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Lecture 9 2/9/26

Last Week:

Monday:
- Finished linear & quadratic drag
  - Found $y(x)$ for linear drag
  - Expanded $y(x)$ & compared to the case w/ no drag

Wednesday:
- Solved a coupled first order dif eq.: Charge in a $\vec{B}$-field:

$$\left.\begin{aligned}\dot{v}_x &= \omega v_y\\ \dot{v}_y &= -\omega v_x\end{aligned}\right\}\quad \eta = v_x + i v_y \Rightarrow \dot{\eta} = -i\omega\eta$$

- reviewed complex numbers: $z = x + iy$ or $z = Re^{i\theta}$
  via $e^{i\theta} = \cos\theta + i\sin\theta$

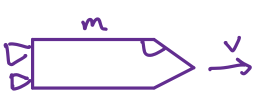
Friday: Started Ch. 3: momentum & $\underbrace{\text{angular momentum}}_{\text{haven't gotten this far yet}}$

If $\vec{F}_{ext} = 0$, $\vec{P}_{total} = \text{const.}$ where $\vec{p} = m\vec{v}$

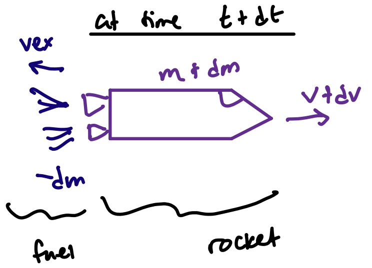
Activity: (1) Found $v(m)$ for a rocket using conservation of momentum
(2) Practiced finding CM of a system

(1) Took a rocket from time t to time $t + dt$
(2) Practiced finding CM of a system

at time t:



at time $t+dt$



$m_{rocket} = m + dm$

$m_{fuel} = -dm$

$V_{rocket} = v + dv$

$V_{fuel} = v + dv - V_{ex}$

$P(t) = mv$

$P(t+dt) = m_{rocket} V_{rocket} + m_{fuel} V_{fuel}$

$P(t+dt) = (m+dm)(v+dv) + (-dm)(v + dv - V_{ex})$

$P(t+dt) = mv + mdv + \cancel{vdm} + \cancel{dmdv}^{\to 0} - \cancel{vdm} - \cancel{dvdm}^{\to 0} + V_{ex}\,dm$

two infinitessimally small numbers

$= mv + mdv - V_{ex}\,dm$

Conservation of momentum: $P(t) = P(t+dt)$

$$\cancel{m}v = \cancel{m}v + m\,dv - v_{ex}\,dm$$

$$v_{ex}\,dm = m\,dv$$

$$\int_{m_0}^{m} \frac{dm'}{m'} = \int_{v_0}^{v} \frac{dv'}{v_{ex}}$$

$$\ln\left(\frac{m}{m_0}\right) = \frac{1}{v_{ex}}(v - v_0)$$

$$v = v_0 - v_{ex}\ln\left(\frac{m}{m_0}\right)$$

Center of mass

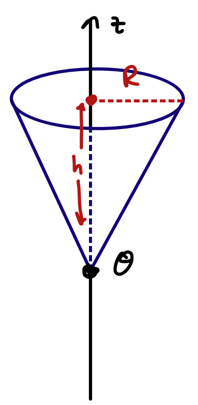
$$\vec{R}_{cm} = \frac{m_\alpha \vec{r}_\alpha}{M} \Rightarrow R_x = \frac{m_\alpha x_\alpha}{M}, \quad R_y = \frac{m_\alpha y_\alpha}{M}, \quad R_z = \frac{m_\alpha z_\alpha}{M}$$

Consider just two particles: $\vec{R}_{cm} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$

For a continuous body:

$\vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$ If density $\rho = \frac{M}{V}$ is uniform, $\vec{R}_{cm} = \frac{1}{M}\int \vec{r}\rho\,dV$

Example: A cone with constant density $\rho = \frac{M}{V}$



$$\vec{R}_{cm} = \frac{1}{M}\int \vec{r}\rho\,dV$$

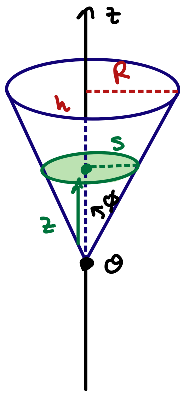
In Cartesian coordinates, $dV = dx\,dy\,dz$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{R}_{cm} = \frac{1}{M}\int (\underbrace{x\hat{x}}_{0} + \underbrace{y\hat{y}}_{0} + z\hat{z})\rho\,dV$$

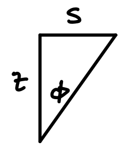
By symmetry, we know $\vec{R} \parallel \hat{z}$, so $R_y = R_x = 0$

$$\vec{R}_{cm} = \frac{1}{M}\hat{z}\int z\rho\,dx\,dy\,dz = \frac{\rho}{M}\hat{z}\int z\,dz \underbrace{\int dx\,dy}_{\text{Do this integral first}}$$



$$\int dx\, dy = A\left[\;\right] = \pi s^2$$

↑ s changes with z



$$\tan\phi = \frac{s}{z} \qquad \text{Also, } \tan\phi = \frac{R}{h}$$

$$\Rightarrow \frac{s}{z} = \frac{R}{h}$$

$$s = \frac{Rz}{h}$$

$$\int dx \int dy = \pi \frac{R^2 z^2}{h^2}$$

$$\vec{R}_{cm} = \frac{\rho}{M}\hat{z}\int z\, dz \int dx\, dy = \frac{\rho\hat{z}}{M}\int z\, dz\left(\frac{\pi R^2 z^2}{h^2}\right) = \frac{\rho}{M}\frac{\pi R^2}{h^2}\hat{z}\int_0^h z^3 dz$$

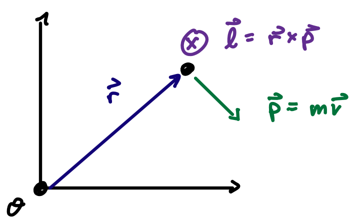
$$= \frac{\rho\pi R^2}{M h^2}\hat{z}\left[\frac{z^4}{4}\right]_0^h = \frac{\rho\pi R^2 h^4}{4 M h^2}\hat{z} \qquad \rho = \frac{M}{V} = \frac{M}{\frac{1}{3}\pi R^2 h}$$

$$= \frac{\cancel{M}\,\cancel{\pi}\cancel{R^2}\,h^2}{4\cdot\frac{1}{3}\cancel{\pi}\cancel{R^2}h\cancel{M}}\hat{z} = \boxed{\frac{3h\hat{z}}{4} = \vec{R}_{cm}}$$

Angular Momentum

Definition: $\vec{\ell} = \vec{r}\times\vec{p}$

↑ depends on origin



$$\dot{\vec{\ell}} = \frac{d}{dt}(\vec{r}\times\vec{p}) = (\dot{\vec{r}}\times\vec{p}) + (\vec{r}\times\dot{\vec{p}})$$

$$= (\dot{\vec{r}}\times m\dot{\vec{r}}) + (\vec{r}\times\dot{\vec{p}})$$

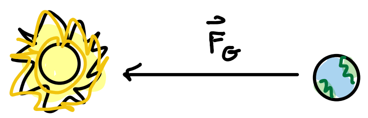
$$= m\underbrace{(\dot{\vec{r}}\times\dot{\vec{r}})}_{=0} + (\vec{r}\times\dot{\vec{p}})$$

$\vec{a}\times\vec{a} = 0$ always

$\Rightarrow \quad \dot{\vec{\ell}} = \vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{\Gamma} = \text{torque} \quad \checkmark\checkmark$

$\dot{\vec{p}} = m\vec{a} = \vec{F}$

<u>Central forces</u>: A force directed along a line between the object exerting the force & the object experiencing the force.



For central force, $\vec{F} \parallel \vec{r}$

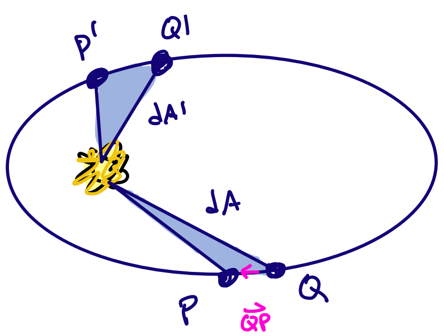
$\Rightarrow \quad \vec{F} \times \vec{r} = 0$

$\Rightarrow \quad \vec{\Gamma} = 0$

$\Rightarrow \quad \vec{\ell} = \text{constant!}$

Angular momentum is conserved when only central forces act on an object.

<u>Kepler's 2nd law</u>: A line drawn from a planet to the sun sweeps out <u>equal areas</u> in <u>equal times</u>.



Kepler's Law restated:

If $t_Q - t_P = t_{Q'} - t_{P'}$, then $dA = dA'$

<u>Show it</u>:

If $dt = t_Q - t_P \ll 1$, then dA & dA' are triangles.

$$dA = \frac{1}{2}bh = \frac{1}{2}\underbrace{|\vec{Q} \times \vec{QP}|}$$

The cross product takes the $\perp$ pieces: $\frac{1}{2}\underbrace{|\vec{Q}|}_{h}\underbrace{|\vec{QP}|\sin\theta}_{b}$

Also, $\vec{Q} = \vec{r}$ and $\vec{QP} = d\vec{r}$

so $\vec{QP} = d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}\,dt$

$$dA = \frac{1}{2}|\vec{r} \times \vec{v}\,dt|$$

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r} \times \frac{\vec{p}}{m}\right| = \frac{|\vec{\ell}|}{2m} = \text{conserved!} \quad \Rightarrow \text{Kepler's 2nd Law}$$