Chaos, Solitons & Fractals 44 (2011) 145–152

Contents lists available at ScienceDirect

Chaos, Solitons & Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Limit of ratio of consecutive terms for general order-k linear homogeneous recurrences with constant coefficients

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ARTICLE INFO

Article history:
Received 15 July 2010
Accepted 12 January 2011
Available online 16 February 2011

ABSTRACT

For complex linear homogeneous recursive sequences with constant coefficients we find a necessary and sufficient condition for the existence of the limit of the ratio of consecutive terms. The result can be applied even if the characteristic polynomial has not necessarily roots with modulus pairwise distinct, as in the celebrated Poincaré's theorem. In case of existence, we characterize the limit as a particular root of the characteristic polynomial, which depends on the initial conditions and that is not necessarily the unique root with maximum modulus and multiplicity. The result extends to a quite general context the way used to find the Golden mean as limit of ratio of consecutive terms of the classical Fibonacci sequence.

1. Introduction

The Fibonacci sequence, which we will denote by (F_n), defined by

$$F_0 = F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n > 1 \tag{1.1}$$

has been of wide interest since his first appearance in the book Liber Abaci published in 1202. A property due to Kepler (in the book "Harmonices Mundi", 1619, p. 273, see [1,2]), is

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi \tag{1.2}$$

where Φ denotes the highly celebrated Golden mean, which appears in several human fields, especially in Art and Architecture, even if sometimes not always appropriately (see e.g. [3,4] and references therein).

The Fibonacci sequence and the Golden mean are often used as a model of recursive phenomena in Botany (see e.g. [5,6]), Chemistry (see e.g. [7]), Physics and Engineering (see e.g. [8] and references therein), Medicine (see e.g. [9]). In such studies it is of a certain relevance the asymptotic behavior of processes which can be described through linear recurrences. This behavior can be understood studying the limit of the ratio of consecutive terms of the related sequences. The aim of this paper is to throw some light on the problem to understand whether a given recurrence develops with a geometric law or not, and, in the positive case, to get informations on the proportion factor. In order to state our results, following the Difference Equations Theory, we recall that a sequence of complex numbers $(F_n)_{n\in\mathbb{N}_0} = (F_n)$ not identically zero (we will write $(F_n) \not\equiv 0$), is said *linear recursive of order k, with constant coefficients*, when there exist $a_0 \neq 0, a_1, \ldots, a_{k-1}$ in $\mathbb{C}$ such that

$$F_n + a_{k-1}F_{n-1} + \cdots + a_0F_{n-k} = 0 \quad \forall n \geqslant k \tag{1.3}$$

The polynomial

$$p(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0$$

is said *characteristic polynomial* of (F_n) and the equation $p(\lambda) = 0$ is said *characteristic equation* of (F_n).

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doi:10.1016/j.chaos.2011.01.003

The explicit expression of sequences (F_n) satisfying (1.3) is given by the following well known theorem (see (4.8) in [10], Theorem 3.6 in [11], or Theorem 2.15 in [12]):

Theorem 1.1. *Let (F_n) be a linear recursive sequence of order k with constant coefficients and let p be its characteristic polynomial. Let $\lambda_1,\ldots,\lambda_h$ be the roots, with respective multiplicity $k_1,\ldots,k_h, k_1+\cdots+k_h=k$. The following equality holds:*

$$F_n = c_{1,1}\lambda_1^n + c_{1,2}n\lambda_1^n + \cdots + c_{1,k_1}n^{k_1-1}\lambda_1^n + \cdots + c_{h,1}\lambda_h^n + c_{h,2}n\lambda_h^n + \cdots + c_{h,k_h}n^{k_h-1}\lambda_h^n \quad \forall n \in \mathbb{N}_0 \tag{1.4}$$

where $c_{i,j}$ are (complex) numbers uniquely determined by $F_0,\ldots,F_{k-1}$.

We will refer to $F_0,\ldots,F_{k-1}$ as the *initial conditions* and to (1.4) as *Binet's formula* for the sequence (F_n).

As a consequence of the above result, we will point out (see Theorem 2.3) the connection, for a given linear recursive sequence (F_n), between the notion of "leading" root of p (see Definition 2.1), and the limit

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \Psi \tag{1.5}$$

Inspired by the classical case of the Fibonacci sequence, we will refer to Ψ as the *Kepler limit* of (F_n). In particular we will show that the unique leading root of p and the Kepler limit coincide.

The problem of the study the ratio of consecutive terms of linear recurrences attracted the attention of several researchers. The first important result in this direction is due to Poincaré (see e.g. [11] or [13, p. 526]), who proved that if the roots of the characteristic polynomial of a homogeneous linear recurrence with constant coefficients (this assumption has been in fact a little bit weakened, the resulting sequences being called "of Poincaré type") have pairwise distinct modulus, then the Kepler limit exists, and it coincides with one of the roots. The condition about the pairwise distinct modulus is somewhat optimal ([13]): if there exist two distinct roots with the same modulus, then it is always possible to consider initial conditions such that the Kepler limit does not exist.

After this result, the attention of the researchers has been devoted to the relation between the characterization of the root of the characteristic polynomial which coincides with the Kepler limit, and how the root may depend on the initial conditions. They studied particular sequences and/or tried to characterize, imposing restrictions on the initial conditions or on the characteristic polynomial, the root which is limit as that one which is the unique with maximum modulus and multiplicity. Without entering, for the moment, into the details of such papers, let us mention here [14] for linear recurrences of order 2, and e.g. [15–20] for more specialistic papers.

In this paper we give, for complex linear homogeneous recursive sequences with constant coefficients, a necessary and sufficient condition for the existence of the limit of the ratio of consecutive terms. The result does not impose restrictions on the initial conditions or on the characteristic polynomial, and, in particular, can be applied even if the characteristic polynomial has not necessarily roots with modulus pairwise distinct, as in the celebrated Poincaré's theorem. In case of existence, the limit is characterized as a particular root of the characteristic polynomial, which depends on the initial conditions and that is not necessarily the unique root with maximum modulus and multiplicity.

2. The main result

Definition 2.1. Let (F_n), $(F_n)\not\equiv 0$, be a linear homogeneous recurrence with constant coefficients, with initial conditions $F_0,\ldots,F_{n-1}$ and characteristic polynomial $p=p(\lambda)$. Let

$$F_n = c_{1,1}\lambda_1^n + c_{1,2}n\lambda_1^n + \cdots + c_{1,k_1}n^{k_1-1}\lambda_1^n + \cdots + c_{h,1}\lambda_h^n + c_{h,2}n\lambda_h^n + \cdots + c_{h,k_h}n^{k_h-1}\lambda_h^n \quad \forall n \in \mathbb{N}_0$$

be the representation of F_n given by Binet's formula (Theorem 1.1), where the coefficients $c_{i,j}$ are not all zero. For any index $i\in\{1,\ldots,h\}$ let $k(i)$ be the greatest index j such that $c_{i,j}\neq 0$; if $c_{i,j}=0$ for any j, let $k(i)=0$. Assuming that

$|\lambda_i|\geqslant|\lambda_j|$, $i<j$ and $k(i)\geqslant k(j)$ when $|\lambda_i|=|\lambda_j|$, $i<j$, let M be the smallest index i such that $c_{i,j}\neq 0$ for some j.

We will say that a root λ_i is a *leading root* of p (*with respect to the initial conditions*), if $|\lambda_i|=|\lambda_M|$ and $k(i)=k(M)$. In particular λ_M is a leading root of p.

We will say that the initial conditions and the characteristic polynomial are *in agreement* when λ_M is the unique leading root of p. Of course this is equivalent to say that $|\lambda_i|<|\lambda_M|$ for any $i>M$ or $k(i)<k(M)$ for any $i>M$ such that $|\lambda_i|=|\lambda_M|$

Remark 2.2. Note that a root of maximum modulus and multiplicity is not necessarily a leading root. Therefore, even if the characteristic polynomial of a sequence (F_n) has a unique root of maximum modulus and multiplicity, this root may be different from the Kepler limit of (F_n) (see next Example 4.5), or the Kepler limit may not exist at all (see next Example 4.6).

Theorem 2.3. *Let (F_n), $(F_n)\not\equiv 0$, be a recursive sequence with characteristic polynomial p. The following are equivalent:*

(i) *The initial conditions and p are in agreement;*
(ii) *There exists $\nu\in\mathbb{N}$ such that $F_n\neq 0$ for every $n\geqslant\nu$ and there exists the Kepler limit of (F_n), different from zero.*

If one of the above holds, the Kepler limit is the unique leading root of p with respect to the initial conditions of (F_n).

The main novelty of our result is that we provide a sufficient condition for the existence of the Kepler limit, without any further assumption on the characteristic polynomial; on the other hand, the novelty of the necessity condition is the fact that the root characterized as limit must be "alone" in modulus and multiplicity, among those ones effectively present in the Binet's formula representation, and that it is not necessarily – in the framework of the maximal generality of linear recurrences – related to the concept of "dominant root" (see e.g. [15]).

Let us stress that the delicate part of the proof of the sufficiency is to estimate the sum (3.16), whose effective number of terms (after grouping the similar monomials) cannot be controlled, especially because, for "few" consecutive terms, such sum could be, a priori, even equal to 0. Usually such estimates can be done splitting the discussion in several cases, or reducing the arguments to recurrences of low order: see e.g. the detailed analysis of positiveness of linear recurrences in [21] or the Poincaré result in [13]. About the statement (i) ⇒ (ii) see also [22].

3. Proof of the main result

We begin with a couple of lemmas, which may have independent interest.

Lemma 3.1. *Let*

$$H_n = \sum_{i=1}^{r} c_i\mu_i^n \quad n = 0,1,\ldots$$

where $r \geqslant 1$, $\mu_i \in \mathbb{C}\setminus\{0\}$ *for any* $i = 1,\ldots,r$, $\mu_i \neq \mu_j$ *if* $i \neq j$, $c_i \in \mathbb{C}$ *for any* $i = 1,\ldots,r$. *If for some* $v \in \mathbb{N}$ *it is*

$$H_v = H_{v+1} = \cdots = H_{v+r-1} = 0 \tag{3.1}$$

then $c_1 = c_2 = \cdots = c_r = 0$.

Proof. Consider the system of r equations in the r unknowns $x_1,\ldots,x_r$:

$$\begin{cases} \mu_1^v x_1 + \mu_2^v x_2 + \cdots + \mu_r^v x_r = 0 \\ \mu_1^{v+1} x_1 + \mu_2^{v+1} x_2 + \cdots + \mu_r^{v+1} x_r = 0 \\ \vdots \\ \mu_1^{v+r-1} x_1 + \mu_2^{v+r-1} x_2 + \cdots + \mu_r^{v+r-1} x_r = 0 \end{cases} \tag{3.2}$$

Since the determinant

$$\begin{vmatrix} \mu_1^v & \mu_2^v & \ldots & \mu_r^v \\ \mu_1^{v+1} & \mu_2^{v+1} & \ldots & \mu_r^{v+1} \\ \vdots & \vdots & \ldots & \vdots \\ \mu_1^{v+r-1} & \mu_2^{v+r-1} & \ldots & \mu_r^{v+r-1} \end{vmatrix} = (\mu_1\mu_2\ldots\mu_r)^v \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \mu_1 & \mu_2 & \ldots & \mu_r \\ \vdots & \vdots & \ldots & \vdots \\ \mu_1^{r-1} & \mu_2^{r-1} & \ldots & \mu_r^{r-1} \end{vmatrix}$$
$$= (\mu_1\mu_2\ldots\mu_r)^v \prod_{i<j}(\mu_j - \mu_i) \neq 0$$

because of the assumptions on the μ_i's, the homogeneous system (3.2) admits only the solution $x_1 = \cdots = x_r = 0$. Since (3.1) is equivalent to the fact that $(c_1,\ldots,c_r)$ solves the system (3.2), the assertion follows. □

Lemma 3.2. *Let*

$$G_n = \sum_{i=1}^{r} c_i\mu_i^n \quad n = 0,1,\ldots \tag{3.3}$$

where $r \geqslant 2$,

$$\mu_i \in \mathbb{C}\setminus\{0\}, \quad \mu_i \neq \mu_j \quad \text{if} \quad i \neq j, \tag{3.4}$$

$$c_i \in \mathbb{C}\setminus\{0\}. \tag{3.5}$$

Then, setting $w_n = G_{n+1}G_{n-1} - G_n^2$, *for every* $n \in \mathbb{N}$ *there exists* $\tau \in \{n, n+1,\ldots,n+r-1\}$ *such that* $w_\tau \neq 0$.

Remark 3.3. Note that the assumption $r \geqslant 2$ in Lemma 3.2 is necessary. In fact, in the case $r = 1$, it is $G_n = c\mu^n$ with $c \neq 0$, $\mu \neq 0$ but

$$w_n = G_{n+1}G_{n-1} - G_n^2 = c\mu^{n+1}\cdot c\mu^{n-1} - (c\mu^n)^2 = 0 \quad \forall n \in \mathbb{N}.$$

Proof of Lemma 3.2. We prove our Lemma by a contradiction argument, so we assume that there exists $v \in \mathbb{N}$ such that

$$G_{n+1}G_{n-1} - G_n^2 = 0 \quad \forall n = v, v+1,\ldots,v+r-1. \tag{3.6}$$

We first observe that it cannot be

$$G_v = G_{v+1} = \cdots = G_{v+r-1} = 0$$

because, applying Lemma 3.1 with G_n in place of H_n, the assumption (3.5) would lead to a contradiction. In the following we may therefore assume that there exists $G_\sigma \neq 0$, for some $\sigma \in \{v, v+1,\ldots,v+r-1\}$.

By (3.6) it is $G_{\sigma+1}G_{\sigma-1} = G_\sigma^2 \neq 0$, and therefore $G_{\sigma-1} \neq 0$, $G_{\sigma+1} \neq 0$. Iterating this argument we get that

$$G_{v-1} \neq 0, \quad G_v \neq 0,\ldots,G_{v+r-1} \neq 0. \tag{3.7}$$

For all $n \in \{v, v+1,\ldots,v+r-1\}$ we have, from (3.6) and (3.7), that

$$\frac{G_n}{G_{n-1}}\left(\frac{G_{n+1}}{G_n} - \frac{G_n}{G_{n-1}}\right) = \frac{G_{n+1}G_{n-1} - G_n^2}{G_{n-1}^2} = 0$$

and therefore, again by (3.7),

$$\frac{G_{n+1}}{G_n} = \frac{G_n}{G_{n-1}} \tag{3.8}$$

$$G_n = G_{v-1}\frac{G_n}{G_{v-1}} = G_{v-1}\frac{G_n}{G_{n-1}}\cdot\frac{G_{n-1}}{G_{n-2}}\cdot\ldots\cdot\frac{G_v}{G_{v-1}}$$
$$= c\rho^{n-v+1} \tag{3.9}$$

where, by (3.7) and (3.8),

$$c = G_{v-1} \neq 0, \quad \rho = \frac{G_v}{G_{v-1}} \neq 0. \tag{3.10}$$

By (3.3) and (3.9) it is

$$\sum_{i=1}^{r} c_i\mu_i^n = c\rho^{1-v}\cdot\rho^n$$

$$\sum_{i=1}^{r} c_i\mu_i^n - (c\rho^{1-v})\rho^n = 0 \quad n = v,\ldots,v+r-1 \tag{3.11}$$

At this point, by assumption (3.4), there are two possibilities:

(A) $\rho \neq \mu_i\ \forall i = 1,\ldots,r$;
(B) $\rho = \mu_{i_0}$ for exactly one $i_0 \in \{1,\ldots,r\}$.

In the case (A), the sum on the left hand side of (3.11) satisfies the assumptions of Lemma 3.1, from which we get that all coefficients in the sum are zero, including $c\rho^{1-v} = 0$, against (3.10). In the case (B), we may again apply Lemma 3.1 to the same sum which possibly has, in fact, $r-1$ terms (this happens exactly in the case $c_{i_0} = c\rho^{1-v}$; notice that since $r \geqslant 2$, by (3.5) at least one term of the sum must survive). The conclusion is that if $j \in \{1,\ldots,r\}$, $j \neq i_0$, it must be $c_j = 0$, and this contradicts assumption (3.5). Lemma 3.2 is therefore proved. □

We are now in position to prove the Theorem 2.3.

Proof (i) ⇒ (ii). With the notation introduced in Definition 2.1, F_n can be written as follows:

$$F_n = c_{M,1}\lambda_M^n + \cdots + c_{M,k(M)}n^{k(M)-1}\lambda_M^n + \sum_{i>M} c_{i,1}\lambda_i^n$$
$$+ \cdots + c_{i,k_i}n^{k_i-1}\lambda_i^n. \tag{3.12}$$

At first we show that (F_n) is definitively not zero. By assumption (i), p has a unique leading root λ_M, of course different from zero. Clearly it is $n^{k(M)-1}\lambda_M^n \neq 0$ for any $n \in \mathbb{N}$ and therefore by (3.12)

$$\left|\left|\frac{F_n}{n^{k(M)-1}\lambda_M^n}\right| - |c_{M,k(M)}|\right| \leqslant \left|\frac{F_n}{n^{k(M)-1}\lambda_M^n} - c_{M,k(M)}\right|$$
$$= \left|\frac{c_{M,1}\lambda_M^n + \cdots + c_{M,k(M)}n^{k(M)-1}\lambda_M^n + \sum_{i>M} c_{i,1}\lambda_i^n + \cdots + c_{i,k_i}n^{k_i-1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}\right.$$
$$\left. - c_{M,k(M)}\right| \leqslant \left|\frac{c_{M,1}\lambda_M^n}{n^{k(M)-1}\lambda_M^n}\right| + \cdots + \left|\frac{c_{M,k(M)}n^{k(M)-1}\lambda_M^n}{n^{k(M)-1}\lambda_M^n} - c_{M,k(M)}\right|$$
$$+ \sum_{i>M}\left|\frac{c_{i,1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}\right| + \cdots + \left|\frac{c_{i,k_i}n^{k_i-1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}\right| \to 0$$

The convergence to zero is readily explained as follows: the first $k(M)$ terms go to zero because the last one is exactly zero and the previous ones, if any, have in the denominator a power of n greater than that one in the numerator; on the other hand, for each $i > M$, there are two possibilities: if $|\lambda_i| < |\lambda_M|$, then for every j, $1 \leqslant j \leqslant k_i$, it is

$$\left|\frac{c_{i,j}n^{j-1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}\right| = |c_{i,j}|n^{j-k(M)}\left(\frac{|\lambda_i|}{|\lambda_M|}\right)^n \to 0;$$

if $|\lambda_i| = |\lambda_M|$ and $j > k(i)$, then $c_{i,j} = 0$ and therefore

$$\left|\frac{c_{i,j}n^{j-1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}\right| = 0;$$

finally, if $|\lambda_i| = |\lambda_M|$ and $j \leqslant k(i)$ then by the assumption of agreement it is $k(i) < k(M)$ so that $j < k(M)$ and

$$\left|\frac{c_{i,j}n^{j-1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}\right| = |c_{i,j}|n^{j-k(M)} \to 0.$$

This shows that

$$\left|\left|\frac{F_n}{n^{k(M)-1}\lambda_M^n}\right| - |c_{M,k(M)}|\right| \to 0$$

and therefore for n sufficiently great it is

$$\left|\frac{F_n}{n^{k(M)-1}\lambda_M^n}\right| \geqslant \left|\frac{c_{M,k(M)}}{2}\right| > 0$$

because $c_{M,k(M)} \neq 0$ by definition of M. It follows that the sequence (F_n) is definitively not zero.

We may now consider the sequence $\left(\frac{F_{n+1}}{F_n}\right)$ and show that it is convergent to λ_M, the unique leading root of p. We have

$$\frac{F_{n+1}}{F_n} = \frac{c_{M,1}\lambda_M^{n+1} + \cdots + c_{M,k(M)}(n+1)^{k(M)-1}\lambda_M^{n+1} + \sum_{i>M} c_{i,1}\lambda_i^{n+1} + \cdots + c_{i,k_i}(n+1)^{k_i-1}\lambda_i^{n+1}}{c_{M,1}\lambda_M^n + \cdots + c_{M,k(M)}n^{k(M)-1}\lambda_M^n + \sum_{i>M} c_{i,1}\lambda_i^n + \cdots + c_{i,k_i}n^{k_i-1}\lambda_i^n}$$

Dividing each term by $n^{k(M)-1}\lambda_M^n$, we have

$$\frac{F_{n+1}}{F_n} = \frac{\frac{c_{M,1}\lambda_M^{n+1}}{n^{k(M)-1}\lambda_M^n} + \cdots + \frac{c_{M,k(M)}(n+1)^{k(M)-1}\lambda_M^{n+1}}{n^{k(M)-1}\lambda_M^n} + \sum_{i>M}\frac{c_{i,1}\lambda_i^{n+1}}{n^{k(M)-1}\lambda_M^n} + \cdots + \frac{c_{i,k_i}(n+1)^{k_i-1}\lambda_i^{n+1}}{n^{k(M)-1}\lambda_M^n}}{\frac{c_{M,1}\lambda_M^n}{n^{k(M)-1}\lambda_M^n} + \cdots + \frac{c_{M,k(M)}n^{k(M)-1}\lambda_M^n}{n^{k(M)-1}\lambda_M^n} + \sum_{i>M}\frac{c_{i,1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n} + \cdots + \frac{c_{i,k_i}n^{k_i-1}\lambda_i^n}{n^{k(M)-1}\lambda_M^n}}$$

Arguing as above, it turns out that the numerator goes to $c_{M,k(M)}\lambda_M$, and the denominator goes to $c_{M,k(M)}$, so that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lambda_M \neq 0. \qquad \square$$

Proof (ii) ⇒ (i). Let λ_M, $k(M)$ be as in Definition 2.1 and let $\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_r}\}$ be the set of the leading roots of the characteristic polynomial of (F_n), i.e. the set of the roots such that $|\lambda_i| = |\lambda_M|$ and $k(i) = k(M)$ for all $i \in \{i_1, i_2, \ldots, i_r\}$. We have to show that $r = 1$.

Using Landau's notation we may write

$$F_n = \sum_{u=1}^{r} c_{i_u,k(M)}n^{k(M)-1}\lambda_{i_u}^n + o\left(n^{k(M)-1}|\lambda_M|^n\right).$$

Observe that setting

$$\mu_{i_u} = \frac{\lambda_{i_u}}{|\lambda_M|} \quad u = 1, \ldots, r$$

$$G_n = \frac{1}{n^{k(M)-1}|\lambda_M|^n}\sum_{u=1}^{r} c_{i_u,k(M)}n^{k(M)-1}\lambda_{i_u}^n = \sum_{u=1}^{r} c_{i_u,k(M)}\mu_{i_u}^n$$
$\forall n \in \mathbb{N}$

it is

$$G_n = \frac{F_n}{n^{k(M)-1}|\lambda_M|^n} + o(1). \tag{3.13}$$

Moreover, since

$$|G_n| = \left|\sum_{u=1}^{r} c_{i_u,k(M)}\mu_{i_u}^n\right| \leqslant \sum_{u=1}^{r}|c_{i_u,k(M)}| < \infty \quad \forall n \in \mathbb{N}$$

it is

$$\left|\frac{F_n}{n^{k(M)-1}|\lambda_M|^n}\right| = |G_n + o(1)| < \infty \quad \forall n \in \mathbb{N} \tag{3.14}$$

and

$$\frac{F_{n+1}F_{n-1} - F_n^2}{n^{2(k(M)-1)}|\lambda_M|^{2n}} = \frac{F_{n-1}^2}{n^{2(k(M)-1)}|\lambda_M|^{2n}}\left[\frac{F_{n+1}}{F_n}\frac{F_n}{F_{n-1}} - \left(\frac{F_n}{F_{n-1}}\right)^2\right] \to 0, \tag{3.15}$$

the convergence being due to the fact that the first factor

$$\frac{F_{n-1}^2}{n^{2(k(M)-1)}|\lambda_M|^{2n}} = \left(\frac{F_{n-1}}{(n-1)^{k(M)-1}|\lambda_M|^{n-1}}\right)^2 \times \frac{(n-1)^{2(k(M)-1)}|\lambda_M|^{2(n-1)}}{n^{2(k(M)-1)}|\lambda_M|^{2n}}$$

is bounded because of (3.14), and the other one is infinitesimal by the existence of the Kepler limit of (F_n).

For every $n \in \mathbb{N}$ we have

$$\begin{aligned} G_{n+1}G_{n-1} - G_n^2 &= \left(\sum_{u=1}^{r} c_{i_u,k(M)}\mu_{i_u}^{n+1}\right)\left(\sum_{u=1}^{r} c_{i_u,k(M)}\mu_{i_u}^{n-1}\right) \\ &- \left(\sum_{u=1}^{r} c_{i_u,k(M)}\mu_{i_u}^{n}\right)^2 = \sum_{1\leqslant\alpha<\beta\leqslant r} c_{i_\alpha,k(M)}c_{i_\beta,k(M)}\left(\frac{\mu_{i_\alpha}}{\mu_{i_\beta}}+\frac{\mu_{i_\beta}}{\mu_{i_\alpha}}-2\right) \\ &\times (\mu_{i_\alpha}\mu_{i_\beta})^n = \sum_{1\leqslant\alpha<\beta\leqslant r} c_{(\alpha,\beta)}z_{(\alpha,\beta)}^n \end{aligned} \tag{3.16}$$

where, for the sake of notation, we set

$$c_{(\alpha,\beta)} = c_{i_\alpha,k(M)}c_{i_\beta,k(M)}\left(\frac{\mu_{i_\alpha}}{\mu_{i_\beta}}+\frac{\mu_{i_\beta}}{\mu_{i_\alpha}}-2\right), \quad z_{(\alpha,\beta)} = \mu_{i_\alpha}\mu_{i_\beta}.$$

Arguing by contradiction, assume that $r \geqslant 2$. By Lemma 3.2, the sequence $w_n = G_{n+1}G_{n-1} - G_n^2$ is not definitively zero and without loss of generality we may write

$$w_n = \sum_{\gamma=1}^{\bar{r}} c_\gamma z_\gamma^n \quad \forall n \in \mathbb{N} \tag{3.17}$$

where $\bar{r} \geqslant 1$, $c_\gamma \neq 0$ for any $\gamma = 1,\dots,\bar{r}$, and the z_γ's are pairwise distinct and such that

$$|z_\gamma| = 1 \quad \forall \gamma = 1,\dots,\bar{r}. \tag{3.18}$$

From the fact that $\bar{r} \geqslant 1$ we will get a contradiction.

The sequence (w_n) is infinitesimal: in fact by (3.13) it is

$$w_n = G_{n+1}G_{n-1} - G_n^2 = \left(\frac{F_{n+1}}{(n+1)^{k(M)-1}|\lambda_M|^{n+1}} + o(1)\right) \times \left(\frac{F_{n-1}}{(n-1)^{k(M)-1}|\lambda_M|^{n-1}} + o(1)\right) - \left(\frac{F_n}{n^{k(M)-1}|\lambda_M|^n} + o(1)\right)^2$$

and by (3.14)

$$\begin{aligned} &= \frac{F_{n+1}}{(n+1)^{k(M)-1}|\lambda_M|^{n+1}} \cdot \frac{F_{n-1}}{(n-1)^{k(M)-1}|\lambda_M|^{n-1}} - \left(\frac{F_n}{n^{k(M)-1}|\lambda_M|^n}\right)^2 + o(1) \\ &= \frac{F_{n+1}F_{n-1}}{n^{2(k(M)-1)}|\lambda_M|^{2n}} \cdot \frac{n^{2(k(M)-1)}}{(n^2-1)^{k(M)-1}} - \left(\frac{F_n}{n^{k(M)-1}|\lambda_M|^n}\right)^2 + o(1) \\ &= \frac{F_{n+1}F_{n-1} - F_n^2}{n^{2(k(M)-1)}|\lambda_M|^{2n}} \cdot \frac{n^{2(k(M)-1)}}{(n^2-1)^{k(M)-1}} \\ &+ \left(\frac{F_n}{n^{k(M)-1}|\lambda_M|^n}\right)^2 \left(\frac{n^{2(k(M)-1)}}{(n^2-1)^{k(M)-1}} - 1\right) + o(1) \to 0, \end{aligned}$$

the last convergence being due to (3.15) and (3.14). We have therefore shown that

$$w_n \to 0 \tag{3.19}$$

and from (3.17)

$$\sum_{\gamma=1}^{\bar{r}} c_\gamma z_\gamma^n = w_n \to 0 \tag{3.20}$$

If $\bar{r} = 1$, this means that $c_1 z_1^n \to 0$, which is absurd because, from (3.18), it is $|c_1 z_1^n| = |c_1|$.

If $\bar{r} \geqslant 2$, fix $n \in \mathbb{N}$ and consider the system of $\bar{r}$ in the $\bar{r}$ unknowns $x_1^{(n)},\dots,x_{\bar{r}}^{(n)}$:

$$\begin{cases} \sum_{\gamma=1}^{\bar{r}} x_\gamma^{(n)} & = w_n \\ \sum_{\gamma=1}^{\bar{r}} z_\gamma x_\gamma^{(n)} & = w_{n+1} \\ \vdots & \\ \sum_{\gamma=1}^{\bar{r}} z_\gamma^{\bar{r}-1} x_\gamma^{(n)} & = w_{n+\bar{r}-1} \end{cases} \tag{3.21}$$

Since the Vandermonde determinant

$$V = \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_{\bar{r}} \\ \vdots & \vdots & \dots & \vdots \\ z_1^{\bar{r}-1} & z_2^{\bar{r}-1} & \dots & z_{\bar{r}}^{\bar{r}-1} \end{vmatrix} \neq 0,$$

the system (3.21) admits the unique solution $\left(x_1^{(n)},\dots,x_{\bar{r}}^{(n)}\right)$ where

$$x_1^{(n)} = \frac{\begin{vmatrix} w_n & 1 & \dots & 1 \\ w_{n+1} & z_2 & \dots & z_{\bar{r}} \\ \vdots & \vdots & \dots & \vdots \\ w_{n+\bar{r}-1} & z_2^{\bar{r}-1} & \dots & z_{\bar{r}}^{\bar{r}-1} \end{vmatrix}}{V} \to 0, \tag{3.22}$$

the convergence being due to (3.19).

On the other hand, by (3.20), a solution of (3.21) is given also by

$$x_1^{(n)} = c_1 z_1^n, \quad x_2^{(n)} = c_2 z_2^n, \dots, x_{\bar{r}}^{(n)} = c_{\bar{r}} z_{\bar{r}}^n$$

and therefore $|x_1^{(n)}| = |c_1 z_1^n| = |c_1|$ which is again a contradiction, because of (3.22). □

4. Examples

In this Section we list some examples and some direct consequences of our Theorem 2.3. The first example is considered here only for completeness: it contains the classical sequences generalized in Examples 4.3, 4.4, and 4.9.

Example 4.1 (*Fibonacci and Lucas sequences*). Consider the sequences

$$\begin{cases} F_0 = F_1 = 1 \\ F_n = F_{n-1} + F_{n-2}, \quad n > 1 \end{cases} \qquad \begin{cases} L_0 = 2, \quad L_1 = 1 \\ L_n = L_{n-1} + L_{n-2}, \quad n > 1 \end{cases} \tag{4.1}$$

Their characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda - 1$, whose roots are

$$1-\Phi = \frac{1-\sqrt{5}}{2}, \quad \Phi = \frac{1+\sqrt{5}}{2}$$

and the Binet's formulas are respectively

$$F_n = \frac{1}{\sqrt{5}}\left[\Phi^{n+1} - (1-\Phi)^{n+1}\right], \quad L_n = \Phi^n + (1-\Phi)^n$$

In both cases Φ it is the unique leading root, and therefore it coincides with their Kepler limit.

By the way, let us recall that the explicit formula for F_n in terms of Φ and $1-\Phi$ (see [23,24] for a nice proof) seems due to Daniel Bernoulli, 1732 (see [25]). In the same paper [25] it is observed that the Kepler limit of any sequence with the same recurrence relation as Fibonacci, but starting from arbitrary integers, is again Φ. Note that the restriction to integers as starting points cannot be removed completely: for any recursive sequence (F_n) not constantly zero such that $F_n = F_{n-1} + F_{n-2}$, $n > 1$, the limit $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}$ exists, and it is Φ, except the case $\frac{F_0}{F_1} = -\Phi$, where the limit is $1-\Phi$ (see Example 4.5).

Example 4.2. Periodic sequences

$$F_n = F_{n-k} \quad \forall n \in \mathbb{N}$$

for some $k \in \mathbb{N}$ are linear homogeneous recurrences, whose characteristic polynomial is

$$p(\lambda) = \lambda^k - 1$$

Since the k roots of unity have the same modulus and multiplicity, if (F_n) is not geometric, by Theorem 1.1 the leading roots must be at least two, and therefore, by Theorem 2.3, the Kepler limit does not exist. This example shows also that the property of a characteristic polynomial to have all simple and pairwise distinct roots does not ensure the existence of the Kepler limit.

Example 4.3 (*k-Fibonacci numbers, [26]*). For any integer $k \geqslant 1$ consider the sequence

$$\begin{cases} F_{k,0} = 0, \quad F_{k,1} = 1 \\ F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n > 1 \end{cases} \tag{4.2}$$

The characteristic polynomial is $p_k(\lambda) = \lambda^2 - k\lambda - 1$, whose roots are

$$\lambda_{k,1} = \frac{k+\sqrt{k^2+4}}{2}, \quad \lambda_{k,2} = \frac{k-\sqrt{k^2+4}}{2}$$

and the Binet's formula is

$$F_{k,n} = \frac{1}{\lambda_{k,1}-\lambda_{k,2}}\lambda_{k,1}^n - \frac{1}{\lambda_{k,1}-\lambda_{k,2}}\lambda_{k,2}^n$$

In this case $\lambda_{k,1}$ it is the unique leading root, and therefore it coincides with the Kepler limit. Incidentally, we recall that this Kepler limit is known in literature as kth *metallic ratio* (see [27,28]): Golden Ratio for $k = 1$, Silver Ratio for $k = 2$, and Bronze Ratio for $k = 3$.

Example 4.4 (*m-extension of Lucas p-numbers, [29–33]*). Let p be a positive integer and m be a positive real number, greater than 2 if $p > 1$. Consider the sequence

$$\begin{cases} L_0^{(p,m)} = p+1 \\ L_n^{(p,m)} = m^n \quad n = 1,\ldots,p \\ L_n^{(p,m)} = mL_{n-1}^{(p,m)} + L_{n-p-1}^{(p,m)} \quad n > p \end{cases} \tag{4.3}$$

The characteristic polynomial is $p(\lambda) = \lambda^{p+1} - m\lambda^p - 1$, which, by our assumptions on m and p, applying statement III, p. 251 of the cornerstone paper by Brauer ([34]), has a dominant real root. Since the Binet's formula reads (see Theorem 2, p. 1901 in [29])

$$L_n^{(p,m)} = \lambda_1^n + \lambda_2^n + \cdots + \lambda_{p+1}^n$$

the dominant root is the unique leading root and therefore by Theorem 2.3 there exists the Kepler limit. For the case $m = 1$ see also [35].

In the paper [15] the following definitions are introduced: for a given sequence (F_n), the characteristic polynomial p is called *asymptotically simple* if, among its roots of maximum modulus, there exists a unique root λ of maximal multiplicity ν, the root and the multiplicity being called *dominant*. They proved (see Theorem 7) that if p is asymptotically simple and $F_n/(n^{\nu-1}\lambda^n)$ converges to a limit $L \neq 0$ (of course depending on the initial conditions), then $L = \lambda$. Therefore in the cases examinated in their paper, the Kepler limit is the dominant root of the characteristic polynomial of (F_n). We stress that our Theorem 2.3 is true for all homogeneous linear recurrences with constant coefficients, and the existence of the Kepler limit is in fact independent of the notion of dominant root. This will be shown by the next four examples.

After the submission of this paper to the journal, the referee suggested to add the following text, before the Example 4.5: "As corollary of Theorem 2.3, if λ is a dominant root of the polynomial p (with respect to the initial conditions), then λ is a leading root of p. Thus, the following problem arise naturally: study under which conditions a leading root is a dominant root".

Example 4.5. In this example we present a sequence for which there exist both the dominant root of the characteristic polynomial and the Kepler limit, but they are different.

Consider the sequence

$$\begin{cases} F_0 = 1, \quad F_1 = 1-\Phi \\ F_n = F_{n-1} + F_{n-2}, \quad n > 1 \end{cases} \tag{4.4}$$

where Φ denotes the Golden mean. For this sequence the Binet's formula reads $F_n = (1-\Phi)^n$ so that (F_n) is geometric and its Kepler limit is $1-\Phi$, which coincides with the unique leading root of the characteristic polynomial p, with respect to the initial conditions $F_0 = 1$, $F_1 = 1-\Phi$; on the other hand, p is asymptotically simple with dominant root Φ.

Example 4.6. In this example we present a sequence for which there exists the dominant root of the characteristic polynomial, but the Kepler limit does not exist.

Consider the sequence

$$\begin{cases} F_0 = 3, \quad F_1 = 1, \quad F_2 = -1, \quad F_3 = 1 \\ F_n = 3F_{n-1} - 3F_{n-2} + 3F_{n-3} - 2F_{n-4}, \quad n > 3 \end{cases} \tag{4.5}$$

The characteristic polynomial is $p(\lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2 - 3\lambda + 2$, whose roots are 2, 1, $-i$, i. The Binet's formula is

$$F_n = 1^n + (-i)^n + i^n.$$

Since there are three leading roots, by Theorem 2.3 the Kepler limit does not exist (notice also that the sequence (F_n) is periodic and not constant). On the other hand, the polynomial p is asymptotically simple with dominant root 2.

Example 4.7. In this example we present a sequence for which it does not exist the dominant root of the characteristic polynomial, but there exists the Kepler limit.

Consider the sequence

$$\begin{cases} F_0 = 1, \quad F_1 = 1, \quad F_2 = 1 \\ F_n = F_{n-1} - 4F_{n-2} + 4F_{n-3}, \quad n > 2 \end{cases} \tag{4.6}$$

The characteristic polynomial is $p(\lambda) = \lambda^3 - \lambda^2 + 4\lambda - 4$, whose roots are $-2i$, $2i$, 1. The Binet's formula is

$$F_n = 1^n.$$

Here the characteristic polynomial has not a dominant root, but there exists the Kepler limit, which is the unique leading root 1 with respect to the initial conditions $F_0 = 1$, $F_1 = 1$, $F_2 = 1$.

Example 4.8. In this example we present another, not constant, sequence for which it does not exist the dominant root of the characteristic polynomial, but there exists the Kepler limit.

Consider the sequence

$$\begin{cases} F_0 = 2, \quad F_1 = 3, \quad F_2 = 5, \quad F_3 = 9 \\ F_n = 3F_{n-1} - 6F_{n-2} + 12F_{n-3} - 8F_{n-4}, \quad n > 3 \end{cases} \tag{4.7}$$

The characteristic polynomial is $p(\lambda) = \lambda^4 - 3\lambda^3 + 6\lambda^2 - 12\lambda + 8$, whose roots are $-2i$, $2i$, 2, 1. The Binet's formula is

$$F_n = 2^n + 1^n.$$

As before, the characteristic polynomial has not a dominant root, but there exists the Kepler limit, which is the unique leading root 2 with respect to the initial conditions $F_0 = 2$, $F_1 = 3$, $F_2 = 5$, $F_3 = 9$.

Example 4.9 (*k-generalized Fibonacci numbers, [36–40,17]*). Let $k \geqslant 2$ be integer. Consider the sequence

$$\begin{cases} F_n^{(k)} = 0 \quad n = 0, \dots, k-2 \\ F_{k-1}^{(k)} = 1 \\ F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad n \geqslant k \end{cases} \tag{4.8}$$

The characteristic polynomial is $p(\lambda) = \lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$. In [36] (see also [41]) it is proved that all but one roots of p lie inside the unit circle, so that there exists a dominant root of p. Note that the same conclusion could have been obtained applying Theorem 2 in [34] (see the last lines of the proof, p. 254). Looking at the Binet's formula (see (2)″ in [36]) it turns out that the dominant root is also the unique leading root and therefore by Theorem 2.3 there exists the Kepler limit (see (3)″ in [36]).

Example 4.10. Looking at the Example 4.6, it may be conjectured that the non-existence of the Kepler limit of a sequence (F_n) implies that there exist a constant subsequence of $\left(\frac{F_n}{F_{n-1}}\right)$, but this is not true.

Consider the sequence

$$\begin{cases} F_0 = 3, \quad F_1 = 1, \quad F_2 = 9 \\ F_n = F_{n-1} + 4F_{n-2} - 4F_{n-3} \quad n > 2 \end{cases} \tag{4.9}$$

The characteristic polynomial is $p(\lambda) = \lambda^3 - \lambda^2 - 4\lambda + 4$, whose roots are 2, -2, 1. The Binet's formula is

$$F_n = 2^n + (-2)^n + 1^n.$$

The Kepler limit, by Theorem 2.3, does not exist. On the other hand,

$$\frac{F_n}{F_{n-1}} = \frac{2^n + (-2)^n + 1}{2^{n-1} + (-2)^{n-1} + 1} = \begin{cases} 2^{n+1} + 1, & n \text{ even} \\ \frac{1}{2^n + 1}, & n \text{ odd} \end{cases}$$

It follows that the terms of the sequence $\left(\frac{F_n}{F_{n-1}}\right)$ are pairwise distinct, so that in particular any of its subsequences cannot be periodic.

Example 4.11. [single nonzero initial condition, [42,20]] For all polynomials having simple roots with pairwise distinct modulus, there exist "universal" initial conditions such that the corresponding sequences have Binet's formula with all nonzero coefficients. In this case, the Kepler limit coincides with the dominant root. Given $a_0 \neq 0$, $a_1, \dots, a_{k-1}$ in $\mathbb{C}$, consider a sequence of the type

$$\begin{cases} F_0 = 1, \quad F_1 = 0, \dots, \quad F_{k-1} = 0 \\ F_n = -a_{k-1}F_{n-1} - \cdots - a_0 F_{n-k} \quad \forall n \geqslant k \end{cases} \tag{4.10}$$

The characteristic polynomial is $p(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0$. Assume that it has simple roots with pairwise distinct modulus. The Binet's formula given by (1.4) has coefficients different from zero: in fact, they are obtained as solutions of a linear system whose coefficient matrix is of Vandermonde type of order k, and $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as column of constant terms; they are ratio whose numerator is of Vandermonde type of order $k-1$, and therefore they must be different from zero. Since there exists the dominant root, then it is the unique leading root, and therefore it coincides with the Kepler limit.

The same argument applies if the vector of the initial conditions is $(0,0,\dots,1)$ instead of $(1,0,\dots,0)$ (see Proposition 2.1, p. 128 in [20]). Note that the same argument cannot be applied if $k > 2$ and the vector of the initial conditions contains the element *1* not exactly in the extremes: it is sufficient to consider

$$\begin{cases} F_0 = 0, \quad F_1 = 1, \quad F_2 = 0 \\ F_n = 2F_{n-1} + F_{n-2} - 2F_{n-3} \quad \forall n > 2 \end{cases} \tag{4.11}$$

where the Binet's formula reads as

$$F_n = \frac{1}{2} + \left(-\frac{1}{2}\right)(-1)^n + 0 \cdot 2^n$$

and therefore even if there exists the dominant root 2, the Kepler limit does not exist.

We conclude with the following remark, which is a consequence of the necessary condition for the existence of the Kepler limit given in Theorem 2.3.

Remark 4.12. If a sequence (F_n) is real, periodic, and such that the Kepler limit exists, then it is constant or there exists c such that $F_n = c(-1)^n$. In fact, by the periodicity (see Example 4.2), the characteristic polynomial is of the type $p(\lambda) = \lambda^k - 1$ and, by the existence of the Kepler limit, it must be $F_n = c\lambda_1^n$ for any $n \in \mathbb{N}$, where λ_1 is a kth root of *1*. As F_n is real, it must be $\lambda_1 = 1$ or $\lambda_1 = -1$. Note that if k is odd, then (F_n) must be constant.

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