Introduction to Lyapunov Stability Theory.

A Geometric Picture.

Up to now we dealt with the problem of ensuring that our nonlinear model $\dot{x} = f(t,x)$, $x(t_0) = x_0$, has a unique solution over a time interval $[t_0, t_0+\delta]$.

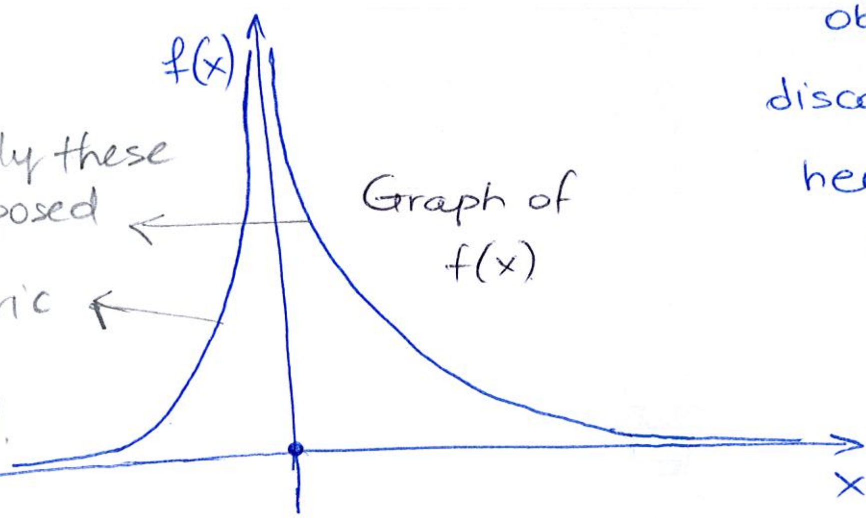
We saw that if the function $f(t,x)$, $t \in [t_0, t_1]$, is locally Lipschitz $\forall\, x, y \in B_r(x_0)$, $\forall\, t$, then we can guarantee that the system has a unique solution over $[t_0, t_0+\delta]$.

Also, if the function $f(t,x)$, $t \in [t_0, t_1]$, is globally Lipschitz $\forall\, x, y \in \mathbb{R}^n$, $\forall\, t$, then we can guarantee that the system has a unique solution over $[t_0, t_1]$

The nature of the theorems is sufficient, not necessary. That means, if the Lipschitz property (locally or globally, respectively) is satisfied, then we can assure that the solution is unique (over $[t_0, t_0+\delta]$ or $[t_0, t_1]$, respectively). If the Lipschitz property is not satisfied, then Theorems 3.1 and 3.2 do not apply, i.e., we can not use Theorems 3.1 and 3.2 for assessing existence and uniqueness of the solution.

Also, that means that if we know that a system has a unique solution over a time interval, then it is not the case that the system is necessarily Lipschitz (locally or globally, depending on the context) This is highlighted in Example 3.5 where the uniqueness of the solution of $\dot{x} = -x^3 = f(x)$, $x(t_0) = x_0$, for all $t \geq t_0$, does not imply that $f(x)$ should be globally Lipschitz. In fact, it is not globally Lipschitz.

Another example. Consider $\dot{x} = \frac{1}{x^2} = f(x)$, $x(t_0) = 0$.



obviously these are supposed to be symmetric wrt the y-axis! :)

obviously $f(t,x)$ is discontinuous at $x=0$, hence not Lipschitz at $x=0$.

However the system has a unique solution given as

$$x(t) = \left(3\,(t-t_0)\right)^{1/3}, \quad \forall\, t \geq t_0, \ \forall\, t_0.$$

So in summary: Continuity of $f(t,x)$ implies at least one solution over a time interval.

[A loose wrap-up]

Lipschitz continuity of $f(t,x)$ Implies a unique solution over a time interval (local or global depends on whether the Lipschitz property holds locally or globally)

remember, Lipschitz continuity is a stronger property than continuity.
Lipschitz continuity $\Rightarrow$ continuity but not vice versa

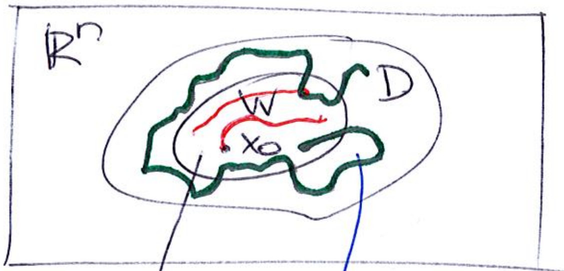
Now, we saw that globally Lipschitz $f(t,x)$ implies existence and uniqueness over a time interval $[t_0, t_1]$ where t_1 can be chosen arbitrarily large. However, many of the models of our physical systems do not possess the globally Lipschitz property, but still have unique solutions!.

Hence it is useful for us to have Theorems that guarantee global (for all $t \geq t_0$) existence and uniqueness of the solution, while requiring the locally Lipschitz property for $f(t,x)$

i.e., requiring locally Lipschitz only.

Of course that comes at a price.

The price is, we have to require something more about the system. Let us consider Theorem 3.3.



It requires (i) $f(t,x)$ locally Lipschitz on D, (ii) a compact (i.e, closed and bounded) subset $W \subset D$, (iii) $x_0 \in W$, AND (iv) knowledge that all solutions starting in W remain in W, i.e, do not escape W!

That reads, we are required to know that the solutions of the system starting from $\boxed{x_0 \in W}$ look like the red trajectories, and there are no solutions like the one in green color.

THEN, the theorem says, the solution from every $x_0 \in W$ is unique and is defined for all $t \geq t_0$.

$f(t,x)$ is is locally Lipschitz on D.

that means in every compact $W \subset D$, $f(t,x)$ is Lipschitz as well. In fact since W is compact (closed and bounded) we can find a sort of "global" Lipschitz constant on W

$\left\{ \begin{array}{l} \text{i.e, a } L \text{ that works} \\ \text{for all } x \in W. \end{array} \right\}$

While the theorem seems to be putting us in a vicious circle – how can we guarantee that the solution $x(t)$ lies entirely in W, $\forall x_0 \in W$, $\forall t$? Wasn't the theorem supposed to tell us if the solution exists and is unique? Well, the theorem requires f to be locally Lipschitz, hence we get confirmation about local existence and uniqueness, and then requires that the solution can not escape a set W. If those two conditions hold, then the theorem tells us the solution is unique and exists for all $t \geq t_0$.

Of course, showing that the solution $x(t, x_0)$ remains in W might not be trivial, but thankfully Lyapunov's methods address this challenge, without solving the system of differential equations!!! The key is to show that the solution $x(t, x_0)$ is trapped in compact level sets of properly defined functions (called Lyapunov functions, to be defined)

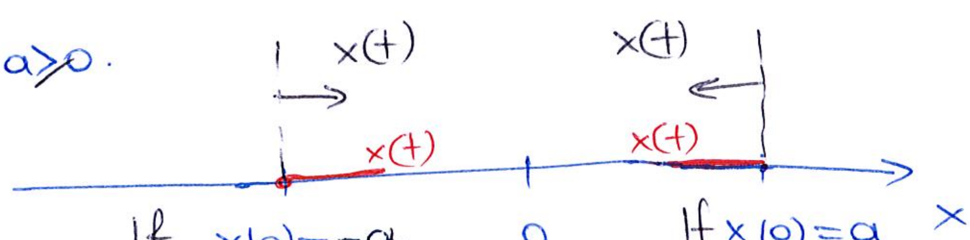
Example (3.6 in Khalil)

$$\dot{x} = -x^3 = f(x)$$

In the meantime, let us see an application of Theorem 3.3.

The function $f(x)$ is locally Lipschitz on $\mathbb{R}$. We want to assess that the solutions starting in a compact set $W \subset \mathbb{R}$ always remain in W. (so that we are able to apply Theorem 3.3)

Let $a > 0$.



If $x(0) = -a$, then $\dot{x}(0) = a^3 > 0$ hence $x(t) > -a$

If $x(0) = a$ then $\dot{x}(0) = -a^3 < 0$ hence $x(t) < a$

That means, the solution is always in $W = \{x \in \mathbb{R} \mid |x| \leq a\}$

Hence, without calculating the solution, we can guarantee that it remains bounded in a closed set W, and the theorem assures us that is unique and exists $\forall\, t \geq 0$.

A conceptual bridge with Lyapunov Stability Theory.

Now, the idea of ensuring that the solutions of the system evolve along (are "trapped") smaller and smaller closed and bounded level sets of what we call Lyapunov functions is fundamental in Lyapunov's stability theory, that studies the stability properties of the equilibrium points of $\dot{x} = f(t,x)$

Example. Let us see the application of Lyapunov's method on a linear system first, to connect stability for LTI with the geometric interpretation of a Lyapunov function.

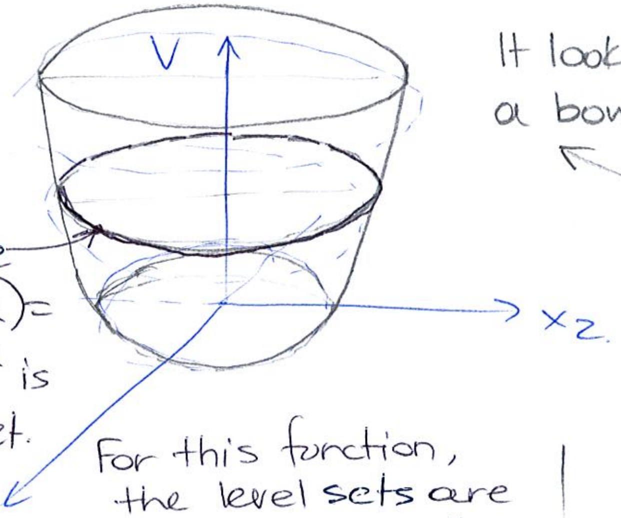
Let us take the harmonic oscillator $\ddot{y} + y = 0$, $y \in \mathbb{R}$.
For state-space form we define $x_1 = y$, $x_2 = \dot{y}$, hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(x), \quad \text{where } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We note that $\det(A) \neq 0$, hence $x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the unique equilibrium.
The eigenvalues of A are $\lambda(A) = \pm j$. Since they are imaginary, we know that the solutions will be periodic trajectories.

Hence we conclude that the system is stable, but not asymptotically stable.

Let us now build the elements of Lyapunov theory for this simple example. (remember this is a linear system)



The surface $V(x_1,x_2) = c^2$ is level set.

It looks like a bowl!

For this function, the level sets are circles around the origin. $x=0$.

The idea is to define an "energy-like" function $V = V(x) =$
$= V(x_1,x_2) = x_1^2 + x_2^2$

Why we call it "energy-like"? Because you can think that x_1 is say the spring displacement x_2 is the mass velocity of the mass-spring system oscillating without friction, hence

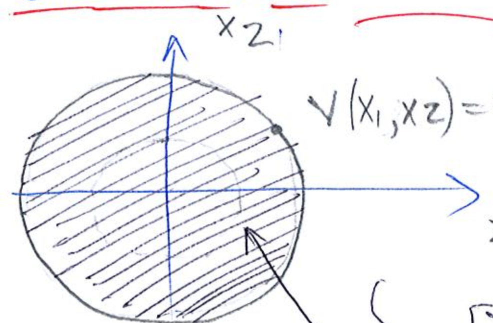
$$\text{total energy} = \underbrace{\text{potential energy}}_{\frac{1}{2}kx_1^2} + \underbrace{\text{kinetic energy}}_{\frac{1}{2}mx_2^2}$$

here we took $k=2$ we took $m=2$.

Some Facts.

- Given a constant $c>0$, we have
  $V(x_1,x_2) \leq c^2 \Rightarrow$
  $x_1^2 + x_2^2 \leq c^2$.

That reads, the sublevel sets of V are circular discs. In fact, they are closed and bounded (i.e compact) sets!!



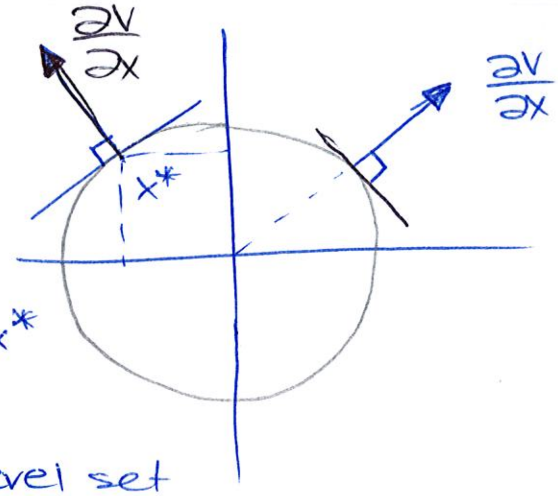
$V(x_1,x_2) = c^2$

$\left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; V(x) \leq c^2 \right\}$

* For this particular choice of function $V(x)$

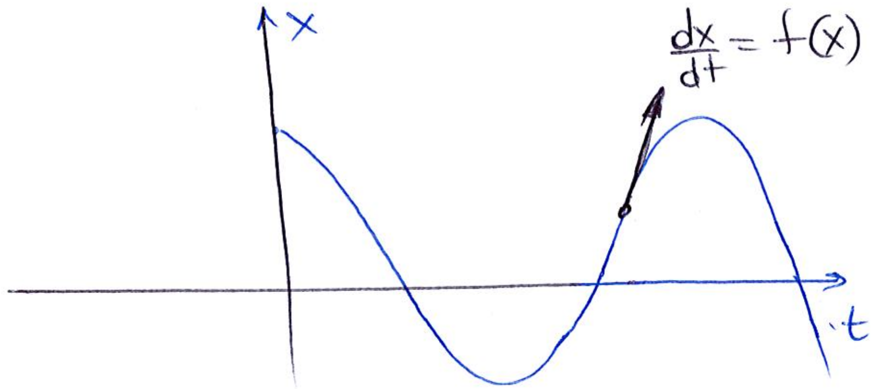
Let us define the gradient of the function $V(x)$

$$\underbrace{\frac{\partial V(x)}{\partial x}}_{\text{Gradient of } V.} = \left[\frac{\partial V(x)}{\partial x_1}, \quad \frac{\partial V(x)}{\partial x_2} \right]$$

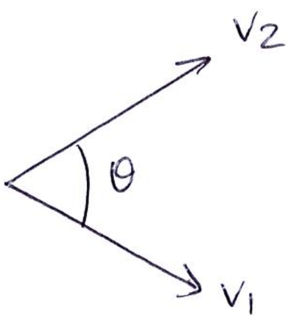


The gradient vector at x^* is orthogonal to the tangent line to the level set $\{x \in \mathbb{R}^2 \mid V(x) = c^2\}$ at x^*

Let us now remember how our $\dot{x}(t) = f(x(t))$ looks like:

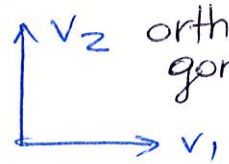


And also let us remember the geometric interpretation of the inner product $\langle \cdot , \cdot \rangle$ between two vectors $v_1, v_2 \in \mathbb{R}^2$

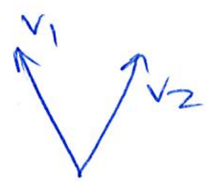


$$\cos\theta = \frac{\langle v_1, v_2 \rangle}{\|v_1\|_2 \, \|v_2\|_2},$$

$\langle v_1, v_2 \rangle = 0 \iff$ orthogonal.

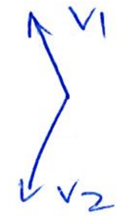


$\langle v_1, v_2 \rangle > 0 \iff$



point along the "same" direction

$\langle v_1, v_2 \rangle < 0 \iff$



vectors point along "opposite" directions

So if we consider the inner product $\left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle \ldots$

$\left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle = 0 \iff$ trajectory evolves tangent to $V(x) = c^2$, i.e, along the direction of <u>constant</u> $V(x)$

$\left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle > 0 \iff$ trajectory evolves in the direction of <u>increasing</u> $V(x)$

$\left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle < 0 \iff$ trajectory evolves in the direction of <u>decreasing</u> $V(x)$

Now let us define $\dot{V}(x(t)) = \frac{d}{dt} V(x(t)) = \frac{\partial}{\partial x} V(x) \Big|_{x=x(t)} \cdot \dot{x}(t)$

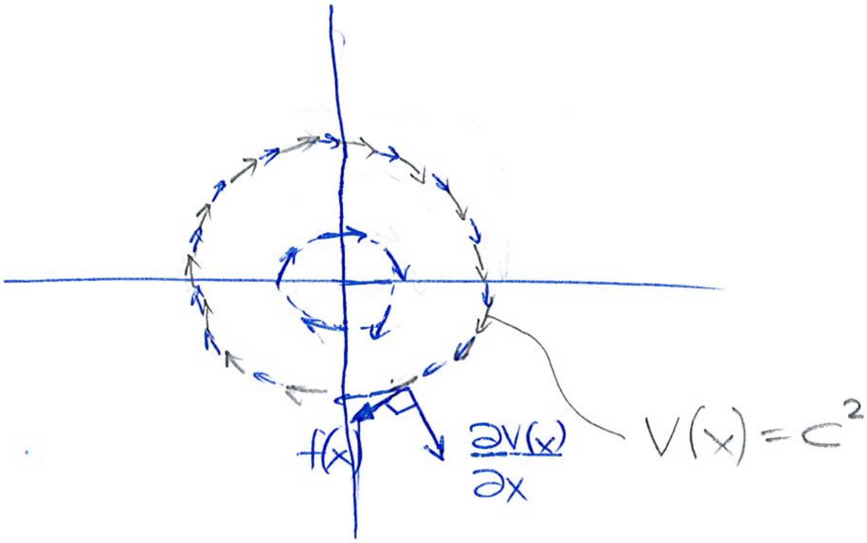
$$= \frac{\partial V(x)}{\partial x}\Big|_{x=x(t)} \cdot f(x(t))$$

or simpler: $\boxed{\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x)}$

And note that $\dot{V}(x) = \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle$

so back to our example: $\dot{V}(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = 0.$

That means, V remains constant along the trajectories of our system, i.e. system evolves <u>along circles</u>.



The phase portrait would look like that.

Indeed, we expected for the equilibrium to be stable, i.e. trajectories remaining close to the equilibrium

The solutions are tangent to the level sets of V, because

$$\dot V(x) = \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle := 0.$$

<u>Second Example</u>.

$$\dot x = \begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(x)$$

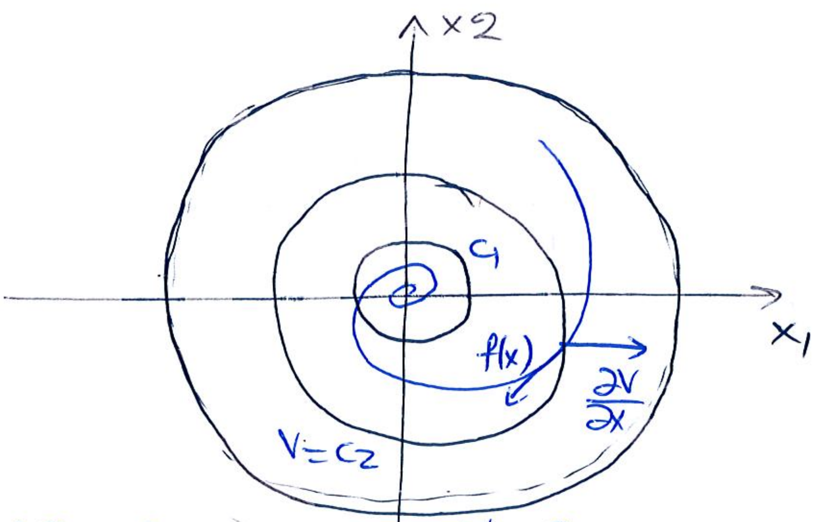
Keep the same $V(x_1, x_2) = x_1^2 + x_2^2$

Compute: $\dot V(x) = [2x_1 \ \ 2x_2] \begin{bmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_1 - \frac{1}{2}x_2 \end{bmatrix} = -x_1^2 - x_2^2 < 0$

for $\begin{Bmatrix} x_1 \neq 0 \\ x_2 \neq 0 \end{Bmatrix}$

Hence we have

$$\left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle < 0, \quad x \neq 0.$$



$c_1 < c_2 < c_3$

That means, the solutions evolve along lower and lower level sets of V, because

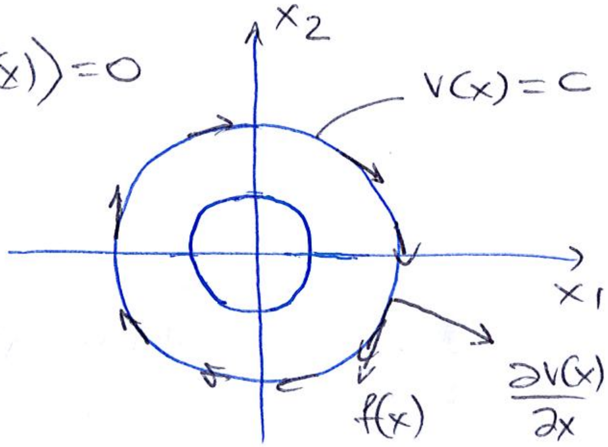
$$\dot V(x) = \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle < 0$$

for $x \neq 0$.

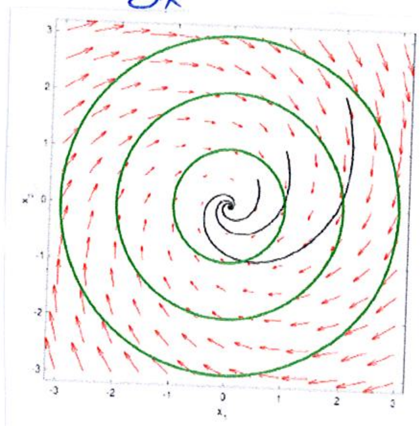
Lyapunov: Definitions and Main Stability Theorem.

Last time we saw an introductory, geometric viewpoint of Lyapunov stability theory. More specifacally, we considered the phase portraits of <u>linear (for simplicity)</u> systems with stable and asymptotically stable equilibria, respectively, and studied their evolution relatively to the level sets of a properly-defined function $V(x)$.

$\langle \frac{\partial V}{\partial x}, f(x) \rangle = 0$
$\forall x$



$\langle \frac{\partial V(x)}{\partial x}, f(x) \rangle < 0$, $\forall x \neq 0$.



Stable equilibrium
[Eigenvalues on the imaginary axis, solutions are periodic trajectories]

↓

Physical Examples
Mass-Spring system without damping.
Pendulum without friction.

Asymptotically stable equilibrium.
[Eigenvalues on the left-half plane, solutions are converging to the equilibrium]

↓

Physical Examples
Mass-Spring-Damper
Pendulum with friction.

<u>Remark</u>: The choice of the function $V(x)$ is <u>not</u> arbitrary. The physical motivation is that $V(x)$ expresses the

total energy of the system. But for applying Lyapunov's method to investigate the stability of the equilibrium, we do not need to define and use the function that expresses the total energy of the system. The energy function might give us some motivation on how to choose the function $V(x)$, but $V(x)$ does not have to coincide with the energy function. We will see guidelines on how to choose candidate Lyapunov functions $V(x)$ later on.

<u>Remark</u>: In the meantime, we have to formally define what we mean by "stability of an equilibrium in the sense of Lyapunov (i.s.L)".

<u>Definition</u>: Let $\dot{x} = f(x)$ be an ODE on $\mathbb{R}^n$ with equilibrium point x_e.
Suppose there exists $\rho > 0$ such that, for <u>each</u> initial condition $x_0 \in B_\rho(x_e)$, a solution $x(t, x_0)$ of the ODE <u>exists on $[0, +\infty)$</u> and is <u>unique</u>. Then, the equilibrium x_e is.

(a) Stable in the sense of Lyapunov (i.s.L) if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$, such that $\|x_0 - x_e\| < \delta \Rightarrow \|x(t, x_0) - x_e\| < \epsilon$, $\forall t \geq 0$.

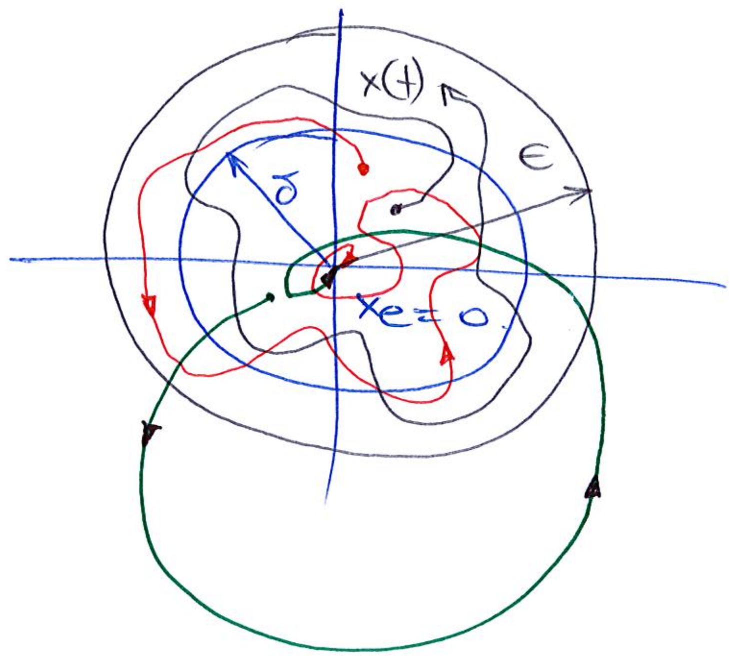
(b) Unstable, if it is not stable i.s.L. Exercise: Negate the stability i.s.L property to write down the instability definition!

(c) Asymptotically stable if it is stable in the sense of

Lyapunov, AND $\exists\, \eta > 0$ such that

$$\|x_0 - x_e\| < \eta \Rightarrow \lim_{t\to\infty} \|x(t, x_0) - x_e\| = 0.$$

Note: "$\exists\, \eta > 0$ such that ..." can be also read as that "δ can be chosen such that ..."



In black: Pictorial representation of stability i.s.L.

In red: Pictorial representation of asymptotic stability i.s.L.

What is the case for the green trajectory?

Remarks. Define $\varphi : \mathbb{R}^n \to C^n[0, +\infty)$ as $\varphi(x_0)(t) = x(t, x_0)$ that is, φ maps initial conditions to solutions of the ODE. Then, stability in the sense of Lyapunov is equivalent to continuity of φ at x_e, when the "sup" norm is used on $C^n[0, +\infty)$

Similarly, unstable is equivalent to φ being discontinuous at x_e.

Remark. The extra condition in (c) is often called attractivity. Hence "asymptotic stability = stability i.s.L + attractivity".

Now, Lyapunov's theorems provide us with the tools to investigate the stability properties of the equilibrium x_e of $\dot{x} = f(x)$.
We will also see how the fundamental existence and uniqueness theorems (Theorems 3.1, 3.2, 3.3, and primarily the latter one) become important and relevant! in stability investigation via Lyapunov's method.
(and are not merely abstract mathematical formalities of no practical implication!)

As always, the best way to understand the physical interpretation of a theorem, is to go through the mathematical proof!

We are now ready to state and prove! the main stability theorem (Theorem 4.1 in your textbook)

Theorem. Let $\dot{x} = f(x)$ be an ODE on $\mathbb{R}^n$ with equilibrium point $x_e = 0$. Assume there exists an open set containing the origin such that:

1. $f: D \to \mathbb{R}^n$ is locally Lipschitz, that is, there exists $L < \infty$ such that $\|f(x) - f(y)\| \leq L \|x - y\|$, $\forall x, y \in D$

and

2. there exists a continuously differentiable function $V: D \to \mathbb{R}$ such that:

(a) $V(0) = 0$
(b) $V(x) > 0$ for $x \in D$, $x \neq 0$
(c) $\dot{V}(x) \leq 0$ for $x \in D$, where $\dot{V}(x) = \frac{\partial V}{\partial x}(x) \cdot f(x)$

Then: The equilibrium point is stable i.s.L.

Moreover, if

(d) $\mathring{V}(x) < 0$ for $x \in D$, $x \neq 0$, then the equilibrium point x_e is asymptotically stable i.s.L.

Remark: A function $V : D \to \mathbb{R}$ that satisfies the conditions of the Theorem is called a Lyapunov function.

Proof of the Theorem.

Part 1: To show stable i.s.L. we must establish that

(a) $\exists\, \rho > 0$ such that for all $x_0 \in B_\rho(0)$, a solution $x(t, x_0)$ exists and is unique on $[0, +\infty)$

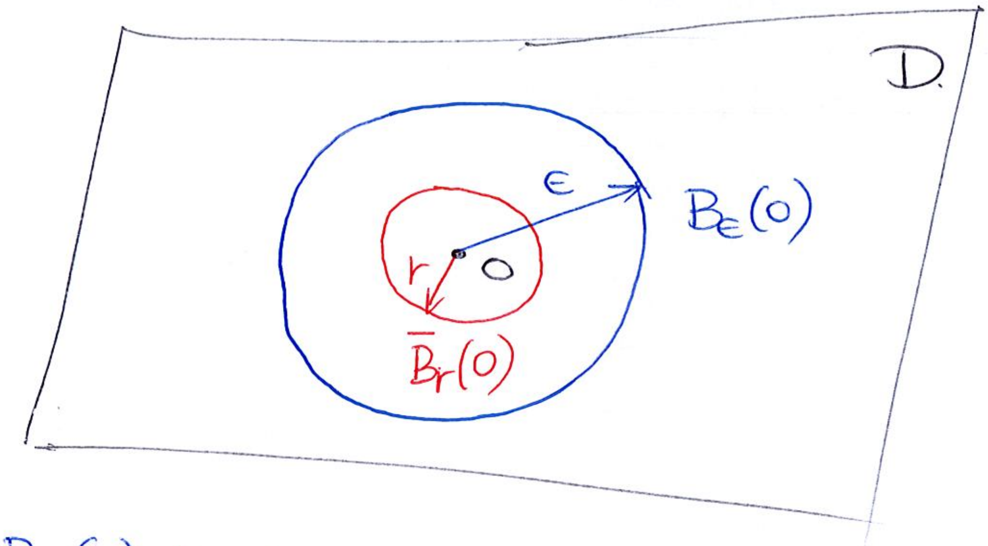
(b) $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that $\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \epsilon$ for all $t \geq 0$.

We will show (a) and (b) together by taking $\rho = \delta$.

Here we go:

Let $\epsilon > 0$ be given. Since D is an open set containing zero, we can choose (i.e, there exists) $0 < r < \epsilon$ such that $\overline{B}_r(0) = \{ x \in \mathbb{R}^n \mid \|x\| \leq r \} \subset D$.

we use overbar $\overline{B}_r(0)$ to denote the closed ball around zero.



$B_\epsilon(0)$ is open: Our objective is to find a ball of radius $\delta > 0$ and satisfying the properties (a) and (b) above.

Now, we choose $\bar{B}_r(0)$ to be closed: The reason is to create a set that is closed and bounded, hence compact. The usefulness of compact sets in our analysis lies in the fact that a continuous function on a compact set attains its max and min values.

Note: We have not formally defined compact sets yet, but definitions and mathematical background are given in the Appendices of the textbook and in the Appendix of this lecture.

Going back to the proof.

Define: $\boxed{\alpha := \min_{\|x\|=r} V(x).}$

[Note: we can define the minimum of $V(x)$ over $\|x\|=r$ since this set is closed and bounded (application of Weierstrass theorem].

Then: $\alpha > 0$, from the definition of $V(x)$.

Let us now choose $0 < \beta < a$ and define the set

$$\Omega_\beta = \{ x \in \bar{B}_r(0) \mid V(x) \leq \beta \}$$

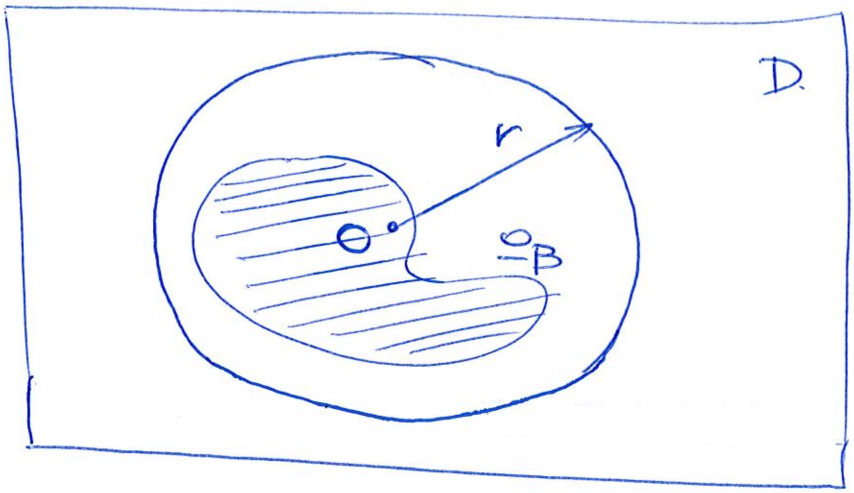


Figure ①

By definition, $\Omega_\beta \subset \bar{B}_r(0)$.

We want to show a stronger condition:

Claim: $\Omega_\beta \subset B_r(0)$, i.e. Ω_β is contained in the open ball of radius r around the origin.

Proof: Assume $\|x\| = r$.

Then $\underbrace{V(x) \geq a}_{\substack{\text{recall} \\ a := \min_{\|x\|=r} V(x)}} > \beta$, which implies that $x \notin \Omega_\beta$.

well, in general of defining Ω_β to be contained in the $\bar{B}_r(0)$.

The importance of the previous claim is the following:

It essentially says that Ω_β looks like the one in the figure ① above, and not like the one in figure ② below.

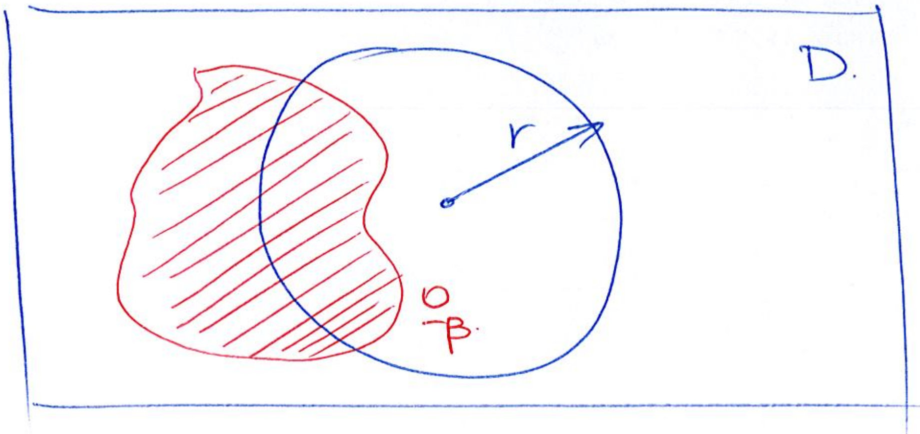


Figure ②

In fact, the situation in Figure ② is "bad" for the following reason: If we have that Ω_β "spills out" of $B_r(0)$, that means that a solution can start with $V(x_0) \leq \beta$, that is, $x_0 \in \Omega_\beta$, continue to satisfy $V(x(t,x_0)) \leq \beta$, i.e, $x(t,x_0) \in \Omega_\beta$, and yet leave the set $B_r(0)$. [But we want to make it get trapped in $B_r(0)$! Keep reading :]

Whereas: the situation in Figure ① is good, because we want to show that a solution that starts with $x_0 \in \Omega_\beta$ will have to stay within Ω_β and as thus it can not leave the set $B_r(0)$.

Claim: Solutions of $\dot{x} = f(x)$ with $x_0 \in \Omega_\beta$ remain in Ω_β. As thus, by Theorem 3.3, we conclude that for all $x_0 \in \Omega_\beta$, solutions exist on $[0, +\infty)$ and are unique.

Proof: Let $\phi(t)$, $t_0 \leq t \leq t_1$ be a solution of $\dot{x} = f(x)$, with $\phi(t_0) \in \Omega_\beta$. We want to show that $\phi(t) \in \Omega_\beta$ for all $t_0 \leq t \leq t_1$.

Since $\mathring{V}(x) \leq 0$, we have that

$V(\phi(t)) \leq V(\phi(t_0))$. Also,

since $\phi(t_0) \in \Omega_\beta$, we have $V(\phi(t_0)) \leq \beta$. Thus:

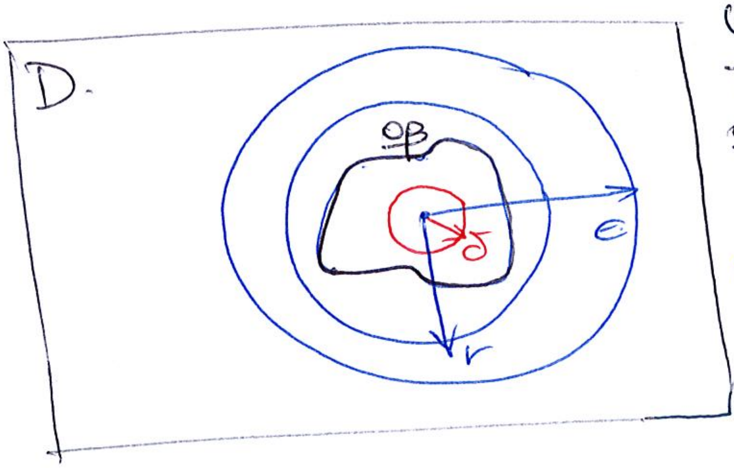
$V(\phi(t)) \leq V(\phi(t_0)) \leq \beta$. Hence we conclude that

$\phi(t) \in \Omega_\beta$, for $t_0 \leq t \leq t_1$.

<u>Remark</u>: So we have:

from Theorem 3.3, that a unique solution $x(t, x_0)$ exists on $[0, +\infty)$, $\forall\ x_0 \in \Omega_\beta$. Also, since it does never escape Ω_β, and since $\Omega_\beta \subset B_r(0) \subset B_\epsilon(0)$, we have that the solution never escapes $B_\epsilon(0)$, i.e. $\|x(t, x_0)\| < \epsilon$, $\forall\ t \geq 0$.

<u>Remark</u>: To finally conclude stability i.s.L, we need to show that there exists $\delta > 0$ such that $B_\delta(0) \subset \Omega_\beta$.



Up to now we have that: Trajectories starting in Ω_β stay in Ω_β, hence stay in $B_\epsilon(0)$ We now need to show that there exists a $B_\delta(0)$ in Ω_β.

Claim. There exists $\delta > 0$ such that $B_\delta(0) \subset \Omega_\beta$.

Proof. Since V is continuous at 0, there exists $0<\delta<r$ such that $\|x-0\| < \delta \Rightarrow |V(x)-V(0)| < \beta$.

Now, since $V(0)=0$, and $V(x) \geq 0$, we can write $\|x\| < \delta \Rightarrow V(x) < \beta$. That is, $B_\delta(0) \subset \Omega_\beta$.

That concludes the part of showing stability i.s.L.

Part 2. Asymptotic stability.

We assume that $\dot{V}(x) < 0$ for $x \in D$ and $x \neq 0$.

We show that $\exists\, \eta > 0$ such that

$$\|x_0\| < \eta \Rightarrow \lim_{t \to \infty} x(t, x_0) = 0.$$

We will in fact show that $\delta = \eta$ works, where δ was defined in Part 1.

Claim. If $\|x_0\| < \delta$, then $\lim\limits_{t \to \infty} V(x(t,x_0)) = 0$.

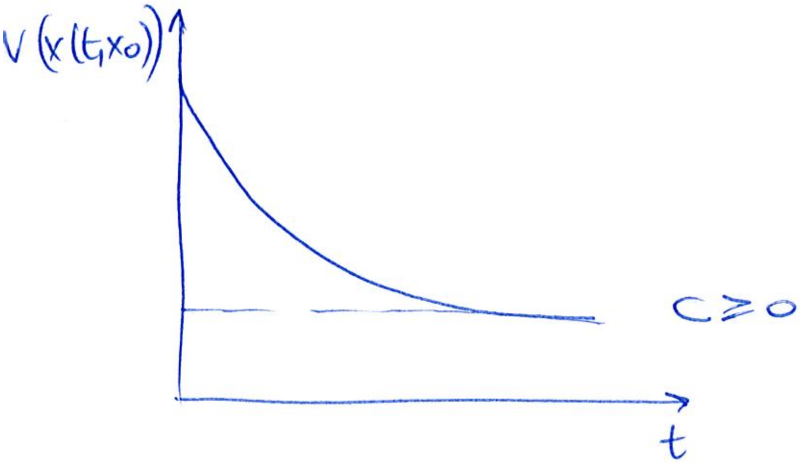
Proof. $\|x_0\| < \delta \Rightarrow x(t,x_0) \in \Omega_\beta \subset D$, for all $t \geq 0$.

This implies that $\dot{V}(x(t,x_0)) \leq 0$, for all $t \geq 0$.

Therefore, $V(x(t,x_0))$ is a non-increasing function of t, that is bounded from below by zero.

Hence, there exists a unique $c \in \mathbb{R}$, $c \geq 0$, such that

$$\lim_{t \to \infty} V(x(t,x_0)) = c.$$ [Why? see next]



Why does the function $V(x(t,x_0))$ have a limit as $t \to \infty$?

See also Appendix

We have that $V(x(t,x_0))$ converges to a constant $c \geq 0$:

Since $\dot{V}(x) \leq 0$, we know that $V(x(t,x_0))$ is non-increasing.
Since $V(x) \geq 0$, we have that $V(x(t,x_0))$ is bounded from below. These two properties guarantee that $V(x(t,x_0))$ has a limit $c \geq 0$.

Now, to conclude that $x(t,x_0) \xrightarrow{t\to\infty} 0$, we need to show that $c = 0$. We prove the argument by contradiction.

Assume that it is the case that $\boxed{c > 0}$ We have that $\boxed{V(x(t,x_0)) \geq 0 \ , \ \forall\, t \geq 0.}$ This is true because $x(t,x_0)$ is trapped in $\Omega_\beta \subset D$, and we know $V(x) \geq 0$ on D.

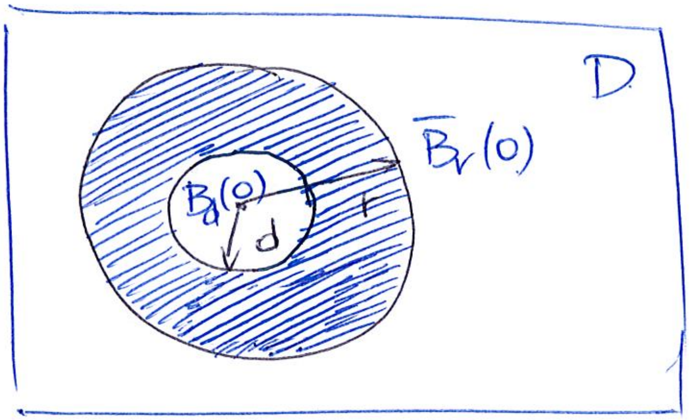
We also have that, from the continuity of V at zero, there exists a $d > 0$ such that

$$\|x\| < d \Rightarrow V(x) < \frac{c}{2}$$ (similarly to the last claim of Part 1), or equivalently, $B_d(0) \subset \Omega_{\frac{c}{2}}$,

where $\Omega_{\frac{c}{2}} := \{x \in \Omega_\beta \mid V(x) \leq \frac{c}{2}\}$.

Now, since $V(x(t,x_0))$ is a <u>decreasing function of t</u> with limit c, we have that $V(x(t,x_0)) \geq c$ for all $t \geq 0$.

This implies that $x(t,x_0)$ never enters $\Omega_{\frac{c}{2}}$, and as thus it never enters $B_d(0)$.....



Now we have that, on the compact set $\{x \in D \mid d \leq x \leq r\}$, $\dot{V}(x)$ is continuous. Hence, by the Weierstrass theorem, it attains a maximum value on that set.

In addition, since $\dot{V}(x) < 0$ everywhere except for the origin, it follows that the maximum value of $\dot{V}(x)$ on the compact set $\{x \in D \mid d \leq x \leq r\}$ is negative.

Denote $-\gamma = \max\limits_{d \leq \|x \leq r} \dot{V}(x) < 0$

Then, using the fundamental theorem of calculus,

$$
\begin{aligned}
V(x(t,x_0)) &= V(x_0) + \int_0^t \dot{V}(x(\tau,x_0))\,d\tau \\
&\leq V(x_0) + \int_0^t -\gamma\,d\tau \\
&\leq V(x_0) - \gamma t,
\end{aligned}
$$

which <u>tends to $-\infty$ as $t\to\infty$!</u>

This implies that $V(x(t,x_0))$ becomes negative for sufficiently large t, which is a contradiction since $x(t,x_0) \subset \Omega_\beta$, and $V(x) \geq 0$ in Ω_β by definition. Hence it can not be the case that $c > 0$, which implies that $c = 0$. This implies that $\lim\limits_{t\to\infty} V(x(t,x_0)) = 0$.

Remark. Note that we used the following three principles when analyzing

$$(**) \quad \dot{x} = f(x), \quad x \in D \subset \mathbb{R}^n, \quad D \text{ an open set,}$$

where $V: D \to \mathbb{R}^n$ is continuously differentiable, and f is locally Lipschitz on D.

Principle 1. Suppose $\dot{V}(x) \leq 0$, for $x \in D$. Then, for all $t \geq 0$ for which a solution to $(**)$ exists and remains in D, we have: $\boxed{V(x(t, x_0)) \leq V(x_0).}$

Principle 2. Let $\beta > 0$ and suppose the sublevel set $\Omega_\beta = \{x \in D \mid V(x) \leq \beta\}$ is compact, and that $\dot{V}(x) \leq 0$ for $x \in D$.

Then, $\forall\ x_0 \in \Omega_\beta$, a solution $x(t, x_0)$ to $(**)$ exists on the unbounded interval $[0, +\infty)$, is unique, and remains in Ω_β for all $t \geq 0$.

Principle 3. Let $C \subset D$ be a compact set such that

(a) $\forall\ x \in C$, $V(x) > 0$ and

(b) $\forall\ x \in C$, $\dot{V}(x) < 0$.

Then any solution of $(**)$ starting in C must exit C. More precisely, for all $x_0 \in C$, there exists $0 < T < +\infty$ such that $x(T, x_0) \notin C$.

Note: We used this last principle in the proof of excluding that $c > 0$. In fact, we proved that the solution can not stay in the compact set $\{x \in D \mid d \leq x \leq r\}$ as that leads to a contradiction (i.e., that $V(x(t, x_0)) < 0$ for large t).

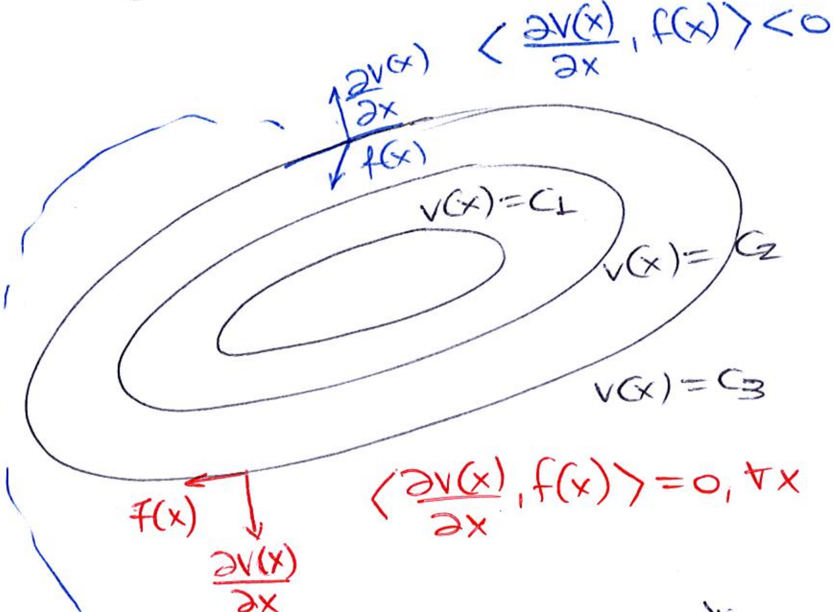
A continuously differentiable function $V: D \to \mathbb{R}$

such that $V(0) = 0$, $V(x) > 0$ on $D/\{0\}$, and

$\dot{V}(x) \leq 0$ on D

is called a Lyapunov function.

The surface $V(x) = c$, $c > 0$, is called Level surface, (or level set) or sometimes Lyapunov surface.



Crossing the surface $V(x) = C_3$* means that the trajectory is trapped within the set $\{x \in D \mid V(x) < C_3\}$

The condition

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} \cdot f(x) \leq 0$$

implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in D \mid V(x) \leq c\}$ and can never come out again!

* i.e. $\frac{\partial V(x)}{\partial x} \cdot f(x) < 0$

In fact, when $\dot{V}(x) < 0$, the trajectory moves along Lyapunov surfaces with smaller and smaller c. As c decreases, the Lyapunov surface shrinks to the origin, showing that the trajectory approaches the origin as time progresses.

If we only know that $\underline{\mathring{V}(x) \leq 0}$ (i.e. not strictly $\mathring{V}(x) < 0$) then the Theorem 4.1 does not tell us if the trajectory approaches zero, i.e., if the origin is asymptotically stable.* For such cases, La Salle's Theorem (to follow) can give us a more powerful tool.

* It can only guarantee that the origin is stable i.s.L.

Definition.

A function $V: D \rightarrow \mathbb{R}$ such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for } x \neq 0$$

is called positive definite.

A function $V: D \rightarrow \mathbb{R}$ such that

$$V(0) = 0 \text{ and } V(x) \geq 0 \text{ for } x \neq 0$$

is called positive semi-definite.

A function $V: D \rightarrow \mathbb{R}$ is called negative (semi-)definite if $-V(x)$ is positive (semi-)definite.

Hence, Lyapunov's theorem can be rephrased as.

"the origin is stable if there is a continuously differentiable positive definite function $V(x)$, so that $\mathring{V}(x)$ is negative semi-definite, and it is asymptotically stable if $\mathring{V}(x)$ is negative definite."

Main Stability Theorem - Examples.

Last time we studied the main theorem for investigation of the stability properties of the equilibrium of $\dot{x} = f(x)$. In summary, the theorem tells us that if we can find a positive definite function $V(x): D \to \mathbb{R}$ *, whose derivative $\dot{V}(x) := \frac{\partial V(x)}{\partial x} \cdot f(x)$ is negative semi-definite, then we can conclude that the equilibrium point is stable in the sense of Lyapunov. If the derivative $\dot{V}(x)$ is negative definite, then the theorem tells us that the equilibrium point is asymptotically stable i.s.L.

* continuously differentiable.

Note: Review the definitions of positive/negative semi-definite functions from the previous lecture

Note: The theorem conditions are sufficient. If we fail to show stability/asymptotic stability with a candidate Lyapunov function $V(x)$, then the theorem does not conclude that the equilibrium point is not stable/asymptotically stable i.s.L. The best we can do in this case is to keep investigating with a different Lyapunov function candidate!

Let us see some illustrative examples.

Example. (4.3 in Khalil.)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin x_1 , \quad a>0. \end{cases}$$

(Pendulum without friction)

Consider the equilibrium $(x_1, x_2) = (0,0)$ We want to investigate its stability properties.

Let us consider the positive definite function

$$\boxed{V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2}$$

Check: Is the function $V(x)$ indeed positive definite?

Well, we have $V(0)=0$, and $V(x)>0$, $x\neq 0$.

Hence yes.

Take the time derivative as:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x) = \begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a\sin x_1 \end{bmatrix} = 0.$$

The time derivative is identically zero. From Theorem 4.1 we conclude the origin is stable i.s.L.

In fact, since $\dot{V}(x) :\equiv 0$, we can conclude that the trajectories do not approach the origin. (they get trapped on the level set $V(x(t_0)) = c$, i.e., on the level set of the function they start from. Remember the geometric representation.)