Appendix: Mathematical background for stability.

Definition. A subset $S \subset \mathbb{R}^n$ is bounded if $\exists K < \infty$ such that $\forall x \in S$, $\|x\| \leq K$.

Equivalently, one can say that $S \subset \mathbb{R}^n$ is bounded if $\exists K < \infty$ such that $S \subset B_K(0)$.

Lemma. Suppose that $V : \mathbb{R}^n \to \mathbb{R}$. Is continuous. Then, $\forall c \in \mathbb{R}$, the set of points (i.e, the sublevel sets)

$$\mathcal{L}(c) = \{ x \in \mathbb{R}^n \mid V(x) \leq c \} \text{ is } \underline{\text{closed}}.$$

Note: → We used that Lemma last time when we were considering the sublevel sets of the function V we had defined.

Proof of the Lemma: To prove the lemma, we will use the sequence characterization of closed sets. Let (x_n) be a sequence of points in $\mathcal{L}(c)$, that is, $\forall n \geq 1$, $V(x_n) \leq c$, and such that $x_n \longrightarrow \bar{x}$, i.e. $\lim_{n\to\infty} x_n = \bar{x}$.

We want to show that $\bar{x} \in \mathcal{L}(c)$, [i.e, that means $V(\bar{x}) \leq c$]

To prove that, we first note that since V is continuous, we have that $V(\bar{x}) = \lim_{n\to\infty} V(x_n)$. [Recall Problem 2 in your HW2.]

Moreover, by definition we have $\forall n \geq 1$, $V(x_n) \leq c$. That means, $V(x_n)$ is a convergent sequence in $(-\infty, c] \subset \mathbb{R}$, where by definition $(-\infty, c]$ is a closed set.

Now, we have that the limit point $V(\bar{x})$ of $V(x_n)$ should belong into $(-\infty, c]$, because if a set S is closed, then it contains its limit points

[Recall Problem 3 in your HW2]

Hence we proved that $\bar{x} \in \mathcal{L}(c)$, i.e., $V(\bar{x}) \leq c$.

Definition. We often denote $\mathbb{R}_+ = [0, +\infty)$

A function $V: \mathbb{R}^n \to \mathbb{R}_+$ is radially unbounded. if $\forall c < \infty$, $\exists K < \infty$ such that $\|x\| > K \Rightarrow V(x) > c$.

Notation $\to$ This definition essentially tells us that:

$$V(x) \to \infty \text{ as } \|x\| \to \infty.$$

Lemma Suppose that $V: \mathbb{R}^n \to \mathbb{R}$ is

(i) continuous

(ii) $V(x) \to \infty$, as $\|x\| \to \infty$.

Then, $\forall c \geq 0$, $\mathcal{L}(c) = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ are closed and bounded.

Proof We proved that $\mathcal{L}(c)$ is closed earlier. It suffices to show that $\mathcal{L}(c)$ is bounded.

Fix $c \geq 0$. We will show that

if $\mathcal{L}(c)$ is <u>NOT</u> bounded, then V is <u>NOT</u> radially unbounded.

If $\mathcal{L}(c)$ is <u>NOT</u> bounded, then $\forall\, k < \infty$, $\exists\, x \in \mathcal{L}(c)$ with $\|x\| > k$. (note we negated the property of bounded set)

That means, $\forall\, k < \infty$, $\exists\, x$ such that $\underbrace{V(x) \leq c}_{\text{since } x \in \mathcal{L}(c)}$ and at the same time $\|x\| \geq k$,

which implies that V is <u>NOT</u> radially unbounded.

This completes the proof.

<u>Definition</u>. A subset $S \subset \mathbb{R}^n$ is compact if it's closed and bounded.

<u>Note</u>: This definition is not valid for infinite-dimensional normed spaces!

<u>Definition</u>. [Weierstrass Theorem.] Suppose that $f: S \to \mathbb{R}$ is continuous and that S is compact. Then f achieves a minimum and a maximum on S; that is, $\exists\, s^* \in S$ and $s_* \in S$ such that

$$f(s^*) = \sup_{x \in S} f(x) = \max_{x \in S} f(x), \qquad f(s_*) = \inf_{x \in S} f(x) = \min_{x \in S} f(x).$$

Non-Examples.

1) $S = [0, +\infty) \subset \mathbb{R}$ is closed, but is <u>not</u> bounded.

Consider the function $f(x) = \frac{1}{1+x^2} > 0 \quad \forall\, x \in S$.

We have $\inf_{x \in S} f(x) = 0$. That means, $\nexists\, s_* \in S$ such that

$$f(s_*) = 0,$$

since $f(x) > 0 \quad \forall\, x \in S$.

In this case, the minimum is not achieved.

Note also that in this case there happens to be a maximum since $\sup_{x \in S} f(x) = 1 = f(0)$, though it is <u>not</u> guaranteed to exist by the theorem.

2) $S = (0,1) \subset \mathbb{R}$ is bounded but it is <u>not</u> closed.

Consider the function $f(x) = x$

We have $\inf_{x \in S} f(x) = 0$ BUT $\nexists\, x_* \in S$ such that $f(x_*) = 0$.

(minimum is not achieved)

$\sup_{x \in S} f(x) = 1$ BUT $\nexists\, x^* \in S$ such that $f(x^*) = 1$

(maximum is not achieved)

Definition. $h: S \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous if $\forall \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that $\forall\, x, y \in S$,

$$\|x-y\| < \delta \Rightarrow \|h(x) - h(y)\| < \epsilon.$$

Key Point: The same δ must work for all x, y! Of course, as ϵ gets smaller, you may require δ to get smaller too, so δ does depend on ϵ.

Non-example. $h(x) = e^x$. It is continuous at every $x \in \mathbb{R}$, but not uniformly continuous.

Proof: We want to show that $h(x) = e^x$ is not uniformly continuous. We start by negating the property of uniform continuity.

$\exists\, \epsilon > 0$ such that $\forall\, \delta(\epsilon) > 0$, $\exists\, x, y \in \mathbb{R}$ such that

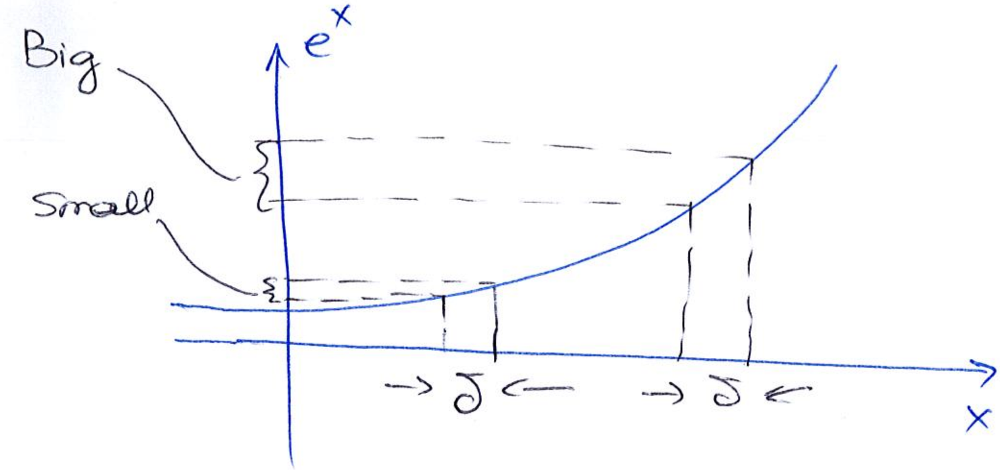
$$|x-y| < \delta \text{ but } |e^x - e^y| \geq \epsilon.$$

Let $\epsilon = 1$, and $\delta > 0$ arbitrary but fixed.

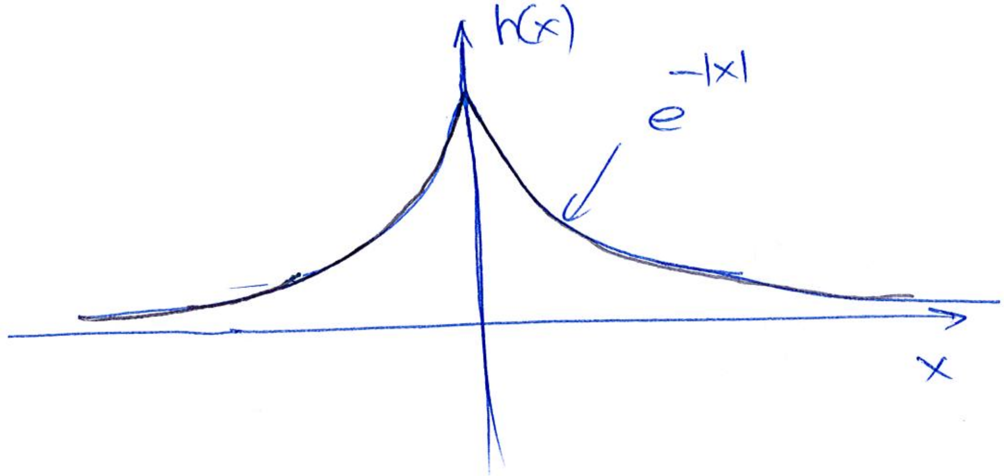
Let $x = -\ln\left(e^{0,9\delta} - 1\right)$ and $y = x + 0,9\delta$. Then,

$|x-y| = 0,9\delta < \delta$ and

$$|e^x - e^y| = |e^x - e^x e^{0,9\delta}| = e^x\left|1 - e^{0,9\delta}\right| = \frac{\left|1 - e^{0,9\delta}\right|}{\left|e^{0,9\delta} - 1\right|} = 1.$$



Example. $h: \mathbb{R} \to \mathbb{R}$ by $x \to e^{-|x|}$ (i.e., $h(x) = e^{-|x|}$)
is uniformly continuous.



Theorem If $h: S \to \mathbb{R}^m$ is continuous and S is compact,
then h is uniformly continuous.

[continuity + compactness] $\Rightarrow$ uniform continuity.

Non-Example: Consider $S = (0, 1]$ and $h(x) = \frac{1}{x}$, then
$h(x)$ is not uniformly continuous over S

Example: Consider $S = [10^{-3}, 1]$ and $h(x) = \frac{1}{x}$, then $h(x)$
is uniformly continuous over S.

Definition. A function $h: \mathbb{R} \to \mathbb{R}$ is bounded from below if $\exists\, m > -\infty$ such that $h(x) \geq m, \ \forall x \in \mathbb{R}$.

Definition A function $h: \mathbb{R} \to \mathbb{R}$ is non-increasing if

$$y \geq x \Rightarrow h(y) \leq h(x).$$

Remark. In a similar manner we can define bounded from above, and non-decreasing.

Examples. Note that ~~to~~ be non-increasing, a function can be constant for a while, then decreasing for a while, and so on. It does not have to be strictly decreasing (. i.e, $x < y \Rightarrow h(x) > h(y)$) in order to be non-increasing. Neither does it have to be continuous

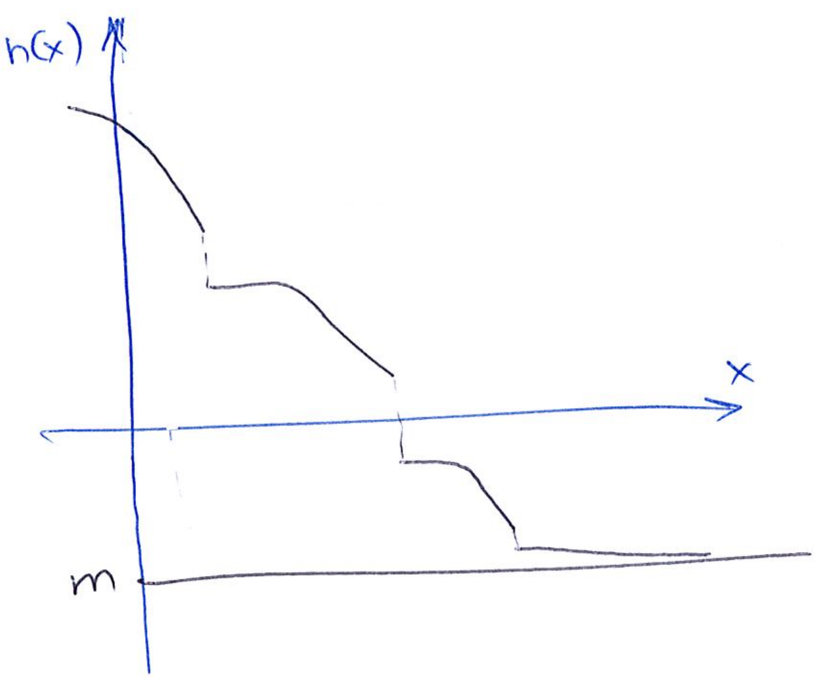


Fig ① Example of bounded from below and non-increasing.

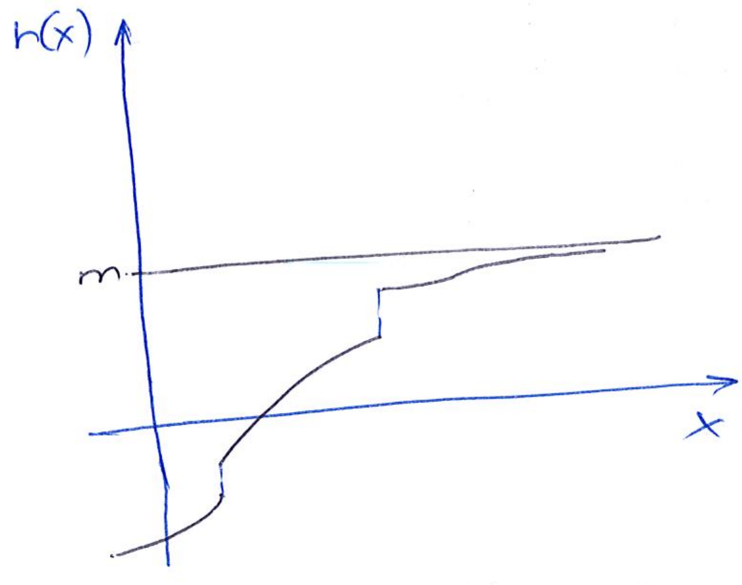


Fig ② Example of bounded from above and non-decreasing.

Theorem If $h: \mathbb{R} \to \mathbb{R}$ is non-increasing and bounded from below then there exists a unique $c \in \mathbb{R}$ such that $\lim_{x \to \infty} h(x) = c$.

Similarly, if h is non-decreasing and bounded from above, then there exists a unique $c \in \mathbb{R}$ such that $\lim_{x \to \infty} h(x) = c$.

Definition. $\lim_{x \to \infty} h(x) = c$ if $\forall \epsilon > 0$, $\exists K < \infty$ such that

$$x \geq K \Rightarrow |h(x) - c| < \epsilon.$$