1)

$\|x\|_\infty = \max_i |x_i| \leq \sum_{i=1}^{n} |x_i| = \|x\|_1$

(as others are +ve)

$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} \max_i (x)$

$= n \|x_i\|_\infty$

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \leq \sqrt{\sum_{i=1}^{n} \max (x_i^2)}$

$\leq \sqrt{n \|x\|_\infty^2}$

$\leq \|x\|_\infty \sqrt{n}$

Similarly $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} > \sqrt{x_{max}^2}$

$\|x\|_2 \geq \|x\|_\infty$

$\|x\|_2^2 = \left(\sqrt{\sum_{i=1}^{n} x_i^2}\right)^2 \leq \max |x_i| \cdot \sum_{i=1}^{n} x_i$

$\|x\|_2^2 \leq \|x\|_\infty \|x\|_1$

as $\|x\|_\infty < \|x\|_2$

$\Rightarrow \|x\|_2 \leq \|x\|_1$

To show $\|x\|_1 \leq \sqrt{n}\,\|x\|_2$

we can use induction

if $\left(\sum_{i=1}^{n-1} |x_i|\right)^2 \leq (n-1) \sum_{i=1}^{n-1} x_i^2$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i|\right)^2 = \left(\sum_{i=1}^{n-1} |x_i|\right)^2 + 2|x_n| \left(\sum_{i=1}^{n-1}\right) + |x_n|^2$$

$$\leq (n-1) \sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} |x_n|^2 + |x_i|^2 + |x_n|^2$$

$$\leq n \sum_{i=1} x_i^2$$

$$\|x\|_1 \leq \sqrt{n}\;\|x\|_2$$

2) if f is a continuous function
&

if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.T $\|x - y\| < \delta$

such that $\| f(x) - f(y) \| \leq \epsilon$

as $\lim x = x_0$

$\forall\ \delta > 0\ \exists\ \forall\ n > N$ & $N < \infty$

$\| x_n - x_0 \| \leq \delta$

$\Rightarrow \forall\ n\ (\geq N)$ & $N < \infty$

$\| f(x_n) - f(x_0) \| \leq \epsilon$

$\lim_{n \to \infty} f(x_n) = f(x_0)$

3) a)

if $\lim_{n\to\infty} x_n = \bar{x}$

$\Rightarrow \forall \epsilon > 0 \; \exists N$ s.t $\forall n \geq N$

$\|\bar{x} - x_n\| \leq \epsilon$

$\Rightarrow$ as $x_n \in S$

$\Rightarrow \forall \epsilon > 0 \; \exists x_n \in S$

such that

$\|\bar{x} - x_n\| \leq \epsilon$

$\Rightarrow d(\bar{x}, S) = 0$

$\Rightarrow \bar{x} \in \bar{S}$ (closure of S)

for closed set $S = \bar{S}$ (given in question)

$\Rightarrow \boxed{\bar{x} \in S}$

4) a) $\dot{x} = -x^2$

The sol^n doesn't exist for all time

$$\frac{\dot{x}}{x^2} = -1$$

$$\frac{dx}{x^2} = -dt$$

$$\frac{1}{x_0} - \frac{1}{x} = t_0 - t$$

∃ sol^n can go to ∞ when values are equal

b) $\frac{dx}{x^3} = -dt$

$$\Rightarrow \left[-3x^{-2}\right]_{x_0}^{x} = \left[-t\right]_{t_0}^{t}$$

$$\Rightarrow 3x^{-2} - 3x_0^{-2} = (t - t_0)$$

$$x^2 = \frac{x_0^2}{3x_0^2(t - t_0) + 1}$$

denominator is always > 0

∴ sol^n always exist

c) $\dot{x} = x^2 \cos(x^2)$

if we consider interval
$$S = (x_{eq} - \delta \ , \ x_{eq} + \delta]$$
if $x_0 \in S$

then derivative at $x_1 = x_{eq} - \delta$
$$\dot{x}\Big|_{x_1} > 0.$$

& $\dot{x}\Big|_{x = x_{eq} + \delta} < 0$

$\Rightarrow$ It is contraction mapping &
Lipschitz continuous locally

so. Doesn't have finite time escape

4) d)

$||\dot{x}|| < 1$

$\therefore$ globally Lipschitz

So the sol^n exist at all time

5)

a) for contraction mapping in $C([0,\delta])$

$$\|f(y) - f(x)\| \leq L\delta \|x - y\|$$

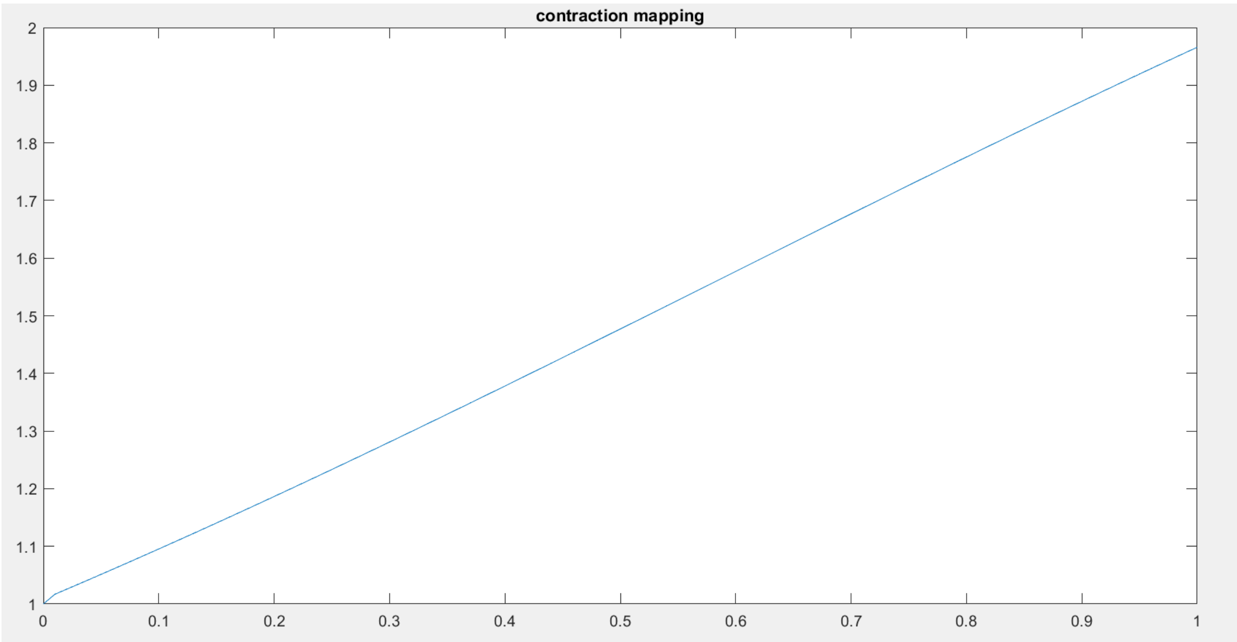
$L\delta \leq 1$ is sufficient condn

as $L = 1$ (globally Lipschitz)

$\Rightarrow \delta \leq 1$

$\therefore$ $\boxed{\delta = 1}$

Both 5b and 5c plots are almost identical.
Problem 5b:



Problem 5c: