Mathematics III

Real Analysis ,Numerical Analysis and Calculus

Unit -I

Sequence and Series

Review: Basics Of sets

Set :A set is a well defined collection of objects.

The individual objects of sets are called elements or members of set

Set :- Denote with Capital letters A, B, C.

Small letters = a, b, c, . . elements of set

element 'a' is a member of of set S  $a \in 'S'$

a is a not a member  $a \notin 'S'$ (does not belong S)

Representation two ways

$S = \{a, b, c \ldots\}$  $S = \{n : n \in N\}$

Typical sets :-

$\mathbb{N} \Rightarrow$ set of all natural number $= \{1, 2, 3 \ldots\ldots\}$

$\mathbb{Z} \Rightarrow$ Set of Integers $\{\ldots -2, -1, 0, 1, 2 \ldots\ldots\}$

$\mathbb{Q} \Rightarrow$ Set of Rational numbers = Integers + fractions  $\sqrt{2}$, $\pi$

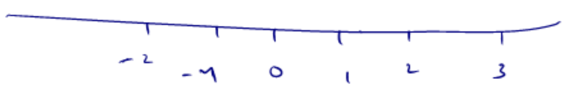
$\mathbb{R} \Rightarrow$ $\boxed{\text{set of real numbers}}$ $\Rightarrow$ rational and irrational

Union:- $A \cup B$ = All elements of A & B

Disjoint set:-

$A \cap B = \phi$

Intersection :- $A \cap B =$



equality of sets

Same elements in both sets

Null set = No elements $= \phi$

Universal set $= U$

$\Rightarrow$ <u>Subset</u> If A and B are two set $x \in A \Rightarrow x \in B$ then A is called subset of A. $A \subseteq B$

<u>Superset</u> :- $B \supseteq A$ (B contains A)

$A \subseteq B, B \subseteq A \Rightarrow A = B$

<u>Proper subset</u> $A \subseteq B$ and $A \neq B$ then A is proper subset of B

$A \subset B$ (There is atleast one element in B which is not in A)

<u>Subtraction</u> :- Inverse of addition composition.

$a - b = a + (-b)$ when $a, -b \in S$

$a + (-b) \in S$

<u>Division</u> = Inverse of Multiplication Composition.

The quotient $\frac{a}{b}$ $(b \neq 0)$ $a \times \frac{1}{b}$ (or) $ab^{-1}$ $a, \frac{1}{b} \in S$

Then $a \times \frac{1}{b} \in S$

Compositions

1) Addition composition : An addition composition is defined in a set S if to each pair of members a,b of S there corresponds a member a+b of S

$a, b \in S$ then $a+b \in S$

ex:- $S = \{1, 2, 3\}$ ⇒ Not addition composition

1,2 $1+2=3 \in S$

2,3 $2+3=5 \notin S$

ex:- $A = \{1, 2, 3, 4, \dots\}$ , $B = \{-1, 0, 1\}$

addition composition

2) Multiplication Composition:A multiplication composition is defined in S if to each pair of members a, b of S there corresponds a member a ab of S

$a, b \in S$ then $ab \in S$

ex:- $A = \{-1, 0, 1\}$ ⇒ Multiplication composition

ex:- $B = \{1, 2, 3\}$

$1,2 \Rightarrow 1 \times 2 = 2 \in B$

$2,3 \Rightarrow 2 \times 3 = 6 \notin B$ Not Multiplication Composition

** Subtraction and division may be defined as inverse operations of addition and multiplication respectively

STRUCTURES:

1. Algebraic structure

2. Field structure

3.Order structure

1.Algebraic structure : A set is said to possess an algebraic structure if the two compositions of Addition and multiplication are defined in the set

$A = \{-1, 0, 1\}$ ⟹ algebraic structure.

addition composition $a+b \in S$

Multiplication composition $a \times b \in S$

2. Field Structure: A set S is said to be a field if two compositions of Addition and muliplication be defined in it such that for all a,b,c belong to S the following properties are satisfied

A-1 : Set S is closed for addition

$a, b \in S \Rightarrow a+b \in S$

A2: Addition is commutative

a+b=b+a ✓

A3: Addition is associative

(a+b)+c = a+(b+c) ✓

A-4: Additive identity exists i.e there exist a member 0 in S such that

a+0= a

$\mathbb{N} = \{1, 2, \dots\}$ ⇒ does not satisfy $0 \notin S$

A5- Additive inverse exist s i.e to each elelment a belongs to S there exists an element -a belongs to S such that

a+(-a) = 0

$a \in S$ $\exists$ $-a \in S$

$\mathbb{Z} \Rightarrow \{\dots -1, 0, 1, 2 \dots\}$ = satisfy

$\mathbb{N} \Rightarrow \{1, 2, \dots\}$ does not satisfy

negative integer $\notin \mathbb{N}$

M1: S is closed for multiplication

$a, b \in S \Rightarrow a \times b \in S$

M2: Multiplication is commutative

ab=ba ✓

M3:Multiplication is associative

(ab)c= a(bc)

M4: Multiplicative Idebtity exists i.e there exists a member 1 in S such that

$1 \in S$

a.1 =a

M5 Multiplicative inverse exists i.e to each $0 \neq a \in S$, $\exists$ an element $a^{-1} \in S$ such that $a a^{-1} = 1$

AM- Multiplication is distributive with respect to addition i.e,

$$a(b+c) = ab+ bc$$

Thus a set S has a field structure if it possesses the two compositions of addition and multiplication and satisfies the eleven properties listed above

3. Order structure :

A field S is an ordered field if it satisfies the following properties:

O1: Law of Trichotomy : For any two elements a, $b \in S$ one and only one of the following is true

$$a>b,\ a= b,\ b>a$$

O2: Transitivity :

O3 : Compatibility of order relation with addition composition

O4: Compatibility of order Relation with multiplication Composition:

Compositions

1) Addition composition : An addition composition is defined in a set S if to each pair of members a,b of S there corresponds a member a+b of S

$a, b \in S$ then $a+b \in S$

ex:- $S=\{1,2,3\} \Rightarrow$ Not addition composition

$1,2 \quad 1+2=3 \in S$

$2,3 \quad 2+3=5 \notin S$

ex:- $A = \{1,2,3,4,\ldots\}$ , $B = \{-1,0,1\}$

addition composition

2) Multiplication Composition:A multiplication composition is defined in S if to each pair of members a, b of S there corresponds a member a ab of S

$a, b \in S$ then $ab \in S$

ex:- $A = \{-1,0,1\} \Rightarrow$ Multiplication composition

ex:-

$B = \{1,2,3\}$

$1,2 \Rightarrow 1\times 2 = 2 \in B$

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** Subtraction and division may be defined as inverse operations of addition and multiplication respectively

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1.Algebraic structure : A set is said to possess an algebraic structure if the two compositions of Addition and multiplication are defined in the set

$A = \{-1, 0, 1\} \Longrightarrow$ algebraic structure.

addition composition $a+b \in S$

Multiplication composition $a\times b \in S$

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A-4: Additive identity exists i.e there exist a member 0 in S such that

a+0= a

$\mathbb{N} = \{1, 2, \ldots\} \Rightarrow$ does not satisfy $\quad 0 \notin S$

A5- Additive inverse exist s i.e to each elelment a belongs to S there exists an element -a belongs to S such that

a+(-a) = 0

$a \in S \quad \exists\ -a \in S$

$\mathbb{Z} \Rightarrow \{\ldots -1, 0, 1, 2 \ldots\}$ = satisfy

$\mathbb{N} \Rightarrow \{1, 2, \ldots\}$ does not satisfy

negative integer $\notin \mathbb{N}$

M1: S is closed for multiplication

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ab=ba ✓

M3:Multiplication is associative

(ab)c= a(bc)

M4: Multiplicative Idebtity exists i.e there exists a member 1 in S such that

$1 \in S$

$\mathbb{N}$ ✓

$\mathbb{Z}$ ✓

$\mathbb{Q}$

a.1 =a

ex:- $A = \{2, 4, 6, 8, \ldots\}$

$2 \in S$

$\frac{1}{2} \in S \quad 2 \times \boxed{\frac{1}{2}} = 1$

M5 Multiplicative inverse exists i.e to each $0 \neq a \in S$, $\exists$ an element $a^{-1} \in S$ such that $aa^{-1} = 1$

AM- Multiplication is distributive with respect to addition i.e,

$$a(b+c) = ab+\cancel{bc}\ ac$$

Thus a set S has a field structure if it possesses the two compositions of addition and multiplication and satisfies the eleven properties listed above

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a>b, a= b, b>a

O2: Transitivity : $\forall a, b, c \in S$

$$a>b \wedge b>c \Rightarrow a>c$$

O3 : Compatibility of order relation with addition composition

$$\forall a, b, c \in S$$

$$a>b$$

$$\Rightarrow a+c > b+c$$

O4: Compatibility of order Relation with multiplication Composition:

$$\forall a, b, c \in S$$

$$a>b \wedge c \geq 0$$

$$ac > bc$$

Set of $\mathbb{N}$ = algebraic structure, Not field structure ( Not Satisfying $A_4$, AS, $M_4$ )

$\mathbb{Z}$ = algebraic, Not field structure ( $M_4$ Donnot Satisfy )

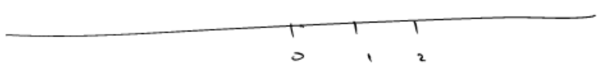
$\mathbb{Q}$ = field Structure
order "

$\mathbb{R}$ = field Structure
order

$\boxed{\text{Completeness property}}$ ( distinguishes $\mathbb{Q}$ from $\mathbb{R}$ )

$\Downarrow$

$\boxed{\text{No gaps}}$



1. Maximum and Minimum

Definition : Let S be a non empty subset of R $S \subseteq \mathbb{R}$

a) If $s_0 \in S$ and $s_0 \geq s \;\forall\; s \in S$ [i.e if set S contains a largest element $s_0$]

then we call '$s_0$' the maximum of S.

$$\max S = s_0$$

(b) If $s_0 \in S$ and $s_0 \leq s \;\forall\; s \in S$ [i.e if S contains a smallest element $s_0$]

then we call $s_0$ the minimum of S.

$$\min S = s_0$$

☆ max & min $\in S$

ⓐ $S = \{1, 2, 3, 4, 5\}$

Max $S = 5$ Min $S = 1$

ⓑ $\mathbb{N}$ : set of natural no's : $\{1, 2, 3, \ldots\ldots\}$

Max = No Min = 1

③ $S_1 = \{x \in \mathbb{R} : a < x \leq b\}$ Max $S_1 = b$ Min $S_1$ = No

$S_2 = \{x \in \mathbb{R} \; a \leq x \leq b\}$ Max $S_2 = b$ Min $S_2 = a$

$S_3 = \{x \in \mathbb{R} \; a < x < b\}$ Max $S_3$ = No Min $S_3$ = No

④ $\mathbb{Z}$ = No max No min

⑤ $\mathbb{Q}$ = No max No min

Upper bound and lower bound

Definition: let $S$ be a non empty subset of R

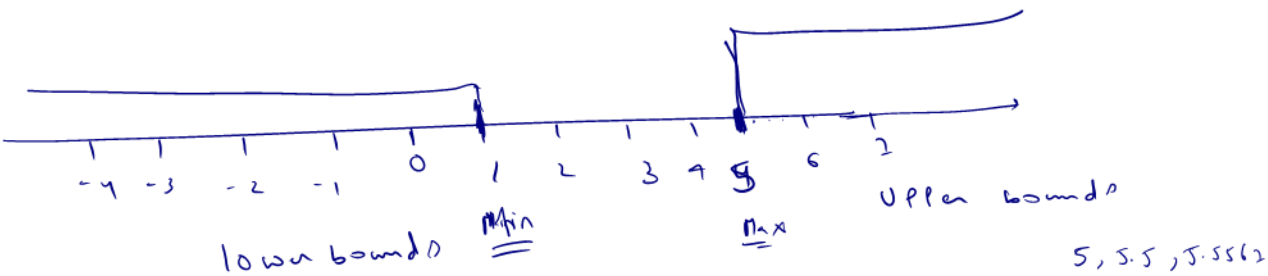
a) If a real number M satisfies $s \le M$ ($M \ge s$) $\forall\, s \in S$, then M is called an upperbound of $S$ and set S is said to bounded above

(b) If a real number m satisfies $m \le s$ $\forall\, s \in S$ then m is called a lower bound of $S$ and $S$ is said to be bounded below.

(c) The set S is said to be bounded if it bounded above and bounded below.

ex: $S = \{1, 2, 3, 4, 5\}$

Max S = 5 Min S = 1




ex: $S_1 = \{x \in \mathbb{R} : a < x \le b\}$

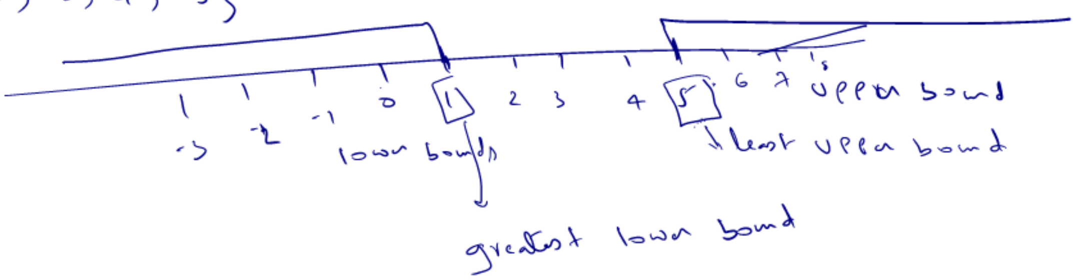
Min S = No Max S = b upper bound: greater than b

lower bound: a, below

$\mathbb{Z} = \{\ldots -1, 0, 1, \ldots\ldots\}$ Not bound above bounded below

$\mathbb{N} = \{1, 2, \ldots\ldots\}$ Not bounded above

ex: $S = \{1, 2, 3, 4, 5\}$



Least upper bound/Supremum and Greatest lower bound/Infimum

Definition:Let S be a non empty subset R

a) If S is bounded above and S has a least upper bound then we call it as supremum of S and it is denoted by Sup S , lub S

b) If S is bounded below and S has a greatest lower bound ,then we call it as infimum of S and it is denoted by inf S , glb S

★ If a set S has max and min

Sup S = Max S

Inf S = Min S

$A = \{1, 2, 3, 4, 5\}$

Max S = 5

Min S = 1

S : $x \in \mathbb{R}$ , $a < x < b$ (3, 8) -1,

$\mathbb{R} = \{ \dots -1, -2, \dots 4, \dots$

$S \subseteq \mathbb{R} = \{ \dots 3, 3.2 \dots, 4, \dots 8 \dots$ (3, 4)

greatest lower bounds

least upper bound

$\mathbb{Z} \subseteq \{ -2, -1, 0, 1, 2, 3 \} \subseteq \mathbb{R}$

$\{ \dots -2, -1, 0, 1, \dots 3 \dots \}$

$\mathbb{N} = \{1, 2, \dots \}$ Not bounded above

$S = \{ x \in \mathbb{R} : a \leq x < b \}$  Max = No

upperbound / Sup S = b, above

$[0, 2]$ = Inf = min S = 0 , Max S = Sup S = 2

lower bounds = $\leq 0$  upper bound $\geq 2$

$[-1, 3)$ = Min = -1, Inf S = -1, Max = No, Sup = 3

$(1, 3)$ = No max, No min

Inf S = 1  Sup S = 3

Review

Maximum s :- greatest element

Minimums : least element

① $\max\{0, \pi, e, 4/3\} = \pi$

② $A = \max\{n \in \mathbb{Z} : -4 \le n \le 100\} = \{-3, -2, -1, 0, \dots, 100\}$

$-4 < n < 100$

$\min A = 1$

$\sup A = 100$

$\operatorname{Max} = 100$

Min $A = -3$


✱ : Every finite non empty subset of $\mathbb{R}$ has Max, min

Upperbound, lowerbound

✱ :- $-4 < n \le 100 = \{-3, -2, -1, 0, \dots 100\}$

$\boxed{\operatorname{Inf} = -3}$ min $= -3$

✱ : If Max and minimum element $\operatorname{Inf} S = \min S$

$\sup S = \max S$

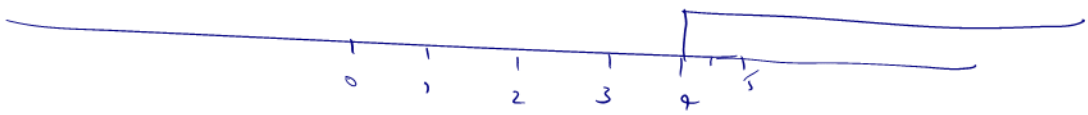
$\boxed{n \in \mathbb{Z}}$

LUB Axiom ( Completeness Axiom)

* Every non empty subset S of R that is bounded above has a least upper bound. In other words Sup S exist and is a real number.

$S \subseteq \mathbb{R}$

$A = \{1, 2, 3, 4\}$




Set of upper bound $\{4, 4.5, 4.625 \ldots \ldots \}$

* Corollary

Every non empty $S \subseteq \mathbb{R}$ that is bounded below has greatest lower bound (or) Inf S

** completeness axiom distinguishes $\mathbb{Q}$ from $\mathbb{R}$.

* Set of real numbers is only set which is complete ordered field.

$\mathbb{Q}$ = Not complete ordered field.

* why set of rational number is not complete ordered field

Set $A = \{x \in \mathbb{Q} : 0 \le x \le \sqrt{2}\} \Rightarrow$ bounded

$\sqrt{2} \notin \mathbb{Q}$ upper bound $= \{\frac{3}{2}$

No sup Not satisfying completeness axiom

$A = \{x \in \mathbb{R} : 0 \le x \le \sqrt{2}\}$

$\sup = \sqrt{2}$

# Real sequences

**Sequence:-** A function whose domain is set $\mathbb{N}$ and range is a set of real numbers $\mathbb{R}$ is called as real sequence

$$\boxed{S : \underline{\underline{N}} \to \mathbb{R}.}$$

Domain : N $\quad S_n : n \in N$

$$\{S_n\} = \{S_1, S_2, S_3, \ldots\ldots S_n \ldots S_m \ldots\ldots\}$$

$S_1$ ↓ 1st term, $S_2$ ↓ 2nd, $S_n$ ↓ nth, $S_m$ ↓ mth term

nth, mth are treated distinct term though $S_m = S_n$

⟹ Sequence is a ordered set of real numbers

⟹ No of terms in a sequence = always infinite

ex:- $\{S_n\}_{n \in N} = \{(-1)^n\} = \{-1, 1, -1, 1, \ldots\ldots\}$

Range $= \{-1, 1\}$ $\quad$ distinct element : $\{-1, 1\}$

ex:- $\{S_n\} = \{\frac{1}{n}\}$ $n \in N$ $\quad n = \{1, 2, 3, \ldots\}$

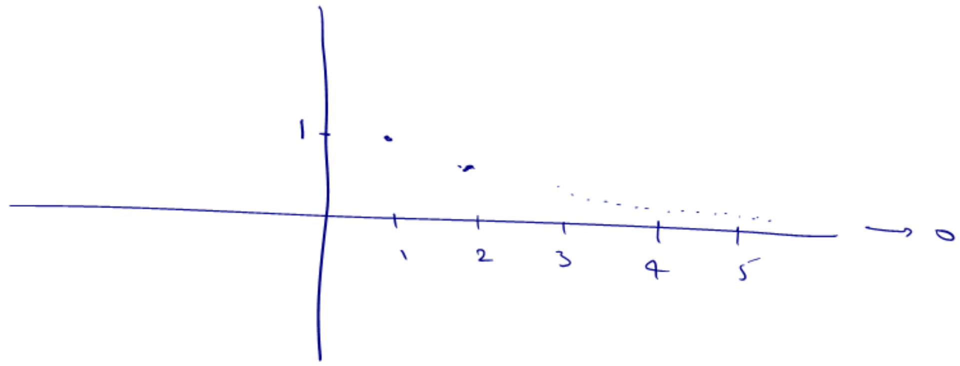
$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\}$

Range $= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\ \}$

**Range:-** The range or range set consists of all the distinct elements of a sequence without repetition and without regard of position.

Range : not set $\phi$

$S_n = \{ \frac{1}{n} \} \quad n \in \mathbb{N} = \{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \ldots \}$




## Bounds of a Sequence

### ① Bounded above sequence

A Sequence $\{S_n\}$ is said to bounded above if $\exists$ real number $K$ such that

$$S_n \le K \quad \forall\, n \in \mathbb{N}$$

$$\{ s_1, s_2, \ldots \ldots \}$$

### bounded below

A Sequence $\{S_n\}$ is said to bounded below if $\exists$ $K \in \mathbb{R}$ such that

$$S_n \ge K \quad \forall\, n \in \mathbb{N}$$

'K', 'k' bounds of sequence

⇒ sequence is bounded if it is both bounded above and bounded below

\# Evidently if sequence is bounded iff range is bounded.

* bounds of the range are bounds of your Sequence.

$S_n = (-1)^n = \{ -1, 1, -1, 1 \ldots \}$

$S_n = \frac{1}{n} = \{ 1, \frac{1}{2}, \frac{1}{3} \ldots \ldots \}$

$S_n = (-1)^n$ = Rang $\{-1, 1\}$

$S_n = \frac{1}{n}$ $\quad \{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots K\}$ $\quad \boxed{K \in R}$

$0$

$S_1, S_2 \ldots\ldots \leq K$ $\quad$ U

$S_n \leq K'$ upper bound

$S_1, S_2 \ldots \geq 'K'$ → lower

## Convergence of Sequences

A sequence $\{S_n\}$ is said converge to a real number 'l'. (to have real number $l$ as its limit) if for each $\varepsilon > 0$ $\exists$ positive integer 'm' (depends on $\varepsilon$) such that

$$|S_n - l| < \varepsilon \quad \forall n \geq m$$

$$l - \varepsilon < S_n < l + \varepsilon$$

$S_n \to l$ as $n \to \infty$

$\lim_{n \to \infty} S_n = l.$

$\varepsilon = 0.1, 0.01$ $\quad \exists m \quad \forall n \geq m \quad m+1, m+2 \ldots\ldots$

$\varepsilon = 0.001$ → find $m$

$|S_n - l| < \varepsilon$

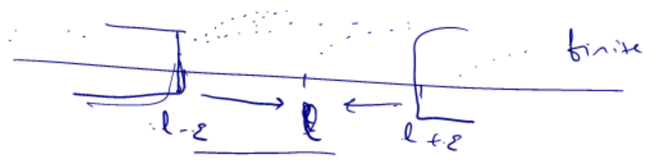
$n \geq m$

$|S_n - l| < \varepsilon\, 0.001$

$n \geq m$ $\quad$ $\boxed{l - \varepsilon < S_n < l + \varepsilon}$ → converges to $l$.

$m+1, m+2 \ldots$

→ Infinite number of terms




finite number of $S_n$

$S_m$

$S_1, S_2, S_3 \ldots\ldots S_{m-1}$

$S_n = \left\{ \frac{3n+1}{7n-4} \right\}$

$n \in N$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{3 + 1/n}{7 - 4/n} = \boxed{\frac{3}{7}} = l.$$

$l = \frac{3}{7}$

$\exists\ \varepsilon > 0$ such that $\forall n \geq m$

$\boxed{|S_n - l| < \varepsilon}$ $\forall n \geq m$

$\varepsilon > 0$ $\exists m$

$n \geq m$

$|S_n - l| < \varepsilon$ $S_{m+1}, \ldots$

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon$$

$$\frac{3}{7} - \varepsilon < S_n < \frac{3}{7} + \varepsilon \qquad n \geq m$$

$\varepsilon = 1$ $\exists\ m$

$|S_n - l| < \varepsilon$

$n > 0$ $\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 1$

$\varepsilon = 0.1$ $m = 4$

$n > 4$ $\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 0.1$

$\varepsilon = 0.01$ $n > 39$ $\left| \frac{3n+1}{7n-1} - \frac{3}{7} \right| < 0.01$

Theorem :-

☆ Every convergent sequence is bounded.

let $\{S_n\}$ converges to $l$

let $\varepsilon > 0$ $\exists m$

$|S_n - l| < \varepsilon$ $\forall n \geq m$

$l - \varepsilon < S_n < l + \varepsilon$ $\forall n \geq m$ $S_{m+1}, S_n$

$g = \min\{l - \varepsilon, S_1, S_2, S_3 \ldots S_{m-1}\}$

$G = \max\{l + \varepsilon, S_1, S_2 \ldots S_{m-1}\}$

$g \le s_n \le G$ bounded

every convergent sequence is bounded

Review

# Sequence of Real Numbers

Sequence : A function whose domain is set of natural numbers and range is set of real numbers

$$S : N \to \mathbb{R}$$

$$\{ S_1, S_2, S_3, \ldots S_n, S_m \ldots \}$$

$S_n = S_m \Rightarrow$ treated as distinct elements

$\{1, 2, 3, \ldots \}$ Infinitely many terms

$\{0.5, \pi, e, 1.3\}$

$\{1, 2, 1, 2, \ldots \}$

* Sequence may have only finite number of distinct elements

$a_n = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3} \ldots \}$

Range: The range or range set is the set consisting of all distinct elements of a sequence without repetition and without regrand to the positionof the term.

Range set can be finite or infinite with out being null set

$a_n = (-1)^n \quad n \in N = \{-1, 1, -1, 1, \ldots \}$ Range $= \{-1, 1\}$

## Bounds of a sequence

Bounded above sequence :

A sequence is said to be bounded above if $\exists K \in \mathbb{R}$ such that

$S_n \leq K \; \forall n \in N$ 'K' upper bound

## Bounded below Sequence

A sequence is said to be bounded below if $\exists K \in \mathbb{R}$ such that

$S_n \geq K \; \forall n \in N$

'K' is called lower bound

* A sequence is said to bounded if it is both bounded above and bounded below.

ex:- $\{a_n\} = \{3 + (-1)^n\}$ $n \in \mathbb{N}$

$\{a_n\} = \{2, 4, 2, 4, \ldots\ldots\}$ Range $= \{2, 4\}$

bounded sequence

(2) $\{b_n\} = \left\{\frac{2n}{1+n}\right\}$ $n \in \mathbb{N}$

$= \left\{1, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \ldots\ldots\right\}$ $\left\{\begin{array}{l}\text{bounded below} = 1 \\ \text{bounded above} = 2\end{array}\right\}$

bounded sequence

(3) $d_n = \left\{\frac{n^2}{1+n}\right\} = \left\{\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots\right\}$

Not bounded sequence $\left\{\begin{array}{l}\text{bounded below} = \frac{1}{2} \\ \text{bounded above} = \text{Not}\end{array}\right\}$

## ✯ Convergence of sequences

A sequence $\{s_n\}$ is said to converge to a real number '$l$' ($l$ as its limit) if for each $\varepsilon > 0$ $\exists$ positive $m$ (depends on $\varepsilon$)

such that $|s_n - l| < \varepsilon \quad \forall n \geq m$ $\quad m+1, m+2, \ldots$

$l - \varepsilon \leq s_n < l + \varepsilon$

$s_1, s_2, s_3 \ldots m-1$

$$s_n \to l \text{ as } n \to \infty$$

$$\lim_{n\to\infty} s_n = l$$


## ✯ Theorem

"Every convergent sequence is bounded"

let sequence $\{s_n\}$ converges to limit "$l$"

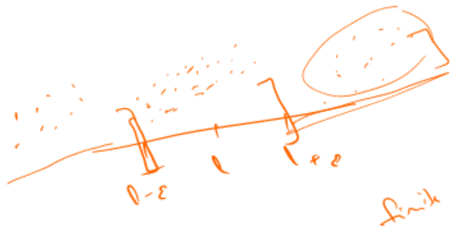
let $\varepsilon > 0$ be a given number so that $\exists$ a positive integer $m$

such that

$$|s_n - l| < \varepsilon \quad \forall n \geq m$$

$m+1, \ldots$

$$l - \varepsilon < s_n < l + \varepsilon$$




finite

$s_1, s_2, \ldots s_{m-1}$

$$g = \min\{l-\varepsilon, s_1, s_2 \ldots s_{m-1}\}$$

$$G = \max\{l+\varepsilon, s_1, s_2 \ldots s_{m-1}\}$$

$$g \leq s_n \leq G \quad \forall n$$

Hence $s_n$ is a bounded sequence

Converse :- Every bounded sequence is convergent. ??

Not true

ex: $a_n = (-1)^n = \{-1, 1, -1, 1, \ldots \ldots\}$

Range : $\{-1, 1\}$

bounded sequence

Assuming $\{s_n\}$ is convergent

Let us assume $\lim_{n \to \infty} (-1)^n = 'a'$ $a \in \mathbb{R}$

let $\varepsilon = 1$ $\exists$ $m > 0$ such that $\forall n \geq m$

$$|s_n - a| < 1$$

$$|(-1)^n - a| < 1$$

n is odd

$$|-1 - a| < 1 \rightarrow (1)$$

n is even

$$|1 - a| < 1 \rightarrow (2)$$

By triangle inequality $|a + b| \leq |a| + |b|$

$$2 = |1 - (-1)| = |1 - a + a - (-1)| \leq |1 - a| + |a - (-1)|$$

$$< 1 + 1 < 2$$

$2 < 2$ Not possible

So we can say that

every bounded sequence is not convergent

## ✪ Theorem

A sequence cannot converge to more than one limit

Proof: $\{S_n\}$ converges to 2 real numbers $l, l'$ $(l \neq l')$

$\exists\, \varepsilon > 0$ $\quad \varepsilon = \frac{1}{3}|l - l'| > 0$ $\quad \exists\, m_1, \; m_2$

$$|S_n - l| < \varepsilon \quad \forall n \geq m_1 \longrightarrow ①$$

$$|S_n - l'| < \varepsilon \quad \forall n \geq m_2 \longrightarrow ②$$

$$|l - l'| = |l - S_n + S_n - l'| \leq |l - S_n| + |S_n - l'| < \varepsilon + \varepsilon = 2\varepsilon$$

$$|l - l'| < \frac{2}{3}|l - l'| \quad \text{Not possible}$$

Contradiction

Sequence cannot converge to more than one limit

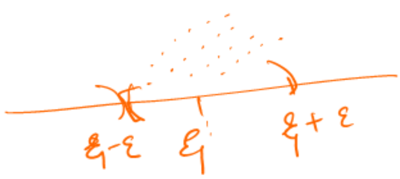
✪ Every convergent sequence is bounded & has a unique limit.

## ✪ Limit Points of Sequence

A real number '$\xi$' eta is said to be a limit point of a sequence $\{S_n\}$. If every neighbourhood of $\xi$ contains an infinite number of members of sequence

Thus $\xi$ is limit point of sequence if $\varepsilon > 0$ → however small

$S_n \in (\xi - \varepsilon, \xi + \varepsilon)$ for infinite number of values of $n$

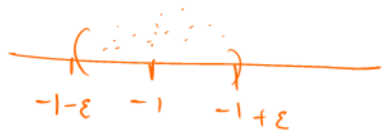


## ★ Not limit point of sequence

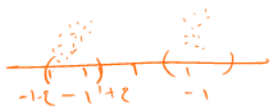
$\xi$ is not limit of sequence $\{S_n\}$ if $\exists\ \varepsilon > 0$ such that $S_n \in (\xi - \varepsilon, \xi + \varepsilon)$ for at most finite values of $n$.

★★ Limit point of range set of a sequence ($S_1, S_2 \ldots S_n \ldots$) is also limit point of sequence. Converse is not true.

★ $a_n = (-1)^n = \{-1, 1, -1, 1, \ldots\ldots\}$




limit points $= -1$ and $1$




Range $= \{-1, 1\}$

Not limit point

**Limit of a sequence**

A sequence $\{S_n\}$ is said to converge to real number '$l$' (limit) if for each $\varepsilon > 0$ $\exists$ $m > 0$ such that

$$|S_n - l| < \varepsilon \quad \forall n \geq m$$

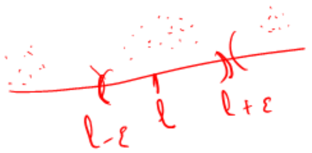
$$\lim_{n \to \infty} S_n = l$$

$\varepsilon = 0.1, 0.01, 0.001$

$m = 100$

$m = 1000$

$l - \varepsilon < S_n < l + \varepsilon$



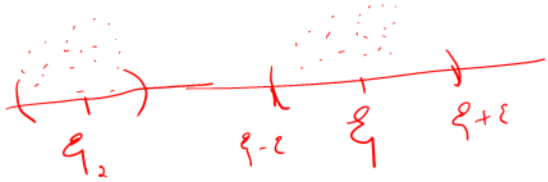

**Limit point of a sequence**

A real number $\xi$ is said to be a limit point of a sequence $\{S_n\}$ if every neighbourhood of $\xi$ contains infinite number of $\{S_n\}$

For every $\varepsilon > 0$

$$|S_n - \xi| < \varepsilon \quad \forall n$$

finite mem by infinite memb




ex:- $S_n = (-1)^n + 1$ , $n \in \mathbb{N}$

$= \{0, 2, 0, 2, 0, 2 \ldots\ldots\}$


limit point of sequence = 0 and 2

limit point of Range set = No limit point $R = \{0, 2\}$

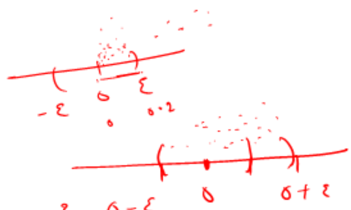
limit of sequence = No


$\forall \varepsilon > 0$ $\exists m$ $n \geq m$

(2) $S_n = \left\{\frac{1}{n}\right\}$ $n \in \mathbb{N}$.

$= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \ldots\right\}$

0.5, 0.33, 0.25, 0.2




limit point of sequence = 0 $\{0, 2, 0, 2, 0, 2 \ldots\}$

$\varepsilon > 0.2$

limit point Range set = 0 $\{0, 2\}$

$|S_n - 0| < 0.2$

$|S_n| < 0.2$

limit of sequence = 0

$m = 6$

$\forall n \geq 6$

✱ unique limit point of sequence is limit of sequence??

Bolzano Weierstrass Theorem: (BW Theorem)

Every bounded sequence has a limit point

$\{S_n\}$ is bounded Sequence

Since $\{S_n\}$ is bounded. Range set is also bounded.

Range set $\{S_n\} = \{0, 2, 0, 2, 0, 2 \cdots\}$

① finite $= \{0, 2\} <$ 0, 2 limit points

② Infinite: $S_n = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\}$

limit point

Range set $: \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

☆ BW theorem for set = a bounded set has atleast one limit point

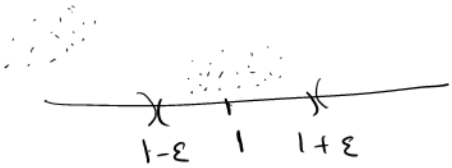
Converse ??

Not true ( unbounded

ex:- $S_n = \{1, 2, 1, 4, 1, 6, 1, 8, \cdots\cdots\}$

limit point = 1

Not bounded = Not bounded above




No limit

☆ Every convergent Sequence is bounded

ex:- $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$

** Bounded sequences may have many limit points

** The greatest and smallest of the limit points of a bounded sequence are respectively called upper and lower limits

upper limit point = $\overline{\lim}\, S_n$ ex:-

lower limit : $\underline{\lim}\, S_n$

Convergent sequence :

Every bounded sequence with a unique limit point is convergent

The necessary and the sufficient condition for the convergent sequence is that it is bounded and has unique limit point

The necessary and the sufficient condition for a sequence $\{S_n\}$ converges to limit '$l$'

For each $\varepsilon > 0$ $\exists$ positive integer $m$ such that

$$|S_n - l| < \varepsilon \quad \forall\, n \geq m$$

Non convergent sequence:

$(-1)^n = \{-1, 1, -1, 1, \dots\}$

1 Bounded: A bounded sequence which does not converge has atleast two limit points is said to oscillate finitely

2 Unbounded:

1 bounded above but not bounded below and has no other limit points $-\infty$

diverges to $-\infty$

2 bounded below but not bounded above, no other limit points $+\infty$

3 diverges to $+\infty$

3 oscillates infinitely

(not diverges to $-\infty$ nor $+\infty$)

ex: $\{1, 2, 3, 1, 2, 3, 1, 2, 3, \dots\}$

set of limit points $= \{1, 2, 3\}$

upper limit $= \overline{\lim}\, S_n = 3$

lower limit $\underline{\lim} = 1$

ex:- 1 $1 + (-1)^n = \{0, 2, 0, 2, \dots\}$

not convergent, bounded = oscillates finitely

2 $n^2 = \{1, 4, 9, 16, \dots\} \Rightarrow$ diverges to $+\infty$

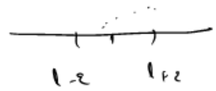
3 $\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots\} \Rightarrow$ oscillates infinitely

lim point $= 0$ $\overline{\lim} = \infty$

**Ex:-** Show that $\lim_{n\to\infty} \dfrac{3+2\sqrt{n}}{\sqrt{n}} = 2.$

$\{S_n\}$ converges to '$l$' then for each $\varepsilon > 0$ $\exists\, m$

such that $|S_n - l| < \varepsilon$ $\forall n \geq m$




let $\varepsilon > 0$ $\exists$ m

$$|S_n - 2| < \varepsilon$$

$$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon \qquad \left| \frac{3}{\sqrt{n}} + 2 - 2 \right| < \varepsilon$$

$$\frac{3}{\sqrt{n}} < \varepsilon$$

$$3+ \quad \frac{3}{\sqrt{n}} < \varepsilon \Rightarrow \frac{3}{\varepsilon} < \sqrt{n}$$

$$n > \frac{9}{\varepsilon^2} \qquad m > \frac{9}{\varepsilon^2}$$

Thus $\varepsilon > 0$ $\exists m$ $(m > 9/\varepsilon^2)$

$$\left| \frac{3+2\sqrt{n}}{\sqrt{n}} - 2 \right| < \varepsilon \quad \forall n \geq m$$

$$\boxed{\lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2}$$

Sandwich Theorem or Squeeze Theorem

If $\{a_n\}, \{b_n\}, \{c_n\}$ are three sequences such that


(i) $\underset{\downarrow l}{a_n} \leq \underset{\downarrow l}{b_n} \leq \underset{\downarrow l}{c_n}$ $\forall n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$ then $\lim_{n\to\infty} b_n = l$

**Proof:-** since $\{a_n\}$, $\{c_n\}$ converges to '$l$'.

Let $\varepsilon > 0$. $\exists$ $m_1, m_2$ such that

$|a_n - l| < \varepsilon \quad \forall n > m_1$ $\qquad$ $|c_n - l| < \varepsilon \quad \forall n > m_2$

$l - \varepsilon < a_n < l + \varepsilon \longrightarrow (1)$ $\qquad$ $l - \varepsilon < c_n < l + \varepsilon \longrightarrow (2)$

let $m = \max(m_1, m_2)$ $\qquad$ and

$a_n \leq b_n \leq c_n$

let $\varepsilon > 0$ $\exists$ $m$ such that

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon$$

$$l - \varepsilon < b_n < l + \varepsilon$$

$$|b_n - l| < \varepsilon \quad \forall n > m$$

$$\therefore \lim_{n \to \infty} b_n = l.$$

'whether (or) diverges

ex: Given sequence converges or diverges

$$\{b_n\} = \frac{\sin n}{n} \qquad n \in \mathbb{N}.$$

Proof: $\qquad -1 \leq \sin n \leq 1 \quad \forall n \in \mathbb{N}$

divide by $n$

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$a_n \qquad b_n \qquad c_n$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{-1}{n} = 0 = \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} c_n$$

By Sandwich theorem

$$\lim_{n \to \infty} b_n = 0 \rightarrow \lim_{n \to \infty} \frac{\sin n}{n} = 0$$

$\{b_n\} = \dfrac{\sin n}{n}$ converges to '0'

Monotonic Sequences

A sequence $\{S_n\}$ is said to be monotonic increasing if $S_{n+1} \geq S_n \quad \forall n$

$\{S_1, S_2, S_3 \ldots\}$

ex :- $\{S_n\} = n^2$

$S_{n+1} \geq S_n$

and monotonic decreasing if $S_{n+1} \leq S_n \quad \forall n$ $\longrightarrow$ ex :- $S_n = \{\frac{1}{n}\}$

It is said to be a monotonic sequence if it is either monotonic increasing or monotonic decreasing

Strictly Increasing and decreasing sequence

A sequence $\{S_n\}$ is said to be strictly increasing if $S_{n+1} > S_n \quad \forall n$

and strictly decreasing if $S_{n+1} < S_n \quad \forall n$

Theorem:

A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.

Proof:- necessary condition :- If monotonic sequence is convergent seq then it is bounded.

Every convergent is bounded.

Sufficient condition :- Monotonic sequence is bounded then it is convergent

Proof:- let $\{S_n\}$ — Monotonic increasing sequence is bounded

Range = S is also bounded $\{1, 4, 9, 16, \ldots\}$

Every bounded above sequence has a supremum = Completeness axiom

let 'M' is supremum. We want to show that

$\lim\limits_{n \to \infty} S_n = M$

let $\varepsilon > 0$

Since M is supremum $S_n \leq M < M + \varepsilon \longrightarrow (1)$

M is supremum, $M - \varepsilon$ is not supremum. $\exists$ atleast $S_m > M - \varepsilon$

$M - \varepsilon < S_n < M + \varepsilon$

$|S_n - M| < \varepsilon \quad \forall n \geq m$ $\qquad S_n$ converges

** Corollary:

1.A monotonic increasing bounded above seqquence converges to its least upper bound ( Supremum) and a monotonic decreasing bounded below to converges to greatest lower bound ( Infimum)

2. Every monotonic increasing sequence which is not bounded above diverges to $+\infty$

3.Every monotonic decreasing sequence which is not bounded above diverges to $-\infty$

ex:- $\{S_n\} = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots \dfrac{1}{n+n} \quad \forall n \in \mathbb{N}$

★ Monotonic increasing (or) decreasing

$S_{n+1} - S_n > 0$ Increasing $\Rightarrow$ bounded above

$S_{n+1} - S_n < 0$ decreasing $\Rightarrow$ bounded below

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n-2} + \frac{1}{n+n-1} + \frac{1}{n+n} < \frac{1}{2n}$$

$$S_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{n+1+n+1-2} + \frac{1}{2(n+1)-1} + \frac{1}{n+1+n+1}$$

$$S_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{(n+1)} = \frac{1}{2(n+1)(2n+1)}$$

$$= \frac{1}{2n+1} + \frac{1}{n+1}\left(\frac{1}{2} - 1\right) = \frac{1}{2n+1} - \frac{1}{2(n+1)} = \frac{2n+2-2n-1}{2(n+1)(2n+1)}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0$$

$S_{n+1} - S_n > 0$ Increasing function

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \cdots \frac{1}{n+n}$$

$$S_n < \frac{1}{n} + \frac{1}{n} + \cdots \cdots \frac{1}{n} = \frac{n}{n} = 1$$

$$\boxed{0 < S_n < 1}$$

Monotonic increasing function and it is bounded.

So $S_n$ converges

Subsequences: If $\{S_n\} = \{S_1, S_2, S_3 \dots\}$ be a sequence then any infinte succession of its terms picked out in any way ( but preserving the original order) is called subsequence of $\{S_n\}$ or in other words if $\{n_k\}$ be srictly monotonic increasing sequence of natural numbers. i.e $n_1 < n_2 < n_3 \dots$ then $\{S_{n_k}\}$ is a subsequence of $\{S_n\}$.

1. $\{S_2, S_4, S_6 \dots S_{2n} \dots\}$

$S_n = \{S_1, S_2, S_3, S_4 \dots\}$

2. $\{S_1, S_4, S_9, \dots S_{n^2}\}$

(1) $S_n = \{\underset{S_1}{1}, \underset{S_2}{5}, \underset{S_3}{9}, \underset{S_4}{10}, \underset{S_5}{2}, \underset{S_6}{4}, \underset{S_7}{6}, \dots\}$

3. $\{S_1, S_5, S_7, S_9 \dots\}$

(i) $S_{n_k} = \{\underset{S_1}{1}, \underset{S_3}{9}, \underset{S_6}{4}, \underset{S_7}{6} \dots\}$ Subsequence

4. $\{S_2, S_3, S_1, \dots\}$

(ii) $S_{n_k} = \{\underset{S_2}{5}, \underset{S_5}{2}, \underset{S_7}{6}, \underset{S_3}{9}, \dots\}$
Not Subsequen

1. A sequence $\{S_n\}$ converges to 'S' if only if its every subsequence converges to 'S'.

→ similarly $\lim S_n = \infty \ (-\infty)$ if and only if every subsequence of $\{S_n\}$ tends to $\infty \ (-\infty)$

2. If '$\xi$' is a limit point of sequence $\{S_n\}$ then $\exists$ subsequence $\{S_{n_k}\}$ of $S_n$ which converges to $\xi$ i.e $\lim_{k \to \infty} S_{n_k} = \xi$

$$S_n = (-1)^n = \{-1, 1, -1, 1, \dots\dots\dots\}$$

limit points of $\{S_n\} = -1, 1$

(1) $S_{n_k} = \{-1, -1, -1, \dots\}$ limit $= -1$

(2) $S_{n_k} = \{1, 1, 1, \dots\}$ limit $= 1$

# Examples

(1) $a_{n_k} = s_n = \left\{ \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2^9}, \cdots \left(\frac{1}{2}\right)^{n^2} \cdots \right\}$

converges (or) diverges

$a_n = \left\{ \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \cdots \right\} = \boxed{\left(\frac{1}{2}\right)^n}$

$\star\star$ $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$

$\star\star$ $\lim_{n\to\infty} \frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

Since $s_n$ is a subsequence of $a_n$ and $a_n$ converges to '0'

So $s_n$ converges to '0'

(2) $\lim_{n\to\infty} \frac{n-1}{n+1} = 1$ (Prove this using definition)

$s_n = \frac{n-1}{n+1}$ $\forall n \in \mathbb{N}$

let $\varepsilon > 0$

$|s_n - 1| < \varepsilon$ $\forall n \geq \boxed{m}$

$$\left| \frac{n-1}{n+1} - 1 \right| = \left| \frac{n-1-n-1}{n+1} \right| = \left| \frac{2}{n+1} \right| < \varepsilon$$

$$\frac{2}{n+1} < \varepsilon \Rightarrow \frac{2}{\varepsilon} < n+1$$

$$n > \frac{2}{\varepsilon} - 1 \longrightarrow m \not> \frac{2}{\varepsilon} - 1$$

let $\varepsilon > 0$ $\exists\, m = \frac{2}{\varepsilon} - 1$ such that

$\forall n \geq m$

$$|s_n - 1| < \varepsilon \quad \text{or} \quad \left|\frac{n-1}{n+1} - 1\right| < \varepsilon \quad \forall n \geq \frac{2}{\varepsilon} - 1$$

$\forall m > \frac{2}{\varepsilon} - 1$ $\quad \exists\, m$

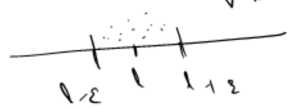
By definition

$$\lim_{n \to \infty} \frac{n-1}{n+1} = 1$$

ex:- Show $\lim_{n \to \infty} \sqrt{n+1} - \sqrt{n} = 0$ (By definition)

let $\varepsilon > 0$

$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon \qquad \forall n \geq m \quad 99 \quad 150$$




$$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}}$$

$$\overbrace{\left|\sqrt{n+1} - \sqrt{n} - 0\right|}^{|s_n - 0|} < \boxed{\frac{1}{2\sqrt{n}} < \varepsilon}$$

$|s_n - l| < \varepsilon \quad \forall n \geq m \quad \exists m$

$$\left|\sqrt{n+1} - \sqrt{n}\right| < \frac{1}{2\sqrt{n}} < \varepsilon$$

$$|s_n - 0| < \varepsilon < \frac{1}{2\sqrt{n}} < \varepsilon$$

$\forall n > \frac{1}{4\varepsilon^2}$

$$\frac{1}{2\sqrt{n}} < \varepsilon \Rightarrow \frac{1}{4\varepsilon^2} < n$$

$$\Rightarrow n > \frac{1}{4\varepsilon^2} = m$$

let $\varepsilon > 0$ $\exists\, m = \frac{1}{4\varepsilon^2}$ such that

$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| < \varepsilon \qquad \forall n > \frac{1}{4\varepsilon^2}$$

ex: $\lim\limits_{n\to\infty} \dfrac{2-\cos n}{n+3}$ converges (or) diverges

By sandwich theorem

$$-1 \leq \cos n \leq 1 \quad \forall n$$

$$1 \geq -\cos n \geq -1 \implies -1 \leq -\cos n \leq 1$$

$$2-1 \leq 2-\cos n \leq 2+1$$

$$\underbrace{\frac{1}{n+3}}_{a_n} \leq \underbrace{\frac{2-\cos n}{n+3}}_{b_n} \leq \underbrace{\frac{3}{n+3}}_{c_n}$$

$a_n \leq b_n \leq c_n$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = l$.

$l = 0$

By sandwich theorem $a_n \leq b_n \leq c_n$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{n+3} = 0 = \lim\limits_{n\to\infty} \dfrac{3}{n+3}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2-\cos n}{n+3} = \underline{\underline{0}}$$

Ex:- $b_n = \left\{ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(2n)^2} \right\}$

By Sandwich theorem

$$\frac{1}{(n+1)^2} < \frac{1}{n^2}$$

$$\frac{1}{(n+2)^2} < \frac{1}{n^2}$$

$$\vdots$$

$$\frac{1}{(2n)^2} < \frac{1}{n^2}$$

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(2n)^2} < \frac{1}{n^2} + \frac{1}{n^2} + \cdots \frac{1}{n^2}$$

$$= \frac{n}{n^2} = \frac{1}{n}$$

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(2n)^2} < \frac{1}{n}$$

$$\longrightarrow \frac{1}{(n+1)^2} > \frac{1}{(2n)^2}$$

$$\frac{1}{(n+2)^2} > \frac{1}{(2n)^2}$$

$$\vdots$$

$$\frac{1}{(2n)^2} \geq \frac{1}{(2n)^2}$$

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(2n)^2} > \frac{n}{(2n)^2}$$

$$\quad '' \quad > \frac{1}{4n}$$

$$\frac{1}{4n} < \underbrace{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(2n)^2}}_{b_n} < \frac{1}{n}$$

$$\frac{1}{4n} \to a_n \to 0, \qquad \frac{1}{n} \to c_n \to 0$$

By Sandwich theorem

$$\lim_{n \to \infty} b_n = 0$$

ex:- $\left\{ \sqrt{30}, \sqrt{30+\sqrt{30}}, \sqrt{30+\sqrt{30+\sqrt{30}}}, \ldots \right\} = S_n$

converges (or) diverges

ex:- $S_n = \left\{ 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4 \ldots \right\}$

converges (or) diverges

limit point $= 0$ , $\lim = \infty$

oscillates infinitely

# Series

Definition: If $u_1, u_2, u_3 \ldots\ldots u_n, \ldots\ldots\ldots$ be an infinite sequence of real numbers then

$u_1 + u_2 + u_3 + \ldots\ldots u_n + \ldots\ldots \infty$ is called Infinite Series.

The Infinite series is denoted by $\sum u_n$ and sum of its first $n$ terms is denoted by $S_n$.

Sequence $\{S_n\}$ is called sequence of partial sums of series.

$$\{S_n\} = \{u_1, u_2, \ldots\ldots u_n \ldots\ldots\}$$

Infinite Series $= \sum u_n = u_1 + u_2 + u_3 \ldots\ldots u_n + \ldots\ldots \infty$

Partial sums

$S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

$\vdots$

$S_n = u_1 + u_2 + \ldots\ldots u_n \Rightarrow$ sum of $n$ terms

# Convergence, divergence and Oscillation of a Series

Consider the infinite series $\sum u_n = u_1 + u_2 + u_3 \cdots\cdots u_n + \cdots \infty$

and let the sum of first $n$ terms be $S_n = u_1 + u_2 + \cdots\cdots u_n$.

Clearly $S_n$ is a function of $n$, and as $n$ increases indefinitely three possibilities arise.

$\lim_{n\to\infty} S_n = 'l'$

1) If $S_n$ tends to a finite limit as $n \to \infty$, the series $\sum u_n$ is said to be convergent

2) If $S_n$ tends to $\pm\infty$ as $n \to \infty$. Then series $\sum u_n$ is said to be divergent

3) If $S_n$ does not tend to a unique limit as $n \to \infty$ then the series $\sum u_n$ is said to be oscillatory (or) nonconvergent

ex:- Examine for convergence of series

(i) $1+2+3+\cdots\cdots n+\cdots\cdots\infty \implies \sum u_n = 1+2+3+\cdots\cdots\infty$

Consider $S_n = 1+2+3+\cdots n = \frac{n(n+1)}{2}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty \quad \text{diverging}$$

$\therefore \sum u_n = 1+2+3+\cdots \infty$ also diverges.

(ii) $5-4-1+5-4-1+\cdots\cdots+\infty$

$$\sum u_n = 5-4-1+5-4-1+\cdots\cdots\infty$$

$$S_n = \overset{u_1+u_2+u_3 \cdots\cdots}{5-4-1+5-4-1\cdots\cdots} \overset{u_n}{n \text{ terms.}}$$

$S_1 = 5$ $\qquad$ 5 10 .

$S_2 = 5-4 = 1$

$S_3 = 5-4-1 = 0$

$S_4 = 5-4-1+5 = 5$

$3m$ — 0
$3m+1$ — 5
$3m+2$ — 1

$\left.\begin{matrix}0\\5\\1\end{matrix}\right\}$ oscillating ex:- Show that the geometric series $1+r+r^2+\cdots\cdots\infty$

converges if $|r|<1$ (ii) diverges $r\geq 1$ (iii) oscillates $r\leq -1$

$$\sum u_n = 1+r+r^2+\cdots\cdots\infty$$

$$S_n = 1+r+r^2+\cdots\cdots r^{n-1} = \frac{1-r^n}{1-r}$$

(i) $|r|<1$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \lim_{n\to\infty} \frac{1}{1-r} - \frac{\overset{|r|<1}{r^n\to 0}}{1-r} = \lim_{n\to\infty} \frac{1}{1-r} = \frac{1}{1-r}$$

$$\lim_{n\to\infty} s_n = \frac{1}{1-r} \quad \text{converges}$$

$\sum u_n$ converges.

(ii) when $r>1$

$$\lim_{n\to\infty} r^n \to \infty$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{r^n-1}{r-1} = \lim_{n\to\infty} \frac{r^n}{r-1} - \frac{1}{r-1} \to \infty$$

diverges

$\sum u_n$ diverges $r>1$

(iii)
when $r=1$

$$S_n = 1+1+\cdots\cdots n \text{ times} = n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n \to \infty \quad \text{diverges}$$

$\sum u_n$ diverges when $r=1$

(iv) when $r=-1$

$$S_n = 1+r+r^2+\cdots\cdots r^{n-1} = 1-1+1-1+1\cdots\cdots$$

$S_1 = 1$

oscillatory

$S_2 = 1-1 = 0$

$S_3 = 1$

Necessary Condition for convergence:

A necessary condition for the convergence of an infinite series $\sum u_n$ is that $\lim_{n\to\infty} u_n = 0$

$u_n = n^{th}$ term

$S_n$ = sum of $n$ terms

Proof:- $\sum u_n = u_1 + u_2 + \dots \infty$

let $S_n = u_1 + u_2 + u_3 \dots u_n$

Let $\sum u_n$ is convergent. Since infinite series is convergent $S_n$ also converges.

let $\lim_{n\to\infty} S_n = 's' \to$ limit

$S_n = u_1 + u_2 + \dots u_n$

$S_{n-1} = u_1 + u_2 + \dots u_{n-1}$

$u_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$\lim_{n\to\infty} u_n = S - S = 0$$

Hence for convergent series, $\lim_{n\to\infty} u_n = 0$

** Simple test to prove divergence of series.

** A series doesnot converge if $\lim_{n\to\infty} u_n \neq 0$

** $\lim_{n\to\infty} u_n = 0$ doesnot prove that a series is convergent.

∴ Series donot converge even though $\lim_{n\to\infty} u_n = 0$

(iv) $r < -1$ Let $r = -p$ where $p > 1$ $r^n = (-1)^n p^n$

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1 - (-1)^n p^n}{1 + p}$$

$\lim_{n\to\infty} p^n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1 - (-1)^n p^n}{1 + p}$$

If $n$ is even $\lim_{n\to\infty} S_n \to -\infty$

$n$ is odd $\lim_{n\to\infty} S_n \to +\infty$

Oscillating

ex:- $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \infty$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}}\right]$$

Geometric series

$1 + r + r^2 + \cdots r^{n-1}$

$|r| < 1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left[\frac{1 - r^n}{1 - r}\right]$$

$$\lim_{n\to\infty} \cancel{\frac{1}{2}}\left[\frac{1 - (1/2)^n}{\cancel{1/2}}\right] = \lim_{n\to\infty} 1 - \left(\frac{1}{2^n}\right)$$

$$= 1$$

$|r| < 1$ $\lim_{n\to\infty} S_n$ converges $\frac{1}{1-r} = \frac{1}{2}\left[\frac{1}{1 - \frac{1}{2}}\right] = 1$

ex:- $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \cdots \infty = \sum u_n$

$S_n = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots \cdots$ $\boxed{\frac{n}{n+1}} = \left( \quad \right)$ $\sum_{n=1}^{\infty}$

$n^{th}$ term $u_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$$

$\lim_{n\to\infty} u_n \neq 0$ diverges

$\sum u_n$ diverges

$\Rightarrow$ $\lim_{n\to\infty} u_n = 0 \Rightarrow$ cannot conclude converges

$\Rightarrow$ $1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots \frac{1}{n} + \cdots \cdots \infty = \sum u_n$

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \cdots \frac{1}{n}$ Harmonic Series

$u_n = \frac{1}{n}$ $S_n = \sum_{n=1}^{n} \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ $\lim_{n\to\infty} u_n = 0$

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \cdots \frac{1}{n}$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2}$

$$S_8 = 1 + \underline{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}} + \frac{1}{5} + \underline{\frac{1}{6} + \frac{1}{7}} + \frac{1}{8} > \frac{3}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2}$$

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \quad \infty \quad \text{diverges}$$

ex :- $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \frac{1}{\sqrt{n}} + \cdots \quad \infty$

$$\boxed{S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \frac{1}{\sqrt{n}}}$$

$U_n = \frac{1}{\sqrt{n}}$ $\qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ $\qquad \lim_{n\to\infty} u_n = 0$

$$S_n = \boxed{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots \frac{1}{\sqrt{n}}} > \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots \cdots \frac{1}{\sqrt{n}}}_{n\ \text{term}}$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} \longrightarrow \infty$$

Necessary condition for convergence of infinite series is $\lim_{n\to\infty} u_n = 0$

divergence test $\lim_{n\to\infty} u_n \neq 0$ diverges

General Properties of series:

1. The convergence or divergence of an infinite series remains unaffected by the addition or removal of finite number of terms

$3+5+2+1+\frac{1}{2}+\frac{1}{2^2}+\dots\infty$ converges $\quad 1+r+r^2+\dots\infty$ $|r|<1$

2. If a series, in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms negative

$1+\frac{1}{2}+\frac{1}{2^2}+\dots\infty$ converges

$-1-\frac{1}{2}-\frac{1}{2^2}+\dots\infty$ converges

3. The convergence or divergence of an infinite series remain unaffected by multiplying each term by a finite number.

$\sum\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\dots$ Harmonic series diverges

$\sum\frac{1}{2n}$ (or) $\sum\frac{3}{n}$ diverges

Series of positive terms:

An infinite series in which all the terms after some particular terms are positive, is a positive term series

ex: -7-5-2+2+7+13+20...............................

A series of positive terms either converge or diverge to + infinity

1. Integral test: A positive term series f(1)+ f(2) + ..............f(n)+.......... where f(n) decreases as n increases converges or diverges as the integral $\int_1^\infty$ f(x) dx is finite or infinite

$\sum\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\dots = (\quad)$

$\sum u_n = \int_1^\infty u_x\,dx = \text{finite}$ converging

P- Test: Test for Comparison

P-series $= \sum_{n=1}^{\infty}\frac{1}{n^p} = 1+\frac{1}{1^p}+\frac{1}{2^p}+\dots\frac{1}{n^p}+\dots = \text{Infinite}$ diverging

$\sum u_n = \sum\frac{1}{n^p}$

$f(x) = \frac{1}{x^p}$ By using integral test this series converges (or) diverge according to $\int_1^\infty \frac{1}{x^p}\,dx$ is finite (or) Infinite

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{m\to\infty}\int_1^m \frac{1}{x^p}\,dx = \lim_{m\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_1^m = \lim_{m\to\infty}\frac{m^{1-p}-1}{1-p}$$

If $p>1$ $\lim_{m\to\infty}\frac{m^{1-p}-1}{1-p} = \frac{-1}{1-p} = \text{finite}$ $\quad m^{1-p}\quad m^{-}\ \boxed{1/m}$

$p<1 \quad = +\infty$

$\sum \frac{1}{n^p}$ converges $\frac{1}{p-1}$ if $p > 1$

diverges if $p \leq 1$

If $p = 1$

$\sum u_n = \sum \frac{1}{n}$ $\qquad f(x) = \frac{1}{x}$

Integral test $= \int_1^\infty \frac{1}{x} dx = \lim_{m \to \infty} \int_1^m \frac{1}{x} dx = \lim_{m \to \infty} [\ln x]_1^m$

$= \lim_{m \to \infty} \ln m - 0 = \infty$

$\sum \frac{1}{n}$ = diverges

ex:- ① $\sum \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ $\qquad p > 1 \quad \sum \frac{1}{n^p}$

$p = 2$ converges

② $\sum \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$ $\qquad p = \frac{1}{2} < 1$

diverge

## Comparison Test :-

I   If two positive term series $\sum u_n$ and $\sum v_n$ be such that

(i) $\sum v_n$ converges   (ii) $u_n \le v_n \ \forall n$ then $\sum u_n$ also converges.

ex:-

$$\sum u_n = \sum \frac{1}{2^n + n}$$

$$\sum v_n = \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

( Geometric series $|r| < 1$, $1 + r + r^2 + \ldots$ converges)

$$2^n + n > 2^n$$

$$\frac{1}{2^n + n} < \frac{1}{2^n}$$

$u_n \le v_n$

and $v_n$ is converging

so $\sum u_n = \sum \frac{1}{2^n + n}$ also converges

II

## Comparison Test

If two positive term series $\sum u_n$ and $\sum v_n$ be such that

(i) $\sum v_n$ diverges (ii) $u_n \geq v_n \; \forall n$ then $\sum u_n$ also diverges.

ex:- $\sum u_n = \sum \frac{1}{n+1}$ converge (or) diverge.

$\sum v_n = \sum \frac{1}{2n}$

$n+1 < n+n$

$\frac{1}{n+1} > \frac{1}{n+n} \Rightarrow \frac{1}{n+1} > \frac{1}{2n}$

we know that $\sum \frac{1}{2n}$ diverges

its $\sum 1$

$\sum v_n = \sum \frac{1}{2n}$ is diverging and

$u_n \geq v_n \; \forall n$

$v = \frac{1}{2^n} \rightarrow$ converging

So $\sum u_n = \sum \frac{1}{n+1}$ also diverges.

$u_n = \frac{1}{n+1} \geq \frac{1}{2^n}$ ($v_n$)

converging

$u_n \geq v_n$ $\quad$ $v_n$ is converging (diverging)

# D'Alembert's Ratio Test :-

In a positive term series $\sum u_n$ if

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda \text{ then series}$$

(i) converges if $\lambda < 1$

(ii) diverges if $\lambda > 1$

(iii) $\lambda = 1$ Test fail.

**

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda$$

$\lambda > 1$ converges

$\lambda < 1$ diverges

$\lambda = 1$ Test fail

ex:- $\sum u_n = \sum \frac{n^2-1}{n^2+1} x^n$ , $x > 0$

Ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}}{\frac{n^2-1}{n^2+1} x^n}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2[(1+1/n^2)-1/n^2]}{n^2[(1+1/n^2)+1/n^2]} x}{\frac{n^2(1-1/n^2)}{n^2(1+1/n^2)}} = x$$

$x < 1$ It converges $\quad x = 1$ Test fail

$x > 1$ diverges

when $x = 1$

$\Rightarrow \sum \frac{n^2-1}{n^2+1} (1^n)$ $\quad u_n = \frac{n^2-1}{n^2+1}$ $\quad \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1 \neq 0$ $\quad \lim_{n\to\infty} u_n \neq 0$

series diverges

## Limit form of comparison test

If two positive series $\sum u_n$ and $\sum v_n$ be such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = k \; (\neq 0) \text{ finite. then } \sum u_n \text{ and } \sum v_n$$

converges (or) diverges together.

ex:- $\sum u_n = \sum \sin \frac{1}{n}$

$\sum v_n = \sum 1/n$ → its diverges

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1 \neq 0$$

Since $\sum v_n$ diverges so $\sum u_n$ also diverges.

# Cauchy's Root test

In a positive series if $\lim_{n\to\infty} (u_n)^{1/n} = \lambda$

(i) $\lambda < 1$ converges

(ii) $\lambda > 1$ diverges

(iii) $\lambda = 1$ Test fails.

ex:- $\sum (\log n)^{-2n} = (\log 1)^{-2} + (\log 2)^{-4} + (\log 3)^{-6} + \cdots\cdots$

$+ \cdots$

$u_n = (\log n)^{-2n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} (\log n)^{-2n \times \frac{1}{n}} = \lim_{n\to\infty} (\log n)^{-2} = \lim_{n\to\infty} \left(\frac{1}{\log n}\right)^2$$

$$= 0 < 1$$

$\sum u_n = \sum (\log n)^{-2n}$ converges

## Alternating series

A series in which terms are alternatively +ve and −ve is called alternating series

## ✱ Leibnitz series

An alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$ converges if

(i) each term is numerically less than its preceeding term

$u_{n+1} < u_n$

(ii) $\lim_{n \to \infty} u_n = 0$

ex:- $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \frac{1}{\sqrt{n}} \cdots \infty = \sum u_n$

① alternate +ve and −ve signs

② $u_{n+1} < u_n$ so $\sum u_n$ converges

③ $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

✱ $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$ $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ cannot conclude

## Series of positive and negative terms

The positive term series and alternating series are special types of these series with arbitrary signs.

## Absolute convergent :-

If the series $u_1 + u_2 + \cdots u_n + \cdots$ ($\sum u_n$) be such that $|u_1| + |u_2| + \cdots |u_n| + \cdots$ ($\sum |u_n|$) is convergent then the series is said to be absolutely convergent.

ex :- $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum u_n$

$$\sum |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

convergent by p-test

$\sum \frac{1}{n^p}$ $p > 1$

$p = 2$ converges

$\sum \frac{1}{n^2}$

By test we can conclude $\sum |u_n|$ converges

$\sum u_n$ is absolutely converging

Note: An absolutely convergent series is necessarily convergent but not conversely

## Conditionally convergent :-

If $\sum |u_n|$ is divergent but $\sum u_n$ is convergent then $\sum u_n$ said to be conditionally convergent.

ex:- $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \infty$

is convergent

using leibnitz test

$$\left.\begin{array}{l} \text{① alternate + and -ve} \\ \text{② } u_{n+1} < u_n \\ \text{③ } \lim\limits_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \sum u_n \text{ converges.}$$

$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \infty$

divergent $\sum \frac{1}{n} \rightarrow$ Harmonic series

$\sum u_n$ is convergent but $\sum |u_n|$ is divergent

So $\sum u_n$ is conditionally convergent.

ex:- Test whether the following series is absolutely convergent or not

$\sum u_n = \sum \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$\sum |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$ $\quad \sum |u_n| = \sum \frac{1}{2n-1}$

## Positive series

1. Comparison test (convergence / divergence)
2. limit form of comparison test
3. Ratio Test
4. Cauchy root test

## 2. Alternating series

Leibnitz series test $u_{n+1} < u_n$

## 3. Absolute convergence

Conditional convergence

Power series:A power series of the form $a_0 + a_1x + a_2x^2 + \dots a^n x^n + \dots$ where all a's are independent of x is called power series of x.

This series may converge (or) diverge for diff values of $x$.

→ Interval of convergence: for what values $x$ the series converges.

Power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots a_nx^n + \dots \infty$

$u_n = a_nx^n$

Ratio:- $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$

$\lambda < 1$ converges

$\lambda > 1$ diverge

$\lambda = 1$ Test fail

✱ Absolute converges ⇒ converges

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right| = \lim_{n\to\infty} |x| \left|\frac{a_{n+1}}{a_n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$

$= |x| l < 1$ converges

$|x| < \frac{1}{l} \Rightarrow \frac{-1}{l} < x < \frac{1}{l}$

Power series converges if $\frac{-1}{l} < x < \frac{1}{l}$

$x = \frac{-1}{l}$ $\qquad$ $x = \frac{1}{l}$

ex: $\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \cdots\cdots \infty$

Interval of convergence ?

$$u_n = \frac{1}{n(1-x)^n}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)(1-x)^{n+1}}}{\frac{1}{n(1-x)^n}}\right| = \lim_{n\to\infty}\left|\frac{n(1-x)^n}{(n+1)(1-x)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\ \frac{1}{(1-x)}\right| = \left|\frac{1}{1-x}\right| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$\downarrow$ 1

$$= \left|\frac{1}{1-x}\right| < 1 \quad \text{converges}$$

when $\frac{1}{|1-x|} < 1 \Rightarrow |1-x| > 1$

$1-x > 1$ (or) $1-x < -1$

$0 > x$ $\qquad$ $2 < x$

$x < 0$ $\qquad$ $x > 2$

$x > 2$ (or) $x < 0$

Series converges

At

$x = 0, \quad x = 2$

$$\frac{1}{1-0} + \frac{1}{2(1-0)^2} + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \infty = \text{HP}$$

At $x = 0$ diverges

At $x = 2$

$$\frac{1}{1-2} + \frac{1}{2(-1)^2} + \frac{1}{3(-1)^3} + \cdots = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$$

Leibnitz series

(1) $-1 + \frac{1}{2} - \frac{1}{3} + \cdots$ alternate $+,-$

$u_{n+1} < u_n$

Series converges $\lim_{n\to\infty} \frac{1}{n} = 0$

$u_n$

at $x = 2$

Cond:

$u_n =$ converges

$|u_n| =$ diverges

Interval of convergence :-

$x \geq 2$ ; (or) $x < 0$

Series converges

## Exponential series

The series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots \infty$

Find the interval of convergence? $u_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)\,n!} \frac{x^{n+1}}{x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = |x| \lim_{n\to\infty} \left|\frac{1}{n(1+1/n)}\right|$$

$$= 0 < 1$$

For all values of $x$ series converges.

# Log Series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \infty$

Find Interval of convergence?

$$u_n = \frac{(-1)^{n+1} x^n}{n}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2} x^{n+1}}{n+1}}{\frac{(-1)^n x^n}{n}} \right| = \lim_{n\to\infty} \left| x \cdot \frac{n}{n+1} \right|$$

$$|x| \lim_{n\to\infty} \left|\frac{n}{n+1}\right| \Rightarrow |x| < 1 \quad \text{Converges}$$

$$-1 < x < 1$$

At $x = 1$

$1 - \frac{1}{2} + \frac{1}{3} + \cdots \infty$

by Leibnitz rule

It converges

At $x = -1$

$-1 - \frac{1}{2} - \frac{1}{3} + \cdots \infty$

diverges

Interval of convergence $\underline{\underline{-1 < x \le 1}}$

## Examples

① $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$ converges (or) diverges

② $\sum \frac{3^n}{4^n+4}$

④ $\sum_{n=0}^{\infty} \frac{n^3 x^{3n}}{n^4+1}$ Interval of convergence

③ $\sum \frac{n!\,(n+1)!}{(3n)!}$

⑤ $\sum (-1)^n \frac{1}{\sqrt{n^2+1}}$ absolutely (or) conditionally

## Solution

① $\sum_{n=1}^{\infty} \frac{n^3}{n^5+3}$ $\quad u_n = \frac{n^3}{n^5+3}$ $\quad \frac{n^3}{n^5+3} < \frac{n^3}{n^5} = \frac{1}{n^2}$

$u_n < v_n$ and $v_n$ converges

$v_n = \frac{1}{n^2}$ by P-test $v_n$ converges

$\sum u_n$ converges

② $\sum \frac{3^n}{4^n+4}$ $\quad u_n = \frac{3^n}{4^n+4} < \frac{3^n}{4^n}$

$v_n = \left(\frac{3}{4}\right)^n = 1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots$

$1 + r + r^2 + \cdots$

$|r| < 1$ converges

$u_n < v_n$ ; $v_n$ converges as it is Geometric Series $|r| < 1$

$\sum u_n$ also converges

③ $\sum \frac{n!\,(n+1)!}{(3n)!}$

Using D'Alembert's ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)!\,(n+2)!}{(3n+3)!}}{\dfrac{n!\,(n+1)!}{(3n)!}}$$

$(n+2)(n+1)$

$$= \lim_{n\to\infty} \frac{(n+1)(n+2)}{(3n+3)(3n+2)(3n+1)} = 0 < 1$$

$\lambda < 1$ converges

④ $\sum_{n=0}^{\infty} \frac{n^3 x^{3n}}{n^4+1}$ Interval of convergence

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^3 x^{3n+3}}{(n+1)^4+1}}{\dfrac{n^3 x^{3n}}{n^4+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3 x^3}{(n+1)^4+1} \times \frac{n^4+1}{n^3} \right|$$

$$= |x|^3 \lim_{n\to\infty} \left| \frac{(n+1)^3 (n^4+1)}{[(n+1)^4+1] \times n^3} \right| \to 1$$

$|x^3| < 1 \implies -1 < x < 1$

A2

$x = 1$

$$\sum \frac{n^3}{n^4+1}$$

$$\frac{n^3}{n^4+1} < \frac{n^3}{n^4} = \frac{1}{n}$$ X

$u_n < v_n \to$ diverging

$$\lim_{n\to\infty} \frac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\cancel{n}(1+1/n^4)} \times \frac{\cancel{n}}{1} = 1$$

$\frac{1}{n}$ diverges so $\sum \frac{n^3}{n^4+1}$ also diverges at $x = 1$

A2

$x = -1$

$$\sum_{n=0}^{\infty} \frac{n^3(-1)^{3n}}{n^4+1}$$

$$1 + \left(\frac{-1}{2}\right) + \frac{8}{17} - \frac{27}{82} + \dots$$

alternating sum

$u_{n+1} < u_n$

$$\lim_{n\to\infty} \frac{n^3}{n^4+1} = 0$$

converging at $x = -1$

Interval of convergence $-1 \le x < 1$

5 $$\sum \frac{(-1)^n}{\sqrt{n^2+1}}$$

$$\sum u_n = \sum \frac{(-1)^n}{\sqrt{n^2+1}} = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{10}} + \dots$$

alternating series

$u_{n+1} < u_n$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$$

By leibnitz test $\sum u_n$ is converging

$\Rightarrow \sum |u_n| = \sum \left|\frac{(-1)^n}{\sqrt{1+n^2}}\right| \Rightarrow |u_n| = \frac{1}{\sqrt{1+n^2}}$ $\quad v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{1+1/n^2}}}{1/n} = 1 \neq 0$$

So $\because 1/n$ diverges

$\sum |u_n|$ diverges

conditionally convergent.

Sequence and series of functions:

Sequence: $\{a_n\}$ $\quad a_n : N \Rightarrow \mathbb{R}$

$\{a_n\} = n^2, \ \frac{n}{n^2+2}$

$f_n : E \xrightarrow{\subseteq \mathbb{C}} \mathbb{C}$

Sequence of functions:

Sequence of functions $E \subseteq R$ for each for each $n \in \mathbb{N}$ $\quad E = [a,b]$

Let $f_n : E \longrightarrow \mathbb{R}$ be a function then $\{f_n\}$ is a sequence of functions on $E$ to $\mathbb{R}$

$$\{f_n(x)\} = \frac{x}{n} \qquad x \in E \subseteq \mathbb{R} \qquad [a,b]$$

$x = 0.4 \in [0,1]$ $\quad f_n(0.4) = \frac{0.4}{n} = \{0.4, \frac{0.4}{2}, \frac{0.4}{3}, \dots\dots\}$

$x = 0.2 \in [0,1]$ $\quad f_n(0.2) = \frac{0.2}{n} = \{0.2, \frac{0.2}{2}, \frac{0.2}{3}, \dots\dots\}$

★ $x_1 \in E$ then $\{f_n(x_1)\}$ is sequence of real no's.

1. Convergence at a point $x_0$
2. Point-wise convergence
3. Uniform convergence

I Convergence at a point $x_0$

we say that sequence of f is convergent at $x = x_0 \in E$ if seq $f_n(x_0)$ is convergent

$f_n(x) = \frac{x}{n}$ $\qquad x \in [0,1]$ $\qquad \lim_{n \to \infty} f_n(\frac{1}{2}) = \lim_{n \to \infty} \frac{1}{2n} = 0$

$x = \frac{1}{2}$ $\quad f_n(\frac{1}{2}) = \frac{1}{2n} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\dots\}$ Convergence at $x = 1/2$

II Point-wise convergence

If $\{f_n(x)\}$ converges $\forall x \in E$ then $f : E \to \mathbb{R}$

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in \mathbb{R} \quad \text{then } \{f_n\}$$

converges to 'f' Pointwise

☆ $f_n \longrightarrow f$ pointwise $\equiv f_n(x) \longrightarrow f(x) \quad \forall x \in E$

$$f_n(x) = \frac{x}{n} \qquad x \in [0,1]$$

$x = 0.1$ $\quad f_n(0.1) = \{0.1, \frac{0.1}{2}, \dots\}$

$x = 0$ $\quad f_n(0) = \{0, 0, \dots\}$

$x = 0.5$

$$\lim_{n\to\infty} \frac{x}{n} = 0$$

$f_n \longrightarrow 0 \quad \forall x \in [0,1]$

Pointwise

ex:- $f_n(x) = x^n \qquad 0 \le x \le 1 \qquad f_n \to f$

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

$x = 0$

$$f_n(x) = f_n(0) = \{0, 0, \dots\} = 0$$

$x = 1/2 \quad f_n(1/2) = \{(1/2)^n\} = \{\frac{1}{2}, (1/2)^2, \dots$

$\Rightarrow E = [-1, 1] \qquad f_n(x) = \frac{x}{n} \qquad f_n \to f$

$$f(x) = 0 \qquad -1 \le x \le 1$$

$E = [-1, 1]$ $\quad f_n(x) = x^n$ $\quad f_n \to f$ pointwise

$$f(x) = 0 \qquad -1 < x < 1$$
$$= 1 \qquad x = 1$$

Pointwise convergence

$f_n(x) =$ seq $\quad x \in E \quad \forall x \in E \quad f_n \to f \ \forall x \in E$

$$\lim_{n \to \infty} f_n(x) = f$$

let $\varepsilon > 0$ $\quad \exists$ $m$ such that

$$|f_n(x) - f| < \varepsilon \quad \forall n \geq m$$

In. $m$ depends both on $x$ & $\varepsilon$.

$x = 0.4$ $\qquad m_1$ at $x = 0.4$

$x = 0.2$ $\qquad m_2 \qquad x = 0.2$ $\qquad \max = \{m_1, m_2, m_3 \ldots\ldots\}$

$\vdots$

$x = c$ $\qquad m_c$

$m_c$

## Uniform convergence

let $f_n : E \longrightarrow R$ be a sequence of function
$f : E \longrightarrow R$ be a function. we say
$f_n \longrightarrow f$ uniformly if $\forall\, \varepsilon > 0\ \exists$
$n_0 \in N$ s.t $\forall n \geq n_0$ ($n_0$ depends only on $\varepsilon$ not on $x$)

$$| f_n(x) - f | < \varepsilon \quad \forall n \geq n_0$$

**ex:-** $f_n(x) = \dfrac{1}{n+x} \qquad x \in [0, 1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n+x} = \lim_{n\to\infty} \frac{1}{n(1+x/n)} = 0$$

$f_n \rightarrow 0 \quad \forall x \in [0,1]$

$f_n \rightarrow 0$ pointwise

## Uniform convergence

$$f_n(x) = \frac{1}{n+x}$$

$\boxed{m = \frac{1}{\varepsilon} - x}$

let $\forall\, \varepsilon > 0 \quad \exists\, m =$

$m = \{ \cdots \}$

$$\left| \frac{1}{n+x} - 0 \right| < \varepsilon$$

$m =$

$\boxed{n > \frac{1}{\varepsilon} - x}$

This $\frac{1}{\varepsilon} - x$ decreases with increase in $x$

$$\frac{1}{n+x} < \varepsilon$$

$m = \frac{1}{\varepsilon}$

max

$$n + x > 1/\varepsilon$$

It's also uniform convergence.

$$f_n(x) = \frac{1}{n+x} \qquad x \in [0,1]$$

$$\left| f_n(x) - f(x) \right| < \left| \frac{1}{n+x} - 0 \right| < \frac{1}{n} < \varepsilon$$

$$\frac{1}{n+x} < \frac{1}{n} < \varepsilon$$

[2·3]: $\qquad n > \frac{1}{\varepsilon}$

let $m = \left[ \frac{1}{\varepsilon} \right] + 1$ then $m \in \mathbb{N}$

Note:-

1. If $f_n \to f$ uniformly then $f_n \to f$ pointwise also

Converse is not true.

2. If $f_n(x) \longrightarrow f(x)$ uniformly $\left[ \forall \varepsilon > 0 \quad \exists\, n_0 \in N \right.$ such that $\left. |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \right]$

then $\{ f_n(x) - f(x) : x \in E \}$ is always bounded

** let $M_n = \operatorname{Sup} \{ |f_n(x) - f(x)| : x \in E \}$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\downarrow$$

$$M_n < \varepsilon \implies |M_n - 0| < \varepsilon$$

$$M_n \longrightarrow 0 \quad \text{as } n \to \infty$$

**

If $f_n(x) \longrightarrow f(x)$ uniformly then $M_n \to 0$ as $n \to \infty$

ex:- $f_n(x) = x^n \quad 0 \leq x \leq 1 \qquad x \in [0,1]$

$$f_n \longrightarrow f$$

$x = 0$ to $1$

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

$$M_n = \operatorname{Sup} \{ |f_n(x) - f(x)| : x \in [0,1) \}$$

$$= \operatorname{Sup} \{ x^n - 0 : 0 \leq x < 1 \}$$

$$M_n = \sup\{x^n : 0 \le x < 1\}$$

$n=1$ $\quad M_1 = \sup\{\underline{x : 0 \le x < 1}\} = 1$

$n=2$ $\quad M_2 = \sup\{(x)^2 : 0 \le x < 1\} = 1$

$\vdots$

$n \to \infty$ $\quad M_n \longrightarrow 1 \ne 0 \quad M_n \not\to 0$ as $n \to \infty$

$f_n \longrightarrow f$ Not uniformly.

3. let $E \subset \mathbb{R}$ and each $n \in \mathbb{N}$ $f_n : E \to \mathbb{R}$ is bounded on $D$. It say $f_n \to f$ uniformly on $D$ then the limit function $f$ is bounded on $D$.

ex:- $\quad f_n(x) = 1 + x + x^2 + \dots\, x^{n-1}$, $x \in [0,1]$

Pointwise

$$\lim_{n\to\infty} f_n(x) = \frac{1}{1-x} \quad x \in [0,1)$$

$1 + r + r^2 + \dots$

$|r| < 1$

$\frac{1}{1-r}$

$f_n \to f$ pointwise

$f_n \to f$ uniformly $\quad f_n$ is bounded

$$f = \frac{1}{1-x} \quad \overset{x\in}{[0,1)} \qquad x \to 1 \quad f \to \infty$$

$$\frac{1}{1-1} = \frac{1}{0} = \infty$$

Not bounded

Not uniformly convergent

$f_n(x) =$

$\boxed{f_n(x)} \longrightarrow f(x)$ pointwise $\lim_{n\to\infty} f_n(x) = f(x)$

$f_n(x)$ bounded $E \subset R$

$f_n(x) \longrightarrow f$ uniformly $f$ is also bounded on $E$.

4. let $E \subseteq R$ and for each $n \in N$ $f_n : E \longrightarrow R$ is continuous on $E$. If the sequence $f_n \to f$ uniformly on $E$. then $f$ is continuous on $E$

$f_n \longrightarrow f$

ex:- $f_n(x) = \dfrac{x^n}{1+x^n}$ $x \in [0, 2]$

$f_n \longrightarrow f$

$$f(x) = 0 \quad 0 \leq x < 1$$
$$= \frac{1}{2} \quad x = 1$$
$$= 1 \quad 1 < x \leq 2$$

$0 \leq x < 1$

$f(x)$

$x = 1/2$

$f_n \to f$ pointwise

At $x = 0$

$f_n(x) = 0$

At $x = 1/2$

$f_n(x) \quad \dfrac{\left(\frac{1}{2}\right)^n}{1+\left(\frac{1}{2}\right)^n} = \dfrac{1}{2^n+1}$

$\lim_{n\to\infty} \dfrac{1}{2^n+1} = 0$

Not uniformly convergent

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \\ 1 & 1 < x \leq 2 \end{cases}$$ not continuous

## Series of functions

let $E \subseteq R$. Let $\{f_n\}$ be a sequence of functions on $E$ to $R$.

Then $f_1(x) + f_2(x) + f_3 + \cdots$ is said to series of functions on $R$

$\sum f_n(x)$

$a_n = \{a_1, a_2, a_3, \ldots$

$S_1 = \sum a_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + \cdots a_n$

$\{S_n\}$ defined by $x \in E$

$S_1(x) = f_1(x)$

$S_2(x) = f_1(x) + f_2(x)$

$S_3(x) = f_1(x) + f_2(x) + f_3(x)$

$\vdots$

$S_n(x) = f_1(x) + f_2(x) + \cdots f_n(x)$

The sequence $\{S_n\}$ is said to sequence of partial sums of infinite series $\sum f_n(x)$

$\Rightarrow$ If $\{S_n\}$ is pointwise convergent on $E$ to a function $S$ then $\sum f_n$ is said to be pointwise convergent on $E$ and '$S$' is said to be sum function of the series.

$S_n \longrightarrow S$ pointwise

$\rightarrow$ If $S_n \longrightarrow S$ uniformly on $E$ then $\sum f_n$ is said to be uniformly convergent on $E$. '$S$' = sum function ex:- Prove that series of $f_n(x) = 1 + x + x^2 + \cdots x^{n-1}, 0 \leq x < 1$ is point wise convergent on $0 \leq x < 1$. but the convergence is not uniform.

$$f_n(x) = 1 + x + x^2 + \cdots x^{n-1}$$

$$S_n(x) = \sum f_n(x)$$

$n = 1 \quad f_1(x) = 1$

$$S_n(x) = 1 + x + x^2 + \cdots \cdots x^{n-1} \quad x \in [0,1)$$

$f_2(x) = 1 + x$

$$\lim_{n \to \infty} S_n(x) = \frac{1}{1-x} \quad x \in [0,1)$$

$S_n \longrightarrow S$ pointwise

$\sum f_n \longrightarrow S$ pointwise

$\sum f_n \longrightarrow S$ not convergent as $S_n$ is bounded
$S$ is not bounded

$$f_n(x) = \frac{x}{n} \qquad f_1(x) = x$$

$$f_2(x) = \frac{x}{2}$$

$$S_n = x + \frac{x}{2} + \frac{x}{3} + \cdots \frac{x}{n} \qquad f_3(x) = \frac{x}{3}$$

$\downarrow$

$\sum f_n$

# Unit-2

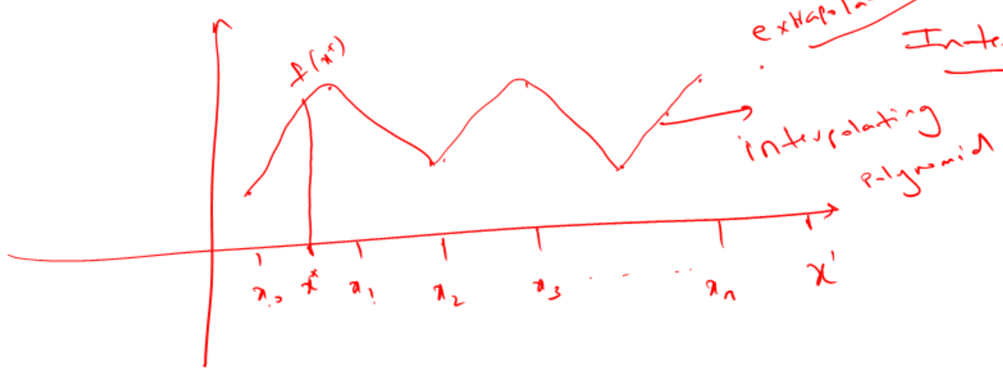
## Numerical Analysis

→ $y = f(x)$ → $y = x^2 + 2$

| $x$ | $x_0$ | $x_1$ | $x_2$ | ... | $x_n$ |
|---|---|---|---|---|---|
| $y = f(x)$ | $f(x_0)$ | $f(x_1)$ | $f(x_2)$ | ... | $f(x_n)$ |

$x_n$ → Nodal values

$f(x_n)$ → functional values at nodes




Interpolation Method

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

✱ $x_0 \quad x_1 \quad x_2 \; \dots \; x_n$ may (or) may not be even spaced

$$x_i - x_{i-1} = h \quad \forall \; i = 1, 2, \dots n$$

$$x_1 = x_0 + h$$

$$x_2 = x_1 + h = x_0 + 2h$$

⟹ **Interpolation with evenly spaced points**

New operators :- Finite difference operators

① Shift operator

② forward difference

③ Backward difference

④ Central difference operator

# Finite difference operator

Given $(x_i, f(x_i))$

$x_0 \overset{h}{\quad} x_1 \overset{h}{\quad} x_2 \overset{h}{\quad} \cdots \; x_n$

$x_i = x_0 + ih$

$i = 0, 1, 2, \ldots n$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$x_0 + h = x_1$

$i = 1, 2, 3,$

① **Shift operator** :- Denoted by 'E'

$$E[f(x_i)] = f(x_i + h) = f(x_{i+1}) = f_{i+1}$$

Applying shift operator on $f(x_i)$ on $x_i$

Functional value shifts to next nodal point

$$E^1[f(x_0)] = f(x_1) = f(x_0 + h)$$

$$E[f(x_1)] = f(x_0 + 2h) = f(x_2) = f(x_1 + h)$$

$$E^2[f(x_i)] = E[E(f(x_i))] = E[f(x_i + h)]$$

$$= f((x_i + h) + h) = f[x_i + 2h]$$

$$E[f(x_{i+1})] = f(x_{i+2})$$

$$E^k[f(x_i)] = f(x_i + kh) = f(x_{i+k})$$

$x_0^2 \quad x_1^2 \quad x_2^2 \quad x_3$

$K = \frac{1}{2}$ any real no.

$$E^K[f(x_i)] = f(x_i + Kh)$$

$$E^{1/2}[f(x_i)] = f\left[x_i + \frac{1}{2}h\right]$$

II <u>Forward difference</u> <u>operator</u> :- denoted by $\Delta$ (del)

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$$

$$= f_{i+1} - f_i$$

$$\Delta f(x_0) = f(x_1) - f(x_0) = f(x_0 + h) - f(x_0)$$

1st forward difference

$$\Delta f(x_1) = f(x_2) - f(x_1) = f(x_0 + 2h) - f(x_0 + h)$$

$$= f(x_1 + h) - f(x_1)$$

2nd forward difference

$$\Delta^2 f(x_i) = \Delta(\Delta f(x_i)) = \Delta[f(x_i + h) - f(x_i)]$$

$$= \Delta f(x_i + h) - \Delta f(x_i)$$

$$= [f(x_i + 2h) - f(x_i + h)] - [f(x_i + h) - f(x_i)]$$

$$\Delta^2 f(x_i) = f(x_i + 2h) - 2f(x_i + h) + f(x_i)$$

$$= f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)$$

$$\Delta^3 f(x_i) = f(x_i + 3h) - 3f(x_i + 2h) + 3f(x_i + h) - f(x_i)$$

Relation between $\Delta$ & $E$ $\qquad$ $E f(x_i) = f(x_i + h)$

$$\Delta f(x_i) = f(x_i + h) - f(x_i)$$

$$= E f(x_i) - f(x_i)$$

$$\Delta f(x_i) = (E-1) f(x_i)$$

$$\Delta \equiv (E-1) \quad \text{(or)} \quad E \equiv \Delta + 1$$

$$\Delta^2 f(x_i) = f(x_i + 2h) - 2f(x_i + h) + f(x_i)$$

$$\boxed{E^k f(x_i) = f(x_i + kh)}$$

$$= E^2 f(x_i) - 2E f(x_i) + f(x_i)$$

$$\underline{\underline{\Delta^2 f(x_i)}} = (E^2 - 2E + 1) f(x_i) = (E-1)^2 f(x_i)$$

$$\Delta^2 \equiv (E-1)^2$$

$$\boxed{\Delta^n f(x_i) = (E-1)^n f(x_i)} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!} f_{i+n-k}$$

# Forward difference table

| $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ |
| --- | --- | --- | --- | --- |
| $x_0$ | $f(x_0)$ | | | |
| | | $\Delta f_0 = f_1 - f_0$ | | |
| $x_1$ | $f(x_1)$ | | $\Delta^2 f_0 = \Delta f_1 - \Delta f_0$ | |
| | | $\Delta f_1 = f_2 - f_1$ | | $\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$ |
| $x_2$ | $f(x_2)$ | | $\Delta^2 f_1 = \Delta f_2 - \Delta f_1$ | |
| | | $\Delta f_2 = f_3 - f_1$ | | |
| $x_3$ | $f(x_3)$ | | | |

forward $\Delta$ dell

$\Longrightarrow$ **Backward difference operator** :- denoted $\nabla$ (nepla)

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1}$$

$$\nabla f(x_1) = f(x_1) - f(x_0)$$

$$\nabla f(x_2) = f(x_2) - f(x_1)$$

$$\nabla^2 f(x_i)$$

Review

① Shift operator

$$E\,f(x_i) = f(x_i + h)$$

$$E f(x_0) = f(x_1)$$

$$E^k f(x_i) = f(x_i + kh)$$

② Forward difference operator : $\Delta$ (Dell)

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i) = f_{i+1} - f_i$$

$$\Delta f(x_1) = f(\overset{x_1+h}{x_2}) - f(x_1)$$

Relation b/w $\Delta$ & $E$

$$\Delta \equiv E - 1 \quad , \quad \Delta^n f(x_i) = (E-1)^n f(x_i - 1)$$

⟹ Backward difference operator : denoted by $\nabla$ (Nebla) $\Delta$ Dell

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1}$$

$$\nabla(f(x_1)) = f(x_1) - f(x_0) = f_1 - f_0$$

2 → 2nd Backward difference

$$\nabla^2 f(x_i) = f(x_i) - 2f(x_{i-1}) + f(x_{i-2})$$

$$\nabla^2 f(x_2) = f(x_2) - 2f(x_1) + f(x_0)$$

$$\nabla(\nabla f(x_i)) = \nabla\big(f(x_i) - f(x_i - h)\big)$$

$$= f(x_i) - f(x_i - h) - \big[f(x_i - h) - f(x_i - h - h)\big]$$

$$\nabla^2 f(x_i) = f(x_i) - 2f(x_i - h) + f(x_i - 2h)$$

$$\nabla^3 f(x_i) = f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})$$

## Relation between $\nabla$ & $E$

$$\nabla f(x_i) = f(x_i) - f(x_i - h)$$

$$= f(x_i) - E^{-1} f(x_i)$$

$$\nabla f(x_i) = (1 - E^{-1}) f(x_i)$$

$$\nabla = (1 - E^{-1})$$

$$E^k f(x_i) = f(x_i + kh)$$

$$\nabla^2 f(x_i) = f(x_i) - 2 f(\overset{x_i - h}{x_{i-1}}) + f(x_{i-2})$$

$$= f(x_i) - 2E^{-1} f(x_i) + E^{-2} f(x_i)$$

$$= (1 - 2E^{-1} - E^{-2}) f(x_i)$$

$$\nabla^n f(x_i) = [1 - E^{-1}]^n f(x_i) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} f_{i-k}$$

## Backward difference table

| $x$ | $f(x)$ | $\nabla f(x_i)$ | $\nabla^2 f$ | $\nabla^3 f$ |
|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | | | |
| | | $\nabla f(x_1)$<br>$\nabla f_1 = f_1 - f_0$ | $f_2 - f_1 - f_1 + f_0$<br>$f_2 - 2f_1 + f_0$ | |
| $x_1$ | $f(x_1)$ | | $\nabla^2 f_2 = \nabla f_2 - \nabla f_1$ | |
| | | $\nabla f_2 = f_2 - f_1$ | | $\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$ |
| $x_2$ | $f(x_2)$ | | $\nabla^2 f_3 = \nabla f_3 - \nabla f_2$ | |
| | | $\nabla f_3 = f_3 - f_2$ | | |
| $x_3$ | $f(x_3)$ | | | |

$$\Rightarrow \Delta f_0 = \nabla f_1, \quad \Delta f_1 = \nabla f_2, \quad \Delta f_2 = \nabla f_3$$

$$\Delta^3 f_0 = \nabla^3 f_3$$

4. Central difference operator :- denoted by delta 'δ'

$$\delta[f(x_i)] = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

$x_0 + \frac{1}{2}$

$\frac{2}{3}$ $\frac{2}{3}$

$\delta[f_i] = f_{\frac{1}{2}} - f$

not node point

$\Delta f(x_i) = f(x_{i+h}) - f(x_i)$

$\nabla f(x_i) = f(x_i) - f(x_{i-1})$

$x_i + h =$

$x_1 + h = x_2$

$x_{i+1} = x_2$

| $x$ | 0 | – | 2 | 3 | 4 |
|---|---|---|---|---|---|
| | $x_0$ | | $x_1$ | | $x_2$ |
| $f$ | $f(0)$ | | $f(2)$ | | $f(4)$ |

$h = 2$

$$\delta[f(x_1)] = f_1(x_1 + 1) = f(3)$$

$$\delta\left[f\left(x_i + \frac{h}{2}\right)\right] = f\left(x_i + \frac{h}{2} + \frac{h}{2}\right) - f\left(x_i + \frac{h}{2} - \frac{h}{2}\right)$$

$$\delta\left[f\left(x_i + \frac{h}{2}\right)\right] = f(x_i + h) - f(x_i)$$

$x_i + h$

$x_{i+1}$ $f_{i+1}$

$x_1$ $x_0$

$f_1$ $f_0$

$$\delta\left[f_{\frac{1}{2}}\right] = \boxed{f_1 - f_0}$$

$$\delta\left[f_{\frac{3}{2}}\right] = f_2 - f_1$$

$f_{\frac{2}{3}}$

Relation between shift operator E and 'δ'

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

$E^k f(x_i) = f(x_i + kh)$

$$= E^{1/2} f(x_i) - E^{-1/2} f(x_i)$$

$$\delta f(x_i) = \left(E^{1/2} - E^{-1/2}\right) f(x_i)$$

$$\delta = E^{1/2} - E^{-1/2} = \frac{(E-1)}{E^{1/2}}$$

$$\delta^n = \underline{\qquad\qquad}$$

$$\boxed{E + \frac{1}{E} - 2}$$

$$\frac{E^2 - 2E + 1}{E^{1/2}}$$

$$\delta^2 f(x_i) = \delta\left[\delta f(x_i)\right]$$

$$= \delta\left[f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)\right]$$

$$= f\left(x_i + \frac{h}{2} + \frac{h}{2}\right) - f\left(x_i + \frac{h}{2} - \frac{h}{2}\right) - f\left(x_i - \frac{h}{2} + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2} - \frac{h}{2}\right)$$

$$\delta^2 f(x_i) = f(x_i + h) - f(x_i) - f(x_i) + f(x_i - h)$$

$$\delta^2 f(x_i) = f(x_i + h) - 2f(x_i) + f(x_i - h)$$

$$= E f(x_i) - 2f(x_i) + E^{-1} f(x_i)$$

$$\delta^2 f(x_i) = \left(E - 2 + E^{-1}\right) = \left(\frac{E - 2E + 1}{E}\right) f(x_i)$$

$$\delta^2 = \left(\frac{E^2 - 2E + 1}{E}\right) = \frac{(E-1)^2}{E}$$

$$\delta^3 f(x_i) = f_{i+\frac{3}{2}} - 3f_{i+\frac{1}{2}} + 3f_{i-\frac{1}{2}} - f_{i-\frac{3}{2}}$$

$$= E^{3/2} - 3E^{1/2} + 3E^{-1/2} - E^{-3/2}$$

$$\delta^3 f(x_i) = \frac{E^3 - 3E^2 + 3E + 1}{E^{3/2}} = \frac{(E-1)^3}{E^{3/2}} f(x_i)$$

$$\delta^n = \frac{(E - 1)^n}{E^{n/2}}$$

$$\left(E - \frac{1}{E}\right)^n \qquad \left(E^{1/2} - E^{-1/2}\right)^n$$

# Central difference table

| $x$ | $f(x)$ | $\delta f$ | $\delta^2 f$ | $\delta^3 f$ |
|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | | | |
| | | $\delta f_{\frac{1}{2}} = f_1 - f_0$ | | |
| $x_1$ | $f(x_1)$ | | $\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$ | |
| | | $\delta f_{\frac{3}{2}} = f_2 - f_1$ | | $\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1$ |
| $x_2$ | $f(x_2)$ | | $\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}$ | |
| | | $\delta f_{\frac{5}{2}} = f_3 - f_2$ | | |
| $x_3$ | $f(x_3)$ | | | |

## Operators

1. Shift $\rightarrow$ $E$
2. forward difference $\rightarrow$ $\Delta$
3. Backward difference $\rightarrow$ $\nabla$
4. Central difference $-$ $\delta$
5. Mean operator : $\mu$

$$\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right]$$

Relation between $\mu$ & $E$

$$\boxed{E^K f(x_i) = f(x_i + Kh)}$$

$$\mu f(x_i) = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-1/2}\right] f(x_i)$$

$$\boxed{\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]}$$

## Note:-

1. Note 1:- Let $P_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots a_n$.

Kth forward difference

$$\Delta^K P_n(x) = 0 \quad , K > n.$$

$$= a_0 n! \, h^n \quad K = n$$

$$= a_0 n! \quad (h = 1)$$

$$\Delta P_n(x) = P_n(x+h) - P_n(x)$$

$$= a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots a_n - (a_0 x^n + a_1 x^{n-1} + \cdots a_n)$$

$$= (n-1) \text{ degree}$$

$$P_2(x) = a_0 x^2 + a_1 x + a_2$$

$$\Delta P_2(x) = P_2(x+h) - P_2(x)$$

$$= a_0 (x+h)^2 + a_1 (x+h) + \cancel{a_2} - a_0 x^2 - a_1 x - \cancel{a_2}$$

$$= a_0 (\cancel{x^2} + h^2 + 2xh) + \cancel{a_1 x} + a_1 h - a_0 \cancel{x^2} - \cancel{a_1 x}$$

$$\Delta P_2(x) = a_0 h^2 + a_0 2xh + a_1 h \longrightarrow$$ $k > n$

$(n-2)$

$$\Delta^2 P_2(x) = \Delta (a_0 h^2 + 2a_0 xh + a_1 h)$$ $k = n$

$$= \Delta a_0 h^2 + 2a_0 h (x + h - \cancel{x}) + \Delta a_1 h$$ $a_1 h - a_1 h$

$f(x+h) - f(x)$

$$= 0 + 2a_0 h^2 + 0$$

$$\Delta^2 P_2(x) = 2 a_0 h^2 = a_0 2! h^2$$ $k = n$

$$\Delta^3 P_2(x) = \Delta 2a_0 h = 2a_0 h^2 - 2a_0 h^2 = 0$$ $\rightarrow k > 2$

$$\Delta^k P_n(x) = \boxed{a_0 n! h^n}$$

$a_0 \cancel{6} h^3 = a_0 3! h^3$

$$P_3(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3 = \Delta^3 P_3(x) = a_0 3! h^3$$

ex:- $P_n(x) = [(1-2x)(1-3x)(1-4x)]$ $h=1$

$= [\underset{\downarrow a_0}{-24x^3} + \ldots\ldots]$

$\Delta^3 P_n(x) = a_0 n! = -24 \times 3! = -144$ $k=3$

$\Delta^4 \underset{=3}{P_n(x)} = 0$ as $k > n$, $k=4$, $n=3$

Note 2:-

① $\Delta^n f_i = \nabla^n f_{i+n} = \delta^n f_{i+\frac{n}{2}}$

Relation

$\boxed{\nabla = 1 - E^{-1}} = \frac{E-1}{E}$

$\Delta = E - 1$

$\delta^n = \left(\frac{E-1}{E^{1/2}}\right)^n$

Proof:-

$\nabla = 1 - E^{-1} = \left(\frac{E-1}{E}\right) = (E-1)E^{-1}$

$\nabla = \Delta E^{-1}$

Taking $n^{th}$ power

$\nabla^n = \Delta^n E^{-n}$

$\nabla^n f_{i+n} = \Delta^n E^{-n} f_{i+n}$

$f_{i+n} = f(x_i + nh)$

$E^{-n} f_{i+n}$

$E^k f(x_i) = f(x_i + kh)$

$E^{-n} f(x_i + nh) = f(x_i + nh - nh)$
$= f(x_i)$

$$\boxed{\nabla^n f_{i+n} = \Delta^n f_i}$$

$\Rightarrow \delta^n = \left(\frac{E-1}{E^{1/2}}\right)^n = \left(E^{1/2} - E^{-1/2}\right)^n$

$\delta^n = \frac{\Delta^n}{E^{n/2}}$

$\delta^n = \Delta^n E^{-n/2}$

$\delta^n f_{i+\frac{n}{2}} = \Delta^n E^{-n/2} f_{i+n/2} = \Delta^n f_i$

$\delta^n f_{i+\frac{n}{2}} = \Delta^n f(x_i)$

$E^{k} f(x_i) = f(x_i + kh)$

$E^{-n/2} f(x_i + \frac{nh}{2}) = f\left(x_i + \cancel{\frac{nh}{2}} - \cancel{\frac{nh}{2}}\right) = f(x_i)$

Note 3

(1) $\delta = \nabla(1-\nabla)^{-1/2}$

(2) $\mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$

(3) $\Delta f_i^2 = (f_i + f_{i+1})\Delta f_i$

$$\boxed{\begin{aligned} \Delta &= E - 1 \\ \nabla &= 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla \\ &\quad = E = (1-\nabla)^{-1} \\ \delta &= \frac{E-1}{E^{1/2}} \end{aligned}}$$

(1) $\delta = \frac{E-1}{E^{1/2}} = E^{-1/2}(E-1) = E E^{-1/2}(1 - E^{-1})$

$= E^{1/2}(1 - E^{-1})$

$[(1-\nabla)^{-1}]^{1/2}\nabla$

$= (1-\nabla)^{-1/2}\nabla$

(2) $\mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$ $\qquad \delta = E^{1/2} - E^{-1/2}$

RHS $\left[1 + \frac{\delta^2}{4}\right]^{1/2} = \left[1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}\right]^{1/2}$

$= \left[1 + \frac{E + E^{-1} - 2}{4}\right]^{1/2}$

$= \left[\frac{4 + E + E^{-1} - 2}{4}\right]^{1/2}$

$= \left[\frac{E + E^{-1} + 2}{4}\right]^{1/2} = \left[\left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2\right]^{1/2} = \frac{E^{1/2} + E^{-1/2}}{2} = \mu$

(3) $\Delta f_i^2 = (f_i + f_{i+1})\,\Delta f_i$

$\Delta f(x_i) = f(x_i+h) - f(x_i)$

$\Delta f_i^2 =$

$\Delta g(x_i) = \Delta f^2(x_i)$

$g(x_i+h) - g(x_i)$

$\Delta f^2(x_i+h) - \Delta f^2(x_i)$

$$\Delta f_i^2 = f^2(x_i+h) - f^2(x_i)$$

$$= \left[f(x_i+h) + f(x_i)\right]\left[f(x_i+h) - f(x_i)\right]$$

$$= \left[f(x_i+h) + f(x_i)\right]\Delta f(x_i)$$

$$= (f_{i+1} + f_i)\,\Delta f_i$$

Note 4 :-

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x)\,g(x+h)}$$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

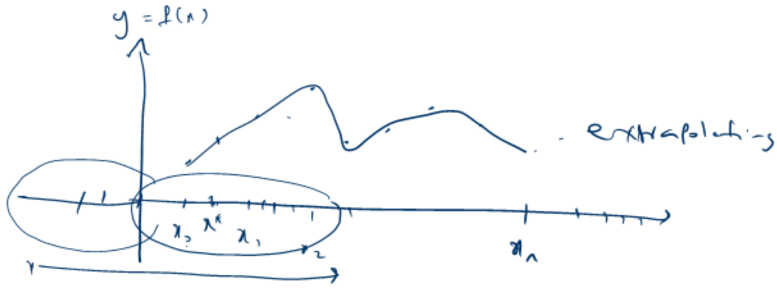
$$= \frac{g(x)\,f(x+h) \overset{-f(x)g(x)}{} - f(x)\,g(x+h) + f(x)g(x)}{g(x)\,g(x+h)}$$

$$= \frac{g(x)\left[f(x+h) - f(x)\right] - f(x)\left[g(x+h) - g(x)\right]}{g(x)\,g(x+h)}$$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x)\,g(x+h)}$$

## Newton's Forward Interpolation :

Let the function y = f(x) take the values $y_0, y_1, y_2 \ldots$ corresponding to the values $x_0, x_0+h$, $x_0+2h, x_0+3h, \ldots$ of $x$.




| $x$ | $f(x)$ |
|---|---|
| $x_0$ | $f(x_0)$ |
| $x_1$ | $f(x_1)$ |
| $x_2$ | $f(x_2)$ |
| $\vdots$ | |
| $x_n$ | $f(x_{n+1})$ |
| $x_{n+1}$ | |

$$f(x_0 + sh)$$

Applicability

(1) Evenly spaced nodd point

$x_i - x_{i-1} = h$ $\quad x_1 = x_0 + h$

$x_2 = x_0 + 2h$

or $x_i = x_0 + ih$ $\quad x_3 = x_0 + 3h$

$i = 0, 1, 2 \ldots n$

(2) This formula is used for the values of $y$ near the begining of set of values of $x$

Proof:- Suppose it required to evaluate $f(x)$ at

$x = x_0 + ph$, $p$ is any real number

$\Rightarrow \dfrac{x - x_0}{h} = p$

$$f(x_0 + ph) = E^p f(x_0)$$

as we know $E^k f(x_i) = f(x_i + kh)$

Relation $E$ & $\Delta$

$\Delta = E - 1 \Rightarrow E = 1 + \Delta$

$$f(x_0 + ph) = (1+\Delta)^p f(x_0)$$

using binomial thrm

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$f(x+ph) =$

$$(1+\Delta)^p f(x_0) = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\cdots\right] f(x_0)$$

$$= f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots\cdots$$

$\boxed{P\text{-form}}$ for NFDI

$$= f(x_0) + \frac{x-x_0}{h}\Delta f(x_0) + \frac{(x-x_0)}{2!\,h}\left(\frac{x-x_0}{h}-1\right)\Delta^2 f(x_0) + \frac{(x-x_0)}{3!\,h}\left(\frac{x-x_0}{h}-1\right)\left(\frac{x-x_0}{h}-2\right)\Delta^3 f(x_0)$$

$$\frac{x-x_0-h}{h} = \frac{x-(x_0+h)}{h} = \frac{x-x_1}{h}$$

$$= f(x_0) + \frac{(x-x_0)}{h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f(x_0) + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}\Delta^3 f(x_0)$$

$$+ \cdots\cdots\cdots \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{n!\,h^n}\Delta^n f(x_0)$$

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!\,h}\Delta f(x_0) + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\Delta^2 f(x_0)$$

$$+ \cdots\cdots\cdots \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{n!\,h^n}\Delta^n f(x_0)$$

☆ NFDI has Permanence property

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!\,h}\Delta f(x_0) + \cdots\cdots \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{n!\,h^n}\Delta^n f(x_0)$$

$$f(x_0) + \frac{(\cancel{x_1-x_0})}{1!\,\cancel{h}}(f_1-f_0)$$

$$+ \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)}{(n+1)!\,h^{n+1}}\Delta^{n+1} f(x_0)$$

EX: Construct a forward difference table for the following data

$x$ : $-1$ $\quad 0 \quad 1 \quad 2$ $\qquad h=1$

$f(x)$ : $-8 \quad 3 \quad 1 \quad 12$

| $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ |
|---|---|---|---|---|
| $-1$ | $-8$ | | | |
| | | $\Delta f_0 = f_1 - f_0$ <br> $3-(-8) = 11$ | | |
| | | | $\Delta^2 f_0 = \Delta f_1 - \Delta f_0$ <br> $-2-11$ <br> $=-13$ | |
| $0$ | $3$ | | | $\Delta^3 f_0 =$ <br> $13-(-13)$ <br> $=26$ |
| | | $\Delta f_1 = f_2 - f_1$ <br> $1-3 = -2$ | | |
| $1$ | $1$ | | $\Delta^2 f_1$ <br> $11-(-2)$ <br> $= 13$ | |
| | | $\Delta f_2 = f_3 - f_2$ <br> $12-1 = 11$ | | |
| $2$ | $12$ | | | |

Ex: Construct forward difference table and find f(-1.5) and Interpolating Polynomial

| | $x$ | $f(x)$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | $\Delta^4 f$ |
|---|---|---|---|---|---|---|
| $x_0$ | $-2$ | $f(x_0)$ $15$ | | | | |
| | | | $\Delta f_0 = -10$ | | | |
| $x_1$ | $-1$ | $f(x_1)$ $5$ | | $\Delta^2 f_0 = 6$ | | |
| | | | $\Delta f_1 = -4$ | | $\Delta^3 f_0$ <br> $0$ | |
| $x_2$ | $0$ | $1$ | | $\Delta^2 f_1 = 6$ | | |
| | | | $\Delta f_2 = 2$ | | $\Delta^3 f_1$ <br> $0$ | |
| $x_3$ | $1$ | $3$ | | $\Delta^2 f_2 = 6$ | | |
| | | | $\Delta f_3 = 8$ | | $\Delta^3 f_2$ <br> $0$ | |
| $x_4$ | $2$ | $11$ | | $\Delta^2 f_3 = 6$ | | |
| | | | $\Delta f_4 = 14$ | | | |
| $x_5$ | $3$ | $25$ | | | | |

$$f(x) = f(x_0) + \frac{(x-x_0)\Delta f_0}{1!\,h} + \frac{(x-x_0)(x-x_1)\Delta^2 f_0}{2!\,h^2} + 0$$

$$= 15 + \left(\frac{x+2}{1!\,(1)}\right)(-10) + \frac{(x+2)(x+1)}{2!\,(1)^2}(6)$$

$$f(x) = 15 - 10(x+2) + \frac{(x+2)(x+1)}{2!} \times 3$$

$$f(x) = 15 - 10x - 20 + 3(x^2 + 3x + 2)$$

$$\underline{\underline{f(x)}} = 3x^2 - x + 1 \longrightarrow \text{quadratic polynomial}$$

$$f(-1.5) = 3(-1.5)^2 - (-1.5) + 1 = \underline{\qquad\qquad}$$

## Backward difference interpolation:

let the function $y = f(x)$ takes values $y_0, y_1, y_2 \dots$ for corresponding values of $x_0, x_0+h, x_0+2h, \dots\dots$

Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, $p$ any real number.

Applicability $\Rightarrow$ $\frac{x - x_n}{h} = p$

1. Evenly spaced nodal point
2. This formula is used for interpolating values of $y$ near end of tabulated values of $x$'s

$$f(x_n + ph) = E^p f(x_n)$$

$$\downarrow$$

$$(1-\nabla)^{-p} f(x_n)$$

$$\Big\downarrow \text{ Binomial term}$$

Relation $E$ & $\nabla$

$$\nabla = \frac{E-1}{E}$$

$$\nabla = 1 - E^{-1}$$

$$E^{-1} = 1 - \nabla$$

$$E = (1-\nabla)^{-1}$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1!}x^2 + \frac{n(n+1)(n+2)}{2!}x^3 + \dots\dots$$

$f(x) =$

$$(1-\nabla)^{-p} f(x_n) = \left[1 + \nabla p + \frac{p(p+1)}{1!}\nabla^2 + \frac{p(p+1)(p+2)}{2!}\nabla^3 + \dots\dots\right] f(x_n)$$

$\downarrow$

$p$ form

$$f(x) = f(x_n) + \frac{(x-x_n)}{h}\nabla f(x_n) + \frac{\overset{p}{(x-x_n)}}{2!\,h}\overset{(p+1)}{\left(\frac{x-x_n}{h}+1\right)}\nabla^2 f(x_n) + \cdots$$

$$p+1 = \frac{x-x_n}{h}+1 = \frac{x-x_n+h}{h} = \frac{x-(x_n-h)}{h}$$

$$\underline{f(x)} = f(x_n) + \frac{(x-x_n)}{h}\nabla f(x_n) + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2}\nabla^2 f(x_n) + \cdots$$

$$\cdots + \frac{(x-x_n)(x-x_{n-1})(x-x_{n-2})\cdots(x-x_1)}{n!\,h^n}\nabla^n f(x_n)$$

**ex:-** Find $f(1.0)$ from the following data

$h=0.2$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 1.1 |
| $f(x)$ : | −1.699 | −1.073 | −0.375 | 0.443 | 1.429 | 2.631 |

| $x$ | $f(x)$ | $\nabla f$ | $\nabla^2 f$ | $\nabla^3 f$ | $\nabla^4 f$ |
|---|---|---|---|---|---|
| $x_0 = 0.1$ | $f(x_0)$ −1.699 | | | | |
| | | $\nabla f_1 = -1.073+1.699 = 0.626$ | | | |
| $x_1 = 0.3$ | $f(x_1)$ −1.073 | | $\nabla^2 f_1 = 0.072$ | | |
| | | $\nabla f_2 = 0.698$ | | $\nabla^3 f_1 = 0.048$ | |
| $x_2 = 0.5$ | $f(x_2)$ −0.375 | | $\nabla^2 f_2 = 0.0120$ | | $\nabla^4 f = 0$ |
| | | $\nabla f_3 = 0.818$ | | $\nabla^2 f_2 = 0.048$ | |
| $x_3 = 0.7$ | $f(x_3)$ +0.443 | | $\nabla^2 f_3 = 0.168$ | | $\nabla^4 f = 0$ |
| | | $\nabla f_4 = 0.986$ | | $\nabla^2 f_3 = 0.048$ | |
| $x_4 = 0.9$ | $f(x_4)$ 1.429 | | $\nabla^2 f_4 = 0.216$ | | |
| | | $\nabla f_5 = 1.202$ | | | |
| $x_5 = 1.1$ | $f(x_5)$ 2.631 | | | | |

$\nabla f(x_5)$ $\quad$ $\nabla^2 f(x_5)$ $\quad$ $\nabla^3 f(x_5)$

$\boxed{h = 0.2}$

$$f(x) = f(x_5) + \frac{(x-x_5)}{1!\,h}\nabla f(x_5) + \frac{(x-x_5)(x-x_4)}{2!\,h^2}\nabla^2 f(x_5) + \frac{(x-x_5)(x-x_4)(x-x_3)}{3!\,h^3}\nabla^3 f(x_5) + 0$$

$$= 2.631 + \frac{(x-1.1)}{1\times 0.2}\times 1.202 + \frac{(x-1.1)(x-0.9)}{2!\,(0.2)^2}\times 0.216 + \frac{(x-1.1)(x-0.9)(x-0.7)}{3!\,(0.2)^3}\times 0.048$$

$$f(1) = 2.631 + \frac{(1-1.1)}{0.2} \times 1.202 + \frac{(1-1.1)(1-0.9)}{2 \times (0.2)^2} \times 0.216 + \frac{(1-1.1)(1-0.9)(1-0.7)}{3! \times (0.2)^3} \times 0.048$$

$$= 2.631 - 0.601 + (-0.027) + (-0.003)$$

$$\simeq \underline{\underline{2}}$$

3. <u>Central difference</u> <u>Interpolation</u> ( <u>Sterling's formula</u> )

Central difference : $\delta f(x_r) = f\left(x_r + \frac{h}{2}\right) - f\left(x_r - \frac{h}{2}\right)$

$$= f_{r+\frac{1}{2}} - f_{r-\frac{1}{2}}$$

Mean operator $\mu f(x_r) = \frac{1}{2}\left[ f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}} \right]$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$

<u>Central difference table</u>

| $x$ | $f(x)$ | $\delta f$ | $\delta^2 f$ | $\delta^3 f$ | $\delta^4 f$ |
|---|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | | | | |
| | | $\delta f_{\frac{1}{2}}$ | | | |
| $x_1$ | $f(x_1)$ | | | | |
| | | $\delta f_{\frac{3}{2}}$ | | | |
| $x_2$ | $f(x_2)$ | | | | |
| $\vdots$ | | $\delta f_{\frac{5}{2}}$ | | | |
| $x_{r-2}$ | $f(x_{r-2})$ | | | | |
| | | $\delta_{r-\frac{3}{2}}$ | | | |
| $x_{r-1}$ | $f(x_{r-1})$ | | $\delta^2 f_{r-1}$ | | |
| | | $\boxed{\delta_{r-\frac{1}{2}}}$ | | $\boxed{\delta^3 f_{r-\frac{1}{2}}}$ | |
| $x_r$ | $f(x_r)$ | | $\delta^2 f_r$ | | $\delta^4 f_r$ |
| | | $\boxed{\delta_{r+\frac{1}{2}}}$ | | $\boxed{\delta^3 f_{r+\frac{1}{2}}}$ | |
| $x_{r+1}$ | $f(x_{r+1})$ | | $\delta^2 f_{r+1}$ | | |
| | | $\delta_{r+\frac{3}{2}}$ | | | |
| $x_{r+2}$ | $f(x_{r+2})$ | | | | |
| $\vdots$ | | | | | |
| $x_n$ | $f(x_n)$ | | | | |

$$f(x_r + kh) = \left[1 + k\mu\delta + \frac{k^2}{2!}\delta^2 + \frac{k(k-1)(k+1)}{3!}\mu\delta^3 + \cdots + \frac{k(k^2-1)}{4!}\delta^4 + \cdots\right] f(x_r)$$

$$\downarrow$$

$$E^k f(x_r)$$

$$= f(x_r) + k\mu\delta f(x_r) + \frac{k^2\delta^2 f(x_r)}{2!} + \frac{k(k-1)(k+1)}{3!}\mu\delta^3 f(x_r) + \frac{k(k^2-1)}{4!}\delta^4 f(x_r) + \cdots$$

$$\mu\delta f(x_r) = \left[\frac{E^{-1/2} + E^{1/2}}{2}\right]\delta f(x_r) = \delta\left[\frac{E^{-1/2} + E^{1/2}}{2}\right] f(x_r)$$

$$= \delta\left[\frac{1}{2}\left(E^{-1/2} f(x_r) + E^{1/2} f(x_r)\right)\right]$$

$$= \delta\left[\frac{f_{r-\frac{1}{2}} + f_{r+\frac{1}{2}}}{2}\right]$$

$f_{r-\frac{1}{2}} \to f(x_r - \frac{h}{2})$, $f_{r+\frac{1}{2}} \to f(x_r + \frac{h}{2})$

$$\mu\delta f(x_r) = \frac{1}{2}\left[\delta f_{r-\frac{1}{2}} + \delta f_{r+\frac{1}{2}}\right]$$

$$\mu\delta f(x_2) = \frac{1}{2}\left[\delta f_{\frac{3}{2}} + \delta f_{\frac{5}{2}}\right]$$

$$\mu\delta^3 f_r = \frac{1}{2}\left[\delta^3 f_{r-\frac{1}{2}} + \delta^3 f_{r+\frac{1}{2}}\right]$$

Q:- find the value of $f(x)$ at $x = 1.3$

| | $x$ | $f(x)$ | $\delta f$ | $\delta^2 f$ | $\delta^3 f$ | $\delta^4 f$ |
|---|---|---|---|---|---|---|
| $x_0$ | 0.1 | 1.1052 | | | | |
| | | | $\delta f_{1/2} =$ 0.7169 | | | |
| $x_1$ | 0.6 | 1.8221 | | $\delta^2 f_1 = \delta f$ 0.4652 | | |
| | | | $\boxed{\delta f_{3/2} = 1.1821}$ | | $\boxed{\delta^3 f_{\frac{3}{2}}\ 0.3015}$ | |
| $x_2$ | 1.1 | $f(x_2)$ 3.0042 | | $\delta^2 f_2$ 0.7662 | | $\delta^4 f_2$ 0.1962 |
| | | | $\boxed{\delta f_{5/2} = 1.9488}$ | | $\boxed{\delta^3 f_{5/2}\ 0.4977}$ | |
| $x_3$ | 1.6 | 4.9530 | | $\delta^2 f_3$ 1.2644 | | |
| | | | $\delta f_{7/2} =$ 3.2132 | | | |
| $x_4$ | 2.1 | 8.1662 | | | | |

0.5

$\delta f_2$

$\mu \delta f(x_2)$

$x = 1.3$

$x_r = 1.1$

$x = x_r + kh$ $\qquad h = 0.5$

$= 1.1 + 0.2$ $\qquad kh = 0.2$

$k = \frac{0.2}{h} = \frac{0.2}{0.5} = \underline{\underline{0.4}}$

$x_r = \underline{\underline{1.1}}$

$$f(x) = f(x_r + kh) = f(x_r) + k\mu\delta f_r + \frac{k^2\delta^2 f_r}{2!} + \frac{k(k-1)(k+1)}{3!}\mu\delta^3 f \quad k(k^2-1)$$

$$+ \frac{k(k^2-1)}{4!}\delta^4 f(x_r) + \cdots \Big]$$

$$f(1.3) = f(x_2 + 0.4 \times 0.5) = 3.0042 + 0.4 \times \frac{1}{2}\left[1.1821 + 1.9488\right] + \frac{(0.4)^2}{2!}(0.7662)$$

$$+ \frac{(0.4)(0.4-1)(0.4+1)}{3!}\left[\frac{0.3015 + 0.4977}{2}\right]$$

$$+ \frac{(0.4)((0.4)^2-1)}{4!} \times 0.1962$$

$$f(1.3) = \underline{\underline{3.6682}}$$

Central difference $\delta f(x_2) = \left[ f\left(x_2 + \frac{h}{2}\right) - f\left(x_2 - \frac{h}{2}\right) \right]$

$f(x_1) - f(x_0)$

$$\delta f(x_1) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$$

| | | |
|---|---|---|
| $x_0$ | $f(x_1)$ | |
| | | $f(x_2)-f(x_1)$ |
| $x_1$ | $f(x_2)$ | |
| | | $f(x_3)-f(x_2)$ |
| 0.3 $x_2$ | $f(x_3)$ | |
| 0.4 — | | |
| 0.5 $x_3$ | | |

$$\delta\left(f\left(x_i + \frac{h}{2}\right)\right) = f\left(x_i + \frac{h}{2} + \frac{h}{2}\right) - f\left(x_i + \frac{h}{2} - \frac{h}{2}\right)$$

$$\delta\left(f x_i + \frac{h}{2}\right) = f(x_i + h) - f(x_i)$$

$$\delta f\left(x_0 + \frac{h}{2}\right) = f(x_1) - f(x_0)$$

$$\delta f_{\frac{1}{2}} = f(x_1) - f(x_0)$$

Numerical Integration:

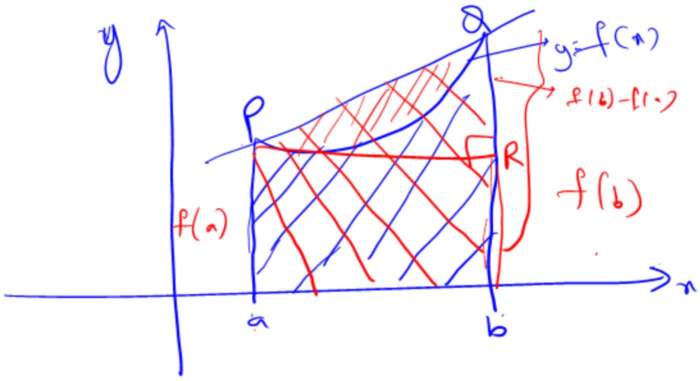
It is required to evaluate the definite integral

$$I = \int_a^b f(x)\,dx .$$ If $f(x)$ is explicitly defined and if it is integrable

$y = f(x)$

Area of quad =

$$\frac{1}{2}(b-a)\left[f(a)+f(b)\right]$$




Area of PQba = Area of rect + area of triangle

$$= (b-a)f(a) + \frac{1}{2}(b-a)\left[f(b)-f(a)\right]$$

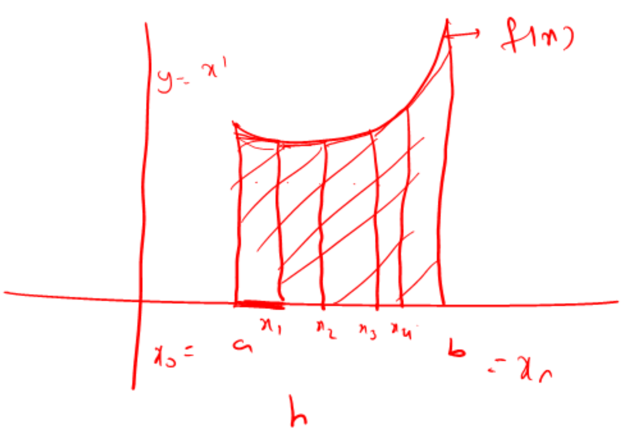
$$\int_a^b f(x)\,dx = \underline{\frac{b-a}{2}\left[f(b)+f(a)\right]} \Longrightarrow \text{Trapezoidal rule}$$

(or) Trapezium rule

## II Composite Trapezoidal rule

$\boxed{\text{order } p}$ $n \le P$

$$I = \int_a^b f(x)\,dx$$




dividing $[a,b]$ into equal subintervals.

$a = x_0$ $\qquad x_2 = x_0 + 2h$

$x_1 = x_0 + h$ $\qquad x_3 = x_0 + 3h$ $\quad \ldots\ldots$ $\quad b = x_n = x_0 + nh$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots\cdots \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \frac{\overset{h}{x_1 - x_0}}{2}\left( f(x_1) + f(x_0) \right) + \frac{\overset{h}{(x_2 - x_1)}}{2}\left( f(x_1) + f(x_2) \right) \cdots\cdots + \frac{x_n - x_{n-1}}{2}\left[ f(x_{n-1}) + f(x_n) \right]$$

$$= \frac{h}{2}\left[ f(x_0) + 2f(x_1) + 2f(x_2) \cdots\cdots 2f(x_{n-1}) + f(x_n) \right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[ f(x_0) + f(x_n) + 2\left( f(x_1) + f(x_2) \cdots f(x_{n-1}) \right) \right]$$

$\downarrow$ composite Trapezoidal

$$= \frac{h}{2}\left[ (\text{sum of first } \& \text{ last}) + 2(\text{remaining terms}) \right]$$

# Newton cotes formula.

This the most popular and widely used integration formula. It forms basis for number of numerical integration methods.

It is required to evaluate $\boxed{\int_a^b f(x)\,dx}$

Given set of $(n+1)$ data points $(x_i, y_i)$ $i = 0, 1, 2 \ldots n$ of the function $y = f(x)$ $\quad x_0, x_1, x_2 \ldots x_n \quad f(x_0)\ f(x_1) \ldots$

Proof:-

Let the interval $[a, b]$ divided into $n$ equal subintervals such that $a = x_0 < x_1 < x_2 \cdots x_n = b$. Then $x_n = x_0 + nh$

$x_1 = x_0 + h, \quad x_2 = x_0 + 2h \quad \ldots \ldots \quad x_n = x_0 + nh.$

By Newtons forward difference formula is

$$y(x) = y(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

when $p = \dfrac{x - x_0}{h}$ $\quad \boxed{p\text{-form}}$ Newton's forward difference

Instead of $f(x)$ we will replace by its interpolating polynomial

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} P_n(x)\,dx.$$ $P_n(x)$ is interpolating polynomial of degree $n$.

$$\int_{x_0}^{x_0+nh} \left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \ldots\right] dx$$

since $x = x_0 + ph \implies \boxed{dx = h\,dp}$

$$\int_{x_0}^{x_n} f(x) = h\int_0^n \Big[ y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\cdots \Big] dp$$

$$= h\left[ y_0(p) + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - \frac{3p^3}{3} + \frac{2p^2}{2}\right)\Delta^3 y_0 + \cdots\cdots \right]_0^n$$

$$\int_{x_0}^{x_n} f(x)\,dx = h\left[ n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\cdots \right]$$

$\uparrow$ $P_n(x)$

$n = 1$

| | | $\Delta f$ | $\Delta^2 f$ |
|---|---|---|---|
| 0 | $f(0)$ | | |
| 1 | $f(1)$ | | |

This is newton's cote's formula.

If $n = 1$.

$$\int_{x_0}^{x_1} f(x)\,dx = h\left[ y_0 + \frac{\Delta y_0}{2} + 0 \cdots\cdots \right]$$

$$\int_{x_0}^{x_1} f(x)\,dx = h\left[\frac{2y_0 + y_1 - y_0}{2}\right] = \frac{h}{2}\left[y_1 + y_0\right] = \frac{x_1 - x_0}{2}\left(f(x_1) + f(x_0)\right)$$

$f(x_1) + f(x_0)$

$\boxed{\mathrm{I}^2}$

$$\int_{x_1}^{x_2} f(x)\,dx = h\left[ y_1 + \frac{\Delta y_1}{2} + \cdots \right] = \frac{h}{2}\left[y_2 + y_1\right]$$

ex:- find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$

using trapezoidal rule with 2, 4, 8 equal subintervals

. find absolute error using exact solution.

$$I = \int_0^1 \frac{dx}{1+x} \longrightarrow f(x) = 1/1+x$$

Trapezoidal rule with 2 subintervals

$[0 \quad 1]$

$N = 2$

$h = \frac{b-a}{N} = \frac{1-0}{2} = \frac{1}{2}$

| $x$ | $x_0 = 0$ | $x_1 = x_0 + h = 0 + 0.5$ <br> $x_1 = 0.5$ | $x_2 = 1$ |
|---|---|---|---|
| $f(x)$ | $f(0) = \frac{1}{1+x}$ <br> $f(0) = \frac{1}{1+0} = 1$ | $f(0.5) = \frac{1}{1+0.5}$ <br> $f(x_1) = 0.667$ | $f(1) = \frac{1}{1+1}$ <br> $f(x_2) = 0.5$ |

$$\int_0^1 f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_2) + 2f(x_1)\right]$$

$$= \frac{0.5}{2}\left[1 + 0.5 + 2 \times 0.667\right] = \boxed{0.708334}$$

$= 17/24$

$N = 4$ Sub intervals $\quad h = \frac{b-a}{4} = \frac{1-0}{4} = 0.25$

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|
| $x$ | 0 | 0.25 | 0.5 | 0.75 | 1 |
| $f(x)$ | 1 | $\frac{1}{1+0.25} = 0.8$ | $\frac{1}{1+0.5}$ <br> $= 0.667$ | $\frac{1}{1+0.75}$ <br> $= 0.57143$ | 0.5 |

$$\int_0^1 f(x)\,dx = \frac{h}{2}\left[ f(x_0) + f(x_4) + 2\left( f(x_1) + f(x_2) + f(x_3)\right)\right]$$

$$= \frac{0.25}{2}\left[ 1 + 0.5 + 2\left( 0.8 + 0.667 + 0.143\right)\right]$$

$$I_{N=4} = \underline{\underline{0.697024}}$$

III $N = 8$ subintervals $h = \frac{b-a}{N} = \frac{1-0}{8} = 1/8 = 0.125$

| $x$ | 0 $x_0$ | 0.125 $x_1$ | 0.25 $x_2$ | 0.375 $x_3$ | 0.5 $x_4$ | 0.625 $x_5$ | 0.75 $x_6$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | $f(x) = \frac{1}{1+x}$ = 1 | 0.89 | 0.8 | 0.73 | 0.667 | 0.6154 | 0.57143 |

| $x$ | 0.875 $x_7$ | 1 $x_8$ |
|---|---|---|
| $f(x)$ | 0.533 | 0.5 |

$h = 0.125$

$$I_{N=8} = \frac{h}{2}\left[ f(x_0) + f(x_8) + 2\left( f(x_1) + f(x_2) + f(x_3) + \cdots f(x_7)\right)\right]$$

$$= \frac{0.125}{2}\left[ 1 + 0.5 + 2\left( 0.89 + 0.8 + \cdots\cdots 0.533 \right)\right.$$

$$= 0.\underline{\underline{694122}}$$

$I_{N=2} = \underline{\underline{0.708334}}$

$I_{N=4} = 0.697024$

$I_{N=8} = 0.694122$

exact solution

$$\int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1$$

$$= \ln 2$$

exact solution $= \underline{\underline{0.693147}}$

error

$$|I_e - \underline{\underline{I_{N=2}}}| = |0.693147 - 0.708334| = 0.015187$$

$$|I_e - \underline{\underline{I_{N=4}}}| = |0.693147 - 0.697024| = 0.003877$$

$$|I_e - \underline{\underline{I_{N=8}}}| = |0.693147 - 0.694122| = 0.000975$$

error is decreasing

No. of subintervals as N increases

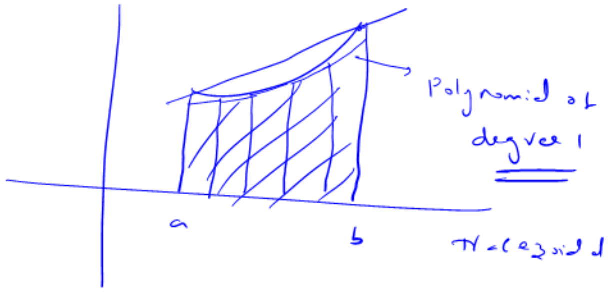
$h$ = decreases

HW

ex:- Evaluate $I = \int_1^2 \frac{dx}{5+2x}$ with 4 & 8 subintervals using trapezoidal rule. Find absolute error.

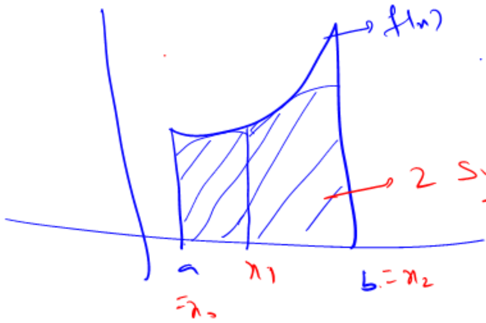
# Simpson's 1/3rd Rule

$$I = \int_a^b f(x)\,dx$$




$n = 2$

approximating using polynomial of degree 2 (parabolas)




2 subintervals equal

By Newton cotes formula → Newton forward interpolating polynomial

$$\int_{x_0}^{x_n} f(x) = h\int_0^n \left[ y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dp$$

$$= h\left[ y_0(p) + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - \frac{3p^3}{3} + \frac{2p^2}{2}\right)\Delta^3 y_0 + \cdots \right]_0^n$$

$$= h\left[ n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots \right]$$

consider

$n = 2$  $P_2(x)$ = polynomial of degree 2

$y = f(x)$

| | $\Delta y_0$ | $\Delta^2 y_0$ |
|---|---|---|
| 0 | | |
| 1 | | |
| 2 | | |

$$\int_{x_0}^{x_n} f(x)\,dx = h\left[ 2y_0 + \frac{2^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{2^3}{3} - \frac{2^2}{2}\right)\Delta^2 y_0 \right]$$

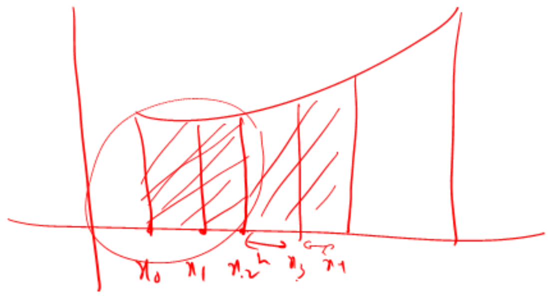
$\Delta y_1 - \Delta y_0$

$$= h\left[ \cancel{2f(x_0)} + 2f(x_1) - \cancel{2f(x_0)} + \frac{1}{2}\left[\frac{8}{3} - \frac{4}{2}\right]\left[f(x_0) - 2f(x_1) + f(x_2)\right] \right]$$

$$= h\left[ 2f(x_1) + \frac{4}{12}\left(f(x_0) - 2f(x_1) + f(x_2)\right) \right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[ f(x_0) + 4f(x_1) + f(x_2) \right]$$ //

Composite Simpson's 1/3rd rule




$$h = \frac{b-a}{2N}$$

$$\int_{x_0}^{x_{2n}} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} + \cdots \int_{x_{2n-2}}^{x_{2n}} f(x)\,dx$$

$$= \frac{h}{3}\Big[\big(f(x_0)+4f(x_1)+f(x_2)\big) + \big(f(x_2)+4f(x_3)+f(x_4)\big) + \big(f(x_4)+4f(x_5)+f(x_6)\big) + \cdots \big(f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})\big)\Big]$$

$$= \frac{h}{3}\Big[f(x_0)+f(x_{2n})+4\big(f(x_1)+f(x_3)+\cdots f(x_{2n-1})\big) + 2\big(f(x_2)+f(x_4)+\cdots f(x_{2n-2})\big)\Big]$$

$$= \frac{h}{3}\Big[\text{Sum of first \& last} + 4(\text{Sum of odd nodes}) + 2(\text{sum of even nodes})\Big]$$

$\rightarrow$ Composite form of $\frac{1}{3}$rd Simpson's rule

Example: Evaluate $\int_0^1 \frac{dx}{1+x}$ using Simpson's $\frac{1}{3}$rd rule with 2, 4, 8 equal subintervals

$\frac{1}{1+x}$

2 Subintervals $2N = 2$ $\frac{1}{3}$rd Simpson's rule.

$a = 0, \quad b = 1$

$$h = \frac{b-a}{2N} = \frac{1}{2} = 0.5$$

| $x$ | $x_0 = 0$ | $x_1 = 0.5$ | $x_2 = 1$ |
|---|---|---|---|
| $f(x) = \frac{1}{1+x}$ | $1.0$ $\frac{1}{1+0}$ | $0.6667$ | $0.5$ |

$\frac{1}{3}$rd Simpson's rule

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{3}\left[f(x_0) + f(x_2) + 4\,f(x_1)\right]$$

$$= \frac{0.5}{3}\left[1 + 0.5 + 4 \times 0.6667\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.69444$$

4 subintervals

$a = 0, \quad b = 1$ $2N = 4$ subintervals

$$h = \frac{b-a}{2N} = \frac{b-a}{4} = \frac{1}{4} = 0.25$$

$h = 0.25$ $\qquad$ 0 to 1

| $x$ | $x_0 = 0$ | $x_1 = 0.25$ | $x_2 = 0.5$ | $x_3 = 0.75$ | $x_4 = 0.5$ |
|---|---|---|---|---|---|
| $f(x) = \frac{1}{1+x}$ | 1 | 0.8 | 0.6667 | 0.57143 | 0.5 |

$$I_4 = \frac{h}{3}\left[f(x_0) + f(x_4) + 4\left(f(x_1) + f(x_3)\right) + 2f(x_2)\right]$$

$$= \frac{0.25}{3}\left[1 + 0.5 + 4(0.8 + 0.57143) + 2(0.6667)\right]$$

$$= \underline{\underline{0.69324}}$$

8 subintervals $\qquad a = 0,\ b = 1 \qquad 2N = 8$ subintervals

$$h = \frac{1-0}{8} = \frac{1}{8} = 0.125$$

| $x$ | $x_0 = 0$ | $x_1$ 0.125 | $x_2$ 0.25 | $x_3$ 0.375 | $x_4$ 0.5 | $x_5$ 0.675 | $x_6$ 0.75 | $x_7$ 0.875 | $x_8$ 1 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | 0.89 | 0.8 | 0.73 | 0.6667 | 0.6154 | 0.87143 | 0.533 | 0.5 |

$$I_8 = \frac{h}{3}\left[f(x_0) + f(x_8) + 4\left(f(x_1) + f(x_3) + f(x_5) + f(x_7)\right) + 2\left(f(x_2) + f(x_4) + f(x_6)\right)\right]$$

$$= \frac{0.125}{3}\left[1 + 0.5 + 4\left[0.89 + 0.73 + 0.6154 + 0.533\right] + 2\left(0.8 + 0.6667 + 0.57143\right)\right]$$

$$I_8 = 0.6\underline{9315}5$$

Exact solution

$$I_E = \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1 = \ln 2 = 0.693147.$$

$I_2 = 0.69444$ , $I_4 = 0.69324$ , $I_8 = 0.693155$

$|I_E - I_2| = |0.693147 - 0.69444| = 0.001297$

$|I_E - I_4| = |0.693147 - 0.69324| = 0.000107$

$|I_E - I_8| = |0.693147 - 0.693155| = 0.000008$

h decreases error in decreasing

Order P :-

$f(x) = \frac{1}{1+x}$ $\le 3$

Numerical integration technique of order P means for that technique the polynomial of degree $\le P$ work fine.

Trapezoidal rule order 1 $\le 1$ $f(x) = 1, x$ $x^2$

Composite trapezoidal rule order $\le 1$

Simpson's 1/3rd rule order 3 $\le 3$

Trapezoidal rule $\boxed{f(x) = x}$ $\frac{1}{1+x}$

order 1 $f(x) = 1, x^1$ $\le 1$

$R(f)$ Global $\int_a^b f(x)\,dx$ − Trapezoidal rule

$= \frac{b^2-a^2}{2} - \frac{b^2-a^2}{2}$

$$\int_a^b f(x)\,dx = \int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$$

$R(f) = 0$

using Trapezoidal rule

$$= \frac{b-a}{2}\left[f(b) + f(a)\right] = \frac{b-a}{2}(b+a) = b^2-a^2/2$$

$\frac{1}{3}$rd Simpson's rule order 3 $f(x) = 1, x, x^2, x^3$ $R[f] = 0$

$f(x) = x^3$

$x^4$

$4r$

$$R = \int_a^b f(x)\,dx - \text{simpson's rule}$$

$$= \int_a^b x^3\,dx - \frac{b-a}{6}\left[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right]$$

error constant $= \dfrac{b^4 - a^4}{4} - \dfrac{b^4 - a^4}{4} = 0$

Error function of numerical integration technique

$$I = \int_a^b f(x)\, w(x)\, dx$$
$w(x) \rightarrow$ weighted function.

$$= \sum_{k=0}^{n} \lambda_k f(x_k)$$

$\lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots$

① Trapezoidal rule

$$= \frac{b-a}{2}\left[ f(x_0) + f(x_1) \right]$$

② Simpson's rule

$$= \frac{h}{3}\left[ f(x_0) + 4f(x_1) + f(x_2) \right]$$

$$\text{Error} = R(f) = \int_a^b w(x)\, f(x)\, dx - \sum_{k=0}^{n} \lambda_k f(x_k)$$
$\rightarrow$ Numerical integration technique with order $p$

degree $\le x^p$

$R(x^{p+1})$ ~~then~~ $f(x) = x^{p+1}$

$$C = \text{error const} = \int_a^b w(x)\, x^{p+1}\, dx - \sum_{n=0}^{n} \lambda_k x_k^{p+1} \rightarrow \text{NI technique}$$

$$\boxed{R(f) = \frac{C}{(p+1)!} f^{p+1}(\xi)}$$

$a < \xi < b$

$0 < \xi < 1$

$p+1$ th derivative of $f(\xi)$

Trapezoidal rule $\quad f(x) = x^2$

$p = 1$ order $= 1$

$f(x) = 1, x$

$$C = \int_a^b x^2\, dx - \frac{b-a}{2}\left[ f(a) + f(b) \right]$$

$$= \frac{b^3 - a^3}{3} - \frac{(b-a)}{2}\left[ a^2 + b^2 \right] = \frac{b^3 - a^3}{3} - \frac{b^3 - a^2 b + ab^2 + a^3}{2}$$

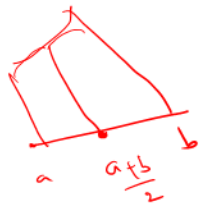
$$C = \frac{(b-a)^3}{6}$$

$$R(f) = \frac{(b-a)^3}{6 \times 2!} f''(\xi) \quad a < \xi < b$$

$$R(f) = \frac{(b-a)^3}{12} f''(\xi) \qquad a < \xi < b$$

$$|R(f)| \leq \frac{(b-a)^3}{12} M_2 \qquad M_2 = \max(f''(x))$$

$$a \leq x \leq b$$




<u>Simpson's rule</u>

<u>order 3 = p</u>

$$f(x) = x^4$$

$$\frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$C = \int_a^b x^4 \, dx - \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]$$

$$\frac{b^5 - a^5}{5} \leftarrow \left.\frac{x^5}{5}\right|_a^b - \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right]$$

$$C = -\frac{(b-a)^5}{120}$$

$$R(f) = \frac{C}{(3+1)!} f^4(\xi) \qquad a < \xi < b$$

$$\frac{C}{(p+1)!} f^{p+1}(\xi)$$

$$= -\frac{(b-a)^5}{120 \times 4!} f^4(\xi)$$

Error for forward difference interpolation

$$f(x) - P_n(x) = E_n(f, x)$$

$$E_n(f,x) = \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)}{(n+1)!} f^{n+1}(\xi)$$

$$0 < \xi < n$$

Error function for Newton backward interpolation

$$E_n(f,x) = f(x) - P_n(x)$$

$$= \frac{(x-x_n)(x-x_{n-1})\ldots\ldots(x-x_0)}{(n+1)!} f^{n+1}(\xi)$$

$$a < \xi < n$$

# Solutions of Nonlinear equations

$3x + 4y = 0$
$3x + 10y = 3$

$f(x) = 0$
↓
Non linear

① Polynomial function

A function of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots a_{n-1} x + a_n$$

$$a_0 \neq 0$$

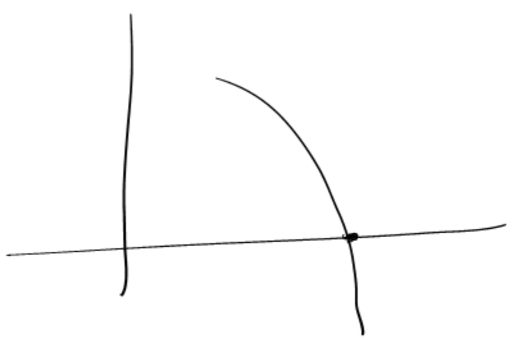
② Algebraic function

A function sum / difference / product of 2 Polynomials is called algebraic function. If it is not algebraic, it
⟹ is Transcedental function (or) non algebraic

$$f(x) = c_1 e^x + c_2 e^{-x}$$

$$f(x) = \cos x - x e^{-x}$$

⟹ Solving equation



$f(x) = 0$

$x = \alpha \qquad f(\alpha) = 0$

$\alpha$ is root of equation
solution of equation
zero of equation

## Types of roots

① **Simple root**

$\alpha$ is a simple root of $f(x)=0$ if $f(\alpha)=0$

& $f'(\alpha) \neq 0$

$$f(x) = (x-\alpha)\, g(x) \quad \text{where } g(\alpha) \neq 0$$

$$f(x) = x^3 + x - 2$$

$$= \underbrace{(x-1)}_{1 \text{ is root}} \underbrace{(x^2 + x + 2)}_{g(x)}$$

$g(x) = x^2 + x + 2$

$g(1) \neq 0$

$f(1) = 1 + 1 - 2 = 0$

$f'(x) = 3x^2 + 1 \Rightarrow f'(1) \neq 0$

⟹ **Multiple root** $\quad \alpha, \alpha, \ldots$ m times

$\alpha$ is a multiple root of multiplicity $m$ of $f(x)=0$

if $f(\alpha) = 0,\ f'(\alpha) = 0,\ \ldots\ f^{m-1}(\alpha) = 0$

and $f^{m}(\alpha) \neq 0$

$$f(x) = (x-\alpha)^m g(x) \qquad g(\alpha) \neq 0$$

$f(x) = x^3 - 3x^2 + 4 = 0$

$f(x) = (x+1)(x-2)^2$

Simple root $= -1$

Multiple root $= 2$ with multiplicity 2

## Finding roots

① **Direct Method** :- exact values of all roots

degree = 2, 3, 4 - $ax^2 + bx + c = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

upto degree 4 we have direct Method

② **Iterative method** ( Polynomial eq with degree above 4 and Transcedental equation)

① let $x_0$ be an approximation to $\alpha$

② $x_{k+1} = \phi(x_k)$ , $\phi(x_k, x_{k-1})$

③ Criteria to terminate Iteration procedure

① $|f(x_k)| \leq \varepsilon$ → 0.0005

$f(x) = 0$

$x_k$ is my root

$f(x_k) = 0$

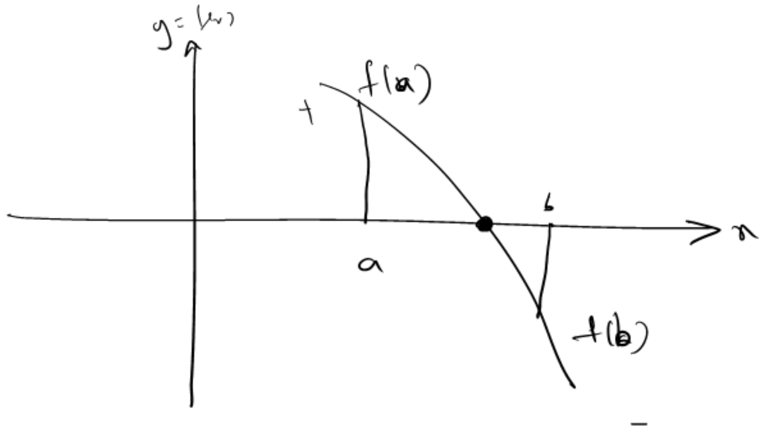
② $|x_{k+1} - x_k| \leq \varepsilon$ → 0.0005

⟹ **Initial guess** (or) **approximation** **for** $\alpha$

→ **Bolzano's theorem**

( Intermediate value theorem)

If the function $f(x)$ is continuous in $[a, b]$ and if $f(a)$ & $f(b)$ are of opposite signs then $\exists$ atleast one real root $f(x) = 0$ between $a$ & $b$




| $x$ | $f(x)$ |
|---|---|
| −3 | + |
| −2 | + |
| −1 | − |
| 0 | + |
| 1 | − |
| 2 | |
| 1 | |

## Method of bisection

It is iterative method

1. Initial approximation
2. $\phi$
3. Criteria for termination

⟹ Bolzano's therm

let $f(x) = 0$ has root in $[a, b]$

$f$ is continuous in $[a, b]$ then $f(a)f(b) < 0$

A real root is between $a$ & $b$

1. from the table find (a) & (b)

$f(a)\,f(b) < 0$


| $x$ | $f(x)$ |
|---|---|
| $a$ | − $f(a)$ |
| $b$ | + $f(b)$ |

2. $x_1 = \dfrac{a+b}{2}$

If $f(x_1) = 0$ $x_1$ is the root of $f(x) = 0$


else $f(x_1)\,f(a) < 0$ (or) $f(x_1)\,f(b) < 0$

✓

$a_1 = a$, $b_1 = x_1$

(2) $a_1$ $b_1$

$x_2 = \frac{a_1 + b_1}{2}$ If $f(x_2) = 0$, $x_2$ is the root

$f(x_2) f(a_1) < 0$ (or) $f(x_2) f(b_1) < 0$

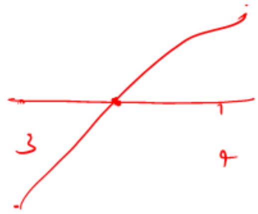
Stoping criteria)

$|x_{k+1} - x_k| < \varepsilon$ $\varepsilon = 0$

**Ex:-** Find by the bisection method a real root of $2x - \log_{10}^{x} = 7$

**Sol:-** $2x - \log_{10}^{x} - 7 = f(x)$

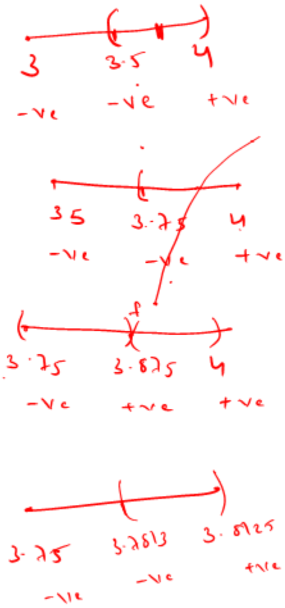
| $x$ | $f(x)$ |
|---|---|
| 1 | $f(1) = 2\times1 - \log_{10}^{1} - 7 = -5$ |
| 2 | $f(2) = -3.3$ |
| 3 | $f(3) = -1.477$ |
| 4 | $f(4) = 0.3977$ |




$a = 3 \qquad b = 4$

$f(x) = 2x - \log_{10}^{x} - 7 = 0$

| $n$ | $a_n$ (-ve) | $b_n$ (+ve) | $x_{n+1} = \frac{a_n + b_n}{2}$ | $f(x_{n+1})$ |
|---|---|---|---|---|
| 0 | 3 | 4 | $\frac{3+4}{2} = 3.5$ | $f(3.5) = -0.5491$ |
| 1 | 3.5 | 4 | $\frac{3.5+4}{2} = 3.75$ | $f(3.75) = -0.0740$ |
| 2. | 3.75 | 4 | $\frac{3.75+4}{2} = 3.875$ | $f(3.875) = 0.1617$ |
| 3. | 3.75 | 3.875 | $\frac{3.75+3.875}{2} = 3.8125$ | $f(3.8125) = 0.0438$ |
| 4 | 3.75 | 3.8125 | 3.7813 | $f(3.7813) = -0.0151$ |
| 5. | 3.7813 | 3.8125 | 3.7969 | 0.0143 |
| 6. | 3.7813 | 3.7969 | 3.7891 | -0.0009 |
| 7 | 3.7891 | 3.7969 | 3.7930 | 0.0070 |
| 8. | 3.7891 | 3.7930 | 3.7910 | 0.0033 |
| 9. | 3.7891 | 3.7910 | 3.790 | $f(3.790)$ |




0.005

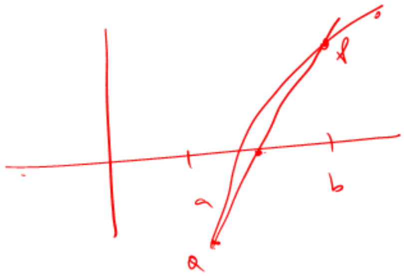
Correcting upto 2 decimal $\alpha = \underline{\underline{3.79}}$

$|x_{k+1} - x_k| < 0.002$

## Chord Methods

We approximate the curve $f(x) = 0$ in a sufficiently small intervals which contains root by a chord (straight line)

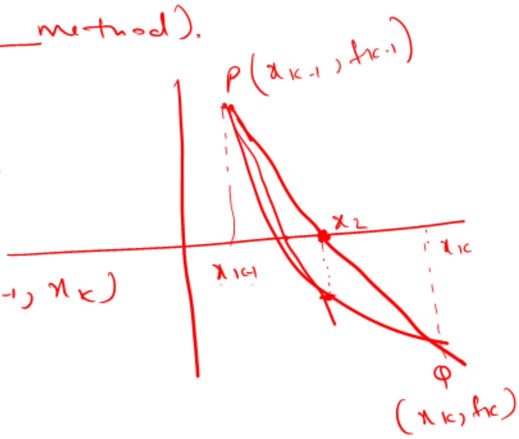
### Two Chord Methods




① Method of false position

② Newton Raphson Method.

## I Method of false Position (Regula falsi method).




Start :- Find the interval in which root lies.

→ let the root of $f(x) = 0$ lies in $(x_{k-1}, x_k)$

i.e $f(x_k)\, f(x_{k-1}) < 0$

→ Then $P(x_{k-1}, f_{k-1})$, $Q(x_k, f_k)$ are two points on the curve.

→ Draw a straight line joining P & Q (approximation of curve in that interval)

Next approximation :- where PQ intersects x axis.

$P = (x_{k-1}, f_{k-1})$ $\quad Q = (x_k, f_k)$

equation of PQ

$$\frac{y - f_k}{f_{k-1} - f_k} = \frac{x - x_k}{x_{k-1} - x_k} \rightarrow$$

PQ meets x axis $y = 0$

$$\frac{0 - f_k}{f_{k-1} - f_k} = \frac{x - x_k}{x_{k-1} - x_k}$$

$$x - x_k = -\frac{f_k (x_{k-1} - x_k)}{f_{k-1} - f_k}$$

$$x = x_k - \frac{f_k (x_{k-1} - x_k)}{f_{k-1} - f_k}$$

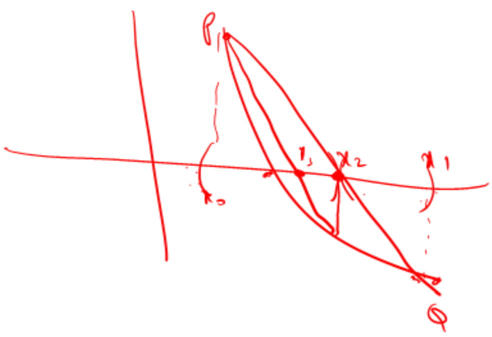
$$x_{k+1} = x_k - \frac{(x_{k-1} - x_k) f_k}{f_{k-1} - f_k}$$

$$\boxed{x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}} \qquad k = 1, 2, \ldots$$

Starting with initial interval $(x_0, x_1)$ $f(x_0) f(x_1) < 1$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

If $f(x_0) f(x_2) < 0$ the root lies between $(x_0, x_2)$ otherwise it lies between $(x_2, x_1)$. Then iteration continued using the interval in which root lies until desired accuracy is achieved.



Ex:- $x^3 - 3x + 1 = 0$. Solve for root correct upto 3 decimal places using Regula falsi method.

$$f(x) = x^3 - 3x + 1$$

| $x$ | $f(x) = x^3 - 3x + 1$ |
|---|---|
| 0 | 1 |
| 1 | $1 - 3 + 1 = -1$ |
| 2 | |

$f(0)\,f(1) < 0$

$f(0)$: +ve, $f(1)$: -ve

root lies between 0 & 1

$x_0 = 0$ $\quad f(x_0) = 1$

$x_1 = 1$ $\quad f(x_1) = -1$


$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{0 - 1}{-1 - 1} = 0.5$$

$f(x) = x^3 - 3x + 1$

$$f(x_2) = f(0.5) = 0.5^3 - 3 \times 0.5 + 1 = -0.375$$

$\Rightarrow (x_0, x_2) = (0, 0.5)$ $\quad f(x_0)\,f(x_2) < 0$

$$x_3 = \frac{x_0 f_2 - x_2 f_0}{f_2 - f_0} = \frac{0 - 0.5}{-0.375 - 1} = \underline{0.36364}$$


$f(x_3) = 0.36364^3 - 3(0.36364) + 1 = \underline{-0.04283}$

$\Rightarrow$

$(x_0, x_3) = (0, 0.36364)$

$$x_4 = \frac{x_0 f_3 - x_3 f_0}{f_3 - f_0} = 0.34870$$

$$f(x_4) = 0.34870^3 - 3(0.34870) + 1 = -0.00370$$

$$\Rightarrow (x_0, x_4) = (0,\ 0.34870)$$

+ve  -ve  +ve

$$x_5 = \frac{x_0 f_4 - x_4 f_0}{f_4 - f_0} = \underline{\underline{0.34741}}$$

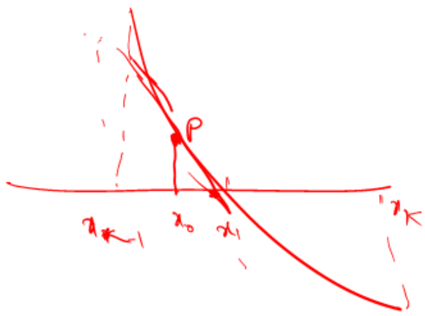
$$|x_6 - x_5| < 0.000$$

$$f(x_5) = -0.00030$$

$$x_6 = \frac{x_0 f_5 - x_5 f_0}{f_5 - f_0} = \underline{\underline{0.347309}}$$

# Newton Raphson Method

(chord method)




Let $x_0$ be the initial approximation of the root of $f(x) = 0$

$P(x_0, f_0)$ is a point on the curve

Draw a tangent to the curve at P.

Approximate the curve in the neighbourhood of root by the tangent to the curve

→ Next approximation : intersection of tangent to x axis.

→ Repeat until required accuracy is obtained.

⇒ Initial point $x_0$

equation of tangent to the curve $y = f(x)$ at point P is given by

$$y - f_0 = (x - x_0) f'(x_0)$$

→ slope of a tangent at $P(x_0, f_0)$

⇒ Set $y = 0$ solve for $x$.

$$0 - f_0 = (x - x_0) f'(x_0)$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

→ $f'(x_0) \neq 0$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad f'(x_k) \neq 0$$

Ex:- Perform 4 iterations of newton raphson method to find a root

$$f(x) = x^3 - 5x + 1$$

| $x$ | $f(x)$ |
|---|---|
| 0 | 1 |
| 1 | $1-5+1 = -3$ |

$f(0)\,f(1) < 0$

→ Pick a $x_0 = 0.5$ $\quad f(x) = x^3 - 5x + 1$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 5x + 1}{3x_k^2 - 5}$$

$$x_{k+1} = \frac{2x_k^3 - 1}{3x_k^2 - 5}$$

$x_0 = 0.5$

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = \frac{2(0.5)^3 - 1}{3(0.5)^2 - 5} = \underline{\underline{0.176471}} \rightarrow x_1$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = \frac{2(0.176471)^3 - 1}{3(0.176471)^2 - 5} = 0.201563$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 5} = \underline{\underline{0.201640}}$$

$$x_4 = \frac{2x_3^3 - 1}{3x_3^2 - 5} = \underline{0.201640}$$

$\alpha = \underline{\underline{0.201640}}$

error of approximation

( Rate of convergence)

$$\varepsilon_k = x_k - \alpha$$

$$|\varepsilon_k| \longrightarrow 0 \quad \text{as } k \to \infty$$

Order $p$ :- Method has rate of convergence of $p$

$$|\varepsilon_{k+1}| \le C |\varepsilon_k|^p$$

$(k+1)^{th}$ $\qquad$ $k^{th}$

Bisection $\quad p = 1$ linear convergence rate $\quad |\varepsilon_{k+1}| \le \frac{1}{2} |\varepsilon_k|^1$ $\quad |x_{k+1} - \alpha| \le \frac{1}{2} |x_k - \alpha|$

Regula falsi: $\quad p = 1$ linear convergence rate

Newton Raphson : $\quad p = 2 \qquad |\varepsilon_{k+1}| \le C |\varepsilon_k|^2 \qquad \varepsilon_k = 0.1$

Quadratic Convergence $\qquad \varepsilon_{k+1} \le \qquad 0.01$