Understanding Navier-Stokes equation

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Overview of the Navier-Stokes equations and essential linear algebra concepts required to understand the NS equation

So everything starts with the Cauchy momentum equation, formulated by Augustin-Louis Cauchy in the 1820s, extends Newton's second law to continuous media by incorporating stress tensors. Building on earlier work by Newton (viscosity) and Euler (inviscid flow), it provides a general framework for fluid and solid mechanics. Later refined by Navier and Stokes to include viscosity, it became the foundation of the Navier-Stokes equations. The Cauchy momentum equation is a partial differential equation that describes non-relativistic momentum transporting in any continuum. The *Cauchy momentum equation* is given by:

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \sigma + B \tag{1}$$

Where,

$\mathbf{u}$ is the velocity vector of the fluid,
$\frac{D\mathbf{u}}{Dt}$ is the material derivative of velocity, defined as $\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$,
σ is the stress tensor,
B represents body forces,

And, $\partial \mathbf{u}/\partial t$ is the time derivative of velocity, it is the unsteady term and $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is the convective acceleration, usually we also see the expression $\mathbf{u} \cdot \nabla \mathbf{u}$ being using for convective acceleration but that is incorrect (refer Appendix, item 4). The Cauchy momentum equation (equation 1) represents a momentum balance on a fluid element, $\Sigma F = ma = m \cdot \Delta \mathbf{u}/\Delta t$, i.e. sum of all forces in the control volume is equal to mass times the acceleration in the system. The general decomposition of stress tensor σ is:

$$\sigma = -p\mathbf{I} + \tau \tag{2}$$

Where, p is the pressure, $\mathbf{I}$ is identity matrix (diagonal ones matrix) and τ is the viscous stress tensor for the fluid element (refer Appendix, item 1 for understanding tensors). The stress tensor has two main contributions - the *pressure force* p, and the *viscous forces* τ, where τ is the viscous stress tensor. In the context of a fluid element, body forces include the gravitational force, which is represented as $\rho\mathbf{g}$, where ρ is the fluid density and $\mathbf{g}$ is the gravitational acceleration.

Divergence of the stress tensor $\nabla \cdot \sigma$, in case of a three dimensional fluid element is the divergence of a rank 2 tensor whose output is a vector (rank 1 tensor) (refer Appendix, item 2 for divergence and gradient operations overview). The divergence of the stress tensor is given by:

$$
\begin{aligned}
\nabla \cdot \sigma &= \nabla \cdot (-p\mathbf{I} + \tau) \\
&= -\nabla \cdot (p\mathbf{I}) + \nabla \cdot \tau \\
&= -\nabla p + \nabla \cdot \tau
\end{aligned} \tag{3}
$$

Here, $\nabla \cdot (p\mathbf{I}) = \nabla p$ is a vector calculus identity (refer Appendix, item 3 for proof). Using equation 3 in the Cauchy stress equation (equation 1) we get:

$$
\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g} \tag{4}
$$

Going forward we will omit gravity for simplicity. The viscous stress tensor τ gives us a complete picture of viscous stress in a fluid control volume, it is given as:

$$
\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}
$$

Here, all the diagonal elements, σ_{xx}, σ_{yy}, σ_{zz}, are normal stresses and the off-diagonal elements are shear stresses.

Now let's walk towards getting the Navier-Stokes (NS) equation. We assume the fluid is **Newtonian** because it provides simple and accurate description of the viscous stress tensor τ for many common fluids under normal conditions. For a Newtonian fluid τ is given by:

$$
\tau = 2\mu\epsilon + \lambda(\nabla \cdot \mathbf{u})\mathbf{I} \tag{5}
$$

Here, μ is the dynamic viscosity, ϵ is the rate-of-strain tensor (also called the deformation rate tensor) it is a symmetric tensor that represents the local rate of deformation of a fluid element, λ is bulk viscosity coefficient, it represents the resistance in volumetric compression/expansion. Both μ and λ have the same units of viscosity (Pa·s). The rate-of-strain tensor ϵ is given as:

$$
\epsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \tag{6}
$$

$$
\epsilon = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\
\frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z}
\end{bmatrix}
$$

Therefore, the viscous stress tensor is given by:

$$
\tau = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \; + \; \lambda(\nabla \cdot \mathbf{u})\mathbf{I} \tag{7}
$$

The viscous stress (equation 7) has two parts, the first term is the **deviatoric stress** (shear component of stress) and the second term is the **hydrostatic stress** (normal stress). The hydrostatic part is just an diagonal matrix. The viscous stress tensor expands as:

$$\begin{aligned}
\tau &= \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \ + \ \lambda(\nabla\cdot\mathbf{u})\mathbf{I} \\
&= \begin{pmatrix} 2\mu\frac{\partial u_x}{\partial x} & \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & 2\mu\frac{\partial u_y}{\partial y} & \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & 2\mu\frac{\partial u_z}{\partial z} \end{pmatrix} + \begin{pmatrix} \lambda(\nabla\cdot\mathbf{u}) & 0 & 0 \\ 0 & \lambda(\nabla\cdot\mathbf{u}) & 0 \\ 0 & 0 & \lambda(\nabla\cdot\mathbf{u}) \end{pmatrix} \\
&= \begin{pmatrix} 2\mu\frac{\partial u_x}{\partial x} + \lambda(\nabla\cdot\mathbf{u}) & \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & 2\mu\frac{\partial u_y}{\partial y} + \lambda(\nabla\cdot\mathbf{u}) & \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & 2\mu\frac{\partial u_z}{\partial z} + \lambda(\nabla\cdot\mathbf{u}) \end{pmatrix}
\end{aligned} \tag{8}$$

We assume the fluid to be incompressible (important assumption for getting the NS equation) i.e. $\nabla\cdot\mathbf{u} = 0$. The Cauchy stress equation (equation 4) requires the divergence of viscous stress tensor:

$$\nabla\cdot\tau = \nabla\cdot\left[\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \ + \ \lambda(\nabla\cdot\mathbf{u})\mathbf{I}\right] \tag{9}$$

Here, $\nabla\mathbf{u}$ is the gradient of velocity (refer Appendix, item 5 for the expression). Since $\nabla\cdot\mathbf{u} = 0$,

$$\nabla\cdot\tau = \mu\nabla\cdot(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) \tag{10}$$

Since $\nabla\cdot(\nabla\mathbf{u})^T = \nabla(\nabla\cdot\mathbf{u})$ (this is a vector calculus identity refer Appendix, item 6 for proof), therefore:

$$\nabla\cdot\tau = \mu(\nabla^2\mathbf{u} + \nabla(\nabla\cdot\mathbf{u})) \tag{11}$$

Again, due to the incompressibility condition ($\nabla\cdot\mathbf{u} = 0$), we get

$$\nabla\cdot\tau = \mu\nabla^2\mathbf{u} \tag{12}$$

The Navier-Stokes equation is a special case of Cauchy momentum equation with some assumptions. The Cauchy momentum equation applies to all continua (solid, liquid and gas), the stress tensor can be specified to any known stress. To recap the assumptions we made here to get the NS equation are:

- Incompressible flow: $\nabla\cdot\mathbf{u} = 0$
- Newtonian fluid: $\nabla\cdot\sigma = -\nabla p + \mu\nabla^2\mathbf{u}$

NOTE: The assumption to omit gravity is for simplicity and not an important assumption for getting the NS equation.

Therefore, the NS equation is given as:

$$\begin{aligned}
\rho\frac{D\mathbf{u}}{Dt} &= -\nabla p + \mu\nabla^2\mathbf{u} + \rho g, \\
\rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\nabla p + \mu\nabla^2\mathbf{u} + \rho g
\end{aligned} \tag{13}$$

We can also define the NS equation in a non-dimensional form by using dimensionless variables:

$$\mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad x^* = \frac{x}{L}, \quad t^* = \frac{t}{L/U}, \quad p^* = \frac{p}{\rho U^2} \quad \& \quad \nabla^* = L\nabla$$

Therefore, the viscous terms become:

$$\begin{aligned} \mu\nabla^2\mathbf{u} &= \mu\frac{\partial^2\mathbf{u}}{\partial x^2} \\ &= \mu\frac{U}{L^2}\nabla^{*2}\mathbf{u}^* \end{aligned} \tag{14}$$

Now the non-dimensional NS equation becomes:

$$\rho\left(\frac{U^2}{L}\frac{\partial\mathbf{u}^*}{\partial t^*} + \frac{U^2}{L}(\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) = -\frac{\rho U^2}{L}\nabla^* p^* + \frac{\mu U}{L^2}\nabla^{*2}\mathbf{u}^* + \rho g \tag{15}$$

diving throughout by $\rho U^2/L$,

$$\frac{\partial\mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* = -\nabla^* p^* + \frac{\mu}{\rho L U}\nabla^{*2}\mathbf{u}^* + \frac{gL}{U^2} \tag{16}$$

and since Reynold number $Re = \rho UL/\mu$ and rewriting the non-dimensional equation without asterisk (*) and omitting gravity:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{u} \tag{17}$$

Now that we have the NS equation, let's spend some time on the mechanics of this equation. This equation is almost impossible to solve analytically due to it's non-linear nature arising because of the convective acceleration term $(\mathbf{u} \cdot \nabla)\mathbf{u}$. The $\partial\mathbf{u}/\partial t$ term is the unsteady term and it vanishes for systems at steady state. The NS equation becomes linear when the convective acceleration term vanishes and it occurs in some cases like uniform flow i.e. velocity field is constant everywhere ($\nabla\mathbf{u} = 0$), Potential (irrotational) flow with no vorticity ($\nabla \times \mathbf{u} = 0$), one dimensional flow ($\mathbf{u}$ = (u(x),0,0)), creeping flow ($Re << 1$). NS equation becomes solvable in its linear form making them tractable with analytical method hence there is usually an emphasis of problems that specifically make NS equations linear. However, in most practical scenarios involving complex geometries, turbulence, or high Reynolds numbers, the nonlinear terms dominate, and analytical solutions are not feasible. In such cases, numerical techniques like Finite Element Analysis (FEA) or Finite Volume Analysis (FVA) are employed to approximate solutions computationally.

We will have a look at the convective acceleration term, and these are the terms involved:

$$\mathbf{u} = (u_x(x,y,z),\ u_y(x,y,z),\ u_z(x,y,z)) \quad \& \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

the convective acceleration is given by:

$$
\begin{aligned}
(\mathbf{u} \cdot \nabla)\mathbf{u} &= \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) \mathbf{u} \\
&= \begin{pmatrix} \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) u_x \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) u_y \\ \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right) u_z \end{pmatrix} \\
&= \begin{pmatrix} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \\ u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \end{pmatrix}
\end{aligned} \tag{18}
$$

Important point to note here is that $\mathbf{u} \cdot \nabla$ is a dot product of the velocity vector and the operator nabla, the component-wise application of the scalar differential operator $\mathbf{u} \cdot \nabla$ is done on the velocity vector field $\mathbf{u}$.

Appendix

1. A **Tensor** is an algebraic object that describes a multi-linear relationship between sets of algebraic objects related to a vector space.

   - A **scalar** is a tensor of **rank 0**. It has only a value and no direction. Examples: temperature, mass.
   - A **vector** is a tensor of **rank 1**. It has both magnitude and direction. Examples: velocity, force.
   - A **matrix** is a tensor of **rank 2**. It can map one vector to another vector (linear map) or alternatively, map two vectors to a scalar (bilinear map). Example: The stress tensor σ_{ij} maps a surface normal vector n_j to a traction force vector f_i, given by $f_i = \sigma_{ij} n_j$. If we analyze a small area with normal in the x-direction, the force can be written as $f_i = \sigma_{i1}$
   - A **3D array** is a tensor of **rank 3**. It maps two vectors to another vector (trilinear map). Example: The piezoelectric tensor d_{ijk} relates stress σ_{jk} to electric polarization P_i: $P_i = d_{ijk}\sigma_{jk}$

2. Overview of outputs for divergence ($\nabla\cdot$) and gradient (∇) operations

Operation	**Input Type**	**Output Type**	**Index Notation**
∇f	Scalar f	Vector	$(\nabla f)_i = \partial_i f$
$\nabla \mathbf{v}$	Vector v_j	Tensor (2^{nd} order)	$(\nabla \mathbf{v})_{ij} = \partial_i v_j$
$\nabla \mathbf{T}$	Tensor T_{jk}	Tensor (3^{rd} order)	$(\nabla \mathbf{T})_{ijk} = \partial_i T_{jk}$
$\nabla \cdot f$	Scalar f	0 (undefined)	$\nabla \cdot f = \partial_i f$ (not meaningful)
$\nabla \cdot \mathbf{v}$	Vector v_i	Scalar	$\nabla \cdot \mathbf{v} = \partial_i v_i$
$\nabla \cdot \mathbf{T}$	Tensor T_{ij}	Vector	$(\nabla \cdot \mathbf{T})_j = \partial_i T_{ij}$

   Gradient of an object outputs an object of a one degree higher rank tensor however, the divergence operation of an object outputs an object of one degree lower rank tensor.

3. To prove $\nabla \cdot (p\mathbf{I}) = \nabla \mathbf{p}$. Using the index notations from item 2 we can say:

$$
\begin{aligned}
\text{LHS} = \nabla \cdot (p\mathbf{I})_i &= \nabla \cdot \mathbf{p}_j \\
&= \partial_i \mathbf{p}_{ij} \\
&= \partial_i \mathbf{p}_j \qquad \text{... since } \mathbf{p} \neq 0 \text{ for i=j} \\
&= \nabla p_j \\
&= \text{RHS}
\end{aligned}
\tag{19}
$$

4. It is very common to see the convective acceleration being written as $\mathbf{u} \cdot \nabla \mathbf{u}$ but $\mathbf{u} \cdot \nabla \mathbf{u} \neq (\mathbf{u} \cdot \nabla)\mathbf{u}$, because when we write $\mathbf{u} \cdot \nabla \mathbf{u}$ the dot product involves multiplying a vector ($\mathbf{u}$) by a matrix ($\nabla \mathbf{u}$) which is not a standard operation in vector calculus and is not equivalent to $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

5. Gradient is typically defined for a scalar field where it produces a vector field, we can also defined gradient of a vector field which gives a rank-2 tensor (as we see in Appendix, item 2). The concept of a gradient generalizes to something called the Jacobian matrix (or derivative matrix).
for scalar field ϕ:
$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \tag{20}$$
gradient for a vector field $\mathbf{u} = (u_x, u_y, u_z)$:
$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_j}{\partial x_i}$$
$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{21}$$

6. To prove $\nabla \cdot (\nabla \mathbf{u})^T = \nabla(\nabla \cdot \mathbf{u})$

Let $\mathbf{u}$ be a vector field with components u_i. The gradient of $\mathbf{u}$ is the Jacobian matrix:
$$\begin{aligned} (\nabla \mathbf{u})_{ij} &= \frac{\partial u_j}{\partial x_i} \\ (\nabla \mathbf{u})^T_{ij} &= \frac{\partial u_i}{\partial x_j} \end{aligned} \tag{22}$$

Now, computing the divergence:
$$(\nabla \cdot (\nabla \mathbf{u})^T)_j = \frac{\partial}{\partial x_i} \left(\frac{\partial u_i}{\partial x_j} \right) = \frac{\partial^2 u_i}{\partial x_j \partial x_i} \tag{23}$$

On the other hand, the divergence of $\mathbf{u}$ is:
$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i}$$

Taking its gradient:
$$(\nabla(\nabla \cdot \mathbf{u}))_j = \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_i} \right) = \frac{\partial^2 u_i}{\partial x_j \partial x_i} \tag{24}$$

Since both expressions are equal, we conclude $\nabla \cdot (\nabla \mathbf{u})^T = \nabla(\nabla \cdot \mathbf{u})$