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3. Random Variables

→ Quantifying a phenomena

Tossing 3 coins

$\Omega = \{ \{TTT\}, \{TTH\}, \{THT\}, \{HTT\}, \{THH\}, \{HTH\}, \{HHT\}, \{HHH\} \}$

No. of heads can be $X = 0, 1, 2$ or 3.

$X(\{TTT\}) = 0$

$X(\{TTH\}, \{THT\}, \{HTT\}) = 1$

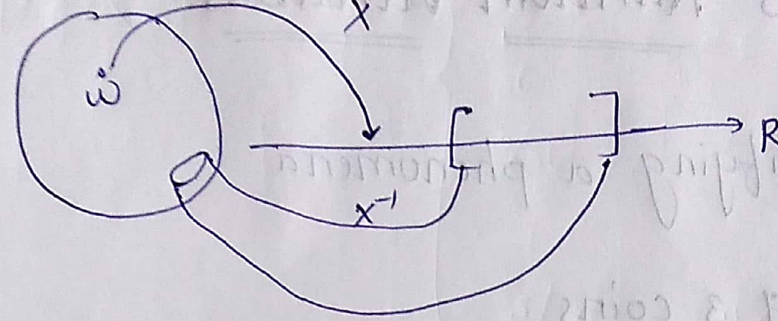
$X(\{THH\}, \{HTH\}, \{HHT\}) = 2$

$X(\{HHH\}) = 3$

→ Random Variable associates a real number to each outcome.

Defn: Let (Ω, S) be a sample space. A function maps Ω into $\mathbb{R}$ is called a Random Variable if the inverse images under X of all Borel sets in $\mathbb{R}$ are events i.e; $\forall$

$X^{-1}(B) = \{ \omega : X(\omega) \in B \} \in S \quad \forall \; B \in \mathcal{B}$



$X^{-1}(x<0) = X^{-1}(-\infty, 0) = \phi$ $\quad X = 0, 1, 2, 3$

$X^{-1}(0 \le x < 1) = \{TTT\}$

$X^{-1}(1 \le x < 2) = \{HTT, THT, TTH\}$

$X^{-1}(2 \le x < 3) = \{HHT, HTH, THH\}$

$X^{-1}(2 < x \le 3) = \{HHH\}$

Note:

$X^{-1}(a \le x \le b) = \{$events i.e. $X(\omega) \in [a,b]\}$

If X is a R.V $\{X = x\}$, $\{a < x \le b\}$, $\{X < x\}$, $\{a \le X < b\}$, $\{a \le X \le b\}$, $\{a \le x \le b\}$ are all events

Ex: Let $\Omega = \{H, T\}$ $\quad X(H) = 1, \; X(T) = 0$

$$X^{-1}(-\infty, x] = \begin{cases} \phi & \text{if } x < 0 \\ \{T\} & \text{if } 0 \le x < 1 \\ \{H, T\} = \Omega & \text{if } 1 \le x \end{cases}$$

Ex: 3 coins are tossed $X(\omega) =$ no. of heads

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Total = 3

Ex: $\Omega = \{HH, TT, HT, TH\}$

$X(\omega) =$ no of heads

$X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$

$$X^{-1}(-\infty, x] = \begin{cases} \phi & x < 0 \\ \{TT\} & 0 \le x < 1 \\ \{HT, TH, TT\} & 1 \le x < 2 \\ \Omega & x \ge 2 \end{cases} \qquad P(X^{-1}) \begin{cases} 0 \\ \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{cases}$$

Def^n The Random Variable X defined on probability space (Ω, S, P) ~~includes~~ induces a probability space called Probability distribution function

$$P(X^{-1}(B)) = P\{\omega : X(\omega) \in B\} \quad \forall B \in \mathcal{B}$$

$\downarrow$ events

Ex: Throwing 2 dice $\Omega = \{(i,j) ; 1 \le i, j \le 6\}$

$X \to$ sum of the two numbers on the dice

Possible values of X: 2, 3, 4, 5, . . . 12

$$X^{-1}(-\infty, x] = \begin{cases} \phi & x < 2 \\ \{(1,1)\} & 2 \le x < 3 \\ \{(1,2),(2,1),(1,1)\} & 3 \le x < 4 \\ \vdots & \vdots \\ \Omega & 12 \le x \end{cases}$$

$$P(X^{-1}(-\infty, x]) = \begin{cases} 0 & x < 2 \\ \frac{1}{36} & 2 \le x < 3 \\ 3/36 & 3 \le x < 4 \\ \vdots & \vdots \\ 35/36 & 11 \le x < 12 \\ 1 & 12 \le x \end{cases}$$

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Cumulative distribution Function:

For any random variable X, cdf is the distribution function defined by

$$F(x) = P(X \le x), \; x \in R$$

Properties:

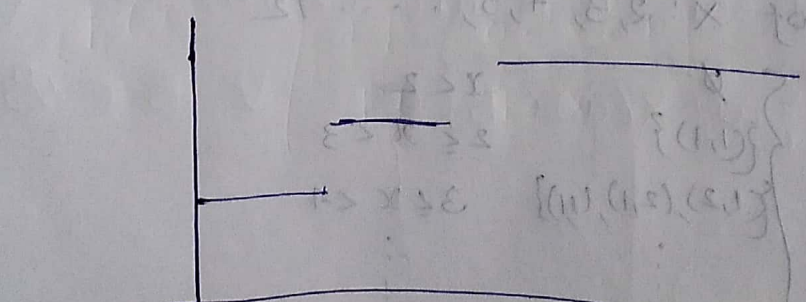
i) $F(x)$ is always non decreasing

$(F(x) \le F(y)$ if $x \le y)$

ii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

iii) $F(x)$ is ~~contir~~ only right continuous.

i.e., $\lim_{x \to c^+} F(x) = F(c)$



Discrete Random Variable: (X takes finite no. of values or countable no. of values)

For discrete R.V the c.d.f is given by

$$F(x) = P(X \le x) = \sum_{u \le x} f(u=x)$$

Ex:

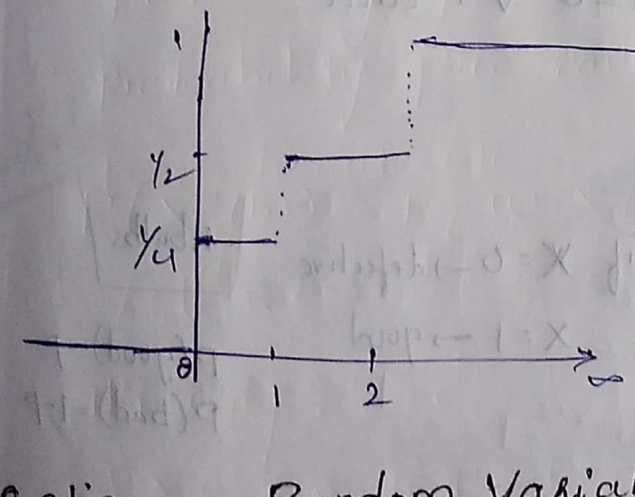
$X = 0, 1, 2, 3, 4$

$$P(X \le 2) = P(X=1) + P(X=2) + P(X=0)$$

Ex: Tossing 2 coins. X = no. of heads

X	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$F(X) = P\left(X^{-1}(-\infty, x)\right) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{3}{4} & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$



Continuous Random Variable: For continous R.V cdf is defined as

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du \quad \text{for } x \in R$$

& interval probability $P(a \le x \le b) = \int_a^b$

Ex: $f(x) = \frac{x}{6}$ interval is $[2,4]$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_2^4 \frac{x}{6}\,dx = \frac{1}{6}\left[\frac{x^2}{2}\right]_2^4$$
$$= 1$$

cdf$(X \le 2.5)$

$$= F(X \le 2.5) = \int_2^{2.5} \frac{x}{6}\,dx = \frac{1}{6}\left[\frac{x^2}{2}\right]_2^{2.5}$$

$$= \frac{6.25 - 4}{12} = \frac{2.25}{12}$$

$$= 0.1875$$

~~Probability Density Funct~~

Probability Mass Function: (Discrete R.V)

Def: Collection of all numbers $\{P_i\}$ is called PMF

if i) $P\{X = x_i\} = P_i \ge 0 \ \forall i$ and

ii) $\sum_{i=1}^{\infty} P_i = 1$

$$P(X=x) = \begin{cases} 1-p & \text{if } X = 0 \to \text{defective} \\ p & X = 1 \to \text{good} \end{cases}$$

n bulbs

$p(\text{good}) = p$

$P(\text{bad}) = 1-p$

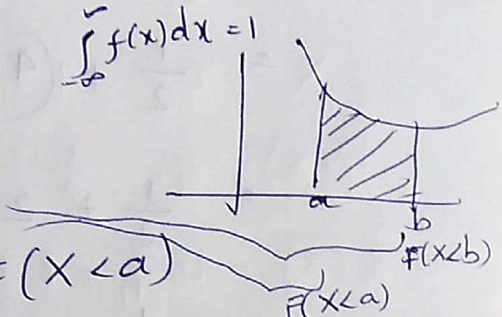
$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Probability Density Function:

For a continous random variable X, if there exists non negative function $f(x)$ such that $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is called pdf of X

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$P(a < x < b) = \int_a^b f(t)\,dt$$



$$= F(x<b) - F(X<a)$$

Ex:

cdf $F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & 1 < x \end{cases}$ $\qquad F'(x) = f(x)$

pdf $f(x) = \begin{cases} 0 & x \le 0 \ \& \ x \ge 1 \\ 1 & 0 < x < 1 \end{cases}$

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$$\underset{\text{cdf}}{F(x)} = \int \underset{\text{pdf}}{f(x)}\,dx$$

discrete → pmf
continous → pdf

$$F'(x) = f(x)$$

pdf $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + \left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2 - \frac{3}{2}$$

$$= 1$$

$\therefore f(x)$ is a pdf

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x t\,dt = \frac{x^2}{2} & 0 \le x < 1 \\ \int_0^1 t\,dt + \int_1^x (2-x)\,dx & 1 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

cdf $$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$P\{0.3 < x < 1.5\}$

$= F(1.5) - F(0.3)$

$= 2(1.5) - \frac{(1.5)^2}{2} - 1 - \frac{(0.3)^2}{2}$

$= 3 - \frac{2.25}{2} - 1 - \frac{0.09}{2}$

$= 2 - \frac{2.34}{2}$

$= 2 - 1.17$

$= 0.83$

Expectation of a Random Variable:

Denoted by $E(x)$ defined as $\mu = \sum_{K=1}^{\infty} x_i P(X=x_i)$ for discrete case

x_i ↓ value of R.V

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{for continuous case.}$$

Ex: Two players A & B

$A \xrightarrow{H/1} B$, $B \xrightarrow{T/1} A$

~~X(H) =~~

Find expected gain of A.

$X(H) = -1, \quad X(T) = 1$

$P(H) = p \quad P(T) = 1-p$

$E(x) = (-1)p + 1(1-p) = 1-2p$

If $p = \frac{1}{2}$ $E(x) = 0$ (fair game)

Thm If X is a random variable & $Y = g(x)$

$$E(Y) = \sum_{j=1}^{\infty} g(x) \cdot P(X = x_j)$$

Ex: Tossing 2 coins X = no. of heads

X	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(x) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$Y = X^2$ $$E(X^2) = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = \frac{3}{2}$$

$$\boxed{E(Y = aX + b) = aE(X) + b}$$

Continous case:

$$f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

$\int x f(x) dx =$ $$E(x) = \int_1^{\infty} x \cdot \frac{2}{x^3} dx = \int_1^{\infty} \frac{2}{x^2} dx$$

$$= \left(-\frac{2}{x}\right)_1^{\infty} = 0 - (-2) = 2$$

$$E(X^2) = \int_1^{\infty} x^2 \cdot \frac{2}{x^3} dx = \int_1^{\infty} \frac{2}{x} dx$$ = undefined does not exists.

<u>Variance</u>: For any random variable X,

$E(X-\mu)^2$ is called variance.

$\sigma^2 = Var(X) = E((X-\mu)^2)$

σ is called standard deviation

$$E((X-\mu)^2) = E(X^2+\mu^2-2X\mu)$$
$$= E(X^2) + E(\mu^2) - 2\mu E(X)$$
$$= E(X^2)+\mu^2 - 2\mu E(X)$$
$$= E(X^2)+\mu^2-2\mu^2$$

$$\boxed{E((X-\mu)^2) = E(X^2) - (E(X))^2}$$

Ex: Calculate $Var(X)$, when X is representing the outcome when a fair die is rolled

X	1	2	3	4	5	6
$P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}$$
$$= 2+\frac{9}{6} = 2+\frac{3}{2} = 7/2$$

~~Var (X)~~

$$E(X^2) = 1^2\cdot\frac{1}{6}+2^2\cdot\frac{1}{6}+3^2\cdot\frac{1}{6}+4^2\cdot\frac{1}{6}+5^2\cdot\frac{1}{6}+6^2\cdot\frac{1}{6}$$
$$= \frac{1}{6}+\frac{4}{6}+\frac{9}{6}+\frac{16}{6}+\frac{25}{6}+\frac{36}{6}$$
$$= \frac{91}{6}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - (7/2)^2$$
$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \qquad \sigma = \sqrt{\frac{35}{12}}$$

$\operatorname{Var}(aX+b) = a^2 \operatorname{Var}(X) + 0$

$= a^2 \operatorname{Var}(X)$

$\operatorname{Var}(X) = E(X-\mu)^2 = E(X^2) - (E(X))^2$

$E(c) = c$, $\operatorname{Var}(c) = 0$ for any $c \in \mathbb{R}$

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Moments:

For any positive integer k (order), & any constant c

$E((X-c)^k)$, is called moment of order k about point c.

* if $c=0$ i.e., $E(X^k)$ is called k^{th} moment about origin $m_i = E(X^i)$ or μ_i'

* if $c=\mu$ $E((X-\mu)^k)$ is called k^{th} moment about mean [$\mu_k = E(X^k)$ is called central moments]

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About origin $(c=0)$ About Mean $(c=\mu)$

$\mu_1' = E(X) = \mu = \text{Mean}$ $\mu_1 = E(X-\mu) = E(X) - E(\mu) = 0$

$\mu_2' = E(X^2)$ $\mu_2 = E((X-\mu)^2) = V(X) = \sigma^2$

$\mu_3' = E(X^3)$ $\mu_3 = E((X-\mu)^3)$

$\mu_4' = E(X^4)$ $\mu_4 = E((X-\mu)^4)$

Ex: $P(X=k) = 1$ & 0 elsewhere Degenerate distribution $\downarrow$ At only 1 point probability $=1$

cdf $F(x) = \begin{cases} 0 & x < k \\ 1 & x \geq k \end{cases}$

$\cancel{E(X^l)}$ $E(X) = \sum x P(X=x_i) = k \cdot 1 = k$

$\mu_2' = E(X^2) = k^2 \cdot 1 = k^2$

$\mu_l' = E(X^l) = k^l \cdot 1 = k^l$

$\mu_1 = 0$ $\mu_2 = E((X-\mu)^2)$

$= E(X^2) - (E(X))^2$

$= k^2 - k^2$

$= 0$

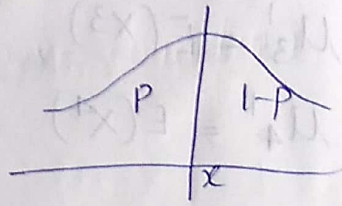
For degenerate distribution $\Rightarrow Var(X) = 0$

Quantiles:

A number x satisfying

$P\{X \le x\} \ge p, \quad P\{X \ge x\} \ge 1-p$, p^{th} quantile

if $p = \frac{1}{2}$ x is called median



For $F(x)$ $\quad p \le F(x) \le p + \{X = x\}$

F is cdf

$P\{X \ge x\} \ge 1-p$

$1 - P\{X \cancel{\le} x\} \ge 1-p$

$-P\{X < x\} \ge -p$

$P\{X < x\} \le p$

$P\{X < x\} + P\{X = x\} \le p + P\{X = x\}$

For $F(x)$ $\boxed{p \le F(x) \le p + \{X = x\}}$ For discrete case

F is cdf

$p \le F(x) \le p + P\{X = x\}$

For continous case $\quad P\{X = x\} = 0$

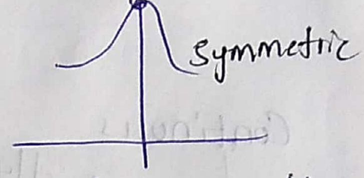
$p \le F(x) \le p$

$F(x) = p$

Median: Let X be a R.V with distribution function $F(x)$. A number x, satisfying

$\frac{1}{2} \le F(x) \le \frac{1}{2} + P\{X = x\}$ For discrete

$F(x) = \frac{1}{2}$ For continuous



Centre is median for symmetric

Ex ① $P\{X = -2\} = P\{X = 0\} = \frac{1}{4}$

$P\{X = 1\} = \frac{1}{3}$, $P\{X = 2\} = \frac{1}{6}$

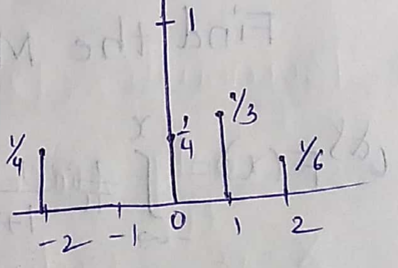
$\underline{x = 0}$:

$P\{X \le 0\} = F(0) = \frac{1}{2}$

$P\{X \ge 0\} = \frac{1}{4} + \frac{1}{3} + \frac{1}{6} = \frac{3}{4} > \frac{1}{2}$

$p = \frac{1}{2}$ $\quad P\{X \le 0\} \ge \frac{1}{2} = p$

$P\{X \ge 0\} = \frac{3}{4} \ge 1 - p$



For any point $0 < x < 1$

$P\{X \le x\} \ge \frac{1}{2}$

$P\{X \ge x\} \ge \frac{1}{2}$

Any $x \in (0,1)$ is a median

$F(-2) = \frac{1}{4} = 0.25$

$F(0) = \frac{1}{2} = 0.5$

$F(1) = \frac{5}{6}$

$F(2) = 1$

if $p = 0.2$ $\quad P\{X \le x\} \ge 0.2$ $\quad P\{X \ge x\} \ge 1 - 0.2 = 0.8$

$P\{X \le -2\} \ge 0.2$ $\quad P\{X \ge -2\} \ge 0.8$

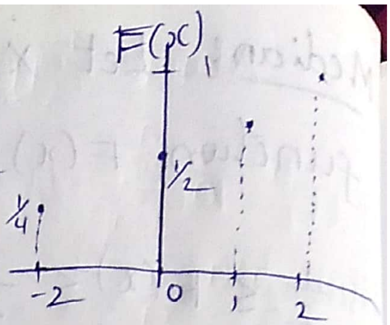
~~Trial~~ Error

$$P \le F(x) \le p + P\{X = x\}$$

Let $x = -2$

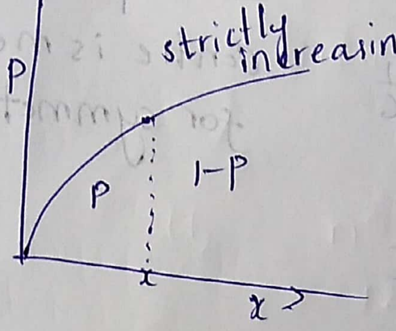
$$0.2 \le F(x) \le 0.2 + 0.25$$

$$0.2 \le F(x) \le 0.45$$

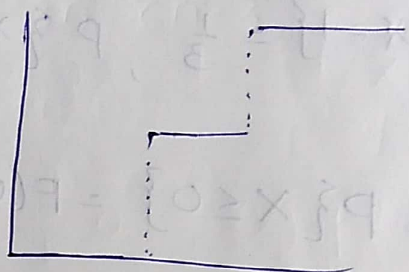


$F(-2) = 0.25$ satisfies the above condition.

Continous



Discrete



For any given p, x is unique

For any given p, there are infinite quantiles.

Ex (2)

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad -\infty < x < \infty$$

Find the Median

cdf $F(x) = \int_{-\infty}^{x} \frac{1}{\pi}\frac{1}{1+x^2}dx = \frac{1}{\pi}\left(\tan^{-1}x\right)_{-\infty}^{x}$

$$= \frac{1}{\pi}\left(\tan^{-1}x + \frac{\pi}{2}\right)$$

$$= \frac{1}{\pi}\tan^{-1}x + \frac{1}{2}$$

For median case $p = \frac{1}{2}$

$x = 0$ is median.

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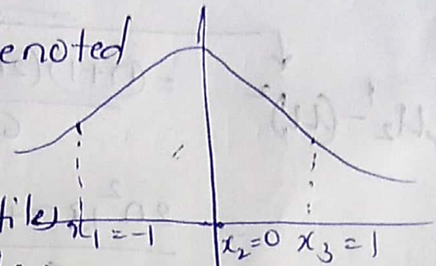
Quartiles: When we divide into 4 parts they arecalled quartiles denoted by Q



For 4 $\quad Q_{\frac{1}{4}} \quad Q_{\frac{1}{2}} \quad Q_{\frac{3}{4}}$ Quartiles

For 10 $\quad Q_{1/10}, Q_{2/10}, \ldots Q_{9/10}$ deciles

For 100 $\quad Q_{1/100}, Q_{2/100}, \ldots Q_{99/100}$ Percentiles

$$F(x) = \frac{1}{\pi}\left[\tan^{-1}x + \frac{\pi}{2}\right]$$

$F(x) = 1/4$ lower quartile

$F(x) = 3/4$ upper quartile

$$F(x) = \frac{1}{4} \Rightarrow x = -1$$

$$F(x) = \frac{3}{4} \Rightarrow x = 1$$

Discrete distribution:

1. **Degenerate distribution:** $P\{X=k\} = 1$,
   $p = 0$ for any other value.

2. **Uniform Discrete Distribution:** The R.V X is said to have uniform distribution on n points $\{x_1, x_2, \ldots x_n\}$ if its PMF is $P\{X = x_i\} = \frac{1}{n}$, $i = 1, 2, \ldots n$

Ex: Tossing coin, $\{H, T\}$ $\quad P(H) = P(T) = \frac{1}{2}$

Throwing dice $\{1,2,3,4,5,6\}$ $\quad P(1) = P(2) \ldots = P(6) = \frac{1}{6}$

$$\mu_1' = E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$$

$$= \sum_{i=1}^{n} x_i \cdot \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$

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$\mu_1 = 0 \quad \mu_2^* = E(X-\mu)^2 \quad\quad \mu_2' = E(X^2)$

$= E(X^2) - (E(X))^2 \quad\quad = \sum_{i=1}^{n} \frac{1}{n} \cdot x_i^2$

$\mu_2' - (\mu_1')^2 \quad = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 \quad\quad = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$

$= \frac{(n+1)(2n+1)}{6}$

$= \frac{2n^2+3n+1}{6} - \frac{(n^2+2n+1)}{4}$

$= \frac{4n^2+6n+2-3n^2-6n-3}{12}$

$\mu_2 = \frac{n^2-1}{12}$

Note: Using central moments we can find non central moments and viceversa.

→ Moment generating function generates non central moments.

3. **Bernoulli's Distribution:**

only success or failure.

$X = 0 \Rightarrow$ failure, $X = 1 \Rightarrow$ Success.

Ex: Tossing coin $\quad P(X = 1 (H)) = \frac{1}{2}$

$P(X = 0 (H)) = \frac{1}{2}$

probability of success is p

failure is $1-p$ or q.

PMF $\quad P\{X=1\} = P \quad P\{X=0\} = 1-P = q$

$\mu_1' = E(X) = \sum x_i P(x) = 1 \cdot P + 0 \cdot (1-P) = P$

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$\mu_1 = 0$

$\mu_2' = E(X^2) = \Sigma x\, p(x)$

$= 1^2 \cdot p + 0^2 (1-p) = p$

$E(X^k) = \mu_k' = p$

$\mu_2 = E(X^2) - (E(X))^2$

$= p - p^2 = p(1-p) = pq$

Binomial Distribution: We say X has Binomial distribution if PMF $\boxed{P(X=k) = {}^nC_k \cdot p^k \cdot (1-p)^{n-k}}$ $k = 0, 1, \ldots, n$

* n trials must be independent.
* success or failure for each trial.
* R.V. X → no. of successes

$P(\text{success}) = p$, $P(\text{failure}) = 1-p$

$\text{pmf} = \sum_{x=0}^{n} {}^nC_x \cdot p^x q^{n-x} = \sum_{x=0}^{n} (p+q)^n = 1$

Ex: 4 coins are tossed $X = 0, 1, 2, 3, 4$

What is probability that $P(2H \& 2T)$

$n = 4$

Treating head as success $k = 2$

$P\{X=2\} = {}^4C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = \frac{\cancel{4}\times 3}{2} \cdot \frac{1}{\cancel{4}} \cdot \frac{1}{4} = \frac{3}{8}$

$P\{X \le 3\} = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 1 - P(X > 3)$

$= 1 - P\{X = 4\}$

$= 1 - {}^4C_4 \cdot \left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16}$

$= \frac{15}{16}$

Moments:

$$E(X) = \sum_{x=0}^{\infty} x \cdot p(X=x)$$

$$= \sum_{x=0}^{\infty} x \cdot {}^nC_x \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{n!}{(n-x)!\,x!} \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{n(n-1)!}{x \cdot (x-1)!\,(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{\infty} \frac{n(n-1)!}{(x-1)!\,(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$= n \cdot \sum_{x=0}^{\infty} \frac{(n-1)!}{(x-1)!\,(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$= n \cdot p \cdot \sum {}^{n-1}C_{x-1}\, p^{x-1} \cdot q^{n-x}$$

$$= n \cdot p \sum (p+q)^{n-1}$$

$$\mu_1' = E(X) = np$$

$$\mu_1 = 0$$

$$\mu_2' = E(X^2) = \sum x^2 \cdot p(X=x)$$

$$= \sum x^2 \cdot \frac{n!}{(n-x)!\,x!} p^x \cdot q^{n-x}$$

$$E(X^2) = E(X(X-1) + X)$$

$$= E(X^2 - X + X)$$

$$= E(X(X-1)) + E(X)$$

$$E(X(X-1)) = \sum x(x-1) \cdot \frac{n!}{x!(n-x)!} \cdot p^x \cdot q^{n-x}$$

$$= \sum \frac{x(x-1) \; n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!} p^x \cdot q^{n-x}$$

$$= n(n-1)p^2 \cdot \sum \frac{(n-2)!}{(x-2)!\,(n-x)!} \cdot p^{x-2} \cdot q^{n-x}$$

$$= n(n-1)p^2.$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$\mu_2' = n(n-1)p^2 + np$$

$$Var = E(X^2) - (E(X))^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$Var = npq$$

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5) Poisson Distribution: A Random Variable is said to have poison distribution if PMF

$$P\{X=x\} = \frac{e^{-\lambda}\lambda^x}{x!}, \quad \lambda>0, \quad x=0,1,2\ldots$$

$$X \sim P(\lambda)$$

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}\cdot e^{\lambda} = 1$$

$$e^{\lambda} = 1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots = \sum_{x=0}^{\infty}\frac{\lambda^x}{x!}$$

→ Independent event

Ex: If no. of accidents on a highway each day is a poisson distribution with parameter $\lambda=3$

P{there is no accident}

Sol Let X be the no. of accidents

$$P\{X=0\} = \frac{e^{-3}(3)^0}{0!} = e^{-3} = (2.718)^{-3} = 0.049$$

Moments

$$E(X) = \sum_{x=0}^{\infty} x\, P(X=x)$$

$$= \sum x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum \frac{x\cdot\lambda^x}{x(x-1)!}$$

$$= e^{-\lambda}\sum \frac{\lambda\cdot\lambda^{x-1}}{(x-1)!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda \cdot e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!} = \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda$$

$$E(X) = \lambda$$

$$E(X^2) = E(X(X-1)+X)$$

$$= E(X(X-1)) + E(X)$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^2 \lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \cdot \lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$= e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda}$$

$$= \lambda^2$$

$$E(X^2) = E(X(X-1)) + E(X)$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$\mu_2 = Var(X) = \lambda$$

For poison distribution Mean & Variance are same i.e., λ

Ex: Suppose that typographical errors on a single page of a book has poison distribution with parameter $\lambda = 1$

P{atleast one error on this page}

X = no. of errors

$$P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= 1 - \frac{e^{-1} \cdot (1)^{0}}{1!}$$

$$= 1 - \frac{1}{e}$$

$$= 0.633$$

6) Geometric Distribution:

Ex: A coin is tossed until a head appears

4th success T T T H

n^{th} success $(n-1)$ failures 1 success

Def: A random variable is said to follow geometric distribution if

$$P\{X = x\} = (1-p)^{x-1} p \qquad x = 1, 2, \ldots n$$

$$= q^{x-1} \cdot p$$

Ex: If your probability of getting success is 0.2 in voting, what is the probability that you get success on third day

sol $X : \cancel{0}, 1, 2, \ldots$

$$P(X=3) = (1-p)^{x-1} \cdot p$$
$$= (0.8)^2 \cdot (0.2)$$
$$= (0.64)(0.2)$$
$$= 0.128.$$

Moments

$$E(X) = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1} \qquad \sum n \cdot x^{n-1} = \frac{1}{(1-x)^2}$$

$$= p \cdot \left(\frac{1}{(1-(1-p))^2} \right) \qquad E(X) = p \frac{-d}{dp}\left(\sum (1-p)^x\right)$$

$$= \frac{p}{p^2} \qquad = p \cdot \frac{-d}{dp} \frac{(1-p)}{p}$$

$$\mu_1' = E(X) = \frac{1}{p} \qquad = p \cdot \frac{d}{dp}\left(1 - \frac{1}{p}\right)$$

$$= p \cdot \frac{1}{p^2} = \frac{1}{p}$$

$$E(X^2) = E(X(X-1) + X)$$

$$E(X(X-1)) = \sum_{x=1}^{\infty} x(x-1)(1-p)^{x-1} \cdot p$$

$$= p \sum_{x=1}^{\infty} x(x-1)(1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} x(x-1)(1-p)(1-p)^{x-2}$$

$$= p(1-p)\sum_{x=1}^{\infty} x(x-1)(1-p)^{x-2}$$

$$= p(1-p)\frac{d^2\sum(1-p)^x}{dp^2}$$

$$= p(1-p)\cdot\frac{d^2}{dp^2}\left(\frac{1-p}{p}\right)$$

$$= p(1-p)\frac{d}{dp}\left(-\frac{1}{p^2}\right)$$

$$= p(1-p)\cdot\left(\frac{2}{p^3}\right)$$

$$E(X(X-1)) = \frac{2(1-p)}{p^2}$$

$$E(X^2) = \frac{2(1-p)}{p^2} + \frac{1}{p}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1-2p}{p^2} + \frac{1}{p} = \frac{1-2p+p}{p^2}$$

$$= \frac{1-p}{p^2} = \frac{q}{p^2}$$

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Continous Distribution:

1) **Uniform distribution:** A R.V is said to follow uniform distribution if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Ex: If X is uniformly distributed over $[0, 10]$ calculate

$$f(x) = \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a) $P\{X < 3\}$

b) $P\{X > 7\}$

c) $P\{1 < x < 6\}$

$$P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{10}\int_0^3 dx = \frac{3}{10}$$

$$P\{X > 7\} = \int_7^{10} f(x)\,dx = \frac{3}{10}$$

$$P\{1 < x < 6\} = \int_1^6 f(x)\,dx = \frac{5}{10}$$

$$E(X) = \int_a^b x \cdot f(x)\,dx = \frac{1}{b-a}\int_a^b x \cdot dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a} \cdot \left[\frac{b^2 - a^2}{2}\right]$$

$$\mu_1' = E(X) = \frac{a+b}{2} \qquad , \mu_1 = 0$$

$$E(X^2) = \int_a^b x^2 \cdot f(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\cdot\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\cdot\frac{b^3-a^3}{3}$$

$$= \frac{(b-a)(a^2+b^2+ab)}{(b-a)\,3}$$

$$\mu_2' = E(X^2) = \frac{a^2+b^2+ab}{3}$$

$$\mu_2 = Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+b^2+ab}{3} - \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$Var(X) = \frac{(b-a)^2}{12}$$

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2) Gamma distribution:

Gamma Function $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad \alpha > 0$

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)\,\Gamma(\alpha-2)$$

$$\vdots$$

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)(\alpha-3)\ldots 1 = (\alpha-1)!$$

$\Gamma(1) = 1 \qquad \Gamma(\alpha) = (\alpha-1)!$

$\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(1+\tfrac{1}{2}\right) = \tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} . e^{-x} . dx \,, \quad \alpha > 0$$

Substitute $x = \dfrac{y}{\beta} \Rightarrow dx = \dfrac{dy}{\beta}$

$$\Gamma(\alpha) = \int_0^\infty \frac{\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y/\beta} . dy}{\beta}$$

$$\Gamma(\alpha) = \int_0^\infty \frac{y^{\alpha-1} . e^{-y/\beta} dy}{\beta^\alpha}$$

$$1 = \frac{1}{\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty y^{\alpha-1} . e^{-y/\beta} . dy$$

A R.V is said to follow gamma distribution if its pdf is given by

pdf $f(y) = \begin{cases} \dfrac{y^{\alpha-1} \cdot e^{-y/\beta}}{\beta^{\alpha} \Gamma(\alpha)} & 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$

denoted by $Y \sim \Gamma(\alpha, \beta)$

Moments

$$\mu_1' = E(Y) = \int_0^\infty y \cdot \frac{y^{\alpha-1} \cdot e^{-y/\beta}}{\beta^{\alpha} \cdot \Gamma(\alpha)} \, dy$$

$$= \frac{1}{\beta^{\alpha} \Gamma(\alpha)} \int_0^\infty y^{\alpha} \cdot e^{-y/\beta} \cdot dy$$

$$= \frac{\beta^{\alpha+1} \Gamma(\alpha+1)}{\beta^{\alpha} \Gamma(\alpha)} \underbrace{\int_0^\infty \frac{y^{\alpha} \cdot e^{-y/\beta}}{\Gamma(\alpha+1) \beta^{\alpha+1}} \, dy}_{1}$$

$$= \frac{\beta^{\alpha+1} \Gamma(\alpha+1)}{\beta^{\alpha} \Gamma(\alpha)}$$

$Y \sim \Gamma(\alpha+1, \beta)$

$$= \frac{\beta (\alpha) \Gamma(\alpha)}{\beta \Gamma(\alpha)}$$

$$\mu_1' = \alpha\beta$$

$$\mu_1 = 0$$

$$\mu_2' = E(Y^2) = \int_0^\infty y^2 \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} dy$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha+1} \cdot e^{-y/\beta}}{} dy$$

$$= \frac{\Gamma(\alpha+2)\beta^{\alpha+2}}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{y^{(\alpha+2)-1} \cdot e^{-y/\beta}}{\Gamma(\alpha+2)\ \beta^{\alpha+2}} dy$$

$$= \frac{\Gamma(\alpha+2)\ \beta^{\alpha+2}}{\beta^\alpha\ \Gamma(\alpha)} \qquad [Y \sim \Gamma(\alpha+2, \beta)]$$

$$= \frac{(\alpha+1)\alpha\ \Gamma(\alpha)\ \beta^{\alpha+2}}{\beta^\alpha\ \Gamma(\alpha)}$$

$$\mu_2' = \alpha(\alpha+1)\beta^2$$

$$Var(x) = \mu_2 = E(X^2) - (E(X))^2$$

$$= \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$

Ex: i) Let the time to fix a car follows gamma distribution with mean = 2 & Variance is 2 hours

sol

$$\text{Mean} = 2, \quad \sigma^2 = 2$$

$$\alpha\beta = 2 \quad \alpha\beta^2 = 2$$

$$\beta = 1$$

$$\alpha = 2$$

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} & 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$f(y) = \begin{cases} \dfrac{y e^{-y}}{\Gamma(2)} & 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} y e^{-y} & 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

ii) What is the chance that the waiting time is atmost 4.5 hours

$$P\{Y \le 4.5\} = \int_0^{4.5} y \cdot e^{-y}\, dy$$

$$= \left[y \cdot (-e^{-y})\right]_0^{4.5} + \int_0^{4.5} e^{-y}\, dy$$

$$= \left[-y e^{-y}\right]_0^{4.5} + \left[-e^{-y}\right]_0^{4.5}$$

$$= -(4.5)e^{-4.5} + \left[-e^{-4.5} + 1\right]$$

$$= 1 - e^{-4.5}(1 + 4.5) = 1 - e^{-4.5}(5.5) = 0.9389$$

$$\cong 0.94$$

3) Exponential Distribution:

Special case of Gamma distribution where $\alpha = 1$

$$f(y) = \begin{cases} \frac{e^{-y/\beta}}{\beta} & 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

moments

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = E(x) = \mu_1'$$

$$= \int_0^\infty x \cdot \frac{1}{\beta} \cdot e^{-x/\beta} \cdot dx$$

$$= \frac{1}{\beta} \int_0^\infty x \, e^{-x/\beta} \, dx$$

$$= \frac{1}{\beta} \left(\underbrace{\left[\frac{x \cdot e^{-x/\beta}}{-\frac{1}{\beta}} \right]_0^\infty}_{=0} - \int_0^\infty \frac{1 \cdot e^{-\frac{x}{\beta}}}{-\frac{1}{\beta}} dx \right)$$

$$= \frac{1}{\beta} \frac{1}{\frac{1}{\beta}} \int_0^\infty e^{-\frac{x}{\beta}} dx$$

$$= \frac{1}{\beta} \frac{1}{\frac{1}{-\beta^2}} \left(e^{-\frac{x}{\beta}} \right)_0^\infty$$

$$= \frac{1}{\beta} \frac{1}{\frac{-1}{\beta^2}} \cdot (0-1)$$

$$\mu_1' = \beta$$

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$\mu_1 = 0$

$Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \frac{1}{\beta}\int_0^\infty x^2 \cdot e^{-x/\beta}\, dx$$

$$= \frac{1}{\beta}\left[x^2 \frac{e^{-x/\beta}}{-\frac{1}{\beta}}\right]_0^\infty - \int_0^\infty 2x\left(\frac{e^{-x/\beta}}{-1/\beta}\right) dx$$

$$= \frac{1}{\beta}\frac{1}{\left(\frac{1}{\beta}\right)} 2 \cdot \int_0^\infty x \cdot e^{-x/\beta}\, dx$$

$$= 2 \cdot \int_0^\infty x \cdot e^{-x/\beta}\, dx$$

$$= 2 \cdot \left[\left(\frac{x\, e^{-x/\beta}}{-1/\beta}\right)_0^\infty - \int_0^\infty \frac{e^{-x/\beta}}{-1/\beta} dx\right]$$

$$= 2\left[\beta \cdot \left(\frac{e^{-x/\beta}}{-1/\beta}\right)_0^\infty\right]$$

$$= -2\beta^2 [0 - 1]$$

$$\mu_2' = 2\beta^2$$

$$Var = E(X^2) - (E(X))^2 = 2\beta^2 - \beta^2$$

$$\mu_2 = \beta^2$$

Ex: Evaluate

$$\int_0^\infty x^6 e^{-5x} dx$$

$$f(y) = \int_0^\infty \frac{y^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha)\beta^\alpha} dy = 1$$

$$I = \int_0^\infty x^6 \cdot e^{-5x} dx$$

$\alpha = 7$

$\beta = \frac{1}{5}$

$$= \left(\frac{1}{5}\right)^7 \cdot \Gamma(7)$$

$\Gamma(x) = (x-1)!$

$$= \frac{1}{5^7} 6!$$

$$= \frac{6\times5\times4\times3\times2}{5^7}$$

$$= \frac{144}{5^5}$$

$$= 0.0092$$

Ex: Let X be amount of time a postal clerk spends on his customer which has exponential distribution with average amount of time is ~~4 months~~ 4 minutes

i) Find Mean & S.D

ii) Formulate PDF

Mean = $\beta = 4$

Variance = $\beta^2 = 16$, S.D = $\sqrt{Var} = 4$

$$f(x) = \frac{1}{\beta} e^{-x/\beta} = 0.25 e^{-0.25x} \quad x > 0$$

iii) Find $f(5)$

$$f(x) = 0.25\, e^{-0.25x}$$

$$f(5) = 0.25\, e^{-0.25(5)}$$

$$= 0.072$$

iv) $P\{4 < x < 5\}$

$$= F(5) - F(4)$$

$$= \int_4^5 0.25\, e^{-0.25x}\, dx$$

$$= 0.25 \left[\frac{e^{-0.25x}}{-0.25}\right]_4^5$$

$$= \frac{0.25}{-0.25}\left[e^{-0.25x}\right]_4^5$$

$$= \left[e^{-0.25x}\right]_5^4$$

$$= 0.3679 - 0.2865$$

$$= 0.0814$$

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4) **Beta Distribution**: Domain $[0, 1]$

Beta Function: $B(\alpha,\beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$, $\alpha,\beta > 0$

$$1 = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$$

defines pdf

for Beta Function

i.e., $f(x) = \begin{cases} \dfrac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha,\beta)} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

we say R.V $X \sim B(\alpha,\beta)$

~~$B(\alpha,$~~

Relation between beta & gamma

$$\boxed{B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}}, \quad \Gamma(\alpha) = (\alpha-1)!$$

$$B(5,6) = \frac{\Gamma(5)\cdot\Gamma(6)}{\Gamma(5+6)}$$

$$= \frac{4!\,5!}{10!}$$

Moments: $E(X) = \mu_1' = \int_0^1 x f(x)\, dx$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha} (1-x)^{\beta-1} dx$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \underbrace{\int_0^1 \frac{x^{\alpha}(1-x)^{\beta-1} dx}{B(\alpha+1,\beta)}}_{\rightarrow ①}$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\dfrac{\Gamma(\alpha+1)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\dfrac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$= \frac{\alpha\,\Gamma(\alpha)}{(\alpha+\beta)\,\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$\mu_1' = \frac{\alpha}{\alpha+\beta}$$

$$\mu_1 = 0$$

$$\mu_1 = E(X^2) - (E(X))^2$$

$$\mu_2' = \frac{1}{B(\alpha,\beta)} \int_0^1 x^2 \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)} \int_0^1 \frac{x^{\alpha+1}(1-x)^{\beta-1}}{B(\alpha+2,\beta)}\,dx$$

$$= \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$= \frac{(\alpha+1)\alpha \cdot \Gamma(\alpha)\ \Gamma(\alpha+\beta)}{(\alpha+1+\beta)(\alpha+\beta)\ \Gamma(\alpha+\beta)\ \Gamma(\alpha)}$$

$$\mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+1+\beta)(\alpha+\beta)}$$

$$\mu_1 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+\beta)(\alpha+1) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha(\alpha^2+\alpha+\alpha\beta+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad \cancel{\frac{\alpha+\beta+\alpha\beta}{(\alpha+1)^2(\alpha+\beta+1)}}$$

$$= \frac{\cancel{\alpha^3} + \cancel{\alpha^2} + \cancel{\alpha^2\beta} + \alpha\beta - \cancel{\alpha^3} - \cancel{\alpha^2\beta} - \cancel{\alpha^2}}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \cancel{\frac{\alpha^2(\alpha+\beta)+\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Ex:

$$f(x) = \begin{cases} \frac{1}{12} x^2(1-x) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

i) Find the parameters α, β

$\alpha = 3, \ \beta = 2$

ii) Find $E(X)$ & Variance

$$E(X) = \frac{\alpha}{\alpha+\beta} = \frac{3}{5}$$

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{6}{25\times 6} = \frac{1}{25}$$

iii) $P\{0.2 < x < 0.5\}$

$$= \int_{0.2}^{0.5} \frac{1}{12} x^2(1-x)\,dx$$

$$= \frac{1}{12} \int_{0.2}^{0.5} (x^2 - x^3)\,dx$$

$$= \frac{1}{12}\left[\frac{x^3}{3} - \frac{x^4}{4}\right)_{0.2}^{0.5}$$

$$= \frac{1}{12}\left[\left(\frac{(0.5)^3}{3} - \frac{(0.5)^4}{4}\right) - \left(\frac{(0.2)^3}{3} - \frac{(0.2)^4}{4}\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{4(0.5)^3 - 3(0.5)^4}{12}\right) - \left(\frac{4(0.2)^3 - 3(0.2)^4}{12}\right)\right]$$

$$= \frac{1}{12}\left[\frac{(0.5)^3[4 - 1.5]}{12} - \frac{(0.2)^3[4 - 0.6]}{12}\right]$$

$$= \frac{1}{12}\left[\frac{(0.5)^3(2.5) - (0.2)^3(3.4)}{12}\right]$$

$$= \frac{0.3125 - 0.0272}{144}$$

$$= \cancel{0.2853}\ 0.00198$$

5) <u>Normal Distribution</u> / <u>Gaussian Distribution</u>:

Any R.V X is said to have normal distribution if its pdf is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)} \quad \text{for } -\infty < x < \infty$$

where μ is mean &

σ is standard deviation

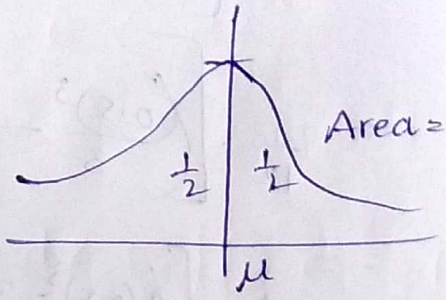
we say $X \sim N(\mu, \sigma^2)$

Properties:

i) Bell shaped curve

ii) symmetric about Mean

iii) Mean & Median are same



μ = Median

iv) Curve is peak at Mean

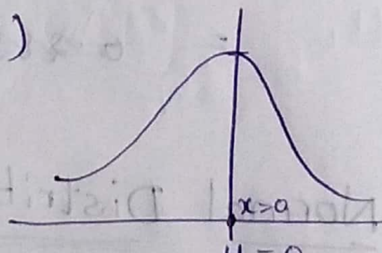
Standard Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty$$

Let $Z = \frac{x-\mu}{\sigma}$

$$f(z) = \phi(z) = \frac{e^{-\frac{z^2}{2}}}{\cancel{\sigma}\sqrt{2\pi}} \quad (\because \sigma = 1)$$

Here the R.V $Z \sim N(0,1)$



changing μ = shift the entire curve on R

changing σ will stretch the curve

pdf $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1$

$Z = \sqrt{2t}$

$\frac{z^2}{2} = t$

$z^2 = 2t$

$2z\,dz = 2\,dt$

$z\,dz = dt$

$$F(z) = \int_{-\infty}^{\infty} \cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{z^2}{2}} dz$$

$$F(-z) = \int_{-\infty}^{\infty} \cancel{\frac{1}{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$F(z) = F(-z)$$

$$F(z) = 2\int_{0}^{\infty} \cancel{\frac{1}{\pi}} e^{-\frac{z^2}{2}} dz$$

$z\,dz = dt$

$\sqrt{2t}\, dz = dt$

$$= 2\int_{0}^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= 2\int_{0}^{\infty} \frac{1}{\sqrt{2t}} e^{-t} . dt$$

$$= \cancel{\frac{1}{\sqrt{2}} \int_{0}^{\infty} \sqrt{t} \frac{e^{-t}}{t} dt}$$

$$= 2\frac{1}{\sqrt{2}} \int_{0}^{\infty} t^{-1/2} e^{-t} dt$$

$\alpha = \frac{1}{2}$

$$= \frac{2}{\sqrt{2}} . \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2}{\sqrt{2}} \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2}{2}} dz}{\sqrt{2\pi}} = 1$$

Moments

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot e^{-\frac{z^2}{2}} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} z \cdot e^{-\frac{z^2}{2}} dz + \int_{0}^{\infty} z \cdot e^{-\frac{z^2}{2}} dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{\infty}^{0} e^{-t} \cdot dt + \int_{0}^{\infty} e^{-t} dt \right] \qquad \frac{z^2}{2} = t \quad 2z\,dz = 2d$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-e^{-t}\right)_{-\infty}^{0} + \left(-e^{-t}\right)_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-e^{-\frac{z^2}{2}}\right)_{-\infty}^{0} + \left(-e^{-\frac{z^2}{2}}\right)_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-1 + 0 + (-0 + 1) \right]$$

$$= \frac{1}{\sqrt{2\pi}} (-1 + 1)$$

$$\mu_1' = E(x) = 0$$

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$$Var(Z) = E(\cancel{Z}^2) - (E(\cancel{Z}))^2$$
$$= E(\cancel{Z}^2)$$

$$\mu_2 = \mu_2'$$

$$E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z^2 \cdot e^{-\frac{Z^2}{2}} dZ$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z \cdot Z e^{-\frac{Z^2}{2}} dZ$$

$$= \frac{1}{\sqrt{2\pi}} \left[Z\left(-e^{-\frac{Z^2}{2}}\right) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 1 \cdot \left(-e^{-\frac{Z^2}{2}}\right) dZ$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{Z^2}{2}} dZ$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$

$$= 1$$

$$Var(Z) = 1$$

$$\frac{X-\mu}{\sigma} = Z, \quad X \sim N(\mu, \sigma^2)$$

$$X = Z\sigma + \mu)$$

$$E(X) = E(Z\sigma) + E(\mu)$$
$$= \sigma E(Z) + \mu$$
$$= \sigma \cdot 0 + \mu$$
$$= \mu$$

$$\text{Var}(X) = \text{Var}(Z\sigma) + \text{Var}(\mu)$$
$$= \sigma^2 \,\text{Var}(Z) + 0$$
$$= \sigma^2 \cdot 1$$
$$= \sigma^2$$

Ex:

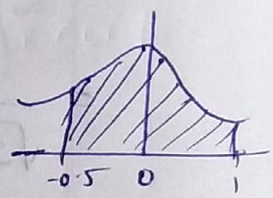
$X \sim N(3,4)$

$\mu = 3, \quad \sigma^2 = 4 \Rightarrow \sigma = 2$

$P\{2 \le X \le 5\}$ $= P\left\{\frac{2-3}{2} \le \frac{X-\mu}{\sigma} \quad \frac{5-3}{2}\right\}$

$Z = \frac{X-\mu}{\sigma}$

$$= P\left\{-\frac{1}{2} \le Z \le 1\right\}$$
$$= P\{-0.5 \le Z \le 1\}$$



$$= \phi(1) - \phi(-0.5)$$
$$= 0.8413 - 0.3085$$
$$= 0.5328$$

Ex: $X \sim N(2,4)$ $\quad \mu = 2, \ \sigma^2 = 4 \Rightarrow \sigma = 2$

i) $P\{X \le 0\}$

$$= P\left\{\frac{X-\mu}{\sigma} \le \frac{0-2}{2}\right)$$
$$= P\{Z \le -1\}$$
$$= 0.1587$$

ii) $P\{|X| \geq 2\}$

$= P\{X \geq 2 \text{ or } X \leq -2\}$

$= P\left\{\frac{X-\mu}{\sigma} \geq \frac{2-2}{2} \text{ or } \frac{X-\mu}{\sigma} \leq \frac{-2-2}{2}\right\}$

$= P\{Z \geq 0\} + P\{Z \leq -2\}$

$= P\{Z \leq -2\} + P\{Z \geq 0\}$

$= \phi(-2) + 1 - P\{Z < 0\}$

$= 0.0228 + 1 - \frac{1}{2}$

$= 0.0228 + 0.5$

$= 0.5228$

Ex:

Scores of an exam are normally distributed with $\mu = 527$ & $\sigma = 112$

What is probability that

$P\{\text{he/she scores more than } 500\}$

$P\{X \geq 500\} = P\left\{\frac{X-\mu}{\sigma} \geq \frac{500-527}{112}\right\}$

$= P\left\{\frac{X-\mu}{\sigma} \geq -0.24\right\}$

$= P\{Z \geq -0.24\}$

$= 1 - P\{Z < -0.24\}$

$= 1 - \phi(-0.24)$

$= 1 - 0.4052$

$= 0.5948$

Ex:

$\mu = 4.11 \quad \sigma = 1.37$

$$P\{X < 3\} = P\left\{\frac{X-\mu}{\sigma} < \frac{3-4.11}{1.37}\right\}$$

$$= P\{Z < -0.81\}$$

$$= \phi(-0.81)$$

$$= 0.2090$$

$P\{X \leq a\}$ & $P\{X < a\}$ For continuous both are same

$\hookrightarrow P\{X < a\} + P\{X = a\}$

$P\{X = a\} = 0$ for continous.

Moment Generating Function:

The function $M_x(t) = E(e^{tx})$ is the MGF of the Random Variable X, if it exists in some neighbourhood of origin

ex:

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot \frac{1}{2} \cdot e^{-\frac{x}{2}} \, dx$$

$$= \frac{1}{2} \int_0^\infty e^{\left(t-\frac{1}{2}\right)x} \, dx$$

$$= \frac{1}{2} \left[\frac{e^{\left(t-\frac{1}{2}\right)x}}{t-\frac{1}{2}} \right]_0^\infty \qquad t < \frac{1}{2}$$

$$= \frac{1}{2} \cdot \left[\frac{0-1}{\left(t-\frac{1}{2}\right)} \right]$$

MGF of PDF $= \dfrac{1}{2\left(\frac{1}{2}-t\right)} = \dfrac{1}{1-2t}$

All the time mgf may not exists

since $\int$ or Σ may diverge sometimes

$M_x(t) = E(e^{tx}) = \dfrac{1}{1-2t}, \; t < \frac{1}{2}$

Thm For given mgf, n^{th} derivative at $t=0$ will give n^{th} order moment about origin

$M_x(t) = E(e^{tx})$

$\frac{d}{dt} M_x(t) = \frac{d}{dt} E(e^{tx})$

$= E(x e^{tx})$

At $t=0$, $\mu_x'(t) = E(x) = \mu_1'$

$\mu_x''(t) = E(x^2 e^{tx})$

At $t=0$ $\mu_x''(t) = E(x^2) = \mu_2'$

Statement
$\mu_x^n = 0$ at $t=0$ will give you n^{th} order moments about origin.

$M_x(t) = \frac{1}{1-2t}$

$M_x'(t) = \frac{+2}{(1-2t)^2}$

At $t=0$ $\mu_x'(t) = 2$

$\mu_1' = E(x) = 2$

$\mu_2' = M_x''(t)\Big|_{t=0} = \frac{8}{(1-2t)^3} = 8$

$Var = \mu_2 = 8 - 2^2 = 4$

Mgf for continous $E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

For discrete $E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot P\{X=x\}$

$Var(X) = \mu_2 = \mu_2' - (\mu_1')^2$

$\mu_3 = E((X-\mu)^3) = E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3)$

$= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3$

$E(X) = \mu = \mu_1'$

$= \mu_3' - 3\mu_1'\mu_2' + 3(\mu_1')^2 \mu_1' - (\mu_1')^3$

$= \mu_3' - 3\mu_1'\mu_2' + 3(\mu_1')^3 - (\mu_1')^3$

$= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

Binomial Distribution:

$$E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot P\{X = x\}$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot {}^nC_x \cdot p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{\infty} {}^nC_x \cdot (pe^t)^x (1-p)^{n-x}$$

$$= (pe^t + 1 - p)^n$$

$$= (pe^t + q)^n$$

Mgf of $X \sim B(n, p)$

$$\mu_x'(t) = n(pe^t + (1-p))^{n-1} \cdot p \cdot e^t$$

$$\mu_x'(t)\Big|_{t=0} = n(p + 1 - p)^{n-1} \cdot p = np$$

$$\mu_1' = E(X) = np$$

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$$M_x'(t) = n(pe^t+(1-p))^{n-1} \cdot pe^t$$

$$= np\left[(pe^t+(1-p))^{n-1} \cdot e^t\right]$$

$$M_x''(t) = np\left[(n-1)(pe^t+(1-p))^{n-2} e^t pe^t + (pe^t+(1-p))^{n-1} \cdot e^t\right]$$

$$M_x''(t)\Big|_{t=0} = np\left[(n-1)(p+(1-p))^{n-2} \cdot p + (p+(1-p))^{n-1}\right]$$

$$= np\left[p(n-1)+1\right]$$

$$M_2'(t) = n(n-1)p^2+np$$

$$M_2 = Var(x) = n(n-1)p^2+np-n^2p^2$$

$$= np(1-p)$$

$$= npq$$

Poisson Distribution

$$E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \qquad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$$

$$= e^{-\lambda} \cdot e^{\lambda \cdot e^t}$$

$$E(e^{tx}) = e^{\lambda(e^t-1)}$$

$$\cancel{M_x'(t) = \frac{e^{\lambda(e^t-1)}}{(e^t-1)}}$$

$$M_x'(t) = e^{-\lambda} \cdot e^{\lambda e^t} \cdot \lambda e^t$$

$$= e^{-\lambda} \cdot \lambda \left[e^{\lambda e^t} \cdot e^t\right]$$

$$M_x'(t)\Big|_{t=0} = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda = E(X) = \mu_1'$$

$$M_x''(t) = e^{-\lambda} \cdot \lambda \left[e^{\lambda e^t} \cdot e^t + \lambda \cdot e^t \cdot e^{\lambda e^t} \cdot e^t\right]$$

$$M_x''(t)\Big|_{t=0} = e^{-\lambda} \cdot \lambda \left[e^{\lambda} + \lambda e^{\lambda}\right]$$

$$= \lambda + \lambda^2 = \mu_2'$$

$$Var = \mu_2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Continuos Distribution

Exponential Distribution:

$$E(e^{tx}) = \int_{x=0}^{\infty} \frac{e^{tx} \cdot e^{-x/\beta}}{\beta} \cdot dx$$

$$= \frac{1}{\beta} \int_0^{\infty} e^{\left(t-\frac{1}{\beta}\right)x} dx$$

$$= \frac{1}{\beta} \cdot \frac{\left[e^{\left(t-\frac{1}{\beta}\right)x}\right]_0^{\infty}}{\cancel{\left(t-\frac{1}{\beta}\right)}\left(t-\frac{1}{\beta}\right)}$$

$$= \frac{1}{\beta}\left[0 - \frac{1}{t-\frac{1}{\beta}}\right]$$

t /

$$= \frac{1}{\beta\left(\frac{1}{\beta} - t\right)}$$

$$M_x'(t) = \frac{1}{\beta\left(\frac{1}{\beta} - t\right)^2}$$

$$M_x'\Big|_{t=0} = \frac{1}{\beta\left(\frac{1}{\beta} - 0\right)^2} = \frac{1}{\beta \cdot \frac{1}{\beta^2}} = \beta = E(X)$$

$$M_x''(t) = \frac{1}{\beta} \cdot \frac{2}{\left(\frac{1}{\beta} - t\right)^3}$$

$$= \frac{2}{\beta\left(\frac{1}{\beta} - t\right)^3}$$

$$M_x''(t)\Big|_{t=0} = \frac{2}{\beta\left(\frac{1}{\beta} - 0\right)^3} = 2\beta^2 = \mu_2'$$

$$\mu_2 = 2\beta^2 - \beta^2$$

$$= \beta^2$$

4/5

Moment Generating function of Normal Distribution:

Let $Z \sim N(0,1)$

$$M_x(t) = E(e^{tx}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2-2tz}{2}\right)} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2-2tz+t^2)}{2}} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} dz$$

$$= e^{t^2/2}$$

$M_x(t)$ for standard Normal

$$X = Z\sigma + \mu$$

$$E(e^{tx}) = E\left(e^{t(z\sigma+\mu)}\right)$$

$$= e^{t\mu} E(e^{tz\sigma}) = e^{t\mu} E(e^{t\sigma z})$$

$$= e^{t\mu} \cdot e^{\frac{\sigma^2 t^2}{2}}$$

$$= e^{\left(\frac{\sigma^2 t^2}{2} + t\mu\right)}$$

is the mgf of $X \sim N(\mu, \sigma^2)$

Mgf of $X \sim N(\mu, \sigma^2)$ is $e^{\frac{\sigma^2 t^2}{2} + t\mu}$

$$M_x'(t) = e^{\left(\frac{\sigma^2 t^2}{2} + t\mu\right)} \cdot \left(\frac{\sigma^2 \cdot 2t}{2} + \mu\right)$$

$$= e^{\left(\frac{\sigma^2 t^2}{2} + t\mu\right)} \cdot (t\sigma^2 + \mu)$$

$$M_x'(t)\Big|_{t=0} = \mu = E(X) = \mu_1'$$

$$\mu_1 = 0$$

$$\mu_x'(t) = e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)} (\sigma^2 t + \mu)$$

$$\mu_x''(t) = e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)} \cdot (\sigma^2) + e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)} (\sigma^2 t + \mu) \cdot (\sigma^2 t + \mu)$$

$$= e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)} \sigma^2 + e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)} \cdot (\sigma^2 t + \mu)^2$$

$$\mu_x''(t)\Big|_{t=0} = \sigma^2 + \mu^2 = E(X^2) = \mu_2'$$

$$Var(X) = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

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Inequalities (unit-6)

1) Markov's Inequality: Suppose X is non -ve random variable then for any $a>0$, we have $P\{X \geq a\} \leq \frac{E(X)}{a}$.

proof: Let X be continous random variable

$$E(X) = \int_0^\infty x f(x)\,dx$$

$$= \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$E(X) \geq \int_a^\infty x f(x)\,dx$$

$$E(x) \geq \int_a^\infty a f(x)\,dx \quad \because x \geq a$$

$$\frac{E(X)}{a} \geq \int_a^\infty f(x)\,dx$$

$$\frac{E(X)}{a} \geq P\{X \geq a\}$$

2) Chebyshew's Inequality: Suppose X is a random variable ~~then for any~~ with mean μ and variance σ^2 then for any $k>0$

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

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Proof:

Since $(x-\mu)^2$ is positive

$$P\{X \geq a\} \leq \frac{E(X)}{a}$$

Let us take $(x-\mu)^2 > 0$ by taking $a = k^2$ in Markov's inequality

$$P\{(X-\mu)^2 \geq k^2\} \leq \frac{E((X-\mu)^2)}{k^2} = \frac{\sigma^2}{k^2}$$

$(x-\mu)^2 \geq k^2$, if $|x-\mu| \geq k$

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$ ($\because$ hence proved).

Ex:

Suppose we know that the no. of items produced in a factory during a week is a R.V with mean 500

i) What can be said about probability that this week's production will be atleast 1000?

Sol:

$$P\{X \geq 1000\} \leq \frac{500}{1000}$$ (By Markov's inequality)

$$P\{X \geq 1000\} \leq \frac{1}{2}$$

$P\{X < 1000\} = ?$

$$-P\{X \geq 1000\} \geq -\frac{E(X)}{a} = -\frac{1}{2}$$ $\rightarrow$ upper bound

$$1 - P\{X \geq 1000\} \geq 1 - \frac{1}{2} = \frac{1}{2}$$ $\leftarrow$ lower bound.

$$P\{X < 1000\} \geq \frac{1}{2}$$

ii) if variance of week's production is 100, what can be said about the probability that the production will be between 400 & 600 ?

sol

$\mu = 500$, $\sigma^2 = 100$

$P\{|x-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

$P\{400 < X < 600\} = ?$

$P\{|x-500| < 100\}$

$P\{-100 < X-500 < 100\}$

$P\{400 < X < 600\}$

Using chebyshew's Inequality

$P\{|x-500| \geq 100\} \leq \frac{100}{(100)^2} = \frac{1}{100}$

$1 - P\{|x-500| \geq 100\} \geq 1 - \frac{1}{100}$

$P\{|x-500| < 100\} \geq \frac{99}{100}$

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5/5/21

1) Consider an experiment which consists of counting α-particles in a one-second interval if we know that on an average 3.2 such α-particles are given, what is the probability that no more than 2 α-particles will appear.

sol

$$P\{X \le 2\} = P\{X=0\} + P\{X=1\} + P\{X=2\}$$

$$= e^{-3.2} + \frac{e^{-3.2}\cdot(3.2)}{1} + \frac{e^{-3.2}\cdot(3.2)^2}{2}$$

$$= e^{-3.2}\left(1 + 3.2 + \frac{(3.2)^2}{2}\right)$$

$$= 0.382$$

2) Assume that p(defective computer component) = 0.02

a) Find P{the 1st defect is caused by 7th component}

b) How many components do you expect to test until one is found to be defective?

sol a)

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{good}}\ \underset{\text{defective}}{\_}$

Follows geometric distribution

$P\{X=7\} = q^{7-1} \cdot p$ sucess - getting defective

$= (0.98)^6 \cdot (0.02)$

$= 0.0177$

b) $E(X) = \frac{1}{p}$

$= \frac{1}{0.02}$

$= 50$

$\sigma^2 = \frac{q}{p^2} = \frac{0.98}{0.02 \times 0.02} = \frac{98}{2 \times 0.02}$

$= 49 \times 50$

$= 2450$

3) An item produced by a machine will be defective with $p = 0.1$ what is the P{atmost one will be defective} in a sample of 3 items.

Sol

$p = 0.1, n = 3$

$P\{X \le 1\} = P\{X=0\} + P\{X=1\}$

$= {}^3C_0 \cdot (0.1)^0 \cdot (0.9)^3 + {}^3C_1 \cdot (0.1)^1 (0.9)^2$

$= (0.9)^3 + 3(0.1)(0.9)^2$

$= 0.972$

4) Time in hours to finish a job follows beta distribution with $\mu = \frac{1}{3}$ and variance $\frac{2}{63}$ hours. Find the P{Job will be finished in 30 min}

Sol For Beta distribution

$$\mu = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}$$

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{2}{63}$$

$3\alpha = \alpha + \beta$

$2\alpha = \beta$

$\beta = 4$

$63\alpha\beta = 2(\alpha+\beta)^2(\alpha+\beta+1)$

$63\alpha(2\alpha) = 2(\alpha+2\alpha)^2(\alpha+2\alpha+1)$

$\cancel{126}\,\cancel{\alpha^2} = \cancel{2}\cdot(9\cancel{\alpha^2})(3\alpha+1)$

$3\alpha+1 = 7$

$3\alpha = 6$

$\alpha = 2$

$$P\{X \le \tfrac{1}{2}\} = \frac{1}{B(2,4)} \int_0^{1/2} x^1 \cdot (1-x)^3 \, dx$$

$$B(2,4) = \frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{1!\,3!}{5!} = \frac{3\times 2}{120} = \frac{1}{20}$$

$$P\{X \le \tfrac{1}{2}\} = \frac{1}{(1/20)} \int_0^{1/2} x \cdot (1 - x^3 + 3x^2 - 3x) \, dx$$

$$= 20 \int_0^{1/2} (x - x^4 + 3x^3 - 3x^2) \, dx$$

$$= 20 \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{3x^4}{4} - \frac{3x^3}{3} \right]_0^{1/2}$$

$$= 20 \left[\frac{1}{8} - \left(\frac{1}{2}\right)^5 \cdot \frac{1}{5} + 3 \cdot \left(\frac{1}{2}\right)^4 \cdot \frac{1}{4} - \cancel{3} \cdot \left(\frac{1}{2}\right)^3 \right]$$

$$= 20 \left[\frac{3}{64} - \frac{1}{32 \times 5} \right] = \cancel{20} \left[\frac{15+2}{\cancel{64 \times 8}} \right] = \frac{13}{16}$$

5) Let waiting time for emails follow exponential distribution if $\beta = 2$; what is chance of waiting atmost 1.1 minutes.

Sol

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\beta = 2$

$$P\{X \leq 1.1\} = \int_0^{1.1} \frac{1}{2} \cdot e^{-x/2} \, dx$$

$$= \frac{1}{2} \int_0^{1.1} e^{-x/2} \, dx$$

$$= \frac{1}{2} \cdot \left[\frac{e^{-\frac{x}{2}}}{-1/2} \right]_0^{1.1}$$

$$= 1 - e^{-1.1/2}$$

$$= 0.423$$

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6) Suppose that reaction time x of a randomly selected individual to a certain stimulus has gamma distribution with $\beta = 1$ & $\alpha = 2$

a) $P\{3 < X < 5\}$ b) $P\{X > 4\}$

<u>sol</u> pdf $f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} y^{\alpha-1} \cdot e^{-y/\beta} dy$

a) $P\{3 < X < 5\} = \frac{1}{\Gamma(2) 1^2} \int_3^5 x \cdot e^{-x} dx$

$= \int_3^5 x \cdot e^{-x} dx$

$= \left[x \cdot (-e^{-x})\right]_3^5 - \int_3^5 (-e^{-x}) dx$

$= -5e^{-5} + 3e^{-3} + (-e^{-x})_3^5$

$= -5e^{-5} + 3e^{-3} \cancel{-2e^{-5}} + (-e^{-5} + e^{-3})$

$= -6e^{-5} + 4e^{-3}$

$= \cancel{0.1598} \cancel{-0.0404 + 0.1991}$

$= 0.1587$

$P\{X > 4\} = 1 - P\{X \leq 4\}$

$= 1 - \int_0^4 x e^{-x} dx = 1 - 0.908$

$= 0.092$

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7) Let $X \sim N(2,4)$ $\mu = 2, \sigma = 2$

a) $P\{1 < x \leq 3\}$

$$= P\left\{\frac{1-2}{2} < Z < \frac{3-2}{2}\right\}$$

$$= P\{-0.5 < Z < 0.5\}$$

$$= \phi(0.5) - \phi(-0.5)$$

$$= 0.6915 - 0.3085$$

$$= 0.3830$$

b) $P\{X \leq 3 / X > 1\}$

$$= \frac{P\{1 \leq X \leq 3\}}{P\{X > 1\}}$$

$$= \frac{0.3830}{0.6915}$$

$$P\{X > 1\} = 1 - P\{X \leq 1\}$$

$$\cancel{= 1 - 0.8413}$$

$$= 1 - P\{Z \leq (-0.5)\}$$

$$= 1 - 0.3085$$

$$= 0.6915$$

$$P\{X \leq 3 / X > 1\} = 0.553$$

18/5/21

Negative Binomial Distribution: $X \sim NB(r, p)$

$p : P(\text{success})$

$1-p = q = P(\text{failure})$

r^{th} success, where $r \geq 1$ fixed integer

X = no. of failures before r^{th} success

where x has the pmf $P\{X = x\} = {}^{x+r-1}C_x \, p^r$

Geometric distribution is special case of NB when $r = 1$

$$Mgf = p^r (1-qe^t)^{-r}$$

$$E(x) = \frac{rq}{p} = \text{Mean} = \mu_1'$$

$$Var = \frac{rq}{p^2} = \mu_2.$$

Ex: John is required to sell candy bars to raise fund for a field trip. There is 40% chance that he sells the candy he has to sell 5 candy bars. What is P(he sells last candy at 11^{th} house)?

sol

$p = 0.4$

$q = 0.6$

$r = 5$

$x + r = 11$

$x = 6$

- - - - - - - - - - $\uparrow$ 5^{th} candy

$$P(X=6) = {}^{x+r-1}C_x \cdot p^r q^x$$

$$= {}^{10}C_6 (0.4)^5 \cdot (0.6)^6$$

$$= {}^{10}C_4 \cdot (0.01024)(0.046656)$$

$$= \frac{10 \times \cancel{9}^{3} \times \cancel{8} \times 7}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} (0.01024)(0.046656)$$

$$= 30 \times 7 \times (0.01024)(0.046656)$$

$$= 0.10032$$

$$\text{Mean} = \frac{rq}{p} = \frac{5 \times (0.6)}{0.4} = \frac{5 \times \cancel{6}^{3}}{\cancel{4}_2} = 7.5$$

$$\text{Variance} = \frac{rq}{p^2} = \frac{5 \times 0.6}{(0.4)^2} = \frac{3}{0.16} = 18.75$$

Hyper Geometric Distribution:

Done Without replacement

'N' marbles → M drawn marking 1, 2, ... M → putting them back

Now it contains 'M' marked marbles → favourable

N-M unmarked

Now again drawing 'n' marbles

x → favourable → no. of marked marbles in 'n'

Def: A random variable is said to follow hypergeometric distribution if its pmf is

$$P\{X = x\} = \frac{{}^{M}C_{x} \cdot {}^{N-M}C_{n-x}}{{}^{N}C_{n}}$$

Mean $\quad E(X) = \frac{n}{N} \cdot M$

Variance $= \frac{nM}{N^2(N-1)} (N-M)(N-n)$

Ex: 25 members, 19 Nurses, 6 doctors

sample of 5 members, $n=5$

P(4 doctors & 1 nurse) = ?

$N = 25$

$$P\{X=1\} = \frac{{}^{19}C_1 \, {}^{6}C_4}{{}^{25}C_5}$$

Ex: There are 50 bulbs among which 10 are defective. 10 random bulbs are taken and tested this lot is accepted if at most one is defective. P(accepting the lot).

Sol: $N = 50, \ n = 10, \ M = 10$

P(accepting the lot) = P(one defective) + P(no defecti

$$= \frac{{}^{10}C_1 \ {}^{40}C_9}{{}^{50}C_{10}} + \frac{{}^{10}C_0 \cdot {}^{40}C_{10}}{{}^{50}C_{10}}$$

Mode:

Value which is more likely to occur.

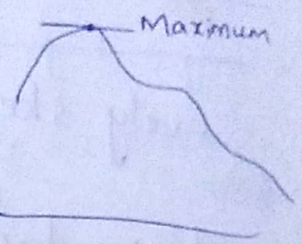
Ex: Sum of two numbers when 2 dice are rolled

X	P
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

$x = 7$ is the mode

For continous function

$$f(x) = \begin{cases} -x^2 + 2x - 1/6 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$



For median $\int_{-\infty}^{x} f(x)\,dx = \frac{1}{2}$

For mode $f'(x) = 0$

$$-2x + 2 = 0$$

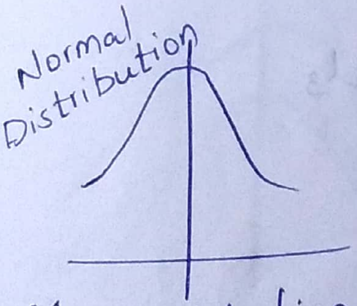
$$\underline{x = 1}$$

Relationship b/w Mean, Mode & Median

Mode = 3(Median) − 2 (Mean)

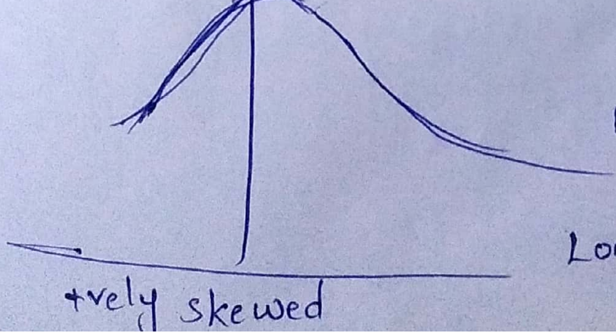
19/5/21

Skewness : Lack of symmetry.



skewness = 0
(since it is symmetrical)

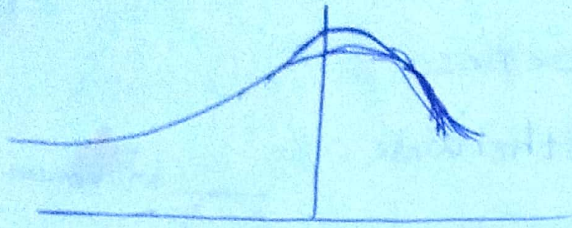
Mean = Median = Mode.



Mode < Median < Mean

Longer tail on right side.

+vely skewed



Mean < Median < Mode.

Longer tail on left side

-vely skewed

3rd standardized non central moment = skewness

$$E\left(\left(\frac{X-\mu}{\sigma}\right)^3\right) = \frac{E(X-\mu)^3}{\sigma^3}$$

$$= \frac{\mu_3}{\sigma^3}$$

skewness $\beta_1 = \dfrac{\mu_3}{\mu_2^{3/2}}$

if $\beta_1 > 0$ it is +vely skewed

$\beta_1 < 0$ it is -vely skewed

$\beta_1 = 0$ symmetric

β_1 is decided by sign of μ_3.

skewness interms of Mean, Median, Mode

$$\beta_1 = \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

$$\beta_1 = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

Mode = 3 Median − 2 Mean

Mostly skewness lies btw −3 and +3

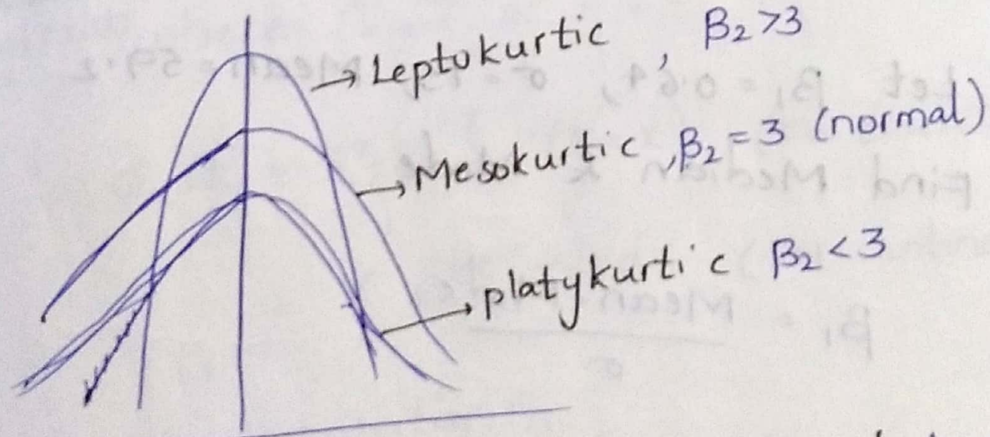
i) If $\beta_1 < -1$ or $\beta_1 > 1$ highly skewed

ii) $-1 < \beta_1 < -0.5$ or $0.5 < \beta_1 < 1$ moderately skewed.

iii) $-0.5 < \beta_1 < 0.5$ Approximately symmetric

Kurtosis: Measure of peakedness

$(-3 < \beta_2 < 3)$



standardized 4^{th} non central moment = kurtosis

$$\beta_2 = E\left(\left(\frac{X-\mu}{\sigma}\right)^4\right) = \frac{E\left((X-\mu)^4\right)}{\sigma^4} = \frac{\mu_4}{\mu_2^2}$$

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Ex: The first four central moments are as follows $0, 2.5, 0.7, 18.75$

Find β_1 & β_2

Sol

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3}{\sigma^3} = \frac{0.7}{(2.5)^{3/2}} = 0.178$$

+vely skewed.

Approximately symmetri[c]

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{18.75}{(2.5)^2} = \frac{18.75}{6.25} = 3$$

$\beta_2 = 3$ Mesokurtic (normal)

$\star$ Extra kurtosis $= \beta_2 - 3$

Ex: Let $\beta_1 = 0.64$, $\sigma = 13$, Mean $= 59.2$

Find Median & Mode.

Sol

$$\beta_1 = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$\text{Mode} = -\beta_1 \sigma + \text{Mean}$$

$$= -(0.64)(13) + 59.2$$

$$= 59.2 - 8.32$$

$$= 50.88$$

$$\beta_1 = \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

$$3\,\text{Median} = 3\,\text{Mean} - \beta_1 \sigma$$

$$\text{Median} = \frac{(3\,\text{Mean} - \beta_1 \sigma)}{3}$$

$$= \frac{3(59.2) - (0.64)(13)}{3}$$

$$= \frac{177.6 - 8.32}{3}$$

$$\text{Median} = 56.42$$

characteristic Function: Let X be a R.V

A complex valued function

→ Generates moments
characteristic fun^c always exists

$$\phi_x(t) = E(e^{itx}) = E(\cos tx + i \sin tx)$$

where $i = \sqrt{-1}$ or $i^2 = -1$

$$\phi_x(t) = \sum e^{itx} P(X=x) \quad \text{(For discrete)}$$

$$= \int e^{itx} f(x)\,dx \quad \text{(For continous)}$$

Poisson distribution:

$$\phi_x(t) = E(e^{itx}) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!} = e^{-\lambda} \cdot e^{+\lambda e^{it}}$$

$$\phi_x(t) = e^{\lambda(e^{it}-1)}$$

statement:

$$\left.\frac{d^n \phi_x(t)}{dt^n}\right|_{t=0} = i^n E(x^n)$$

$$\phi_x(t) = e^{\lambda(e^{it}-1)}$$

$$\frac{d\,\phi_x(t)}{dt} = e^{-\lambda}\, e^{\lambda \cdot e^{it}} \cdot \lambda e^{it} \cdot i$$

$$\left.\frac{d}{dt}\phi_x(t)\right|_{t=0} = e^{-\lambda} \cdot e^{\lambda} \cdot \lambda i$$

$$i E(x) = \lambda i$$

$$E(X) = \lambda$$

Uniform

$$f(x) = \frac{1}{b-a} \quad \text{for } a \le x \le b$$

$$\phi_x(t) = \int_a^b \frac{1}{b-a} \cdot e^{itx}\, dx$$

$$= \frac{1}{(it)(b-a)} \left[e^{itx}\right]_a^b$$

$$= \frac{1}{it(b-a)} \left[e^{itb} - e^{ita}\right]$$

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standard Normal:

$$\phi_x(t) = \int \frac{e^{itx} \cdot e^{-\frac{x^2}{2}} dx}{\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x^2 - 2itx - t^2)} \cdot dx$$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int_{2}^{\infty} e^{-\frac{1}{2}(x-it)^2} dx$$

$$\phi_x(t) = e^{-t^2/2}$$

Normal $E(e^{itz}) = E\left(e^{it\left(\frac{x-\mu}{\sigma}\right)}\right)$

$$= E\left(e^{\frac{itx}{\sigma}} \cdot e^{-\frac{it\mu}{\sigma}}\right)$$

$$\phi_x(t) = e^{-it\mu} \cdot e^{\frac{it^2\mu}{\sigma}}$$

→ characteristic function always exists unlike mgf i.e., $M_x(t)$.

→ It is always continous

→ It generates moments

Ex: $M_x(t) = \int_{-\infty}^{\infty} e^{it} \frac{1}{(1+x^2)} = \infty$

Here mgf does not exist we use characteristic function

20/5/21

Random process / Stochastic Process:

Random process is a function of time /

Random variable which is function of time /
indexed by time.

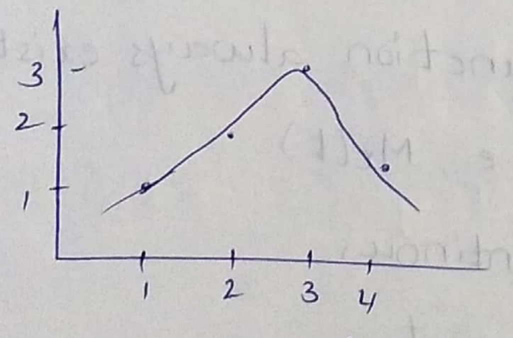
Def:
Collection of Random variables that are indexed by time $X(t)$.

$X_0, X_1, X_2, X_3 \ldots$ for distrete

$\{X_t\}_{t \geq 0}$ for continous.

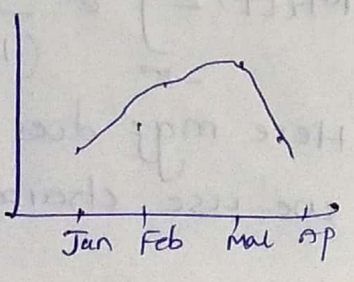
Ex: i) There is a hospital

$N(t)$: No. of patients arrive at time t



ii) No. of international calls in a time interval

iii) Price of an item



Pdf $f(x,t)$

pmf $P(x,t)$

Ex:1 $x(t) = \begin{cases} \sin(\pi t) & \text{if head occurs} \\ 2t & \text{if tail occurs} \end{cases}$

$P(x(t) = \sin(\pi t)) = \frac{1}{2}$

$P\{x(t) = 2t\} = \frac{1}{2}$

$E(x(t)) = \frac{1}{2} \cdot \sin\pi t + \frac{1}{2} . 2t$

$= \frac{\sin\pi t}{2} + t$

$t = 0.25$ has

$P\{x(t) = \sin\pi(0.25)\} = \frac{1}{2}$

$P\{x(t) = 2(0.25)\} = \frac{1}{2}$

$P\{x(t) = \frac{1}{\sqrt{2}}\} = \frac{1}{2}$ $P\{x(t) = \frac{1}{2}\} = \frac{1}{2}$

$E(x)$ at $t = 0.25 = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{2\sqrt{2}}$

$P\{x = \frac{1}{\sqrt{2}}\} = \frac{1}{2}$, $P\{x = \frac{1}{2}\} = \frac{1}{2}$

cdf $F(x,t) = \begin{cases} 0 & x < \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < x < \frac{1}{\sqrt{2}} \\ 1 & x > \frac{1}{\sqrt{2}} \end{cases}$

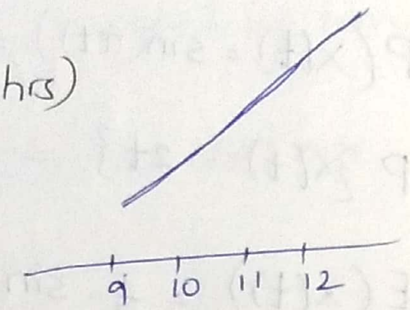
Ex2

N(t) denotes no. of failures in a system over $[0,t)$ described by a poisson random process with no. of failures for every 4 hours.

9'o clock

$N(10) = 9$ to 10

$N(11) = 9$ to 11

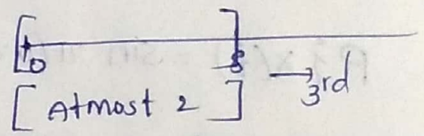


P(3rd failure occurs after 8 hrs)

$\lambda = \frac{1}{4}$

$$P(x,t) = \frac{e^{-\lambda t} \cdot (\lambda t)^x}{x!}$$

P(getting at most 2 errors in $[0,8)$] 3rd failure after 8 hrs



$t = 8$

$x = 0,\ x = 1,\ x = 2$

0 1 2 3

1 2 3

2 3

P(getting at most 2 errors]

$$= P(x=0, t=8) + P(x=1, t=8) + P(x=2, t=8)$$

$$= e^{-(0.25)8} + e^{-(0.25)8}(0.25)(8) + \frac{e^{-(0.25)8}\,((0.25)(8))^2}{2!}$$

$$= 0.677$$