The Art of Concession in General Lotto Games*

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Abstract. Success in adversarial environments often requires investment into additional resources in order to improve one's competitive position. But, can intentionally decreasing one's own competitiveness ever provide strategic benefits in such settings? In this paper, we focus on characterizing the role of "concessions" as a component of strategic decision making. Specifically, we investigate whether a player can gain an advantage by either conceding budgetary resources or conceding valuable prizes to an opponent. While one might naïvely assume that the player cannot, our work demonstrates that – perhaps surprisingly – concessions do offer strategic benefits when made correctly. In the context of General Lotto games, we first show that neither budgetary concessions nor value concessions can be advantageous to either player in a 1-vs.-1 scenario. However, in settings where two players compete against a common adversary, we find opportunities for one of the two players to improve her payoff by conceding a prize to the adversary. We provide necessary and sufficient conditions on the parameters for which such concessions exist, and identify the optimal prize value to concede.

Keywords: Game theory · Resource allocation · General Lotto games

1 Introduction

Strategic advantages are often held by competitors that possess more budgetary resources that can be invested in more advanced technology, research, or surveillance in order to improve one's competitive position against opponents. Such factors are central to many domains that feature competitive interactions, such as airport security [19,24], wildlife protection [27], market economics [17], and political campaigning [23]. In this paper, we analyze "concessions" as a viable, alternative component of strategic decision-making in adversarial environments. In particular, we seek to identify whether or not conceding one's competitive position can ever be advantageous. Intuitiviely, concessions would appear to be contradictory to the conventional wisdom on how to gain a strategic advantage, e.g., investing in more resources or information, as concessions weaken one's competitive position. Nonetheless, this paper demonstrates that such intuition is false as appropriately chosen concessions can often be strategically beneficial.

Within the framework of General Lotto games, we study two types of concessions. The first type, which we term *budgetary concessions*, involves willingly reducing one's resource budget. The act of "money burning" serves as an analogy for this type of concession. The second type of concession, which we term *battlefield concessions*, involves voluntary non-participation on a non-zero

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valued battlefield. An appropriate analogy for this type of concession is market abandonment from economics. In these scenarios, we assume that concessions are announced to all other players, such that the other players can respond strategically to the modified competitive environment.

General Lotto games, Colonel Blotto games, and other contest models offer a flexible framework to generate basic insights about the interplay between a competitor's performance guarantees and the amount of resources reserved for competition [3,6,8,12,13,20]. In common formulations, two opposing players have limited resource budgets to allocate to multiple battlefields. A player wins a battlefield and its associated value if she sends more resources than her opponent. To study the role of concessions as a strategic component, we continue this section with a brief overview of General Lotto games and describe our extensions that allow us to study concessions under this model. We also provide a summary of our contributions, namely, the identification of settings where concessions are beneficial. Finally, we draw connections between our work and the related literature.

1.1 General Lotto games with concessions

The General Lotto game is played between two opposing players, A and B, who each have a limited budget of resources $X_A, X_B \geq 0$. The players compete over a set of n battlefields $\mathcal{B} = \{1, \dots, n\}$, where a player wins a battlefield $b \in \mathcal{B}$ and its value $v_b \geq 0$ by allocating more resources to b than the opponent. The players make moves simultaneously (i.e., a one-shot game). Each player can use randomized allocations such that the resources spent do not exceed its limited budget in expectation. We denote an instance of the General Lotto game with $\text{GL}(X_A, X_B, \mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$ is the vector of battlefield valuations. The equilibrium strategies and payoffs in any instance are characterized in the existing literature [12,14], and we reproduce these in Section 2.

We consider the following extension in order to study the strategic role of concessions in General Lotto games: One of the players, say player B, has the option to either voluntarily reduce her own resource budget, or to voluntarily withdraw completely from a chosen battlefield, before engaging with A in the resulting General Lotto game. Specifically, B selects one of the following options:

- *Budgetary concession:* Player B selects some nonzero value $x \in [0, X_B]$, whereupon her resource budget is reduced from X_B to $X_B - x$.
- *Battlefield concession:* Player B selects a battlefield $b \in \mathcal{B}$. The value of the battlefield, v_b, is immediately awarded to player A.

The complete competitive interaction between A and B occurs in two stages. In Stage 0, B decides to concede either budgetary resources or a battlefield to A, as described above. Player B's decision in this stage then becomes binding and common knowledge. Subsequently, in Stage 1, the players engage in the resulting General Lotto game. If a budgetary concession of $x \in [0, X_B]$ was made in Stage 0, the game $\text{GL}(X_A, X_B - x, \mathbf{v})$ is played and the players receive their respective equilibrium payoffs. If a battlefield concession of $b \in \mathcal{B}$ was made in Stage 0, the value v_b is immediately awarded to player A, and the game $\text{GL}(X_A, X_B, \mathbf{v}_{-b})$ is played. Here, $\mathbf{v}_{-b}$ is the vector of valuations for the battlefields $\mathcal{B} \backslash \{b\}$. We say that a player has a *beneficial concession* if there exists any concession such that the player secures a strictly higher payoff than her payoff in the nominal General Lotto game (i.e. without concessions). For example, if player B has a beneficial budgetary concession in the General Lotto game, then there exist parameters $X_A, X_B > 0$, $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$ and $x \in [0, X_B]$ such that B's equilibrium payoff is greater in $\text{GL}(X_B - x, X_A, \mathbf{v})$ than in $\text{GL}(X_B, X_A, \mathbf{v})$. Our first contribution is as follows:

Contribution 1. *There never exist concessions of either type that improve a player's payoff in the General Lotto game (Proposition 1).*

1.2 Three-player General Lotto games with concessions

Contribution #1 conforms with the conventional intuition that concessions only ever weaken one's position in competitive scenarios. We thus seek to address whether this phenomenon holds more generally. To that end, we shift our focus to a three-player setting, in which players B and C compete in General Lotto games against a common adversary A over two disjoint sets of battlefields $\mathcal{B}_B, \mathcal{B}_C$ whose valuations are given by the vectors $\mathbf{v}_B, \mathbf{v}_C$, respectively. This formulation was first proposed and studied in [15]. The top diagram in Figure 1a depicts a *nominal* three-player Lotto game (under no concession options). We consider the case where only player B has the option to make concessions. The competitive interaction occurs over three stages as follows, where players' actions become binding and common knowledge in subsequent stages:

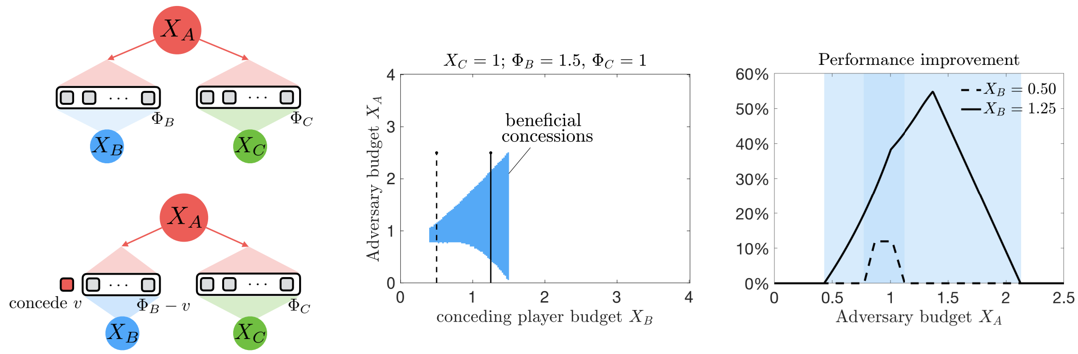
– *Stage 0:* Player B decides to make either a budgetary or battlefield concession;
– *Stage 1:* Player A deploys resources $X_{A,B}, X_{A,C} \geq 0$ to the two competitions against B and C, where $X_{A,B} + X_{A,C} \leq X_A$ must be satisfied; and,
– *Stage 2:* Player A engages in the two resulting General Lotto games. If a budgetary concession of $x \in [0, X_B]$ was made in Stage 0, then she plays the game $\mathrm{GL}(X_{A,B}, X_B - x, \mathbf{v}_B)$ against player B. Else, if a battlefield concession of $b \in \mathcal{B}_B$ was made in Stage 0, then she plays $\mathrm{GL}(X_{A,B}, X_B, \mathbf{v}_{B,-b})$ against B, where $\mathbf{v}_{B,-b}$ denotes the vector of valuations for battlefields $\mathcal{B} \backslash \{b\}$. The game $\mathrm{GL}(X_{A,C}, X_C, \mathbf{v}_C)$ is played against player C.

The bottom diagram in Figure 1a depicts the scenario following a battlefield concession. In Stage 1, we assume player A employs an optimal division of resources such that her cumulative payoff from the two General Lotto games in Stage 2 is maximized. Such optimal divisions are characterized in the literature by [15], and we reproduce these results in the forthcoming Section 2. In this three-player setting, we say that B has a *beneficial concession* if there exist any concessions such that B secures a strictly higher payoff than her payoff in the nominal three-player General Lotto game, i.e. if B were to make no concession in Stage 0. Our second contribution is as follows:

Contribution 2. *In three-player General Lotto games, there never exist budgetary concessions that improve a player's payoff (Theorem 1); however, there* do *exist battlefield concessions that can improve a player's payoff. Theorem 2 provides a full characterization of when such opportunities are available, and of the optimal battlefield concession.*

In the standard, two-player General Lotto game, we observe that beneficial concessions do not exist, and, indeed, our result concerning budgetary concessions in the three-player General Lotto game further supports this naïve intuition. However, our results show that beneficial battlefield concessions do exist in three-player General Lotto games, contradicting the conventional wisdom on what constitute viable mechanisms for gaining strategic advantages. More generally, our results suggest that concessions do, in fact, represent reasonable strategic options in competitive interactions. In the following example, we identify the occurrence of beneficial battlefield concessions for player B, and the magnitude of B's payoff improvement under various parameterizations of the three-player General Lotto game:

Consider a three-player General Lotto game in which players' initial budget endowments satisfy $X_A, X_B \in [0, 4]$ and $X_C = 1$, and where the cumulative battlefield values in fronts $\mathcal{B}_B$ and $\mathcal{B}_C$ are $\Phi_B = 1.5$ and $\Phi_C = 1$, respectively. For every such game, we compare player B's payoff in the nominal game against her payoff after a battlefield concession of each value $v \in [0, \Phi_B]$, and identify the optimal battlefield concession value v^{opt}. Figure 1b illustrates the regime of initial



(a) (b) (c)

Fig. 1: (a) The top diagram depicts the nominal three player General Lotto game, where the adversary (A) must decide how to divide its endowment to two separate fronts of battlefields, with cumulative values of Φ_B and Φ_C, respectively. The optimal division for A and the resulting payoffs are well-known from the literature [15]. The bottom diagram shows a scenario where player B concedes a battlefield of value $v \in [0, \Phi_B]$. The adversary responds by re-calculating her optimal division based on the modified environment. We seek to answer whether B can benefit from concessions. (b) The parameter region (in blue) where player B has an incentive to concede battlefields. Here, we set $X_C = 1$ and the total valuations of the two fronts are $\Phi_B = 1.5$, $\Phi_C = 1$. (c) The percentage improvement over player B's payoff in the nominal three player game (without concessions) associated with the optimal battlefield concession. We plot the improvements when $X_B = 0.5$ (dashed line) and $X_B = 1.25$ (solid line) for all values $X_A \in [0, 2.5]$, as depicted in (b).

player budgets in which there exist battlefield concessions of any value $v \in [0, \Phi_B]$ such that player B's payoff in the resulting game is strictly higher than her payoff in the corresponding nominal game (i.e., the regime where there exists a beneficial battlefield concession for B). Figure 1c shows the percentage improvement over player B's payoff in the nominal game associated with conceding the corresponding optimal battlefield concession value v^{opt}. We plot this percentage improvement for $X_A \in [0, 2.5]$, and $X_B = 0.5$ (dashed line) or $X_B = 1.25$ (solid line).

Intuitively, our results illustrate that battlefield concessions in the three-player General Lotto game – if done properly – can redirect more of player A's budget toward player C's set of battlefields, rather than drawing more of A's budget to B's remaining set of battlefields, as the remaining value on B's set of battlefields, $\Phi_B - v^{\text{opt}}$, becomes less of a priority for A. In a sense, the conceding player "appeases" the common adversary by freely offering up a portion of the cumulative battlefield value, and faces less competition as a result. The presence of the additional player C is critical for there to be benefits derived from such concessions. In contrast, budgetary concessions invite A to further pursue her contest against the weakened player B. A budgetary concession reduces B's strength with no change to the cumulative value of the battlefields. This increases the ratio between value and strength on $\mathcal{B}_B$, and leads to A seeking even more value from that front.

1.3 Related Works

A primary line of research in Colonel Blotto games focuses on characterizing its equilibria. Since Borel's initial study [3], many works have advanced this thread over the last one hundred years [2,8,14,16,20,21,25]. However, solutions to the most general settings remain as open problems. As such, there are several variants of the Colonel Blotto game that have been studied extensively, none more so than the General Lotto game [1,12,14,18]. Notably, the players' equilibrium payoffs in the General Lotto games have been fully characterized [12,14]. Due to its tractability, the General Lotto game is often adopted in studies of more complex adversarial environments, including engineering domains such as network security [7,9,22] and the security of cyber-physical systems [5,11].

Our work in this paper is closest to a recent thread in the literature on similar sequential Colonel Blotto and General Lotto games, where players have the option to publicly announce their strategic intentions ahead of play. The three-player General Lotto game was first introduced in [15], who study their own variant model where in Stage 0, players B and C have the opportunity to form an alliance that takes the form of a unilateral budgetary transfer between the players. It is shown that there are cases in which the two players can make unilateral budgetary transfers that are mutually beneficial. Subsequent work in [10,11] considers similar settings where the two players can decide to add battlefields in addition to transferring resources amongst each other. Counter-intuitively, under this model, both the players achieve better payoff if the transfers are publicly announced to their adversary. The authors of [4] identify a sufficient condition for when publicly pre-committing resources to battlefields offers strategic advantages in the same three player setting. Pre-commitments are a broader class of concessions, where instead of giving away value, the pre-committing player puts a price in terms of budgetary resources on a battlefield. The pre-commitment of resources is also studied in [26], but in a different context that involves favouritism. In that work, a one-shot Colonel Blotto game is studied where resources are pre-allocated non-strategically over the various battlefields.

The formation of alliances such as those studied in [10,11,15] is often not possible, either because mechanisms for coordination between the agencies are not available or because the agencies' budgets are not directly transferable. In contrast, concessions offer a means for a player to improve her competitive position, even when mutual coordination is not possible. Another notable difference between concessions and alliances is that, while alliances can only lead to mutually beneficial outcomes for the players involved, our results suggest that any benefits derived from concessions by one player must come at an expense to the other.

2 Model

In this section, we review useful background on the standard, two-player General Lotto game, then formalize the three-player General Lotto game model.

2.1 Background on General Lotto games

The standard General Lotto game consists of two players A and B with respective, fixed budgets $X_A, X_B > 0$ competing over the set of n battlefields $\mathcal{B} = \{1, \ldots, n\}$ (i.e., front). A player wins on a battlefield b by allocating more budget to b than her opponent, and otherwise loses on b.[3] For each battlefield $b \in \mathcal{B}$, the winning player receives her value $v_b \geq 0$, while the losing player receives zero. Let $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$ denote the vector of battlefield values. An allocation is any vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, where

[3] In the case that the players allocate the same amount of budget to a battlefield, the player with higher overall budget is conventionally made to win. However, the choice of tie-breaking rule has no effect on equilibrium characterizations of General Lotto games [14], and hence, our results.

x_b denotes the amount of budget allocated to battlefield b. An admissible strategy for each player $i \in \{A, B\}$ is an n-variate distribution F_i on $\mathbb{R}^n_{\geq 0}$ that satisfies the following budget constraint:

$$\mathbb{E}_{\mathbf{x}\sim F_i}\left[\sum_{b\in\mathcal{B}} x_b\right] \leq X_i. \tag{1}$$

Intuitively, a player may select any distribution over vectors $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ such that the budget expenditure does not exceed her budget *in expectation*. Each player aims to maximize the expected value won over the battlefields. We observe that the game is a two-player, constant-sum game played in a single stage (Stage 1), and that an instance of the game can be succinctly denoted as $\text{GL}(X_A, X_B, \mathbf{v})$. The General Lotto game is a relaxation of the Colonel Blotto game [3], in which the players' allocations must satisfy the budget constraint with probability 1.

The equilibrium characterization of the General Lotto game is well-understood [12,14], and each instance $\text{GL}(X_A, X_B, \mathbf{v})$ is known to admit unique equilibrium payoffs as follows:

Fact 1. *Let* $\text{GL}(X_A, X_B, \mathbf{v})$ *denote an instance of the General Lotto game, and* $\Phi = \sum_{b\in\mathcal{B}} v_b$. *The equilibrium payoff to player* $i \in \{A, B\}$ *is* $\Phi \cdot L(X_i, X_{-i})$, *where*

$$L(X_i, X_{-i}) = \begin{cases} \frac{X_i}{2X_{-i}} & \text{if } X_i \leq X_{-i} \\ 1 - \frac{X_{-i}}{2X_i} & \text{if } X_i > X_{-i}, \end{cases} \tag{2}$$

and $-i \in \{A, B\} \setminus \{i\}$ *is the opposing player.*

As discussed in Section 1.1, concessions in the two player General Lotto game can be considered by introducing an additional stage (Stage 0) that occurs before the players engage in the General Lotto game (Stage 1). Recall that, in Stage 0, player B makes either a budgetary concession or battlefield concession, which then becomes binding and common knowledge before Stage 1 is played. In the following proposition, we show that neither type of concession can ever increase a player's payoff over her payoff in the nominal General Lotto game:

Proposition 1. *Consider the General Lotto game with* $X_A, X_B \geq 0$ *and* $\Phi \geq 0$. *Neither player can benefit from either a budgetary or battlefield concession.*

Proof. We consider the scenario where player B makes either a budgetary or battlefield concession in Stage 0. Since we make no assumption on the players' relative strengths, considering player B's perspective is without loss of generality.

Firstly, from the equilibrium payoffs identified in Fact 1, if player B makes a budgetary concession, i.e., $X'_B \leq X_B$, then it follows that $\Phi \cdot L(X'_B, X_A) \leq \Phi \cdot L(X_B, X_A)$ since, for fixed y, $L(x, y)$ is monotonically increasing in x. Second, and finally, if player B makes a battlefield concession, i.e., $\Phi' \leq \Phi$, then $\Phi' \cdot L(X_B, X_A) \leq \Phi \cdot L(X_B, X_A)$ since L is nonnegative. □

2.2 Three-player General Lotto games with concessions

We have shown that concessions cannot provide payoff improvements in the two-player General Lotto game. Thus, we consider the three-player game model proposed in [15] for the remainder of this manuscript. This game consists of players A, B and C with respective budgets $X_A, X_B, X_C > 0$. Player A is engaged in simultaneous General Lotto games against the players B and C over the respective, disjoint fronts $\mathcal{B}_B$ and $\mathcal{B}_C$. The game is played in two stages: in Stage 1, player A allocates her budget between the two fronts; and, in Stage 2, the two resulting General Lotto games are played. In Stage 2, players B and C receive the payoffs from their respective General Lotto games, and player A receives the sum of her expected payoffs from both General Lotto games. An instance of the game can be succinctly denoted as $3\mathrm{GL}(X_A, X_B, X_C, \mathbf{v}_B, \mathbf{v}_C)$, where $\mathbf{v}_i$ denotes the vector of battlefield values in front $\mathcal{B}_i$, $i \in \{B, C\}$. As we have already done with the standard General Lotto game, we propose a variation on the three-player General Lotto model that includes a preliminary stage (Stage 0) in which player B makes either a budgetary or battlefield concession. Below, we formalize the three stages of this variant, which we term the *three-player General Lotto game with concessions*, where it is assumed that the players' actions in each stage become binding and common knowledge in subsequent stages:

- *Stage 0:* Player B selects one of the following concession formats:
  - *Budgetary concession:* Player B discards a portion of her budget $x \in (0, X_B]$; or,
  - *Battlefield concession:* Player B commits to allocating zero budget to a battlefield $b \in \mathcal{B}_B$.
- *Stage 1:* Player A allocates $X_{A,B}, X_{A,C} \geq 0$ of her budget to the fronts $\mathcal{B}_B$ and $\mathcal{B}_C$, respectively, such that $X_{A,B} + X_{A,C} \leq X_A$ holds.
- *Stage 2:* Player A engages players B and C in the two resulting General Lotto games. If B made a budgetary concession of $x \in (0, X_B]$ in Stage 0, then A and B play the game $\mathrm{GL}(X_{A,B}, X_B - x, \mathbf{v}_B)$. Else, if B made a battlefield concession of $b \in \mathcal{B}_B$, then A and B play the game $\mathrm{GL}(X_{A,B}, X_B, \mathbf{v}_{B,-b})$, where $\mathbf{v}_{B,-b}$ denotes the vector of valuations for battlefields $\mathcal{B}_B \setminus \{b\}$. Players A and C play the game $\mathrm{GL}(X_{A,C}, X_C, \mathbf{v}_C)$. Player A's payoff is the sum of her expected payoffs in the two General Lotto games, and of v_b only if player B selected to concede the battlefield b in Stage 0. Each player $i \in \{B, C\}$ receives the expected payoff from her corresponding General Lotto game against A.

In order to identify player B's optimal strategy in Stage 0 of the game we must first understand player A's strategic behaviour in Stage 1. The allocation rule that maximizes A's cumulative payoff in Stage 2 was characterized by Kovenock and Roberson [15]. We summarize this result below:

Fact 2. *Consider Stage 1 of the three-player General Lotto game where the players' budgets are normalized (w.l.o.g.) such that $X_A = 1$ and $X_B, X_C > 0$. Let $\Phi_B, \Phi_C > 0$ denote the cumulative value of non-conceded battlefields in the fronts $\mathcal{B}_B$ and $\mathcal{B}_C$, respectively. Define $\mathcal{R}_{1i}$, $\mathcal{R}_{2i}$, $\mathcal{R}_{3i}$ and $\mathcal{R}_4$, $i \in \{B, C\}$ as the following regions:*

$$
\begin{aligned}
\mathcal{R}_{1i}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ s.t.\ \Phi_i/\Phi_{-i} > \max\{(X_i)^2, 1\}/(X_i X_{-i})\} \\
&\quad \cup \{(X_i, X_{-i}) \ s.t.\ X_i < 1 \ and\ \Phi_i/\Phi_{-i} = 1/(X_i X_{-i})\} \\
\mathcal{R}_{2i}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ s.t.\ \Phi_i/\Phi_{-i} > X_i/X_{-i} \ and\ 0 < 1 - \sqrt{\Phi_i X_i X_{-i}/\Phi_{-i}} \leq X_{-i}\} \\
\mathcal{R}_{3i}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ s.t.\ \Phi_i/\Phi_{-i} \geq X_i/X_{-i} \ and\ 1 - \sqrt{\Phi_i X_i X_{-i}/\Phi_{-i}} > X_{-i}\} \\
\mathcal{R}_{4}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ s.t.\ \Phi_i/\Phi_{-i} = X_i/X_{-i} \ and\ X_i + X_{-i} \geq 1\}.
\end{aligned}
$$

Player A's optimal allocation $X_{A,i}$ is determined in closed-form as follows:

- *If $(X_i, X_{-i}) \in \mathcal{R}_{1i}(\Phi_i, \Phi_{-i})$, then $X_{A,i} = 1$.*
- *If $(X_i, X_{-i}) \in \mathcal{R}_{2i}(\Phi_i, \Phi_{-i})$, then $X_{A,i} = \sqrt{\Phi_i X_i X_{-i}/\Phi_{-i}}$.*
- *If $(X_i, X_{-i}) \in \mathcal{R}_{3i}(\Phi_i, \Phi_{-i})$, then $X_{A,i} = \sqrt{\Phi_i X_i}/(\sqrt{\Phi_i X_i} + \sqrt{\Phi_{-i} X_{-i}})$.*

– *If* $(X_i, X_{-i}) \in \mathcal{R}_4(\Phi_i, \Phi_{-i})$*, then any* $X_{A,i} \in [1 - X_{-i}, X_i]$ *is optimal,*

where $X_{A,-i} = 1 - X_{A,i}$ *in all the above cases.*

Observe that the result above can be applied in Stage 1, whether or not player B makes a concession. If player B makes no concessions (i.e., the nominal game), then her payoff in Stage 2 is $\Phi_B \cdot L(X_B, X_{A,B})$, where we use $X_{A,B}$ here to denote player A's optimal allocation to the front $\mathcal{B}_B$ in Stage 1 when there is no concession. Otherwise, if player B makes a budgetary concession of $x \in (0, X_B]$, then her payoff in Stage 2 is $\Phi_B \cdot L(X_B - x, X'_{A,B})$, and if player B makes a battlefield concession of $b \in \mathcal{B}_B$, then her payoff in Stage 2 is $(\Phi_B - v_b) \cdot L(X_B, X''_{A,B})$, where we use $X'_{A,B}$ and $X''_{A,B}$ here to denote player A's optimal allocation in Stage 1 to the General Lotto game against player B in response to the budgetary and battlefield concessions, respectively. Crucially, observe that player B translates the point (X_B, X_C) to the left by making budgetary concessions, and alters the parametric regions identified in Fact 2 by making battlefield concessions.

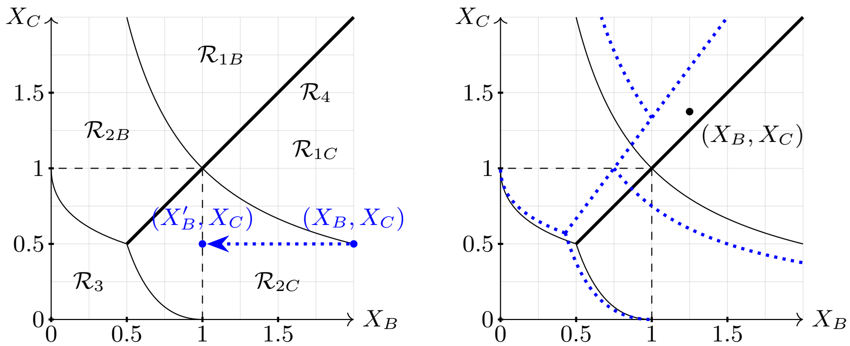
The following observations will be important in the proofs of the forthcoming results:

i. $X_{A,B} > X_B$ holds in regions $\mathcal{R}_{1B}$ (if $X_B < 1$), $\mathcal{R}_{2B}$, $\mathcal{R}_{3B}$ and $\mathcal{R}_{3C}$, while $X_{A,B} \leq X_B$ holds in regions $\mathcal{R}_{1B}$ (if $X_B \geq 1$), $\mathcal{R}_{1C}$, $\mathcal{R}_{2C}$ and $\mathcal{R}_4$.
ii. The closed-form expressions of player A's optimal allocation – and, thus, all of the players' payoffs – are identical in regions $\mathcal{R}_{3B}$ and $\mathcal{R}_{3C}$. Thus, it is equivalent to denote the union of the parametric regions $\mathcal{R}_{3i}(\Phi_i, \Phi_{-i})$, $i \in \{B, C\}$, simply by $\mathcal{R}_3(\Phi_B, \Phi_C)$. Further, note that any point (X_B, X_C) in $\mathcal{R}_3$ must satisfy $X_B + X_C < 1$ since $X_{A,i} > X_i$, $i \in \{B, C\}$.
iii. As shown in Figure 2(a), the point (X_B, X_C) translates to the left in the (X_B, X_C)-plane following a budgetary concession by player B. On the other hand, the regions identified in Fact 2 remain unperturbed.
iv. In contrast, the concession of a battlefield with value $\hat{v}$ does not translate the point (X_B, X_C), but rather modifies the regions identified in Fact 2, as shown in Figure 2(b). In particular, the line $X_C = \Phi_C X_B / \Phi_B$ which serves as the boundary between the regions $\mathcal{R}_{1B} \cup \mathcal{R}_{2B}$ and $\mathcal{R}_{1C} \cup \mathcal{R}_{2C}$ (i.e., the *median line*) rotates counterclockwise about the origin following a battlefield concession by player B. Crucially, since the points on the (X_B, X_C)-plane remain stationary, a point (X_B, X_C) can move from one region to another (i.e., *transit*) following a battlefield concession by either player.

Throughout this paper, we will use $\pi_i^j(X_B, X_C, \Phi_B, \Phi_C)$ to denote the payoff to player $i \in \{B, C\}$ in the region $\mathcal{R}_j$ for all $j \in \{1B, 1C, 2B, 2C, 3, 4\}$. For example, if player B makes a budgetary concession of $x \in (0, X_B]$ in Stage 0, the payoff to each player $i \in \{B, C\}$ in Stage 2 is equal to $\pi_i^j(X_B - x, X_C, \Phi_B, \Phi_C)$, and the payoff to player A in Stage 2 is $(\Phi_B + \Phi_C - \sum_{i \in \{B,C\}} \pi_i^j(X_B - x, X_C, \Phi_B, \Phi_C))$, which all depend on what region $\mathcal{R}_j$ contains the point $(X_B - x, X_C)$ in Stage 1.

3 Main Results

All of the results in this section focus on concessions in the three-player General Lotto game from the perspective of player B. However, by flipping the players' labels, all the results apply identically to concessions from the perspective of player C. Throughout this section, we refer to concessions that strictly improve the player's payoff above her payoff in the nominal setting as beneficial budgetary and battlefield concessions.



a) Budgetary concession b) Battlefield concession

Fig. 2: *The regions dividing the possible player budgets* (X_B, X_C) *in Stage 2, as derived in [15] and reviewed in Fact 2.* (a) Illustration of the Stage 1 regions in the three-player General Lotto game with $\Phi_B = \Phi_C$. The solid, black lines depict the borders between the labelled regions. In blue, we depict the impact of a budgetary concession: the point $(X_B, X_C) = (2, 0.5)$ translates to the left to $(X_B - x, X_C) = (1, 0.5)$ after player B makes a budgetary concession of $x = 1$. (b) We depict the impact of a battlefield concession within the same setting as (a), but where B makes a battlefield concession of b with $v_b = \Phi_C/4$. The solid, black lines depict the borders between the regions for no concession (i.e., $\mathcal{R}_j(\Phi_B, \Phi_C)$), while the dotted, blue lines depict the borders between the regions after the concession of b (i.e., $\mathcal{R}_j(\Phi_B - v_b, \Phi_C)$). Observe that all points on the plot, including $(X_B, X_C) = (1.25, 1.375)$, remain stationary, while the regions change. Notably, (X_B, X_C) is in region $\mathcal{R}_{1B}$ if no concession is made, but is in $\mathcal{R}_{1C}$ after the concession of battlefield b.

Budgetary concessions. We first focus on budgetary concessions, and show that players cannot improve their payoffs by making such concessions.

Theorem 1. *Consider the three-player General Lotto game with* $X_A = 1$, $X_B, X_C \geq 0$ *and* $\Phi_B, \Phi_C \geq 0$. *Player* B *cannot benefit from a budgetary concession.*

Proof. The proof amounts to showing that player B's payoff is nonincreasing for any budgetary concession $x \in (0, X_B]$ such that $(X_B - x, X_C)$ is in any of the regions $\mathcal{R}_j$.

We first consider the scenario where $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$. Recall that, in this scenario, player A commits no budget to the battlefields in the front $\mathcal{B}_B$. Thus, player B's payoff before the concession is Φ_B, the highest possible payoff. Furthermore, $(X_B - x, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$ can only hold if $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$ as well, since the value $1 - \sqrt{\Phi_C(X_B - x)X_C/\Phi_B}$ is increasing in x. If $(X_B - x, X_C) \in \mathcal{R}_4(\Phi_B, \Phi_C)$, then any budgetary concession $x' < x$ would be in either $\mathcal{R}_{1C}$ or $\mathcal{R}_{2C}$, since $X_B + X_C \geq 1$ in $\mathcal{R}_4$, while any budgetary concession $x' > x$ would be in either $\mathcal{R}_{1B}$, $\mathcal{R}_{2B}$ or $\mathcal{R}_3$. Thus, conceding any amount $x' < x$ would guarantee B greater payoff since $X_{A,B} = 1 - X_{A,C}$, and $X_{A,C} = 1$ in $\mathcal{R}_{1C}$ and $X_{A,C} > X_C$ in $\mathcal{R}_{2C}$ whereas $X_{A,C} \in [1 - X_B, \min\{1, X_C\}]$ in $\mathcal{R}_4$. Further, conceding any amount $x' > x$ cannot guarantee B greater payoff since $X_{A,B} = 1$ in $\mathcal{R}_{1B}$, and $X_{A,B} > X_B$ in $\mathcal{R}_{2B}$ and $\mathcal{R}_3$, whereas $X_{A,B} \in [1 - X_C, \min\{1, X_B\}]$ in $\mathcal{R}_4$. In all other regions, we show that player B's payoff is strictly decreasing in x by checking the partial derivative with respect to $x \geq 0$:

If $(X_B - x, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ and $X_B - x > 1$, then

$$\frac{\partial}{\partial x}\Phi_B\left[1 - \frac{1}{2(X_B - x)}\right] = -\frac{\Phi_B}{2(X_B - x)^2} < 0.$$

Else, if $(X_B - x, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ and $X_B - x \leq 1$, then

$$\frac{\partial}{\partial x}\frac{\Phi_B(X_B - x)}{2} = -\frac{\Phi_B}{2} < 0.$$

If $(X_B - x, X_C) \in \mathcal{R}_{2B}(\Phi_B, \Phi_C)$, then

$$\frac{\partial}{\partial x}\frac{\Phi_B(X_B - x)}{2\sqrt{\frac{\Phi_B(X_B - x)X_C}{\Phi_C}}} = -\frac{\Phi_B}{4\sqrt{\frac{\Phi_B(X_B - x)X_C}{\Phi_C}}} < 0.$$

If $(X_B - x, X_C) \in \mathcal{R}_{2C}(\Phi_B, \Phi_C)$, then

$$\frac{\partial}{\partial x}\Phi_B\left[1 - \frac{1 - \sqrt{\frac{\Phi_C(X_B - x)X_C}{\Phi_B}}}{2(X_B - x)}\right] = -\frac{\Phi_B\left[2 - \sqrt{\frac{\Phi_C(X_B - x)X_C}{\Phi_B}}\right]}{4(X_B - x)^2} < 0,$$

which is strictly negative as the condition $1 - \sqrt{\Phi_C(X_B - x)X_C/\Phi_B} \geq 0$ must hold in $\mathcal{R}_{2C}$. Finally, if $(X_B - x, X_C) \in \mathcal{R}_3(\Phi_B, \Phi_C)$, then

$$\frac{\partial}{\partial x}\frac{\Phi_B(X_B - x)}{2\frac{\sqrt{\Phi_B(X_B - x)}}{\sqrt{\Phi_B(X_B - x)} + \sqrt{\Phi_C X_C}}} = -\frac{\Phi_B}{2} - \frac{\Phi_B\sqrt{\Phi_C X_C}}{4\sqrt{\Phi_B(X_B - x)}} < 0.$$

This concludes the proof. □

The proof of the above theorem is fairly technical, so we provide an intuitive interpretation for the reader's convenience. Suppose player B makes a budgetary concession of $x \in (0, X_B]$. Observe that the budgetary concession leaves player B more vulnerable to attacks from player A, since her budget is lowered, but the cumulative value of the battlefields in front $\mathcal{B}_B$ remains unchanged. As a result, the adversary will seek either the same or greater payoff from the front $\mathcal{B}_B$. In the best case, the amount of payoff that adversary extracts from the front $\mathcal{B}_B$ will stay the same, as is the case if either the pairs (X_B, X_C) and $(X_B - x, X_C)$ are both in $\mathcal{R}_{1C}(\Phi_B, \Phi_C)$, i.e., player A still sends no budget to $\mathcal{B}_B$, or if $(X_B, X_C) \in \mathcal{R}_4(\Phi_B, \Phi_C)$ initially with $X_B \geq 1$ and $X_{A,B} = 1$, such that $(X_B - x, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ after the concession. In all other settings, player B's payoff will strictly decrease after a budgetary concession.

Battlefield concessions. Next, we focus on battlefield concessions. Here, we are concerned with identifying the instances in which a battlefield concession is beneficial for player B, i.e., B's resulting payoff is higher than in the nominal game. In particular, we seek conditions on the budgets X_A, X_B, and X_C, and the players' front values Φ_B and Φ_C for which there exists a beneficial battlefield concession. Note here that we are not concerned with the particular vectors of battlefield valuations $\mathbf{v}_B, \mathbf{v}_C$ that constitute each front. As such, we allow player B to have full choice over the conceded value $v \in [0, \Phi_B]$. Our next result identifies necessary and sufficient conditions for the existence of beneficial battlefield concessions in any three-player General Lotto game, as well as the optimal value v^{opt} to concede.

Theorem 2. *Consider the three-player General Lotto game with* $X_A = 1$, $X_B, X_C \geq 0$ *and* $\Phi_B, \Phi_C \geq 0$. *Let* $v_1^* = \Phi_B - \Phi_C X_B / X_C$, $v_2^* = \Phi_B - \Phi_C X_B X_C / (2 - 4X_B)^2$, $v_3^* = \Phi_B - \Phi_C X_B X_C$ *and* $v_4^* = \Phi_B - \Phi_C X_B X_C / (1 - X_B)^2$. *The following conditions characterize the optimal beneficial battlefield concession value* $v^{\text{opt}} \in [0, \Phi_B]$*:*

(i) If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C) \cup \mathcal{R}_{2B}(\Phi_B, \Phi_C)$ *and* $X_C \geq 1$, *then* $v^{\text{opt}} = v_1^*$*;*
(ii) If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C) \cup \mathcal{R}_{2B}(\Phi_B, \Phi_C)$, $X_C < 1$, $X_B + X_C \geq 1$ *and*

$$\frac{\partial}{\partial v} \pi_B^{2C}(\Phi_B - v, \Phi_C, X_B, X_C) \bigg|_{v = v_1^*} \leq 0,$$

then $v^{\text{opt}} = v_1^*$*;*
(iii) If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C) \cup \mathcal{R}_{2B}(\Phi_B, \Phi_C)$, $X_B + X_C \geq 1$ *and*

$$\frac{\partial}{\partial v} \pi_B^{2C}(\Phi_B - v, \Phi_C, X_B, X_C) \bigg|_{v = v_1^*} > 0,$$

then $v^{\text{opt}} = \min\{v_2^*, v_3^*\}$*;*
(iv) If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C) \cup \mathcal{R}_{2B}(\Phi_B, \Phi_C) \cup \mathcal{R}_3(\Phi_B, \Phi_C)$, $X_B + X_C < 1$ *and*

$$\frac{\partial}{\partial v} \pi_B^{2C}(\Phi_B - v, \Phi_C, X_B, X_C) \bigg|_{v = v_4^*} > 0,$$

then $v^{\text{opt}} = \min\{v_2^*, v_3^*\}$*;*
(v) If $(X_B, X_C) \in \mathcal{R}_{2C}(\Phi_B, \Phi_C)$ *and*

$$\frac{\partial}{\partial v} \pi_B^{2C}(\Phi_B - v, \Phi_C, X_B, X_C) \bigg|_{v = 0} > 0,$$

then $v^{\text{opt}} = \min\{v_2^*, v_3^*\}$*;*
(vi) Otherwise, $v^{\text{opt}} = 0$.

Furthermore, there exists a beneficial battlefield concession for player B if and only if conceding a battlefield with value v^{opt} *improves her payoff.*

We present the proof of Theorem 2 in Appendix A, for ease of presentation. In place of the proof, we devote the remainder of this section to developing the reader's intuition about the conditions provided in Theorem 2, and the optimal battlefield concessions in the various settings.

First, we explain the significance of the values v_1^*, v_2^*, v_3^* and v_4^* defined in the claim of Theorem 2. The value v_1^* is precisely the battlefield value that satisfies $(\Phi_B - v)/X_B = \Phi_C / X_C$. Thus, when $X_B + X_C \geq 1$, player B can concede $v_1^* + \epsilon$, $\epsilon \to 0^+$, in order to "hop" from the regions $\mathcal{R}_{1B}$ and $\mathcal{R}_{2B}$ to the regions $\mathcal{R}_{1C}$ (when $X_C \geq 1$) or $\mathcal{R}_{2C}$ (when $X_C < 1$). The value v_2^* satisfies $\partial/(\partial v)[\pi_B^{2C}(X_B, X_C, \Phi_B - v, \Phi_C)] = 0$, which maximizes player B's payoff in $\mathcal{R}_{2C}$ as the second partial derivative of $\pi_B^{2C}(X_B, X_C, \Phi_B - v, \Phi_C)$ with respect to v is strictly negative, i.e., $\pi_B^{2C}(X_B, X_C, \Phi_B - v, \Phi_C)$ is concave down in v. When $X_C < 1$, the value v_3^* satisfies $1 - \sqrt{\Phi_C X_B X_C / (\Phi_B - v)} = 0$. After conceding v_3^*, observe that the point (X_B, X_C) is placed at the boundary between the regions $\mathcal{R}_{1C}$ and $\mathcal{R}_{2C}$. Note that if $v_2^* < v_3^*$, then player B can concede v_2^* without leaving $\mathcal{R}_{2C}$, and thus attain the global maximum of her payoff in $\mathcal{R}_{2C}$. Otherwise, if

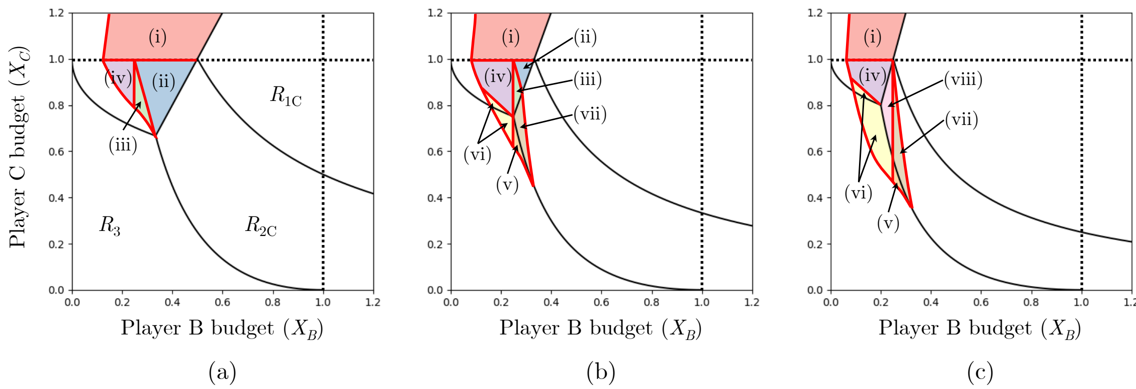


Fig. 3: *Existence of and optimal beneficial battlefield concessions for player B.* We identify the optimal battlefield concession for player B under normalized player budgets (i.e., $X_A = 1$), for $X_B, X_C \in [0, 1.2]$ and (a) $\Phi_B = 1, \Phi_C = 2$, (b) $\Phi_B = 1, \Phi_C = 3$, and (c) $\Phi_B = 1, \Phi_C = 4$. The white area indicates where player B has no beneficial battlefield concession, and thus the optimal battlefield concession is $\hat{b} = \emptyset$. The solid, black lines divide the (X_B, X_C)-plane into the various regions, where $\mathcal{R}_{1C}$, $\mathcal{R}_{2C}$ and $\mathcal{R}_3$ are as labelled in plot (a), $\mathcal{R}_{2B}$ is at the top left of each plot (not labelled), and $\mathcal{R}_{1B}$ does not appear. The optimal battlefield concession in each of the coloured areas coincides with Cases (i)–(viii), as described in the text.

$v_2^* > v_3^*$, then conceding v_2^* would take the point (X_B, X_C) into the interior of $\mathcal{R}_{1C}$, and she can obtain strictly better payoff by conceding v_3^* instead, since $\partial/(\partial v)[\pi_B^{1C}(X_B, X_C, \Phi_B - v, \Phi_C)] = -1 < 0$. Finally, when $X_B + X_C < 1$, the value v_4^* satisfies $1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)} = X_B$. Thus, after conceding v_4^*, the point (X_B, X_C) is placed at the boundary between regions $\mathcal{R}_{2C}$ and $\mathcal{R}_3$.

Next, we consider simulation results identifying the parameter regime in which beneficial battlefield concessions exist as well as the optimal battlefield concessions. In Figure 3, we plot player B's optimal battlefield concession, where the players' budgets are normalized such that $X_A = 1$, and $X_B, X_C \in [0, 1.2]$. In each of the panels, the cumulative values of battlefields in the two fronts are as follows: Figure 3a has $\Phi_B = 1, \Phi_C = 2$, 3b has $\Phi_B = 1, \Phi_C = 3$, and 3c has $\Phi_B = 1, \Phi_C = 4$. Player B has no beneficial concession in the white area. The various regions $\mathcal{R}_j$, defined in Fact 2, are divided by solid, black lines. Each of the coloured areas coincides with a unique optimal battlefield concession, according to its label. Cases (i)–(viii) in Figure 3 are described as follows:

(i) Player B's optimal concession is v_1^*. Since $X_C \geq 1$, the value v_1^* is just enough such that $(X_B, X_C) \in \mathcal{R}_{1C}(X_B, X_C, \Phi_B - v_1^*, \Phi_C)$, thereby forcing player A to allocate all of her budget to $\mathcal{B}_C$.

(ii) Player B's optimal concession is v_1^*. Since $X_C < 1$ and $X_B + X_C \geq 1$, the value v_1^* is just enough such that $(X_B, X_C) \in \mathcal{R}_{2C}(X_B, X_C, \Phi_B - v_1^*, \Phi_C)$, where player A allocates more than X_C to $\mathcal{B}_C$, and less than X_B to $\mathcal{B}_B \setminus \{\hat{b}\}$.

(iii) Player B's optimal concession is v_2^*. Here, $X_B + X_C \geq 1$, the partial derivative of $\pi_B^{2C}(X_B, X_C, \Phi_B - v, \Phi_C)$ with respect to v is positive at $v = v_1^*$, and $v_2^* < v_3^*$. Thus, the optimal battlefield concession places (X_B, X_C) in the interior of $\mathcal{R}_{2C}$.

(v) Player B's optimal concession is v_2^*. Here, $X_B + X_C < 1$, the partial derivative of $\pi_B^{2C}(X_B, X_C, \Phi_B - v, \Phi_C)$ with respect to v is positive at $v = v_4^*$, and $v_2^* < v_3^*$. Thus, the optimal battlefield concession places (X_B, X_C) in the interior of $\mathcal{R}_{2C}$.

(vii) Player B's optimal concession is v_2^*. Here, $(X_B, X_C) \in \mathcal{R}_{2C}(X_B, X_C, \Phi_B, \Phi_C)$, the partial derivative of $\pi_B^{2C}(X_B, X_C, \Phi_B - v, \Phi_C)$ with respect to v is positive at $v = 0$, and $v_2^* < v_3^*$. Thus, the optimal battlefield concession places (X_B, X_C) in the interior of $\mathcal{R}_{2C}$.

Cases (iv), (vi) and (viii) are identical to Cases (iii), (v) and (vii), respectively, except that $v_2^* \geq v_3^*$. Thus, player B's optimal concession is v_3^*. This places (X_B, X_C) at the boundary between $\mathcal{R}_{1C}$ and $\mathcal{R}_{2C}$, and forces player A to allocate all of her budget to $\mathcal{B}_C$.

As seen in Figure 3 and the descriptions of Cases (i)–(viii) above, player B's set of beneficial battlefield concessions predominantly appear in regions where the ratio between the cumulative value of the battlefields in front $\mathcal{B}_B$ and B's initial budget endowment, Φ_B/X_B, is greater than the ratio Φ_C/X_C, and the players B and C together possess enough budget to force A to prioritize one of her General Lotto games over the other (i.e., $\Phi_B/X_B > \Phi_C/X_C$ and $X_B + X_C \geq 1$). In such scenarios, player A primarily pursues her General Lotto game against B in the nominal game. By conceding enough battlefield value $v \in [0, \Phi_B]$ such that $(\Phi_B - v)/X_B < \Phi_C/X_C$, player B can force A to prioritize her game against C instead. Observe that, if the difference between Φ_B/X_B and Φ_C/X_C is moderate, then the gains from shifting A's attention will outweigh the forfeited battlefield value. If the difference between Φ_B/X_B and Φ_C/X_C is too high, however, then too much value v must be conceded by B to minimize her conflict with A, and the gains will not outweigh the losses. In the other regions where beneficial battlefield concessions can exist, either player A is already preoccupied with her General Lotto game against C in the nominal game, or player A's budget is higher than $X_B + X_C$. Nonetheless, if Φ_B is significantly lower than Φ_C and the ratios Φ_B/X_B and Φ_C/X_C are sufficiently close (see Figures 3b and 3c), then player B can benefit from further nudging A's interest toward her game against C.

In Section 1.3, we briefly describe the variant of the three-player General Lotto game studied in [15]. Recall that, in their variant, the players B and C have the opportunity to negotiate an alliance which entails a unilateral transfer of budgetary resources in Stage 0 of the game, and that cases are identified in which forming an alliance is mutually beneficial for B and C. The results in [15] suggest that mutually beneficial alliances only occur when the difference between the ratios Φ_B/X_B and Φ_C/X_C is sufficiently large. In contrast, our findings show that beneficial battlefield concessions only exist when the ratios Φ_B/X_B and Φ_C/X_C are close. This comparison suggests that, if there are significant asymmetries in the players' strengths relative to the values of their respective contests, then cooperative mechanisms, such as alliances, provide strategic advantages; meanwhile, if differences in players' relative strengths are small, then unilateral mechanisms such as battlefield concessions prevail.

4 Conclusions and Future Work

In this paper, we considered the viability of "concessions" as a component of strategic decision-making in adversarial environments. We considered two types of concessions: *budgetary concessions*, where a competitor voluntarily reduces one's resource budget, and *battlefield concessions*, where a player voluntarily forfeits a certain prize to her adversary. Intuitively, concessions should not offer strategic advantages as they weaken one's competitive position. However, we demonstrated that they do offer benefits if made correctly. We studied concessions under the framework of General Lotto games, where we showed that neither type of concession offers benefits under the two-player model. However, in settings where two players compete against a common adversary, we showed that one of the two players can often improve her payoff by conceding a battlefield to the adversary.

This work provides several avenues for future study. First, we have shown that conceding battlefields is beneficial when General Lotto games are the underlying model of conflict. However, we suspect this phenomenon is robust to larger classes of models, e.g., Tullock and other contest success functions. Second, considering a richer setting wherein both players can simultaneously make concessions to the adversary opens questions of what strategic outcomes are possible. Finally, though we have studied concessions as a strategic component in two- and three-player settings, broader forms of strategic pre-commitments and more general interaction networks could be considered.

A Proof of Theorem 2

Before presenting the proof, we note that, in the case of battlefield concessions, we can disregard the scenario when $(X_B, X_C) \in \mathcal{R}_4(\Phi_B - v, \Phi_C)$, for any $v \in [0, \Phi_B]$. To see why, consider a battlefield concession of value v such that (X_B, X_C) is in $\mathcal{R}_4$, i.e., $X_B + X_C \geq 1$ and $(\Phi_B - v)/X_B = \Phi_C/X_C$. Observe that by conceding a battlefield of value slightly greater than v (i.e., $v' = v + \epsilon$ for $\epsilon \to 0^+$), player B obtains strictly higher payoff as (X_B, X_C) now falls in region $\mathcal{R}_{1C}$ (if $X_C \geq 1$) or region $\mathcal{R}_{2C}$ (if $X_C < 1$). Thus, in the following proof, we assume that any point (X_B, X_C) with $X_B + X_C \geq 1$ transits directly from $\mathcal{R}_{1B}$ (if $X_B \geq 1$) or $\mathcal{R}_{2B}$ (if $X_B < 1$), to $\mathcal{R}_{1C}$ (if $X_C \geq 1$) or $\mathcal{R}_{2C}$ (if $X_C < 1$), without first passing through $\mathcal{R}_4$.

Proof. The proof is presented in two parts. In Part 1, we consider the scenario when the point (X_B, X_C) falls in each of the regions $\mathcal{R}_j$ (defined in Fact 2 for the nominal game) except $\mathcal{R}_4$, and demonstrate if and when a battlefield concession can provide an improvement to player B's payoff. This allows us to rule out battlefield concessions that transit within and between particular regions $\mathcal{R}_j$. Then, in Part 2, we identify the optimal battlefield concession value v^{opt}, and thus the maximum payoff that player B can achieve under a battlefield concession. Thus, the necessary and sufficient conditions in the claim amount to verifying that player B's maximum achievable payoff under a battlefield concession is strictly greater than player B's payoff under no concession.

Part 1. Recall that, following a battlefield concession, the point (X_B, X_C) can "transit" between the regions $\mathcal{R}_j$ as the ratio $(\Phi_B - v)/X_B$ is decreasing in v.

We begin this part of the proof by making some important observations about the first and second partial derivatives in the various parametric regions. The partial derivative of player B's payoff with respect to $v_b > 0$ is always negative when the point (X_B, X_C) in the regions $\mathcal{R}_{1B}$, $\mathcal{R}_{1C}$, $\mathcal{R}_{2B}$ and $\mathcal{R}_3$:

$$
\begin{aligned}
\frac{\partial}{\partial v}\pi_B^{1B}(\Phi_B - v, \Phi_C, X_B, X_C) &= \begin{cases} -\frac{X_B}{2} & \text{if } X_B \leq 1 \\ -\frac{1}{2X_B} & \text{if } X_B > 1 \end{cases} \\
\frac{\partial}{\partial v}\pi_B^{1C}(\Phi_B - v, \Phi_C, X_B, X_C) &= -1 \\
\frac{\partial}{\partial v}\pi_B^{2B}(\Phi_B - v, \Phi_C, X_B, X_C) &= -\frac{X_B}{4\sqrt{\frac{(\Phi_B - v)X_B X_C}{\Phi_C}}} \\
\frac{\partial}{\partial v}\pi_B^{3}(\Phi_B - v, \Phi_C, X_B, X_C) &= -\frac{X_B}{2}\left[1 + \frac{1}{2}\frac{\sqrt{(\Phi_B - v)X_B}}{\sqrt{\Phi_C X_C}}\right]
\end{aligned}
$$

Furthermore, the second partial derivative of player B's payoff with respect to $v_b > 0$ is always negative in region $\mathcal{R}_{2C}$:

$$
\frac{\partial^2}{\partial v^2}\pi_B^{2C}(\Phi_B - v, \Phi_C, X_B, X_C) = -\frac{\sqrt{\frac{\Phi_C X_B X_C}{\Phi_B - v}}}{8(\Phi_B - v)X_B} < 0.
$$

Now we are ready to examine the possible transitions between the regions following a battlefield concession, and the corresponding impact on player B's payoff.

Consider the scenario when $(X_B, X_C) \in \mathcal{R}_3(\Phi_B, \Phi_C)$. Observe that from region $\mathcal{R}_3$, the only possible transitions are to regions $\mathcal{R}_{1C}$ and $\mathcal{R}_{2C}$ since the values $1 - \sqrt{(\Phi_B - v)X_B X_C/\Phi_C}$ and $1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)}$ are increasing and decreasing in v, respectively. Furthermore, the players' payoffs in the transition from region $\mathcal{R}_3$ to region $\mathcal{R}_{2C}$ are continuous but not necessarily smooth, i.e., the partial derivatives with respect to v need not be continuous. Since the first partial derivative of player B's payoff with respect to v is negative in region $\mathcal{R}_3$, it follows that all beneficial budget concessions in this setting must transit the point (X_B, X_C) to either $\mathcal{R}_{1C}$ or $\mathcal{R}_{2C}$.

Next, consider the scenario when either $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ or $(X_B, X_C) \in \mathcal{R}_B(\Phi_B, \Phi_C)$. In this setting, note that if $X_B > 1$ then (X_B, X_C) must be in $\mathcal{R}_{1B}$ and will cross the median line $X_C = \Phi_C X_B/(\Phi_B - v)$ without transitting through $\mathcal{R}_{2B}$, and that if $X_C > 1$ then (X_B, X_C) transits to $\mathcal{R}_{1C}$ directly when it crosses the median line, without passing through $\mathcal{R}_{2C}$. Further, note that in the scenarios where either $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ or $(X_B, X_C) \in \mathcal{R}_{2B}(\Phi_B, \Phi_C)$ and the point (X_B, X_C) transits through $\mathcal{R}_3$ as v is increased (i.e., when $X_B + X_C \leq 1$), the partial derivative of player B's payoff with respect to v must remain negative at least until it reaches the boundary between $\mathcal{R}_{2C}$ and $\mathcal{R}_3$. Thus, there can be no beneficial budget concession such that (X_B, X_C) transits from either $\mathcal{R}_{1B}$ or $\mathcal{R}_{2B}$ to $\mathcal{R}_3$. Likewise, there is no beneficial budget concession such that (X_B, X_C) transits from $\mathcal{R}_{1B}$ to $\mathcal{R}_{2B}$. Thus, for all beneficial battlefield concessions from either $\mathcal{R}_{1B}$ or $\mathcal{R}_{2B}$, if $X_C > 1$, then (X_B, X_C) must transit to $\mathcal{R}_{1C}$, and, if $X_C < 1$, then (X_B, X_C) must transit to either $\mathcal{R}_{1C}$ or $\mathcal{R}_{2C}$.

Finally, consider the scenarios when either $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$ or $(X_B, X_C) \in \mathcal{R}_{2C}(\Phi_B, \Phi_C)$. Since $(\Phi_B - v)/X_B$ is decreasing in v, note that if $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$, then (X_B, X_C) remains in $\mathcal{R}_{1C}(\Phi_B - v, \Phi_C)$ as v is increased since $1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)}$ is decreasing in v. Similarly, $(X_B, X_C) \in \mathcal{R}_{2C}(\Phi_B, \Phi_C)$ will remain in $\mathcal{R}_{2C}$ until it transits to $\mathcal{R}_{1C}$ as v is increased. We showed above that the partial derivative of player B's payoff with respect to v in $\mathcal{R}_{1C}$ is negative, and so there can be no beneficial battlefield concession if $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$.

Part 2. In the previous part of the proof, we observed that transitions between regions can occur following battlefield concessions, as the median line $X_C = \Phi_C X_B/(\Phi_B - v)$ rotates counter-clockwise about the origin as v is increased, and the values $1 - \sqrt{(\Phi_B - v)X_B X_C/\Phi_C}$ and $1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)}$ are increasing and decreasing in v, respectively. In particular, we demonstrate that v cannot be a beneficial battlefield concession value for player B if either (X_B, X_C) is in $\mathcal{R}_{1B}(\Phi_B - v, \Phi_C)$, $\mathcal{R}_{2B}(\Phi_B - v, \Phi_C)$ or $\mathcal{R}_3(\Phi_B - v, \Phi_C)$, or $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$. We will use further insights garnered from the previous part further on in this proof.

Consider the scenario where $X_C \geq 1$ and (X_B, X_C) is in either $\mathcal{R}_{1B}(\Phi_B, \Phi_C)$ or $\mathcal{R}_{2B}(\Phi_B, \Phi_C)$. Observe that in this setting, the battlefield concession v_1^* satisfies $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B - v_1^*, \Phi_C)$. For $v < v_1^*$, (X_B, X_C) remains in either $\mathcal{R}_{1B}(\Phi_B - v, \Phi_C)$ or $\mathcal{R}_{2B}(\Phi_B - v, \Phi_C)$ and, thus, player B cannot derive any benefit by the previous lemma. Further, player B receives strictly lower payoff for the concession of $v > v_1^*$ than the concession of v_1^* because the partial derivative of her payoff with respect to v is strictly negative in $\mathcal{R}_{1C}$. Since $X_C > 1$, it must be that $X_B + X_C > 1$ and so (X_B, X_C) cannot transit to $\mathcal{R}_3$ for any v. It follows that the optimal battlefield concession value is $v^{\text{opt}} = v_1^*$ in this setting, as in Condition (i) and (iv) of the claim.

Next, consider the scenario where (X_B, X_C) is in either $\mathcal{R}_{1B}(\Phi_B, \Phi_C)$ or $\mathcal{R}_{2B}(\Phi_B, \Phi_C)$, $X_C < 1$ and $X_B + X_C \geq 1$ (i.e., (X_B, X_C) still cannot transit to $\mathcal{R}_3$ for any v). Here, the battlefield concession v_1^* satisfies $(X_B, X_C) \in \mathcal{R}_{2C}(\Phi_B - v_1^*, \Phi_C)$. Once again, (X_B, X_C) remains in $\mathcal{R}_{1B}$ or $\mathcal{R}_{2B}$ after a concession of any value $v < v_1^*$, which cannot provide payoff improvements to player B. Next, recall from the proof of the previous part that the second partial derivative of player B's payoff with respect to v is strictly negative in $\mathcal{R}_{2C}$ (i.e., player B's payoff is concave down in v). This means that if the first partial derivative of player B's payoff with respect to v is strictly negative or zero at $v = v_1^*$, then the optimal battlefield concession value is $v^{\text{opt}} = v_1^*$ as the derivative will remain nonpositive in $\mathcal{R}_{2C}$ as well as in $\mathcal{R}_{1C}$. Observe that this outcome corresponds with Condition (ii) of the claim. However, if the first partial derivative of player B's payoff with respect to v is strictly positive when $v = v_1^*$, then there are two possibilities: (1) that there is a value v for which player B's payoff is maximized (i.e., partial derivative with respect to v is zero) within $\mathcal{R}_{2B}$, or (2) that the first partial derivative remains positive up until the boundary between $\mathcal{R}_{1C}$ and $\mathcal{R}_{2C}$. The first partial derivative of player B's payoff with respect to v is zero after conceding the value $v \geq 0$ that satisfies $4X_B = 2 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)}$ which is $v = \Phi_B - \Phi_C X_B X_C/(2 - 4X_B)^2$, precisely the definition of v_2^*. Furthermore, (X_B, X_C) is at the boundary between $\mathcal{R}_{1C}$ and $\mathcal{R}_{2C}$ after conceding the value $v \geq 0$ that satisfies $1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)} = 0$ which is $v = \Phi_B - \Phi_C X_B X_C$, precisely the definition of v_3^*. Thus, if the first partial derivative of player B's payoff with respect to v is strictly positive when $v = v_1^*$, then the optimal battlefield concession value is $v^{\text{opt}} = v_2^*$ if $v_2^* < v_3^*$ (Possibility 1), or $v^{\text{opt}} = v_3^*$ if $v_2^* \geq v_3^*$ (Possibility 2). Observe that this is equivalent to writing $v^{\text{opt}} = \min\{v_2^*, v_3^*\}$ as in Condition (iii) of the claim.

Next, consider the scenario where $X_B + X_C < 1$ and (X_B, X_C) is in either $\mathcal{R}_{1B}(\Phi_B, \Phi_C)$, $\mathcal{R}_{2B}(\Phi_B, \Phi_C)$ or $\mathcal{R}_3(\Phi_B, \Phi_C)$. Observe that the point (X_B, X_C) is at the boundary between $\mathcal{R}_{2C}$ and $\mathcal{R}_3$ after conceding the value $v \geq 0$ that satisfies $1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v)} = X_B$ which is $v = \Phi_B - \Phi_C X_B X_C/(1 - X_B)^2$, precisely the definition of v_4^*. Since the point (X_B, X_C) transits through $\mathcal{R}_3$, the partial derivative of player B's payoff with respect to v remains strictly negative for all $v < v_4^*$. Furthermore, since the players' payoffs are continuous as we transit from $\mathcal{R}_3$ to $\mathcal{R}_{2C}$, there can be no benefit to transitting only to the boundary between $\mathcal{R}_3$ and $\mathcal{R}_{2C}$. Thus, if the first partial derivative of player B's payoff with respect to v is nonpositive after conceding $v = v_4^*$, the partial derivative will remain nonpositive in $\mathcal{R}_{2C}$ as well as in $\mathcal{R}_{1C}$. However, if the derivative is strictly positive after conceding $v = v_4^*$, then we consider Possibilities 1 and 2 as in the above paragraph, and obtain an optimal battlefield concession value of $v^{\text{opt}} = \min\{v_2^*, v_3^*\}$ as in Condition (iv) of the claim.

Finally, consider the scenario where (X_B, X_C) is in $\mathcal{R}_{2C}(\Phi_B, \Phi_C)$. Here, if the first partial derivative of player B's payoff with respect to v is strictly negative or zero when $v = 0$, then there is no beneficial battlefield concession as the partial derivative will remain nonpositive in $\mathcal{R}_{2C}$ as well as in $\mathcal{R}_{1C}$. However, if the first partial derivative of player B's payoff with respect to v is strictly positive when $v = 0$, then we consider Possibilities 1 and 2 once again, an optimal battlefield concession value of $v^{\text{opt}} = \min\{v_2^*, v_3^*\}$ as in Condition (v) of the claim. $\square$

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