Exam Number:______

EECS 562 EXAM-I

TUESDAY, March 17, 2009, 5:40-7:30 PM
Room 1200 EECS

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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SIGNATURE
(Sign **after** the exam is completed)

Solution
________________________, ________________
LAST NAME (PRINTED) FIRST NAME

RULES:
1. OPEN TEXT BOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. ONE SHEET OF NOTE PAPER
6. NO CALCULATORS

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. Fifty percent of the points on the exam are NO PARTIAL CREDIT GIVEN and fifty percent are PARTIAL CREDIT GIVEN. For those problems that allow partial credit, show your work clearly on this booklet. **May your wisdom pour forth!!**

Record Answers Here	
	Your Answer
Problem 1	T F
Problem 2	T F
Problem 3	(a) (b) (c) (d)
Problem 4	(a) (b) (c) (d)
Problem 5	(a) (b) (c) (d)
Problem 6	(a) (b) (c) (d)
Problem 7-1	(a) (b) (c) (d)
Problem 7-2	(a) (b) (c) (d)
Problem 7-3	(a) (b) (c) (d)
Problem 7-4	(a) (b) (c) (d)
Problem 7-5	(a) (b) (c) (d)

Scores (Filled in by Instructor)		
	Your Score	Max Score
Problems 1-6		30
Problem 7		10
Problem 8		20
Problem 9		20
Total		**80**

Multiple-Choice True-False Section of the Exam. There are six questions. Each question is worth five (5) points. Respond as appropriate. You are NOT asked to show your work. **Record all answers on page 2.**

1. __F__ **(T or F):** For a continuously differentiable function $V : \mathbb{R}^2 \to [0, \infty)$ and a constant $c > 0$, define $\mathcal{L}(c) := \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$. Suppose that for all $x_2 \neq 0$, $\lim_{|x_1| \to \infty} V(x_1, x_2) = \infty$ and for all $x_1 \neq 0$, $\lim_{|x_2| \to \infty} V(x_1, x_2) = \infty$. Then you can conclude that $\mathcal{L}(c)$ is compact.

2. __F__ **(T or F):** Consider a differential equation $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f(0) = 0$. Suppose that $A = \frac{\partial f}{\partial x}(0)$ has an e-value of $+1$. Then for every $\epsilon > 0$ and $\delta > 0$, there exists an initial condition x_0 such that $||x_0|| < \delta$ and $\sup_{t>0} ||x(t, x_0)|| \geq \epsilon$.

3. For which of the scalar differential equations below does every initial condition give rise to a solution defined for all $t \in [0, \infty)$? (circle **ALL** that apply):

(a) $\dot{x} = -x^2$.

(b) $\dot{x} = x^2 \cos(x^3)$.

(c) $\dot{x} = x \ \mathrm{sat}(x^4)$.

(d) none of the above.

4. Which of the functions $f : \mathbb{R} \to \mathbb{R}$ given below is (are) <u>locally</u> Lipschitz continuous? (circle **ALL** that apply):

(a) $f(x) = x^2 + |x^3|$.

(b) $f(x) = \mathrm{sgn}(x) \cos(x)$.

(c) $f(x) = \sqrt{(x^2 + 1)}$.

(d) none of the above.

5. Consider a time-invariant differential equation $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f(0) = 0$. Write $f(x) = Ax + R(x)$, where $A = \frac{\partial f}{\partial x}(0)$, and let $P > 0$ satisfy $A^T P + PA = -C^T C$, where (A, C) is observable. It follows that $V(x) = x^T P x$ results in $\dot{V}(x) = -x^T C^T C x + 2x^T P R(x)$. With this information, the **strongest statement** you can make about stability is:

(a) $x_e = 0$ is stable i.s.L.

(b) $x_e = 0$ is asymptotically stable i.s.L.

(c) $x_e = 0$ is locally exponentially stable.

(d) none of the above.

Next Page for Problem 6

6. Consider a time-varying differential equation $\dot{x} = f(t,x)$, where $f : [0,\infty) \times I\!\!R^n \to I\!\!R^n$ is p.w. continuous in t, locally Lipschtiz in x, and $f(t,0) = 0,\ t \geq 0$. Suppose that there exists a continuously differentiable function $V : [0,\infty) \times I\!\!R^n \to I\!\!R$ such that

i) $\forall t \geq 0,\ \ V(t,0) = 0$

ii) $\forall x \neq 0,\ \ \inf_{t\geq 0} V(t,x) > 0$

iii) $\forall x \neq 0,\ \ \sup_{t\geq 0} V(t,x) < \infty$

iv) $\forall x \neq 0,\ t \geq 0,\ \ \dot{V}(t,x) < 0.$

Then the **strongest statement** you can make about stability is:

(a) $x_e = 0$ is uniformly stable i.s.L.

(b) $x_e = 0$ is asymptotically stable i.s.L.

(c) $x_e = 0$ is uniformly asymptotically stable i.s.L.

(d) The correct answer is not in the above list.

Note: You would select (d), for example, if you think the answer is $x_e = 0$ is merely stable i.s.L., or if you think that nothing can be said, or if you think the answer is GAS, etc.

Reasons: Not required by you

1. F See Prob. 4.9, page 182 Khalil, 3rd Edition. See Fig 4.4 page 123 after making the substitution $\bar{x}_1 = x_1 + x_2$ and $\bar{x}_2 = x_1 - x_2$.

2. F The given ε-δ condition implies that solutions starting near the origin can become arbitrarily large. The van der Pol oscillator in Fig 2.20 shows that Re{e-value} > 0 may yield (globally) bounded solutions.

3. (b) and (c).

(a). Finite escape time for $x_0 = -1$

(b) Solutions go to equilibria

and hence remain bounded.

(c) For $|x| \leq 1$, $sat(x^4) = x^4$.

Hence, for $|x| \leq 1$, $\left| \frac{d}{dx}(x^5) \right| = |5x^4| \leq 5$

$\Rightarrow$ Lipschitz constant of 5.

For $|x| > 1$, $sat(x^4) = 1$, hence,

$$\left| \frac{d}{dx}(x) \right| = 1$$

$$\therefore \quad L = \max_{x \in \mathbb{R}} \left| \frac{d}{dt} x\, sat(x^4) \right| \leq 5 \Rightarrow$$

Globally Lipschitz.

4.

(a) $f(x)$ is clearly cont. diff for all $x \neq 0$,

$\Rightarrow$ Loc. Lip for all $x \neq 0$. We only need to check at $x = 0$. For $x \geq 0$,

(4d)

$f(x) = x^2 + x^3 \quad \Rightarrow \quad \frac{df}{dx}(x) = 2x + 3x^2$

$\Rightarrow \frac{df}{dx}(0) = 0.$ (derivative from the right)

For $x \leq 0$, $f(x) = x^2 - x^3 \Rightarrow \frac{df}{dx}(x) =$

$2x - 3x^2 \Rightarrow \frac{df}{dx}(0) = 0$ (deriv. from left)

Derivatives from left & right agree $\Rightarrow$ cont. diff. $\Rightarrow$ Loc. Lip.

(b) There is a jump at the origin $\Rightarrow$ discontinuous $\Rightarrow$ NOT Locally Lip. Cont.

(c) $\frac{df}{dx}(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

which is cont $\Rightarrow$ Loc. Lip. cont.

<u>Remark:</u> $\sup_x \left| \frac{df}{dx}(x) \right| \leq 1 \Rightarrow$ Is even globally Lipschitz.

PLACE NAME OR INITIALS HERE:____________________

4e

5. By Prob. 4.22, Page 185 Khalil 3rd Edition, we deduce that A is Hurwitz. Because the linearization $\dot{x}=Ax$ is Hurwitz, we deduce that $x_e=0$ is loc. exp stable for $\dot{x}=f(x)$.

Prob 6 Solution: To prove $x_e = 0$ is uniformly stable i.s.L., we note that (i), (ii), and (iii), by results from the class notes, imply that V is locally positive definite and decrescent. By (iv), $\forall\, t \geq 0,\ \dot{V}(t,x) \leq 0$.. Hence, by Theorem 4.8, page 151, the origin is uniformly stable i.s.L.

This example shows that you cannot conclude asymptotic stability. Consider $\dot{x} = 0$ on $\mathbb{R}$ and $V(t,x) := x^2(1+e^{-t})$. Then all of the hypotheses are easily checked; indeed, $\forall t \geq 0,\ V(t,0) = 0$; $\forall x \neq 0,\ \inf_{t\geq 0} V(t,x) = x^2 > 0$; $\forall x \neq 0,\ \sup_{t\geq 0} V(t,x) \leq (1+e)x^2 < \infty$; and $\forall x \neq 0,\ t \geq 0,\ \dot{V}(t,x) = 2x(1+e^{-t})\cdot 0 + x^2(-e^{-t}) = -x^2e^{-t} < 0$. Yet, clearly, the origin is not asymptotically stable.

Thus, the **strongest** statement we can make is indeed, $x_e = 0$ is uniformly stable i.s.L.

PLACE NAME OR INITIALS HERE:____________________

5

Problem 7:(10 points; i.e., 2 points for each problem.) Circle all that apply, no reasons are necessary. Note that (1) the list of answers is immediately BELOW the given function, and(2), if you circle positive definite, then you should also circle locally positive definite.

In the following, $x \in \mathbb{R}^2$ and $t \geq 0$. To be very clear, (pos. def) $\equiv$ (globally pos. def.)

1. $V_1(x) = x^T \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} x$

(a)locally positive definite (b) positive definite (c) radially unbounded (d) none of these

2. $V_2(x) = \frac{1}{2}(x_1)^2 + \int_0^{x_2} h(\sigma)d\sigma$, where $h(\sigma) = \sigma(3 + \cos(\sigma^3))$

(a)locally positive definite (b) positive definite (c) radially unbounded (d) none of these

3. $V_3(x) = (x_1)^2 + 4x_1x_2 + (x_2)^2$

(a)locally positive definite (b) positive definite (c) radially unbounded (d) none of these

4. $V_4(t, x) = (x_1)^4 + \frac{1+t}{1+t+t^2}(x_2)^2$

(a) positive definite (b) radially unbounded (c) decrescent (d) none of these

5. $V_5(t, x) = (x_1 - x_2)^2 + \frac{1+t}{1+2t}(x_2)^4$

(a) positive definite (b) radially unbounded (c) decrescent (d) none of these

Reasons: NOT Required

1. $P = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$ is pos. def $\Leftrightarrow$ $2>0$

and $2\cdot 3-(-2)(-2) > 0$. The latter equals

$6-4=2>0$ $\therefore$ P is pos def. Hence, $V(x)=x^TPx$

is pos. def and radially unbounded.

2. Because $|2\sigma| \le |h(\sigma)| \le |4\sigma|$,

it follows that $(x_2)^2 \le \int_0^{x_2} h(\sigma)d\sigma \le 2(x_2)^2$

Hence, $\frac{1}{2}(x_1)^2 + (x_2)^2 \le V_2(x) \le \frac{1}{2}(x_1)^2 + 2(x_2)^2$

$\therefore$ $V_2(x)$ is pos. def. and radially unbounded.

3. $$V_3(x) = [x_1, x_2]\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{P}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To check is P is pos def, we check $1>0$ and

$(1)(1)-(2)(2) = 1-4=-3$ $\Rightarrow$ NOT POS. DEF.

5c

To see that $V_3(x)$ is NOT radially unbounded, set $x_2 = -x_1$. Then

$$V(x_1, -x_1) = 2(x_1)^2 - 4(x_1)^2 \xrightarrow[|x_1| \to \infty]{} -\infty$$

$\Rightarrow$ NOT Radially Unbounded.

4.

$$\bar{W}_4(x) = \inf_{t \geq 0} V_4(t,x) = (x_1)^4$$

because $\lim_{t \to \infty} \frac{1+t}{1+t+t^2} = 0$

Because $\bar{W}_4(x_1,x_2)$ is not loc. pos. def, neither is $V_4(t,x)$. This also shows not radially unbounded.

To check decrescent, we evaluate if

$$\sup_{\|x\| \leq \rho} \sup_{t \geq 0} V_4(t,x) < \infty.$$

$\frac{1+t}{1+t+t^2}$ is montonically decreasing,

as can be checked by computing

$$\frac{d}{dt}\frac{1+t}{1+t+t^2} = \frac{-2t-t^2}{(1+t+t^2)^2} < 0 \quad \text{for all } t > 0.$$

Hence, $\sup_{t\geq 0} V_4(t,x) = V_4(0,x) = (x_1)^4 + (x_2)^2$,

and thus $\sup_{\|x\|\leq\rho} \sup_{t\geq 0} V_4(t,x) = \sup_{\|x\|\leq\rho}\left[(x_1)^4 + (x_2)^2\right] \leq \rho^4 + \rho^2$

$< \infty \Rightarrow$ decresent.

5. Because $\frac{1}{2} \leq \frac{1+t}{1+2t} \leq 1$,

$$(x_1-x_2)^2 + \frac{1}{2}(x_2)^4 \leq V_5(t,x) \leq (x_1-x_2)^2 + (x_2)^4$$

From these bounds, the rest follows easily because $(x_1-x_2)^2 + \frac{1}{2}(x_2)^4$ is pos. for all $x \neq 0$, and $\lim_{\|x\|\to\infty}\left[(x_1-x_2)^2 + \frac{1}{2}(x_2)^4\right] = \infty$. $\square$

End of Short Answer Section of the Exam

and Beginning of

Partial Credit Section of the Exam

For the next problems, partial credit is awarded. You MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know!",

because then, to be fair, I must do so for everyone! Hence, please do not come up and ask me. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture or the book, you do not have to re-derive it. If you do re-derive it, that will be a sign that you did not know what we actually covered in class, and I may mark off for that.

8. (20 points) Show your work. A right answer is only worth something if supported by adequate reasoning. Give an estimate of the region of attraction of the origin, $\mathcal{R}_A(0)$, for the following system using the proposed Lyapunov function. Sketch your estimate of $\mathcal{R}_A(0)$. Your score will be related to the soundness and clarity of your reasoning as well as to the size of your estimated region.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -3 \\ 1 & -\frac{1}{4} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + R(x) \tag{1}$$

(a) **20 point version** $R(x) = \begin{bmatrix} -(x_1)^3 \\ \frac{1}{2}(x_1)^4 x_2 \end{bmatrix}$

$V(x) = x^T P x$,

where, $A^T P + PA = -Q$, and

$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$Q > 0$ because
$2 > 0$ and $2 \cdot 1 - 1 \cdot 1 = 1 > 0$.

(b) **15 point version** $R(x) = \begin{bmatrix} 0 \\ \frac{x_1 x_2}{0.4} \end{bmatrix}$

$V(x) = x^T P x$,

where, $A^T P + PA = -Q$,

$P = \begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 3.2 \end{bmatrix}$ $\quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$[V, D] = \text{eig(P)}$ yields $V = \begin{bmatrix} -0.9860 & 0.1666 \\ 0.1666 & 0.9860 \end{bmatrix}$,

$D = \begin{bmatrix} 0.8324 & 0 \\ 0 & 3.2676 \end{bmatrix}$

To aid me in grading, and to structure your thinking, please record the following answers here (put all supporting computations on the following pages):

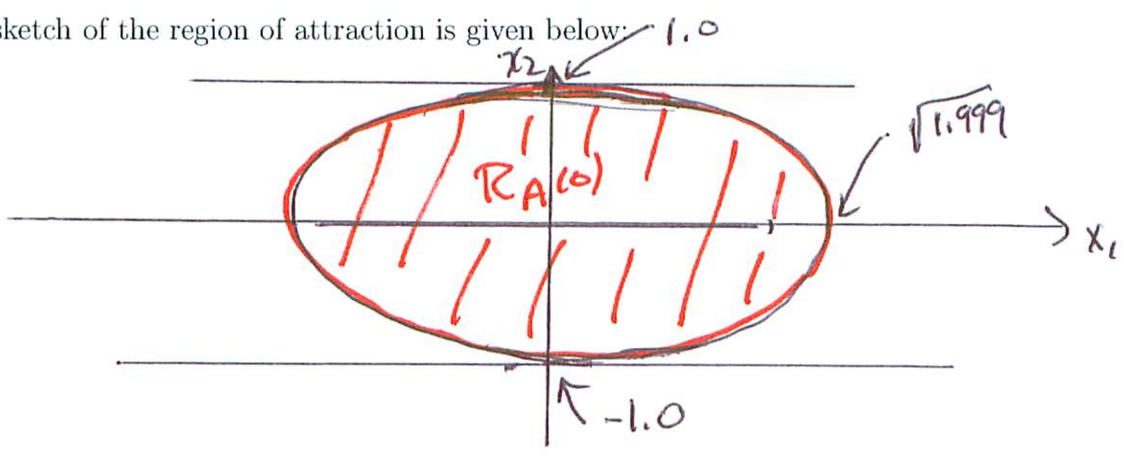
(a) I am working version **(a)** or **(b)** (circle one); WORK ONLY ONE version.

(b) $\dot{V}(x) = -x^T Q x - 2(x_1)^4 (1 - (x_2)^2)$

(c) A domain $\mathcal{D}$ where $\dot{V}$ is negative definite is $\mathcal{D} = \{ x \in \mathbb{R}^2 \mid |x_2| < 1 \}$

(d) My estimate of the region of attraction is $\mathcal{R}_A(0) = \{ x \in \mathbb{R}^2 \mid (x_1)^2 + 2(x_2)^2 \leq 1.999 \}$

(e) My sketch of the region of attraction is given below:



Extra Page for Problem 8: Do NOT FORGET to record your answers as requested.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -(x_1)^3 \\ \frac{1}{2}(x_1)^4 x_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{x_1 x_2}{0.4} \end{bmatrix}$$

Start work here that supports the answers given on the previous page:

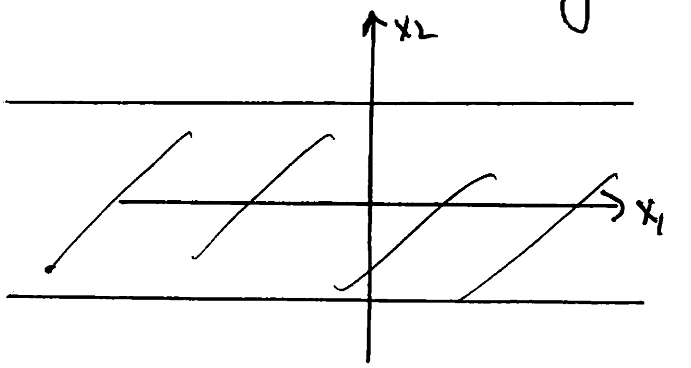
(a) $\mathring{V}(x) = -x^T Q x + 2R^T(x) P x$

$$= -x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x + 2\left[-(x_1)^3 \quad \tfrac{1}{2}(x_1)^4 x_2\right] \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$$

$$= -x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x + 2\left(-(x_1)^4 + (x_1)^4 (x_2)^2\right)$$

$$= -x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x - 2(x_1)^4 \left(1 - (x_2)^2\right)$$

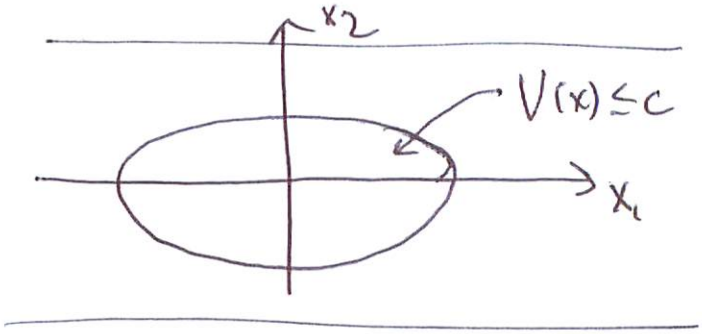
$\therefore \mathring{V}$ is neg. def on $\mathcal{D} = \{(x_1, x_2) \mid |x_2| < 1\}$



To find an estimate of $R_A(0)$, we seek the largest sublevel set of $V(x) = (x_1)^2 + 2(x_2)^2$ contained in $\mathcal{D}$.

Extra Page for Problem 8: Do NOT FORGET to record your answers as requested.

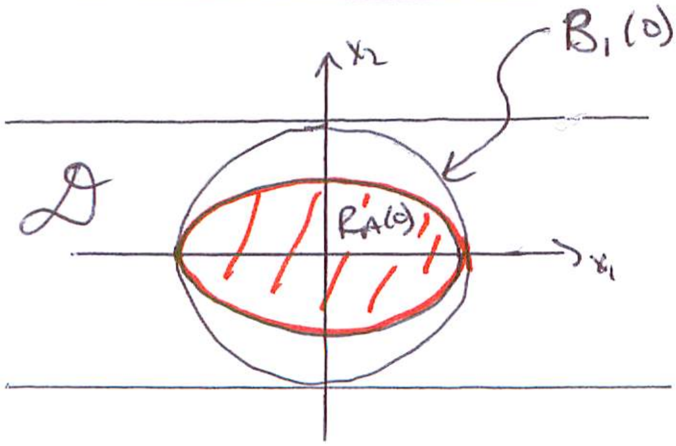
The e-vector of $\lambda_{min}=1$ is $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and hence, the sublevel sets look like this



$\therefore$ $R_A(0) = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$, $0 < c < 2$

works.

Alternative Estimate (fewer points)



Fit circle of radius 1 in $\mathcal{D}$. Fit ellipse in circle

$r^2 \lambda_{min} = c^*$

$\Rightarrow c^* = 1$

$R_A(0) = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$, $0 < c < 1$.

6

8. (20 points) Show your work. A right answer is only worth something if supported by adequate reasoning. Give an estimate of the region of attraction of the origin, $\mathcal{R}_A(0)$, for the following system using the proposed Lyapunov function. Sketch your estimate of $\mathcal{R}_A(0)$. Your score will be related to the soundness and clarity of your reasoning as well as to the size of your estimated region.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -3 \\ 1 & -\frac{1}{4} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + R(x) \qquad (1)$$

(a) **20 point version** $R(x) = \begin{bmatrix} -(x_1)^3 \\ \frac{1}{2}(x_1)^4 x_2 \end{bmatrix}$

$V(x) = x^T P x$,

where, $A^T P + PA = -Q$, and

$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(b) **15 point version** $R(x) = \begin{bmatrix} 0 \\ \frac{x_1 x_2}{0.4} \end{bmatrix}$

$V(x) = x^T P x$,

where, $A^T P + PA = -Q$,

$P = \begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 3.2 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$[V, D] = \text{eig(P)}$ yields $V = \begin{bmatrix} -0.9860 & 0.1666 \\ 0.1666 & 0.9860 \end{bmatrix}$,

$D = \begin{bmatrix} 0.8324 & 0 \\ 0 & 3.2676 \end{bmatrix}$

To aid me in grading, and to structure your thinking, please record the following answers here (put all supporting computations on the following pages):

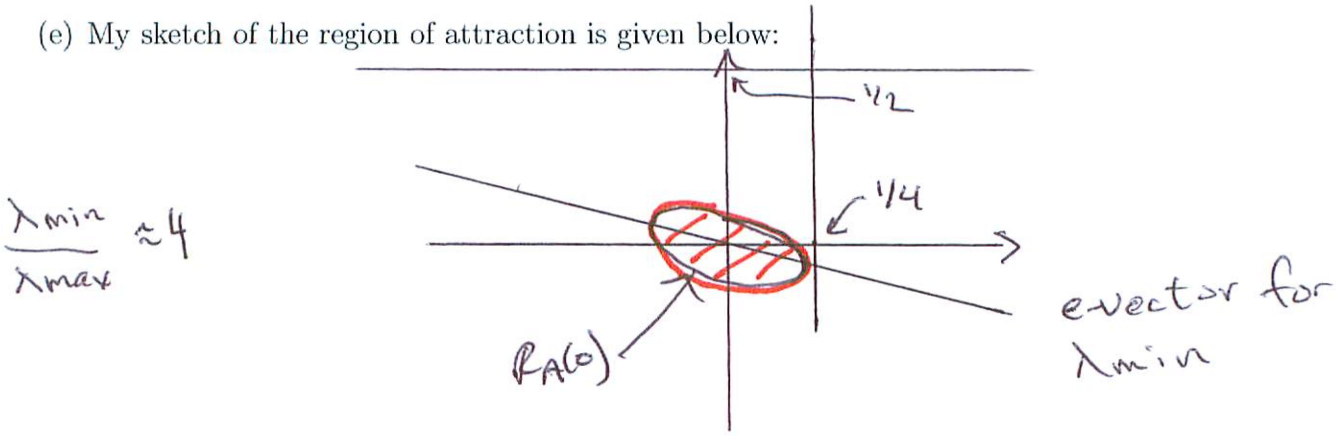
(a) I am working version (a) or (b) (circle one); WORK ONLY ONE version.

(b) $\dot{V}(x) = -(x_1)^2 - 4(x_2)^2 + 2(x_1)^2 x_2 + 16 x_1 (x_2)^2$
$= -(x_1)^2[1 - 2x_2] - 4(x_2)^2[1 - 4x_1]$

(c) A domain $\mathcal{D}$ where $\dot{V}$ is negative definite is $\mathcal{D} = \{x \in \mathbb{R}^2 \mid x_2 < 1/2,\ x_1 < 1/4\}$

(d) My estimate of the region of attraction is $\mathcal{R}_A(0) = \{x \in \mathbb{R}^2 \mid V(x) \le 0.05\}$

(e) My sketch of the region of attraction is given below:



PLACE NAME OR INITIALS HERE:____________________

8b

Extra Page for Problem 8: Do NOT FORGET to record your answers as requested.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -(x_1)^3 \\ \frac{1}{2}(x_1)^4 x_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{x_1 x_2}{0.4} \end{bmatrix}$$

Start work here that supports the answers given on the previous page:

$$\dot{V}(x) = -x^T Q x + 2x^T P R(x)$$

$$= -(x_1)^2 - 4(x_2)^2 + 2[x_1 \;\; x_2]\begin{bmatrix} x_1 x_2 \\ 8 x_1 x_2 \end{bmatrix}$$

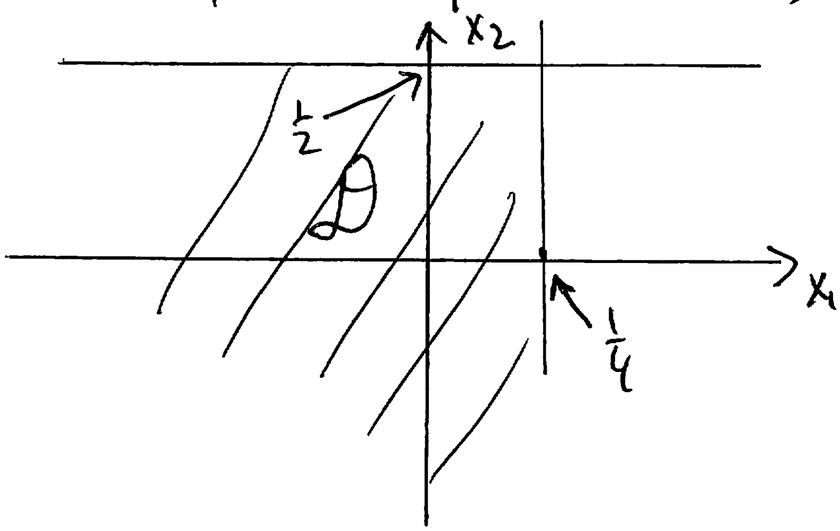
$$= -(x_1)^2 - 4(x_2)^2 + 2(x_1)^2 x_2 + 16 x_1 (x_2)^2$$

$$= -(x_1)^2[1 - 2x_2] - 4(x_2)^2[1 - 4x_1]$$

$\therefore$ Can select $\mathcal{D}$ as

$$\mathcal{D} = \{ x \in \mathbb{R}^2 \mid 1 - 2x_2 > 0 , \; 1 - 4x_1 > 0 \}$$

$$= \{ x \in \mathbb{R}^2 \mid x_2 < 1/2 , \; x_1 < 1/4 \}$$



PLACE NAME OR INITIALS HERE:________________________ 9b

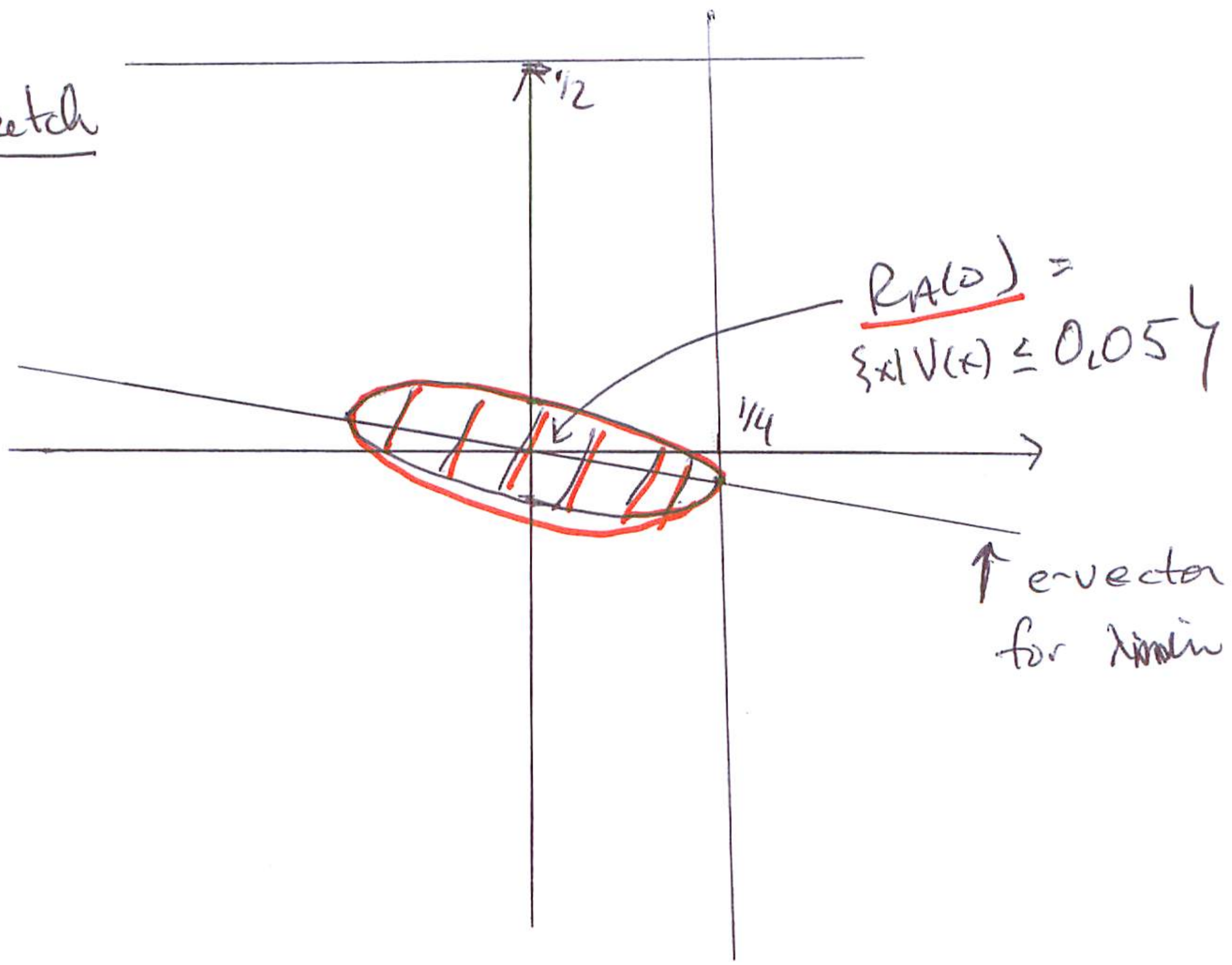
Extra Page for Problem 8: Do NOT FORGET to record your answers as requested.

Estimate of $R_A(0)$: Consider a circle of radius $r = \frac{1}{4}$. We seek the largest ellipse in the circle and obtain

$$R_A(0) = \{ x \in \mathbb{R}^2 \mid V(x) \le c \}, \quad 0 < c < c^*$$

where $c^* = \lambda_{min}(r)^2 = 0.83 \cdot \left(\frac{1}{4}\right)^2 \approx 0.052$

Sketch



9. (20 points) Show your work. A right answer is only worth something if supported by adequate reasoning. Make the strongest statement possible about the stability properties of the origin for the following system, using the proposed Lyapunov function. Of course, it is possible that nothing can be said on the basis of the given Lyapunov function; this is a valid response. In all cases, your conclusions must be backed up by solid reasoning and/or calculations. Since the exam is open-book, quote the number of the theorem you are applying; if you are using a theorem from the lectures, state it as best as you can. Solid reasoning includes verifying the hypotheses of the theorems you use! If you invent your own Lyapunov function, I'll be impressed, but will still give you zero points.

$$\begin{aligned}
\dot{x}_1 &= -(x_1)^2 + x_2\cancel{x_2} \\
\dot{x}_2 &= 2(x_1)^3 \\
V(x) &= (x_1)^4 - (x_2)^2 \\
\dot{V}(x) &= -4(x_1)^5
\end{aligned}$$

To aid me in grading, and to structure your thinking, record the following answers here:

(a) The strongest statement I can make about the stability of $x_e = 0$ is: UNSTABLE

(b) I am relying on the following: if from the book, give Theorem, Lemma, Proposition, Corollary number(s) and page number(s); otherwise, state you are using class notes or a handout and summarize the result as best you can IN YOUR PROBLEM SOLUTION.

Handout on proving instability

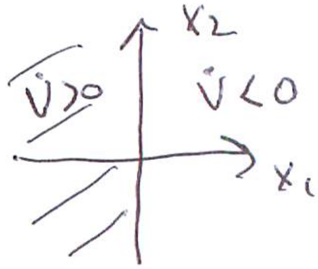
Start work here that supports the above answers:

Remarks: V is not loc. pos. def., so we cannot establish stability.

V takes on pos. values near the origin, so we may be able to prove instability.

$$\dot{V}(x) = 4(x_1)^3\,\dot{x}_1 - 2x_2\,\dot{x}_2$$

$$= -4(x_1)^5$$



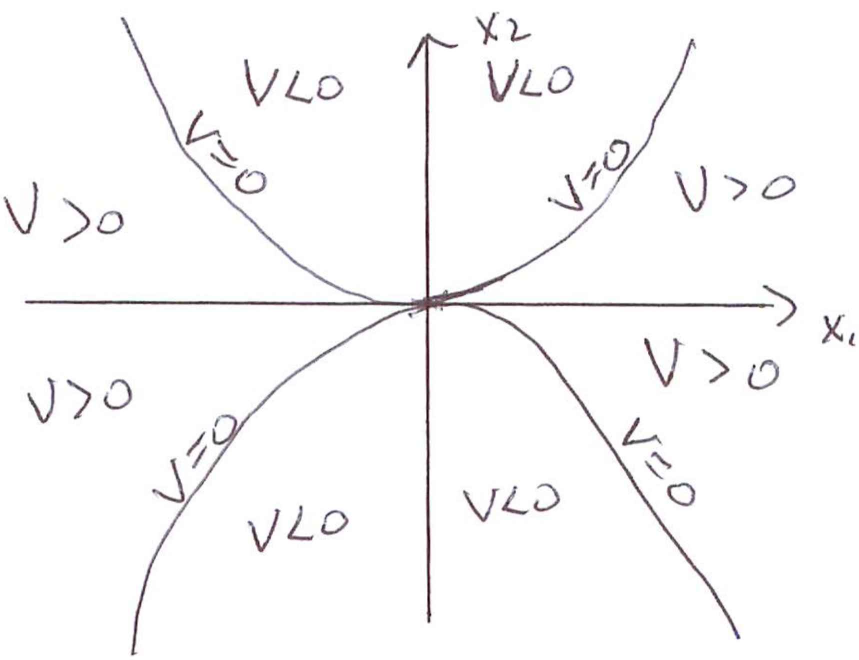
$$\therefore \{x \mid \dot{V}(x) > 0\} = \{x \mid x_1 < 0\}$$

Extra Page for Problem 9: Do NOT FORGET to answer the questions on the previous page.

$$
\begin{aligned}
\dot{x}_1 &= -(x_1)^2 + x_2 u \\
\dot{x}_2 &= 2(x_1)^3 \\
V(x) &= (x_1)^4 - (x_2)^2 \\
\dot{V}(x) &= -4(x_1)^5
\end{aligned}
$$

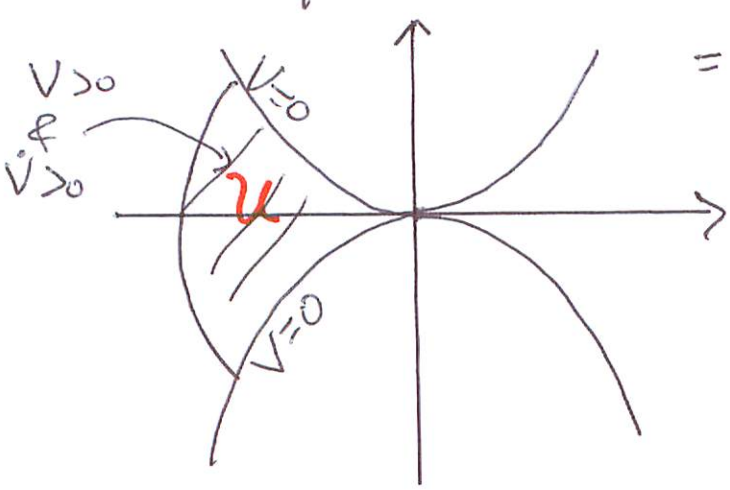
Next, we sketch where $V=0$, $V>0$, $V<0$:

$$\{x \in \mathbb{R}^2 \mid V(x) = 0\} = \{x \in \mathbb{R}^2 \mid |x_2| = (x_1)^2\}$$



Hence if we let $\mathcal{U} = \{x \in B_1(0) \mid V(x) > 0,\ x_1 < 0\}$

$= \{x \in B_1(0) \mid |x_2| < (x_1)^2,\ x_1 < 0\}$



Then, we have

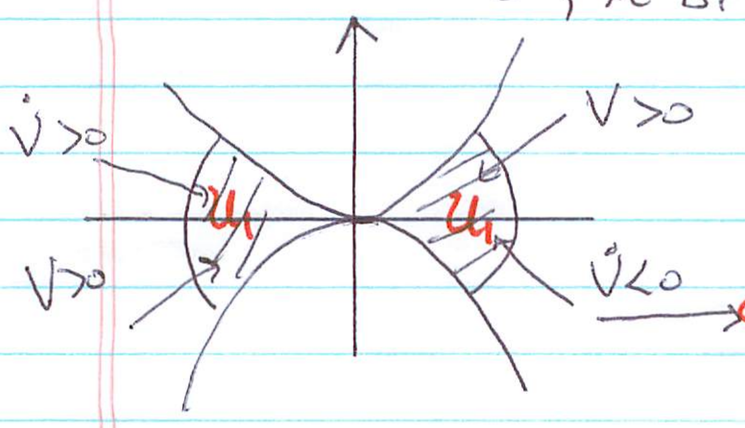
(a) $V(0) = 0$
(b) $0 \in \partial U$
(c) $\dot{V}(x) > 0 \quad \forall x \in U$
(d) $\forall \delta > 0 \ \exists \ x_0 \in U$ s.t. $\|x_0\| < \delta$ & $V(x_0) > 0$
(e) $\forall x \in \partial U, \|x\| < 1 \Rightarrow V(x) = 0$

$\therefore$ By the handout $x_e = 0$ is unstable

The following solution of "No Conclusion Can Be Made" is worth $\boxed{\text{15 points}}$ IF WORKED FULLY as SHOWN. Reasoning that is less complete will receive fewer points.

Check Thm 4.3, page 125: For $r>0$, we define $\mathcal{U}_1(r) = \{x \in B_r(0) \mid V(x) > 0\}$

$$= \{x \in B_r(0) \mid |x_2| < (x_1)^2\}$$



$\dot{V}(x)$ is NOT pos. def. on $\mathcal{U}_1(r)$ for any $r>0$

$\therefore$ Hence, the hypotheses of Thm 4.3 cannot be met.

Exercise 4.11, page 183: Cannot be applied either because $\dot{V}(x)$ is NOT locally pos. def.

Exercise 4.12, page 183: To apply this result, we have to be able to write

$$\dot{V}(x) = \lambda V(x) + W(x), \quad \lambda > 0$$

and $W(x)$ loc. pos. semi-def. To see if this is possible, let $\lambda > 0$ and note that $W(x)$ must satisfy

$$W(x) = \dot{V}(x) - \lambda V(x) = -4(x_4)^5 - \lambda\left[(x_1)^4 - (x_2)^2\right].$$

But $W(x_1, 0) = -4(x_1)^5 - \lambda(x_4)^4$, $\lambda > 0$

is negative for all $x_1 > 0$, and hence W is not. loc. pos. semi def. for any $\lambda > 0$.

$\boxed{\text{No conclusion possible using results in book.}}$