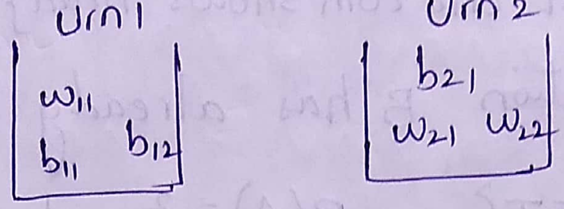
06/4/21

2. CONDITIONAL PROBABILITY

i) <u>Ex:</u>



A fair coin is tossed if H $\rightarrow$ urn 1

if T $\rightarrow$ urn 2

Let E be the event that the ball which is drawn is black.

$\Omega = \{(HW_{11}, HB_{11}, HB_{12}), (TW_{21}, TW_{22}, TB_{21})\}$

$E = \{HB_{11}, HB_{12}, TB_{21}\}$

$P(E) = \frac{n(E)}{n(\Omega)} = \frac{3}{6} = \frac{1}{2}$

ii) Conditional information:

Head has appeared on tossing $\Omega = \{HB_{11}, HB_{12}, HW_{11}\}$

P(getting black ball after getting head on tossing)

$= \frac{2}{3} = \frac{2/6}{3/6} = \frac{P(E \cap H)}{P(H)}$

$$\boxed{P(A/H) = \frac{P(A \cap H)}{P(H)}}$$

Conditional infoor

H has already happened

Ex: Toss two coins.

$\Omega = \{HH, HT, TH, TT\}$

A = {both coins show same face}

B = {atleast one coin shows Head}

Conditional information B has already happened

$A = \{HH, \cancel{H} TT\}$ $\quad P(A) = \frac{2}{4} = \frac{1}{2}$

$B = \{HT, TH, HH\}$ $\quad P(B) = \frac{3}{4}$

$P(A \cap B) = \frac{1}{4}$

$\Omega' = \{HT, TH, HH\}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = 1/3$$

Def: Let (Ω, S, P) be probability space

$H \in S$ with $P(H) > 0$ for arbitrary event

$A \in S$, $P(A/H) = \dfrac{P(A \cap H)}{P(H)}$

is called conditional probability of A given H

$$P(A \cap H) = P(H) \cdot P(A/H)$$

$$P(H \cap A) = P(A) \cdot P(H/A)$$

H.W

A lottery consists of 1 to 50 members what is the probability of getting a 3 multiple or a 5 multiple

$$P(3mul \cup 5mul) = P(3mul) + P(5mul) - P(3mul \cap 5mul)$$

$$= \frac{16}{50} + \frac{10}{50} - \frac{3}{50}$$

$$= \frac{23}{50}$$

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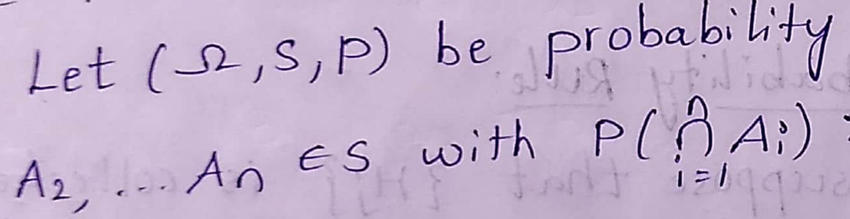
Multiplication Rule:

Let (Ω, S, P) be probability space

$A_1, A_2, \ldots A_n \in S$ with $P(\bigcap_{i=1}^{n} A_i) > 0$ then

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1) \cdot P\{A_2/A_1\} \cdot P\{A_3/A_1 \cap A_2\} \cdots \cdots \cdot P\{A_n / \bigcap_{j=1}^{n-1} A_j\}$$

is called Multiplication rule.



A die is rolled if 1, 2 or 3 $\rightarrow$ Urn 1
4 $\rightarrow$ Urn 2
5 or 6 $\rightarrow$ Urn 3

A marble is drawn at random from selected urn.

A be the event that marble drawn is white if U, V & W denote the events that the urn selected is 1, 2, & 3 respectively.

$$A = (A \cap U) \cup (A \cap V) \cup (A \cap W)$$

$$P(A) = P(A \cap U) + P(A \cap V) + P(A \cap W)$$

$$P(A \cap U) = P(U) \cdot P(A/U) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$P(A \cap V) = P(V) \cdot P(A/V) = \frac{1}{6} \cdot \frac{2}{3} = \frac{1}{9}$$

$$P(A \cap W) = P(W) \cdot P(A/W) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

$$P(A) = \frac{1}{6} + \frac{1}{9} + \frac{1}{6} = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

Total Probability Rule:

Let us suppose that $\{H_j\}$ is collection of events in S such that $H_j \cap H_k = \phi$ & $\bigcup_{j=1} H_j = \Omega$ & $P(H_j) > 0, \forall j$ then

$$P(A) = \sum_{j=1}^{n} P(H_j) \cdot P(A/H_j)$$

for $A \in S$

$$A = \sum_{j=1}^{n} A \cap H_j$$

Urn 1: w, b b

Urn 2: b, w w

Urn 3: b b b, w w w

A die is rolled if 1, 2 or 3 $\rightarrow$ Urn 1

4 $\rightarrow$ Urn 2

5 or 6 $\rightarrow$ Urn 3

A marble is drawn at random from selected urn

$$P(V/A) = \frac{P(V \cap A)}{P(A)}$$

$\hookrightarrow$ already got a marble

what is the probability that it is from urn 2?

$$P(V \cap A) = \frac{1}{9}$$

$$P(A) = \frac{4}{9}$$

$$P(V/A) = \frac{\frac{1}{9}}{4/9} = \frac{1}{4}$$

$$P(U/A) = \frac{P(U \cap A)}{P(A)} = \frac{\frac{1}{6}}{\frac{4}{9}} = \frac{3}{8}$$

$$P(W/A) = \frac{P(W \cap A)}{P(A)} = \frac{\frac{1}{6}}{\frac{4}{9}} = \frac{3}{8}$$

Baye's Rule: Let $\{H_n\}$ be the disjoint sequence of events such that $P(H_n) > 0, \forall n$ and $\bigcup_{i=1}^{n} H_i = \Omega$ for any event $B \in S$, with $P(B) > 0$ then

$\rightarrow$ total probability

$$P\{H_i/B\} = \frac{P(H_i \cap B)}{P(B)} \quad \text{(using conditional probability)}$$

$\cancel{\sum_{i=1} P(H_i) P(B/H_i)}$

$$= \frac{P(H_i) \cdot P(B/H_i)}{\sum_{i=1} P(H_i) \cdot P(B/H_i)}$$

Independence of events: $P(B/A) = \frac{P(B \cap A)}{P(A)}$

Ex: Let 2 fair coins are tossed

$\Omega = \{HH, HT, TH, TT\}$

$A = \{\text{head on second throw}\} = \{HH, TH\}$

$B = \{\text{head on first throw}\} = \{HH, HT\}$

$P(A) = P(B) = \frac{1}{2}$, $A \cap B = \{H, H\}$

$P(A \cap B) = \frac{1}{4}$

$$P(A \cap B) = \underbrace{P(A) \cdot P(B/A)}_{\text{dependent}} = \underbrace{P(A) \cdot P(B)}_{\text{independent}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B)$$

Def Two events A & B are independent iff

$P(A \cap B) = P(A) \cdot P(B)$

~~then~~ if A & B are independent

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B) \cdot P(A)}{P(A)} = P(B)$$

Ex: A card is chosen at random from deck of 52 cards

$A = \{\text{card is an ace}\}$

$B = \{\text{card is club}\}$

Sol $P(A) = \frac{4}{52} = \frac{1}{13}$

$P(B) = \frac{13}{52} = \frac{1}{4}$

$A \cap B = \{\text{ace of clubs}\}$

$P(A \cap B) = \frac{1}{52}$

$P(A \cap B) = \frac{1}{52} = \frac{1}{13} \cdot \frac{1}{4} = P(A) \cdot P(B)$

Ex: Consider a family with 2 children

$\Omega = \{BB, BG, GB, GG\}$

E = Family has atleast one girl $= \{BG, GG, GB\}$

$P(E) = \frac{3}{4}$

F = Family has children of both gender $= \{GB, BG\}$

$P(F) = \frac{2}{4} = \frac{1}{2}$

$E \cap F = \{GB, BG\}$

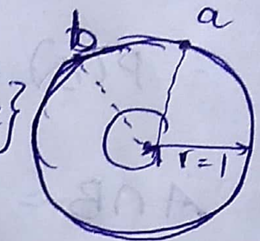
$P(E \cap F) = \frac{2}{4} = \frac{1}{2}$

$P(E) \cdot P(F) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} \neq \frac{1}{2}$

$\therefore$ E & F are dependent.

09/4/21 Tutorial-1

1) Consider a pointer that is free to spin about the centre of the circle with $r = 1$. If the pointer is spun by an impulse it will come to rest at some point.



$\Omega = \{$all points on circumference$\}$

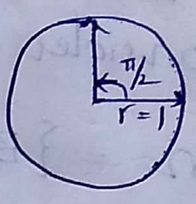
$0 \le x < 2\pi$

$(a, b) \subset [0, 2\pi)$

$P(a, b) = \dfrac{b-a}{2\pi} > 0$

↓

Probability of any point in the interval.



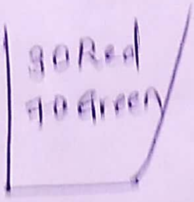
$P(0, \pi/2) = \dfrac{\pi/2 - 0}{2\pi} = \dfrac{1}{4}$

2) 60% of students wear neither ring nor a necklace. 20% wear ring & 30% wear necklace if one of the student is chosen randomly. What is the probability that the student wear

1) ring or necklace 40% $= \dfrac{40}{100} = \dfrac{2}{5}$

2) ring and necklace 10% $= \dfrac{10}{100} = \dfrac{1}{10}$

3) An urn consists of 30 Red & 70 green balls. P(exactly k red balls in a sample of 20 if sampling is done without Replacement).



$$P = \frac{{}^{30}C_k \cdot {}^{70}C_{20-k}}{{}^{100}C_{20}}$$

4) Birthday problem: Suppose there are n persons in a party $n \leq 365$ & no person has birthday on Feb 29^{th}. What is P(atleast 2 persons share same birthday).

$$P(A^c) = \frac{{}^{365}P_n}{(365)^n}$$

$$P(A) = 1 - P(A^c) = 1 - \frac{{}^{365}P_n}{(365)^n}$$

5) Consider a hand of 5 cards in a game if 5 cards are drawn randomly from deck of 52. $P(B/A) = ?$ where

$A = \{$at least 3 cards are spades$\}$

$B = \{$all the 5 are spades$\}$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} \neq \cancel{\frac{{}^{13}C_5 / {}^{52}C_5}{{}^{13}C_3 + {}^{13}C_4 + {}^{13}C_5}}$$

$$= \frac{{}^{13}C_5 / {}^{52}C_5}{\frac{{}^{13}C_3 \cdot {}^{39}C_2 + {}^{13}C_4 \cdot {}^{39}C_1 + {}^{13}C_5}{{}^{52}C_5}}$$

$$P(B/A) = \frac{{}^{13}C_5}{{}^{13}C_3 \cdot {}^{39}C_2 + {}^{13}C_4 \cdot {}^{39}C_1 + {}^{13}C_5}$$

6) An insurance company believes that people are divided into 2 classes. Accident prone & non accident prone. Statistics shows that an accident prone person will have an accident within a fixed 1 year period with probability 0.4 and it is 0.2 for non accident prone. If 30% of population is accident prone, What is P(a new policy holder will have an accident within an year of purchasing policy)?

$A_1 \rightarrow$ be the event that person will have an accident

$P(A_1/A) = \frac{P(A_1 \cap A)}{P(A)}$ $A \rightarrow$ policy holder is accident prone

$P(A_1) = P(A \cap A_1) + P(A^c \cap A_1)$

$= P(A)\, P(A_1/A) + P(A^c)\, P(A_1/A^c)$

$= \frac{3}{10} \cdot \frac{4}{10} + \frac{7}{10} \cdot \frac{2}{10} = \frac{26}{100} = 0.26 \rightarrow$ Total probability

b) Suppose that a new policy holder has an accident within the fixed period. What is the probability that he/she is accident prone.

$$P(A/A_1) = \frac{P(A \cap A_1)}{P(A_1)}$$

$$= \frac{P(A) \cdot P(A_1/A)}{P(A_1)}$$

$$= \frac{\frac{3}{10} \cdot \frac{4}{10}}{\frac{26}{10}}$$

$$= \frac{12}{26} = \frac{6}{13}$$

7) Two dice are thrown

$A = \{$ Sum of two numbers is 6 $\}$ $P(A) = \frac{5}{36}$

$B = \{$ first number is 4 $\}$ $P(B) = \frac{1}{6}$

$C = \{$ Sum of two numbers is 7 $\}$ $P(C) = \frac{1}{6}$

check if a) A & B are Independent
b) B & C are independent

$P(A \cap B) = \frac{1}{36}$ $\qquad P(A) \cdot P(B) = \frac{5}{36} \cdot \frac{1}{6} = \frac{5}{216}$

A & B are dependent

$P(B \cap C) = \frac{1}{36}$ $\qquad P(B) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

B & C are Independent

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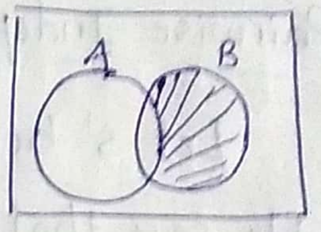
Theorem

If A and B are independent, then A & B^c, A^c and B and A^c & B^c are also independent

Proof: Given that A & B are independent

We need to P.T A^c & B are independent



$$A^c \cap B = B - (A \cap B)$$

$$P(A^c \cap B) = P(B - (A \cap B))$$

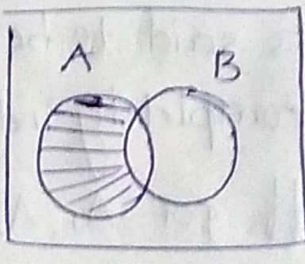
$$= P(B) - P(A \cap B) \quad \because A \cap B \subset B$$

$$= P(B) - P(A) \cdot P(B) \quad \because (A \,\&\, B \text{ are independent})$$

$$= P(B)\left[1 - P(A)\right]$$

$$P(A^c \cap B) = P(A^c) \cdot P(B)$$

$\therefore$ A^c & B are independent.



$$A \cap B^c = A - (A \cap B)$$

$$P(A \cap B^c) = P(A - (A \cap B))$$

$$= P(A) - P(A \cap B)$$

$$= P(A) - P(A)\,P(B)$$

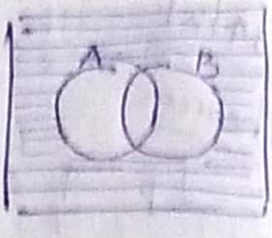
$$= P(A)\left[1 - P(B)\right]$$

$$P(A \cap B^c) = P(A) \cdot P(B^c)$$

$\therefore$ A & B^c are independent

$A^c \cap B^c = U - (A \cup B)$

$P(A^c \cap B^c) = 1 - P(A \cup B)$



$= \underbrace{1 - P(A)} - P(B) + P(A \cap B)$

$= P(A^c) - P(B) + P(A) \cdot P(B)$

$= P(A^c) - P(B)[1 - P(A)]$

$= P(A^c)[1 - P(B)]$

$P(A^c \cap B^c) = P(A^c) P(B^c)$

A^c & B^c are independent

Pairwise Independence:

Let s' be the collection of events from S. We say that the events in s' are pairwise independent if

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{for } A, B \in s'$$

Mutually Independent / Completely Independent

Def: A collection of events in s' is said to be mutually / completely independent iff for $A_1, A_2, A_3 \ldots A_n \in s'$

$P(A_1 \cap A_2 \cap \ldots A_n)$

$= \prod_{i=1}^{n} P(A_i)$

$= P(A_1) \cdot P(A_2) \ldots P(A_n)$

A_1, A_2, A_3, A_4

P.I	M.I
A_1A_2	$A_1A_2, A_1A_3,$
A_1A_3	$A_1A_4, A_2A_3,$
A_1A_4	A_2A_4, A_3A_4
A_2A_3	$A_1A_2A_3, A_2A_3A_4$
A_2A_4	$A_1A_3A_4, A_1A_2A_4$
A_3A_4	$A_1A_2A_3A_4$

$P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ where $i \neq j$

$P(A_i \cap A_j \cap A_k) = P(A_i)\, P(A_j)\, P(A_k)$ $\quad i \neq j \neq k$

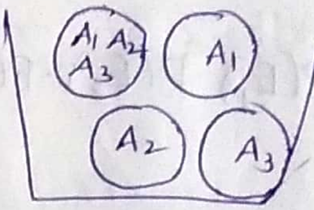
$\vdots$

$P(A_1 \cap A_2 \cap \cdots A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n)$

Ex:

1) Take 4 identical Marbles. Placing these in an urn

Let E_i denote that the symbol A_i appears on drawn marble



$E_1 = \{A_1$ on drawn marble$\}$.

$E_2 = \{A_2$ on drawn marble$\}$

$E_3 = \{A_3$ on drawn marble

$P(E_1) = \frac{1}{2} \quad P(E_2) = \frac{1}{2} \quad P(E_3) = \frac{1}{2}$

$P(E_1 \cap E_2) = \frac{1}{4} \quad P(E_2 \cap E_3) = \frac{1}{4} \quad P(E_3 \cap E_1) = \frac{1}{4}$

$P(E_1) \cdot P(E_2) = \frac{1}{4} \quad P(E_2) \cdot P(E_3) = \frac{1}{4} \quad P(E_3) \cdot P(E_1) = \frac{1}{4}$

So E_1, E_2, E_3 are pairwise independent

$P(E_1 \cap E_2 \cap E_3) = \frac{1}{4}$

$P(E_1) \cdot P(E_2) \cdot P(E_3) = \frac{1}{8}$

$E_1, E_2, \& E_3$ are not mutually independent.

2) Let $\Omega = \{1, 2, 3, 4\}$ Let P_i be the probability assigned to $\{i\}$ $P(1) = \frac{\sqrt{2}}{2} - \frac{1}{4}$; $P(2) = \frac{1}{4}$

$P(3) = \frac{3}{4} - \frac{\sqrt{2}}{2}$; $P(4) = \frac{1}{4}$

$E_1 = \{1, 3\}$ $E_2 = \{2, 3\}$ $E_3 = \{3, 4\}$

sol)

$$P(E_1 \cap E_2 \cap E_3) = P(3) = \frac{3}{4} - \frac{\sqrt{2}}{2}$$

$$P(E_1) \cdot P(E_2) \cdot P(E_3)$$

$$= [P(1) + P(3)] [P(2) + P(3)] \cdot [P(3) + P(4)]$$

$$= \left(\frac{\sqrt{2}}{2} - \frac{1}{4} + \frac{3}{4} - \frac{\sqrt{2}}{2}\right)\left(\frac{1}{4} + \frac{3}{4} - \frac{\sqrt{2}}{2}\right)\left(\frac{3}{4} - \frac{\sqrt{2}}{2} + \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} - \sqrt{2}\right)$$

$$= \frac{3}{4} - \frac{\sqrt{2}}{2}$$

$$P(E_1 \cap E_2) = P(3) = \frac{3}{4} - \frac{\sqrt{2}}{2}$$

$$P(E_1) \cdot P(E_2) = \frac{1}{2} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{\sqrt{2}}{4} \neq P(E_1 \cap E_2)$$

E_1, E_2, & E_3 are not pairwise independent hence not mutually independent.

Throwing two dice

$A = \{$ Sum of two points is 7 $\}$

$B = \{$ first throw has 3 $\}$

$C = \{$ 2nd throw has 4 $\}$

Check mutually independence

$P(A) = \frac{1}{6}$ $\quad P(B) = \frac{1}{6}$ $\quad P(C) = \frac{1}{6}$

$P(A \cap B) = \frac{1}{36}$ $\quad P(B \cap C) = \frac{1}{36}$ $\quad P(A \cap C) = \frac{1}{36}$

$P(A) \cdot P(B) = \frac{1}{36}$ $\quad P(B) \cdot P(C) = \frac{1}{36}$ $\quad P(A) \cdot P(C) = \frac{1}{36}$

$P(A \cap B \cap C) = \frac{1}{6}$

$P(A) \cdot P(B) \cdot P(C) = \frac{1}{216}$

A, B & C are pairwise independent but not mutually independent

* Mutually exclusive events are never independent.