

$$1) \text{ def } V(u) = u_1^2 + u_2^2$$

$$\text{given } \dot{u}_1 = -u_1 + u_1 u_2$$

$$(u_1 - u_2)^2 = u_1^2 - 2u_1 u_2 + u_2^2$$

$$\begin{aligned} \dot{V}(u) &= [2u_1 + 2u_2] \begin{cases} -u_1 + u_1 u_2 \\ -u_2 \end{cases} \\ &= -2u_1^2 + 2u_1 u_2 - 2u_2^2 \end{aligned}$$

$$\text{if } |u_1| < r \Rightarrow \dot{V}(u) < -2u_1^2 - 2u_2^2 + 2u_1 u_2$$

$$\text{as } |ab| \leq \frac{1}{2}(a^2 + b^2)$$

$$\therefore \dot{V}(u) \leq -2u_1^2 - 2u_2^2 + 2(u_1^2 + u_2^2)$$

$$\rightarrow r < 2 \text{ for } \dot{V}(u) < 0.$$

$\{0\}$ belongs to the domain $V(u) > 0$
 $\dot{V}(u) < 0$.

$\{u=0\}$ is eq^m Point

so. the system is locally asymptotically stable at zero.

global:

$$\dot{x}_1 = x_2$$

$$\Rightarrow x_2 = x_{20} e^{-t}$$

$$\dot{x}_1 = -x_1 (1 + x_2)$$

$$\Rightarrow \ln\left(\frac{x_1}{x_{10}}\right) = -\left(t + x_{20}(e^{-t} - 1)\right) > 0$$

$$\therefore \text{as } t \rightarrow \infty, x_1, x_2 \rightarrow 0$$

$(x_1, x_2) \neq 0$ is a globally asymptotically stable

$$0 > f(x) > 0 \Rightarrow f'$$

0 < f'(x) < 0 \Rightarrow \text{stable point}

$$f'(x) < 0 \Rightarrow \text{stable}$$

the derivative of a function at a point

is called the slope

$$3) V(n) = m_1^2 + m_2^2 \quad \frac{\partial V}{\partial n_1} = -m_1 \sqrt{-m_1} (\sigma - m_1^2 - m_2^2)$$

$$\dot{m}_1 = m_2 (1 - m_1^2) \quad \dot{m}_2 = - (m_1 + m_2) (1 - m_1^2)$$

$$\dot{m}_2 = - (m_1 + m_2) (1 - m_1^2)$$

$$\dot{V}(n) = \begin{bmatrix} 2m_1 & 2m_2 \end{bmatrix} \begin{bmatrix} m_2(1-m_1^2) \\ -(m_1+m_2)(1-m_1^2) \end{bmatrix}$$

$$\begin{aligned} \dot{V}(n) &= 2 \left(m_1 m_2 \cancel{+ m_1^3 m_2 + m_2^3 m_1} - \cancel{m_1 m_2} \right. \\ &\quad \left. - m_2^2 + m_2^2 m_1^2 \right) \\ &= 2 (m_2^2 m_1^2 - m_2^2) \end{aligned}$$

$$\dot{V}(n) < 0 \Rightarrow (m_1^2 - 1) < 0 \Rightarrow |m_1| < 1$$

$$V(n) > 0 \quad \& \quad \dot{V}(n) < 0 \quad \& \quad \exists \, \bar{n} \in \text{Domain}$$

$$\text{Domain} = \{ |m_1| < 1 \}$$

\Rightarrow ~~stable~~ $\& \dot{m}_1 = 0 \quad \dot{m}_2 = 0$ when $m_1, m_2 = 0$

$\therefore \dot{x} = 0$ is locally asymptotically stable

$\{$ in the Domain $\}$

$m \geq 0$ is not the only m

when $M_1 \geq 1$ then $M_1 = 0$

$\therefore m \geq 0$ is not globally

asymptotically stable

is first ($\alpha > 0$) for α fixed

among all β 's satisfying $\alpha < \beta$

β is not a number

not unique right β exists

exists, not

that vanishes when $\alpha = 0$

4)

$$\ddot{m}_1 = m_1 - m_1 - m_2^2 \approx -m_2^2 \approx -10$$

$$\ddot{m}_2 = 2m_1 - m_2^3 \approx 2m_1$$

$$\Rightarrow V(m) \approx m_1^2 + m_2^2$$

$$\Rightarrow V(m) = [2m_1 \quad 2m_2] \begin{pmatrix} -m_1 & -m_2 \\ 2m_1 & -m_2^3 \end{pmatrix}$$

$$= -2(-m_1^2 - m_1 m_2 + m_1 m_2 - m_2^2)$$

$$V(m) \leq 0 \text{ at } m_1, m_2$$

$$\therefore \text{as } m \in \text{domain} \quad \left\{ \begin{array}{l} \ddot{m}_1 = 0 \\ \ddot{m}_2 = 0 \end{array} \right.$$

$$\text{when } m_1 = 0, m_2 = 0$$

$\Rightarrow m = 0$ is locally asymptotically stable

$$\left\{ \begin{array}{l} \ddot{m}(m) \leq 0 \\ \ddot{m} = 0 \end{array} \right.$$

$$\Rightarrow \ddot{m}_1 = 0 \Rightarrow m_1 = m_2$$

$$\ddot{m}_2 = 0 \Rightarrow 2m_1 - m_2^3 = 0$$

$$\Rightarrow 2m_1 + m_2^2 = 0$$

$$m_1 = 0 / \quad \downarrow m_2^2 = -2$$

not possible

$\therefore m = 0$ is only eq^m

$\{$ as $n \rightarrow \infty$ then $N(n) \rightarrow \infty$

$$v_{\text{cm}} = m_1^{-1} + m_2^{-1}$$

c. v is radially unbounded

d. x_{∞} is globally asymptotically stable

stable stationary points

$$2) \text{ if } m_1 = m_2 = 0 \quad v(n) = \frac{(m_1+m_2)^2}{(m_1+m_2)^2} + (m_1-m_2)^2$$

a) if $m_1 > 0$

$$\Rightarrow v(n) = \frac{m_1^2}{1+m_2^2} + m_2^2 \rightarrow 0$$

term a: $\frac{m_1^2}{1+m_2^2}$ as $m_2 \rightarrow \infty$ $\Rightarrow \frac{2m_1^2}{2m_2} = 1$

term b: m_2^2 as $m_2 \rightarrow \infty \Rightarrow m_2^2 \rightarrow \infty$

$$\Rightarrow v(n) \rightarrow \infty \text{ as } \|m\| \rightarrow \infty \text{ along } m_1 > 0$$

similarly if $m_2 > 0$

$$\Rightarrow v(n) = \frac{m_1^2}{1+m_2^2} + m_1^2$$

term a: $m_1 \rightarrow \infty = \frac{m_1^2}{1+m_2^2} = \frac{2m_1^2}{2m_1} = 1$

term b: $m_1 \rightarrow \infty : 2m_1^2 \rightarrow \infty$

$$\Rightarrow v(n) \rightarrow \infty$$

$$\therefore v(n) \rightarrow \infty \text{ as } \|m\| \rightarrow \infty \text{ along } m_1 > 0, m_2 > 0$$

$$b). \quad V(m) = \frac{(m_1 + m_2)^2}{1 + (m_1 + m_2)^2} + (m_1 - m_2)^2$$

$V(n)$ is a increasing function w.r.t $|m|$

so. $\inf V(m)$ will be at $|m| = k$

at $|m| = k$ $\Rightarrow \inf V(m)$

\Rightarrow thus is possible when $m_1 = m_2$

let $\alpha \geq 1$

$$\Rightarrow V(m) = \frac{(2m)^{\alpha}}{1 + (2m)^{\alpha}} + (0)^2$$

as $|m| \rightarrow \infty \Rightarrow m_1 \rightarrow \infty$

$$\Rightarrow \lim_{m_1 \rightarrow \infty} V(m) = \lim_{m_1 \rightarrow \infty} \frac{4m_1^{\alpha}}{1 + 4m_1^{\alpha}}$$

$$= \lim_{m_1 \rightarrow \infty} \frac{8m_1}{8m_1} = 1$$

\therefore as $|m| \rightarrow \infty V(m) \rightarrow 1$

\therefore By defn $V(m)$ is not radially bounded

3) if $w(m)$ is positive definite

let $\alpha(p) := \inf_{\substack{m \in M \\ p \leq \|m\| \leq r}} w(m) \quad 0 \leq p < r$

a) $\alpha(0) = \inf_{\substack{m \in M \\ 0 \leq \|m\| \leq r}} w(m)$ as $w(m)$ is positive def

$$\min(w(m)) = 0 \leq \|m\| \leq r$$

$$\alpha(0) = 0$$

for any p $\|m\|$ belongs to compact set $[p, r]$

b) w continuous $\Rightarrow \inf_{\substack{p \leq \|m\| \leq r}} w(m) > 0$

$\Rightarrow \alpha(p) > 0$

c) $\alpha(p) = \inf_{\substack{m \in M \\ \|m\| \leq \|y\| \leq r}} w(m)$

$\Rightarrow \alpha(\|m\|) \leq w(m)$ this is

uses positivity of w

c) α : non-decreasing & continuous,

$$S(p) = \{m \in M \mid p \leq \|m\| \leq r\}$$

If $p_1 \leq p_2 \leq r \Rightarrow S(p_2) \subset S(p_1)$

$\Rightarrow \alpha$ is the right-continuous

as W is Fredholm & $S(P_1), S(P_2)$ are compact sets

$$\inf_{m \in S(P_1)} W(m) \leq \inf_{m \in S(P_2)} W(m)$$

$$\Rightarrow d(CP) \leq d(P_2) \text{ where } P_1 \subset P_2$$

$\therefore d$ is non-decreasing.

To prove ~~continuous~~, continuity

if $\alpha, \beta \in \mathbb{R}$ such that

$$\|P_1 - P_2\| \leq \rho \Rightarrow \|d(CP_1) - d(CP_2)\| < \epsilon$$

using Lemma 5: Suppose $0 \in P_1 \subseteq P_2 \subset Y$

then if $x \in S(P_1)$, $d(x, S(P_2)) \leq P_2 - P_1$

where $d(x, s) := \inf_{y \in s} \|x - y\|$

since $B_{r(0)}$ is compact set & W is continuous

$\Rightarrow \forall m_1, m_2 \in B_r(0)$

if $\|m_1 - m_2\| \leq \delta$

$$\Rightarrow \|W(m_1) - W(m_2)\| \leq \frac{\epsilon}{2}$$

$$\text{If } p_1 \leq p_2 \leq r$$

$$\Rightarrow p_2 - p_1 < \frac{\epsilon}{2}$$

Since $d(p_1) = \inf_{m \in S(p_1)} w(m)$

\Rightarrow let's choose $m_1 \in S(p_1)$ such that

$$w(m_1) - d(p_1) < \frac{\epsilon}{2}$$

\Rightarrow from lemma 5 we know

$$d(m_1, S(p_2)) \leq p_2 - p_1 < \frac{\epsilon}{2}$$

\Rightarrow we can choose $m_2 \in S(p_2)$ such that

$$w(m_2) - d(p_1) \geq |w(m_2) - d(p_1)|$$

$$\text{notably } \text{such that} \quad \leq |w(m_2) - w(m_1)|$$

$$\text{but } |w(m_1) - d(p_1)| + |w(m_1) - d(p_2)|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

\Rightarrow ~~$d(p_2)$~~ $\Rightarrow \inf_{m \in S(p_2)} w(m) \leq w(m_2) \leq d(p_1) + \epsilon$

$$\Rightarrow d(p_2) - d(p_1) \leq \epsilon$$

\Rightarrow d is continuous.

if $\alpha \in \mathcal{A}$ as shown

\Rightarrow given $w(\alpha) > 0$

$\{ \quad \alpha(CP) > 0 \quad \forall \quad 0 \leq p \leq \bar{p}$

$\Rightarrow \{ \quad \alpha(\|m\|) \leq w(m)$

as $\alpha(\|m\|) > 0$

$w(m) \geq 0$

$\{ \quad \text{as} \quad \alpha(\|m\|) \leq w(m)$

as α is nondecreasing continuous

$w(m) \geq 0$

hence w is positive definite function
locally

$$3 - \frac{2}{3} + \frac{2}{3} - \frac{2}{3}$$

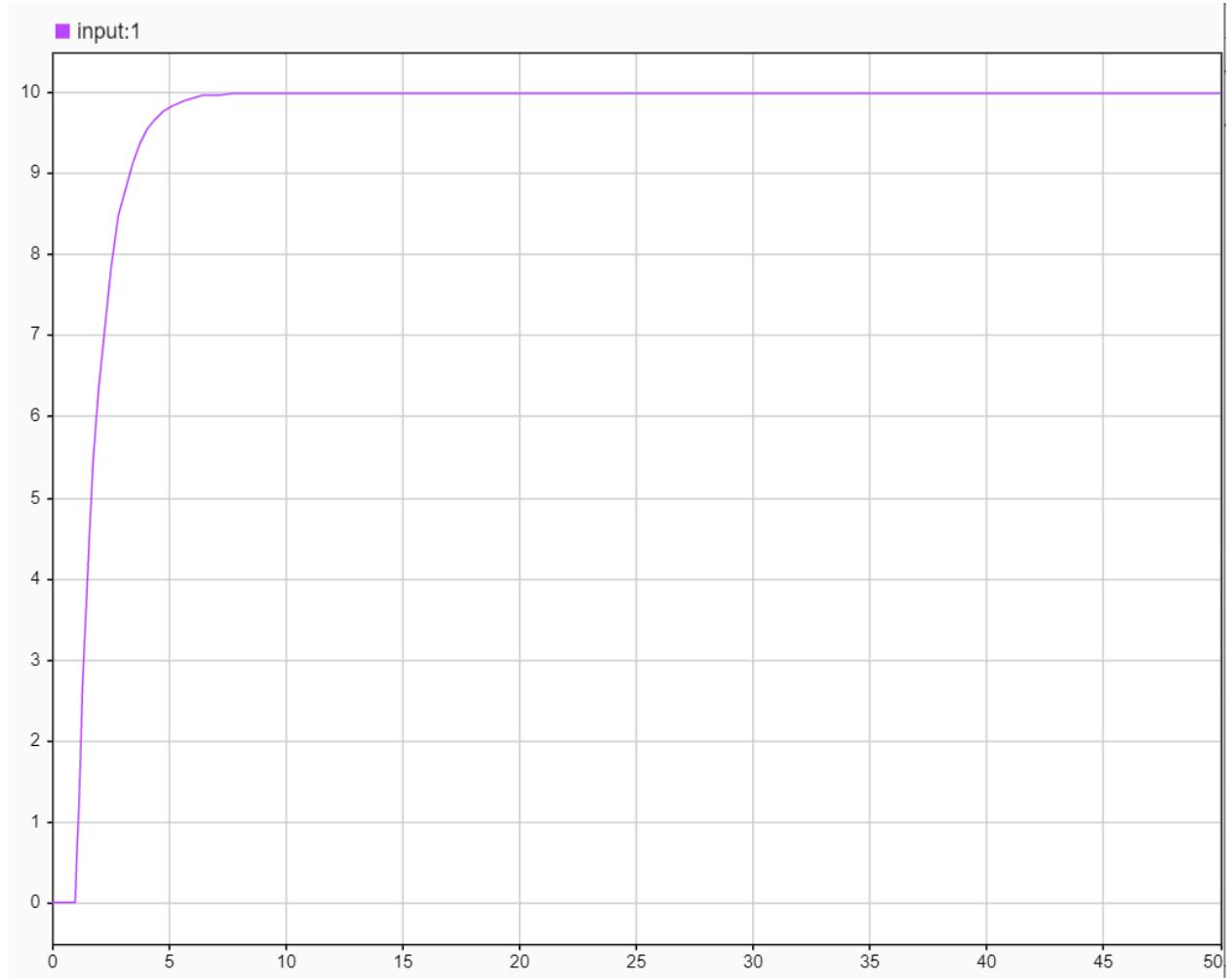
\rightarrow $3 - \frac{2}{3} = \frac{7}{3}$ $\geq \frac{1}{3}$ $\geq \frac{1}{3}$ $\geq \frac{1}{3}$

$$3 - \frac{2}{3} = \frac{7}{3} \quad (\geq \frac{1}{3})$$

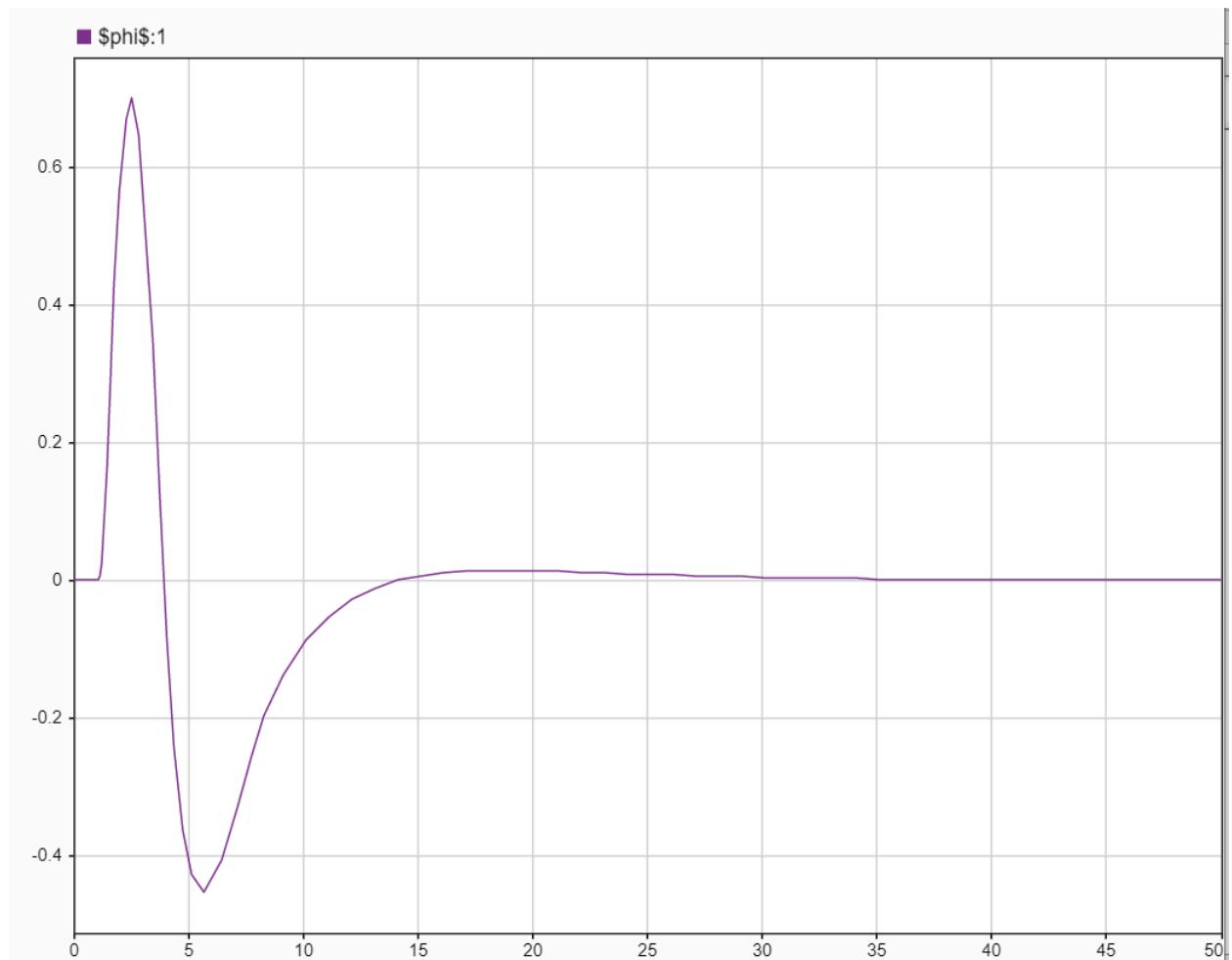
Equality is in \mathbb{C}

Question 5:

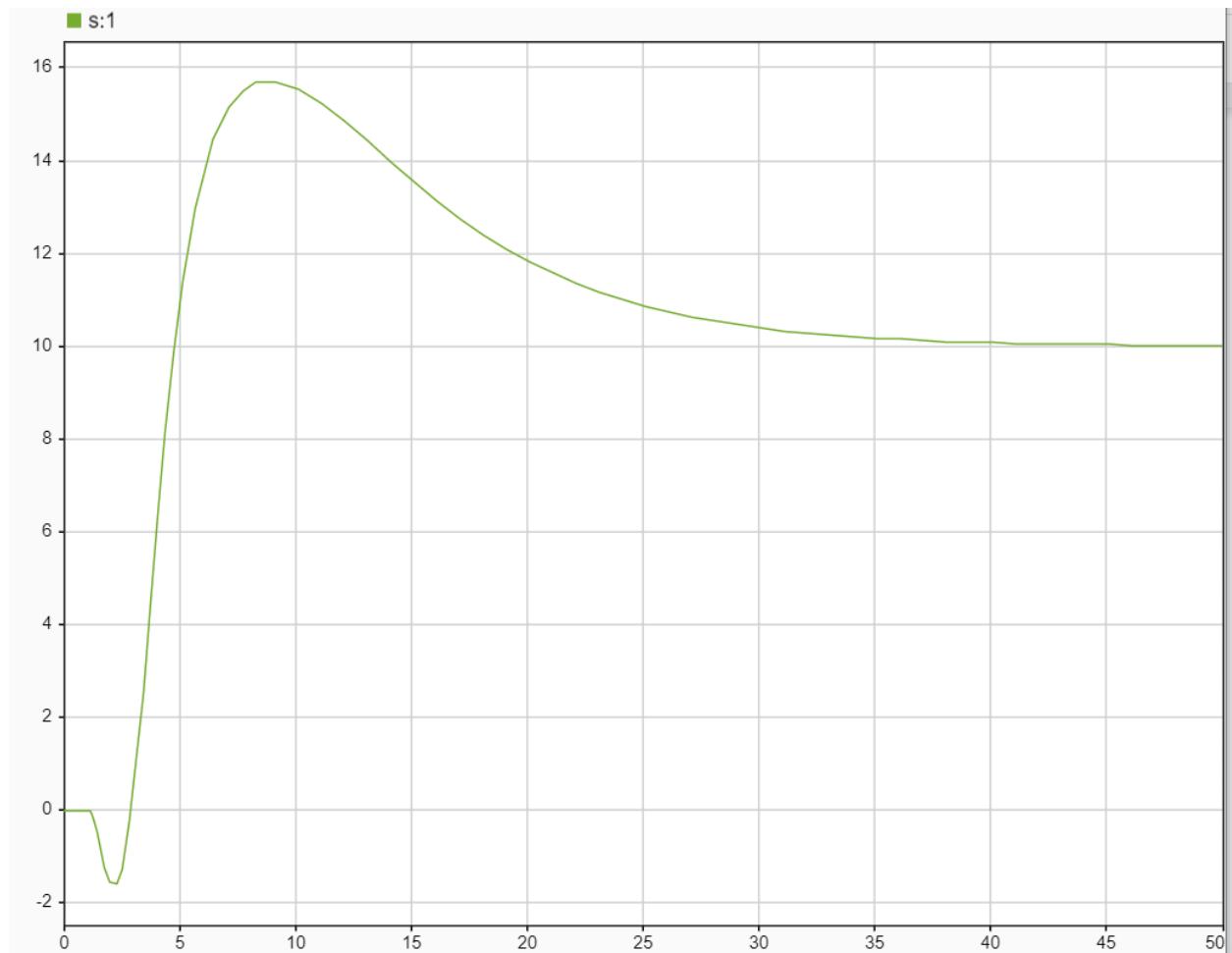
Input:



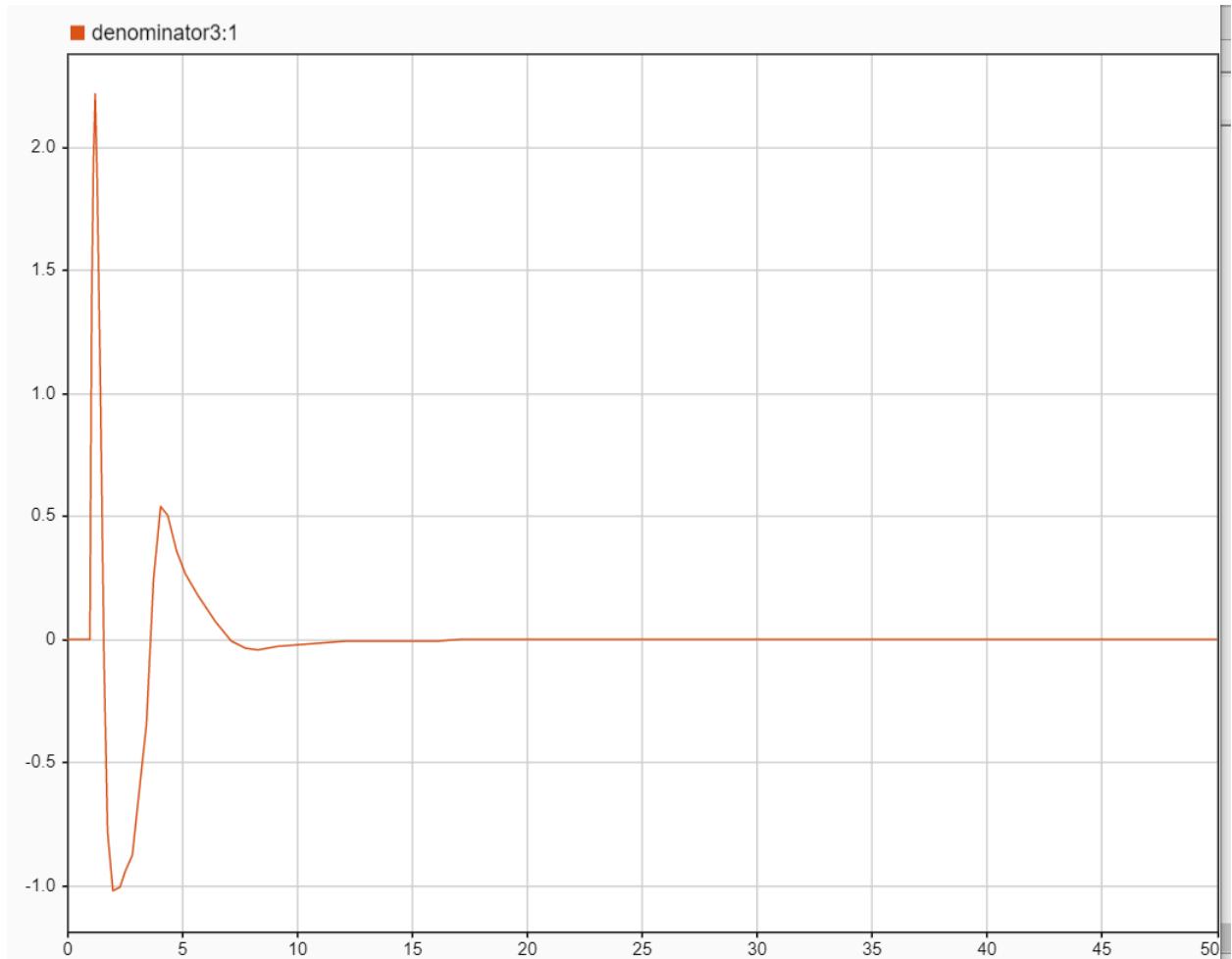
Phi:



S_bar:



Phi_dot:



S_bar_dot:

