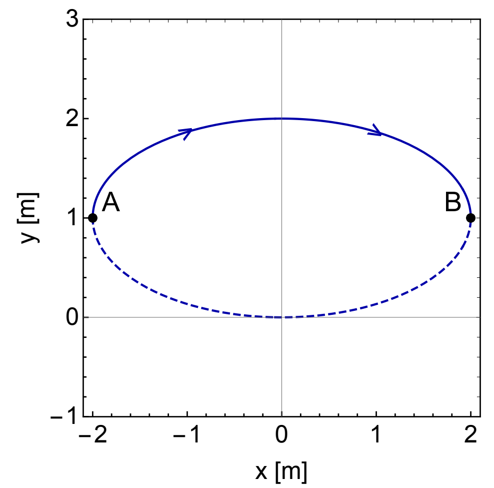


Figure 3: The path along which the ball slides. Distances are measured in units of meters (m).

(path-independent)

Since $\vec{F}$ is conservative, the work done by $\vec{F}$ from $A \to B$ is path-independent. So I'll choose to find it by going along a straight horizontal line from $A \to B$:

0, horizontal path at y=1

$$W = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot(dx\,\hat{x} + \cancel{dy\,\hat{y}}) = \int_{-2}^{2} F_x\,dx = \int_{-2}^{2} \tfrac{1}{2} q \kappa y^2 dx \Big|_{y=1} = \tfrac{1}{2} q \kappa y^2 \Big|_{y=1} (x)_{-2}^{2} = \tfrac{1}{2} q \kappa (1)\,[2-(-2)] = \tfrac{1}{2} q\kappa \cdot 4 = 2q\kappa$$

(c) The ball is released from point A from rest. By the the time gets to point B, the speed of the ball is $v_B = \sqrt{q\kappa/M}$. Given v_B, did the ball lose energy to friction?

From $A \to B$:

$$\Delta E = \Delta T + \Delta U = W_{friction}$$

$$= \tfrac{1}{2} M v_B^2 + (-2q\kappa) = W_{friction}$$

$$= \tfrac{1}{2} M \left(\tfrac{q\kappa}{M}\right) - 2q\kappa = W_{friction}$$

$$= \tfrac{1}{2} q\kappa - 2q\kappa = W_{friction}$$

$$-\tfrac{3}{2} q\kappa = W_{friction} \Rightarrow$$ energy left the system $(\Delta E < 0)$ due to friction.

Formulas (just for the Practice Midterm)

Newton's 2nd Law

$$\vec{F} = m\ddot{\vec{r}}$$

$$(\text{Cartesian}) \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases}$$

$$(\text{Polar}) \begin{cases} F_r = m(\ddot{r} - r\dot{\phi}^2) \\ F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \end{cases}$$

$$(\text{Cylindrical Polar}) \begin{cases} F_\rho = m(\ddot{\rho} - \rho\dot{\phi}^2) \\ F_\phi = m(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}) \\ F_z = m\ddot{z} \end{cases}$$

Coordinate Systems

$$\begin{aligned} \text{Cartesian:} \quad \vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ d\vec{r} &= dx\hat{x} + dy\hat{y} + dz\hat{z} \end{aligned}$$

$$\begin{aligned} \text{Cylindrical :} \quad \vec{r} &= \rho\hat{\rho} + z\hat{z} \\ d\vec{r} &= d\rho\hat{\rho} + \rho d\phi\hat{\phi} + dz\hat{z} \end{aligned}$$

$$\begin{aligned} \text{Spherical:} \quad \vec{r} &= r\hat{r} \\ d\vec{r} &= dr\hat{r} + rd\theta\,\hat{\theta} + r^2\sin\theta\,d\phi\,\hat{\phi} \end{aligned}$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Momentum and Force

$$\vec{p} = m\vec{v}$$

$$\dot{\vec{P}} = \vec{F}^{\text{ext}}$$

Lorentz Force Law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Angular Momentum and Torque

$$\vec{l} = \vec{r} \times \vec{p}$$

$$\dot{\vec{L}} = \vec{\Gamma}^{\text{ext}}$$

$$\vec{\Gamma} = \vec{r} \times \vec{F}$$

$$L = I\omega$$

Work

$$W = \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

Conservative forces

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

$$F = -\vec{\nabla}U$$

$$\vec{\nabla} \times \vec{F} = 0$$

Useful Integrals

$$\int \frac{du}{u} = \ln(u)$$

$$\int \frac{dx}{1 + (x/a)^2} = a\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{1 - (x/a)^2} = a\tanh^{-1}\left(\frac{x}{a}\right)$$