# Control Lyapunov Functions.

Up to now we have seen tools for analyzing the stability properties of the equilibria of nonlinear systems.

Towards the second half of the course we will learn tools on control design/synthesis: our objective is to create closed-loop systems with desirable stability properties for their equilibria. In fact, we will see how the analysis tools that we have learned so far can guide the development and use of the control design tools.

The first tool towards control design can be developed as the extension of the Lyapunov function concept, called the Control Lyapunov Function (CLF).

**Problem**: Given $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $f(0,0) = 0$ we want to design a feedback control law $u = a(x)$ for the control variable $u$, such that the equilibrium $x = 0$ of the closed-loop system $\dot{x} = f(x, a(x))$ is G.A.S.

**Approach**: We can pick a Lyapunov function candidate $V(x)$, and require that its derivative along the system trajectories satisfy:

$$\dot{V}(x) \leq -W(x), \text{ where } W(x) \text{ is a positive definite function.}$$

In other words, we need to find $a(x)$ to guarantee that

$$\forall\ x \in \mathbb{R}^n, \qquad \frac{\partial V}{\partial x}(x)\, f(x, a(x)) \leq -W(x).$$

In general, this is a difficult task. A stabilizing control law may exist, but we may fail to satisfy the above inequality due to poor choice of $V(x)$ and $W(x)$. If $V(x)$ and $W(x)$ can be found, then we say that the system possesses a Control Lyapunov Function. (CLF)

<u>Definition</u>. Let $\dot{x} = f(x, u)$, where $f(0,0) = 0$, $f$ locally Lipschitz, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. Then, the continuously differentiable, radially unbounded, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ is called a Control Lyapunov Function (CLF) of the system if

$$\inf_{u \in \mathbb{R}} \left\{ \frac{\partial V}{\partial x}(x)\, f(x, u) \right\} < 0, \quad \forall\ x \neq 0.$$

In other words, the definition tells us that a system has a CLF $V(x)$, if at each $x \neq 0$, we can find (there exists) a control input that will reduce the "energy" $V$.

Intuitively, this tells us that if at each state $x \neq 0$, we can find a way, i.e, a control input $u \in \mathbb{R}$ to reduce the energy $V$, then we should eventually be able to bring the energy to zero, i.e, bring the system state trajectories to a stop.

For the case of systems of the form

$$\boxed{\dot{x} = f(x) + g(x)u, \quad f(0) = 0.}$$

(called affine in control, or control-affine)

we can obtain a direct formula for a stabilizing control law $u = a(x)$, using the concept of CLF.

Namely, we want to impose $\dot{V}(x) \leq -W(x)$.

We have $\quad \frac{\partial V}{\partial x}\left(f(x) + g(x)u\right) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) a(x)$

$$\leq -W(x). \qquad (A)$$

A control $a(x)$, that is smooth for all $x \neq 0$, is then given by Sontag's formula.

$$\boxed{u = a_s(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\frac{\partial V}{\partial x} g}, & \frac{\partial V}{\partial x} g \neq 0 \\ 0, & \frac{\partial V}{\partial x} g = 0. \end{cases}}$$

In fact, the condition (A) can be satisfied only if

$$\frac{\partial V}{\partial x} g(x) = 0 \Rightarrow \boxed{\frac{\partial f}{\partial x} f(x) < 0, \quad \forall x \neq 0.}$$

and in this case, the choice of the control input from Sontag's formula yields. $W(x) = \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}, \quad \forall x \neq 0.$

However, the main deficiency of the clf concept as a design tool* is that for most nonlinear systems, a CLF is not known.

* ie. as a way of finding the control input $u$ that will stabilize the origin of the nonlinear system.

The task of finding an appropriate CLF may be as complex as that of designing a stabilizing control law. [and we need both in order to apply the condition (A)!]

Fortunately, for several important classes of nonlinear systems, we can solve these two tasks simultaneously, using what is called a backstepping procedure or technique; that is going to be the topic of the following lectures.

Examples of Nonlinear Control Designs

Let the scalar system

$$\dot{x} = \cos x - x^3 + u.$$

Find a feedback control law that creates and globally stabilizes the equilibrium at $x=0$.

We will compare three different designs.

(1) Feedback linearization. (recall also your pendulum project)

In a feedback linearization design, our approach is to choose the control law such that it cancels out the nonlinearities, and results in a linear closed-loop system. Here, we can choose

$$u = -\cos x + x^3 - x \triangleq a(x)$$ , so that the

resulting feedback system is

$$\boxed{\dot{x} = -x.}$$

Then, taking $V(x) = \frac{1}{2}x^2$ and its time derivative along the closed-loop system trajectories, yields

$\dot{V}(x) = x(-x) = -x^2 \leq -x^2.$ , where $W(x) = x^2$.

Hence $V(x)$ is a CLF for the given system

However, the choice of the control law cancels the $-x^3$ nonlinearities of the system, which is unnecessary since they are stabilizing to the origin. In addition, the presence of $x^3$ in the control law in principle leads to large magnitudes of $u$, and may cause nonrobustness.

(ii) Can we do something better? Let us investigate the control law

$$u = -\cos x - x \triangleq a(x)$$

The resulting feedback system is $\boxed{\dot{x} = -x^3 - x}$

Take $V(x) = \frac{1}{2}x^2$, and the time derivative along the closed-loop trajectories as

$$\dot{V}(x) = x(-x^3 - x) = -x^4 - x^2 \leq -W(x),$$

where $W(x) = x^2 + x^4$.

Hence, again $V(x)$ is a CLF, and with this design, the magnitude of $u$ grows only linearly with $|x|$.

(iii) Let us now apply Sontag's formula.

The formula is based on the assumption that $f(0) = 0$. Hence we first cancel $\cos x$ by introducing

$$u = -\cos x + u_s.$$

Then the system reads.

$$\dot{x} = -x^3 + u_s$$

Our goal is to design $u_s$ out of Sontag's formula for the above system, where $f(x) = -x^3$, $g(x) = 1$.

Pick $V(x) = \frac{1}{2}x^2$ as the clf. We have.

$$u_s = a_s(x) = -\frac{-x^4 + \sqrt{(x^4)^2 + x^4}}{x} =$$

$$= \frac{x^4 - \sqrt{x^4(x^4+1)}}{x} = x^3 - x\sqrt{x^4+1}.$$

Remarkably, as $|x| \to \infty$, we have that $\underline{a_s(x) \to 0}$.

That means that for large $|x|$, the control law for $u$ reduces to the term $-\cos x$ required to place the equilibrium at $x=0$. The rationale is the following: for large $|x|$, the internal nonlinear feedback "takes over" and forces $x$ towards zero. This way, the control effort is not wasted in achieving a property that is already present in the system! On the other hand, for small $|x|$, we have $a_s \simeq -x$, which is the same as in the previous control laws.

The time derivative of $V(x)$ along the closed-loop system trajectories reads.

$$\dot{V}(x) = x\left(-x^3 + x^3 - x\sqrt{x^4+1}\right) = -x^2\sqrt{x^4+1} \leq -W(x)$$

where $W(x) = x^2\sqrt{x^4+1}$, hence as expected, $V(x)$ is a CLF.

In summary, the control law given out of Sontag's formula is superior than the other two as it requires <u>less control effort.</u>

Note: More worked examples on CLFs, Sontag's formula and their usefulness in estimating the RoA are given in Prof. Grizzle's notes, that follow right after.

# EECS 562 Handout: Grizzle
# Control Lyapunov Functions and Example

**Def.** Let $\dot{x} = f(x, u)$ be a control system on $\mathbb{R}^n$ with $u \in \mathbb{R}^m$. A continuously differentiable, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ is a <u>control Lyapunov function (clf)</u> if, for all $x \neq 0$,

$$\inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x}(x) f(x, u) \right\} < 0.$$

**Def.** A control system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

is said to be <u>affine</u>. Sometimes it is convenient to write the system as $\dot{x} = f(x) + G(x)u$, where $G(x) = [g_1(x), \cdots, g_m(x)]$.

**Notation** Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Then

$$L_f V(x) = \frac{\partial V}{\partial x}(x) f(x),$$

$$L_{g_i} V(x) = \frac{\partial V}{\partial x}(x) g_i(x),$$

**Lemma** For a single-input affine control system, the following are equivalent

(i) $\forall\ x \neq 0, \quad \inf_{u \in \mathbb{R}} \{L_f V(x) + L_g V(x) u\} < 0$

(ii) $(\forall\ x \neq 0$ and $\ L_g V(x) = 0) \Rightarrow L_f V(x) < 0$.

**Sontag's feedback** (single-input case) Assume $V$ is a clf:

$$u = \alpha_S(x) = \begin{cases} -\dfrac{L_fV(x) + \sqrt{(L_fV(x))^2 + (L_gV(x))^4}}{L_gV(x)} & , \quad L_gV(x) \neq 0 \\ 0 & , \quad L_gV(x) = 0 \end{cases}$$

**Remark:** Here is how $\dot{V}(x)$ works out

(i) Assume $L_gV(x) \neq 0$. Then

$$\begin{aligned} \dot{V}(x) &= L_fV(x) + L_gV(x)\alpha_S(x) \\ &= L_fV(x) - L_fV(x) - \sqrt{(L_fV(x))^2 + (L_gV(x))^4} \\ &= -\sqrt{(L_fV(x)^2 + (L_gV(x))^4} \end{aligned}$$

(ii) Assume $L_gV(x) = 0$. Note that then, because $V$ is a clf, $L_fV(x) < 0$. For any real number $y < 0$, $y = -|y| = -\sqrt{y^2}$. Using this fact, we obtain

$$\begin{aligned} \dot{V}(x) &= L_fV(x) + L_gV(x)\alpha_S(x) \\ &= L_fV(x) \\ &= -\sqrt{(L_fV(x))^2} \\ &= -\sqrt{(L_fV(x)^2 + (L_gV(x))^4} \end{aligned}$$

where the last line is true because $L_gV(x) = 0$.

**Lemma** [Given for completeness. You only need to know the single-input case for the Final Exam] For a multi-input affine control system, the following are equivalent

(i) $\forall\ x \neq 0, \quad \inf_{u \in I\!R^m} \{L_f V(x) + \sum_{i=1}^m L_{g_i} V(x) u_i\} < 0$

(ii) $(\forall\ x \neq 0$ and $\ [L_{g_1} V(x), \cdots, L_{g_m} V(x)] = 0) \Rightarrow L_f V(x) < 0$

(iii) $\left(\forall\ x \neq 0 \text{ and } \ \sum_{i=1}^m \left(L_{g_i} V(x)\right)^2 = 0\right) \Rightarrow L_f V(x) < 0$

(iv) $(\forall\ x \neq 0$ and $\ \Gamma(x) = 0) \Rightarrow L_f V(x) < 0$, where

$$\Gamma(x) = [L_{g_1} V(x), \cdots, L_{g_m} V(x)]$$

**Sontag's feedback** (multi-input case) Assume $V$ is a clf:

$$u = \alpha_S(x) = \begin{cases} -\dfrac{L_f V(x) + \sqrt{(L_f V(x))^2 + (||\Gamma(x)||_2)^4}}{(||\Gamma(x)||_2)^2}\ \Gamma'(x) & , \quad \Gamma(x) \neq 0 \\ 0 & , \quad \Gamma(x) = 0 \end{cases}$$

**Exercise:** $\dot{V}(x)$ works out to be

$$\dot{V}(x) = -\sqrt{(L_f V(x))^2 + (||\Gamma(x)||_2)^4}$$

## Powerful Design Method

Consider

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u$$

and let

$$\dot{x} = Ax + Bu$$

be the linearization about $x_e = 0$. That is,

$$A = \frac{\partial f}{\partial x}(x_e) \quad \text{and} \quad B = [g_1(x_e), \cdots, g_m(x_e)] = G(x_e).$$

(i) Let $u = Kx$ be a stabilzing feedback, that is $\Re e\{\lambda_i(A + BK)\} < 0$.

(ii) Let $V = x^T P x$ be a quadratic Lyapunov function for $\dot{x} = (A + BK)$, that is $(A + BK)^T P + P(A + BK) = -Q$ for some $Q > 0$.

(iii) **Observation 1:** $V$ is therefore a clf for $\dot{x} = Ax + Bu$

(iv) **Observation 2:** $V$ is therefore a LOCAL clf for $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u$, that is, there exists $\sigma > 0$ such that, for all $x \in B_\sigma(0)$, $x \neq 0$, we have

$$\inf_{u \in I\!R^m} \left\{ L_f V(x) + \sum_{i=1}^{m} L_{g_i} V(x) u_i \right\} < 0$$

(v) **Fact:** Compute Sontag's feedback for $V$. It has a very nice property, namely, let $c > 0$ and let $\Omega_c$ be a sublevel set of $V$. Then $\forall\, x \in \Omega_c$, $x \neq 0$, the following two statements are equivalent:

(a) $\inf_{u \in I\!R^m} \{L_f V(x) + \sum_{i=1}^{m} L_{g_i} V(x) u_i\} < 0$

(b) $L_{f_{cl}} V(x) < 0$, where

$$f_{cl}(x) = f(x) + G(x)\alpha_S(x).$$

In other words, for the given Lyapunov function, if ANY feedback can render $\dot{V}$ negative definite on $\Omega_c$, then Sontag's feedback will also do it!

It follows that, for the given Lyapunov function, Sontag's feedback gives you the largest $\Omega_c$ that you can prove to be in the region of attraction.

In particular, $\Omega_c$ for Songtag's feedback is at least as large as the $\Omega_c$ you can find for the linear feedback. On particular examples, you will often find that $\Omega_c$ for Sontag's feedback is quite a bit larger than what you can prove to work for the original linear feedback.

For particular systems, you may be able to find a "simpler" feedback that will render $\dot{V}$ negative definite on $\Omega_c$. Finding such a feedback usually depends on physical insight for the problem at hand.

**Summary of the philosophy:** Linearize your system. Apply linear feedback design tools to stabilize your equilibrium point. Compute a Lyapunov function for the linear closed-loop system. *Throw away the linear feedback and only keep the Lyapunov function, which will be a local clf for the nonlinear system*! Apply Sontag's feedback.

**An example follows, using our pendulum and cart dynamics.**

# Example of CLF vs Linearization

Consider the model of the pendulum w/o the cart:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = f_2(x_1,x_2) + g_2(x_1,x_2)u \qquad = f(x) + g(x)u$$

Where,

$$f_2(x_1,x_2) = \frac{-\frac{1}{4}(x_2)^2 \sin(2x_1) + \sin(x_1)}{1 + \frac{1}{2}\sin^2(x_1)}$$

$$g_2(x_1,x_2) = \frac{-\cos(x_1)}{1 + \frac{1}{2}\sin^2(x_1)}$$

Linearized system about the origin:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{B} u$$

Find feedback to place the e-values at

$(-2, -3) \quad \Rightarrow \quad u = kx \ , \ k = [7, 5]$

$$(A + BK) =: \bar{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

To find a Lyapunov function, solve

$\bar{A}^T P + P\bar{A} = -I$ (for example)

$$\therefore \quad P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix}$$

$$[V, D] = \mathrm{eig}(P) \quad \Rightarrow \quad V = \begin{bmatrix} -0.9966 & 0.0824 \\ -0.0824 & -0.9966 \end{bmatrix}$$

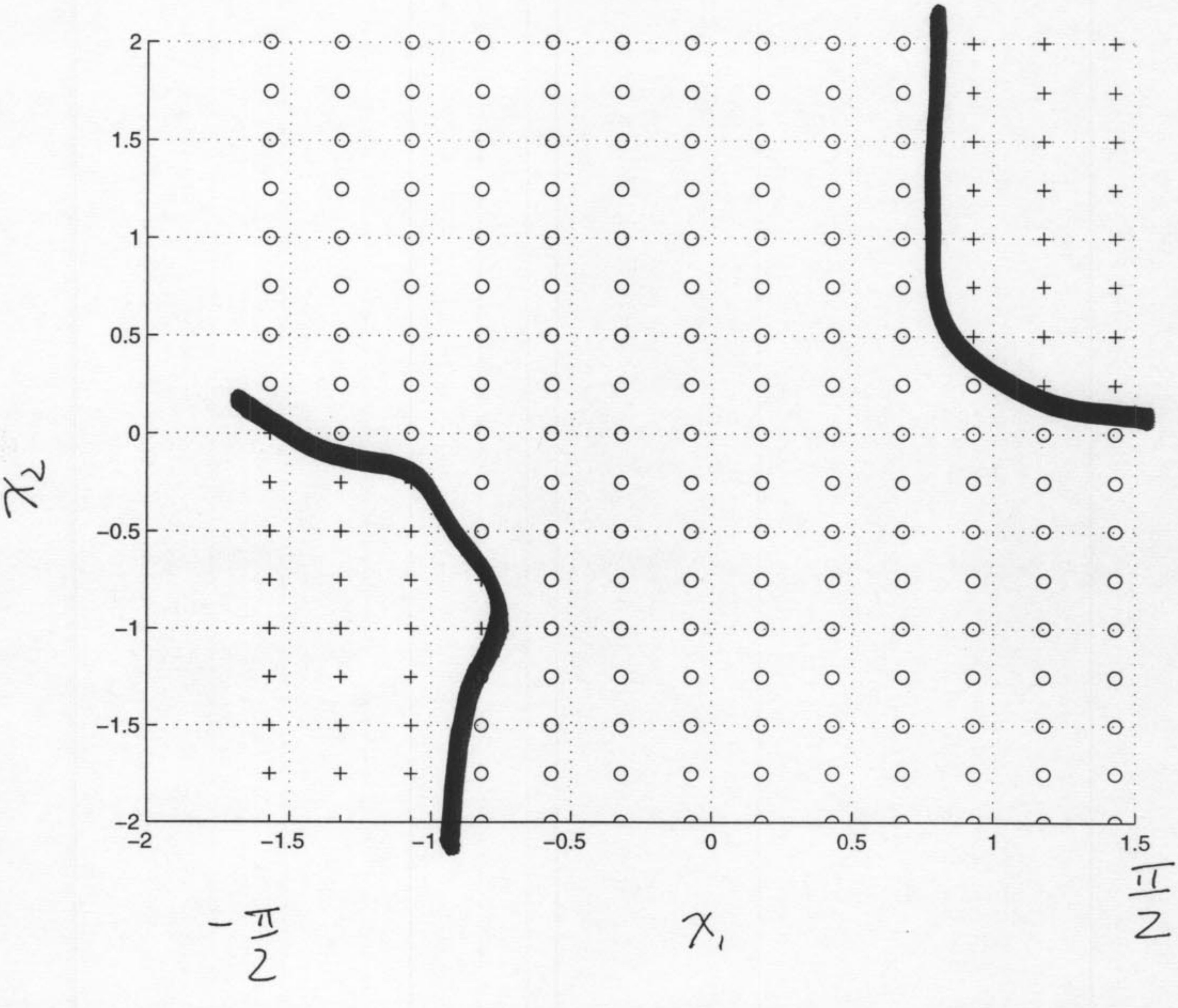
$$D = \begin{bmatrix} 1.1236 & 0 \\ 0 & 0.1098 \end{bmatrix}$$

To estimate the region of attraction, we want to determine the largest $c > 0$ such that $\{x \mid V(x) \le c\} \subset \{x \mid \dot{V}(x) < 0\} \cup \{0\}$, where $\dot{V}$ is evaluated for the closed-loop system, using $u = kx$.

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \Big|_{u = kx}$$

For this example, let's take a numerical approach. The next page is a plot of the set where $\dot{V} < 0$.

o if $\dot{V} \leq 0$
+ if $\dot{V} > 0$



$\{x \mid V(x) \le 0.8\}$ superimposed on $\dot{V} < 0$

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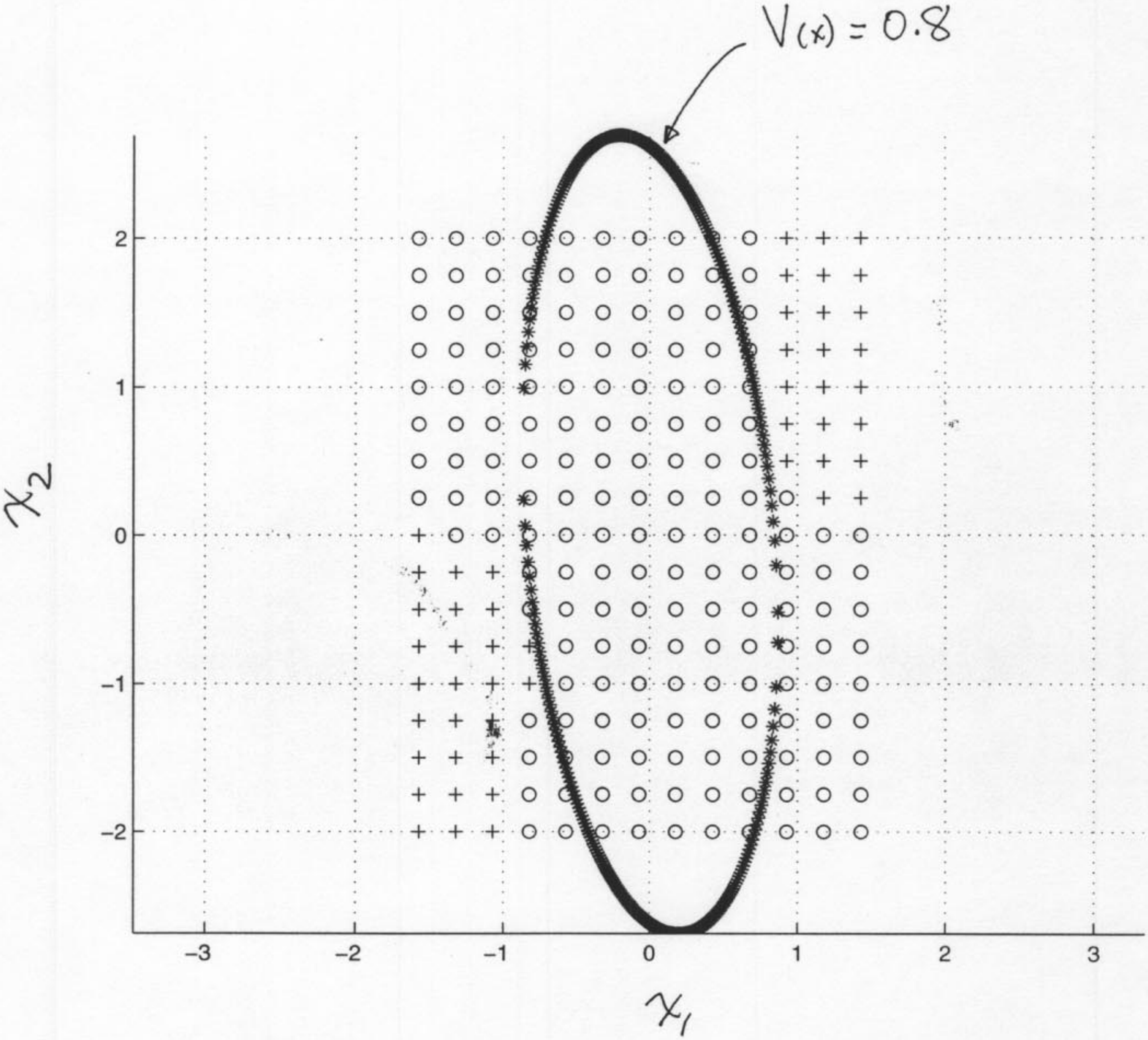

# The Nonlinear Perspective

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$$V(x) = \frac{1}{2} x^T P x \quad , \quad P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2) u \end{aligned} \quad = f(x) + g(x) u$$

$$f_2(x_1, x_2) = \frac{-\frac{1}{4}(x_2)^2 \sin(2x_1) + \sin(x_1)}{1 + \frac{1}{2}\sin^2(x_1)}$$

$$g_2(x_1, x_2) = \frac{-\cos(x_1)}{1 + \frac{1}{2}\sin^2(x_1)}$$

$$\frac{\partial V}{\partial x} \cdot f(x) = (1.1167\, x_1 + 0.0833\, x_2)\, x_2 \quad +$$

$$+ (0.0833\, x_1 + 0.1167\, x_2)\, f_2(x_1, x_2)$$

$$\frac{\partial V}{\partial x} g(x) = (0.0833 x_1 + 0.1167 x_2)\, g_2(x_1, x_2)$$

In what region is $V(x)$ a CLF?

$$\frac{\partial V}{\partial x} g(x) = 0 \iff x_2 = \frac{-0.0833}{0.1167} x_1 \quad \text{OR}$$

$$x_1 = \pm \frac{\pi}{2} \pm k\pi, \quad k = 0, 1, 2, \ldots$$

Must check $\frac{\partial V}{\partial x} f(x) < 0$ on the above set (except $x=0$).

$$\left.\frac{\partial V}{\partial x} f(x)\right|_{x_2 = -0.7143 x_1} = \left(1.1167 x_1 + 0.0833[-.07143] x_1\right) \cdot$$

$$\cdot \left(-.0.7143 x_1\right) \quad + \left\{ 0.0833 x_1 - 0.1167[-0.7143 x_1] \right\} \cdot$$

$$\cdot \frac{\frac{1}{4}\left(-0.7143 x_1\right)^2 \sin(2x_1) + \sin(x_1)}{1 + \frac{1}{2}\sin^2(x_1)}$$

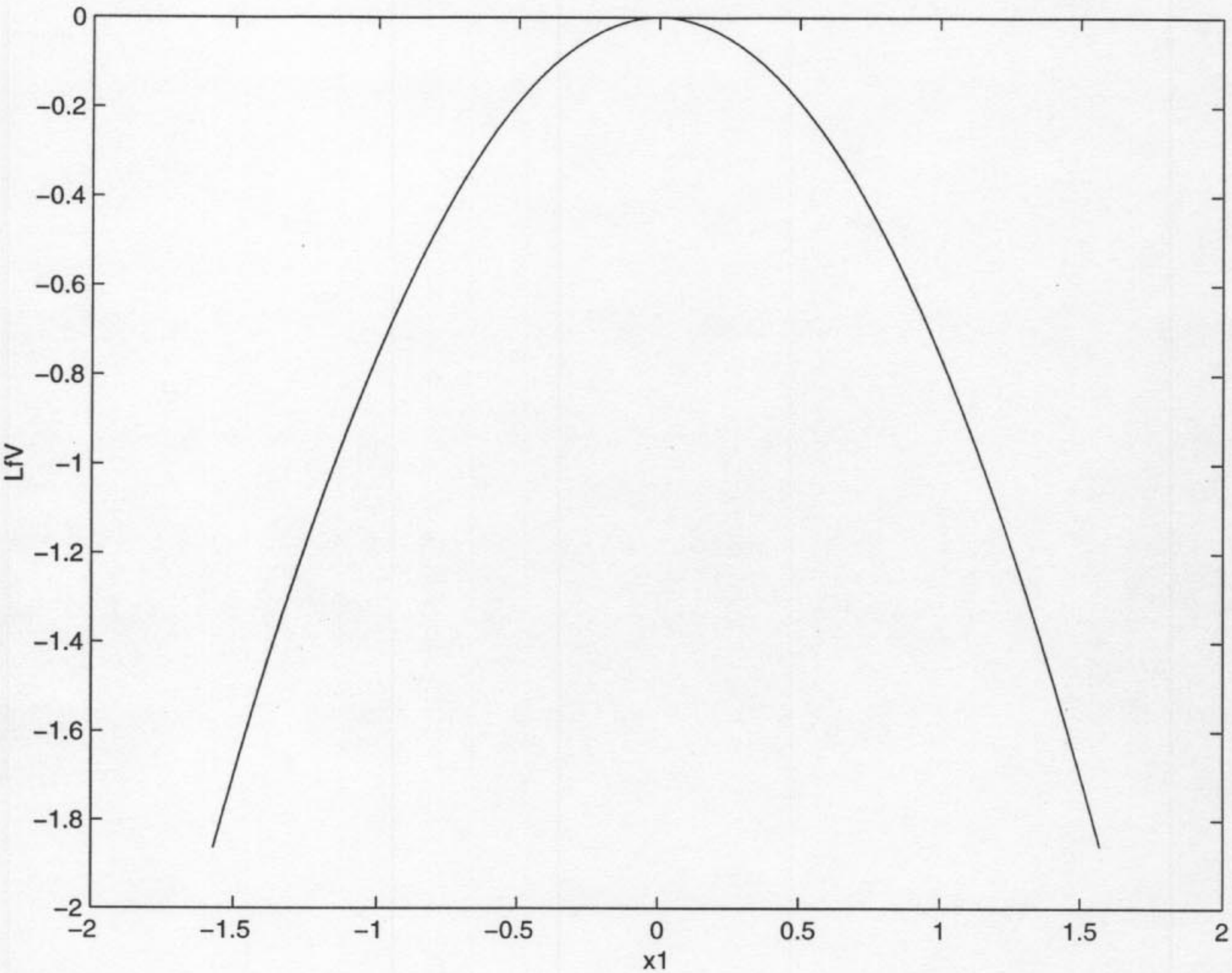
This is plotted on the next page, showing that for all $x \in \mathbb{R}^2$, $|x_1| < \frac{\pi}{2}$, $L_g V(x) = 0$

$\Longrightarrow L_f V(x) < 0$ (except $x=0$)

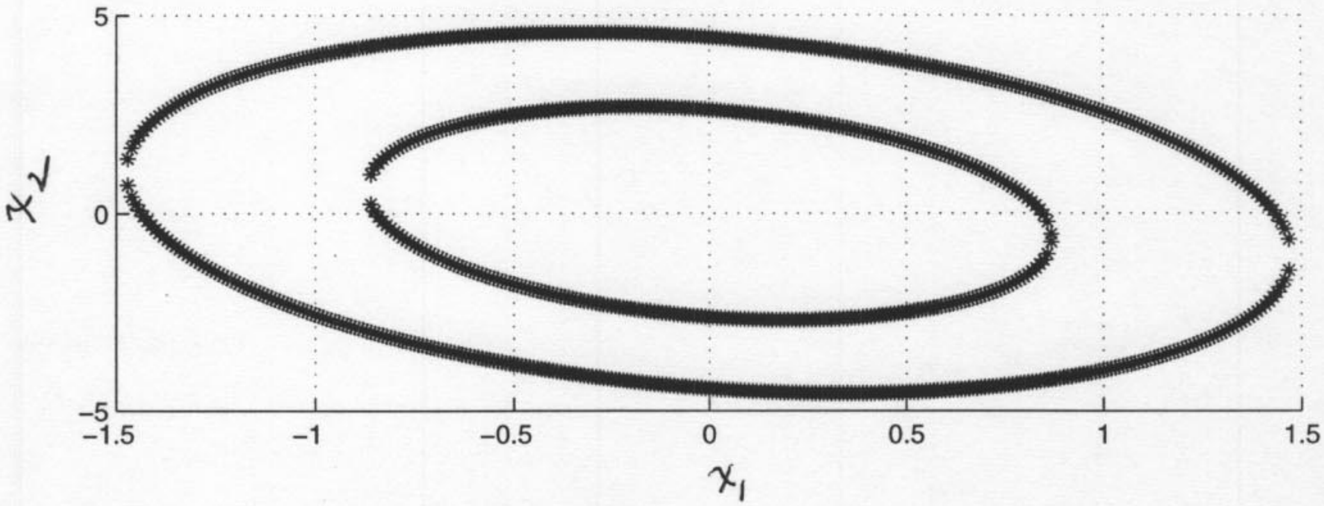
$\therefore \; \exists \; u = \alpha(x)$ s.t. $\dot{V} = L_f V + L_g V \alpha < 0$

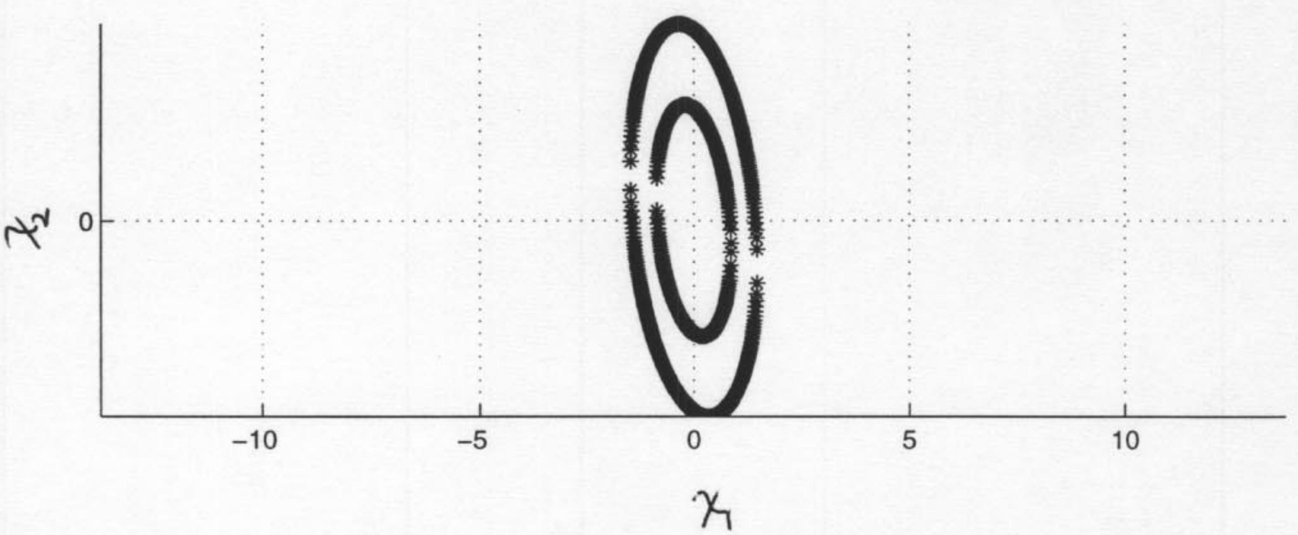
for $|x_1| < \frac{\pi}{2}$, $x \neq 0$.

Checking $L_g V(x) = 0 \Rightarrow L_f V(x) < 0$



Plots of $V(x) = 2.3$ & $V(x) = 0.8$




A NL controller can be found to give $R_A \supset \{x \mid V(x) \leq 2.3\}$ while the linear feedback $u = 7x_1 + 5x_2$ gives $R_A \supset \{x \mid V(x) \leq 0.8\}$, which is 70% smaller.

## Possible NL controllers

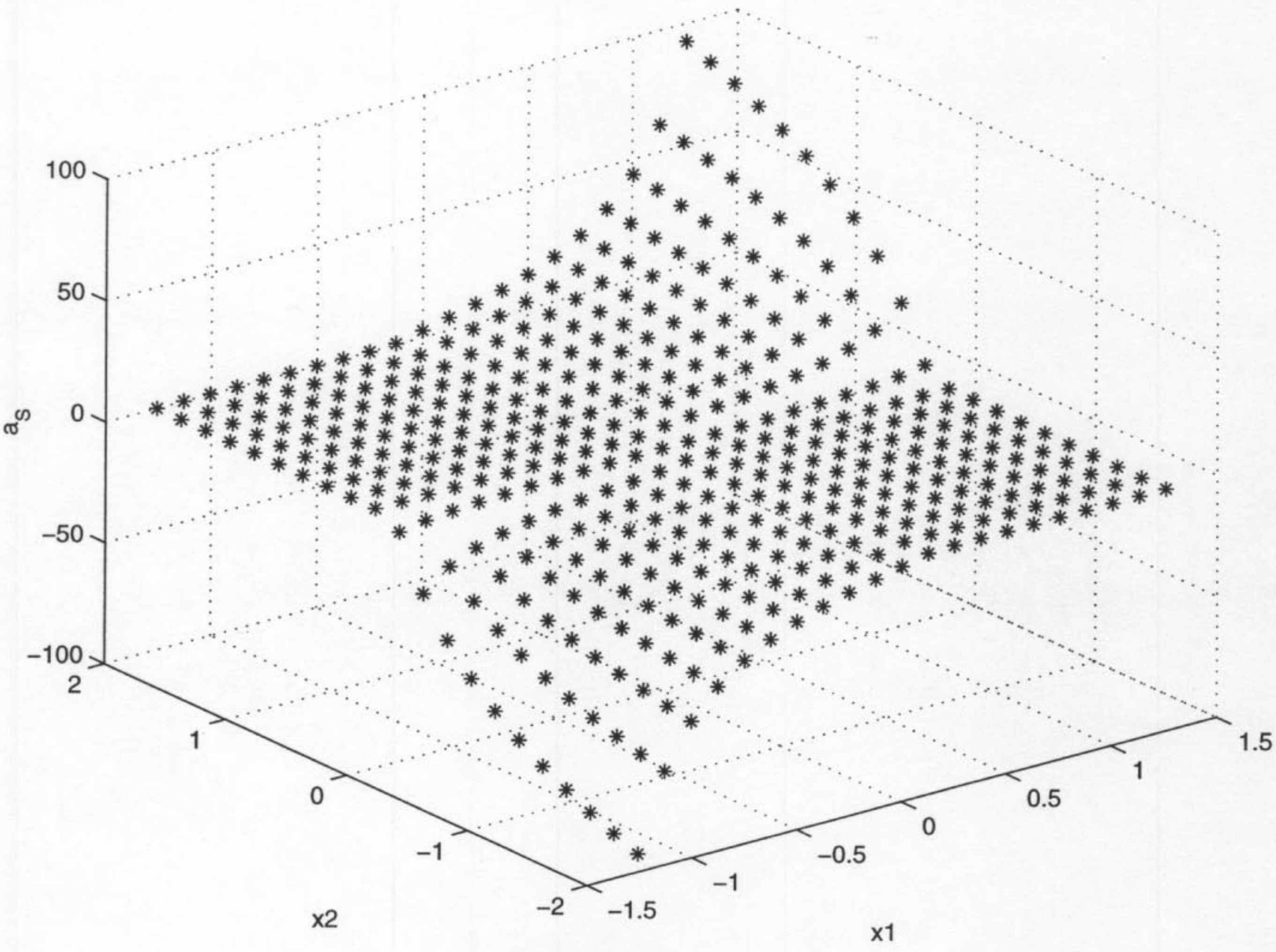
Sontag's formula:

$$\alpha_{S'}(x) = \begin{cases} -\dfrac{L_fV(x) + \sqrt{(L_fV(x))^2 + (L_gV(x))^4}}{L_gV(x)} & , \ L_gV(x) \neq 0 \\ 0 & , \ L_gV(x) = 0 \end{cases}$$

This gives a very complicated controller in this case. However, it could be implemented as a table-look up or with a "low order" polynomial fit.

Illustrations follow.

Sontag's feedback as a function of $(x_1, x_2)$



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a_S_m =

| | | | | | | |
|---|---|---|---|---|---|---|
| -92.4958 | -78.1197 | -55.2989 | -5.2044 | -0.0000 | 0.0000 | 0.0000 |
| -29.2101 | -25.0946 | -18.5541 | -1.4790 | 0.0000 | 0.0000 | 0.0002 |
| -11.6973 | -10.0897 | -7.8228 | -0.4855 | 0.0001 | 0.0011 | 0.0037 |
| -2.8741 | -1.9161 | -0.9580 | 0 | 0.9580 | 1.9161 | 2.8741 |
| -0.0037 | -0.0011 | -0.0001 | 0.4855 | 7.8228 | 10.0897 | 11.6973 |
| -0.0002 | -0.0000 | -0.0000 | 1.4790 | 18.5541 | 25.0946 | 29.2101 |
| -0.0000 | -0.0000 | 0.0000 | 5.2044 | 55.2989 | 78.1197 | 92.4958 |

x(:,:,1) =

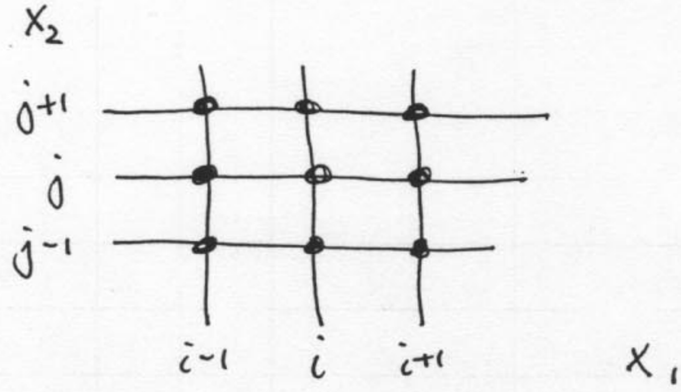
| | | | | | | |
|---|---|---|---|---|---|---|
| -1.2566 | -1.2566 | -1.2566 | -1.2566 | -1.2566 | -1.2566 | -1.2566 |
| -0.8378 | -0.8378 | -0.8378 | -0.8378 | -0.8378 | -0.8378 | -0.8378 |
| -0.4189 | -0.4189 | -0.4189 | -0.4189 | -0.4189 | -0.4189 | -0.4189 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.4189 | 0.4189 | 0.4189 | 0.4189 | 0.4189 | 0.4189 | 0.4189 |
| 0.8378 | 0.8378 | 0.8378 | 0.8378 | 0.8378 | 0.8378 | 0.8378 |
| 1.2566 | 1.2566 | 1.2566 | 1.2566 | 1.2566 | 1.2566 | 1.2566 |

x(:,:,2) =

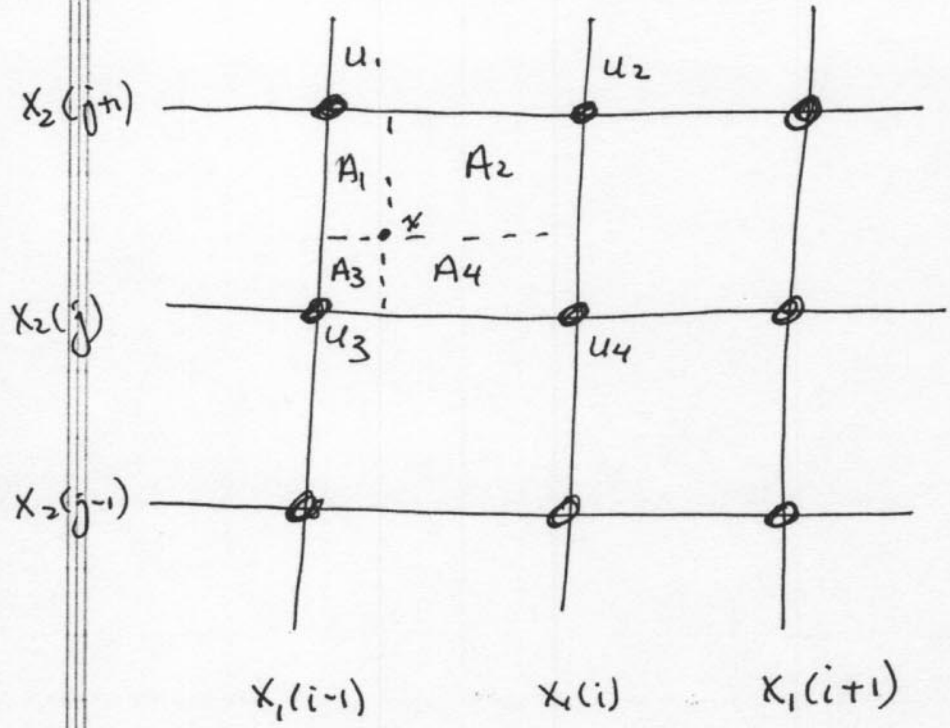
| | | | | | | |
|---|---|---|---|---|---|---|
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |

## An aside on interpolating a 2-D table.

Suppose you have stored $u(x)$, $x \in \mathbb{R}^2$, $u \in \mathbb{R}$ at a number of grid points.




Typically, you will want to compute $u(x)$ at an $x$ which is NOT a grid point. The following is a common way of doing this.




$x = (x_1, x_2)$

Interpolated value of $u$ is

$$u = \frac{u_1 A_4 + u_2 A_3 + u_3 A_2 + u_4 A_1}{A}$$

where

$$A = [x_2(j+1) - x_2(j)]\,[x_1(i) - x_1(i-1)]$$

$$A_1 = [x_2(j+1) - x_2]\,[x_1 - x_1(i-1)]$$

$$A_2 = [x_2(j+1) - x_2]\,[x_1(i) - x_1]$$

$$A_3 = [x_2 - x_2(j)]\,[x_1 - x_1(i-1)]$$

$$A_4 = [x_2 - x_2(j)]\,[x_1(i) - x_1]$$

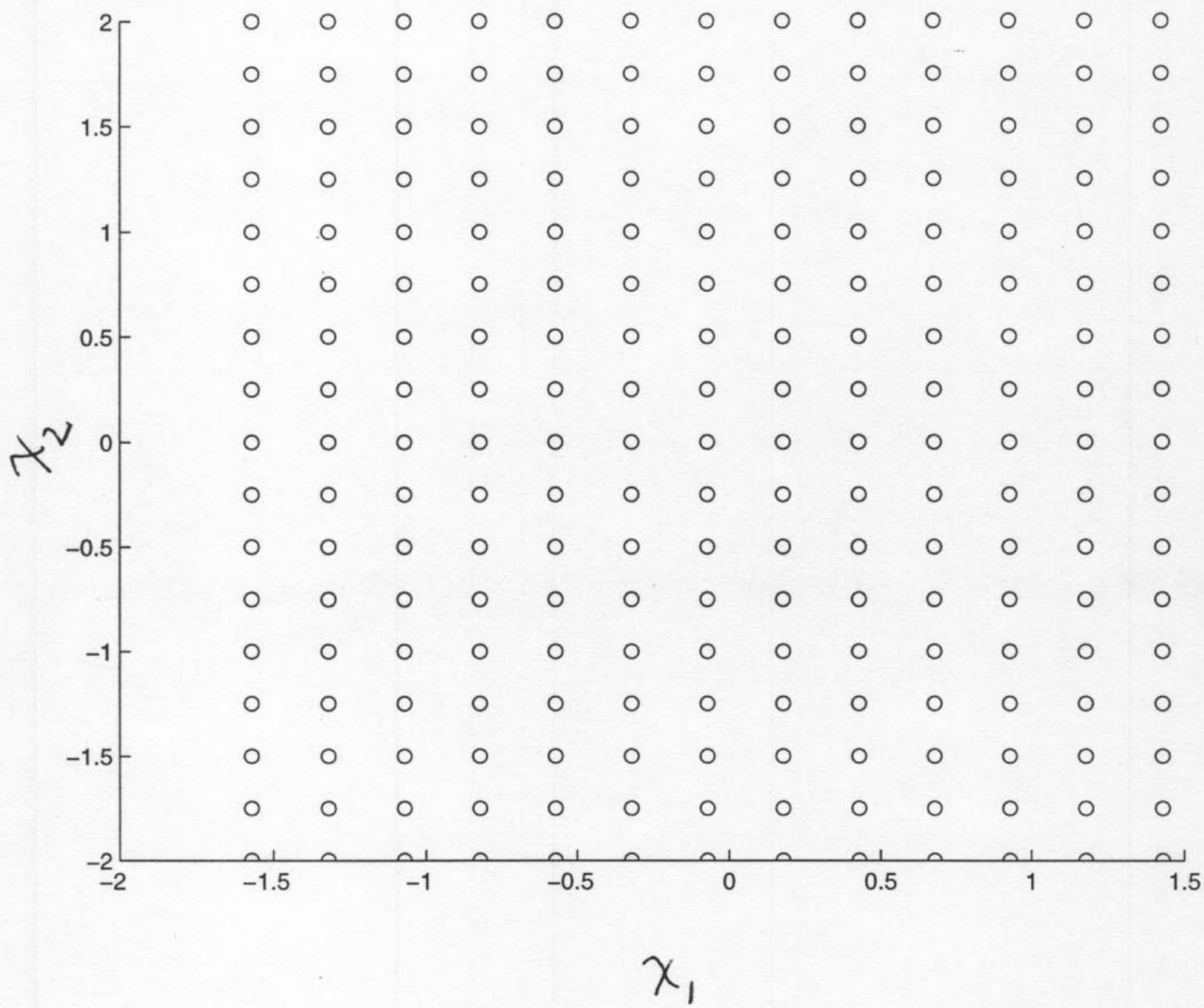
An Alternate Feedback

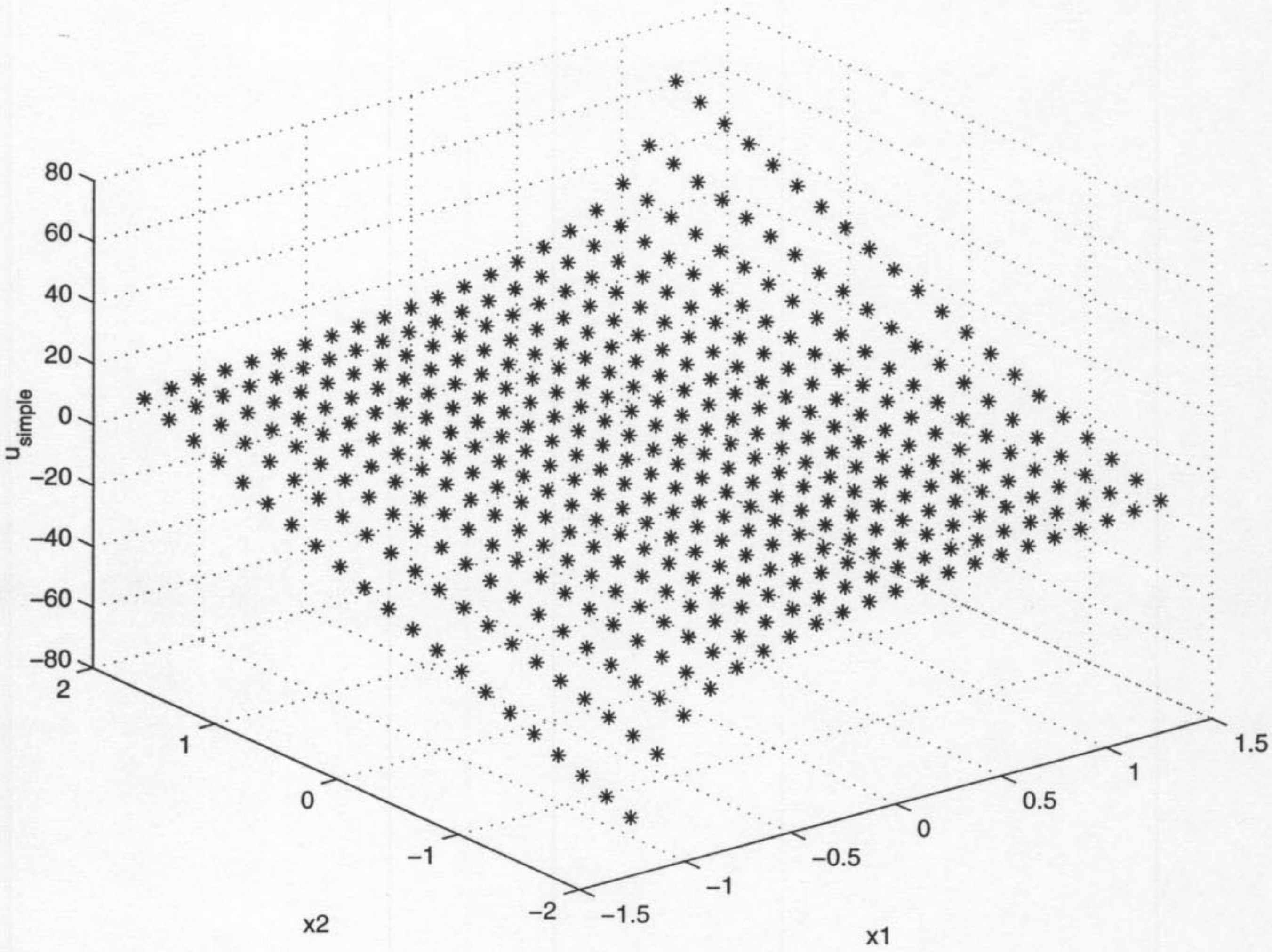


$\dot{V}$ for $u = \dfrac{7x_1 + 5x_2}{\cos(x_1)}$

O for $\dot{V} < 0$

+ for $\dot{V} > 0$




80
60
40
20
0
−20
−40
−60
−80
$u_{simple}$
2
1
0
−1
−2
x2
−1.5
−1
−0.5
0
0.5
1
1.5
x1

u_Simple=

| | | | | | | |
|---|---|---|---|---|---|---|
| -60.8266 | -50.0397 | -39.2528 | -28.4659 | -17.6790 | -6.8922 | 3.8947 |
| -23.7088 | -18.7272 | -13.7457 | -8.7641 | -3.7825 | 1.1991 | 6.1807 |
| -14.1560 | -10.5072 | -6.8584 | -3.2096 | 0.4391 | 4.0879 | 7.7367 |
| -10.0000 | -6.6667 | -3.3333 | 0 | 3.3333 | 6.6667 | 10.0000 |
| -7.7367 | -4.0879 | -0.4391 | 3.2096 | 6.8584 | 10.5072 | 14.1560 |
| -6.1807 | -1.1991 | 3.7825 | 8.7641 | 13.7457 | 18.7272 | 23.7088 |
| -3.8947 | 6.8922 | 17.6790 | 28.4659 | 39.2528 | 50.0397 | 60.8266 |

x(:,:,1) =

| | | | | | | |
|---|---|---|---|---|---|---|
| -1.2566 | -1.2566 | -1.2566 | -1.2566 | -1.2566 | -1.2566 | -1.2566 |
| -0.8378 | -0.8378 | -0.8378 | -0.8378 | -0.8378 | -0.8378 | -0.8378 |
| -0.4189 | -0.4189 | -0.4189 | -0.4189 | -0.4189 | -0.4189 | -0.4189 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.4189 | 0.4189 | 0.4189 | 0.4189 | 0.4189 | 0.4189 | 0.4189 |
| 0.8378 | 0.8378 | 0.8378 | 0.8378 | 0.8378 | 0.8378 | 0.8378 |
| 1.2566 | 1.2566 | 1.2566 | 1.2566 | 1.2566 | 1.2566 | 1.2566 |

x(:,:,2) =

| | | | | | | |
|---|---|---|---|---|---|---|
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |
| -2.0000 | -1.3333 | -0.6667 | 0 | 0.6667 | 1.3333 | 2.0000 |

## Key Points:

(1) Sontag's formula is a useful starting point.

(2) Sontag's formula provides only <u>ONE</u> of an infinite number of possible feedbacks that impose $\dot{V} < 0$

(3) Physical insight on real problems may help you to find other feedbacks.

(4) Don't be afraid to do some numerical computations. $\nabla$

# Review on Control Lyapunov Functions.

Ⓐ Control Lyapunov Functions (CLFs) vs Lyapunov Functions.
What is the difference?

Throughout the course we talked a lot about Lyapunov functions, but also about Control Lyapunov functions. Let us try to clarify any confusion on these two notions.

- We studied Lyapunov functions as the most important necessary and sufficient condition for the stability of equilibria of nonlinear systems.
  The sufficiency was proved by Lyapunov and is expressed through our basic theorems (Theorem 4.1, 4.2)
  Now, necessity was established half a century later with the appearance of the so-called <u>converse theorems</u> (which we didn't cover in class, but are available in chapter 4 of our textbook for interested readers.)

- Having been developed much earlier than control theory, Lyapunov theory deals with the stability properties of dynamical systems without inputs.
  Now, throughout the years, researchers started using candidate Lyapunov functions in <u>feedback design</u> by "making the Lyapunov derivative negative" when choosing

the control. Such ideas were made precise with the introduction of Control Lyapunov Functions for systems with control inputs.

To wrap-up: Lyapunov functions were defined initially for dynamical systems without control inputs. Control Lyapunov Functions extended the concept to dynamical systems with control inputs.

Definition A control Lyapunov function (CLF) for a system of the form

$$\dot{x} = f(x,u)$$

is a continuously differentiable, positive definite, radially unbounded function $V(x)$ such that

$$(*) \qquad \forall\, x \neq 0, \Rightarrow \inf_{u \in \mathbb{R}} \frac{\partial V(x)}{\partial x} \cdot f(x,u) < 0.$$

In other words, a CLF is a candidate Lyapunov function whose derivative can be made negative pointwise (i.e, at each $x \neq 0$) by the choice of some control value.

For single input-affine systems, i.e, systems of the form $\dot{x} = f(x) + g(x)u$, $f(0) = 0$, $u \in \mathbb{R}$, condition $(*)$ is equivalently stated as:

$$\boxed{\left(\forall\, x \neq 0 \text{ and } \frac{\partial V}{\partial x} g(x) = 0.\right) \Rightarrow \frac{\partial V}{\partial x} f(x) < 0}$$

**Example** $\dot{x} = -x(2-x) + (1-x^2)u$, $x \in \mathbb{R}$.

Investigate whether $V(x) = \frac{1}{2}x^2$ is a CLF.

Old Exam Problem

Consider the time derivative of $V(x)$ along the system trajectories as

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)\, u, \quad \text{where} \quad \begin{aligned} f(x) &= -x(2-x) \\ g(x) &= 1-x^2 \end{aligned}$$

Now, $V(x)$ is a CLF if $\boxed{\left(\forall\, x \neq 0,\ \frac{\partial V}{\partial x}(x)\, g(x) = 0\right) \Rightarrow \frac{\partial V(x)}{\partial x} f(x) < 0.}$

We have $\frac{\partial V}{\partial x}(x)\, g(x) = x(1-x^2) = 0 \Rightarrow$

$x = 0$, or $x = -1$, or $x = 1$

Then, for $x \neq 0$, we have

$$\left.\frac{\partial V}{\partial x} f(x)\right|_{x=1} = \left. -x^2(2-x)\right|_{x=1} = -1 < 0$$

and $\left.\frac{\partial V}{\partial x} f(x)\right|_{x=-1} = -3 < 0$

Hence, per Sontag's result, $V(x) = \frac{1}{2}x^2$ is a CLF for the system.

NOTE!!! The analysis does NOT provide any control law, only the conclusion that the system possesses a CLF.

In other words, the above analysis concludes that the equilibrium of the system is Globally Asymptotically Stabilizable!!!

[Trajectories can be driven asymptotically to the equilibrium under some control input].

Now, the "classical" Lyapunov function concept is used in the problem of: "Given $\dot{x} = f(x,u)$, find a stabilizing feedback control law $u = a(x)$ so that the equilibrium of the closed-loop system $\dot{x} = f(x, a(x))$ is Globally Asymptotically Stable.

And we use our usual approach: We choose a Lyapunov function candidate $V(x)$, and require that its derivative along the system trajectories satisfies

$$\boxed{\dot{V}(x) = \frac{\partial V}{\partial x}(x)\, f(x, a(x)) \leq -W(x)} \quad (**)$$

Obviously, $V(x)$ under (**) is proved to be ~~a~~ CLF for the system $\dot{x} = f(x,u)$! Since it proves that there does exist a controller, (in fact, $u = a(x)$!) that renders condition (*) true $\forall\, x \neq 0$!

- So, as usual, finding a control Lyapunov function and a stabilizing controller for a system of the form $\dot{x} = f(x,u)$ is not a trivial task.

- Things become easier if the system is control-affine. Then, Sontag's formula provides a stabilizing control law:

$$u = a_s(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\frac{\partial V}{\partial x} g} & \text{if } \frac{\partial V}{\partial x} g(x) \neq 0 \\ 0 & \text{if } \frac{\partial V}{\partial x} g(x) = 0 \end{cases}$$

For the previous example, we get

$$u = a_s(x) = \begin{cases} -\dfrac{-x^2(2-x) + \sqrt{x^4(2-x)^2 + x^4(1-x^2)^4}}{x(1-x^2)} & \text{if } x(1-x^2) \neq 0 \\ 0 & \text{if } x(1-x^2) = 0 \end{cases}$$

- Other techniques we can use depending on the structure of the system: Backstepping, Feedback Linearization.

Methodological Difference Between Control Lyapunov Function and "classical" Lyapunov Function.

- The Control Lyapunov Function allows us to conclude the STABILIZABILITY of a system with an undefined feedback control

where as

- The classical Lyapunov function allows us to conclude the STABILITY of the closed-loop system generated by a predetermined feedback control.

## Feedback Linearization. (Continued, see also Chapter 13.1 in textbook.)

Motivation Example: Design a control law $u$ for

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\left(\sin(x_1+\delta) - \sin\delta\right) - bx_2 + cu. \end{cases}$$

to stabilize the origin of the closed-loop system.

Approach: Key observation: setting the control law

$$\boxed{u = \frac{a}{c}\left(\sin(x_1+\delta) - \sin\delta\right) + \frac{v}{c}}$$

results in

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -bx_2 + v, \end{cases}$$

which now has a linear form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

Then we can design $\boxed{v = -k_1 x_1 - k_2 x_2}$ to locate the eigenvalues of the closed-loop system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1 x_1 - (k_2+b)x_2 \end{cases}$$

on the open left-hand plane.

The control law is $\boxed{u = \frac{a}{c}\left(\sin(x_1+\delta) - \sin\delta\right) - \frac{1}{c}\left(k_1 x_1 - k_2 x_2\right)}$

How general can be this idea of cancelling the nonlinearities?

Not every nonlinearity can be canceled. The nonlinear system must have a certain structure. More specifically, the ability to use feedback to convert a nonlinear system into a controllable. linear system by cancelling nonlinearities requires the nonlinear system to be of the form:

$$\dot{x} = Ax + B\gamma(x)\big(u - \alpha(x)\big) \qquad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $\gamma : \mathbb{R}^n \to \mathbb{R}^{p \times p}$, $\alpha : \mathbb{R}^n \to \mathbb{R}^p$, $(A, B)$ controllable, $\alpha$ and $\gamma$ are defined over a domain $D$ that contains the origin, and $\gamma(x)$ is nonsingular (i.e its inverse exists) for every $x \in D$.

Then the nonlinear system (1) can be linearized via the state feedback.

$$u = \alpha(x) + \beta(x)\, v \qquad (2), \quad \text{where } \beta(x) = \gamma^{-1}(x)$$

to obtain the linear system

$$\dot{x} = Ax + Bv \qquad (3)$$

For stabilization, we design $\boxed{v = -Kx}$ such that $(A - BK)$ is Hurwitz. The overall nonlinear stabilizing state feedback is $\boxed{u = \alpha(x) - \beta(x) Kx.}$

What if the nonlinear system does not have the structure (1)? Does this mean we can <u>not</u> linearize the system by state feedback? The answer is no. Recall that the state model of a system is <u>not</u> unique; it depends on the choice of state variables. So, even if the state equation does not have the form (1) for one choice of state variables, it might do so for another choice.

<u>Example</u>

$$\begin{cases} \dot{x}_1 = a \sin x_2 \\ \dot{x}_2 = -x_1^2 + u \end{cases}$$

We can first perform a change of variables as follows:

Let $\begin{cases} z_1 = x_1 \\ z_2 = \dot{x}_1 = a \sin x_2 \end{cases}$, then we have

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = a \cos x_2 \, \dot{x}_2 = a \cos x_2 \left( -x_1^2 + u \right) \end{cases}$$

where now the nonlinearities can be canceled via

$$\boxed{u = x_1^2 + \frac{1}{a \cos x_2} v}$$

which is well-defined for $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$

(remember the transformation must be valid in a domain that contains the origin!)

Then the system takes the form

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \end{cases}$$

and the problem reduces to designing $v = -k_1 z_1 - k_2 z_2$ such that the closed-loop dynamics of the linear system are asymptotically stable.

Note that the transformed state equations read

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = a \cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)(-z_1^2 + u) \end{cases}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{a \cos\left(\sin^{-1} \frac{z_2}{a}\right)}_{\gamma(z)} (u - \underbrace{z_1^2}_{\alpha(z)})$$

i.e. it is in the form (1)! So, in general, for applying feedback linearization, we must be able to find a coordinate transformation $z = T(x)$ where the map $T: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, i.e. a continuously differentiable map with a continuously differentiable inverse. This holds in a neighborhood around $x_0$ if the Jacobian $\left[\frac{\partial T}{\partial x}\right]_{x_0}$ is non-singular.

Example $\begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u. \end{cases}$

Stabilize the origin using a control law for $u$ designed via feedback linearization.

Let $z_1 = x_1$

$z_2 = \dot{x}_1 = x_1^2 + x_2$

Then $\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = 2x_1\dot{x}_1 + \dot{x}_2 = 2z_1z_2 + u. \end{cases}$

which reads.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{(2z_1z_2 + u)}_{-\alpha(z)}$$

Then $\boxed{u = -2z_1z_2 + v}$

and the closed-loop linear system reads.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} v$$

where we set $v = -Kz$ so that $(A - BK)$ is Hurwitz.

The overall control input is

$\boxed{u = -2x_1(x_1^2 + x_2) - Kz}$, where $z = \begin{bmatrix} x_1 \\ x_1^2 + x_2 \end{bmatrix}$