EECS 562 - Nonlinear Systems and Control
HW #11

Due on Thursday, April 13th, 2023
By 11:59pm, on Canvas

Remarks:

- Problems 1 and 2 are important for the final exam. Problem 3 is interesting because it shows how La Salle can be related to instability. Problem 4 is super cool, on the pendulum, where you can spin it up from near the downward equilibrium. Problem 5 extends that idea. **You get full credit for the HW set if you turn in four of the five problems:** Problems 1, 2 and 4, and one of Problems 3 and 5.
- Problems 4 and 5 present feedback methods that you may find useful in the future. They require simulations on the inverted pendulum. **If you skip Problem 5, at least check out the solution, as the feedback design used there is super cool.**
- This is your last formal HW assignment.

1. Use backstepping to design a clf and a globally asymptotically stabilizing feedback for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - (x_1)^3 + x_2 \\ x_2 + u \end{bmatrix}$$

   Express the final answer in terms of the original coordinates. Try not to cancel "friendly" (useful) nonlinearities! The answer to the problem is not unique because you will make design choices along the way.

2. Use backstepping to design a clf and a globally asymptotically stabilizing feedback for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (x_1)^2 x_2 \\ u \end{bmatrix}$$

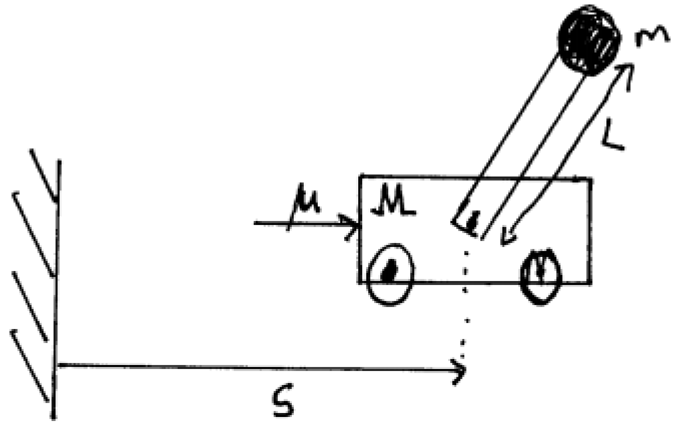
   Express the final answer in terms of the original coordinates. **Note:** For this system, the linearization is not stabilizable, so you could do nothing using linear control theory.

3. **[Optional]** The goal of this problem is to develop another condition for INSTABILITY. Consider a differential equation on $\mathbb{R}^n$, $\dot{x} = f(x)$, where $f(0) = 0$ and f is locally Lipschitz.

   (a) Prove that if the time-reversed system, $\dot{\bar{x}} = -f(\bar{x})$, is asymptotically stable, then $\dot{x} = f(x)$ is unstable. (Recall HW #1 for the time-reversed system.)

   (b) Deduce that if there exists an open set D about the origin and a continuously differentiable, locally positive definite function $V : D \longrightarrow \mathbb{R}$ such that $\dot{V} \geq 0$ on D, and the only solution belonging to the set where $\dot{V} = 0$ is the trivial solution, then the origin is unstable.

Return of "The Pendulum on a Cart": The goal this time is to study the swing-up problem, using the pendulum's energy as a control objective. To make things easier, we are going to work with a simpler model of the pendulum, namely, a bob of mass m attached to a massless rod of length L, plus the usual cart. Also, this time **we will assume the pendulum is attached in such a manner that it can swing freely in a full circle about its hinged point:**



The (Lagrangian) equations of motion are then

$$mL^2\ddot{\phi} + mL\ddot{s}\cos(\phi) - mLg\sin(\phi) = 0$$
$$mL\ddot{\phi}\cos(\phi) + (M+m)\ddot{s} - mL\left(\dot{\phi}\right)^2\sin(\phi) = \mu$$

where as before, M is the mass of the cart, m is the mass of the pendulum, L is the length of the rod, μ is the force applied to the cart and g represents gravity. For simplicity, set all constants equal to 1 (even g) and apply a preliminary feedback

$$\mu = \cos(\phi)\sin(\phi) - \dot{\phi}^2\sin(\phi) + \left(2 - \cos^2(\phi)\right)u,$$

which corresponds to taking the new input, u, as the acceleration of the cart. Choosing the states as $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = s$, $x_4 = \dot{s}$; you should end up with

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - \cos(x_1)u \\ x_4 \\ u \end{bmatrix} = f(x) + g(x)u$$

(if not, pretend that you did and proceed with the above equations). You do not have to turn in a derivation of this model. Just use it!

4. We concentrate for now on the pendulum equations only:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - \cos(x_1)u \end{bmatrix}.$$

When $u = 0$, the total energy (potential plus kinetic) of the pendulum is $E = \cos(x_1) - 1 + \frac{x_2^2}{2}$. The reference for the potential energy has been chosen so that it is zero at the upright equilibrium position; the potential energy is then -2 at the downward equilibrium.

(a) Verify the well-known fact that, when $u = 0$, the total energy is conserved along solutions of the equations. That is, show that $u = 0 \implies \frac{dE}{dt} = 0$. [Note, if this were not true, the technology of clocks would have suffered a major set-back.]

If the energy of the pendulum is not initially zero, we may seek to employ feedback control to force it to zero asymptotically. The motivation for doing this is that the orbit of the pendulum for $E = 0$ contains the upright equilibrium point. Thus, imagine that the initial angle is $\frac{3\pi}{4}$, with zero angular velocity, or maybe $\frac{\pi}{2}$ with some large angular velocity. Such initial conditions are outside of the region of attraction of all of our previous controllers. If we can drive the pendulum energy to zero, however, the pendulum will then pass near the upright equilibrium point, at which time we can switch to one of our previous controllers to stabilize it!

(b) Let $V_p = \frac{1}{2}E^2$, and show that $\dot{V}_p = -Ex_2 \cos(x_1)\, u$.
This suggests the feedback $u = Ex_2 \cos(x_1)$, since it renders $\dot{V}_p$ negative semi-definite. You should note that V_p is NOT a locally positive definite function since it is the square of the sum of a negative and positive function, E. Indeed, V_p vanishes along the free pendulum motion corresponding to $E = 0$.

(c) First check that the down equilibrium point is unstable under the feedback law $u = Ex_2 \cos(x_1)$.

(d) **[Do not turn in]** Use LaSalle's theorem (Theorem 4.4) to analyze the asymptotic behavior of the trajectories under the feedback $u = Ex_2 \cos(x_1)$.

(e) Simulate the above feedback law on the pendulum (only) equations and provide a simulation for $x_1(0) = \frac{3\pi}{4}$ and $x_2(0) = 0$. Remember, in your simulation, that MATLAB will not automatically do its computations modulo 2π, so you must interpret your plot with this in mind.

Remark: At this point, we could proceed as in HW #9, Problem 5, to build an overall switching feedback for the pendulum and cart. You may wish to do this just for fun. Instead, to further illustrate the control of energy, we will develop an idea due to Prof. Mark Spong.

5. **[Optional]** We next look at the pendulum plus the cart velocity:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - \cos(x_1)\, u \\ u \end{bmatrix}.$$

Define a new energy related function by $V = V_p + \frac{\sigma}{2}(x_4)^2$, $\sigma > 0$, which is based on the energy of the pendulum plus the kinetic energy of the cart. Once again, V is not locally positive definite.

(a) Verify that $\dot{V} = -(Ex_2 \cos(x_1) - \sigma x_4)\, u$, and thus $u = Ex_2 \cos(x_1) - \sigma x_4$ renders $\dot{V} \leq 0$.
Facts: **(1)** One can prove that the down equilibrium point is unstable under this feedback law; the idea is similar to the computation in Problem 4. **(2)** One can also use LaSalle's theorem to determine that all solutions converge to

$$I = \{x_1 = \{0, \pi\},\ x_2 = 0,\ x_4 = 0\} \cup \{x_4 = 0,\ E = 0\}$$

(where x_1 is modulo 2π), and that $u(t) \longrightarrow 0$.

(b) Apply $u = Ex_2 \cos(x_1) - \sigma x_4$ to the **full** cart and pendulum model. What happens to the cart position? Why?

(c) Implement a switching feedback law that will drive the cart and pendulum to the origin for all initial conditions except the downard equilibrium point. Note that when you switch to a stabilizing feedback law about the "origin," you will have to compute x_1 modulo 2π. Furthermore, you will want to have $x_1 \in [-\pi, \pi)$ so that zero is the upward position of the pendulum.

Remark: The advantage of the approach in Problem 5 is that the cart energy is taken into account, so the controller naturally tries to keep the cart velocity small, which in turn, keeps the cart position from growing too rapidly. To really see the difference with respect to what we had done previously, just start the pendulum at $\frac{\pi}{2} - 0.1$, and see how the controller responds. You may think that the next step is to include some form of cart potential energy in the energy function, V, and control the cart position as well. If you figure out how to do this, let me know!

Hints for Problem 2: The problem is extremely similar to an example worked in lecture!

Hints for Problem 3: (a) First write down mathematically what you need to show in terms of ϵ, δ, etc. Then write down what you know about the solutions of the time reversed system (due to asymptotic stability). Then relate solutions of the original system and the time-reversed system to complete your proof. (b) The time-reversed system then satisfies the conditions of LaSalle's theorem, so (a) implies the result.

Hints for Problem 4: (c) You can use the result of Problem 3 to do this. Try using the locally (around the downward equilibrium) positive definite function $\bar{E} = 2 + E = 1 + \cos(x_1) + \frac{x_2^2}{2}$.