EECS 562 - Nonlinear Systems and Control
HW #6

Due on Thursday, February 23rd, 2023
by 11:59pm on Canvas

need to solve!!

1. (20 points) Khalil, Nonlinear Systems, 3^{rd} Edition, Page 185, Prob. 4.22.

2. (20 points) Khalil, Nonlinear Systems, 3^{rd} Edition, Page 185, Prob. 4.23 The problem has two parts (1) You prove the result when $Q > 0$ and (2) you prove the result under the weaker condition, $Q = C^\top C$ and the pair (A, C) is observable.

   Remark on gain margin: You can use the Lyapunov function of this problem to show that the system $\dot{x} = \left(A - \alpha B R^{-1} B^T P\right) x$ is GAS for $0.5 < \alpha < \infty$.

   Remark: You may wish to look over Khalil, Nonlinear Systems, 3^{rd} Edition, Page 185, Prob. 4.24 if you know something about optimal control.

3. (Old Exam Problem) (20 points) Make the strongest statement possible about the stability properties of the origin for the following system, using the proposed Lyapunov function. **Draw a box around your answer.** Of course, it is possible that nothing can be said on the basis of the given Lyapunov function; this is a valid response. In all cases, your conclusions must be backed up by solid reasoning and/or calculations.

   can't comment anything
   v_dot is not <= 0

$$\begin{aligned} \dot{x}_1 &= -(x_2)^3 \\ \dot{x}_2 &= -(x_1)^3 \\ V(x) &= \frac{1}{4}\left((x_1)^4 + (x_2)^4\right) \end{aligned}$$

4. (Old Exam Problem) (20 points) Make the strongest statement possible about the stability properties of the origin for the following system, using the proposed Lyapunov function. **Draw a box around your answer.** Of course, it is possible that nothing can be said on the basis of the given Lyapunov function; this is a valid response. In all cases, your conclusions must be backed up by solid reasoning and/or calculations.

   origin is globally asym stable

   Corollary 4.2: origin is only point in the invariant set

$$\begin{aligned} \dot{x}_1 &= x_2 - \frac{1}{2}(x_1)^3 \\ \dot{x}_2 &= -\frac{1}{2}x_2 \\ V(x) &= \frac{1}{2}(x_1)^4 + (x_2)^2 \end{aligned}$$

5. (20 points) Khalil, Nonlinear Systems, 3^{rd} Edition, Page 185, Prob. 4.28 done. ez of all!!!

1 ok
2 ok
3 ok
4 no, if system starts in x1x2>2 then it will stay in that set only, never reach origin

Hints for Problem 1: For the sufficiency: You want to show that A is Hurwitz (all e-values have negative real part). This is equivalent to showing the origin is an asymptotically stable equilibrium point of $\dot{x} = Ax$. Use the Lyapunov function $V(x) = x^\top Px$ and follow the book's hint on applying La Salle's Corollary. For the necessity, you need to show that if A is Hurwitz, then there exists a positive definite solution to the Lyapunov equation. Try the observability grammian[1]:

$$P = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau.$$

To prove that P satisfies the Lyapunov equation, it is worth noting that the chain rule and $\frac{d}{d\tau} e^{B\tau} = Be^{B\tau} = e^{B\tau}B$ for any square matrix B give

$$\frac{d}{d\tau}\left(e^{A^T\tau} C^T C e^{A\tau}\right) = A^T e^{A^T\tau} C^T C e^{A\tau} + e^{A^T\tau} C^T C e^{A\tau} A.$$

To show that P is indeed positive definite, you have to use observability. Feel free to appeal to known theorems from linear system theory; you may just quote the results correctly and give a reference. I normally use C.T. Chen, but J.S. Bay or a web link are fine as well.

Hints for Problem 2: Rewrite the Riccati equation as

$$\left(A - BR^{-1}B^T P\right)^T P + P\left(A - BR^{-1}B^T P\right) = -Q - PBR^{-1}B^T P$$

and observe that $PBR^{-1}B^T P \geq 0$.

FYI: $R > 0$, (A, Q) observable, and (A, B) controllable imply the existence of a positive definite solution to the Riccati equation.

Hint for Problem 4: $(a^3 - b)^2 = a^6 - 2a^3b + b^2$.

Hint for Problem 5: For part c), positively invariant means trajectories starting in the set remain in the set for all $t \geq 0$, i.e., the trajectories never cross the boundary of the set. Choose a function $V = x_1x_2 - 2$ and study its time derivative along the boundary of the set Γ.

[1] See C.T. Chen. If you have not had the Linear Systems Course, then you'll want to do a bit of searching on the web. MATLAB provides some information as well.