Positive Definite Matrices and the Lyapunov Equation

Linearization (Indirect Method of Lyapunov)

We have already seen examples from our textbook where the use of positive definite matrices in the definition of our Lyapunov function candidates was useful in establishing the time derivative of the function negative definite (and hence in being able to conclude asymptotic stability)

Let us summarize/review the properties of positive definite matrices and see their usefulness in finding Lyapunov functions for linear systems.

Definition Let P be a real, $n \times n$, symmetric $(P^T = P)$ matrix.

(i) P is positive semi-definite if $x^T P x \geq 0$, for all $x \in \mathbb{R}^n$

Notation: $P \geq 0$.

(ii) P is positive definite if $x^T P x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.

Notation: $P > 0$.

(iii) P is negative definite if $x^T P x < 0$, for all $x \in \mathbb{R}^n$, $x \neq 0$.

Notation: $P < 0$

(iv) P is negative semi-definite if $x^T P x \leq 0$, for all $x \in \mathbb{R}^n$

Notation: $P \leq 0$.

Remark !!! $P > 0$ does not mean that each of the entries of P are positive !!!

Remark. Let M be an arbitrary, real, $n \times n$ matrix.

Write M as $M = \frac{M+M^T}{2} + \frac{M-M^T}{2}$.

Claim: (i) $x^T (M-M^T) x = 0$, for all $x \in \mathbb{R}^n$

(ii) $\frac{M+M^T}{2}$ is symmetric and is called the symmetric part of M.

(iii) $\frac{M-M^T}{2}$ is skew-symmetric and is called the skew-symmetric part of M.

Proof: We prove only the first claim; the rest are left as an exercise for the reader.

We note that $x^T (M-M^T) x$ is a scalar, hence equal to its transpose. Therefore, for any $x \in \mathbb{R}^n$ we have

$$x^T (M-M^T) x = x^T (M^T - M) x = - x^T (M-M^T) x$$

which proves the result, since the only real number equal to its negative is zero.

Remark 2: From the above we conclude that

$$x^T M x = x^T \left(\frac{M+M^T}{2}\right) x.$$

As thus, symmetry is always assumed as part of the definition of positive and negative (semi-) definite matrices.

Facts. Let P be real, $n\times n$ matrix, and let $\{\lambda_1, \dots, \lambda_n\}$ denote its eigenvalues.

(i) If $P = P^T$ then

(a) all eigenvalues are real,

(b) the Jordan canonical form is trivial (i.e, all blocks are 1×1)

(c) the eigenvectors are mutually orthogonal

(ii) $P \geq 0 \iff P = P^T$ and $\lambda_i \geq 0$, $i = 1, \dots n$

(iii) $P > 0 \iff P = P^T$ and $\lambda_i > 0$, $i = 1, \dots, n$

(iv) $P \leq 0 \iff -P \geq 0$

(v) $P < 0 \iff -P > 0$

(vi) Let N be an $m\times n$ real matrix. Then $N^T N \geq 0$, and moreover $N^T N > 0 \iff \text{rank}(N) = n$

(vii) $P \geq 0 \iff \exists N$ that is $m\times n$ such that $P = N^T N$ and $m = \text{rank}(P)$

(viii) $P > 0 \iff \exists N$ that is $n\times n$ and invertible such that

$$P = N^T N$$

(ix) $P > 0 \iff P^{-1} > 0$.

(x) Suppose that $P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} = P^T$

Then $P > 0$ if and only if its leading principal minors are positive, which means.

$$p_{11} > 0$$

$$\begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0$$

$$\begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{vmatrix} > 0$$

$$\vdots$$

$$\begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{vmatrix} > 0$$

Caveat. Checking positive-semidefiniteness is subtle via principal minors.

For more background material on useful results / properties of matrices, see also Handouts 8, 9 of Professor Grizzle's notes.

Let us now start building our Lyapunov functions!

$\longrightarrow$

Important Remark. Consider $\dot{x} = Ax$, $x \in \mathbb{R}^n$

with $V(x) = x^T P x$

From linear systems theory we know that the origin is asymptotically stable if and only if the matrix A is Hurwitz, i.e., $Re(\lambda_i(A)) < 0$, $i = 1, \ldots, n$.

We can investigate asymptotic stability using the Lyapunov method as well.

Let us consider the positive definite function

$$V(x) = x^T P x, \quad \text{where } P > 0 \text{ (positive definite.)}$$

and take its time derivative as

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x =$$

$$= x^T \underbrace{(A^T P + PA)}_{-Q} x = -x^T Q x$$

So, we can now observe that if $A^T P + PA < 0$, then we can conclude that the origin is asymptotically stable.

The equation $A^T P + PA = -Q$ is called the Lyapunov equation.

In fact, we can observe also the following: If we go "backwards" and assume we have a positive definite matrix Q, solve the Lyapunov equation and obtain a P that is positive definite, then we can conclude that the origin is asymptotically stable! We can summarize that by saying that

$\dot{x} = Ax$ is asymptotically stable $\Leftrightarrow$

$$\exists \; V = x^T P x, \; P > 0, \text{ such that } \dot{V} = x^T (A^T P + PA) x$$

is negative definite

<u>Remark</u>. The same result in different wording is given in Theorem 4.6 of our textbook.

<u>Theorem 4.6</u>: A matrix A is Hurwitz (aka $\dot{x} = Ax$ is globally asymptotically stable) if and only if for any given $Q > 0$, $\exists \; P > 0$ such that $A^T P + PA = -Q$. Moreover, if A is Hurwitz, then P is the unique solution of $A^T P + PA = -Q$.

<u>Proof</u>: We will go through the steps of the proof as an exercise on how to prove sufficiency and necessity (i.e an "if and only if" statement).

To prove sufficiency: The sufficiency statement is

"if $\forall \; Q > 0$, $\exists \; P > 0$ such that $A^T P + PA = -Q$, then A is Hurwitz".

To prove that, we apply Theorem 4.1: Let us consider $V(x) = x^T P x$, $P > 0$, and

$$\dot{V}(x) = \cdots = x^T(A^TP + PA)x = -x^TQx < 0$$

Hence the origin is asymptotically stable, hence A is a Hurwitz matrix.

That proves the sufficiency part.

The necessity statement is "If the matrix is Hurwitz, then for all $Q > 0$, $\exists P > 0$ such that $A^TP + PA = -Q$".

To prove this argument, the idea is to start with the fact that A is Hurwitz, and construct/find a $P > 0$ that is a solution to the Lyapunov equation.

A Hurwitz means that $Re(\lambda_i(A)) < 0$.

We define the matrix

$$P = \int_0^{\infty} e^{A^Tt} Q e^{At} dt$$

[we know that the integrand exists, i.e, is not infinite, since the terms in the integrand are sum of terms of the form $t^{k-1} \exp(\lambda_i t)$, where $Re(\lambda_i) < 0$]

~~the~~ So the matrix P we defined does exist. The next step is to show that $P > 0$ (positive definite)

We will prove that by contradiction, i.e we will assume P is not positive definite.

That would imply that for some $x \neq 0$, we would have

$$x^T P x = 0.$$ However,

$$x^T P x = 0 \Rightarrow \int_0^\infty x^T e^{A^T t} Q e^{At} x \, dt = 0 \Rightarrow$$

$$e^{At} x \equiv 0, \ \forall t \geq 0 \ (\text{since } Q > 0) \Rightarrow$$

$$x = 0, \text{ since } e^{At} \text{ is nonsingular } \forall t.$$

which is a contradiction to our assumption that $x \neq 0$.

Hence P is positive definite., $P > 0$.

Now we need to prove that it is the solution of the Lyapunov equation. We have

$$PA + A^T P = \int_0^\infty e^{A^T t} Q e^{At} \cdot A \, dt + \int_0^\infty A^T e^{A^T t} Q e^{At} \, dt =$$

$$= \int_0^\infty \left(e^{A^T t} Q e^{At} \cdot A + A^T e^{A^T t} Q e^{At} \right) dt$$

$$= \int_0^\infty \frac{d}{dt} \left(e^{A^T t} Q e^{At} \right) dt = e^{A^T t} Q e^{At} \Big|_0^\infty = -Q.$$

Hence, we derived that indeed P is a solution of the Lyapunov equation.

That concludes the necessity part.

To prove the last argument, i.e. that P is the unique solution, we again use the proof by contradiction and assume that there is another solution $\tilde{P} \neq P$.

Then.

$$(P-\tilde{P})A + A^T(P-\tilde{P}) = 0.$$

Premultiplying by e^{A^Tt} and postmultiplying by e^{At} yields.

$$e^{A^Tt}\left[(P-\tilde{P})A + A^T(P-\tilde{P})\right]e^{At} = 0 \quad \Rightarrow$$

$$\frac{d}{dt}\left(e^{A^Tt}(P-\tilde{P})e^{At}\right) = 0 \Rightarrow$$

$$e^{A^Tt}(P-\tilde{P})e^{At} = c, \quad \text{constant } \forall t \geq 0.$$

In fact for $t=0$ we have $e^{A\cdot 0} = I$; Hence

$$P-\tilde{P} = e^{A^Tt}(P-\tilde{P})e^{At}, \quad \forall t \geq 0.$$

This can hold true for all $t \geq 0$ only when $P = \tilde{P}$.

Hence we proved that $P = \tilde{P}$. ∎

Remark. The above result pertains to linear systems!

So, what is its usefulness, if any, for nonlinear systems?!

So, the Lyapunov equation provides a procedure for finding a Lyapunov function for any linear system $\dot{x} = Ax$, when A is Hurwitz.

Now, the existence of such Lyapunov function will allow us to draw conclusions about the system when the right-hand side of the equation is perturbed, i.e if the coefficients of A are linearly perturbed or if the perturbation is a nonlinear one.

This becomes clearer through Theorem 4.7 of our textbook, that gives us conditions under which we can draw conclusions about the stability of the origin as an equilibrium of a nonlinear system by investigating its stability as an equilibrium point of a linear system!!!

Let us first see what Theorem 4.7 tells us, and how this connects to the Lyapunov equation.

Theorem 4.7. (Lyapunov's Indirect Method)

Let $x = 0$ be the equilibrium of the nonlinear system $\dot{x} = f(x)$, where $f: D \to \mathbb{R}^n$ is continuously differentiable and D a neighborhood of the origin. Let

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$ be the Jacobian matrix, evaluated at the equilibrium!

Then:

1) The origin is asymptotically stable if $\mathrm{Re}\,\lambda_i < 0$ for all eigenvalues of A.

2) The origin is unstable if $\mathrm{Re}\,\lambda_i > 0$ for at least one eigenvalue of A.

Proof. We ~~pp~~ provide the main steps; the full proof is available in Khalil's book.

First, we notice that from the Mean Value Theorem (see Appendix in the textbook) we can write.

$$f_i(x) = f_i(0) + \left.\frac{\partial f}{\partial x}\right|_{z_i} (x - 0),$$ where z_i is a point on the line segment connecting x to the origin.

Then, since $f(0) = 0$, we have:

$$f_i(x) = \left.\frac{\partial f_i}{\partial x}\right|_{z_i} x = \left.\frac{\partial f_i}{\partial x}\right|_0 x + \underbrace{\left.\frac{\partial f_i}{\partial x}\right|_{z_i} x - \left.\frac{\partial f_i}{\partial x}\right|_0 x}_{g_i(x)},$$

or in vector form.

$$f(x) = Ax + g(x), \quad \text{where} \quad A = \left.\frac{\partial f}{\partial x}\right|_0$$ (Jacobian evaluated at the equilibrium)

We have

$$g_i(x) = \left(\left.\frac{\partial f_i}{\partial x}\right|_{z_i} - \left.\frac{\partial f_i}{\partial x}\right|_0\right) x \Rightarrow$$

$$|g_i(x)| \leq \left\| \left.\frac{\partial f_i}{\partial x}\right|_{z_i} - \left.\frac{\partial f_i}{\partial x}\right|_0 \right\| \|x\|.$$

Now from the continuity of $\frac{\partial f}{\partial x}$, we have that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to 0$$ (to see why, consider that the first term of the right-hand side is a continuous function; then as $\|x\| \to 0$, $\| \left.\frac{\partial f}{\partial x}\right|_x \| \to 0$. The left-hand side being $\frac{\|g(x)\|}{\|x\|}$ is then bounded by something that goes to zero, hence it has to go to zero as well.)

Now, $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$ means that in a small neighborhood around the origin, we can approximate the nonlinear system as

$$\dot{x} = Ax, \quad \text{where} \quad A = \frac{\partial f}{\partial x}\Big|_0$$

[Note: The above analysis offers theoretical justification on why the linearized system $\dot{x} = Ax$, where $A = \frac{\partial f}{\partial x}\Big|_0$ is considered a valid approximation of the nonlinear system $\dot{x} = f(x)$ in a neighborhood of the origin]

So we have $\dot{x} = f(x) = Ax + g(x)$, where

$$A = \frac{\partial f}{\partial x}\Big|_0, \qquad \frac{\|g(x)\|_2}{\|x\|_2} \to 0 \quad \text{as } \|x\|_2 \to 0.$$

To proceed with proving the first claim of the theorem, let A be a Hurwitz matrix.

Then by Theorem 4.6, we know that for any $Q > 0$, the solution P of the Lyapunov equation

$$A^T P + PA = -Q \quad \text{is positive definite, } P > 0.$$

We use $V(x) = x^T P x$ as the Lyapunov function candidate for the nonlinear system.

The time derivative reads.

$$\dot{V}(x) = x^T P f(x) + f^T(x) P x = x^T P \left(Ax + g(x)\right) +$$

$$+ \left(x^T A^T + g^T(x)\right) P x =$$

$$= x^T \left(PA + A^T P\right) x + 2 x^T P g(x) =$$

$$= -x^T Q x + 2 x^T P g(x)$$

The first term on the right-hand side is negative definite, but the second one is in general indefinite.

Now, we have that $\dfrac{\|g(x)\|_2}{\|x\|_2} \to 0$ as $\|x\|_2 \to 0$

That means that for any $\gamma > 0$, there exists $r > 0$ such that $\|g(x)\|_2 < \gamma \|x\|_2$, $\forall \ \|x\|_2 < r$.

Hence.

$$\boxed{\dot{V}(x) \leq -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2 , \quad \forall \ \|x\|_2 < r}$$

We also have $Q > 0$, so its eigenvalues are real and positive, and also $\underline{x^T Q x \geq \lambda_{min}(Q) \|x\|_2^2}$

Thus :

$$\dot{V}(x) \leq -\left(\lambda_{min}(Q) - 2\gamma \|P\|_2\right) \|x\|_2^2 , \quad \forall \ \|x\|_2 < r$$

It follows that if we choose $\gamma < \dfrac{\lambda_{min}(Q)}{2\|P\|_2}$, then $\dot{V}(x)$ is negative definite. By Theorem 4.1, we complete the proof of the first claim. ✓

<u>Note</u>: In fact, we also proved that the Lyapunov function $V(x) = x^T P x$, where P is the solution of the Lyapunov equation $A^T P + PA = -Q$ for any $Q > 0$, <u>is</u> a <u>local</u> Lyapunov function for the nonlinear system (i.e, valid on a domain D)

Now, for the proof of the second part of the theorem, I will let you study it from your textbook. The key idea is that we can find a non-singular matrix T such that

$$TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \text{where } A_1, A_2 \text{ Hurwitz.}$$

Then we define the two Lyapunov equations corresponding to each of the matrices A_1, A_2, and define the function

$$V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2. \quad \text{(in general indefinite)}$$

Our goal is to apply the instability theorem 4.3. In fact, we prove that on a set $\mathcal{U}$ where $V(z)>0$, we also have $\dot{V}(z)>0$, hence the origin unstable.

WRAP-UP of Theorem 4.7.

- If $A = \left.\frac{\partial f}{\partial x}\right|_0$ has all its e-values with $Re(\lambda_i)<0$, then the origin is <u>asymptotically stable</u>.
- If $A = \left.\frac{\partial f}{\partial x}\right|_0$ has at least one e-value with $Re(\lambda_i)>0$, then the origin is <u>unstable</u>.
- If $A = \left.\frac{\partial f}{\partial x}\right|_0$ has eigenvalues such that $Re(\lambda_i)\le 0$, i.e, imaginary e-values, then the theorem is inconclusive – linearization fails to determine the stability of the origin in this case.