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1. Axioms of Probability

Topics: Sample space, Events, Axioms, Probability rules & def^ns

Random Experiment:

i) all outcomes are known in advance.

ii) for any performance the result is unknown

iii) Experiment can be repeated under identical conditions.

Ex: Throwing a dice, drawing a card, tossing a coin etc.

Sample space:

It is a pair (Ω, S) where ~~omega~~ sigma

i) Ω is set of all possible outcomes

ii) S is set of all subsets of Ω

If Cardinality of $\Omega = n$

Cardinality of $S = 2^n$

Elements of Ω are called sample points.

Ex i) Tossing a coin $\Omega = \{H, T\}$

$S = \{\{H\}, \{T\}, \phi, \{H, T\}\}$

ii) Tossing a coin twice

$\Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

$S = \{\phi, \Omega, \{(H,H)\}, \{(H,T)\}, \{(T,H)\}, \{(T,T)\},$
$\{(H,H),(H,T)\}, \{(H,H),(T,H)\}, \{(H,H),(T,T)\},$
$\{(T,H)(T,T)\}, \{(H,T),(T,H)\}, \{(H,T),(T,T)\}, \{(H,H),(H,T),(T,H)\},$
$\{(H,H)(T,H)(T,T)\}, \{(H,H),(H,T),(T,T)\}, \{(H,T),(T,H),(T,T)\}$

iii) Throwing a dice

$\Omega = \{1,2,3,4,5,6\}$, $|\Omega| = 6$

$S = \{\phi, \Omega, \{1\}, \{2\}, \{3\}, \{4\}, \dots\}$

$|S| = 2^6 = 64$

<u>Event</u>: Any element $A \in S$ is called event (set)

→ Each one point set (set of cardinality 1) is called <u>elementary</u> <u>event</u>

<u>Ex</u>: Getting at least one head when a coin is tossed twice $A = \{(H,T), (T,H), (H,H)\}$

<u>Ex</u>: Get Gender of a baby $\Omega = \{g, b\}$

$S = \{\phi, \{g,b\}, \{g\}, \{b\}\}$

Let E be the event that baby is a girl.

$E = \{g\}$

<u>Impossible</u> <u>event</u>: ϕ : getting a 1 when you toss a coin

$P(\phi) = 0$

<u>Sure</u> <u>event</u>: Getting a head or tail when a coin is tossed

$E = \{H, T\}$

$P(E) = 1$

Ex: A coin is tossed until head appears → countable & infinite

$\Omega = \{H, TH, TTH, TTTH, \ldots\}$

S = set of all subsets of Ω

Axioms of Probability:

Let (Ω, S) be the sample space.

A function P is called probability if it satisfies the following.

i) $P(A) \geq 0 \quad \forall$ elements in S

ii) $P(\Omega) = 1$

iii) For any disjoint sets $\{A_j\}$, $j = 1, 2, \ldots$

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

→ $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$ is called countable additive property

$$\bigcap_{j=1}^{\infty} A_j = \phi$$

Probability: For any sample point ω_i for n possible outcomes, $P(\omega_i) = \frac{1}{n}$

Ex: $\{H, T\}$

$P(H) = \frac{1}{2}$

$P(T) = \frac{1}{2}$

Ex 2: Throwing a dice

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$P(1) = P(2) = P(3) = \ldots = P(6) = \frac{1}{6}$

Ex 3: $P(\omega_i)$ throwing a dice twice: $P(\omega_i) = \left(\frac{1}{36}\right)$

$\Omega = \{(i, j) : 1 \leq i \leq 6,\ 1 \leq j \leq 6\}$

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Probability of an event $A \in S$, where

A contains m elements, $1 \leq m \leq n$

$$P(A) = \frac{m}{n}$$

Ex 1: Tossing coin twice $|\Omega| = 4$

A: Getting atleast one head

$A = \{(H,H), (H,T), (T,H)\}$

$P(A) = \frac{3}{4}$

Ex 2: Getting same number when you throw dice twice.

$A = \{(i,i) \,/\, 1 \leq i \leq 6\}$

$= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$P(A) = \frac{6}{36} = \frac{1}{6}$

Mutually exclusive events: The events which cannot occur at same time.

Ex: $E_1 = \{H\}$, $E_2 = \{T\}$

Events are disjoint sets. $E_1 \cap E_2 = \phi$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Def^n: Probability of not happening of an event denoted by A^c

$P(A^c) = 1 - P(A)$

Def^n: Inclusion & Exclusion Principle (Addition rule)

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

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Ex: Coin is tossed thrice

$\Omega = \{(H,H,H), (H,H,T), (H,T,T),$
$(T,T,T), (H,T,H), (T,H,T)$
$(T,H,H)\ (T,T,H)\}$

$|\Omega| = 8 \quad |S| = 2^8$

A be the event that atleast one head shows up.

$P(A) = 1 - P(A^c)$

$A^c = \{(T,T,T)\}$

$= 1 - P(\text{no heads})$

$= 1 - \frac{1}{8}$

$P(A) = \frac{7}{8}$

Ex: Die is rolled twice

$|\Omega| = 36 \qquad \Omega = \{(i,j) : 1 \le i,j \le 6\}$

A_1 be the event that first throw $i \le 2$

A_2 be the event that second throw $j \ge 5$

Find Probability of happening of either $i \le 2$ or $j \ge 5$

$A_1 = \{(1,1)(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2)$
$(2,3), (2,4), (2,5), (2,6)\}$

$P(A_1) = \frac{12}{36}$

$A_2 = \{(1,5), (1,6), (2,5), (2,6), (3,5), (3,6)$
$(4,5), (4,6), (5,5), (5,6), (6,5), (6,6)\}$

$P(A_2) = \frac{12}{36}$

$A_1 \cap A_2 = \{(1,5), (1,6), (2,5), (2,6)\}$

$P(A_1 \cap A_2) = \frac{4}{36}$

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = \frac{12}{36} + \frac{12}{36} - \frac{4}{36} = \frac{5}{9}$

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Principle of Inclusion Exclusion:

Let $A_1, A_2, \ldots A_n$ be n events

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) - \sum_{k_1<k_2}^{n} P(A_{k_1} \cap A_{k_2}) +$$

$$\sum_{k_1<k_2<k_3}^{n} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) - \ldots + (-1)^{n+1} P\left(\bigcap_{k=1}^{n} A_k\right)$$

For $n = 3$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2)$$

$$- P(A_2 \cap A_3) - P(A_3 \cap A_2) + P(A_1 \cap A_2 \cap A_3)$$

Combinatorics & Probability - 4 Rules

Rule 1: Given collection of

$a_{11}, a_{12}, a_{13}, \ldots a_{1n_1}$ of n_1 elements

$a_{21}, a_{22}, a_{23} \ldots a_{2n_2}$ of n_2 elements

$\vdots$

$a_{k1}, a_{k2}, a_{k3} \ldots a_{kn_k}$ of n_k elements,

it is always possible to form $n_1 \cdot n_2 \cdots n_k$ ordered k-tuples of the form $(a_{1j_1}, a_{2j_2}, a_{3j_3}, \ldots, a_{kj_k})$ containing one element of each kind.

Ex: 'r' distinguishable balls to be placed in 'n' cells.

n^r possible cases.

Rule 2:

Ordered samples let $a_1, a_2 \ldots a_n$ be the set of n elements. Any ordered arrangement $\underbrace{(a_{i_1}, a_{i_2}, \ldots a_{i_r})}_{\text{sample}}$ of r of these n elements is called ordered sample.

i) Sampling with replacement
ii) Sampling without replacement

| $b_1\ b_2\ b_3\ b_4$ |

$r = 2 \quad n = 4 \quad$ SWR $\quad 4 \times 4 = 16 = n^r$

$r = 2 \quad n = 4 \quad$ SWOR $\quad 3 \times 4 = {}^nP_r$

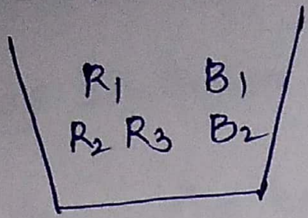
Sampling with replacement different samples n^r
Sampling without replacement different samples ${}^nP_r$

Ex1: Consider set of n elements. A sample of size 'r' drawn at random with replacement. The probability that no element appears more than once $= \dfrac{{}^nP_r}{n^r}$

Ex 2: r students birthdays

$P(\text{all students have diff birthdays}) = \dfrac{{}^{365}P_r}{(365)^r}$

Ex:



With replacement

P(1st ball is Red & 2nd ball is blue) $= \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

Without replacement

P(1st ball is Red & 2nd ball is blue) $= \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$

Rule 3: There are ^nC_r different sub populations of size $r \le n$, from a population of n elements

where $^nC_r = \dfrac{n!}{r!(n-r)!}$

Ex: Distribution of r balls in n cells. A_k be the event that a specified cell has k balls

$k = 0, 1, \ldots r$.

$$P(A_k) = {}^rC_k \cdot \frac{(n-1)^{r-k}}{n^r}$$

Rule 4: A population of n elements

The no. of ways the population can be partitioned into k subpopulation of sizes $r_1, r_2, \ldots r_k$ such that $r_1 + r_2 + \ldots + r_k = n$

$$^{n}C_{r_1 r_2 \ldots r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}$$

↓

Multinomial coefficients

Ex: 8 dishes. You want to choose 3 for lunch, 3 for dinner & 2 for breakfast $^{8}C_3 \times {}^{5}C_3 \times {}^{2}C_2$

The no. of ways to choose $= \frac{8!}{3!3!2!}$

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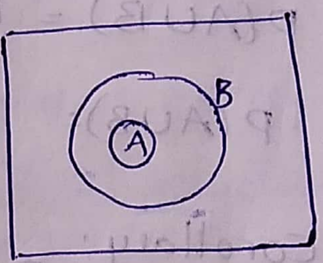
Probability Rules:

1. Probability is Monotone and subtractive

i.e., $A, B \in S$, and $A \subseteq B$ then $P(A) \leq P(B)$ (Monotone)

and $P(B-A) = P(B) - P(A)$ (subtractive)

Proof: $A \subseteq B$



$B = (A \cap B) \cup (B - A)$
(union of exclusive events)

By 3rd axiom $P(B) = P(A \cap B) + P(B-A)$

$P(B) = P(A) + P(B-A)$

$P(B-A) = P(B) - P(A)$ → subtractive

$P(A) = P(B) - P(B-A)$

$P(A) \leq P(B)$ → Monotone property

Corollary: For any event $A \in S$,

$$P(A) \le P(B), \quad B = \Omega$$

$$P(A) \le P(\Omega) = 1$$

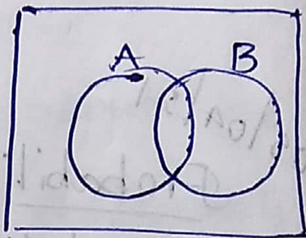
$$\boxed{0 \le P(A) \le 1}$$

2. **Addition Rule:** If $A, B \in S$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof:

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$$



By Third axiom

$$P(A \cup B) = P(A - B) + P(A \cap B) + P(B - A)$$

$$A = (A \cap B) \cup (A - B) \qquad B = (A \cap B) \cup (B - A)$$

$$A - B = A - (A \cap B) \qquad B - A = B - (A \cap B)$$

$$P(A \cup B) = P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Corollary: P is subadditive i.e., for $A, B \in S$

$$\boxed{P(A \cup B) \le P(A) + P(B)}$$

Generalization:

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

(we can prove this by mathematical induction)

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Corollary 2: $B = A^c$, A & B are disjoint

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$B = A^c \Rightarrow P(A \cup A^c) = P(A) + P(A^c) - P(A \cap A^c)$$

$$1 = P(A) + P(A^c) - 0$$

$$P(A) + P(A^c) = 1$$

$$\boxed{P(A^c) = 1 - P(A)}$$

Inequalities: (unit-6)

1. Bonferroni's Inequality:

Given n events $A_1, A_2, \ldots A_n$

$$\underset{\text{Lower bound}}{\sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j)} \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \underset{\text{Upper bound}}{\sum_{i=1}^{n} P(A_i)}$$

R.H.S L.H.S

Proof:

L.HS $P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$ can be proved using mathematical induction

R.H.S. $\sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right)$

can be proved by mathematical Induction.

For $n=2$, $P(A_1) + P(A_2) - P(A_1 \cap A_2) = P(A_1 \cup A_2)$

The condition is true (by addition rule)

For $n=3$

$$P(A_1)+P(A_2)+P(A_3)-P(A_1\cap A_2)-P(A_2\cap A_3)-P(A_1\cap A_3)$$

$$\leq P(A_1\cup A_2\cup A_3)$$

is true by inclusion exclusion principle for 3 sets.

We P.T this inequality is ~~true~~ holds for $3<m\leq n-1$,

We assume that if it is true for m & P.T it is true for m+1

For $n=m$, $$P\left(\bigcup_{i=1}^{m} A_i\right) \geq \sum_{i=1}^{m} P(A_i) - \sum_{i<j}^{m} P(A_i\cap A_j) \quad (1)$$

Now consider

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i \cup A_{m+1}\right)$$

$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right)$$

(By addition rule)

Using eqn (1) or assumption

$$\cancel{P\left(\bigcup_{i=1}^{m+1} A_i\right) \geq \sum_{i<j}^{m} (A_i\cap A_j)}$$

$$P\left(\bigcup_{i=1}^{m+1} A_i\right) \geq \sum_{i=1}^{m} P(A_i) - \sum_{i<j}^{m} P(A_i\cap A_j) + P(A_{m+1})$$

$$- P\left(A_{m+1} \cap \bigcup_{i=1}^{m} A_i\right)$$

$$\geq \sum_{i=1}^{m+1} P(A_i) - \sum_{i<j}^{m} P(A_i\cap A_j) - P\left(\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right)$$

$$\geq \sum_{i=1}^{m+1} P(A_i) - \sum_{i<j}^{m} P(A_i \cap A_j) - \sum_{i=1}^{m} P(A_i \cap A_{m+1})$$

$$\geq \sum_{i=1}^{m+1} P(A_i) - \sum_{i<j}^{m+1} P(A_i \cap A_j)$$

2. Boole's Inequality:

For any 2 events A & B

$$P(A \cap B) \geq 1 - P(A^c) - P(B^c)$$

Proof: Consider

$$P(A \cap B) = 1 - P((A \cap B)^c) \quad \text{(using complement rule)}$$

$$= 1 - P(A^c \cup B^c) \quad \text{(using Demorgan's)}$$

$$= \cancel{1} - [\cancel{P(A^c) + P(B^c)}$$

By subadditive rule

$$P(A^c \cup B^c) \leq P(A^c) + P(B^c)$$

$$-P(A^c \cup B^c) \geq -P(A^c) - P(B^c)$$

$$P(A \cap B) \geq 1 - P(A^c) - P(B^c)$$

Generalization: For a countable sequence of events

Infinite $P(\bigcap_{i=1}^{\infty} A_i) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$

Finite $P(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$

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1) Bonferroni's inequality: $\_\_ \leq P(\cup A_i) \leq \_\_$

Let us suppose that there are 5 events

A_1, A_2, A_3, A_4, A_5, $P(A_i) = 0.05$

$P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } A_4 \text{ or } A_5) = P(\bigcup_{i=1}^{5} A_i)$

$= P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)$

Upper Bound $P(\bigcup_{i=1}^{5} A_i) \leq P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5)$

$\leq 5(0.05)$

$P(\bigcup_{i=1}^{5} A_i) \leq 0.25$

Lower Bound $P((A_i) \cap (A_j)) = 0.01$, if $i \neq j$, $1 \leq i, j \leq 5$

$\sum_{i<j} P(A_i \cap A_j) = 10(0.01) = 0.1$

Lower Bound $\sum_{i=1}^{5} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$

$= 0.25 - 0.1$

$= 0.15$

$0.15 \leq P(\bigcup_{i=1}^{5} A_i) \leq 0.25$

2) Boole's Inequality: $P(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$

Let A_i, $i = 1$ to 10 with $P(A_i) = 0.99$

$P(A_1 \cap A_2 \cap \ldots A_{10}) \geq 1 - 0.01$

≥ 0.9

$P(A_i^c) = 1 - P(A_i)$

$= 1 - 0.99$

$= 0.01$

Finite Sample Space

Ex: 1) 2 dice are thrown, Let A be the event that sum of the numbers on two throws is 7

$n(\Omega) = 36$

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$P(A) = \frac{6}{36} = \frac{1}{6}$

Countable & Infinite Sample Space.

2) Let a coin is tossed untial head appears

$\Omega = \{H, TH, TTH, TTTH, TTTTH, \ldots\}$

position of head

$= \{1, 2, 3, \ldots\}$

$P(1) = \frac{1}{2}$, $P(2) = \frac{1}{4}$, $P(3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$P(i) = \frac{1}{2^i}$ $\quad P(i) \geq 0$

$P(\Omega) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$

P(getting a head at 3^{rd} or 5^{th} position)

$$= P(3) + P(5)$$

$$= \frac{1}{2^3} + \frac{1}{2^5}$$

Uncountable space.

Ex: Life time of a battery

$\Omega = (0, \infty)$ S = Borel field is set of all intervals of $(0, \infty)$

$$P(I) = P((a,b))$$

$$= \int e^{-x} dx$$

$b > a$

$$P(a,b) = \int_a^b e^{-x} = \left[-e^{-x}\right]_a^b = \frac{1}{e^a} - \frac{1}{e^b} > 0$$

$$P(\Omega) = P(0,\infty)$$

$$= \int_0^\infty e^{-x} dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= -\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = 1$$

$$P(2,5) = \frac{1}{e^2} - \frac{1}{e^5}$$

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Ex: Probability of getting a queen or a diamond when a card is picked at random from a deck of 52 card

sol: $P(Q \cup D) = P(Q) + P(D) - P(Q \cap D)$

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$

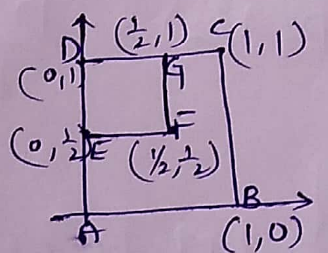
$= \frac{16}{52} = \frac{4}{13}$

Measures:

$P(A) = \frac{\text{Measure}(A)}{\text{Measure}(\Omega)}$ Area, Volume, Length

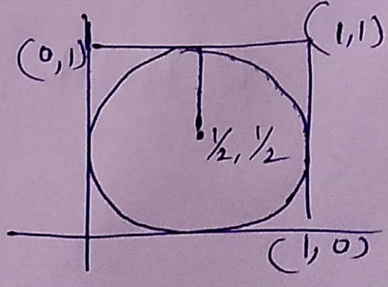
Ex.

1) Let A be the event that point is from $\{(x,y) / 0 \le x \le 1/2, \frac{1}{2} \le y \le 1\}$



$P(A) = \frac{\text{Area of DGFE}}{\text{Area of ABCD}} = \frac{\frac{1}{2} \times \frac{1}{2}}{1 \times 1} = \frac{1}{4}$

2) A be the event that the point is from a circle with centre $(1/2, 1/2)$ are $r = 1/2$



$P(A) = \frac{\text{Area of circle}}{\text{Area of square}}$

$= \frac{\pi(\frac{1}{2})(\frac{1}{2})}{1 \times 1} = \frac{\pi}{4}$