Graph Colouring and its applications

Report submitted to University of Kerala in partial fulfilment of the requirement for the award of the degree of Bachelor of Science.

Submitted by,

Name	Candidate Code
1. **Anagha R S**	: 22018124006
2. **Ashok U R**	: 22018124010
3. **Athulya Kishor**	: 22018124012
4. **Geethu Mohan**	: 22018124017
5. **Gayathri Gopal**	: 22018124034
6. **Pavithra J R**	: 22018124039
7. **R Rahul Varma**	: 22018124040
8. **Syama M S**	: 22018124049

Examination Code: 22018601

Subject Code: MM 1646

Under the guidance of

Mr. Saravanan B

Head of Department

Department of Mathematics

Name of College: N S S College, Nilamel

Month and year: May 2021

Certificate

This is to certify that this dissertation entitled, **'Graph colouring and its applications'** is a record of bonafide project work submitted by the following students in partial fulfilment of requirements for the award of the Degree of Bachelor of Science of University of Kerala under my supervision and guidance.

1. **Anagha R S**
2. **Ashok U R**
3. **Athulya Kishor**
4. **Geethu Mohan**
5. **Gayathri Gopal**
6. **Pavithra J R**
7. **R Rahul Varma**
8. **Syama M S**

Counter signed by:

Mr.B Saravanan

Head of the Department

Department of Mathematics

NSS College, Nilamel

HOD, DEPT. OF MATHEMATICS
NILAMEL
N.S.S. COLLEGE

ACKNOWLEDGEMENT

We would like to thank the almighty God for showering all his heavenly blessings upon us for the successful completion of this project.

We are thankful to our principal Dr. Devika.S, N.S.S College, Nilamel for providing us with the valuable guidance and support.

We express our heartfelt thanks to Mr. B.Saravanan, Head of the department, Department of Mathematics, N.S.S College, Nilamel for the valuable support and guidance throughout this project. His advice and involvement were a major source of encouragement and inspiration throughout the project work.

Last but not least we express our deepest gratitude to our beloved parents, classmates and our teachers for their constant encouragement for the completion of this project.

Contents

Chapter	Chapter Name	Page No.
Introduction		1
Preliminary		2
Chapter 1	Vertex Colouring	3-5
Chapter 2	Chromatic Number	6-12
Chapter 3	Brook's Theorem	13-15
Chapter 4	Critical Graphs	16-19
Chapter 5	Chromatic Polynomial	20-22
Chapter 6	Application of Vertex Colouring	23-24
Chapter 7	Edge Colouring	25-26
Chapter 8	Application of Edge Colouring	27-28
Chapter 9	Four Colour Theorem and Heawood Five Colour Theorem	29-30
Conclusion		31
Bibliography		32

Introduction

This project deals with graph colouring and its applications. In graph theory, graph colouring is a special case of graph labelling; it is an assignment of labels traditionally called "colours" to elements of a graph subject to certain constraints. The convention of using colours originated from colouring the countries of a map, where each face is literally coloured. This was generalized to colouring the faces of a graph embedded in the plane.

The project is divided into 9 chapters. Each chapter discusses different topics. Chapter 1 discusses about vertex colouring and the minimization problem for vertex colouring and some definitions related to vertex colouring.

The second chapter discusses about chromatic number, its basic principles and its related proposition, corollary, theorems and definitions. This chapter also discusses about the chromatic numbers for common graph families.

The third chapter mainly discusses about Brooke's theorem. Chapter 4 discusses about critical graphs and their related propositions, corollary and theorem.

Chapter 5 discusses about chromatic polynomials and an illustration of a chromatic polynomial of a graph. The sixth chapter is related to the application of vertex colouring.

The seventh chapter discusses about edge colouring, time table problem and Konig theorem. Chapter 8 is related to the applications of edge colouring.

Preliminary

Graph :- A graph $G(V,E)$ consists of a set of objects $V=\{v_1,v_2,v_3,\ldots\}$ called vertices, and another set $E=\{e_1,e_2,e_3,\ldots\}$ elements called edges. The edges are denoted as uv.

Sub-graph:- A graph g is said to be a sub-graph of a graph G if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G.

Complete Graph:- A simple graph in which there exist an edge between every pair of vertices is called complete graph. A complete graph is also called universal graph or a clique. A complete graph is denoted by K_n.

Bipartite Graph:- A graph G is bipartite if its vertex set can be partitioned into two non-empty subset X and Y such that each edge of G has one end in X and other in Y.

Connected and disconnected Graph:- A graph G is said to be connected if G is disconnected. A disconnected graph consists of two or more connected graph. each of this connected sub-graphs is called a component. A component itself is a graph.

Tree:- Tree is a connected graph without any circuits.

Cycle:- In graph theory, two types of object are commonly called cycles. One type of cycle, more commonly called a closed walk, consists of a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph. The other type of cycle, sometimes called a simple cycle, circuit, circle, or polygon, is a closed walk with no repetitions of vertices or edges allowed, other than the repetition of the starting and ending vertex.

Wheel:- A wheel graph W_n is a graph with n vertices ($n \geq 4$), formed by connecting a single vertex to all vertices of an $(n-1)$-cycle.

Cubic graph:- A cubic graph is a graph in which all vertices have degree three. In other words a cubic graph is a 3-regular graph. They are also called trivalent graphs.

CHAPTER 1

VERTEX COLOURING

A graph is said to be k vertex colouring (or k-colourable) if it is possible to assign one colour from a set of k colours to each vertex such that no two adjacent vertices have the same colour. If the graph G is k-colourable but not (k-1) colourable, we say that G is a k-chromatic graph and that its chromatic number $\chi(G)$ is k. So the chromatic is the minimum number k such that G is k-colourable. Hence, graph G is k-colourable if and only if $\chi(G) \leq k$. In other words, a k-chromatic graph is a graph that needs at least k colours, whereas a k-colourable graph is a graph that does not need more than k colours.

The Minimization Problem for Vertex Colouring

In the most common kind of graph colouring, colours are assigned to the vertices. From a standard mathematical prospective, the subset comprising all the vertices of a given colour would be regarded a cell of a partition of the vertex-set. Drawing the graph with colours on the vertices is simply an intuitive way to represent such a partition.

We begin with a practical application of graph colouring known as the storage problem. Suppose the Department of Chemistry of a college wants to store its chemicals. It is a quite probable that some chemicals cause violent reactions when brought together. Such chemicals are incompatible chemicals. For safe storage, incompatible chemicals should be kept in distinct rooms. The easiest way to accomplish this is, of course, to store one chemical in each room. But this is certainly not the best way of doing it since we will be using more rooms than are really needed (unless, of course, all

the chemicals are mutually incompatible!). So we ask: What is the minimum number of rooms required to store all the chemicals so that in each room only compatible chemicals are stored?

We convert the above storage problem into a problem in graphs. Form a graph $G = G(V,E)$ by making V correspond bijectively to the set of available chemicals and making u adjacent to v if and only if the chemicals corresponding to u and v are incompatible. Then, any set of compatible chemical corresponds to a set of independent vertices of G. Thus a safe storing of chemicals corresponds to a partition of V into independent subsets of G. The cardinality of such a minimum partition of V is then the required number of rooms. This minimum cardinality is called the chromatic number of the graph G.

The chromatic number, $\chi(G)$, of a graph G is the minimum number of independent subsets that partition the vertex set of G. Any such minimum partition is called a chromatic partition of V(G).

The storage problem just described is actually a vertex colouring problem of G. A vertex colouring of G is a map $f: V(G) \rightarrow C$, where C is a set of distinct colours; it is proper if adjacent vertices of G receive distinct colours of C; that is, if $uv \in E(G)$ then $f(u) \neq f(v)$. Thus $\chi(G)$ is the minimum cardinality of C for which there exists a proper vertex colouring of G by

Definition 1.1: The chromatic number of a graph G is the minimum number of colours needed for a proper vertex colouring of G. If $\chi(G) = k$, G is said to be k-chromatic

For example, the chromatic number of the graph in Figure 1.1

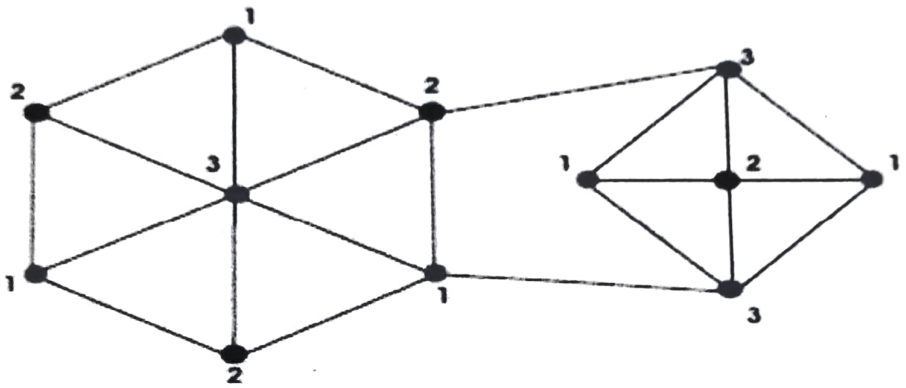

Figure 1.1: A 3-chromatic graph

Definition 1.2: A k-colouring of a graph G is a vertex colouring of G that uses k colours.

Definition 1.3: A graph G is said to be k-colourable, if G admits a proper vertex colouring using k colours.

Thus, $\chi(G) = k$ if graph G is k-colourable but not $(k - 1)$-colourable.

In considering the chromatic number of a graph only the adjacency of vertices is taken into account. A graph with a self-loop is regarded as un-colourable, since the endpoint than of the self-loop is adjacent to itself. Moreover, a multiple adjacency has no more effect on the colours of its endpoints than a single adjacency. As a consequence, we may restrict ourselves to simple graphs when dealing with chromatic numbers.

It is clear that $\chi(G) = 1$ if and only if G has no edges and $\chi(G) = 2$ if and only if G is bipartite.

CHAPTER 2

CHROMATIC NUMBER

Basic Principles for Calculating Chromatic Numbers

A few basic principles recur in many chromatic-number calculations. Now, we will try to find upper and lower bound to provide a direct approach to the chromatic number of a given graph.

Upper bound: Show $\chi(G) \leq k$ by exhibiting a proper k-colouring of G.

Lower bound: Show $\chi(G) \geq k$ by using properties of graph G, most especially, by finding a sub-graph that requires k-colours.

Proposition 2.1: Let G be a graph with k-mutually adjacent vertices. Then $\chi(G) \geq k$.

Proof: Using fewer than k colours on graph G would result in a pair from the mutually adjacent set of k vertices being assigned the same colour.

Proposition 2.2: Let H be a sub-graph of G. Then $\chi(G) \geq \chi(H)$.

Proof: Whatever colours are used on the vertices of sub-graph H in a minimum colouring of G can also be used in colouring of H by itself.

Corollary 2.1: Let G be a graph. Then $\chi(G) \geq \omega(G)$.

Proof: Since clique is a sub-graph of G, we get this inequality.

The 4-colouring of the graph G shown in Figure 2.1 establishes that $\chi(G) \leq 4$, and the K_4-subgraph (drawn in bold) shows that $\chi(G) \geq 4$. Hence, $\chi(G) = 4$.

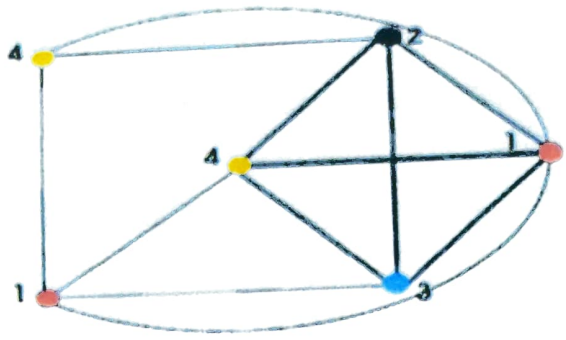

Figure 2.1: A 4-colourable graph

Proposition 2.3: Let G be any graph. Then $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$

Proof: Given a k-colouring of G, the vertices being coloured with the same colour form an independent set. Let G be a graph with n vertices and c a k-colouring of G. We define

$$V_i = \{v \mid c(v) = i\} \text{ for } i = 0, 1, \ldots, k.$$

Each V_i is an independent set. Let $\alpha(G)$ be the independence number of G, we have $V_i \leq \alpha(G)$.

[An independent set of vertices in a graph is a set of mutually non-adjacent vertices. The independence number of a graph G is the maximum cardinality of an independent set of vertices. It is denoted by $\alpha(G)$.]

Since, $$n = |V(G)| = |V_1| + |V_2| + \ldots + |V_k| \leq k\,\alpha(G) = \chi(G)\,\alpha(G)$$

We have $$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

Definition 2.1: The greedy colouring relative to a vertex ordering $v_1, v_2, \ldots, v_n$ of V (G) is obtained by colouring vertices in order $v_1, v_2, \ldots, v_n$, assigning to v_i the smallest-indexed colour not already used on its lower-indexed neighbours.

Unlike the n=2 case there is no easy characterisation of graphs with chromatic index 3, or for that matter, higher index. However there are various results which gave upper bounds for the chromatic under g an arbitrary graph G, provided we know the degree of all their vertices of G. For the first of these we need some notation.

For a graph G we get

$$\Delta(G)=\max\{d(v): v \text{ is a vertex of } G\}$$

Thus $\Delta(G)$ is the max vertex degree of G.

Theorem 2.1: For any graph G, $\chi(G) \leq \Delta(G)+1$

Proof: We use induction on n, the number of vertices in G. The theorem is clearly true, for n=1 since here $G=k_1$, $\chi(G)=1$ and $\Delta(G)=0$.

Now suppose that the result is true for all graphs with (n-1) vertices, where n is a fixed integer greater than 1, and let G be some graph with n vertices. Choose a vertex v of G. Then the sub-graph G-v has (n-1) vertices and so, by the induction assumption, $\chi(G-v) \leq \Delta(G-v)+1$.

This allows us to choose a vertex colouring of G-v involving $(\Delta(G-v)+1)$ colours. Now our vertex v has at most $\Delta(G)$ neighbours in G and so these neighbours use up at most $\Delta(G)$ colours in the colouring of G-v. Thus if $\Delta(G) = \Delta(G-v)$ there is at least one colour not used by v's neighbours and we can use such a colour for v, giving a $(\Delta(G)+1)$ colouring for G. On the other hand, if $\Delta(G)\neq\Delta(G-v)$ then $\Delta(G-v) < \Delta(G)$ and simply colouring v with a brand new colour gives a $(\Delta(G-v)+2)$ colouring of G which is good enough since $\Delta(G-v)+2 \leq \Delta(G)+1$

Hence, in both cases, we have

Chromatic Numbers for Common Graph Families

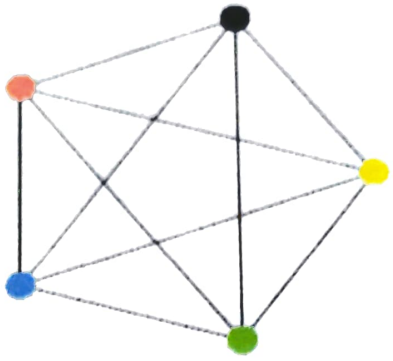

Figure 2.2: A Complete Graph

Proposition 2.4: For complete graphs, $\chi(K_n) = n$.

Proof: Since n complete graphs have n mutually adjacent vertices using fewer than n colours result in a pair of mutually adjacent vertices being assigned the same colour.

Proposition 2.5: For bipartite graphs, $\chi(G) = 2$.

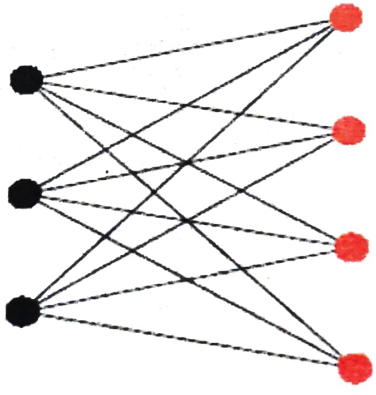

Figure 2.3: A Complete Bipartite Graph

Corollary 2.3: For path graphs, we have $\chi(P_n) = 2$.

Figure 2.4: A Path Graph

Corollary 2.4: Since trees are bipartite, $\chi(T) = 2$.

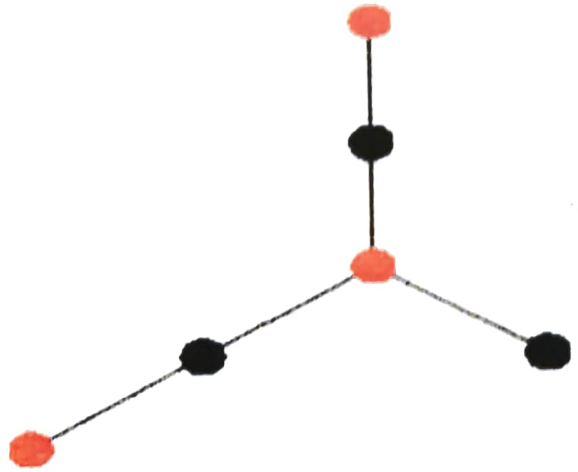

Figure 2.5: A Tree

Corollary 2.5: Cube graphs are bipartite graphs so we have $\chi(Q_n) = 2$.

Proposition 2.6: For odd-cycle graphs, $\chi(C_{2n+1}) = 3$.

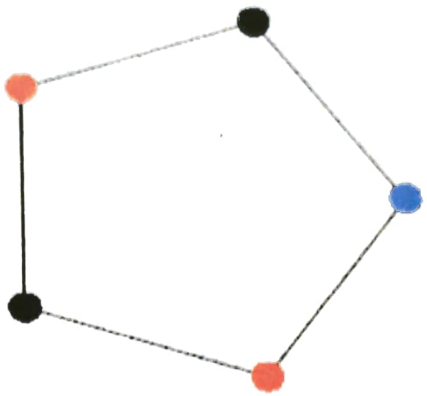

Figure 2.6: An Odd-Cycle

Proof: Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices of cycle graph C_{2n+1}. If two colours were to suffice, then they would have to alternate around the cycle. Thus, the odd-subscripted vertices would have to be one colour and the even-subscripted ones the other. But vertex v_{2n+1} is adjacent to v_1, which means that the odd cycle graph C_{2n+1} is not 2-colourable.

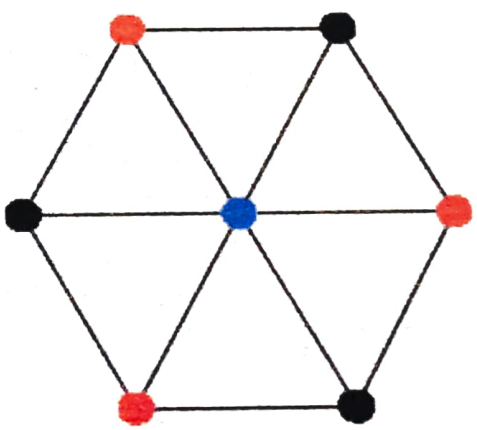

Figure 2.7: An Odd-Order Wheel

Proposition 2.7: For odd-order wheel graphs, $\chi(W_{2m+1}) = 3$.

Proof: Using the fact that the wheel graph $\chi(W_{2m+1})$ is the join of an even cycle C_{2m} and complete graph K_1, by the proposition we have

$$\chi(W_{2m+1}) = \chi(C_{2m} + K_1) = \chi(C_{2m}) + \chi(K_1) = 2 + 1 = 3.$$

Proposition 2.8: For even-order wheel graphs, $\chi(W_{2m}) = 4$.

Proof: The wheel graph $\chi(W_{2m})$ is the join of an odd-cycle C_{2m+1} and the complete graph K_1. Then we have

$$\chi(W_{2m}) = \chi(C_{2m+1} + K_1) = \chi(C_{2m+1}) + \chi(K_1) = 3 + 1 = 4.$$

CHAPTER 3

BROOK'S THEOREM

Theorem 3.1: (Brooks' Theorem) If a connected graph G is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

Proof: If $\chi(G) \leq 2$, then G is either a path or a cycle. For a path G (other than K_1 and K_2), and for an even cycle G, $\chi(G) = 2 = \Delta(G)$. According to our assumption G is not an odd cycle. So let $\Delta(G) \geq 3$.

The proof is by contradiction. Suppose the result is not true. Then there exists a minimal graph G of maximum degree $\Delta(G) \geq 3$ such that G is not Δ-colourable, but for any vertex v of G, $G - v$ is $\Delta(G - v)$-colourable and therefore Δ-colourable.

Claim 1: Let v be any vertex of G. Then in any proper Δ-colouring of $G - v$, all the Δ-colours must be used for colouring the neighbours of v in G. Otherwise, if some colour I is not represented in $N_G(v)$, then v could be coloured using i, and this would give a Δ-colouring of G, a contradiction to the choice of G. Thus, G is a regular graph satisfying claim 1.

For $v \in V(G)$, let $N(v) = v_1, v_2, \ldots, v_n$. In a proper Δ-colouring of $G - v = H$, let v_i receive colour i, $1 \leq i \leq \Delta$. For $i \neq j$, let H_{ij} be the subgraph of H induced by the vertices receiving the ith and j^{th} colours.

Claim 2: v_i and v_j belong to the same component of H_{ij}. Otherwise, the colours i and j can be interchanged in the component of H_{ij} that contains the vertex v_j. Such an interchange of colours once again yields a proper Δ-colouring of H. In this new colouring, both v_i and v_j receive the same colour, namely, i, a contradiction to claim 1. This proves claim 2.

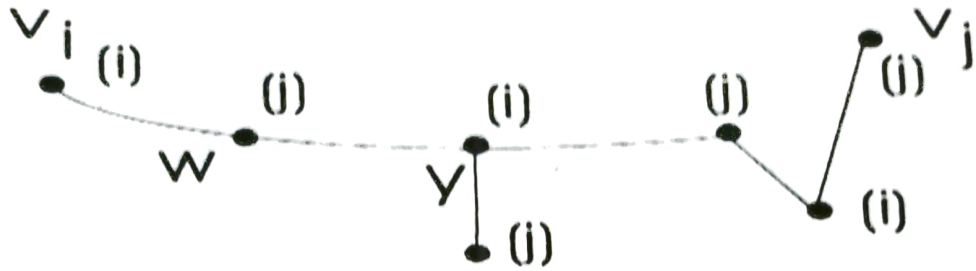

Figure 3.1: Brooks' Theorem (a)

Claim 3: If C_{ij} is the component of H_{ij} containing v_i and v_j, then C_{ij} is a path in H_{ij}. As before, $N_H(v_i)$ contains exactly one vertex of colour j. Further, C_{ij} cannot contain a vertex, say y, of degree at least 3; for, if y is the first such vertex on a $v_i - v_j$ path in C_{ij} that has been coloured, say, with i, then at least three neighbours of y in C_{ij} have the colour j. Hence, we can recolour y in H with a colour different from both i and j, and in this new colouring of H, v_i and v_j would belong to distinct components of H_{ij} .(See Figure 3.1.)Note that by our choice of y, any $v_i - v_j$ path in H_{ij} must contain y. But this contradicts claim 2.

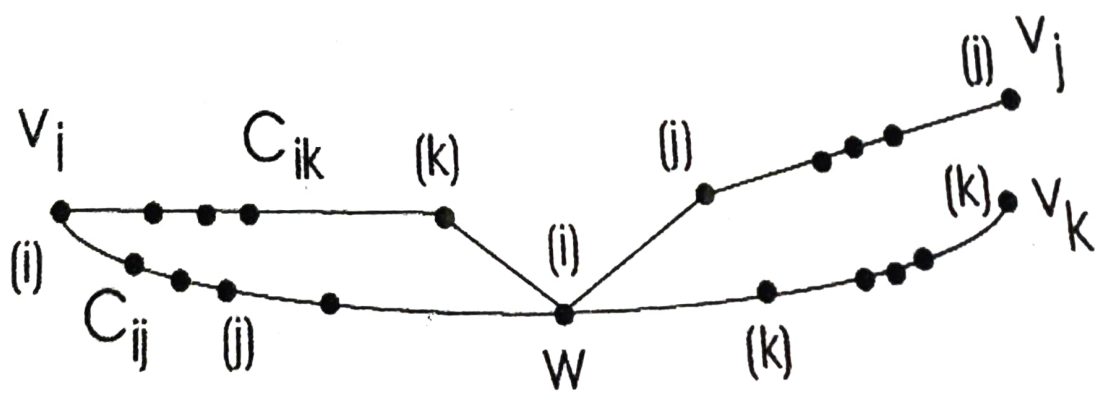

Figure 3.2: Brooks' Theorem (b)

w in H. In this new colouring of H, v_i and v_j belong to distinct components of H_{ij}, a contradiction to claim 1. This completes the proof of claim 4.

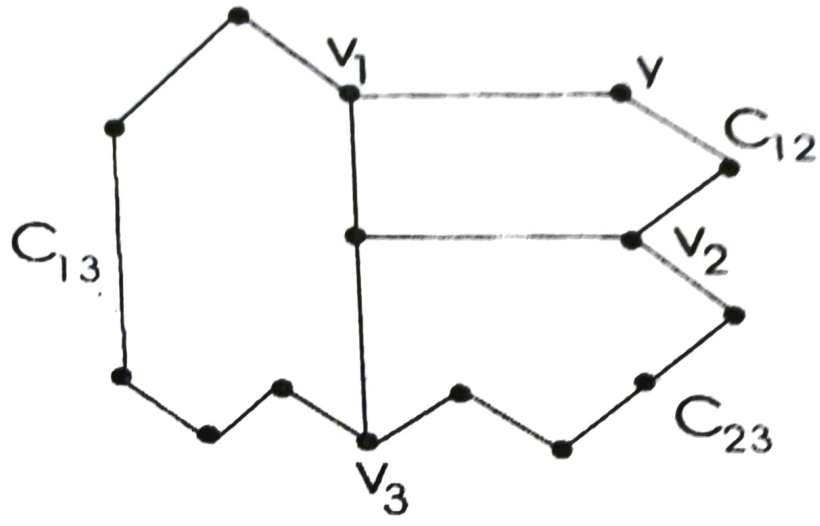

Figure 3.3: Brooks' Theorem (c)

We are now in a position to complete the proof of the theorem. By hypothesis, G is not complete. Hence, G has a vertex v, and a pair of nonadjacent vertices v_1 and v_2 in $N_G(v)$. Then the $v_1 - v_2$ path C_{12} in H_{12} of $H = G - v$ contains a vertex $y(\neq v_2)$ adjacent to v_1. Naturally, y would receive colour 2. Since $\Delta \geq 3$, there exists a vertex $v_3 \in N_G(v)$. Now interchange colours 1 and 3 in the path C_{13} of H_{13}. This would result in a new colouring of $H = G - v$. Denote the $v_i - v_j$ path in H under this new colouring by C'_{ij}. (See Figure 3.3).Then $y \in C'_{23}$, since v_1

CHAPTER 4

CRITICAL GRAPHS

Definition 4.1: A k-chromatic graph G is said to be a critically k-chromatic (or a k- critical) graph if the vertex-deletion sub-graph G−v is (k-1)-colourable and $\chi(G-v) = k-1$ for every vertex of G.

Definition 4.2: A k-chromatic graph G is said to be a minimally k-chromatic graph if $\chi(G - e) = k - 1$ for every edge of G.

Proposition 4.1: Let G be a k-critical graph.

(a) For $v \in V(G)$, there is a proper k-colouring of G in which the colour on v appears nowhere else, and the other $(k - 1)$ colours appear on N(v).

(b) For $e \in E(G)$, every proper $(k - 1)$ colouring of G−e gives the same colour to the two endpoints of e.

Proof:

(a) Given a proper $(k - 1)$-colouring f of $G - v$, adding colour k on v alone completes a proper k-colouring of G. The other colours must all appear on N(V), since otherwise assigning a missing colour to v would complete a proper $(k - 1)$-colouring of G.

(b) If some proper $(k - 1)$-colouring of $G - e$ gave distinct colours to the endpoints of e, then adding e would yield a proper $(k - 1)$-colouring of G.

Theorem 4.1: Every k-chromatic graph contains a critically (minimally) k-chromatic graph.

Proof: Suppose the k-chromatic graph G is not critically k-chromatic. Then there exists a

Otherwise, there is a vertex w in G − v such that (G − v) − w is k-chromatic. Continue this process. We eventually arrive at a critically k-chromatic graph. The proof is similar for the case of minimally k-chromatic graphs.

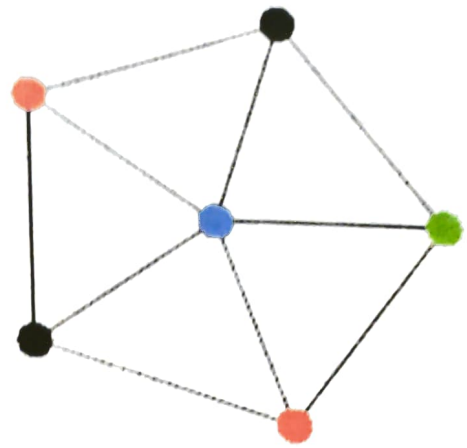

Figure 4.1: W_6 is both critically and minimally k-chromatic.

Proposition 4.2: Every minimally k-chromatic graph (without isolated vertices) is critically k-chromatic.

The converse is not true in general. For k = 2 and k = 3 it is true. In fact, K_2 is the only critically 2-chromatic graph as well as the only minimally 2-chromatic graph without isolated vertices; while the odd cycles are the only critically 3-chromatic graphs and the only minimally 3-chromatic graphs having no isolated vertices. For $k \geq 4$, neither the critically k-chromatic graphs nor the minimally k-chromatic

Figure 4.2: A critically 4-chromatic graph that is not minimally 4-chromatic

A k-chromatic sub-graph of G of minimum order is critically k-chromatic, while a k-chromatic sub-graph of G of minimum size is minimally k-chromatic. Every critically k-chromatic graph is non-separable and every minimally k-chromatic graph without isolated vertices is non-separable.

Theorem 4.2: If G is a critically k-chromatic graph then G is $(k-1)$-edge connected. (Equivalently, any connected minimally k-chromatic graph is $(k-1)$-edge connected.)

Proof: If $k = 2$, the graph is K_2, which is 1-edge-connected. If $k = 3$, the graph is an odd cycle, which is 2-edge connected. Let $k \geq 4$. Suppose G is not $(k-1)$-edge connected. Then there is a partition of the vertex set of G into two sets X and Y such that the cardinality of cut [X, Y] is less than $(k-1)$. So the sub-graphs induced by X and Y are $(k-1)$-colourable. Since the chromatic number of G is k, cut [X, Y] should have at least $(k-1)$ edges. This contradiction establishes that G is $(k-1)$-edge connected.

Theorem 4.3: If G is critically k-chromatic (or connected and minimally k-chromatic), then no vertex of G has degree less than $k-1$, i.e. $\delta(G) \geq k-1$.

Proof: Suppose that v were a vertex of degree less than $k-1$. Since G is k-critical, the vertex deletion sub-graph G–v is $(k-1)$-colourable. The colours assigned to the neighbours of v would not include all the colours of $(k-1)$-colouring because vertex v

has fewer than, $(k-1)$ neighbours. Thus, if v were restored to the graph, it could be coloured with any one of the $k-1$ colours that was not used on any of its neighbours. This would achieve a $(k-1)$-colouring of G, which is a contradiction. Thus $\delta(G) \geq k-1$.

Corollary 4.1: Every k-chromatic graph has at least k vertices of degree at least $k-1$.

Proof: Let G be a k-chromatic graph and let H be a k-critical sub-graph of G. By the Theorem 4.3, each vertex of H has degree at least $k-1$ in H, and hence also in G. Since H is k-chromatic, clearly has an at least k vertex.

Proposition 4.3: A critically k-chromatic graph G with exactly one vertex whose degree exceeds $(k-1)$ is minimally k-chromatic.

Proof: Since G is critically k-chromatic, the degree of each vertex is at least $(k-1)$. In this case, the degree of each vertex is $(k-1)$ except for one vertex. If e is any edge of G, $\delta(G-e) = k-2$. Then $\chi(G-e) \leq 1 + \max \delta(G')$, where G' is any induced sub-graph of $(G-e)$. Thus $\chi(G-e) \leq 1 + (k-2) = k-1$. Since the minimum degree in $(G-e)$ is $k-2$, $\chi(G-e) = k-1$.

CHROMATIC POLYNOMIAL

In general, a given graph G of n vertices can be properly coloured in many different ways using a sufficiently large number of colours. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows:

The value of the chromatic polynomial $p_n(\lambda)$ of a graph with n vertices gives the number of ways of properly colouring the graph, using λ or fewer colours.

Let c_I be the different ways of properly colouring G using exactly 'i' different colours. Since I colours can be chosen out of λ colours in $\binom{\lambda}{i}$ different ways, there are $c_i\binom{\lambda}{i}$ different ways of properly colouring G using exactly I colours out of λ colours.

Since i can be any positive integer from 1 to n(it is not possible to use more than n colours on n vertices), the chromatic polynomial is a sum of these terms;

ie, $p_n(\lambda)=\sum_{i=1}^{n} c_i \binom{\lambda}{i}$

$$= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots + c_n\frac{\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)}{n!}$$

As an illustration, let us find the chromatic polynomial of the graph

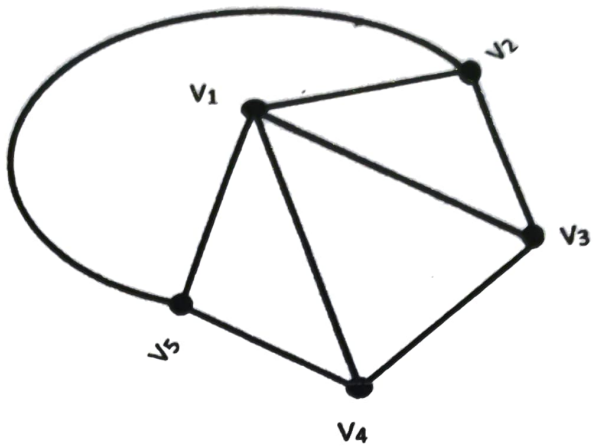

Figure 5.1

$$p_5(\lambda)= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

Since the graph in figure 14 has a triangle, it will require at least three different colours for proper colouring.

$$\therefore c_1 = c_2 = 0 \text{ and } c_5 = 5!$$

Moreover, to evaluate c_3, suppose that we have three colours x, y, and z. these three colours can be assigned properly to vertices v_1,v_2, and v_3 in 3!=6 different ways. Having done that, we have no more choices left, because vertex v_5, must have the same colour as v_3, and v_4 must have the same colour as v_2.

$$\therefore c_3 = 6$$

Similarly, with four colours v_1,v_2 and v_3 can be properly coloured in 4×6=24 different ways. The 4th colour can be assigned to v_4 or v_5, thus providing two choices. The fifth vertex provides no additional choice.

$$\therefore c_4 = 24 \times 2 = 48$$

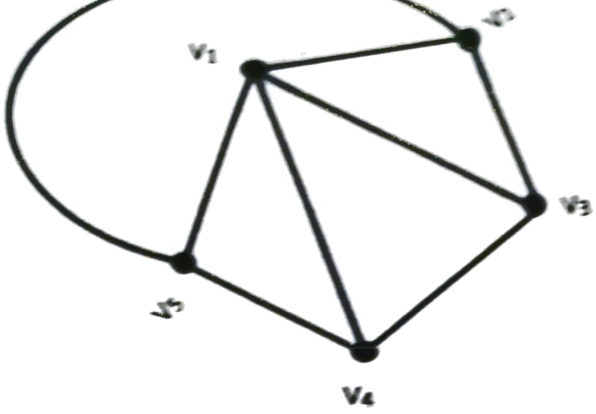

Figure 5.1

$$p_5(\lambda) = c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

Since the graph in figure 14 has a triangle, it will require at least three different colours for proper colouring.

$$\therefore c_1 = c_2 = 0 \text{ and } c_5 = 5!$$

Moreover, to evaluate c_3, suppose that we have three colours x, y, and z. these three colours can be assigned properly to vertices v_1,v_2, and v_3 in 3!=6 different ways. Having done that, we have no more choices left, because vertex v_5, must have the same colour as v_3, and v_4 must have the same colour as v_2.

$$\therefore c_3 = 6$$

Similarly, with four colours v_1,v_2 and v_3 can be properly coloured in 4×6=24 different ways. The 4th colour can be assigned to v_4 or v_5, thus providing two choices. The fifth vertex provides no additional choice.

$$\therefore c_4 = 24 \times 2 = 48$$

Substituting these co-efficients in $p_S(\lambda)$, we get for the graph in figure 5.1.

$$p_S(\lambda) = \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

The presence of factors (λ-1) and (λ-2) indicates that G is at least 3-chromatic.

CHAPTER 6

APPLICATION OF VERTEX COLOURING

Some scheduling problems induce a graph colouring, i.e., an assignment of positive integers (colours) to vertices of a graph. We discuss a simple example for colouring the vertices of a graph with a small number k of colours and present computational results for calculating the chromatic number, i.e., the minimal possible value of such a k.

Draw up an examination schedule involving the minimum number of days for the following problem:

Set of students: $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9$

Examination subjects for each group: {algebra, real analysis, and topology}, {algebra, operations research, and complex analysis }, {real analysis, functional analysis, and topology }, {algebra, graph theory, and combinatorics }, {combinatorics, topology, and functional analysis }, {operations research, graph theory, and coding theory }, {operations research, graph theory, and number theory }, {algebra, number theory, and coding theory }, {algebra, operations research, and real analysis}.

Let S be a set of students, P = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} be the set of examinations respectively algebra, real analysis, topology, operational research, complex analysis, functional analysis, graph theory, combinatorics, coding theory, and number theory. S(p) be the set of students who will take the examination $p \in P$. Form a graph $G = G(P,E)$, where $a, b \in P$ are adjacent if and only if $S(a) \cap S(b) \neq \emptyset$. Then each proper vertex colouring of G yields an examination schedule with the vertices in any colour class representing the schedule on a particular day. Thus $\chi(G)$ gives the minimum number of days required for the examination schedule.

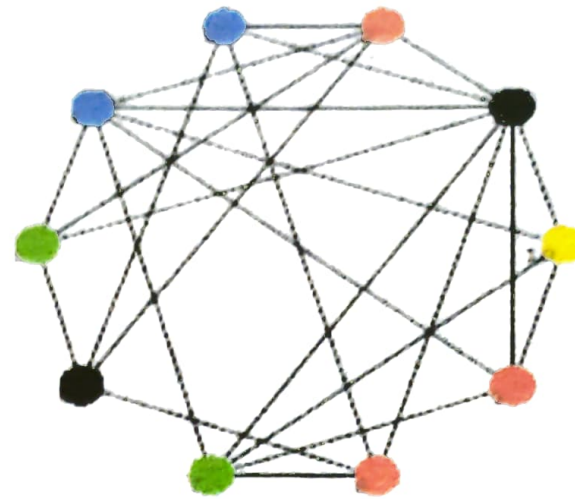

Figure 6.1: The output of the application

5 days are required and you can see below the lessons in the same parenthesis which are on the same day

{{1, 6}, {2, 8, 9}, {3, 4}, {5, 7}, {10}}

CHAPTER 7

EDGE COLOURING

Timetable problem

Suppose in a school there are r teachers $T_1, T_2, \ldots, T_r$ and 5 classes $C_1, C_2, \ldots, C_5$. Each teacher T_i is expected to teach the class C_j for P_{ij} periods. It is clear that during any particular period, no more than one teacher can handle a particular class and no more than one class be engaged by any teacher. Our aim is to draw up a timetable for the day that requires only the minimum number of periods. This problem is known as the "timetable problem".

To convert this problem into a graph, theoretic one, we form the bipartite graph G=G(T,C) with bipartite (T,C) where T represents the set of teachers T_i and C represents the set of classes C_j. Further, T_i is made adjacent to C_j in G with P_{ij} edges if and only if teachers T_i is to handle class C_j for P_{ij} periods. Now colour the edges of G so that no two adjacent edges receive the same colour. Then the edges in a particular colour class, that is, the edges in that colour form a matching in G and correspond to a schedule of work for a particular period. Hence, the minimum number of periods required is the minimum number of colours in an edge colouring of G in which adjacent edges receive distant colours; in other words it is the edge chromatic number of G. we now present these notations as formal definitions.

Definition 7.1: An edge colouring of a loopless graph G is a function $\pi: E(G) \to S$, where S is the set of distinct colours; it is proper if no two adjacent edges receive the same colour. Thus a proper edge colouring π of G is a function $\pi: E(G) \to S$ such that $\pi(e) \neq \pi(e')$ when eved edge e and e' are adjacent in G.

Definition 7.2: The minimum k for which a loopless graph G has a proper k-edge colouring is called the edge chromatic number or chromatic index of G. it is denoted by

$\chi'(G)$. G is k edge chromatic if $\chi'(G) = k$

Further, if an edge uv is coloured by colour c, we say that c is represented at both u and v. If G has a proper k edge colouring, E(G) is partitioned into k edge disjoint matchings.

It is clear that for any (loop-less) graph G $\chi'(G) \geq \Delta(G)$ since the $\Delta(G)$ edges incident at a vertex v of maximum degree $\Delta(G)$ must all receive distinct colours for bipartite graphs, however a quality holds.

Theorem 7.1 (Konig): If G is a loop-less bipartite graph $\chi'(G) = \Delta(G)$.

Proof: The proof is by induction on the size (i.e., number of edges) m of G. the result is true m=1. Assume the result for bipartite graphs of size at most m-1. Let G have m edges, let e = uv $\in$ E(G). Then G-e has (since $\Delta(G-e) \leq \Delta(G)$) a proper Δ-edge colouring, say c out of these Δ colours. Suppose that, one particular colour is not represented at both u and v. Then edge uv can be coloured with this colour and G proper Δ-edge colouring of G is obtained.

In the other case (i.e., in the case for which each of the Δ colours represented either at u or at v) since the degrees of u and v in G-e are at most Δ-1, there exists a colour out of the Δ colours that is not represented at u, and similarly there exist a colour out of the Δ colours that is not represented at v. Thus if colour j is not represented at u in c, then j is represented at v in c, and if colours is not represented at v in c and if colours i is not represented at v in c, then I is represented at G in c. Since G is bipartite and u and v are not in the same parts of the bipartition there can m exist no G-v path in G which the colours alternate between i and j.

Let P be a maximal path in G-e starting from G in which the colours of the edges alternate between i and j interchange the colours i and j in P. This would still yield an edge colouring of G-e using the Δ colours in which colour i is not represented at both u and v. Now, colour the edge uv by the colour i. This results in a proper Δ-edge colouring of G.

CHAPTER 8

APPLICATION OF EDGE COLOURING

Edge colourings of complete graphs may be used to schedule a round-robin tournament into as few rounds as possible so that each pair of competitors plays each other in one of the rounds; in this application, the vertices of the graph correspond to the competitors in the tournament, the edges correspond to games, and the edge colours correspond to the rounds in which the games are played. Similar colouring techniques may also be used to schedule other sports pairings that are not all-play-all; for instance, in the National Football League, the pairs of teams that will play each other in a given year are determined, based on the teams' records from the previous year, and then an edge colouring algorithm is applied to the graph formed by the set of pairings in order to assign games to the weekends on which they are played.

Open shop scheduling is a problem of scheduling production processes, in which there are a set of objects to be manufactured, each object has a set of tasks to be performed on it (in any order), and each task must be performed on a specific machine, preventing any other task that requires the same machine from being performed at the same time. If all tasks have the same length, then this problem may be formalized as one of edge colouring a bipartite multi-graph, in which the vertices on one side of the bipartition represent the objects to be manufactured, the vertices on the other side of the bipartition represent the manufacturing machines, the edges represent tasks that must be performed, and the colours represent time steps in which each task may be performed. Since bipartite edge colouring may be performed in polynomial time, the same is true for this restricted case of open shop scheduling.

Gandham, Dawande & Prakash study the problem of link scheduling for time division multiple access network communications protocols on sensor networks as a variant of edge colouring. In this problem, one must choose time slots for the edges of a wireless communications network so that each node of the network can communicate with each neighbouring node without interference. Using a strong edge colouring (and using two time slots for each edge colour, one for each direction) would solve the

problem but might use more time slots than necessary. Instead, they seek a colouring of the directed graph formed by doubling each undirected edge of the network, with the property that each directed edge uv has a different colour from the edges that go out from v and from the neighbours of v. They propose a heuristic for this problem based on a distributed algorithm for $(\Delta + 1)$-edge-colouring together with a post processing phase that reschedules edges that might interfere with each other.

In fibre-optic communication, the path colouring problem is the problem of assigning colours (frequencies of light) to pairs of nodes that wish to communicate with each other, and paths through a fibre-optic communications network for each pair, subject to the restriction that no two paths that share a segment of fibre use the same frequency as each other. Paths that pass through the same communication switch but not through any segment of fibre are allowed to use the same frequency. When the communications network is arranged as a star network, with a single central switch connected by separate fibres to each of the nodes, the path colouring problem may be modelled exactly as a problem of edge colouring a graph or multi-graph, in which the communicating nodes form the graph vertices, pairs of nodes that wish to communicate form the graph edges, and the frequencies that may be used for each pair form

CHAPTER 9

FOUR COLOUR THEOREM AND HEAWOOD FIVE COLOUR THEOREM

What is the minimum number of colours required to colour the world map of countries so that no two countries having a common boundary receive the same colour? This simple looking problem manifested itself into one of the most challenging the four colour conjecture.

It was young Francis Guthrie were sufficient to colour the world map so that adjacent countries receive distinct colours. This conjecture was communicated by his brother to De Morgan in 1852. The conjecture of Guthrie is equivalent to the conjecture that any plane graph is 4-face colourable. The latter conjecture is equivalent to the conjecture: Every planar graph is 4-vertex colourable.

Ever since the conjecture was first published in 1852 many eminent mathematicians especially graph theorists attempted to settle the conjecture. In the process of setting the conjecture, many equivalent formulations of this conjecture were found. Assaults on the conjecture were made using such varied branches of mathematics as algebra, number theory, and finite geometries. The solution found the light of the day when K. Appel, W Haken and J Koch of the University of Illinois established the validity of conjecture in 1976 with the aid of computers. The proof includes, among other things 10^{10} units of operations, amounting to a staggering 1200 hours of computer time on the high speed computer available at that time.

Although the computer oriented proof of Appel, Haken and Koch settled the conjecture in 1976 and has stood the test of time, a theoretical proof of the four colour problem is still to

Even though the solution of the 4-colour conjecture has been formidable task, it is rather easy to establish that every planar graph is 6-vertex colourable.

Corollary 9.1: Every planar graph contains a vertex of degree 5 or less.

Theorem 9.2: Every planar graph is 6-vertex colourable.

Proof: The proof is by induction on n, the number of vertices of the graph. The result is trivial for planar graphs with at most 6 vertices. Assume the result for planar graphs with n-1, $n \geq 7$, vertices. Let G be a planar graph with n vertices. By corollary, $\delta(G) \leq 5$ and thus G has a vertex v of degree at most 5. By hypothesis, G-v is 6-vertex colourable in any 6-vertex colouring of G-v the neighbours of v in G would have used only at most five colours and hence v can be coloured by a used colour. In other words, G is 6-vertex colourable.

Theorem 9.1(Heawood Five colour Theorem): Every planar graph is 5-colourable.

Conclusion

Graph colouring is a very interesting aspect in the modern world. The first results about graph colouring deal almost exclusively with planar graphs in the form of the colouring of maps, while trying to colour a map of the counties of England.

We can easily understand any problem using graph colouring. It has great significance in modern world. There are different types of graph colouring but we mainly discussed about vertex and edge colouring and their applications in the world.

I learned that graph colouring enjoys many practical applications as well as theoretical challenges. Besides the classical types of problems, different limitations can also be set on the graph, or on the way a colour is assigned, or even on the colour itself. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku. Graph colouring is still a very active field of research.

Bibliography

1. Graph Theory ,V.K.Balakrishnan

2. www.wikipedia.org

3. Graph Theory with Applications to Engineering and Computer Science, Narsingh Rao