EECS 562 - Nonlinear Systems and Control
HW #7

Due on Thursday, March 9th, 2023
by 11:59pm on Canvas

1. (10pt) Consider the linear, scalar, time-varying differential equation

$$\dot{x}(t) = a(t)\, x(t), \quad x(t_0) = x_0 \tag{0.1}$$

where $a : [0, \infty) \longrightarrow \mathbb{R}$ is continuous. Prove that the origin is asymptotically stable i.s.L. if, and only if, it is globally asymptotically stable i.s.L.

example 2. (10 pt)Khalil, Nonlinear Systems, 3^{rd} Edition, Page 188, Prob. 4.36.

3. (20 pt) (Old Exam Problem) Estimate the region of attraction R_A of the origin for the asymptotically stable system given below, using the provided data. Sketch an estimate of R_A. Your exam score would depend on the size of your estimate; here, that will not be the case.

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + \left[x_1^4 - 2\right] x_2 \end{aligned}$$

Facts that may be useful to you when working Problem 2 (were also provided on the exam):

ok

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad A^T P + PA = -Q \quad P = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[V, D] = \text{eig}(P); \quad V = \begin{bmatrix} -0.9239 & 0.3827 \\ 0.3827 & 0.9239 \end{bmatrix} \quad D = \begin{bmatrix} 3.414 & 0 \\ 0 & 0.5858 \end{bmatrix}$$

4. (30 pt)Estimate the region of attraction R_A of the origin for the asymptotically stable system given below

ok

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \left(x_1 - x_1^3\right) \end{aligned}$$

Note: You are <u>not</u> required to sketch your estimate of R_A .

5. (30 pt) Consider the system given below, which represents a damped, driven pendulum:

$$\dot{x} = f(x) := \begin{bmatrix} x_2 \\ -2\sin(x_1) - 0.5x_2 + 1.3 \end{bmatrix}$$

(a) Find all the equilibrium points of the system. For each, state and prove the strongest conclusions about stability or instability of that equilibrium point that you can.

(b) You should find exactly one stable equilibrium point with x_1 in the interval $[-2.5, 3.5]$ (although you may have found stronger conclusions than just Lyapunov stability). Use the candidate Lyapunov function below to numerically estimate the region of attraction of this stable equilibrium point. Plot the region of attraction estimate on the same plot as the true region of attraction. Let $x^s = (x_1^s, x_2^s)$ denote the stable equilibrium point. The Lyapunov function candidate is:

$$V(x) = \frac{1}{2}(x_2 - x_2^s)^2 - 2(\cos(x_1) - \cos(x_1^s)) - 1.3(x_1 - x_1^s)$$

(c) What is the maximum value of the Lyapunov function inside your region of attraction estimate, and how does it compare to the values of the Lyapunov function at the equilibrium points with x_1 in the interval $[-2.5, 3.5]$?

Remark: You should notice that the best region of attraction estimate you obtain in this problem is very conservative - it hardly resembles the true region of attraction at all. If the standard Lyapunov function for such a simple, well-known physical system - the nonlinear pendulum - gives such conservative estimates, you can imagine how conservative such estimates might be for more complicated, higher dimensional systems. In general, estimating the region of attraction is very challenging and it is very difficult to assess how good a particular estimate might be.

Hints for Problem 1: One direction is easy! For the other direction, recalling linear systems theory write the solution $x(t)$ to the system using the state transition matrix from time t_0 to time t (in this problem the state transition matrix is a scalar, but this same technique would work for higher dimensions as well). Now use linearity: observe that scaling the initial condition causes the solution to scale by the same amount. Think about scaling the initial condition so that its norm is less than δ.

Hints for Problem 3: Do exactly as we did in lecture.

Hints for Problem 4:

Hint 1: Consider the candidate Lyapunov function (recall kinetic energy + potential energy in mass-spring example)

$$\begin{aligned} V(x) &= \frac{1}{2}x_2^2 + \int_0^{x_1} \left(y - y^3\right) dy \\ &= \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 \end{aligned}$$

Hint 2: Don't be trapped into following the exact same analysis that we did for the example in class. Think a bit here about all the tools at your disposal for showing that an equilibrium point is asymptotically stable i.s.L.! Try combining a few ideas together. I will NOT give any further hints on this one in office hours. Study the solution carefully for the exam.

Hints for Problem 5: (a) There are multiple ways to prove stability or instability of the equilibrium points you find. You are free to choose whichever method you prefer.

(b) Use the meshgrid command to create a dense grid of points over the intervals $[-2.3, 3.5]$ in x_1 and $[-4, 2]$ in x_2. The intervals could be generated using the linspace command. Then compute the values of V and $\dot{V}$ at each point of the grid, and follow the standard procedure for estimating the region of attraction from Lyapunov functions. As you do this problem, think carefully about the region of attraction estimation procedure. There is (at least one) technical detail which you may find confusing or misleading if you are not thinking carefully - it is recommended that you visualize your region of attraction estimates as you go to see what is going on.

Use plot_boundary.m to plot the true region of attraction boundary using the data file boundary_data.mat.