1)

$$\dot{x}_1 = -x_1 - x_2 + \frac{x_1 x_2}{12}$$

$$\dot{x}_2 = u$$

$$f(x) = \begin{bmatrix} -x_1 - x_2 + \frac{x_1 x_2}{12} \\ 0 \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

WKT $\left(x \neq 0 \;\; L_g V(x) = 0\right) \Rightarrow L_f\left(V(x)\right) < 0.$

$\Rightarrow$ $L_g V(x) = 0$ is our domain

$$L_g V(x) = \frac{\partial V}{\partial x} g(x) = 2x^T P g(x) = 2\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 2 & 1.5 \\ 1.5 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{L_g V(x) = 3x_1 + 6x_2} \qquad L_g V(x) = 0 \Rightarrow \boxed{x_1 = -2x_2}$$

ii)

$$L_f V(x) < 0 \Rightarrow 2x^T P f(x)$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1.5 \\ 1.5 & 3 \end{bmatrix} \begin{bmatrix} -x_1 - x_2 + \frac{x_1 x_2}{12} \\ 0 \end{bmatrix}$$

$$= (2x_1 + 1.5x_2)\left(-x_1 - x_2 + \frac{x_1 x_2}{12}\right)$$

$$x_1 = -2x_2$$

$$\Rightarrow = -2.5x_2\left(x_2 - \frac{x_2^2}{6}\right) = \left(\frac{x_2}{6} - 1\right)x_2^2 \leq 0$$

$$\Rightarrow \quad x_2 < 6 \quad \& \quad x_1 > -12$$

$\therefore$ Open set is $D = \{ x \in \mathbb{R}^2 \mid x_1 > -\frac{1}{2}$ & $x_2 < 6 \}$

b) we can use Sontags formula here for control input over the Domain.

$$u = \begin{cases} \dfrac{-L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^4}}{L_g V(x)}, & L_g V(x) \neq 0 \\ 0, & L_g V(x) = 0 \end{cases}$$

2)

$$\dot{x}_1 = x_2$$

$$v_1(x_1) = \frac{x_1^2}{2} = x_1 \dot{x}_1$$

$\Rightarrow$ let $\phi_1(x_1) = -x_1$

$$\Rightarrow \dot{x}_1 = -x_1 + (x_2 + x_1).$$

$$z_1 = x_2 + x_1$$

$$\Rightarrow \dot{z}_1 = \dot{x}_2 + \dot{x}_1$$

$$= x_3 - x_1 + z_1$$

$$\dot{v}_2(x_1, z_1) = x_1(-x_1 + z_1) + z_1(\dot{z}_1)$$

$$= -x_1^2 + z_1 x_1 + \cancel{z_1(\ldots - x_1 + z_1)}$$

$$+ z_1 \phi_2$$

$$\Rightarrow \phi_2 = -x_1 - z_1$$

$$\Rightarrow \dot{z}_1 = (-x_1 - z_1) + \underbrace{(x_3 + 2z_1)}_{z_2} = -x_1 - z_1 + z_2$$

$$\Rightarrow z_2 = x_3 + 2z_1$$

$$\dot{z}_2 = \dot{x}_3 + 2\dot{z}_1$$

$$\dot{V}(x_1, z_1, z_2) = x_1\dot{x}_1 + z_1\dot{z}_1 + z_2\dot{z}_2$$

$$= -x_1^2 + z_1 x_1 - z_1^2 - z_1 x_1 + z_1 z_2$$

$$+ z_2(\dot{x}_3 + 2\dot{z}_1)$$

$\dot{V} = -x_1^2 - z_1^2 + z_1 z_2$

$+ z_2 u + 2 z_2 (-x_1 - \cancel{z_1} + z_2)$

$\dot{V} = -x_1^2 - z_1^2 + z_2 u + (-2 z_2 x_1 - z_1 z_2 + 2 z_2^2)$

$\Rightarrow$ for liner setting

$\boxed{u = -3z_2 + z_1 + 2x_1}$

for non liner setting

$\boxed{u = -z_2^3 - 3z_2 + z_1 + 2x_1}$

So that $\dot{V} = -x_1^2 - z_1^2 - z_2^2 - z_2^4$