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Logarithmic spirals and continue triangles

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ABSTRACT

In this article we will use some special triangles, to construct polygonal chains that describe the families of logarithmic spirals, among which are the celebrated Golden Spiral, Spira solaris and Pheidia Spiral.

1. Introduction

A *logarithmic spiral* is a plain curve whose equation in polar coordinate (ρ, θ) is $\rho = te^{(h\theta)}$. The term h is a positive number called the *growth constant* of the spiral, and t is a *constant of the spiral* depending on the choice of the initial condition $\theta = 0$. Note that θ increases anti-clockwise. The Cartesian representation of a logarithmic spiral is

$$\begin{cases} x(\theta) = \rho(\theta)\cos(\theta) = te^{(h\theta)}\cos(\theta) \\ y(\theta) = \rho(\theta)\sin(\theta) = te^{(h\theta)}\mathrm{sen}(\theta), \end{cases} \tag{1.1}$$

thus the distance from the origin (Pole) of $(x(\theta), y(\theta))$ increases exponentially when θ increases (anti-clockwise). Sometimes this kind of spiral is more precisely called a *Left hand logarithmic spiral*, to distinguish it from a *Right hand logarithmic spiral*, whose equation is of the type $\rho = te^{(-h\theta)}$. For the latter type of spirals the distance from the Pole of $(x(\theta), y(\theta))$ decreases exponentially when θ increases.

The most celebrated curve of this type is certainly the *Golden Spiral*, which is a logarithmic spiral whose growth constant is $(2/\pi)\lg(\Phi)$, where Φ is the "Golden Mean".

Looking at the equation of the Golden Spiral

$$\rho = e^{(2/\pi)\lg(\Phi)\theta} \quad \text{with starting point } (1, 0),$$

we note that for $\theta = 0$ we have $\rho = 1$, and for $\theta = \pi/2$ we have $\rho = \Phi$. More generally it can be easily seen that, a golden spiral gets wider (or further from its origin) by a factor of Φ for every quarter turn it makes; therefore "Φ^4" gives a measurement of the *growth factor* of this spiral after a complete turn around the pole.

The Golden Spiral was first described by Descartes, and then studied by the Swiss mathematician Jakob Bernoulli (1654–1705), who called it *Spira mirabilis*, and dedicated to it the famous motto "*Eadem Mutata resurgo*" which is inscribed on his tombstone.

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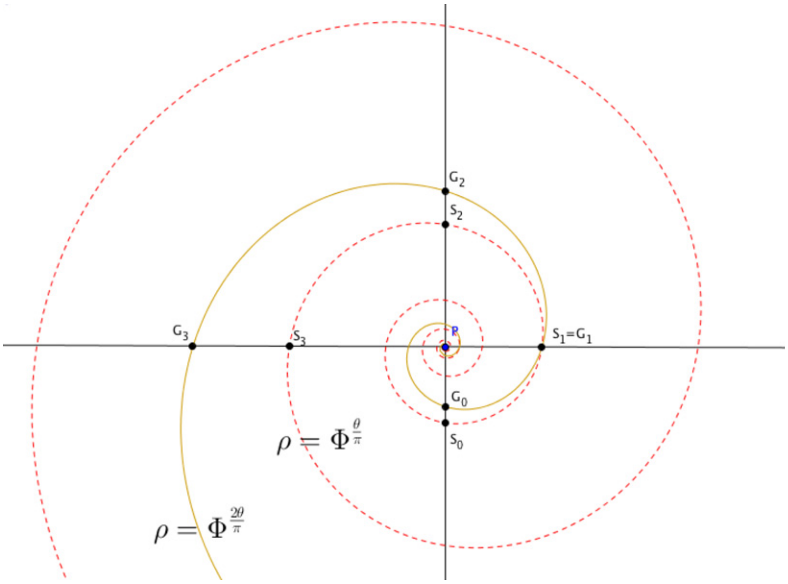

Fig. 1.1. Dotted Red line is the Spira Solaris, Gold line is the Golden Spiral.

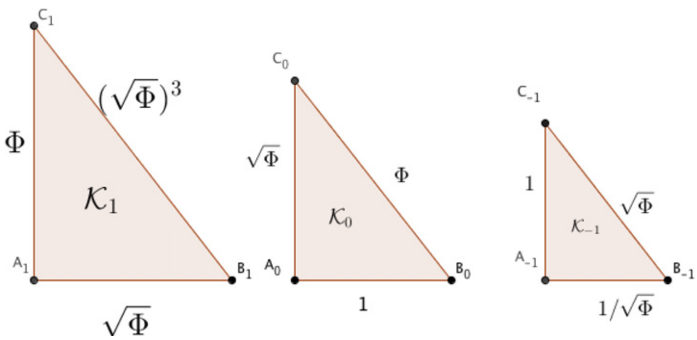

Fig. 1.2. Examples of Kepler triangles.

Approximations of logarithmic spirals can occur in nature (for example, the arms of spiral galaxies or phyllotaxis of leaves); golden spirals are one special case of these. It is sometimes stated that spiral galaxies and nautilus shells get wider in the pattern of a golden spiral, and hence are related to both Φ and the Fibonacci series. In truth, spiral galaxies and nautilus shells (and many mollusk shells) exhibit logarithmic spiral growth, but at a variety of angles usually distinctly different from that of the golden spiral. This pattern allows the organism to grow without changing shape (see [1,2]).

Also, we note a very interesting connection between the *DNA*-spiral and some Fibonacci-like sequences (see [3]). This highlights the relevant connection between recursive sequences and spirals (see also [4–7] and the references therein).

The German Mathematician Johannes Kepler (1571–1630) was the first to study the nature of the logarithmic spirals, and its possible *discretizations*. He was also attracted by their shape and their applications in astronomy. For this purpose he considered a logarithmic spiral, with a lower factor growth (Φ^2): The *Spira Solaris*, whose equation is given by (see Fig. 1.1):

$$\rho = e^{(1/\pi)\lg(\Phi)\theta} = \Phi^{\theta/\pi} \quad \text{with starting point } (1, 0).$$

Another celebrated logarithmic spiral is the *Pheidia Spiral*, whose equation is:

$$\rho = e^{(1/2\pi)\lg(\Phi)\theta} = \Phi^{\theta/2\pi}, \quad \text{with starting Point } (1, 0).$$

Note that Pheidia Spirals, Spira Solaris, and Golden Spiral have respectively the following "growth": Φ, Φ^2, Φ^4.

Connected to the Spira Solaris, there are the *Golden right triangle* or *Kepler triangle*, that is any right triangle $\mathcal{K}$ the lengths of whose sides, $a > b > c$ satisfy the proportion: $a{:}b = b{:}c$. Fig. 1.2 shows examples of Kepler triangles. For every integer n, we will denote by $\mathcal{K}_n$ a triangle the measurement of whose sides are $(\sqrt{\Phi})^n$, $(\sqrt{\Phi})^{n+1}$ and $(\sqrt{\Phi})^{n+2}$ respectively (Φ is the golden mean). It is easy to check that each $\mathcal{K}_n$ is a Kepler triangle.

Following Pennisi (see [8]), we will say that a triangle (not necessarily right) is *continue* if the lengths of whose sides, $a > b > c$ satisfy the proportion: $a{:}b = b{:}c$. In [8] Pennisi studied the connection between Kepler triangles and spirals, and

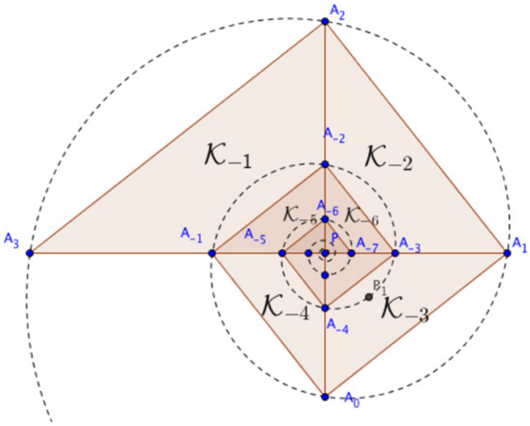

Fig. 1.3. $\mathcal{K}_{n-1} \cup \mathcal{K}_{n-2}$, is congruent to $\mathcal{K}_n$.

highlighted that (see Fig. 1.3) the polygonal chain

"$\ldots A_{-4}, A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3, \ldots$"

can describe (or *discretize*) the Spira Solaris (in his paper he called it *continue spiral*). In the figure it is represented by a dotted line.

There is ample literature from Descartes to the present, that also shows how some polygonal chains can describe other celebrated logarithmic curves, and their impact on sciences such as Astronomy, Physics, Biology, Botany, Medicine (see [9,2,10,11] and reference therein). The applications of logarithmic spirals are very relevant also in Architecture and Engineering.

It arises the natural question: Are there other logarithmic spirals that can be described by using continue triangles?

In this article we will give some positive answers. We will investigate the properties of polygonal chains constructed by continue triangles. We will use a new computational techniques (see Lemma 3.1 and Corollary 3.2), in order to arrive at a description of the family of logarithmic spirals (see Theorem 4.1 and Section 5). In Section 2 we recall the main properties of a continue triangle (not necessarily right). In Section 3 we show the geometric algorithm for the construction of some polygonal chains: the (r, k)-*male spirals*. In Section 4 we will see how some families of logarithmic spirals, among which Pheidia Spiral, Spira Solaris, and Golden Spiral, can be described by these "special" polygonal chains. The paper is as self-contained as possible, and has been written providing a soft approach to the study of logarithmic spirals and encouraging the use of professional mathematical software.

2. Chains of continue triangles

Consider a triangle $\mathcal{T}$ whose vertices are A, B and C, and define a, b, c as the measurement of the sides that are opposite respectively to A, B and C. In the following we will denote triangle $\mathcal{T}$ by (A, B, C) or (a, b, c).

Clearly a triangle is continue, if the measurement of its sides are three consecutive numbers of a geometric progression, but the converse is not true: if we consider three numbers in geometric progression, we cannot say that they are the length of the sides of a continue triangle. To this end, it is enough to consider the numbers 2, 4, 8: they do not satisfy the triangular property. Moreover, if $a > b > c$ and (a, b, c) is a continue triangle, then also (b, c, d) is a continue triangle, having chosen d such that $b{:}c = c{:}d$. Clearly the triangles (a, b, c) and (b, c, d) are similar and have two sides congruent. Following R.T. Jones and B.B. Peterson we will call such pairs of triangles *almost congruent* (see [12]).

By definition and triangular inequality we have two elementary properties of continue triangles (see also [12, Theorem 1], and [8, page 22]).

Theorem 2.1. *Let $\mathcal{T} = (a, b, c)$ be a triangle. If $\mathcal{T}$ is a continue triangle, then a, b, c are in geometric progression of mean lying in $(1/\Phi, \Phi) \setminus \{1\}$. Conversely for every geometric progression kr, kr^2, kr^3, where k is a positive real number and the mean r lies in $(1/\Phi, \Phi) \setminus \{1\}$, the triangle (kr, kr^2, kr^3) is continue.*

Remark 2.1. In the above statement it is easy to see that if $r \in (1/\Phi, 1/\sqrt{\Phi}) \cup (\sqrt{\Phi}, \Phi)$ the corresponding triangles are obtuse; and if $r \in (1/\sqrt{\Phi}, \sqrt{\Phi}) \setminus \{1\}$ the corresponding triangles are acute. Moreover, if r is exactly $\sqrt{\Phi}$ we have that (kr, kr^2, kr^3) define a Kepler triangle, for every positive k.

Fig. 2.1 shows examples of pairs of almost congruent triangles, with integer sides (see [13] and [12, p. 182]):

Let $r \in (1/\Phi, \Phi)$, and for each integer n consider the continue triangle $\mathcal{T}_n = (r^n, r^{n+1}, r^{n+2})$. In the following, by writing "$\mathcal{T}_n < \mathcal{T}_m$" we mean that the area of $\mathcal{T}_n$ is less than the area of $\mathcal{T}_m$.

Clearly, for every $r \in (1/\Phi, \Phi)$ the sets $\{\mathcal{T}_n\}_{n\in\mathbb{Z}} = \{(r^n, r^{n+1}, r^{n+2})\}_{n\in\mathbb{Z}}$ and $\{\mathcal{T}'_n\}_{n\in\mathbb{Z}} = \{((\frac{1}{r})^n, (\frac{1}{r})^{n+1}, (\frac{1}{r})^{n+2})\}_{n\in\mathbb{Z}}$ coincide.

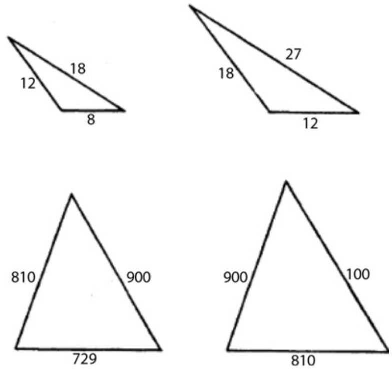

Fig. 2.1. Pairs of almost congruent triangles with integer sides. Each of them is a continue triangle.

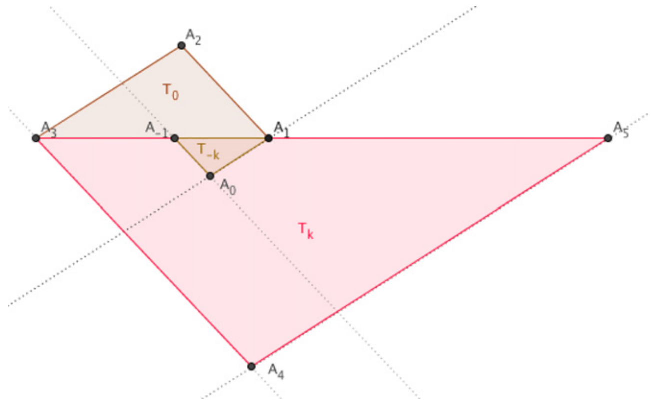

Fig. 3.1. The construction of the (r, k)-male spiral.

These two chains of continue triangles exhibit opposite behavior:

If $1 < r$, then the chain $(\mathcal{T}_n)$ is ascendant, while $(\mathcal{T}'_n)$ is descendant;
If $r < 1$, then the chain $(\mathcal{T}_n)$ is descendant and $(\mathcal{T}'_n)$ is ascendant.

Thus we may restrict our considerations to the case of $r \in (1, \Phi)$. In particular we have:

$$\mathcal{T}_n < \mathcal{T}_m \quad \text{and} \quad \mathcal{T}'_n > \mathcal{T}'_m, \quad \forall n < m \in \mathbb{Z}.$$

3. The class of (r, k)-male spirals

Let $r \in (1, \Phi)$, and consider the chain $\{\mathcal{T}_n\}_{n\in\mathbb{Z}}$ as defined in Section 2. In the following, for every integer n we set $A_{n+1}, A_{n+2}, A_{n+3}$, as the vertices of $\mathcal{T}_n$. Thus

$$\mathcal{T}_n = (r^n, r^{n+1}, r^{n+2}) = (A_{n+1}, A_{n+2}, A_{n+3}).$$

For every positive integer k we can consider the subchain $\{\mathcal{T}_{nk}\}_{n\in\mathbb{Z}}$ and make the following construction:
Starting from $\mathcal{T}_0 = (1, r^1, r^2) = (A_1, A_2, A_3)$, for every integer k we can consider the triangles $\mathcal{T}_k = (r^k, r^{k+1}, r^{k+2}) = (A_3, A_4, A_5)$ and $\mathcal{T}_{-k} = (r^{-k}, r^{-k+1}, r^{-k+2}) = (A_1, A_0, A_{-1})$, and by using their similarity we can draw them as in Fig. 3.1:
Now we can join triangle $\mathcal{T}_{-k} = (A_1, A_0, A_{-1})$, with the smaller

$$\mathcal{T}_{-2k} = (r^{-2k}, r^{-2k+1}, r^{-2k+2}) = (A_{-1}, A_{-2}, A_{-3});$$

and triangle $\mathcal{T}_k = (A_3, A_4, A_5)$ with the larger one

$$\mathcal{T}_{2k} = (r^{2k}, r^{2k+1}, r^{2k+2}) = (A_5, A_6, A_7).$$

The result of this iterated process is a polygonal chain that we call (r, k)-*male spiral*. We will denote it by $\mathcal{P}_{r,k}$. For this "spiral" we have the following result:

Lemma 3.1. *Let $r \in (1, \Phi)$ and k be a positive integer, and consider the (r, k)-male spiral $\mathcal{P}_{r,k}$. Then all the vertices A_{2n}, indexed by an even number, lie on the same line s. In particular, if we define P as the intersection between s and the line containing all the vertices A_{2n+1}, then each A_{1-4n} lies to the right of P and each A_{3-4n} lies to the left of P, for every integer n. Moreover: the following relations hold:*

(a) $|\overline{A_3P}| = \sum_{n=0}^{\infty} |\overline{A_{3-4n}A_{3-4(n+1)}}| = |\overline{A_3A_{-1}}| + |\overline{A_{-1}A_{-5}}| + \cdots = \frac{r^{2+k}}{r^k+1}$;

(b) $|\overline{A_1P}| = \sum_{n=0}^{\infty} |\overline{A_{1-4n}A_{1-4(n+1)}}| = |\overline{A_1A_{-3}}| + |\overline{A_{-3}A_{-7}}| + \cdots = \frac{r^2}{r^k+1}$;

(c) $|\overline{A_2P}| = \sum_{n=0}^{\infty} |\overline{A_{2-4n}A_{2-4(n+1)}}| = |\overline{A_2A_{-2}}| + \cdots = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}$;

(d) $|\overline{A_0P}| = \sum_{n=0}^{\infty} |\overline{A_{4n}A_{-4(n+1)}}| = |\overline{A_0A_{-4}}| + \cdots = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{r^k(r^k+1)}$.

Proof. We start by showing that A_2, A_0 and A_4 are collinear. To this end, consider the line a through the vertices A_1 and A_0, and define F as the intersection of a with $\overline{A_3A_4}$.

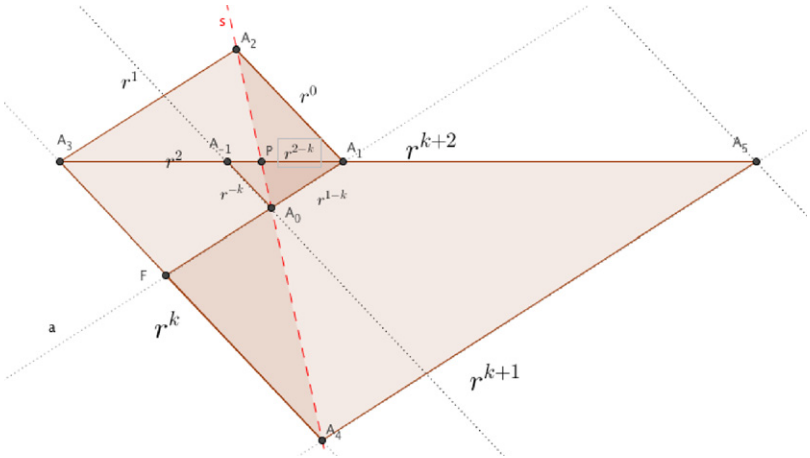

Fig. 3.2. The pole of the (r, k)-male spiral.

The angles $A_2\hat{A_1}A_0$ and $A_0\hat{F}A_4$ are congruent; moreover, the following proportion holds: $\overline{A_2A_1} : \overline{A_1A_0} = \overline{FA_4} : \overline{FA_0}$, in fact, the measurements of these segments result as:

$$1 : r^{1-k} = r^k - 1 : r - r^{1-k},$$

so that the triangles (A_2, A_1, A_0) and (A_0, F, A_4) are similar. In particular, $A_1\hat{A_0}A_2$ and $F\hat{A_0}A_4$ are congruent. We have proved that three vertices A_0, A_2, A_4, of the three consecutive continue triangles $\mathcal{T}_{-k}$, $\mathcal{T}_0$ and $\mathcal{T}_k$ are collinear. Iterating the process "up and down" we find that all A_{2n} are collinear.

In particular, for each integer n, the vertex A_{1-4n} lies to the right of the line s, and A_{3-4n} lies to the left.

Now we note that P is the intersection of all $\mathcal{T}_k$, so that:

$$\bigcup_{n\in\mathbb{N}_0} \overline{A_{3-4n}A_{3-4(n+1)}} = \overline{A_3P}; \qquad \bigcup_{n\in\mathbb{N}_0} \overline{A_{1-4n}A_{1-4(n+1)}} = \overline{A_1P};$$

$$\bigcup_{n\in\mathbb{N}_0} \overline{A_{4-4n}A_{4-4(n+1)}} = \overline{A_4P}; \qquad \bigcup_{n\in\mathbb{N}_0} \overline{A_{2-4n}A_{2-4(n+1)}} = \overline{A_2P}.$$

It follows that:

(a) $|\overline{A_3A_{-1}}| = r^2 - r^{2-k}$, $|\overline{A_{-1}A_{-5}}| = r^{2-2k} - r^{2-3k}$ and in general for every integer $n \geq 0$ $|\overline{A_{3-4n}A_{3-4(n+1)}}| = r^{2-2nk} - r^{2-(2n+1)k}$. Therefore

$$\begin{aligned}\overline{A_3P} &= \sum_{n=0}^{\infty} |\overline{A_{3-4n}A_{3-4(n+1)}}| = \sum_{n=0}^{\infty} r^{2-2nk} - r^{2-(2n+1)k} = \sum_{n=0}^{\infty} r^{2-2nk}(1-r^{-k}) \\ &= r^2(1-r^{-k})\sum_{n=0}^{\infty} r^{-2nk} = r^2(1-r^{-k})\sum_{n=0}^{\infty}\left(\frac{1}{r^{2k}}\right)^n = r^2(1-r^{-k})\frac{1}{1-1/r^{2k}} \\ &= \frac{r^{2+k}(r^k-1)}{r^{2k}-1} = \frac{r^{2+k}}{r^k+1}\end{aligned}$$

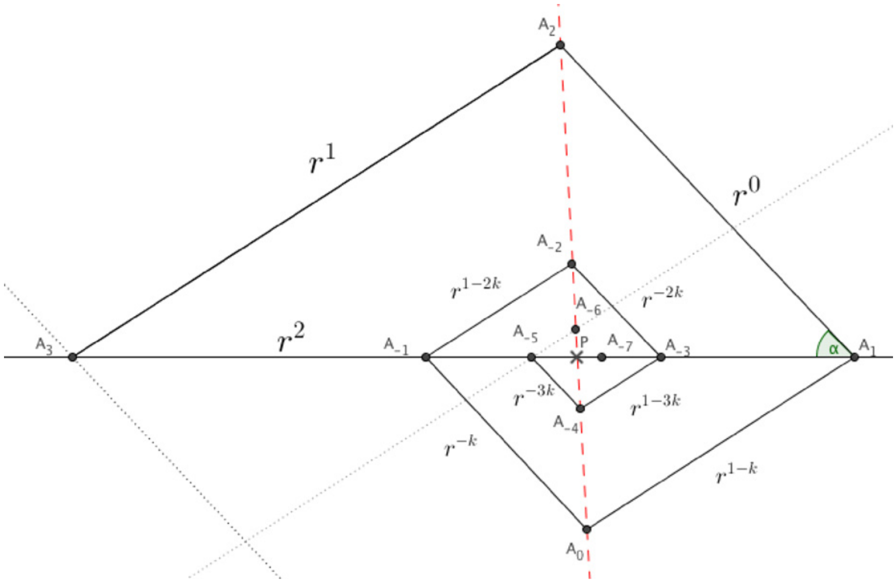

Fig. 3.3. The growth of the (r, k)-male spiral.

(b) It follows from (a) that: $\overline{A_1P} = \overline{A_1A_3} - \overline{A_3P} = r^2 - \frac{r^{2+k}}{r^k+1} = \frac{r^2}{r^k+1}$.

(c) To verify this item, we will make double use of Carnot' theorem. Following Fig. 3.3, we set $\alpha = A_2\hat{A}_1A_3$. Then
(i) $|\overline{A_3A_2}|^2 = |\overline{A_3A_1}|^2 + |\overline{A_2A_1}|^2 - 2|\overline{A_3A_1}|\,|\overline{A_2A_1}|\cos(\alpha)$, so we have

$$\cos(\alpha) = \frac{|\overline{A_3A_1}|^2 + |\overline{A_2A_1}|^2 - |\overline{A_3A_2}|^2}{2|\overline{A_3A_1}|\,|\overline{A_2A_1}|} = \frac{r^4 + r^0 - r^2}{2r^2r^0} = \frac{r^4 + 1 - r^2}{2r^2}.$$

(ii)

$$\begin{aligned}
|\overline{PA_2}|^2 &= |\overline{PA_1}|^2 + |\overline{A_2A_1}|^2 - 2|\overline{PA_1}|\,|\overline{A_2A_1}|\cos(\alpha) \\
&= \left(\frac{r^2}{r^k+1}\right)^2 + 1 - 2\frac{r^2}{r^k+1}\, r^0\, \frac{r^4+1-r^2}{2r^2} \\
&= \frac{r^4}{(r^k+1)^2} + 1 - \frac{r^4+1-r^2}{r^k+1} = \frac{r^4}{(r^k+1)^2} + \frac{r^{2k}+2r^k+1}{(r^k+1)^2} - \frac{(r^k+1)(r^4+1-r^2)}{(r^k+1)^2} \\
&= \frac{r^4 + r^{2k} + 2r^k + 1 - r^{k+4} - r^k + r^{k+2} - r^4 - 1 + r^2}{(r^k+1)^2} = \frac{r^{2k} + r^k - r^{k+4} + r^{k+2} + r^2}{(r^k+1)^2}.
\end{aligned}$$

Therefore

$$|\overline{PA_2}| = \frac{\sqrt{r^{2k} + r^k - r^{k+4} + r^{k+2} + r^2}}{(r^k+1)}.$$

(d) The triangles (A_{-1}, P, A_0) and (A_1, P, A_2) are similar so that:

$$\overline{A_2A_1} : \overline{A_{-1}A_0} = \overline{A_2P} : \overline{PA_0} \quad \text{and} \quad r^0 : r^{-k} = \frac{\sqrt{r^{2k} + r^k - r^{k+4} + r^{k+2} + r^2}}{(r^k+1)} : \overline{PA_0}.$$

Therefore $\overline{PA_0} = \frac{r^{-k}\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}$, and the last relation d is verified. □

In the next we will refer to the point P of the statement above as *the pole* of the male spiral.

Example 3.1. As a particular male spiral we may consider the $(\sqrt{\Phi}, 2)$-male spiral. Using (a) and (c) of Lemma 3.1 we have:

$$|\overline{A_1P}| = \frac{r^2}{r^k+1} = \frac{\sqrt{\Phi}^2}{\sqrt{\Phi}^2 + 1} = \frac{\Phi}{\Phi + 1} = \frac{1}{\Phi};$$

$$
\begin{aligned}
|\overline{A_2P}| &= 1/\sqrt{\Phi} = \frac{\sqrt{\Phi^2+\Phi-\Phi^3+\Phi^2+\Phi}}{(\Phi+1)} = \frac{\sqrt{2(\Phi^2+\Phi)-\Phi(\Phi+1)}}{(\Phi+1)} \\
&= \sqrt{\frac{\Phi(\Phi+1)}{(\Phi+1)^2}} = \sqrt{\frac{\Phi}{\Phi+1}} = 1/\sqrt{\Phi}.
\end{aligned}
$$

Thus (A_1, P, A_2) gives Kepler triangle $\mathcal{K}_{-2}$ (see also Fig. 1.3). Here the pole P is exactly the intersection of the height of (A_1, A_2, A_3) with its hypotenuse.

Corollary 3.2. *Let $r \in (1, \Phi)$ and k be a positive integer, and consider the (r, k)-male spiral, $\mathcal{P}_{r,k}$, as in the lemma above (see Fig. 3.3). Then the distance of any A_m from the pole P of $\mathcal{P}_{r,k}$ increases by a factor of r^{2k} for each (positive) turn around the pole.*

Proof. We will examine four cases.

(a) A_m is to the left of P, that is $m \equiv 3 \pmod 4$, or in other words $m = 3 - 4n$ for some $n \in \mathbb{Z}$. Starting from $\overline{A_3P}$, we have: $|\overline{A_{-1}P}| = |\overline{A_3P}| - |\overline{A_3A_{-1}}| = \frac{r^{2+k}}{r^k+1} - r^2 + r^{2-k} = \frac{r^{2-k}}{r^k+1}$, and hence

$$
\frac{|\overline{A_3P}|}{|\overline{A_{-1}P}|} = \frac{r^{2+k}}{r^k+1}\,\frac{r^k+1}{r^{2-k}} = r^{2k}. \tag{3.1}
$$

On the other hand, by construction (see also item (a) of Lemma 3.1), $\frac{|\overline{A_7A_3}|}{|\overline{A_3A_{-1}}|} = r^{2k}$. Using the elementary properties of the proportions, we have: $\overline{A_3P} : \overline{A_{-1}P} = \overline{A_7A_3} : \overline{A_3A_{-1}}$, thus
$\overline{A_3P} : \overline{A_{-1}P} = \overline{A_7A_3} + \overline{A_3P} : \overline{A_3A_{-1}} + \overline{A_{-1}P}$, that is $\overline{A_3P} : \overline{A_{-1}P} = \overline{A_7P} : \overline{A_3P}$. In particular

$$
\frac{|\overline{A_7P}|}{|\overline{A_3P}|} = r^{2k}. \tag{3.2}
$$

Now by iterating this process, we see that we can extend Eqs. (3.1) and (3.2) to any $n \equiv 3 \pmod 4$:

$$
\frac{|\overline{A_{3-4n}P}|}{|\overline{A_{3-4(n+1)}P}|} = r^{2k}. \tag{3.3}
$$

(b) A_m is on the right of P, that is $m \equiv 1 \pmod 4$. Note that:

$$
\begin{aligned}
|\overline{A_{1-4n}A_{1-4(n+1)}}| &= r^{2-(2n+1)k} - r^{2-(2n+2)k} = r^{2-(2n+1)k}(1-r^{-k}) \\
&= r^{2-k}(1-r^{-k})r^{-(2n)k} = r^{2-k}(1-r^{-k})\left(\frac{1}{r^{2k}}\right)^n.
\end{aligned} \tag{3.4}
$$

Starting from $\overline{A_1P}$, by Lemma 3.1(b), we have:
$|\overline{A_{-3}P}| = |\overline{A_1P}| - |\overline{A_1A_{-3}}| = \frac{r^2}{r^k+1} - r^{2-k}(1-r^{-k}) = \frac{r^2-r^2+r^{2-k}-r^{2-k}+r^{2-2k}}{r^k+1} = \frac{r^{2-2k}}{r^k+1}$. It follows that

$$
\frac{|\overline{A_1P}|}{|\overline{A_{-3}P}|} = \frac{r^2}{r^k+1}\,\frac{r^k+1}{r^2r^{-2k}} = r^{2k}. \tag{3.5}
$$

On the other hand, by Eq. (3.4) $\frac{|\overline{A_5A_1}|}{|\overline{A_1A_{-3}}|} = \frac{r^{2-k}(1-r^{-k})(\frac{1}{r^{2k}})^{-1}}{r^{2-k}(1-r^{-k})(\frac{1}{r^{2k}})^0} = r^{2k}$.
Proceeding as in the second part of the proof of item (a), we obtain

$$
\frac{|\overline{A_5P}|}{|\overline{A_1P}|} = r^{2k}, \quad \text{and more generally } \frac{|\overline{A_{1-4n}P}|}{|\overline{A_{1-4(n+1)}P}|} = r^{2k}. \tag{3.6}
$$

(c) A_m is above P, that is $m \equiv 2 \pmod 4$. Set $m = 2 - 4n$. Note that the triangles $(A_{2-4n}PA_{3-4n})$ and $(A_{2-4(n+1)}PA_{3-4(n+1)})$ are similar. Then:

$$
\frac{|\overline{A_{3-4n}P}|}{|\overline{A_{3-4(n+1)}P}|} = \frac{|\overline{A_{2-4n}P}|}{|\overline{A_{2-4(n+1)}P}|},
$$

and by (a)

$$
\frac{|\overline{A_{2-4n}P}|}{|\overline{A_{2-4(n+1)}P}|} = r^{2k}.
$$

(d) A_m is below P, that is $m \equiv \pmod 4$. In this case we use the similarity of the triangles $(A_{1-4n}PA_{-4n})$ and $(A_{-3-4n}PA_{-4(n+1)})$ and the item (b). $\square$

Remark 3.1. Clearly we may consider the construction of a male spiral by starting from any triangle; but in the general case the above lemma does not hold, as simple considerations show (see Fig. 3.4). For example, starting from the triangle $\mathcal{T}_0 = (A_1, A_2, A_3)$ where $\overline{A_1A_2} = 3$, $\overline{A_2A_3} = 4$ and $\overline{A_1A_3} = 6$, we can consider the similar smaller triangle $\mathcal{T}_{-1}$ the lengths of whose sides are $3/2, 2, 3$, and the larger one $\mathcal{T}_1$ the lengths of whose sides are $6, 8, 12$. If we make the same construction as in Fig. 3.2, we find that the vertices A_0, A_2, A_4 are not collinear.

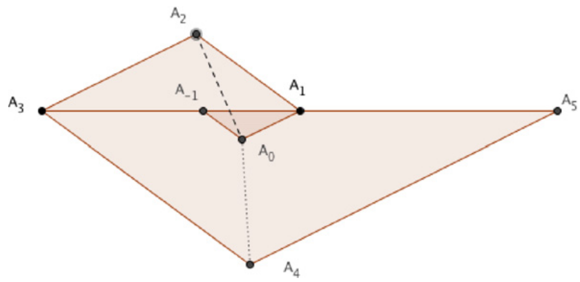

Fig. 3.4. The construction starting from a non continue triangle.

4. (r, k)-male spiral and logarithmic spirals

Let $r \in (1, \Phi)$, and k be a positive integer, and consider the logarithmic spiral $\mathcal{S}$ whose equation is:

$$\rho = te^{(1/\pi)\lg(r^k)\theta} = tr^{k\theta/\pi}, \quad \text{where } t \text{ depends on the starting point } (\rho(0), 0). \tag{4.1}$$

The factor growth of $\mathcal{S}$, or more simply the *growth of* $\mathcal{S}$, is the term r^{2k}.

The next result shows that the (r, k)-male spiral is connected to a pair of logarithmic spirals: $\mathcal{S}_1 = \mathcal{S}_1(r, k)$ and $\mathcal{S}_2 = \mathcal{S}_2(r, k)$ (see Fig. 4.1).

Theorem 4.1. *Let $r \in (1, \Phi)$, and k be a positive integer. Then all the vertices A_{1-2n} (indexed by odd numbers) of the (r, k)-male spiral $\mathcal{P}_{r,k}$, lie on a logarithmic spiral $\mathcal{S}_1$ of growth r^{2k}, with starting point $A_1 = (\frac{r^2}{r^k+1}, 0)$, and all the even points A_{2n} lie on a logarithmic spiral $\mathcal{S}_2$ of the same growth r^{2k}, with a suitable starting point H depending on r and k.*

Proof. By virtue of Corollary 3.2, for every $m \in \mathbb{Z}$, the distance of any A_m from the pole P of $\mathcal{P}_{r,k}$ increases by a factor of r^{2k} for each (positive) turn around the pole.

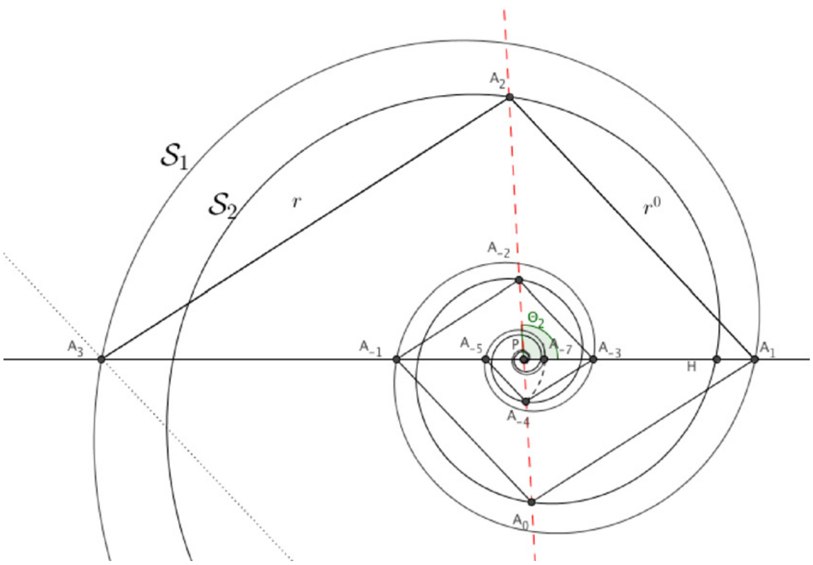

Fig. 4.1. Logarithmic spirals associated with the (r, k)-male spiral.

From item (b) of Lemma 3.1, $|\overline{A_1P}| = \frac{r^2}{r^k+1}$, so that the equation

$$\rho(\theta) = \frac{r^2}{r^k+1} e^{(1/\pi)\lg(r^k)\theta} = \frac{r^2}{r^k+1} r^{k\theta/\pi}, \tag{4.2}$$

defines a logarithmic spiral $\mathcal{S}_1$ to which all the vertices A_{1-2n} of the (r, k)-male spiral belong.

In order to determine the equation of the logarithmic spiral $\mathcal{S}_2$ to which all the A_{2n} of the (r, k)-male spiral belong, it suffices to determine the constant t' in the equation:

$$\rho(\theta) = t' e^{(1/\pi)\lg(r^k)\theta} = t' r^{k\theta/\pi}. \tag{4.3}$$

From item (c) of Lemma 3.1 we have $|\overline{A_2P}| = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}$.

As an application of Carnot' Theorem, we will determine that angle $\theta_2 := A_1\hat{P}A_2$:
$|A_2A_1|^2 = |PA_2|^2 + |PA_1|^2 - 2|PA_2||PA_1|\cos(\theta_2)$. It follows

$$\begin{aligned}
\cos(\theta_2) &= \frac{|PA_2|^2 + |PA_1|^2 - |A_2A_1|^2}{2|PA_2|\,|PA_1|} \\
&= \left(\frac{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}{(r^k+1)^2} + \frac{r^4}{(r^k+1)^2} - 1\right)\frac{1}{2}\,\frac{(r^k+1)}{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}\,\frac{r^k+1}{r^2} \\
&= \frac{1}{2r^2}\,\frac{-r^{k+4}-r^k+r^{k+2}+r^4+r^2-1}{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}.
\end{aligned}$$

Thus

$$\theta_2 = \arccos\left(\frac{1}{2r^2}\,\frac{-r^{k+4}-r^k+r^{k+2}+r^4+r^2-1}{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}\right). \tag{4.4}$$

Now we are in a position to determine t' in equation $\rho = t'e^{(h\theta)}$, which describes this new spiral. For this purpose, we substitute θ_2 in Eq. (4.3), and we have:

$$|PA_2| = \rho(\theta_2) = t' r^{k\theta_2/\pi} \quad \text{and hence; } t' = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}\,\frac{1}{r^{k\theta_2/\pi}}.$$

Thus the equation of logarithmic spiral $\mathcal{S}_2$ also described by vertices A_{2n} of the (r, k)-male is:

$$\rho(\theta) = \frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}\,\frac{1}{r^{k\theta_2/\pi}}\,r^{k\theta/\pi}, \quad \text{where } \theta_2 \text{ is given by Eq. (4.4).} \tag{4.5}$$

Thus the starting point of $\mathcal{S}_2$ is $H = (\frac{\sqrt{r^{2k}+r^k-r^{k+4}+r^{k+2}+r^2}}{(r^k+1)}\frac{1}{r^{k\theta_2/\pi}}, 0)$. $\square$

The following remarks follow from Theorem 4.1:

Remark 4.1. Note that $\mathcal{S}_1 = \mathcal{S}_1(\sqrt{\Phi}, 2) = \mathcal{S}_2 = \mathcal{S}_2(\sqrt{\Phi}, 2)$ is the same spira Solaris, Indeed it can be checked that $A_1 = H \equiv (1/\Phi, 0)$.

Remark 4.2. Note that $\mathcal{S}_1 = \mathcal{S}_1(\sqrt{\Phi}, 1)$ and $\mathcal{S}_2 = \mathcal{S}_2(\sqrt{\Phi}, 1)$ are two Pheidia Spirals, with different starting points.

Remark 4.3. Note that $\mathcal{S}_1 = \mathcal{S}_1(\sqrt{\Phi}, 4)$ and $\mathcal{S}_2 = \mathcal{S}_2(\sqrt{\Phi}, 4)$ are two Golden Spirals, with different starting points.

Remark 4.4. Let $\mathcal{P}_{r,k}$ be an (r, k)-male spiral. If $r \neq \sqrt{\Phi}$, there is no logarithmic spiral through all the vertices of $\mathcal{P}_{r,k}$.

5. Conclusions

Note that starting from an assigned segment $\mathbf{r} = \overline{A_1A_2}$ of length $r \in (1/\Phi, \Phi)$, the construction that we have presented (see 3.1) can be made by ruler and compass. In fact all the powers r^n, and all the n-parts of $\mathbf{r}$ can be constructed, so that it is possible to construct any (r, k)-mail spiral, for every positive integer k.

In Section 1 we have remarked that logarithmic spirals appear in nature, and some of them can be discretized by triangles. On the other hand, we have seen that every (r, k)-male spiral is connected to a pair of logarithmic spirals: $\mathcal{S}_1 = \mathcal{S}_1(r, k)$ and $\mathcal{S}_2 = \mathcal{S}_2(r, k)$. Conversely, here we may highlight that every logarithmic spiral can be discretized by an (r, k)-male spiral. To this end, it is enough to see that

$$\forall h \in \mathbb{R}^+ \quad \lim_n \sqrt[n]{e^{h\pi}} = 1, \tag{5.1}$$

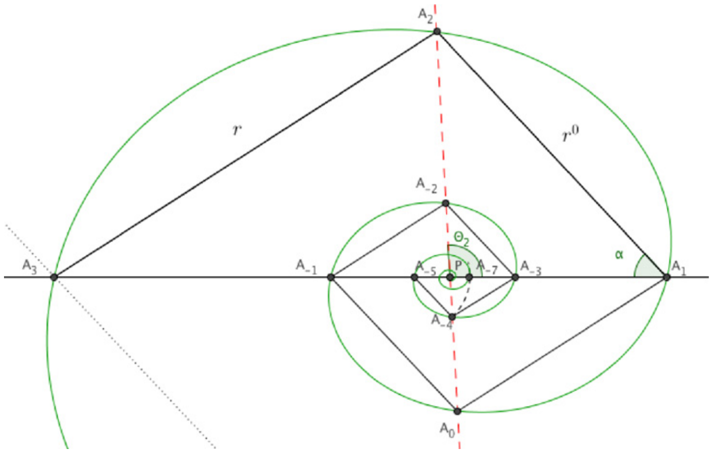

Fig. 5.1. Elliptic logarithmic spiral approximately through all vertices of an assigned (1.35, 2)-male spiral.

so that there exist infinite pairs $(k, \sqrt[k]{e^{h\pi}})$, such that k is a positive integer and $\sqrt[k]{e^{h\pi}}$ lies in $(1, \Phi)$. Therefore, if $\mathcal{S}$ is a logarithmic spiral, then we can write its equation as follows:

$$\rho = te^{h\theta} = tr^{\frac{k\theta}{\pi}} \quad \text{where } r = \sqrt[k]{e^{h\pi}}. \tag{5.2}$$

An application of Theorem 4.1 shows that the odd vertices (resp. the even vertices) of the $(\sqrt[k]{e^{h\pi}}, k)$-male spiral discretize $\mathcal{S}$.

We note that the discretization of logarithmic spirals provided by Theorem 4.1 has a low implementation difficulty, as the vertices of the male spiral are easily computable (see Lemma 3.1 and Corollary 3.2). This could help the study of all those phenomena related to logarithmic spirals.

The property of *self-similarity* of a logarithmic spiral leads to the consideration of these curves as a model in the study of the Fractal process. This connection is related for example to the notion of *dissipative quantum interference phase* (see [14]).

The use of the discretization of the curve can be useful in the learning of the Digital design Process (see [15], Introduction and Part II) which is an important skill for the experience of an architect (see also [16] and [17]). The discretization of a logarithmic spiral also has applications in seismology for the prevention of collapse mechanisms (see [18]).

It is natural to ask whether there exist some kind of "good" spirals through all the vertices of an (r, k)-male spiral, $\mathcal{P}_{r,k}$. Our conjecture is that the answer is positive.

We will define the *elliptic logarithmic spiral* and we denote it by $\mathcal{E}$, the curve whose equation is:

$$\begin{cases} x(\theta) = t_1 r^{h\theta/\pi} \cos(\theta) \\ y(\theta) = t_2 r^{h\theta/\pi} \sin(\theta), \end{cases} \quad \text{where } t_1 \text{ and } t_2 \text{ depend on the initial condition.} \tag{5.3}$$

Note that when $t_1 = t_2$ we have the equation of a logarithmic spiral, so we may think of $\mathcal{E}$ as an elliptic curve with an exponential growth around its pole P. In general $\mathcal{E}$ is not equiangular like the logarithmic spirals; on the other hand it preserves the proportion growth.

The problem is: given $r \in (1, \Phi)$, and a positive integer k, can we determine h, t_1, t_2 depending on r and k, such that Eq. (5.3) is satisfied? In other words: is there an elliptic logarithmic spiral, $\mathcal{E}_{r,k}$, through all the vertices of the (r, k)-spiral?

We represent the problem by Fig. 5.1.

Note that this last family of spirals also occurs in nature. It is known (see [19, p. 113]) that the two-thirds of all galaxies have a spiral structure. Among these there are some, for example the "S0 type" galaxies, that are placed in between the elliptical and logarithmic spiral galaxies. (See [20, page 390].) Also in Architecture the elliptic logarithmic spirals could be employed to realize special curves and surfaces (see for example [21–23]). Here their discretization by triangles suggest important informations.

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