Methodologies for Quantifying and Optimizing the Price of Anarchy

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***Abstract*— The use of Game Theory for the analysis and design of socio-technological systems has received significant attention as it enables the quantification of the system-level performance while accounting for the presence of individual decision makers. In this context, a popular metric for analyzing the system-level performance is the Price of Anarchy (PoA), which measures the loss in efficiency caused by distributed decision making. Over the past two decades, significant effort has been devoted to developing analytical and computational methodologies for characterizing this metric, with remarkable success. However, existing approaches for obtaining PoA bounds either require the solution of intractable programs, or provide bounds that are not tight. Motivated by this gap, our work develops a computationally efficient framework to tightly characterize the PoA in a broad class of problems. Our framework not only recovers and generalizes many existing PoA results, but it also enables the efficient computation of decision-making rules that optimize the PoA – a central component in the design of socio-technological systems.**

***Index Terms*— game theory, multiagent systems, price of anarchy, congestion games**

I. Introduction

AMONG the emerging engineering marvels of the near-future are large-scale socio-technological systems in which technological components coordinate in real-time while interacting seamlessly with human users. For example, advances in computer vision and robotics have led to the development of autonomous vehicles which will revolutionize how we approach human mobility. Moreover, the widespread proliferation of smart devices has dramatically improved the efficiency and reliability of networked infrastructure, including information, traffic, and power networks.

A major challenge associated with designing and implementing such systems lies in coordinating the decision making of the various technological and human components (agents). This task is particularly challenging as socio-technological systems often rely on *distributed* decision making, which may either be imposed by design considerations (scalability, modularity, security), or result from humans' self-interested behaviour. In this context, it is well-known that, when the agents' decision making is not well-aligned with the system-level goal, the emergent behaviour can result in wide-ranging inefficiencies [2]–[5].

In this manuscript, we seek to mitigate this fundamental issue by carefully re-aligning the agents' decision making with the system-level goal. To that end, we consider a game-theoretic setting where the agents are modelled as strategic players, each equipped with a local utility or objective function. The emergent behaviour is then modelled as an equilibrium of the resulting game, and the system-level performance associated with such behaviour is measured through a welfare function.[1] Within this model, our goal amounts to designing the players' local utilities to induce equilibria with high social welfare, a research theme that falls under the broader agenda of *utility design* in games [6], [7]. Many works have pursued a similar approach toward quantifying and optimizing the system-level efficiency associated with distributed decision making, see, e.g., [8]–[12]. While different performance metrics can be considered to assess the quality of an equilibrium, we adopt the widely employed notion of *Price of Anarchy* (PoA) [13], measuring the worst-case ratio between the social welfare at an equilibrium of the game and at the corresponding system optimum.

Given the importance of the research agenda described above, many works have sought to establish analytical and computational techniques for characterizing and optimizing the PoA over the past two decades, see, e.g., [14]–[17]. Crucially, the applicability of these methodologies to the design of players' utilities depends on whether these approaches provide a *tight* characterization of the PoA.

In this context, the most widely-used approach, termed *smoothness* [17], translates the derivation of a PoA bound to the satisfaction of a crucial inequality (the smoothness condition). Computational approaches then incorporate this inequality in a suitably defined optimization problem, whose solution provides a bound on the price of anarchy. While *primal-dual* approaches are available to solve the resulting optimization problem [15], [16], such techniques provide a tight bound on the PoA *only* when the original smoothness condition does so. Interestingly, [17] shows that this holds

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[1] In this respect, a parallel research effort has successfully identified simple, distributed learning dynamics which *guarantees* convergence to an equilibrium of a game. Examples include the convergence of best-response dynamics to a Nash equilibrium of any potential game, and of no-regret dynamics to a coarse correlated equilibrium of any matroid game.

true for congestion games, a well-studied model for resource sharing in which the social objective is identically equal to the sum of players' local objectives [18]. However, such tightness guarantees no longer hold when the social objective is not equal to the sum of players' local objectives, as is common in many of the application areas we consider (e.g., congestion games with incentives). In these settings, existing approaches that optimize the PoA bound obtained through smoothness necessarily give suboptimal results.

Thus, we aim to develop novel methodologies for quantifying and optimizing the exact PoA, even in settings where the social objective is not equal to the sum of agents' local objectives. Our main contributions are as follows: leftmargin=*,itemsep=0.5em

1) In Section II, we propose a generalization of the smoothness framework. We show that for *any* cost minimization game (or analogous welfare maximization game), this new framework yields an improved PoA bound when compared to the original smoothness notion (Proposition 1).
2) Our second result focuses on an important generalization of the congestion game model, which we term *generalized congestion games*, and includes congestion games with incentives. Here, we demonstrate that the newly proposed framework *tightly* characterizes the PoA (Theorem 1) while the bound associated with the standard notion of smoothness does not.
3) Our third result shifts attention to the efficient computation of the PoA. For any set of generalized congestion games, we show that the (intractable) problem of finding optimal parameters governing the generalized smoothness bound can be reduced to a tractable linear program (Corollary 1).
4) Our fourth result focuses on optimizing the PoA. Here, we show that the linear program for computing the PoA can be modified to derive local objectives that minimize the PoA in any generalized congestion games (Theorem 2).
5) Lastly, we discuss the applicability of our methodology to the problems of incentive design in congestion games and utility design in distributed welfare games (Section VI). Additionally, we show how our approach recovers and unifies a variety of existing results in the literature.

A. Related Work

There is an extensive body of literature focusing on quantifying the price of anarchy for different classes of problems, most notably in congestion games and distributed welfare games, e.g., [19]–[26]. The approaches followed in these works, and others alike, can be broadly classified as *analytical*, where ad-hoc arguments are required to obtain the desired PoA bound, and *computational*, where suitable optimization problems are utilized to achieve a similar goal.

Our work touches upon both these streams in that we propose a general condition that ensures improved estimates of the PoA (generalized smoothness), and show how to encode this condition in a tractable linear program. Firstly, the notion of generalized smoothness that we propose builds upon the seminal work of [27], though providing provably tighter bounds. Secondly, we improve upon the existing literature in that we exactly characterize the PoA for a broad classes of problems (including congestion games with incentives) through the solution of a *tractable* linear program. Within this second domain, perhaps the closest works to ours are those of [15], [16], [28]. However, the linear program proposed in [16] has exponentially many constraints (and, thus, cannot be solved efficiently), while that in [15] does not provide a tight characterization of the PoA as we demonstrate at the end of Section III. In this regard, the novelty of our work lies in identifying a game parameterization ensuring that the resulting linear program is i) tractable, and ii) tightly characterizes the PoA. As for [28], we note their approach is severely constrained to problems where all resources have an identical cost. Finally, we observe that none of these works introduces a general condition for deriving tighter PoA bounds as we do herein.

Moving beyond the mere characterization of PoA, we note that the question of *optimizing* the PoA is much less common in the literature, with solutions available only for special settings subsumed and generalized by this work, e.g., covering problems [19], polynomial congestion games [29], or distributed welfare games with identical resource costs [28]. Our work provides a significant contribution to this growing field.

Finally, we note that this manuscript differs substantially from its conference version [1] in that: i) It provides extensions of the results on cost minimization games and generalized congestion games in Sections III and IV to welfare maximization games and distributed welfare games in Section V; ii) It contains additional claims that shed further light on the differences between the notions of smoothness and its generalized counterpart (e.g., Observations #1-#3 in Section II-C); iii) It contains three novel illustrative examples with corresponding numerical analyses (Section VI, including empirical analysis of PoA surrogate metrics, PoA in variants of congestion games, utility design in welfare games); iv) It contains all proofs supporting the main claims, which are not present in [1].

Smoothness bounds, and the computational techniques derived thereof, have also received significant attention beyond the study of the PoA, e.g., in comparisons of the approximation guarantees of centralized and distributed algorithms for NP-hard problems [30], in the study of novel no-regret learning dynamics and their convergence guarantees [31], [32], and in the design of polynomial-time algorithms for computing approximate pure Nash equilibria in congestion games [33].[2]

B. Outline

Section II defines the class of games and the performance metrics that we consider throughout this paper, reviews the

[2]The problem of computing a pure Nash equilibrium, or even approximations thereof, in a congestion game is PLS-complete in general [34].

original notion of smoothness [17] and defines the novel generalized smoothness argument. Section III refines our study to the class of generalized congestion games, presents our results relating to the characterization of tight and tractable bounds on the PoA using the primal-dual approach in conjunction with a novel game parameterization and the derivation of optimal local objectives. Section V presents analogous results for the welfare maximization problem setting. Section VI applies our theoretical results to the problems of incentive design in congestion games and utility design in distributed welfare games. Section VII includes conclusions and a brief discussion on future work.

II. Generalized Smoothness in Cost Minimization Games

This section introduces the class of games and performance metrics used throughout this paper. We proceed to review the smoothness framework proposed by [17] and highlight its limitations. We then introduce a revised framework, termed generalized smoothness, that alleviates these limitations and improves upon the characterization of the equilibrium efficiency provided by the original smoothness framework.

A. Cost minimization games

We consider the class of cost minimization games in which there is a set of agents $N = \{1, \dots, n\}$, and where each agent $i \in N$ is associated with a given action set $\mathcal{A}_i$ and a cost function $J_i : \mathcal{A} \to \mathbb{R}$. The system cost induced by a collective action $a = (a_1, \dots, a_n) \in \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is measured by the function $C : \mathcal{A} \to \mathbb{R}_{>0}$, and an optimal allocation satisfies

$$a^{\rm opt} \in \arg\min_{a \in \mathcal{A}} C(a). \tag{1}$$

We represent a cost minimization game as defined above as a tuple $G = (N, \mathcal{A}, C, \mathcal{J})$, where $\mathcal{J} = \{J_1, \dots, J_n\}$. Here, G is a normal-form game augmented with a function $C : \mathcal{A} \to \mathbb{R}_{>0}$ that measures the quality of each collective action $a \in \mathcal{A}$. Importantly, the agents' cost functions $J_1, \dots, J_n$ and the system cost function C need not necessarily share *any* dependence with one another, though such dependence may stem from the problem setting of interest. For example, the system cost is often taken as the *social welfare*, i.e., $C(a) = \sum_{i=1}^n J_i(a)$, for all $a \in \mathcal{A}$, in the study of congestion games [14], [21], [24] and network games [35], [36].

The main focus of this work is on characterizing and minimizing the degradation in system-wide performance resulting from self-interested decision making. To that end, we focus on the solution concept of (pure) Nash equilibrium as a model of the emergent behaviour in such systems. A Nash equilibrium is defined as any allocation $a^{\rm ne} \in \mathcal{A}$ such that

$$J_i(a^{\rm ne}) \le J_i(a_i, a^{\rm ne}_{-i}) \quad \forall a_i \in \mathcal{A}_i, \forall i \in N. \tag{2}$$

For a given game G, let $\mathrm{NE}(G)$ denote the set of all allocations $a \in \mathcal{A}$ that satisfy (2). Assuming the set $\mathrm{NE}(G)$ is non-empty, we define the *Price of Anarchy* (PoA) of the game G as

$$\mathrm{PoA}(G) := \frac{\max_{a \in \mathrm{NE}(G)} C(a)}{\min_{a \in \mathcal{A}} C(a)} \ge 1. \tag{3}$$

The PoA represents the ratio between the costs of the worst-performing pure Nash equilibrium in the game G, and the optimal allocation. For a given class of cost minimization games $\mathcal{G}$, which may contain infinitely many game instances, we further define the PoA as,

$$\mathrm{PoA}(\mathcal{G}) := \sup_{G \in \mathcal{G}} \mathrm{PoA}(G) \ge 1. \tag{4}$$

Observe that a lower PoA corresponds to an improvement in worst-case equilibrium performance and $\mathrm{PoA}(\mathcal{G}) = 1$ implies that all Nash equilibria of all games $G \in \mathcal{G}$ are optimal.

B. Smoothness in cost minimization games

Though the instance-based PoA introduced in (3) is a powerful performance metric, its *direct* computation is extremely difficult in general. In fact, the problems of computing the minimum achievable system cost, and of computing a pure Nash equilibrium are both associated with pessimistic hardness results even in restricted settings [37]–[39]. For this reason, researchers have developed analytical techniques aimed at characterizing the PoA over various classes of games, i.e., in the spirit of (4). One such technique that is widely used in the existing literature is (λ, μ)-smoothness, which was formally defined in [17]. A cost minimization game G is termed (λ, μ)-smooth if the following two conditions are met:

(i) For all $a \in \mathcal{A}$, we have $\sum_{i=1}^n J_i(a) \ge C(a)$;
(ii) For all $a, a' \in \mathcal{A}$, there exist $\lambda > 0$ and $\mu < 1$ such that

$$\sum_{i \in N} J_i(a'_i, a_{-i}) \le \lambda C(a') + \mu C(a). \tag{5}$$

If a game G is (λ, μ)-smooth, [17] shows that the PoA of game G is upper bounded by

$$\mathrm{PoA}(G) \le \frac{\lambda}{1 - \mu}.$$

Informally, condition (5) can be thought of as a disentangling inequality, where the sum of the players costs $J_i(a'_i, a_{-i})$ at the entangled allocations (a'_i, a_{-i}) are decoupled into two quantities each depending only on the profile a or a'.

Naturally, if all the games in a class of games $\mathcal{G}$ are shown to be (λ,μ)-smooth, then the PoA of the class – $\mathrm{PoA}(\mathcal{G})$ – is also upper bounded by $\lambda/(1-\mu)$. Following the literature, we refer to the best upper bound obtainable using the smoothness argument in (5) on a given class of games $\mathcal{G}$ as the *Robust Price of Anarchy* (RPoA), i.e.,

$$\mathrm{RPoA}(\mathcal{G}) := \inf_{\lambda > 0, \mu < 1} \left\{ \frac{\lambda}{1 - \mu} \text{ s.t. (5) holds } \forall G \in \mathcal{G} \right\}, \tag{6}$$

also denoted with $\mathrm{RPoA}(G)$ if the class is composed of a single game G. It is important to note that the RPoA represents only an upper bound on the PoA, i.e., for any class of (λ, μ)-smooth games $\mathcal{G}$, it holds that $\mathrm{PoA}(\mathcal{G}) \le \mathrm{RPoA}(\mathcal{G})$. However, for certain classes of games this inequality is tight, i.e., $\mathrm{PoA}(\mathcal{G}) = \mathrm{RPoA}(\mathcal{G})$, while for others the inequality might be strict, i.e., $\mathrm{PoA}(\mathcal{G}) < \mathrm{RPoA}(\mathcal{G})$. We showcase this in the following two examples.

Example 1 (Congestion games): A congestion game is a cost minimization game G played over a ground set of resources $\mathcal{R}$, where the admissible actions of each agent $i \in N$ are elements from the power set of $\mathcal{R}$, i.e., $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$. Each resource $r \in \mathcal{R}$ is associated with a congestion function $c_r : \{1, \ldots, n\} \to \mathbb{R}$. For a collective action $a \in \mathcal{A}$, each agent $i \in N$ perceives a cost of $J_i(a) = \sum_{r \in a_i} c_r(|a|_r)$, and the system cost is equal to the cumulative cost experienced by the agents, i.e., $C(a) = \sum_{i=1}^n J_i(a)$, where $|a|_r$ denotes the number of agents selecting r in the collective action a. For the family of all congestion games $\mathcal{G}$ with affine congestion functions (i.e., in each game $G \in \mathcal{G}$, each $r \in \mathcal{R}$ has $c_r(x) = a_r \cdot x + b_r$ with $a_r, b_r \geq 0$), the optimal smoothness parameters are $\lambda = 5/3$ and $\mu = 1/3$, such that $\mathrm{RPoA}(\mathcal{G}) = 5/2$, which is equal to $\mathrm{PoA}(\mathcal{G})$ as shown in [21], [24].

Example 2 (Congestion games with incentives [40]): A more recently studied alternative to the classical congestion game model considers the impact of incentives on the PoA. Accordingly, in a congestion game with incentives, each resource $r \in \mathcal{R}$ is associated with both a congestion function $c_r : \{1, \ldots, n\} \to \mathbb{R}$ as well as an incentive function $\tau_r : \{1, \ldots, n\} \to \mathbb{R}$ such that each agent $i \in N$ perceives a cost of $J_i(a) = \sum_{r \in a_i} [c_r(|a|_r) + \tau_r(|a|_r)]$, while the system cost remains $C(a) = \sum_{r \in \cup a_i} [c_r(|a|_r) \cdot |a|_r]$. Note that $\sum_{i=1}^n J_i(a) \neq C(a)$ unless the incentives are all equal to zero. For the family of congestion games $\mathcal{G}$ with affine congestion functions under Pigouvian (marginal cost) tolls (i.e., $\tau_r(x) = c_r(x) \cdot x - c_r(x-1) \cdot (x-1)$ for all $r \in \mathcal{R}$), the optimal smoothness parameters can be shown to be $\lambda = 17/5$ and $\mu = 2/5$, such that $\mathrm{RPoA}(\mathcal{G}) = 17/3 \approx 5.67$.[3] However, $\mathrm{PoA}(\mathcal{G}) = 3$ [41] which is nearly 50% smaller than $\mathrm{RPoA}(\mathcal{G})$. Thus, ($\lambda$,$\mu$)-smoothness does not give a tight characterization of the PoA in this case.

The above examples establish that the RPoA and PoA do not always match. As a result, it follows that any analytical approach for quantifying and/or optimizing the PoA that is based on smoothness (e.g., [14], [42], [43]) is *inadequate* in these settings. Based on this observation, in the forthcoming section, we introduce a novel notion of smoothness that improves upon the PoA bound provided by the RPoA.

C. Generalized smoothness in cost minimization games

In this section, we provide a generalization of the smoothness framework, termed *generalized smoothness*. We will then proceed to show how this new framework provides tighter efficiency bounds and covers a broader spectrum of problem settings than the original smoothness framework.

Definition 2.1 (Generalized smoothness): The cost minimization game G is (λ, μ)-generalized smooth if, for any two allocations $a, a' \in \mathcal{A}$, there exist $\lambda > 0$ and $\mu < 1$ satisfying,

$$\sum_{i=1}^{n} \Big[J_i(a_i', a_{-i}) - J_i(a) \Big] + C(a) \leq \lambda C(a') + \mu C(a). \tag{7}$$

Note that we maintain the notation of (λ, μ) as in the original notion of smoothness for ease of comparison. In the specific case when $\sum_{i=1}^n J_i(a) = C(a)$ for all $a \in \mathcal{A}$, observe that the smoothness conditions in (7) are equivalent to the original smoothness conditions in (5). As with (6), we define the *Generalized Price of Anarchy* (GPoA) of a class of cost minimization games $\mathcal{G}$ as the best upper bound obtainable using a generalized smoothness argument, i.e.,

$$\mathrm{GPoA}(\mathcal{G}) := \inf_{\lambda > 0, \mu < 1} \left\{ \frac{\lambda}{1-\mu} \text{ s.t. (7) holds } \forall G \in \mathcal{G} \right\}, \tag{8}$$

also denoted with $\mathrm{GPoA}(G)$ if the class is composed of a single game G.

In the following we show that: (i) PoA bounds under the generalized smoothness framework follow in the same way as the original smoothness framework without the restriction that $\sum_{i=1}^n J_i(a) \geq C(a)$ for all $a \in \mathcal{A}$; (ii) The generalized smoothness framework provides stronger bounds on the PoA than the original smoothness framework whenever both are defined. While the proof of (i) follows readily from [17], the novelty of the result is in (ii), which establishes an ordering between RPoA, GPoA and PoA.

Proposition 1: The following statements hold:

1) If game G is (λ,μ)-generalized smooth, then its PoA is upper bounded as $\mathrm{PoA}(G) \leq \lambda/(1-\mu)$.
2) For any game G s.t. $\sum_{i=1}^n J_i(a) \geq C(a)$ for all $a \in \mathcal{A}$, it is $\mathrm{RPoA}(G) \geq \mathrm{GPoA}(G) \geq \mathrm{PoA}(G)$. Furthermore, if $\sum_{i=1}^n J_i(a) > C(a)$ for all $a \in \mathcal{A}$, then $\mathrm{RPoA}(G) > \mathrm{GPoA}(G) \geq \mathrm{PoA}(G)$.

Proof: The proof is provided in Appendix I. ■

Further comparisons between the RPoA and GPoA can be made, as summarized by the following observations, whose proof can be found in Appendix II:

– *Observation #1*: The PoA and GPoA are shift-, and scale-invariant, i.e., for any given $\gamma > 0$ and $(\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$,

$$\mathrm{PoA}((N, \mathcal{A}, C, \{J_i\})) = \mathrm{PoA}((N, \mathcal{A}, C, \{\gamma J_i + \delta_i\})),$$
$$\mathrm{GPoA}((N, \mathcal{A}, C, \{J_i\})) = \mathrm{GPoA}((N, \mathcal{A}, C, \{\gamma J_i + \delta_i\})).$$

Neither of these properties hold in general for the RPoA, i.e., for any given $\gamma > 0$ and $(\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$,

$$\mathrm{RPoA}((N, \mathcal{A}, C, \{J_i\})) \neq \mathrm{RPoA}((N, \mathcal{A}, C, \{\gamma J_i + \delta_i\})),$$

except when $\gamma = 1$ and $(\delta_1, \ldots, \delta_n) = 0$.

– *Observation #2*: The RPoA is optimized by budget-balanced agent cost functions, i.e., $\sum_{i \in N} J_i(a) = C(a)$ for all $a \in \mathcal{A}$. In general, this does not hold for the PoA and GPoA.

– *Observation #3*: For a given cost minimization game G, a coarse correlated equilibrium is any probability distribution $\sigma \in \Delta(\mathcal{A})$ satisfying, for all $a' \in \mathcal{A}$,

$$\mathbb{E}_{a \sim \sigma}\left[J_i(a)\right] \leq \mathbb{E}_{a \sim \sigma}\left[J_i(a_i', a_{-i})\right], \quad \forall i \in N. \tag{9}$$

Observe that the set of coarse correlated equilibria contains all of the game's pure Nash equilibria, mixed Nash equilibria, and correlated equilibria, since the equilibrium conditions for these more restrictive notions of equilibrium imply the inequality in (9) [17]. For any cost minimization game G, $\mathrm{RPoA}(G)$ (when admissible) and $\mathrm{GPoA}(G)$ provide upper bounds not

[3]This lower bound on the RPoA was computed by adapting the linear program from the forthcoming Corollary 1 to the RPoA. For the reader's convenience, we reproduce this linear program in Appendix IV.

only on the efficiency of the game's pure Nash equilibria (i.e., $\mathrm{PoA}(G)$), but also on the efficiency of the game's coarse correlated equilibria. However, recall that the GPoA's bound applies even when the RPoA is inadmissible, and that the GPoA always provides the stronger bound (see Proposition 1).

So far, we have presented two different smoothness bounds aimed at quantifying the PoA: (i) RPoA, and (ii) GPoA. The result in Proposition 1 shows that the GPoA always provides better (i.e., tighter) bounds on the PoA than the RPoA. Recall that our primary reasoning for considering smoothness bounds like RPoA and GPoA is that computing the PoA directly is a difficult problem, so we wish to consider a surrogate metric that is both simpler to characterize and sufficiently representative of the PoA. Though the earlier proposition and observations show that the GPoA is more representative of the PoA than the RPoA, it remains to be shown that either one of these upper bounds is actually simpler to compute than the PoA. To resolve these two concerns, we wish to address the following questions:

1) *Does the GPoA ever tightly characterize the PoA?*; and,
2) *Are there tractable techniques for quantifying the GPoA?*

In the next section, we will identify a broad class of games for which we can achieve both.

III. Generalized Smoothness in Generalized Congestion Games

The previous section introduced the framework of generalized smoothness and showed that the corresponding GPoA provides improved bounds on the PoA when compared with the RPoA. However, it remains to be shown whether the GPoA bound is tight for a meaningful class of problems, and whether it can be computed efficiently. In this section, we show that the GPoA tightly characterizes the PoA for a generalization of the well-studied class of congestion games (including congestion games with incentives). Furthermore, we demonstrate that the optimal parameters λ and μ for this important class of games can be computed as solutions of a tractable linear program. Finally, we provide a tractable linear program for deriving the optimal local objectives.

A. Generalized congestion games

In this section, we consider a generalization of the classical congestion game model that we reviewed in Example 1. In a *generalized congestion game*, we are given an agent set $N = \{1, \dots, n\}$ and a resource set $\mathcal{R}$ where each agent $i \in N$ has a feasible action set $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$. Each resource $r \in \mathcal{R}$ is now associated with both a resource-cost function $C_r : \{1, \dots, n\} \to \mathbb{R}_{\geq 0}$ and an agent-cost function $F_r : \{1, \dots, n\} \to \mathbb{R}$. Given an allocation $a \in \mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$, the costs experienced by the system and the agents have the following structure:

$$C(a) = \sum_{r \in \cup a_i} C_r(|a|_r), \tag{10}$$

$$J_i(a_i, a_{-i}) = \sum_{r \in a_i} F_r(|a|_r). \tag{11}$$

We will denote a generalized congestion game by the tuple $G = (N, \mathcal{R}, \mathcal{A}, \{C_r, F_r\}_{r \in \mathcal{R}})$. We observe that this game model covers many of the existing models studied in the literature, including the classical congestion games and congestion games with incentives that we reviewed in Section II-B.

Example 3 (Classical congestion games): In congestion games, each resource $r \in \mathcal{R}$ is associated with a congestion function c_r. In the equivalent generalized congestion game, each $r \in \mathcal{R}$ has resource-cost and agent-cost functions $C_r(k) = k \cdot c_r(k)$ and $F_r(k) = c_r(k)$ for $k = 1, \dots, n$. Note that $C_r(k) = k \cdot F_r(k)$ for this case, and, thus, $C(a) = \sum_i J_i(a)$ for all $a \in \mathcal{A}$. Hence, the definitions of smoothness and generalized smoothness coincide.

Example 4 (Congestion games with incentives): In congestion games with incentives, each resource $r \in \mathcal{R}$ is associated with both a congestion function c_r and an incentive function τ_r. In the equivalent generalized congestion game, each $r \in \mathcal{R}$ has resource-cost and agent-cost functions $C_r(k) = k \cdot c_r(k)$ and $F_r(k) = c_r(k) + \tau_r(k)$ for $k = 1, \dots, n$. Here, the definitions of smoothness and generalized smoothness no longer coincide.

B. Tight, tractable PoA in generalized congestion games

In this section, we are interested in characterizing the PoA associated with the family of generalized congestion games, $\mathcal{G}_{\mathcal{P}}$, where each pair of resource-cost and agent-cost function belongs to a given set of function pairs $\mathcal{P}$.[4] For ease of notation, we denote the family $\mathcal{G}_{\mathcal{P}}$ as $\mathcal{G}$ when the dependence on the set $\mathcal{P}$ is clear. Our next result shows that the GPoA provides a tight bound on the PoA associated with any family of generalized congestion games $\mathcal{G}_{\mathcal{P}}$ (proof in Appendix III).

Theorem 1: For any set of resource-cost, agent-cost function pairs $\mathcal{P}$ and positive integer n, let $\mathcal{G}$ denote the family of all generalized congestion games with a maximum of n agents in which each resource $r \in \mathcal{R}$ satisfies $\{C_r, F_r\} \in \mathcal{P}$. It holds that $\mathrm{PoA}(\mathcal{G}) = \mathrm{GPoA}(\mathcal{G})$.

Theorem 1 highlights that the GPoA represents a tight bound on the PoA for the family $\mathcal{G}_{\mathcal{P}}$ under any set of resource-cost, agent-cost function pairs $\mathcal{P}$. Therefore, for this broad class of problems, there is no loss in characterizing the PoA using the generalized smoothness bound. However, it remains to be shown whether $\mathrm{GPoA}(\mathcal{G}_{\mathcal{P}})$ can be quantified efficiently.

In many commonly studied settings, we can leverage the structure of the set $\mathcal{P}$ to efficiently quantify the GPoA of the family of all generalized congestion games under $\mathcal{P}$. For instance, in the forthcoming Corollary 1, we show that the quantity $\mathrm{GPoA}(\mathcal{G}_{\mathcal{P}})$ in (8) can be computed efficiently when the number of resource-cost, agent-cost function pairs in $\mathcal{P}$ and the maximum number of agents are finite. Moreover, the GPoA may be computable even when the size of $\mathcal{P}$ is not finite, but all its pairs of resource-cost and agent-cost function are obtained from linear combination of finitely many basis functions as, e.g., for the widely studied class of polynomial congestion games. Specifically, let $\mathcal{P} = \{\{C^1, F^1\}, \dots, \{C^m, F^m\}\}$ denote any (finite) set of m resource-cost, agent-cost function

[4] It immediately follows that classical congestion games, and congestion games with incentives can be represented under this notation.

pairs, and let $\Delta(\mathcal{P})$ denote the set of all resource-cost, agent-cost function pairs $\{C, F\}$ that can be represented as

$$C(k) = \sum_{j=1}^{m} \alpha^j \cdot C^j(k), \quad F(k) = \sum_{j=1}^{m} \alpha^j \cdot F^j(k), \quad k = 1, \ldots, n,$$

with $\alpha^1, \ldots, \alpha^m \geq 0$.[5] In the proof of Theorem 1, we establish that the GPoA of the family $\mathcal{G}_{\Delta(\mathcal{P})}$ is equal to the GPoA of $\mathcal{G}_{\mathcal{P}}$, and, thus, can also be computed as the solution of a tractable linear program. We state and prove this result in the following corollary. Towards this goal, it is convenient to define $C^j(0) = F^j(0) = F^j(n+1) = 0$, for $j = 1, \ldots, m$, and introduce the set of integer tuples

$$\mathcal{I}(n) := \{(x, y, z) \in \mathbb{N}^3 \,|\, 1 \leq x + y - z \leq n \text{ and } z \leq \min\{x, y\}\},$$
$$\mathcal{I}_{\mathcal{R}}(n) := \{(x, y, z) \in \mathcal{I}(n) \text{ s.t. } x + y - z = n\} \cup \{(x, y, z) \in \mathcal{I}(n) \text{ s.t. } (x - z)(y - z)z = 0\}.$$

Corollary 1: For any set of resource-cost, agent-cost function pairs $\mathcal{P} = \{\{C^1, F^1\}, \ldots, \{C^m, F^m\}\}$ and positive integer n, let $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{G}_{\Delta(\mathcal{P})}$ denote the families of all generalized congestion games with a maximum of n agents under their specified sets of function pairs. Then, $\mathrm{PoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})})$.

Further, let ρ^{opt} be the optimal value of the following (tractable) linear program:

$$\begin{aligned} \rho^{\mathrm{opt}} = \underset{\nu \in \mathbb{R}_{\geq 0}, \rho \in \mathbb{R}}{\text{maximize}} \quad & \rho \quad \text{subject to:} \\ C^j(y) - \rho C^j(x) + \nu[(x-z)F^j(x) - (y-z)F^j(x+1)] &\geq 0, \\ \forall j = 1, \ldots, m, \quad \forall (x, y, z) &\in \mathcal{I}_{\mathcal{R}}(n). \end{aligned} \tag{12}$$

Then, $\mathrm{PoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{GPoA}(\mathcal{G}_{\mathcal{P}}) = 1/\rho^{\mathrm{opt}}$.

The linear program in (12) has two decision variables and $\mathcal{O}(mn^2)$ constraints. Thus, for m and n finite, there are computationally efficient approaches for characterizing the PoA of generalized congestion games. For example, the worst-case runtime of linear program solvers that use interior-point methods is polynomial in the size of the input (i.e., the number of decision variables, and the number of constraints) [44].

The joint proof of Theorem 1 and Corollary 1 is provided in Appendix III. However, for the reader's convenience, we provide a proof outline summarizing the four steps of the formal proof:

– *Step 1* : For given set of resource-cost, agent-cost function pairs $\mathcal{P}$, we show that the PoA of the family $\mathcal{G}_{\mathcal{P}}$ is equal to the PoA of the family $\mathcal{G}_{\Delta(\mathcal{P})}$. We do so by showing that for every game $G \in \mathcal{G}_{\Delta(\mathcal{P})}$, there is a game $G' \in \mathcal{G}_{\mathcal{P}}$ such that $\mathrm{PoA}(G) = \mathrm{PoA}(G')$.

– *Step 2* : We define a game parameterization, which represents any generalized congestion game $G \in \mathcal{G}_{\Delta(\mathcal{P})}$ with $\mathcal{O}(mn^3)$ parameters $\theta(x, y, z, j) \geq 0$ corresponding with basis pairs $\{(C^j, F^j)\}$, $j = 1, \ldots, m$, and triplets $x, y, z \in \{0, \ldots, n\}$ such that $1 \leq x + y - z \leq n$ and $z \leq \min\{x, y\}$.

– *Step 3* : For any family of generalized congestion games $\mathcal{G}$, we observe that an upper bound on the GPoA can be computed as a fractional linear program with $\mathcal{O}(mn^2)$ constraints under the game parameterization presented in Step 2.

– *Step 4* : Following a change of variables, we observe that the linear program in (12) is equivalent to the fractional program from Step 3. We then construct a game $G \in \mathcal{G}_{\Delta(\mathcal{P})}$ with PoA equal to the upper bound on the GPoA from Step 3, implying that $\mathrm{PoA}(G) \geq \mathrm{GPoA}(\mathcal{G})$. Since $\mathrm{PoA}(G) \leq \mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})}) \leq \mathrm{GPoA}(\mathcal{G}_{\mathcal{P}}) \leq \mathrm{GPoA}(\mathcal{G}_{\Delta(\mathcal{P})})$, it must then be that $\mathrm{PoA}(G) = \mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})}) = \mathrm{GPoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{GPoA}(\mathcal{G}_{\Delta(\mathcal{P})})$, concluding the proof.

Observe that Step 1 of the proof establishes that $\mathrm{PoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})})$, while the remaining Steps 2–4 show that $\mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})}) = \mathrm{GPoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{GPoA}(\mathcal{G}_{\Delta(\mathcal{P})})$. Theorem 1 immediately follows from this series of equalities, i.e., $\mathrm{PoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{GPoA}(\mathcal{G}_{\mathcal{P}})$. Furthermore, Corollary 1 follows as a consequence of this proof, since we have shown that $\mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})}) = \mathrm{GPoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{GPoA}(\mathcal{G}_{\Delta(\mathcal{P})})$, and that solving the linear program in (12) is equivalent to computing $\mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})})$.

Remark 1: Note that both the smoothness and generalized smoothness conditions can be written as linear programs for any family of games $\mathcal{G}$. For example, observe that $\mathrm{GPoA}(\mathcal{G}) = 1/\rho^{\mathrm{opt}}$, where

$$\begin{aligned} \rho^{\mathrm{opt}} = \underset{\nu \in \mathbb{R}_{\geq 0}, \rho \in \mathbb{R}}{\text{maximize}} \quad & \rho \quad \text{subject to:} \\ C(a') - \rho C(a) + \nu \left[\sum_{i=1}^{n} J_i(a) - \sum_{i=1}^{n} J_i(a_i', a_{-i}) \right] &\geq 0, \\ \forall a, a' \in \mathcal{A}, \forall G &\in \mathcal{G}, \end{aligned} \tag{13}$$

which follows by definition from (7) and (8) for the change of variables $\nu = 1/\lambda$ and $\rho = (1 - \mu)/\lambda$. However, solving (13) is intractable as there are exponentially many constraints (one for each pair of allocations a, a'). The novelty of the results in Theorem 1 and Corollary 1 is in identifying a game parameterization allowing for a concise representation of (13) while preserving the tightness of the PoA bound. In this respect, the authors of [15] also propose a tractable linear program to upper bound the PoA. However, their program does not provide a tight characterization, as we show next by considering affine congestion games with n agents. Specifically, [15] computes an upper bound γ^{opt} on the PoA as follows:

$$\begin{aligned} \gamma^{\mathrm{opt}} = \underset{\kappa \in \mathbb{R}_{\geq 0}, \gamma \in \mathbb{R}}{\text{maximize}} \quad & \gamma \quad \text{subject to:} \\ \gamma y^2 - x^2 + \kappa[x^2 - (x+1)y] \geq 0, & \forall x, y \in \{0, 1, \ldots, n\}. \end{aligned} \tag{14}$$

Solving (14) with $n = 2$ returns $\gamma^{\mathrm{opt}} = 2.5$. However, the solution of (12) gives $\mathrm{PoA}(\mathcal{G}) = 2$, so that $\mathrm{PoA}(\mathcal{G}) < \gamma^{\mathrm{opt}}$.[6]

IV. Optimizing the PoA

The previous section focused on how to characterize the PoA in any set of generalized congestion games. In this

[5] For example, the family of affine congestion games (studied in, e.g., [21], [24]) is equivalent to the family $\mathcal{G}_{\Delta(\mathcal{P})}$ with $m = 2$, where $\{C^1(k), F^1(k)\} = \{k, 1\}$ and $\{C^2(k), F^2(k)\} = \{k^2, k\}$, $k = 1, \ldots, n$.

[6] One can verify that the solution to the linear program in (12) is $(\nu^{\mathrm{opt}}, \rho^{\mathrm{opt}}) = (0.5, 0.5)$, while the solution to the linear program in (14) is $(\kappa^{\mathrm{opt}}, \gamma^{\mathrm{opt}}) = (1.5, 2.5)$.

section, we shift our focus to the derivation of agent-cost functions that optimize the PoA. That is, given a set of resource-cost functions $C^1, \dots, C^m$, what is the corresponding set of agent-cost functions $F^1, \dots, F^m$ that minimizes the resulting PoA? Recall from the introduction that this line of questioning is relevant to the design of agents' local decision-making rules in systems under distributed decision making.

The following theorem provides a tractable and scalable methodology for computing the set of agent-cost functions that minimize the PoA:

Theorem 2: Let $C^1, \dots, C^m$ denote a set of resource-cost functions defined for n agents, and let $(F^{\mathrm{opt},j}, \rho^{\mathrm{opt},j})$, $j = 1, \dots, m$, be solutions to the following m linear programs:

$$\begin{aligned} &\underset{F\in\mathbb{R}^n,\rho\in\mathbb{R}}{\text{maximize}} \quad \rho \quad \text{subject to:} \\ &C^j(y) - \rho C^j(x) + (x-z)F(x) - (y-z)F(x+1) \ge 0, \\ &\qquad\qquad\qquad\qquad \forall (x,y,z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned} \tag{15}$$

Then the agent-cost functions $F^{\mathrm{opt},1}, \dots, F^{\mathrm{opt},m}$ minimize the PoA. Further, the PoA of the family $\mathcal{G}$ of all generalized congestion games with a maximum of n agents under function pairs $\{C^j, F^{\mathrm{opt},j}\}$, $j = 1, \dots, m$, satisfies

$$\mathrm{PoA}(\mathcal{G}) = \max_{j\in\{1,\dots,m\}} \frac{1}{\rho^{\mathrm{opt},j}}.$$

Proof: The proof is provided in Appendix V, for ease of presentation. ■

Theorem 2 states that we can derive agent-cost functions $F^{\mathrm{opt},1}, \dots, F^{\mathrm{opt},m}$ that minimize the PoA by solving m independent linear progams, where each $F^{\mathrm{opt},j}$ can be derived using only information about the corresponding resource-cost function C^j. Accordingly, the PoA of this optimized system corresponds to the worst PoA associated with any single pair $\{C^j, F^{\mathrm{opt},j}\}$, i.e.,

$$\mathrm{PoA}(\mathcal{G}) = \max_{j\in\{1,\dots,m\}} \mathrm{PoA}(\mathcal{G}^j),$$

where $\mathcal{G}^j \subseteq \mathcal{G}$ represents the family of all generalized congestion games with a maximum of n agents under the single pair $\{C^j, F^{\mathrm{opt},j}\}$. Observe that this statement is not true in general for an arbitrary set of function pairs, i.e., there exist sets of function pairs $\{C^j, F^j\}$, $j = 1, \dots, m$, such that[7]

$$\mathrm{PoA}(\mathcal{G}) > \max_{j\in\{1,\dots,m\}} \mathrm{PoA}(\mathcal{G}^j).$$

However, when we restrict our attention to optimal agent-cost functions for each C^j, the above strict inequality turns into an equality. This is the key technical insight in the proof of Theorem 2.

V. Generalized Smoothness in Welfare Maximization Games

Although the primary focus of this paper is on cost minimization settings, many of the results that we obtain can be analogously derived for welfare maximization problems. A welfare maximization problem consists of a set $N = \{1, \dots, n\}$ of agents, where each agent $i \in N$ is associated with a finite action set $\mathcal{A}_i$. The global objective is to *maximize* the system's welfare, which is measured by the welfare function $W : \mathcal{A} \to \mathbb{R}_{>0}$, i.e. we wish to find the allocation $a^{\mathrm{opt}} \in \mathcal{A}$, such that $a^{\mathrm{opt}} \in \arg\max_{a\in\mathcal{A}} W(a)$. As with cost minimization problems, we consider a game-theoretic model where each agent $i \in N$ is associated with a utility function $U_i : \mathcal{A} \to \mathbb{R}$, which it uses to evaluate its own actions against the collective actions of the other agents. A *welfare maximization game* is a tuple $G = (N, \mathcal{A}, W, \{U_i\})$.

Given a welfare maximization game G, a pure Nash equilibrium is an allocation $a^{\mathrm{ne}} \in \mathcal{A}$ such that $U_i(a^{\mathrm{ne}}) \ge U_i(a_i, a^{\mathrm{ne}}_{-i})$ for all $a \in \mathcal{A}_i$, and all $i \in N$. The PoA in welfare maximization games is defined similarly to (3) and (4),[8]

$$\begin{aligned} \mathrm{PoA}(G) &:= \frac{\max_{a\in\mathcal{A}} W(a)}{\min_{a\in\mathrm{NE}(G)} W(a)} \ge 1, \\ \mathrm{PoA}(\mathcal{G}) &:= \sup_{G\in\mathcal{G}} \mathrm{PoA}(G) \ge 1, \end{aligned}$$

where a lower PoA corresponds to better performance.

A. Generalized smoothness in welfare maximization games

We begin with the definition of generalized smoothness and then provide the analogue of Proposition 1.

Definition 5.1: The welfare maximization game G is (λ, μ)-generalized smooth if, for any two allocations $a, a' \in \mathcal{A}$, there exist $\lambda > 0$ and $\mu > -1$ satisfying,

$$\sum_{i=1}^n U_i(a'_i, a_{-i}) - \sum_{i=1}^n U_i(a) + W(a) \ge \lambda W(a') - \mu W(a). \tag{16}$$

Proposition 2: The PoA of a (λ, μ)-generalized smooth welfare maximization game G is upper bounded as,

$$\mathrm{PoA}(G) \le \frac{1+\mu}{\lambda}.$$

We define the GPoA of a set of welfare maximization games $\mathcal{G}$ as

$$\mathrm{GPoA}(\mathcal{G}) := \inf_{\lambda>0,\mu>-1} \left\{ \frac{1+\mu}{\lambda} \text{ s.t. (16) holds } \forall G \in \mathcal{G} \right\}. \tag{17}$$

As with cost minimization games, all efficiency guarantees also extend to average coarse-correlated equilibria (as in Observation #3) and there are also provable advantages of generalized smoothness over the original smoothness framework in terms of characterizing PoA bounds (as in Proposition 1). We do not state these to avoid redundancy.

[7] For example, consider the family of generalized congestion games $\mathcal{G}$ with a maximum of $n = 3$ agents under pairs $\{\{C^1, F^1\}, \{C^2, F^2\}\}$, where $\{C^1(k), F^1(k)\} = \{k^2, k\}$ and $\{C^2(k), F^2(k)\} = \{k, k\}$ for $k = 1, \dots, n$. Using the linear program in (12), we get $\mathrm{PoA}(\mathcal{G}^1)) = 2.5$, $\mathrm{PoA}(\mathcal{G}^2)) = 2.0$, and $\mathrm{PoA}(\mathcal{G}) = 2.6$. For this particular choice of $\mathcal{G}$, observe that $\mathrm{PoA}(\mathcal{G}) > \max_{j\in\{1,\dots,m\}} \mathrm{PoA}(\mathcal{G}^j)$.

[8] For consistency with the previous sections, we opt to define the PoA in welfare maximization games as the ratio between the welfare at the optimal allocation and the system welfare at the worst performing Nash equilibrium, in contrast with previous works [17], [19]. This is achieved by inverting the ratio, i.e., defining the PoA as the worst case ratio between the welfare at optimum, and the welfare at the equilibria in $\mathrm{NE}(G)$, and thereby retaining the overall objective of *minimizing* the system's PoA.

B. Generalized smoothness in distributed welfare games

In this section, we consider *distributed welfare games* [26], which are the welfare maximization analogue to generalized congestion games. In these games, there is a set of agents $N = \{1, \ldots, n\}$ and a finite set of resources $\mathcal{R}$, where each agent $i \in N$ has an associated set of admissible actions $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ and each resource $r \in \mathcal{R}$ is associated with a resource welfare function $W_r : \{1, \ldots, n\} \to \mathbb{R}_{\geq 0}$ and a utility generating function $F_r : \{1, \ldots, n\} \to \mathbb{R}$. The system welfare and agent utility functions are defined as

$$W(a) = \sum_{r \in \cup a_i} W_r(|a|_r), \qquad U_i(a) = \sum_{r \in a_i} F_r(|a|_r).$$

For the remainder of this section, given function pairs $\{W^j, F^j\}$, $j = 1, \ldots, m$, we define the family of all local welfare maximization games $\mathcal{G}$ in the same fashion as for generalized congestion games given in Section III. Distributed welfare games have been used to model many problems of interest [19], [22], [26], [43], [45].

The following theorem provides the analagous results derived for generalized congestion games to the domain of distributed welfare games. As before, we define $W^j(0) = F^j(0) = F^j(n+1) = 0$, for $j = 1, \ldots, m$, for ease of notation. Theorem 3 is stated without proof as the reasoning follows almost identically to the proofs of Corollary 1 and theorem 2.

Theorem 3: Let $\mathcal{G}$ denote the set of all distributed welfare games with a maximum of n agents under function pairs $\{W^j, F^j\}$, $j = 1, \ldots, m$. The following statements hold true:
(i) $\mathrm{PoA}(\mathcal{G}) = \mathrm{GPoA}(\mathcal{G})$.
(ii) Let ρ^{opt} be the value of the following linear program:

$$\begin{aligned} &\rho^{\mathrm{opt}} = \min_{\nu \in \mathbb{R}_{\geq 0}, \rho \in \mathbb{R}} \rho \quad \text{subject to:} \\ &W^j(y) - \rho W^j(x) + \nu \left[(x-z)F^j(x) - (y-z)F^j(x+1)\right] \leq 0 \\ &\qquad\qquad \forall j = 1, \ldots, m, \quad \forall (x, y, z) \in \mathcal{I}_{\mathcal{R}}(n), \end{aligned} \tag{18}$$

Then, it holds that $\mathrm{PoA}(\mathcal{G}) = \rho^{\mathrm{opt}}$.
(iii) Let $(F^{\mathrm{opt},j}, \rho^{\mathrm{opt},j})$, $j = 1, \ldots, m$, be solutions to the following m linear programs:

$$\begin{aligned} &(F^{\mathrm{opt},j}, \rho^{\mathrm{opt},j}) \in \operatorname*{arg\,min}_{F \in \mathbb{R}^n, \rho \in \mathbb{R}} \rho \quad \text{subject to:} \\ &W^j(y) - \rho W^j(x) + (x-z)F(x) - (y-z)F(x+1) \leq 0, \\ &\qquad\qquad \forall (x, y, z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned} \tag{19}$$

Then, the utility generating functions $F^{\mathrm{opt},1}, \ldots, F^{\mathrm{opt},m}$ minimize the PoA, and the PoA corresponding to function pairs $\{W^j, F^{\mathrm{opt},j}\}$, $j = 1, \ldots, m$, satisfies

$$\mathrm{PoA}(\mathcal{G}) = \max_{j \in \{1, \ldots, m\}} \rho^{\mathrm{opt},j}.$$

VI. Illustrative Examples

In this section, we demonstrate the depth and breadth of our approach by: i) Offering a numerical comparison between the PoA, RPoA and GPoA in cost-minimization games; ii) Showcasing the applicability of our approach to congestion games and their variants; iii) Assessing the performance of optimally designed utilities in distributed welfare games.

A. PoA, RPoA and GPoA in cost-minimization games

A central objective in our study of smoothness arguments has been to identify computationally efficient alternatives to upper bound the actual PoA. In this paper, we have covered two such metrics: the Robust Price of Anarchy (RPoA) and the Generalized Price of Anarchy (GPoA), both of which require solutions of linear programs with equally many decision variables and constraints (see (6) and (8)). Contrary to their similarity in computational complexity, we have seen in Section II (Proposition 1) that the GPoA always provides a better bound on the PoA. Specifically, for any cost minimization game G for which $\mathrm{RPoA}(G)$ is well-defined, we have that $\mathrm{PoA}(G) \leq \mathrm{GPoA}(G) \leq \mathrm{RPoA}(G)$.

In this section, we supplement the analytical comparisons in Section II with a simulation example comparing the PoA, RPoA and GPoA. Towards this goal, we randomly generate cost minimization games for which the PoA and RPoA are well-defined as in the following:[9]

1) We consider games with n agents, where each agent has only two actions, i.e., $\mathcal{A}_i = \{a_i^{(1)}, a_i^{(2)}\}$ for each $i \in N$.
2) For the system cost function $C : \mathcal{A} \to \mathbb{R}$ and agent cost functions $J_i : \mathcal{A} \to \mathbb{R}$, $i = 1, \ldots, n$, randomly sample the values $C(a)$ and $J_i(a)$, $i = 1, \ldots, n$, from the uniform distribution $U(0, 1)$ for all $a \in \mathcal{A}$.
3) If there exists $a \in \mathcal{A}$ such that $\sum_{i=1}^n J_i(a) > C(a)$, then discard the game G and return to Step 2.
4) Compute the set of pure Nash equilibria $\mathrm{NE}(G)$. If $\mathrm{NE}(G) = \emptyset$, discard the game G and return to Step 2.

Observe that upon completing Step 4, above, we have a cost-minimization game G for which the PoA and RPoA are both well-defined, since the set of pure Nash equilibria $\mathrm{NE}(G)$ is non-empty, and $C(a) \geq \sum_{i=1}^n J_i(a)$ for all $a \in \mathcal{A}$.

Following the above, we generate 10^5, 3-agent cost-minimization games. For each of these games, we compute the PoA as the ratio between $\max_{a \in \mathrm{NE}(G)} C(a)$ and $\min_{a \in \mathcal{A}} C(a)$, and we compute the RPoA and GPoA by solving (6) and (8), respectively.

Among the 10^5 randomly generated games, the maximum ratio between RPoA and PoA is 30,815, the maximum ratio between GPoA and PoA is 21,959, and the maximum ratio between RPoA and GPoA is 1037.4. Despite the magnitude of the maximum ratios, the 75-th percentiles are 4 orders of magnitude lower than the maximum, and the 95-th percentiles are 3 orders of magnitude lower. This suggests that the data is *extremely* left-skewed. Accordingly, in Figure 1a, we plot the distributions of the ratios $\mathrm{RPoA}(G)/\mathrm{PoA}(G)$ and $\mathrm{GPoA}(G)/\mathrm{PoA}(G)$ but limit the horizontal axis to values below the 95-th percentile of $\mathrm{RPoA}(G)/\mathrm{PoA}(G)$. We observe that the RPoA is within a factor of 1.001 of the PoA for 12.09% of the generated cost-minimization games, while the GPoA is within this factor for 12.24% of the games. Similarly, in Figure 1b, we plot the distribution of the ratio $\mathrm{RPoA}(G)/\mathrm{GPoA}(G)$, limiting the horizontal axis to values below the 95-th percentile of $\mathrm{RPoA}(G)/\mathrm{GPoA}(G)$. We

[9]Recall from Definition 2.1 that the GPoA is well-defined for any cost-minimization game, in contrast with the PoA and RPoA.

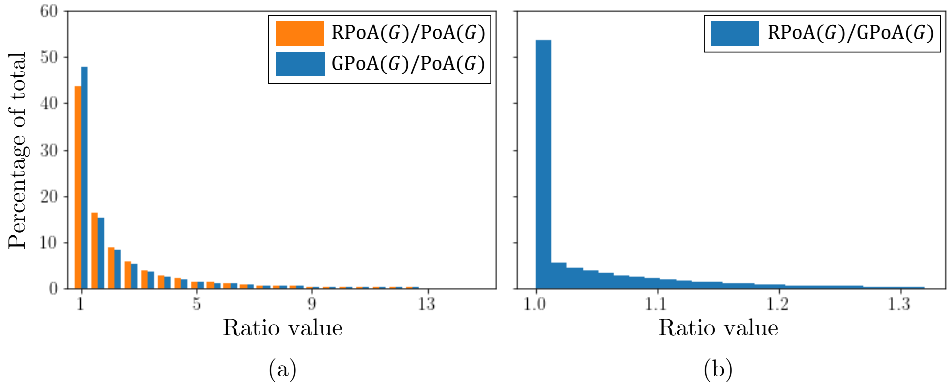


Fig. 1. *Comparison of PoA, RPoA and GPoA in cost-minimization games.* The PoA, RPoA and GPoA values used for the above figures were taken from the same set of $\mathbf{10^5}$, 3-player cost-minimization games randomly generated following Steps 1–4 in Section VI.A. *Left:* Empirical distribution of the ratios between RPoA and PoA, and GPoA and PoA. Although the maximum ratio $\mathbf{RPoA}(\boldsymbol{G})/\mathbf{PoA}(\boldsymbol{G})$ observed was 30,815, we plot the values below the 95-th percentile since the distributions are extremely skewed. Notably, the RPoA (resp., GPoA) is within a factor of 1.001 of the PoA for 12.09% (resp., 12.24%) of the generated games. *Right:* Empirical distribution of the ratio between RPoA and GPoA. Similar to the left figure, the maximum ratio $\mathbf{RPoA}(\boldsymbol{G})/\mathbf{GPoA}(\boldsymbol{G})$ observed was 1,037.4, but we plot the values below the 95-th percentile. Among the cost-minimization games generated, approximately 46.69% have RPoA within a factor of 1.001 of the GPoA.

observe that approximately 46.69% of the cost-minimization games generated have RPoA within a factor of 1.001 of the GPoA. As a final note on the empirical distributions in Figure 1, observe thaat each of the games generated agrees with the ordering $\mathrm{PoA}(G) \leq \mathrm{GPoA}(G) \leq \mathrm{RPoA}(G)$ that we derived analytically in Proposition 1.

Recall from our discussion in Section II that the RPoA (when well-defined) and the GPoA are valid upper-bounds on the worst-case efficiency of mixed Nash, correlated, coarse correlated and aggregate equilibria in cost-minimization games. An interesting extension of this study would compare the worst-case efficiency of these generalized equilibrium notions to the RPoA and GPoA in cost-minimization games. Intuitively, the RPoA and GPoA should provide better bounds as the equilibrium notion of interest becomes more general.

B. PoA in congestion games and their variants

Corollary 1 provides a methodology to efficiently determine the exact PoA in any problem that can be cast as a generalized congestion game. We showed in Example 3 that classical congestion games constitute a subclass of problems to which the methodology applies. Hence, we are able to compute their PoA by simply solving the linear program in Theorem 1. As a special case, we subsume well-known PoA results for classical congestion games with polynomial congestion functions of maximum degree d [14], [21], [24], i.e., congestion games where the resource congestion is obtained by non-negative linear combinations of monomials $1, x, \ldots, x^d$. Although the bounds provided in previous works are exact (for large n), their authors had to combine traditional smoothness arguments with nontrivial worst-case game constructions.[10] In contrast, the linear program in Theorem 1 provides exact PoA values for all n, can be solved efficiently, and does not require ad-hoc constructions of worst-case instances. For example, one can verify that the PoA values for $n = 5$ players are $2.5, 9.58, 41.54, 267.64$ for $d = 1, ..., 4$, respectively, which already match the bounds previously obtained for large n, see third and fourth panels in Figure 2. Beyond that, the methodology provided in Corollary 1 can be leveraged to compute the PoA of classical congestion games under *any* family of congestion functions $\mathcal{C}$, even congestion functions employed in applied settings. This includes the well-studied Bureau of Public Roads (BPR) congestion function [46], defined as $c_r(x) = T_r \cdot \left[1 + 0.15 \cdot (x/K_r)^4\right]$ for each $r \in \mathcal{R}$, where $T_r \geq 0$, $K_r \in \mathbb{N}_{\geq 1}$ are the free flow congestion and capacity of resource r. Solving the corresponding linear program in Theorem 1, one obtains a PoA of approximately 36.09 for $n = 50$ agents and $K_r \in \{1, \ldots, 50\}$. This highlights that, although the BPR congestion functions are polynomials of order $d = 4$, their special structure admits significant reductions in the PoA (from 267.64 to 36.09).

[10]Limited to classical congestion games, recall that the smoothness and generalized smoothness inequalities coincide since the system cost equals the sum of the agents' costs.

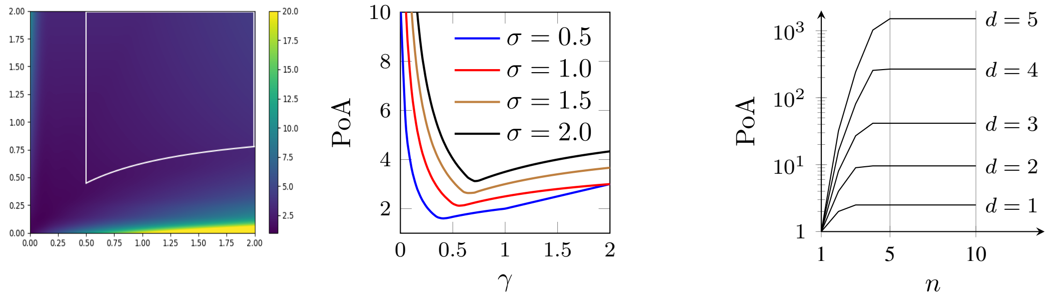
Of course, the generalized congestion game model includes many other classes of games. For instance, consider the (σ,γ)-*perception-parameterized* congestion game model that was proposed by [25] and unifies the study of risk sensitivity [47], [48], probabilistic participation [49] and altruism [23], [50] in affine congestion games. Here, $\sigma, \gamma \geq 0$ parameterize how the system and agents perceive the load on resources in the game, respectively. It is straightforward to verify that any (σ,γ)-perception-parameterized congestion game under congestion functions $c_r \in \mathcal{C}$, $r \in \mathcal{R}$, is equivalent to a generalized congestion game with resource-cost, agent-cost function pairs

$$C_r(k) = k \cdot c_r(1 + \sigma(k-1)), \quad F_r(k) = c_r(1 + \gamma(k-1)),$$

such that the system and agent cost functions are

$$C(a) = \sum_{r \in \cup a_i} C_r(|a|_r) = \sum_{r \in \cup a_i} |a|_r \cdot c_r(1 + \sigma(|a|_r - 1)),$$
$$J_i(a) = \sum_{r \in a_i} F_r(|a|_r) = \sum_{r \in a_i} c_r(1 + \gamma(|a|_r - 1)).$$

Thus, evaluating the PoA of perception-parameterized congestion games – which remains an open problem even in the



d	PoA, $n = 5$
1	2.50
2	9.58
3	41.54
4	267.64
5	1513.57

Fig. 2. Exact price of anarchy for perception-parameterized affine congestion games with $n = 20$ users and $\sigma, \gamma \in [0, 2]$, computed via Corollary 1 (first panel). Corresponding values for fixed $\sigma \in \{0.5, 1, 1.5, 2\}$ (second panel). Kleer and Schäfer [25] give asymptotic values limited to the region enclosed in the white perimeter, which we recover exactly and generalize. Evolution of the price of anarchy in polynomial congestion games of order $d = 1, \dots, 5$ as a function of the number of users (third panel). These values were obtained by solving the corresponding linear program. Observe that the price of anarchy plateaus at $n = 5$, matching the asymptotic bounds ($n \to \infty$) previously obtained in the literature [14], [21], [24]. This suggests that small instances are sufficient to produce highly inefficient equilibria.

affine case – is equivalent to solving the linear program in Corollary 1. We do so and present the results in the first and second panel of Figure 2.

C. Optimal utility design in distributed welfare games

While a similar approach to that taken in the previous section can be followed for welfare maximization problems – and in fact, recent results in welfare maximization games build on top of this work to derive state-of-the-art approximation algorithms with explicit guarantees [22], [33], [45] – we purposely take a different perspective in this section. Specifically, we aim at demonstrating the *robustness* of the proposed approach and the *quality* of the corresponding results.

Toward this goal, rather than specifying a set of welfare functions, we consider a general setting, whereby $W_r : \{1, \dots, n\} \to \mathbb{R}$ merely satisfies two properties: non-decreasingness, i.e., $W_r(k+1) \geq W_r(k)$, and concavity (or diminishing returns property), i.e., $W_r(k+1) - W_r(k) \leq W_r(k) - W_r(k-1)$, for all k and r. Observe that these properties are commonly encountered in application areas including vehicle-target assignment problems [51], [52], multiwinner elections [53] and sensor coverage [19], [54]. For any given welfare function satisfying non-decreasingness and concavity, we compare the performance (as captured by the PoA) obtained by *optimal utilities* against that of the commonly-advocated-for *identical interest* design, whereby each agent's utility coincides with the system welfare, i.e., $U_i(a) = W(a)$. We do so for 10^5 unique resource welfare functions W_r, randomly generated by sorting 10 values sampled independently from a uniform distribution over $[0, 1]$ from largest to smallest, and setting $W_r(k)$ to be the sum over the first k sorted values. Nondecreasingness and concavity of W_r follow readily. For each generated resource welfare function, we determine the PoA of the corresponding family of distributed welfare games under the identical interest utility design, and the PoA of the corresponding family of distributed welfare games under the optimal utility design through the solutions of the linear programs in (18) and (19), respectively.[11] The left panel in Figure 3 depicts the resulting empirical distribution of the PoA values, while the right reports the ratio between the PoA in the identical interest and optimal settings (this ratio is never lower than one, as expected). We observe that, whilst the identical interest design may provide an intuitive and appealing option – and, indeed, it optimizes the best-case equilibrium efficiency – Figure 3 highlights that strictly better worst-case equilibrium efficiency guarantees can be readily achieved using the machinery developed in this work.

VII. Conclusion and Future Directions

Though well-studied, the PoA can still be difficult to compute, and ad-hoc approaches are often used in the existing literature. As a result, the design of incentives that optimize this metric is particularly challenging, with only few results available. Motivated by this observation, we develop a framework achieving two goals: to tightly characterize and optimize the PoA through a computationally tractable approach.

Toward this end, we first introduced the notion of *generalized smoothness*, which we showed always produces tighter or equal PoA bounds compared to the original smoothness approach. We proved that such bounds are *exact* for generalized congestion and local welfare maximization games, unlike those obtained through a classical smoothness argument. Additionally, we showed that the problems of computing and optimizing the PoA can be posed (and solved) as tractable linear programs, when considering these broad problem classes. Finally, we demonstrated the ease of applicability, strength and breadth of our approach by showing that it recovers and generalizes existing results on the computation of the PoA, and that it can be applied to the problems of incentive design

[11] Observe that the PoA of the family of games under the identical interest utility design is equivalent to the PoA of $\mathcal{G}_{\mathcal{P}}$ such that $\mathcal{P} = \{(W_r, F_r^{\mathrm{mc}})\}$, where $F_r^{\mathrm{mc}}(k) = W_r(k) - W_r(k-1)$ with $W_r(0) = 0$. In other words, we can consider the PoA under the marginal contribution utility (taking the form $U_i^{\mathrm{mc}}(a) = W(a) - W(\emptyset_i, a_{-i}) = \sum_{r \in a_i} F_r^{\mathrm{mc}}(|a|_r)$) in place of the PoA under the identical interest utility, as these two values coincide. To see this note that the two utilities have the same underlying set of Nash equilibria since $W(a) - W(a_i', a_{-i}) \geq 0 \iff U^{\mathrm{mc}}(a) - U^{\mathrm{mc}}(a_i', a_{-i}) \geq 0$. Here, $W(\emptyset_i, a_{-i})$ denotes the system welfare if agent i were to be removed from the game while the remaining agents selected their action in a.

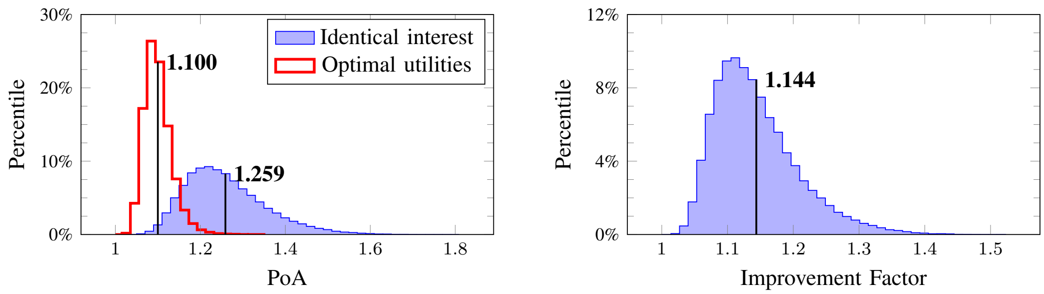


Fig. 3. *PoA for identical interest and optimal utilities in distributed welfare games. Left:* Empirical distribution of the PoA with optimal utilities vs. identical interest design. *Right:* Improvement factor when using the optimal design in place of the identical interest design. The mean of each distribution is indicated by a bisecting solid black line. Observe that the optimal utilities offer significant improvement over the identical interest utilities, reducing the PoA by a factor of approximately 1.144 on average, and by more than a factor of 1.4 in some cases. The histograms' values were obtained using the linear programs in Theorem 3 for nondecreasing, concave resource welfare functions.

in congestion games and utility design in distributed welfare games. In this regard, the list of illustrative example provided in Section VI is certainly non-exhaustive.

Overall, we feel the proposed approach has significant potential, especially since it can be used as a "black box" to compute and optimize the exact PoA in many settings of interest. The linear programs derived here can be used, for example, as "computational companions" to support the analytical study of the PoA, e.g, by providing analytical performance guarantees in practical applications. For this reason, we include a software package that implements our techniques in the hope that it spurs new research to come.[12]

Future directions: We observe that the PoA represents but one of many metrics for measuring an algorithm's performance. Nevertheless, we believe that the techniques introduced here can be suitably extended to analyze different metrics (e.g., the Price of Stability) and to understand whether optimizing for the PoA has any unintended consequences on them. This is especially relevant in light of the results in Section VI.C; the identical interest utilities optimize the best-case equilibrium efficiency of any game, but – in the setting considered – do so at a cost to the worst-case equilibrium efficiency.

In Section VI.A, we observe the empirical distribution of the ratios between PoA, RPoA and GPoA in cost-minimization games. A surprising takeaway was that the maximum realized ratio between RPoA and PoA, GPoA and PoA, and RPoA and GPoA for a randomly generated game was up to 4 orders of magnitude greater than the same ratio for the vast majority (i.e., 95%) of realized games. In fact, the PoA, GPoA and RPoA were practically equal (i.e., within a factor of 1.001) for nearly one in every eight of the randomly generated games, and the GPoA and RPoA were practically equal for nearly half the games. An interesting direction is to investigate equilibrium efficiency where games are extracted according to a probability distribution, as opposed to adversarially selected.

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Appendix I
Proof of Proposition 1

For the proof of statement (i), observe that, for all $a^{\rm ne} \in {\rm NE}(G)$ and $a^{\rm opt} \in \mathcal{A}$,

$$\begin{aligned} C(a^{\rm ne}) &\leq \sum_{i=1}^{n} \Big[J_i(a_i^{\rm opt}, a_{-i}^{\rm ne}) - J_i(a^{\rm ne}) \Big] + C(a^{\rm ne}) \\ &\leq \lambda\, C(a^{\rm opt}) + \mu\, C(a^{\rm ne}). \end{aligned} \tag{20}$$

The inequalities hold by (2) and (7), respectively. Rearranging gives the result.

The remainder of the proof focuses on statement (ii). Since the condition $\sum_{i=1}^{n} J_i(a) \geq C(a)$ for all $a \in \mathcal{A}$ implies that any pair of (λ, μ) satisfying (5) necessarily satisfies (7), we note that the GPoA is less than or equal to the RPoA, i.e., ${\rm RPoA}(G) \geq {\rm GPoA}(G) \geq {\rm PoA}(G)$.

Note that for any game $G = (N, \mathcal{A}, \mathcal{C}, \mathcal{J})$ with $\sum_{i=1}^{n} J_i(a) > C(a)$ for all $a \in \mathcal{A}$ there must exist a uniform scaling factor $0 < \gamma < 1$ such that $\sum_{i=1}^{n} \gamma J_i(a) \geq C(a)$, but for which the PoA remains the same, i.e., for $G' = (N, \mathcal{A}, \mathcal{C}, \mathcal{J}')$ where $\mathcal{J}' = \{\gamma J_1, \dots, \gamma J_n\}$, it holds that ${\rm PoA}(G') = {\rm PoA}(G)$. The PoA remains the same despite the rescaling, because the inequalities in (2) are unaffected by a positive scaling factor (i.e., ${\rm NE}(G) = {\rm NE}(G')$), and because the optimal cost remains unchanged since the scaling does not impact the system cost. Further, one can verify from (5) that ${\rm RPoA}(G) > {\rm RPoA}(G')$, and thus ${\rm RPoA}(G) > {\rm RPoA}(G') \geq {\rm PoA}(G') = {\rm PoA}(G)$. Finally, we know that ${\rm GPoA}(G')$ is less than or equal to ${\rm RPoA}(G')$ and can verify from (7) that ${\rm GPoA}(G) = {\rm GPoA}(G')$. Thus, ${\rm RPoA}(G) > {\rm RPoA}(G') \geq {\rm GPoA}(G') = {\rm GPoA}(G) \geq {\rm PoA}(G)$. ■

Appendix II
Proofs of Observations #1–#3

Proof of Observation #1: Given a cost-minimization game $G = (N, \mathcal{A}, C, \{J_i\}_{i\in N})$, we denote $\tilde{G} = (N, \mathcal{A}, C, \{\gamma J_i + \delta_i\}_{i\in N})$ for given parameters $\gamma > 0$ and $(\delta_1, \dots, \delta_n) \in \mathbb{R}^n$. We first show that the PoA and GPoA are shift- and scale-invariant. Observe that if pure Nash equilibrium condition in (2) holds for an allocation a in the game G, it must identically hold for the same allocation in $\tilde{G}$, since

$$J_i(a) \geq J_i(a_i', a_{-i}) \iff \gamma J_i(a) + \delta_i \geq \gamma J_i(a_i', a_{-i}) + \delta_i.$$

Thus, G and $\tilde{G}$ have the same set of pure Nash equilibria, i.e., $\mathrm{NE}(G) = \mathrm{NE}(\tilde{G})$. Since the games G and $\tilde{G}$ also have the same optimum a^{opt} (both games have the same system cost function C, and allocation set $\mathcal{A}$), it must also be that $\mathrm{PoA}(G) = \mathrm{PoA}(\tilde{G})$. Similarly, since

$$\begin{aligned}
&\sum_{i=1}^n [J_i(a_i', a_{-i}) - J_i(a)] + C(a) \leq \lambda C(a') + \mu C(a) \\
\iff &\sum_{i=1}^n [\gamma J_i(a_i', a_{-i}) + \delta_i - \gamma J_i(a) - \delta_i] + \gamma C(a) \\
&\leq \gamma\lambda C(a') + \gamma\mu C(a),
\end{aligned}$$

then, if $\mathrm{GPoA}(G) = \lambda/(1-\mu)$, it must be that $\mathrm{GPoA}(\tilde{G}) = \gamma\lambda/(\gamma - \gamma\mu) = \mathrm{GPoA}(G)$.[13]

We now show that RPoA is not shift- and scale-invariant. Firstly, note that if γ and $\delta_1, \dots, \delta_n$ violate $\sum_{i=1}^n [\gamma J_i(a) + \delta_i] \geq C(a)$ for any allocation $a \in \mathcal{A}$, then the game $\tilde{G}$ violates Condition (i) in the definition of (λ,μ)-smoothness (see Section II.B), and the RPoA is undefined. Otherwise, we are interested in the values $\lambda > 0$ and $\mu < 1$ that minimize the expression $\lambda/(1-\mu)$ while satisfying (5), i.e.,

$$\sum_{i=1}^n [\gamma J_i(a_i', a_{-i}) + \delta_i] \leq \lambda C(a') + \mu C(a),$$

for all $a, a' \in \mathcal{A}$. Note that $\mathrm{RPoA}(G)$ coincides with parameter values $\gamma = 1$ and $\delta_i = 0$, $i = 1, \dots, n$. Note that the condition above becomes strictly tighter for all $a, a' \in \mathcal{A}$ if we set $\gamma > 1$ or $\delta_i = \delta > 0$, $i = 1, \dots, n$. In this case, $\mathrm{RPoA}(\tilde{G}) > \mathrm{RPoA}(G)$. Furthermore, if we make the condition strictly looser for all $a, a' \in \mathcal{A}$ by setting $\gamma < 1$ or $\delta_i = \delta < 0$, $i = 1, \dots, n$, then $\mathrm{RPoA}(\tilde{G}) < \mathrm{RPoA}(G)$. ■

Proof of Observation #2: Recall from the proof of Observation #1, above, that reducing the quantity on the left-hand side of (5) – $\sum_{i=1}^n J_i(a_i', a_{-i})$ – corresponds with an improved (i.e., reduced) RPoA. This coincides with reducing the value $J_i(a)$ for all $a \in \mathcal{A}$. Since Condition (i) in the definition of (λ, μ)-smoothness requires that $\sum_{i=1}^n J_i(a) \geq C(a)$ be satisfied for all $a \in \mathcal{A}$, the lowest admissible values of agent cost satisfy $\sum_{i=1}^n J_i(a) = C(a)$ for all $a \in \mathcal{A}$.

On the contrary, in previous studies of polynomial congestion games [19], [29], [55], the authors identify agent cost functions that coincide with strictly lower PoA than budget-balanced agent cost functions. Furthermore, Theorem 1 and Corollary 1 state that GPoA tightly characterizes PoA in such scenarios. Thus, PoA and GPoA are not optimized by budget-balanced agent cost function in some settings. ■

[13] See proof of Proposition 1 in Section I for further details.

Proof of Observation #3: The proof that GPoA also bounds the efficiency of a cost minimization game's coarse correlated equilibria follows closely to the proof of Proposition 1 in Appendix I. Observe that, for any coarse correlated equilibrium $\sigma \in \Delta(\mathcal{A})$,

$$\begin{aligned}
&\mathbb{E}_{a\sim\sigma}[C(a)] \\
&\leq \mathbb{E}_{a\sim\sigma}\left[\sum_{i=1}^n \left[J_i(a_i^{\mathrm{opt}}, a_{-i}) - J_i(a)\right]\right] + \mathbb{E}_{a\sim\sigma}[C(a)] \\
&= \mathbb{E}_{a\sim\sigma}\left[\sum_{i=1}^n \left[J_i(a_i^{\mathrm{opt}}, a_{-i}) - J_i(a)\right] + C(a)\right] \\
&\leq \mathbb{E}_{a\sim\sigma}\left[\lambda\, C(a^{\mathrm{opt}}) + \mu\, C(a)\right] \\
&= \lambda\, C(a^{\mathrm{opt}}) + \mu\, \mathbb{E}_{a\sim\sigma}[C(a)].
\end{aligned} \tag{21}$$

Both equalities hold by the linearity of the expectation operator, and the inequalities hold by (9) and (7), respectively. Rearranging gives the desired result. ■

Appendix III
Joint Proof of Theorem 1 and Corollary 1

For any given set $\mathcal{P}$, the proof shows that $\mathrm{PoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})}) = \mathrm{GPoA}(\mathcal{G}_{\Delta(\mathcal{P})})$. The proof is shown in four steps, as summarized in the outline above.

– *Step 1* : We first show that any game G in the family $\mathcal{G}_{\Delta(\mathcal{P})}$ is (strategically) equivalent to a game $\hat{G}$ in $\mathcal{G}_{\mathcal{P}}$ with potentially a much larger resource set, and, thus, that $\mathrm{PoA}(\mathcal{G}_{\mathcal{P}}) = \mathrm{PoA}(\mathcal{G}_{\Delta(\mathcal{P})})$. For example, consider a game $G \in \mathcal{G}_{\Delta(\mathcal{P})}$ with rational coefficients $\alpha_r^1, \dots, \alpha_r^m \geq 0$, $r \in \mathcal{R}$. Let LCD denote the lowest common denominator across all coefficients $\alpha_r^1, \dots, \alpha_r^m \geq 0$, $r \in \mathcal{R}$, and observe that $\mathrm{LCD} \cdot \alpha_r^j$ is an integer for each $j \in \{1, \dots, m\}$ and $r \in \mathcal{R}$. Now consider a game $\hat{G} \in \mathcal{G}_{\mathcal{P}}$ with the same number of agents as G. For each resource r in game G, define $\hat{R}_j(r)$, $j = 1, \dots, m$, as a distinct set of $\mathrm{LCD} \cdot \alpha_r^j$ resources each with function pair $\{C^j, F^j\}$. We define the resource set of the game $\hat{G}$ as the union of all the distinct sets $\hat{R}_j(r)$ over $j = 1, \dots, m$, and over all the resource r in game G. Finally, each agent i in the game $\hat{G}$ has the same number of actions as its corresponding agent i in game G. When agent i plays the action a_i in game G, the corresponding agent i in $\hat{G}$ plays the action $\hat{a}_i = \cup_{r\in a_i} \cup_{j=1}^m \hat{R}_j(r)$. In other words, the action $\hat{a}_i$ is the union over all the resources replacing the resources in a_i. It follows from the construction that the system cost function and agents' cost functions in $\hat{G}$ amount to a uniform rescaling by a factor LCD of the system cost function and agents' cost functions in G, and, thus, the PoA remains unchanged (i.e., $\mathrm{PoA}(G) = \mathrm{PoA}(\hat{G})$). In the case where one or more of the coefficients $\alpha_r^1, \dots, \alpha_r^m \geq 0$, $r \in \mathcal{R}$, are irrational, we can approximate these from above or below (as required) to arbitrary precision using rational numbers, and then use the above approach to obtain an equivalent game in $\mathcal{G}_{\mathcal{P}}$. The above reasoning is further elaborated in [17], [56].

– *Step 2* : For a given game $G \in \mathcal{G}_{\Delta(\mathcal{P})}$, our game parameterization is defined as follows for allocations $a, a' \in \mathcal{A}$: For every resource $r \in \mathcal{R}$, we define integers $x_r, y_r, z_r \geq 0$ where $x_r = |a|_r$ is the number of agents that select r in a, $y_r = |a'|_r$ is the number of agents that select r in a' and $z_r = |\{i \in N \text{ s.t. } r \in a_i\} \cap \{i \in N \text{ s.t. } r \in a'_i\}|$ is the number of agents that select r in both a and a'. Note that $1 \leq x_r + y_r - z_r \leq n$ and $z_r \leq \min\{x_r, y_r\}$ must hold for all $r \in \mathcal{R}$. For all $x, y, z \geq 0$ such that $1 \leq x + y - z \leq n$ and $z \leq \min\{x, y\}$, and all $j = 1, \ldots, m$, we define the parameters

$$\theta(x, y, z, j) = \sum_{r \in \mathcal{R}(x,y,z)} \alpha_r^j, \tag{22}$$

where $\mathcal{R}(x, y, z) = \{r \in \mathcal{R} \text{ s.t. } (x_r, y_r, z_r) = (x, y, z)\}$, and $\alpha_r^j \geq 0$, $j = 1, \ldots, m$, are the coefficients in the representation of the resource-cost, agent-cost function pair $\{C_r, F_r\}$. Although the parameterization into values $\theta(x, y, z, j) \geq 0$ is of size $\mathcal{O}(mn^3)$, we show in Step 2 that only $\mathcal{O}(mn^2)$ parameters are needed in the computation of the PoA.

– *Step 3* : For any generalized congestion game $G \in \mathcal{G}_{\Delta(\mathcal{P})}$, we denote an optimal allocation as a^{opt}, and a Nash equilibrium as a^{ne}, i.e. $a^{\mathrm{ne}} \in \mathrm{NE}(G)$ such that $\mathrm{PoA}(G) \geq C(a^{\mathrm{ne}})/C(a^{\mathrm{opt}})$. We observe that using the above definitions of (x_r, y_r, z_r) for $a = a^{\mathrm{ne}}$ and $a' = a^{\mathrm{opt}}$, it follows that

$$\sum_{i=1}^{n} J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) = \sum_{r \in \mathcal{R}} \Big[(y_r - z_r) F_r(x_r + 1) + z_r F_r(x_r) \Big].$$

Informally, if an agent $i \in N$ selects a given resource $r \in \mathcal{R}$ in both a_i^{ne} and a_i^{opt}, then by deviating from a_i^{ne} to a_i^{opt}, the agent does not add to the load on r, i.e., $|a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}|_r = |a^{\mathrm{ne}}|_r = x_r$. However, if $r \in a_i^{\mathrm{opt}}$ and $r \notin a_i^{\mathrm{ne}}$, then $|a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}|_r = |a^{\mathrm{ne}}|_r + 1 = x_r + 1$.

Recall that for all $r \in \mathcal{R}$, it must hold that $z_r \leq \min\{x_r, y_r\}$, and $1 \leq x_r + y_r - z_r \leq n$. We define the set of triplets $\mathcal{I}(n) \subseteq \{0, 1, \ldots, n\}^3$ as

$$\mathcal{I}(n) := \{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x + y - z \leq n \text{ and } z \leq \min\{x, y\}\}, \tag{23}$$

and $\gamma(\mathcal{G}_{\Delta(\mathcal{P})})$ as the value of the following fractional program:

$$\begin{aligned} \gamma(\mathcal{G}_{\Delta(\mathcal{P})}) := \inf_{\lambda > 0, \mu < 1} \quad & \frac{\lambda}{1 - \mu} \quad \text{subject to:} \\ (z - x) F^j(x) + (y - z) F^j(x+1) + C^j(x) \leq \lambda C^j(y) + \mu C^j(x), \\ \forall j = 1, \ldots, m, \quad \forall (x, y, z) \in \mathcal{I}(n). \end{aligned} \tag{24}$$

Observe that, by (22), the condition in (7) can be rewritten for a given generalized congestion game as

$$\begin{aligned} &\sum_{\mathcal{I}(n)} \sum_{j=1}^{m} \Big[(x - z) F^j(x) - (y - z) F^j(x+1) + C^j(x) \Big] \theta(x, y, z, j) \\ &\leq \sum_{\mathcal{I}(n)} \sum_{j=1}^{m} \Big[\lambda C^j(y) + \mu C^j(x) \Big] \theta(x, y, z, j) \end{aligned}$$

It must then hold that for any pair (λ, μ) in the feasible set of the fractional program in (24), all games $G \in \mathcal{G}_{\Delta(\mathcal{P})}$ are (λ, μ)-generalized smooth, i.e., $\gamma(\mathcal{G}_{\Delta(\mathcal{P})}) \geq \mathrm{GPoA}(\mathcal{G}_{\Delta(\mathcal{P})})$. This is because the generalized smoothness condition can be expressed as a weighted sum with positive coefficients over a subset of the constraints in (24).

To conclude Step 2 of the proof, we show that it is sufficient to define $\gamma(\mathcal{G}_{\Delta(\mathcal{P})})$ in (24) over the reduced set of constraints corresponding to $j \in \{1, \ldots, m\}$ and triplets in $\mathcal{I}_{\mathcal{R}}(n) \subseteq \mathcal{I}(n)$, where $\mathcal{I}(n)$ is defined as in (23) and

$$\begin{aligned} \mathcal{I}_{\mathcal{R}}(n) := &\{(x, y, z) \in \mathcal{I}(n) \text{ s.t. } x + y - z = n\} \\ &\cup \{(x, y, z) \in \mathcal{I}(n) \text{ s.t. } (x - z)(y - z)z = 0\}. \end{aligned} \tag{25}$$

For each $j \in \{1, \ldots, m\}$ and any $(x, y, z) \in \mathcal{I}(n)$, observe that the constraint in (24) is equivalent to $yF^j(x + 1) - xF^j(x) + z[F^j(x) - F^j(x + 1)] \leq \lambda C^j(y) + (\mu - 1)C^j(x)$. If $F^j(x + 1) \geq F^j(x)$, the strictest condition on λ and μ corresponds to the lowest value of z. Thus, $z = \max\{0, x+y-n\}$, and either $(x-z)(y-z)z = 0$ or $x+y-z = n$. Otherwise, if $F^j(x + 1) < F^j(x)$, then the largest value of z is strictest, i.e., $z = \min\{x, y\}$ which satisfies $(x - z)(y - z)z = 0$.

– *Step 4* : In order to derive the game instances with PoA matching $\gamma(\mathcal{G}_{\Delta(\mathcal{P})})$, it is convenient to perform the following change of variables: $\nu(\lambda, \mu) := 1/\lambda$ and $\rho(\lambda, \mu) := (1-\mu)/\lambda$. For ease of notation, we will refer to the new variables simply as ν and ρ, respectively, i.e., $\nu = \nu(\lambda, \mu)$ and $\rho = \rho(\lambda, \mu)$. For each $j \in \{1, \ldots, m\}$ and each $(x, y, z) \in \mathcal{I}_{\mathcal{R}}(n)$, it is straightforward to verify that the constraints in (24) can be rewritten in terms of ν and ρ as

$$C^j(y) - \rho C^j(x) + \nu[(x - z)F^j(x) - (y - z)F^j(x + 1)] \geq 0.$$

Thus, the value $\gamma(\mathcal{G}_{\Delta(\mathcal{P})})$ must be equal to $1/\rho^{\mathrm{opt}}$, where ρ^{opt} is the value of the following linear program:

$$\begin{aligned} \rho^{\mathrm{opt}} = \underset{\nu \in \mathbb{R}_{\geq 0}, \rho \in \mathbb{R}}{\text{maximize}} \quad & \rho \quad \text{subject to:} \\ C^j(y) - \rho C^j(x) + \nu[(x - z)F^j(x) - (y - z)F^j(x+1)] \geq 0, \\ \forall j = 1, \ldots, m, \quad \forall (x, y, z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned} \tag{26}$$

Note that while $\gamma(\mathcal{G}_{\Delta(\mathcal{P})})$ is the infimum of a fractional program (see, e.g., (24)), the value ρ^{opt} can be computed as a maximum because the feasible set is bounded and closed. Firstly, since $\gamma(\mathcal{G}_{\Delta(\mathcal{P})})$ is an upper bound on the PoA, its inverse (i.e., ρ) must be in the bounded and closed interval $[0, 1]$. Additionally, one can verify that ν is not only bounded from below by 0, but also from above by the quantity

$$\begin{aligned} \bar{\nu} := \min_{j \in \{1, \ldots, m\}} \underset{(x,y,z) \in \mathcal{I}_{\mathcal{R}}(n)}{\text{minimize}} & \frac{C^j(y)}{(y - z)F^j(x+1) - (x - z)F^j(x)} \\ \text{s.t. } (x - z)F^j(x) - (y - z)F^j(x + 1) < 0, & C^j(x) = 0, \end{aligned} \tag{27}$$

which comes from the constraints in (26) corresponding to triplets $(x, y, z) \in \mathcal{I}_{\mathcal{R}}(n)$ such that $C^j(x) = 0$ and $(x - z)F^j(x) - (y - z)F^j(x + 1) < 0$. Such a value must exist, as we define $C^j(0) = 0$. One can verify that any $j \in \{1, \ldots, m\}$ and $(x, y, z) \in \mathcal{I}_{\mathcal{R}}(n)$ such that $C^j(x) = 0$ and $(x-z)F^j(x) - (y-z)F^j(x+1) \geq 0$ correspond to constraints that are satisfied trivially in (26) since $\nu \geq 0$, by definition, and $C^j(y) \geq 0$ for all $y = 0, 1, \ldots, n$, by assumption.

We denote with $\mathcal{H}^j(x, y, z)$ the halfplane of (ν, ρ) values

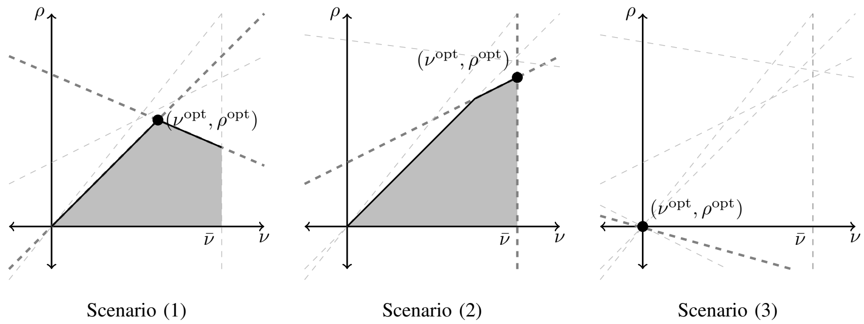

Fig. 4. *The three different scenarios in which optimal solutions* $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ *to* (26) *can exist.* We illustrate the reasoning behind each of the three scenarios for optimal solutions $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ to the linear program in (26). Since the objective of (26) is to maximize $\boldsymbol{\rho}$, the optimal values will be at the (upper) boundary of the feasible set, illustrated with a solid, bolded line in each of the examples above. Additionally, the optimal solution $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ is marked by a solid, black dot in the illustrations above. In Scenario (1), on the left, $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ lie on the intersection of a boundary line with positive slope and a boundary line with nonpositive slope. In Scenario (2), centre, $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ lie on the intersection of a boundary line with positive slope at $\boldsymbol{\nu = \bar{\nu}}$, which is defined in (27). In Scenario (3), on the right, there exists a halfplane boundary line with nonpositive slope and $\boldsymbol{\rho}$-intercept equal to zero, and so $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}}) = (\mathbf{0}, \mathbf{0})$. Using the parameters corresponding to the halfplanes on which the pair $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ lays, we can construct games $\boldsymbol{G \in \mathcal{G}}$ with $\mathbf{PoA}(\boldsymbol{G}) = \mathbf{1}/\boldsymbol{\rho^{\text{opt}}}$ in each of these scenarios.

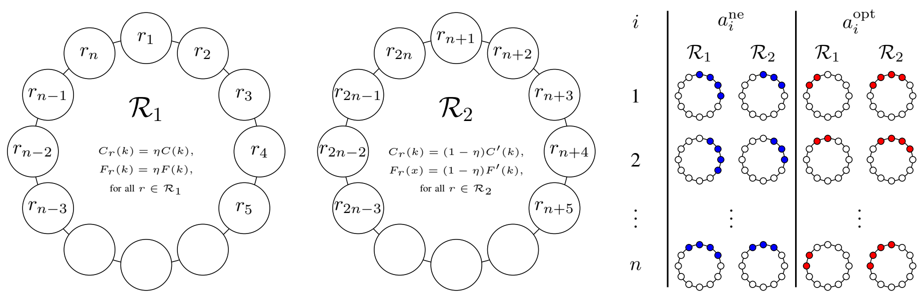

Fig. 5. *The game instance construction* $\boldsymbol{G}$ *consisting of* $\boldsymbol{n}$ *agents, and two disjoint cycles* $\boldsymbol{\mathcal{R}_1}$ *and* $\boldsymbol{\mathcal{R}_2}$*, as described in the proof of Corollary 1, Step 2 for Scenarios (1) and (2).* Consider the family of games $\boldsymbol{\mathcal{G}_{\Delta(\mathcal{P})}}$, where $\boldsymbol{n}$ is the maximum number of agents and $\boldsymbol{\mathcal{P}}$ is the set of basis functions pairs, and suppose that $(\boldsymbol{\nu^{\text{opt}}}, \boldsymbol{\rho^{\text{opt}}})$ satisfy the conditions of Scenarios (1) or (2). Further, suppose that the parameters for which (28) and (29) hold are $\boldsymbol{C, F, C', F' \in \mathcal{Z}}$, $(\boldsymbol{x, y, z}) = (\mathbf{4, 2, 0}), (\boldsymbol{x', y', z'}) = (\mathbf{3, 4, 2}) \in \boldsymbol{\mathcal{I}_{\mathcal{R}}(n)}$ and $\boldsymbol{\eta} \in [\mathbf{0, 1}]$. In the above figure, we illustrate the game $\boldsymbol{G \in \mathcal{G}_{\Delta(\mathcal{P})}}$ such that $\mathbf{PoA}(\boldsymbol{G}) = \mathbf{PoA}(\boldsymbol{G'}) = \mathbf{1}/\boldsymbol{\rho^{\text{opt}}}$ according to the reasoning for constructing game instances in Scenarios (1) and (2). Observe that each resource $\boldsymbol{r \in \mathcal{R}_1}$ has $\boldsymbol{C_r(k) = \eta C(k)}$, and $\boldsymbol{F_r(k) = \eta F(k)}$, whereas each resource $\boldsymbol{r \in \mathcal{R}_2}$ has $\boldsymbol{C_r(k) = (1-\eta)C'(k)}$, and $\boldsymbol{F_r(x) = (1-\eta)F'(k)}$, for all $\boldsymbol{k} \in \{\mathbf{1}, \ldots, \boldsymbol{n}\}$. Each agent $\boldsymbol{i \in N}$ has two actions $\boldsymbol{a_i^{\text{ne}}}$ and $\boldsymbol{a_i^{\text{opt}}}$, as defined in the table on the right. Observe that every resource in $\boldsymbol{\mathcal{R}_1}$ is selected by 4 agents in the allocation $\boldsymbol{a^{\text{ne}}} = (\boldsymbol{a_1^{\text{ne}}}, \ldots, \boldsymbol{a_n^{\text{ne}}})$, and 3 agents in $\boldsymbol{a^{\text{opt}}} = (\boldsymbol{a_1^{\text{opt}}}, \ldots, \boldsymbol{a_n^{\text{opt}}})$, where no agent $\boldsymbol{i \in N}$ has a common resource between its actions $\boldsymbol{a_i^{\text{ne}}}$ and $\boldsymbol{a_i^{\text{opt}}}$, i.e., $\boldsymbol{x_r = 4 = x}$, $\boldsymbol{y_r = 3 = y}$, and $\boldsymbol{z_r = 0 = z}$ for all $\boldsymbol{r \in \mathcal{R}_1}$. Similarly, $\boldsymbol{x_r = 3 = x'}$, $\boldsymbol{y_r = 4 = y'}$, and $\boldsymbol{z_r = 2 = z'}$, for each resource $\boldsymbol{r \in \mathcal{R}_2}$.

that satisfy the constraint corresponding to $j \in \{1, \ldots, m\}$ and $(x, y, z) \in \mathcal{I}_{\mathcal{R}}(n)$, i.e.,

$$\begin{aligned}\mathcal{H}^j(x,y,z) &:= \Big\{(\nu,\rho) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \text{ s.t.} \\ \rho &\leq \frac{C^j(y)}{C^j(x)} + \frac{1}{C^j(x)}\nu\left[(x-z)F^j(x) - (y-z)F^j(x+1)\right]\Big\}.\end{aligned}$$

The set of feasible (ν, ρ) is the intersection of these $m \times |\mathcal{I}_{\mathcal{R}}(n)|$ halfplanes. Since the objective is to maximize ρ, any solution $(\nu^{\text{opt}}, \rho^{\text{opt}})$ to the linear program in (26) must be on the (upper) boundary of the feasible set. We argue below that a solution $(\nu^{\text{opt}}, \rho^{\text{opt}})$ can only exist in one of the three following scenarios: (1) at the intersection of two halfplanes' boundaries, where one halfplane has boundary line with positive slope, and the other has boundary line with nonpositive slope; (2) on a halfplane boundary line with positive slope at $\nu^{\text{opt}} = \bar{\nu}$; or (3) at $(\nu^{\text{opt}}, \rho^{\text{opt}}) = (0, 0)$.

We denote with $\partial\mathcal{H}^j(x,y,z)$ the boundary line of the halfplane $\mathcal{H}^j(x,y,z)$, i.e., the set of $(\nu, \rho) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ such that the inequality in the definition of $\mathcal{H}^j(x,y,z)$ holds with equality. Observe that the boundary lines of halfplanes corresponding to the choice $y = z = 0$ have ρ-intercept equal to zero and slope $xF^j(x)/C^j(x)$. If $F^j(x) \leq 0$ for any $j \in \{1, \ldots, m\}$ and $x \in \{1, \ldots, n\}$, then an optimal pair (ν, ρ) is trivially at the origin, i.e., $(\nu^{\text{opt}}, \rho^{\text{opt}}) = (0, 0)$ (i.e., Scenario (3) above). Note that the ρ-intercept of any halfplane boundary cannot be below 0, as we only consider cost functions such that $C^j(k) \geq 0$ for all k and all j. Otherwise, the maximum value of ρ occurs at the intersection of a boundary line with positive slope and a boundary line with nonpositive slope (i.e., Scenario (1) above) or on a boundary line with positive slope at $\nu = \bar{\nu}$ (i.e., Scenario (2) above). We illustrate this reasoning in Figure 4.

Observe that for Scenarios (1) and (2), the pair $(\nu^{\text{opt}}, \rho^{\text{opt}})$ is at the intersection of two boundary lines, which we denote as $\partial\mathcal{H}^j(x,y,z)$ and $\partial\mathcal{H}^{j'}(x',y',z')$. The parameters $j, j' \in \{1, \ldots, m\}$ and $(x,y,z), (x',y',z') \in \mathcal{I}_{\mathcal{R}}(n)$ satisfy

$$\rho^{\mathrm{opt}}C^j(x) - C^j(y) = \nu^{\mathrm{opt}}[(x{-}z)F^j(x) - (y{-}z)F^j(x{+}1)],$$
$$\rho^{\mathrm{opt}}C^{j'}(x'){-}C^{j'}(y') = \nu^{\mathrm{opt}}[(x'{-}z')F^{j'}(x'){-}(y'{-}z')F^{j'}(x'{+}1)], \tag{28}$$

because $(\nu^{\mathrm{opt}}, \rho^{\mathrm{opt}})$ is on both boundary lines. Further, there must exist $\eta \in [0,1]$ such that

$$\begin{aligned} 0 =& \eta \left[(x{-}z)F^j(x) - (y{-}z)F^j(x{+}1)\right] \\ &+ (1{-}\eta)\left[(x'{-}z')F^{j'}(x') - (y'{-}z')F^{j'}(x'{+}1)\right]. \end{aligned} \tag{29}$$

(29) holds in Scenario (1) because one of the boundary lines has positive slope, i.e., $(x-z)F^j(x) - (y-z)F^j(x+1) > 0$, while the other has nonpositive slope, and in Scenario (3) because one boundary line has positive slope while the other is the vertical line $\nu = \bar{\nu}$ which corresponds to a particular choice of $j \in \{1,\dots,m\}$ and $(x,y,z) \in \mathcal{I}_{\mathcal{R}}(n)$ such that $(x-z)F^j(x) - (y-z)F^j(x+1) < 0$ by (27).

Next, for the parameters $j, j' \in \{1,\dots,m\}$, $(x,y,z),(x',y',z') \in \mathcal{I}_{\mathcal{R}}(n)$, and $\eta \in [0,1]$ obtained above, we construct a game instance $G \in \mathcal{G}_{\Delta(\mathcal{P})}$ such that $\mathrm{PoA}(G) = 1/\rho^{\mathrm{opt}}$. Let $\mathcal{R}_1 = \{r_1,\dots,r_n\}$ and $\mathcal{R}_2 = \{r_{n+1},\dots,r_{2n}\}$ denote two disjoint cycles of resources. Every resource $r \in \mathcal{R}_1$ has cost function $C_r(k) = \eta C^j(k)$, and agent-cost function $F_r(k) = \eta F^j(k)$ for all k. Meanwhile, every $r \in \mathcal{R}_2$ has cost function $C_r(k) = (1-\eta)C^{j'}(k)$, and cost generating function $F_r(k) = (1-\eta)F^{j'}(k)$ for all k. We define the agent set $N = \{1,\dots,n\}$, where each agent $i \in N$ has action set $\mathcal{A}_i = \{a_i^{\mathrm{ne}}, a_i^{\mathrm{opt}}\}$. In action a_i^{ne}, the agent i selects x consecutive resources in $\mathcal{R}_1$ starting with r_i, i.e. $\{r_i, r_{(i \bmod n)+1}, \dots, r_{((i+x-2) \bmod n)+1}\}$, and x' consecutive resources in $\mathcal{R}_2$ starting with resource r_{n+i}. In a_i^{opt}, agent i selects y consecutive resources in $\mathcal{R}_1$ ending with resource $r_{((i+z-2) \bmod n)+1}$, i.e. $\{r_{((i+z-y-1) \bmod n)+1}, \dots, r_{((i+z-2) \bmod n)+1}\}$, and y' consecutive resources in $\mathcal{R}_2$ ending with resource $r_{n+((i+z'-2) \bmod n)+1}$. We provide an illustration of this game construction in Figure 5. Observe that $a^{\mathrm{ne}} = (a_1^{\mathrm{ne}},\dots,a_n^{\mathrm{ne}})$ satisfies the conditions for a Nash equilibrium,

$$\begin{aligned} J_i(a^{\mathrm{ne}}) &= \eta x F^j(x) + (1-\eta)x' F^{j'}(x') \\ &= \eta[zF^j(x) + (y-z)F^j(x+1)] \\ &\quad + (1-\eta)[z'F^{j'}(x') + (y'-z')F^{j'}(x'+1)] \\ &= J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}), \end{aligned}$$

which holds by (29). Then, by the above equality and (28),

$$\begin{aligned} 0 &= \sum_{i=1}^n J_i(a_i^{\mathrm{opt}}, a_{-i}^{\mathrm{ne}}) - \sum_{i=1}^n J_i(a^{\mathrm{ne}}) \\ &= \frac{1}{\nu^{\mathrm{opt}}}\Big[n\cdot\eta\left[\rho^{\mathrm{opt}}C^j(x) - C^j(y)\right] \\ &\qquad + n\cdot(1-\eta)\left[\rho^{\mathrm{opt}}C^{j'}(x') + C^{j'}(y')\right]\Big] \\ &= \frac{1}{\nu^{\mathrm{opt}}}\left[\rho^{\mathrm{opt}}C(a^{\mathrm{ne}}) - C(a^{\mathrm{opt}})\right], \end{aligned}$$

where $a^{\mathrm{opt}} = (a^{\mathrm{opt}})_{i=1}^n$. Thus, $\mathrm{PoA}(G) = 1/\rho^{\mathrm{opt}}$. For Scenario (3), observe that $\rho^{\mathrm{opt}} = 0$, and so $1/\rho^{\mathrm{opt}}$ is unbounded. Recall that, in this scenario, there exist $j \in \{1,\dots,m\}$ and $x \in \{1,\dots,n\}$ such that $F^j(x) \le 0$. We use the basis function pair $\{C^j, F^j\}$ to construct a game G with unbounded PoA. Consider a game instance with x agents and resource set $\mathcal{R} = \{r_1, r_2\}$, where $x \in \{1,\dots,n\}$ is the value that minimizes the function $F(x)$, i.e., $F^j(x) = \min_{k\in\{1,\dots,n\}} F^j(k) \le 0$. Every agent $i \in \{1,\dots,x\}$ has action set $\mathcal{A}_i = \{\{r_1\},\{r_2\}\}$. The resource r_1 has cost function $C_r(k) = \eta C^j(k)$ and agent-cost function $F_r(k) = \eta F^j(k)$ for all k. Similarly, the resource r_2 has cost function $C_r(k) = (1-\eta)C^j(k)$ and agent-cost function $F_r(k) = (1-\eta)F(k)$. It is immediate to verify that, for η approaching 0 from above, the allocation in which all agents select r_1 is an equilibrium and the PoA is unbounded. ■

Appendix IV
Linear program in Corollary 1 adapted to RPoA

Following the same steps as in Appendix III but adapting them to the original smoothness notion and the RPoA, one obtains the following linear program for computing the RPoA of the family of generalized congestion games $\mathcal{G}_{\mathcal{P}}$ for given set of resource-cost, agent-cost function pairs $\mathcal{P} = \{\{C_1,F_1\},\dots,\{C_m,F_m\}\}$ and positive integer n:

$$\begin{aligned} \rho^{\mathrm{opt}} = &\underset{\nu\in\mathbb{R}_{\ge 0},\rho\in\mathbb{R}}{\text{maximize}} \quad \rho \quad \text{subject to:} \\ &C^j(y) - \rho C^j(x) + \nu[C^j(x) - (y{-}z)F^j(x{+}1) - zF^j(x)] \ge 0, \\ &\qquad \forall j = 1,\dots,m, \quad \forall (x,y,z) \in \mathcal{I}_{\mathcal{R}}(n), \end{aligned} \tag{30}$$

Then, $\mathrm{RPoA}(\mathcal{G}_{\mathcal{P}}) = 1/\rho^{\mathrm{opt}}$, and the optimal smoothness parameters (λ,μ) can be uniquely obtained by inverting the change of variables $\nu = 1/\lambda$ and $\rho = (1-\mu)/\lambda$ from Step 4 of Appendix III.

Appendix V
Proof of Theorem 2

For each $j \in \{1,\dots,m\}$, the function $F^{\mathrm{opt},j}$ maximizes $\rho^{\mathrm{opt},j}$ by the following reasoning: For each resource-cost function C^j, we wish to find the function $F^{\mathrm{opt},j}$ that maximizes ρ in (12). Finding such a function is equivalent to solving

$$\begin{aligned} (F^{\mathrm{opt},j}, \nu^{\mathrm{opt},j}, \rho^{\mathrm{opt},j}) \in &\underset{F\in\mathbb{R}^n,\nu\in\mathbb{R}_{\ge 0},\rho\in\mathbb{R}}{\arg\max} \quad \rho \quad \text{subject to:} \\ &C^j(y) - \rho C^j(x) + \nu[(x-z)F(x) - (y-z)F(x+1)] \ge 0, \\ &\qquad \forall (x,y,z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned}$$

It is important to note that an optimal function $F^{\mathrm{opt},j}$ must exist since the above program is feasible for $F^j(k) = 0$, $k = 1,\dots,n$, $\nu = 1$ and $\rho \le \min_{x,y} C^j(y)/C^j(x)$, and is bounded since any pair $\{C^j, F^{\mathrm{opt},j}\}$ generates a set of games $\mathcal{G}^j$ so $\rho^{\mathrm{opt},j} = 1/\mathrm{PoA}(\mathcal{G}^j) \in [0,1]$ must hold by Corollary 1.

To obtain a linear program, we combine the decision variables ν and F in $\tilde{F}(k) := \nu F(k)$ to get

$$\begin{aligned} (\tilde{F}^j_{\mathrm{opt}}, \tilde{\rho}^j_{\mathrm{opt}}) \in &\underset{F\in\mathbb{R}^n,\rho\in\mathbb{R}}{\arg\max} \quad \rho \quad \text{subject to:} \\ &C^j(y) - \rho C^j(x) + (x-z)F(x) - (y-z)F(x+1) \ge 0, \\ &\qquad \forall (x,y,z) \in \mathcal{I}_{\mathcal{R}}(n). \end{aligned}$$

Note that $\tilde{F}^j_{\rm opt} \in \mathbb{R}^n$ must be feasible as $\tilde{F}^{{\rm opt},j}(k) = \nu^{{\rm opt},j} F^{{\rm opt},j}(k)$, and we know that $F^{{\rm opt},j} \in \mathbb{R}^n$ exists. Further, $\tilde{\rho}^{{\rm opt},j} = \rho^{{\rm opt},j}$, as equilibrium conditions are invariant to scaling of F.

For the set of generalized congestion games $\mathcal{G}$ induced by n and basis function pairs $\{C^j, \tilde{F}^{{\rm opt},j}\}$, $j = 1, \ldots, m$, and the set of games $\mathcal{G}^j$ induced by n and the basis function pair $\{C^j, \tilde{F}^{{\rm opt},j}\}$, it holds that ${\rm PoA}(\mathcal{G}) \geq \max_{j\in\{1,\ldots,m\}} {\rm PoA}(\mathcal{G}^j)$. We conclude by proving that the converse also holds, i.e., ${\rm PoA}(\mathcal{G}) \leq \max_{j\in\{1,\ldots,m\}} {\rm PoA}(\mathcal{G}^j)$. Simply note that the values $(\nu, \rho) = (1, \rho^{\rm opt})$ are feasible in the linear program in (12) for the function pairs $\{C^j, \tilde{F}^{{\rm opt},j}\}$, $j = 1, \ldots, m$, where $\rho^{\rm opt} := \min_j \rho^{{\rm opt},j}$. This implies that ${\rm PoA}(\mathcal{G}) \leq 1/\rho^{\rm opt}$. Observing that $1/\rho^{\rm opt} = \max_{j\in\{1,\ldots,m\}} {\rm PoA}(\mathcal{G}^j)$ concludes the proof. ■

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