EECS562 HW # 10 Solutions

Problem 1:

$$f(x) = \begin{bmatrix} -x_1 - x_2 + \frac{x_1 x_2}{12} \\ 0 \end{bmatrix} \ , \ g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(a) Seek a set about the origin in which

$$L_g V(x) = 0 \ \Rightarrow L_f V(X) < 0$$

$$L_g V(x) = 2x^T P g(x) = 2x^T \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} = 3x_1 + 6x_2$$

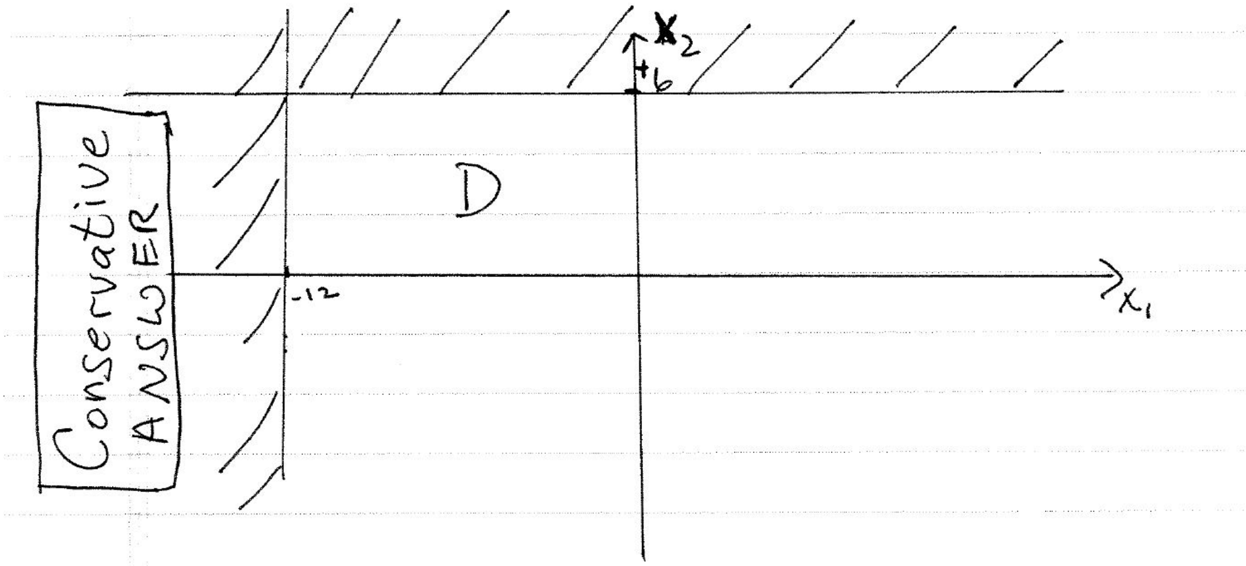
$$\therefore \{x \in \mathbb{R}^2 \mid L_g V(x) = 0 \ \} = \{x \in \mathbb{R}^2 \mid x_1 = -2x_2 \ \}$$

$$\begin{aligned} L_f V(x) = 2x^T P f(x) = 2x^T &\begin{bmatrix} -2x_1 - 2x_2 + \frac{x_1 x_2}{6} \\ -1.5x_1 - 1.5x_2 + \frac{x_1 x_2}{8} \end{bmatrix} \\ &= 2[-2(x_1)^2 - 2x_1 x_2 + \frac{(x_1)^2 x_2}{6} - 1.5x_1 x_2 - 1.5(x_2)^2 + \frac{x_1 (x_2)^2}{8}] \\ &= -4(x_1)^2 - 7x_1 x_2 - 3(x_2)^2 + \frac{(x_1)^2 x_2}{3} + \frac{x_1 (x_2)^2}{4} \end{aligned}$$

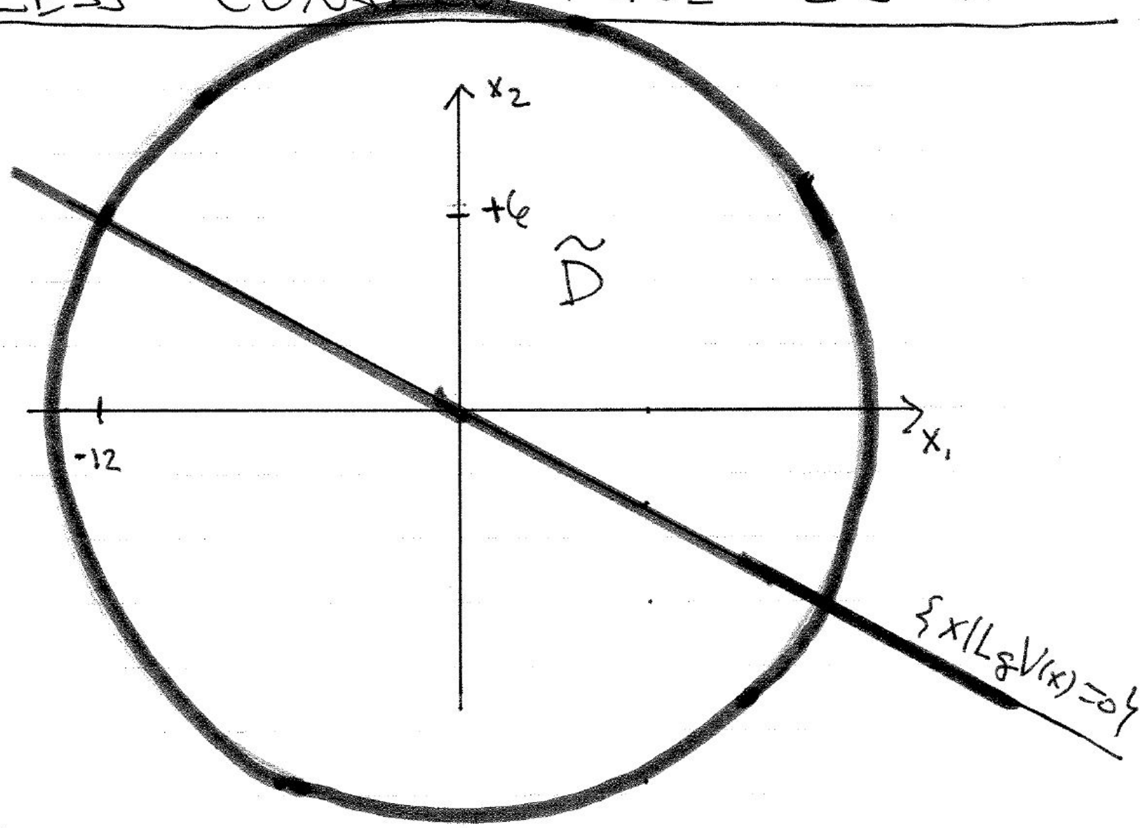
$$\begin{aligned} \therefore \quad L_f V(x) \Big|_{x_1 = -2x_2} &= -16(x_2)^2 + 14(x_2)^2 - 3(x_2)^2 + \frac{4}{3}(x_2)^3 - \frac{1}{2}(x_2)^3 \\ &= -5(x_2)^2 + \frac{5}{6}(x_2)^3 \\ &= -5(x_2)^2[1 - \frac{1}{6}x_2] \end{aligned}$$

$$\Rightarrow \ L_f V(x)\Big|_{x_1 = -2x_2} \ < \ 0 \quad \text{for} \quad x_2 < 6 \ , \ x_2 \neq 0$$

Thus, for $D = \{x \in \mathbb{R}^2 \mid x_1 > -12 \text{ and } x_2 < 6\}$, we have that $\inf\limits_{u \in \mathbb{R}} \dot{V}(x, u) < 0 \quad \forall \ x \in D \ , \ x \neq 0$



A LESS CONSERVATIVE ESTIMATE



$$\tilde{D} = \{x \in \mathbb{R}^2 \mid \|x\|^2 < 6^2 + 12^2\} = B_{6\sqrt{5}}(0)$$

Note: $\forall\ x \in \tilde{D}\ ,\ x \neq 0$, we can make $\dot{V} < 0$! If $x \notin L_g V(x) = 0$, then there is no problem. And for every point $x \in\ \tilde{D} \cap \{x \mid L_g V(x) = 0\},\ x \neq 0$, we have $L_f V(x) < 0$. We will see that $\tilde{D}$ will yield a larger estimated region of attraction.

(b) One possible controller is given by Sontag's formula :

$$u = \alpha_S(x) = \begin{cases} -\frac{L_f V(x) + \sqrt{(L_f V(x))^2 + (L_g V(x))^4}}{L_g V(x)} & , L_g V(x) \neq 0 \\ 0 & , L_g V(x) = 0 \end{cases}$$

with $L_f V(x)$ and $L_g V(x)$ as calculated in (a).

From class, we know that this feedback gives us

$$\dot{V}(x) = -\sqrt{(L_f V(x))^2 + (L_g V(x))^4} < 0 \quad \forall x \in D\ , x \neq 0$$

Remark: For a "real" system, one could either look for a simpler control law, OR do a polynomial approximation to α_S, OR do a table look-up implementation.

Problem 2:

(a) We need to find the "largest" compact sub-level set of $V(x) = x^T P x$ contained in D. As usual, we will find a conservative estimate ☹.

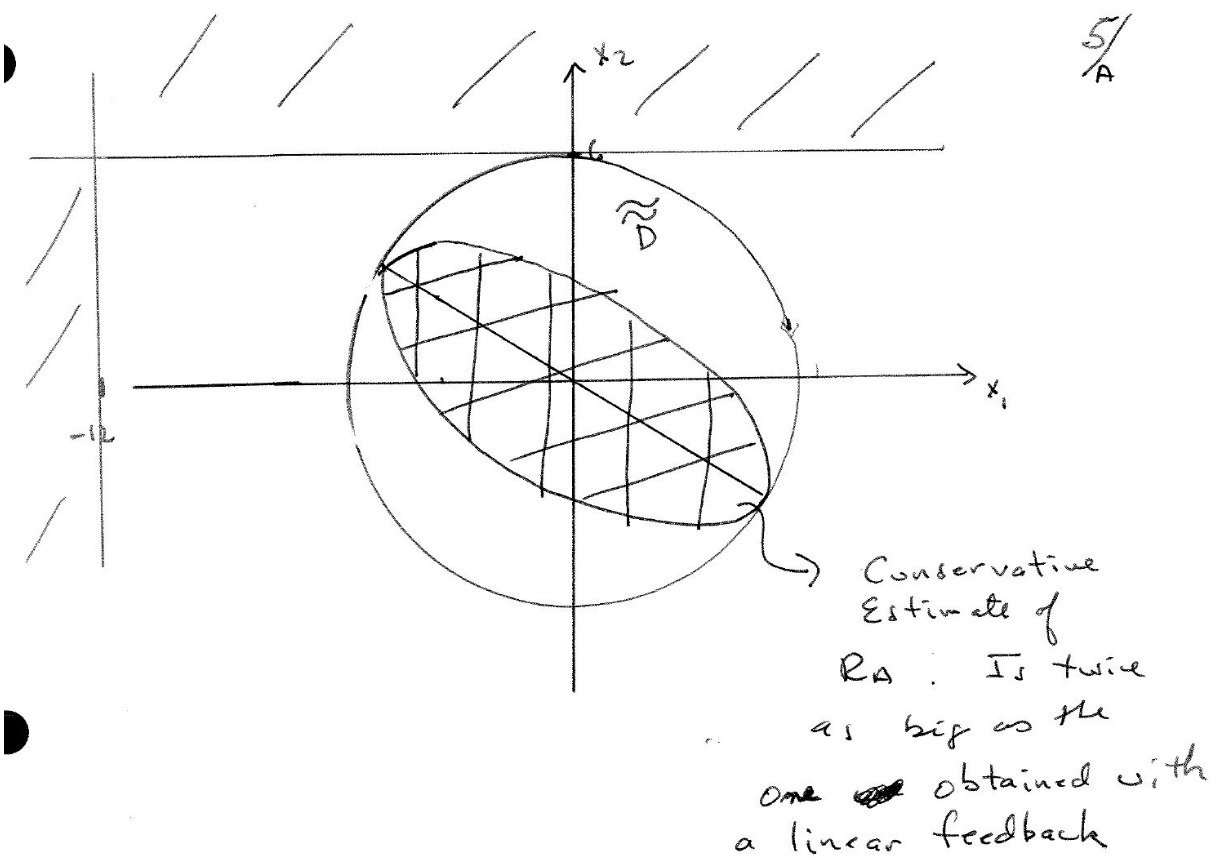
Let $\widetilde{\widetilde{D}} = \{x \in \mathbb{R}^2 \mid \|x\|_2 < 6\}$. Then $V : \widetilde{\widetilde{D}} \to \mathbb{R}$ is positive definite and $\dot{V} : \widetilde{\widetilde{D}} \to \mathbb{R}$ is negative definite. Seek "largest" compact level set of V contained in $\widetilde{\widetilde{D}}$. From class,

$$c^* = (6)^2 \lambda_{min} = 36(0.9819) \approx 35.348\ldots$$

Thus, $\forall 0 < c < c^*$,

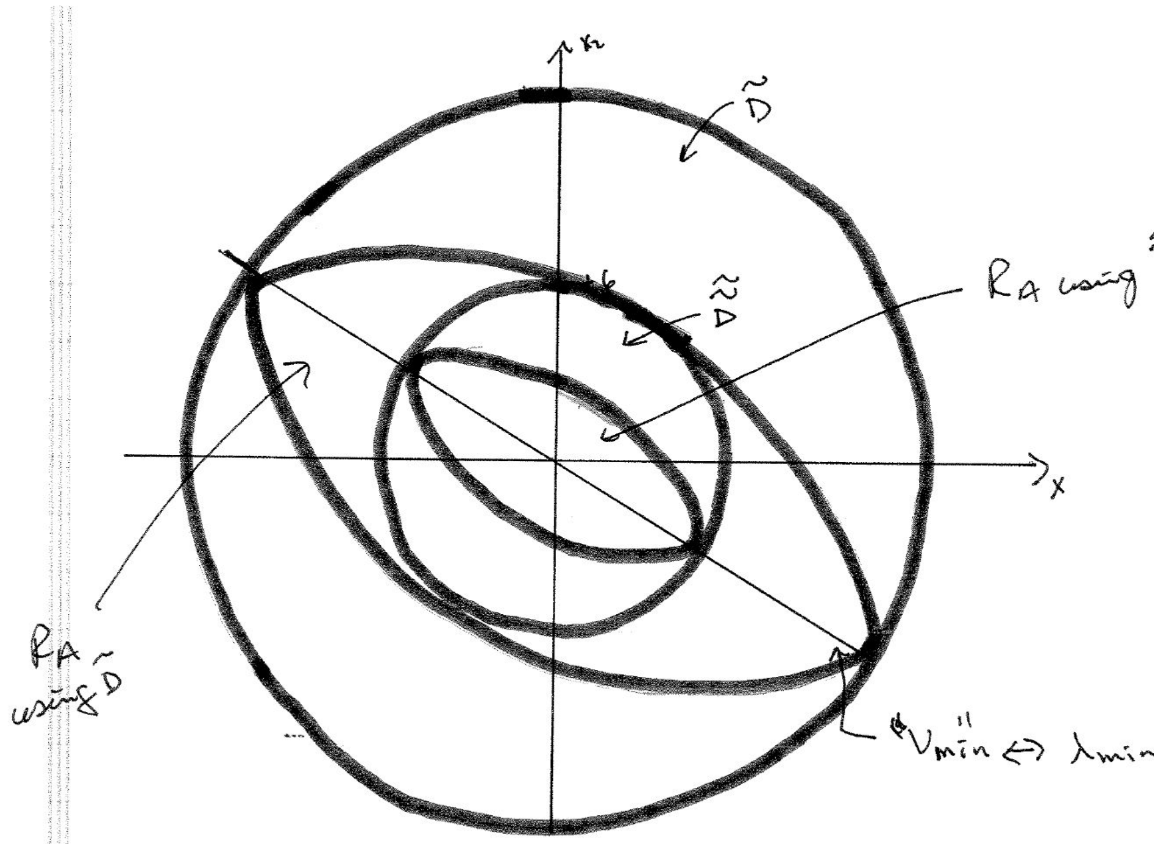
$$\Omega_c = \{x \in \widetilde{\widetilde{D}} \mid x^T P x \le C\}$$

$\therefore \widetilde{\widetilde{R}}_A \subset \{x \in \widetilde{\widetilde{D}} \mid x^T p x \le 35.3\}$



Conservative estimate of R_A Is twice as big as the one obtained with a linear feedback

Now, if we use the set $\widetilde{D}$ instead of $\widetilde{\widetilde{D}}$, we have $\bar{C}^* = (6\sqrt{5})^2 \lambda_{min} = 5C^*$!
$\therefore \widetilde{R}_A = \{x \in \widetilde{D} \mid x^T P x \leq 176.5\}$



The above are just rough sketches !

(b)

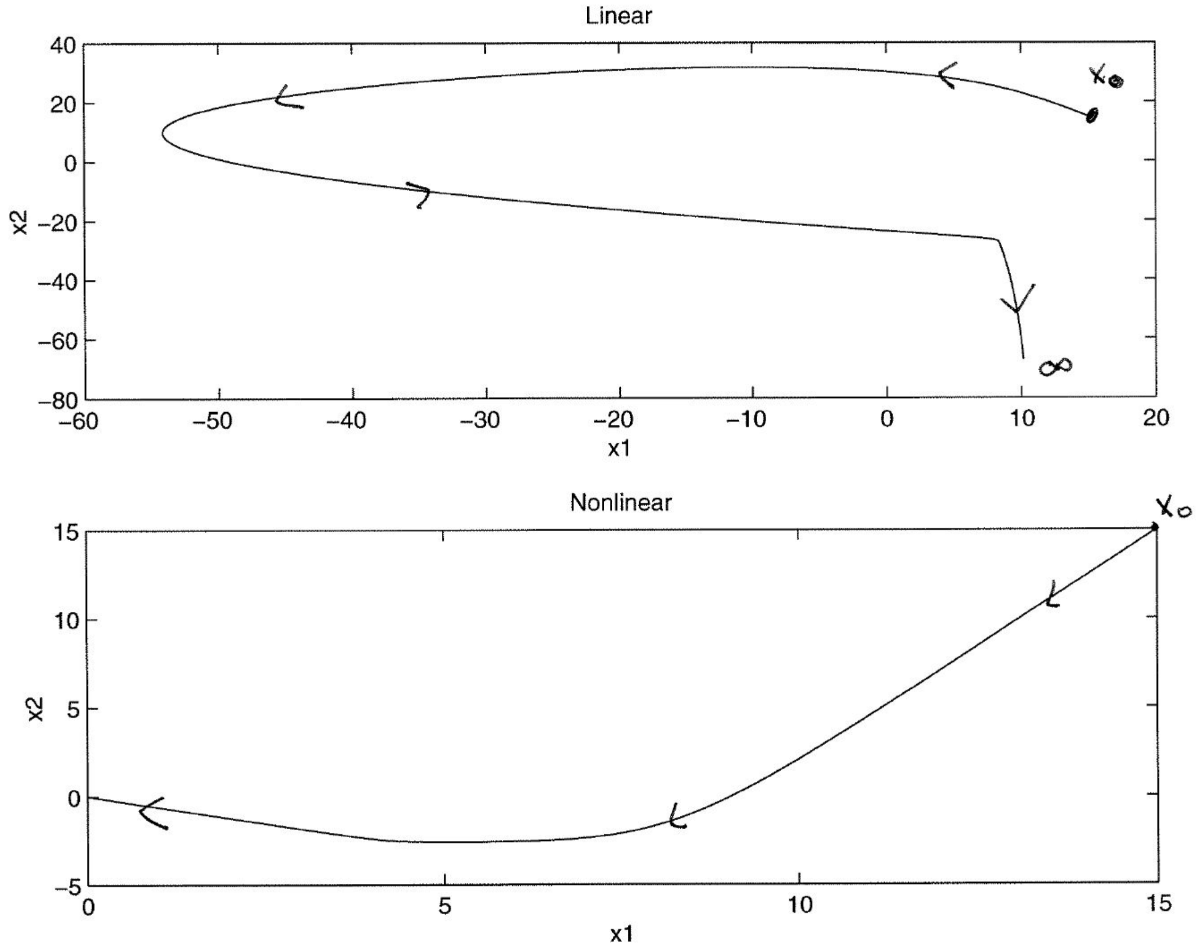
$$u = x_1 + \frac{1}{3}x_2 \Rightarrow A = \begin{bmatrix} -1 & -1 \\ 1 & \frac{1}{3} \end{bmatrix}$$

$$\Rightarrow eig(A) = [-0.33 \pm j0.745] \Rightarrow \text{asymp stable}$$

$$\Rightarrow \text{linear feedback is OK locally}$$

.

Simulations are attached



$x_0 = \begin{bmatrix} 15 \\ 15 \end{bmatrix}$ for both cases. The linear feedback leads to an unbounded trajectory.

Problem 3:

<u>**Step 1:**</u> $\dot{x}_1 = x_2$.
View x_2 as a virtual control
$V_1(x_1) = \frac{(x_1)^2}{2} \Rightarrow \dot{V}(x_1, x_2) = x_1 x_2$
$\Rightarrow x_2 = -c_1 x_1, \quad c_1 > 0$

<u>**Step 2:**</u> $\dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
Do a change of coordinates, $z_2 = x_2 + c_1 x_1$
$\Rightarrow$

$$\begin{aligned} \dot{x}_1 &= z_2 - c_1 x_1 \\ \dot{z}_2 &= x_3 + c_1(z_2 - c_1 x_1) \end{aligned}$$

View x_3 as a virtual control, and use the Lyapunov function:

$$\begin{aligned} V_2(x_1, z_2) &= \frac{(x_1)^2}{2} + \frac{(z_2)^2}{2} \\ \dot{V}_2(x_1, z_2, x_3) &= x_1 \dot{x}_1 + z_2 \dot{z}_2 \\ &= x_1(z_2 - c_1 x_1) + z_2(x_3 + c_1 z_2 - (c_1)^2 x_1) \\ &= -c_1 (x_1)^2 + z_2 \underbrace{(x_1 - (c_1)^2 x_1 + c_1 z_2 + x_3)}_{-c_2 z_2 \ , \ c_2 > 0} \\ \Rightarrow \qquad x_3 &= -c_2 z_2 - (1 - (c_1)^2) x_1 - c_1 z_2 \\ &= -(c_1 + c_2) z_2 - (1 - (c_1)^2) x_1 \end{aligned}$$

<u>**Step 3:**</u> $\dot{x}_1 = z_2 - c_1 x_1$
$\dot{z}_2 = x_3 + c_1(z_2 - c_1 x_1)$
$\dot{x}_3 = u$
Do a change of coordinates, $z_3 = x_3 - (-(c_1 + c_2) z_2 - (1 - (c_1)^2) x_1)$

$$\Rightarrow \qquad \begin{aligned} \dot{x}_1 &= z_2 - c_1 x1 \\ \dot{z}_2 &= z_3 - c_2 z_2 - x_1 \\ \dot{z}_3 &= u + (2c_1 + c_2 - c_1^3) x_1 + (1 - c_1 - (c_1)^2 - c_2) z_2 + (c_1 + c_2) z_3 \end{aligned}$$

u is now the "real" control, and use the Lyapunov function

$$\begin{aligned} V(x_1, z_2, z_3) &= \frac{(x_1)^2}{2} + \frac{(z_2)^2}{2} + \frac{(z_3)^2}{2} \\ \dot{V}(x_1, z_2, z_3, u) &= x_1 \dot{x}_1 + z_2 \dot{z}_2 + z_3 \dot{z}_3 \\ &= x_1(z_2 - c_1 x_1) + z_2(z_3 - c_2 z_2 - x_1) + z_3(u + (2c_1 + c_2 - (c_1)^3)) x_1 \\ &+ (1 - c_1 - (c_1)^2) - c_2) z_2 + (c_1 + c_2) z_3 \\ &= -c_1 (x_1)^2 - c_2 (z_2)^2 + z_3 \{ \underbrace{z_2 + u + (2c_1 + c_2 - (c_1)^3) x_1 + (1 - c_1 - (c_1)^2 - c_2) z_2 + (c_1 + c_2) z_3}_{= \alpha(x_1, z_2, z_3)} \} \end{aligned}$$

Setting $\quad \alpha(x_1, z_2, z_3) = -c_3 z_3, \quad c_3 > 0$ yields a linear controller :

$$\boxed{u = -(c_1 + c_2 + c_3) z_3 - (2 - c_1 - (c_1)^2 - c_2) z_2 - (2c_1 + c_2 - (c_1)^3) x_1}$$

* Going back to the original coordinates :

$$z_3 = x_3 + (c_1 + c_2)z_2 + (1 - (c_1)^2)x_1$$

⇒

$$\begin{aligned} u = &- (c_1 + c_2 + c_3)x_3 - \{(c_1 + c_2 + c_3)(c_1 + c_2) + (2 - c_1 - (c_1)^2 - c_2)\}z_2 \\ &- \{(c_1 + c_2 + c_3)(1 - (c_1)^2) + (2c_1 + c_2 - (c_1)^3)\}x_1 \end{aligned}$$

$$z_2 = x_2 + c_1 x_1$$

⇒

$$u = \text{ well, you get the idea !}$$

Setting $\alpha(x_1, z_2, z_3) = -c_3(z_3)^3,\ c_3 > 0$
would yield a NL controlller

$\boxed{\text{Alternative Step 3}}$ From Step 2,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \quad =: f(x) + g(x)u$$

is stabilized by $u = \underbrace{-(1 + c_1c_2)x_1 - (c_1 + c_2)x_2}_{\alpha_2(x_1,x_2)}$;

and has Lyapunov function: $V_2(x_1, x_2) = \frac{(x_1)^2}{2} + \frac{(x_2+c_1x_1)^2}{2}$
consider

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned}$$

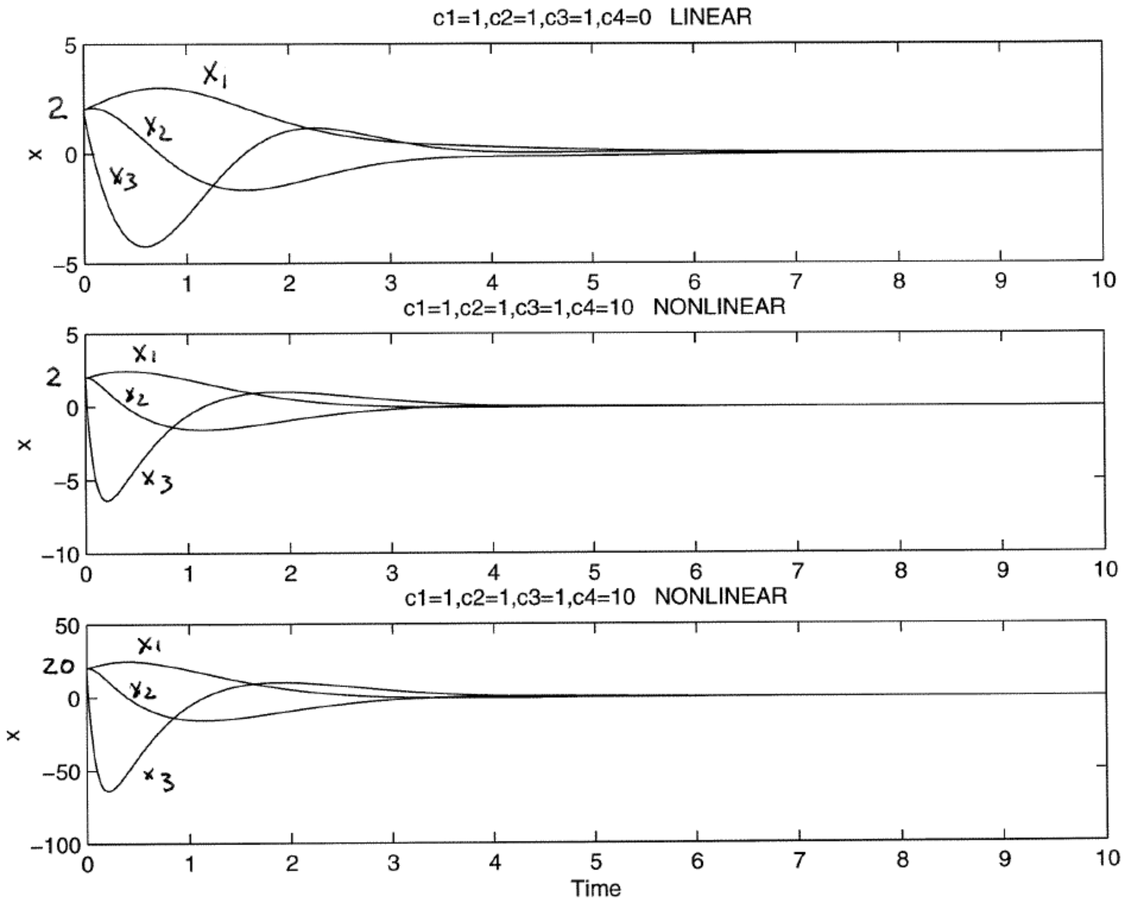
By the integrator backstepping Lemma, $V_3(x_1, x_2, x_3) = V_2(x_1, x_2) + \frac{(x_3-\alpha_2(x_1,x_2))^2}{2}$ and a linear GA stabilizing feedback is:

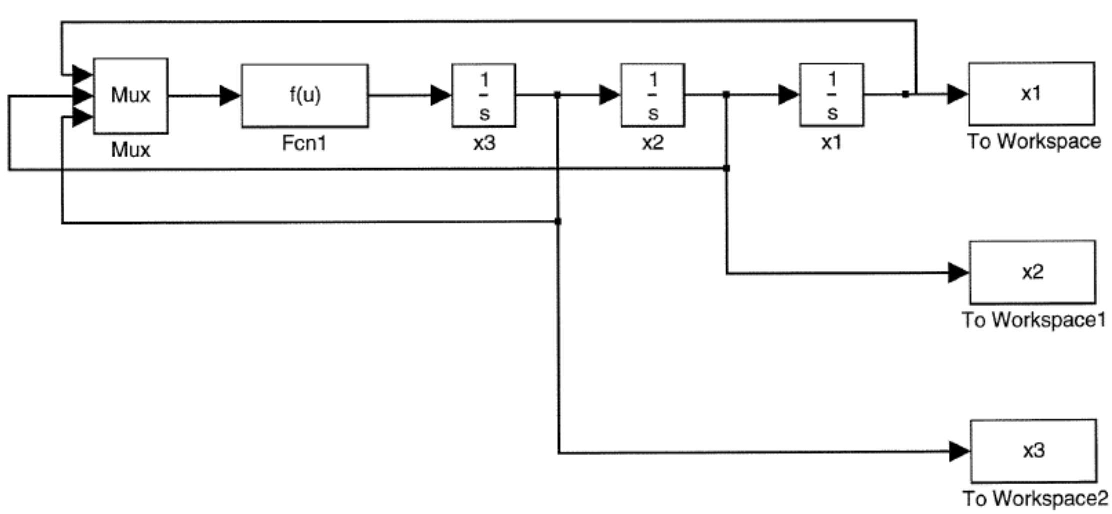
$$\begin{aligned} u_L &= \alpha_3(x_1, x_2, x_3) \\ &= -c_3(x_3 - \alpha_2(x_1, x_2)) + \frac{\partial \alpha_2}{\partial(x_1, x_2)}(f(x) + g(x)x_3) - \frac{\partial V_2}{\partial(x_1, x_2)}g(x),\ c_3 > 0 \\ &= -c_3(x_3 - \alpha_2(x_1, x_2)) - (1 + c_1c_2)x_2 - (c_1 + c_2)x_3 - (x_2 + c_1x_1) \\ &= -(c_1 + c_2 + c_3)x_3 - c_3(1 + c_1c_2)x_1 - c_3(c_1 + c_2)x_2 - (1 + c_1c_2)x_2 - (x_2 + c_1x_1),\ c_1, c_2, c_3 > 0 \\ or &\ \boxed{u_L = -(c_1 + c_2 + c_3)x_3 - [c_3(c_1 + c_2) + 2 + c_1 + c_2]x_2 - [c_1 + c_3(1 + c_1c_2)]x_1\ c_1, c_2, c_3 > 0} \end{aligned}$$

A NL feedback would be $\boxed{u_{NL} = u_L - c_4(x_3 - \alpha_2(x_1, x_2))^3, \quad c_4 > 0}$

∴ The computation seem to be easier if the integrator backstepping Lemma is applied recursively.

Problem 4:


c1=1,c2=1,c3=1,c4=0 LINEAR
c1=1,c2=1,c3=1,c4=10 NONLINEAR
c1=1,c2=1,c3=1,c4=10 NONLINEAR
x
Time
x1
x2
x3
2
20

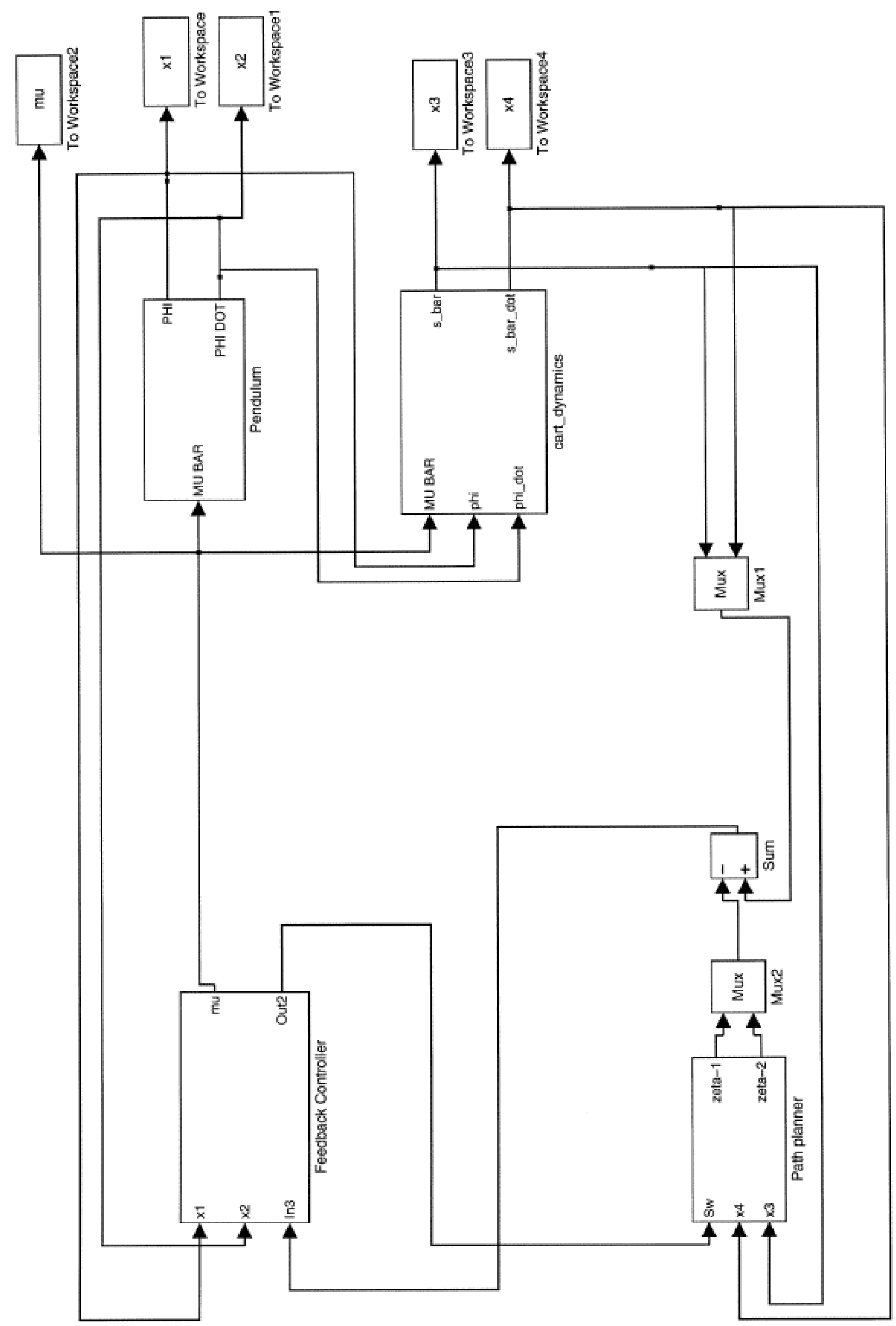

Mux
Mux
f(u)
Fcn1
1/s
x3
1/s
x2
1/s
x1
x1
To Workspace
x2
To Workspace1
x3
To Workspace2

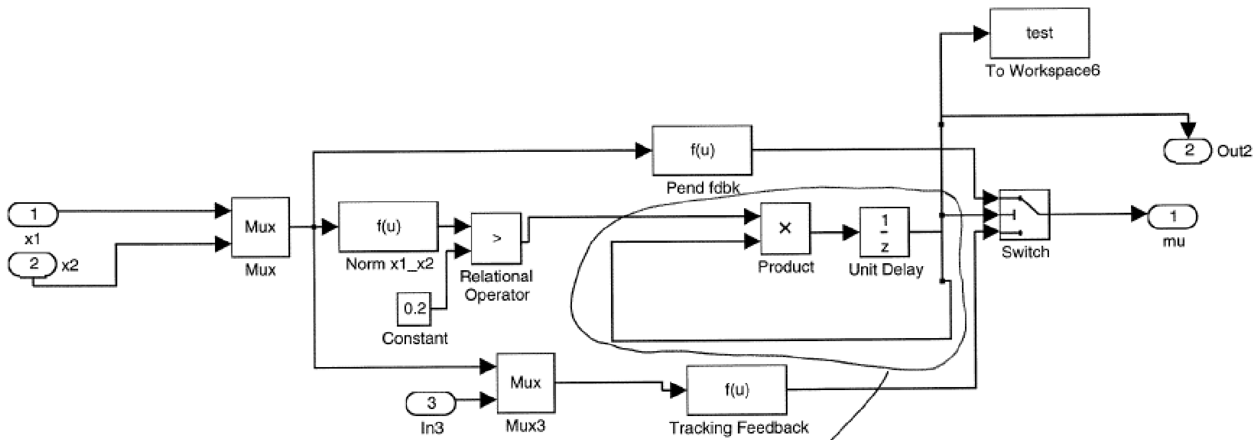
Problem 5:

$|x_1(0)| < \frac{\pi}{2}$

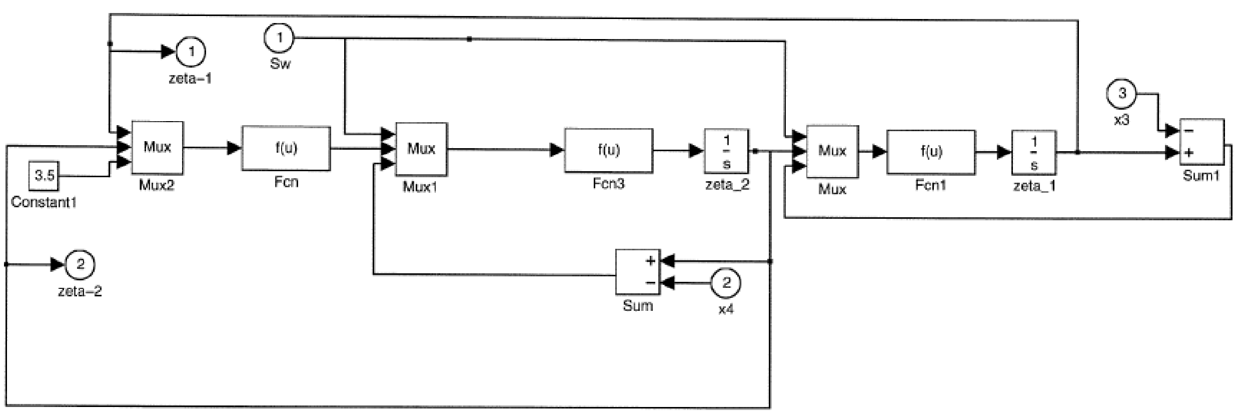
I checked it up to $\frac{\pi}{2.1}$!

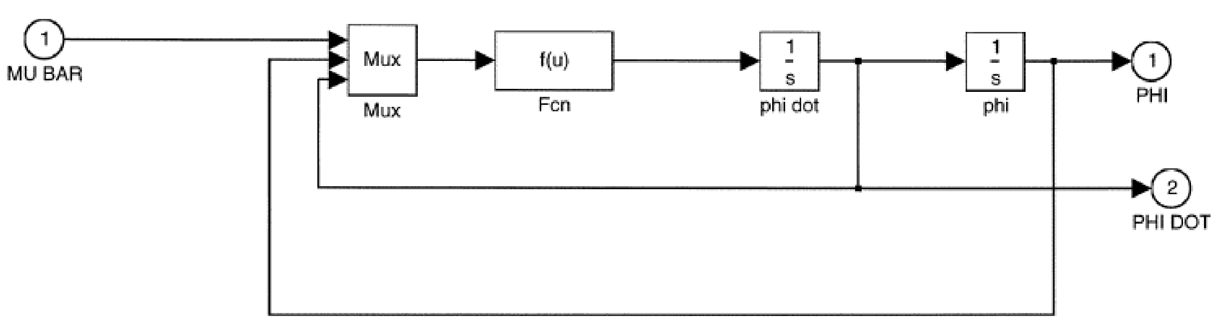
Attached simulation is for $\frac{\pi}{2.2}$!

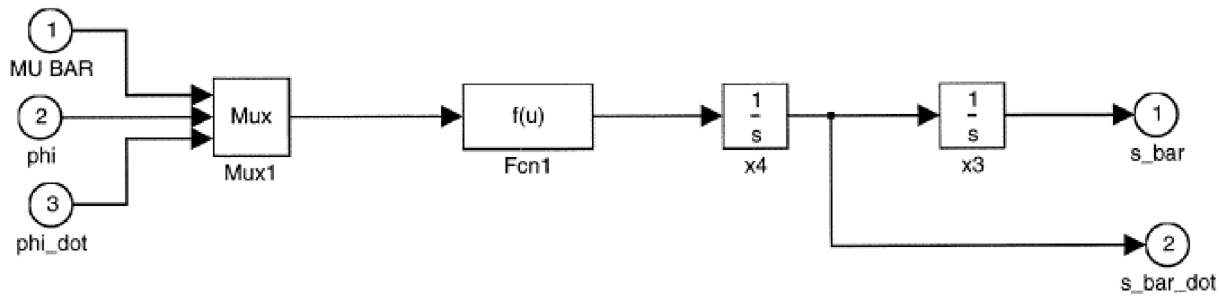


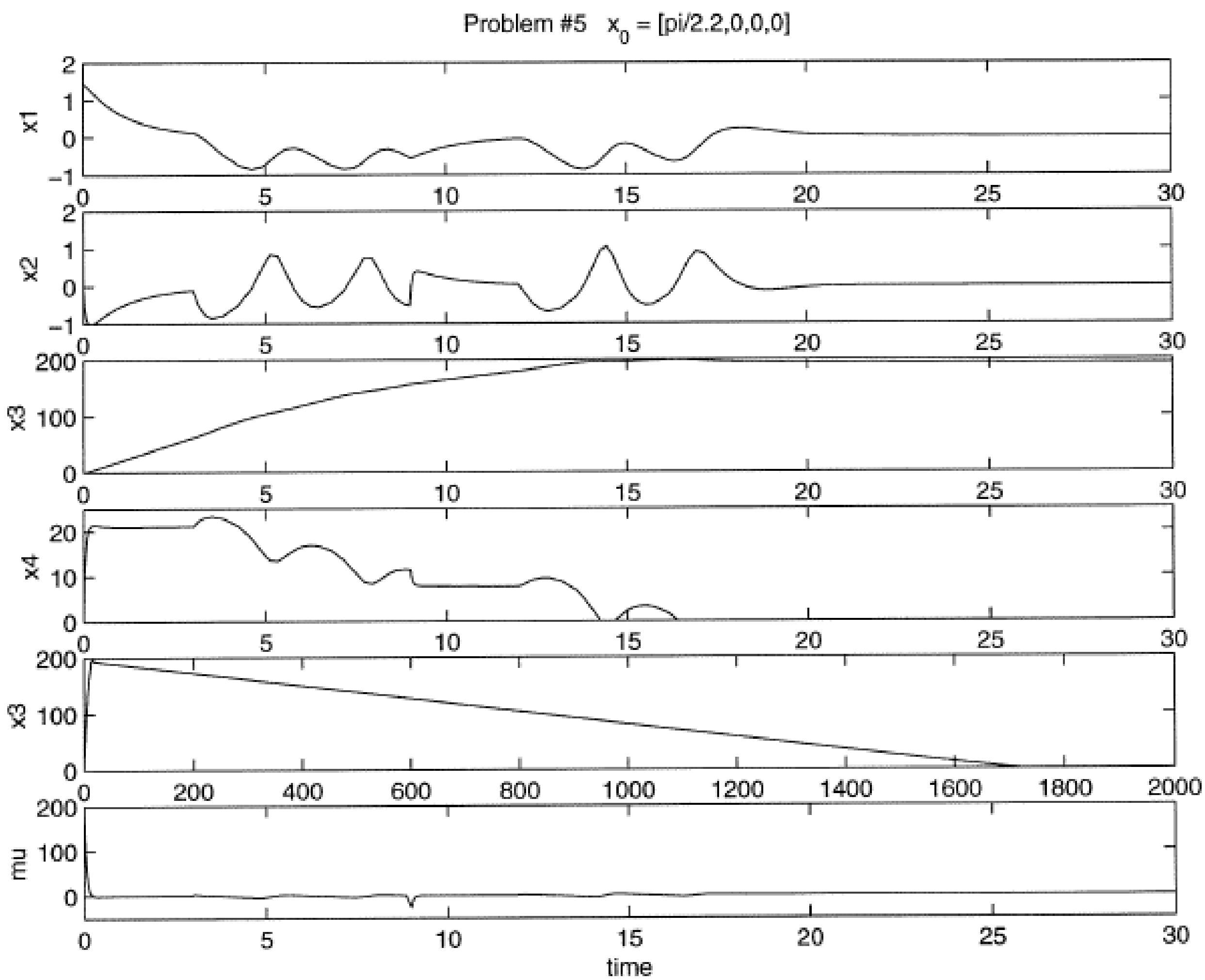


So that it only
switches once.





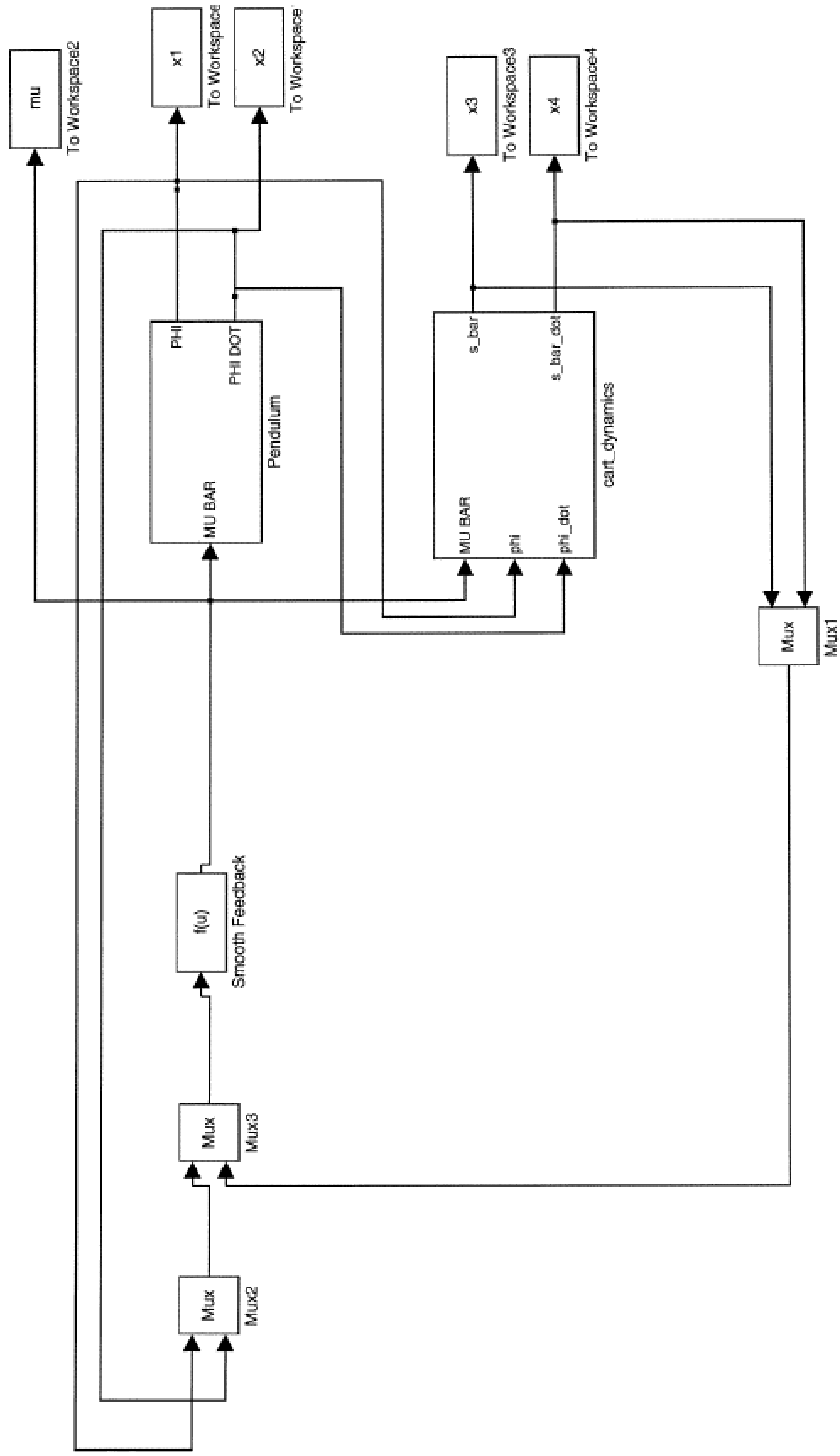

1
MU BAR
2
phi
3
phi_dot
Mux
Mux1
f(u)
Fcn1
1
s
x4
1
s
x3
1
s_bar
2
s_bar_dot

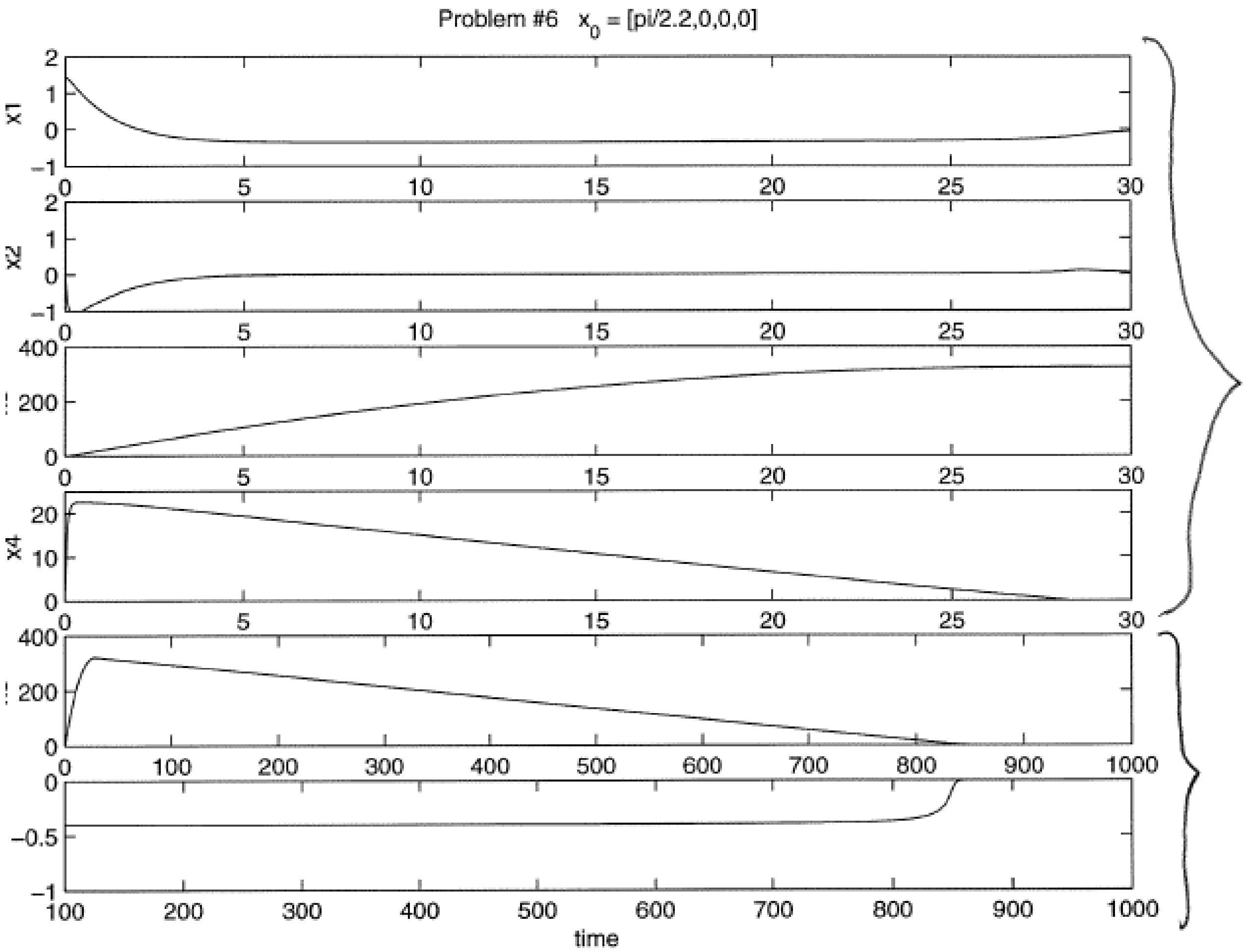

Problem #5 x0 = [pi/2.2,0,0,0]
x1
x2
x3
x4
x3
mu
time

Problem 6:

$|x_1(0)| < \frac{\pi}{2}$

Once again, I checked it for $\frac{\pi}{2.1}$, and repeat a simulation for $\frac{\pi}{2.2}$





Note different time scales for the two groups of plots.

EECS 562 HW Sol10
Additional Information Related to HW #10

A. Formal justification of tracking slowly verying $\bar{x}(t)$

B. Still another solution of HW Problem #3 using the Integrator Backstepping Lemma

Inverted Pendulum on a Cart

In HWs#9 and #10, we designed "tracking controllers" to cause $x(t)$, the state of the pendulum on a cart, to converge to a trajectory of the form $\bar{x}(t) = \begin{bmatrix} 0 \\ 0 \\ P + tv \\ v \end{bmatrix}$, p and v constant as long as $\|x(t_0) - \bar{x}(t_0)\|$ was sufficient small. We then NOTED and USED the property that if $\bar{x}(t)$ were transformed sufficient slowly, then $x(t)$ could be made to approximately track $\bar{x}(t)$, and this led us to slowly modifying $\bar{x}(t)$ into a trajectory that brought us back to the origin.

The purpose of this HANDOUT is to prove that this will work. You have observed that this idea works well by doing MATALB simulation, the question before us now is, can we prove this mathematically?

The answer of course, is YES, and this is how we do it.

Let $\boxed{\dot{x}(t) = f(x(t)) + g(x(t))u(t)}$ be the model of the inverted pendulum on a cart.

Rewrite $\bar{x}(t,p,v) = \begin{bmatrix} 0 \\ 0 \\ p + tv \\ v \end{bmatrix}$ as $\bar{x}(t,p,v) = \begin{bmatrix} 0 \\ 0 \\ p + \int_0^t v(\tau)d\tau \\ v(t) \end{bmatrix}$

If p and v are constant, these two trajectories are the same.

For p and v constant, we verified in HW#8 that:

$$\boxed{\frac{d}{dt}(x(t) - \bar{x}(t,p,v)) = f(x(t) - \bar{x}(t,p,v)) + g(x(t) - \bar{x}(t,p,v))u(t)}$$

I will leave it as an exercise to verify that if p =constant BUT $v = v(t)$, then

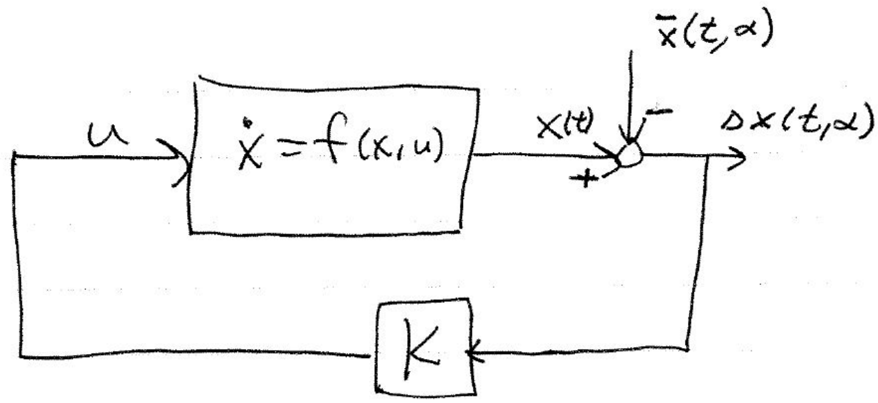
$$\boxed{\frac{d}{dt}(x(t) - \bar{x}(t,p,v)) = f(x(t) - \bar{x}(t,p,v)) + g(x(t) - \bar{x}(t,p,v))u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{v}(t) \end{bmatrix}}$$

As a second exercise [see last HW problem of HW #8], you can verify that $\forall p_0 \in \mathbb{R}, v_0 \in \mathbb{R}$ and $\epsilon > 0$, $\exists v(t)$ such that $v_0 = v(0)$, $\sup_{t \geq 0} |\dot{v}(t)| < \epsilon$ and :

1. $\begin{bmatrix} p_0 + \int_0^t v(\tau)d\tau \\ v(t) \end{bmatrix} \xrightarrow[t \to \infty]{} 0$
2. $\dot{v}(t) \xrightarrow[t \to \infty]{} 0$

Theorem:

Suppose that $f(x,u)$ is continuously differentiable in x and u, and suppose that $u = Kx$ renders the origin exponentially stable for $\dot{x} = f(x, Kx)$. Consider the feedback configuration



$(\star)$

Where $\bar{x}(t,\alpha)$ is a state trajectory to be tracked, depending upon t and α, where $\alpha \in \mathbb{R}^p$. Suppose that $\exists$ a function Γ such that for any continuously differentiable $\alpha(t)$, and any solution $(x(t), u(t))$ of $\dot{x} = f(x,u)$

$$\frac{d}{dt}(\ x(t) - \bar{x}(t,\alpha(t))\ ) = f(\ x(t) - \bar{x}(t,\alpha(t))\ ,\ u(t)\ ) + \Gamma(\dot{\alpha}(t))$$

Where $\Gamma(0) = 0$ and $\|\Gamma(\dot{\alpha})\| \le L\|\dot{\alpha}\|$ for some finite constant L, and all $\dot{\alpha} \in \mathbb{R}^m$.

THEN, for the feedback system $(\star)$:

① $\exists\ \varepsilon > 0\ ,\ n > 0\ ,\ 0 \le k < \infty\ ,\ \gamma > 0$ such that if $\|\dot{\alpha}(t)\|_\infty < \varepsilon$ and $\|x(t_0) - \bar{x}(t_0, \alpha(t_0))\| < n$,

(a) the solution $x(t)$ of $(\star)$ exists for all $t \ge t_0$; and

(b) for all t_0 and $T > t_0$ finite, for all $t_0 \le t \le T$

$$\|x(t) - \bar{x}(t,\alpha(t))\|_2^2 \le e^{-\gamma(t-t_0)}\|x(t_0) - \bar{x}(t_0,\alpha(t_0))\|_2^2 + \frac{1 - e^{-\gamma(t-t_0)}}{\gamma} k \|\ \dot{\alpha}(t)|_{[t_0,T]}\ \|_\infty$$

② If in addition $\dot{\alpha}(t) \xrightarrow[t\to\infty]{} 0$, then $\|x(t) - \bar{x}(t,\alpha(t))\|_2 \xrightarrow[t\to\infty]{} 0$

③ And finally, if also $\bar{x}(t,\alpha(t)) \xrightarrow[t\to\infty]{} 0$ (in addition to all the above), then $x(t) \xrightarrow[t\to\infty]{} 0$

Proof:

Let $\tilde{A} := A + BK$ where $A = \frac{\partial f}{\partial x}(0,0)$ & $B = \frac{\partial f}{\partial u}(0,0)$. Define $P = P^T > 0$ by $\tilde{A}^T P + P\tilde{A} = -I$, and set $V(x) = x^T P x$. Also, let $g(x) := f(x, Kx) - \tilde{A}x$. Then, $\exists r > 0$ s.t. $\forall x \in B_r(0)$

$$\|2x^T P g(x)\|_2 \le \frac{1}{2}\|x\|_2^2$$

Then,

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x) f(x,Kx) = -x^T x + 2x^T P g(x) \le -x^T x + \frac{1}{2}\|x\|_2^2 = -\frac{1}{2}x^T x \quad \forall x \in B_r(0)$$

Let λ_{min} & λ_{max} be minimum and maximum e-values of P so that $\lambda_{min} x^T x \le x^T P x \le \lambda_{max} x^T x$.

$$\therefore -x^T x^T \le -\frac{1}{\lambda_{max}} x^T P x \le -\frac{1}{\lambda_{max}} V(x),$$

from which we deduce that

$$\dot{V}(x) \le -\frac{1}{2\lambda_{max}} V(x) \quad , \quad \forall x \in B_r(0)$$

$$\boxed{\dot{V}(x) \le -\gamma V(x) \quad , \quad x \in B_r(0) \quad \text{where } \gamma := \tfrac{1}{2\lambda_{max}}} \tag{1}$$

Consider now,

$$\begin{aligned}
\frac{d}{dt}V(\ x(t) - \bar{x}(t,\alpha(t))\ ) &:= \frac{\partial V}{\partial x}(x(t) - \bar{x}(t)) \cdot \frac{d}{dt}[x(t) - \bar{x}(t,\alpha(t))] \\
&= \frac{\partial V}{\partial x}(x(t) - \bar{x}(t))[f(\ x(t) - \bar{x}(t,\alpha(t))\ ,\ K(x(t) - \bar{x}(t,\alpha(t))) + \Gamma(\dot{\alpha}(t))\ )] \\
&= \dot{V}(x(t) - \bar{x}(t,\alpha(t))) + \frac{\partial V}{\partial x}(\ x(t) - \bar{x}(t,\alpha(t))\ )\Gamma(\dot{\alpha}(t))
\end{aligned}$$

Let $K := \sup_{\|x\| \le r} \|\frac{\partial V}{\partial x}(x)\|_2$. Then,

$$\frac{d}{dt}V(\ x(t) - \bar{x}(t,\alpha(t))\ ) \le \dot{V}(\ x(t) - \bar{x}(t,\alpha(t))\ ) + KL\|\dot{\alpha}(t)\|_2$$

and for $x(t) - \bar{x}(t,\alpha(t)) \in B_r(0)$,

$$\boxed{\tfrac{d}{dt}V(\ x(t) - \bar{x}(t,\alpha(t))\ ) \le -\gamma V(\ x(t) - \bar{x}(t,\alpha(t))\ ) + KL\|\dot{\alpha}(t)\|_2} \tag{2}$$

Hence,for any $[t_0, T]$ for which a solution to ($\star$) exists, $\quad \forall\ t_0 \le t \le T$

$$\begin{aligned}
&\boxed{V(\ x(t) - \bar{x}(t,\alpha(t))\ ) \le e^{-\gamma(t-t_0)}V(\ x(t_0) - \bar{x}(t_0,\alpha(t_0))\ ) + \textstyle\int_{t_0}^{t} e^{-\gamma(t-\tau)}KL\|\dot{\alpha}(\tau)\|d\tau} \\
&\qquad\qquad \boxed{\le e^{-\gamma(t-t_0)}V(\ x(t_0) - \bar{x}(t_0,\alpha(t_0))) + \tfrac{1-e^{-\gamma(t-t_0)}}{\gamma}KL\|\dot{\alpha}|_{[t_0,T]}\|_\infty)}
\end{aligned} \tag{3}$$

$$\boxed{\therefore V(\ x(t) - \bar{x}(t,\alpha(t))\ ) \le V(\ x(t_0) - \bar{x}(t_0,\alpha(t_0))\ ) + \tfrac{KL}{\gamma}\|\dot{\alpha}\|_\infty} \tag{4}$$

Since $e^{-\gamma(t-t_0)} \le 1$ and $\frac{1-e^{-\gamma(t-t_0)}}{\gamma} \le \frac{1}{\gamma}$.

It is now shown that for appropriate choices of $n > 0$ and $\varepsilon > 0$, (4) implies that $\|x(t) - \bar{x}(t,\alpha(t))\| \le \frac{\gamma}{2}$ for all $t_0 \le t \le T$. This will imply that all solutions lies in a compact set and hence existence & uniqueness on $[0,\infty)$ follows.

Let

$$\begin{aligned}
&\delta := \lambda_{min}\frac{r^2}{4} \\
&\lambda_{max}n^2 = \frac{\delta}{2} \qquad (\Rightarrow n = \frac{r}{2}\sqrt{\frac{\lambda_{min}}{2\lambda_{max}}}) \\
&\frac{LK}{\gamma}\cdot\varepsilon = \frac{\delta}{2} \qquad (\Rightarrow \varepsilon = \frac{\lambda_{min}r^2}{8LK})
\end{aligned}$$

Then (4) implies that

$$\lambda_{min}||x(t)-\bar{x}(t,\alpha(t))||_2^2 \leq \lambda_{max}||x(t_0)-\bar{x}(t_0,\alpha(t_0))||_2^2 + \frac{KL}{\gamma}||\dot{\alpha}||_\infty \tag{5}$$

$\Rightarrow \lambda_{min}||x(t)-\bar{x}(t,\alpha(t))||_2^2 \leq \frac{\delta}{2}+\frac{\delta}{2} = \delta = \lambda_{min}\frac{r^2}{4}$ whenever $||x(t_0)-\bar{x}(t_0,\alpha(t_0)) \leq n$ & $||\dot{\alpha}||_\infty \leq \varepsilon$
$\Rightarrow ||x(t)-\bar{x}(t,\alpha(t))||_2 \leq \frac{r}{2}$ as promised
$\therefore$ 1 (a) & (b) are shown.

It is now shown that 2 and 3 are true.
Since $\dot{\alpha}(t) \xrightarrow[t\to\infty]{} 0$, it follows that $\| \dot{\alpha}|_{[\frac{T}{2},T]} \|_\infty \xrightarrow[T\to\infty]{} 0$ (exercise)
In (3), let $t_0 = \frac{T}{2}$ and $t = T$ Then

$$\begin{aligned} V(\ x(T)-\bar{x}(T,\alpha(T))\ ) &\leq e^{-\frac{\gamma}{2}T}V(\ x(\frac{T}{2})-\bar{x}(\frac{T}{2},\alpha(\frac{T}{2}))\ ) + \frac{1-e^{-\gamma\frac{T}{2}}}{\gamma}KL\| \dot{\alpha}|_{[\frac{T}{2},T]} \|_\infty \\ &\leq e^{-\frac{\gamma}{2}T}\frac{r}{2} + \frac{KL}{\gamma}\| \dot{\alpha}|_{[\frac{T}{2},T]} \|_\infty \end{aligned}$$

$e^{-\frac{\gamma}{2}T} \xrightarrow[T\to\infty]{} 0$ and $\| \dot{\alpha}|_{[\frac{T}{2},T]} \|_\infty \xrightarrow[T\to\infty]{} 0$ and thus $V(\ x(T)-\bar{x}(T,\alpha(T))\ ) \xrightarrow[T\to\infty]{} 0$,
showing 2 and 3 is then immediate.

More on HW # 10, Prob. # 3

$$\text{Use backstepping to G.A.S.} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases}$$

Lets write the system as

$$\dot{x} = f_0(x) + g_0(x)\xi$$
$$\dot{\xi} = u$$

Where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\xi = x_3$, $f_0(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $g_0(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
By the Routh criterion, we know that

$$\xi = \alpha_0(x_1, x_2) = -x_1 - x_2$$

will yield G.A.S of the origin for

$$\dot{x} = f_0(x) + g_0(x)\alpha_0(x) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

What is an appropriate Lyapunov function ?

Let $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, and solve for $P > 0$ s.t. $A_0^T P + P A_0 = -I$
we obtain: $P = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$, yielding

$$V_0(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

By the Integrator Backstepping lemma, we propose the candidate c.l.f.

$$V_a(x, \xi) = V_0(x) + \frac{1}{2}(\xi - \alpha_0(x))^2,$$

and we compute

$$u = -c(\xi - \alpha_0(x)) + \frac{\partial \alpha_0(x)}{\partial x}(f_0(x) + g_0(x)\xi) + (-\frac{\partial V_0(x)}{\partial x} g_0(x)) \quad , \quad c > 0$$

Since:

$$\alpha_0(x) = -x_1 - x_2$$
$$\frac{\partial \alpha_0}{\partial x}(x) = \begin{bmatrix} -1 & -1 \end{bmatrix}$$
$$f_0(x) + g_0(x)\xi = \begin{bmatrix} x_2 \\ \xi \end{bmatrix}$$
$$\frac{\partial V_0}{\partial x}(x) = 2x^T P = 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{aligned} \frac{\partial V_0}{\partial x}(x) g_0(x) &= 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \\ &= x_1 + 2x_2 \end{aligned}$$

$$
\begin{aligned}
\therefore \quad u &= -c(\xi + x_1 + x_2) + \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi \end{bmatrix} - (x_1 + 2x_2) \\
&= -c(\xi + x_1 + x_2) - x_2 - \xi - x_1 - 2x_2
\end{aligned}
$$

Substituting $\xi = x_3$ yields

$$
\boxed{u = -c(x_1 + x_2 + x_3) - (x_1 + x_2 + x_3) - 2x_2 \quad , \quad c > 0}
$$