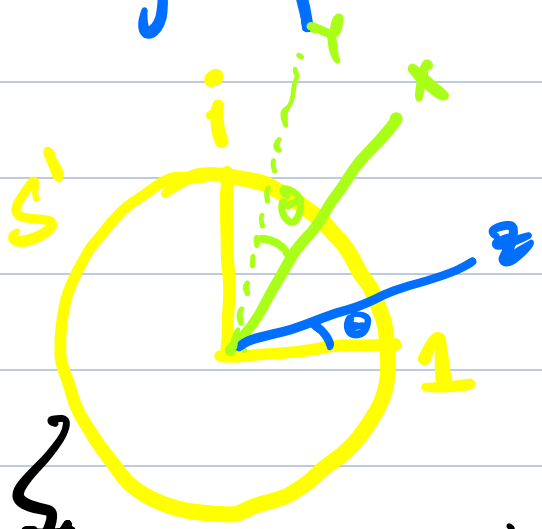
Lie Theory:

- Old & abstract theory involving sets & geometry.
- Useful for roboticists for performing a lot of geometric operations
- Can do derivatives, integration, etc etc,

Lie group: S^1 → The set of unit complex nos



Unit circle comprising of all complex nos z where $|z| = 1$

Say $x, y \in C$

Action: $y = z \cdot x$

→ x is rotated by θ when z operator is applied to it.

Constraint: $z^* \cdot z = 1$

Topology: unit circle S^1

Elements: $z = \cos\theta + i\sin\theta$

Inverse: $z^* = \cos\theta - i\sin\theta$

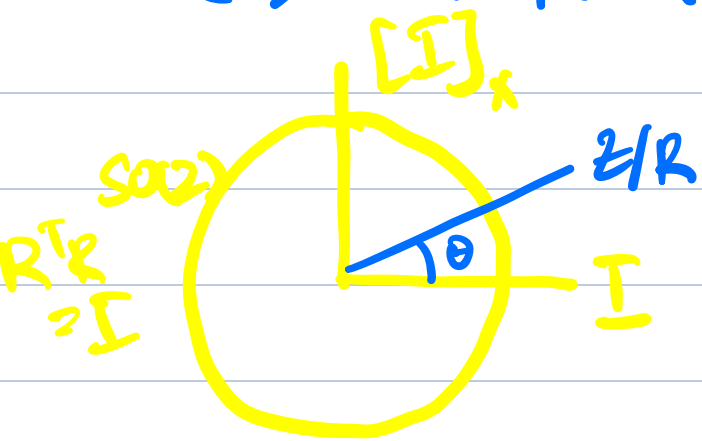
Composition: $z_1 \cdot z_2$

We can apply a composition of operators. Example.

$y = (z_1 \cdot z_2)x$: we rotate x by z_2 first then rotate it by z_1.

Let's extend that to another group, 2D Rot Mat

SO(2): 2D Rotation Matrices

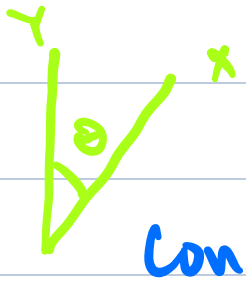


We draw the same circle as before, but replace complex nos with rotation matrices.

$1 \leftrightarrow I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$i \leftrightarrow [I]_x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$R = z = \cos\theta\, I + \sin\theta\, [I]_x$



Action: $y = R \cdot x$

→ x is rotated by θ when 'R' is applied

Constraint: $R^TR = I$

Topology: "Circle" of SO(2)

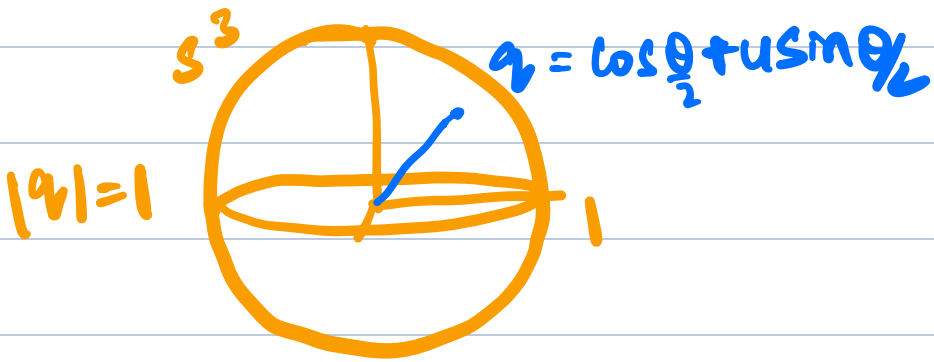
Elements: $R = \cos\theta\, I + \sin\theta\, [I]_x = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$

Inverse: R^T $(R^{-1} = R^T)$

Composition: $R_1(R_2 x)$: rotate using R_2, then R_1.

Note: x & y are vectors, they are on a different set when compared to Rs. or SO(2)

S^3: Unit Quaternions

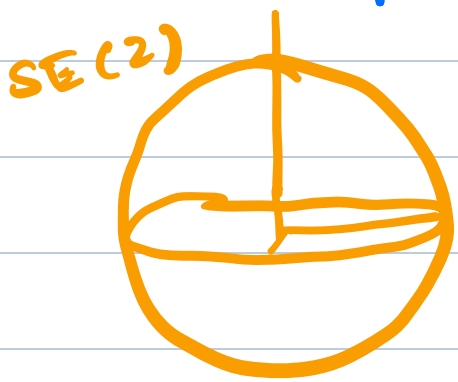


$q = \cos\frac{\theta}{2} + u\sin\theta/2$

A unit quaternion q is able to rotate a vector x about axis $\bar{u}$ by an angle θ.
qxq^* is the rotated vector.

Pose of a robot in a plane SE(2)

This is considered a special case called Special Euclidean group.



All elements of SE(2) are

$$\mathcal{X}(t) = \begin{bmatrix} R(t) & P(t) \\ 0 & 1 \end{bmatrix}$$

where $R(t)$ is a 2x2 rotation matrix
$P(t)$ is a 2x1 vector.

Both $R(t)$, $P(t)$ are functions over time and each element defined by $\mathcal{X}(t)$ is an element of SE(2)
Each element is a point in the weird space of SE(2).

Geometrically, each position and orientation makes sense in a 2D space. But each pose orien in 2D space is a point in SE(2). Its an abstraction
Abstracting in SE(2) gives us a lot of advantages.

Why Lie groups?

- Abstract: Can apply to many things and helps us think in a mathematical fashion.
- Rigorous: The axioms and theorems developed are rigorous, so no approximations
- Principled: There are axioms and theorems that allow us to do many things. Like apply operators, calc inverse, interpolate in b/w etc.,

act on other

Invert, compose, interpolate

Integrate, differentiate, uncertainty estimation

We can do things like: estimation, planning, control, learning etc..

Example

Group	Set	Operation
1-D vector	$\mathbb{R}_3$	+
Unit complex number / Circle	S_1	·
Rotmat 2D	$SO(2)$	·
Rigid motion 2D	$SE(2)$	·
Quaternion	S_2	·
Rotmat 3D	$SO(3)$	·
Rigid motion 3D	$SE(3)$	·

	size	dim	$x \in \mathcal{M}$	constraint	$\tau^\wedge \in \mathfrak{m}$	$\tau \in \mathbb{R}^m$	$\exp(\tau)$	comp	Action
	n	n	$v \in \mathbb{R}^n$	$v - v = 0$	$v \in \mathbb{R}^n$	$v \in \mathbb{R}^n$	$v = \exp(v)$	$v_1 + v_2$	$v + x$
	2	1	$z \in \mathbb{C}$	$z^* z = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$	$z = \exp(i\theta)$	$z_1 \cdot z_2$	$z \cdot x$
	4	1	R	$R^T R = I$	$[\theta]_\times \in so(2)$	$\theta \in \mathbb{R}$	$R = \exp([\theta]_\times)$	$R_1 \cdot R_2$	$R \cdot x$
.	9	3	$M = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$	$R^T R = I$	$\begin{bmatrix} [\theta]_\times & \rho \\ 0 & 0 \end{bmatrix} \in se(3)$	$\begin{bmatrix} \rho \\ \theta \end{bmatrix} \in \mathbb{R}^3$	$\exp\left(\begin{bmatrix} [\theta]_\times & \rho \\ 0 & 0 \end{bmatrix}\right)$	$M_1 M_2$	$M x = Rx + t$
.	4	3	$q \in \mathbb{H}$	$q^* q = 1$	$\theta/2 \in \mathbb{H}_p$	$\theta \in \mathbb{R}^3$	$q = \exp(u\theta/2)$	$q_1 q_2$	$q^* x q$
.	9	3	R	$R^T R = I$	$[\theta]_\times \in so(3)$	$\theta \in \mathbb{R}^3$	$R = \exp([\theta]_\times)$	$R_1 R_2$	$R \cdot x$
.	16	6	$M = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$	$R^T R = I$	$\begin{bmatrix} [\theta]_\times & \rho \\ 0 & 0 \end{bmatrix} \in se(3)$	$\begin{bmatrix} \rho \\ \theta \end{bmatrix} \in \mathbb{R}^6$	$\exp\left(\begin{bmatrix} [\theta]_\times & \rho \\ 0 & 0 \end{bmatrix}\right)$	$M_1 M_2$	$M x = Rx + t$

mples of Lie Groups used in Robotics.

Groups:

Defn: Groups are set of elements 'G' with an operator '$\cdot$' such that

If $x, y \in G$ → 4 axioms

Composition: $x \cdot y \in G$

Identity: Element 'E' such that $E \cdot x = x$ is in G

Inverse: $x^{-1} \cdot x = E = x \cdot x^{-1}$, $x^{-1} \in G$

Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Note: In many groups, '$\cdot$' is non-commutative

Lie Group:

Defn: A group that is also a smooth manifold.

Manifold is a surface of 'n' dimensions that is embedded in a space on 'n+1' dimensions

Smooth Manifold | Non-smooth Manifold

Lie group is a smooth manifold whose elements satisfy the 4 group axioms

Group Action:

Defn: A group can act on another set V such that it can transform V's elements.

Given $x, y \in G$, $v \in V$ action is such that:

① Identity is null action

$E \in G$ is identity, $E \cdot v = v$

② Compatibility with composition

$$(x \cdot y) \cdot v = x \cdot (y \cdot v)$$

If it satisfies above property only then it is defined as an action.

Note: Lie groups were formerly known as Continuous Transformation Groups

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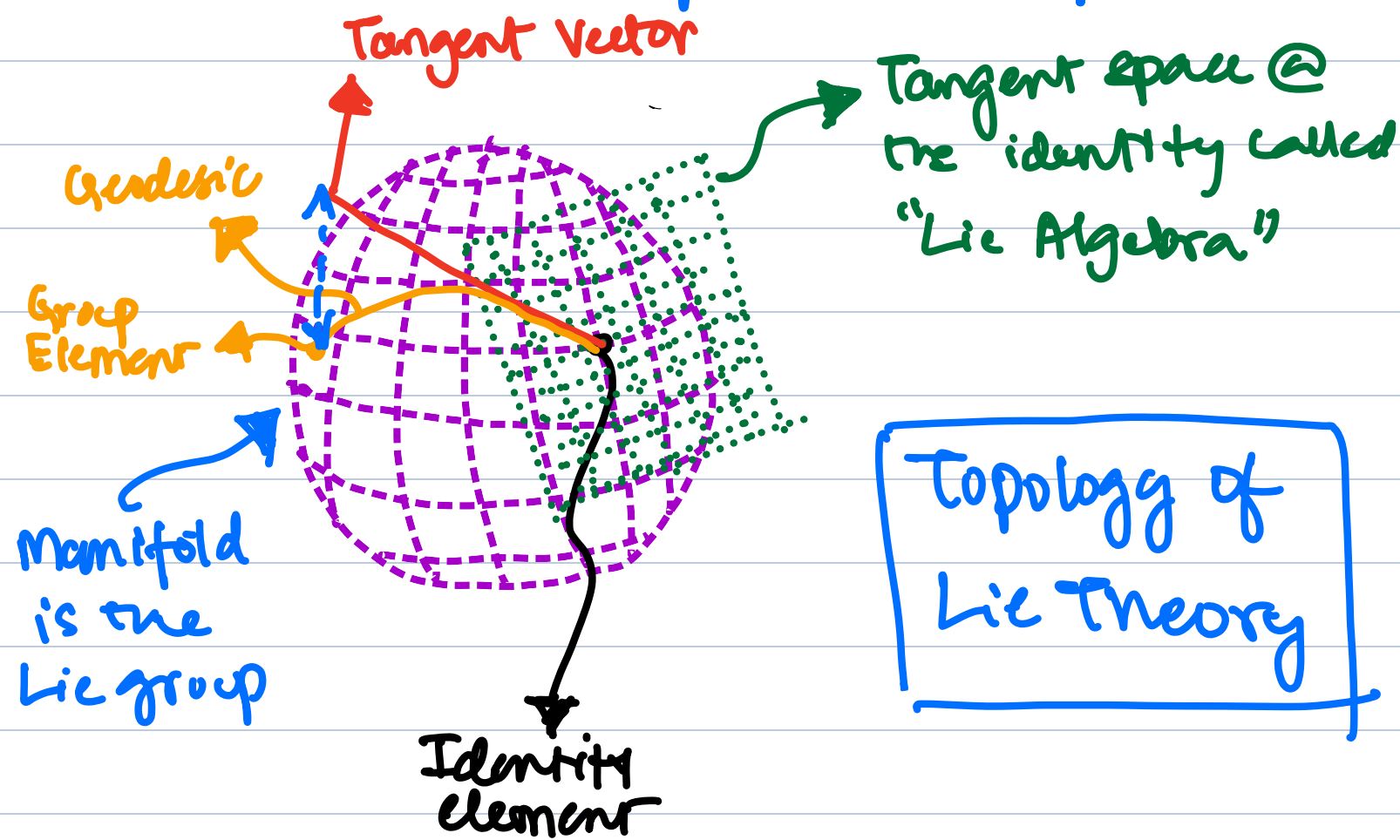
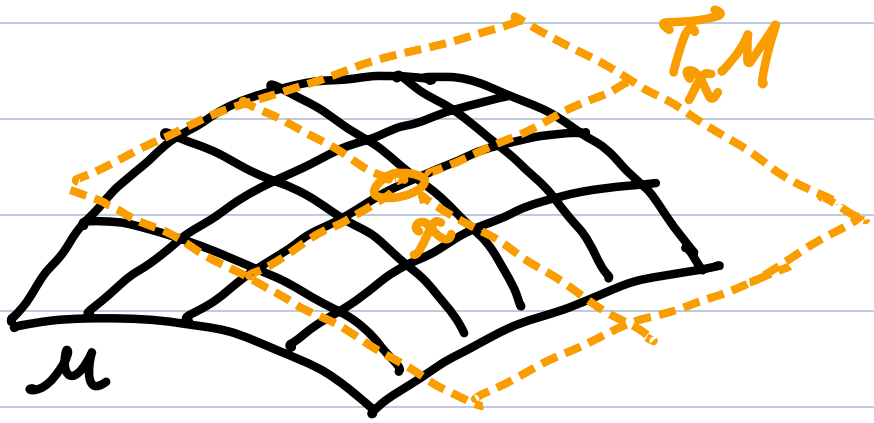


$\boxed{\text{Topology of Lie Theory}}$

- In the diagram above, the surface of the sphere (only surface) is the Lie Group.
- Then we define an identity element in the group 'I'.
- The tangent space at 'I' is called Lie Algebra
- The tangent line at the 'I' identity element is the Tangent Vector
- Assume the tangent vector is a string. If we curve the string along the surface, we get a Geodesic.
- For a point on the tangent vector/line, we can establish an equivalence to a point on the Geodesic/Group. This is called Exponential Mapping
- The opposite of that, establishing equivalence from a point in the group to a point on the tangent vector is Logarithmic Mapping.

Tangent Space & the Lie Algebra



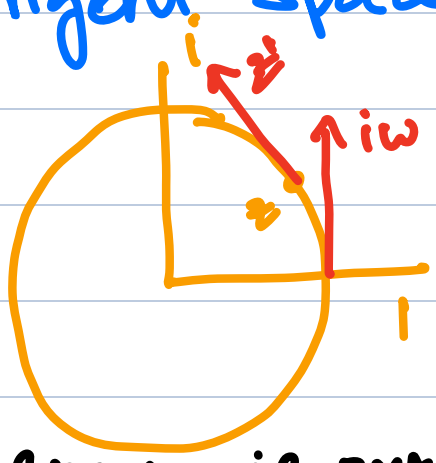
Say M is a manifold & x is a point on M
Tangent space at x is the tangential surface at x for the manifold M. It is denoted as

$T_x M$ → The manifold
→ The point

Properties of Tangent Space.

- For a given manifold a tangent space at any given point on the manifold is unique
- Tangent space is a vector space. We can do calculus in that space.
- # of dimensions of tangent space is equal to the number of degrees of freedom of the manifold.
- Tangent space @ the identity is called the Lie Algebra. If the tangent space is not @ identity, we don't call it the Lie Algebra.

Tangent space of S^1.



Consider velocity of a point $z(\dot{z})$

$z' \in$ Tangent space of the manifold.

@ Identity (1), the tangent space is purely imaginary numbers.

How are z, $\dot{z}$ & ω related?

Take the constraint $z^* z = 1$ & diff wrt. time

$$\frac{d}{dt}(z^* z) = 0 \Rightarrow \dot{z}^* z + z^* \dot{z} = 0$$

$$\Rightarrow z^* \dot{z} = -\dot{z}^* z$$

$$\Rightarrow z^* \dot{z} = -(z^* \dot{z})^* \quad (AB)^* = B^* A^*)$$

This is only possible when $(z^* \dot{z}) \in i\mathbb{R}$

$\therefore$ $z^* \dot{z}$ must be of the form $i\omega \in i\mathbb{R}$

Relationship:

$$z^* \dot{z} = i\omega$$

We represent the tangent space in 2 forms.

① Lie Algebra ② Cartesian.

Lie Algebra:

By definition, lie algebra is the tangent space @ the identity. Plug in identity in the relationship established previously and we get..

$$\dot{z} = i\omega$$

writing $\dot{z}$ as $\omega^\wedge$, we have

$$\omega^\wedge = i\omega \in i\mathbb{R}$$

Note: In general, elements of Lie algebra are represented with a $^\wedge$. Hence the above representation, $\omega^\wedge$. ω is just a real no. $\mathbb{R}$.

Cartesian:

Instead, one could just represent the tangent space for S^1 using just the number

$$\omega \in \mathbb{R}.$$

Given ω, we have the same information as $i\omega$ & vice versa.

Isomorphism:

Because there are 2 representations, we'll have a conversion mechanism!

Hat: $\omega^\wedge = i\omega$

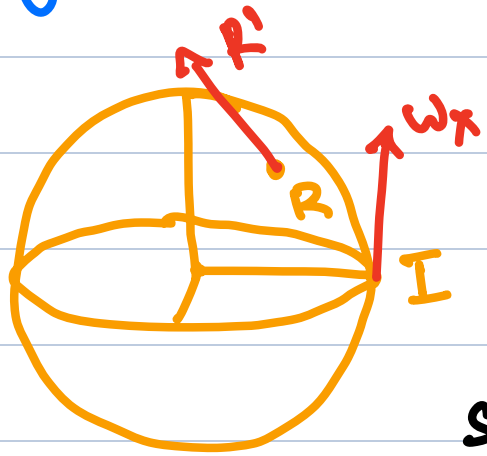
Vee: $\omega = (\omega^\wedge)^\vee = -i\omega^\wedge$

To get Lie Algebra $i\omega$, we apply $^\wedge$ on $\omega^\wedge$. Representation is $\omega^\wedge = i\omega$.

To get cartesian from $\omega^\wedge$, we apply $^\vee$ (reverse of $^\wedge$) on $\omega^\wedge$. $(\omega^\wedge)^\vee$ is represented as $-i\omega^\wedge$.

$(\omega)^\vee$ would be $-i\omega$.

Tangent Space of SO(3): [Rot mat in 3D]



Note: The example surface drawn is a 2D surface but for SO(3) it is a 3D surface. This is just a representⁿ.

We do the same as before:

$$\frac{d}{dt}(R^T R) = I \Rightarrow \dot{R}^T R + R^T \dot{R} = 0$$

$$\Rightarrow R^T \dot{R} = -\dot{R}^T R$$

$$\Rightarrow R^T \cdot \dot{R} = -(R^T \cdot \dot{R})^T \rightarrow ①$$

What is the solution for ①? What type of matrices have that property? It turns out..

All skew symmetric matrices!

Solution for ① is matrices of the form:

$$R^T \dot{R} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in so(3)$$

For the lie algebra; $R = I$;

$$\dot{R} = \omega_x \triangleq \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

We can always construct it back as:

$$w_x = w_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + w_y \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + w_z \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 basic matrices.

Cartesian: $\mathbb{R}^3$

We can instead choose the representation

$$w = [w_x \; w_y \; w_z]^T \in \mathbb{R}^3$$

$$\Rightarrow w = w_x [1\,0\,0]^T + w_y [0\,1\,0]^T + w_z [0\,0\,1]^T$$

Given cartesian or lie algebra, we have the same representation.

Isomorphism: $so(3) \leftrightarrow \mathbb{R}^3$

Hat: $w^\wedge = w_x$

vee: $w = (w_x)^\vee$

Exponential & Logarithmic Mapping:

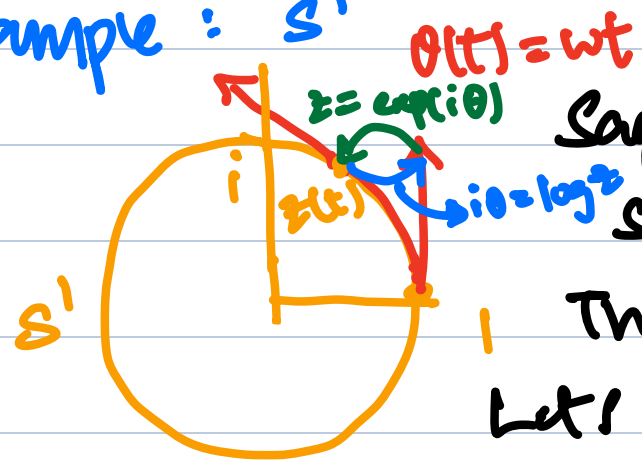
exp: mapping from tangential space to the manifold

wrap the tangent onto the geodesic

log: mapping from manifold to tangential

unwrap the geodesic onto the tangential.

Example: S^1



Say a point is rotating on S^1 (unit circle)

The velocity is $\dot{z} = dz/dt$

Let's say the angle $\theta(t) = \omega t$.

Previously we proved;

$$z^* \dot{z} = i\omega$$

$$\Rightarrow \quad \dot{z} = z i\omega$$

Solving for the diff equation, we have:

$$z(t) = z_0 \exp(i\omega t)$$

we assumed $\omega t = \theta$

$$\Rightarrow \quad z(t) = z_0 \exp(i\theta)$$

Expand taylor series for $\exp(i\theta)$

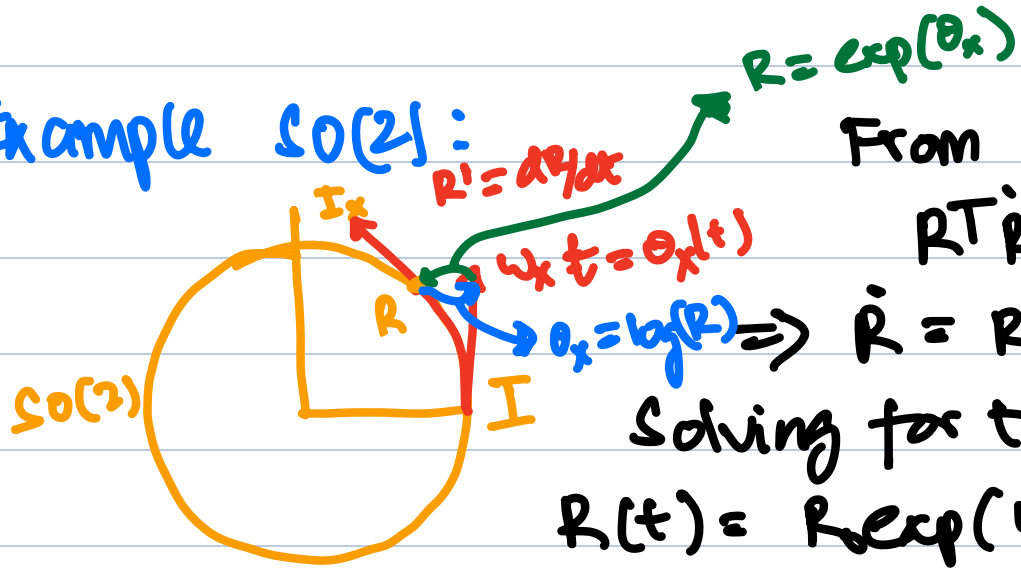
$$\exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2} + \cdots\right) + \left(i\theta - \frac{i\theta^3}{3!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\therefore \quad z(t) = z_0(\cos\theta(t) + i\sin\theta(t)) = z_0\exp(i\theta(t))$$

Example SO(2):



From previous proof

$$R^T\dot{R} = \omega_x$$

$$\Rightarrow \dot{R} = R\omega_x$$

Solving for the ODE:

$$R(t) = R_0\exp(\omega_x t)$$

If $R_0 = R(0)$; $R_0 = I$

$$\omega_x t = \theta_x = \theta \cdot I_x$$

Consider $\exp(\omega_x t) = \exp(\theta I_x)$

similar taylor series expansion;

$$\exp(\theta I_x) = (\theta I_x)^0 + (\theta I_x)^1 + \frac{(\theta I_x)^2}{2} + \frac{(\theta I_x)^3}{3!} + \cdots$$

$$(\theta I_x)^2 = -\theta^2 I, \quad (\theta I_x)^3 = -\theta^2 I_x$$

$$\Rightarrow \exp(\theta I_x) = I + \theta I_x - \frac{\theta^2 I}{2} - \frac{\theta^3 I_x}{3!} + \cdots$$

$$\Rightarrow \exp(\theta I_x) = \left(I - \frac{\theta^2 I}{2} + \cdots\right) + I_x\left(\theta - \frac{\theta^3 I}{3!} + \cdots\right)$$

$$= \cos\theta\, I + \sin\theta\, I_x = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore R(t) = R_0\exp(\theta I_x) = R_0(\cos\theta\, I + \sin\theta\, I_x)$$

Note: What if the constraint ODE does not have a closed form solution?

- The taylor series will make sure the terms get near 0 very fast. But @ the end of the day it is an approximation. It won't be the perfect solution we have, but it is something..

Notes on some notation.

In SO(2), we observed

$R = R_0 \exp(\theta_x)$. Lets drop the R_0 by assuming $R_0 = I$. We have

$R = \exp(\theta_x)$ which also is

$R = \exp(\theta I_x)$

This expression brings about a nice way of looking at it like $\exp(i\theta)$. So lets only use the scalar θ and denote

$$R = \exp(\theta_x) = \exp(\theta I_x) = \mathrm{Exp}(\theta)$$

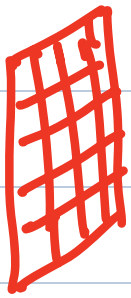
It basically is the same thing but Exp is more like a compound operator. Similarly for log.

$$\theta_x = \log R \Rightarrow \theta I_x = \log R = \theta = \mathrm{Log}(R)$$

Log is the compound operator.

Tangent Space

$T_z M$



$\tau^\wedge \in M$ Lie Algebra

$(\cdot)^\wedge$ ↑ ↓ $(\cdot)^\vee$

$\tau \in \mathbb{R}^m$ Cartesian

exp →, ← log

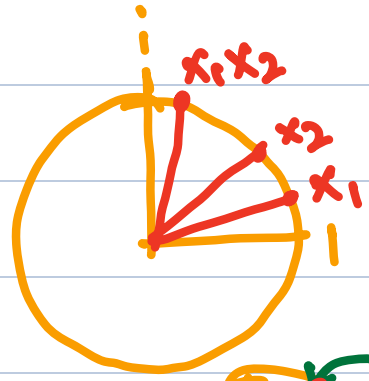
Exp →, ← Log

Manifold M

Plus and Minus Operators:

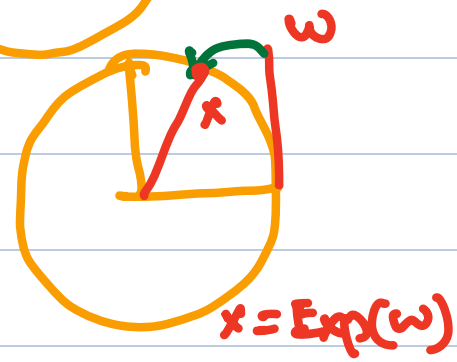
We have:

① Composition $x_1 \cdot x_2$ →



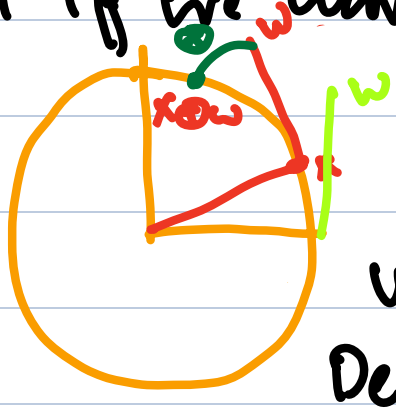
② Exponential mapping

$x = \mathrm{Exp}(w)$ →



Note: The circles drawn are just for geometric idea. They can be any Lie Groups.

What if we want to combine ① & ②?



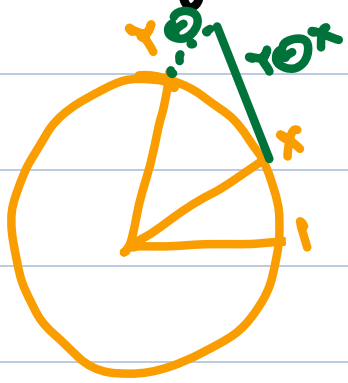
we define such operator as $\oplus$

$$x \oplus w \triangleq x \cdot \mathrm{Exp}(w)$$

Useful in:

Derivatives, Covariance matrices, errors, propagation etc.,

Similarly we define $\ominus$ operator

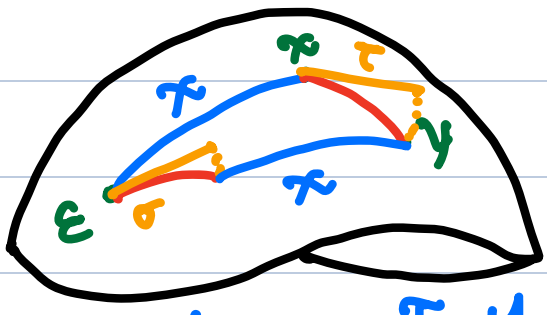


This is basically the reverse of $\oplus$ operator.

$$Y \ominus X = \mathrm{Log}(X^{-1} \cdot Y)$$

Note: In $\oplus$, we "add" a group element to a vector in tangent space and get another group element. In $\ominus$, we find the "difference" b/w 2 group elements & express that difference in the tangent space aka. vector.

Adjoint Matrix:



$\sigma \in T_E M$; $\tau \in T_x M$

Say there is a Lie group & E is the identity in the group. To go to x from E; it is simple. $E \cdot x = x$. From x to y; we take a tangential space route and do $x \oplus \tau = y$.

But we could've gone the other way too. Take a tangential route to some point using 'σ' and then take x to get to y.

$$\Rightarrow \sigma \oplus x = x \oplus \tau.$$

Note: Since $\oplus$ is non-commutative; $\sigma \neq \tau$.

From $\sigma \oplus x = x \oplus \tau$, we get

$$\sigma^{\wedge} = x \tau^{\wedge} x^{-1} \rightarrow \text{Lie Algebra}$$

$$\Rightarrow \sigma = Adj_x \cdot \tau \rightarrow \text{Cartesian}$$

Adj_x here is the adjoint Matrix. It is a matrix that maps from one tangential space to another in the lie group. It is a linear operator.

$$T_x M : \longrightarrow T_\varepsilon M \quad \text{(mapping)}$$

It is a matrix that relates tangent vector at one point to tangent vector at Identity.

Calculus on Lie Groups:

- For performing calculus we use $\oplus$ & $\ominus$ operators.
- We use Cartesian expression for Tangent spaces & define Jacobians of functions.

Jacobians in vector spaces:

$$J = \frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

That's the def^n for jacobian in vector space

For Lie groups, we have:

$$J = \frac{Df(x)}{Dx} = \lim_{\tau \to 0} \frac{f(x \oplus \tau) \ominus f(x)}{\tau}$$

Here $f(\cdot)$ could be any function. x is a member of a manifold & $f(x)$ is a member of a different (or same) manifold.

Example:

Action of $SO(3)$ on $\mathbb{R}^2$. Some rotation.

$$f: SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3 \; ; \; (R, p) \to f(R, p) = R \cdot p$$

function uses an element from $SO(3)$ & $\mathbb{R}^3$ to provide a result in $\mathbb{R}^3$.

Let's find Jacobian for such a function. first wrt R then wrt p.

Because its wrt R, we consider a small increment $(R \oplus \theta)$ where $\theta \to 0$. $R \cdot p$ is a vector in $\mathbb{R}^3$, $(R \oplus \theta) \cdot p$ is also a vector in $\mathbb{R}^3$.

$$\frac{df}{dR} = \lim_{\theta \to 0} \frac{(R \oplus \theta) \cdot p - R \cdot p}{\theta}$$

$$= \lim_{\theta \to 0} \frac{(R\,\mathrm{Exp}(\theta))\cdot p - R\cdot p}{\theta}$$

$\mathrm{Exp}(\theta) = \exp(\theta I_x) = \cos\theta\, I + \sin\theta\, I_x$

But around $\theta \to 0$, $\cos\theta = 1$, $\sin\theta = \theta$.

$$\therefore \frac{df}{dR} = \lim_{\theta \to 0} \frac{R(I + \theta I_x)\cdot p - R\cdot p}{\theta}$$

$$= \lim_{\theta \to 0} \frac{R(I + \theta_x)\cdot p - R\cdot p}{\theta}$$

$$= \lim_{\theta \to 0} \frac{R\,\theta_x\, p}{\theta}$$

$\theta_x \cdot p = -p_x \theta$ (from cross product matrix)

$$\frac{df}{dR} = \lim_{\theta \to 0} \frac{-R\,p_x\,\theta}{\theta} = -R\cdot p_x$$

We can compute for $\frac{df}{dp}$; but that's easy since p operates on $\mathbb{R}^3$.

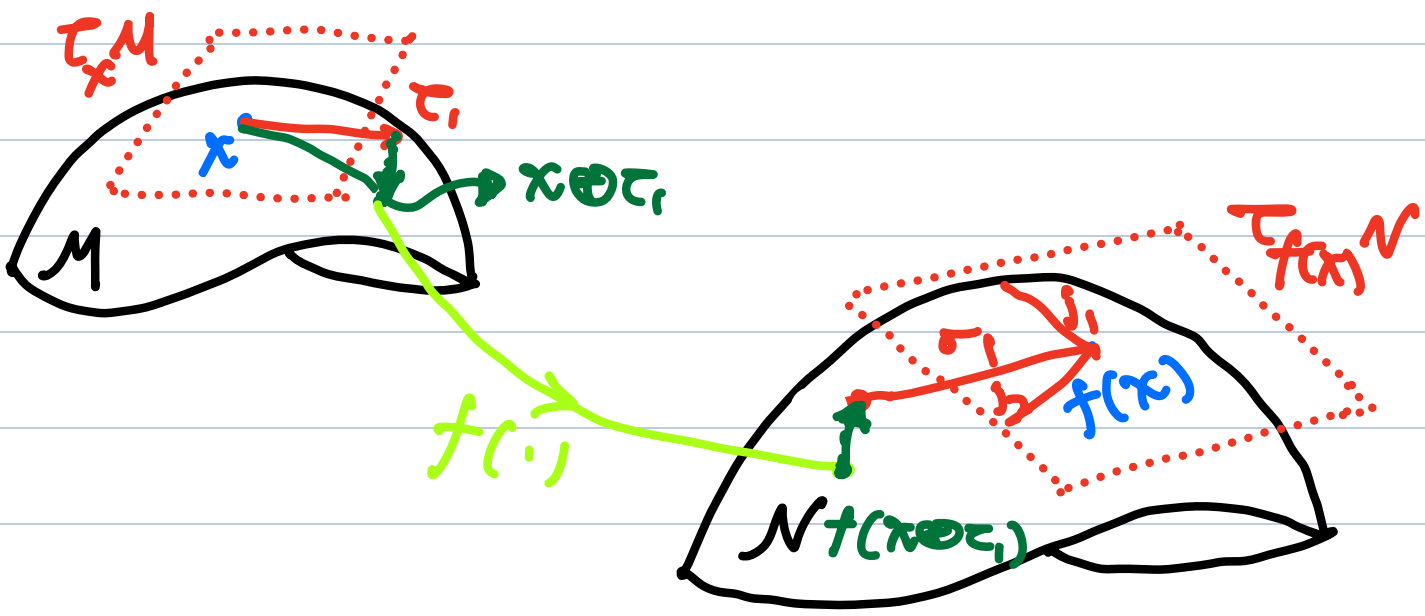
$$\frac{df}{dp} = \lim_{k \to 0} \frac{R(p+k) - Rp}{k} = R.$$

Jacobian Interpretations:

Jacobians on Lie groups can be interpreted as mapping from one manifold's tangent space to the others. Specifically

$$J : T_x M \to T_{f(x)} N$$

where M, N are two different manifolds & $f(x)$ is the function whose Jacobian we found.



Differentiation rules (without proof):

$$\frac{Dx^{-1}}{Dx} = -\mathrm{Adj}_x \qquad \frac{D\,xy}{Dx} = \mathrm{Adj}_y^{-1}$$

$$\frac{D\,xy}{Dy} = I$$

$$\frac{D\,\mathrm{Log}(x)}{Dx} = J_r(\mathrm{Log}(x))^{-1}$$

$$\frac{D\,x \oplus \tau}{Dx} = \mathrm{Adj}_{\mathrm{Exp}(\tau)}^{-1} \qquad \frac{D\,x \oplus \tau}{D\tau} = J_r(\tau)$$

$$J_r = \frac{D\,\mathrm{Exp}(\tau)}{D\tau} \quad \text{(Right Jacobian)}$$

Chain rules for other diff!

Example:

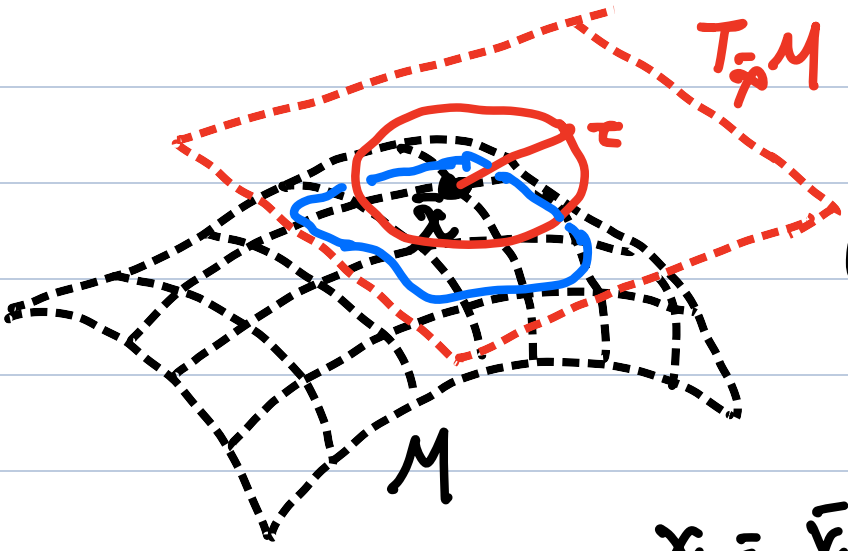
$$\frac{D\,R^T p}{DR} = \frac{D\,R^T p}{DR^T} \cdot \frac{DR^T}{DR}$$

Since $R^T = R^{-1}$, $\frac{DR^T}{DR} = -\mathrm{Adj}\,R$

We proved $\frac{DRp}{DR} = -Rp_\times$ $\therefore$ $\frac{DR^T p}{DR^T} = -R^T p_\times$

$$\therefore \frac{DR^T p}{DR} = -R^T p_\times \cdot \mathrm{Adj}_R = R^T p_\times R = (R^T p)_\times$$

Perturbations on Lie Groups:



$T_{\bar{x}}M$ is the tangent space @ $\bar{x}$

Lets say there is a small perturbation on $\bar{x}$ called x.

$$x = \bar{x} \oplus \tau \; ; \; \tau = x \ominus \bar{x}$$

Covariance of x; expressed in terms of τ is

$$P \triangleq E[\tau \cdot \tau^T] \rightarrow \text{standard def of covariance}$$
$$= E[(x \ominus \bar{x})(x \ominus \bar{x})^T]$$

What if we want to propogate covariance?

Say we have a mapping

$$y = f(x) \quad \& \quad J = \frac{Dy}{Dx}, \; P_x \text{ is covariance}$$

$$P_y = J P_x J^T$$