Nonlinear Models and Nonlinear Phenomena

In this course, we deal with dynamical systems modeled as a finite number of coupled differential equations

$$\left.\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_p) \\ &\vdots \\ \dot{x}_n &= f_n(t, \underbrace{x_1, x_2, \ldots, x_n}_{\text{state variables}}, \underbrace{u_1, \ldots, u_p}_{\text{input variables}}) \end{aligned}\right\} \text{ system equations}$$

We will often use the vector notation to write the system equations in compact form:

$$\boxed{\dot{x} = f(t, x, u)} \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix},$$

$$f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix}$$

This is often called the state equation, and $x \in \mathbb{R}^n$ is called the state of the system.

We will also often consider the output equation

$$\boxed{y = h(t, x, u)} \quad \text{where} \quad y \in \mathbb{R}^q, \; h: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$$
$$x \in \mathbb{R}^n, \; u \in \mathbb{R}^p$$

with the output vector $y \in \mathbb{R}^q$ comprising variables of particular interest in the analysis of the dynamical system. (e.g., variables that are physically measured, or variables that are required to behave in a specified manner)

A good part of the analysis we will do in this course, the state equation will not have an explicit presence of an input u., that is, we will study the unforced state equation $\boxed{\dot{x} = f(t,x)}$ This does not mean that the input u is necessarily zero. It can also mean that the input has already been specified as a function of time, $u = \gamma(t)$, or as a function of state, $u = \gamma(x)$, or both, $u = \gamma(t, x)$, and has already been substituted in the original state equation to yield a (closed-loop) unforced equation.

A special case of the unforced equation $\dot{x} = f(t,x)$ is when the function $f(\cdot,\cdot)$ does not depend explicitly on time, $\underline{\dot{x} = f(x)}$. The system is called autonomous or time-invariant.

Equilibrium of the State Equation. $\rightarrow$ One of the key concepts in the study of dynamical systems. A point x^* in the state space is an equilibrium point of $\dot{x} = f(t,x)$ if it has the property that whenever the state of the system starts at x^*, it will remain at x^* for all future times.

Example $\rightarrow$ For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real roots of the equation $f(x) = 0$.

Equilibrium points can be isolated, or they can be in a continuum.

Nonlinear vs Linear Systems.

We recall the linear time-varying system

$$\begin{cases} \dot{x} = A(t)\,x + B(t)\,u \\ y = C(t)\,x + D(t)\,u \end{cases}$$

and its solution concept via the State Transition Matrix.

$$x(t) = \Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} \underbrace{\Phi(t,\tau)}_{\text{State Transition Matrix}} B(\tau)u(\tau)\,d\tau$$

where the STM is the solution of the system:

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,\tau) = A(t)\,\Phi(t,\tau) \\ \Phi(\tau,\tau) = I \end{cases}$$

In fact, for Linear Time-Invariant (LTI) systems

$\boxed{\dot{x} = Ax, \quad x(t_0) \in \mathbb{R}^n,}$ where $A \in \mathbb{R}^{n\times n}$ constant,

the S.T.M. reads $\Phi(t,\tau) = e^{A(t-t_0)}$, and as thus the solution of the system is obtained as:

$$\boxed{x(t) = e^{A(t-t_0)}\,x(t_0)}$$

We have the following facts regarding this solution.

① It is defined (i.e., it exists) for all $-\infty < t < +\infty$

② It is unique

③ Set of equilibrium points is in the nullspace $\mathcal{N}(A)$ of the matrix A, which is a subspace. Hence, equilibrium points are connected, and not isolated.

④ If all trajectories converge to a given bounded set, then they all converge to the origin.

⑤ If all trajectories are bounded, then they converge to a periodic solution.

⑥ Solutions can be written down in closed-form:

$$x(t) = e^{A(t-t_0)} x(t_0), \quad t \geq t_0.$$

We also recall the superposition principle being the fundamental property of linear systems.

Now, as we move to the nonlinear regime, things become more challenging. First, the superposition principle does no longer hold. Second, the analysis requires more advanced mathematics and tools tailored specifically to nonlinear systems. The reason is twofold.

① While we can in principle linearize the nonlinear model around an operating point, and use linear systems tools to analyze the behavior of the linearized system, this method alone is not sufficient: Linearization provides an approximation in the neighborhood of the operating point, and as thus it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It can not predict the "nonlocal" behavior far away from that operating point, and certainly not the "global" behavior throughout the state space.

② The dynamics of nonlinear systems are much richer than the dynamics of a linear system. There are what is called "nonlinear phenomena" that can not be described or predicted with linear models (hence they can not be analyzed and treated with linear systems tools). Below we give some examples.

Examples of Nonlinear Phenomena.

① Finite Escape Time

Recall that the state of an unstable linear system goes to infinity as time approaches infinity. A nonlinear system's state, however, can go to infinity in finite time.

Example: Consider $\dot{x} = 1 + x^2$, where $x \in \mathbb{R}$.

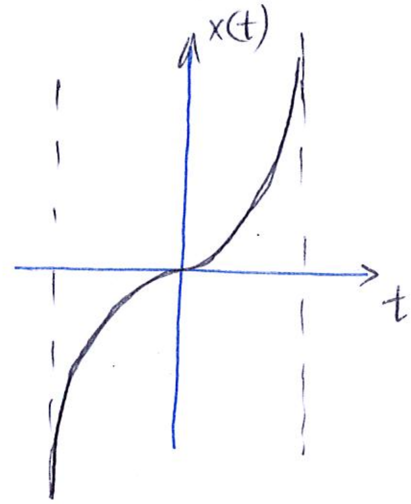
Take $x_0 = 0$, $t_0 = 0$.

The solution is given as

$$\frac{dx}{dt} = 1 + x^2 \Rightarrow \frac{dx}{1+x^2} = dt \Rightarrow \int_{x_0}^{x(t)} \frac{dx}{1+x^2} = \int_{t_0}^{t} d\tau$$

$$\Rightarrow \arctan(x(t)) - \underbrace{\arctan(x_0)}_{=0} = (t - \underbrace{t_0}_{=0}) \Rightarrow$$

$$\Rightarrow x(t) = \tan(t)$$



The solution exists only on a bounded interval of time!

The phenomenon can occur for systems with $n \geq 1$.

② Non-uniqueness of solutions.

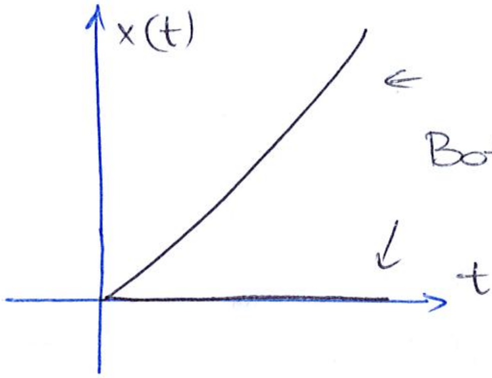
Consider for instance $\dot{x} = x^{2/3}$, $x_0 = 0$, $x \in \mathbb{R}$.

Then you can verify that

$$\begin{cases} x(t) = 0, & t \geq 0 \\ \text{and} & \\ x(t) = \left(\frac{t}{3}\right)^3, & t \geq 0 \end{cases}$$

are both solutions of the system!

To verify that, plug the solution into the system equation, and you will see that the left hand side equals the right hand side, and the initial condition is satisfied.



Both are solutions!

The phenomenon can occur for systems with $n \geq 1$.

③ Multiple isolated Equilibria.

A linear system can only have <u>one</u> <u>isolated</u> equilibrium point; that is, it can only have only one steady-state operating point that attracts the state of the system irrespectively of the initial conditions.

A nonlinear system can have <u>more than one isolated</u> equilibria. The state may converge to one of several steady-state operating points depending on the initial state of the system.

Example: $\dot{x} = \sin x$, $x \in \mathbb{R}$. The equilibria are given as the solutions of the equation

$$\sin(x_e) = 0, \Rightarrow x_e = k\pi, \; k \in \mathbb{Z}.$$

Can occur for systems with $n \geq 1$

④ Limit Cycles.

A linear time-invariant system oscillates if it has a pair of eigenvalues on the imaginary axis. This is a non-robust condition to maintain in the presence of perturbations. In addition, the amplitude of the oscillation depends on the initial conditions. In real life, a stable oscillation can be produced by a nonlinear system. There are nonlinear systems that can go into an oscillation of fixed amplitude and frequency, irrespectively of the initial condition. This type of oscillation is called a <u>limit cycle</u>.

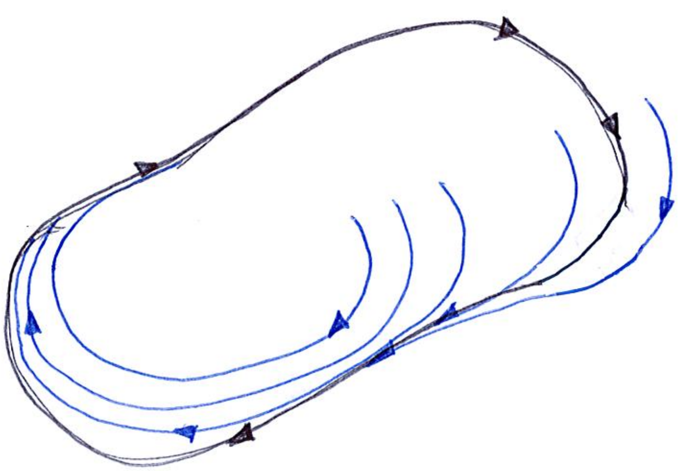
Example. $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0, \quad \mu > 0$

(Van der Pol's Oscillator)

Let $x_1 = x$, $x_2 = \dot{x}$, so that we can write the system's equation in state space form:

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu(x_1^2-1)x_2 - x_1 \end{aligned}}$$

The solutions of this system look like this:



That is, the solutions converge to a bounded set, but do not converge to the origin.

What they converge to is called a limit cycle. It is an asymptotically stable oscillator.

Can occur for systems with $n \geq 2$.

5. Strange attractor or chaos. A nonlinear system can have a more complicated steady-state behavior, that is not equilibrium, periodic or almost periodic solution. Such behavior is called chaos.

Example: Lorenz equations.
$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \beta x - y - xz \\ \dot{z} = -\beta x + xy \end{cases} \qquad \beta, \sigma > 0.$$

The solutions of this system are bounded, but they do not converge to anything. Graphical representations and animations are available on the web.

The phenomenon can occur for systems with $n \geq 3$.

(6) Solutions can not be written always in closed-form.

Example: Poincare's "Three-Body Problem".

Can occur for systems with $n \geq 1$

(7) Subharmonic, harmonic, or almost-periodic solutions

A stable linear system under a periodic input produces an output of the same frequency.

A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation (e.g., the sum of periodic oscillations that are not multiples of each other).

<u>Bottom line</u>: For a nonlinear system $\dot{x} = f(x)$, it does <u>not</u> automatically follow that solutions exist, are unique etc. And, when we can show that a solution exists, we will not in general be able to compute it in closed form!

Hence our analysis has to be indirect: We will deduce properties of the solution by analyzing the ODE model instead of analyzing the solution itself. In the first part of the course, we develop the mathematical tools that allow us to do this analysis.