Exam Number:______

EECS 562 FINAL EXAM

FRIDAY, APRIL 24, 2009, 1:40 to 3:30 PM

Room 1200 EECS

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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SIGNATURE

(Sign **after** the exam is completed)

Solutions

LAST NAME (PRINTED) , FIRST NAME

RULES:

1. OPEN TEXTBOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. TWO SHEETS OF NOTE PAPER
6. NO CALCULATORS, CELL PHONES, PDAs, etc.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. Fifty percent of the points on the exam are NO PARTIAL CREDIT GIVEN and fifty percent are PARTIAL CREDIT GIVEN. For those problems that allow partial credit, show your work clearly on this booklet.

Record Answers Here	
	Your Answer
Problem 1	T F
Problem 2	T F
Problem 3	T F
Problem 4	T F
Problem 5	(a) (b) (c) (d) (e)
Problem 6	(a) (b) (c) (d) (e)
Problem 7	(a) (b) (c) (d) (e)
Problem 8	(a) (b) (c) (d) (e)

Scores (Filled in by Instructor)		
	Your Score	Max Score
Problems 1-8		40
Problem 9		20
Problem 10		20
Total		**80**

Multiple-Choice True-False Section of the Exam. There are eight (8) questions. Each question is worth five (5) points. You are NOT asked to show your work and no partial credit is possible. If a question is multiple choice, then choose **the single best** answer. **Record all answers on page 2.**

1. __F__ **(T or F):** There exists a continuously differentiable state variable feedback rendering the origin of the system below *locally* exponentially stable.

$$\begin{aligned} \dot{x}_1 &= (x_1)^2 + x_1 x_2 \\ \dot{x}_2 &= u \end{aligned}$$

2. __T__ **(T or F):** There exists a continuously differentiable dynamic output feedback rendering the origin of the system below *locally* exponentially stable.

$$\begin{aligned} \dot{x}_1 &= x_2 + (x_1)^2 + x_1 x_2 \\ \dot{x}_2 &= u \\ y &= x_1 + (x_2)^3 \end{aligned}$$

3. __F__ **(T or F):** Suppose that $V(x)$ is continuously differentiable and positive definite on $\mathbb{R}^n$. Suppose further that the control system $\dot{x} = f(x) + g(x)u$, $u \in \mathbb{R}$, satisfies f and g are continuously differentiable, $f(0) = 0$ and for all $x \neq 0$, $L_g V(x) = 0 \implies L_f V(x) < 0$. Then there exists a continuous state feedback controller $u = \alpha(x)$ that renders the origin GAS.

4. __F__ **(T or F):** Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies the global Lipschitz condition $|\varphi(x) - \varphi(y)| \leq 5|x - y|$, for all $x, y \in \mathbb{R}$. Then $\varphi \in [-5, 5]$, that is, φ lies in the sector $[-5, 5]$.

5. __d__ For the given system and Lyapunov function,

$$\begin{aligned} \dot{x}_1 &= -(x_1)^2 + x_2 \\ \dot{x}_2 &= 2(x_1)^3 \\ V(x) &= (x_1)^4 - (x_2)^2 \\ \dot{V}(x) &= -4(x_1)^5 \end{aligned}$$

choose the strongest statement that can be made about the stability of the origin:

(a) nothing can be said.

(b) asymptotically stable.

(c) stable i.s.L.

(d) unstable.

(e) antistable [Note: An equilibrium point of $\dot{x} = f(x)$ is antistable if it is asymptotically stable for $\dot{x} = -f(x)$.]

6. ___d___ Consider the SISO transfer function $g(s) = +108\frac{s+1}{(s-2)(s+4)(s+6)}$, which has the Bode plot shown in Figure 1. The transfer function induces a mapping on (extended) input signals through convolution: $y = h * u$, that is, $y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$ for $u \in L_e^2$, where $h(t)$ is the impulse response. The L^2-norm of this input-output system is

(a) $\int_0^\infty |h(\tau)|d\tau$.

(b) $\sup_{t\geq 0} |h(t)|$.

(c) approximately 3.0

(d) not defined (though it is not the same thing, if you think the answer is ∞ then you probably mean this answer).

(e) none of the above (select this answer if you think the norm is defined but is not given by (a), (b), or (c)).

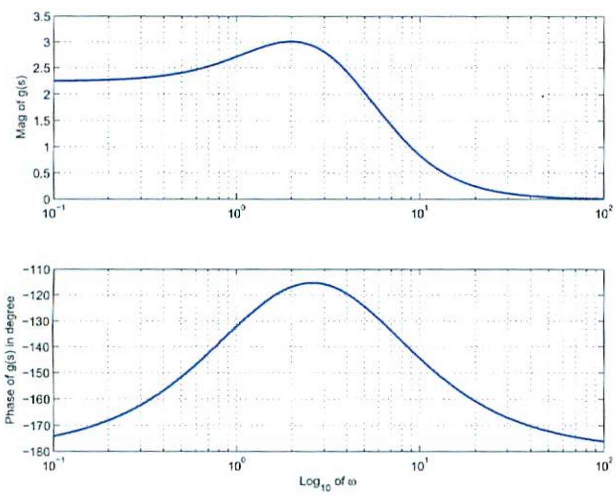


Fig. 1. This figure is used for questions 5 and 6. Note that the magnitude $|g(j\omega)|$ is plotted in a linear scale and is NOT in dB.

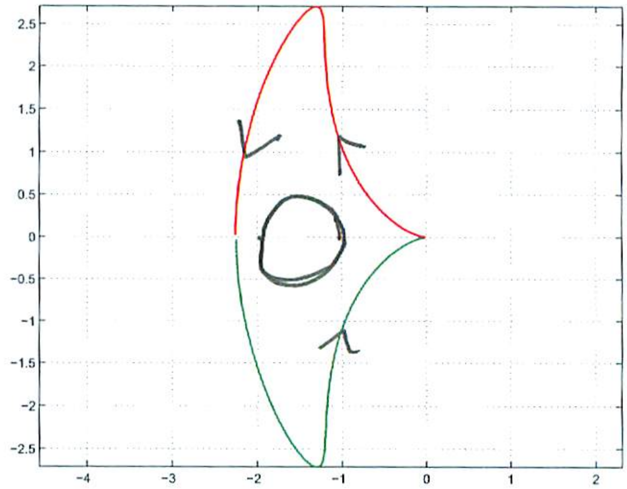


Fig. 2. This figure is used for questions 5 and 6. Note the units on the horizontal and vertical axes are the same. Hence, a circle will be round in such a plot! The arrows are not shown; they are in the counterclockwise direction.

7. ___a___ Consider the system

$$\Sigma := \begin{cases} \dot{x} &= Ax + bu \\ y &= cx \\ u &= -\varphi(y), \end{cases}$$

where (A, b, c) is minimal. Suppose further that the transfer function is given by $c(sI - A)^{-1}b = g(s) = +108\frac{s+1}{(s-2)(s+4)(s+6)}$, with Bode and Nyquist plots given in Figures 1 and 2, and that φ is locally Lipschitz continuous. Then the origin of Σ is Globally Exponentially Stable for

(a) $\varphi \in [\frac{1}{2}, 1]$

(b) $\varphi \in [-\frac{1}{2}, -1]$

(c) $\varphi \in [1, 5]$

(d) $\varphi \in [-1, -5]$

(e) none of the above.

8. ______ Which parts of the course were **most useful** for you? Circle exactly two selections.

(a) Fundamental theory (existence and uniqueness of solutions, etc.)

(b) Lyapunov methods for analyzing stability

(c) CLF and backstepping methods.

(d) Input-output stability, describing functions, circle criterion

(e) HW problems on the inverted pendulum and the Segway.

End of Short Answer Section of the Exam

Be Sure to Record All Answers on Page 2!

Reasons (not required)

1. $\boxed{F}$

Linearization $\dot{x}_1 = 0$, $\dot{x}_2 = u$ $\Rightarrow$ not stabilizable.

2. $\boxed{T}$ Linearization

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$
$$y = x_1$$

is both controllable & observable

$\therefore$ Can use observer-based feedback as in the pendulum project.

3. $\boxed{F}$ System may not satisfy the small control property, such as for $\dot{x} = x + x^2 u$, where, near the origin, u must grow like $\frac{1}{x}$.

4. $\boxed{F}$ Must also impose $\varphi(0) = 0$! Hence, $\varphi(x) = 3x + 4$ satisfies $|\varphi(x) - \varphi(y)| = 3|x-y| \leq 5|x-y|$, but $\varphi \notin [-5, 5]$.

5. $\boxed{d}$ See exam 1.

6. $\boxed{d}$ System is unstable, due to the pole at +2, hence (d).

7. $\boxed{a}$ Because the system is open-loop unstable, (b) and (d) are impossible. We thus check (a) and (c).

(a) $\varphi \in [\frac{1}{2}, 1] \Rightarrow D[-2, -1]$ which is

clearly contained within the Nyquist plot and is encircled -1 times, which is precisely -1 times the number of poles in the CRHP $\Rightarrow$ G.E.S. by the Circle Criterion.

(c) $\varphi \in [1, 5] \Rightarrow D[-1, -\frac{1}{5}]$ which intersects

the Nyquist plot $\Rightarrow$ Circle Criterion is not satisfied.

8. Freeby.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded. You MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know!",

so please do not come up and ask me. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture or the book, you do not have to re-derive it. If you do re-derive it, that will be a sign that you did not know what we actually covered in class, and I will mark off for that.

9. (20 points) Show your work. A correct answer is only worth something if supported by adequate reasoning. For each part below, determine if V is a global CLF. If it is, and you have documented your derivation correctly, then you are done. If it is NOT a global CLF, then find the **largest** sublevel set of V where it does satisfy the CLF property, that is, find a set D such that $D = \{x | V(x) < c\}$, for an appropriate choice of constant c, AND

$$\forall x \in D, x \neq 0, \ \inf_{u \in I\!R} \dot{V}(x, u) < 0.$$

(a) (8 points) $V(x) = \frac{1}{2}x^2$, where $x \in I\!R$, and the system is $\dot{x} = -x(2-x) + (1-x^2)u$.

(b) (12 points) $V(x) = \frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2$, where $x = (x_1, x_2) \in I\!R^2$ and the system is

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 3x_2 + x_1 e^{x_2} + u \\ \dot{x}_2 &= -3x_1 - x_2 + \tfrac{1}{2}(x_2)^2 + (x_1)^4. \end{aligned}$$

Please put work for part (a) here and start (b) on the next page.

(a) $\boxed{V(x) = \frac{x^2}{2} \ \underline{\text{is}} \text{ a global CLF}}$

$g(x) = 1 - x^2 \Rightarrow L_gV(x) = x(1-x^2)$

$\therefore L_gV(x) = 0 \iff x = 0$, or $x = \pm 1$.

$f(x) = -x(2-x) \Rightarrow L_fV(x) = -x^2(2-x)$

Evaluating $L_fV(x)$ at points where $L_gV(x)$ vanishes, $x \neq 0$, yields:

$x = +1 \Rightarrow L_fV(x)\big|_{x=1} = -1 < 0$

$x = -1 \Rightarrow L_fV(x)\big|_{x=-1} = -3 < 0$

$\therefore$ By Sontag's characterization, V is a global CLF. $\square$

Extra Page for Problem 9: Please box your answers.

(b) $V(x) = \frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2$, where $x = (x_1, x_2) \in I\!R^2$ and the system is

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 3x_2 + x_1 e^{x_2} + u \\ \dot{x}_2 &= -3x_1 - x_2 + \tfrac{1}{2}(x_2)^2 + (x_1)^4. \end{aligned}$$

$\boxed{V(x) = \frac{(x_1)^2 + (x_2)^2}{2} \text{ is NOT a global CLF}}$

Largest sublevel set where V is a CLF is

$$\boxed{D = \{x \in \mathbb{R}^2 \mid V(x) < 2\} = \{(x_1, x_2) \mid (x_1)^2 + (x_2)^2 < 4\}}$$

Solution $\quad g(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$L_gV(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1$$

$$\therefore L_gV(x) = 0 \iff x_1 = 0 \iff \{(x_1, x_2) \mid x_1 = 0, x_2 \in \mathbb{R}\}$$

Next, we compute $L_fV(x)$ and evaluate it on the set where $L_gV(x)$ vanishes

8b

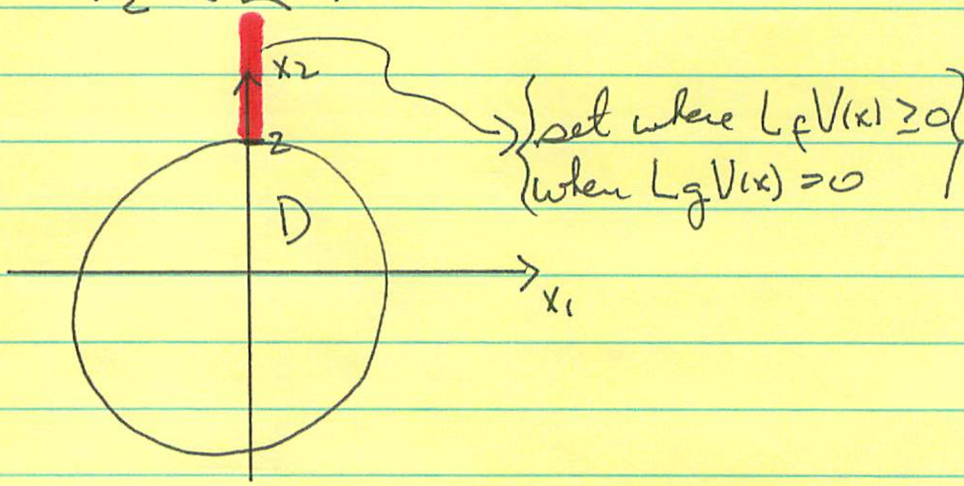
$$L_f V(x)\Big|_{x_1=0} = \begin{bmatrix} 0 & x_2 \end{bmatrix} \begin{bmatrix} 3x_2 \\ -x_2 + \frac{1}{2}(x_2)^2 \end{bmatrix}$$

$$= -(x_2)^2 + \frac{1}{2}(x_2)^3$$

$$= -(x_2)^2 \left[1 - \frac{1}{2} x_2 \right]$$

$\therefore$ $L_g V(x) = 0$, $x \neq 0 \Rightarrow L_f V(x) < 0$

<u>only</u> for $x_2 < 2$.



$\therefore$ $D = \left\{ x \in \mathbb{R}^2 \;\middle|\; \frac{(x_1)^2}{2} + \frac{(x_2)^2}{2} < 2 \right\}$

10. (20 points) Show your work. A correct answer is only worth something if supported by adequate reasoning. Find a continuously differentiable feedback that will make the origin GAS. Determine and use an appropriate Lyapunov function to prove that your feedback really works! To be extra clear, your solution must provide (i) a feedback, and (ii) a Lyapunov function with appropriate properties. Either by direct computation or by appealing to known results, you need to show that $\dot{V}$ is negative definite. You may of course use freely any results from class.

Notes: If you do not give your feedback and Lyapunov function in the original coordinates, you will be penalized for five (5) points.

(a) **20 point version**

$$\begin{aligned}\dot{x}_1 &= \frac{x_2}{1+5(x_2)^2}\\ \dot{x}_2 &= u\end{aligned}$$

Hint: Hypotheses of integrator backstepping lemma are not met for the 20 point problem.

(b) **15 point version**

$$\begin{aligned}\dot{x}_1 &= \frac{1}{2}(x_1)^3 + x_1x_2\\ \dot{x}_2 &= \frac{u}{1+5(x_2)^2}\end{aligned}$$

Remark: Work only one problem!

(b) We first apply a preliminary feedback

$$V = \frac{u}{1+5(x_2)^2} \quad , \text{ yielding}$$

$$\begin{cases}\dot{x}_1 = \frac{1}{2}(x_1)^3 + x_1x_2\\ \dot{x}_2 = V\end{cases}$$

so that the Integrator Back Stepping Lemma applies. We first consider

$\dot{x}_1 = \frac{1}{2}(x_1)^3 + x_1x_2$ and note that

$$\boxed{x_2 = \alpha(x_1) = -(x_1)^2} \Rightarrow \boxed{\dot{x}_1 = -\frac{1}{2}(x_1)^3}$$

Hence $V(x_1) = \frac{(x_1)^2}{2} \Rightarrow \dot{V}(x_1) = -\frac{1}{2}(x_1)^4 < 0$

and G.A.S. is proven. The Integrator Back Stepping Lemma then yields

$$V_a(x_1, x_2) = \frac{(x_1)^2}{2} + \frac{(x_2 - \alpha(x_1))^2}{2}$$

$$= \frac{(x_1)^2}{2} + \frac{(x_2 + (x_1)^2)^2}{2}$$

is positive definite and the feedback

$$v = -c\left(x_2 - \alpha(x_1)\right) + \frac{\partial\alpha}{\partial x_1}(x_1)\left[\frac{1}{2}(x_1)^3 + x_1 x_2\right] - x_1(x_1)$$

as a G.A. stabilizing feedback, for any $c > 0$. Setting $c = 1$, we compute

10c

$$V = -(x_2 + (x_1)^2) - 2x_1\left[\frac{1}{2}(x_1)^3 + x_1 x_2\right] - (x_1)^2$$

or

$$\boxed{V = -2(x_1)^2 - x_2 - (x_1)^4 - 2(x_1)^2 x_2}$$

Finally

$$\boxed{u = \left[1 + 5(x_2)^2\right] V}$$

(a) Because the top equation is not affine in x_2, our virtual control, the Integrator Back Stepping Lemma cannot be applied. The method of Back Stepping still works.

We consider first

$$\dot{x}_1 = \frac{x_2}{1 + 5(x_2)^2}$$

and observe that $x_2 = \alpha(x_1) = -x_2$,

$V(x_1) = \dfrac{(x_1)^2}{2}$ yield

$$\dot{V}(x_1) = x_1 \left[\frac{-x_1}{1 + 5(x_1)^2} \right] = \frac{-(x_1)^2}{1 + 5(x_1)^2} < 0$$

proving the origin is G.A.S.

We introduce

$$Z = x_2 - \alpha(x_1) = x_2 + x_1$$

yielding $x_2 = Z - x_1$ and

$$\boxed{\dot{x}_1 = \frac{Z - x_1}{1 + 5(Z - x_1)^2}}$$

$$\dot{Z} = \dot{x}_2 + \dot{x}_1 = u + \frac{x_2}{1 + 5(x_2)^2}$$

$$\therefore \boxed{\dot{Z} = \frac{Z - x_1}{1 + 5(Z - x_1)^2} + u}$$

We propose $V_a(x_1, z) = \frac{(x_1)^2}{2} + \frac{(z)^2}{2}$

and compute

$$\dot{V}_a(x_1, z) = x_1\left[\frac{Z - x_1}{1 + 5(Z - x_1)^2}\right] + Z\left[\frac{Z - x_1}{1 + 5(Z - x_1)^2} + u\right]$$

$$= \frac{-(x_1)^2}{1 + 5(Z - x_1)^2} + Z\left[\frac{+(x_1)}{1 + 5(Z - x_1)^2} + \frac{Z - x_1}{1 + 5(Z - x_1)^2} + u\right]$$

$\therefore$

$$\dot{V}_a(x_1, z) = \frac{-(x_1)^2}{1+5(z-x_1)^2} + z\underbrace{\left[\frac{z}{1+5(z-x_1)^2} + u\right]}_{-z}$$

$\therefore$ $\dfrac{z}{1+5(z-x_1)^2} + u = -z$

$$\iff \boxed{u = -z - \frac{z}{1+5(z-x_1)^2}}$$

yields $\boxed{\dot{V}_a(x_1, z) = \dfrac{-(x_1)^2}{1+5(z-x_1)^2} - z^2 < 0}$

proving G.A.S.

Going back to the original coordinates yields

$$\boxed{\begin{aligned} u &= -(x_1+x_2) - \frac{(x_1+x_2)}{1+5(x_2)^2} \\ V(x_1, x_2) &= \frac{(x_1)^2}{2} + \frac{(x_1+x_2)^2}{2} \end{aligned}}$$

Alternative Solutions

A

We set $\frac{z}{1+5(z-x_1)^2} + u = \frac{-z}{1+5(z-x_1)^2}$

$$\Rightarrow u = \frac{-2z}{1+5(z-x_1)^2}$$

and $\dot{V}_a(x_1, z) = -\frac{(x_1)^2}{1+5(z-x_1)^2} - \frac{(z)^2}{1+5(z-x_1)^2} < 0$

In the original coordinates

$$\boxed{u = \frac{-2(x_1+x_2)}{1+5(x_2)^2}}$$

B We note that $\frac{1}{1+5(z-x_1)^2} \leq 1$

Hence, we can choose

$$\boxed{u = -2z}$$

because, then

$$z\left[\frac{z}{1+5(z-x_1)^2} - 2z\right] = -z^2 - z^2\underbrace{\left[1 - \frac{1}{1+5(z-x_1)^2}\right]}_{\geq 0}$$

$$\therefore \dot{V}_a(x_1, z) \leq \frac{-(x_1)^2}{1+5(z-x_1)^2} - z^2$$

In the original coordinates,

$$\boxed{u = -2(x_1 + x_2)}$$

a linear feedback!

C) We propose $V = \frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2$, differentiate along solutions and hope for the best! Doing so yields

$$\dot V(x_1, x_2, u) = x_1 \dot x_1 + x_2 \dot x_2$$

$$= x_1 \left[\frac{x_2}{1+5(x_2)^2} \right] + x_2 u$$

$$= x_2 \underbrace{\left[\frac{x_1}{1+5(x_2)^2} + u \right]}_{-x_2} = -(x_2)^2$$

therefore, we select u as

$$-x_2 = \frac{x_1}{1+5(x_2)^2} + u$$

$$\therefore \boxed{u = \frac{-x_1}{1+5(x_2)^2} - x_2}$$

To prove G.A.S., we apply La Salle:

$$S = \{ x \mid \dot V(x) = 0 \} = \{ (x_1, 0) \mid x_1 \in \mathbb{R} \}$$

$$x(t) \in S \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot x_2(t) \equiv 0$$

10j

$$\therefore \quad 0 \equiv u(t) = \frac{-x_1(t)}{1 + 5(\cancelto{0}{x_2(t)})^2} - \cancelto{0}{x_2(t)}$$

$\therefore \quad x_1(t) \equiv 0$

$\therefore$ G.A.S. by La Salle.