Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Determine whether systems are causal or anti-causal

$$(a)\;\; h[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$(b)\;\; h[n] = \left(\frac{1}{2}\right)^n u[n-1]$$

$$(c)\;\; h[n] = \left(\frac{1}{2}\right)^{|n|}$$

$$(d)\;\; h[n] = u[n+2] - u[n-2]$$

$$(e)\;\; h[n] = \left(\frac{1}{3}\right)^n u[n] + 3^n\, u[-n-1]$$

$|x| = x$ if x is positive,
$|x| = -x$ if x is negative (in which case $-x$ is positive),
$|0| = 0$

Stability

A system is said **to be stable** if bounded input (BI) (finite) produces the bounded output (BO) (finite)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

For discrete case: if $|x[n]| \leq M_x < \infty$

$\rightarrow |y[n]| \leq M_y < \infty$ for *all* n

For continuous case: if

$$|x(t)| \leq M_x < \infty$$

$\rightarrow |y(t)| \leq M_y < \infty$ for *all* t

Determine whether the following system is stable ?

$(a)\ y(t) = x(t)\cos\omega_c t$ for all t

$(b)\ y[n] = x[n-1]$

$$|y(t)| = |x(t)\cos\omega_c t| \leq |x(t)|\,|\cos\omega_c t| \qquad |\cos\omega_c t| \leq 1$$

$$\leq x(t)$$

If input $x(t)$ is **bounded** → $y(t)$ will be **bounded** i.e. BIBO

$(b)\ y[n] = x[n-1]$

$$|y[n]| = |x[n-1]| \leq k \qquad if\ |x[n]| \leq k \quad \text{for } all\ n$$

If input $x(n)$ is **bounded (finite)** → $y(n)$ will be **bounded (finite)** i.e. BIBO

the system is BIBO stable

Condition for LTI system to be Stable

We know for the LTI system **input-output** is related

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

If $x[n]$ is **bounded (finite)** say, $M_x \rightarrow x[n-k]$ will bounded

For the bounded value of $y[n]$, $\sum_{k=-\infty}^{\infty} |h[k]|$ should be (finite) $< \infty$

Therefore, a LTI system is **stable if** its impulse response $h[n]$ is absolutely summable

- Determine the range of values of the parameter a *for which the linear time-invariant* system with impulse response is stable.

$$h(n) = a^n u(n)$$

For the stability of LTI system, it should satisfy $$\sum_{k=-\propto}^{\propto} |h[k]| < \propto$$

Using the relation,

$$\sum_{k=-\propto}^{\propto} |h(k)| = \sum_{k=-\propto}^{\propto} \left|a^k u(k)\right| = \sum_{k=0}^{\propto} \left|a^k\right| <\propto$$

If **|a| < 1** i.e. **sum able value will decreases as k→** ∞

Therefore, the system is stable if $|a| < 1$

Thank you!

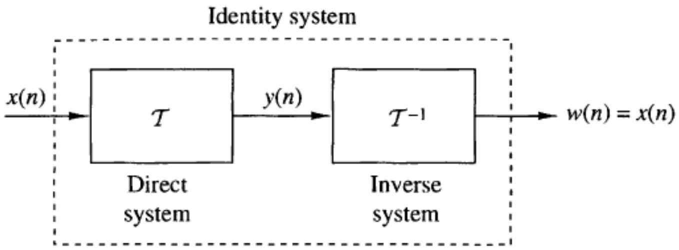
Warning notification!!!!!

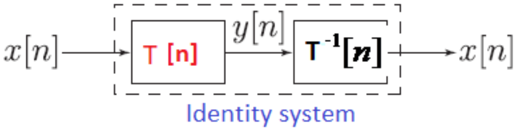
- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

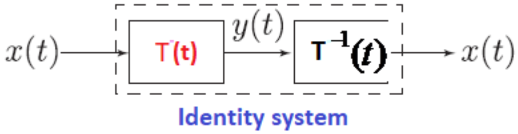
Invertibility

- A system is **invertible**, if an inverse system exists that when cascaded with the original system yields an output equal to input
- A system is invertible if **distinct inputs lead** to **distinct outputs**





Continuous case:



Determine if the following systems are invertible or not

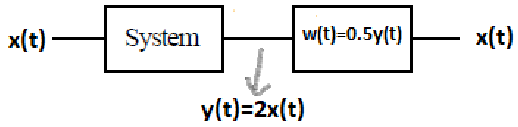
$(a)\ y(t) = 2\,x(t)$

$(b)\ y(t) = \ x^2(t)$

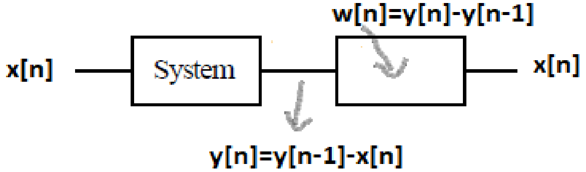
$(c)\ y[n] = \sum_{k=-\infty}^{\infty} x[k]$

$(d)\ y[n] = 0$

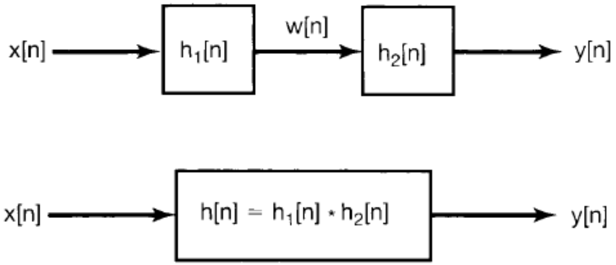
(a)



$$(c)\quad y[n] = \sum_{k=-\infty}^{n} x[k] = [\cdots x[n-2] + x[n-1]] + x[n] = y[n-1] + x[n]$$



Invertibility – LTI system



$$h[n] * h_1[n] = \delta[n]$$

(identity)

- Consider an LTI system with impulse response $h[n] = u[n]$. Determine whether inverse system of it is exist.

We know for the LTI system input-output is related

$$y[n] = \sum_{k=-\propto}^{\propto} x[k]\, h[n-k] = \sum_{k=-\propto}^{\propto} x[k]\, u[n-k]$$

$$y[n] = \sum_{k=-\propto}^{n} x[k] \quad \ldots (\mathbf{1})$$

for $(n-k) > 0\,, u[n-k] = 1$

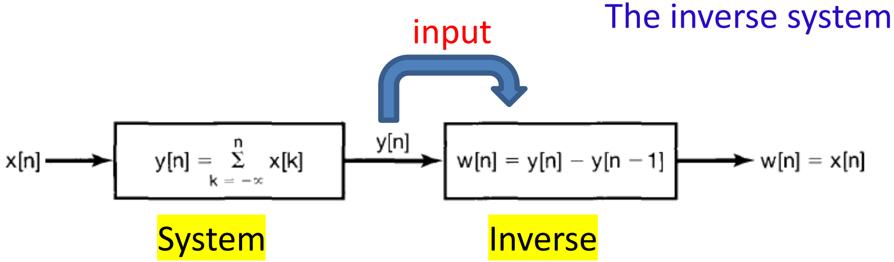
$$y[n] = \sum_{k=-\propto}^{n} x[k] \quad \ldots (\mathbf{1})$$

$$y[n-1] = \sum_{k=-\propto}^{n-1} x[k] \quad \ldots (\mathbf{2})$$

Using Eq. (1) & (2)

$$y[n] - y[n-1] = \sum_{k=-\propto}^{n} x[k] - \sum_{k=-\propto}^{n-1} x[k] \quad = x[n]$$

The inverse system is $y[n] - y[n-1] = x[n]$



To evaluate the impulse response of **the inverse system**, consider input to the system $y[n] = \delta[n]$

$$h_1[n] = \tau\{\delta[n]\} = \delta[n] - \delta[n-1]$$

Cross-check: $h_I[n] = h[n] * h_1[n] \quad = u[n] * (\delta[n] - \delta[n-1]) \quad = u[n] * \delta[n] - u[n] * \delta[n-1]$

$= u[n] * \delta[n] - u[n] * \delta[n-1]$

$1 \text{ for } [n-k] = 0,$
$\rightarrow$ k=n

$1 \text{ for } [n-k-1] = 0,$
$\rightarrow k = n - 1$

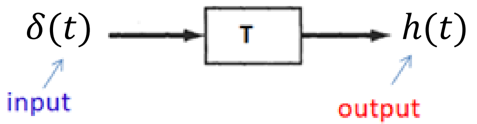
$$= \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k] - \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k-1]$$

$= u[n] - u[n-1]$

$= \delta[n]$

$h_I[n] = h[n] * h_1[n] = \delta\,[n]$

Response of LTI systems to complex exponentials (continuous-time signal)



From convolution integral of LTI continuous-time system, we can write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

Let's consider complex exponential input to the system $x(t) = e^{st}$

Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{st}e^{-s\tau}\, d\tau \quad = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\, d\tau$$

$$y(t) = e^{st}H(s)$$

(response in the form of e^{st})

Eigen function

Eigen value

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

Shows that **complex exponentials** are **eigenfunctions** of LTI systems

The input-output relation of a continuous-time LTI system is given by

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} x(\tau) d\tau$$

(a) Evaluate the impulse response $h(t)$ of the system
(b) Show that complex exponential function e^{st} is an eigen function of the system
(c) Evaluate the eigen function of the system for e^{st} using the impulse response obtained in (a)

Impulse-response nothing but the output of the system for input $x(t) = \delta(t)$

(a)

$$x(t) = \delta(t) \rightarrow y(t) = h(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} \delta(t)\, d\tau \;\; = \; e^{-(t-\tau)}|_{\tau=0}$$

$$= e^{-t}, \qquad (say\ for\ t > 0)$$

$$h(t) = e^{-t} u(t)$$

(b) Let $x(t) = e^{st}$

Using convolution integral, we know $y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau \; = \int_{-\infty}^{\infty} e^{-\tau}\, u(\tau)\, e^{s(t-\tau)}\, d\tau$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Decomposing of signals in terms of eigenfunctions

Let's $x(t)$ is a linear combination of three exponential signals

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$
$$= x_1(t) + x_2(t) + x_3(t)$$
$$= \sum_{k=1}^{3} a_k x_k(t)\, e^{s_k t}$$

If $x(t)$ is applied to a LTI system, the response of the system ?

Using the eigen function concept of continuous LTI system, we can write

$$x_1(t) = a_1 e^{s_1 t} \rightarrow y_1(t) = e^{s_1 t} a_1 H(s_1)$$

$$x_2(t) = a_2 e^{s_2 t} \rightarrow y_2(t) = e^{s_2 t} a_2 H(s_2)$$

$$x_3(t) = a_3 e^{s_3 t} \rightarrow y_3(t) = e^{s_3 t} a_3 H(s_3)$$

Using the superposition property

$$x(t) \rightarrow y(t) = y_1(t) + y_2(t) + y_3(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t} = \sum_{k=1}^{3} a_k H(s_k)\, e^{s_k t}$$

Observation:

If the **input** to a continuous-time LTI system is a linear combination of complex exponentials i.e.

$$x(t) = \sum_k a_k x_k(t)\, e^{s_k t}$$

$$\Downarrow$$

Output:

$$y(t) = \sum_k a_k H(s_k)\, e^{s_k t}$$

(a linear combination of the input complex exponential signals)

Harmonically Related Complex Exponentials

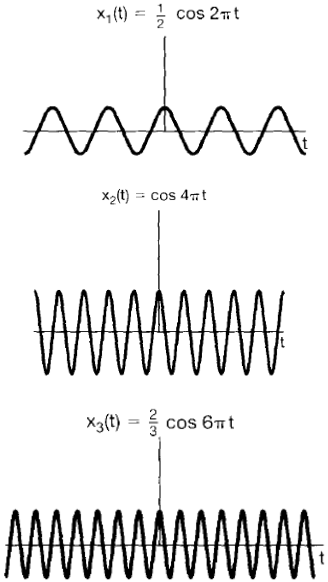
Let's consider a signal,

$$\emptyset_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$k = 0 \rightarrow (t) = e^{j.0.\omega_0 t}$$

$$k = 1 \rightarrow (t) = e^{j.1.\omega_0 t}$$

$$k = 2 \rightarrow (t) = e^{j.2.\omega_0 t}$$



Each of these signals has a fundamental frequency that is a **multiple of ω_0** → **called harmonically related**

$|k| \geq 2,$ the fundamental period of $\emptyset_k(t)$ is fraction of T

The fundamental period of the signal $\emptyset_k(t)$ *is* T

Fourier-series representation

- A linear combination of harmonically related **complex exponentials can be written as,**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

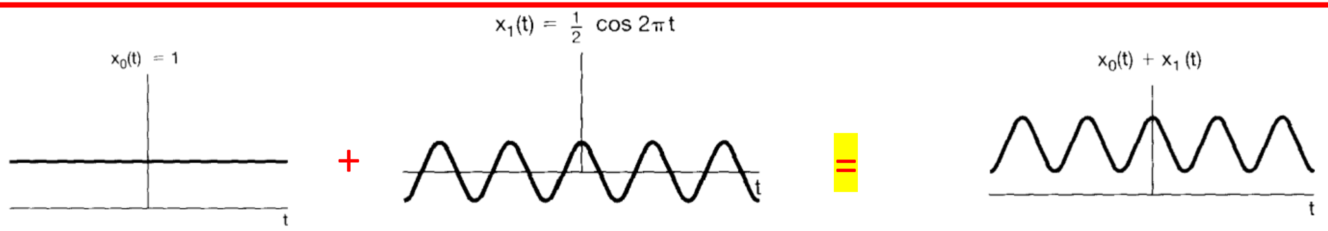
, ω_0 is the fundamental frequency and $T = \frac{2\pi}{\omega_0}$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

(called synthesis equation)

$$= \ldots + a_{-1}e^{j.-1.\left(\frac{2\pi}{T}\right)t} + a_0 e^{j.0.\left(\frac{2\pi}{T}\right)t} + a_1 e^{j.1.\left(\frac{2\pi}{T}\right)t} + a_2 e^{j.2.\left(\frac{2\pi}{T}\right)t} + \ldots\ldots$$

$\boldsymbol{a_k}$ are called the *Fourier series coefficients*



Fourier-series (analysis equation)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

(synthesis equation)

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(\frac{2\pi}{T})t}dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk.2\pi f_0 t}dt$$

(Analysis equation)

Derivation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.\, e^{-jn\omega_0 t}$$

(multiplying both side $e^{-jn\omega_0 t}$)

$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.\, e^{-jn\omega_0 t}\, dt$$

(Integrating both side from 0 $to\ T$)

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

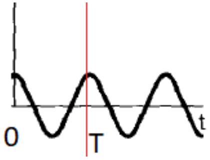
$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_0^T \cos(k-n)\omega_0 t\ dt + j \int_0^T \sin(k-n)\omega_0 t\ dt \right)$$

For $k = n$ $\left(\int_0^T \cos(k-n)\omega_0 t\ dt + j \int_0^T \sin(k-n)\omega_0 t\ dt \right) = T$

For $k \neq n$ $\left(\int_0^T \cos(k-n)\omega_0 t\ dt + j \int_0^T \sin(k-n)\omega_0 t\ dt \right) = 0$

(integration -> area under function)



In one period,
$+\ ve$ and $-\ ve$ value ,
area will zero

$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt = a_n T$$

$$a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

(**Analysis equation**)

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

The input-output relation of a continuous-time LTI system is given by

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} x(\tau) d\tau$$

(a) Evaluate the impulse response $h(t)$ of the system

(b) Show that complex exponential function e^{st} is an eigen function of the system

(c) Evaluate the eigen value of the system for e^{st} using the impulse response obtained in (a)

Impulse-response nothing but the output of the system for input $x(t) = \delta(t)$

(a)

$$x(t) = \delta(t) \rightarrow y(t) = h(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} \delta(t) \, d\tau \;\; = \;\; e^{-(t-\tau)}|_{\tau=0}$$

$$= e^{-t}, \qquad (say \; for \; t > 0)$$

$$h(t) = e^{-t} u(t)$$

(b) Let $x(t) \;=\; e^{st}$

Using convolution integral, we know $y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \, d\tau \;\; = \int_{-\infty}^{\infty} e^{-\tau} \, u(\tau) \, e^{s(t-\tau)} \, d\tau$

$$y(t) = \int_0^{\infty} e^{-\tau}\, e^{st} e^{-s\tau}\, d\tau$$

$$= e^{st} \int_0^{\infty} e^{-(1+s)\tau}\, d\tau$$

$$= e^{st} . \frac{e^{-(1+s)\tau}}{-(1+s)} \Big|_0^{\tau=\infty}$$

$$= \frac{e^{st}}{-(1+s)} [e^{-(1+s).\infty} - e^{-(1+s).0}]$$

$$= \frac{e^{st}}{-(1+s)}. [0 - 1]$$

$$= e^{st}. \underbrace{\left(\frac{1}{1+s}\right)}_{\text{Eigen-value}}$$

(c) Using the relation

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$Since, \quad h(t) = e^{-t} u(t)$$

$$= \int_{-\infty}^{\infty} e^{-\tau} u(\tau).\, e^{-s\tau} d\tau$$

$$= \int_0^{\infty} e^{-\tau}. e^{-s\tau}\, d\tau$$

$$= \int_0^{\infty} e^{-(1+s)\tau}\, d\tau$$

$$= \frac{1}{1+s}$$

Decomposing of signals in terms of eigenfunctions

Let's $x(t)$ is a linear combination of three exponential signals

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$
$$= x_1(t) + x_2(t) + x_3(t)$$
$$= \sum_{k=1}^{3} a_k\, e^{s_k t}$$

If $x(t)$ is applied to a LTI system, the response of the system ?

Using the eigen function concept of continuous LTI system, we can write

$$x_1(t) = a_1 e^{s_1 t} \rightarrow y_1(t) = e^{s_1 t}\, a_1\, H(s_1)$$

$$x_2(t) = a_2 e^{s_2 t} \rightarrow y_2(t) = e^{s_2 t}\, a_2\, H(s_2)$$

$$x_3(t) = a_3 e^{s_3 t} \rightarrow y_3(t) = e^{s_3 t}\, a_3\, H(s_3)$$

Using the superposition property

$$x(t) \rightarrow y(t) = y_1(t) + y_2(t) + y_3(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t} = \sum_{k=1}^{3} a_k H(s_k)\, e^{s_k t}$$

Observation:

If the **input** to a continuous-time LTI system is a linear combination of complex exponentials i.e.

$$x(t) = \sum_k a_k \, e^{s_k t}$$

$$\Downarrow$$

Output:

$$y(t) = \sum_k a_k H(s_k) \, e^{s_k t}$$

(a linear combination of the input complex exponential signals)

Harmonically Related Complex Exponentials

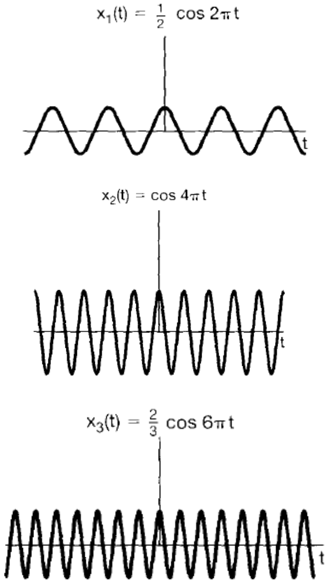
Let's consider a signal,

$$\emptyset_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$k = 0 \rightarrow \emptyset_0(t) = e^{j.0.\omega_0 t}$$

$$k = 1 \rightarrow \emptyset_1(t) = e^{j.1.\omega_0 t}$$

$$k = 2 \rightarrow \emptyset_2(t) = e^{j.2.\omega_0 t}$$



Each of these signals has a fundamental frequency that is a **multiple of ω_0** → **called harmonically related**

$|k| \geq 2$, the fundamental period of $\emptyset_k(t)$ is fraction of $\boldsymbol{T}$

The fundamental period of the signal $\emptyset_k(t)$ *is* T

Fourier-series representation

- A **linear combination of harmonically** related **complex exponentials can be written as,**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

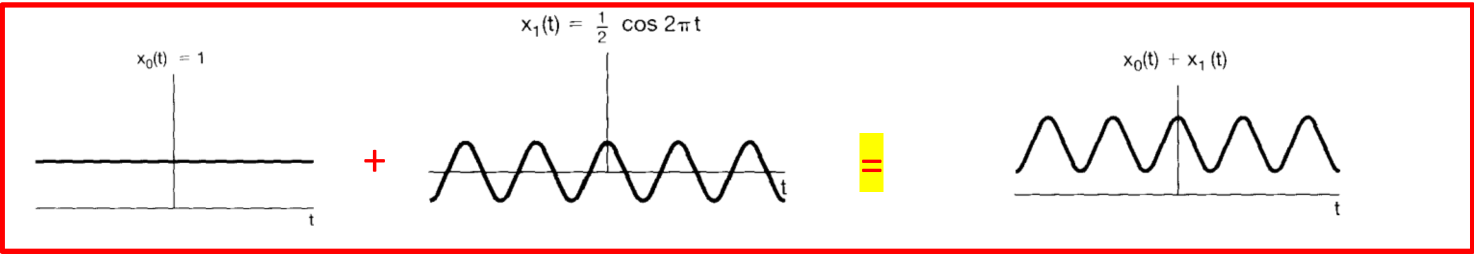
, ω_0 is the fundamental frequency and $T = \frac{2\pi}{\omega_0}$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

(called **synthesis equation**)

$$= \ \dots + a_{-1} e^{j.-1.\left(\frac{2\pi}{T}\right)t} + a_0 e^{j.0.\left(\frac{2\pi}{T}\right)t} + a_1 e^{j.1.\left(\frac{2\pi}{T}\right)t} + a_2 e^{j.2.\left(\frac{2\pi}{T}\right)t} + \dots\dots$$

$\boldsymbol{a_k}$ are called the *Fourier series coefficients*



- Determine the Fourier series coefficient $x(t) = \sin \omega_0 t$

From **Fourier-series synthesis equation**, we know

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \ldots + a_{-1} e^{j.-1.\omega_0 t} + a_0 e^{j.0.\omega_0 t} + a_1 e^{j\omega_0 t} + \ldots..$$

$$= \ldots + a_{-1} e^{-j.\omega_0 t} + a_0 e^{0} + a_1 e^{j\omega_0 t} + \ldots..$$

Expanding the given equation,

$$x(t) = \sin \omega_0 t$$

$$= \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$$

$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing two equations

$$a_{-1} = -\frac{1}{2j}, \quad a_1 = \frac{1}{2j}, \quad a_k = 0, \quad k \neq \pm 1$$

Determine the Fourier series coefficient

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$$

From **Fourier-series synthesis equation**, we know

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \ldots + a_{-1}e^{-j.\omega_0 t} + a_0 e^0 + a_1 e^{j\omega_0 t} + \ldots..$$

Comparing two equations

Expanding the given equation $x(t)$

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\ [e^{j(2\omega_0 t + \pi/4)} +\ e^{-j(2\omega_0 t + \frac{\pi}{4})}]$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1\ -\ \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{\frac{j\pi}{4}}\right)e^{j2\omega_0 t}\ + (\frac{1}{2}e^{\frac{j\pi}{4}})\ e^{j2\omega_0 t}$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$a_0 = 1,\ a_1 = \left(1 + \frac{1}{2j}\right), a_{-1} = \left(1 - \frac{1}{2j}\right), a_2 = \frac{1}{2}e^{j\pi/4}, a_{-2} = \frac{1}{2}\ e^{-j\pi/4},\ a_k = 0,\ \text{for } |k| > 2$$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

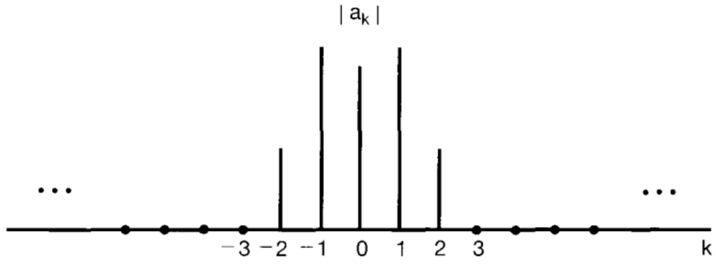
- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

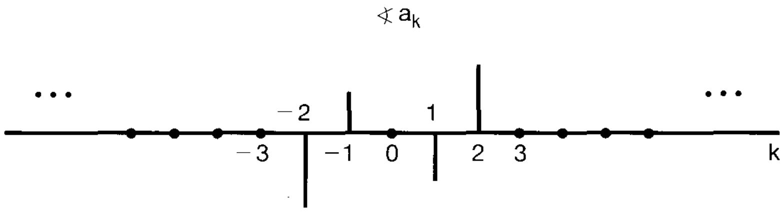
Cont..

- Magnitude and phase plot of a_k

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right),$$
$$a_2 = \frac{1}{2}\, e^{j\pi/4},$$
$$a_{-2} = \frac{1}{2}\, e^{-j\pi/4},$$

$$a_k = 0, \text{ for } |k| > 2$$





$$z = x + jy$$
$$Magnitude\ |z| = \sqrt{x^2 + y^2}$$
$$Phase\ \measuredangle z = \tan^{-1}\frac{y}{x}$$

Fourier-series (analysis equation)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right) t}$$ (synthesis equation)

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(\frac{2\pi}{T})t}dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk.2\pi f_0 t}dt$$ (Analysis equation)

Derivation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right) t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\rightarrow x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. e^{-jn\omega_0 t}$ (multiplying both side $e^{-jn\omega_0 t}$)

$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. e^{-jn\omega_0 t}\, dt$ (Integrating both side from 0 to T)

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

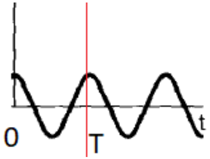
$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . e^{-jn\omega_0 t}\, dt \quad = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_0^T \cos(k-n)\omega_0 t\ dt + j \int_0^T \sin(k-n)\omega_0 t\ dt \right)$$

For $k = n$ $\quad \left(\int_0^T \cos(k-n)\omega_0 t\ dt + j \int_0^T \sin(k-n)\omega_0 t\ dt \right) = T$

For $k \neq n$ $\quad \left(\int_0^T \cos(k-n)\omega_0 t\ dt + j \int_0^T \sin(k-n)\omega_0 t\ dt \right) = 0$

(integration -> area under function)



In one period,
$+ve$ and $-ve$ value ,
area will zero

$$\rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt = a_n T$$

$$a_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

(**Analysis equation**)

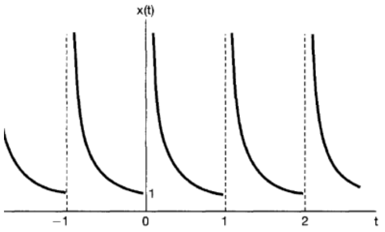
Fourier -series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

(synthesis equation)

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

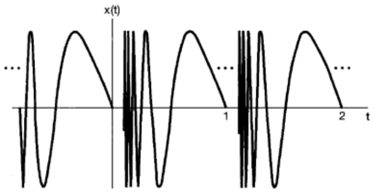
(Analysis equation)



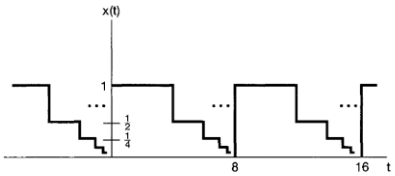
Convergence of Fourier Series (Dirichlet condition):

1. $x(t)$ is absolutely integrable over any period, that is,

$$\int_{T_0} |x(t)|\, dt < \infty$$



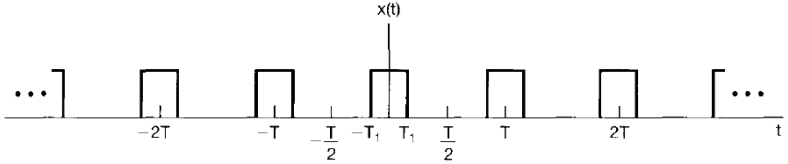
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.



- The periodic square wave defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Determine the Fourier series coefficients for $x(t)$.



Fundamental period $\boldsymbol{T}$,

Fundamental frequency $\omega_0 = 2\pi F_0 = \frac{2\pi}{T}$

We have to calculate,

$$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

n=0

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1.\, e^{-j.(n=0).\omega_0 t}\, dt = \frac{2T_1}{T}$$

n ≠ 0

$$a_n = \frac{1}{T}\int_{-T_1}^{T_1} 1.\, e^{-j.n.\omega_0 t}\, \mathrm{dt}$$

$$= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T_1}^{T_1}$$

n ≠ 0

$$a_n = \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{-jn\omega_0 T}\left[e^{-jn\omega_0 T_1} - e^{jn\omega_0 T_1}\right]$$

$$= \frac{2}{n\omega_0 T}\left[\frac{e^{jn\omega_0 T_1} - e^{-jn\omega_0 T_1}}{2j}\right]$$

$$= \frac{2}{n\omega_0 T}\sin n\omega_0 T_1$$

$$= \frac{2}{n2\pi}\sin n\omega_0 T_1$$

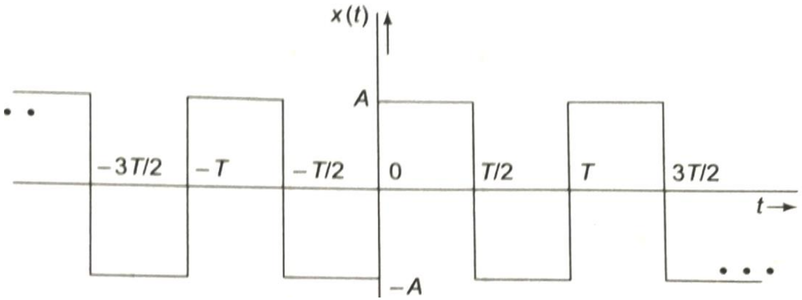
$$= \frac{\sin n\omega_0 T_1}{n\pi}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$as\ \omega_0 = 2\pi F_0 = \frac{2\pi}{T}$$

Determine the Fourier series coefficients



Fundamental period $\boldsymbol{T}$,

Fundamental frequency $\omega_0 = 2\pi F_0 = \frac{2\pi}{T}$

We have to calculate, $$a_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

n=0 $$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t).\, e^{-j.(n=0).\omega_0 t}\, dt \quad = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} -A.\,1dt + \int_0^{\frac{T}{2}} A.\,1dt\right] = 0$$

n ≠ 0

$$a_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}\,dt = \frac{1}{T}\left[\int_{-\frac{T}{2}}^{0} -A.\,e^{-jn\omega_0 t}\,dt + \int_{0}^{\frac{T}{2}} A.\,e^{-jn\omega_0 t}\,dt\right]$$

$$= \frac{-A}{T}\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_{\frac{-T}{2}}^{0} + \frac{A}{T}\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_{0}^{T/2}$$

$$as\ \omega_0 = 2\pi F_0 = \frac{2\pi}{T}$$

$$= \frac{A}{jn2\pi F_0.T}\left[1 - e^{j\pi n}\right] + \frac{A}{-jn2\pi F_0 T}\left[1 - e^{-j\pi n}\right]$$

$$= \frac{A}{j2\pi n}\left[2 - \left(e^{j\pi n} + e^{-j\pi n}\right)\right]$$

$$= \frac{A}{j\pi n}\left[1 - \cos\pi\, n\right]$$

Properties

Linearity: Let $x(t)$ and $y(t)$ denote two periodic signals with *period* T and which have *Fourier-series coefficients*

$$a_k, b_k$$

$$x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\alpha\, x(t) + \beta\, y(t) \xrightarrow{FS} \alpha\, a_k + \beta\, b_k$$

Proof:

$$\alpha\, a_k + \beta\, b_k = \frac{1}{T}\int_T \alpha\, x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_T \beta\, y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T [\alpha\, x(t) + \beta\, y(t)]\, e^{-jk\omega_0 t}\, dt \xrightarrow{FS} \alpha\, a_k + \beta\, b_k$$

Cont..

Time-shift:

If $\quad x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$ Then, $\quad x(t-t_0) \xrightarrow{FS} a_k e^{-jk\omega_0 t_0}$

Let, $y(t) = x(t-t_0)$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

$$t - t_0 = \tau; dt = d\tau; t = \tau + t_0$$

$$= \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau+t_0)}\, d\tau \quad = e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau}\, d\tau \quad = a_k e^{-jk\omega_0 t_0}$$

Cont..

Frequency shift:

$$If \qquad x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \qquad \text{Then,} \qquad e^{jL\omega_0 t}\, x(t) \xrightarrow{FS} a_{k-L}$$

Proof: Let, $y(t) = e^{jL\omega_0 t} x(t)$

$$y(t) \overset{FS}{\rightarrow} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T e^{jL\omega_0 t}\, x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\, e^{-j(k-L)\omega_0 t}\, dt \xrightarrow{FS} a_{k-L}$$

Cont..

Convolution:

If

$$x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

Then, $z(t) = x(t) * y(t)$

$$z(t) \xrightarrow{FS} a_k b_k T$$

$x(t)$ and $y(t)$ denote two periodic signals with *period* T

Proof:

$$z(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt \quad = \frac{1}{T}\int_T \left[\int x(\tau) y(t-\tau)\, d\tau\right] e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int x(\tau) \left[\int_T y(t-\tau) e^{-jk\omega_0 t}\, dt\right] d\tau$$

$$\left(z(t) = x(t) * y(t) = \int x(\tau)\, y(t-\tau)\, d\tau \right)$$

$$t - \tau = L; dt = dL; t = \tau + L$$

$$= \frac{1}{T}\int x(\tau) \left[\int_T y(L) e^{-jk\omega_0(\tau+L)}\, dL\right] d\tau = \frac{1}{T}\int x(\tau) \left[e^{-jk\omega_0 \tau} \int_T y(L) e^{-jk\omega_0 L}\, dL\right] d\tau = T\, a_k b_k$$

Multiplication:

If

$$x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$y(t) \xrightarrow{FS} b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$x(t)$ and $y(t)$ denote two periodic signals with *period T*

Then, $z(t) = x(t)\, y(t)$

$$z(t) \xrightarrow{FS} c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l\, b_{k-l}$$

(Proof yourself)

Differentiation:

If $x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

Then

$$y(t) = \frac{d}{dt}x(t) \xrightarrow{FS} j\,k\,\omega_0 a_k = 2\pi k F_0 a_k$$

Proof:

As per synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right]$$

$$= \sum_{k=-\infty}^{\infty} a_k . jk\omega_0 . e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (a_k . jk\omega_0) . e^{jk\omega_0 t}$$

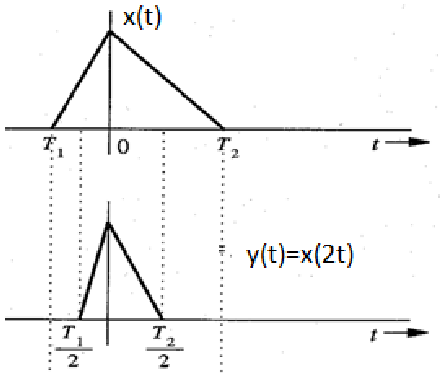
$$= \sum_{k=-\infty}^{\infty} b_k\, e^{jk\omega_0 t}$$

Comparing equation

$$y(t) = \frac{d}{dt}x(t) \xrightarrow{FS} j\,k\,\omega_0 a_k = 2\pi k F_0 a_k$$

Scaling:

If $\quad x(t) \xrightarrow{FS} a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$ **Then** $\quad y(t) = x(at) \xrightarrow{FS} ?$



If $x(t)$ has period T, then the period of $x(at)$ =? $\quad T_1 = \frac{T}{a}$

$$F_1 = \frac{1}{T_1} = \frac{1}{T/a} = \frac{a}{T} = aF_0$$

$$\omega_1 = 2\pi F_1 = 2\pi\, aF_0 = a\omega_0$$

$$y(t) = x(at) \xrightarrow{FS} \frac{1}{T_1}\int_{T_1} x(at)\, e^{-jk\omega_1 t}\, dt$$

$$= \frac{1}{T/a}\int x(\tau)\, e^{-jka.\omega_0 \frac{\tau}{a}} . \frac{1}{a}\, d\tau = \frac{1}{T}\int x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau = a_k$$

Let, $at = \tau,\ \ dt = \frac{1}{a} d\tau,$ t= τ/a

Fourier coefficients **does not changed**

Time Reversal:

If $x(t) \xrightarrow{FS} a_k$ Then $x(-t) \xrightarrow{FS} a_{-k}$

As per synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 -t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j(-k)\omega_0 t}$$

Replacing $-k$ by m

$$= \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

$$x(-t) \xrightarrow{FS} a_{-k}$$

Conjugate:

As per synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$ (using time-reversal property)

If $x(t)$ is **real valued** → $\overset{*}{x}(t) = x(t)$

$$\rightarrow \overset{*}{x}(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\rightarrow a_{-k}^* = a_k$

Try yourself for if $x(t)$ is pure imaginary

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

- A periodic signal $x(t)$ with fundamental period T_0 has complex-exponential Fourier- Series coefficients a_k. Express the following signal in terms of a_k.

(a) $x(t-t_0)$

(b) $\dfrac{dx(t)}{dt}$

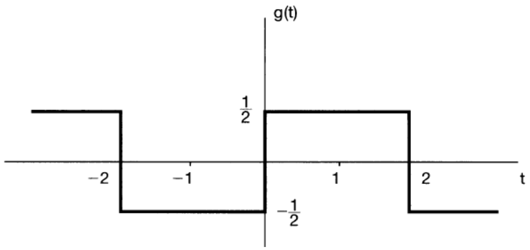
(a) As per time-shift property, we know $if\ x(t) \xrightarrow{FS} a_k$ $then,\ x(t-t_0) \xrightarrow{FS} b_k = a_k e^{-jk\omega_0 t_0}$

$$y(t) = x(t-t_0) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k . e^{-jk\omega_0 t_0} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k . e^{jk\omega_0 (t-t_0)}$$

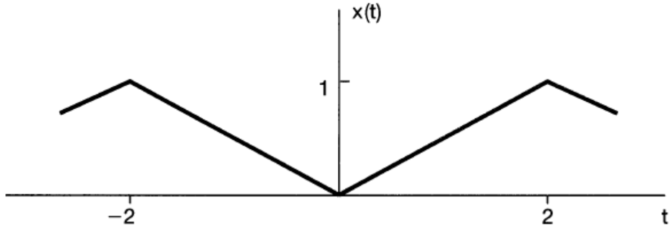
(b) Let, $y(t) = \dfrac{dx(t)}{dt}$ As per derivative property, we know $if\ x(t) \xrightarrow{FS} a_k$

$then, y(t) = \dfrac{d}{dt} x(t) \xrightarrow{FS} b_k = j\,k\,\omega_0 a_k = j2\pi k F_0 a_k$

So, $y(t) = \dfrac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (jk\omega_0 a_k) e^{jk\omega_0 t}$



$$d_k = \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, \quad \text{for } k \neq 0$$



- The triangular wave signal $x(t) with\ period\ T = 4$, Evaluate the F.S. Coefficient of above signal?

Using derivative property, $y(t) = \frac{d}{dt} x(t) \xrightarrow{FS} b_k\,(of\ derivative) = j\,k\,\omega_0 a_k$ (without derivative)

$$d_k = jk\omega_0 a_k$$

$$\rightarrow a_k = \frac{d_k}{jk\omega_0} = \frac{d_k}{jk\frac{2\pi}{T}} = \frac{2d_k}{jk\pi} \ ; k \neq 0$$

$$\rightarrow a_k = \frac{2\sin(\frac{\pi k}{2})}{j(k\pi)^2} e^{-j\pi/2}$$

$$a_0 = \frac{1}{T}\int_T x(t) e^{-jn=0.\omega_0}\, dt$$

$$= 1/2$$

A periodic signal $x(t)$ with fundamental period T_0 has complex-exponential Fourier- Series coefficients a_k. Express the following signal in terms of a_k.

(a) $x^*(t)$

By the definition of Fourier-series for $given\ x(t)$ and a_k, we can write

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Let $y(t) = x^*(t)$

Now

$$y(t) = x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

(using conjugate property)

Parseval's theorem (continuous-time periodic signals)

- The average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$ is

$$P = \frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof: $|x(t)|^2 = x(t)\,x^*(t) \quad = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t}\right)$

$$P = \frac{1}{T}\int_T |x(t)|^2\,dt = \frac{1}{T}\int_T \left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t}\right)\right\} dt$$

$$\int_T e^{j(k-L)\omega_0 t}\,dt = T, \qquad if\ k = L$$
$$= 0, \qquad k \neq L$$

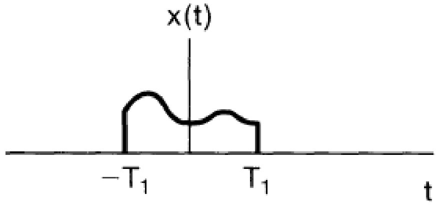
$$= \frac{1}{T}\left[\sum_{k=-\infty}^{\infty} a_k \sum_{L=-\infty}^{\infty} a_L^* \left\{\int_T e^{j(k-L)\omega_0 t}\,dt\right\}\right] = \frac{1}{T}.T\sum_{k=-\infty}^{\infty}|a_k|^2 \quad = \sum_{k=-\infty}^{\infty}|a_k|^2$$

Fourier Transform – (continuous signal)

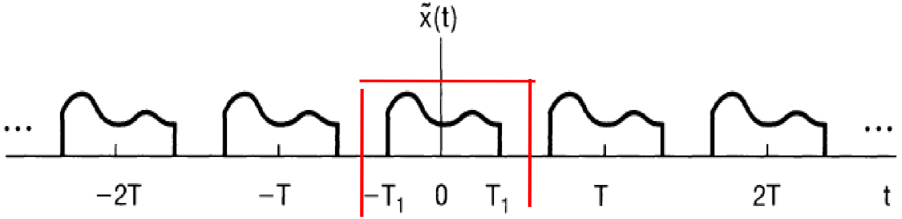
Fourier series of periodic continuous-time signal (with period T):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

Consider a aperiodic signal:



we can **construct a periodic signal** for which $x(t)$ *is one period*



$-\frac{T}{2}$ $\frac{T}{2}$

Observations:

1. **If** $T \to \infty$ $x(t) = \tilde{x}(t)$ i.e. $x(t)$ repeat itself in infinite

2. $x(t) = \tilde{x}(t)$, $|t| < \frac{T}{2}$ *since* $x(t) = 0$ *outside this* interval

Cont..

As $\tilde{x}(t)$ is a periodic signal with $T \rightarrow \infty$, so using the concept of Fourier series, we can write

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow a_k = \frac{1}{T}\int_T \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

Using the observation 2. $x(t) = \tilde{x}(t)$, $\quad |t| < \frac{T}{2}$ $\quad$ *since* $x(t) = 0$ *outside this* interval

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} X(\omega_0 k)$$

Where, $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

Fourier-transform:
Forward Fourier Transform
Analysis Equation

$$a_k = \frac{1}{T} X(\omega_0 k)$$

(**Relation** between Fourier series and X($\boldsymbol{\omega}$))

Cont..

$a_k = \frac{1}{T} X(\omega_0 k)$ i.e. we can get Fourier-series (FS) from Fourier-transform (FT) $a_k = \frac{1}{T} X(\omega)|_{\omega = k\omega_0}$

$x(t) \xrightarrow{FT} \mathrm{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ **(Fourier-transform)**

$FT[x(t)] = \mathrm{X}(\omega)$

- Now, we would like to *derive* $\boldsymbol{x(t)} \leftarrow \boldsymbol{X(\omega)}$

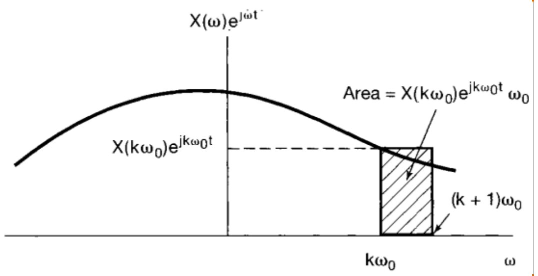
Now consider again

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega_0 k) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(\omega_0 k) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$

(replacing a_k by $\frac{1}{T} X(\omega_0 k)$) (replacing $\mathrm{T} = \frac{2\pi}{\omega_0}$)

Observation 1.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$



$T \to \infty$ $\quad x(t) = \tilde{x}(t)$

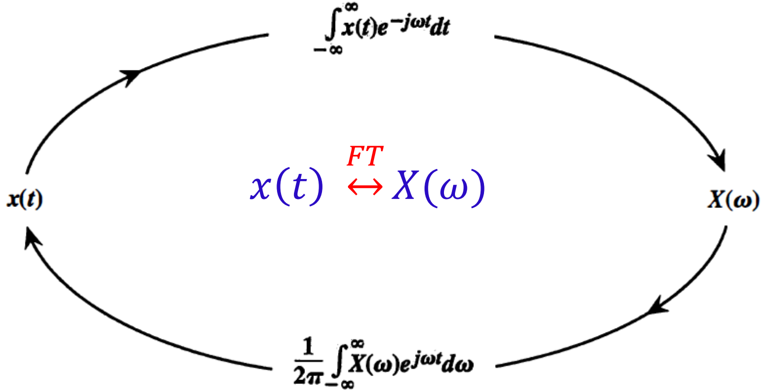
i.e. $x(t)$ repeat itself in infinite

(passes to integral)
as $T \to \infty$, $\omega_0 \to 0$

$$x(t) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$x(t) \quad = Inverse\ FT\ [X(\omega)] = FT^{-1}[X(\omega)]$$

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$



Properties of Fourier Transform

- **Linearity**

If

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$y(t) \xleftrightarrow{FT} Y(\omega)$$

Then

$$ax(t) + by(t) \xleftrightarrow{FT} aX(\omega) + bY(\omega)$$

- **Time shifting**

If

$$x(t) \xleftrightarrow{FT} X(\omega)$$

Then

$$x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0}\, X(\omega)$$

$$FT[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)e^{-j\omega t} dt \quad = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}.e^{-j\omega t_0} dt$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0}\ X(\omega)$$

- **Frequency shift**

If $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(t)\, e^{j\omega_c t} \overset{FT}{\leftrightarrow} X(\omega - \omega_c)$

$$x(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] \overset{FT}{\leftrightarrow} X(\omega - \omega_c) + X(\omega + \omega_c)$$

- **Time scaling** **If** $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(at) \xleftrightarrow{FT} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

$$\mathfrak{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt.$$

Using the substitution $\tau = at$, we obtain

$$\mathfrak{F}\{x(at)\} = \begin{cases} \dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ -\dfrac{1}{a}\displaystyle\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases},$$

Differentiation (time domain)

If $x(t) \overset{FT}{\leftrightarrow} X(\omega)$ **then**

We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$ $\quad$ $i.e.\ \ x(t) = Inverse\ FT\ [X(\omega)] = FT^{-1}[X(\omega)]$

$$\rightarrow \frac{d}{dt}x(t) = \frac{d}{dt}[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega\ X(\omega)]\ e^{j\omega t} d\omega \ = FT^{-1}[j\omega\ X(\omega)]$$

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(\omega)}$$

- **Differentiation (frequency domain)**

We know, $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \ = \int_{-\infty}^{\infty} x(t).-jt.e^{-j\omega t}\ dt \ = -jt.FT\{x(t)\}$$

$$\boxed{-jt\ x(t) \overset{FT}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}}$$

▪ Convolution (time)

If $x(t) \overset{FT}{\longleftrightarrow} X(\omega)$

$h(t) \overset{FT}{\longleftrightarrow} H(\omega)$

Then $y(t) = x(t) * h(t) \overset{FT}{\longleftrightarrow} Y(\omega) = ?$

From convolution theorem, we can write $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$Y(\omega) = FT[y(t)] = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \quad = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, FT[h(t-\tau)]\, d\tau$$

$$= H(\omega) \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = H(\omega)X(\omega)$$

$$y(t) = x(t) * h(t) \overset{FT}{\longleftrightarrow} X(\omega)H(\omega)$$

- Conjugate property

If $x(t) \xleftrightarrow{FT} X(\omega)$ **then** $x^*(t) \xleftrightarrow{FT} X^*(-\omega)$

If x(t) is real $X^*(\omega) = X(-\omega)$

Example: $x(t) = e^{-at}u(t)$

$$X(\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{(a + j\omega)(a - j\omega)} = \frac{a}{a^2 + \omega^2} + j\frac{-\omega}{a^2 + \omega^2}$$

$$x(t) \xleftrightarrow{FT} X(\omega) = \frac{1}{a + j\omega}$$

$$X(-\omega) = \frac{1}{a + j.-\omega} = \frac{a + j\omega}{(a + j\omega)(a - j\omega)} = \frac{a}{a^2 + \omega^2} + j\frac{\omega}{a^2 + \omega^2}$$

$$X(-\omega) = \frac{1}{a - j\omega} = X^*(\omega)$$

$$= \mathrm{Re}\{X(\omega)\} - \mathrm{Im}\{X(\omega)\}$$

If x(t) real: $\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$

$$x(t) = x_e(t) + x_o(t).$$

$$\mathfrak{F}\{x(t)\} = \mathfrak{F}\{x_e(t)\} + \mathfrak{F}\{x_o(t)\},$$

$$x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(\omega),$$

$$\mathcal{E}v\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} \Re e\{X(\omega)\},$$

$$\mathcal{O}d\{x(t)\} \overset{\mathfrak{F}}{\longleftrightarrow} j\Im m\{X(\omega)\}$$

Given, $e^{-at}u(t) \overset{FT}{\longleftrightarrow} \dfrac{1}{a+j\omega}$

$$x(t) = e^{-a|t|}$$

$$= e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$

$$= 2\;Even\;\{e^{-at}u(t)\}$$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Parseval's theorem (continuous-time periodic signals)

- The average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$ is

$$P = \frac{1}{T}\int_{T} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof: $|x(t)|^2 = x(t)\, x^*(t) \quad = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t} \right)$

$$P = \frac{1}{T}\int_{T} |x(t)|^2\, dt = \frac{1}{T}\int_{T} \left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \left(\sum_{L=-\infty}^{\infty} a_L^* e^{-jL\omega_0 t} \right) \right\} dt$$

$$\int_T e^{j(k-L)\omega_0 t}\, dt = T, \qquad if\ k = L$$
$$= 0, \qquad k \neq L$$

$$= \frac{1}{T}\left[\sum_{k=-\infty}^{\infty} a_k \sum_{L=-\infty}^{\infty} a_L^* \left\{ \int_T e^{j(k-L)\omega_0 t}\, dt \right\} \right] = \frac{1}{T}.T \sum_{k=-\infty}^{\infty} |a_k|^2 \quad = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Evaluate the complex-exponential Fourier-series expansion of the signal

$$x(t) = 2 + 3\cos 2\pi t + 4\sin 3\pi t$$

And then verify the Parseval's theorem.

By the definition of synthesis equation, we can write $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

ω_0 is $unknown$ i.e. we have to $first$ $determine$ the T

$$2 \rightarrow periodic \text{ any value of } T$$

$$\cos 2\pi t \rightarrow T_1 = 1$$

$$sin\, 3\pi t \rightarrow T_2 = \frac{2}{3}$$

$$T = \text{Least} - \text{common multiplier}\left(1, \frac{2}{3}\right) = 2$$

$$\omega_0 = 2\pi F_0 = 2\pi.\frac{1}{2} = \pi$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk.\pi.t} = \ldots + a_{-2}e^{-2j\pi t} + a_{-1}e^{-j\pi t} + a_0 + a_1 e^{j\pi t} + a_2 e^{2j\pi t} + \ldots.$$

Using Euler relation, we can expand the following equation $x(t) = 2 + 3\cos 2\pi t + 4\sin 3\pi t$

$$x(t) = 2 + 3.\frac{1}{2}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + 4.\frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$$= 2 + \frac{3}{2}e^{-j2\pi t} + \frac{3}{2}e^{-j2\pi t} - \frac{4}{2j}e^{-j3\pi t} + \frac{4}{2j}e^{j3\pi t}$$

Comparing two equations

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \ldots + a_{-2}e^{-2j\pi t} + a_{-1}e^{-j\pi t} + a_0 + a_1 e^{j\pi t} + a_2 e^{2j\pi t} + \ldots.$$

Fourier-series expansion

$$a_{-2} = \frac{3}{2} = a_2$$
$$a_{-1} = 0 = a_1$$
$$a_0 = 2$$
$$a_3 = \frac{4}{2j}$$
$$a_{-3} = -\frac{4}{2j}$$

To verify Parseval's theorem:

$x(t)$ has *period* 2

$$a_{-2} = \frac{3}{2} = a_2$$
$$a_{-1} = 0 = a_1$$
$$a_0 = 2$$

$$a_3 = \frac{4}{2j}$$
$$a_{-3} = -\frac{4}{2j}$$

As per definition of power of a signal, we can write:

$$P = \frac{1}{T}\int_0^T |x(t)|^2 dt \qquad = \frac{1}{T}\int_0^T |2 + 3\cos 2\,\pi t + 4\sin 3\,\pi t|^2\, dt \qquad =?$$

From Parseval's theorem:

$$\sum_{k=-\propto}^{\propto} |a_k|^2 = \sum_{k=-3}^{3} |a_k|^2 = |a_{-3}|^2 + |a_{-2}|^2 + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2$$

$$= 2^2 + \left(\frac{3}{2}\right)^2 + 0^2 + 2^2 + 0^2 + \left(\frac{3}{2}\right)^2 + 2^2$$

$$= \frac{33}{2}$$

$$= 16.2$$

PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration	$\int_{-\infty}^{t} x(t)\, dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\left\{\begin{array}{ll} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Trigonometric Fourier-series

$$x(t) = a_0 + \sum_{n=1}^{\propto} a_n \cos n\,\omega_0 t + \sum_{n=1}^{\propto} b_n \sin n\,\omega_0 t;\ \omega_0 = \frac{2\pi}{T};\ t_1 < t < t_1 + T$$

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos n\,\omega_0 t dt\ ;\ n = 1,2,3,\ldots$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin n\,\omega_0 t dt\ ;\ n = 1,2,3,\ldots$$

If x(t) is even

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos\omega_0 t dt \qquad b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin\omega_0 t dt = 0$$

(odd function)

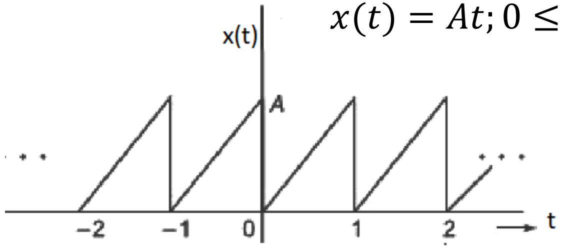
If x(t) is odd

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos\omega_0\, t dt = 0$$

(odd function)

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin\omega_0\, t dt$$

Find the trigonometric Fourier series coefficient:



$$x(t) = At; 0 \le t \le 1$$

$$T = 1$$

$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos n\,\omega_0 t dt\,; n = 1,2,3,\ldots$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin n\,\omega_0 t dt\,; n = 1,2,3,\ldots$$

$$a_0 = \frac{1}{1}\int_0^1 Atdt = \frac{A}{2}$$

$$a_n = \frac{2}{1}\int_0^1 At\cos n\,\omega_0 tdt = 2A\left[t\frac{\sin n\,\omega_0 t}{n\omega_0}\Big|_0^1 - \int_0^1 1.\frac{\sin n\,\omega_0 t}{n\omega_0}dt\right] \quad = 0$$

$$\int F(x)G'(x)\,dx = F(x)G(x) - \int F'(x)G(x)\,dx + C.$$

$$b_n = 2A\int_0^1 t\sin n\,\omega_0 tdt = 2A\left[\frac{t.-\cos n\,\omega_0 t}{n\omega_0}\Big|_0^1 - \int_0^1 1.-\frac{\cos n\,\omega_0 t}{n\omega_0}dt\right]$$

$$= -2.\frac{A}{n\omega_0}\cos 2\,\pi n + \frac{2A}{n\omega_0}.0 = -\frac{2A}{n.2\pi} = -\frac{A}{n\pi}$$

Fourier Series versus Fourier Transform

	Continuous time	Discrete time
Periodic	Fourier Series	Discrete Fourier Transform
Aperiodic	Continuous Fourier Transform	Discrete Fourier Transform

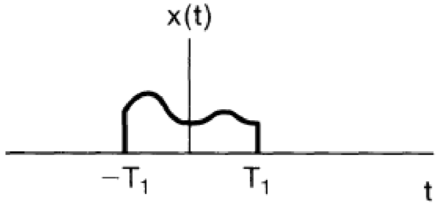
A **spectrum analyzer** measures the magnitude of an input signal versus frequency within the full frequency range of the instrument. It measures frequency, power, harmonics, distortion, noise, spurious signals and bandwidth.

Fourier Transform – (continuous signal)

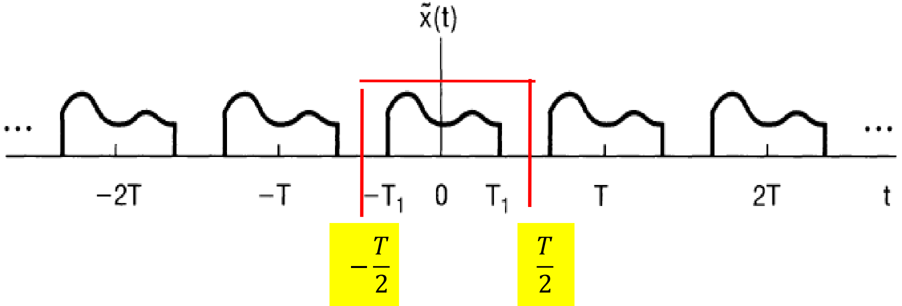
Fourier series of periodic continuous-time signal (with period T):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right) t}$$

Consider a aperiodic signal:



we can **construct a periodic signal** for which $x(t)$ *is one period*



Observations:

1. If $T \rightarrow \propto$ $x(t) = \tilde{x}(t)$ i.e. $x(t)$ repeat itself in infinite

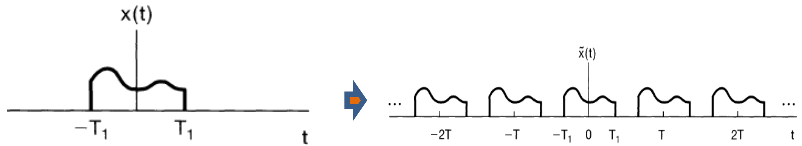
2. $x(t) = \begin{cases} \tilde{x}(t), & |t| < \frac{T}{2} \\ 0, else \end{cases}$ *since* $x(t) = 0$ *outside this* interval

Cont..

As $\tilde{x}(t)$ is a periodic signal with $T \rightarrow \infty$, so using the concept of Fourier series, we can write

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow a_k = \frac{1}{T}\int_T \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt \quad = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

Using the observation 2. $x(t) = \tilde{x}(t), \quad |t| < \frac{T}{2}$



since $x(t) = 0$ *outside this* interval

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} X(\omega_0 k)$$

Where, $\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt}$

Fourier-transform:
Forward Fourier Transform
Analysis Equation

$$\boxed{a_k = \frac{1}{T} X(\omega_0 k)}$$

(**Relation** between Fourier series and $X(\boldsymbol{\omega})$)

Cont..

$a_k = \frac{1}{T} X(\omega_0 k)$ i.e. we can get Fourier-series (FS) from Fourier-transform (FT) $a_k = \frac{1}{T} X(\omega)|_{\omega = k\omega_0}$

$x(t) \xrightarrow{FT} \mathrm{X}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ $FT[x(t)] = \mathrm{X}(\omega)$ **(Fourier-transform)**

- Now, we would like to derive $\boldsymbol{x(t)} \leftarrow \boldsymbol{X(\omega)}$

Now consider again

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega_0 k) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{\omega_0}{2\pi} X(\omega_0 k) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$

(replacing a_k by $\frac{1}{T} X(\omega_0 k)$) (replacing $\mathrm{T} = \frac{2\pi}{\omega_0}$)

Observation 1.

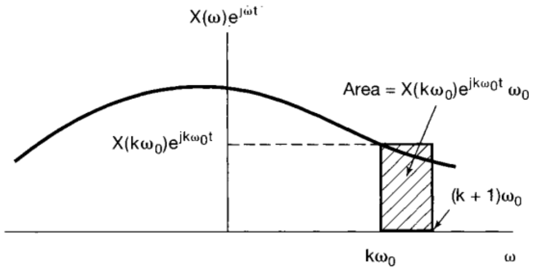
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega_0 k) e^{jk\omega_0 t} \omega_0$$

$T \rightarrow \propto$ $x(t) = \tilde{x}(t)$

i.e. $x(t)$ repeat itself in infinite

(passes to integral)

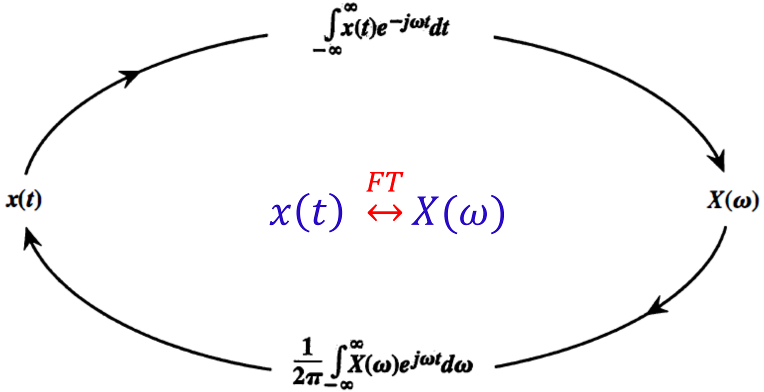
as $T \rightarrow \propto, \ \omega_0 \rightarrow 0$



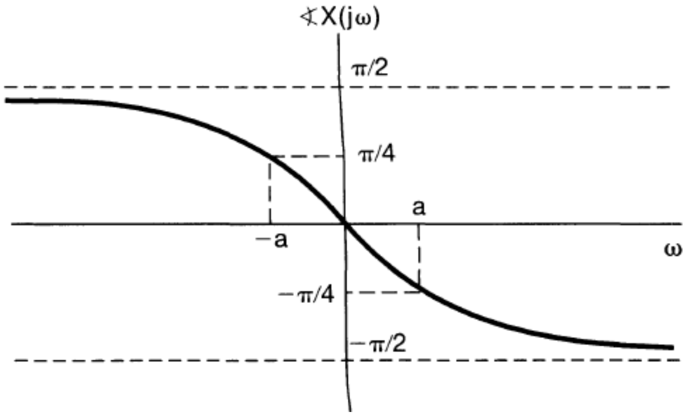
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$x(t) = Inverse\ FT\,[X(\omega)] = FT^{-1}[X(\omega)]$$

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

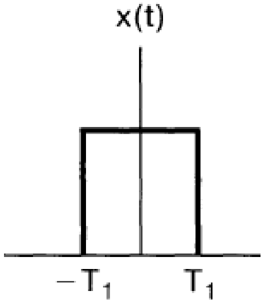


$$X(\omega) = \frac{1}{a + j\omega} \quad = \overbrace{\frac{a}{a^2+\omega^2}}^{x} + j\,\overbrace{\frac{-\omega}{a^2+\omega^2}}^{y}$$



Phase: $\angle X(\omega) = -\tan^{-1}\left(\frac{y}{x}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$

(b) $x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases}$

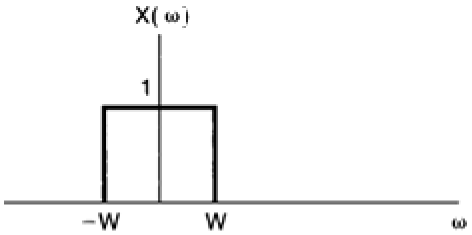


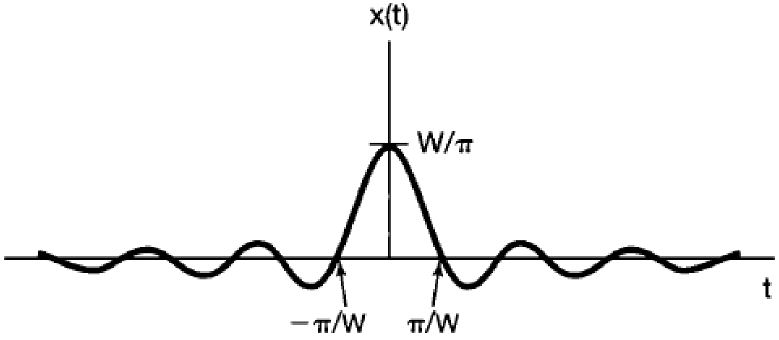
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \; = \int_{-T_1}^{T_1} 1.\, e^{-j\omega t}\; dt$$

$$= -\frac{1}{j\omega}\; e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{-j\omega.-T_1}\right]$$

$$= 2\frac{\sin\omega T_1}{\omega}$$





$$\frac{\sin w\, t}{\pi t} = \frac{w}{\pi}\frac{\sin(wt)}{wt} = \frac{w}{\pi}\sin c\,(wt)$$

$$\sin c\,(x) = 0; \quad x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots.$$

Properties of Fourier Transform

- **Linearity**

If
$$x(t) \xleftrightarrow{FT} X(\omega)$$
$$y(t) \xleftrightarrow{FT} Y(\omega)$$

Then
$$ax(t) + by(t) \xleftrightarrow{FT} aX(\omega) + bY(\omega)$$

- **Time shifting**

If
$$x(t) \xleftrightarrow{FT} X(\omega)$$

Then
$$x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0}\, X(\omega)$$

$$FT[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt \quad = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} . e^{-j\omega t_0} dt$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0}\ X(\omega)$$

- **Frequency shift**

If $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(t)\, e^{j\omega_c t} \overset{FT}{\leftrightarrow} X(\omega - \omega_c)$

$$x(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] \overset{FT}{\leftrightarrow} X(\omega - \omega_c) + X(\omega + \omega_c)$$

- **Time scaling** **If** $x(t) \xleftrightarrow{FT} X(\omega)$ **Then** $x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$$\mathfrak{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt.$$

Using the substitution $\tau = at$, we obtain

$$\mathfrak{F}\{x(at)\} = \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau, & a < 0 \end{cases},$$

(b) Scaling property: $x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$$\Rightarrow \quad x(-t) \xleftrightarrow{FT} \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right) = X(-\omega)$$ (for $a=-1$) [folding]

$$x(-t+1) \overset{FT}{\leftrightarrow} \int_{-\infty}^{\infty} x(-t+1)\, e^{-j\omega t} dt$$

(replacing $-t+1 = \tau,\ dt = -d\tau$)

$$\overset{FT}{\leftrightarrow} \int_{+\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau+1)} \,.\, -d\tau$$

$$\overset{FT}{\leftrightarrow}\ e^{-j\omega} \int_{+\infty}^{-\infty} x(\tau)\, e^{j\omega\tau} \,.\, -d\tau$$

$$\overset{FT}{\leftrightarrow}\ e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{-j.(-\omega).\tau}\, d\tau = e^{-j\omega} X(-\omega)$$

Using Time shift property (delay):

$$x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

$$x(-t+1) \overset{FT}{\leftrightarrow} e^{-j\omega.1} X(-\omega)$$

(c) Using Time shift property (delay): $x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0}\, X(\omega)$

$$y(t) = x(t-2) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega.2}\, X(\omega)$$ [shifting]

Using scaling property $x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ [Scaling]

$$y\left(\frac{t}{2}\right) = x\left(\frac{t}{2}-2\right) \overset{FT}{\leftrightarrow} \frac{1}{1/2}\, Y\left(\frac{\omega}{\frac{1}{2}}\right) = 2.Y(2\omega) = 2e^{-j.2\omega.2}X(\omega)$$

Differentiation (time domain)

If $x(t) \overset{FT}{\leftrightarrow} X(\omega)$ **then**

We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$ $\quad$ $i.e.\ \ x(t) = Inverse\ FT\ [X(\omega)] = FT^{-1}[X(\omega)]$

$$\rightarrow \frac{d}{dt}x(t) = \frac{d}{dt}[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[j\omega\, X(\omega)]\, e^{j\omega t} d\omega \quad = FT^{-1}[j\omega\, X(\omega)]$$

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(\omega)}$$

- ## Differentiation (frequency domain)

We know, $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \quad = \int_{-\infty}^{\infty} x(t).-jt.e^{-j\omega t}\ dt \quad = -jt.FT\{x(t)\}$$

$$\boxed{-jt\, x(t) \overset{FT}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}}$$

▪ Convolution (time)

If $x(t) \xleftrightarrow{FT} X(\omega)$

$h(t) \xleftrightarrow{FT} H(\omega)$

Then $y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(\omega) = ?$

From convolution theorem, we can write $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$Y(\omega) = FT[y(t)] = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \quad = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, FT[h(t-\tau)]\, d\tau$$

$$= H(\omega) \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau = H(\omega) X(\omega)$$

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} X(\omega) H(\omega)$$

Partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]\big|_{s=-1} = 1,$$
$$B = [(s+2)X(s)]\big|_{s=-2} = -1.$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

- **Multiplication (time)**

If

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$y(t) \xleftrightarrow{FT} Y(\omega)$$

Then $$z(t) = x(t)y(t) \xleftrightarrow{FT} Z(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau)H(\omega - \tau)d\tau$$

Example: $$x(t) = \frac{\sin(t)\sin(\frac{t}{2})}{\pi t^2}$$

$$= \pi \left(\frac{\sin t}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$= \pi \underbrace{\left(\frac{\sin t}{\pi t}\right)}_{signal-1}\underbrace{\left(\frac{\sin(t/2)}{\pi t}\right)}_{signal-2}$$

$$x(t) \xleftrightarrow{FT} X(\omega) = \frac{1}{2\pi}.\pi.(\text{signal-1}) * (\text{signal-2})$$

$$= \frac{1}{2} FT\left\{\frac{\sin t}{\pi t}\right\} * FT\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Properties of the Fourier Transform

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\pi X(0)\delta(\omega)+\frac{1}{j\omega}X(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Real signal	$x(t)=x_e(t)+x_o(t)$	$X(\omega)=A(\omega)+jB(\omega)$ $X(-\omega)=X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\}=A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\}=jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\,d\lambda=\int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\,d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\lvert t\rvert},\ a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2},\ a>0$	$\sqrt{\frac{\pi}{a}}\,e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2a\frac{\sin\omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$\mathrm{sgn}\,t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0),\ \omega_0=\frac{2\pi}{T}$

Duality

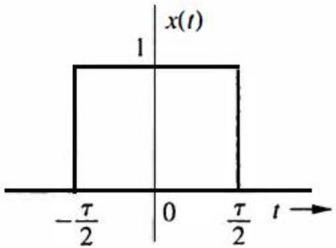
If

$$x(t) \xleftrightarrow{FT} X(\omega)$$

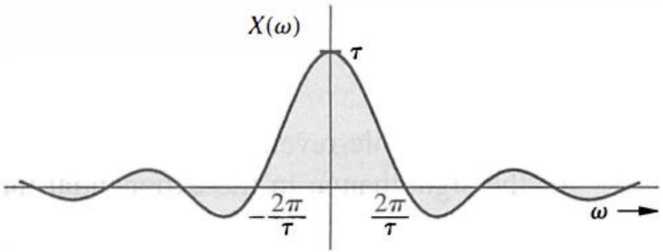
$$\boldsymbol{X(t)} \xleftrightarrow{FT} ? \qquad \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = ?$$

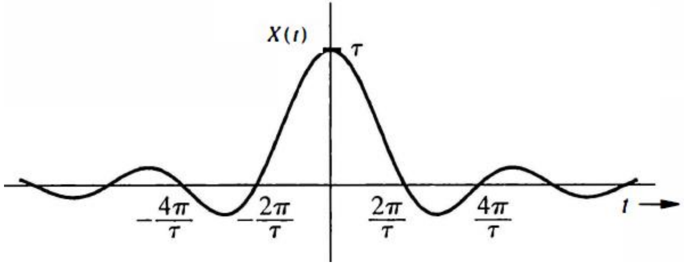
Called duality of time and frequency.

The duality principle may be compared with a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, a negative can be obtained from the photograph



$\Longleftrightarrow$





$\Longleftrightarrow$

?

Proof: We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}\, d\tau$

$$2\,\pi\, x(t) = \int_{-\infty}^{\infty} X(\tau)e^{j\tau t}\, d\tau$$

Putting, $t = -\omega$

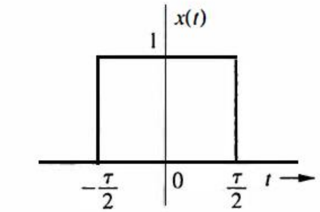
$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(\tau)e^{j\tau.-\omega}\, d\tau$$

Putting, $\tau = t$

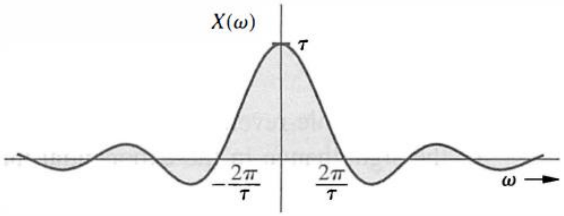
$$= \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\, dt$$

$$= FT\{X(t)\}$$

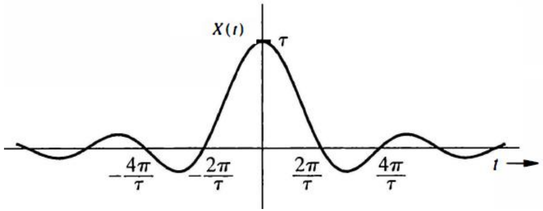
$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = x(-f)$$



$\Longleftrightarrow$



$$x(t) = \begin{cases} 1, |t| < \frac{\tau}{2} \\ 0, else \end{cases} \longleftrightarrow X(\omega) = \tau\frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}} = \tau \sin c\left(\frac{\omega\tau}{2}\right)$$

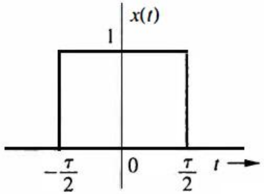


$\Longleftrightarrow$

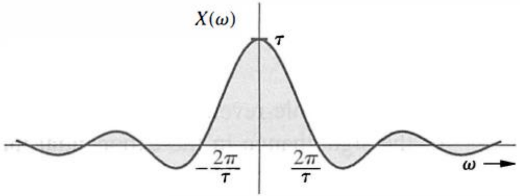
$X(t)$ is the same as $X(\omega)$ with ω replaced by t

$$x(t) = \begin{cases} 1, |t| < \dfrac{\tau}{2} \\ 0, else \end{cases}$$

$$x(t) = rect(\frac{t}{\tau})$$

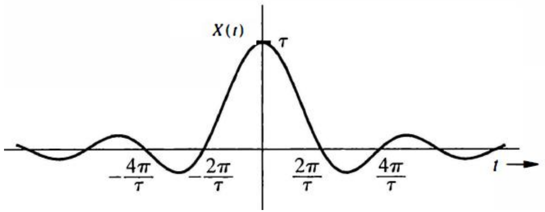


$\Longleftrightarrow$



$$X(\omega) = \tau \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}}$$

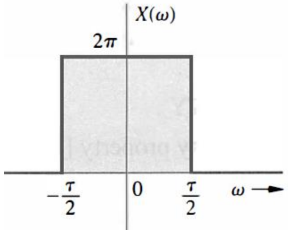
$$= \tau \sin c\left(\frac{\omega\tau}{2}\right)$$



$\Longleftrightarrow$

$$2\pi x(-\omega)$$

$x(-\omega)$ is the same as $x\ (t)$ with t replaced by $-\omega$.



$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = x(-f)$$

$$\underbrace{\tau \operatorname{sinc}\left(\frac{\tau t}{2}\right)}_{X(t)} \Longleftrightarrow \underbrace{2\pi \operatorname{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi x(-\omega)} = 2\pi \operatorname{rect}\left(\frac{\omega}{\tau}\right)$$

- Evaluate Fourier transform of the signal $x(t) = \dfrac{1}{1+t^2}$

$$y(t) = e^{-a|t|}; a > 0$$

$$y(t) \xleftrightarrow{FT} Y(\omega) = \frac{2a}{a^2+\omega^2}$$

$$a = 1$$

$$y(t) = e^{-|t|} \xleftrightarrow{FT} Y(\omega) = \frac{2}{1+\omega^2}$$

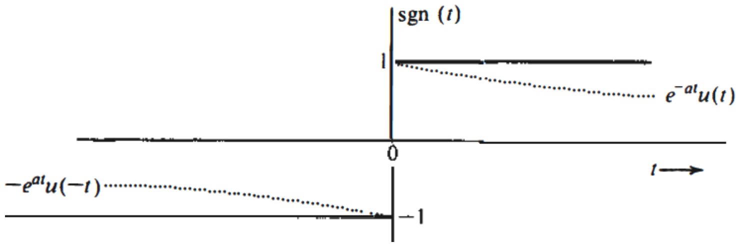
Using dual property: $z(t) \xleftrightarrow{FT} 2\pi z(-\omega)$

$$Y(t) = \frac{2}{1+t^2} \overset{FT}{\leftrightarrow} 2\,\pi\, \boldsymbol{y(-\omega)}$$

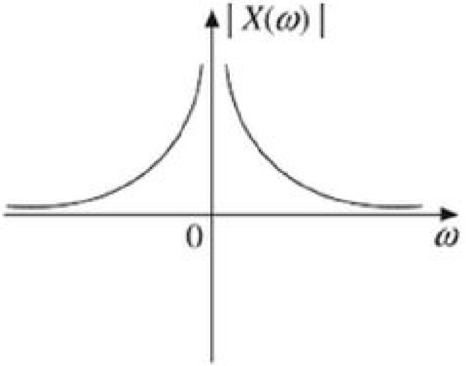
$$\Rightarrow \frac{1}{1+t^2} \overset{FT}{\leftrightarrow} \pi\ e^{-a|-\omega|}$$

$$\Rightarrow \frac{1}{1+t^2} \overset{FT}{\leftrightarrow} \pi\ e^{-a|\omega|}$$

- Evaluate the Fourier transform of the signal



$$\operatorname{sgn} t = \lim_{a\to 0} [e^{-at}u(t) - e^{at}u(-t)]$$



$$\begin{aligned}\mathcal{F}[\operatorname{sgn}(t)] &= \lim_{a\to 0} \{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\} \\ &= \lim_{a\to 0} \left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) \\ &= \lim_{a\to 0} \left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f} = \frac{2}{j\omega}\end{aligned}$$

Perseval's theorem/ Energy spectrum

Energy of the signal $x(t)$can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad = \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \int_{-\infty}^{\infty} X^*(\omega) d\omega \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right]$$

$$= \int_{-\infty}^{\infty} X^*(\omega) . \frac{1}{2\pi} X(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Express the principle conversation of energy in time and frequency domains

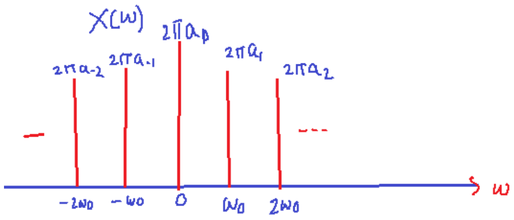
Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Inverse FT of linear combination of impulse signal



As per the definition of Inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

$$\rightarrow x(t) = FT^{-1}\,[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} 2\pi\, a_k\delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\delta(\omega - k\omega_0) \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies

Determine the FT of $\cos\omega_0 t$?

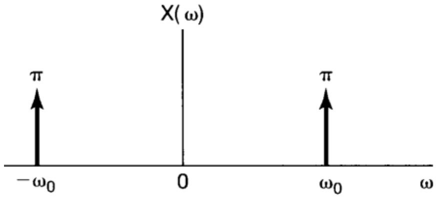
$$\cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$e^{j\omega_0 t} \overset{FT}{\leftrightarrow} ?$$

$$1.\ e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

$$\cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

$$FT\,[\cos\omega_0 t] = \frac{1}{2}.2\pi\delta(\omega - \omega_0) + \frac{1}{2}.2\pi\delta(\omega + \omega_0)$$



Fourier series coefficient

$$a_1 = \frac{1}{2}$$

$$a_{-1} = \frac{1}{2}$$

$$a_k \neq 1, -1$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k \delta(\omega - k\omega_0)$$

$$\leftrightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$X(\omega) = 2\pi\, a_1 \delta(\omega - \omega_0) + 2\pi\, a_{-1}\delta(\omega + \omega_0)$$

$$= 2\pi\, \frac{1}{2}\delta(\omega - \omega_0) + 2\pi \frac{1}{2}\delta(\omega + \omega_0)$$

$$= \pi\, \delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Properties of the Fourier Transform

Property	Signal	Fourier transform
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}X\left(\dfrac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\dfrac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\dfrac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\pi X(0)\delta(\omega)+\dfrac{1}{j\omega}X(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\dfrac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Real signal	$x(t)=x_e(t)+x_o(t)$	$X(\omega)=A(\omega)+jB(\omega)$ $X(-\omega)=X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\}=A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\}=jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\,d\lambda=\int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\,d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,d\omega$$

Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\dfrac{1}{j\omega}$
$e^{-at}u(t),\ a>0$	$\dfrac{1}{j\omega+a}$
$te^{-at}u(t),\ a>0$	$\dfrac{1}{(j\omega+a)^2}$
$e^{-a\lvert t\rvert},\ a>0$	$\dfrac{2a}{a^2+\omega^2}$
$\dfrac{1}{a^2+t^2}$	$e^{-a\lvert\omega\rvert}$
$e^{-at^2},\ a>0$	$\sqrt{\dfrac{\pi}{a}}\,e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \lvert t\rvert<a\\ 0 & \lvert t\rvert>a\end{cases}$	$2a\dfrac{\sin\omega a}{\omega a}$
$\dfrac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \lvert\omega\rvert<a\\ 0 & \lvert\omega\rvert>a\end{cases}$
$\mathrm{sgn}\, t$	$\dfrac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0),\ \omega_0=\dfrac{2\pi}{T}$

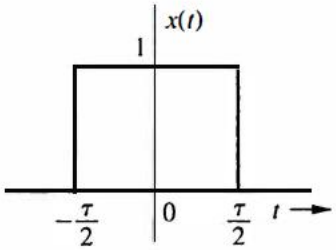
Duality

If $x(t) \xleftrightarrow{FT} X(\omega)$

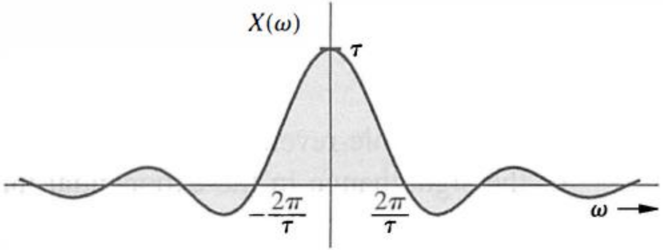
$\boldsymbol{X(t)} \xleftrightarrow{FT} ?$ $\quad \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt = ?$

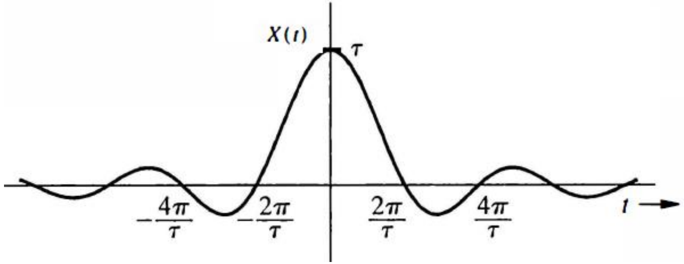
The duality principle may be compared with a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, a negative can be obtained from the photograph

Called duality of time and frequency.



$\Longleftrightarrow$





$\Longleftrightarrow$?

Proof: We know, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}\, d\tau$

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(\tau)e^{j\tau t}\, d\tau$$

Putting, $t = -\omega$

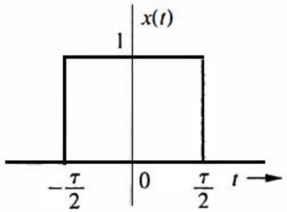
$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(\tau)e^{j\tau.-\omega}\, d\tau$$

Putting, $\tau = \mathrm{t}$

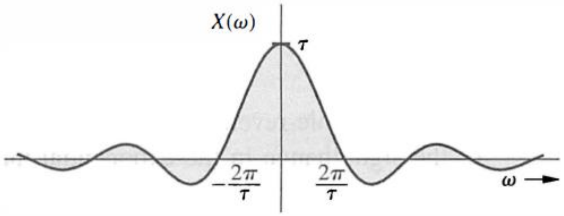
$$= \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\, dt$$

$$= FT\{X(t)\}$$

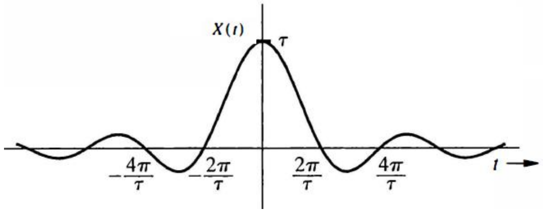
$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = x(-f)$$



$\Longleftrightarrow$



$$x(t) = \begin{cases} 1, |t| < \frac{\tau}{2} \\ 0, else \end{cases} \longleftrightarrow \boldsymbol{X(\omega)} = \tau \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}} = \tau \sin c\left(\frac{\omega\tau}{2}\right)$$

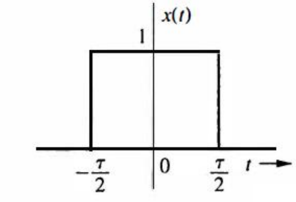


$\Longleftrightarrow$

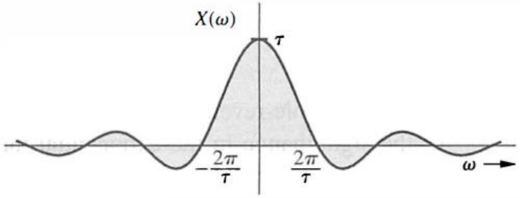
$\boldsymbol{X(t)}$ is the same as $\boldsymbol{X(\omega)}$ with ω replaced by $\boldsymbol{t}$

$$x(t) = \begin{cases} 1, |t| < \dfrac{\tau}{2} \\ 0, else \end{cases}$$

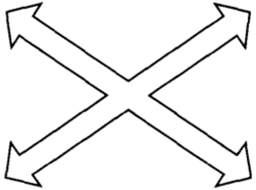
$$x(t) = rect(\frac{t}{\tau})$$

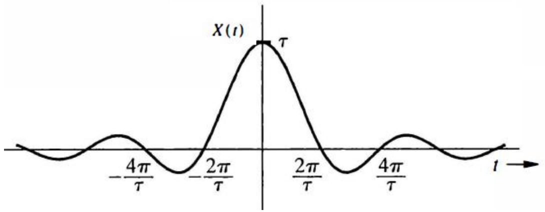


$\Longleftrightarrow$



$$X(\omega) = \tau \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\omega \frac{\tau}{2}}$$
$$= \tau \sin c\left(\frac{\omega\tau}{2}\right)$$

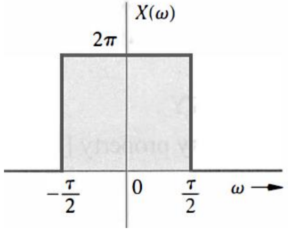




$\Longleftrightarrow$

$$2\pi x(-\omega)$$

$x(-\omega)$ is the same as $x\ (t)$ with t replaced by $-\omega$.



$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega) = x(-f)$$

$$\underbrace{\tau \text{ sinc}\left(\frac{\tau t}{2}\right)}_{X(t)} \Longleftrightarrow \underbrace{2\pi \text{ rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi x(-\omega)} = 2\pi \text{ rect}\left(\frac{\omega}{\tau}\right)$$

- Evaluate Fourier transform of the signal $x(t) = \dfrac{1}{1+t^2}$

$$y(t) = e^{-a|t|};\, a > 0$$

$$y(t) \xleftrightarrow{FT} Y(\omega) = \frac{2a}{a^2+\omega^2}$$

$$a = 1$$

$$y(t) = e^{-|t|} \xleftrightarrow{FT} Y(\omega) = \frac{2}{1+\omega^2}$$

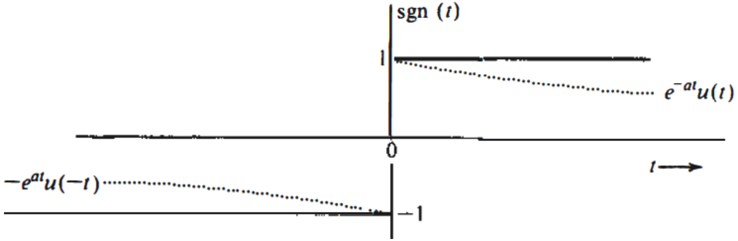
Using dual property: $z(t) \xleftrightarrow{FT} 2\pi z(-\omega)$

$$Y(t) = \frac{2}{1+t^2} \overset{FT}{\leftrightarrow} 2\,\pi\, \boldsymbol{y(-\omega)}$$

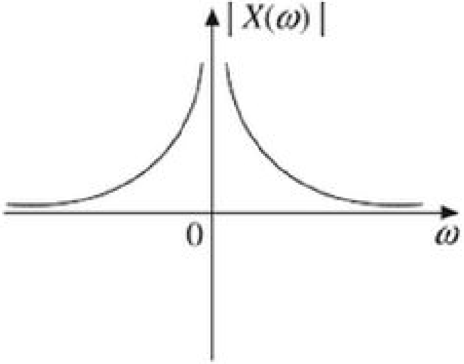
$$\Rightarrow \frac{1}{1+t^2} \overset{FT}{\leftrightarrow} \pi\ e^{-a|-\omega|}$$

$$\Rightarrow \frac{1}{1+t^2} \overset{FT}{\leftrightarrow} \pi\ e^{-a|\omega|}$$

- Evaluate the Fourier transform of the signal



$$\operatorname{sgn} t = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$



$$\mathcal{F}[\operatorname{sgn}(t)] = \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\}$$

$$= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right)$$

$$= \lim_{a\to 0}\left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f} = \frac{2}{j\omega}$$

Perseval's theorem/ Energy spectrum

Energy of the signal $x(t)$can be defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t} d\omega\right] dt$$

$$= \int_{-\infty}^{\infty} X^*(\omega)d\omega \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt\right]$$

$$= \int_{-\infty}^{\infty} X^*(\omega).\frac{1}{2\pi} X(\omega)d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad \boxed{= \int_{-\infty}^{\infty} |X(f)|^2 df}$$

Express the principle conversation of energy in time and frequency domains

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Given, $X(\omega)=\dfrac{1}{(1+\omega^2)}e^{-\frac{2\omega^2}{(1+\omega^2)}}$. **Determine the Fourier transform** of the following signals:

$(a)\ x(t-2)e^{jt}$

$(b)\ x(1-t)$

$(c)\ x\left(\frac{t}{2}-2\right)$

(a) Time shift property: $x(t-t_0)\overset{FT}{\leftrightarrow} e^{-j\omega t_0}X(\omega)$ ⇒ $y(t)=x(t-2)\overset{FT}{\leftrightarrow} Y(\omega)=e^{-j\omega.2}X(\omega)$

Frequency shift property: $x(t)\ e^{j\omega_c t}\ \overset{FT}{\leftrightarrow}\ X(\omega-\omega_c)$ ⇒ $y(t)e^{j.\omega_c=1.t}\ \overset{FT}{\leftrightarrow}\ Y(\omega-\omega_c)$

$y(t)\ e^{j.\omega_c=1.t}=x(t-2)e^{j\omega_c=1.t}\ \overset{FT}{\leftrightarrow}\ Y(\omega-\omega_c)=e^{-j2(\omega-\omega_c)}X(\omega-\omega_c)$ where $\omega_c=1$

(b) Scaling property: $x(at) \overset{FT}{\longleftrightarrow} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

$$\Rightarrow \quad x(-t) \overset{FT}{\longleftrightarrow} \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right) = X(-\omega)$$ (for $a=-1$) [folding]

$$x(-t+1) \overset{FT}{\leftrightarrow} \int_{-\infty}^{\infty} x(-t+1)\, e^{-j\omega t} dt$$

(replacing $-t+1 = \tau,\ dt = -d\tau$)

$$\overset{FT}{\leftrightarrow} \int_{+\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau+1)} \,.\, -d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{+\infty}^{-\infty} x(\tau)\, e^{j\omega\tau} \,.\, -d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{-j.(-\omega).\tau}\, d\tau = e^{-j\omega} X(-\omega)$$

Using Time shift property (delay):

$$x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

$$x(-t+1) \overset{FT}{\leftrightarrow} e^{-j\omega.1} X(-\omega)$$

(c) Using Time shift property (delay): $x(t - t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$

$$y(t) = x(t-2) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega.2} X(\omega)$$ [shifting]

Using scaling property $x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ [Scaling]

$$y\left(\frac{t}{2}\right) = x\left(\frac{t}{2} - 2\right) \overset{FT}{\leftrightarrow} \frac{1}{1/2} Y\left(\frac{\omega}{\frac{1}{2}}\right) = 2.Y(2\omega) = 2e^{-j.2\omega.2} X(2\omega)$$

Given, $X(\omega)=\dfrac{1}{(1+\omega^2)}e^{-\frac{2\omega^2}{(1+\omega^2)}}$. **Determine the Fourier transform** of the following signals:

$(a)\ x(t-2)e^{jt}$

$(b)\ x(1-t)$

$(c)\ x\left(\dfrac{t}{2}-2\right)$

(a) Time shift property: $x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0}X(\omega)$ $\Rightarrow$ $y(t)=x(t-2) \overset{FT}{\leftrightarrow} Y(\omega)=e^{-j\omega.2}X(\omega)$

Frequency shift property: $x(t)\ e^{j\omega_c t} \overset{FT}{\leftrightarrow} X(\omega-\omega_c)$ $\Rightarrow$ $y(t)e^{j.\omega_c=1.t} \overset{FT}{\leftrightarrow} Y(\omega-\omega_c)$

$y(t)\ e^{j.\omega_c=1.t}=x(t-2)e^{j\omega_c=1.t} \overset{FT}{\leftrightarrow} Y(\omega-\omega_c)=e^{-j2(\omega-\omega_c)}X(\omega-\omega_c)$ where $\omega_c=1$

(b) Scaling property: $x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$$\Rightarrow \quad x(-t) \overset{FT}{\longleftrightarrow} \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right) = X(-\omega)$$ (for *a=-1*) [folding]

$$x(-t+1) \overset{FT}{\leftrightarrow} \int_{-\infty}^{\infty} x(-t+1)\, e^{-j\omega t} dt$$

(replacing $-t+1 = \tau,\ dt = -d\tau$)

$$\overset{FT}{\leftrightarrow} \int_{+\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau+1)} . -d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{+\infty}^{-\infty} x(\tau)\, e^{j\omega\tau} . -d\tau$$

$$\overset{FT}{\leftrightarrow} e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau = e^{-j\omega} \int_{-\infty}^{\infty} x(\tau)\, e^{-j.(-\omega).\tau}\, d\tau = e^{-j\omega} X(-\omega)$$

Using Time shift property (delay):

$$x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

$$x(-t+1) \overset{FT}{\leftrightarrow} e^{-j\omega.1} X(-\omega)$$

(c) Using Time shift property (delay): $x(t-t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0}\, X(\omega)$

$$y(t) = x(t-2) \overset{FT}{\leftrightarrow} Y(\omega) = e^{-j\omega.2}\, X(\omega)$$

[shifting]

Using scaling property $x(at) \overset{FT}{\longleftrightarrow} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

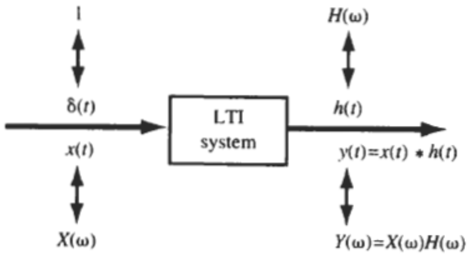
[Scaling]

$$y\left(\frac{t}{2}\right) = x\left(\frac{t}{2} - 2\right) \overset{FT}{\leftrightarrow} \frac{1}{1/2}\, Y\left(\frac{\omega}{\frac{1}{2}}\right) = 2.\,Y(2\omega) = 2e^{-j.2\omega.2} X(2\omega)$$

- Evaluate the response of the LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0$$

To the input signal $\quad x(t) = e^{-bt}u(t),\ b > 0$



From convolution theorem, we can write $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

(using convolution property) $\quad y(t) = x(t) * h(t) \xleftrightarrow{FT} \boldsymbol{X(\omega)H(\omega)}$

Fourier transform of $x(t)$ and $h(t)$ as, $\quad X(\omega) = \dfrac{1}{b + j\omega}$

$$\mathrm{H}(\omega) = \frac{1}{a+j\omega}$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

Partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]|_{s=-1} = 1,$$
$$B = [(s+2)X(s)]|_{s=-2} = -1.$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

- The input output of a system is described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Compute the impulse response h(t) of the system

Taking Fourier-transform both side of the equation

$$y'(t) + 2y(t) = x(t) + x'(t)$$

$$\rightarrow \ j\omega Y(\omega) + 2\,Y(\omega) = X(\omega) + j\omega X(\omega)$$

$$\rightarrow (j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1+j\omega}{2+j\omega} = \frac{2+j\omega-1}{2+j\omega} = 1 - \frac{1}{2+j\omega}$$

$$Taking\ Inverse:\ \ h(t) = \delta(t) - e^{-2t}u(t)$$

- **Multiplication (time)**

If

$$x(t) \overset{FT}{\longleftrightarrow} X(\omega)$$
$$y(t) \overset{FT}{\longleftrightarrow} Y(\omega)$$

Then $z(t) = x(t)y(t) \overset{FT}{\longleftrightarrow} Z(\omega) = \dfrac{1}{2\pi} X(\omega) * Y(\omega)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau)H(\omega - \tau)d\tau$$

Example:

$$x(t) = \frac{\sin(t)\sin(\frac{t}{2})}{\pi t^2}$$

$$= \pi\left(\frac{\sin t}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

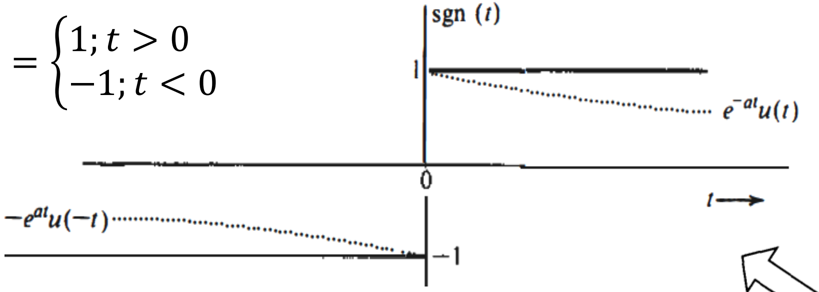
$$= \pi \underbrace{\left(\frac{\sin t}{\pi t}\right)}_{signal-1} \underbrace{\left(\frac{\sin(t/2)}{\pi t}\right)}_{signal-2}$$

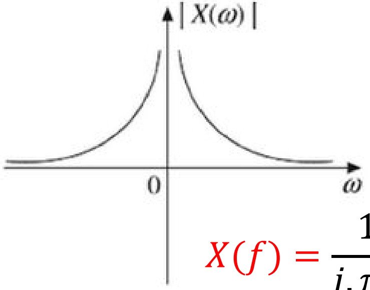
$x(t) \overset{FT}{\longleftrightarrow} X(\omega) = \frac{1}{2\pi} . \pi.$(signal-1)$*$ (signal-2)

$$= \frac{1}{2} FT\left\{\frac{\sin t}{\pi t}\right\} * FT\left\{\frac{\sin(t/2)}{\pi t}\right\}$$

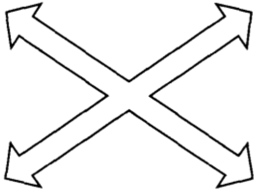
Hilbert Transform (concept)

$$\operatorname{sgn} t = \begin{cases} 1; t > 0 \\ -1; t < 0 \end{cases}$$





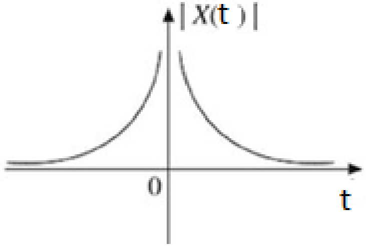
$$X(f) = \frac{1}{j.\pi f} = \frac{2}{j\omega}$$



$$2x(-f) = \operatorname{sgn}(-f) = -\operatorname{sgn}(f)$$

$$= \begin{cases} -1, f > 0 \\ 1, f < 0 \end{cases}$$

Using dual property,



$$X(t) = \frac{1}{j\pi t} \overset{FT}{\leftrightarrow} \operatorname{sgn}(-f) = -\operatorname{sgn} f$$

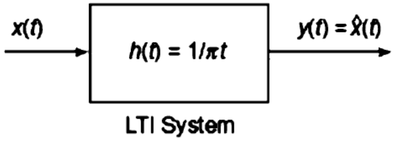
$$\frac{1}{\pi t} \overset{FT}{\leftrightarrow} -j\, sgn(f)$$

($\operatorname{sgn}(f)$ **odd function**)

- Hilbert transform of a **signal $\boldsymbol{x(t)}$** is defined as,

(time domain)

$$\hat{x}(t) = HT[x(t)] = x(t) * \frac{1}{\pi t}$$



$$= \int_{-\propto}^{\propto} x(\tau).\frac{1}{\pi(t-\tau)}d\tau$$

$$= \frac{1}{\pi}\int_{-\propto}^{\propto} x(t-\tau).\frac{1}{\tau}d\tau \quad \text{(putting } t-\tau=\tau)$$

(Frequency domain)

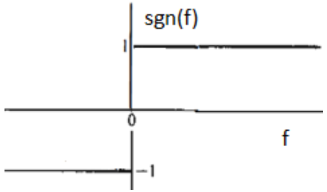
$$FT\left\{\hat{x}(t)\right\} = FT\{x(t)\}.\,FT\left\{\frac{1}{\pi t}\right\}$$

$$\hat{X}(f) = \quad X(f). \quad -j\ \mathrm{sgn}\ f$$

$$FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\ \mathrm{sgn}(-f) = -j\ \mathrm{sgn}\ f$$

$$= \begin{cases} -j, f > 0 \\ j, f < 0 \end{cases}$$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

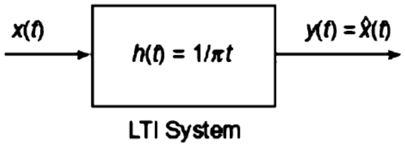


Hilbert transform

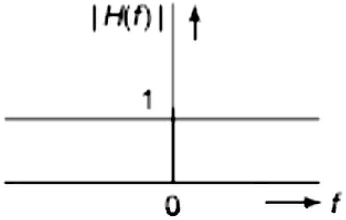
$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

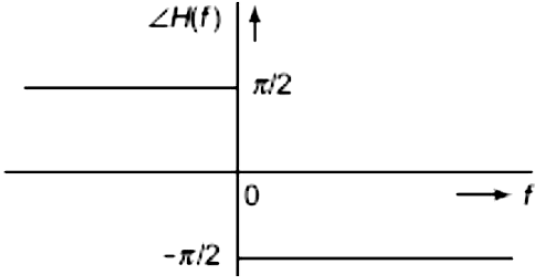
$$\left|\hat{X}(f)\right| = |X(f)|$$

(magnitude unchanged)



$$h(t) = FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\,f = \begin{cases} j, f > 0 \\ -j, f < 0 \end{cases}$$





Applications:

Phase shift, representation of band pass signals, single side-band, band pass to low pass etc

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

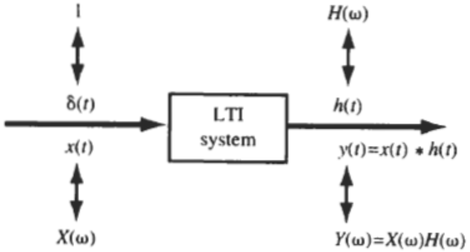
- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

- Evaluate the response of the LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0$$

To the input signal $\quad x(t) = e^{-bt}u(t), \; b > 0$

From convolution theorem, we can write $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$



(using convolution property) $\quad y(t) = x(t) * h(t) \xleftrightarrow{FT} \boldsymbol{X(\omega)H(\omega)}$

Fourier transform of $x(t)$ and $h(t)$ as, $\quad X(\omega) = \dfrac{1}{b + j\omega}$

$$\mathrm{H}(\omega) = \frac{1}{a+j\omega}$$

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(b+j\omega)(a+j\omega)}$$

Partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]|_{s=-1} = 1,$$
$$B = [(s+2)X(s)]|_{s=-2} = -1.$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

- The input output of a system is described by

$$y'(t) + 2y(t) = x(t) + x'(t)$$

Compute the impulse response h(t) of the system

Taking Fourier-transform both side of the equation

$$y'(t) + 2y(t) = x(t) + x'(t)$$

$$\rightarrow j\omega Y(\omega) + 2\,Y(\omega) = X(\omega) + j\omega X(\omega)$$

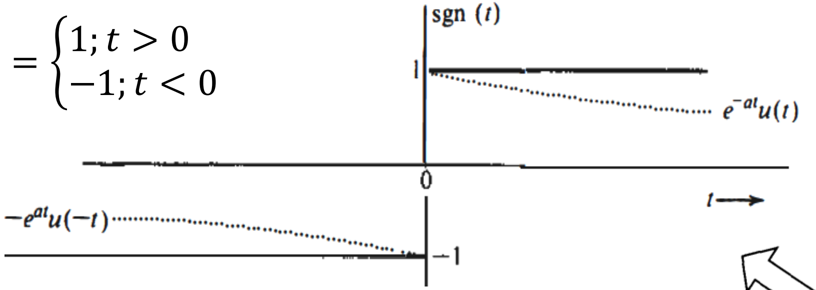
$$\rightarrow (j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$$

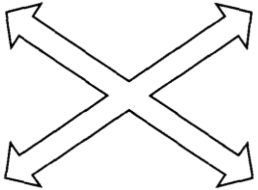
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1+j\omega}{2+j\omega} = \frac{2+j\omega-1}{2+j\omega} = 1 - \frac{1}{2+j\omega}$$

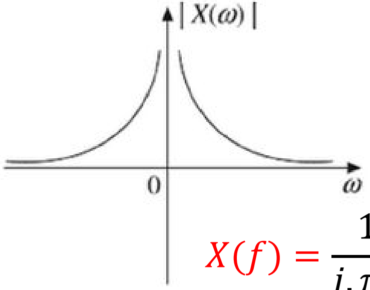
$$Taking\ Inverse:\ \ h(t) = \delta(t) - e^{-2t}u(t)$$

Hilbert Transform (concept)

$$\operatorname{sgn} t = \begin{cases} 1; t > 0 \\ -1; t < 0 \end{cases}$$





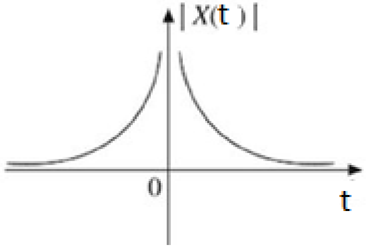


$$X(f) = \frac{1}{j.\pi f} = \frac{2}{j\omega}$$

$$2x(-f) = \operatorname{sgn}(-f) = -\operatorname{sgn}(f)$$

$$= \begin{cases} -1, f > 0 \\ 1, f < 0 \end{cases}$$

Using dual property,



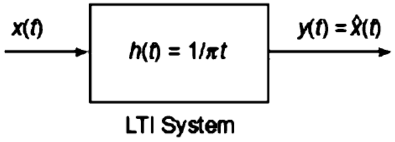
$X(t) = \frac{1}{j\pi t} \overset{FT}{\leftrightarrow} \operatorname{sgn}(-f) = -\operatorname{sgn} f$ $\quad \frac{1}{\pi t} \overset{FT}{\leftrightarrow} -j\, sgn(f)$

($\operatorname{sgn}(f)$ odd function)

- Hilbert transform of a **signal $x(t)$** is defined as,

(time domain)

$$\hat{x}(t) = HT[x(t)] = x(t) * \frac{1}{\pi t}$$



$$= \int_{-\infty}^{\infty} x(\tau).\frac{1}{\pi(t-\tau)}\,d\tau$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} x(t-\tau).\frac{1}{\tau}\,d\tau \quad \text{(putting } t-\tau=\tau)$$

(Frequency domain)

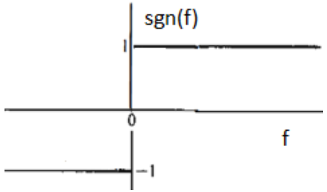
$$FT\{\hat{x}(t)\} = FT\{x(t)\}.\,FT\left\{\frac{1}{\pi t}\right\}$$

$$\hat{X}(f) = \quad X(f). \quad -j\,\mathrm{sgn}\,f$$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

$$FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\,f$$

$$= \begin{cases} -j, f > 0 \\ j, f < 0 \end{cases}$$

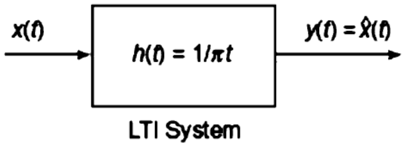


Hilbert transform

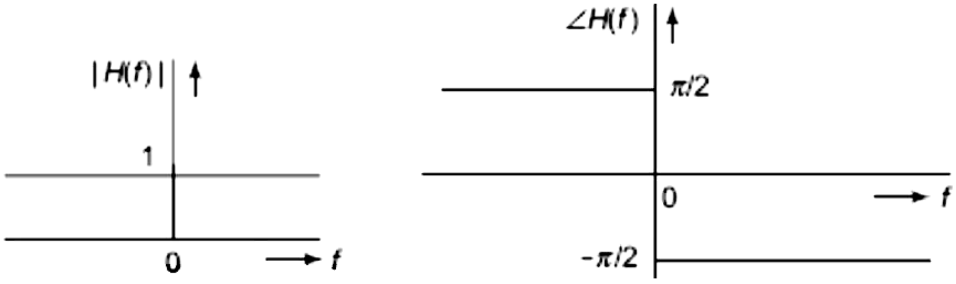
$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

$$\left|\hat{X}(f)\right| = |X(f)|$$

(magnitude unchanged)



$$h(t) = FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\,f = \begin{cases} j, f > 0 \\ -j, f < 0 \end{cases}$$



Applications:

Phase shift, representation of band pass signals, single side-band, band pass to low pass etc

- **Evaluate the Hilbert transform** of the signal $x(t) = \sin \omega_0 t$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

Using Euler's relation, $\sin \omega_0 t$

$$x(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

Hilbert transform: $FT\left\{\hat{x}(t)\right\} = FT\{x(t)\}.FT\left\{\frac{1}{\pi t}\right\}$

$$\mathrm{FT}[\mathrm{x(t)}] = \mathrm{FT}[\sin \omega_0 t] = \frac{1}{2j}\, FT\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2j}\, FT\left[e^{j2\pi f_0 t}\right] - FT\left[e^{-j2\pi f_0 t}\right]$$

We know,

$$1.\, e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$$

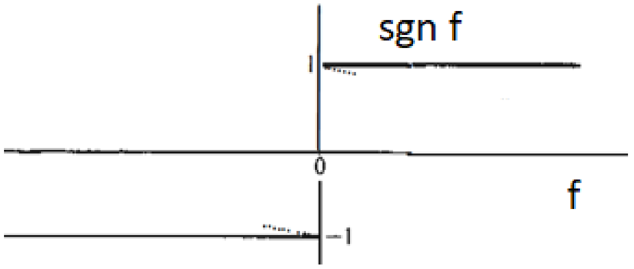
$$X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$\rightarrow \hat{X}(f) = -j\,\mathrm{sgn}\, f\,.\,X(f)$$

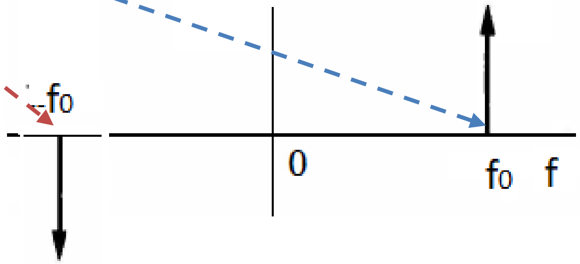
$$\rightarrow \hat{X}(f) = -j\,\mathrm{sgn}\, f\,.\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$= -\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]\,\mathrm{sgn}\, f$$

$$\hat{X}(f) = -\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]\,\mathrm{sgn}\,f$$



$$\hat{X}(f) = -\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$



We know, $1.\; e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$

With inverse FT $\hat{x}(t) = -\frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] = -\cos\omega_0\, t$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

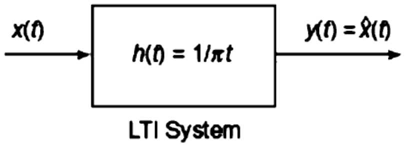
- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Hilbert transform

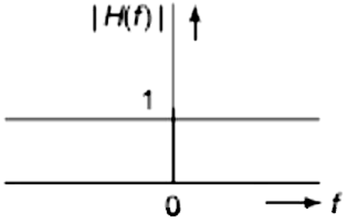
$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

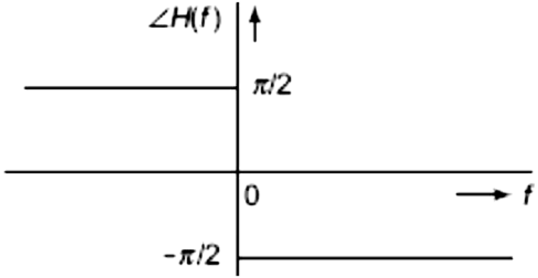
$$\left|\hat{X}(f)\right| = |X(f)|$$

(magnitude unchanged)



$$h(t) \;=\; FT\left\{\frac{1}{\pi t}\right\} \overset{FT}{\leftrightarrow} j\,\mathrm{sgn}(-f) = -j\,\mathrm{sgn}\,f \quad = \begin{cases} j, f > 0 \\ -j, f < 0 \end{cases}$$





Applications:

Phase shift, representation of band pass signals, single side-band, band pass to low pass etc

- **Evaluate the Hilbert transform** of the signal $x(t) = \sin \omega_0 t$

$$\hat{X}(f) = \begin{cases} -j\,X(f) & ; f > 0 \\ j\,X(f) & ; f < 0 \end{cases}$$

Using Euler's relation, $\sin \omega_0 t$

$$x(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$= \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

Hilbert transform: $FT\{\hat{x}(t)\} = FT\{x(t)\}.FT\left\{\frac{1}{\pi t}\right\}$

$$\mathrm{FT}[\mathrm{x(t)}] = \mathrm{FT}[\sin \omega_0 t] = \frac{1}{2j}\, FT\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2j}\, FT\left[e^{j2\pi f_0 t}\right] - FT\left[e^{-j2\pi f_0 t}\right]$$

We know, $1.\, e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$

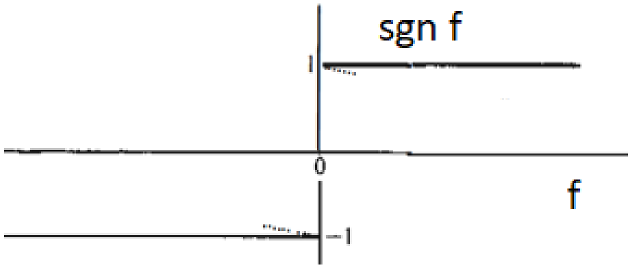
$$X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$\rightarrow \hat{X}(f) = -j \operatorname{sgn} f \,.\, X(f)$$

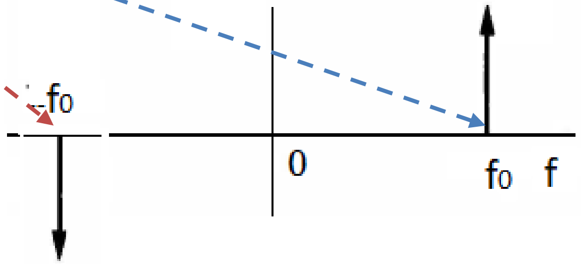
$$\rightarrow \hat{X}(f) = -j \operatorname{sgn} f \,.\, \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

$$= -\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)] \operatorname{sgn} f$$

$$\hat{X}(f) = -\frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]\,\mathrm{sgn}\, f$$



$$\hat{X}(f) = -\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$



We know, $1.\, e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)$

With inverse FT $\hat{x}(t) = -\frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] = -\cos\omega_0\, t$

Properties:

- **Linearity:**

$$HT\{\alpha x(t) + \beta y(t)\} = \alpha \hat{x}(t) + \beta \hat{y}(t)$$

- **Time shift:**

$$HT\{x(t - t_0)\} = \hat{x}(t - t_0)$$

- **Conjugate :**

$$x(t) \xleftrightarrow{FT} X(f)$$

If $x(t)$ is real $\quad X^*(f) = X(-f)$

$$\left\{\hat{X}(f)\right\}^* = \{-j \operatorname{sgn} f \,.\, X(f)\}^* \quad = \begin{Bmatrix} -j.1.X(f), & f > 0 \\ -j.-1.X(f), & f < 0 \end{Bmatrix}^* \quad = \begin{cases} -j.\overset{*}{X}(f) \\ j.\overset{*}{X}(f) \end{cases}$$

- **Derivative:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[g(t)] = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\frac{g(t-\tau)}{\tau}\,\mathrm{d}\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g'(t-\tau)}{\tau}\,\mathrm{d}\tau = \mathcal{H}[g'(t)]$$

- **Convolution:**

$$HT\{x(t) * y(t)\} = x(t) * y(t) * \frac{1}{\pi t}$$

$$= \left[x(t) * \frac{1}{\pi t}\right] * y(t) = x(t) * \left[y(t) * \frac{1}{\pi t}\right]$$

$$= \hat{x}(t) * y(t) \quad = x(t) * \hat{y}(t)$$

- **Perseval's theorem:** $\int_{-\infty}^{\infty} x(t)\overset{*}{\hat{x}}(t)dt = ?$

$$\int_{-\infty}^{\infty}\left[\int X(f)e^{j2\pi f t}df \int \{X(f)e^{j2\pi f t}df\}^{*}\right]dt = \int X(f)\overset{*}{\hat{X}}(f)df$$

$$\int_{-\infty}^{\infty} x(t)\overset{*}{\hat{x}}(t)dt = \int_{-\infty}^{\infty} X(f)\left\{\hat{X}(f)\right\}^{*} df \quad = \int_{-\infty}^{\infty} X(f)\{-j\,\mathrm{sgn}(f)\,X(f)\}^{*}df$$

$$= \int_{-\infty}^{\infty} X(f)\, j\,\mathrm{sgn}(f)\, X^{*}(f)df = \int_{-\infty}^{\infty} |X(f)|^{2}\, j\,\mathrm{sgn}(f)\, df$$

$$= \int_{-\infty}^{0} -j|X(f)|^{2}df + \int_{0}^{\infty} j|X(f)|^{2}df = \mathbf{0}$$

$x(t)$ and $\hat{x}^{*}(t)$ orthogonal

Pre-envelope/Analytical signal

Pre-envelope / Analytical signal of x(t) **real** valued signal is defined as,

$$x_+(t) = x(t) + j\hat{x}(t)$$

Real Part itself

Imaginary part: Hilbert transform of x(t)

In frequency domain:

$$X_+(f) = X(f) + j\hat{X}(f)$$

$$= X(f) + j[-j \operatorname{sgn} f \; X(f)]$$

$$= \begin{cases} X(f) + j.-j.X(f), & f \geq 0 \\ X(f) + j.j.X(f), & f < 0 \end{cases}$$

$$= \begin{cases} X(f) + X(f), f \geq 0 \\ X(f) - X(f), f < 0 \end{cases} \quad = \begin{cases} 2X(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

$$\hat{X}(f) = -j \operatorname{sgn} f \,.\, X(f)$$

$$= \begin{cases} -j.1.X(f) = -jX(f), f \geq 0 \\ -j.-1.X(f) = jX(f), f < 0 \end{cases}$$

$$X_+(f) = X(f) + j\hat{X}(f)$$

$$= \begin{cases} 2X(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

Similarly, we can defined as,

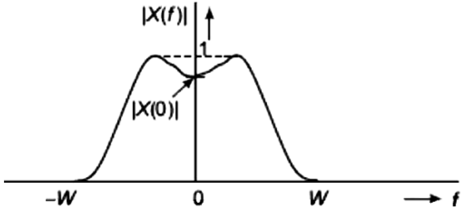
$$x_-(t) = x(t) - j\hat{x}(t)$$

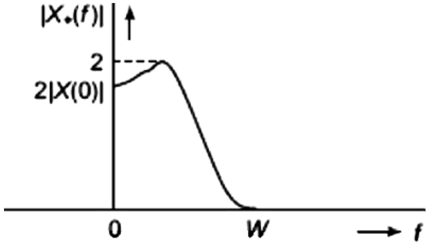
$$X_-(f) = \begin{cases} 0, f > 0 \\ 2X(f), f \leq 0 \end{cases}$$

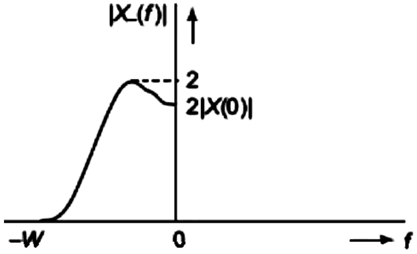
(Negative frequency pre-envelope)

Observations:

1. Spectrum of $x_+(t)$/analytical signal = 2 × positive frequency part of spectrum x(t)
2. Spectrum of $x_+(t)$/analytical signal = zero for all negative frequencies
3. Called positive frequency pre-envelope

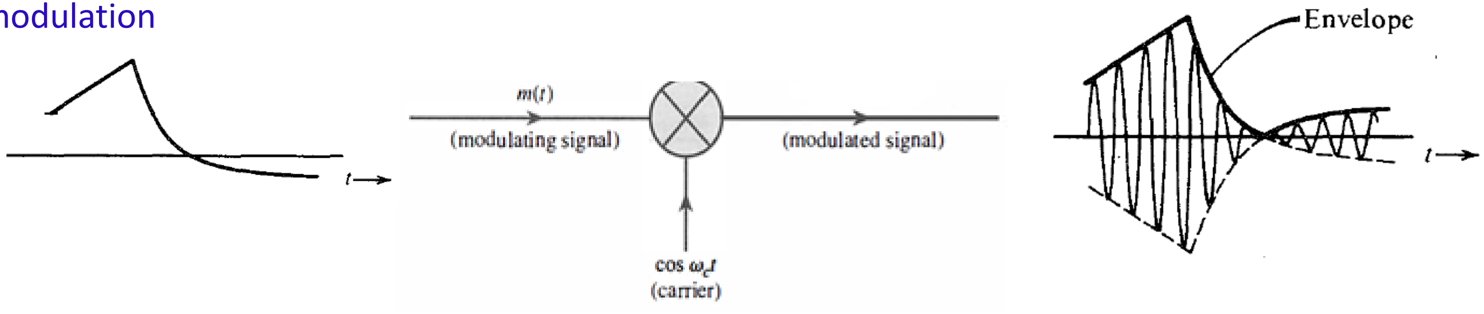




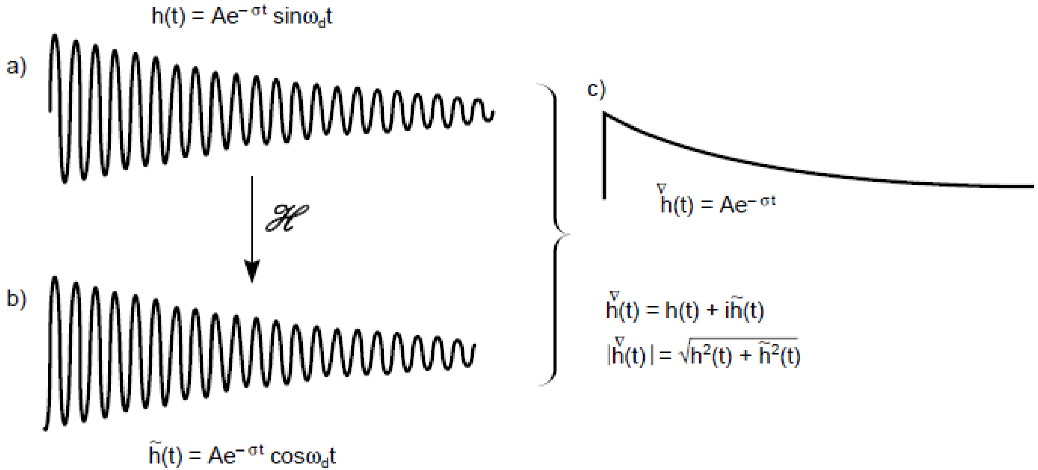


Envelope detector:

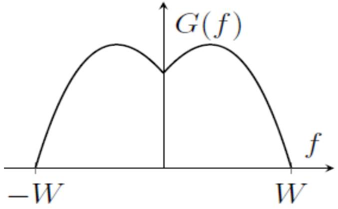
- Amplitude modulation

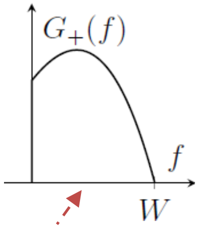


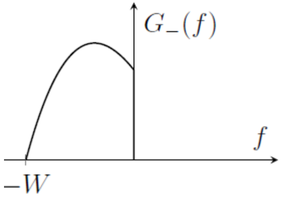
- Removal of the oscillations



- Single-sideband Modulation: Let $g(t)$ is the message signal





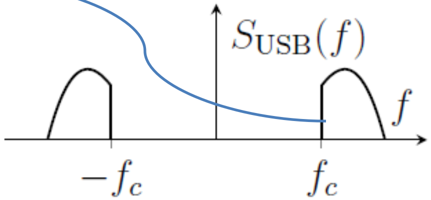


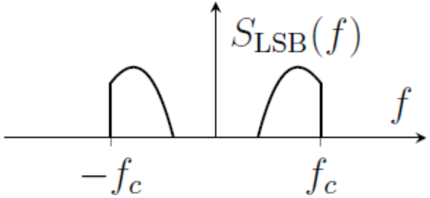
Analytical signals:

$$g_+(t) = \frac{1}{2}[g(t) + \mathrm{j}\hat{g}(t)] \quad \overset{FT}{\longleftrightarrow} \quad G_+(f) = G(f) + j\hat{G}(f) = \begin{cases} G(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

$$g_-(t) = \frac{1}{2}[g(t) - \mathrm{j}\hat{g}(t)] \quad \overset{FT}{\longleftrightarrow} \quad G_-(f) = \begin{cases} 0, f > 0 \\ G(f), f \leq 0 \end{cases}$$

$$g_+(t)\, e^{j2\pi f_c t}$$





Thank you!

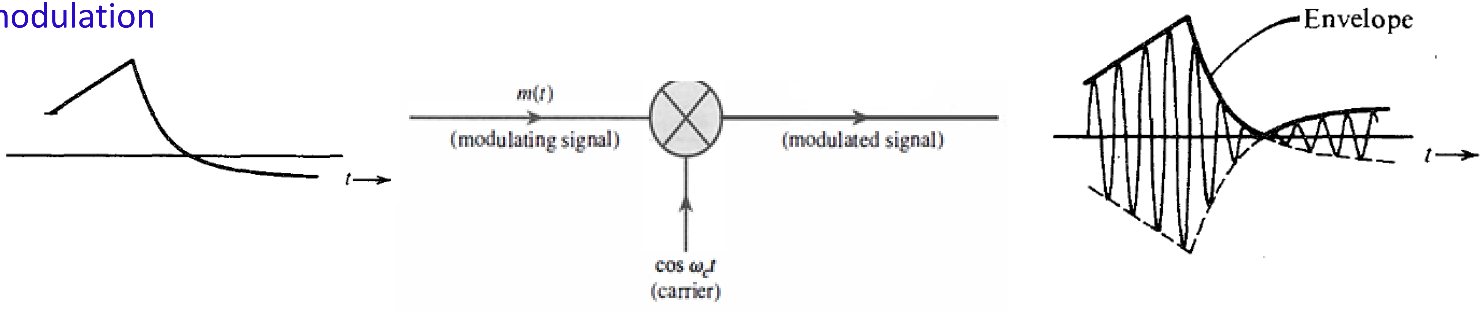
Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

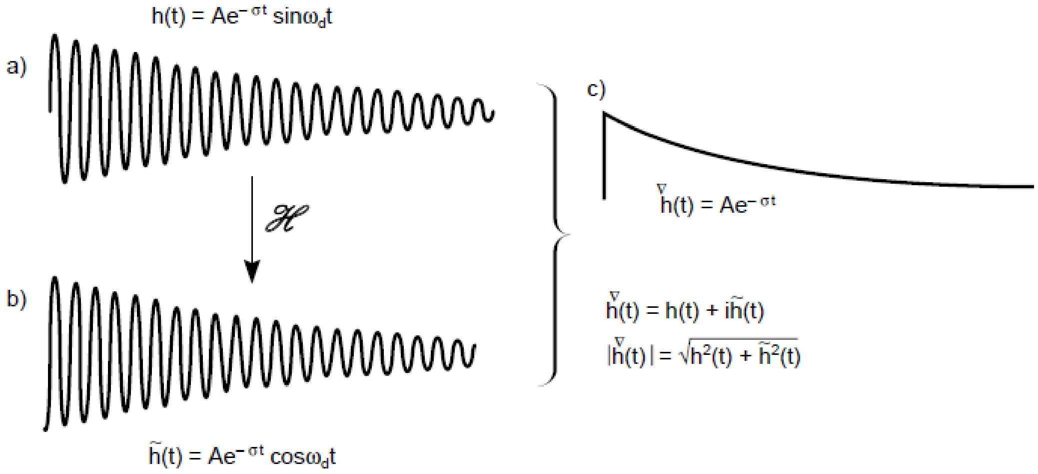
- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Envelope detector:

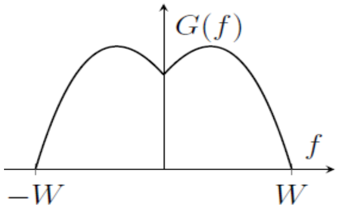
- Amplitude modulation



- Removal of the oscillations



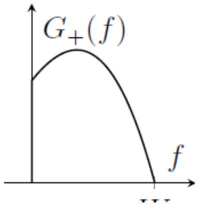
Let $g(t)$ is the message signal, with spectrum

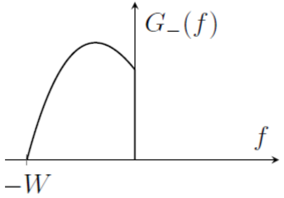


Analytical signals:

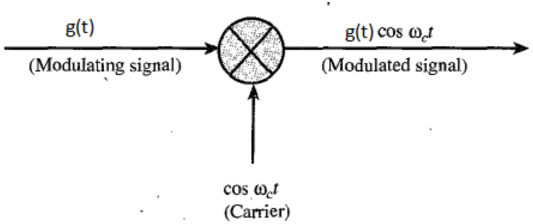
$$g_+(t) = \frac{1}{2}[g(t) + \mathrm{j}\hat{g}(t)] \quad \overset{FT}{\leftrightarrow} \quad G_+(f) = G(f) + j\hat{G}(f) = \begin{cases} G(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

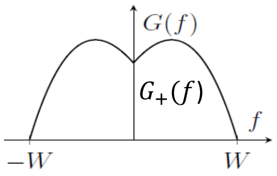
$$g_-(t) = \frac{1}{2}[g(t) - \mathrm{j}\hat{g}(t)] \quad \overset{FT}{\leftrightarrow} \quad G_-(f) = \begin{cases} 0, f > 0 \\ G(f), f \leq 0 \end{cases}$$



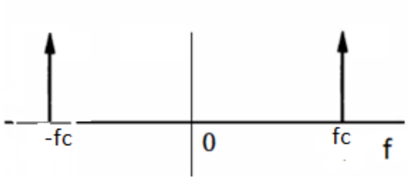


Single-side band modulation





$$\mathrm{FT}[\cos(\omega_c t)] = \frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)$$

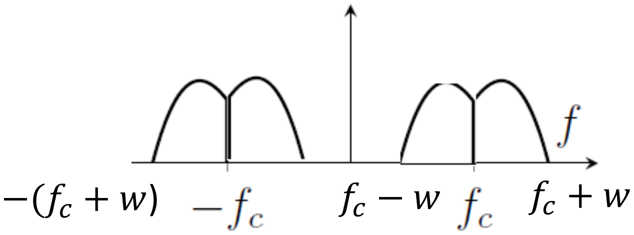


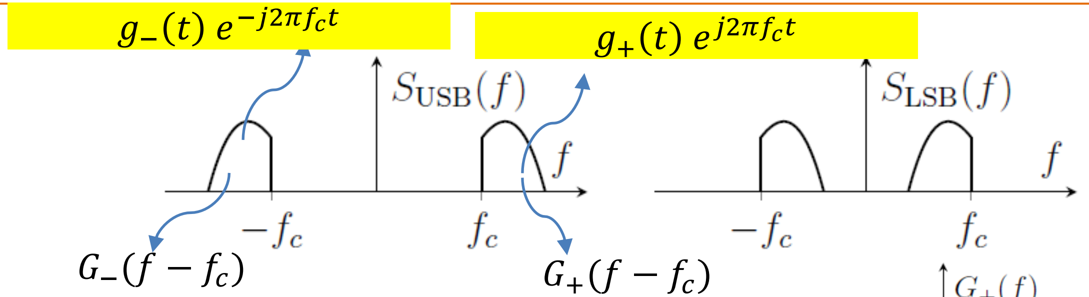
$$g(t)\cos(\omega_c t) \overset{FT}{\leftrightarrow} \frac{1}{2} FT\ [g(t)e^{-j\omega_c t} + g(t)e^{j\omega_c t}]$$

$$= \frac{1}{2}[FT\{g(t)e^{-j\omega_c t}\} + FT\{g(t)e^{j\omega_c t}\}]$$

$$= \frac{1}{2}[G(f + f_c) + \ G(f - f_c)]$$

As $e^{j\omega_c t} \leftrightarrow 2\pi\delta(\omega - \omega_c) = \delta(f - f_c)$





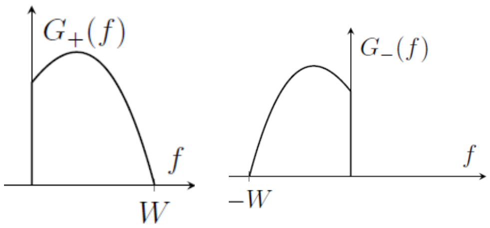
Analytical signals:

$$g_+(t) = \frac{1}{2}[g(t) + \mathrm{j}\hat{g}(t)]$$

$$g_-(t) = \frac{1}{2}[g(t) - \mathrm{j}\hat{g}(t)]$$

$\overset{FT}{\longleftrightarrow}$

$$G_+(f) = G(f) + j\hat{G}(f) = \begin{cases} G(f), f \geq 0 \\ 0, f < 0 \end{cases}$$

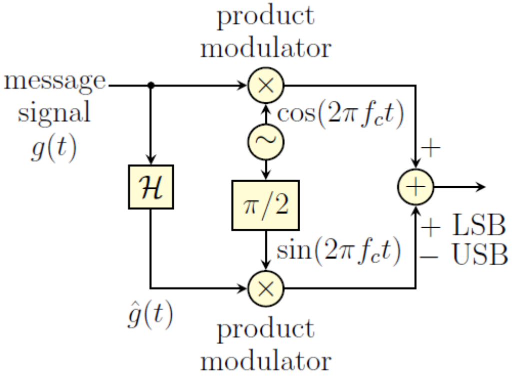


$$s_{usb}(t) = g_+(t)e^{2j\pi f_c t} + g_-(t)e^{-2j\pi f_c t}$$

$$= \frac{1}{2}[g(t) + j\hat{g}(t)]e^{2j\pi f_c t} + \frac{1}{2}[g - j\hat{g}(t)]e^{-2j\pi f_c t}$$

$$= g(t)\frac{1}{2}\left[e^{2j\pi f_c t} + e^{-2j\pi f_c t}\right] + \hat{g}(t)\frac{1}{2}\left[je^{2j\pi f_c t} - je^{-2j\pi f_c t}\right]$$

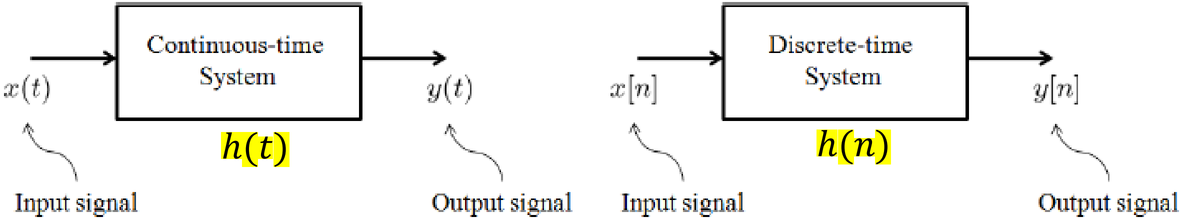
$$= g(t)\cos(2\pi f_c t) - \hat{g}(t)\sin(2\pi f_c t)$$



Hilbert transform pairs.

Time-domain signal	Hilbert transform
$g(t)$	$\hat{g}(t)$
$a_1 g_1(t) + a_2 g_2(t);\ a_1, a_2 \in \mathbb{C}$	$a_1 \hat{g}_1(t) + a_2 \hat{g}_2(t)$
$h(t - t_0)$	$\hat{h}(t - t_0)$
$h(at);\ a \neq 0$	$\operatorname{sgn}(a)\hat{h}(at)$
$\frac{\mathrm{d}}{\mathrm{d}t} h(t)$	$\frac{\mathrm{d}}{\mathrm{d}t} \hat{h}(t)$
$\delta(t)$	$\frac{1}{\pi t}$
$e^{\mathrm{j}t}$	$-\mathrm{j}e^{\mathrm{j}t}$
$e^{-\mathrm{j}t}$	$\mathrm{j}e^{-\mathrm{j}t}$
$\cos(t)$	$\sin(t)$
$\operatorname{rect}(t)$	$\frac{1}{\pi} \ln \lvert (2t+1)/(2t-1) \rvert$
$\operatorname{sinc}(t)$	$\frac{\pi t}{2} \operatorname{sinc}^2(t/2) = \sin(\pi t/2) \operatorname{sinc}(t/2)$
$1/(1+t^2)$	$t/(1+t^2)$

Phase, Group delay

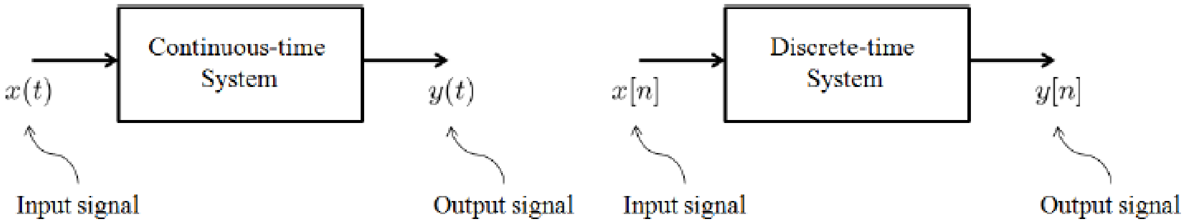


$$y(t) = x(t) * h(t)$$
$$Y(\omega) = X(\omega)H(\omega)$$

Magnitude and phase response can be written:

$$|Y(\omega)|e^{\varphi(\omega)} = |X(\omega)||H(\omega)|e^{j[\varphi_g(\omega)+\varphi_h(\omega)]}$$

$$\begin{cases} |Y(\omega)| = |X(\omega)||H(\omega)| & \text{(Magnitude)} \\ \varphi(\omega) = [\varphi_x(\omega) + \varphi_h(\omega)] & \text{(Phase)} \end{cases}$$



$$|Y(\omega)|e^{\varphi(\omega)} = |X(\omega)||H(\omega)|e^{j[\varphi_g(\omega)+\varphi_h(\omega)]}$$

$$|Y(\omega)| = |X(\omega)||H(\omega)|$$ (Magnitude)

$$\varphi(\omega) = [\varphi_x(\omega) + \varphi_h(\omega)]$$ (Phase)

Observation:

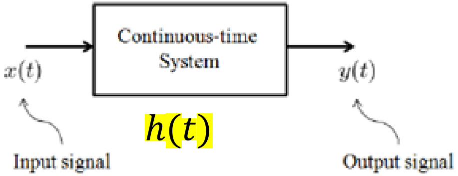
- During transmission, the input signal amplitude spectrum $|G(\omega)|$ is changed to $|G(\omega)||H(\omega)|$

  Input signal spectrum components of freq. ω is **modified** by a factor of |H(ω)|

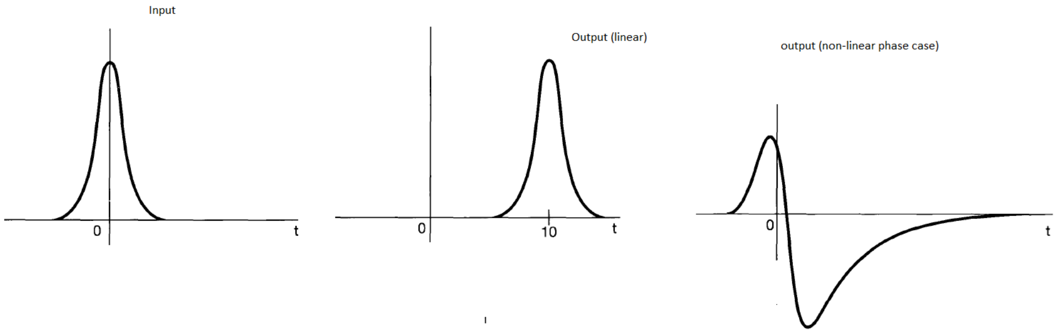
  i.e. during transmission some frequency may boosted in amplitude and other may attenuated

- Input signal phrase spectrum $ø(\omega)$ is changed to $[\varphi_x(\omega) + \varphi_h(\omega)]$
- Similarly, **phase of the various components** also change.

Linear and non-linear Phase



$h(t)$



$Let's$ consider Frequency response of LTI system $h(t) \overset{FT}{\leftrightarrow} H(\omega) = e^{-j\omega_0 t}$

In polar form, $H(\omega)$=$|H(\omega)|e^{j\theta}$ ➡ $|H(\omega)| = 1,$
$\measuredangle\, H(\omega) = -\omega_0 t$

Output of system for input $x(t)$

$(in\ freq.domain)\ \ Y(\omega) = X(\omega)H(\omega)$

$= X(\omega)|H(\omega)|\, e^{-j\omega_0 t} \ = 1.\ [X(\omega)e^{-j\omega_0 t}]$

$(in\ time\ domain)\ \ y(t) = x(t - t_0)]$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Laplace transform

- The Fourier transform is applicable to a large variety of functions and is widely used as a mathematical tool in engineering science, when it satisfy the relation

$$\int_{-\propto}^{\propto} |f(t)|\, dt < \propto$$

- Many functions which are interest in engineering work cannot be handled by this method

  - Ramp, parabolic etc (integral is not converging and functions are not Fourier transformable)

- In order handle these functions, the Laplace transform is proposed by introducing a convergence factor $\boldsymbol{e^{-\sigma}}$, σ where is real number and large enough to ensure absolute convergence

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\propto}^{\propto} x(t)e^{-\sigma t}\, e^{-j\omega t}dt$$

called the **bilateral (or two-sided**)

$$s = \sigma + j\omega$$

Unilateral (or one-sided) Laplace transform

$$X(s) = \int_0^{\propto} x(t)e^{-\sigma t}\, e^{-j\omega t} dt = \int_0^{\propto} x(t)e^{-st}\, dt$$

Clearly the bilateral and unilateral transforms are equivalent only if x(t) = **0 for t < 0.**

- **The Region of Convergence (ROC):** The range of values of the complex variables $\boldsymbol{s}$ for which the Laplace transform converges is called the region of convergence (ROC).

➢ ROC of the Laplace-transform (LS) of $x(t)$ consists of those value of **s** for which x(t)$e^{-\sigma t}$ is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t} dt < \infty$$

This condition depends on $\sigma, real\ part\ of\ s,\ \ Re\{s = \sigma + j\omega\}$

- Evaluate the Laplace transform of signal

$$x(t) = e^{-at}u(t) \quad , a \text{ real}$$

As per definition of Laplace transform,

$$X(s) = \int_{-\propto}^{\propto} x(t)\; e^{-st} dt$$

$$= \int_{-\propto}^{\propto} e^{-at}\, u(t) e^{-st} dt = \int_{0}^{\propto} e^{-at}\, e^{-st} dt$$

$$= \int_{0}^{\propto} e^{-(a+s)t}\, dt$$

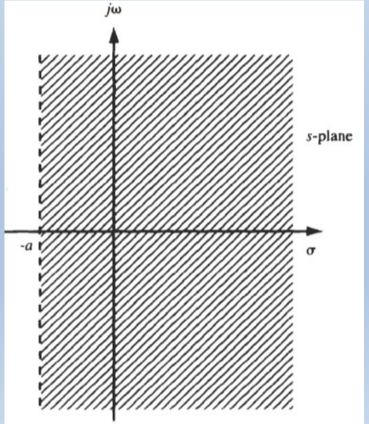
$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$

For Convergence, $\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty$

$$e^{-(s+a)t} \xrightarrow{t\to\propto} 0 \quad \text{if} \quad \text{Re}\{s+a\} > 0$$

$$\to \text{Re}\{s\} > -a$$

ROC



- Evaluate the Laplace transform of signal

$$x(t) = -e^{-at}u(-t) \quad , \text{a real}$$

As per definition of Laplace transform,

$$X(s) = \int_{-\propto}^{\propto} x(t)e^{-\sigma t}\, e^{-j\omega t} dt$$

$$= \int_{-\propto}^{\propto} -e^{-at}\, u(-t)e^{-st} dt = \int_{-\propto}^{0} -e^{-at}\, e^{-st} dt$$

$$= \int_{-\propto}^{0} -e^{-(a+s)t}\, dt$$

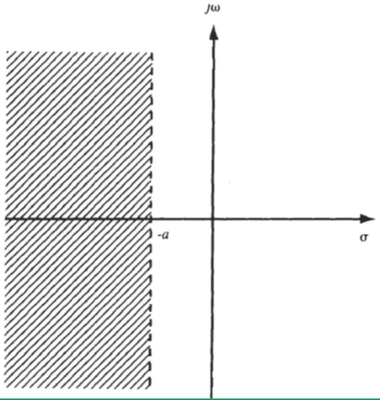
$$= \frac{1}{s+a}, \qquad Re\{s\} < -a$$

For Convergence, $\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t} dt < \infty$

$$e^{-(s+a)t} \xrightarrow{t \to -\propto} 0 \quad \textbf{if} \quad \mathrm{Re}\{s+a\} < 0$$

$$\to \mathrm{Re}\{s\} < -a$$

ROC



- Evaluate the Laplace-transform (LS) of the signal $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

As per definition of Laplace transform,

$$X(s) = \int_{-\propto}^{\propto} x(t)e^{-st}\,dt = \int_{-\propto}^{\propto} [3e^{-2t}u(t) - 2e^{-t}u(t)]\,e^{-st}dt$$

$$= 3\int_0^{\propto} e^{-2t}\,e^{-st}dt - 2\int_0^{\infty} e^{-t}\,e^{-st}dt$$

$$= 3\int_0^{\propto} e^{-(s+2)t}\,dt - 2\int_0^{\infty} e^{-(s+1)t}\,dt$$

For Convergence, $\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$

$e^{-(s+2)t} \xrightarrow{t\to\propto} 0$ if $\mathrm{Re}\{s+2\} > 0$

$\rightarrow \mathrm{Re}\{s\} > -2$

$e^{-(s+1)t} \xrightarrow{t\to\propto} 0$ if $\mathrm{Re}\{s+1\} > 0$

$\rightarrow \mathrm{Re}\{s\} > -1$

$Re\{s\} \succ -1$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

Pole and Zeros

- A **rational** X(s) can be written as,

$$X(s) = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n} = \frac{a_0}{b_0} \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

- The coefficients a_k, and b_k, are real constants, and m *and* n are positive integers.
- The X(s) is called a proper rational function if $\boldsymbol{n} > \boldsymbol{m}$, and an improper rational function if $\boldsymbol{n} \leq$ m.
- The roots of the numerator polynomial are called, z the zeros of X(s) because X(s) = **0** for those values of s.
- The roots of the denominator polynomial are called poles because X(s) is infinite for those values of s
- The **poles of X(s) lie outside** the ROC since X(s) does not converge at the poles
- The **zeros** may lie inside or outside the ROC

$$X(s) = \frac{2s + 4}{s^2 + 4s + 3} = 2\frac{s + 2}{(s + 1)(s + 3)} \qquad \mathrm{Re}(s) > -1$$

X(s) has one zero at s = - 2 and two poles at s = - 1 and s = - 3

PROPERTIES OF THE LAPLACE TRANSFORM

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{Re}\{s\} > 0\}$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \longrightarrow \infty} x(t) = \lim_{s \longrightarrow 0} sX(s)$$

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{Re}\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\mathcal{Re}\{s\} > 0$
13	$[e^{-\alpha t} \cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
14	$[e^{-\alpha t} \sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$	$\mathcal{Re}\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\} > 0$

Inverse Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Putting $s = \sigma + j\omega$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}\,e^{-j\omega t}dt = FT\{x(t)e^{-\sigma t}\}$$

We can invert this relationship using the inverse Fourier transform as given $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$

$$x(t)e^{-\sigma t} = FT^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}\,d\omega$$

$s = \sigma + j\omega, ds = jd\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

Evaluate inverse Laplace transform

$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \Re e\{s\} > -1$$

we first perform a partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

we can evaluate the coefficients A and B by

$$A = [(s+1)X(s)]|_{s=-1} = 1$$

$$B = [(s+2)X(s)]|_{s=-2} = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Since the ROC for $X(s)$ is $\Re e\{s\} > -1$ ROC is to the right of the pole.

$$e^{-at}u(t) \overset{L}{\leftrightarrow} \frac{1}{s+a}, Re\{S\} \succ -a$$

$x(t)$ = Inverse Laplace (X(s))= $e^{-t}u(t)$ - $e^{-2t}u(t)$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Evaluate inverse Laplace transform

$$X(s) = \frac{1}{(s+1)(s+2)};\quad Re\{s\} < -2$$

we first perform a partial-fraction expansion

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = [(s+1)X(s)]|_{s=-1} = 1$$

$$B = [(s+2)X(s)]|_{s=-2} = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

As $x(t) = -e^{-at}u(-t)$ $\rightarrow \mathrm{Re}\{s\} < -a$

$$-e^{-t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+1}, Re\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{L}{\leftrightarrow} \frac{1}{s+2}, Re\{s\} < -2$$

$$x(t) = [-e^{-t} + e^{-2t}]u(-t)$$

- Evaluate the Inverse Laplace transform

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad -2 < \Re e\{s\} < -1$$

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

$$-e^{-t}u(-t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \Re e\{s\} < -1$$

$$e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \Re e\{s\} > -2.$$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \overset{\mathfrak{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad -2 < \Re e\{s\} < -1.$$

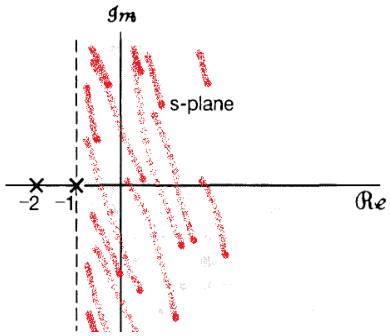
Linearity:

If

$$x_1(t) \longleftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

Then $$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(s) + a_2 X_2(s) \qquad R' \supset R_1 \cap R_2$$



Example:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \int_{-\propto}^{\propto} x(t)e^{-st}\,dt = \int_{-\propto}^{\propto} [3e^{-2t}u(t) - 2e^{-t}u(t)]\,e^{-st}dt$$

$$= 3\int_0^{\propto} e^{-2t}\,e^{-st}dt - 2\int_0^{\infty} e^{-t}\,e^{-st}dt$$

$$= 3\int_0^{\propto} e^{-(s+2)t}\,dt - 2\int_0^{\infty} e^{-(s+1)t}\,dt$$

$e^{-(s+2)t} \xrightarrow{t\to\propto} 0$ if $\text{Re}\{s+2\} > 0$
$\to \text{Re}\{s\} > -2$

$e^{-(s+1)t} \xrightarrow{t\to\propto} 0$ if $\text{Re}\{s+1\} > 0$
$\to \text{Re}\{s\} > -1$

$\text{Re}\{s\} > -1$

Properties

Time-shift:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(t - t_0) \leftrightarrow e^{-st_0} X(s) \qquad R' = R$$

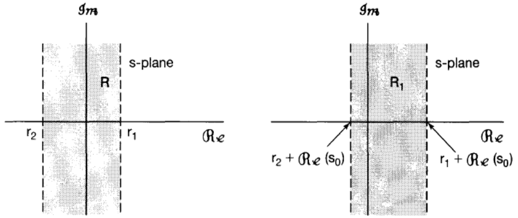
Shift in S-domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \text{Re}(s_0)$$

ROC is shifted by



Scaling:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR$$

Time reversal:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $\quad x(-t) \leftrightarrow X(-s) \qquad R' = -R$

Differentiation in S-domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $\quad -tx(t) \leftrightarrow \dfrac{dX(s)}{ds} \qquad R' = R$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} (-t)x(t)e^{-st}\,dt = \int_{-\infty}^{\infty} [-tx(t)]e^{-st}\,dt$$

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

Differentiation in time domain:

If

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

then $\quad \dfrac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$

Conjugate:

$$x(t) \longleftrightarrow X(s), \quad \text{with ROC} = R$$

$$x^*(t) \longleftrightarrow X^*(s^*), \quad \text{with ROC} = R.$$

$$X(s) = X^*(s^*) \quad \text{when } x(t) \text{ is real.}$$

Integration:

If

$$x(t) \longleftrightarrow X(s) \qquad \text{ROC} = R$$

then
$$\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{s} X(s) \qquad R' = R \cap \{\mathrm{Re}(s) > 0\}$$

$$\mathfrak{L}\left\{\int_{0^-}^{t} f(\tau)\tau\right\} = \int_{0^-}^{\infty} \underbrace{\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)}_{u(t)} \underbrace{e^{-st}}_{v'(t)}\mathrm{d}t$$

$$= \left[\left(\int_{0^-}^{t} f(\tau)\mathrm{d}\tau\right)\left(-\frac{1}{s}e^{-st}\right)\right]_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\left(-\frac{1}{s}e^{-st}\right)\mathrm{d}t$$

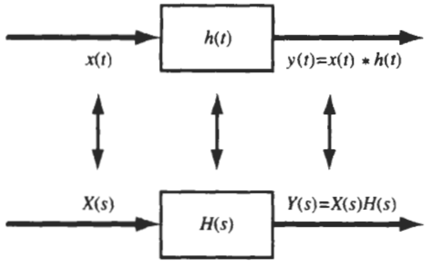
Convolution:

If

$$x_1(t) \longleftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \longleftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

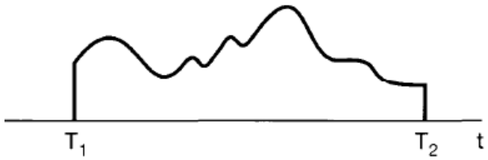
then
$$x_1(t) * x_2(t) \longleftrightarrow X_1(s)X_2(s) \qquad R' \supset R_1 \cap R_2$$



Properties

1. The ROC does not contain any poles
2. If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane

$x(t)$ is absolutely integrable for any value of σ



$$x(t) = \begin{cases} e^{-at}, 0 < t < T \\ 0, else \end{cases}$$

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]$$

It looks $X(s)$ has a pole at $s = -a$

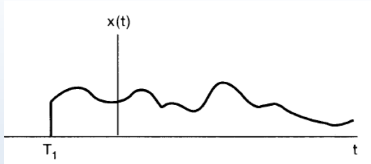
To find the value of $X(s)$ *at* $s = -a$, consider L'Hopital rule

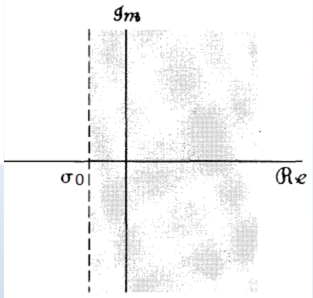
$$\lim_{s \to -a} X(s) = \lim_{s \to -a} \left[\frac{\frac{d}{ds}\left(1 - e^{-(s+a)T}\right)}{\frac{d}{ds}(s+a)} \right] = \lim_{s \to -a} T e^{-aT} e^{-sT} = T \text{ (finite)}$$

the ROC is the entire s-plane

Properties

3. If $x(t)$ **is right sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**,

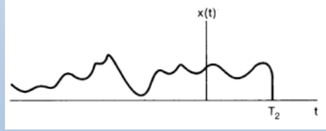


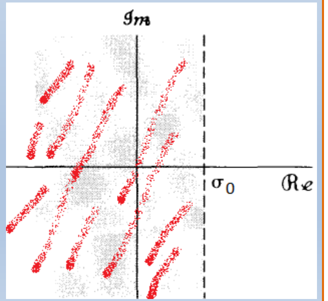


→ then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC

$$if\ \sigma_1 > \sigma_0 \qquad \int_{-\infty}^{\infty} |x(t)|e^{-\sigma_1 t}dt < \int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t}dt < \infty$$

4. If $x(t)$ **is left sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**



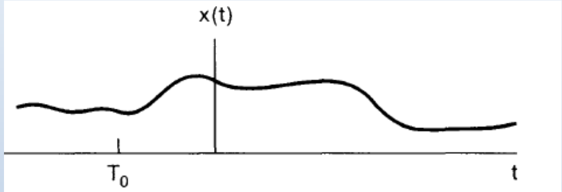


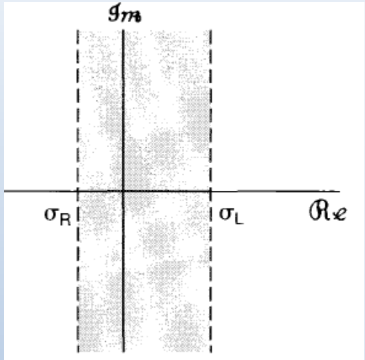
→ then all values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC

Properties

5. If $x(t)$ is **both sided**, and if the line $Re\{s\} = \sigma_0$ is in the **ROC**,

→ then the ROC consist of **a strip** in the s-plane that includes the line $Re\{s\} = \sigma_0$





5. If x(t) **is right sided** and its Laplace transform $X(s)$ is rational, then the ROC in s-plane is right of the **rightmost** pole

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{s+2}{(s+1)(s+3)} \qquad \mathrm{Re}(s) > -1$$

X(s) has one zero at s = - 2 and two poles at s = - 1 and s = - 3

Properties

6. If x(t) **is left sided** and its Laplace transform $X(s)$ is rational, then the ROC in s-plane is right of the **leftmost pole**

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the **Laplace transform of u(t)**

$(a)\ \delta(t)$
(b) $\delta'(t)$
(c) $e^{-at}u(t)$
(d) t$e^{-at}u(t)$
(e) cos$\omega_0 t\ u(t)$

Laplace transform of u(t):

$$U(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt \quad = \int_{0}^{\infty} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty} = \frac{1}{s};\ \ \mathrm{Re}\{s\} > 0$$

(a) $$\delta(t) = \frac{du(t)}{dt}$$ Using differentiation property $$\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$$ $$\delta(t) \leftrightarrow sU(s) = s.\frac{1}{s} = 1$$ /ROC all **S**

(b) $$\delta'(t) = \frac{d}{dt}\left(\frac{du(t)}{dt}\right)$$ Using differentiation property $$\delta'(t) \leftrightarrow s(sU(s)) = s.s.\frac{1}{s} = s$$ ROC all **S**

(c) Using shifting property $e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s-s_0)$ $e^{-at}u(t) \overset{L}{\leftrightarrow} U(s+a) = \frac{1}{s+a}$, $Re\{S\} \succ -a$

(d) Using differentiation property $-tx(t) \overset{L}{\leftrightarrow} \dfrac{dX(s)}{ds}$

We know, $e^{-at}u(t) \overset{L}{\leftrightarrow} \dfrac{1}{s+a}$, $Re\{S\} \succ -a$

$$t[e^{-at}u(t)] \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right); \quad \mathrm{Re}\{s\} > -a$$

(e) $\cos(\omega_0 t)\, u(t) = \frac{1}{2}\left(e^{j\omega_0 t+} e^{-j\omega_0 t}\right) u(t) = \frac{1}{2} e^{j\omega_0 t}u(t) + \frac{1}{2} e^{-j\omega_0 t}\mathrm{u(t)}$

Using shifting property $e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s-s_0)$ ⇒ $e^{j\omega_0 t}u(t) \overset{L}{\leftrightarrow} \mathrm{U(s\text{-}j}\omega_0) = \frac{1}{s-j\omega_0}$

$$X(s) = \frac{1}{2}\frac{1}{s-j\omega_0} + \frac{1}{2}\frac{1}{s+j\omega_0} = \frac{s}{s^2+{\omega_0}^2}, \quad \mathbf{Re\{s\}>0}$$

Thank you!

Warning notification!!!!!

- The PPTs are prepared for the offline interactive teaching in the class using the materials from different books and web. The author may not have the legal permission for online sharing those materials via social media/web/email or use in business etc in public domain.

- Therefore, students are requested not to share the PPTs outside the class/institute, which can violation the copy-write related issues.

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the **Laplace transform of u(t)**

$(a)\ \delta(t)$
(b) $\delta'(t)$
(c) $e^{-at}u(t)$
(d) t$e^{-at}u(t)$
(e) cos$\omega_0 t\ u(t)$

Laplace transform of u(t):

$$U(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt \quad = \int_{0}^{\infty} e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty} = \frac{1}{s};\ \ \mathrm{Re}\{s\} > 0$$

(a) $\delta(t) = \dfrac{du(t)}{dt}$ Using differentiation property $\dfrac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s)$ $\delta(t) \leftrightarrow sU(s) = s.\dfrac{1}{s} = 1$ /ROC all **S**

(b) $\delta'(t) = \dfrac{d}{dt}\left(\dfrac{du(t)}{dt}\right)$ Using differentiation property $\delta'(t) \leftrightarrow s(sU(s)) = s.s.\dfrac{1}{s} = s$ ROC all **S**

(c) Using shifting property $e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s-s_0)$ $e^{-at}u(t) \overset{L}{\leftrightarrow} U(s+a) = \frac{1}{s+a}$, $Re\{S\} \succ -a$

(d) Using differentiation property $-tx(t) \overset{L}{\leftrightarrow} \dfrac{dX(s)}{ds}$

We know, $e^{-at}u(t) \overset{L}{\leftrightarrow} \dfrac{1}{s+a}$, $Re\{S\} \succ -a$

$$t[e^{-at}u(t)] \leftrightarrow -\frac{d}{ds}\left(\frac{1}{s+a}\right); \quad \mathrm{Re}\{s\} > -a$$

(e) $\cos(\omega_0 t)\, u(t) = \frac{1}{2}\left(e^{j\omega_0 t +} e^{-j\omega_0 t}\right) u(t) = \frac{1}{2} e^{j\omega_0 t}u(t) + \frac{1}{2} e^{-j\omega_0 t}$u(t)

Using shifting property $e^{s_0 t}x(t) \overset{L}{\leftrightarrow} X(s-s_0)$ ⇨ $e^{j\omega_0 t}u(t) \overset{L}{\leftrightarrow}$ U(s-jω_0) $= \frac{1}{s-j\omega_0}$

$X(s) = \frac{1}{2}\frac{1}{s-j\omega_0} + \frac{1}{2}\frac{1}{s+j\omega_0} = \frac{s}{s^2+{\omega_0}^2}$, **Re{s}>0**

- The input-output of a LTI system is represented by $y'(t) + 3\,y(t) = x(t)$.

  Determine the impulse response of the system

Taking Laplace transform both side of the equation

$$sY(s) - \boldsymbol{y}(\mathbf{0}+) + 3Y(s) = X(s)$$

$$Y(s)[s+3] - y(0+) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}, \qquad \text{considering initial condition zero}$$

$$= \frac{1}{s+3}$$

$$\boxed{h(t) = e^{-3t}u(t)}$$

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

$$u = x(t), \qquad du = \left[\frac{dx(t)}{dt}\right]dt$$

$$dv = e^{-st}dt, \qquad v = \frac{1}{-s}e^{-st}$$

$$\int_0^\infty u dv = \mathrm{uv}|_0^\infty - \int_0^\infty v\, du$$

$$X(s) = -\frac{1}{s}x(t)e^{-st}|_0^\infty - \frac{1}{-s}\int_0^\infty e^{-st}\left[\frac{dx(t)}{dt}\right]dt$$

$$\left[\frac{dx(t)}{dt}\right] \overset{L}{\leftrightarrow} -\mathrm{x}(0+) + sX(s)$$

$Task:\quad y''(t) + 3y'(t) + 2\,y(t) = x'(t) + 3\,x(t)$

- Express $X = \frac{3x+1}{(x-1)^2(s+2)}$ as sum of partial fractions

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$= \frac{5}{9(x-1)} + \frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}$$

$$3x + 1 = A(x-1)(x+2) + B(x+2) + c(x-1)^2$$

If $x = 1$, $3 \times 1 + 1 = A \times 0 \times 3 + B \times 3 + C \times 0 = 3B$, $B = \frac{4}{3}$

If $x = -2$, $3 \times -2 + 1 = A \times -3 \times 0 + B \times 0 + C \times 9 = 9C$ C= $\frac{-5}{9}$

$$3x + 1 = A(x-1)(x+2) + B(x+2) + c(x-1)^2$$
$$= A(x^2 + x - 2) + B(x+2) + c(x^2 - 2x + 1)$$
$$= (A + C)x^2 + (A + B - 2C)x + (-2A + 2B + C)$$

A= $\frac{5}{9}$

Find the Laplace transform and the associated ROC for each of the following signals

(a) $x(t) = \delta(t - t_0)$

(b) $x(t) = u(t - t_0)$

(c) $x(t) = e^{-2t}[u(t) - u(t-5)]$

(d) $x(t) = \sum_{k=0}^{\infty} \delta(t - kT)$

(e) $x(t) = \delta(at + b)$, a, b real constants

(e)

$$x(t) = \delta(t) \rightarrow X(s) = 1, \forall s$$

$$x(t+b) = \delta(t-(-b)) \rightarrow e^{-(-b)s} X(s)$$

$$x(at+b) = \delta(at+b) \rightarrow \frac{1}{|a|} X(\frac{s}{a}) = \frac{1}{|a|} e^{b\frac{s}{a}}$$

ROC S all

(d)

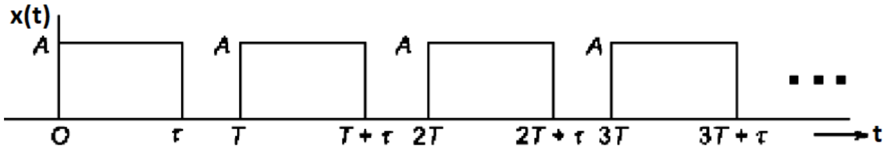
$$X(s) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \delta(t-kT)\, e^{-st} dt \quad = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \delta(t-kT)\, e^{-st} dt \quad = \sum_{k=0}^{\infty} e^{-skT} = \frac{1}{1-e^{-sT}}$$

$$\mathrm{Re}\{sT\} > 0$$
$$\mathrm{Re}\{s\} > 0$$

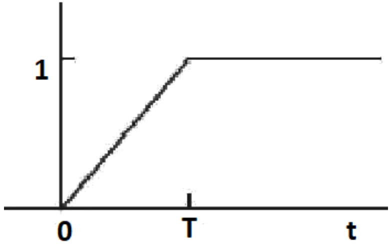
Evaluate the Laplace transform and ROC of following signals

$(a)\ x(t) = 5e^{-3t}$
$(b)\ x(t) = (2e^{-2t} + 3e^{-3t})u(t)$
$(c)\ x(t) = tu(t)$

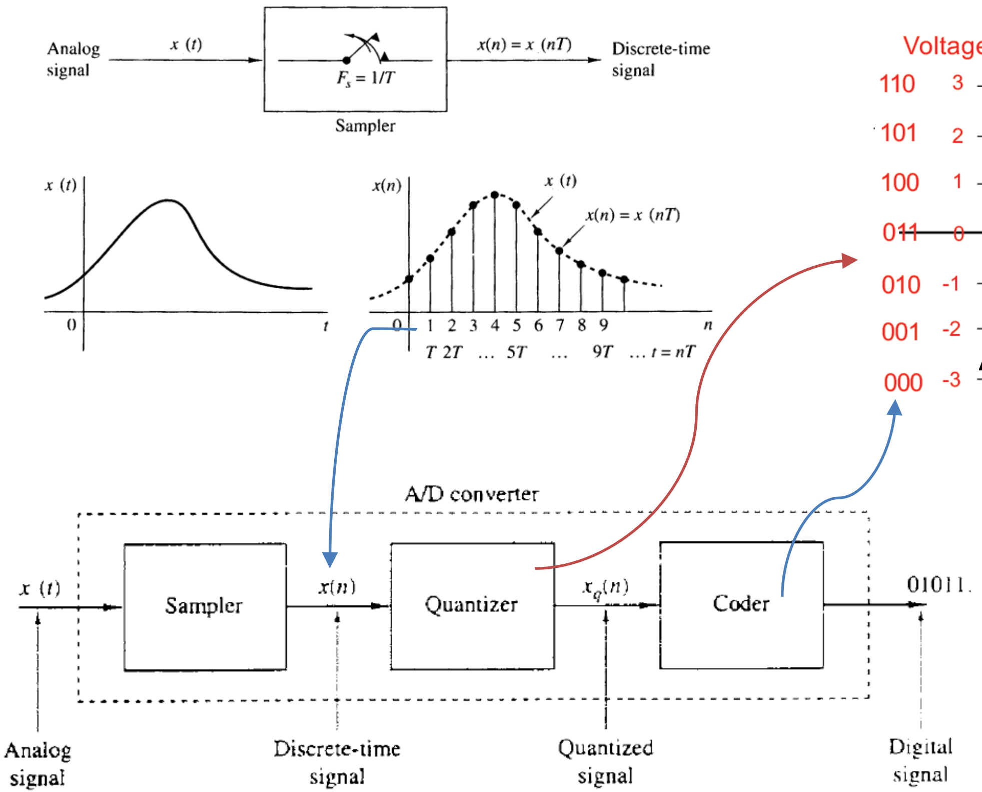
(d)

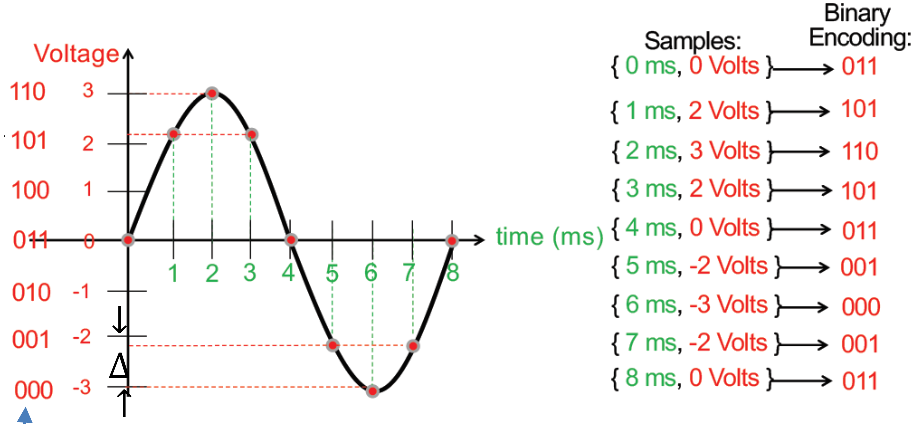


(c)



Periodic sampling of an analog signal



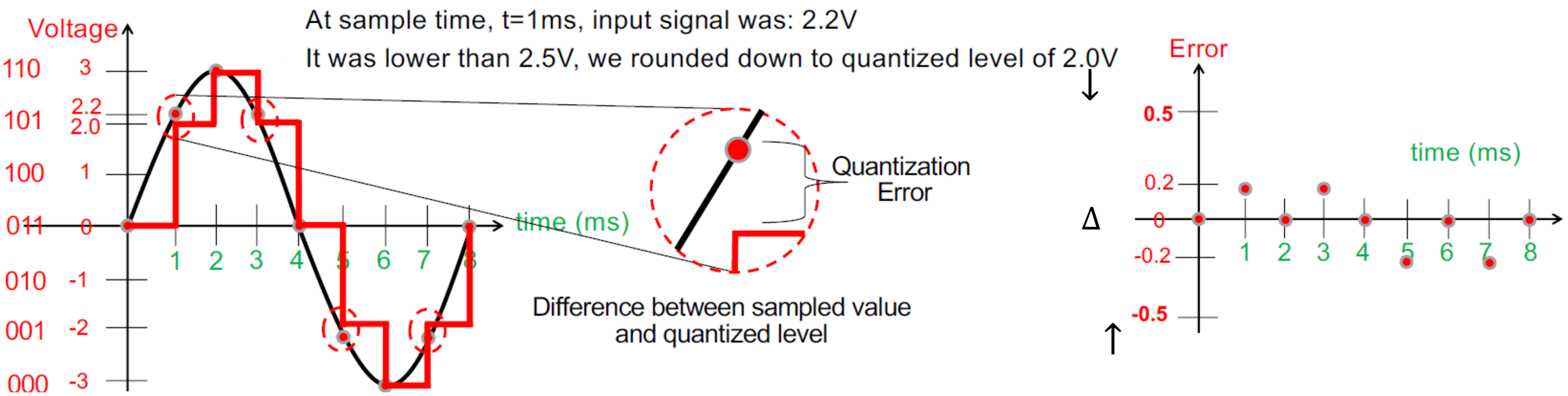


$$\Delta = resolution(quantization\ step) = \frac{v_{max}-V_{min}}{L-1}$$

$$L = Number\ of\ quantization\ levels$$

$$Number\ of\ bits\ required\ to\ each\ level\ (b) \geq \log_2 L$$

Quantization error



Mean-square quantization error: $\frac{\Delta^2}{12}$

$$\Delta = resolution(quantization\ step) = \frac{v_{max-}V_{min}}{L-1}$$

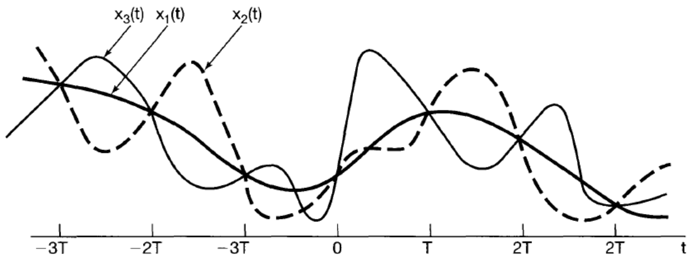
Sampling theorem

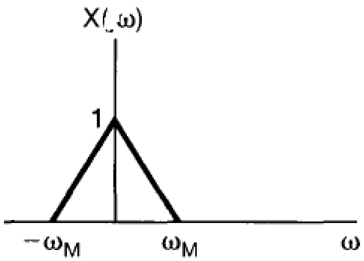
- **A continuous-time signal**
  - Can be completely represented by and recoverable from knowledge of its values, or *samples*, *at* **points equally spaced in time** - *sampling theorem*

✓ If a signal is band limited -i.e., if its Fourier transform is zero outside a finite band of frequencies

✓ If the samples are taken sufficiently close together in relation to the highest (2x) frequency present in the signal,

-- then the samples *uniquely specify* the signal, and we **can reconstruct it perfectly**





Example: Consider the analog signal

$$x_a(t) = 3\cos 2\,000\pi t + 5\sin 6\,000\pi t + 10\cos 1\,2000\pi t$$

(a) What is the *Nyquist /sampling rate required* for this signal to reconstruct at receiver?

The frequency existing in the analog signal are

$$\cos 2\,000\pi t \rightarrow \omega = 2\pi F_1 = 2000\pi \rightarrow F_1 = \frac{2000\pi}{2\pi} = 1000 = 1kHz$$

$$\sin 6\,000\pi t \rightarrow \omega = 2\pi F_2 = 6000\pi \rightarrow F_2 = \frac{6000\pi}{2\pi} = 3000 = 3kHz$$

$$\cos 1\,2000\pi t \rightarrow \omega = 2\pi F_3 = 12000\pi \rightarrow F_2 = \frac{12000\pi}{2\pi} = 6000 = 6kHz$$

$$F_{\max} = 6kHz$$

According to sampling theorem, sampling rate should be (called Nyquist rate) $F_s > 2F_{\max} = 12kHz$

Thank you!