Time-varying Systems (A Quick Introduction)

Let the non-autonomous (time-varying) system

$$\dot{x} = f(t,x) \qquad (*)$$

where $f: [0,\infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0,\infty) \times D$, and $D \subset \mathbb{R}^n$ be a domain that contains the origin $x=0$. The origin is an equilibrium point of the above system at time $t=0$ if $f(t,0)=0, \ \forall t \geq 0$.

- The solution of an autonomous system depends only on $(t-t_0)$
- The solution of a non-autonomous system depends on both t and t_0!
- Hence the stability of the equilibrium point will be, in general, dependent on t_0!

Overview of Stability Definitions

(Definition 4.4 in our textbook) The equilibrium point of $(*)$ is

- stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall t \geq t_0 \geq 0$.
- uniformly stable if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$, independent of t_0, such that
$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall t \geq t_0 \geq 0.$$
- unstable if it is not stable.
- asymptotically stable if it is stable and there exists a $c = c(t_0) > 0$ such that
$$x(t) \to 0 \text{ as } t \to \infty, \ \forall \|x(t_0)\| < c.$$

- **uniformly asymptotically stable** if it is uniformly stable and there exists $c>0$, independent of t_0, such that

$$x(t) \to 0 \quad \text{as } t\to\infty, \quad \forall \, \|x(t_0)\| < c, \quad \text{uniformly in } t_0$$

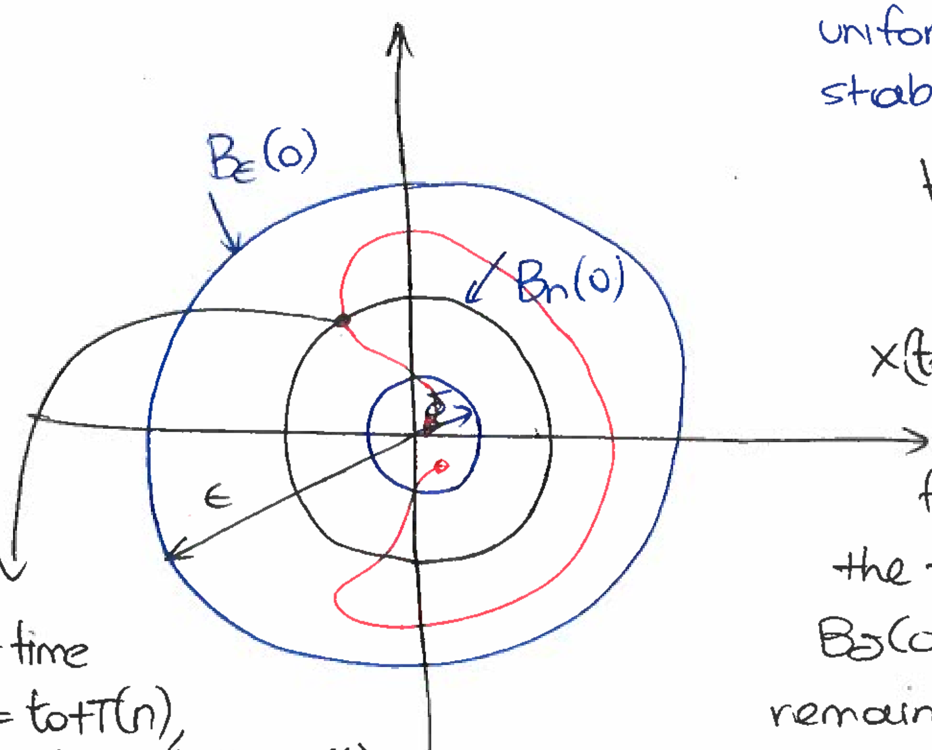
[Note that the above condition equivalently reads that $\forall\, \eta>0$, there is a $T = T(\eta) > 0$ such that

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta), \quad \forall\, \|x(t_0)\| < c \,]$$

- globally uniformly asymptotically stable if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$, and for each pair of $\eta>0$, $c>0$, there exists $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta, c), \quad \forall\, \|x(t_0)\| < c.$$

Pictorially the concept of a uniformly asymptotically stable equilibrium:



At time $t = t_0 + T(\eta)$, the trajectories $x(t)$ enter the ball $B_\eta(0)$.

$\forall\, \epsilon>0$ we can find $\delta = \delta(\epsilon)$ such that

$$x(t_0) \in B_\delta(0) \Rightarrow x(t) \in B_\epsilon(0)$$

AND

for all $t \geq t_0 + \boxed{T(\eta)}$ (independent of initial time) the trajectories starting $B_\delta(0) = B_c(0)$ enter and remain in $B_\eta(0)$, i.e. We must be able to find a $T(\eta)$ independent of t_0 after which the trajectories are confined in $B_\eta(0)$, no matter how small η is chosen.

Time-Varying Systems. (Continued)

Overview of Stability Theorems.

Lyapunov theory for autonomous systems can be extended to non-autonomous systems, i.e.

Compared to autonomous systems, one key difference is how to choose an appropriate Lyapunov candidate function.

We review the main stability theorems and provide some examples.

Theorem 4.8 Let $x=0$ be an equilibrium point of $\dot{x} = f(t,x)$, where $f: [0,\infty)\times D \to \mathbb{R}^n$ is piecewise continuous in t, and locally Lipschitz wrt x on $[0,\infty)\times D$, and $D \subset \mathbb{R}^n$ is a domain that contains $x=0$.

Let $V: [0,+\infty)\times D \to \mathbb{R}$ be a continuously differentiable function such that

(i) $W_1(x) \le V(t,x) \le W_2(x)$ → reads: $V(t,x)$ is decrescent.

reads: $V(t,x)$ is locally positive definite

(ii) $\dot{V}(t,x) = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} f(t,x) \le 0,$

for all $t \ge 0$, $x \in D$, where $W_1(x), W_2(x)$ are continuous positive definite functions on D.

Then, $x=0$ is uniformly stable.

Theorem 4.9 Suppose the assumptions of Theorem 4.8 are satisfied with (ii) strengthened to

$$\text{(ii)} \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq - W_3(x),$$

$\forall\ t \geq 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then

$x = 0$ is uniformly asymptotically stable.

Finally, if $D = \mathbb{R}^n$, and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable. ■

Remarks The key point is to make sure that the candidate Lyapunov function $V(t,x)$ is positive definite (on D) and decrescent.

Note: Positive definite (on D) is often called "locally" positive definite.

Let us recall the definitions of positive definitiness for a function $V(t,x) : [0, +\infty) \times D \to \mathbb{R}$.

Note the dependence of $V(t,x)$ on time t

Definition A function $V: [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is locally positive definite if $\exists\, r>0$ and a continuous, non-decreasing function $a: [0,r) \to \mathbb{R}$ such that.

① $V(t,0) = 0, \ \forall\, t \geq 0$.

② $a(0) = 0, \ a(p) > 0, \quad 0<p<r$.

③ $a(\|x\|) \leq V(t,x), \ \forall\, t \geq 0, \ \forall\, x \in B_r(0)$
(hence, "locally")

Note: A continuous function $a: [0,r) \to [0,+\infty)$ is said to belong to class K if it is strictly increasing and $a(0) = 0$.
It is said to belong to class K_∞ if $r = \infty$ and

$$a(w) \xrightarrow[w \to \infty]{} \infty$$

Remark: Hence, we can restate as.
A function $V: [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is locally positive definite if

ⓘ $V(t,0) = 0, \ \forall\, t \geq 0$, and

ⓘⓘ $\exists\, r>0$ and a class K function $a: [0,r) \to [0,+\infty)$ such that $a(\|x\|) \leq V(t,x)$, $\forall\, t \geq 0, \ \forall\, x \in B_r(0)$

Remark The following are equivalent.

(i) $V(t,x)$ is locally positive definite.

Equivalently, (ii) $\exists\, W(x)$ locally positive definite such that

$$W(x) \leq V(t,x), \quad \forall\, t \geq 0, \ \forall\, x \in B_r(0)$$

(iii) $\inf\limits_{t \geq 0} V(t,x) := \bar{W}(x)$ is locally positive definite.

Example: Recall the non-example function that we used in the previous lecture, to demonstrate that $V(t,x)$ should be chosen carefully.

The function was $V(t,x) = e^{-2t} x^2$

We note that $\inf\limits_{t \geq 0} \left(e^{-2t} x^2\right) \equiv 0$, which is not locally positive definite. Hence,

$V(t,x) = e^{-2t} x^2$ is NOT locally positive definite!

Definition. A continuous function $V : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is decrescent if $\exists\, \sigma > 0$ and a class K function $\gamma : [0,\sigma) \to [0,+\infty)$

such that: $V(t,x) \leq \gamma(\|x\|)$,

$$\forall\, t \geq 0, \ \forall\, x \in B_\sigma(0).$$

Facts (a) If $V(t,x) = V(x)$ independent of t, and $V(x)$ is locally positive definite, then $V(x)$ is automatically decrescent.

(b) $V(t,x)$ is decrescent if and only if $\exists\ \sigma > 0$ such that $\forall\ 0 < p < \sigma$

$$\sup_{\|x\| < p} \ \sup_{t \geq 0} V(t,x) < +\infty .$$

Example 4.20.

$$\dot{x}_1 = -x_1 - g(t)\, x_2$$
$$\dot{x}_2 = x_1 - x_2 \qquad ,$$

where $g(t)$ continuously differentiable such that $0 \leq g(t) \leq k$, $\dot{g}(t) \leq g(t)$, $\forall\ t \geq 0$.

Consider $V(t,x) = x_1^2 + (1 + g(t))\, x_2^2$

Then:

$$\underbrace{x_1^2 + x_2^2}_{W_1(x)} \leq \underbrace{x_1^2 + (1 + g(t))\, x_2^2}_{\text{since } g(t) \geq 0} , \text{ and}$$

$$x_1^2 + \underbrace{(1 + g(t))}_{\text{since } g(t) \leq k} x_2^2 \leq \underbrace{x_1^2 + (1 + k)\, x_2^2}_{W_2(x)} , \ \forall\ x \in \mathbb{R}^2$$

Hence $W_1(x) \leq V(t,x) \leq W_2(x)$, i.e, $V(t,x)$ is positive definite, decrescent, and radially unbounded (the latter since $W_1(x)$ is radially unbounded)

The time derivative of $V(t,x)$ along the system trajectories reads.

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - (2+2q(t) - \dot{q}(t))\, x_2^2$$

We have $\dot{q}(t) \le q(t) \Rightarrow$

$$2+2q(t) - \dot{q}(t) \ge 2+2q(t) - q(t) =$$

$$2+q(t) \ge 2$$

Then $\dot{V}(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 =$

$$-\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{-x^T Q x}_{-W_3(x)}$$

where

Q is positive definite.

Therefore, $\dot{V}(t,x)$ is negative definite.

Then all the assumptions of Theorem 4.9 are satisfied globally, with positive definite functions $W_1(x)$, $W_2(x)$, $W_3(x)$, i.e, the origin is globally uniformly asymptotically stable.

Time-Varying (Non Autonomous) Systems.

Let us now consider the time-varying system

$$(*) \qquad \dot{x} = f(t, x), \text{ where}$$

$f: [0, +\infty) \times D \to \mathbb{R}^n$ is

- piecewise continuous in t
- locally Lipschitz on $[0, +\infty) \times D$
- D is a domain that contains the origin $x = 0$.

Definition. The origin is an equilibrium point of $(*)$ at $t = 0$ if

$$f(t, 0) = 0, \quad \forall\, t \geq 0.$$

Remark. Suppose that $\bar{x}(t, t_0, x_0)$ is a solution of $(*)$ for $t \geq t_0$. Let $\boxed{z(t) = x(t) - \bar{x}(t, t_0, x_0)}$

Then:

$$\dot{z}(t) = f(t, x(t)) - f(t, \bar{x}(t, t_0, x_0)) =$$
$$= f(t, z(t) + \bar{x}(t, t_0, x_0)) - f(t, \bar{x}(t, t_0, x_0)) =$$
$$=: \bar{f}(t, z(t))$$

Then, we argue that $z(t) = 0$ is an equilibrium of

$$\dot{z}(t) = \bar{f}\left(t, z(t)\right) \quad \text{Indeed,}$$

$$\bar{f}(t, 0) = f\left(t, \bar{x}(t, t_0, x_0)\right) - f\left(t, \bar{x}(t, t_0, x_0)\right)$$
$$= 0, \quad \forall\, t \geq t_0.$$

→ This means that we can shift any solution to the origin

We can proceed by defining stability, asymptotic stability and so on. Note that compared to autonomous systems, there is a <u>key difference</u>.

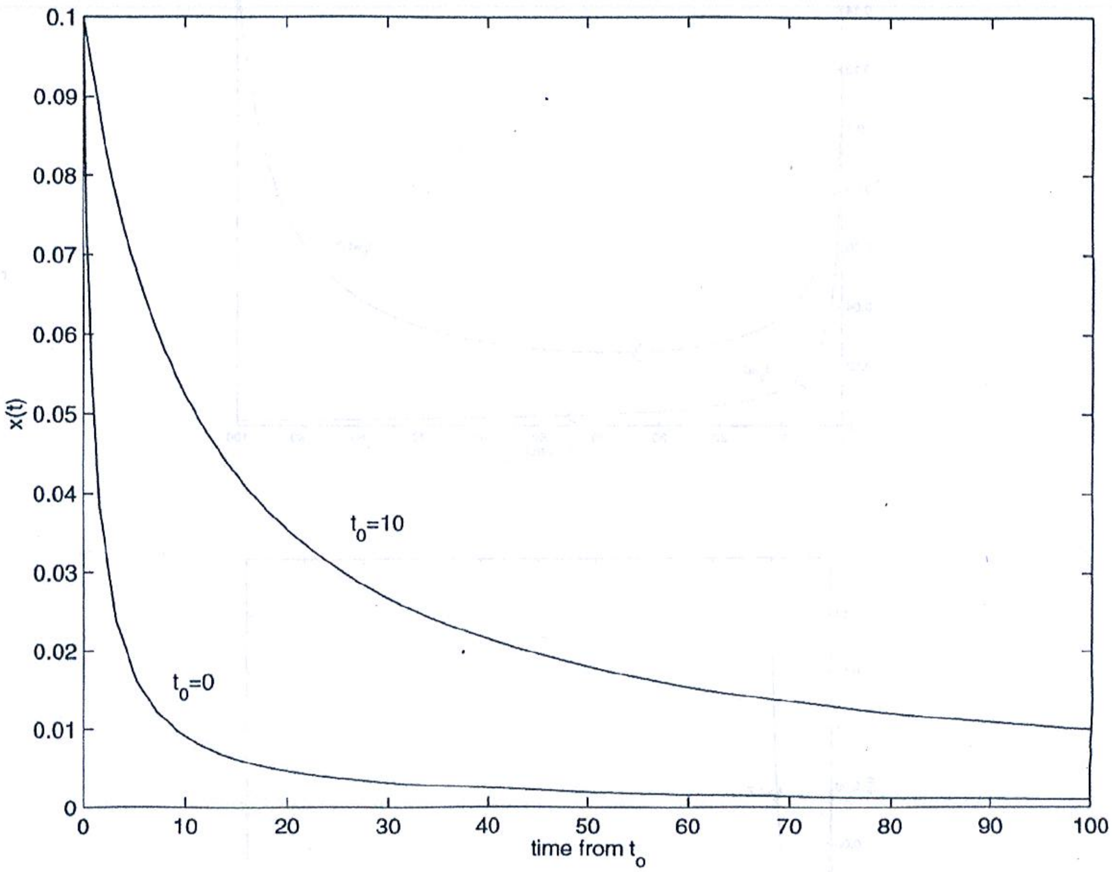
Fact: The solution of an autonomous system depends only on $(t - t_0)$

Fact: The solution of a non-autonomous system may depend on both t and t_0.

Consequence: Hence, the stability of the equilibrium point of a non-autonomous system will, in general, <u>be dependent on t_0</u>.

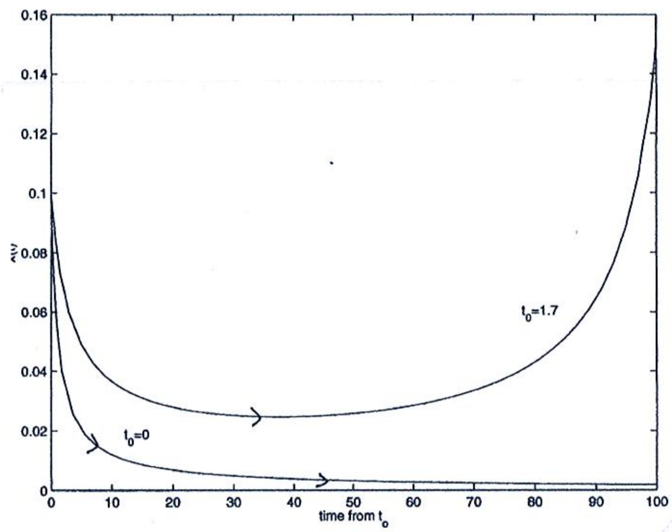
Why “uniform” is important

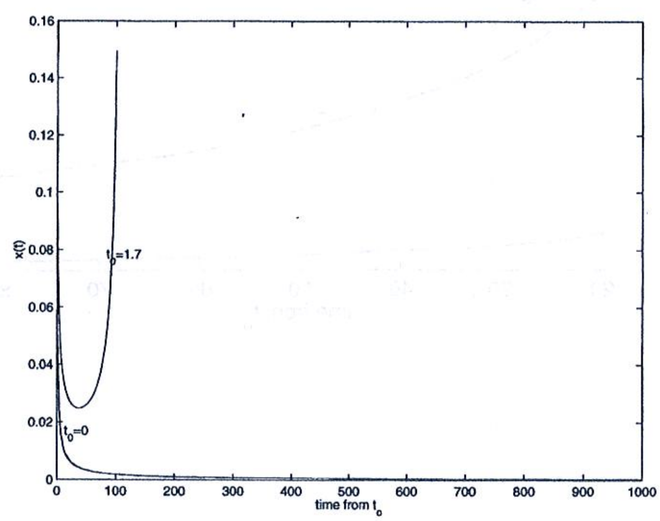
$\dot{x} = -\frac{1}{1+t}x$ is asymptotically stable but not uniformly asymptotically stable. Indeed, the "rate" that $\lim_{t \to \infty} x(t, t_0, x_0)$ goes to zero depends on t_0. {See Example 4.18}



Now,consider a NL system with linearization $\dot{x} = -\frac{1}{1+t}x$

$$\boxed{\dot{x} = -\frac{1}{1+t}x + x^2} \; ; \; \boxed{x_0 = 0.1}$$





We see that the region of attraction, R_A, gets <u>smaller</u> for <u>larger</u> t_0 ! This will NOT happen with a uniformly asymptotically stable equilibrium point.

Definitions. The equilibrium point $x_e = 0$ of (*) is

- Stable if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that:

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall\, t \geq t_0 \geq 0.$$

- Uniformly stable if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall\, t \geq t_0 \geq 0.$$

- Unstable, if not stable.

- Asymptotically stable if it is stable and there exists $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$.

- Uniformly asymptotically stable if it is uniformly stable and there exists $c > 0$, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there exists $T = T(\eta) > 0$ such that

$$\|x(t)\| < \eta, \ \forall\, t \geq t_0 + T(\eta), \ \forall\, \|x(t_0)\| < c.$$

- Globally uniformly asymptotically stable if it is uniformly stable and, for each pair of positive numbers η and c, there exists $T = T(\eta, c) > 0$, such that $\|x(t)\| < \eta, \ \forall\, t \geq t_0 + T(\eta, c)$,

$$\forall\, \|x(t_0)\| < c$$

- **Bounded** if $\forall x_0$, $\exists K(x_0, t_0)$ such that
$$\| x(t, t_0, x_0) \| < K, \quad \forall t \geq t_0.$$

- **Uniformly bounded** if $\forall x_0$, $\exists K(x_0)$, independent of t_0, such that $\| x(t, t_0, x_0) \| < K$, $\forall t \geq t_0$.

Why "uniform" is important.

Example. 4.17. Consider $\dot{x} = (6t \sin t - 2t)x$

The solution is

$$x(t) = x(t_0)\, e^{\int_{t_0}^{t} (6\tau \sin\tau - 2\tau)\, d\tau} \Rightarrow$$

$$x(t) = x(t_0) \exp\big(6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6 t_0 \cos t_0 + t_0^2\big)$$

Now we see that for any t_0, the term $-t^2$ will eventually dominate. This implies that the exponential term is bounded for all $t \geq t_0$ by some constant $c(t_0)$ that is dependent on t_0. Hence we have:

$$\boxed{|x(t)| < c(t_0)\, |x(t_0)|, \quad \forall t \geq t_0.}$$

Hence we also have that for any $\epsilon > 0$, the choice of $\delta = \dfrac{\epsilon}{c(t_0)}$ implies that $|x(t_0)| < \delta \Rightarrow$

$$\Rightarrow c(t_0)\,|x(t_0)| < \epsilon \Rightarrow |x(t)| < \epsilon,$$

i.e, that the origin is stable.

Let us now consider $t_0 = 0, 2\pi, 4\pi, 6\pi, \cdots$

and that $x(t)$ is evaluated π seconds later in each case.

In other words, $x(t_0+\pi) = x(t_0)\exp\left((4n+1)(6-\pi)\pi\right)$

We note that $\dfrac{x(t_0+\pi)}{x(t_0)} \to \infty$ as $n \to \infty$, [for $x(t_0) \neq 0$]

Hence, given $\epsilon > 0$, there is no δ independent of t_0, that would satisfy the stability requirement uniformly in t_0.

Example 4.18. Consider $\dot{x} = -\dfrac{x}{1+t}$

The solution is $x(t) = x(t_0)\exp\left(\int_{t_0}^{t} -\dfrac{1}{1+\tau}\,d\tau\right) =$

$$= x(t_0)\,\frac{1+t_0}{1+t}$$

We have that $|x(t)| \leq |x(t_0)|$, $\forall\ t \geq t_0$. Hence the origin is stable. In fact, given $\epsilon > 0$, we can choose $\delta > 0$ independent of t_0.

Also we have $x(t) \to 0$ as $t \to \infty$. Consequently, the origin is asymptotically stable.

However: The origin in not uniformly asymptotically stable; recall that this would require that given $\eta > 0$, there exists $T = T(\eta, t_0) > 0$ such that $|x(t)| < \eta$ for all $t \geq t_0 + T$. Although this is true for every t_0, the constant T can not be chosen independent of t_0.

Summary of Key Points so far.

Given: $\dot{x}(t) = f(t, x(t))$, $t \geq 0$

$0 = f(t, 0)$, $t \geq 0$.

Definition of stable i.s.L: $\forall\ \epsilon_1 > 0$ and $t_0 \geq 0$, $\exists\ \delta_1(\epsilon_1, t_0) > 0$ such that $x_0 \in B_{\delta_1}(0) \Rightarrow x(t, t_0, x_0) \in B_{\epsilon_1}(0)$, $\forall\ t \geq t_0$.

Potential "Problem": (a) $\delta_1(\epsilon_1, t_0) \xrightarrow[t_0 \to \infty]{} 0$

Definition of Asymptotically Stable i.s.L.

Assume (1) $x_e = 0$ is stable i.s.L. and

(2) $\forall\ t_0 \geq 0$, $\exists\ \delta_2(t_0) > 0$ such that

$x_0 \in B_{\delta_2}(0) \Rightarrow x(t, t_0, x_0) \xrightarrow[t \to \infty]{} 0$.

Part (2) means that $\forall\ t_0 \geq 0$, $\exists\ \delta_2(t_0) > 0$ such that $\forall \eta_2 > 0$, $\exists\ T(\eta_2, t_0) < \infty$ such that: $\forall\ x_0 \in B_{\delta_2}(0)$, $\forall\ t \geq t_0 + T$, the solution satisfies $x(t, t_0, x_0) \in B_{\eta_2}(0)$.

note the dependence on t_0!

Potential "Problems". (b) $\delta_2(t_0) \xrightarrow[t_0 \to \infty]{} 0$

(c) The rate that $x(t, t_0, x_0)$ converges to the equilibrium point decreases as $t_0 \to \infty$; in other words, the time it takes to enter and remain in the ball of radius η_2 becomes larger and larger as t_0 increases. It is possible that $T(\eta_2, t_0) \to \infty$ as $t_0 \to \infty$.

Pictorial Representations of Stability.

① Stability i.s.L.

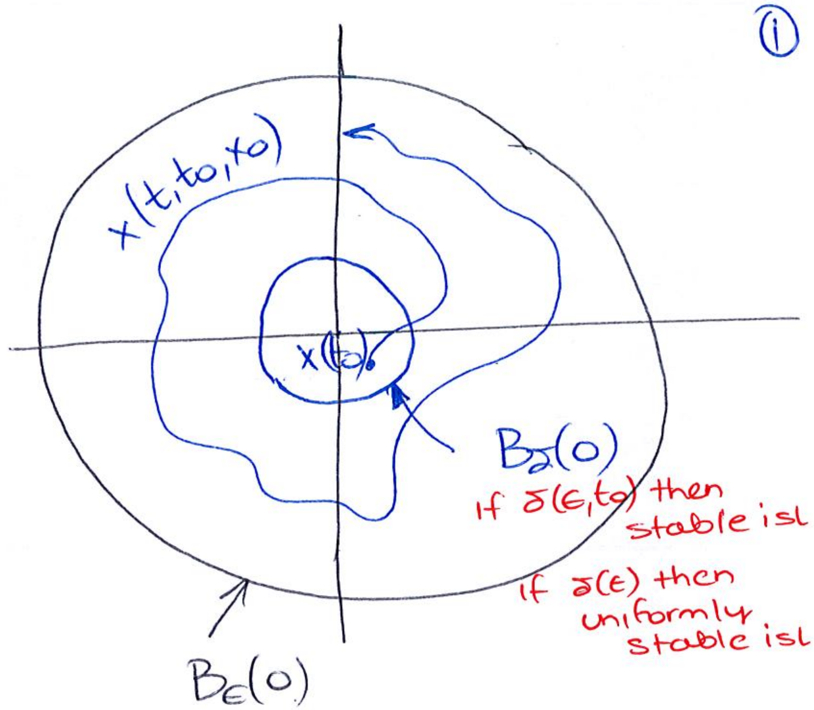
$\|x(t_0)\| < \delta(t_0, \epsilon) \Rightarrow$

$\|x(t)\| < \epsilon, \; \forall \; t \geq t_0 \geq 0$

$\delta(t_0, \epsilon)$ may be a function of ϵ and t_0.

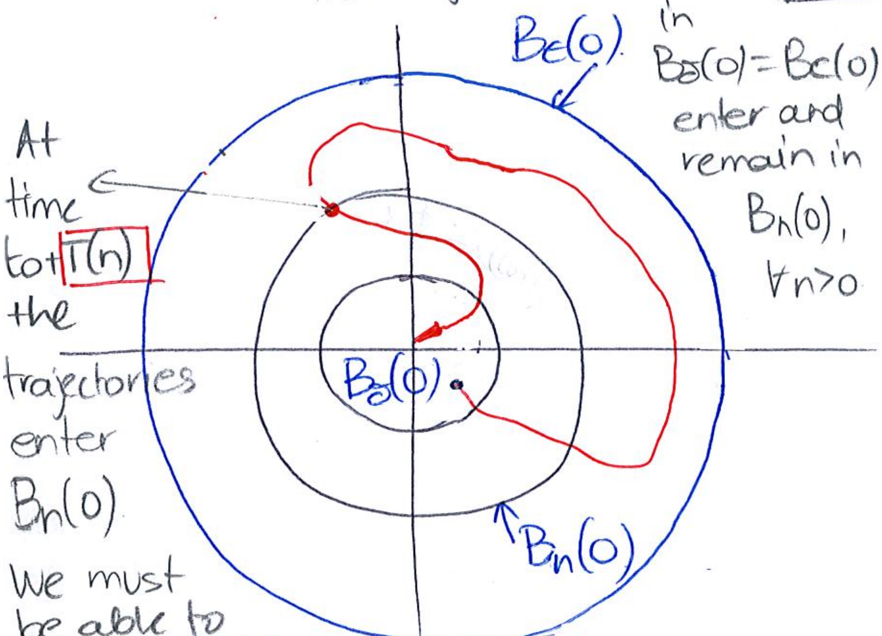
If we can find a $\delta(t_0, \epsilon)$, dependent on t_0, $\forall \epsilon > 0$, then we have stability isL.

If we can find a $\delta(\epsilon)$, independent of t_0, $\forall \epsilon > 0$, then we have uniform stability isL.



③ Uniform asymptotic stability

If uniform stable (i.e, for each $\epsilon > 0$, we can find $\delta(\epsilon)$ such that $x_0 \in B_\delta(0) \Rightarrow x(t) \in B_\epsilon(0)$) AND for $t \geq t_0 + \boxed{T(n)}$ the trajectories starting in $B_\delta(0) = B_c(0)$ enter and remain in $B_n(0)$, $\forall n > 0$



We must be able to find a $\boxed{T(n)}$ independent of t_0, after which the trajectories are confined in $B_n(0)$, no matter how small $n > 0$ might be chosen.

② Asymptotic stability isL.

If stable isL and in addition $\exists \; c = c(t_0)$ such that

$x(t) \to 0$ as $t \to \infty$,

for all $\|x(t_0)\| < c(t_0)$.

We can consider

$B_\delta(0) := B_c(0)$.

The definitions of uniform stability and uniform asymptotic stability are formulated to avoid the potential "problems" labeled (a), (b), (c).

They do so by making appropriate quantities in the definitions INDEPENDENT of t_0.

Lyapunov Stability Theory for Time-Varying Systems.

Idea: For the time-varying system $\dot{x}(t) = f(t, x(t))$ with the equilibrium at the origin $f(t,0) = 0, \forall t \geq 0$,

with (i) $V(t,x)$ locally positive definite

(ii) $\dot{V}(t,x) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}\dot{x}(t)$ locally negative definite

we wish to conclude that $x_e = 0$ is asymptotically stable i.s.L.

Question: How to correctly define "locally positive definite", etc. for time-varying systems.

Example. $\dot{x} = x$, $x \in \mathbb{R}$. Time-invariant.

The origin is unstable.

$$V(t,x) = e^{-3t}x^2; \quad V(t,x) > 0, \ \forall t \geq 0, x \neq 0.$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\,\dot{x}(t) =$$

$$= -3e^{-3t}x^2 + 2x\,e^{-3t}x = -e^{-3t}x^2 < 0,$$

$$\forall\, t \geq 0,\ x \neq 0.$$

But clearly, we can <u>not</u> conclude that the origin is asymptotically stable !

$$\boxed{\text{Source of the problem: } V(t, x(t)) \xrightarrow[t\to\infty]{} 0 \not\Rightarrow x(t) \xrightarrow[t\to\infty]{} 0}$$

<u>Indeed</u>, $x(t) = e^{t}x_0 \xrightarrow[t\to\infty]{} \infty$, despite the fact that

$$V(t, x(t)) = e^{-t}x_0^2 \xrightarrow[t\to\infty]{} 0, \quad \forall\, x_0 \in \mathbb{R}.$$

<u>Definition.</u> The function $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is locally positive definite, if $\exists\, r > 0$ and a continuous nondecreasing function $a : [0, r) \to \mathbb{R}$ such that
(recall HW #4, Problem #3)

① $V(t, 0) = 0, \quad \forall\, t \geq 0,$

② $a(0) = 0, \quad a(p) > 0, \quad 0 < p < r$

③ $a(\|x\|) \leq V(t, x), \quad \forall\, t \geq 0, \quad \forall\, x \in B_r(0).$

Fact: The following are equivalent.

① $V(t,x)$ is locally positive definite.

② $\exists\, W(x)$ locally positive definite such that

$W(x) \leq V(t,x), \quad \forall t \geq 0, \; x \in B_r(0)$

} in other words, $V(t,x)$ is bounded from below by a loc. pos. def. fun.

③ $\bar{W}(x) := \inf_{t \geq 0} V(t,x)$ is locally positive definite.

Note: $V(t,x) = e^{-2t} x^2$ is NOT locally positive definite

because $\bar{W}(x) = \inf_{t \geq 0} e^{-3t} x^2 \equiv 0$

is NOT locally positive definite.

Definition. A continuous function $\alpha : [0,r) \to [0,\infty)$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$.

It is said to belong to class K_∞ if $r = \infty$ and $\alpha(w) \xrightarrow[w \to \infty]{} \infty$.

Lemma. Suppose $\alpha : [0,a) \to [0,+\infty)$ continuous, nondecreasing, $\alpha(0) = 0$, and $\alpha(p) > 0$, $0 < p < a$.

Then $\exists\, \tilde{\alpha} \in K$ such that $\tilde{\alpha}(p) \leq \alpha(p)$, $0 \leq p \leq a$.

Corollary. $V: [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is locally positive definite if and only if.

① . $V(t,0) = 0$, $\forall\, t \geq 0$, and

② . $\exists\, r > 0$ and a class $\mathcal{K}$ function

$\alpha: [0,r) \to [0,+\infty)$, such that $\alpha(\|x\|) \leq V(t,x)$, $\forall\, t \geq 0$, $\forall\, x \in B_r(0)$.

Definition. A continuous function $V: [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ is decrescent if $\exists\, \sigma > 0$ and $\gamma: [0,\sigma) \to [0,+\infty)$ in class $\mathcal{K}$ such that $V(t,x) \leq \gamma(\|x\|)$, $\forall\, t \geq 0$, $x \in B_\sigma(0)$.

Facts (a) V is decrescent if and only if, $\exists\, \sigma > 0$ such that $\forall\; 0 \leq p < \sigma$,

$$\boxed{\sup_{\|x\| \leq p} \sup_{t \geq 0} V(t,x) < \infty} \qquad (*)$$

(b) $(*) \Rightarrow \beta(p) := \sup_{\|x\| \leq p} \sup_{t \geq 0} V(t,x)$ is a continuous, nondecreasing function of p.

(c) If $V(t,x) = V(x)$ which is independent of t, and V is locally positive definite, then V is automatically decrescent.

Definining $\mathring{V}$ It can be readi'ly shown that

$$\boxed{\mathring{V}(t,x) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\, f(t,x).}$$

Theorem. Consider $\mathring{x} = f(t,x)$, and suppose that $f(t,0) = 0$, $\forall\, t \geq 0$, and that $f(t,x)$ is locally Lipschitz in x, and piecewise continuous in t.

The equilibrium point $x_e = 0$ is (uniformly) stable i. s. L. if there exists a continuously differentiable, (decrescent), locally positive definite function $V: [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ and a constant $\rho > 0$ such that $\mathring{V}(t,x) \leq 0$, $\forall\, t \geq 0$, $\forall\, x \in B_\rho(0)$.

Example.

$$\left.\begin{aligned} \mathring{x}_1 &= x_2 \\ \mathring{x}_2 &= -x_2 - (2+\sin t)\, x_1 \end{aligned}\right\} = f(t,x) \qquad [LTV]$$

Let us try $V_1(t,x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, positive definite and decrescent.

We have $\mathring{V}(t,x_1,x_2) = -x_2^2 - x_1 x_2 (1+\sin 2t)$

This is NOT ≤ 0 for $x \in B_\rho(0)$, for any $\rho > 0$.

Another try: $V_2(t,x_1,x_2) = x_1^2 + \dfrac{x_2^2}{2+\sin t}$ Then

$$\underbrace{\frac{x_1^2 + x_2^2}{3}}_{a(\|x\|),\ a(p) = \frac{1}{3}p^2} \leq x_1^2 + \frac{x_2^2}{2+\sin t} \leq \underbrace{x_1^2 + x_2^2}_{\beta(\|x\|),\ \beta(p) = p^2}$$

That proves that V is locally positive definite and decrescent.

Performing the calculations.

$$\dot{V}(t, x_1, x_2) = - x_2^2 \quad \frac{4 + 2\sin t + \cos t}{(2+\sin t)^2} \leq 0,$$

$$\forall\, t \geq 0, \ \forall\, x \in \mathbb{R}^2.$$

Hence the equilibrium point $x_e = 0$ is uniformly stable.

Theorem. Consider $\dot{x} = f(t,x)$ and suppose that $f(t,0) = 0$, $\forall\, t \geq 0$, and that $f(t,x)$ is locally Lipschitz in x, and piecewise continuous in t. The equilibrium point $x_e = 0$ is (uniformly) asymptotically stable, i.s.L. if there exists a continuously differentiable (decresent), locally positive definite function $V: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ such that $-\dot{V}$ is locally positive definite.

The previous theorems are stated as Theorems 4.8 and Theorem 4.9 in our textbook.

Theorem 4.8. Let $x=0$ be an equilibrium point for

$$\dot{x} = f(t,x),$$

where $f:[0,\infty)\times D \to \mathbb{R}^n$ piecewise continuous in t, and locally Lipschitz wrt x on $[0,+\infty)\times D$, and $D \subset \mathbb{R}^n$ a domain containing $x=0$.

Let $V:[0,\infty)\times D \to \mathbb{R}$ be a continuously differentiable function such that

(i) $W_1(x) \le V(t,x) \le W_2(x)$ → reads: $V(t,x)$ is decrescent

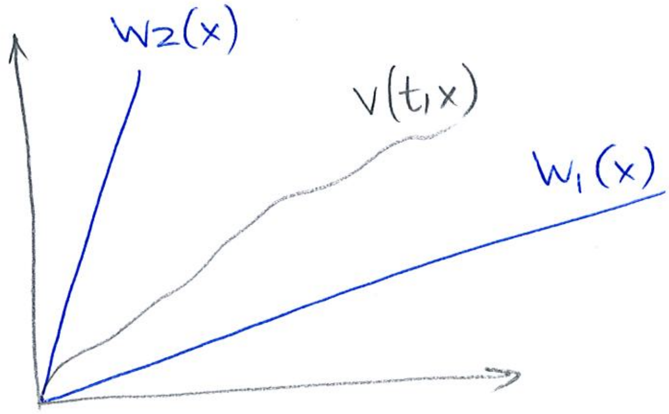
reads: $V(t,x)$ is locally positive definite

(ii) $\dot{V}(t,x) = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} f(t,x) \le 0,$

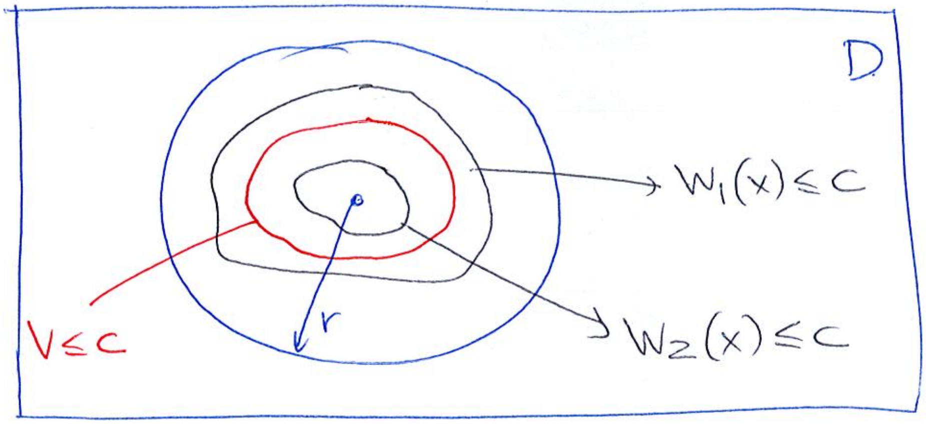
for all $t \ge 0$ and $x \in D$, where $W_1(x), W_2(x)$ are continuous positive definite functions on D.

Then $x=0$ is uniformly stable.

Pictorial Representation.



Proof.



Choose $r > 0$ such that $B_r(0) \subset D$.

Consider $\min\limits_{\|x\|=r} W_1(x) := a$, and choose $c < a$ such that the set $\{x \in B_r \mid W_1(x) \leq c\}$ is in the interior of $B_r(0)$.

Define $\Omega_{t,c} = \{x \in B_r(0) \mid V(t,x) \leq c\}$

Since $W_2(x) \leq c \Rightarrow V(t,x) \leq c$, the set $\Omega_{t,c}$ contains the set $\{x \in B_r(0) \mid W_2(x) \leq c\}$.

In addition, since $V(t,x) \leq c \Rightarrow W_1(x) \leq c$, the set $\Omega_{t,c}$ is a subset of $\{x \in B_r(0) \mid W_1(x) \leq c\}$.

Thus we have the five nested sets as in the Figure.

$$\{x \in B_r(0) \mid W_2(x) \leq c\} \subset \Omega_{t,c} \subset \{x \in B_r(0) \mid W_1(x) \leq c\} \subset B_r(0) \subset D.$$

for all $t \geq 0$.

Note that the setup is similar to the proof of Theorem 4.1, with the difference that the set $\Omega_{t,c}$ is now dependent on t, and is surrounded by time-invariant surfaces $W_1(x) = c$, $W_2(x) = c$.

Now, since $\dot{V}(t,x) \leq 0$ on D, for any $t_0 \geq 0$ and $x_0 \in \Omega_{t_0,c}$, the solution starting at (t_0,x_0) stays in $\Omega_{c,t}$ for all $t \geq t_0$.

Hence the solution starting in $\{x \in B_r(0) \mid W_2(x) \leq c\}$ stays in $\Omega_{c,t}$, and consequently in $\{x \in B_r(0) \mid W_1(x) \leq c\}$, for all future time. Hence the solution is bounded and defined for all $t \geq t_0$.

Now, since $\dot{V}(t,x) \leq 0$, we have $V(t,x(t)) \leq V(t_0,x(t_0))$, $\forall t \geq t_0$

By Lemma 4.3 in the textbook, there exist class $\mathcal{K}$ functions $\alpha_1 : [0,r] \to [0,+\infty)$, $\alpha_2 : [0,r] \to [0,+\infty)$, such that

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|)$$

Now we have $\alpha_1(\|x\|) \leq V(t,x) \Rightarrow$

$$\|x(t)\| \leq \alpha_1^{-1}\left(V(t_0,x(t_0))\right)$$

Similarly $V(t,x) \leq V(t_0,x(t_0)) \leq \alpha_2(\|x\|) \Rightarrow$

$$V(t_0,x(t_0)) \leq \alpha_2(\|x(t_0)\|)$$

Hence $\|x(t)\| \leq \alpha_1^{-1}\left(\alpha_2(\|x(t_0)\|)\right)$

since $(\alpha_1^{-1} \circ \alpha_2)$ is a class $\mathcal{K}$ function, [by Lemma 4.2] we have that the origin is uniformly stable. [by Lemma 4.5 that expresses stability using class $\mathcal{K}$ functions]

Theorem 4.9. Suppose the assumptions of Theorem 4.8 are satisfied with (ii) strengthened to

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x),$$

$\forall\, t \geq 0$ and $\forall\, x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, $x=0$ is uniformly asymptotically stable.

Finally, if $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $x=0$ is globally uniformly asymptotically stable.

Example. $\dot{x} = -(1+g(t))\,x^3$, $g(t) \geq 0$ for all $t \geq 0$, continuous function

Consider $V(x) = \frac{1}{2}x^2$

$$\dot{V}(t,x) = x\left(-(1+g(t))\right)x^3 = -x^4\,(1+g(t)) \leq -x^4$$

$\forall\, x \in \mathbb{R},\ \forall\, t \geq 0.$

Hence the assumptions of Theorem 4.9 are satisfied with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$.

Hence the origin is globally uniformly asymptotically stable.