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4. Jointly Distributed Random Variables

Def: Let $X_1, X_2, \ldots X_n$ be the random variables for (Ω, S, P), then

$(X_1(\omega), X_2(\omega), X_3(\omega) \ldots X_n(\omega))$ is multiple

Random variables/ Random vector

2 variables

Joint Pmf $\quad p(x,y) = P\{X=x, Y=y)$ $\quad$ (height, weight) (dice, coin)

Joint Pdf $\quad f(x,y) = \int_{y \text{ interval}} \int_{x \text{ interval}} f(x,y)\,dx\,dy$

Joint cdf: ~~for dis~~

$F(x,y) = P(X \le x, Y \le y)$ $\quad$ for discrete

$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\,dx\,dy$ $\quad$ for continuous

Ex:1 Let X = Value on a throw of dice

Y = 0 if tail occurs
1 if head occurs

Sample space = $\{(1,0), (2,0), (3,0), (4,0), (5,0), (6,0)$
$(1,1), (2,1), (3,1), (4,1), (5,1), (6,1)\}$

$|\Omega| = 12$

$$P_{ij} = P(x,y) = P(X=x, Y=y) \quad \begin{matrix} 1 \le i \le 6 \\ 1 \le j \le 1 \end{matrix}$$

$$P(X=6, Y=1) = \frac{1}{12}$$

$$P(X \le 3, Y=0) = \frac{3}{12} = \frac{1}{4}$$

$$P(X=5, Y \le 1) = \frac{2}{12} = \frac{1}{6}$$

Ex: 2

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0<x<\infty \\ & 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

cdf

$$F(x,y) = \int_0^x \int_0^y e^{-(x+y)} \, dx \, dy$$

$$= \int_0^x e^{-x} dx \cdot \int_0^y e^{-y} dy$$

X & Y are independent

$$= (-e^{-x})_0^x \cdot (e^{-y})_0^y$$

$$= \cancel{(-e^{-x})}($$

$$= \left[-(e^{-x}-1)\right]\left[-(e^{-y}-1)\right]$$

$$= (1-e^{-y})(1-e^{-x}) \quad \begin{matrix} 0<x<\infty \\ 0<y<\infty \end{matrix}$$

$$P(X \le 9.25, Y \le 10)$$

$$= \int_0^{10} \int_0^{9.25} e^{-(x+y)} \, dx \, dy$$

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Ex:

X = no. of heads in 3 tossings

Y = diff b/w heads & tails

$Y \backslash X$	0	1	2	3
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$

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Joint Probability Distribution:

Joint PMF $P\{X = x, Y = y\} = P_{ij}$

$$\sum\sum P_{ij} = 1$$

$$P_{ij} \geq 0$$

Joint PDF $f(x,y) = f(x,y)\,dx\,dy$

$$f(xy) \geq 0$$

$$\iint f(x,y)\,dx\,dy = 1$$

Ex:

X - no. of heads in 3 tossings

Y - absolute diff b/w no. of heads & tails

$X = 0, 1, 2, 3.$

$Y = 1, 3$

TTT, THT, HTT, TTH, HHT, HTH, THH, HHH

(0,3) (1,1) (1,1) (1,1) (2,1) (2,1) (2,1) (3,3)

Y \ X	0	1	2	3	$P(Y=y)$
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\frac{6}{8}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{2}{8}$
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

$P(Y=y) \rightarrow$ Marginal PMF of Y

Y	P
1	$\frac{6}{8}$
3	$\frac{2}{8}$

$P(X=x)$ ↓ Marginal PMF of X

X	P
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

Collection of all P_i's $\quad P_i = \sum_{j=1}^{\infty} P_{ij} \quad$ PMF of X

P_j's $\quad P_j = \sum_{i=1}^{\infty} P_{ij} \quad$ PMF of Y

Joint conditional Distribution

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P\{X=x_i / Y=y_j\} = \frac{P\{X=x_i, Y=y_j\}}{P\{Y=y_j\}}$$ $\sim$ intersection

↓
Condition pmf of X given $Y=y_j$

$$P\{Y=y_j / X=x_i\} = \frac{P\{X=x_i, Y=y_j\}}{P\{X=x_j\}}$$

↓
Conditional pmf of Y given $X=x_i$

Ex:

$y \backslash x$	0	1	2	3	y
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\frac{6}{8}$
3	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{2}{8}$
x	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

$$P\{X=1/Y=1\} = \frac{P\{X=1, Y=1\}}{P\{Y=1\}}$$

$$= \frac{\frac{3}{8}}{\frac{6}{8}} = \frac{1}{2}$$

$$P\{X=i/Y=1\} = P\begin{cases} 0 & \text{if } x=0,3 \\ \frac{1}{2} & \text{if } x=1,2 \end{cases}$$

$$P\{X=i/Y=3\} = \begin{cases} \frac{1}{2} & \text{if } x=0,3 \\ 0 & \text{if } x=1,2 \end{cases}$$

$$P\{y=j/X=0\} = \begin{cases} 0 & \text{if } y=1 \\ 1 & \text{if } y=3 \end{cases}$$

Ex

$y \backslash x$	0	1	PMF of y
0	$1/3$	$1/3$	$2/3$
1	$1/3$	0	$1/3$
PMF of X	$2/3$	$\frac{1}{3}$	1

$$P(X/Y=0) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{1}{2} & x=1 \end{cases}$$

$$P(Y/X=0) = \begin{cases} \frac{1}{2} & y=0 \\ \frac{1}{2} & y=1 \end{cases}$$

$$P(X/Y=1) = \begin{cases} 1 & x=0 \\ 0 & x=1 \end{cases}$$

$$P(X/Y=0) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{1}{2} & x=1 \end{cases}$$

Marginal pdf & Joint Conditional Pdf

Marginal PDF of X $\quad f(x) = \int f(x,y)\,dy$ ↷ y limits

PDF of Y $\quad f(y) = \int f(x,y)\,dx$

Joint conditional PDF of X given Y $\quad f(x/y) = \frac{f(x,y)}{f(y)}$

PDF of Y given X $\quad f(y/x) = \frac{f(x,y)}{f(x)}$

Ex:

$$f(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

i) Find c.

$$\int_0^1 \int_0^1 (x + cy^2)\,dx\,dy = 1 \quad \text{(Total probability)}$$

$$\int_0^1 \left(\frac{x^2}{2} + cxy^2\right)_0^1 dy = 1$$

$$\int_0^1 \left(\frac{1}{2} + cy^2\right) dy = 1$$

$$\left[\frac{y}{2} + c\frac{y^3}{3}\right]_0^1 = 1$$

$$\frac{1}{2} + \frac{c}{3} = 1$$

$$\frac{c}{3} = \frac{1}{2}$$

$$\boxed{c = 3/2}$$

ii) $p\left(0<x<\frac{1}{2}, 0<y<\frac{1}{2}\right)$

$$= \int_{x=0}^{1/2}\int_{y=0}^{1/2}\left(x+\frac{3}{2}y^2\right)dy\,dx$$

$$= \int_0^{1/2}\left[xy+\frac{3}{2}\cdot\frac{y^3}{3}\right]_0^{1/2} dx$$

$$= \int_0^{1/2}\left[\frac{x}{2}+\frac{1}{16}\right]dx$$

$$= \left[\frac{x^2}{4}+\frac{x}{16}\right]_0^{1/2}$$

$$= \left[\frac{1}{16}+\frac{1}{32}\right]$$

$$= \frac{3}{32}$$

iii) Marginal PDF of x

$$f(x) = \int_0^1\left(x+\frac{3}{2}y^2\right)dy$$

$$= \left[xy+\frac{y^3}{2}\right]_0^1$$

$$f(x) = \begin{cases} x+\frac{1}{2} & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal PDF of y

$$f(y) = \int_0^1\left(x+\frac{3}{2}y^2\right)dx$$

$$f(y) = \begin{cases} \frac{3}{2}y^2+\frac{1}{2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

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Ex:

$$f(x,y) = \begin{cases} \frac{x+y}{3} & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

i) $P(x>y) = \int_{y=0}^{1}\int_{x=y}^{2} \left(\frac{x+y}{3}\right) dx\, dy$ $\quad y \to 0$ to 1, $x \to y$ to 2

$$= \int_0^1 \left[\frac{x^2}{6} + \frac{xy}{3}\right]_y^2 dy$$

$$= \int_0^1 \cancel{\frac{y^2}{6}} \left[\frac{4}{6} + \frac{2y}{3} - \frac{y^2}{6} - \frac{y^2}{3}\right] dy$$

$$= \int_0^1 \left[\frac{2}{3} + \frac{2y}{3} - \frac{y^2}{2}\right] dy$$

$$= \left[\frac{2}{3}y + \frac{2}{3}\cdot\frac{y^2}{2} - \frac{1}{2}\cdot\frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3} + \frac{1}{3} - \frac{1}{6}$$

$$= 1 - \frac{1}{6}$$

$$= \frac{5}{6}$$

(or)

$$P(x>y) = 1 - P(x<y)$$

$$= 1 - \int_{y=0}^{1}\int_{x=0}^{y} \frac{x+y}{3}\, dx\, dy$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

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ii) Marginal PDF of X $= \int_{y=0}^{1} \frac{x+y}{3}\,dy$

$$= \left[\frac{xy}{3} + \frac{1}{3}\cdot\frac{y^2}{2}\right]_0^1$$

$$= \frac{x}{3} + \frac{1}{6}$$

Marginal PDF of Y $= \int_{x=0}^{2} \left(\frac{x+y}{3}\right) dx$

$$= \left[\frac{x^2}{6} + \frac{xy}{3}\right]_0^2$$

$$= \frac{4}{6} + \frac{2y}{3}$$

$$= \frac{2}{3}(1+y)$$

Ex:

$$f(x,y) = \begin{cases} 2x - xy & 0<x<1 \\ & 0<y<2 \\ 0 & \end{cases}$$

$$P\left(X<\frac{1}{2} / Y<1\right) = \frac{P\left(X<\frac{1}{2}, Y<1\right)}{P(Y<1)}$$

$$f(y) = \int_0^1 (2x - xy)\,dx = \left[\frac{2x^2}{2} - y\cdot\frac{x^2}{2}\right]_0^1$$

$$= 1 - \frac{y}{2} \quad 0<y<2$$

$$P(Y<1) = \int_0^1 \left(1 - \frac{y}{2}\right) dy = \left(y - \frac{y^2}{4}\right)_0^1$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

$$\frac{P\left(X<\frac{1}{2}, Y<1\right)}{P(Y<1)} = \frac{\int_{x=0}^{1/2}\int_{y=0}^{1}(2x-xy)\,dy\,dx}{\frac{3}{4}}$$

$$= \frac{\int_{x=0}^{1/2}\left[2xy - x\cdot\frac{y^2}{2}\right]_0^1 dx}{\frac{3}{4}}$$

$$= \frac{4}{3}\int_0^{1/2}\left(2x-\frac{x}{2}\right)dx$$

$$= \frac{4}{3}\cdot\left[x^2-\frac{x^2}{4}\right]_0^{1/2}$$

$$= \frac{4}{3}\cdot\left[\frac{1}{4}-\frac{1}{16}\right]$$

$$= \frac{4}{3}\left[\frac{4-1}{16}\right]$$

$$= \frac{4}{3}\cdot\frac{3}{16}$$

$$P\left(X<\tfrac{1}{2} / Y<1\right) = \frac{1}{4}$$

25/5/2

Joint Independent R.V

The necessary and sufficient condition for any two R.V's to be independent

$P\{X=x, Y=y\} = P\{X=x\} \cdot P\{Y=y\}$ For discrete

Marginal PMF's

$f(x,y) = f(x) \cdot f(y)$ For continous

Marginal PDF's.

When X & Y are independent

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{f(x) \cdot f(y)}{f(y)} = f(x)$$

Similarly $f(y/x) = f(y)$

Ex:

Y \ X	0	1	P(Y)
0	1/4	1/4	$\frac{1}{2}$
1	1/4	1/4	$\frac{1}{2}$
P(X)	$\frac{1}{2}$	$\frac{1}{2}$	1

$P(X=0, Y=0) = P(X=0) \cdot P(Y=0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(X=0, Y=0) = \frac{1}{4} \implies \frac{1}{4} = \frac{1}{4}$

$P(X=i, Y=j) = P(X=i) \cdot P(Y=j)$

$\therefore$ X & Y are independent.

$X \rightarrow$ no. of heads

$Y \rightarrow$ diff b/w no. of heads & tails

Not independent

$\exists$ x & y for which

$$P\{X=x, Y=y\} \neq P\{X=x\} \cdot P\{Y=y\}$$

Ex:

$$f(x,y) = 4xy \qquad 0<x<1, \; 0<y<1$$

$$f(x) = \int_{y=0}^{1} 4xy \, dy = \left[2x y^2\right]_0^1 = 2x$$

$$f(y) = \int_{x=0}^{1} 4xy \, dx = 2y$$

$$f(x,y) = f(x) \cdot f(y)$$

$$= 2x \cdot 2y$$

$$= 4xy$$

$\therefore$ X & Y are independent

Ex:

$$f(x,y) = x+y \qquad 0<x<1, \; 0<y<1$$

Marginal PMF of X

$$f(x) = \int_{y=0}^{1} (x+y) \, dy = \left(xy + \frac{y^2}{2}\right)_0^1 = x + \frac{1}{2} \qquad 0<x<1$$

$$f(y) = \int_0^1 (x+y)\,dx = y + \frac{1}{2} \qquad 0 < y < 1$$

$$(x+y) \neq \left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)$$

$\therefore$ X & Y are not independent.

Mutually / Complete Independence:

Let $X_1, X_2, \dots X_n$ be the random variable we say they are mutually independent if

$$f(X_1, X_2, \dots X_n) = f(X_1) f(X_2) \cdots f(X_n)$$

should be true for all combinations.

& it is pairwise independent if

$$f(X_i, X_j) = f(X_i) f(X_j) \quad \text{for } i,j$$

Moments for Joint Random Variables.

Non Central Moments: $E(X^i Y^j) < \infty$, for i&j is called non central moment of order $i+j$

$$\mu'_{ij} = E(x^i y^j)$$

$$\mu'_{10} = E(x) \qquad \mu'_{20} = E(x^2)$$

$$\mu'_{01} = E(Y) \qquad \mu'_{02} = E(Y^2)$$

$$\mu'_{11} = E(XY)$$

Central Moments:

$E\left((X-\underbrace{E(X)}_{\mu'_{10}})^i (Y-\underbrace{E(Y)}_{\mu'_{01}})^j\right)$ is said to be central moment of order $i+j$

$\mu_{ij} = \left(E\,(X-E(X))^i (Y-E(Y))^j\right)$

$\mu_{10} = 0$, $\mu_{01} = 0$

$\mu_{20} = Var(X)$, $\mu_{02} = Var(Y)$

$\mu_{11} = E\left((X-E(X))(Y-E(Y))\right)$

$\mu_{11} = E\left((X-\mu'_{10})(Y-\mu'_{01})\right)$

$= E\left(XY - X\mu'_{01} - Y\cdot\mu'_{10} + \mu'_{10}\cdot\mu'_{01}\right)$

$= E(XY) - \mu'_{01}\,E(X) - \mu'_{10}\,E(Y) + \mu'_{10}\cdot\mu'_{01}$

$= E(XY) - \mu'_{01}\,\mu'_{10} - \mu'_{10}\cdot\mu'_{01} + \mu'_{10}\cdot\mu'_{01}$

$= E(XY) - \mu'_{01}\cdot\mu'_{10}$

$\mu_{11} = Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$ is called covariance.

Properties:

1) a) if $cov(X,Y) > 0$, X & Y are +vely related (i.e; in same direction)

b) if $cov(X,Y) < 0$, X & Y are -vely related (different direction)

2) $cov(X,Y) = 0$ no relation btw X & Y

3) $-\infty < cov(X,Y) < \infty$

4) Higher the value of covariance stronger relationship
Lower the value → weak relation.

5) $Cov(aX+b, cY+d) = ac\, Cov(X,Y)$

6) $Cov(X,X) = Var(X)$

$\cancel{Cov(X,X)} = \cancel{Cov(X \cdot X) - Cov(X)}$

$Cov(X,X) = E(X \cdot X) - E(X) \cdot E(X)$

$= Var(X)$

7) When X & Y are independent $cov(X,Y) = 0$

converse need not be true

if $cov(X,Y) = 0$ may not be independent.

Correlation coefficient:

We define correlation coefficient

$$\rho_{xy} = \frac{Cov(X,Y)}{S.D(X) \cdot S.D(Y)}$$

$$\frac{E(XY) - E(X) \cdot E(Y)}{\sqrt{E(X^2) - (E(X))^2}\ \sqrt{(E(Y^2) - (E(Y))^2)}} = \frac{Cov(X,Y)}{\sqrt{\sigma_x^2}\ \sqrt{\sigma_y^2}}$$

1) Value of ρ lies b/w -1 & 1

2) if $\rho = 0 \Rightarrow Cov(X,Y) = 0$ no relation

3) $\rho = +ve$ → +ve relation

$= -ve$ → −ve relation

$0.5 < |\rho| < 1$ strong

$0.30 < |\rho| < 0.49$ Medium/Moderate

$0 < |\rho| < 0.29$ small/less.

Ex.

$y \backslash x$	0	1	2	$P(y)$
0	$1/8$	$1/8$	0	$\frac{1}{4}$
1	$1/8$	$2/8$	$1/8$	$\frac{1}{2}$
2	0	$1/8$	$1/8$	$\frac{1}{4}$
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

$$E(x) = \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + 2\cdot\frac{1}{4} = 1$$

$$E(y) = \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + 2\cdot\frac{1}{4} = 1$$

$$E(xy) = \sum\sum xy\,P(x,y)$$

$$= 1\cdot 1\cdot\frac{2}{8} + 1\cdot 2\cdot\frac{1}{8} + 2\cdot 1\cdot\frac{1}{8} + 2\cdot 2\cdot\frac{1}{8}$$

$$= \frac{5}{4}$$

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$

$$= \frac{5}{4} - 1\cdot 1$$

$$= \frac{1}{4} = 0.25$$

$$E(x^2) = 1^2\cdot\frac{1}{2} + 2^2\cdot\frac{1}{4} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$E(y^2) = 1^2\cdot\frac{1}{2} + 2^2\cdot\frac{1}{4} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$Var(x) = E(x^2) - (E(x))^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

$$\rho_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\frac{1}{4}}{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Ex: $f(x,y) = \begin{cases} c(2x+y) & 2<x<6,\ 0<y<5 \\ 0 \end{cases}$

~~$c = 2$~~

$c = \frac{1}{210}$

$f(x) = \frac{1}{210} \int_0^5 (2x+y)\,dy$

$= \frac{1}{210} \left[2xy + \frac{y^2}{2}\right]_0^5$

$= \frac{1}{210} \cdot \cancel{10y} + \left[10x + \frac{25}{2}\right]$

$f(y) = \frac{1}{210} \int_2^6 (2x+y)\,dx$

$= \frac{1}{210} \left(x^2 + xy\right)_2^6$

$= \frac{1}{210} \left[36 + 6y - 4 - 2y\right]$

$= \frac{1}{210} \left[32 + 4y\right]$

$E(X) = \frac{1}{210} \int_2^6 x \cdot \left(10x + \frac{25}{2}\right) dx$

$= \frac{1}{210} \int_2^6 \left[10x^2 + \frac{25}{2}x\right] dx$

$= \frac{1}{210} \left[10 \cdot \frac{x^3}{3} + \frac{25}{2} \cdot \frac{x^2}{2}\right]_2^6$

$= \frac{1}{210} \left[\frac{10 \cdot (6)^3}{3} + \frac{25}{4}(6)^2 - \frac{10}{3}(2)^3 - \frac{25}{4}(2)^2\right]$

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$$= \frac{1}{210}\left[\cancel{948}\ 720 + 225 - \frac{80}{3} - 25\right]$$

$$= \frac{1}{210}\left(\frac{2680}{3}\right)$$

$$E(X) = \frac{268}{63}$$

$$E(Y) = \frac{1}{210}\int_0^5 y\cdot[32 + 4y]\,dy$$

$$= \frac{1}{210}\int_0^5 \left[32y + 4y^2\right] dy$$

$$= \frac{1}{210}\left[32\cdot\frac{y^2}{2} + 4\cdot\frac{y^3}{3}\right]_0^5$$

$$= \frac{1}{210}\left[16y^2 + \frac{4}{3}\cdot y^3\right]_0^5$$

$$= \frac{1}{210}\left[16(5)^2 + \frac{4}{3}(5)^3\right]$$

$$= \frac{1}{210}\left[400 + \frac{500}{3}\right]$$

$$= \frac{1}{21\cancel{0}}\left(\frac{170\cancel{0}}{3}\right)$$

$$E(Y) = \frac{170}{63}$$

$$E(X^2) = \frac{1}{210}\int_2^6 x^2\left(10x + \frac{25}{2}\right)dx$$

$$= \frac{1}{20}\int_2^6 \left(10x^3 + \frac{25}{2}x^2\right)dx$$

$$= \frac{1}{210} \left[10 \cdot \frac{x^4}{4} + \frac{25}{2} \cdot \frac{x^3}{3}\right]_2^6$$

$$= \frac{1}{210} \left[\frac{10}{4} \cdot (6)^4 + \frac{25}{6} \cdot (6)^3 - \frac{10}{4} \cdot (2)^4 - \frac{25}{6} \cdot (2)^3\right]$$

$$= \frac{1}{210} \left[3240 + 900 - 40 - \frac{100}{3}\right]$$

$$= \frac{1}{210} \left[\frac{12200}{3}\right]$$

$$E(x^2) = \frac{1220}{63}$$

$$E(y^2) = \frac{1}{210} \int_0^5 y^2 \cdot [32 + 4y] \, dy$$

$$= \frac{1}{210} \int_0^5 [32y^2 + 4y^3] \, dy$$

$$= \frac{1}{210} \left[32 \cdot \frac{y^3}{3} + 4 \cdot \frac{y^4}{4}\right]_0^5$$

$$= \frac{1}{210} \left[\frac{32}{3} (5)^3 + (5)^4\right]$$

$$= \frac{1}{210} \left(\frac{5875}{3}\right)$$

$$E(y^2) = \frac{1175}{126}$$

$$E(xy) = \frac{1}{210} \int_{x=2}^{6} \int_{y=0}^{5} xy (2x + y) \, dy \, dx$$

$$= \frac{1}{210} \int_2^6 \int_0^5 \cancel{xy}\left(2x^2y + xy^2\right) dy\, dx$$

$$= \frac{1}{210} \int_2^6 \left[\frac{2x^2y^2}{2} + x \cdot \frac{y^3}{3}\right]_0^5 dx$$

$$= \frac{1}{210} \int_2^6 \left(25x^2 + \frac{125}{3}x\right) dx$$

$$= \frac{1}{210} \left[25 \cdot \frac{x^3}{3} + \frac{125}{3} \cdot \frac{x^2}{2}\right]_2^6$$

$$= \frac{1}{210} \left[\frac{25}{3}(6)^3 + \frac{125}{6}(6)^2 - \frac{25}{3}(2)^3 - \frac{125}{6}(2)^2\right]$$

$$= \frac{1}{210} \left[1800 + 750 - \frac{200}{3} - \frac{250}{3}\right]$$

$$= \frac{1}{210} \left[2550 - 150\right]$$

$$= \frac{240\cancel{0}}{21\cancel{0}}$$

$$E(XY) = \frac{80}{7}$$

$$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$= \frac{80}{7} - \frac{268}{63} \cdot \frac{170}{63}$$

$$= \frac{45360 - 45560}{63 \times 63}$$

$$= -\frac{200}{3969} = -0.0503$$

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$$\rho_{xy} = \frac{Cov(X,Y)}{\sqrt{\sigma_x^2}\,\sqrt{\sigma_y^2}}$$

$$= \frac{\frac{-200}{3969}}{\sqrt{\frac{5036}{(63)^2}}\cdot\sqrt{\frac{16225}{2\times(63)^2}}}$$

$$\rho_{xy} = -0.03129$$

$$Var(X) = E(X^2) - (E(X))^2$$
$$= \frac{1220}{63} - \left(\frac{268}{63}\right)^2$$
$$= \frac{5036}{(63)^2}$$

$$Var(Y) = E(Y^2) - (E(Y))^2$$
$$= \frac{1175}{126} - \left(\frac{170}{63}\right)^2$$
$$= \frac{74025 - 57800}{2\times(63)^2}$$
$$= \frac{16225}{2\times(63)^2}$$

Ex

Y \ X	-2	-1	0	1	2	P(Y)
0	0	0	1/5	0	0	1/5
1	0	1/5	0	1/5	0	2/5
4	1/5	0	0	0	1/5	2/5
P(X)	1/5	1/5	1/5	1/5	1/5	1

$$E(X) = (-2)\frac{1}{5} + (-1)\frac{1}{5} + 1\left(\frac{1}{5}\right) + 2\left(\frac{1}{5}\right) = 0$$

$$E(Y) = 0\cdot\frac{1}{5} + 1\cdot\frac{2}{5} + 4\cdot\frac{2}{5} = 2$$

$$E(XY) = 1(-1)\cdot\frac{1}{5} + (1)(1)\frac{1}{5} + 4(-2)\cdot\frac{1}{5} + (2)(4)\cdot\frac{1}{5}$$

$$= 0$$

$$Cov(X,Y) = E(XY) - E(X)\cdot E(Y) = 0$$

~~$P(X=-2/Y=0) = P(X=$~~

$P(X=-2, Y=0) = 0$

$P(X=-2) = \frac{\cancel{1}}{5}$ $P(Y=0) = \frac{1}{5}$

$P(X=-2, Y=0) \neq P(X=-2) \cdot P(Y=0)$

$\therefore$ X & Y are dependent even if $cov(X,Y)=0$.

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Bivariate & Multinomial Distribution:

Normal Distribution pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad X \sim N(\mu, \sigma^2)$$

For n R.V's

$$X_{n\times1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \qquad \mu_{n\times1} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

$$\Sigma_{n\times n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

σ_i SD of X_i

σ_{ij} $cov(X_i, X_j)$

Inverse of Σ should exist.

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pdf of multinormal distribution

$$f(x_1, x_2 \dots x_n) = \frac{1}{(\sqrt{2\pi})^n \sqrt{|\Sigma|}} e^{-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

we say $X \sim N(\mu, \Sigma)$

where μ is mean vector &

Σ is the matrix of covariances.

→ Marginals follow normal distribution

↓

$f(x_1), f(x_2), \dots, f(x_n)$

$$f(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \qquad N(\mu_1, \sigma_1^2)$$

Bivariate:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\sigma_{11} = \sigma_1^2 = Var(X_1)$$

$$\sigma_{22} = \sigma_2^2 = Var(X_2)$$

Correlation coefficient $\rho = \dfrac{Cov(X, Y)}{\sigma_x \sigma_y}$

$$\rho_{x_1 x_2} = \frac{Cov(X_1, X_2)}{\sigma_1 \cdot \sigma_2} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2}$$

$$\boxed{\sigma_{12} = \rho \cdot \sigma_1 \sigma_2}$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \qquad |\Sigma| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2$$

$$= \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$f(x_1, x_2) = \frac{1}{(\sqrt{2\pi})^2\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} e^{-\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \Sigma^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}}$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

$$\boxed{f(x_1, x_2) = \frac{1}{2\pi\cdot\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & -\rho/\sigma_1\sigma_2 \\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}}}$$

$$\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix}_{1\times 2} \begin{bmatrix} \frac{1}{\sigma_1^2} & -\rho/\sigma_1\sigma_2 \\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(x-\mu_1)}{\sigma_1^2} - \frac{\rho(x_2-\mu_2)}{\sigma_1\sigma_2} & \frac{-\rho(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)}{\sigma_2^2} \end{bmatrix}_{1\times 2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}_{2\times 1}$$

$$= \frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{\rho(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2} - \frac{\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}$$

$$= \frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_2-\mu_2)(x_1-\mu_1)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}$$

pdf of bivariate

$$\boxed{f(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}}$$

Marginal PDF

$$f(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \qquad x_1 \sim N(\mu_1, \sigma_1^2)$$

$$f(x_2) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} \qquad x_2 \sim N(\mu_2, \sigma_2^2)$$

<u>Note:</u>

$$(X_1+X_2) \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)$$

$$E(X_1+X_2) = E(X_1) + E(X_2)$$

$$Var(X_1+X_2) = Var(X_1) + Var(X_2) + 2\,Cov(X_1,X_2)$$

Conditional Dist's

$$f(X/Y=y) = \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left(x-\left(\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^2}$$

$$(X/Y=y) \sim N\left(\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right),\ \sigma_1^2(1-\rho^2)\right)$$

$$f(Y/X=x) = \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left(y-\left(\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)\right)\right)^2}$$

$$(Y/X=x) \sim N\left(\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right),\ \sigma_2^2(1-\rho^2)\right)$$

$$E(X/Y=y) = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(y-\mu_2)$$

$$E(Y/X=x) = \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x-\mu_1)$$

$$Var(X/Y=y) = (1-\rho^2)\sigma_1^2$$

$$Var(Y/X=x) = (1-\rho^2)\sigma_2^2$$

Ex:1 $\mu_x = 1,\ \sigma_x^2 = 1,\ \mu_y = 0,\ \sigma_y^2 = 4,\ \rho = \frac{1}{2}$

a) $P(2X+Y \leq 3)$

Let $2X+Y = V$

$E(V) = E(2X+Y) = 2E(X)+E(Y) = 2(1)+0 = 2$

$Var(V) = Var(2X+Y) = 4\,Var(X) + Var(Y) + 2\,Cov(2X, Y)$

$Cov(aX, bY) = ab\,Cov(X, Y)$

$$\mathrm{cov}(2X, Y) = 2\,\mathrm{cov}(X,Y)$$

$$= \cancel{2\left(E(XY) - E(X)\cdot E(Y)\right)}$$

$$\rho = \frac{\mathrm{cov}(X,Y)}{\sigma_x \cdot \sigma_y}$$

$$\frac{1}{2} = \frac{\mathrm{cov}(X,Y)}{1\cdot 2} \implies \mathrm{cov}(X,Y) = 1$$

$$\mathrm{Var}(V) = 4\cdot 1 + 4 + 2\cdot 2(1)$$

$$\mathrm{Var}(V) = 12$$

$$V \sim N(2, 12)$$

$$P(V \leq 3) = P\left(\frac{V-2}{\sqrt{12}} \leq \frac{3-2}{\sqrt{12}}\right)$$

$$= P\left(Z \leq \frac{1}{\sqrt{12}}\right)$$

$$= \phi\left(\frac{1}{\sqrt{12}}\right)$$

$$= \phi(0.29)$$

$$P(2X+Y \leq 3) = 0.6141$$

b) $P(Y>1 / X=2)$

$$E(Y/X=2) = \mu_y + \frac{\rho\,\sigma_y}{\sigma_x}\cdot(x - \mu_x)$$

$$= 0 + \frac{1}{2}\cdot\frac{2}{1}\cdot(2-1)$$

$$= 1$$

$$Var(Y/X=2) = \left(1-\left(\tfrac{1}{2}\right)^2\right)4$$

$$= \frac{3}{4} \cdot 4$$

$$= 3$$

$$P(Y>1/X=2) = 1 - P(Y<1/X=2)$$

$$= 1 - P\left(Z < \frac{1-1}{\sqrt{3}}\right)$$

$$= 1 - \phi(0)$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

Ex. 2

	Exam 1	Exam 2
Math	X	Y

$\mu_1 = 70$ $\mu_2 = 60$

$\sigma_1 = 10$ $\sigma_2 = 15$

$\rho = 0.6$

a) $P(Y > 75) \neq \cancel{P\left(\frac{Y-60}{15}\right.}$

$$= 1 - P(Y < 75)$$

$$= 1 - P\left(\frac{Y-60}{15} < \frac{75-60}{15}\right)$$

$$= 1 - \phi(1) = 1 - 0.8413$$

$$= 0.1586$$

b) $P\left(Y>75 / X=80\right)$

$$E\left(Y/X=80\right) = \mu_2 + \rho \cdot \frac{\sigma_2}{\sigma_1} (x-\mu_1)$$

$$= 60 + 0.6 \cdot \frac{15}{10} \cdot (80-70)$$

$$= 60 + 9$$

$$= 69$$

$$V(Y/X=x) = (1-\rho^2)\,\sigma_2^2$$

$$= (1-(0.6)^2)(15)^2$$

$$= (1-0.36)(225)$$

$$= 144$$

$$P\left(Y>75 / X=80\right) = 1-\left(P\left(Z<\frac{75\tfrac{1}{2}-69}{12}\right)\right)$$

$$= 1 - \phi(0.5)$$

$$= 1 - 0.6915$$

$$= 0.3085$$

~~Ex:3~~ c) $P(X+Y>150)$

$$E(X+Y) = E(X) + E(Y) = 70+60 = 130$$

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$

$$= 100 + 225 + 2(90)$$

$$= 505$$

$$\rho = \frac{Cov(X,Y)}{\sigma_1 \cdot \sigma_2} \Rightarrow Cov(X,Y) = (0.6) \cdot 10 \cdot 15 = 90$$

$$P(X+Y>150) = 1 - P\left(Z < \frac{150-130}{\sqrt{505}}\right)$$

$$= 1 - P(Z < 0.89)$$

$$= 1 - \phi(0.89)$$

$$= 1 - 0.8133$$

$$= 0.1867$$

d) $P((5X-4Y) > 150)$

$$E(5X-4Y) = 5E(X) - 4E(Y)$$

$$= 5(70) - 4(60)$$

$$= 350 - 240$$

$$= 110$$

$$Var(5X-4Y) = 25\,Var(X) + 16\,Var(Y) + 2\,Cov(5X, -4Y)$$

$$= 25\,Var(X) + 16\,Var(Y) - 40\,Cov(X,Y)$$

$$= 25(100) + 16(225) - 40(90)$$

$$= 2500$$

$$P(5X-4Y > 150) = 1 - P\left(Z < \frac{150-110}{\sqrt{2500}}\right)$$

$$= 1 - P\left(Z < \frac{40}{50}\right)$$

$$= 1 - \phi(0.80)$$

$$= 1 - 7881$$

$$= 0.2119$$

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Example on Multivariate Normal Distribution

A candy company makes 3 size candy bars small X_1, Medium X_2, Large X_3

Assume the weights of candy bars (X_1, X_2, X_3) follow multivariate normal distribution with the parameters

$$\mu = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{bmatrix} \qquad \begin{matrix} \sigma_1^2 = 4 \\ \sigma_2^2 = 4 \\ \sigma_3^2 = 9 \end{matrix}$$

$\mu_1 = 5$

$\mu_2 = 3$

$\mu_3 = 7$

a) What is P(weight of small size candy > 8)

$$P(X_1 > 8) = 1 - P(X_1 < 8)$$
$$= 1 - P\left(Z_1 < \frac{8-5}{2}\right)$$
$$= 1 - P(Z_1 < 1.5)$$
$$= 1 - \phi(1.50)$$
$$= 1 - 0.9332$$
$$= \cancel{0.9332}\ 0.0668$$

b) $P(4X_1 - 3X_2 + 5X_3 < 63)$

$$Y = 4X_1 - 3X_2 + 5X_3$$

$$E(Y) = 4E(X_1) - 3E(X_2) + 5E(X_3)$$
$$= 4(5) - 3(3) + 5(7) = 20 - 9 + 35$$
$$= 55 - 9$$
$$= 46$$

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$$Var(Y) = Var(4X_1 - 3X_2 + 5X_3)$$

★★★

$$\mu \xrightarrow{} E(aX_1+bX_2+cX_3)$$

$$\mu = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

$$\sigma^2 \xrightarrow{} Var(aX_1+bX_2+cX_3)$$

$$\sigma^2 = \begin{bmatrix} a & b & c \end{bmatrix} \Sigma \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$Var\ Y = \begin{bmatrix} 4 & -3 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & -6 & 39 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$$

$$= 289$$

$$P(4X_1 - 3X_2 + 5X_3 < 63) = P(Y < 63)$$

$$= P\left(\frac{Y-46}{\sqrt{289}} < \frac{63-46}{\sqrt{289}}\right)$$

$$= P(Z < 1)$$

$$= \phi(1)$$

$$= 0.8413$$