Solutions

Quiz 5

1. Test whether the following force is path-independent using $\vec{\nabla} \times \vec{F}$:

$$\vec{F}_{\text{quiz}} = -y\hat{x} \tag{1}$$

$(\vec{\nabla}\times\vec{F})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k$

$(\vec{\nabla}\times\vec{F})_1 = \varepsilon_{123} \frac{\partial}{\partial y}(0) + \varepsilon_{132} \frac{\partial}{\partial z}(0) = 0$

$(\vec{\nabla}\times\vec{F})_2 = \varepsilon_{213} \frac{\partial}{\partial x}(0) + \varepsilon_{231} \frac{\partial}{\partial z}(-y) = 0 + 0 = 0$

$(\vec{\nabla}\times\vec{F})_3 = \varepsilon_{312} \frac{\partial}{\partial x}(0) + \varepsilon_{321} \frac{\partial}{\partial y}(-y)$

$= 0 + \varepsilon_{321}(-1) = -\varepsilon_{321}$, $\quad \varepsilon_{321} = -\varepsilon_{123} = -1$

$= -(-1) = +1 \neq 0$

$(\vec{\nabla}\times\vec{F}) = +\hat{z} \neq 0$

$\boxed{\text{not conservative}}$

2. The gradient of a function f in spherical coordinates is:

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Fill in the blanks:

$$d\vec{r} = (\ \ 1\ \ )\,\hat{r} + (\ \ r\ \ )\,\hat{\theta} + (\ r\sin\theta\ )\,\hat{\phi}$$

3. Let $U = mgz = mgr\cos\theta$. Find $\vec{F} = -\vec{\nabla}U$

Two ways. 1) Cartesian:

$\vec{\nabla}U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$

$= 0 + 0 + (mg)\hat{z} = mg\hat{z} = \vec{\nabla}U$

$\boxed{\vec{F} = -mg\hat{z}}$

2) Spherical:

$\vec{\nabla}U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$

$= (mg\cos\theta)\hat{r} + \frac{1}{r}(-mgr\sin\theta)\hat{\theta} + \frac{1}{r\sin\theta}(0)$

$= mg(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$

$$\boxed{\vec{F} = \underbrace{mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}_{-\hat{z}}}$$