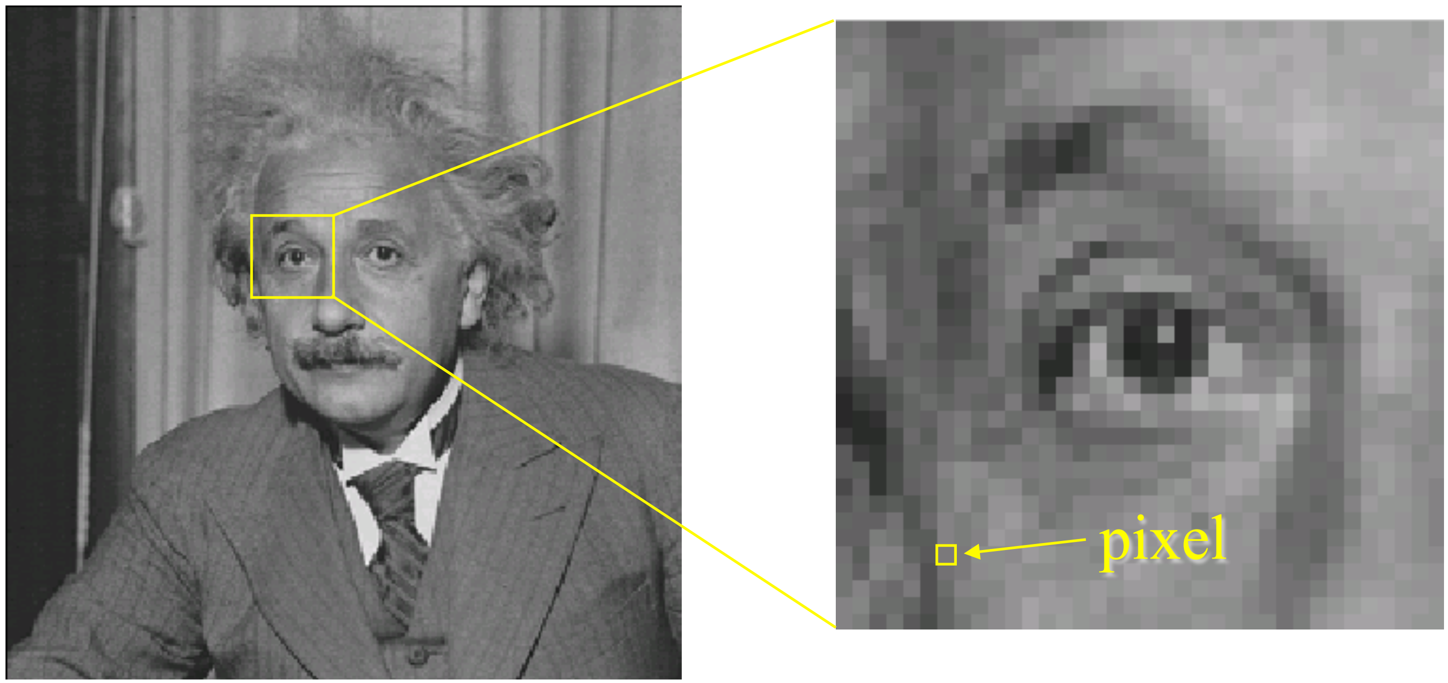
Topic 4:

Local analysis of image patches

- What do we mean by an image “patch”?
- Applications of local image analysis
- Visualizing 1D and 2D intensity functions

Local Image Patches

So far, we have considered pixels completely independently of each other (as a 2D array of numbers or RGB values)



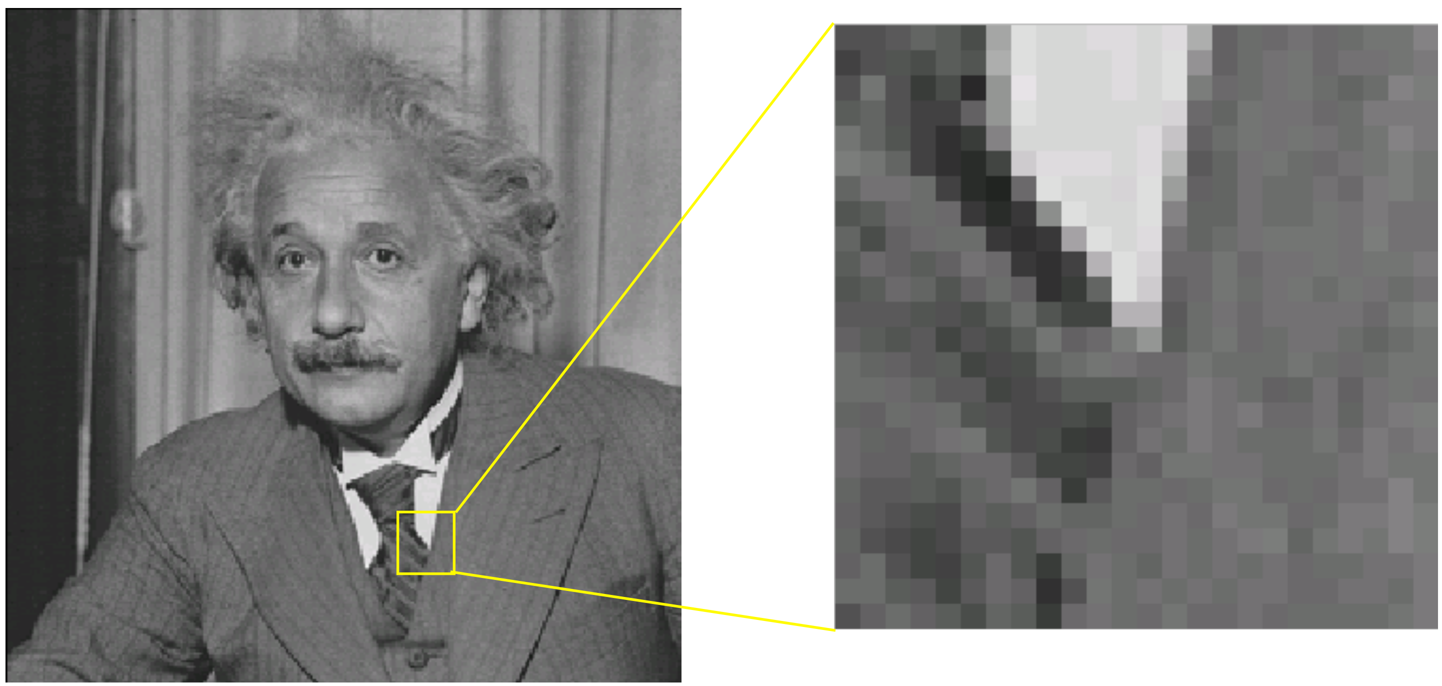
In reality, photos have a great deal of structure

This structure can be analyzed at a local level (eg., small groups of nearby pixels) or a global one (eg. entire image)

Local Image Patches

There are many different types of patches in an image

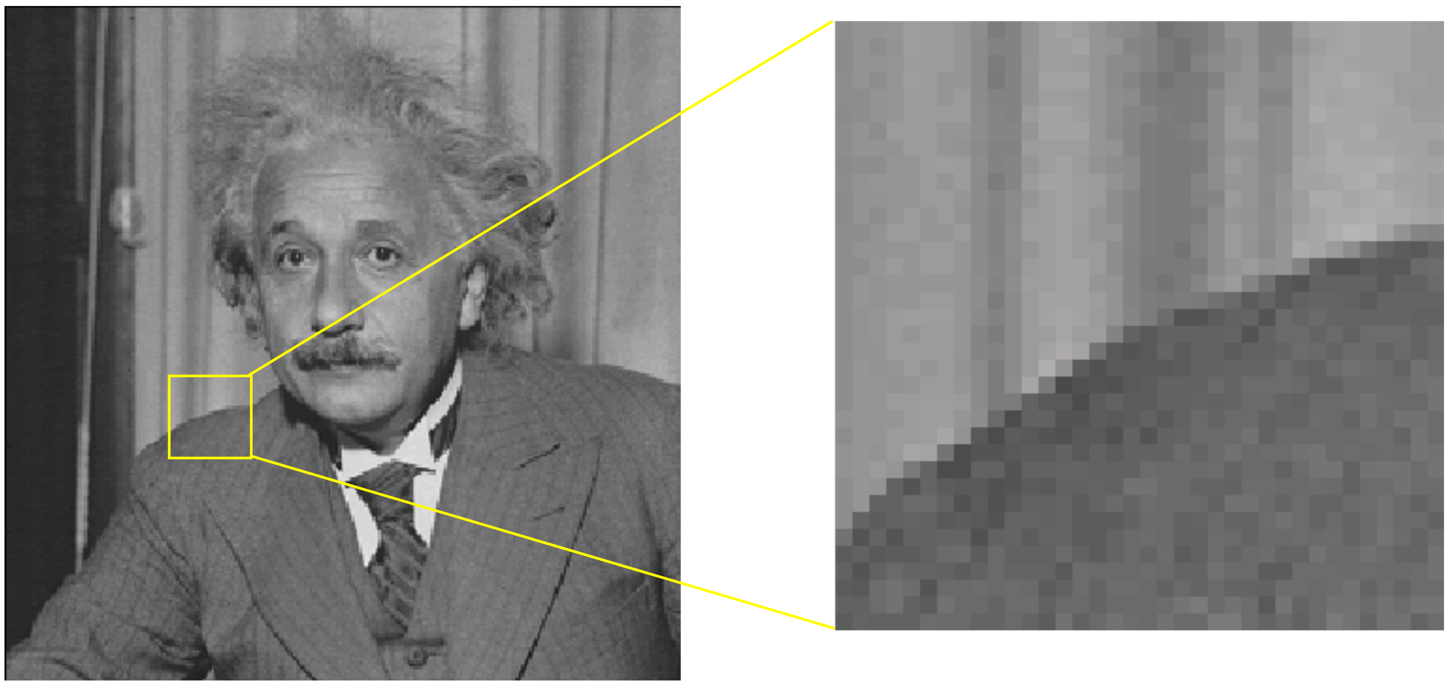
Patches corresponding to a “corner” in the image



Local Image Patches

There are many different types of patches in an image

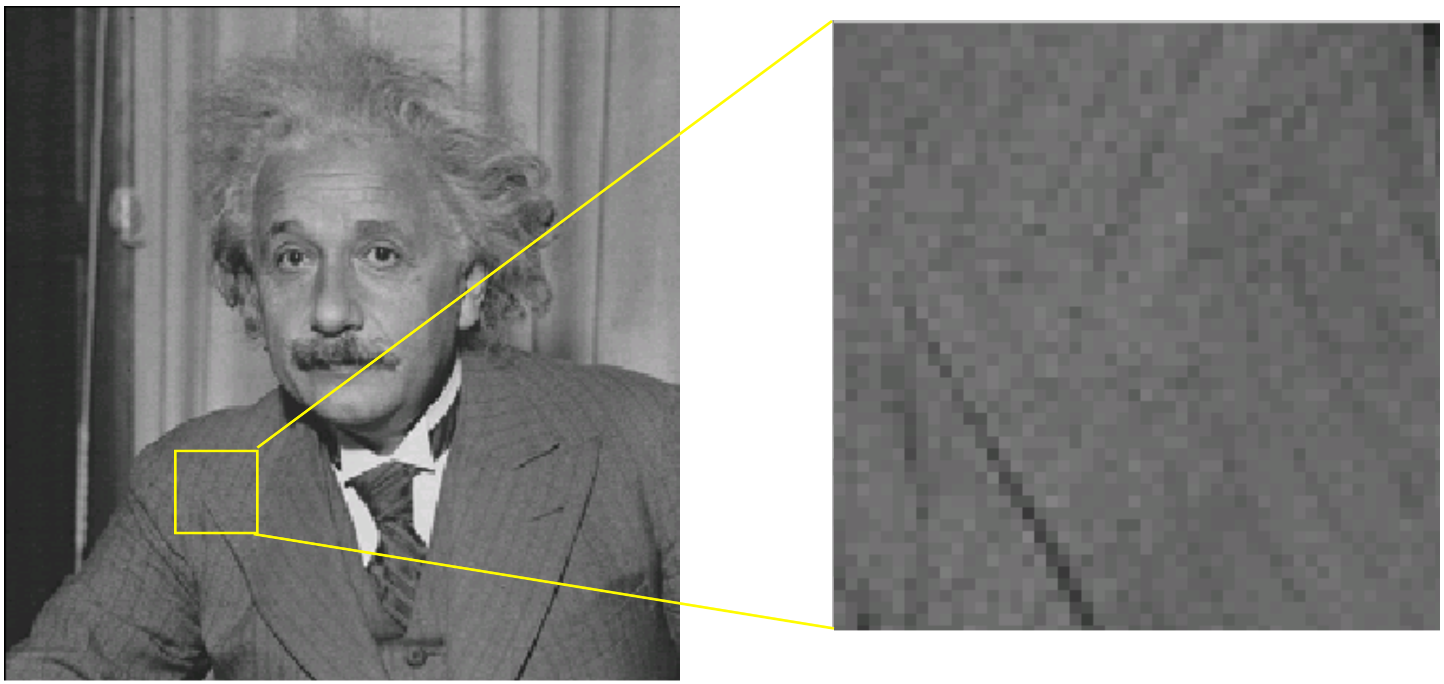
Patches corresponding to an “edge” in the image



Local Image Patches

There are many different types of patches in an image

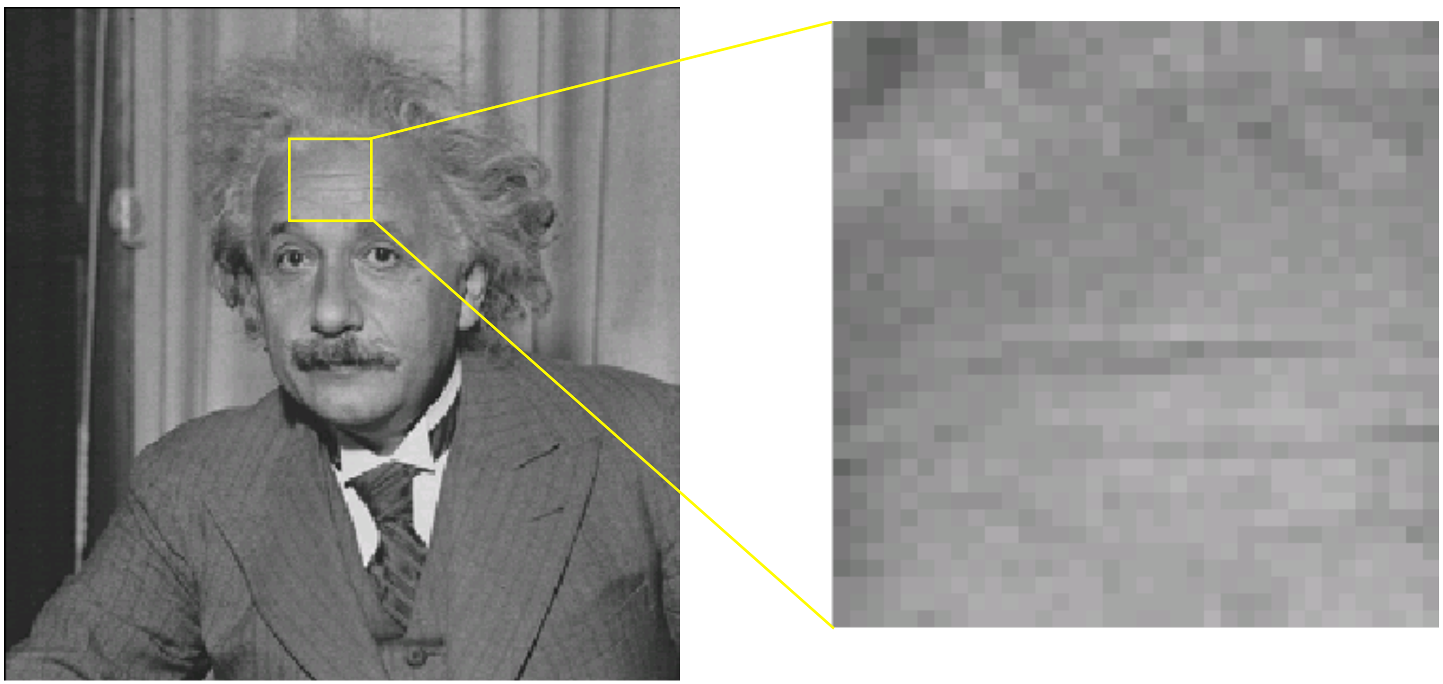
Patches of uniform texture



Local Image Patches

There are many different types of patches in an image

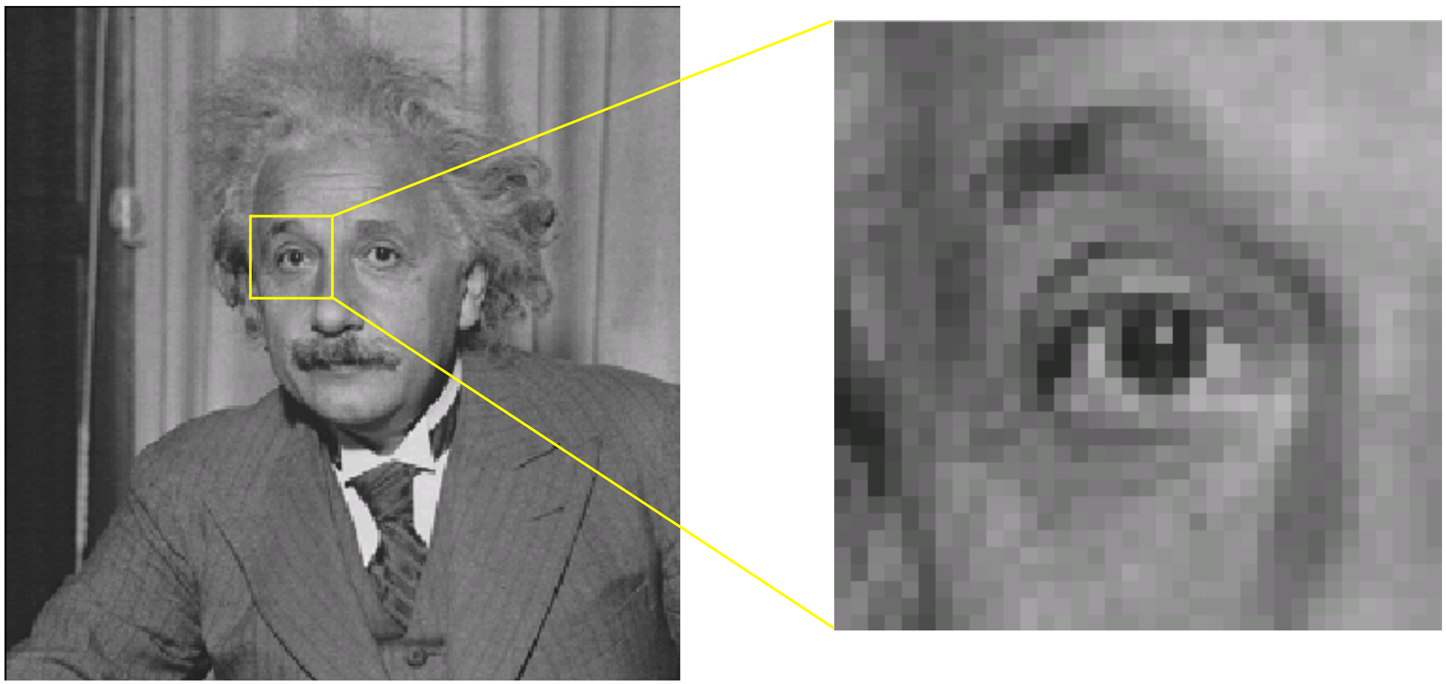
Patches associated with a single surface



Local Image Patches

There are many different types of patches in an image

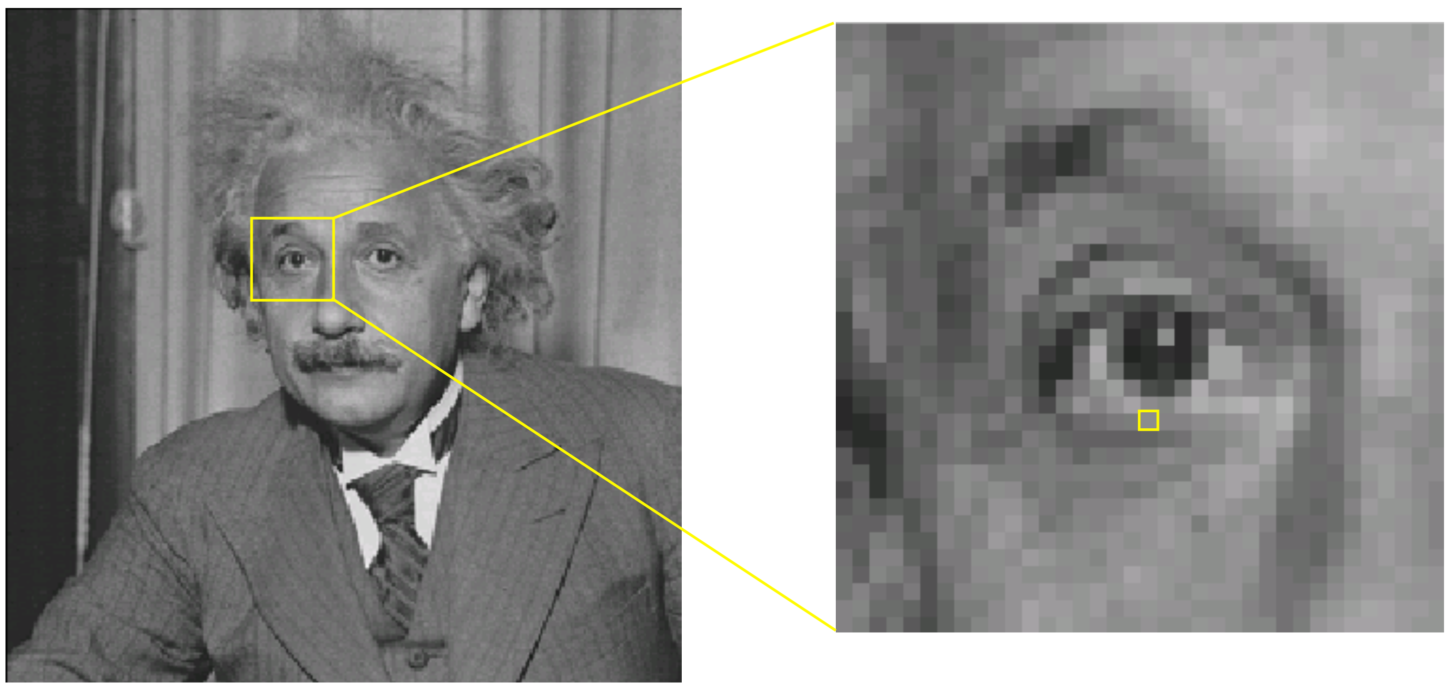
Perceptually-significant “features”



Local Image Patches

When is a group of pixels considered a local patch?

There is no answer to this question!

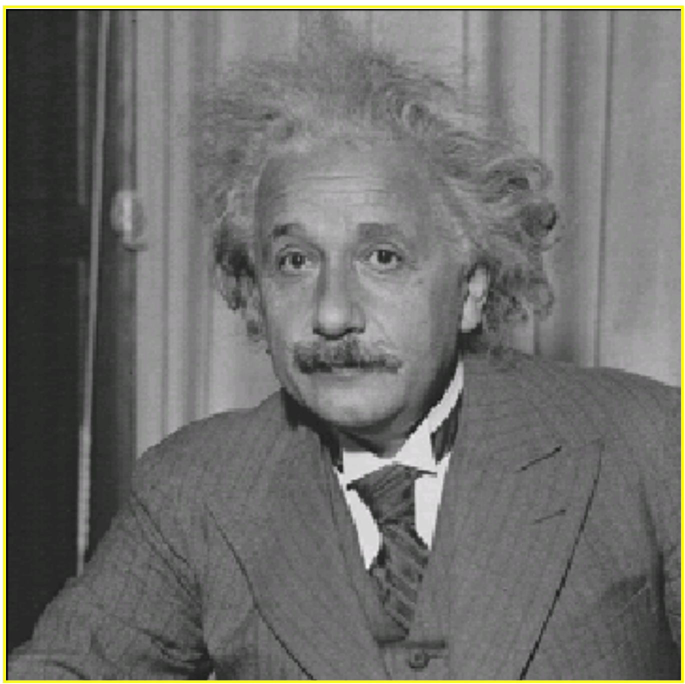


The notion of a patch is relative---it can be a single pixel

Local Image Patches

When is a group of pixels considered a local patch?

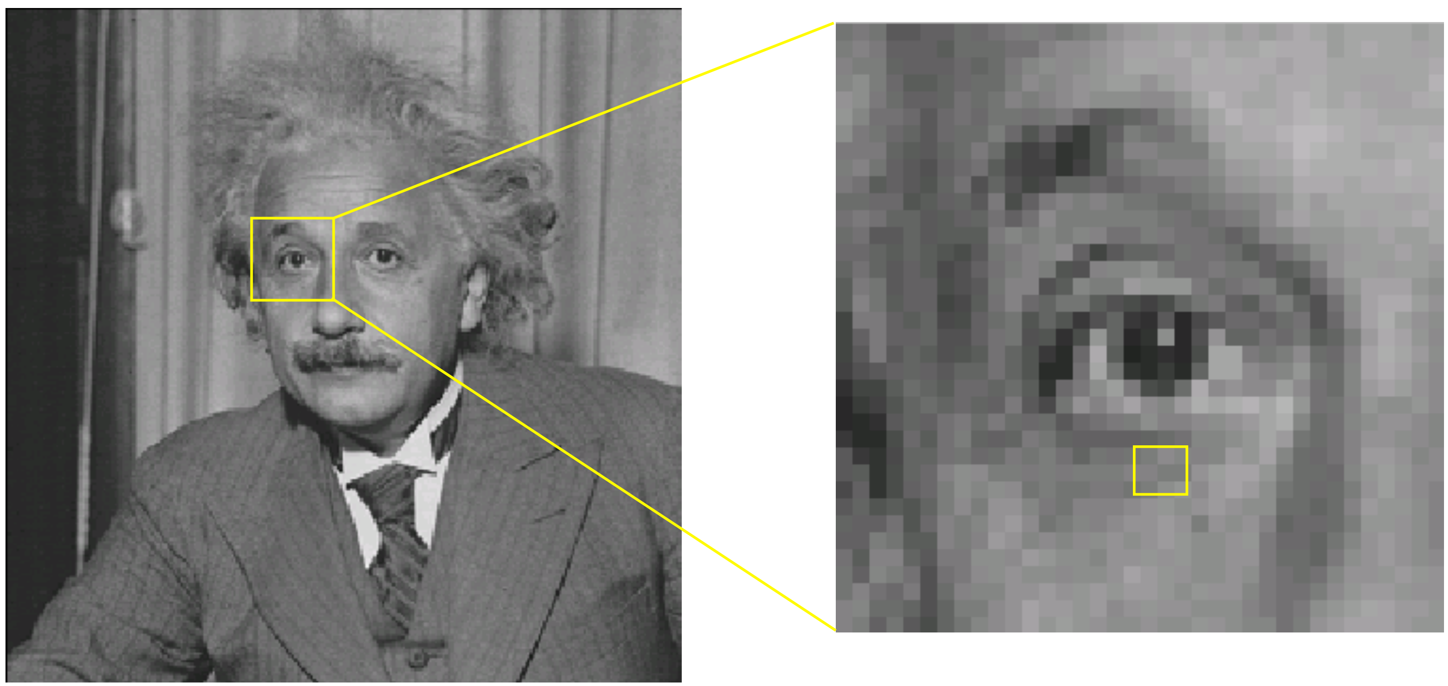
There is no answer to this question!



The notion of a patch is relative---it can be the entire image

Local Image Patches

We will begin with mathematical descriptions that apply mostly to very small patches (e.g., 3x3)



… and eventually consider descriptions that apply to entire images

Topic 4:

Local analysis of image patches

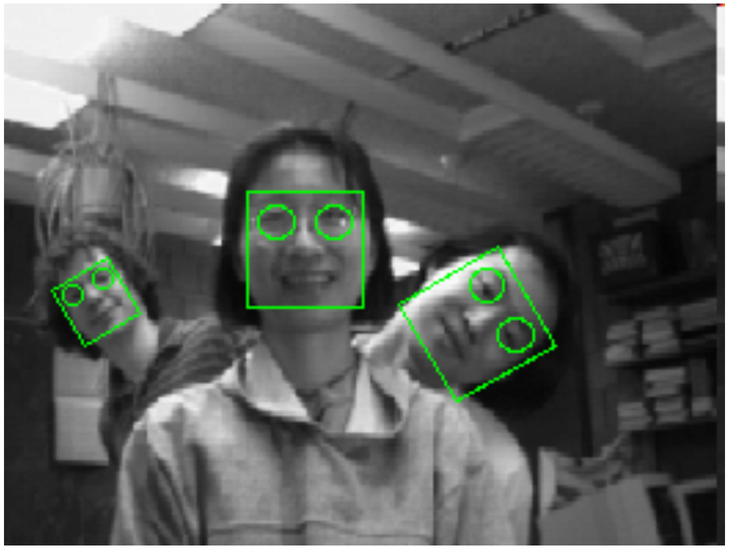
- What do we mean by an image “patch”?
- Applications of local image analysis
- Visualizing 1D and 2D intensity functions

Why Do We Care?

Many applications…

- Recognition
- Inspection
- Video-based tracking
- Special effects

Recognition & Tracking



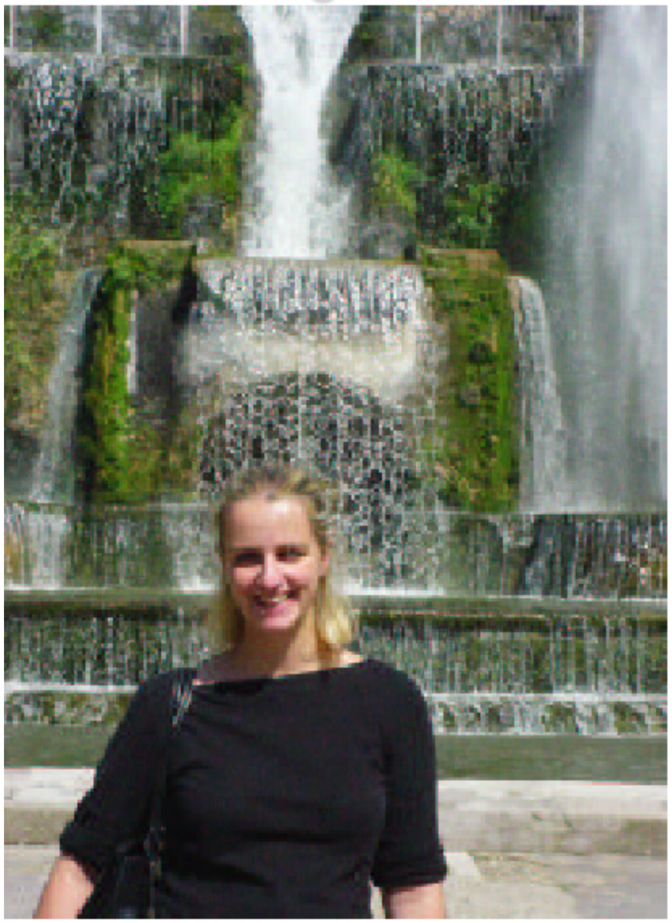
(Rowley et al, PAMI'98)

(El-Maraghi et al, CVPR'01)

Editing & Manipulating Photos

Object removal from a photo

Original



New

(Criminisi et al, CVPR 2003)

Editing & Manipulating Photos

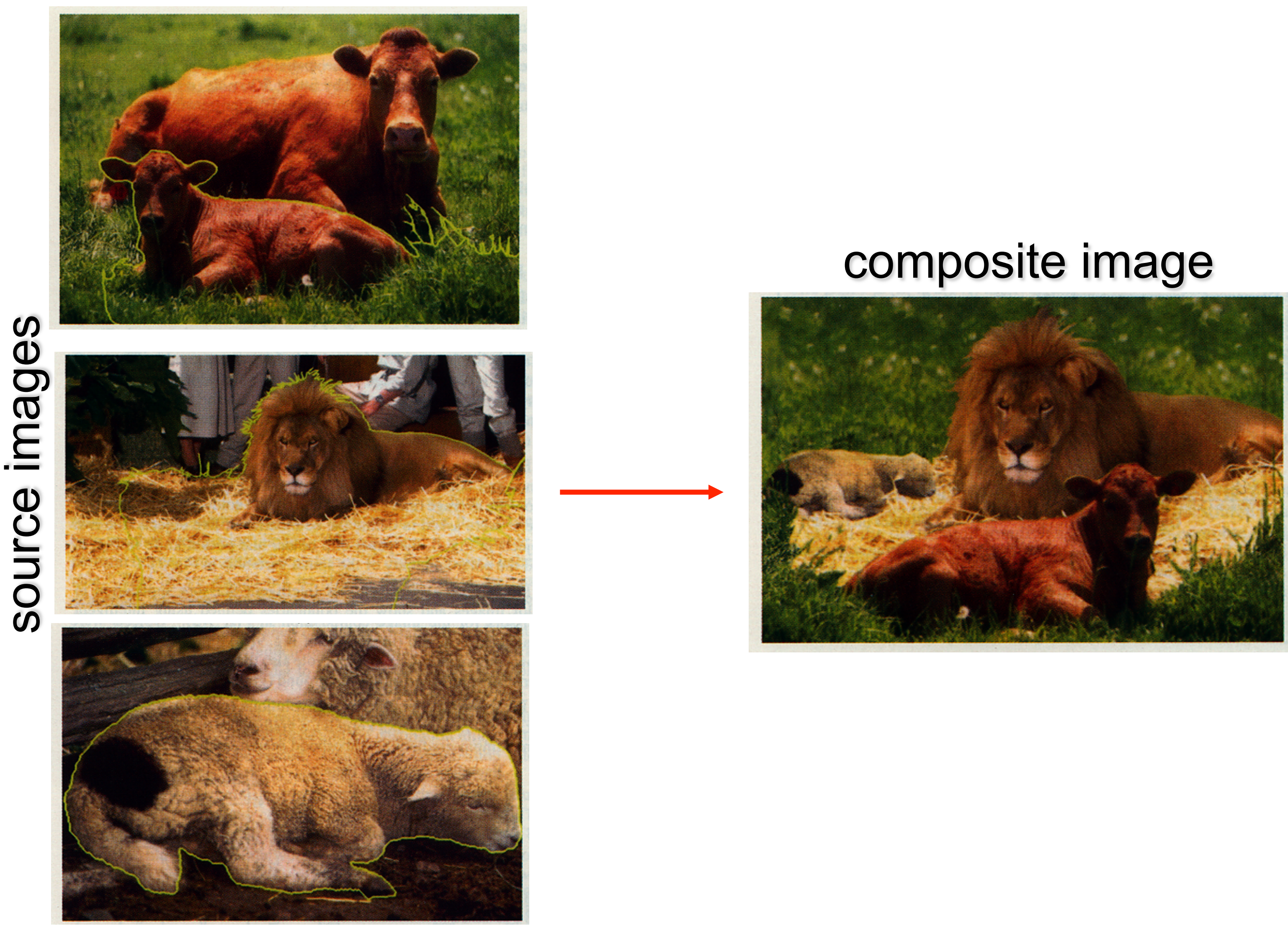
Colorization of black and white photos

Original (B&W) New (Color)

(Levin & Weiss, SIGGRAPH 2004)

Editing & Manipulating Photos

Scissoring objects from a photo



Giving Photos a "Painted" Look

From P. Litwinowicz's SIGGRAPH'97 paper
"Processing Images and Videos for an Impressionist Effect"

Topic 4:

Local analysis of image patches

- What do we mean by an image “patch”?
- Applications of local image analysis
- Visualizing 1D and 2D intensity functions

Image ⇔ Surface in 3D

Gray-scale image $I(x,y)$

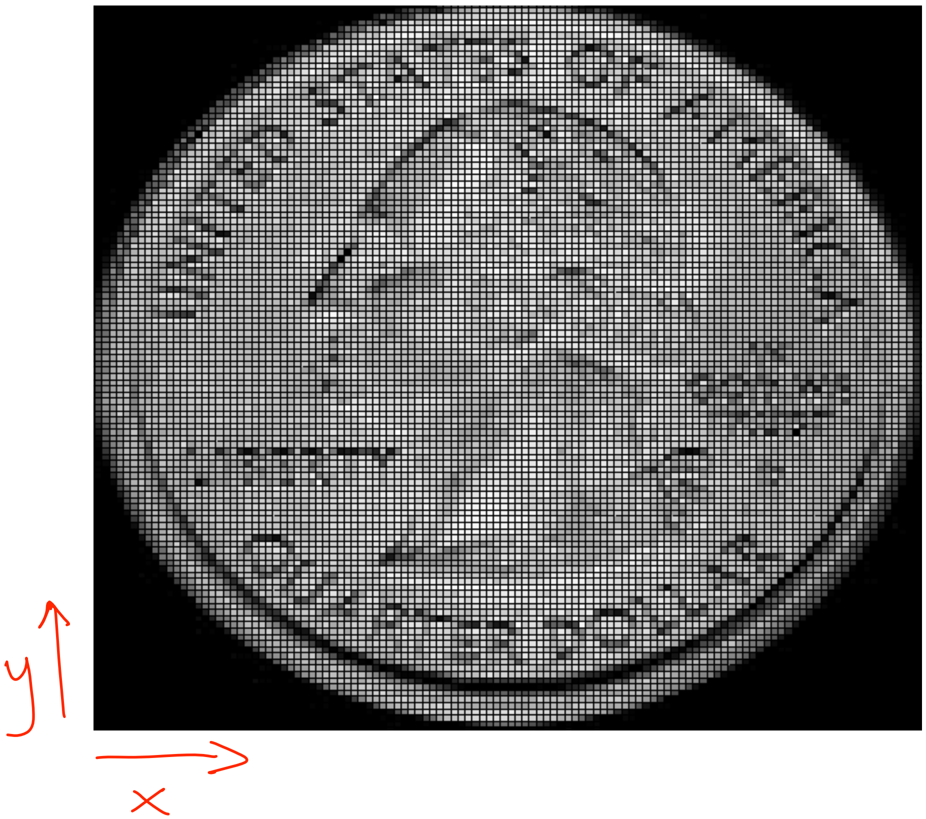
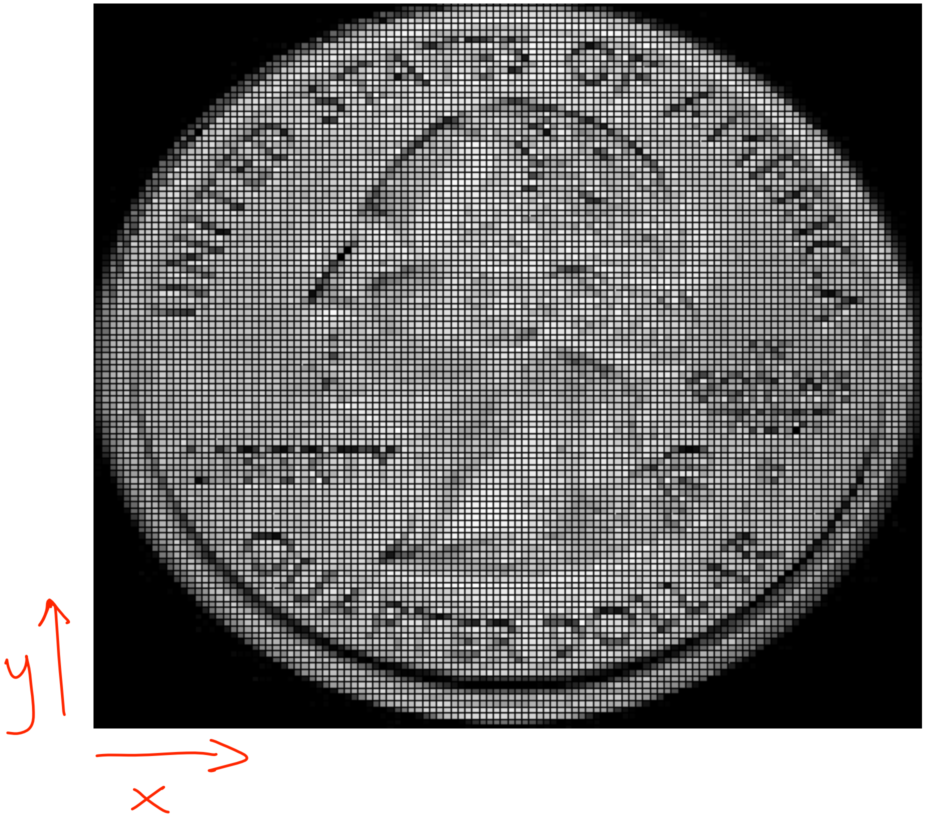


Image ⇔ Surface in 3D

Gray-scale image $I(x,y)$



Surface $z = I(x,y)$

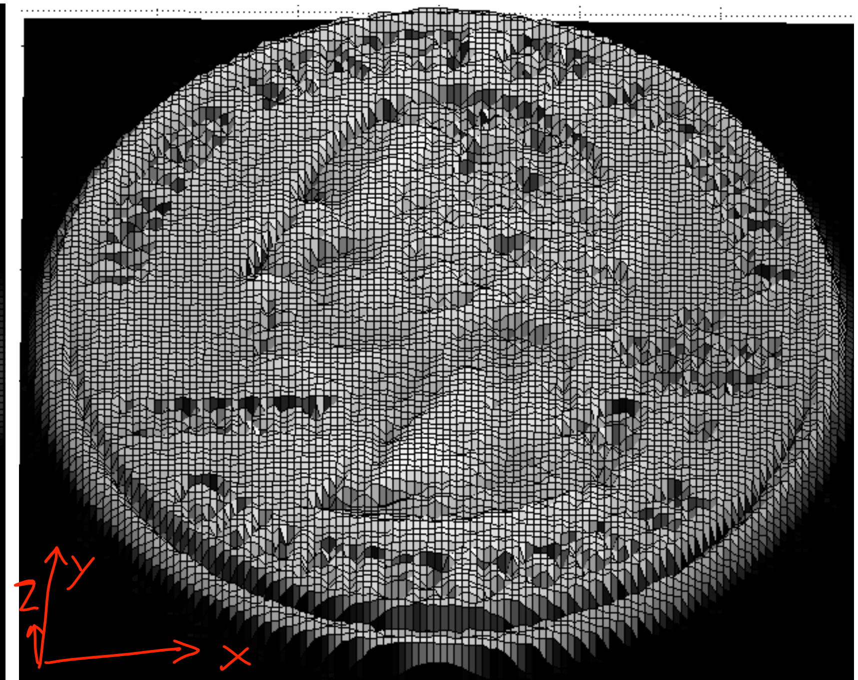
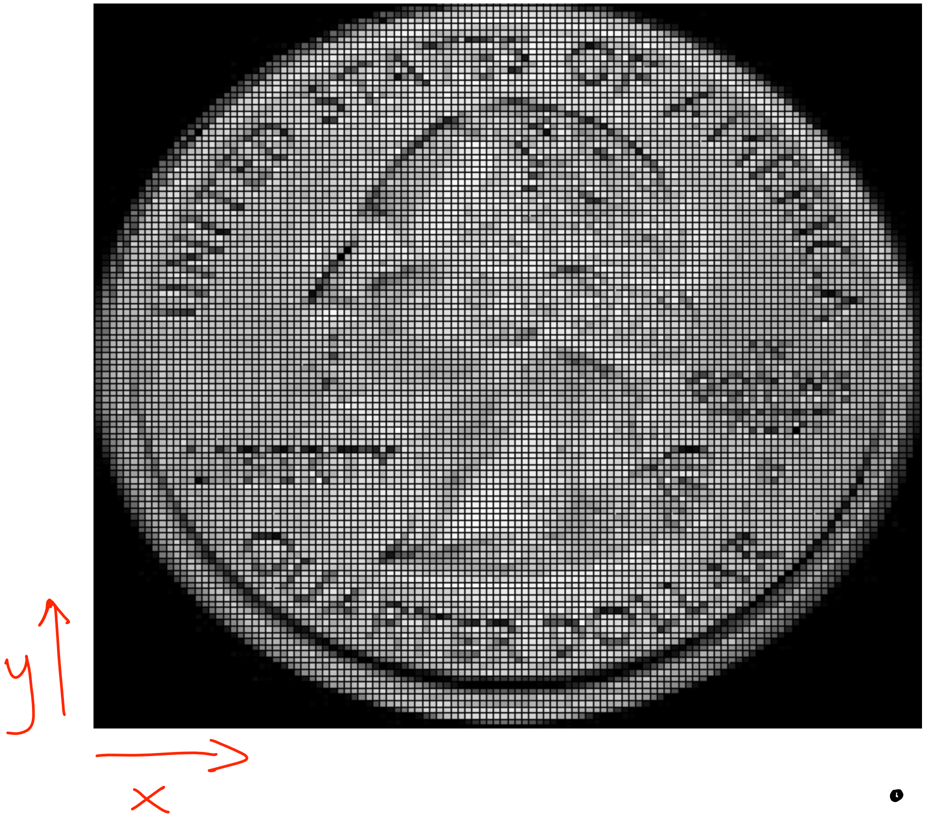
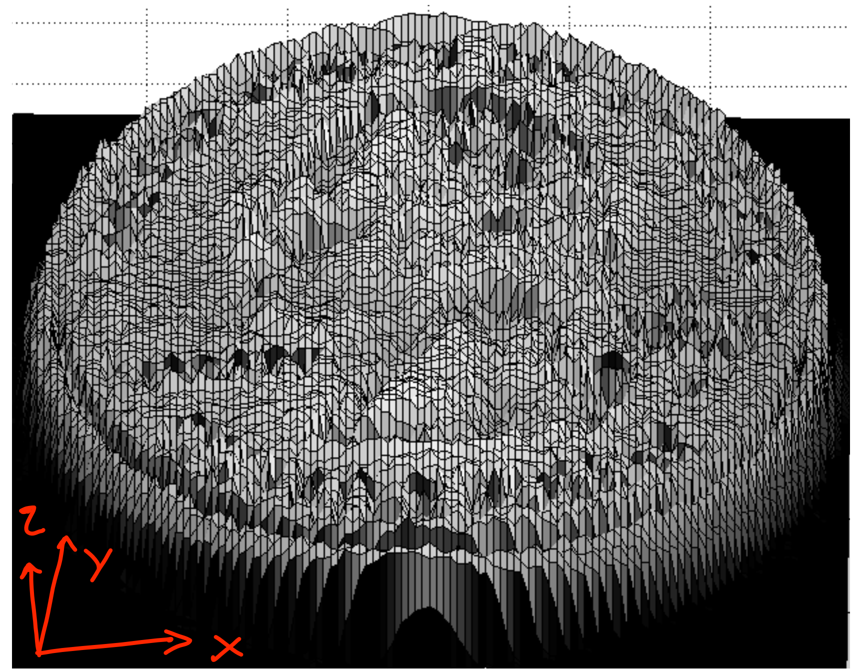


Image ⇔ Surface in 3D

Gray-scale image $I(x,y)$



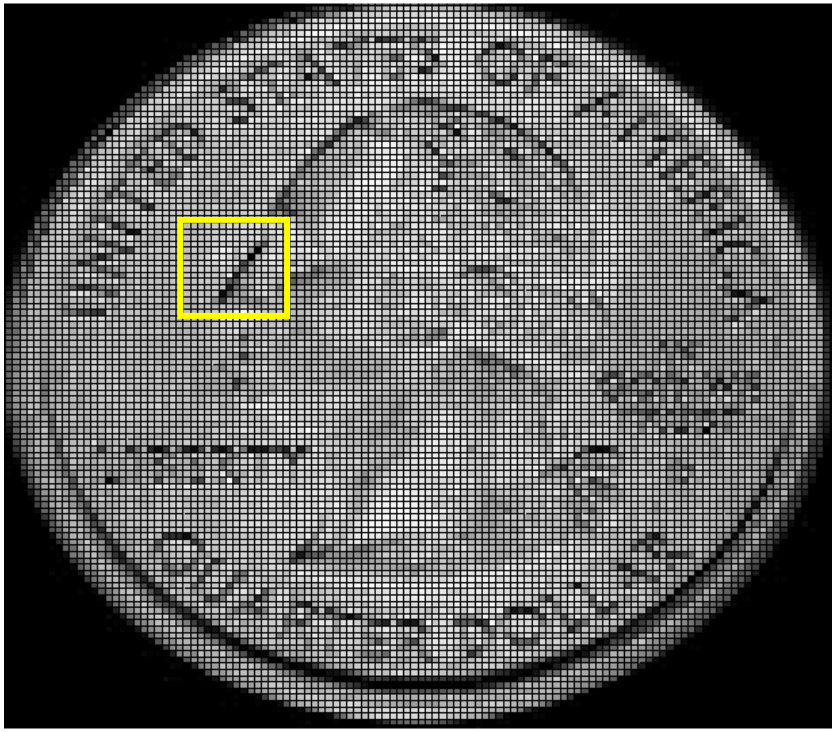
Surface $z = I(x,y)$



- The height of the surface at (x,y) is $I(x,y)$
- The surface contains point $(x, y, I(x,y))$

Image ⇔ Surface in 3D

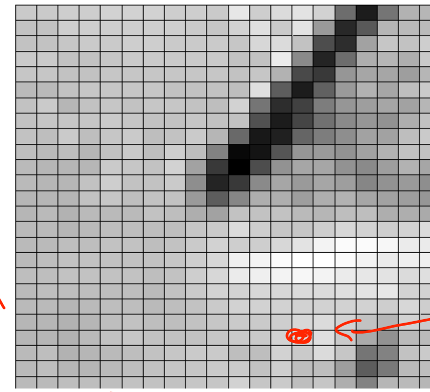
Gray-scale image



y

x

Image patch



Surface patch $Z = I(x,y)$

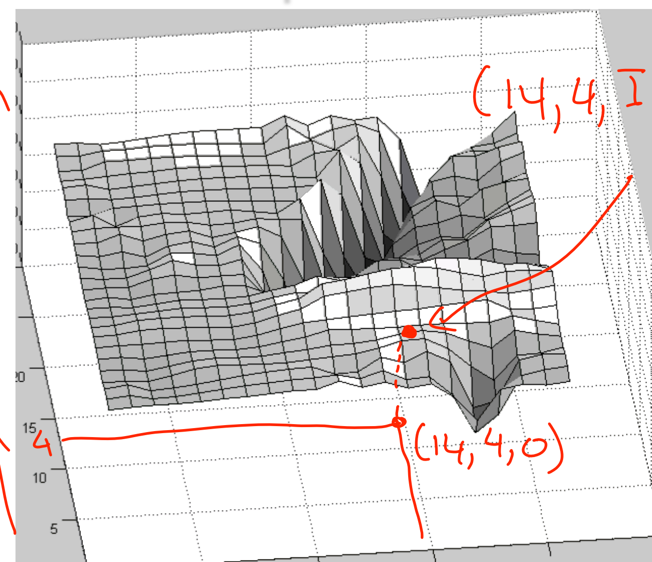
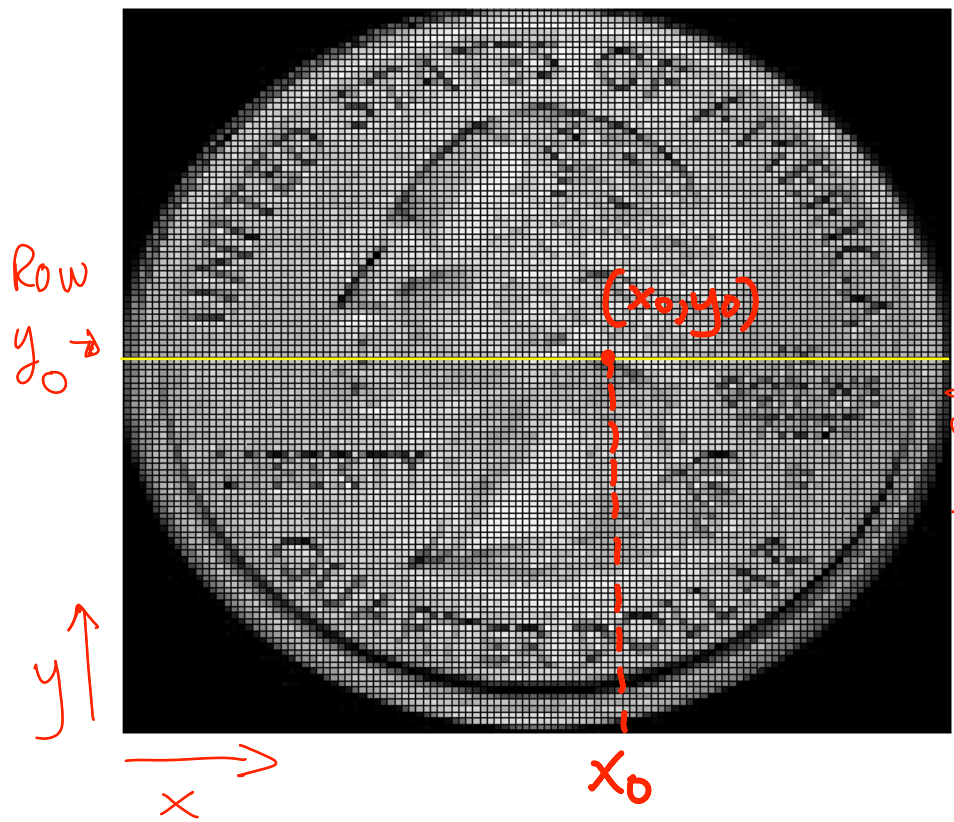


Image row or column ⇔ Graph in 2D

Gray-scale image



Graph in 2D

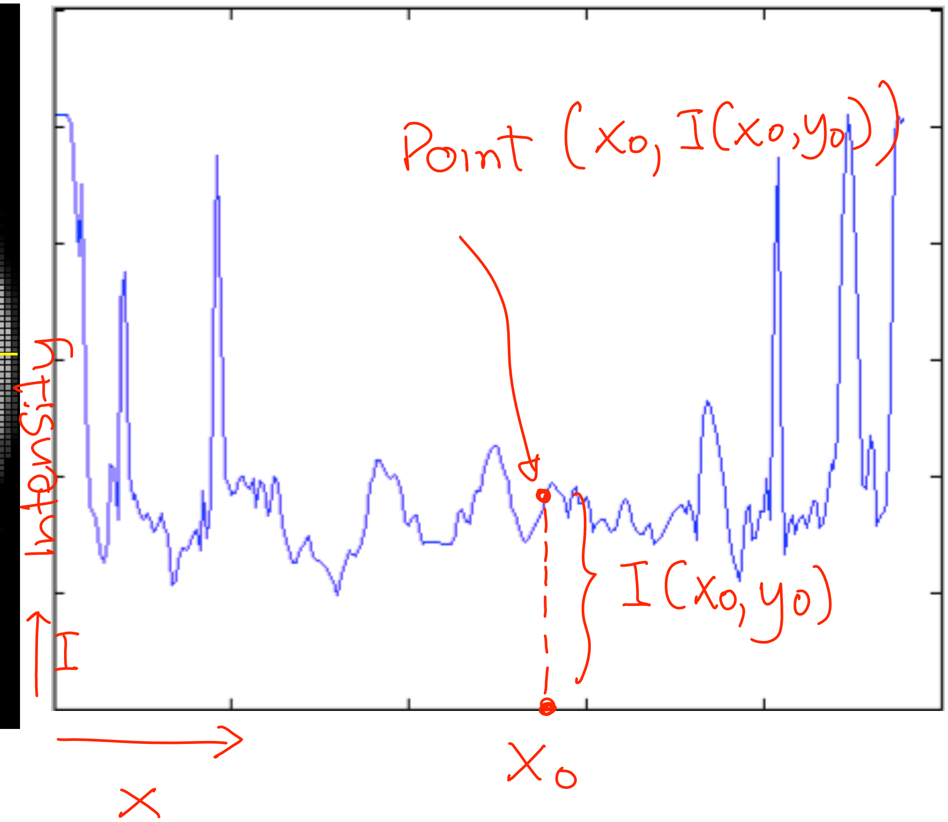
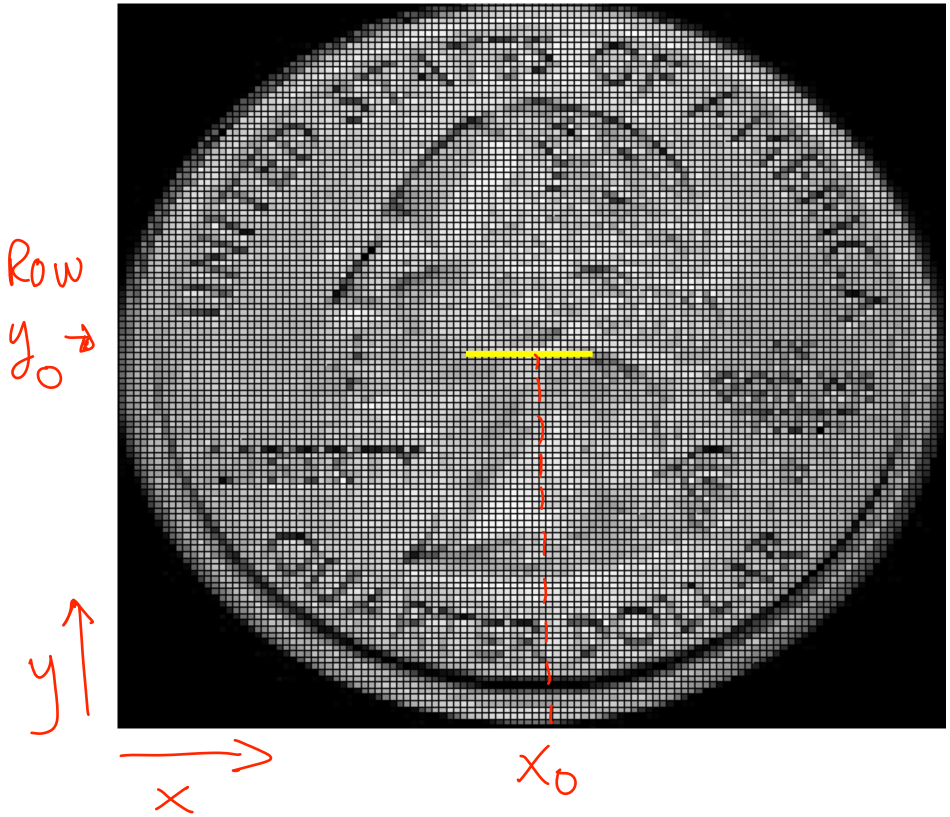
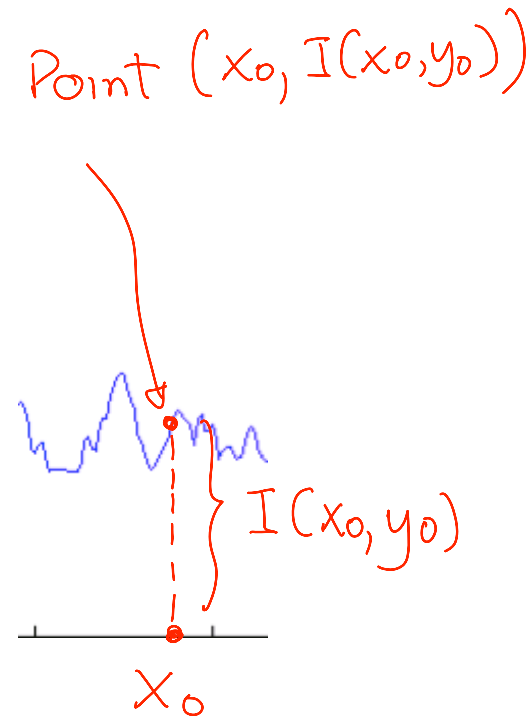


Image row or column ⇔ Graph in 2D

Gray-scale image

Graph in 2D





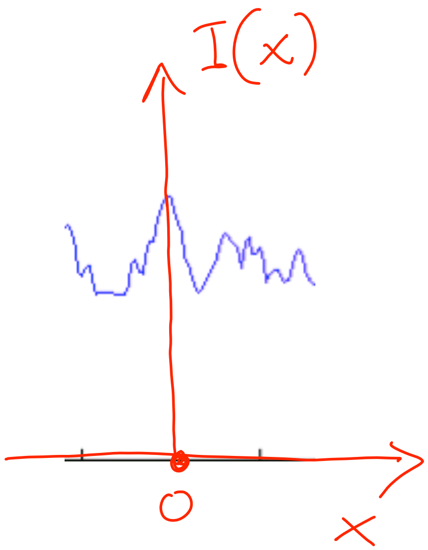
Topic 4:

Local analysis of image patches

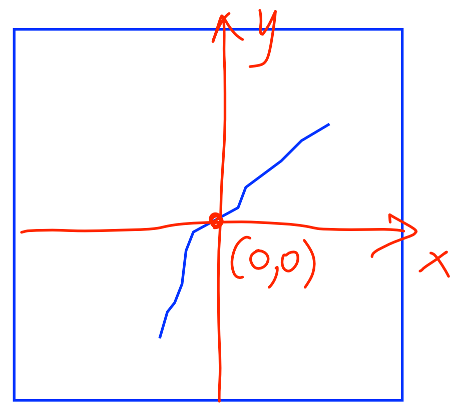
- Subtopics:
    1. Local analysis of 1D image patches
    2. Local analysis of 2D curve patches
    3. Local analysis of 2D image patches

Local Analysis of Image Patches: Outline

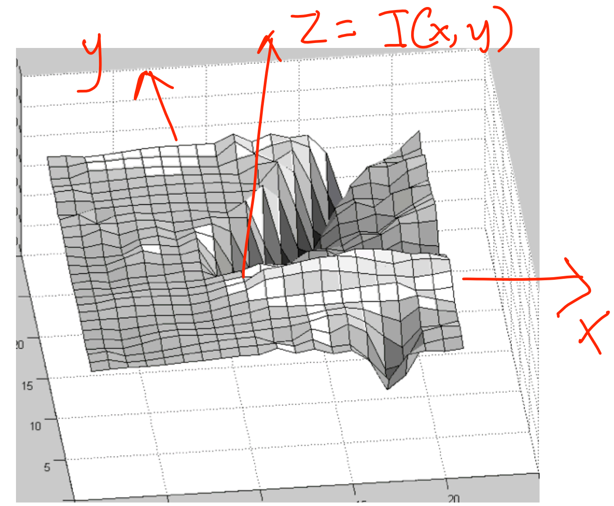
As graph in 2D

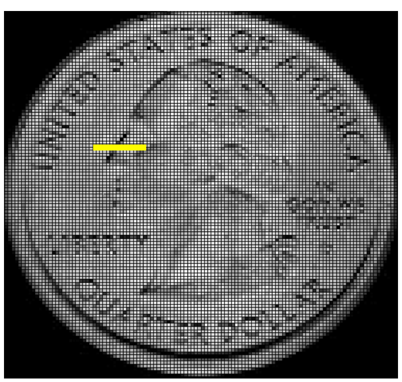


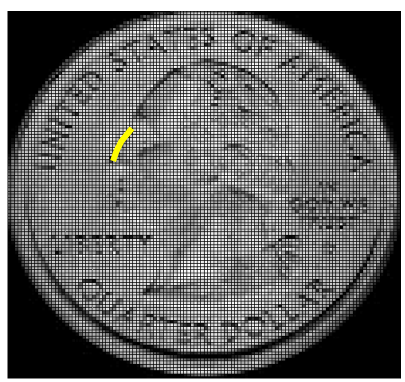
As curve in 2D

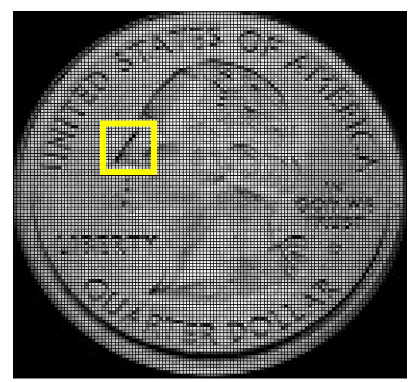


As surface in 3D









Topic 4.1:

Local analysis of 1D image patches

- Taylor series approximation of 1D intensity patches
- Estimating derivatives of 1D intensity patches:
  - Least-squares fitting
  - Weighted least-squares fitting
  - Robust polynomial fitting: RANSAC

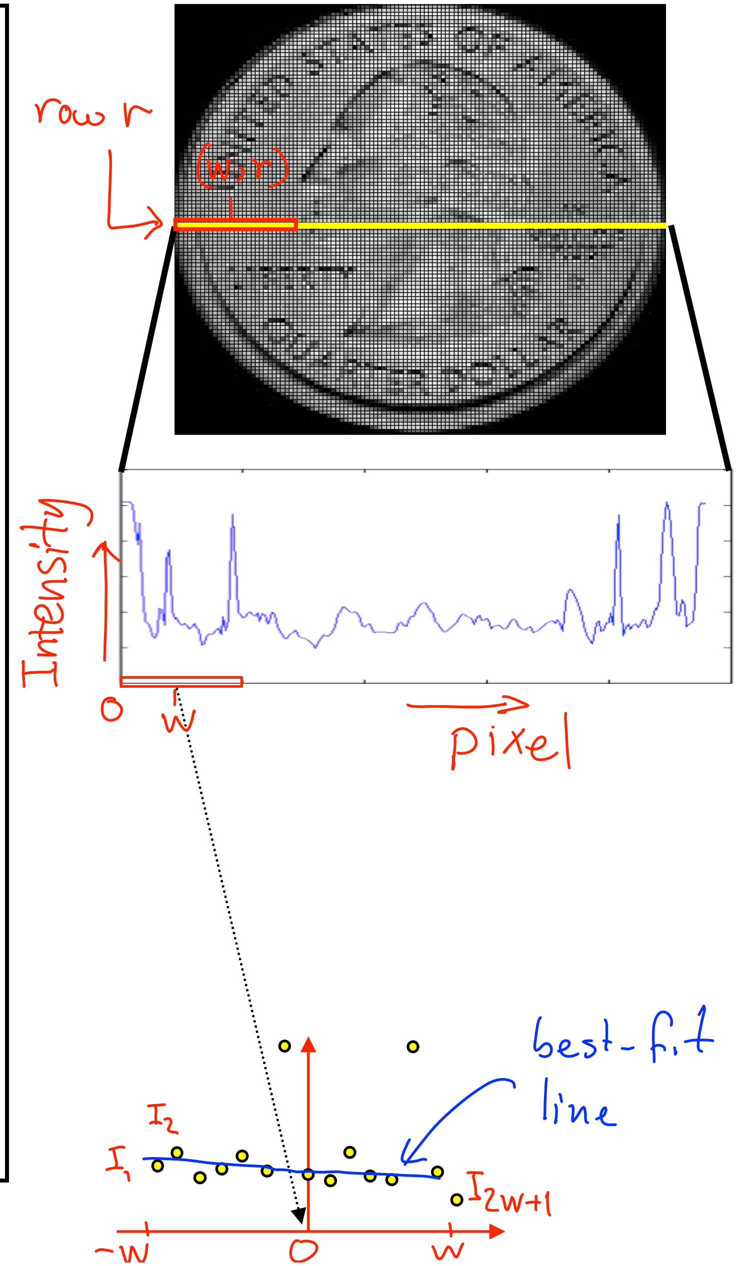
Estimating Derivatives For Image Row r

"Sliding window" algorithm:

- Define a "pixel window" centered at pixel (w,r)
- Fit n-degree poly to window's intensities (usually n=1 or 2)
- Assign the poly's derivatives at x=0 to pixel at window's center

$$\underbrace{\frac{dI}{dx}(w)}_{\text{image}} \longleftrightarrow \underbrace{\frac{dI}{dx}(0)}_{\text{patch}}$$

- "Slide" window one pixel over, so that it is centered at pixel (w+1,r)
- Repeat 1-4 until window reaches right image border



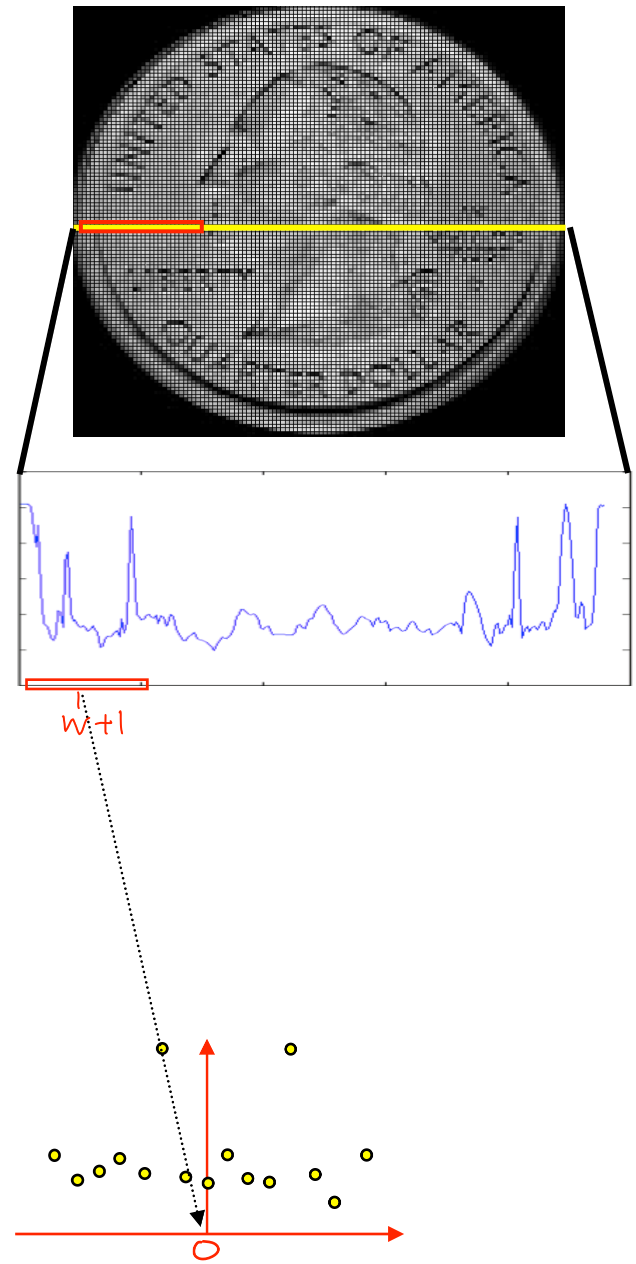
Estimating Derivatives For Image Row r

"Sliding window" algorithm:

- Define a "pixel window" centered at pixel (w,r)
- Fit n-degree poly to window's intensities (usually n=1 or 2)
- Assign the poly's derivatives at x=0 to pixel at window's center

$$\underbrace{\frac{dI}{dx}(w)}_{\text{image}} \longleftrightarrow \underbrace{\frac{dI}{dx}(0)}_{\text{patch}}$$

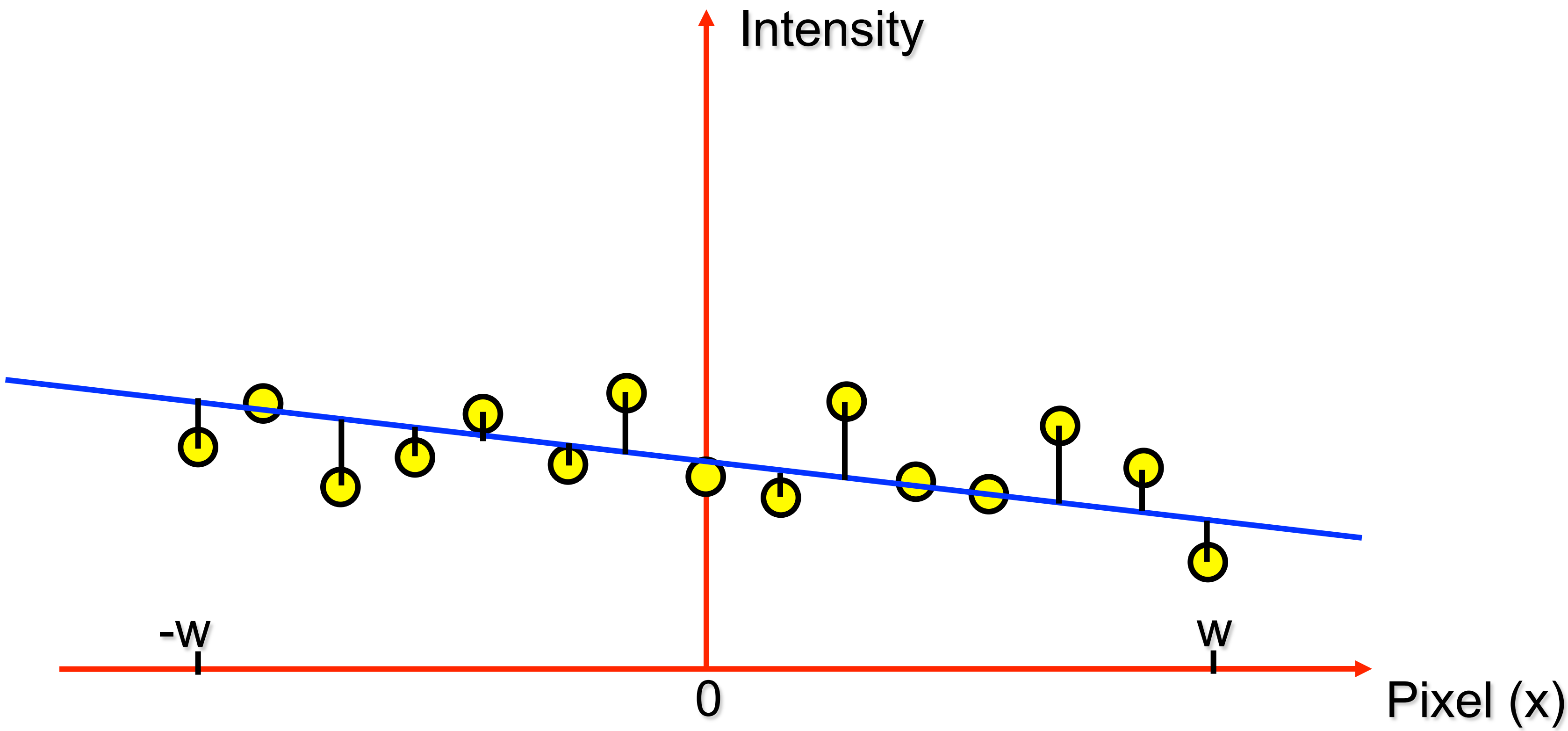
- "Slide" window one pixel over, so that it is centered at pixel (w+1,r)
- Repeat 1-4 until window reaches right image border



Least-Squares Polynomial Fitting

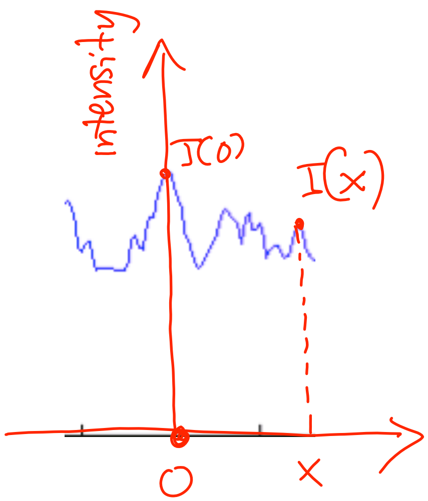
Scenario #1:

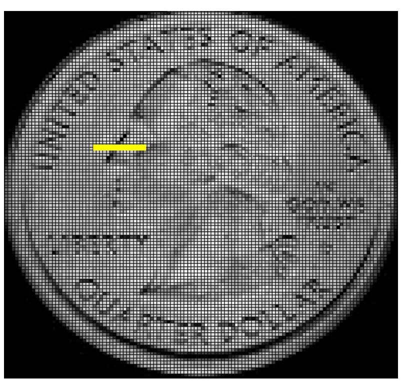
- Fit polynomial to ALL pixel intensities in a patch
- All pixels contribute equally to estimate of derivative(s) at patch center (i.e., at x=0)



Taylor-Series Approximation of I(x)

As graph in 2D





Taylor series expansion of $I(x)$ near the "patch" center 0

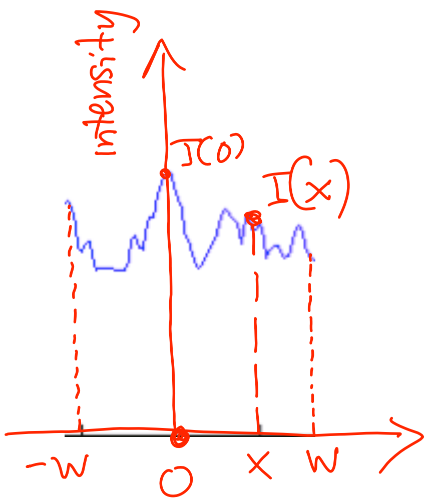
$$I(x) = \underbrace{\underbrace{\underbrace{\underbrace{I(0)}_{\text{0-th order approximation}} + x \cdot \frac{dI}{dx}(0)}_{\text{1st-order approx. of } I} + \frac{1}{2}x^2\frac{d^2I}{dx^2}(0)}_{\text{2nd-order approx of } I} + \cdots + \frac{1}{n!}x^n\frac{d^nI}{dx^n}(0)}_{\text{n-th order approx}} + R_{n+1}(x)$$

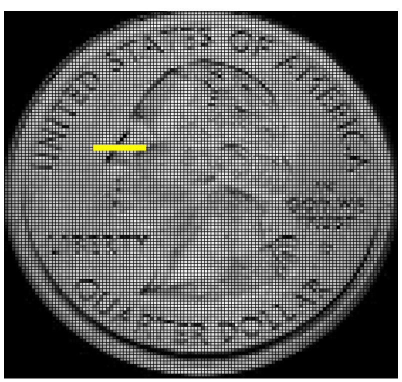
The residual $R_{n+1}(x)$ satisfies

$$\lim_{x \to 0} R_{n+1}(x) = 0$$

Taylor-Series Approximation of I(x)

As graph in 2D





Taylor series expansion of $I(x)$ near the "patch" center 0

n^{th}-order approximation in matrix notation:

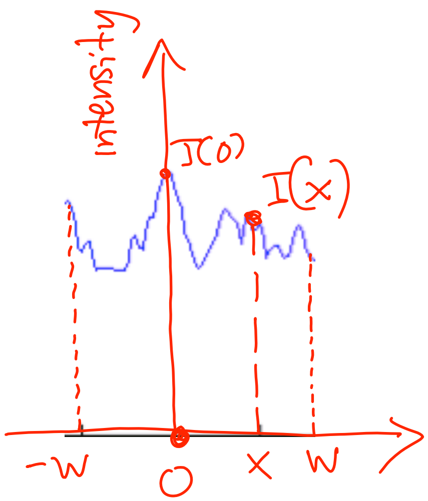
for $x \in (-w, w)$

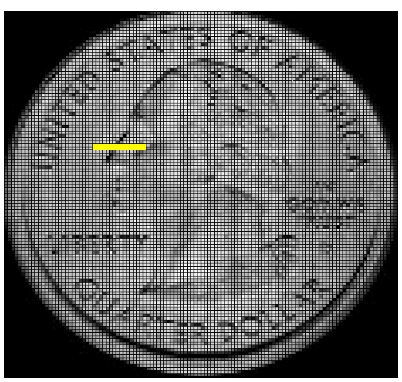
$$I(x) \simeq \left[1 \quad x \quad \frac{1}{2}x^2 \quad \frac{1}{6}x^3 \quad \cdots \quad \frac{1}{n!}x^n \right] \underbrace{\begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{dI}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}}$$

for a given x, approximation depends on (n+1) constants corresponding to the intensity derivatives at the patch origin

Taylor-Series Approximation of I(x)

As graph in 2D

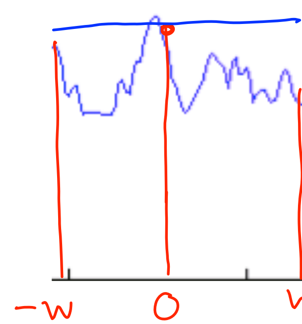




Taylor series expansion of $I(x)$ near the "patch" center O

Example: 0th-order approx

$$I(x) = I(0)$$

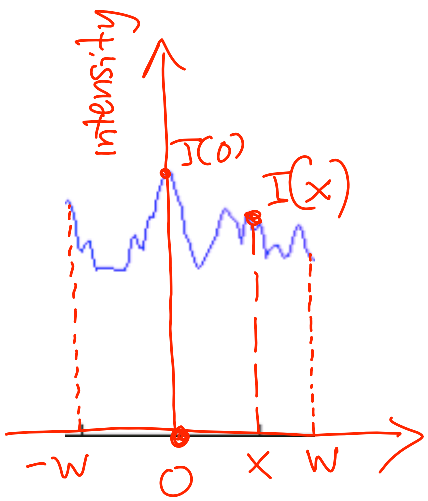


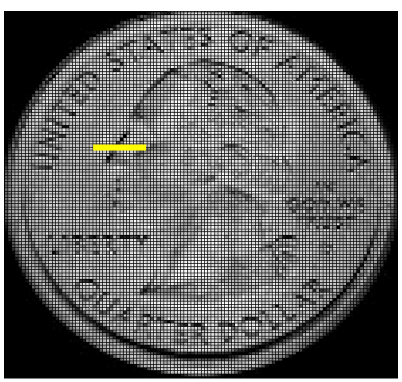
0-th order approximation of I

$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{d^2I}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

Taylor-Series Approximation of I(x)

As graph in 2D



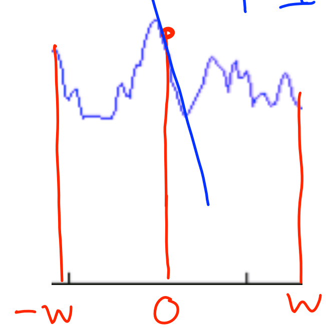


Taylor series expansion of I(x) near the "patch" center 0

Example: 1st-order approx

$$I(x) = I(0) + x \cdot \frac{dI}{dx}(0)$$

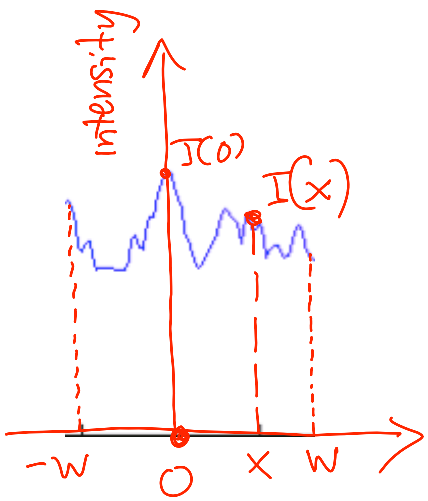
← 1st-order approximation of I

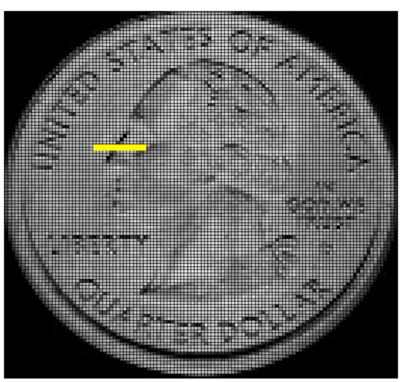


$$I(x) \simeq \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{d^2I}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

Taylor-Series Approximation of I(x)

As graph in 2D



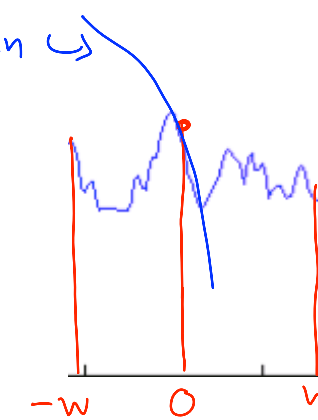


Taylor series expansion of $I(x)$ near the "patch" center 0

Example: 2nd-order approx

$$I(x) = I(0) + x \cdot \frac{dI}{dx}(0) + \frac{x^2}{2}\frac{d^2 I}{dx^2}(0)$$

2nd-order approximation of I



$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{d^2I}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

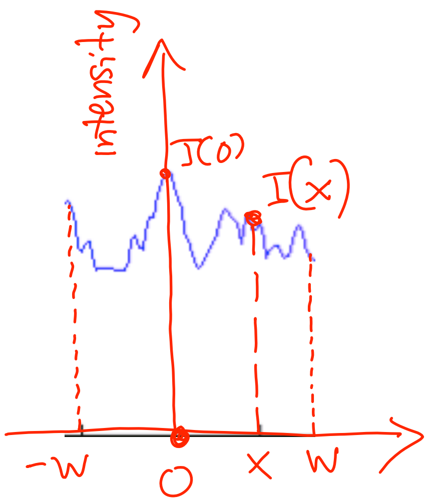
Topic 4.1:

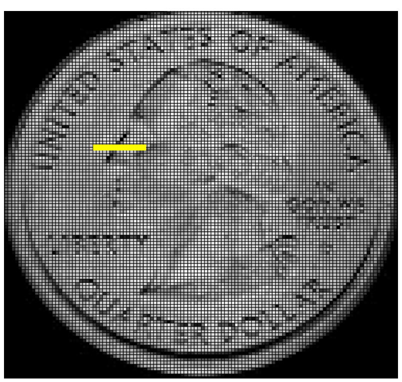
Local analysis of 1D image patches

- Taylor series approximation of 1D intensity patches
- Estimating derivatives of 1D intensity patches:
  - Least-squares fitting
  - Weighted least-squares fitting
  - Robust polynomial fitting: RANSAC

Least-Squares Polynomial Fitting of I(x)

As graph in 2D





Our first "patch descriptor":
<u>intensity derivatives</u>

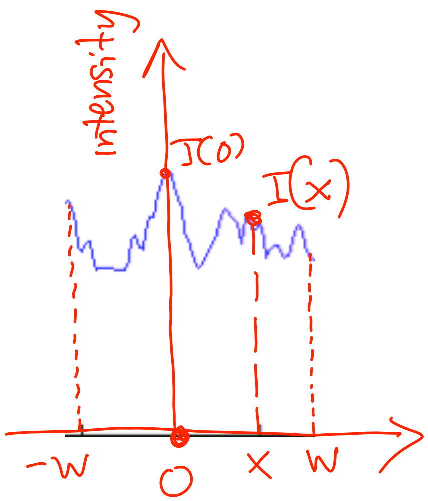
To compute the n derivatives at pixel 0:

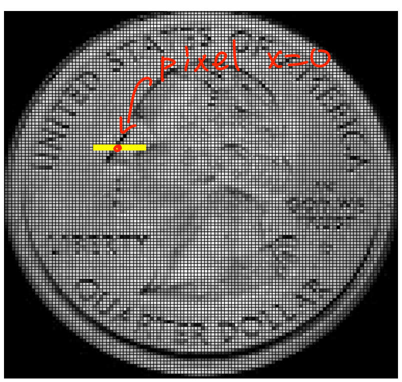
$\boxed{\text{fit a polynomial of degree } n \text{ to the patch intensities}}$

$$I(x) \simeq \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{dI}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

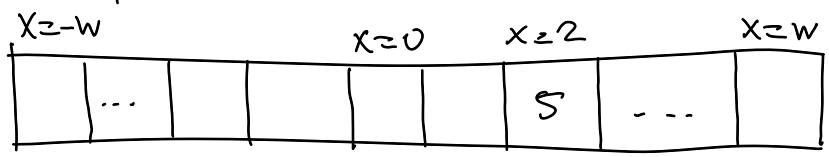
Least-Squares Polynomial Fitting of I(x)

As graph in 2D





Patch (2w+1 pixels)



1 linear eq. for every pixel

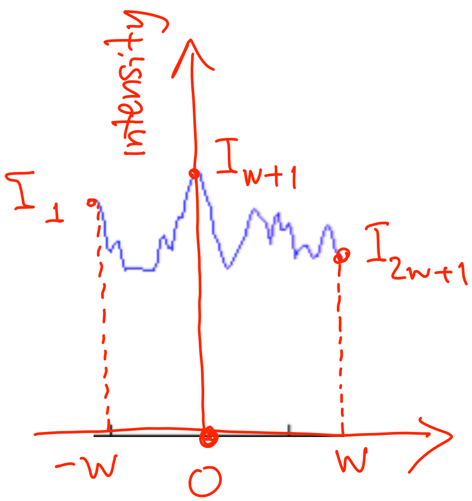
$$5 = \begin{bmatrix} 1 & 2 & \frac{1}{2}2^2 & \frac{1}{6}2^3 & \cdots & \frac{1}{n!}2^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{d^2I}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

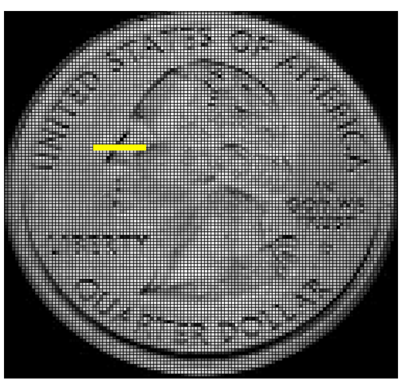
$\Leftrightarrow$ have 2w+1 eqs for the 2w+1 pixels

$$I(x) \simeq \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{d^2I}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

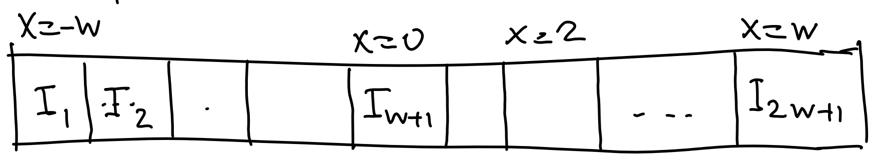
Least-Squares Polynomial Fitting of I(x)

As graph in 2D





Patch (2w+1 pixels)

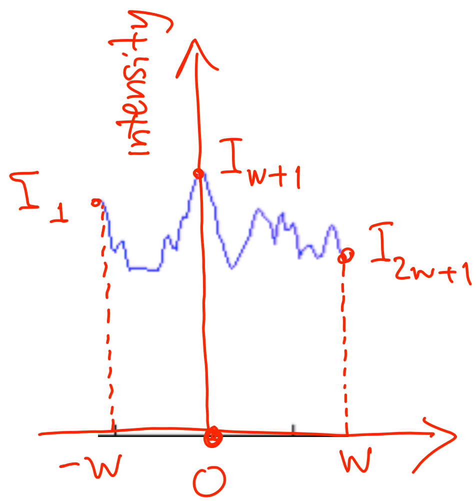


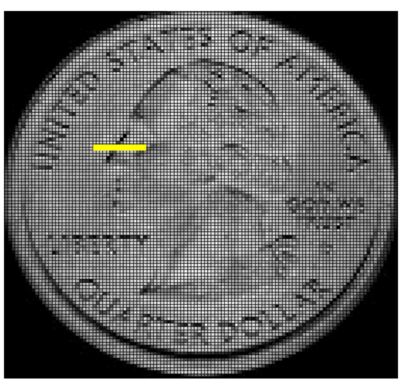
$$I_{(2w+1)\times 1} = X_{(2w+1)\times(n+1)}\, d_{(n+1)\times 1}$$

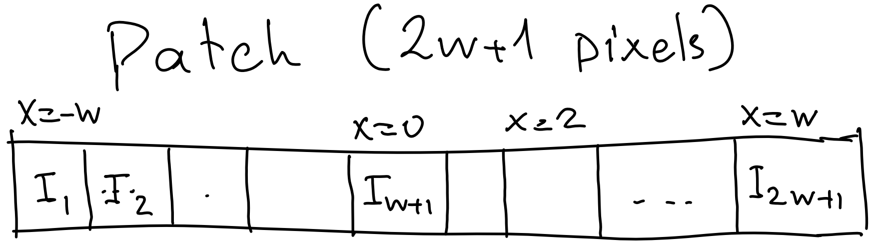
$$I(x) \approx \begin{bmatrix} 1 & x & \frac{1}{2}x^2 & \frac{1}{6}x^3 & \cdots & \frac{1}{n!}x^n \end{bmatrix} \begin{bmatrix} I(0) \\ \frac{dI}{dx}(0) \\ \frac{dI}{dx^2}(0) \\ \vdots \\ \frac{d^nI}{dx^n}(0) \end{bmatrix}$$

Least-Squares Polynomial Fitting of I(x)

As graph in 2D







$$I_{(2w+1)\times 1} = X_{(2w+1)\times(n+1)}\, d_{(n+1)\times 1}$$

I	X	d
intensities (known)	positions (known)	derivatives (unknown)

Solving linear system in terms of d minimizes the "fit error"

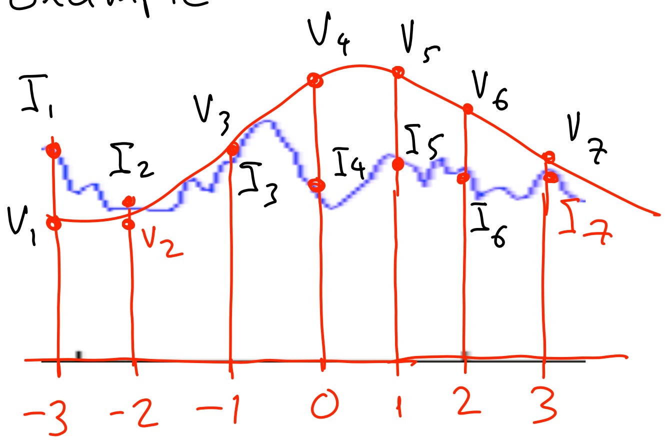
$$\| I - Xd \|^2$$

Definition (α-norm $\|v\|^\alpha$ of vector v)

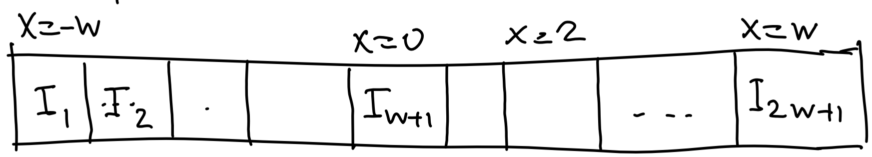
for $v = [v_1\ v_2 \ldots v_m]$, $\|v\|^\alpha = \left(\sum_{i=1}^{m} (v_i)^\alpha \right)^{1/\alpha}$

Least-Squares Polynomial Fitting of I(x)

Example



Patch (2w+1 pixels)



$$I_{(2w+1)\times 1} = X_{(2w+1)\times(n+1)}\, d_{(n+1)\times 1}$$

↑	↑	↑
intensities (known)	positions (known)	derivatives (unknown)

- For the solution d, the vector $v = Xd$ gives us the values of the polynomial at $(-w, \ldots, 0, \ldots, w)$
- This solution minimizes the 2-norm (i.e. the length) of the error vector $(I-v)$:

$$\left(\sum_{i=1}^{2w+1} (I_i - v_i)^2\right)^{1/2}$$

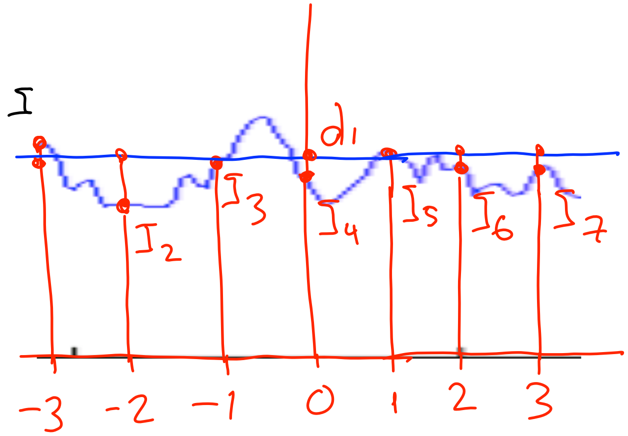
Solving linear system in terms of d minimizes the "fit error"

$$\|I - Xd\|^2$$

Solution d is called a <u>least-squares fit</u>

0^{th}-Order (Constant) Estimation of I(x)

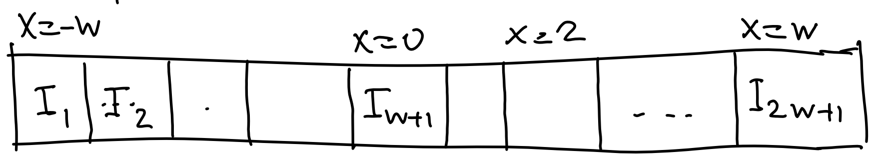
Special case:



- Solution minimizes
$$\sum_{i=1}^{2w+1} (I_i - d_1)^2$$

- Solution is the mean intensity of the patch:
$$d_1 = \frac{1}{2w+1} \sum_{i=1}^{2w+1} I_i$$

Patch (2w+1 pixels)



$$I_{(2w+1)\times 1} = X_{(2w+1)\times 1} \; d_{1\times 1}$$

intensities (known) — positions (known) — one unknown (equal to I(0))

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [d_1]$$

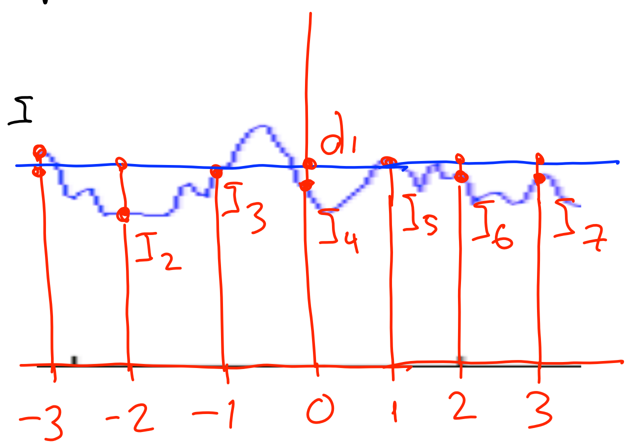
I(0)

Solving linear system in terms of d minimizes the "fit error"

$$\|I - Xd\|^2$$

0^{th}-Order (Constant) Estimation of I(x)

Special case:



- Solution minimizes

$$\sum_{i=1}^{2w+1} (I_i - d_1)^2$$

- Solution is the mean intensity of the patch:

$$d_1 = \frac{1}{2w+1} \sum_{i=1}^{2w+1} I_i$$

Proof

- Let $E(x) = \sum_{i=1}^{2w+1} (I_i - x)^2$

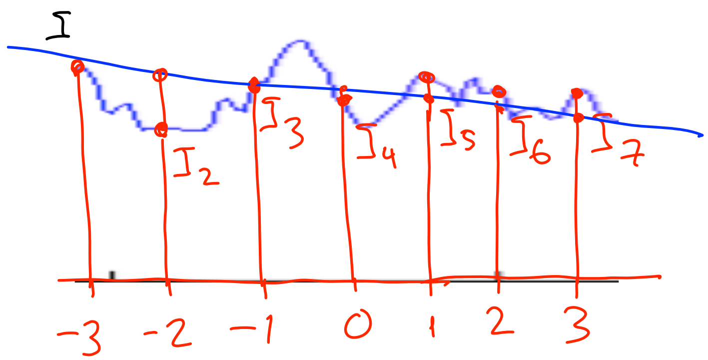
- At the minimum of $E(x)$, the derivative $\frac{d}{dx}E(x)$ must be zero

- $$\frac{d}{dx}E(x) = \sum_{i=1}^{2w+1} \frac{d}{dx}\left[(I_i - x)^2\right]$$
$$= \sum_{i=1}^{2w+1} 2(I_i - x)\cdot(-1)$$
$$= -2\left[\sum_{i=1}^{2w+1} (I_i - x)\right]$$
$$= -2\left(\sum_{i=1}^{2w+1} I_i\right) + 2(2w+1)x$$

- $\frac{d}{dx}E(x) = 0 \iff x = \frac{1}{2w+1}\left(\sum_{i=1}^{2w+1} I_i\right)$

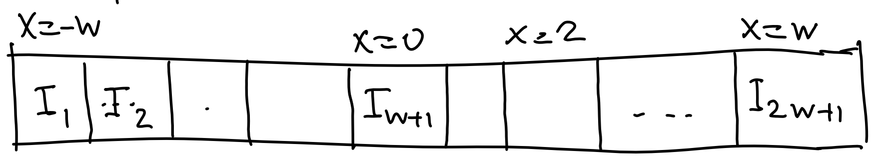
1^{st}-Order (Linear) Estimation of I(x)

Special case:



- Solution minimizes sum of "vetrical" distances between line and image intensities
- Gives us an estimate of $I(0)$ and $\frac{dI}{dx}(0)$ (i.e. value & derivative at 0)

Patch (2w+1 pixels)



$$I_{(2w+1)\times 1} = X_{(2w+1)\times 2} \, d_{2\times 1}$$

intensities (known) — positions (known) — two unknowns

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ \vdots & \vdots \\ 1 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

d_1 ← $I(0)$

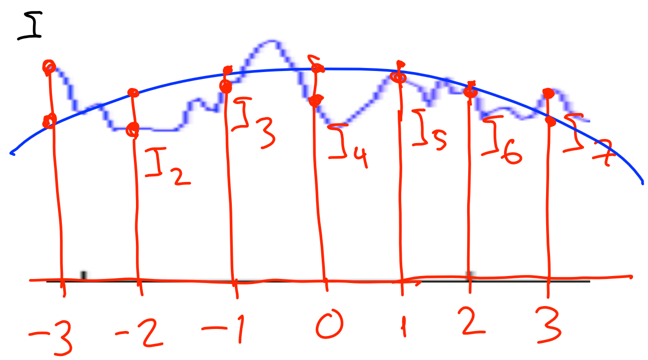
d_2 ← $\frac{dI}{dx}(0)$

Solving linear system in terms of d minimizes the "fit error"

$$\| I - Xd \|^2$$

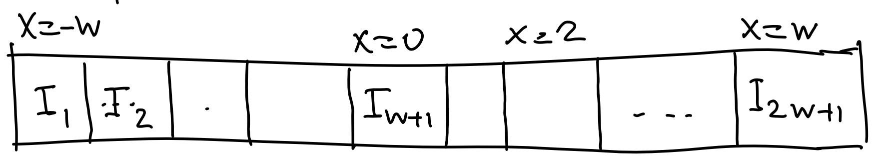
2^{nd}-Order (Quadratic) Estimation of I(x)

Special case:



- Fits a parabola/ hyperbola/ ellipse
- Gives us an estimate of 1st & 2nd image derivative at patch center

Patch (2w+1 pixels)



$$I_{(2w+1)\times 1} = X_{(2w+1)\times 3} \; d_{3\times 1}$$

intensities (known) — positions (known) — 3 unknowns

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} 1 & -3 & 9/2 \\ 1 & -2 & 2 \\ \vdots & \vdots & \\ 1 & 3 & 9/2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\frac{d^2 I}{dx^2}(0)$