EECS 562/AE 551 - Nonlinear Systems and Control
HW #5

Due on Thursday, February 9th, 2023
by 11:59pm on Canvas

1. (20 points) Do either Khalil, Nonlinear Systems, 3^{rd} Edition, Page 183, Prob. 4.11 **or** Khalil, Nonlinear Systems, 3^{rd} Edition, Page 183, Prob. 4.12.

cools

   Remark: You can keep your proofs quite short. Do NOT reprove things known from class or the textbook.

2. (20 points) Show that the origin of the system below is unstable.

$$\dot{x}_1 = -x_1 + x_2^6$$
$$\dot{x}_2 = x_2^3 + x_1^6$$

hints

3. (20 points) Khalil, Nonlinear Systems, 3^{rd} Edition, Page 184, Prob. 4.15. ok

4. (20 points)

   Consider the ordinary differential equation $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$, and suppose f has an equilibrium point at the origin. If f is an analytic function, then applying the multivariable version of Taylor's Theorem near the origin it can be written $f(x) = f(0) + Ax + r(x)$ where $A = \frac{\partial f}{\partial x}|_{x=0}$ and $r(x)$ are higher order terms which satisfy $\lim_{x\to 0} \frac{||r(x)||}{||x||} = 0$.

   (a) Prove that if A is negative definite, the linear system $\dot{z} = Az$ is globally asymptotically stable.

   (b) Let $V(x) = x^T x$ be a candidate Lyapunov function. What theorem could be used to prove (local) asymptotic stability of the origin of the nonlinear system $\dot{x} = f(x)$ using this (potential) Lyapunov function?

b,c check

   (c) What conditions on V and $\dot{V}$ are required in the statement of the theorem identified in part (b)?

   (d) Still assuming that A is negative definite and using the Lyapunov function candidate V from part (b), prove that there exists some $r > 0$ such that the conditions on V and $\dot{V}$ you identified in part (c) are satisfied at every x in $B_r(0)$.

   Remark: The result of Problem 4 shows that if the linear system $\dot{z} = Az$, obtained by linearizing about an equilibrium point of the nonlinear system, is globally asymptotically stable, then the equilibrium point of the nonlinear system is locally asymptotically stable. In other words, it relates local (asymptotic) stability of the nonlinear system to global (asymptotic) stability of the linear system. This relationship justifies linear control theory! A stronger result, which we will not prove, is the Hartman-Grobman theorem, which also relates the global behavior of the linear system to the local behavior of the nonlinear system near an equilibrium point in certain (more general) cases.

5. (20 points)

Let

$$V(x) = x^T P x, \quad P = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

and let

$$f(x) = \begin{bmatrix} -2.5 & -2.5 \\ -2.5 & -7.5 \end{bmatrix} x + \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \begin{bmatrix} x_1^3 \\ 0 \end{bmatrix}$$

ok

(a) Sketch the following set: $\left\{x \in \mathbb{R}^2 | V(x) = 0.5\right\}$ (you may use MATLAB or do it by hand). You should obtain an ellipse. How are the major axis and minor axis of the ellipse (the longest and shortest diameters, respectively) related to the eigenvectors of P?

need to solve

(b) Prove that the origin of the system $\dot{x} = f(x)$ is asymptotically stable on the domain $\left\{x \in \mathbb{R}^2 | V(x) < 0.5\right\}$ using the candidate Lyapunov function V given in the problem statement.

(c) (OPTIONAL) Find the maximum value of $c > 0$ such that it can be shown that the origin of the system $\dot{x} = f(x)$ is locally asymptotically stable on the domain $\left\{x \in \mathbb{R}^2 | V(x) < c\right\}$ using the candidate Lyapunov function V given in the problem statement. Note that other choices of Lyapunov functions may provide larger domains, but this problem asks what can you show with this particular Lyapunov function candidate.

Remark: The result of Problem 5 shows that the open ellipse corresponding to $\{x \in \mathbb{R}^2 : V(x) < c\}$ is contained in the region of attraction of the equilibrium point at the origin and that, for $c = c_{\max}$, no larger ellipse lies in the region of attraction. Soon we will see how this set $\{x \in \mathbb{R}^2 : V(x) < c_{\max}\}$, called a sublevel set, is used to estimate regions of attraction more generally.

Hints for Problem 1: The wording of the Prob. 4.11 may be somewhat confusing. Focus on the condition, "not negative semi-definite arbitrarily near the origin" and use this to prove instability using Theorem 4.3.

The next hint is to write down carefully what "not negative semi-definite arbitrarily near the origin" means, and then the solution will become more clear.

To get you going, I will do the case "not negative definite arbitrarily near the origin" and then let you handle the case "not negative semi-definite arbitrarily near the origin". Recall that $V_1 : \mathbb{R}^n \to \mathbb{R}$ is negative definite if $V_1(x) < 0$ for $x \neq 0$. Negative definite near the origin is the same as locally negative definite, that is, there exists $r > 0$ such that $V_1(x) < 0$ for all $x \in B_r(0)$, $x \neq 0$. Hence, not negative definite near the origin means that, there does not exist $r > 0$ such that $V_1(x) < 0$ for all $x \in B_r(0)$, $x \neq 0$. In other words, for every $r > 0$, there exists $x_0 \in B_r(0)$, $x_0 \neq 0$, such that $V_1(x_0) \geq 0$.

Hints for Problem 2: Try $V(x) = -\frac{1}{6}x_1^6 + \frac{1}{4}x_2^4$, and see Problem 1.

Remark: Your $\dot{V}$ is not the following, but it is something like this. Suppose

$$\dot{V}(x) = (x_1)^4 + (x_2)^4 - x_1 x_2 (x_1)^4$$

Then, $|-x_1 x_2 (x_1)^4| \leq \frac{1}{4}(x_1)^4$ for $(x_1, x_2) \in B_{1/2}(0)$, where the open ball is defined with the max-norm $||x||_\infty$. Hence,

$$\dot{V}(x) = (x_1)^4 + (x_2)^4 - x_1 x_2 (x_1)^4 \geq (1 - \frac{1}{2})(x_1)^4 + (x_2)^4$$

for $(x_1, x_2) \in B_{1/2}(0)$.

Hints for Problem 4:

(a) This can be shown either by verifying the definition of asymptotic stability directly, or by choosing a suitable Lyapunov function and applying a theorem from Khalil.

(d) Note that $\frac{\partial x^T x}{\partial x} = 2x^T$. Imagine that $f_i(x) = A_i x + x^T B_i x$ for $1 \leq i \leq n$ where $f_i(x)$ denotes the ith entry of the vector $f(x)$. Then $r_i(x) = x^T B_i x$ for $1 \leq i \leq n$. Try to see if you can prove the result for this particular choice of f (and for any possible set of B_is). If you can complete this part for $f_i(x) = A_i x + x^T B_i x$ then you will receive full credit, although the general case above is not much more difficult.

Hints for Problem 5:

(b) Note that $\frac{\partial x^T P x}{\partial x} = x^T(P + P^T)$. The most straighforward way to show this problem is to note that the set $\{x \in \mathbb{R}^2 : V(x) < 0.5\}$ is contained in $B_1(0)$. Then, proving that the system is asymptotically stable on $B_1(0)$ implies that it is asymptotically stable on $\{x \in \mathbb{R}^2 : V(x) < 0.5\}$.

Suppose you have an expression $-5x_1^2 - 5x_2^2 + x_1^4$ and you want to show that, over a set of possible values of (x_1, x_2), this expression is negative. To do so, you might show that the maximum value of the expression over that set is negative. For any fixed x_1, the larger in absolute value x_2 becomes the more negative the expression becomes. Therefore, for a fixed x_1, the maximum value of the expression corresponds to the case where $x_2 = 0$. So, to maximize that expression over some set of possible values, you might consider maximizing the simpler expression $-5x_1^2 + x_1^4$, or at least showing that it is negative over some set of interest.

(c) The most straightforward way to achieve this would be to vary c over a range of potential values in MATLAB and, for each c to sample a large number of points on the ellipse given by $\{x \in \mathbb{R}^2 : V(x) = c\}$. Testing V and $\dot{V}$ on these sample points should allow you to estimate the maximum value of c.