Generating Kepler Triangles and Logarithmic Spiral with Recurrent Functions

Rahul Gohil

Student, B.E(Computer Engineering), Thadomal Shahani Engineering College, Mumbai, India
rahul.gohil@thadomal.org

Abstract

This paper presents some recurrent functions which we can use to make Kepler Triangles and through their special arrangement we can obtain logarithmic spirals, we also obtain a way get their discrete form.

Keywords — Kepler Triangles, Logarithmic Spirals, Recurrent Functions

1 Introduction

G. Anatriello, G. Vincenzi [1] discussed about continue triangles and how they discretize a logarithmic spiral by polygon chains and obtained different discrete spirals known as (r, k)-male spirals which discretize well known logarithmic spirals. Here we give a different approach for discretization of logarithmic spiral using kepler triangles.

1.1 Defining the Recurrent Function

Let's take the Fibonacci sequence, $f'(n) = f'(n-1) + f'(n-2)$, where $f'(0) = 0$ & $f'(1) = 1$.
It's easy to see that for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^n = \begin{bmatrix} f'(n+1) & f'(n) \\ f'(n) & f'(n-1) \end{bmatrix}$$

Binet's formula for nth Fibonacci number is $f'(n) = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$, where $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$
[2].
$\psi = 1 - \varphi = \dfrac{-1}{\varphi}$, therefore Binet's formula can also be written as $f'(n) = \dfrac{\varphi^{2n} - (-1)^n}{\varphi^n \sqrt{5}}$
Let's take a function f where $f(n) = \sqrt{f(n-1)^2 + f(n-2)^2}$, $f(0) < f(1)$ and $f(0), f(1) \in \mathbb{R}^+$.

We can define this sequence in matrix form as

$$\begin{bmatrix} f(n)^2 \\ f(n-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f(n-1)^2 \\ f(n-2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} f(1)^2 \\ f(0)^2 \end{bmatrix}$$

$$= A^{n-1} \begin{bmatrix} f(1)^2 \\ f(0)^2 \end{bmatrix}$$

$$\begin{bmatrix} f(n)^2 \\ f(n-1)^2 \end{bmatrix} = \begin{bmatrix} f'(n) & f'(n-1) \\ f'(n-1) & f'(n-2) \end{bmatrix} \begin{bmatrix} f(1)^2 \\ f(0)^2 \end{bmatrix}$$

Therefore, $f(n) = \sqrt{f'(n)f(1)^2 + f'(n-1)f(0)^2}$

Theorem 1.1.

$$\lim_{n \to \infty} \frac{f(n+a)}{f(n+b)} = \varphi^{\left(\dfrac{a-b}{2}\right)}$$

Proof.

$$\frac{f(n+a)}{f(n+b)} = \frac{\sqrt{f(n+a-1)^2 + f(n+a-2)^2}}{\sqrt{f(n+b-1)^2 + f(n+b-2)^2}}$$

$$= \frac{\sqrt{f'(n+a)f(1)^2 + f'(n+a-1)f(0)^2}}{\sqrt{f'(n+b)f(1)^2 + f'(n+b-1)f(0)^2}}$$

$$= \frac{\sqrt{f'(n+a-1)\left(\dfrac{f'(n+a)f(1)^2}{f'(n+a-1)} + f(0)^2\right)}}{\sqrt{f'(n+b-1)\left(\dfrac{f'(n+b)f(1)^2}{f'(n+b-1)} + f(0)^2\right)}}$$

$$= \frac{\sqrt{\dfrac{\varphi^{2(n+a)} - (-1)^{n+a}}{\varphi^{n+a}\sqrt{5}}\left(\dfrac{f'(n+a)f(1)^2}{f'(n+a-1)} + f(0)^2\right)}}{\sqrt{\dfrac{\varphi^{2(n+b)} - (-1)^{n+b}}{\varphi^{n+b}\sqrt{5}}\left(\dfrac{f'(n+b)f(1)^2}{f'(n+b-1)} + f(0)^2\right)}}$$

$$\lim_{n\to\infty} \frac{f(n+a)}{f(n+b)} = \lim_{n\to\infty} \frac{\sqrt{\dfrac{\varphi^{2(n+a)} - (-1)^{n+a}}{\varphi^{n+a}\sqrt{5}}\left(\dfrac{f'(n+a)f(1)^2}{f'(n+a-1)} + f(0)^2\right)}}{\sqrt{\dfrac{\varphi^{2(n+b)} - (-1)^{n+b}}{\varphi^{n+b}\sqrt{5}}\left(\dfrac{f'(n+b)f(1)^2}{f'(n+b-1)} + f(0)^2\right)}}$$

$$= \frac{\sqrt{\dfrac{\varphi^{2(n+a)}}{\varphi^{n+a}}\left(\varphi f(1)^2 + f(0)^2\right)}}{\sqrt{\dfrac{\varphi^{2(n+b)}}{\varphi^{n+b}}\left(\varphi f(1)^2 + f(0)^2\right)}}$$

$$= \sqrt{\varphi^{2(n+a)-n-a-2(n+b)+n+b}}$$
$$= \sqrt{\varphi^{2n+2a-n-a-2n-2b+n+b}}$$
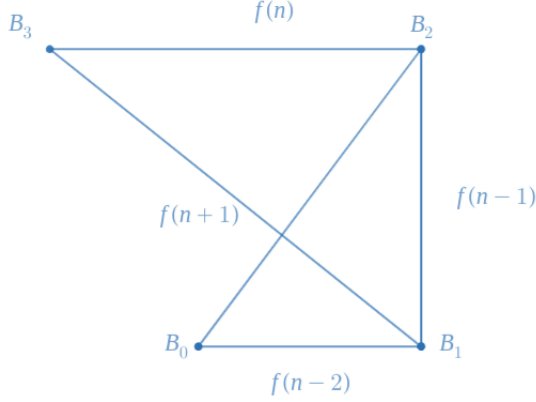$$\lim_{n\to\infty} \frac{f(n+a)}{f(n+b)} = \varphi^{\left(\dfrac{a-b}{2}\right)}$$
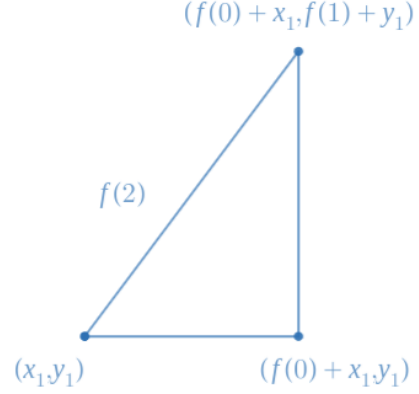
Figure 1: nth and (n+1)th triangle

Figure 2: Initial Triangle

1.2 Geometric Interpretation

The function $f(n)$ can be geometrically defined as length of a hypotenuse of a triangle whose base has length $f(n-2)$ and height of length $f(n-1)$. Now if we assume a line perpendicular to line B_1B_2 in Figure 1 at 1.2 passing through point B_2 and mark a point at length $f(n)$ from B_2 we get B_3 and then connect B_1 and B_3 we get it's length to be $f(n+1)$. Similarly we can generate more points, lines and triangles using this method.

Let (x_1, y_1) be any point on the 2D Plane, $x_1 < f(0)$, $y_1 < f(1)$. Let $(f(0) + x_1, y_1)$ and $(f(0) + x_1, f(1) + y_1)$ be points on the plane, by connecting them we get the triangle illustrated in Figure 2 at 1.2 which has it's length of hypotenuse as $f(2)$.

Let $I_0 = (x_1, y_1)$ and $I_1 = (f(0) + x_1, y_1)$, we define a function P which is point on the 2D Plane. Let $P(0) = (f(0) + x_1, f(1) + y_1)$, the P function has 2 class variables(a class variable is used to denote a variable associated with the object of that class in programming languages), namely x and y, which denote the x co-ordinate and y co-ordinate of that point respectively.

We define $P(n)$ as

$$P(n) = \begin{cases} (P(n-1).x - f(n+1), P(n-1).y) & n \equiv 1 \mod 4 \\ (P(n-1).x, P(n-1).y - f(n+1)) & n \equiv 2 \mod 4 \\ (P(n-1).x + f(n+1), P(n-1).y) & n \equiv 3 \mod 4 \\ (P(n-1).x, P(n-1).y + f(n+1)) & n \equiv 0 \mod 4 \end{cases}$$

We define a function L which is a line on the 2D Plane.

Let $L(0) = (I_0, P(0))$, this means that $L(0)$ is a line between points I_0 and $P(0)$, $L(1) = (I_1, P(1))$. L function has 5 class variables, namely *point*1, *point*2, *slope*, *constant*, *length* denoting the first point in the tuple, the second point in the tuple, slope of the line, the constant associated with it

and the length of the line respectively.
We define $L(n)$ as
$$L(n) = (L(n-2).point2, P(n))$$

Essentially $L(n)$ is the hypotenuse of the nth triangle. We also define a function L' which describes base and height of a triangle which has same class variables as L. Let $L'(0) = (I_0, I_1)$ and $L'(1) = (I_1, P(0))$.
$$L'(n) = (L'(n-1).point2, P(n-1))$$

Let's also define a function for a triangle T, $T(0) = (L'(0), L'(1), L(0))$. T function has 3 class variables $line1$, $line2$ and $line3$, denoting base, height and hypotenuse respectively.

$$T(n) = (T(n-1).line2, L'(n+1), L(n))$$

For each n, $L'(n)$ is the base, $L'(n+1)$ is the height and $L(n)$ is the hypotenuse of triangle $T(n)$. Through out this paper, we are going to assume n being an integer such that $n \equiv 0 \mod 4$
Theorem 1.2.
$$\lim_{n \to \infty} P(n).x - x_1 = f(n-2)$$

Proof. $P(n).x$ can alternatively be defined as

$$P(n).x = x_1 + \sum_{i=0}^{n/2} (-1)^i f(2i)$$

.

$$P(n).x - x_1 = \sum_{i=0}^{n/2} (-1)^i f(2i)$$

Let $g(n) = \dfrac{P(n).x - x_1}{f(n-2)} - 1$

$$g(n) = \frac{\displaystyle\sum_{i=0}^{n/2} (-1)^i f(2i)}{f(n-2)} - 1$$
$$= \sum_{i=0}^{n/2} \frac{(-1)^i f(2i)}{f(n-2)} - 1$$

Using 1.1
$$g(n) = \sum_{i=0}^{n/2} (-1)^i \varphi^{(\frac{2i-n+2}{2})} - 1$$

Let $S(n) = \displaystyle\sum_{i=0}^{n/2} (-1)^i \varphi^{(\frac{2i-n+2}{2})}$

The terms of $S(n)$ are $\varphi^{(\frac{-(n-2)}{2})} - \ldots + \varphi^{-3} - \varphi^{-2} + \varphi^{-1} - \varphi^0 + \varphi$.

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Therefore $S(n)$ can also be defined as

$$S(n) = \sum_{i=0}^{n/2} (-1)^i \varphi^{1-i}$$

Claim 1.

$$S(n) = \begin{cases} f'(n+1)\varphi^{-n} + (f'(n)+1)\varphi^{-n-1} & n \equiv 0 \mod 4 \text{ or } n \equiv 2 \mod 4 \\ f'(n+1)\varphi^{-n} + (f'(n)-1)\varphi^{-n-1} & n \equiv 1 \mod 4 \text{ or } n \equiv 3 \mod 4 \end{cases}$$

Proof. Let's prove this by mathematical induction.
Base Case $n = 0$, $S(n) = \varphi$ is our left side, $f'(0+1)\varphi^0 + (f'(0)+1)\varphi^{-1} = 1 + \varphi^{-1}$ is our right side.
Using the formula $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$ [3], right side can be written as $\varphi^0 + \varphi^{-1} = \varphi$, which is equal to left side.
So the theorem holds for $n = 0$
Inductive hypothesis : Suppose the proposition holds for all values of n upto some k, $k \geq 1$
Inductive step : Let $n = k + 1$, and $k \equiv 0 \mod 4$ then our left side is,

$$S(k+1) = f'(k+2)\varphi^{-k-1} + (f'(k+1)-1)\varphi^{-k-2}$$

Our right side is,

$$\begin{aligned} S(k+1) &= \varphi - \varphi^0 + \varphi^{-1} - \ldots + \varphi^{1-k} - \varphi^{1-(k+1)} \\ &= f'(k+1)\varphi^{-k} + (f'(k)+1)\varphi^{-k-1} - \varphi^{-k} \\ &= f'(k+1)\varphi^{-k-1} + f'(k+1)\varphi^{-k-2} + f'(k)\varphi^{-k-1} + \varphi^{-k-1} - \varphi^{-k} \\ S(k+1) &= f'(k+2)\varphi^{-k-1} + (f'(k+1)-1)\varphi^{-k-2} \end{aligned}$$

Since $f'(k+2) = f'(k+1) + f'(k)$ and $\varphi^{-k} = \varphi^{-k-1} + \varphi^{-k-2}$, which is our left side.
So the theorem holds for $n = k + 1$, therefore by the principle of mathematical induction, the theorem holds for all $n \in \mathbb{N}$ □

$$g(n) = S(n) - 1$$

$$S(n) = f'(n+1)\varphi^{-n} + (f'(n)+1)\varphi^{-n-1}$$
$$= f'(n+1)\varphi^{-n} + f'(n)^{-n-1} + \varphi^{-n-1}$$

$$= \frac{\varphi^{2(n+1)} - (-1)^{n+1}}{\varphi^{n+1}\sqrt{5}\ \varphi^n} + \frac{\varphi^{2(n)} - (-1)^n}{\varphi^n\sqrt{5}\ \varphi^{n+1}} + \varphi^{-n-1}$$

$$= \frac{\varphi^{2(n+1)} + \varphi^{2n}}{\varphi^{2n+1}\sqrt{5}} + \varphi^{-n-1}$$

$$= \frac{\varphi^{2n}(\varphi^2 + 1)}{\varphi^{2n+1}\sqrt{5}} + \varphi^{-n-1}$$

$$= \frac{\varphi^2 + 1}{\varphi\sqrt{5}} + \varphi^{-n-1}$$
$$= \frac{5 + \sqrt{5}}{\varphi\, 2\sqrt{5}} + \varphi^{-n-1}$$
$$= \frac{1 + \sqrt{5}}{2\varphi} + \varphi^{-n-1}$$
$$S(n) = 1 + \varphi^{-n-1}$$
$$\lim_{n\to\infty} S(n) = 1$$
$$\lim_{n\to\infty} g(n) = \lim_{n\to\infty} S(n) - 1$$
$$\lim_{n\to\infty} g(n) = 0$$

$$\lim_{n\to\infty} \frac{P(n).x - x_1}{f(n-2)} - 1 = 0$$
$$\lim_{n\to\infty} \frac{P(n).x - x_1}{f(n-2)} = 1$$
$$\lim_{n\to\infty} P(n).x - x_1 = f(n-2)$$

$\square$

Similarly we can prove that,

$$\lim_{n\to\infty} P(n).y - y_1 = f(n-1)$$
$$\lim_{n\to\infty} P(n+1).x - x_1 = -f(n)$$

Theorem 1.3. *As $n \to \infty$, angle between $L(n)$ and $L(n+1)$ tends $\to 90°$*

Proof. Let $A(n)$ be angle between $L(n)$ and $L(n+1)$

$$A(n) = \tan^{-1}\left(\left|\frac{L(n+1).slope - L(n).slope}{1 + (L(n+1).slope \cdot L(n).slope)}\right|\right)$$

Let $1 + (L(n+1).slope \cdot L(n).slope) = a$
$L(n+1) = (P(n-1), P(n+1))$ and $L(n) = (P(n-2), P(n))$

$$L(n+1).slope = \frac{a-1}{L(n).slope}$$

$$\frac{P(n+1).y - P(n-1).y}{P(n+1).x - P(n-1).x} = \frac{(a-1)(P(n).x - P(n-2).x)}{P(n).y - P(n-2).y}$$

$P(n+1) = (P(n).x - f(n+2), P(n).y)$
$P(n) = (P(n-1).x, P(n-1).y + f(n+1))$
$P(n-1) = (P(n-2).x + f(n), P(n-2).y)$

$$\frac{P(n).y - P(n-2).y}{P(n).x - P(n-2).x - f(n+2) - f(n)} = \frac{(a-1)(P(n).x - P(n-2).x)}{P(n).y - P(n-2).y}$$

$P(n).x = P(n-1).x = P(n-2).x + f(n)$, therefore $P(n).x - P(n-2).x = f(n)$
$P(n.y) = P(n-1).y + f(n-1)$ and $P(n-1).y = P(n-2).y$, therefore $P(n).y - P(n-2) = f(n+1)$

$$\frac{f(n+1)}{f(n) - f(n+2) - f(n)} = \frac{(a-1)(f(n))}{f(n+1)}$$

$$\frac{-f(n+1)^2}{f(n+2)f(n)} = a - 1$$

Let $g'(n) = \dfrac{-f(n+1)^2}{f(n+2)f(n)}$, using 1.1,

$$\lim_{n\to\infty} g'(n) = \frac{-\sqrt{\varphi}}{\sqrt{\varphi}}$$

$$\lim_{n\to\infty} g'(n) = -1$$

$$\lim_{n\to\infty} g'(n) = a - 1$$

$$-1 = a - 1$$

$$a = 0$$

Therefore,

$$\lim_{n\to\infty} 1 + (L(n+1).slope \cdot L(n).slope) = 0$$

Since, $\tan^{-1}\dfrac{1}{0} = 90°$

$$\lim_{n\to\infty} A(n) = 90°$$

Therefore, as $n \to \infty$, angle between $L(n)$ and $L(n+1)$ tends $\to 90°$ $\qquad\square$

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Theorem 1.4. *As $n \to \infty$, $P(n+2)$ tends to satisfy $L(n)$*

Proof. Let's satisfy $P(n+2)$ in $L(n)$ equation.

$$S'(n) = P(n+2).y - L(n).slope \cdot P(n+2).x - L(n).constant$$

$L(n) = (P(n-2), P(n))$

$L(n).slope = \dfrac{P(n).y - P(n-2).y}{P(n).x - P(n-2).x} = \dfrac{f(n+1)}{f(n)}$ as seen in 1.3

$L(n).constant = P(n).y - L(n).slope \cdot P(n).x = P(n).y - \dfrac{f(n+1)}{f(n)} P(n).x$

$P(n+2) = (P(n+1).x, P(n+1).y - f(n+3))$

$P(n+1) = (P(n).x - f(n+2), P(n).y)$

$$S'(n) = P(n).y - f(n+3) - \frac{f(n+1)}{f(n)}(P(n).x - f(n+2)) - \left(P(n).y - \frac{f(n+1)}{f(n)} P(n).x \right)$$

$$= \frac{f(n+2)f(n+1)}{f(n)} - f(n+3)$$

$$= \frac{f(n+2)f(n+1)}{f(n)} - \frac{f(n+3)f(n+2)}{f(n+2)}$$

Using 1.1

$$\lim_{n\to\infty} S'(n) = \sqrt{\varphi} f(n+2) - \sqrt{\varphi} f(n+2)$$

$$\lim_{n\to\infty} S'(n) = 0$$

Therefore, as $n \to \infty$, $P(n+2)$ tends to satisfy $L(n)$ $\qquad\square$

Corollary 1.4.1. *As $n \to \infty$, all $P(n)$ with even n tend to lie on the same line, same is true for all $P(n)$ with odd n.*

Proof. $L(n) = (P(n-2), P(n))$, therefore, for some n, $(P(n-2), P(n), P(n+2))$, all lie on the same line.

Similarly $P(n+4)$ satisfies $L(n+2)$, where $(P(n), P(n+2), P(n+4))$ lie on the same line. Since $P(n-2)$ and $P(n)$ lie on the same line, $P(n-2)$ and $P(n+4)$ must lie on the same line.

Therefore

$$(\ldots, P(n-2), P(n), P(n+2), P(n+4), \ldots)$$

lie on the same line. $\qquad\square$

2 Generating Kepler Triangles

A Kepler triangle is a right angled triangle whose edge-lengths are in geometric progression with $\sqrt{\varphi}$ as the common ratio and can be written as $1 : \sqrt{\varphi} : \varphi$ [4].

Theorem 2.1. *As $n \to \infty$, $T(n)$ is a kepler triangle.*

Proof. $T(n) = (T(n-1).line2, L'(n+1), L(n))$

$T(n-1) = (T(n-2).line2, L'(n), L(n))$

Therefore, $T(n) = (L'(n), L'(n+1), L(n))$

$$\frac{L'(n+1).length}{L'(n).length} = \frac{\sqrt{(P(n).x - P(n-1).x)^2 + (P(n).y - P(n-1).y)^2}}{\sqrt{(P(n-1).x - P(n-2).x)^2 + (P(n-1).y - P(n-2).y)^2}}$$

$$\frac{L'(n+1).length}{L'(n).length} = \frac{f(n+1)}{f(n)}$$

Using 1.1

$$\lim_{n\to\infty} \frac{L'(n+1).length}{L'(n).length} = \sqrt{\varphi}$$

$$\frac{L(n).length}{L'(n).length} = \frac{\sqrt{(P(n).x - P(n-2).x)^2 + (P(n).y - P(n-2).y)^2}}{\sqrt{(P(n-1).x - P(n-2).x)^2 + (P(n-1).y - P(n-2).y)^2}}$$

$$\frac{L(n).length}{L'(n).length} = \frac{f(n+2)}{f(n)}$$

Using 1.1

$$\lim_{n\to\infty} \frac{L(n).length}{L'(n).length} = \varphi$$

Therefore, as $n \to \infty$

$$\frac{L'(n).length}{L'(n).length} : \frac{L'(n+1).length}{L'(n).length} : \frac{L(n).length}{L'(n).length} = 1 : \sqrt{\varphi} : \varphi$$

Therefore, as $n \to \infty$, $T(n)$ is a kepler triangle. $\qquad\square$

Corollary 2.1.1. *As $n \to \infty$, $T(n)$ is similar to $T(n+1)$*

Proof. Since both $T(n)$ and $T(n+1)$ are kepler triangles, when we scale $T(n)$ by $\sqrt{\varphi}$ we get $T(n+1)$ or use 1.3 and use SAS test. $\qquad\square$

3   Logarithmic Spiral

Equation of a logarithmic spiral in polar form is,

$$r = ae^{k\theta}$$

 where a and k are constants [5].

To determine a and k, consider the equations,

$$r_n = ae^{k\theta_n}$$

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$$r_{n+1} = ae^{k\theta_{n+1}}$$

θ_n can be defined as,

$$\theta_n = \begin{cases} \lfloor \frac{n}{4} \rfloor 2\pi + \tan^{-1}(L(I_0, P(n)).slope) & n \equiv 0 \mod 4 \\ \lfloor \frac{n}{4} \rfloor 2\pi + \tan^{-1}(L(I_0, P(n)).slope) + \pi & n \equiv 1 \mod 4 \text{ or } n \equiv 2 \mod 4 \\ \lfloor \frac{n}{4} \rfloor 2\pi + \tan^{-1}(L(I_0, P(n)).slope) + 2\pi & n \equiv 3 \mod 4 \end{cases}$$

where $L(I_0, P(n))$ is a line which passes through I_0 and $P(n)$.
r_n can be defined as,

$$r_n = \sqrt{(P(n).x - x_1)^2 + (P(n).y - y_1)^2}$$

Now,

$$\frac{r_{n+1}}{r_n} = e^{k(\theta_{n+1}-\theta_n)}$$

$$k = \frac{\ln\left(\dfrac{r_{n+1}}{r_n}\right)}{\theta_{n+1}-\theta_n}$$

$$= \frac{\ln\left(\dfrac{\sqrt{(P(n+1).x - x_1)^2 + (P(n+1).y - y_1)^2}}{\sqrt{(P(n).x - x_1)^2 + (P(n).y - y_1)^2}}\right)}{\lfloor\frac{n+1}{4}\rfloor 2\pi + \tan^{-1}(L(I_0, P(n+1)).slope) + \pi - \lfloor\frac{n}{4}\rfloor 2\pi - \tan^{-1}(L(I_0, P(n)).slope)}$$

$$= \frac{\ln\left(\dfrac{\sqrt{(P(n+1).x - x_1)^2 + (P(n).y - y_1)^2}}{\sqrt{(P(n).x - x_1)^2 + (P(n).y - y_1)^2}}\right)}{\tan^{-1}\left(\dfrac{P(n+1).y - y_1}{P(n+1).x - x_1}\right) - \tan^{-1}\left(\dfrac{P(n).y - y_1}{P(n).x - x_1}\right) + \pi}$$

Using 1.2

$$\lim_{n\to\infty} k = \frac{\ln\left(\dfrac{\sqrt{(-f(n))^2 + f(n-1)^2}}{\sqrt{f(n-2)^2 + f(n-1)^2}}\right)}{\tan^{-1}\left(\dfrac{f(n-1)}{-f(n)}\right) - \tan^{-1}\left(\dfrac{f(n-1)}{f(n-2)}\right) + \pi}$$

$$= \frac{\ln\left(\dfrac{f(n+1)}{f(n)}\right)}{\tan^{-1}\left(\dfrac{f(n-1)}{-f(n)}\right) - \tan^{-1}\left(\dfrac{f(n-1)}{f(n-2)}\right) + \pi}$$

Using 1.1

$$= \frac{\ln\left(\sqrt{\varphi}\right)}{\tan^{-1}\left(\dfrac{-1}{\sqrt{\varphi}}\right) - \tan^{-1}\left(\sqrt{\varphi}\right) + \pi}$$

$$= \frac{\ln\left(\sqrt{\varphi}\right)}{-\tan^{-1}\left(\dfrac{1}{\sqrt{\varphi}}\right) - \tan^{-1}\left(\sqrt{\varphi}\right) + \pi}$$

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$$= \frac{\ln\left(\sqrt{\varphi}\right)}{-\tan^{-1}\left(\dfrac{\dfrac{1}{\sqrt{\varphi}} + \varphi}{1 - \dfrac{\sqrt{\varphi}}{\sqrt{\varphi}}}\right) + \pi}$$

$$= \frac{\ln\left(\sqrt{\varphi}\right)}{\pi - \dfrac{\pi}{2}}$$

$$= \frac{2\ln\left(\sqrt{\varphi}\right)}{\pi}$$

Therefore,

$$k = \frac{\ln\left(\varphi\right)}{\pi}$$

Now, $r = ae^{k\theta} = ae^{\frac{\ln \varphi}{\pi}\theta} = a\varphi^{\frac{\theta}{\pi}}$

$$a = r\varphi^{-\frac{\theta}{\pi}}$$

$$= r_n\varphi^{-\frac{\theta_n}{\pi}}$$

$$= r_n\varphi^{-\left(\frac{\lfloor \frac{n}{4}\rfloor 2\pi + \tan^{-1}\left(\frac{P(n).y - y_1}{P(n).x - x_1}\right)}{\pi}\right)}$$

$$= r_n\varphi^{-\left(\frac{n}{2} + \frac{\tan^{-1}\left(\frac{P(n).y - y_1}{P(n).x - x_1}\right)}{\pi}\right)}$$

$$= \frac{r_n}{\varphi^{\frac{n}{2}}}\varphi^{-\left(\frac{\tan^{-1}\left(\frac{P(n).y - y_1}{P(n).x - x_1}\right)}{\pi}\right)}$$

$$= \sqrt{\frac{(P(n).x - x_1)^2 + (P(n).y - y_1)^2}{\varphi^n}}\;\varphi^{-\left(\frac{\tan^{-1}\left(\frac{P(n).y - y_1}{P(n).x - x_1}\right)}{\pi}\right)}$$

Using 1.2

$$= \sqrt{\frac{f(n-2)^2 + f(n-1)^2}{\varphi^n}}\;\varphi^{-\left(\frac{\tan^{-1}\left(\frac{f(n-1)}{f(n-2)}\right)}{\pi}\right)}$$

Using 1.1

$$= \sqrt{\frac{f(n-2)^2(1+\varphi)}{\varphi^n}}\;\varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

$$= \sqrt{\frac{f'(n-2)f(1)^2 + f'(n-3)f(0)^2}{\varphi^n}}\;\sqrt{1+\varphi}\;\varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

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$$= \sqrt{\frac{f'(n-3)}{\varphi^n}} \; \sqrt{(\varphi f(1)^2 + f(0)^2)(1+\varphi)} \; \varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

$$= \sqrt{\frac{\varphi^{2(n-3)} - (-1)^{n-3}}{\varphi^{n-3}\sqrt{5}\,\varphi^n}} \; \sqrt{(\varphi f(1)^2 + f(0)^2)(1+\varphi)} \; \varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

$$= \sqrt{\frac{\varphi^{2n-6}}{\varphi^{2n-3}\sqrt{5}}} \; \sqrt{(\varphi f(1)^2 + f(0)^2)(1+\varphi)} \; \varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

$$a = \sqrt{\frac{(\varphi f(1)^2 + f(0)^2)(1+\varphi)}{\varphi^3\sqrt{5}}} \; \varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

Therefore,

$$a = \sqrt{\frac{\varphi f(1)^2 + f(0)^2}{\varphi\sqrt{5}}} \; \varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

Therefore, we obtained the constants

$$k = \ln\frac{\varphi}{\pi} \quad \text{and} \quad a = \sqrt{\frac{\varphi f(1)^2 + f(0)^2}{\varphi\sqrt{5}}} \; \varphi^{-\left(\frac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$$

The equation of logarithmic spiral is

$$r = a\varphi^{\frac{\theta}{\pi}}$$

Theorem 3.1. *As $n \to \infty$ $P(n)$ satisfies the logarithmic spiral*

Proof. Consider cartesian equation of logarithmic spiral,

$$y = r\sin\theta \quad \text{and} \quad x = r\cos\theta$$

$$y_n = r_n\sin\theta_n \quad \text{and} \quad x_n = r_n\cos\theta_n$$

$r_n = \sqrt{(P(n).x - x_1)^2 + (P(n).y - y_1)^2}$, using 1.2 $\lim_{n\to\infty} r_n = f(n)$

Let $s = L(I_0, P(n)).slope$

$$\theta_n = \lfloor\frac{n}{4}\rfloor 2\pi + \tan^{-1}(L(I_0, P(n)).slope) = \frac{n\pi + 2\tan^{-1}(s)}{2}$$

Using the trigonometric relation $2\tan^{-1}(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\theta_n = \frac{n\pi + \cos^{-1}\left(\frac{1-s^2}{1+s^2}\right)}{2}$$

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Since $s > 1$, $1 - s^2 < 0$, therefore $\cos^{-1}\left(\frac{1-s^2}{1+s^2}\right) = \cos^{-1}\left(\frac{(-1)(s^2-1)}{1+s^2}\right) = \pi - \cos^{-1}\left(\frac{(s^2-1)}{1+s^2}\right)$

$$\theta_n = \frac{(n+1)\pi - \cos^{-1}\left(\frac{(s^2-1)}{1+s^2}\right)}{2}$$

$$\sin\theta_n = \sin\left(\frac{(n+1)\pi - \cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)}{2}\right) \quad \text{and} \quad \cos\theta_n = \cos\left(\frac{(n+1)\pi - \cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)}{2}\right)$$

Using the trignometric relations $\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1-\cos(x)}{2}}$ and $\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos(x)}{2}}$

$$\sin\theta_n = \sqrt{\frac{1 - \cos\left((n+1)\pi - \cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)\right)}{2}} \quad \text{and} \quad \cos\theta_n = \sqrt{\frac{1 + \cos\left((n+1)\pi - \cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)\right)}{2}}$$

Using the trignometric relation $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

$$\cos\left((n+1)\pi - \cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)\right) = \cos((n+1)\pi)\cos\left(\cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)\right) +$$

$$\sin((n+1)\pi)\sin\left(\cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)\right) \quad (1)$$

Using $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$, $\cos((n+1)\pi) = (-1)$, $\sin((n+1)\pi) = 0$

$$\cos\left((n+1)\pi - \cos^{-1}\left(\frac{s^2-1}{1+s^2}\right)\right) = (-1)\left(\frac{s^2-1}{1+s^2}\right)$$

Therefore,

$$\sin\theta_n = \sqrt{\frac{1 + \left(\frac{s^2-1}{1+s^2}\right)}{2}} \quad \text{and} \quad \cos\theta_n = \sqrt{\frac{1 - \left(\frac{s^2-1}{1+s^2}\right)}{2}}$$

$s = L(I_0, P(n)).slope = \frac{P(n).y - y_1}{P(n).x - x_1}$, using 1.1 and 1.2, $\lim_{n\to\infty} s = \sqrt{\varphi}$

$$\lim_{n\to\infty} \sin\theta_n = \sqrt{\frac{1 + \left(\frac{\varphi-1}{1+\varphi}\right)}{2}} \quad \text{and} \quad \lim_{n\to\infty} \cos\theta_n = \sqrt{\frac{1 - \left(\frac{\varphi-1}{1+\varphi}\right)}{2}}$$

$$\lim_{n\to\infty} \sin\theta_n = \sqrt{\frac{1 + \left(\frac{\varphi-1}{\varphi^2}\right)}{2}} \quad \text{and} \quad \lim_{n\to\infty} \cos\theta_n = \sqrt{\frac{1 - \left(\frac{\varphi-1}{\varphi^2}\right)}{2}}$$

$$\lim_{n\to\infty} \sin\theta_n = \sqrt{\frac{\varphi^3 + 1}{2\varphi^3}} \quad \text{and} \quad \lim_{n\to\infty} \cos\theta_n = \sqrt{\frac{\varphi^3 - 1}{2\varphi^3}}$$

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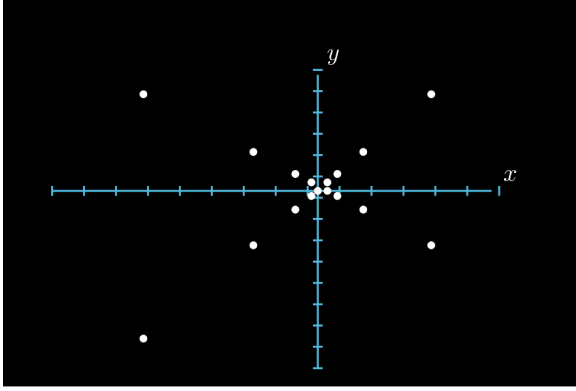Figure 3: Plot of $P(n)$

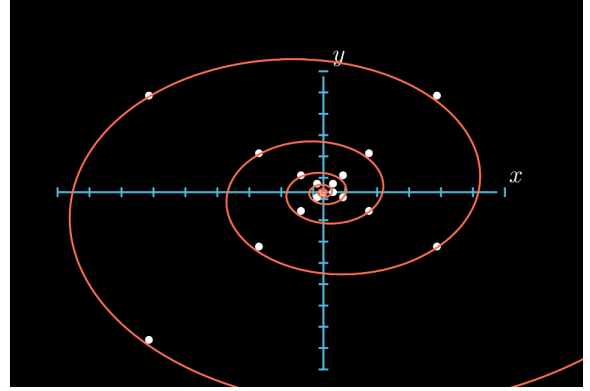Figure 4: Logarithmic Spiral

$\varphi^3 + 1 = \varphi^2 + \varphi + 1 = 2\varphi^2$
$\varphi^3 - 1 = \varphi^2 + \varphi - 1 = 1 + \varphi + \varphi - 1 = 2\varphi$

$$\lim_{n \to \infty} \sin \theta_n = \frac{1}{\sqrt{\varphi}} \quad \text{and} \quad \lim_{n \to \infty} \cos \theta_n = \frac{1}{\varphi}$$

$$\lim_{n \to \infty} r_n \sin \theta_n = \frac{f(n)}{\sqrt{\varphi}} \quad \text{and} \quad \lim_{n \to \infty} r_n \cos \theta_n = \frac{f(n)}{\varphi}$$

Using 1.1

$$\lim_{n \to \infty} r_n \sin \theta_n = f(n-1) \quad \text{and} \quad \lim_{n \to \infty} r_n \cos \theta_n = f(n-2)$$

Using 1.2

$$\lim_{n \to \infty} r_n \sin \theta_n = P(n).y - y_1 \quad \text{and} \quad \lim_{n \to \infty} r_n \cos \theta_n = P(n).x - x_1$$

For $x_1 = 0$ and $y_1 = 0$

$$\lim_{n \to \infty} y_n = P(n).y \quad \text{and} \quad \lim_{n \to \infty} x_n = P(n).x$$

Therefore, as $n \to \infty$ $P(n)$ satisfies the logarithmic spiral $\qquad\square$

Figure 6 at 3 describes plot of $L'(n)$ as a discrete form of the logarithmic spiral.
The following plots are visualized using GFKTv1.0.4 [6], it is made using Python3 and uses manimlib, available at https://github.com/rahul-gohil/GFKT, archived at doi:10.5281/zenodo.3779050
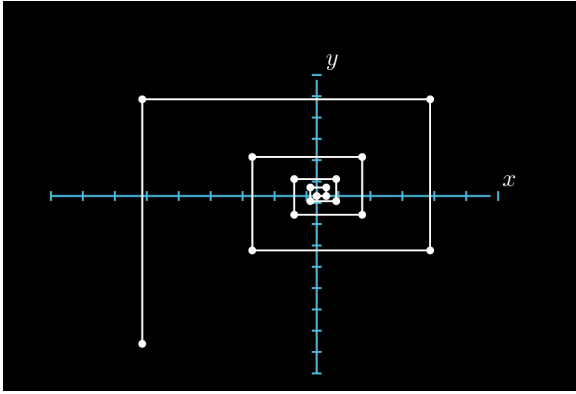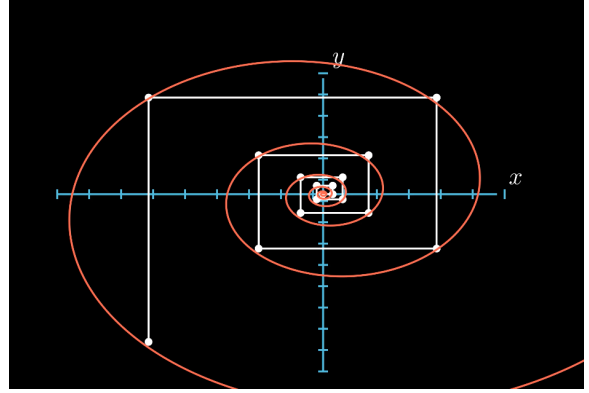
Figure 5: Plot of $L'(n)$

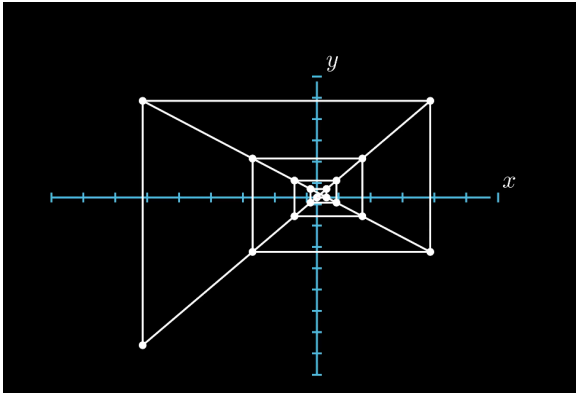Figure 6: $L'(n)$ as discrete form of logarithmic spiral
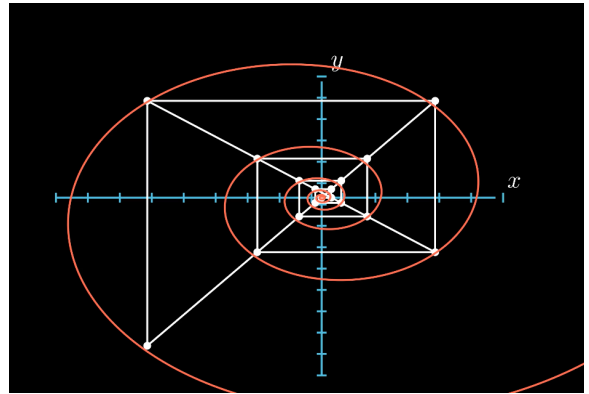

Figure 7: Plot of $T(n)$

Figure 8: Complete Plot

4 Conclusion

Thus we made use of the recurrent function $f(n) = \sqrt{f(n-1)^2 + f(n-2)^2}$, where $f(0)$ and $f(1) \in \mathbb{R}^+$

to generate Kepler Triangles through the recurrent functions $P(n)$, $L'(n)$, $L(n)$ and
$T(n) = (L'(n), L'(n+1), L(n))$ at 1.2, and through their geometrical arrangement obtained the logarithmic spiral

$$r = a\varphi^{\dfrac{\theta}{\pi}}$$

where $a = \sqrt{\dfrac{\varphi f(1)^2 + f(0)^2}{\varphi \sqrt{5}}} \; \varphi^{-\left(\dfrac{\tan^{-1}\left(\sqrt{\varphi}\right)}{\pi}\right)}$

We obtained $L'(n)$ as the discrete form of logarithmic spiral and for small values of x_1 and y_1 as $n \to \infty$ $P(n)$ satisfies the logarithmic spiral.

References

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[2] Wikipedia Article on Fibonacci Numbers

[3] Powers of Phi by Gary Meisner,https://www.goldennumber.net/powers-of-phi/

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