15/2/21

5. TRANSFORMATIONS

- Functions of Random Variables

Ex: X is a R.V

$|X|$ - errors

$\log X$

X^2

$\sin X$

$aX+b$ . . .

→ For R.V X $y=g(x)$ is also a R.V

we find pdf, cdf ... for y.

Ex:1 $Y = aX+b$

cdf

$$F_Y(y) = P(Y \le y) = P(aX+b \le y)$$

$$= P\left(X \le \frac{y-b}{a}\right)$$

$$= F_X\left(\frac{y-b}{a}\right)$$

Ex:2 $Y = |X|$

cdf

$$F_Y(y) = P(Y \le y) = P(|X| \le y)$$

$$= P(-y \le X \le y)$$

$$F_Y(y) = F_X(y) - F_X(-y)$$

$P(a<x<b) = F(b)-F(a)$

On differentiating cdf of Y i.e., $F_Y(y)$ we get pdf of Y.

Ex:

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\cdot\lambda^{x}}{x!} & x = 0, 1, 2, \ldots \end{cases}$$

$$y = x^2 + 3$$

$$P(Y=y) = P(x^2+3 = y)$$
$$= P(X = \sqrt{y-3})$$
$$= \frac{e^{-\lambda}\cdot\lambda^{\sqrt{y-3}}}{(\sqrt{y-3})!} \qquad y = 3, 4, 7, \ldots$$

$$P(Y=y) = \begin{cases} \dfrac{e^{-\lambda}\cdot\lambda^{\sqrt{y-3}}}{(\sqrt{y-3})!} & , y = 3, 4, 7 \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Ex.

$P(X=-2) = \frac{1}{5}$, $P(X=-1) = \frac{1}{6}$

$P(X=0) = \frac{1}{5}$, $P(X=1) = \frac{1}{15}$

$P(X=2) = \frac{11}{30}$

$A = \{-2, -1, 0, 1, 2\}$
↓
values of x.

$y = x^2$

$B = \{0, 1, 4\}$
↓
values of y.

$$P(Y=y) = \begin{cases} \frac{1}{5} & y = 0 \\ \frac{1}{6}+\frac{1}{15} = \frac{7}{30} & y = 1 \\ \frac{1}{5}+\frac{11}{30} = \frac{17}{30} & y = 4 \end{cases}$$

y	P(Y=y)
0	$\frac{1}{5}$
1	$\frac{7}{30}$
4	$\frac{17}{30}$

$E(Y) = 0 \cdot \frac{1}{5} + 1 \cdot \frac{7}{30} + 4 \cdot \frac{17}{30}$

$= \frac{7}{30} + \frac{68}{30}$

$= \frac{75}{30}$

$= \frac{5}{2}$

1/6/21

Ex.

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$Y = x^2$

$F_Y(y) = P(Y \le y) = P(x^2 \le y)$

$= P(x \le y^{1/2})$

$$F_x(\sqrt{y}) = \int_0^{\sqrt{y}} f(x)\,dx$$

$$= \int_0^{\sqrt{y}} 3x^2\,dx = \left[\frac{3x^3}{3}\right]_0^{\sqrt{y}}$$

$F_x(\sqrt{y}) = y^{3/2}$

cdf $F_Y(y) = y^{3/2}$ $0 < y < 1$

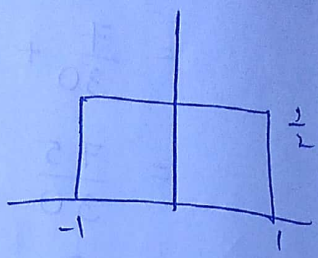
0 otherwise

pdf $$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{3}{2} y^{1/2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

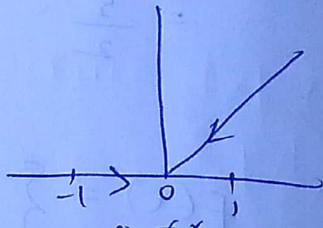
Ex.

$X \sim U[-1, 1]$

$$f(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$Y = X^+ = \begin{cases} x & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$0 \neq x$

At $x = 0$ discontinuous

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2} & y = 0 \\ \frac{1}{2} + \frac{y}{2} & 0 < y < 1 \\ 1 & y > 1 \end{cases}$$

*** (Even if x is continous y may not be continuous).

Theorem:

Let X be R.V of continous type with pdf $f(x)$. Let $y = g(x)$ be a differentiable function for all x. and $g'(x) < 0$ $\forall x$ or $g'(x) > 0$ $\forall x$ then $y = g(x)$ is also R.V of continous type. pdf which is given by

$$f(y) = \begin{cases} f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \alpha < y < \beta \\ 0 & \text{otherwise} \end{cases}$$

$\alpha = \min(g(-\infty), g(\infty))$

$\beta = \max(g(-\infty), g(\infty))$

(it can also be a finite interval)

if $g'(x) > 0$ $\quad g(x)$ is always strictly increasing

$g'(x) < 0$ $\quad g(x)$ is strictly decreasing.

$g(x)$ should be monotonic.

Ex:

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad y = e^x$$

$g'(x) = e^x > 0 \ \forall x$

so $g(x)$ is continuous & strictly increasing

$$f(y) = f(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

$y = e^x$

$\log y = x$

$g^{-1}(y) = \log y$

$$f(y) = f(\log y) \left| \frac{d}{dy}(\log y) \right|$$

$$= \begin{cases} 1 \cdot \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

$0 < x < 1$

$e^0 < e^x < e^1$

$1 < y < e$

Ex:

$X \sim U(0, 1)$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y = g(x) = \frac{X}{1+X}$$

Scanned with CamScanner

$$g(x) = \frac{1+x-1}{1+x}$$

$$g(x) = 1 - \frac{1}{1+x}$$

$$g'(x) = \frac{1}{(1+x)^2} > 0$$

$g(x)$ is strictly increasing

$$y = \frac{x}{1+x}$$

$$y + xy = x$$

$$x(y-1) = -y$$

$$x = \frac{y}{1-y}$$

$$g^{-1}(y) = \frac{y}{1-y}$$

$$f(y) = f(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right|$$

$0 < x < 1$

$$= 1 \cdot \left|\frac{d}{dy}\left(\frac{y}{1-y}\right)\right|$$

$$0 < \frac{x}{1+x} < \frac{1}{2}$$

$$0 < y < \frac{1}{2}$$

$$= \left|\frac{d}{dy} \frac{-(1-y+1)}{(1-y)}\right|$$

$$= \left|\frac{d}{dy}\left(1 + \frac{1}{(1-y)}\right)\right|$$

$$f(y) = \begin{cases} \frac{1}{(1-y)^2} & 0 < y < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Ex:
$$f(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty$$

$$y = g(x) = \tan^{-1} x$$

$$g^{-1}(y) = x$$

$$g^{-1}(y) = \tan y$$

$$f(y) = f(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= f(\tan y) \left| \frac{d}{dy} \tan y \right| \qquad -\infty < x < \infty$$

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$= \frac{1}{\pi(1+\tan^2 y)} \cdot |\sec^2 y| \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$= \frac{1}{\pi \cdot (\sec^2 y)} |\sec^2 y| \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$f(y) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < y < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Ex:
$$f(x) = 5x^4 \qquad 0 < x \le 1$$

$$g(x) = 1 - x^2 = y \implies g'(x) = -2x$$

$$x^2 = 1 - y$$

$$x = \sqrt{1-y}$$

g(x) is decreasing for the given interval.

$$g^{-1}(y) = \sqrt{1-y}$$

$$f(y) = f(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= f(\sqrt{1-y}) \left| \frac{d}{dy} (\sqrt{1-y}) \right|$$

Scanned with CamScanner

$$= 5\left(\sqrt{1-y}\right)^4 \cdot \left|\frac{-1}{2\sqrt{1-y}}\right|$$

$$= 5\left(\sqrt{1-y}\right)^4 \cdot \frac{1}{2\cdot\sqrt{1-y}}$$

$$f(y) = \begin{cases} \frac{5}{2}\left(\sqrt{1-y}\right)^3 & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Functions of Random Vectors / Several Random Variables

Let $X_1, X_2, \dots X_n$ be the random variables defined on (Ω, S, P)

Ex: Functions $X_1 - X_2$, $X_3 + 2X_4 - X_5$, $X_1 \cdot X_2$, ...

we study the joint distributions of functions of R.V.

→ Functions are denoted by $Y = g(X_1, X_2, \dots X_n)$

cdf's $P(Y=y) = P(g(X_1, X_2, \dots X_n) \leq y)$

$$\begin{cases} \displaystyle\sum_{\{(x_1, x_2, \dots x_n) / g(x_1, x_2 \dots x_n) < y\}} P(X_1 = x_1, X_2 = x_2 \dots X_n = x_n) & \text{For discrete} \\ \displaystyle\int_{(x_1, x_2 \dots x_n) / g(x_1, x_2 \dots x_n) \leq y} f(x_1, x_2, \dots x_n)\, dx_1 dx_2 \dots dx_n & \text{For continous} \end{cases}$$

2/6/21 Example on function of Random Variable

$X \sim U[-1, 1]$

pdf of X $f(x)$ is continous

$g(x) = y$

$$F_Y(y) = D \cdot F = \begin{cases} 0 & y < 0 \\ \frac{1}{2} & y = 0 \\ \frac{y+1}{2} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

D·F → distribution function.

$$g'(x) = \begin{cases} \frac{1}{2} & y = 0 \\ \frac{1}{2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Functions of Random Vectors

$X_1, X_2, \ldots X_n$

$g_1(x_1, x_2, \ldots x_n), \; g_2(x_1, x_2, \ldots x_n)$

$g_n(x_1, x_2, \ldots x_n)$

Ex-1

Y \ X	-1	0	1
-2	1/6	1/12	1/6
1	1/6	1/12	1/6
2	1/12	0	1/12

$U = |X|$ $\quad V = Y^2$

$U = 0, 1$ ($0 \to 0$, $1 \to -1, 1$)

$V = 1, 4$ ($1 \to 1$, $4 \to -2, 2$)

$P(U=0, V=1) = P(X=0, Y=1) = \frac{1}{12}$

$P(U=0, V=4) = P(X=0, Y=-2) + P(X=0, Y=2)$

$= \frac{1}{12} + 0 = \frac{1}{12}$

$P(U=1, V=1) = P(X=-1, Y=1) + P(X=1, Y=1)$

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Scanned with CamScanner

$P(U=1, V=4) = P(X=-1, Y=-2) + P(X=-1, Y=2)$

$\downarrow \qquad \downarrow$

$X=-1,1 \qquad -2,2$

$+ P(X=1, Y=-2) + P(X=1, Y=2)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12}$

$= \frac{1}{2}$

V \ U	0	1
1	$\frac{1}{12}$	$\frac{1}{3}$
4	$\frac{1}{12}$	$\frac{1}{2}$

Ex 2

$$f(x,y) = \begin{cases} \frac{1+xy}{4} & |x|<1, \ |y|<1 \\ & -1<x<1 \quad -1<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$U = X^2, \quad V = Y^2$

$F_{u,v}(u,v) = P(U \leq u, V \leq v)$

$= P(X^2 \leq u, Y^2 \leq v)$

$= P(-\sqrt{u} < X < \sqrt{u}, -\sqrt{v} < Y < \sqrt{v})$

$$= \int_{x=-\sqrt{u}}^{\sqrt{u}} \int_{y=-\sqrt{v}}^{\sqrt{v}} \frac{1+xy}{4} \, dy \cdot dx.$$

$$= \int_{x=-\sqrt{u}}^{\sqrt{u}} \left[\frac{y}{4} + \frac{x}{4}\cdot\frac{y^2}{2}\right]_{-\sqrt{v}}^{\sqrt{v}} \cdot dx$$

$$= \frac{1}{4}\int_{x=-\sqrt{u}}^{\sqrt{u}} \left[\sqrt{v} + x\cdot\frac{v}{2} - \left(-\sqrt{v} + \frac{xv}{2}\right)\right] dx$$

$$= \frac{1}{4}\cdot\int_{x=-\sqrt{u}}^{\sqrt{u}} \left(\sqrt{v} + \frac{xv}{2} + \sqrt{v} - \frac{xv}{2}\right) dx$$

$$= \frac{1}{4}\int_{x=-\sqrt{u}}^{\sqrt{u}} 2\sqrt{v}\, dx$$

$$= \frac{1}{2}\cdot\int_{x=-\sqrt{u}}^{\sqrt{u}} \sqrt{v}\, dx$$

$$= \frac{1}{2}\cdot\left[\sqrt{v}\cdot x\right]_{-\sqrt{u}}^{\sqrt{u}}$$

$$= \frac{1}{2}\cdot\sqrt{v}\cdot\left(2\cdot\sqrt{u}\right)$$

$$F_{u,v}(U,V) = \sqrt{v}\cdot\sqrt{u}$$

$U = X^2 \quad V = Y^2$

$|X| \le 1 \quad |Y| \le 1$

$0 \le u \le 1 \quad 0 \le v \le 1$

$$\frac{\partial^2 F_{u,v}(U,V)}{\partial V\, \partial U} = \frac{\partial}{\partial U}\left(\frac{\partial F}{\partial V}\right)$$

$$= \frac{\partial}{\partial U}\left(\frac{\sqrt{u}\cdot 1}{2\sqrt{v}}\right)$$

$$= \frac{1}{2\sqrt{v}\cdot 2\sqrt{u}}$$

$$= \frac{1}{4\sqrt{u}\sqrt{v}}$$

$0 \le u \le 1$

$0 \le v \le 1$

Theorem:

Let $(X_1, X_2, \ldots X_n)$ be Random Vector of dimension of n, which is continous type with pdf or $f(x_1, x_2, \ldots x_n)$

Let
$$y_1 = g_1(x_1, x_2 \ldots x_n)$$
$$y_2 = g_2(x_1, x_2 \ldots x_n)$$
$$\vdots \qquad \vdots$$
$$y_n = g_n(x_1, x_2 \ldots x_n)$$

be a mapping from R_n to R_n

$(x_1 \; x_2 ; x_3)$ $\downarrow$ $(y_1 \; y_2 \; y_3)$, $\uparrow g_1 \; \uparrow g_2 \; \uparrow g_3$

there exists inverse mapping
$$x_1 = h_1(y_1 y_2, \ldots y_n)$$
$$x_2 = h_2(y_1 y_2, \ldots y_n)$$
$$\vdots \qquad \vdots$$
$$x_n = h_n(y_1, y_2 \ldots y_n)$$

Conditions:

① Assume both mapping $(g_i(x))$ & inverse $(h(y))$ are continous

② Assume that partial derivatives

$$\frac{\partial x_i}{\partial y_j}, \quad 1 \le i \le n, \quad 1 \le j \le n \quad (n^2 \text{ partial derivatives})$$

should exist & continous

③ Assume that Jacobian of inverse mapping

will be formed only for square matri...

$$J = \frac{\partial(x_1, x_2 \ldots x_n)}{\partial(y_1, y_2 \ldots y_n)}$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}_{n\times n} = \text{function of } y$$

is non zero for $(y_1, y_2 \ldots y_n)$

then $(Y_1, Y_2 \ldots Y_n)$ has a joint continous Distribution function with PDF

$$w(y_1, y_2 \ldots y_n) = |J| \; f(h_1(y_1, y_2 \ldots y_n), h_2(y_1, y_2 \ldots y_n) \ldots h_n(y_1, y_2 \ldots y_n))$$

Ex:1 X_1, X_2, X_3 be independent R.V's with common exponential distribution

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f(h_1, h_2, h_3) = f(h_1) \cdot f(h_2) f(h_3)$$

$$\boxed{\begin{aligned} &\text{Ex: if } f(x_1, x_2, x_3) = x_1 + x_2^2 + x_3 \\ &\text{then } f(h_1, h_2, h_3) = h_1 + h_2^2 + h_3 \end{aligned}}$$

$$y_1 = x_1 + x_2 + x_3 , \quad y_2 = \frac{x_1 + x_2}{x_1 + x_2 + x_3} , \quad y_3 = \frac{x_1}{x_1 + x_2}$$

sol

$$y_2 = \frac{x_1 + x_2}{y_1} \qquad y_2 = \frac{x_1 + x_2}{y_1}$$

$$y_2 y_1 = y_1 y_2 y_3 + x_2 \qquad y_2 = \frac{x_1}{y_1 y_3}$$

$$x_2 = y_2 y_1 - y_1 y_2 y_3 \qquad \boxed{x_1 = y_1 y_2 y_3}$$

$$\boxed{x_2 = y_1 y_2 (1 - y_3)}$$

$$y_1 = x_1 + x_2 + x_3$$

$$y_1 = \cancel{x_1} y_2 y_3 + y_1 y_2 (1 - y_3) + x_3$$

$$x_3 = y_1 - y_1 y_2 y_3 - y_1 y_2 (1 - y_3)$$

$$= y_1 - y_1 y_2$$

$$\boxed{x_3 = y_1 (1 - y_2)}$$

$$x_1 = y_1 y_2 y_3 \qquad x_2 = y_1 y_2 (1 - y_3) \qquad x_3 = y_1 (1 - y_2)$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1 y_2 \\ 1 - y_2 & -y_1 & 0 \end{vmatrix}$$

$$= y_2 y_3 (-y_1^2 y_2) - y_1 y_3 (y_1 y_2 (1 - y_2)) + y_1 y_2$$

$$[-y_1 \, y_2 (1-y_3) - (1-y_2) (y_1 (1-y_3)$$

$$= -y_1^2 y_2^2 y_3 - y_1^2 y_2 y_3 + y_1^2 y_2^2 y_3$$
$$+ y_1 y_2 \left[-y_1 y_2 + y_1 y_2 y_3 - (y_1 - y_1 y_2)(1-y_3) \right]$$

$$= -y_1^2 y_2 y_3 - y_1^2 y_2^2 + y_1^2 y_2^2 y_3 + -y_1 y_2 (y_1 - y_1 y_3 - y_1 y_2 + y_1 y_2 y_3)$$

$$= -y_1^2 y_2^2 - y_1^2 y_2 + y_1^2 y_2^2 y_3 + y_1^2 y_2^2 - y_1^2 y_2^2 y_3$$

$$= -y_1^2 y_2$$

$$J = -y_1^2 y_2$$

$$|J| = y_1^2 y_2$$

Joint pdf of $w(y_1, y_2, y_3) = |J| \, f(h_1, h_2, h_3)$

$$= y_1^2 y_2 \; f(h_1) \cdot f(h_2) \cdot f(h_3)$$

$$= y_1^2 y_2 \cdot f(y_1 y_2 y_3) \, f(y_1 y_2 (1-y_3)) \cdot f(y_1(1-y_2))$$

$$= y_1^2 y_2 \cdot e^{-y_1 y_2 y_3} \cdot e^{-y_1 y_2 (1-y_3)} \cdot e^{-y_1 (1-y_2)}$$

$$= y_1^2 y_2 \cdot e^{-y_1}$$

$0 < y_1 < \infty$

$0 < y_2 < 1$

$0 < y_3 < 1$

$x_1 > 0$

$x_2 > 0$

$x_3 > 0$

$y_1 = x_1 + x_2 + x_3$

$y_1 > 0$

$y_2 = \dfrac{x_1 + x_2}{x_1 + x_2 + x_3}$

↳ fraction $\Rightarrow 0 < y_2 < 1$

$y_3 = \dfrac{x_1}{x_1 + x_2}$

↳ fraction

$0 < y_2 < 1$

solution

$$w(y_1, y_2, y_3) = \begin{cases} y_1^2 y_2 e^{-y_1} & \text{for } 0 < y_1 < \infty \\ & 0 < y_2 < 1 \\ & 0 < y_3 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Ex: 2

Let X_1, X_2 be two independent R.V

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_1 = X_1 + X_2 \qquad Y_2 = X_1 - X_2$$

sol

$$Y_1 + Y_2 = 2X_1 \qquad Y_1 - Y_2 = 2X_2$$

$$X_1 = \frac{Y_1 + Y_2}{2} \qquad X_2 = \frac{Y_1 - Y_2}{2}$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$|J| = \frac{1}{2}$$

$$w(y_1, y_2) = \frac{1}{2} \cdot f(h_1, h_2)$$

$$= \frac{1}{2} f\left(\frac{y_1 + y_2}{2}\right) \cdot f\left(\frac{y_1 - y_2}{2}\right)$$

$= \frac{1}{2} \cdot 1 \cdot 1$

$0 < x_1 < 1 \quad 0 < x_2 < 1$

$= \frac{1}{2}$

$y_1 = x_1 + x_2 \quad y_2 = x_1 - x_2$

$0 < y_1 < 2 \quad -1 < y_2 < 1$

$$w(y_1, y_2) = \begin{cases} \frac{1}{2} & 0 < y_1 < 2, \; -1 < y_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Scanned with CamScanner

4/6/21 Tutorial problems

Bivariate normal distribution.

(1) Let x be the height of father

y be the height of son

Assume x & y follow bivariate normal distribution.

Assume that $E(x) = 68$ inches

$E(y) = 69$

$\sigma_x = \sigma_y = 2$

$\rho = 0.5$

Given $x = 80$, find the Mean & variance of y.

sol

$$E(y/x=80) = \mu_y + \frac{\rho\,\sigma_y}{\sigma_x}(x-\mu_x)$$

$$= 69 + 0.5\left(\frac{2}{2}\right)(80-68)$$

$$= 69 + \frac{1}{2}(12)$$

$$= 69 + 6$$

$$= 75$$

$$Var(y/x=80) = (1-\rho^2)\,\sigma_y^2$$

$$= (1-(0.5)^2)(2)^2$$

$$= (4)\left(1-\frac{1}{4}\right)$$

$$= 4\left(\frac{3}{4}\right)$$

$$= 3.$$

② Measurements taken on 'n' heart attack patients on their cholestrol levels. For each patient measurements were taken 0, 2, & 4 days after attack.

Sample mean.

0th	X_1	259.5
2nd	X_2	230.8
4th	X_3	221.5

Covariance matrix

	0	2	4
0	2276	1508	813
2	1508	2206	1349
4	813	1349	1865

$\boxed{\text{Covariance matrix is always symmetric}}$

Find Mean and Variance of $X_1 - X_2$.

Coefficient vector $= (1, -1, 0)$

$$E(X_1 - X_2) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 259.5 \\ 230.8 \\ 221.5 \end{pmatrix}$$

$$= 259.5 - 230.8$$

$$= 28.7$$

$$\text{Var} = \underset{1\times3}{\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}} \underset{3\times3}{\begin{pmatrix} 2276 & 1508 & 813 \\ 1508 & 2206 & 1349 \\ 813 & 1349 & 1865 \end{pmatrix}} \underset{3\times1}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 768 & -698 & -536 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1466$$

③ $f(x,y) = \begin{cases} \frac{xy}{96} & 0<x<4, \quad 1<y<5 \\ 0 & \text{otherwise} \end{cases}$

Find $E(X)$, $E(Y)$, $E(XY)$, $Cov(X,Y)$, $E(2X+3Y)$

sol

$$f_x(x) = \int_1^5 f(x,y)\,dy = \int_1^5 \frac{xy}{96}\,dy$$

$$= \frac{x}{96} \cdot \left[\frac{y^2}{2}\right]_1^5$$

$$= \frac{x}{192}(25-1)$$

$$= \frac{24x}{192}$$

Marginal $f_x(x) = \begin{cases} \frac{x}{8} & 0<x<4 \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = \int_0^4 x \cdot \frac{x}{8}\,dx = \frac{1}{8}\int_0^4 x^2\,dx$$

$$= \frac{1}{8} \cdot \left[\frac{x^3}{3}\right]_0^4 = \frac{1}{24} \cdot 64$$

$$= \frac{8}{3}$$

$$f_y(y) = \int_0^4 f(x,y)\,dx = \int_0^4 \frac{xy}{96}\,dx$$

$$= \frac{y}{96} \cdot \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{y}{192} \cdot 16$$

$$f_Y(y) = \begin{cases} \dfrac{y}{12} & 1 < y < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$E(Y) = \int_1^5 y \cdot \frac{y}{12} \, dy$$

$$= \frac{1}{12} \cdot \left[\frac{y^3}{3}\right]_1^5$$

$$= \frac{1}{36} \cdot (125 - 1)$$

$$= \frac{124}{36}$$

$$E(Y) = \frac{31}{9}$$

$$E(XY) = \int_{x=0}^{4} \int_{y=1}^{5} xy \cdot \frac{xy}{96} \, dy \, dx$$

$$= \int_{x=0}^{4} \int_{y=1}^{5} \frac{x^2 y^2}{96} \, dy \, dx$$

$$= \int_{x=0}^{4} \frac{x^2}{96} \cdot \left[\frac{y^3}{3}\right]_1^5 dx$$

$$= \int_{x=0}^{4} \frac{x^2}{96} \cdot \left(\frac{124}{3}\right) \cdot dx$$

$$= \cancel{\int_{0}^{4} \frac{x^3}{3}} = \frac{124}{3 \times 96} \int_{0}^{4} x^2 \, dx$$

$$= \frac{124}{3 \times 96} \cdot \left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{124}{3 \times 96} \times \frac{64}{3}$$

$$E(XY) = \frac{248}{27}$$

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \frac{248}{27} - \frac{8}{3}\,\frac{31}{9}$$

$$Cov(X,Y) = 0$$

$$E(2X+3Y) = 2E(X) + 3E(Y)$$

$$= 2 \cdot \frac{8}{3} + 3 \cdot \frac{31}{9}$$

$$= \frac{16}{3} + \frac{31}{3}$$

$$E(2X+3Y) = \frac{47}{3}$$

Scanned with CamScanner

4) Let Y be a continous Random Variable.

where $f(y) = \begin{cases} \frac{3}{2} y^2 & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

Determine density funtion of $U = 3 - y$.

sol

$U = 3 - Y$ $\qquad\qquad U = g(y)$

$g(y) = 3 - y$ $\qquad\qquad g^{-1}(u) = y$

$g'(y) = -1 < 0$ $\qquad\qquad g^{-1}(u) = 3 - u$

$\therefore$ function is strictly decreasing

$$f(u) = f(g^{-1}(u)) \left| \frac{d}{du} \cdot g^{-1}(u) \right|$$

$$= f(3-u) \cdot \left| \frac{d}{du} (3-u) \right|$$

$$= \frac{3}{2} \cdot (3-u)^2 \cdot |-1|$$

$-1 \le y \le 1$

$-1 \le -y \le 1$

$2 \le 3 - y \le 4$

$2 \le u \le 4$

$$f(u) = \begin{cases} \frac{3}{2} (3-u)^2 & 2 \le u \le 4 \\ 0 & \text{otherwise} \end{cases}$$

5) $f(y) = \begin{cases} \frac{1}{2} & 9 \le y \le 11 \\ 0 & \text{otherwise} \end{cases}$

$U = 2Y^2$

$g(y) = U = 2y^2 \Rightarrow g'(y) = 4y > 0 \ \forall [9, 11]$

$g^{-1}(u) = y = \sqrt{\frac{u}{2}}$

$$f(u) = f\left(g^{-1}(u)\right)\left|\frac{d}{du}\cdot g^{-1}(u)\right|$$

$$= f\left(\sqrt{\frac{u}{2}}\right)\left|\frac{d}{du}\cdot\sqrt{\frac{u}{2}}\right|$$

$$= \frac{1}{2}\cdot\left|\frac{1}{2\sqrt{\frac{u}{2}}}\right|\cdot\frac{1}{\cancel{2}}$$

$9 \leq y \leq 11$

$81 \leq y^2 \leq 121$

$162 \leq 2y^2 \leq 242$

$162 \leq u \leq 242$

$$f(u) = \begin{cases} \frac{1}{4\sqrt{2}\sqrt{u}} & 162 \leq u \leq 242 \\ 0 & \text{otherwise} \end{cases}$$

(6)

$$f(x) = \begin{cases} 2^{-x} & x = 1, 2, 3 \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$U = X^4 + 1$$

$$g(x) = u$$

$$g^{-1}(u) = x = (u-1)^{1/4} = \sqrt[4]{(u-1)}$$

$x = A = 1, 2, 3 \ldots$

$u = B = 2, 17, 82, \ldots$

$$g(u) = P(U = u) = P\left(X^4 + 1 = u\right)$$

$$= P\left(X = \sqrt[4]{u-1}\right)$$

$$= f\left((u-1)^{1/4}\right)$$

$$g(u) = \begin{cases} 2^{-(u-1)^{1/4}} & u = 2, 17, 82, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Scanned with CamScanner

(7) If X and Y are independent gamma distributions

Compute joint density function of $U = X+Y$

$V = \frac{X}{X+Y}$

Sol $f_{XY} = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)}$

$U = X+Y \qquad V = \frac{X}{X+Y}$

$X = UV$

$Y = U - X$

$= U - UV$

$Y = U(1-V)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$|J| = |-uv - u(1-v)|$

$= |-uv - u + uv|$

$= |-u|$

$= u$

$f(u,v) = |J| \; f(h_1, h_2)$

$= u \cdot f(uv, \; u(1-v))$

$= u \cdot f(uv) \cdot f(u(1-v))$

$=$

$$= \frac{u \cdot \lambda e^{-\lambda u v} (\lambda u v)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{\lambda \cdot e^{-\lambda(u(1-v))} \cdot (\lambda \cdot u(1-v))^{\beta-1}}{\Gamma(\beta)}$$

$$= \frac{\cancel{u \cdot \lambda \cdot \lambda^{\alpha-1} \cdot \lambda \cdot \lambda^{\beta-1}}}{\cancel{\Gamma(\alpha) \cdot \Gamma(\beta)}}$$

$$= \frac{u \cdot \lambda^2 \cdot (\lambda u v)^{\alpha-1} \cdot (\lambda u)^{\beta-1} \cdot (1-v)^{\beta-1} \cdot e^{-\lambda u v - \lambda u + \lambda u v}}{\Gamma(\alpha) \cdot \Gamma(\beta)}$$

$$= \frac{u \cdot \lambda^{\alpha+\beta} \cdot u^{\alpha+\beta-2} \cdot v^{\alpha-1} \cdot (1-v)^{\beta-1} \cdot e^{-\lambda u}}{\Gamma(\alpha) \cdot \Gamma(\beta)}$$

$$f(u,v) = \frac{\lambda^{\alpha+\beta} \cdot u^{\alpha+\beta-1} \cdot v^{\alpha-1} \cdot (1-v)^{\beta-1} \cdot e^{-\lambda u}}{\Gamma(\alpha) \cdot \Gamma(\beta)} \quad \begin{matrix} 0<u<1 \\ 0<v<1 \end{matrix}$$

H.W

① $f(x) = \begin{cases} x^2/81 & -3 < x < 6 \\ 0 & \text{otherwise} \end{cases}$

$U = \frac{1}{3}(12 - x)$

Determine density function of U.

$g(x) = u$ $\qquad u = 4 - \frac{x}{3}$

$g^{-1}(u) = x$ $\qquad \frac{x}{3} = 4 - u$

$g^{-1}(u) = 3(4-u)$

$g(x) = u = 4 - \frac{x}{3}$

$g'(x) = -\frac{1}{3} < 0$

Function is stritcly decreasing

$$f(u) = f(g^{-1}(u)) \left| \frac{d}{du} \cdot g^{-1}(u) \right|$$

$$= f(3(4-u)) \left| \frac{d}{du} \cdot 3(4-u) \right|$$

$$= \frac{3^2 (4-u)^2}{81} \cdot |-3|$$

$$= \frac{9}{81} \cdot (4-u)^2 \cdot 3$$

$-3 < x < 6$

$-6 < -x < 3$

$6 < 12 - x < 15$

$2 < \frac{12-x}{3} < 5$

$2 < u < 5$

$$f(u) = \begin{cases} \frac{1}{3} \cdot (4-u)^2 & 2 < u < 5 \\ 0 & \text{otherwise} \end{cases}$$

(2) X_1, X_2, X_3 be R.V

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \qquad -\infty < x < \infty$$

$$Y_1 = \frac{X_1 - X_2}{\sqrt{2}}, \qquad Y_2 = \frac{X_1 + X_2 - 2X_3}{\sqrt{6}}$$

$$Y_3 = \frac{X_1 + X_2 + X_3}{\sqrt{3}}$$

$f(Y_1, Y_2, Y_3) = ?$

$$y_1 = \frac{x_1 - x_2}{\sqrt{2}} \quad , \quad y_2 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}$$

$$x_1 - x_2 = \sqrt{2}\, y_1 \rightarrow ① \quad , \quad x_1 + x_2 - 2x_3 = \sqrt{6}\, y_2 \rightarrow ②$$

$$y_3 = \frac{x_1 + x_2 + x_3}{\sqrt{3}}$$

$$x_1 + x_2 + x_3 = \sqrt{3}\, y_3 \rightarrow ③$$

③ - ② $\quad 3x_3 = \sqrt{3}\, y_3 - \sqrt{6}\, y_2$

$$x_3 = \frac{\sqrt{3}\, y_3 - \sqrt{6}\, y_2}{3}$$

$$x_3 = \frac{y_3 - \sqrt{2}\, y_2}{\sqrt{3}}$$

Put x_3 in ③

$$x_1 + x_2 + \frac{y_3 - \sqrt{2}\, y_2}{\sqrt{3}} = \sqrt{3}\, y_3$$

$$x_1 + x_2 = \sqrt{3}\, y_3 - \frac{y_3 - \sqrt{2}\, y_2}{\sqrt{3}}$$

$$= \frac{3y_3 - y_3 + \sqrt{2}\, y_2}{\sqrt{3}}$$

$$x_1 + x_2 = \frac{2y_3 + \sqrt{2}\, y_2}{\sqrt{3}} \rightarrow ④$$

① + ④ $\Rightarrow \quad 2x_1 = \sqrt{2}\, y_1 + \dfrac{2y_3 + \sqrt{2}\, y_2}{\sqrt{3}}$

$$x_1 = \frac{\sqrt{6}\, y_1 + 2y_3 + \sqrt{2}\, y_2}{2\sqrt{3}}$$

(1) $\Rightarrow$ $$2X_2 = \frac{2Y_3 + \sqrt{2}Y_2}{\sqrt{3}} - \sqrt{2}Y_1$$

$$2X_2 = \frac{2Y_3 + \sqrt{2}Y_2 - \sqrt{6}Y_1}{\sqrt{3}}$$

$$X_2 = \frac{2Y_3 + \sqrt{2}Y_2 - \sqrt{6}Y_1}{2\sqrt{3}}$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \frac{\partial X_1}{\partial Y_3} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \frac{\partial X_2}{\partial Y_3} \\ \frac{\partial X_3}{\partial Y_1} & \frac{\partial X_3}{\partial Y_2} & \frac{\partial X_3}{\partial Y_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\sqrt{6}}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-\sqrt{6}}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{vmatrix}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{\sqrt{6}}{2\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{\sqrt{6}}{2\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\left(\frac{\sqrt{6}}{2\sqrt{3}}\cdot\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\cdot\frac{\sqrt{6}}{2\sqrt{3}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{12}}$$

$$= \frac{2\sqrt{2}}{3\sqrt{2}} + \frac{2}{6}$$

$$= \frac{2}{3} + \frac{1}{3}$$

$J = 1$

$|J| = 1$

$$f(y_1, y_2, y_3) = |J| \cdot f(h_1, h_2, h_3)$$

$$= 1 \cdot f\left(\frac{\sqrt{6}y_1 + 2y_3 + \sqrt{2}y_2}{2\sqrt{3}}\right) \cdot f\left(\frac{2y_3 + \sqrt{2}y_2 - \sqrt{6}y_1}{2\sqrt{3}}\right) \cdot f\left(\frac{y_3 - \sqrt{2}y_2}{\sqrt{3}}\right).$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{\sqrt{6}y_1 + 2y_3 + \sqrt{2}y_2}{2\sqrt{3}}\right)^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\cdot\left(\frac{2y_3 + \sqrt{2}y_2 - \sqrt{6}y_1}{2\sqrt{3}}\right)^2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\cdot\left(\frac{y_3 - \sqrt{2}y_2}{\sqrt{3}}\right)^2}$$

$$f(y_1 y_2 y_3) = \begin{cases} \frac{1}{(\sqrt{2\pi})^3} \cdot e^{-\frac{1}{2}(y_1^2 + y_2^2 + y_3^2)} & -\infty < y_1 < \infty \\ & -\infty < y_2 < \infty \\ & -\infty < y_3 < \infty \\ 0 & \text{otherwise} \end{cases}$$

Distributions of sums of Random Variable

Random variables are independent

Discrete: (For 2 R.V's)

If X & Y are 2 random variables then the function of sum of these two R.V's X & Y

i.e., $Z = X + Y$, then $P(Z = X+Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(X = k, Y = z-k)$

if they are independent

$$P(Z = X+Y) = \sum P(X=k) \cdot \sum P(Y = z-k)$$

Ex:1 $P(X=0) = \frac{1}{2}$ $P(X=1) = \frac{1}{2}$

$P(Y=0) = \frac{1}{2}$ $P(Y=1) = \frac{1}{2}$

$Z = X+Y$)

Distribution of Z ?

Sol Z takes $0, 1, 2$

$P(Z=0) = P(X=0, Y=0) = P(X=0) \cdot P(Y=0) = \frac{1}{4}$

$P(Z=1) = \sum P(X=k, Y=1-k)$

$= P(X=0, Y=1) + P(X=1, Y=0)$

$= P(X=0) \cdot P(Y=1) + P(X=1) \cdot P(Y=0)$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$P(Z=2) = \sum P(X=k, Y=2-k) = P(X=1, Y=1)$

$= P(X=1) \cdot P(Y=1) = \frac{1}{4}$

$Z=z$	0	1	2
$P(Z=z)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Ex:2

2 dice are thrown $Z=X+Y$ where X is the number which appears on first throw & Y is the number on second throw. $Z=X+Y$.

Distribution of Z?

sol

$$Z=2 = P(X=1,Y=1) = P(X=1)\cdot P(Y=1) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$$

$$3 = P(X=1,Y=2)+P(X=2,Y=1) = \frac{1}{6}\cdot\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{6} = \frac{2}{36}$$

$$4 = P(X=1,Y=3)+P(X=2,Y=2)+P(X=3,Y=1)$$

$$= \frac{3}{36}$$

$\vdots$

Ex: Poisson Distribution

Let X & Y be two R.V's which are independent & follow poisson distribution with respective means λ_1 & λ_2. Distribution of $X+Y$?

sol

$$X+Y=n$$

$$P(X+Y=n) = \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

$$= \sum_{k=0}^{n} \frac{e^{-\lambda_1}\cdot\lambda_1^{k}}{k!}\cdot\frac{e^{-\lambda_2}\cdot\lambda_2^{n-k}}{(n-k)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \cdot \lambda_2^{n-k}}{k!(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \underbrace{\sum_{k=0}^{n} \frac{n! \, \lambda_1^k \cdot \lambda_2^{n-k}}{k!(n-k)!}}$$

$$P(X+Y=n) = \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n}{n!}$$

Mean $= \lambda_1 + \lambda_2$.

Continous case: Let X & Y are two R.V's sum of these two R.V's X+Y's distribution is defined as follows.

cdf $P(Z \le z) = P(X+Y \le z)$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f(x,y) \, dx \, dy$$

since X & Y are independent

$$P(X+Y \le z) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_X(x) \cdot f_Y(y) \, dx \, dy$$

$$= \int_{y=-\infty}^{\infty} \left(\int_{x=-\infty}^{z-y} f_X(x) \, dx \right) f_Y(y) \, dy$$

$$= \int_{-\infty}^{\infty} F_X(z-y) \cdot f_Y(y) \, dy$$

$$F_{X+Y}(z) = \int_{-\infty}^{\infty} F_x(z-y) \cdot f_y(y)\, dy$$

is called cumulative D.F of X+Y which is called convolution when X & Y are independent

By differentiating

Pdf $\quad f_{X+Y}(z) = \int_{-\infty}^{\infty} f_x(z-y) \cdot f_y(y) \cdot dy$

$$= \int_{y=-\infty}^{\infty} f_x(z-y) \cdot f_y(y) \cdot dy$$

Ex: Sum of two independent uniform Random variables X & Y follows uniform distribution

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the distribution of $Z = X+Y$.

Sol
$$f_{(x+y)}(z) = \int_{y=0}^{1} f_x(z-y) \cdot f_y(y)\, dy$$

$0 \le z \le 2$

$$= \int_{y=0}^{1} f_x(z-y) \cdot dy$$

$x+y=z$

$0 \le z \le 1$

$$\int_{y=0}^{z} f_x(z-y)\,dy = \int_{y=0}^{z} 1 \cdot dy = z \qquad \begin{array}{l} z-y \ge 0 \\ z \ge y \end{array}$$

$1 \le z \le 2$

$$z-y \le 1$$
$$z \le 1+y$$
$$y \ge \cancel{1+z}\ z-1$$

$$\cancel{\int_{y=0}^{1+z} f_x(z-y)}$$

$$\int_{y=z-1}^{1} f_x(z-y)\,dy = \int_{y=z-1}^{1} 1 \cdot dy$$

$$= 1-z+1$$

$$= 2-z$$

$$f_z(z) = \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where $z = x+y$

Ex:

$$f_x(x) = \lambda \cdot e^{-\lambda x} \text{ if } x \ge 0$$
$$f_y(y) = \lambda \cdot e^{-\lambda y} \text{ if } y \ge 0$$

$$f_z(z) = \int_{y=0}^{z} f_x(z-y)\, f(y)\,dy \qquad \begin{array}{l} z-y \ge 0 \\ z \ge y \\ y \le z \end{array}$$

$$= \int_{y=0}^{z} \lambda \cdot e^{-\lambda(z-y)} \cdot \lambda\, e^{-\lambda y} \cdot dy.$$

$$= \lambda^2 \cdot \int_{y=0}^{z} e^{-\lambda z} \, dy$$

$$= \lambda^2 \cdot e^{-\lambda z} \cdot (z)$$

$$f_Z(z) = \begin{cases} \lambda^2 z \cdot e^{-\lambda z} & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2)$$

when independent $Var(X_1 + X_2) = Var(X_1) + Var(X_2)$

$$S.D(X_1 + X_2) = \sqrt{Var(X_1) + Var(X_2)}$$

$$= \sqrt{\sigma_1^2 + \sigma_2^2}$$

Moments of $X+Y$

$$M_{x+y}(t) = E\left(e^{t(x+y)}\right)$$

$$= \int_x \int_y e^{t(x+y)} f_{xy}(x, y) \, dx\,dy$$

$$= \int_x \int_y e^{tx} \cdot e^{ty} \cdot f_x(x) \cdot f_y(y) \cdot dx\,dy$$

$$= \int_x e^{tx} f_x(x) \cdot dx \int_y e^{ty} \cdot f_y(y)\,dy$$

Scanned with CamScanner

$$M_{x+y}(t) = M_x(t) \cdot M_y(t)$$

Binomial:

$$X \sim B(n, p) \quad Y \sim B(m, p)$$

$$M_{x+y}(t) = M_x(t) \cdot M_y(t)$$

$$= (p \cdot e^t + q)^n \cdot (pe^t + q)^m$$

$$= (pe^t + q)^{n+m}$$ → Moment generating fun of $X+Y$.

$$X + Y \sim B(n+m, p)$$

Normal Dist.

$$X \sim N(\mu_1, \sigma_1^2)$$

$$Y \sim N(\mu_2, \sigma_2^2)$$

$$Z = X + Y$$

$$M_{x+y}(t) = M_x(t) \cdot M_y(t)$$

$$= \exp\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right) \cdot \exp\left(\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right)$$

$$= \exp\left(\frac{(\sigma_1^2 + \sigma_2^2) t^2}{2} + (\mu_1 + \mu_2) t\right)$$

Scanned with CamScanner