REGISTRATION NUMBER

180128022

Ans. 1. (a)

	$x=1$	$x=2$	$x=3$
$y=1$	0.2	0.1	0
$y=2$	0.1	0.2	0.1
$y=3$	0	0.1	0.2

$P(X=1) = 0.2+0.1+0 = 0.3$

$P(X=2) = 0.1+0.2+0.1 = 0.4$

$P(X=3) = 0+0.1+0.2 = 0.3$

Hence, $P_X(x) = \begin{cases} 0.3, & x=1 \\ 0.4, & x=2 \\ 0.3, & x=3 \end{cases}$

Ans. 1. (b) $E(X) = 1\times 0.3 + 2\times 0.4 + 3\times 0.3$

$E(X) = 0.3+0.8+0.9$

$\boxed{E(X) = 2}$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = (1)^2\times 0.3 + (2)^2 \times 0.4 + (3)^2\times 0.3$

$= 4.6$

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$$Var(X) = E(X^2) - [E(X)]^2$$
$$= 4.6 - (2)^2$$
$$= 4.6 - 4$$
$$\boxed{Var(X) = 0.6}$$

Ans. 1.(c) $P(X+Y=3) = P(X=1, Y=2) + P(X=2, Y=1)$
$$P(X+Y=3) = 0.1 + 0.1$$
$$\boxed{P(X+Y=3) = 0.2}$$

Ans. 1.(d) $P(Y=2 \mid X=2) = \frac{P(X=2 \cap Y=2)}{P(X=2)} = \frac{0.2}{0.4}$ 0.5

$$\boxed{P(Y=2 \mid X=2) = 0.5}$$

Ans. 1.(e) To check if X and Y are independent, the following condition must be true:
$$P(X \cap Y) = P(X)\,P(Y)$$
Now, at $X=1$ and $Y=1$;
$$P(X \cap Y) = 0.2, \quad P(X) = 0.3, \quad P(Y) = 0.3$$
$$\therefore P(X \cap Y) \neq P(X)\,P(Y)$$

Hence, X and Y are not independent.

Ans. 2(a)(i)

$$f_1(x) = \begin{cases} x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

To check whether the above function is a probability density function for a continuous random variable, we need to check if it follows the two conditions stated below:

i) $f(x) \ge 0 \quad \forall x$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Now, $\int_0^2 f_1(x)\,dx = \int_0^2 x\,dx = \frac{1}{2}\left[x^2\right]_0^2 = \frac{1}{2}(2^2 - 0^2) = \frac{1}{2} \times 4 = 2$

So, $\int_0^2 f_1(x)\,dx = 2$

As, the ~~upon~~ integration $\ne 1$, we can say that $f_1(x)$ is not a probability density function for a continuous random variable.

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Ans. 2.(a).(ii)

$$f_2(x) = \begin{cases} x - \frac{1}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

For the above function, we can see that the condition $f(x) \geq 0 \ \forall x$ does not get satisfied for the range $\left(0 \le x < \frac{1}{2}\right)$.

Hence, $f_2(x)$ is not a probability density function for a continuous random variable.

Ans. 2.(b).(i) $f_X(x) = \begin{cases} \frac{3}{2}x^2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

In order to find the cumulative distribution function of X, we can write

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \qquad -\infty < x < \infty$$

$$F_X(x) = \int_{-1}^{x} \frac{3}{2} t^2\,dt$$

$$= \frac{\cancel{3}}{2} \cdot \frac{1}{\cancel{3}} \left[t^3\right]_{-1}^{x} = \frac{1}{2}\left[x^3 - (-1)^3\right]$$

$$= \frac{1}{2}(x^3 + 1)$$

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{1}{2}(x^3+1) & -1 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

$$P\left(X \le -\frac{1}{2}\right) = F\left(-\frac{1}{2}\right) = \frac{1}{2}\left(\left(-\frac{1}{2}\right)^3 + 1\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{8}\right)$$

$$= \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$$

$$\boxed{P\left(X \le -\frac{1}{2}\right) = \frac{7}{16}}$$

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Ans. 2.(b)(ii)

We know that,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

In our case,

$$E(X) = \int_{-1}^{1} x \cdot \frac{3}{2} x^2 \cdot dx$$

$$= \frac{3}{2} \times \frac{1}{4} \left[x^4\right]_{-1}^{1}$$

$$= \frac{3}{8}(1-1) = 0.$$

Hence $\boxed{E(X) = 0.}$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= \int_{-1}^{1} \frac{3}{2} x^4\, dx = \frac{3}{2} \times \frac{1}{5} \left[x^5\right]_{-1}^{1}$$

$$= \frac{3}{10}(1+1) = \frac{3}{5}$$

$$\therefore E(X^2) = \frac{3}{5}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{5} - 0 = \frac{3}{5}$$

Hence $\boxed{Var(X) = 0.6}$

Ans. 3 (a) $T = \{(x, y) ; 0 < |y| < x < \infty\}$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4} y^2 e^{-x} & \text{for } (x, y) \in T \\ 0 & \text{otherwise} \end{cases}$$

Marginal probability density function

$$f_X(x) = \int_{-x}^{x} f_{X,Y}(x, y)\, dy \quad \text{as } 0 < |y| < x$$

$$= \int_{-x}^{x} \frac{1}{4} y^2 e^{-x}\, dy = \frac{1}{4} e^{-x} \int_{-x}^{x} y^2\, dy$$

$$= \frac{1}{4} e^{-x} \cdot \frac{1}{3} \left[y^3 \right]_{-x}^{x} = \frac{1}{12} e^{-x} \left(x^3 - (-x)^3 \right)$$

$$\boxed{f_X(x) = \frac{1}{6} e^{-x} \cdot x^3}$$

Ans. 3(b) Conditional probability density function of Y given that X = x can be expressed as,

$$h(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \left(\frac{1}{4}y^2 e^{-x}\right) \Big/ \left(\frac{1}{6}e^{-x} \cdot x^3\right)$$

$$\boxed{h(y|x) = \frac{3}{2}\frac{y^2}{x^3}}$$

Ans. 3. (c) $E[Y|X] = \int_{-x}^{x} y \cdot \frac{3}{2}\frac{y^2}{x^3}\,dy$

$$= \frac{3}{2} \cdot \frac{1}{x^3} \cdot \int_{-x}^{x} y^3\,dy$$

$$= \frac{3}{2} \cdot \frac{1}{4} \cdot \frac{1}{x^3}\left[x^4 - (-x)^4\right]$$

$$\boxed{E[Y|X] = 0}$$

Ans. 3.(d) If X and Y are independent, then we know that $P(Y|X) = P(Y)$.

However, just because the value of $E(Y|X)$ does not depend on X, we cannot attribute it to the fact that X and Y are independent.

So, I do not agree with the statement.

Ans. 4. (a) $X = (X, Y)^T$

$\mu = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$U = X + Y$ $V = 2X - Y - 1$

$\mu(U) = 1 + (-1)$ [putting the values of μ]
$= 1 - 1$
$= 0$

Similarly $\mu(V) = 2 \cdot 1 - (-1) - 1$
$= 2 + 1 - 1$
$= 2$

We know that $Cov(a + BX) = B \Sigma B^T$
The matrix B is made up of the values from the co-efficients of X and Y for U and V.

$\therefore B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ $a = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Now, $B \Sigma B^T$ [as mentioned earlier]

$B^T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

So, $B \Sigma B^T = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 3+2 & 2+3 \\ 6-2 & 4-3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5+5 & 10-5 \\ 4+1 & 8-1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 7 \end{bmatrix}$$

So, mean vector $\mu_{u,v} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Covariance matrix $\Sigma_{u,v} = \begin{bmatrix} 10 & 5 \\ 5 & 7 \end{bmatrix}$

Ans. 4.(b) For a marginal distribution of a multivariate normal distribution, we write,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$$

$$\left.\begin{aligned} \sigma_1^2 &= 10 \\ \sigma_1 &= \sqrt{10} \\ \mu_1 &= 0 \end{aligned}\right\} \text{(from the results obtained)}$$

Putting the values,

$$f_U(u) = \frac{1}{\sqrt{2\pi}\sqrt{10}} e^{-\frac{(u-0)^2}{2(10)}}$$

$$f_U(u) = \frac{1}{\sqrt{2\pi}\sqrt{10}} e^{-\frac{(u)^2}{20}}$$

Ans. 5.(a)

$$f_{X,Y}(x,y) = \frac{1}{\pi}(x^2+y^2)\, e^{-(x^2+y^2)} \text{ [given]}$$

$U = X+Y$ — ① $V = X - Y$ — ②

Now, solving for X and Y,

$$\begin{aligned} X + \cancel{Y} &= U \text{ — ①} \\ X - \cancel{Y} &= V \text{ — ②} \\ (+)\quad (+)\quad & (+) \\ \hline 2X &= U+V \\ X &= \frac{U+V}{2} \end{aligned}$$

$$\begin{aligned} \cancel{X} + Y &= U \text{ — ①} \\ \cancel{X} - Y &= V \text{ — ②} \\ (-)\quad (-)\quad & (-) \\ \hline 2Y &= U - V \\ Y &= \frac{U-V}{2} \end{aligned}$$

Substituting the values of X and Y in the joint probability density function,

$$\Rightarrow \frac{1}{\pi}\left[\left(\frac{U+V}{2}\right)^2 + \left(\frac{U-V}{2}\right)^2\right] e^{-\left[\left(\frac{U+V}{2}\right)^2 + \left(\frac{U-V}{2}\right)^2\right]} \text{ — ③}$$

We know that, (from Block A Notes)

Jacobian (generalisation of derivative)

$$J(y) = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

where $J(y) \neq 0$ for invertibility.

$$f_Y(y) = f_X(G_1(y), \dots G_m(y)) \, |J(y)| \quad (4)$$

$$x = \frac{u}{2} + \frac{v}{2} \qquad y = \frac{u}{2} - \frac{v}{2}$$

$$\frac{dx}{du} = \frac{1}{2} \qquad \frac{dy}{du} = \frac{1}{2}$$

$$\frac{dx}{dv} = \frac{1}{2} \qquad \frac{dy}{dv} = -\frac{1}{2}$$

Taking Jacobian,

$$J = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -1/4 - 1/4 = -2/4 = -1/2$$

Putting the values in eq-(4) from eq-(3)

$$f_{U,V}(u,v) \Rightarrow \frac{1}{\pi}\left[\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right] e^{-\left[\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right]} \times \left(-\frac{1}{2}\right)$$

$$f_{U,V}(u,v) = -\frac{1}{2\pi}\left[\frac{u^2+v^2+2uv}{4} + \frac{u^2+v^2-2uv}{4}\right] e^{-\left[\frac{u^2+v^2+2uv}{4} + \frac{u^2+v^2-2uv}{4}\right]}$$

$$f_{U,V}(u,v) = -\frac{1}{2\pi}\left(\frac{u^2+v^2+\cancel{2uv}+u^2+v^2-\cancel{2uv}}{4}\right) e^{-\left(\frac{u^2+v^2+\cancel{2uv}+u^2+v^2-\cancel{2uv}}{4}\right)}$$

$$\boxed{f_{U,V}(u,v) = -\frac{1}{2\pi}\left(\frac{u^2+v^2}{2}\right) e^{-\left(\frac{u^2+v^2}{2}\right)}}$$

In order to find the region on which it is non-zero, we can write:

$$\frac{u^2+v^2}{2}=0$$

or, $\frac{u^2}{2} = -\frac{v^2}{2}$

or, $u^2 = -v^2$

Now, this can only happen if $u=v=0$. So, the region on which it is non-zero is where $u \neq 0$ and $v \neq 0$.

Ans. 5.(b) We need to find the cumulative distribution function $F_W(w)$ of $W = Z^2$, where Z is uniformly distributed on the interval $(-1, 1)$.

We know, CDF is given by,

$$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b) \\ 1 & x \geq b \end{cases}$$

Here $b = 1$, $a = -1$.

Hence, $$F_Z(z) = \begin{cases} 0 & z < -1 \\ \frac{z+1}{2} & -1 \leq z < 1 \\ 1 & z \geq 1 \end{cases}$$

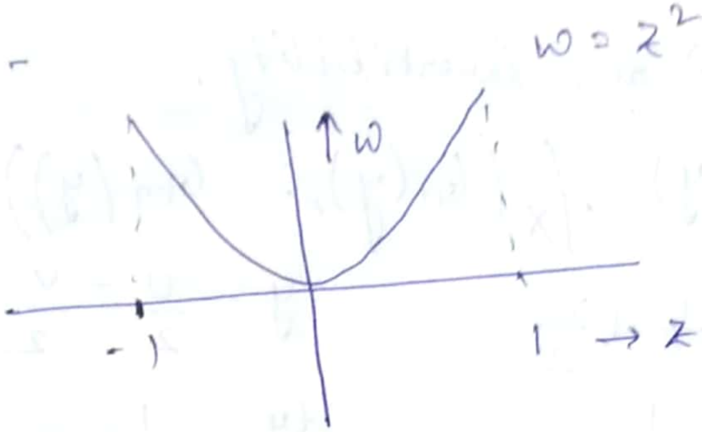
Transforming the function,

$$F_W(w) = P(W \leq w) = P(Z^2 \leq w) = P(Z \leq \sqrt{w}) = F_Z(\sqrt{w})$$

Hence $$F_W(w) = F_Z(\sqrt{w}) = \begin{cases} 0 & w < 0 \\ \frac{\sqrt{w}+1}{2} & 0 \leq w < 1 \\ 1 & w \geq 1 \end{cases}$$

Here, the limits are changed because W will have the lower limit $= 0$, as for any value of Z from -1 to 1, the lowest value W can assume is 0.

For eg -



Putting some values:

$-1 \ldots -0.4 \; -0.2 \; 0 \; 0.2 \; 0.4 \; 0.6 \; 0.8 \; 1$

z	from -1 to 1
w	from $(-1)^2$ to $(1)^2$ $(-1)^2 \ldots (0.4)^2 \ldots (-0.2)^2 \ldots (0)^2 \; (0.4)^2 \ldots 1$.

Hence, we can see that the lowest value w can assume is 0.

Ans. 6.(a) X and Y are independent variables.

$E[X] = E[Y]$ — ①, $var(X) = var(Y)$ — ②

$U = X - Y$

$V = XY$

$Cov(U,V) = E[UV] - E[U]E[V]$ [from Block A notes]

$$Cov(U,V) = cov[X-Y, XY]$$

$$= Cov(X, XY) - Cov(Y, XY)$$

$$= E[X.XY] - E[X]E[XY] - \left(E[Y.XY] - E[Y]E[XY]\right)$$

$$= E[X^2Y] - E[X]E[XY] - E[XY^2] + E[Y]E[XY]$$

$$= E[X^2]E[Y] - \left(E[X]\right)^2 E[Y] - E[X]E[Y^2] + E[X]\left(E[Y]\right)^2$$

$$= E[Y]\left[E[X^2] - \left(E[X]\right)^2\right] - E[X]\left[E(Y^2) - \left(E[Y]\right)^2\right]$$

$$= E[Y]\,Var(X) - E[X]\,Var(Y)$$

$Cov(U,V) = E[Y]Var(X) - E[Y]Var(X)$ [From ① & ②]

$$\boxed{Cov(U,V) = 0}$$

Ans. 6.(b) $U = X - Y$

$V = XY$

$UV = X^2Y - XY^2$

$P(UV) = P(X^2Y) - P(XY^2)$ [X and Y are independent]

$= P(X)P(X)P(Y) - P(X)P(Y)P(Y)$

$= \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^3$ $\left[P(X) = P(Y) = \frac{1}{2} \right]$

$= 0$ ——— (1)

$P(U) = P(X) - P(Y)$

$= \frac{1}{2} - \frac{1}{2}$

$= 0$ ——— (2)

$P(V) = P(XY) = P(X) \cdot P(Y)$

$= \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{4}$ ——— (3)

$\therefore$ $P(U) \cdot P(V) = 0 \times \frac{1}{4} = 0$ [From eqn. (2) & (3)]

and $P(UV) = 0$ [from eqn (1)]

$\therefore$ $P(UV) = P(U)P(V)$

Hence, we can say that U and V are independent.

Ans. 7.(a) $\bar{X} = (x_1, \dots x_n)$ is a vector of independent, identically distributed samples from the $Be(\theta, 1)$ distribution.

We know,

$$B(q, r) = \int_0^1 x^{q-r} (1-x)^{r-1} dx$$

$$B(\theta, 1) = \int_0^1 x^{\theta-1} (1-x)^{1-1} dx$$

$$= \int_0^1 x^{\theta-1} (1-x)^0 dx = \int_0^1 x^{\theta-1} \cdot 1 \, dx$$

$$= \int_0^1 x^{\theta-1} dx$$

$$= \left[\frac{x^{\theta-1+1}}{\theta-1+1}\right]_0^1$$

$$= \frac{1}{\theta} \left[x^\theta\right]_0^1$$

$$= \frac{1}{\theta} \left[1^\theta - 0^\theta\right]$$

$$= \frac{1}{\theta} \cdot 1$$

$$\boxed{B(\theta, 1) = \frac{1}{\theta}}$$ (Proved)

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Ans. 7. (b) We know that,

$$P_{x_i}(x_i) = \frac{x_i^{\theta-1}(1-x_i)^{1-1}}{B(\theta,1)}$$ [From Introductory Material]

$$= x_i^{\theta-1} \cdot \theta$$

$$L(\theta;x) = \prod_{i=1}^{n} P_{x_i}(x_i)$$

$$\boxed{L(\theta;x) = \prod_{i=1}^{n} \theta \cdot x_i^{\theta-1}}$$

$$l(\theta;x) = \log_e(L(\theta;x))$$

$$= \sum_{i=1}^{n} \log_e(\theta \cdot x_i^{\theta-1})$$

$$= \sum_{i=1}^{n} \ln(\theta) + \sum_{i=1}^{n} (\theta-1)\ln(x_i)$$

$$\boxed{l(\theta;x) = n\ln(\theta) + (\theta-1)\sum_{i=1}^{n}\ln(x_i)}$$

Ans. 7. (c)

$$\frac{dl(\theta;x)}{d\theta} = \frac{d}{d\theta}\left[n\ln(\theta) + (\theta-1)\sum_{i=1}^{n}\ln(x_i)\right]$$

$$0 = \frac{n}{\hat{\theta}} + 1 \cdot \sum_{i=1}^{n}\ln(x_i)$$

$$\frac{n}{\hat{\theta}} = -\sum_{i=1}^{n}\ln(x_i)$$

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln(x_i)}$$

In order to find the maximum likelihood estimator $\hat{\theta}$ for θ,

$$\frac{d}{d\theta}\frac{dl(\theta;x)}{d\theta} = n\frac{d}{d\theta}(\theta)^{-1} + \frac{d}{d\theta}\sum_{i=1}^{n}\ln(x_i)$$

$$= n(-1)\theta^{-2} + 0$$

$$= -\frac{n}{\theta^2}$$

Since $-\frac{n}{\theta^2} < 0$, ~~we~~ we can say that it is the maxima.

Reference List

(1) Block A and Block B Notes

(2) "Khan Academy" (2018). Beta Function [online] khanacademy.org. Available at https://www.khanacademy.org/math/statistics-probability [Accessed 6 Oct, 2018]