① Linearization Method / Lyapunov's <u>Indirect Method</u>.
② <u>LaSalle's Method</u>. (Continued)

Last time we studied the linearization method for drawing conclusions for the stability properties of the equilibrium point of a nonlinear system.

Our textbook has some examples (4.14, 4.15) on the application of the method. Let's see one more.

<u>Example</u> (<u>Linearization Method for Stability</u>)

Let us consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 = -x_1 + x_2^2 \end{cases}$$

[see trajectories at the end of lecture]

Let us investigate the stability properties of the origin.

We have $A = \left.\frac{\partial f}{\partial x}\right|_0 = \left.\underbrace{\begin{bmatrix} -1+x_2 & 1+x_1 \\ -1 & 2x_2 \end{bmatrix}}_{\frac{\partial f}{\partial x}}\right|_{\substack{x_1=0 \\ x_2=0}} =$

$$= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues are $\lambda_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$

Hence the origin is asymptotically stable. [As an exercise, is it the only equilibrium?]

Note Theorem 4.7 is known as the Indirect Method, and Theorems 4.1, 4.2 are known as the Direct Method of Lyapunov's Stability Theory.

Note We have already experienced the limitations of Lyapunov's Theorem 4.1, that requires $\dot{V}(x) < 0$, for $x \neq 0$ (negative definite) for concluding asymptotic stability. [Remember the example 4.4]

Fortunately there are other tools that may help us establish asymptotic stability in cases where $\dot{V}(x) \leq 0$, i.e, negative semi-definite.

One of the most popular is LaSalle's Invariance Principle. The core idea is to establish that the only trajectory that can stay identically on the set $\{x \in D \mid \dot{V}(x) = 0\}$ is the trivial/zero trajectory.

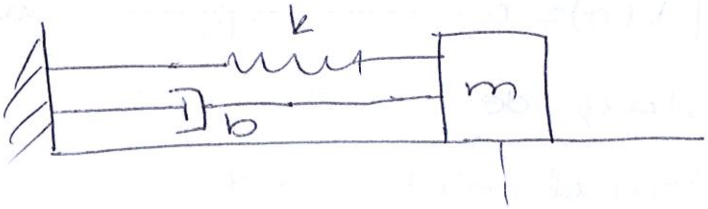
In the sequel we will see a motivating example, and how LaSalle's Theorem is formally stated.

Remark The (not so new, as you will see) concept in our investigation is the concept of an invariant set and the concept of a positively invariant set.

LaSalle's Invariance Principle or LaSalle's Method

(Both terms used interchangeably).

Motivating Example: Nonlinear Mass-Spring-Damper



$x_1 = 0$
$x_2 = 0$

The second order dynamics are modeled as

$$m\ddot{\bar{x}} + \underbrace{b|\dot{\bar{x}}|\,\dot{\bar{x}}}_{\text{nonlinear damping}} + \underbrace{k_0\bar{x} + k_1\bar{x}^3}_{\text{nonlinear spring}} = 0$$

where $\bar{x}$ is the displacement, $\dot{\bar{x}}$ the velocity of the mass.

Letting $x_1 = \bar{x}$, $x_2 = \dot{\bar{x}}$ we obtain the state-space representation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{m}|x_2|x_2 - \frac{k_0}{m}x_1 - \frac{k_1}{m}x_1^3 \end{cases}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let us take the total energy function as a Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}m x_2^2 + \int_0^{x_1}(k_0\sigma + k_1\sigma^3)\,d\sigma$

$$\Rightarrow \boxed{V(x_1, x_2) = \frac{1}{2}m x_2^2 + \frac{1}{2}k_0 x_1^2 + \frac{1}{4}k_1 x_1^4}$$

The time derivative reads:

$$\dot{V}(x_1, x_2) = m x_2 \dot{x}_2 + k_0 x_1 \dot{x}_1 + k_1 x_1^3 \dot{x}_1 = -b x_2^2 |x_2| - x_2(k_0 \cancel{x_1 + k_1 x_1^3}) + x_2(k_0 \cancel{x_1 + k_1} x_1^3) = -b x_2^2 |x_2|$$

The function is negative semidefinite.

Note that this example is reminiscent of the pendulum example in our textbook, that we studied earlier.

The time derivative $\dot{V}(x_1,x_2)$ is negative semi-definite, hence we can only conclude that the origin is stable i.s.L.

Well, we expect from physical intuition that the origin is asymptotically stable as well. LaSalle's Invariance Principle, or LaSalle's Theorem, can help us establish the result. To proceed let us observe the following.

We have $\boxed{\dot{V}(x_1,x_2)=0 \quad \text{if } \{x_2=0,\, x_1\in\mathbb{R}\}}$

For $(x_1,x_2)=(x_1,0)$, $x_1\neq 0$, the system equation reads

$$m\dot{x}_2+k_0x_1+k_1x_1^3=0 \quad \Rightarrow \quad m\dot{x}_2=-x_1(k_0+k_1x_1^2)$$

$$\Rightarrow \dot{x}_2\neq 0 \text{ since } x_1\neq 0.$$

But $\dot{x}_2\neq 0$ implies that x_2 does not remain constant!

In other words, at the next time we will have $x_2\neq 0$.

Then, from $\dot{x}_1=x_2\neq 0$, we have that x_1 will change too! And since $\dot{V}(x_1,x_2)=-b|x_2|x_2^2<0$ for $x_2\neq 0$, we have that the system trajectories do move along lower level sets of $V(x_1,x_2)$!

Geometrically, that means that when $\dot{V}(x_1,x_2)=\dot{V}(x_1,0)=0$, $x_1\neq 0$, the vector field $f(x_1,x_2)$ is tangent to the level set $V(x_1,0)$.

The phase portrait of our system, with level surfaces of the Lyapunov function overlaid on it.

The handdrawn arrows represent the vector field of the system

$$\begin{cases} \mathring{x}_1 = x_2 \\ \mathring{x}_2 = -\frac{b}{m}|x_2|x_2 - \frac{k_0}{m}x_1 - \frac{k_1}{m}x_1^3 \end{cases}$$

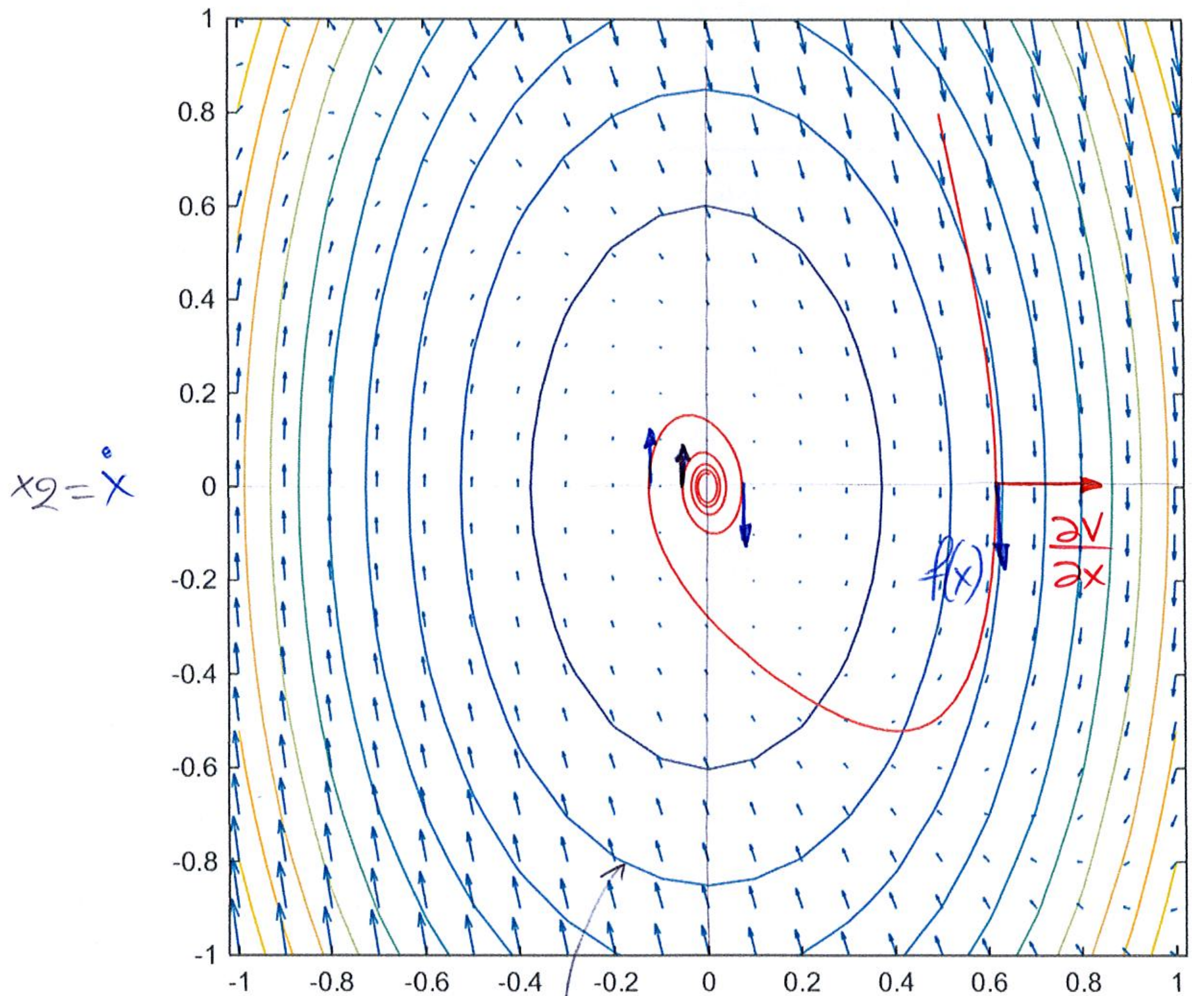
$\boxed{\text{when } \{x_1 \neq 0,\ x_2 = 0\}}$

At these points we have $\mathring{V}(x_1, x_2) = 0$

From the system dynamics we have $\begin{Bmatrix} \mathring{x}_1 = 0 \\ \mathring{x}_2 \neq 0. \end{Bmatrix}$

That implies that $x_2 = 0$ will not stay identically zero.

From $\mathring{x}_1 = x_2$, we have that x_1 will be forced to change as well.



$X = x_1$

Level sets of

$$V(x_1, x_2) = \frac{1}{2}mx_2^2 + \frac{1}{2}k_0 x_1^2 + \frac{1}{4}k_1 x_1^4$$

Then, $\mathring{V}(x_1, x_2) = -b|x_2|\, x_2^2$ will take some negative value, implying that the system trajectories will keep evolving along lower level sets of $V(x_1, x_2)$.

The above example illustrates the core concept used in LaSalle's Invariance Principle, i.e, the concept on an invariant set.

<u>Definition.</u> Consider a nonlinear system $\dot{x} = f(x)$, where $f: D \to \mathbb{R}^n$ locally Lipschitz.

A set M is called an <u>invariant set with respect to $\dot{x} = f(x)$</u> if $x(0) \in M \Rightarrow x(t) \in M$, <u>$\forall\, t \in \mathbb{R}$.</u>

That reads, if the solution belongs to M at some time instant, then it belongs to M for all <u>future</u> and <u>past time</u>.

<u>Remark.</u> Note that the invariance concept for a set is defined with respect to the system $\dot{x} = f(x)$, in the sense that $x(t)$ is the solution of the given ODE. In the sequel we drop this notation for the sake of brevity.

<u>Definition.</u> A set M is called positively invariant if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \geq 0.$$

<u>Remark.</u> In fact, we have already seen quite many examples of invariant and positively invariant sets. $\rightarrow$

Examples : (i) The equilibrium point and the limit cycle are invariant sets since any solution starting in either set remains in the set $\forall\, t \in \mathbb{R}$.

(ii) The set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ with

$$\dot{V}(x) \leq 0 \quad \forall\, x \in \Omega_c \text{ is a}$$

positively invariant set, since as we saw in the proof of Theorem 4.1, any solution starting in Ω_c remains in Ω_c $\forall\, t \geq 0$.

We are now ready to state LaSalle's Theorem (or Invariance Principle per other textbooks)

Theorem 4.4. Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$, where $f: D \to \mathbb{R}^n$ locally Lipschitz. Let $V: D \to \mathbb{R}^n$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then, every solution starting in Ω approaches M as $t \to \infty$.

Remark. In the above definition, largest is understood in the sense of set theory, i.e, M is the union of all invariant sets in $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$.

Remark In the requirements of Theorem 4.4, the function V does not have to be positive definite! This is one of the key extensions of LaSalle's principle relative to Lyapunov's Theorem.

Application Let us formally apply LaSalle's Theorem 4.4 to the mass-spring-damper system, so that we illustrate the construction of all the sets in the theorem's assumptions.

We had $V(x_1, x_2) = \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{4} k_1 x_1^4$ | Denote $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Note that this function is positive definite, but LaSalle's theorem does not require so. Nevertheless, the fact that is positive definite allows us to say that the level set $\Omega_c = \{x \in D \mid V(x) \leq c\}$ is compact for sufficiently small c (remember the proof of Theorem 4.1) In fact, this happens to be a radially unbounded function, hence its level sets are compact sets for any $c \in \mathbb{R}^+$.

Let $\Omega = \Omega_c$. Since $\dot{V}(x) \leq 0$ on Ω, we have that any solution starting in Ω_c remains in Ω_c per the proof of Theorem 4.1.

Hence we have $\underline{\Omega_c = \{x \in D \mid V(x) \leq c\}}$ being a $\underline{\text{positively invariant set}}$. (with respect to $\dot{x} = f(x)$)

and $\underline{\dot{V}(x) = -b|x_2|x_2^2 \leq 0}$ negative semi-definite on Ω_c

Denote $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$.

In our case $E = \{x \in \Omega \mid \{x_1 \in \mathbb{R}, x_2 = 0\}\} \Rightarrow$

$$\boxed{E = \{x \in \Omega \mid \underbrace{\{x_1 \neq 0, x_2 = 0\}}_{E_1} \cup \underbrace{\{x_1 = 0, x_2 = 0\}}_{E_2}\}}$$

Now, La Salle's Theorem says that every solution starting in Ω will approach the largest invariant set M in E. Hence, for concluding convergence of the solutions to the origin, we need to show that the origin $\underline{\underline{\text{is}}}$ the largest invariant set M in E. That reads, it suffices to show that $E_1 = \{x \in \Omega \mid \{x_1 \neq 0, x_2 = 0\}\}$ is $\underline{\text{not}}$ an invariant set, while $E_2 = \{x \in \Omega \mid x_1 = 0, x_2 = 0\}$ $\underline{\text{is}}$ an invariant set. From the system equations we trivially verify that E_2 is indeed an invariant set. Now, we claim that E_1 is not invariant. We prove the claim by contradiction. Assume E_1 is invariant. Then any trajectory $x(0) \in E_1 \Rightarrow x(t) \in E_1$, $\underline{\forall t \in \mathbb{R}}$. However, from the system dynamics we have $\dot{x}_2(t) \neq 0$, hence $x_2(t) \neq x_2(0)$, $t > 0$. That means, trajectories in E_1

do not stay in E_1 $\forall\, t \geq 0$, a contradiction to our assumption. Hence E_1 is not invariant. This in turn implies that the largest invariant set $M = \{0\}$. Hence from Theorem 4.4 we have that the system trajectories approach the origin.

Remark: Notice our wording: "approach the origin" in Theorem 4.4.

Q. Can you think why we use this wording and not "the origin is asymptotically stable"?

A. Yes, it has to do with the math in the proof of Theorem 4.4. The proof shows that the solution approaches the positive limit set of Ω. No ε-δ balls were constructed.

Well, in the example we studied, we indeed had a positive definite function. In that case, La Salle's Theorem is specialized into the following corollaries.

<u>Corollary 4.1.</u> Let $x=0$ be the equilibrium of $\dot{x}=f(x)$ $f: D \to \mathbb{R}^n$ locally Lipschitz.

Let $V: D \to \mathbb{R}$ be

(i) continuously differentiable

(ii) positive definite on D, $V(x)>0$ for $x \neq 0$ and $V(0)=0$.

(iii) $\dot{V}(x) \leq 0$ on D

Let $S=\{x \in D \mid \dot{V}(x)=0\}$ and suppose that no solution can stay identically on S other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.

<u>Corollary 4.2.</u> Let $x=0$ be the equilibrium for $\dot{x}=f(x)$, $f: D \to \mathbb{R}^n$ locally Lipschitz. Let $V: \mathbb{R}^n \to \mathbb{R}$ be

(i) continuously differentiable

(ii) radially unbounded

(iii) positive definite, such that

(iv) $\dot{V}(x) \leq 0$, $\forall x \in \mathbb{R}^n$

Let $S=\{x \in \mathbb{R}^n \mid \dot{V}(x)=0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is globally asymptotically stable.

Remark. Corollaries 4.1 and 4.2 are known as the theorems of Barbashin and Krasovskii, who proved them before the introduction of LaSalle's Invariance Principle.

Remark. When $\dot{V}(x)$ is negative definite, $S = \{0\}$, then the Corollaries 4.1 and 4.2 coincide with Theorems 4.1 and 4.2, respectively.

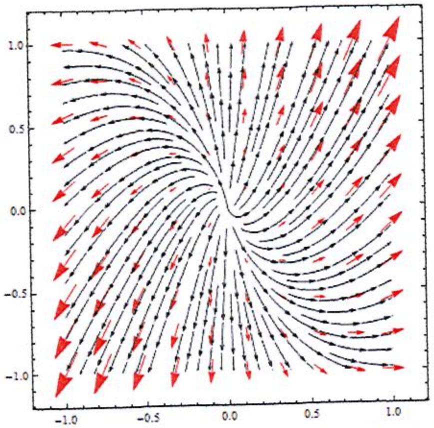
Examples See examples (4.8–4.11) in our textbook.

In summary. What we gain with LaSalle's Theorem?

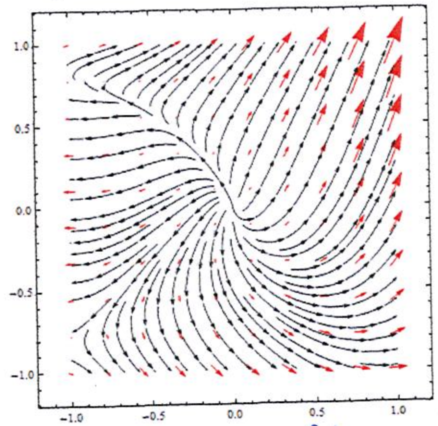
1. It relaxes the negative definiteness requirement of Lyapunov's Theorem.
2. It does not require $V(x)$ to be positive definite [See example 4.11 for an application]
3. It can be used when the system has an equilibrium set instead of isolated equilibria.
   [See example 4.10 for an application]
4. It gives an estimate of the region of attraction, as Ω in Theorem 4.4 can be any compact positively invariant set. Note that Ω does not have to be of the form $\Omega_c = \{x \in \mathbb{R} \mid V(x) \le c\}$. More in our next lecture!

Lyapunov's 1st Method - Example

Trajectories of the *linearized* system



Trajectories of the *nonlinear* system



[See example in the beginning of lecture)

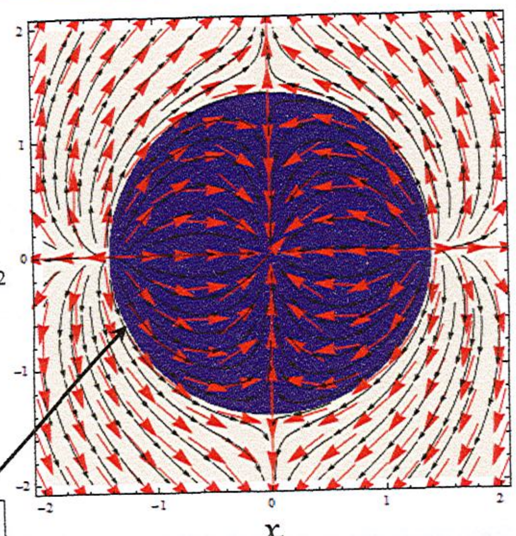
Example: Asymptotic Stability

- System Dynamics: $\boxed{\begin{aligned}\dot{x}_1 &= x_1\left(x_1^2+x_2^2-2\right)-4x_1x_2^2\\ \dot{x}_2 &= x_2\left(x_1^2+x_2^2-2\right)+4x_1^2x_2\end{aligned}}$
- Lyapunov function candidate: $\boxed{V(x_1,x_2)=x_1^2+x_2^2}$
  - positive definite
  - time-derivative is *negative definite* in the 2-dimensional ball defined by $\boxed{x_1^2+x_2^2<2}$

    $\boxed{\dot{V}(x_1,x_2)=2\left(x_1^2+x_2^2\right)\left(x_1^2+x_2^2-2\right)<0}$
- *Conclusion*: The origin is asymptotically stable in $D=\{x \mid x\in B_{\sqrt{2}}(0)\}$

Example: Asymptotic Stability

- System trajectories starting out of the ball $B(0,\sqrt{2})$ are *unstable*
- System trajectories starting in $B(0,\sqrt{2})$ are *asymptotically stable*



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In fact, the ball $B_{\sqrt{2}}(0)$ is the Region of Attraction. But in general, $D \neq$ RoA !!!

Example: Global Asymptotic Stability

- System Dynamics: $\boxed{\begin{aligned}\dot{x}_1 &= x_2-x_1\left(x_1^2+x_2^2\right)\\ \dot{x}_2 &= -x_1-x_2\left(x_1^2+x_2^2\right)\end{aligned}}$
- Lyapunov function candidate: $\boxed{V(x_1,x_2)=x_1^2+x_2^2}$
  - Positive definite and radially unbounded
  - Time derivative is negative definite

    $\boxed{\dot{V}=-2\left(x_1^2+x_2^2\right)^2}$
- *Conclusion*: The origin is a *globally asymptotically stable* equilibrium, RoA $=\mathbb{R}^n$