Control Design under Uncertainty:

Nonlinear Damping and Adaptive Backstepping.

Today we will review 2 methods for control design under uncertainty, as they are presented in the textbook "Nonlinear and Adaptive Control Design" by Krstic, Kanellakopoulos, Kokotovi

The purpose is to motivate why control should be done carefully, and also to see some of the principles of adaptive control, which are not covered in our main textbook.

We start with a simple scalar system

$\dot{x} = u + \phi(x)\Delta(t)$, where $\phi(x)$ a known function and $\Delta(t)$ a disturbance.

To make things even simpler, let's assume that the disturbanc is a vanishing function of time, $\Delta(t) = \Delta(0)e^{-kt}$, $k>0$.

We might wonder whether this disturbance could cause harm to the system, given that eventually its effects fade away. Let us consider that we want to apply a linear control law

$\boxed{u = -cx, \quad c>0}$, and let us further consider that

$\boxed{\phi(x) = x^2}$

Then the solution to the closed-loop dynamics

$\dot{x} = -cx + x^2\Delta(0)e^{-kt}$ reads:

$$\boxed{x(t) = \frac{x(0)(c+k)}{\left[(c+k) - \Delta(0)x(0)e^{-kt}\right]e^{ct} + \Delta(0)x(0)e^{-kt}}}$$

Note that if $\Delta(0)x(0) < c+k$, then the solutions remain bounded and converge to zero asymptotically.

However, for $\Delta(0)x(0) > c+k > 0$, the solutions starting from such initial conditions diverge to infinity and in fact do so in finite time:

$$x(t) \to \infty \quad \text{as} \quad t \to t_f = \frac{1}{c+k} \ln\left(\frac{\Delta(0)\,x(0)}{\Delta(0)x(0) - (c+k)} \right)$$

Note that this finite escape time can not be eliminated by making c larger: For any values c and k, and any non-zero $\Delta(0)$, there exists an initial condition $x(0)$ such that

$$\Delta(0) + x(0) > c+k.$$

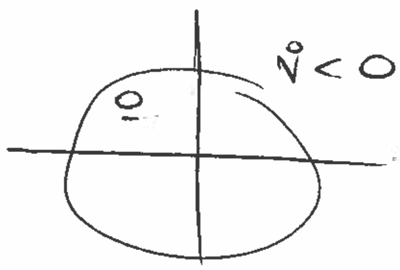
To overcome this problem, and guarantee that $x(t)$ will remain bounded for all bounded $\Delta(t)$, and for all $x(0)$, we consider the control law

$$\boxed{u = -cx - \underbrace{s(x)\cdot x}_{\text{nonlinear damping}}}$$

The question now is how to design $s(x)$. We can consider a Lyapunov-based approach: Take the Lyapunov function candidate $\boxed{V(x) = \frac{1}{2}x^2}$ and consider the time derivative along the trajectories of the closed-loop system as.

$$\dot{V}(x) = x\left(-cx - s(x)x + \phi(x)\,\Delta(t)\right) = -cx^2 - s(x)x^2 + \phi(x)\Delta(t)x$$

Now, recall that the objective is to guarantee boundedness of the trajectories in a compact domain, which is equivalent to rendering $\dot{V}(x)$ negative outside a compact region/domain.



So if we choose

$$s(x) = k\phi^2(x), \quad k > 0$$

we have

$$\dot{V}(x) = -cx^2 - kx^2\phi^2(x) + x\phi(x)\Delta(t)$$

$$= -cx^2 - k\left(x\phi(x) - \frac{\Delta(t)}{2k}\right)^2 + \frac{\Delta^2(t)}{4k}$$

$$\le -cx^2 + \frac{\Delta^2(t)}{4k}$$

So $\dot{V}$ is negative whenever

$$-cx^2 + \frac{\Delta^2(t)}{4k} \leq 0 \implies |x(t)| \geq \frac{\Delta(t)}{2\sqrt{kc}}$$

In other words, $\dot{V}$ is negative outside the compact set

$$\underline{\Omega} = \left\{ x \in \mathbb{R} \;\middle|\; |x| < \frac{\|\Delta\|_\infty}{2\sqrt{kc}} \right\}.$$

Hence, $\|x\|_\infty = \max\left\{ \frac{\|\Delta\|_\infty}{2\sqrt{kc}}, |x(0)|. \right\}$ i.e, the trajectories $x(t)$ remain bounded for all t.

To generalize this control design technique, we review Lemma 2.6 (Nonlinear Damping) from Krstic et al.

Let $\boxed{\dot{x} = f(x) + g(x)\left(u + \phi(x)^T \Delta(x,u,t)\right)}$ (*)

where $\phi(x) \in \mathbb{R}^{q\times 1}$ are known, smooth nonlinear functions

$\Delta(x,u,t) \in \mathbb{R}^{q\times 1}$ is uncertain functions that are uniformly bounded for all values of x, u, t.

<u>If</u> for the nominal system $\dot{x} = f(x) + g(x)u$, $f(0)=0$, it holds that there exists a continuously differentiable, feedback control law $u = a(x)$, $a(0) = 0$, and a smooth, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(x)\left(f(x) + g(x)\,a(x)\right) \leq -W(x) \quad \forall x \in \mathbb{R}^n,$$ where $W(x)$ is positive definite,

<u>Then</u> the control law

denotes Euclidean norm.

$$\boxed{u = a(x) - k\,\frac{\partial V}{\partial x}(x)\,g(x)\,\|\phi(x)\|_2^2, \quad k>0}$$

renders the perturbed system (*) Input-to-State-Stable (ISS) with respect to the disturbance input $\Delta(x,u,t)$, and therefore guarantees global uniform boundedness of $x(t)$, and convergence to the set

$$\underline{o} = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\| < \gamma_1^{-1} \circ \gamma_2 \circ \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4k} \right) \right\},$$

where $\gamma_1, \gamma_2, \gamma_3$ are class-K_∞ functions such that

$$\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$$
$$\gamma_3(\|x\|) \leq W(x)$$

[The proof is in the attached copy of the Krstic et.al. textbook. Interested readers are also further encouraged to study this chapter for more details on the ISS interpretation and the gain interpretation. The application to backstepping under uncertainty is also given!]

We will now continue with a special case of the original system $\boxed{\dot{x} = u + \theta\phi(x)}$, where θ is an unknown, constant parameter (instead of a function of time)

We can apply the nonlinear damping design

$$\boxed{u = -cx - kx\phi^2(x), \quad k > 0}$$

To obtain the closed-loop system $\boxed{\dot{x} = -cx - kx\phi^2(x) + \theta\phi(x)}$, which under the Lyapunov function candidate $V(x) = \frac{1}{2}x^2$ yields $\dot{V}(x) \leq -cx^2 + \frac{\theta^2}{4k}$ (same procedure as before)

which implies that the trajectories $x(t)$ remain bounded and converge to $\underline{o} = \left\{ x \;\middle|\; |x| < \frac{|\theta|}{2\sqrt{kc}} \right\}$

However, for non-zero θ, the size of this set can not be reduced to zero. We can reduce it by increasing $k_1 c$ but increasing the gains too much is not desirable either.

Hence we seek an alternative design where we consider an adaptation law for the unknown parameter θ. The idea is the following. If we consider $\boxed{u = -\theta\phi(x) - c_1 x, \quad c_1 > 0}$

then for the candidate Lyapunov function

$V_0(x) = \frac{1}{2}x^2$, we would have $\dot{V}_0(x) = -c_1 x^2 < 0, \forall x \neq 0$

(negative definite)

The problem is that the above controller can not be implemented since θ is unknown! Hence we consider

$$\boxed{u = -\hat{\theta}\phi(x) - c_1 x, \quad c_1 > 0}$$

where $\hat{\theta}$ is an estimate of θ, and $\tilde{\theta} = \theta - \hat{\theta}$ is the estimation error. So now we have that

$$\dot{V}_0(x) = x\left(-\hat{\theta}\phi(x) - c_1 x + \theta\phi(x)\right) = -c_1 x^2 + x\phi(x)\tilde{\theta}$$

The second term is indefinite and contains the unknown parameter error $\tilde{\theta}$, so we can not conclude anything about the stability of the origin $x = 0$. The objective is to design an update law for $\hat{\theta}$, so that we can say something about $\tilde{\theta}$. To design such law, we again follow a Lyapunov approach. Consider.

$V_1(x, \tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\tilde{\theta}^2$, where $\gamma > 0$ an adaptation gain.

Then

$$\dot{V}_1(x, \tilde{\theta}) = x\dot{x} + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} = -c_1 x^2 + x\phi(x)\tilde{\theta} - \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \Rightarrow$$

recall $\tilde{\theta} = \theta - \hat{\theta} \Rightarrow$
$\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$

$$\dot{V}_1(x,\tilde{\theta}) = -c_1 x^2 + \tilde{\theta}\left(x\phi(x) - \frac{1}{\gamma}\dot{\hat{\theta}}\right)$$

So now we can cancel the effect of the unknown $\tilde{\theta}$ by selecting the parameter update law as

$$\boxed{\dot{\hat{\theta}} = \gamma x \phi(x)}, \quad \gamma > 0.$$

Then $\dot{V}_1(x,\tilde{\theta}) = -c_1 x^2 \leq 0$ (negative semi-definite.)

Then the equilibrium $(x=0, \tilde{\theta}=0)$ is stable in the sense of Lyapunov, while application of LaSalle's - Yoshizawa's Theorem implies that $\lim_{t\to\infty} x(t) = 0$.

(To verify that, see statement of Theorem 2.1 in Krstic book, or apply LaSalle's principle as an exercise.).

Example of Adaptive Backstepping

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(x_1) \\ \dot{x}_2 = \theta\phi_2(x) + u. \end{cases}$$

In the nominal case where θ would be known, we would have

$$a_1(x_1) = -c_1 x_1 - \phi_1(x_1)$$

$$V_0(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2 - a_1(x_1)\right)^2.$$

Taking the time derivative for $V_a(x)$ would yield the design

$$\boxed{u = -c_2(x_2 - a_1) - x_1 + \frac{\partial a_1}{\partial x_1}(x_2 + \phi_1) - \theta\phi_2(x)}$$

in order to have

$$\dot{V}_0(x) = -c_1 x_1^2 - c_2 (x_2 - a_1)^2$$

(see the backstepping process in earlier lecture notes)

However we do not know θ, so we have to replace it with its estimate $\hat{\theta}$, to obtain the control law.

$$\boxed{u = -c_2(x_2 - a_1) - x_1 + \frac{\partial a_1}{\partial x_1}(x_2 + \phi_1) - \hat{\theta}\phi_2(x)}$$

We still need to design the parameter update law.

Consider $z_1 = x_1$, $z_2 = x_2 - a_1$ to write the system equations in a more compact form:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_2(x) \end{bmatrix} \tilde{\theta}$$

and consider the augmented Lyapunov function candidate.

$$V_1(z_1, z_2, \tilde{\theta}) = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma} \tilde{\theta}^2$$

Take its time derivative

$$\dot{V}_1(z_1, z_2, \tilde{\theta}) = \underline{-c_1 z_1^2 - c_2 z_2^2} + \tilde{\theta}\left(z_2\phi_2 - \frac{1}{\gamma}\dot{\hat{\theta}}\right)$$

and choose $\boxed{\dot{\hat{\theta}} = \gamma z_2 \phi_2}$

to render $\dot{V}_1(z_1, z_2, \tilde{\theta}) = -c_1 z_1^2 - c_2 z_2^2 \leq 0$ (negative semi-definite.)

and with application of LaSalle's – Yoshizawa's Theorem, obtain $\lim_{t\to\infty} z_1(t) = 0 \Rightarrow \lim_{t\to\infty} x_1(t) = 0$

$$\lim_{t\to\infty} z_2(t) = 0 \Rightarrow \lim_{t\to\infty} x_2(t) = \lim_{t\to\infty}\left(-c_1 x_1(t) - \phi_1(x_1(t))\right) = 0$$

when the reference signal is constant, and *tracking* when it is a time-varying signal. For convergence analysis, a powerful tool is the following theorem due to LaSalle [110] and Yoshizawa [201]:

Theorem 2.1 (LaSalle-Yoshizawa) *Let $x = 0$ be an equilibrium point of (2.1) and suppose f is locally Lipschitz in x uniformly in t. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable, positive definite and radially unbounded function $V(x)$ such that*

$$\dot{V} = \frac{\partial V}{\partial x}(x) f(x,t) \le -W(x) \le 0\,, \quad \forall t \ge 0,\ \forall x \in \mathbb{R}^n\,, \tag{2.6}$$

where W is a continuous function. Then, all solutions of (2.1) are globally uniformly bounded and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0\,. \tag{2.7}$$

In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ is globally uniformly asymptotically stable (GUAS).

Because of its importance, a more general version of this theorem and its proof are included in Appendix A (Theorem A.8), along with a frequently used technical lemma due to Barbalat [155] (Lemma A.6). The LaSalle-Yoshizawa theorem is applicable to time-varying systems and allows us to establish convergence to the set E where $W(x) = 0$. For most of our design tasks, we will construct $V(x)$ such that the set E consists solely of the trajectories which meet the tracking objective, that is, along which the tracking error is zero.

For the regulation task, the designed system is usually time-invariant,

$$\dot{x} = f(x)\,, \tag{2.8}$$

in which case we are interested in its *invariant sets*. A set M is called an invariant set of (2.8) if any solution $x(t)$ that belongs to M at some time instant t_1 must belong to M for all future and past time:

$$x(t_1) \in M \;\Rightarrow\; x(t) \in M\,, \quad \forall t \in \mathbb{R}\,. \tag{2.9}$$

A set Ω is *positively invariant* if this is true for all future time only:

$$x(t_1) \in \Omega \;\Rightarrow\; x(t) \in \Omega\,, \quad \forall t \ge t_1\,. \tag{2.10}$$

Can we guarantee convergence to a desired invariant set? A rewarding answer to this question is provided by LaSalle's Invariance Theorem and its asymptotic stability corollary:

Theorem 2.2 (LaSalle) *Let Ω be a positively invariant set of (2.8). Let $V : \Omega \to \mathbb{R}_+$ be a continuously differentiable function $V(x)$ such that $\dot{V}(x) \le 0$, $\forall x \in \Omega$. Let $E = \{x \in \Omega \mid \dot{V}(x) = 0\}$, and let M be the largest invariant set contained in E. Then, every bounded solution $x(t)$ starting in Ω converges to M as $t \to \infty$.*

Corollary 2.3 (Asymptotic Stability) *Let $x = 0$ be the only equilibrium of (2.8). Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable, positive definite, radially unbounded function $V(x)$ such that $\dot{V}(x) \le 0$, $\forall x \in \mathbb{R}^n$. Let $E = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$, and suppose that no solution other than $x(t) \equiv 0$ can stay forever in E. Then the origin is globally asymptotically stable (GAS).*

These invariance results will motivate us to closely examine the invariant subsets of E. As we shall see, the convergence properties of the designed system are stronger if the dimension of M is lower. In the most favorable case of asymptotic stability, the largest invariant subset M of E is just the origin $x = 0$. Our aim will thus be to render the dimension of M as low as possible.

Input-to-State Stability. Another stability concept which is used throughout the book is that of input-to-state stability (ISS), introduced by Sontag [173]. The system

$$\dot{x} = f(x,u) \tag{2.11}$$

is said to be *input-to-state stable (ISS)* if for any $x(0)$ and for any input $u(\cdot)$ continuous and bounded on $[0,\infty)$ the solution exists for all $t \ge 0$ and satisfies

$$|x(t)| \le \beta(|x(0)|,t) + \gamma\left(\sup_{0\le\tau\le t}|u(\tau)|\right), \quad \forall t \ge 0\,, \tag{2.12}$$

where $\beta(s,t)$ and $\gamma(s)$ are strictly increasing functions of $s \in \mathbb{R}_+$ with $\beta(0,t) = 0$, $\gamma(0) = 0$, while β is a decreasing function of t with $\lim_{t\to\infty}\beta(s,t) = 0$, $\forall s \in \mathbb{R}_+$.

This definition of input-to-state stability is appropriate for nonlinear systems since it explicitly incorporates the effect of the initial conditions $x(0)$: (2.12) shows that the norm of the state $x(t)$ depends not only on the input $u(\tau)$, but also includes an asymptotically decaying contribution from $x(0)$. A more extensive treatment of ISS is given in Appendix C.

2.1.2 Control Lyapunov functions (clf)

This book is about control *design*: Our objective is to create closed-loop systems with desirable stability properties, rather than analyze the properties of a given system. For this reason, we are interested in an extension of the Lyapunov function concept, called a *control Lyapunov function* (clf).

Hence our control law can be selected to be linear,

$$u = -\phi + c_0\psi + \left(c_2 + \frac{9c_0}{8c_1}\right)\tilde{\psi}, \qquad c_2 > 0, \tag{2.233}$$

and yield

$$\dot{V}_2 \leq -c_1\phi^2 - c_2\tilde{\psi}^2. \tag{2.234}$$

This proves that the equilibrium $\phi = 0, \psi = 0$ is globally asymptotically stable. Denoting

$$k_1 = 1 + c_2c_0 + \frac{9c_0^2}{8c_1}, \qquad k_2 = c_2 + c_0 + \frac{9c_0}{8c_1}, \tag{2.235}$$

we rewrite (2.233) in the more compact form

$$u = -k_1\phi + k_2\psi \tag{2.236}$$

and obtain the closed-loop system

$$\dot{\phi} \;=\; -\frac{1}{2}\phi^3 - \frac{3}{2}\phi^2 - \psi \tag{2.237}$$

$$\dot{\psi} \;=\; k_1\phi - k_2\psi. \tag{2.238}$$

For comparison, we also derive a feedback linearizing controller,

$$u \;=\; -k_1\phi + \left(k_2 - 3\phi - \frac{3}{2}\phi^2\right)\left(\psi + \frac{3}{2}\phi^2 + \frac{1}{2}\phi^3\right), \tag{2.239}$$

which makes the system (2.218), (2.219) appear linear in the coordinates $\chi_1 = \phi$ and $\chi_2 = \dot{\phi}$. The controller simplification achieved with backstepping is impressive: While the linearizing control (2.239) grows as ϕ^5 and $\psi\phi^2$, the backstepping controller (2.236) is linear. The improvement over the control law (2.216) in Section 2.4 is also significant: (2.216) grows as ϕ^4 and $\psi\phi$ because the quadratic nonlinearity was cancelled at the first step, so the cancellation could not be avoided at the second step.

In the remainder of the book we will not assume the presence of useful nonlinearities. However, it should always be understood that whenever such additional information is available, backstepping designs should incorporate it.

2.5 Stabilization with Uncertainty

The full power of backstepping is exhibited in the presence of uncertain nonlinearities and unknown parameters, because for such applications no other systematic design procedure exists. We now begin the study of such design problems which are the main subject of this book. The first of the design tools that will be used to counteract uncertainty is *nonlinear damping*.

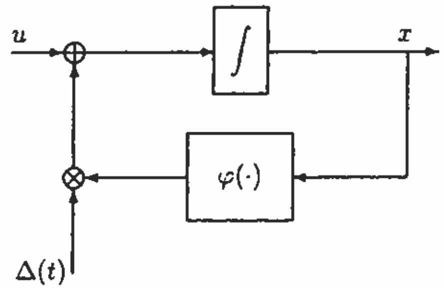


Figure 2.9: A system with "matched" uncertainty $\Delta(t)$.

2.5.1 Nonlinear damping

We introduce nonlinear damping for systems with "matched" uncertainty, in which both the uncertainty and the control appear in the same equation. The simplest example is the scalar nonlinear system depicted in Figure 2.9:

$$\dot{x} = u + \varphi(x)\Delta(t), \tag{2.240}$$

where $\varphi(x)$ is a known smooth nonlinearity, and $\Delta(t)$ is a bounded function of t. Let us first examine the case when $\Delta(t)$ is an exponentially decaying disturbance:

$$\Delta(t) \;=\; \Delta(0)e^{-kt}. \tag{2.241}$$

Can such an innocent-looking uncertainty cause harm? One might be tempted to ignore it and use the linear control $u = -cx$, which results in the closed-loop system

$$\dot{x} \;=\; -cx + \varphi(x)\Delta(0)e^{-kt}. \tag{2.242}$$

While this design may be satisfactory when $\varphi(x)$ is bounded by a constant or a linear function of x, it is inadequate if $\varphi(x)$ is allowed to be any smooth nonlinear function. For example, when $\varphi(x) = x^2$ we have

$$\dot{x} \;=\; -cx + x^2\Delta(0)e^{-kt}. \tag{2.243}$$

As we saw in Chapter 1, equations (1.29)–(1.32), the solution $x(t)$ of this system can be calculated explicitly using the change of variable $w = 1/x$:

$$\dot{w} = -\frac{1}{x^2}\dot{x} = c\frac{1}{x} - \Delta(0)e^{-kt} = cw - \Delta(0)e^{-kt}, \tag{2.244}$$

which yields

$$w(t) \;=\; \left[w(0) - \frac{\Delta(0)}{c+k}\right]e^{ct} + \frac{\Delta(0)}{c+k}e^{-kt}. \tag{2.245}$$

The substitution $w = 1/x$ gives

$$x(t) \;=\; \frac{x(0)(c+k)}{[c+k-\Delta(0)x(0)]e^{ct} + \Delta(0)x(0)e^{-kt}}. \tag{2.246}$$

From (2.246) we see that the behavior of the closed-loop system (2.243) depends critically on the initial conditions $\Delta(0), x(0)$:

(i) If $\Delta(0)x(0) < c + k$, the solutions $x(t)$ are bounded and converge asymptotically to zero.

(ii) The situation changes dramatically when $\Delta(0)x(0) > c + k > 0$. The solutions $x(t)$ which start from such initial conditions not only diverge to infinity, but do so in *finite time:*

$$x(t) \to \infty \quad \text{as } t \to t_f = \frac{1}{c+k} \ln\left\{\frac{\Delta(0)x(0)}{\Delta(0)x(0) - (c+k)}\right\}. \tag{2.247}$$

Note that this finite escape cannot be eliminated by making c larger: For any values of c and k and for any nonzero value of $\Delta(0)$ there exist initial conditions $x(0)$ which satisfy the inequality $\Delta(0)x(0) > c + k$. This example shows that in a nonlinear system, neglecting the effects of exponentially decaying disturbances or nonzero initial conditions can be catastrophic.

To overcome this problem and guarantee that $x(t)$ will remain bounded for all bounded $\Delta(t)$ and for all $x(0)$, we augment the control law $u = -cx$ with a *nonlinear damping term* $-s(x)x$:

$$u = -cx - s(x)x\,. \tag{2.248}$$

We design $s(x)$ for the system (2.240), using the quadratic function $V(x) = \frac{1}{2}x^2$ whose derivative is

$$\begin{aligned} \dot V &= x\,u + x\,\varphi(x)\Delta(t) \\ &= -cx^2 - x^2 s(x) + x\,\varphi(x)\Delta(t)\,. \end{aligned} \tag{2.249}$$

The objective of guaranteeing global boundedness of solutions can be equivalently expressed as rendering $\dot V$ negative outside a compact region. This is achieved with the choice

$$s(x) = \kappa\varphi^2(x)\,, \quad \kappa > 0\,, \tag{2.250}$$

which yields the control

$$u = -cx - \kappa\, x\,\varphi^2(x) \tag{2.251}$$

and the derivative

$$\begin{aligned} \dot V &= -cx^2 - \kappa\, x^2\varphi^2(x) + x\,\varphi(x)\Delta(t) \\ &= -cx^2 - \kappa\left[x\,\varphi(x) - \frac{\Delta(t)}{2\kappa}\right]^2 + \frac{\Delta^2(t)}{4\kappa} \\ &\le -cx^2 + \frac{\Delta^2(t)}{4\kappa}\,. \end{aligned} \tag{2.252}$$

Comparing (2.249) with (2.252) we see that the nonlinear damping term (2.250) is chosen to allow the completion of squares in (2.252). In more complicated situations we can use *Young's Inequality*, which, in a simplified form, states that if the constants $p > 1$ and $q > 1$ are such that $(p-1)(q-1) = 1$, then for all $\varepsilon > 0$ and all $(x, y) \in \mathbb{R}^2$ we have

$$xy \le \frac{\varepsilon^p}{p}|x|^p + \frac{1}{q\varepsilon^q}|y|^q\,. \tag{2.253}$$

Choosing $p = q = 2$ and $\varepsilon^2 = 2\kappa$, (2.253) becomes

$$xy \le \kappa x^2 + \frac{1}{4\kappa}y^2\,, \tag{2.254}$$

which is the inequality used in (2.252):

$$x\,\varphi(x)\Delta(t) \le \kappa x^2\varphi^2(x) + \frac{\Delta^2(t)}{4\kappa}\,. \tag{2.255}$$

Global boundedness and convergence. Returning to (2.252), we see that $\dot V$ is negative whenever $|x(t)| \ge \frac{\Delta(t)}{2\sqrt{\kappa c}}$. Since $\Delta(t)$ is a bounded disturbance, we conclude that $\dot V$ is negative outside the compact residual set

$$\mathcal{R} = \left\{x : |x| \le \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}}\right\}. \tag{2.256}$$

Recalling that $V(x) = \frac{1}{2}x^2$, we conclude that $|x(t)|$ decreases whenever $x(t)$ is outside the set $\mathcal{R}$, and hence $x(t)$ is bounded:

$$\|x\|_\infty \le \max\left\{|x(0)|, \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}}\right\}. \tag{2.257}$$

Moreover, we can draw some conclusions about the asymptotic behavior of $x(t)$. Let us rewrite (2.252) as

$$\frac{d}{dt}\left(\frac{1}{2}x^2\right) \le -cx^2 + \frac{\Delta^2(t)}{4\kappa}\,. \tag{2.258}$$

To obtain explicit bounds on $x(t)$, we consider the signal $x(t)e^{ct}$. Using (2.258) we get

$$\begin{aligned} \frac{d}{dt}\left(x^2e^{2ct}\right) &= \frac{d}{dt}\left(x^2\right)e^{2ct} + 2cx^2e^{2ct} \\ &\le -2cx^2e^{2ct} + \frac{\Delta^2(t)}{2\kappa}e^{2ct} + 2cx^2e^{2ct} \\ &= \frac{\Delta^2(t)}{2\kappa}e^{2ct}\,. \end{aligned} \tag{2.259}$$

Integrating both sides over the interval $[0, t]$ yields

$$
\begin{aligned}
x^2(t)e^{2ct} &\le x^2(0) + \int_0^t \frac{1}{2\kappa}\Delta^2(\tau)e^{2c\tau}d\tau \\
&\le x^2(0) + \frac{1}{2\kappa}\left[\sup_{0\le\tau\le t}\Delta^2(\tau)\right]\int_0^t e^{2c\tau}d\tau \\
&= x^2(0) + \frac{1}{4\kappa c}\left[\sup_{0\le\tau\le t}\Delta^2(\tau)\right]\left(e^{2ct}-1\right). \qquad (2.260)
\end{aligned}
$$

Multiplying both sides with e^{-2ct} and using the fact that $\sqrt{b^2+c^2} \le |b| + |c|$, we obtain an explicit bound for $x(t)$:

$$
\begin{aligned}
|x(t)| &\le |x(0)|e^{-ct} + \frac{1}{2\sqrt{\kappa c}}\left[\sup_{0\le\tau\le t}|\Delta(\tau)|\right]\left(1-e^{-2ct}\right)^{\frac{1}{2}} \\
&\le |x(0)|e^{-ct} + \frac{1}{2\sqrt{\kappa c}}\left[\sup_{0\le\tau\le t}|\Delta(\tau)|\right]. \qquad (2.261)
\end{aligned}
$$

Since $\sup_{0\le\tau\le t}|\Delta(\tau)| \le \sup_{0\le\tau<\infty}|\Delta(\tau)| \triangleq \|\Delta\|_\infty$, (2.261) leads to

$$
|x(t)| \le |x(0)|e^{-ct} + \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}}, \qquad (2.262)
$$

which shows that $x(t)$ converges to the compact set $\mathcal{R}$ defined in (2.256):

$$
\lim_{t\to\infty} \operatorname{dist}\{x(t), \mathcal{R}\} = 0. \qquad (2.263)
$$

We reiterate that these properties of boundedness (cf. (2.257)) and convergence (cf. (2.263)) are guaranteed for any bounded disturbance $\Delta(t)$ and for any smooth nonlinearity $\varphi(x)$, including $\varphi(x) = x^2$. Furthermore, the nonlinear control law (2.251) does not assume knowledge of a bound on the disturbance, nor does it have to use large values for the gains κ and c. Indeed, the residual set $\mathcal{R}$ defined in (2.256) is compact for any finite value of $\|\Delta\|_\infty$ and for any positive values of κ and c. Hence, *global boundedness is guaranteed in the presence of bounded disturbances with unknown bounds, regardless of how small the gains κ and c are chosen.* While the size of $\mathcal{R}$ cannot be estimated *a priori* if no bound for $\|\Delta\|_\infty$ is given, it can be reduced *a posteriori* by increasing the values of κ and c.

This property is achieved by the "nonlinear damping" term $-\kappa x\varphi^2(x)$ in (2.251), which renders the effective gain of (2.251) "selectively high:" When κ and c are chosen to be small, the gain is low around the origin, but it becomes high when x is in a region where $\varphi(x)$ is large enough to make the term $\kappa\varphi^2(x)$ large. If we interpret the nonlinearity $\varphi(x)$, which multiplies the disturbance $\Delta(t)$, as the "disturbance gain," we see that the term $-\kappa\varphi^2(x)$ causes the control gain to become large when the disturbance gain is large.

Finally, we should note that, if the disturbance $\Delta(t)$ converges to zero in addition to being bounded, then the control (2.251) guarantees convergence of $x(t)$ to zero in addition to global boundedness. To show this, let $\bar{\Delta}(t)$ be a continuous nonnegative *monotonically decreasing* function such that $|\Delta(t)| \le \bar{\Delta}(t)$ and $\lim_{t\to\infty}\bar{\Delta}(t) = 0$. Then, starting with the first inequality from (2.260) and using an argument almost identical to the proof of Lemma 2.24, we obtain

$$
|x(t)| \le |x(0)|e^{-ct} + \frac{1}{2\sqrt{\kappa c}}\left(\bar{\Delta}(0)e^{-\frac{c}{2}t} + \bar{\Delta}(t/2)\right). \qquad (2.264)
$$

Since $\lim_{t\to\infty}\bar{\Delta}(t/2) = 0$, we see that $\lim_{t\to\infty} x(t) = 0$.

ISS interpretation. For interpreting the effect of the nonlinear damping term $-\kappa x\varphi^2(x)$ in (2.251) from an input-output point of view, it is very convenient to use the concept of input-to-state stability: (cf. Appendix C) This κ-term renders the closed-loop system ISS with respect to the disturbance input $\Delta(t)$. To show that the ISS inequality (2.12) holds for our closed-loop system with $u(\tau)$ replaced by the disturbance $\Delta(\tau)$, we repeat the argument that led from (2.259) to (2.261), this time integrating over the interval $[t_0, t]$. The result is

$$
|x(t)| \le |x(t_0)|e^{-c(t-t_0)} + \frac{1}{2\sqrt{\kappa c}}\left[\sup_{t_0\le\tau\le t}|\Delta(\tau)|\right], \qquad (2.265)
$$

which is identical to (2.12) with $\beta(r,s) = re^{-cs}$, $\gamma(r) = \frac{1}{2\sqrt{\kappa c}}r$, $r = |x(t_0)|$ and $s = t - t_0$.

Operator gain interpretation. It is also convenient to interpret the effect of nonlinear damping from an operator point of view on the basis of (2.257) and Figure 2.10. For all initial conditions $x(0)$ such that $|x(0)| < \frac{\|\Delta\|_\infty}{2\sqrt{\kappa c}}$, we obtain

$$
\|x\|_\infty \le \frac{1}{2\sqrt{\kappa c}}\|\Delta\|_\infty, \qquad (2.266)
$$

which shows that the nonlinear operator K mapping the disturbance $\Delta(t)$ to the output $x(t)$ is bounded, and its $\mathcal{L}_\infty$-induced gain is

$$
\|K\|_{\infty\,\mathrm{ind}} \le \frac{1}{2\sqrt{\kappa c}}. \qquad (2.267)
$$

The nonlinear damping term renders the operator K bounded for *any* positive values of κ and c. Note, however, that (2.266) does not provide a complete description of this operator because, unlike (2.257), it hides the effect of initial conditions, which can be quite dangerous for nonlinear systems.

The following lemma recapitulates the properties achieved with nonlinear damping as a design tool.

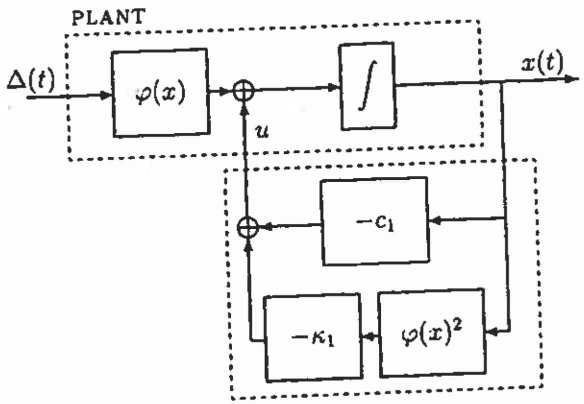


Figure 2.10: The bounded nonlinear operator $K: \Delta(t) \to x(t)$.

Lemma 2.26 (Nonlinear Damping) *Let the system (2.48) be perturbed as in*

$$\dot{x} = f(x) + g(x)\left[u + \varphi(x)^{\mathrm{T}}\Delta(x,u,t)\right], \tag{2.268}$$

where $\varphi(x)$ is a $(p \times 1)$ vector of known smooth nonlinear functions, and $\Delta(x,u,t)$ is a $(p \times 1)$ vector of uncertain nonlinearities which are uniformly bounded *for all values of x, u, t.*

If Assumption 2.7 is satisfied with $W(x)$ positive definite and radially unbounded, then the control

$$u = \alpha(x) - \kappa\frac{\partial V}{\partial x}(x)g(x)|\varphi(x)|^2, \quad \kappa > 0, \tag{2.269}$$

when applied to (2.268), renders the closed-loop system ISS with respect to the disturbance input $\Delta(x,u,t)$ and hence guarantees global uniform boundedness of $x(t)$ and convergence to the residual set

$$\mathcal{R} = \left\{x : |x| \le \gamma_1^{-1}\circ\gamma_2\circ\gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right)\right\}, \tag{2.270}$$

where $\gamma_1, \gamma_2, \gamma_3$ are class-$\mathcal{K}_\infty$ functions such that[11]

$$\begin{aligned} \gamma_1(|x|) \;\le\; V(x) \;&\le\; \gamma_2(|x|) && (2.271)\\ \gamma_3(|x|) \;\le\; W(x)&. && (2.272)\end{aligned}$$

[11]Since $V(x)$ and $W(x)$ are positive definite and radially unbounded and $V(x)$ is smooth, there exist class-$\mathcal{K}_\infty$ functions $\gamma_1, \gamma_2, \gamma_3$ satisfying (2.271) and (2.272).

Proof. The derivative of $V(x)$ is

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x}[f + gu] + \frac{\partial V}{\partial x}g\varphi^{\mathrm{T}}\Delta\\
\text{by (2.269)} \quad &= \frac{\partial V}{\partial x}[f + g\alpha] - \kappa\left(\frac{\partial V}{\partial x}g\right)^2|\varphi|^2 + \frac{\partial V}{\partial x}g\varphi^{\mathrm{T}}\Delta\\
\text{by (2.50)} \quad &\le -W(x) - \kappa\left(\frac{\partial V}{\partial x}g\right)^2|\varphi|^2 + \frac{\partial V}{\partial x}g\varphi^{\mathrm{T}}\Delta\\
&\le -W(x) - \kappa\left(\frac{\partial V}{\partial x}g\right)^2|\varphi|^2 + \left|\frac{\partial V}{\partial x}g\right||\varphi|\|\Delta\|_\infty\\
\text{by (2.254)} \quad &\le -W(x) + \frac{\|\Delta\|_\infty^2}{4\kappa}.
\end{aligned} \tag{2.273}$$

From (2.273) it follows that $\dot{V}$ is negative whenever $W(x) > \frac{\|\Delta\|_\infty^2}{4\kappa}$. Combining this with (2.272) we conclude that

$$|x(t)| > \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right) \Rightarrow \dot{V} < 0. \tag{2.274}$$

This means that if $|x(0)| \le \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right)$, then

$$V(x(t)) \le \gamma_2\circ\gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right), \tag{2.275}$$

which in turn implies that

$$|x(t)| \le \gamma_1^{-1}\circ\gamma_2\circ\gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right). \tag{2.276}$$

If, on the other hand, $|x(0)| > \gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right)$, then $V(x(t)) \le V(x(0))$, which implies

$$|x(t)| \le \gamma_1^{-1}\circ\gamma_2(|x(0)|). \tag{2.277}$$

Combining (2.276) and (2.277) leads to the global uniform boundedness of $x(t)$:

$$\|x\|_\infty \le \max\left\{\gamma_1^{-1}\circ\gamma_2\circ\gamma_3^{-1}\left(\frac{\|\Delta\|_\infty^2}{4\kappa}\right),\ \gamma_1^{-1}\circ\gamma_2(|x(0)|)\right\}, \tag{2.278}$$

while (2.274) and (2.271) prove the convergence of $x(t)$ to the residual set defined in (2.270). Finally, the ISS property of the closed-loop system with respect to the disturbance input $\Delta(x,u,t)$ follows from Theorem C.2. □

2.5.2 Backstepping with uncertainty

Lemma 2.26 deals with the case where the uncertainty is in the span of the control u, i.e, the *matching condition* is satisfied. Combining Lemma 2.26 with Lemma 2.8 allows us to go beyond the matching case, as the following example illustrates.

Example 2.27 Consider the system

$$\begin{aligned} \dot{x} &= \xi + x^2 \arctan \xi \Delta_0(t) && (2.279a)\\ \dot{\xi} &= (1+\xi^2)u + e^{x\xi}\Delta_0(t), && (2.279b) \end{aligned}$$

where $\Delta_0(t)$ is a bounded time-varying disturbance. Clearly, the uncertain terms in (2.279) are not in the span of the control u. Therefore, we will design a static nonlinear controller in two steps, combining nonlinear damping and backstepping.

Step 1. The starting point is equation (2.279a) and the choice of a virtual control variable. Clearly, ξ is the only choice. The fact that ξ is also present in the uncertain term does not present a problem, since it enters that term through the bounded function arctan(·). In the notation of (2.268), we have

$$x^2 \arctan \xi \Delta_0(t) \triangleq x^2 \Delta_1(\xi, t) = \varphi_1(x)\Delta_1(\xi,t). \tag{2.280}$$

The uncertain nonlinearity $\Delta_1(\xi, t)$ is bounded:

$$\|\Delta_1(\xi,t)\|_\infty = \|\Delta_0 \arctan \xi\|_\infty \le \frac{\pi}{2}\|\Delta_0\|_\infty. \tag{2.281}$$

Hence, Lemma 2.26 can be used to design a stabilizing function for ξ. The unperturbed system in this case would be the integrator $\dot{x} = \xi$, for which a clf is given by $V(x) = \frac{1}{2}x^2$ and the corresponding control is $\alpha(x) = -c_1 x$. From (2.269) we have

$$\alpha_1(x) = -c_1 x - \kappa_1 x \varphi_1^2(x), \tag{2.282}$$

which results in

$$\dot{x} = -c_1 x + z - \kappa_1 x \varphi_1^2(x) + \varphi_1(x)\Delta_1(\xi,t), \tag{2.283}$$

with the error variable z defined as in Lemma 2.8:

$$z = \xi - \alpha_1(x). \tag{2.284}$$

The derivative of $V(x)$ along (2.283) is

$$\begin{aligned} \dot{V} &= zx - c_1x^2 - \kappa_1 x^2\varphi_1^2 + x^3 \arctan\xi\Delta_0(t)\\ \text{by (2.280)} \quad &\le zx - c_1x^2 - \kappa_1 x^2 \varphi_1^2 + |x\varphi_1(x)|\|\Delta_1\|_\infty\\ \text{by (2.254)} \quad &= zx - c_1x^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1}, \end{aligned} \tag{2.285}$$

which confirms that if $z \equiv 0$, that is, if ξ were the actual control, then (2.282) would guarantee global uniform boundedness of x.

Step 2. Using the error variable z from (2.284), the system (2.279) is rewritten as

$$\begin{aligned} \dot{x} &= -c_1 x + z - \kappa_1 x\varphi_1^2(x) + \varphi_1(x)\Delta_1(\xi,t) && (2.286a)\\ \dot{z} &= (1+\xi^2)u + e^{x\xi}\Delta_0(t) - \frac{\partial\alpha_1}{\partial x}\left[\xi + x^2\arctan\xi\Delta_0(t)\right]\\ &= (1+\xi^2)u - \frac{\partial\alpha_1}{\partial x}\xi + \left[e^{x\xi} - \frac{\partial\alpha_1}{\partial x}x^2\arctan\xi\right]\Delta_0(t), && (2.286b) \end{aligned}$$

where the partial $\frac{\partial\alpha_1}{\partial x}$ is computed from (2.280) and (2.282):

$$\frac{\partial\alpha_1}{\partial x_1} = -c_1 - \kappa_1\frac{\partial}{\partial x}\left[x\varphi_1^2(x)\right] = -c_1 - 5\kappa_1 x_1^4. \tag{2.287}$$

If the $\Delta_0(t)$-term were not present in (2.286b), then Lemma 2.8 would dictate the Lyapunov function

$$V_2(x,\xi) = \frac{1}{2}x^2 + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}\left[\xi - \alpha_1(x)\right]^2 \tag{2.288}$$

and the following choice of control:

$$u = \bar{\alpha}(x,\xi) = \frac{1}{1+\xi^2}\left[-c_2 z + \frac{\partial\alpha_1}{\partial x}\xi - x\right]. \tag{2.289}$$

To compensate for the presence of the $\Delta_0(t)$-term in (2.286b), Lemma 2.26 is used again. From (2.269) we obtain

$$u = \frac{1}{1+\xi^2}\left\{-c_2 z + \frac{\partial\alpha_1}{\partial x}\xi - x - \kappa_2 z\left[e^{x\xi} - \frac{\partial\alpha_1}{\partial x}x^2\arctan\xi\right]^2\right\}, \tag{2.290}$$

which renders the derivative of $V_2(x,\xi)$ negative outside a compact set, thus guaranteeing boundedness of $x(t)$ and $\xi(t)$:

$$\begin{aligned} \dot{V}_2 &= \dot{V} + z\dot{z}\\ \text{by (2.285)} \quad &\le zx - c_1x^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} + z\dot{z}\\ \text{by (2.286b) and (2.290)} \quad &= -c_1x^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1}\\ &\quad + z\left\{-c_2 z - \kappa_2 z\left[e^{x\xi} - \frac{\partial\alpha_1}{\partial x}x^2\arctan\xi\right]^2\right.\\ &\quad \left. + \left[e^{x\xi} - \frac{\partial\alpha_1}{\partial x}x^2\arctan\xi\right]\Delta_0(t)\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq -c_1x^2 - c_2z^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} \\
&\quad -\kappa_2 z^2\left[e^{x\xi} - \frac{\partial \alpha_1}{\partial x}x^2 \arctan\xi\right]^2 \\
&\quad +|z|\left|e^{x\xi} - \frac{\partial \alpha_1}{\partial x}x^2 \arctan\xi\right| \|\Delta_0\|_\infty \\
\text{by (2.254)} \quad &\leq -c_1x^2 - c_2z^2 + \frac{\|\Delta_1\|_\infty^2}{4\kappa_1} + \frac{\|\Delta_0\|_\infty^2}{4\kappa_2}\,. \qquad (2.291)
\end{aligned}
$$

◇

The combination of Lemmas 2.8 and 2.26, illustrated in the above example, is now formulated as another design tool.

Lemma 2.28 (Boundedness via Backstepping) *Consider the system*

$$\dot{x} = f(x) + g(x)u + F(x)\Delta_1(x,u,t)\,, \qquad (2.292)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $F(x)$ is an $(n \times q)$ matrix of known smooth nonlinear functions, and $\Delta_1(x,u,t)$ is a $(q \times 1)$ vector of uncertain nonlinearities which are uniformly bounded for all values of x,u,t. Suppose that there exists a feedback control $u = \alpha(x)$ that renders $x(t)$ globally uniformly bounded, and that this is established via positive definite and radially unbounded functions $V(x), W(x)$ and a constant b, such that

$$\frac{\partial V}{\partial x}(x)\left[f(x) + g(x)\alpha(x) + F(x)\Delta_1(x,u,t)\right] \leq -W(x) + b\,. \qquad (2.293)$$

Now consider the augmented system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\xi + F(x)\Delta_1(x,u,t) \qquad (2.294a)\\
\dot{\xi} &= u + \varphi(x,\xi)^{\mathrm{T}}\Delta_2(x,\xi,u,t)\,, \qquad (2.294b)
\end{aligned}
$$

where $\varphi(x,\xi)$ is a $(p \times 1)$ vector of known smooth nonlinear functions, and $\Delta_2(x,\xi,u,t)$ is a $(p\times 1)$ vector of uncertain nonlinearities which are uniformly bounded for all values of x,ξ,u,t. For this system, the feedback control

$$
\begin{aligned}
u = {}& -c\left[\xi - \alpha(x)\right] + \frac{\partial \alpha}{\partial x}(x)\left[f(x) + g(x)\xi\right] - \frac{\partial V}{\partial x}(x)g(x) \\
& -\kappa\left[\xi - \alpha(x)\right]\left\{|\varphi(x,\xi)|^2 + \left|\frac{\partial \alpha}{\partial x}(x)F(x)\right|^2\right\} \qquad (2.295)
\end{aligned}
$$

guarantees global uniform boundedness of $x(t)$ and $\xi(t)$ with any $c > 0$ and $\kappa > 0$.

Proof. Using the error variable

$$z = \xi - \alpha(x)\,, \qquad (2.296)$$

the system (2.294) is rewritten as

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\left[\alpha(x) + z\right] + F(x)\Delta_1(x,u,t) \qquad (2.297a)\\
\dot{z} &= u + \varphi(x,\xi)^{\mathrm{T}}\Delta_2(x,\xi,u,t) \\
&\quad -\frac{\partial \alpha}{\partial x}(x)\left[f(x) + g(x)\xi + F(x)\Delta_1(x,u,t)\right]\,. \qquad (2.297b)
\end{aligned}
$$

The derivative of

$$V_2(x,\xi) = V(x) + \frac{1}{2}\left[\xi - \alpha(x)\right]^2 = V(x) + \frac{1}{2}z^2 \qquad (2.298)$$

along the solutions of (2.297) with the control (2.295) is

$$
\begin{aligned}
\dot{V}_2 &= \frac{\partial V}{\partial x}(f + g\alpha + F\Delta_1) + \frac{\partial V}{\partial x}gz \\
&\quad +z\left[u + \varphi^{\mathrm{T}}\Delta_2 - \frac{\partial \alpha}{\partial x}(f + g\xi + F\Delta_1)\right] \\
&\leq \frac{\partial V}{\partial x}(f + g\alpha + F\Delta_1) + z\left[u - \frac{\partial \alpha}{\partial x}(f + g\xi) + \frac{\partial V}{\partial x}g\right] \\
&\quad +z\left[\varphi^{\mathrm{T}}\Delta_2 - \frac{\partial \alpha}{\partial x}F\Delta_1^{\mathrm{T}}\right] \\
\text{by (2.293)} \quad &\leq -W(x) + b + z\left[u - \frac{\partial \alpha}{\partial x}(f + g\xi) + \frac{\partial V}{\partial x}g\right] \\
&\quad +z\left[\varphi^{\mathrm{T}}\Delta_2 - \frac{\partial \alpha}{\partial x}F\Delta_1\right] \\
\text{by (2.295)} \quad &= -W(x) + b - cz^2 - \kappa z^2\left[|\varphi|^2 + \left|\frac{\partial \alpha}{\partial x}F\right|^2\right] \\
&\quad +|z|\,|\varphi|\,\|\Delta_2\|_\infty + |z|\left|\frac{\partial \alpha}{\partial x}F\right| \|\Delta_1\|_\infty \\
\text{by (2.254)} \quad &= -W(x) - cz^2 + b + \frac{\|\Delta_1\|_\infty^2}{4\kappa} + \frac{\|\Delta_2\|_\infty^2}{4\kappa}\,. \qquad (2.299)
\end{aligned}
$$

The radial unboundedness of $W(x)$ combined with (2.299) implies that $\dot{V}_2$ is negative outside a compact set, which in turn implies that $x(t)$ and $\xi(t)$ are globally uniformly bounded. □

2.5.3 Robust strict-feedback systems

Just as we generalized Lemma 2.8 to strict-feedback systems in Section 2.3.1 and Lemma 2.25 to block-strict-feedback systems in Section 2.3.3, we can generalize Lemma 2.28 to broader classes of uncertain nonlinear systems.

We consider systems in the *robust strict-feedback form*:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1^{\mathrm{T}}(x_1)\Delta(x,u,t) \\
\dot{x}_2 &= x_3 + \varphi_2^{\mathrm{T}}(x_1,x_2)\Delta(x,u,t) \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}^{\mathrm{T}}(x_1,\ldots,x_{n-1})\Delta(x,u,t) \\
\dot{x}_n &= \beta(x)u + \varphi_n^{\mathrm{T}}(x)\Delta(x,u,t),
\end{aligned} \tag{2.300}$$

where $\beta(x) \neq 0$, $\forall\, x \in \mathbb{R}^n$, $\varphi_i(x_1,\ldots,x_i)$ is a $(p\times 1)$ vector of known smooth nonlinear functions, and $\Delta(x,u,t)$ is a $(p\times 1)$ vector of uncertain nonlinearities which are *uniformly bounded* for all values of x, u, t.

Corollary 2.29 (Robust Strict-Feedback Systems) *The state $x(t)$ of the system (2.300) will be globally uniformly bounded if the control is chosen as*

$$u = \frac{1}{\beta(x)}\alpha_n(x), \tag{2.301}$$

where the function $\alpha_n(x)$ is defined by the following recursive expressions for $i = 1,\ldots,n$ (where we denote $z_0 \equiv \alpha_0 \equiv 0$):

$$z_i = x_i - \alpha_{i-1}(x_1,\ldots,x_{i-1}) \tag{2.302}$$

$$\alpha_i(x_1,\ldots,x_i) = -c_i z_i - z_{i-1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} - \kappa_i z_i\left|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right|^2, \tag{2.303}$$

with c_i, κ_i, $i = 1,\ldots,n$ positive design constants.

Proof. Using the definitions (2.302) and (2.303) and denoting $x_0 \equiv \alpha_0 \equiv 0$, $x_{n+1} \equiv \beta(x)u$, the derivative of the error variable z_i, $i = 1,\ldots,n$, becomes

$$\begin{aligned}
\dot{z}_i &= \dot{x}_i - \dot{\alpha}_{i-1}(x_1,\ldots,x_{i-1}) \\
&= x_{i+1} + \varphi_i^{\mathrm{T}}\Delta - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\left(x_{j+1} + \varphi_j^{\mathrm{T}}\Delta\right) \\
&= \alpha_i + z_{i+1} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}x_{j+1} + \left(\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^{\mathrm{T}}\Delta \\
&= -c_i z_i - z_{i-1} + z_{i+1} + \left(\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^{\mathrm{T}}\Delta - \kappa_i z_i\left|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right|^2.
\end{aligned} \tag{2.304}$$

The choice of control (2.301) guarantees that $z_{n+1} \equiv 0$. The closed-loop error system can therefore be expressed as

$$\begin{aligned}
\dot{z}_1 &= -c_1 z_1 + z_2 + \varphi_1^{\mathrm{T}}\Delta - \kappa_1 z_1|\varphi_1|^2 \\
\dot{z}_2 &= -c_2 z_2 - z_1 + z_3 + \left(\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right)^{\mathrm{T}}\Delta - \kappa_2 z_2\left|\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right|^2 \\
&\vdots \\
\dot{z}_{n-1} &= -c_{n-1}z_{n-1} - z_{n-2} + z_n + \left(\varphi_{n-1} - \sum_{j=1}^{n-2}\frac{\partial\alpha_{n-2}}{\partial x_j}\varphi_j\right)^{\mathrm{T}}\Delta \\
&\quad -\kappa_{n-1}z_{n-1}\left|\varphi_{n-1} - \sum_{j=1}^{n-2}\frac{\partial\alpha_{n-2}}{\partial x_j}\varphi_j\right|^2 \\
\dot{z}_n &= -c_n z_n - z_{n-1} + \left(\varphi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j\right)^{\mathrm{T}}\Delta - \kappa_n z_n\left|\varphi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\varphi_j\right|^2.
\end{aligned} \tag{2.305}$$

Now we can use the quadratic Lyapunov function

$$V_n(z_1,\ldots,z_n) = \frac{1}{2}\sum_{i=1}^n z_i^2 \tag{2.306}$$

to prove global uniform boundedness. Indeed, the derivative of (2.306) along the solutions of (2.305) is

$$\begin{aligned}
\dot{V}_n &= \sum_{i=1}^n\left[-c_i z_i^2 + z_i\left(\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right)^{\mathrm{T}}\Delta - \kappa_i z_i^2\left|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right|^2\right] \\
&\leq \sum_{i=1}^n\left[-c_i z_i^2 + |z_i|\left|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right|\|\Delta\|_\infty - \kappa_i z_i^2\left|\varphi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_j\right|^2\right] \\
&\leq \sum_{i=1}^n\left[-c_i z_i^2 + \frac{\|\Delta\|_\infty^2}{4\kappa_i}\right].
\end{aligned} \tag{2.307}$$

The last inequality implies that $z(t)$ is globally uniformly bounded. But from (2.303) we see that, since the α_i's are smooth functions, x_i can be expressed as a smooth function of $z_1,\ldots,z_i$:

$$x_1 = z_1, \quad x_i = z_i + \bar{\alpha}_{i-1}(z_1,\ldots,z_{i-1}),\ i = 2,\ldots,n. \tag{2.308}$$

Hence, $x(t)$ is globally uniformly bounded and, furthermore, converges to the compact residual set

$$\mathcal{R} = \left\{x \in \mathbb{R}^n : \sum_{i=1}^n c_i z_i^2 \leq \sum_{i=1}^n \frac{\|\Delta\|_\infty^2}{4\kappa_i}\right\}, \tag{2.309}$$

whose size is unknown since the bound $\|\Delta\|_\infty$ is unknown. □

estimates. The extended-matching design, presented in Section 3.4, is of interest as a transition between overparametrized and minimal-order designs. It also contains the first adaptive performance results.

3.1 Adaptation as Dynamic Feedback

The difference between a static and a dynamic (that is, adaptive) design will first be illustrated on the simplest nonlinear system:

$$\dot{x} = u + \theta\varphi(x) . \tag{3.1}$$

This is the special case of the system (2.240), where the uncertainty $\Delta(t)$ is the unknown constant parameter θ.

Even if we do not know a bound for θ, we can use Lemma 2.26 to design a static nonlinear controller which guarantees global boundedness of $x(t)$. The nonlinear damping design (2.251) applies also here. The corresponding static controller is

$$u = -cx - \kappa x\varphi^2(x) , \tag{3.2}$$

and the resulting closed-loop system is of first order:

$$\dot{x} = -cx - \kappa x\varphi^2(x) + \theta\varphi(x) . \tag{3.3}$$

According to (2.252), the derivative of $V = \frac{1}{2}x^2$ satisfies

$$\dot{V} \le -cx^2 + \frac{\theta^2}{4\kappa} , \tag{3.4}$$

which means that $x(t)$ converges to the interval

$$|x| \le \frac{|\theta|}{2\sqrt{\kappa c}} . \tag{3.5}$$

This interval can be reduced by increasing the gains κ and c, but $x(t)$ will not converge to zero if θ is a nonzero constant. Excessive increase of these gains enlarges the system bandwidth, which is undesirable. Our task is therefore to achieve $\lim_{t\to\infty} x(t) = 0$ without increasing κ and c. In fact, we will first accomplish this task with $\kappa = 0$, and then use $\kappa > 0$ for further improvement of transients.

To achieve regulation of $x(t)$, we design a dynamic feedback controller, that is, we employ adaptation.

If θ were known, the control

$$u = -\theta\varphi(x) - c_1 x , \quad c_1 > 0 \tag{3.6}$$

would render the derivative of $V_0(x) = \frac{1}{2}x^2$ negative definite: $\dot{V}_0 = -c_1x^2$. Of course the control law (3.6) can not be implemented, since θ is unknown.

Instead, one can employ its *certainty-equivalence* form in which θ is replaced by an estimate $\hat{\theta}$:

$$u = -\hat{\theta}\varphi(x) - c_1 x . \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\dot{x} = -c_1 x + \tilde{\theta}\varphi(x) , \tag{3.8}$$

where $\tilde{\theta}$ is the *parameter error:*

$$\tilde{\theta} = \theta - \hat{\theta} . \tag{3.9}$$

The derivative of $V_0(x) = \frac{1}{2}x^2$ becomes

$$\dot{V}_0 = -c_1x^2 + \tilde{\theta}x\varphi(x) . \tag{3.10}$$

Since the second term is indefinite and contains the unknown parameter error $\tilde{\theta}$, we can not conclude anything about the stability of (3.6). We make the controller dynamic with an update law for $\hat{\theta}$. To design this update law, we augment V_0 with a quadratic term in the parameter error $\tilde{\theta}$,

$$V_1(x,\tilde{\theta}) = \frac{1}{2}x^2 + \frac{1}{2\gamma}\tilde{\theta}^2 , \tag{3.11}$$

where $\gamma > 0$ is the *adaptation gain*. The derivative of this function is

$$\begin{aligned} \dot{V}_1 &= x\dot{x} + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -c_1x^2 + \tilde{\theta}x\varphi(x) + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -c_1x^2 + \tilde{\theta}\left[x\varphi(x) + \frac{1}{\gamma}\dot{\tilde{\theta}}\right] . \end{aligned} \tag{3.12}$$

The second term is still indefinite and contains $\tilde{\theta}$ as a factor. However, the situation is much better than in (3.10), because we now have the dynamics of $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ at our disposal. With the appropriate choice of $\dot{\hat{\theta}}$ we can cancel the indefinite term. Thus, we choose the update law

$$\dot{\hat{\theta}} = -\dot{\tilde{\theta}} = \gamma x\varphi(x) , \tag{3.13}$$

which yields

$$\dot{V}_1 = -c_1x^2 \le 0 . \tag{3.14}$$

The resulting adaptive system consists of (3.1) with the control (3.7) and the update law (3.13), and is shown in Figure 3.1. In Figure 3.2, this system is redrawn in its closed-loop form consisting of (3.8) and (3.13), namely

$$\begin{aligned} \dot{x} &= -c_1x + \tilde{\theta}\varphi(x) \\ \dot{\tilde{\theta}} &= -\gamma x\varphi(x) . \end{aligned} \tag{3.15}$$

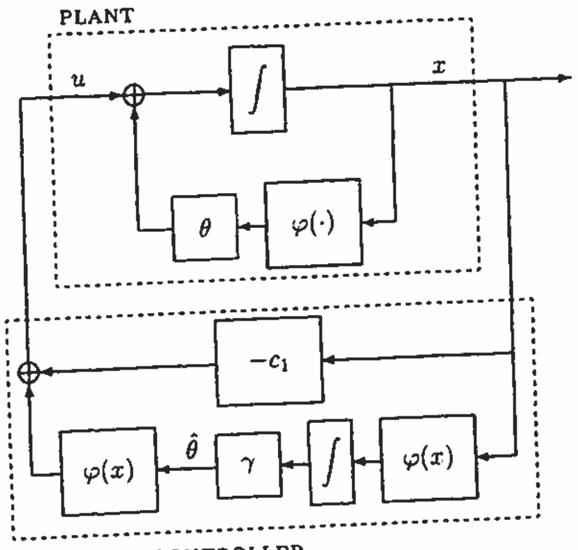


Figure 3.1: The closed-loop adaptive system (3.15).

Because $\dot{V}_1 \leq 0$, the equilibrium $x = 0$, $\tilde{\theta} = 0$ of (3.15) is globally stable. In addition, the desired regulation property $\lim_{t\to\infty} x(t) = 0$ follows from the LaSalle-Yoshizawa theorem (Theorem 2.1). The adaptive nonlinear controller which guarantees these properties is given by (3.8) and (3.13):

$$\begin{aligned} u &= -c_1 x - \hat{\theta}\varphi(x) \\ \dot{\hat{\theta}} &= \gamma\, x\varphi(x) . \end{aligned} \tag{3.16}$$

One may think that the above adaptive design is so straightforward because (3.1) is a first-order system. In fact, this is due to the *matching condition*: The terms containing unknown parameters in (3.1) are in the span of the control, that is, they can be directly cancelled by u when θ is known. To illustrate this point, let us consider the following second-order system, where again the uncertain term is "matched" by the control u:

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1) \\ \dot{x}_2 &= \theta\varphi_2(x) + u . \end{aligned} \tag{3.17}$$

If θ were known, we would be able to apply Lemma 2.8: First view x_2 as the virtual control, design the stabilizing function

$$\alpha_1(x_1) = -c_1 x_1 - \varphi_1(x_1) , \tag{3.18}$$

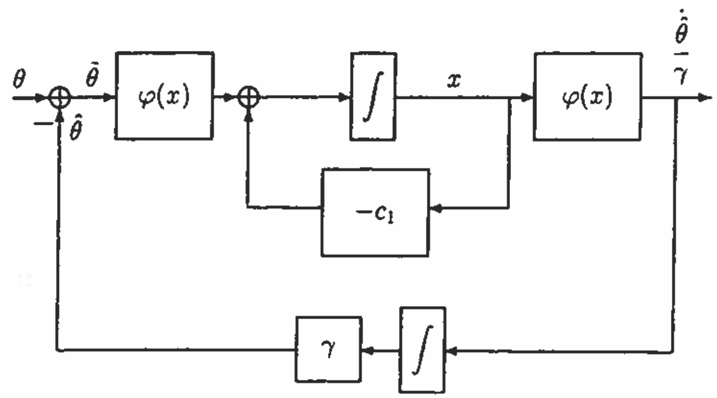


Figure 3.2: An equivalent representation of (3.15).

and then form the Lyapunov function

$$V_c(x) = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2 - \alpha_1(x_1)\right)^2 , \tag{3.19}$$

whose derivative is rendered negative definite

$$\dot{V}_c(x) = -c_1 x_1^2 - c_2\left(x_2 - \alpha_1\right)^2 \tag{3.20}$$

by the control

$$u = -c_2\left(x_2 - \alpha_1\right) - x_1 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + \varphi_1) - \theta\varphi_2(x) . \tag{3.21}$$

Since θ is unknown, we again replace it with its estimate $\hat{\theta}$ in (3.21) to obtain the adaptive control law:

$$u = -c_2\left(x_2 - \alpha_1\right) - x_1 + \frac{\partial \alpha_1}{\partial x_1}(x_2 + \varphi_1) - \hat{\theta}\varphi_2(x) . \tag{3.22}$$

This results in the error system ($z_1 = x_1$, $z_2 = x_2 - \alpha_1$):

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_2(x) \end{bmatrix}\tilde{\theta} . \tag{3.23}$$

Then we augment (3.20) with a quadratic term in the parameter error $\tilde{\theta}$ to obtain the Lyapunov function:

$$V_1(z, \tilde{\theta}) = V_c + \frac{1}{2\gamma}\tilde{\theta}^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{\theta}^2 . \tag{3.24}$$

Its derivative is

$$\dot{V}_1 = -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}\left[z_2\varphi_2 - \frac{1}{\gamma}\dot{\hat{\theta}}\right] . \tag{3.25}$$

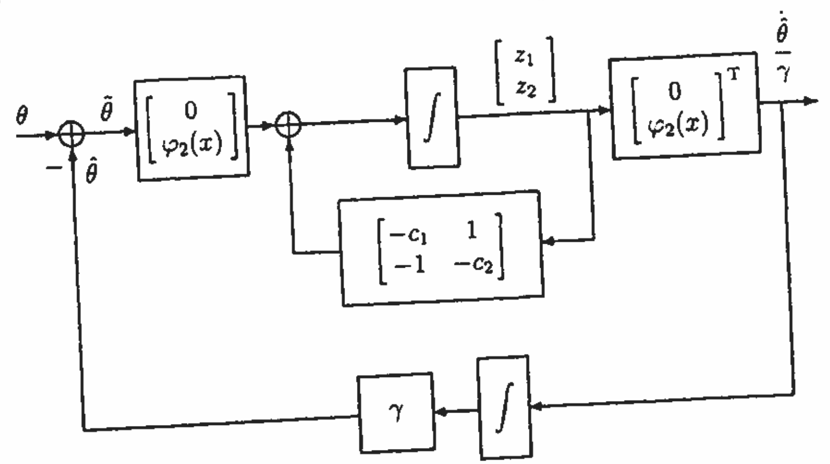


Figure 3.3: The closed-loop adaptive system (3.28).

The choice of update law

$$\dot{\hat\theta} = \gamma\varphi_2 z_2 \tag{3.26}$$

eliminates the $\tilde\theta$-term in (3.25) and renders the derivative of the Lyapunov function (3.24) nonpositive:

$$\dot V_1 = -c_1 z_1^2 - c_2 z_2^2 \le 0\,. \tag{3.27}$$

This implies that the $z = 0, \tilde\theta = 0$ equilibrium point of the closed-loop adaptive system consisting of (3.23) and (3.26) (see block diagram in Figure 3.3)

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}z_1\\ z_2\end{bmatrix} &= \begin{bmatrix}-c_1 & 1\\ -1 & -c_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}0\\ \varphi_2(x)\end{bmatrix}\tilde\theta \\ \dot{\tilde\theta} &= -\gamma\begin{bmatrix}0 & \varphi_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}\end{aligned} \tag{3.28}$$

is globally stable and, in addition, $x(t) \to 0$ as $t \to \infty$.

3.2 Adaptive Backstepping

3.2.1 Adaptive integrator backstepping

The adaptive design in the above examples was simple because of the matching: The parametric uncertainty was in the span of the control. We now move to the more general case of *extended matching*, where the parametric uncertainty enters the system one integrator before the control does:

$$\begin{aligned}\dot x_1 &= x_2 + \theta\varphi(x_1) &&(3.29a)\\ \dot x_2 &= u\,. &&(3.29b)\end{aligned}$$

We use this example to introduce *adaptive backstepping*.

If θ were known, we would apply Lemma 2.8 to design the stabilizing function for x_2

$$\alpha_1(x_1,\theta) = -c_1x_1 - \theta\varphi(x_1)\,, \tag{3.30}$$

with the Lyapunov function

$$V_c(x,\theta) = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2 - \alpha_1(x_1,\theta)\right)^2\,, \tag{3.31}$$

whose derivative is rendered negative definite

$$\dot V_c(x,\theta) = -c_1x_1^2 - c_2\left(x_2 - \alpha_1(x,\theta)\right)^2 \tag{3.32}$$

by the control

$$u = -c_2\left(x_2 - \alpha_1(x_1,\theta)\right) - x_1 + \frac{\partial\alpha_1}{\partial x_1}(x_2 + \theta\varphi)\,. \tag{3.33}$$

Since θ is unknown and appears one equation before the control does, we can not apply Lemma 2.8 because the dependence of $\alpha_1(x_1) = -c_1x_1 - \theta\varphi(x_1)$ on the unknown parameter makes it impossible to continue the procedure. However, we can utilize the idea of integrator backstepping.

Step 1. If x_2 were the control, an adaptive controller for (3.29a) would be given by (3.16):

$$\begin{aligned}\alpha_1(x_1,\vartheta_1) &= -c_1z_1 - \vartheta_1\varphi(x_1) &&(3.34)\\ \dot\vartheta_1 &= \gamma z_1\varphi(x_1)\,. &&(3.35)\end{aligned}$$

In the above equations we have replaced the parameter estimate $\hat\theta$ with the estimate ϑ_1, which denotes the estimate generated in this design step. As we will see, there will be another estimate generated in the next step. With (3.34) and the new error variable $z_2 = x_2 - \alpha_1$, the $\dot z_1$-equation becomes

$$\dot z_1 = -c_1z_1 + z_2 + (\theta - \vartheta_1)\varphi\,. \tag{3.36}$$

The derivative of the Lyapunov function

$$V_1(x,\vartheta_1) = \frac{1}{2}z_1^2 + \frac{1}{2\gamma}(\theta - \vartheta_1)^2 \tag{3.37}$$

along the solutions of (3.36) is

$$\begin{aligned}\dot V_1 &= z_1\dot z_1 - \frac{1}{\gamma}(\theta - \vartheta_1)\dot\vartheta_1\\ &= z_1z_2 - c_1z_1^2 + (\theta - \vartheta_1)\left(\varphi_1 z_1 - \frac{1}{\gamma}\dot\vartheta_1\right)\\ &= z_1z_2 - c_1z_1^2\,.\end{aligned} \tag{3.38}$$

Step 2. The derivative of z_2 is now expressed as

$$\begin{aligned}\dot z_2 &= \dot x_2 - \dot\alpha_1\\ &= u - \frac{\partial\alpha_1}{\partial x_1}\dot x_1 - \frac{\partial\alpha_1}{\partial\vartheta_1}\dot\vartheta_1 .\end{aligned}$$

Substituting (3.29a) and the update law (3.35) results in

$$\begin{aligned}\dot z_2 &= u - \frac{\partial\alpha_1}{\partial x_1}(x_2+\theta\varphi) - \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1\\ &= u - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 - \theta\frac{\partial\alpha_1}{\partial x_1}\varphi .\end{aligned} \tag{3.39}$$

At this point we need to select a Lyapunov function and design u to render its derivative nonpositive. Our first attempt is the augmented Lyapunov function

$$V_2(z_1,z_2,\vartheta_1) = V_1(z_1,\vartheta_1) + \frac{1}{2}z_2^2 ,$$

whose derivative, using (3.38) and (3.39), is

$$\begin{aligned}\dot V_2 &= \dot V_1 + z_2\dot z_2\\ &= -c_1z_1^2 + z_2\left[z_1 + u - \frac{\partial\alpha_1}{\partial x_1}x_2 - \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 - \theta\frac{\partial\alpha_1}{\partial x_1}\varphi\right] .\end{aligned}$$

The control u should now be able to cancel the indefinite terms in $\dot V_2$. To deal with the terms containing the unknown parameter θ, we will try to employ the existing estimate ϑ_1:

$$u = -z_1 - c_2z_2 + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 + \vartheta_1\frac{\partial\alpha_1}{\partial x_1}\varphi_1 .$$

From the resulting derivative

$$\dot V_2 = -c_1z_1^2 - c_2z_2^2 - (\theta-\vartheta_1)\frac{\partial\alpha_1}{\partial x_1}\varphi_1 z_2 ,$$

we see that we have no design freedom left to cancel the $(\theta-\vartheta_1)$-term. To overcome this difficulty, we replace ϑ_1 in the expression for u with a *new* estimate ϑ_2:

$$u = -z_1 - c_2z_2 + \frac{\partial\alpha_1}{\partial x_1}x_2 + \frac{\partial\alpha_1}{\partial\vartheta_1}\gamma\varphi z_1 + \vartheta_2\frac{\partial\alpha_1}{\partial x_1}\varphi . \tag{3.40}$$

With the choice (3.40), the $\dot z_2$-equation becomes

$$\dot z_2 = -c_2z_2 - z_1 - (\theta-\vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi . \tag{3.41}$$

The presence of the new parameter estimate ϑ_2 suggests the following augmentation of the Lyapunov function:

$$\begin{aligned}V_2(z_1,z_2,\vartheta_1,\vartheta_2) &= V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}(\theta-\vartheta_2)^2\\ &= \frac{1}{2}\left(z_1^2+z_2^2\right) + \frac{1}{2\gamma}\left[(\theta-\vartheta_1)^2 + (\theta-\vartheta_2)^2\right] .\end{aligned} \tag{3.42}$$

The derivative of V_2 is

$$\begin{aligned}\dot V_2 &= \dot V_1 + z_2\dot z_2 - \frac{1}{\gamma}(\theta-\vartheta_2)\dot\vartheta_2\\ &= z_1z_2 - c_1z_1^2 + z_2\left[-c_2z_2 - z_1 - (\theta-\vartheta_2)\frac{\partial\alpha_1}{\partial x_1}\varphi\right] - \frac{1}{\gamma}(\theta-\vartheta_2)\dot\vartheta_2\\ &= -c_1z_1^2 - c_2z_2^2 - (\theta-\vartheta_2)\left(\frac{\partial\alpha_1}{\partial x_1}\varphi + \frac{1}{\gamma}\dot\vartheta_2\right) .\end{aligned} \tag{3.43}$$

Now the $(\theta-\vartheta_2)$-term can be eliminated with the update law

$$\dot\vartheta_2 = -\gamma\frac{\partial\alpha_1}{\partial x_1}\varphi z_2 , \tag{3.44}$$

which yields

$$\dot V_2 = -c_1z_1^2 - c_2z_2^2 . \tag{3.45}$$

The equations (3.41) and (3.44) along with (3.36) and (3.35) form the error system representation of the resulting closed-loop adaptive system:

$$\begin{aligned}\dot z_1 &= -c_1z_1 + z_2 + (\theta-\vartheta_1)\varphi\\ \dot z_2 &= -c_2z_2 - z_1 - (\theta-\vartheta_2)\tfrac{\partial\alpha_1}{\partial x_1}\varphi\\ \dot\vartheta_1 &= \gamma\varphi z_1\\ \dot\vartheta_2 &= -\gamma\tfrac{\partial\alpha_1}{\partial x_1}\varphi z_2 .\end{aligned} \tag{3.46}$$

The matrix form of this system,

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} z_1\\ z_2\end{bmatrix} &= \begin{bmatrix} -c_1 & 1\\ -1 & -c_2\end{bmatrix}\begin{bmatrix} z_1\\ z_2\end{bmatrix} + \begin{bmatrix} \varphi & 0\\ 0 & -\frac{\partial\alpha_1}{\partial x_1}\varphi\end{bmatrix}\begin{bmatrix} \theta-\vartheta_1\\ \theta-\vartheta_2\end{bmatrix}\\ \frac{d}{dt}\begin{bmatrix} \vartheta_1\\ \vartheta_2\end{bmatrix} &= \gamma\begin{bmatrix} \varphi & 0\\ 0 & -\frac{\partial\alpha_1}{\partial x_1}\varphi\end{bmatrix}\begin{bmatrix} z_1\\ z_2\end{bmatrix} ,\end{aligned} \tag{3.47}$$

makes its properties more visible:

- The constant system matrix has negative terms along its diagonal, while its off-diagonal terms are skew-symmetric, and
- the matrix that multiplies the parameter errors in the $\dot z$-equation is used in the update laws for the parameter estimates.

The stability properties of (3.47) follow from (3.42) and (3.45): The LaSalle-Yoshizawa theorem (Theorem 2.1) establishes that $z_1, z_2, \vartheta_1, \vartheta_2$ are bounded, and $z \to 0$ as $t \to \infty$. Since $z_1 = x_1$, x_1 is also bounded and converges to zero. The boundedness of x_2 then follows from the boundedness of α_1 (defined in (3.34)) and the fact that $x_2 = z_2 + \alpha_1$. Using (3.40) we conclude that the control u is also bounded. Finally, we note that the regulation of z and x_1 does *not* imply the regulation of x_2: From $z_2 = x_2 - \alpha_1$ and (3.34) we see that $x_2 + \vartheta_1\varphi(0)$ will converge to zero. Thus, x_2 is not guaranteed to converge to zero unless $\varphi(0) = 0$. However, x_2 will converge to a constant value:

$$\lim_{t\to\infty} \;=\; -\theta\varphi(0) \triangleq x_2^e . \tag{3.48}$$

This can be seen from (3.29a): Since x_1 and $\dot{x}_1$ converge to zero, so does $x_2 + \theta\varphi(0)$.

With the above example we have illustrated the idea of adaptive backstepping. To formulate it as a design tool analogous to Lemma 2.8, we start with the assumption that an adaptive controller is known for an initial system.

Assumption 3.1 *Consider the system*

$$\dot{x} \;=\; f(x) + F(x)\theta + g(x)u , \tag{3.49}$$

where $x \in \mathbb{R}^n$ is the state, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, and $u \in \mathbb{R}$ is the control input. There exist an adaptive controller

$$\begin{aligned} u &= \alpha(x,\vartheta) \\ \dot{\vartheta} &= T(x,\vartheta) , \end{aligned} \tag{3.50}$$

with parameter estimate $\vartheta \in \mathbb{R}^q$, and a smooth function $V(x,\vartheta) : \mathbb{R}^{(n+q)} \to \mathbb{R}$ which is positive definite and radially unbounded in the variables $(x, \vartheta - \theta)$ such that for all $(x,\vartheta) \in \mathbb{R}^{(n+q)}$:

$$\frac{\partial V}{\partial x}(x,\vartheta)\left[f(x) + F(x)\theta + g(x)\alpha(x,\vartheta)\right] + \frac{\partial V}{\partial \vartheta}(x,\vartheta)T(x,\vartheta) \;\le\; -W(x,\vartheta) \le 0, \tag{3.51}$$

where $W : \mathbb{R}^{n+q} \to \mathbb{R}$ is positive semidefinite. □

Under this assumption, the control (3.50), applied to the system (3.49), guarantees global boundedness of $x(t), \vartheta(t)$ and, by the LaSalle-Yoshizawa theorem (Theorem 2.1), regulation of $W(x(t), \vartheta(t))$. Adaptive backstepping allows us to achieve the same properties for the augmented system.

Lemma 3.2 (Adaptive Backstepping) *Let the system (3.49) be augmented by an integrator,*

$$\dot{x} \;=\; f(x) + F(x)\theta + g(x)\xi \tag{3.52a}$$
$$\dot{\xi} \;=\; u , \tag{3.52b}$$

where $\xi \in \mathbb{R}$. Consider for this system the dynamic feedback controller

$$\begin{aligned} u &= -c(\xi - \alpha(x,\vartheta)) + \frac{\partial \alpha}{\partial x}(x,\vartheta)\left[f(x) + F(x)\bar{\vartheta} + g(x)\xi\right] \\ &\quad + \frac{\partial \alpha}{\partial \vartheta}T(x,\vartheta) - \frac{\partial V}{\partial x}(x,\vartheta)g(x) , \quad c > 0 \end{aligned} \tag{3.53}$$

$$\dot{\vartheta} \;=\; T(x,\vartheta) \tag{3.54}$$

$$\dot{\bar{\vartheta}} \;=\; -\Gamma\left[\frac{\partial \alpha}{\partial x}(x,\vartheta)F(x)\right]^{\mathrm{T}}(\xi - \alpha(x,\vartheta)) , \tag{3.55}$$

where $\bar{\vartheta}$ is a new estimate of θ, $\Gamma = \Gamma^{\mathrm{T}} > 0$ is the adaptation gain matrix. Under Assumption 3.1, this adaptive controller guarantees global boundedness of $x(t)$, $\xi(t)$, $\vartheta(t)$, $\bar{\vartheta}(t)$ and regulation of $W(x(t), \vartheta(t))$ and $\xi(t) - \alpha(x(t), \vartheta(t))$. These properties can be established with the Lyapunov function

$$V_a(x,\xi,\vartheta,\bar{\vartheta}) \;=\; V(x,\vartheta) + \frac{1}{2}\left[\xi - \alpha(x,\vartheta)\right]^2 + \frac{1}{2}(\theta - \bar{\vartheta})^{\mathrm{T}}\Gamma^{-1}(\theta - \bar{\vartheta}) . \tag{3.56}$$

Proof. With the error variable $z = \xi - \alpha(x,\vartheta)$, (3.52) is rewritten as

$$\dot{x} \;=\; f(x) + F(x)\theta + g(x)\left[\alpha(x,\vartheta) + z\right] \tag{3.57a}$$

$$\dot{z} \;=\; u - \frac{\partial \alpha}{\partial x}(x,\vartheta)\left[f(x) + F(x)\theta + g(x)\left(\alpha(x,\vartheta) + z\right)\right] - \frac{\partial \alpha}{\partial \vartheta}T(x,\vartheta) . \tag{3.57b}$$

Note that in (3.57b) the derivative of ϑ was replaced by the update law (3.54). Introducing a new parameter estimate $\bar{\vartheta}$, we augment the Lyapunov function:

$$V_a(x,\xi,\vartheta,\bar{\vartheta}) \;=\; V(x,\vartheta) + \frac{1}{2}z^2 + \frac{1}{2}(\theta - \bar{\vartheta})^{\mathrm{T}}\Gamma^{-1}(\theta - \bar{\vartheta}) . \tag{3.58}$$

Using (3.51), it is easy to show that the derivative of (3.58) satisfies

$$\begin{aligned} \dot{V}_a &= \frac{\partial V}{\partial x}(f + F\theta + g\alpha + gz) + \frac{\partial V}{\partial \vartheta}T \\ &\quad + z\left[u - \frac{\partial \alpha}{\partial x}\left(f + F\theta + g\left(\alpha + z\right)\right) - \frac{\partial \alpha}{\partial \vartheta}T\right] - \dot{\bar{\vartheta}}^{\mathrm{T}}\Gamma^{-1}(\theta - \bar{\vartheta}) \\ &= \frac{\partial V}{\partial x}(f + F\theta + g\alpha) + \frac{\partial V}{\partial \vartheta}T \\ &\quad + z\left[u - \frac{\partial \alpha}{\partial x}\left(f + F\theta + g(\alpha + z)\right) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g\right] - \dot{\bar{\vartheta}}^{\mathrm{T}}\Gamma^{-1}(\theta - \bar{\vartheta}) \\ &\le -W(x,\vartheta) + z\left[u - \frac{\partial \alpha}{\partial x}\left(f + F\bar{\vartheta} + g(\alpha + z)\right) - \frac{\partial \alpha}{\partial \vartheta}T + \frac{\partial V}{\partial x}g\right] \\ &\quad - \left[\frac{\partial \alpha}{\partial x}Fz + \dot{\bar{\vartheta}}^{\mathrm{T}}\Gamma^{-1}\right](\theta - \bar{\vartheta}) . \end{aligned} \tag{3.59}$$

The $(\theta - \bar{\vartheta})$-term is now eliminated with the update law (cf. (3.55))

$$\dot{\bar{\vartheta}} \;=\; -\Gamma\left(\frac{\partial \alpha}{\partial x}F\right)^{\mathrm{T}} z , \tag{3.60}$$

and the control (3.53) is chosen to make the bracketed term multiplying z in (3.59) equal to $-cz$ (cf. (2.54)):

$$u = -cz + \frac{\partial\alpha}{\partial x}\left(f + F\tilde{\vartheta} + g(\alpha+z)\right) + \frac{\partial\alpha}{\partial\vartheta}T - \frac{\partial V}{\partial x}g. \tag{3.61}$$

This results in the desired nonpositivity of $\dot{V}_a$:

$$\dot{V}_a \le -W(x,\vartheta) - cz^2 \le 0. \tag{3.62}$$

From (3.56) and (3.62) we conclude that $V(x,\vartheta)$, $\tilde{\vartheta}$ and z are bounded. By Assumption 3.1, this means that $x(t)$ and $\vartheta(t)$ are bounded. Hence, $\xi = z + \alpha(x,\vartheta)$ and u are bounded. By Thorem 2.1, the boundedness of all the signals combined with (3.62) proves the regulation of $W(x(t),\vartheta(t))$ and $z(t)$. □

3.2.2 Adaptive block backstepping

We now extend the Adaptive Backstepping Lemma (Lemma 3.2) by augmenting the initial system with a relative-degree-one nonlinear system whose zero dynamics subsystem is ISS, just like we did in Chapter 2, Lemmas 2.8 and 2.25. The adaptive counterpart of Assumption 2.7 was Assumption 3.1. We now formulate the adaptive counterpart of Assumption 2.21, with analogous changes in the properties of $V(x,\vartheta)$ from Assumption 3.1.

Assumption 3.3 *Suppose Assumption 3.1 is valid, but $V(x,\vartheta)$ is only positive semidefinite, and the closed-loop system (3.49) with the adaptive controller (3.50) has the property that $x(t)$ and $\vartheta(t)$ are bounded if $V(x(t),\vartheta(t))$ is bounded.* □

Under this assumption, the control (3.50), applied to the system (3.49), guarantees global boundedness of $x(t), \vartheta(t)$ and, by Lemma A.6, regulation of $W(x(t),\vartheta(t))$.

Lemma 3.4 (Adaptive Block Backstepping) *Let the system (3.49) be augmented by a nonlinear system which is linear in the unknown parameter vector θ,*

$$\dot{x} = f(x) + F(x)\theta + g(x)y \tag{3.63a}$$

$$\dot{\xi} = m(x,\xi) + M(x,\xi)\theta + \beta(x,\xi)u, \quad y = h(\xi), \tag{3.63b}$$

where $\xi \in \mathbb{R}^q$, and suppose that (3.63b) has relative degree one uniformly in x and that its zero dynamics subsystem is ISS with respect to y and x. Under Assumption 3.3, the feedback control

$$u = \left[\frac{\partial h}{\partial \xi}(\xi)\beta(x,\xi)\right]^{-1}\left\{-c(y-\alpha(x,\vartheta)) - \frac{\partial h}{\partial\xi}(\xi)\left[m(x,\xi) + M(x,\xi)\bar{\vartheta}\right] + \frac{\partial\alpha}{\partial x}(x,\vartheta)\left[f(x) + F(x)\bar{\vartheta} + g(x)y\right] + \frac{\partial\alpha}{\partial\vartheta}T(x,\vartheta) - \frac{\partial V}{\partial x}(x,\vartheta)g(x)\right\}, \tag{3.64}$$

with $c > 0$ and $\bar{\vartheta}$ a new estimate of θ, along with the update laws

$$\dot{\vartheta} = T(x,\vartheta) \tag{3.65}$$

$$\dot{\bar{\vartheta}} = \Gamma\left[\frac{\partial h}{\partial\xi}(\xi)M(x,\xi) - \frac{\partial\alpha}{\partial x}(x,\vartheta)F(x)\right]^{\mathrm{T}}(y-\alpha(x,\vartheta)), \tag{3.66}$$

with the adaptation gain matrix $\Gamma = \Gamma^{\mathrm{T}} > 0$, guarantees global boundedness of $x(t)$, $\xi(t)$, $\vartheta(t)$, $\bar{\vartheta}(t)$ and regulation of $W(x(t),\vartheta(t))$ and $\xi(t) - \alpha(x(t),\vartheta(t))$.

Proof. As in Lemma 2.25, we employ the change of coordinates $(y,\zeta) = (h(\xi),\phi(x,\xi))$, with $\frac{\partial\phi}{\partial\xi}\beta \equiv 0$, to transform (3.63b) into the normal form

$$\dot{y} = \frac{\partial h}{\partial\xi}(\xi)\left[m(x,\xi) + M(x,\xi)\theta + \beta(x,\xi)u\right] \tag{3.67a}$$

$$\begin{aligned}\dot{\zeta} &= \frac{\partial\phi}{\partial x}(x,\xi)\left[f(x) + F(x)\theta + g(x)y\right] + \frac{\partial\phi}{\partial\xi}(x,\xi)\left[m(x,\xi) + M(x,\xi)\theta\right] \\ &\triangleq \Phi_0(x,y,\zeta) + \Phi(x,y,\zeta)\theta.\end{aligned} \tag{3.67b}$$

Introducing a new parameter estimate $\bar{\vartheta}$, we use the feedback transformation

$$u = \left(\frac{\partial h}{\partial\xi}\beta\right)^{-1}\left\{v - \frac{\partial h}{\partial\xi}\left[m + M\bar{\vartheta}\right]\right\} \tag{3.68}$$

to rewrite (3.63a) and (3.67a) as

$$\dot{x} = f(x) + F(x)\theta + g(x)y \tag{3.69a}$$

$$\dot{y} = v + \frac{\partial h}{\partial\xi}(\xi)M(x,\xi)(\theta - \bar{\vartheta}). \tag{3.69b}$$

We now apply Lemma 3.1 to (3.69). The only difference between (3.69) and (3.52) is the presence of the additional parameter error term $\frac{\partial h}{\partial\xi}M(\theta-\bar{\vartheta})$ in (3.69b). This term can be eliminated in $\dot{V}_a$ by adding the term $-\Gamma[\frac{\partial h}{\partial\xi}M]^{\mathrm{T}}(y-\alpha)$ to the update law (3.55). Combining this modification with (3.68), we see that the resulting adaptive controller is given by (3.64)–(3.66). This guarantees the boundedness of x, ϑ, $\bar{\vartheta}$, z and the regulation of $W(x,\vartheta)$ and z. Hence, $y = z + \alpha(x,\vartheta)$ is bounded. Then, from (3.67b) and the ISS property of the zero dynamics, ζ is also bounded, and thus ξ and u are bounded. □

3.3 Recursive Design Procedures

3.3.1 Parametric strict-feedback systems

Through repeated application of Lemma 3.2, the backstepping design procedure is now generalized to nonlinear systems which can be transformed[1] into

[1] The coordinate-free characterization of these systems in terms of differential geometric conditions is given in Appendix G, Corollary G.15.

Control Design under Uncertainty (Continued)

Last time we investigated a class of uncertain systems of the form $\dot{x} = f(x) + g(x)\left(u + \phi(x)^T \Delta(t)\right)$, where

$\phi(x) \in \mathbb{R}^{q\times 1}$ known function

$\Delta(t) \in \mathbb{R}^{q\times 1}$ uncertain function

from Krstic et al. textbook. We studied the design method called nonlinear damping. In fact, the same technique is presented in our main textbook, Chapter 14.2.2

Let us review the notation because we will use it for introducing another technique, called sliding-mode control, shortly.

Let the system $\dot{x} = f(x) + G(x)\left(u + \Gamma(x,t)\,\delta_0(t,x,u)\right)$

where f, G, Γ known functions, piecewise continuous in t and locally Lipschitz in x and u, for all $(t,x,u) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^p$,

and δ_0 is unknown, uniformly bounded for all (t,x,u)

Let $\psi(t,x)$ be a control law such that the origin of the nominal system (i.e. for $\delta_0 \equiv 0$) is globally uniformly asymptotically stable.

Then the proposed control law (nonlinear damping controller) is $u = \psi(t,x) - k \cdot \frac{\partial V}{\partial x} G(x) \, \|\Gamma(t,x)\|_2^2$, $k > 0$.

in the Euclidean norm

where $V(t,x)$ is a locally positive definite and decrescent function that is a Lyapunov function for the nominal system.

i.e. such that $\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big(f(x) + G(x)\,\psi(x)\Big) \leq$

$$\leq -a_3(\|x\|)$$

where a_3 is a class-K_∞ function.

so that along the trajectories of the perturbed system we have

$$\dot{V}(t,x) \leq -a_3(\|x\|) + \frac{k_0^2}{4k}, \text{ where } k_0 = \sup\|\delta_0\|$$

With this we conclude boundedness of the trajectories for all initial conditions $x(t_0)$, i.e. $\sup_{t \geq t_0} \|x(t)\| < \infty$.

In fact, the nonlinear damping term $v = -k\frac{\partial V}{\partial x} G(x)\, \|r(t,x)\|_2^2$ has the nice property of establishing that the trajectories will remain bounded without knowing the actual bound k_0 of the disturbance δ_0! In other words, the nonlinear damping controller achieves boundedness and convergence to a set $\Omega = \{\|x\| \in \mathbb{R}^n \mid a_1^{-1} \circ a_2 \circ a_3^{-1}\left(\frac{k_0^2}{4k}\right)\}$

where a_1, a_2 class-K_∞ functions such that

$$a_1(\|x\|) \leq V(t,x) \leq a_2(\|x\|)$$

without using any knowledge about the disturbance δ_0!

In fact, this is a special case of a more general technique called Lyapunov Redesign, presented in Chapter 14.2.1.

Lyapunov Redesign Technique (Chapter 14.2)

The notion of ultimate boundedness is particularly useful in the so-called Lyapunov redesign technique.

The problem is formulated as follows. We consider the system

$$\dot{x} = f(t,x) + G(t,x)\left[u + \delta(t,x,u)\right] \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the control input.
The functions f, G, δ are defined for $[0,+\infty) \times D \times \mathbb{R}^p$, where $D \subset \mathbb{R}^n$ is a domain that contains the origin.

Assumptions.

(i) f, G, δ are piecewise continuous in t, and locally Lipschitz in x and u.

(ii) f, G are known precisely.

(iii) δ is unknown, and lumps together uncertainty due to modeling simplifications, disturbances etc.
Note that δ satisfies the so-called matching condition, i.e, appears exactly at the same place as the control input u.

(iv) The nominal system is taken as

$$\dot{x} = f(t,x) + G(t,x)u. \qquad (2)$$

Suppose that we have designed a feedback control law $u = \varphi(t,x)$ such that the origin of the closed-loop nominal system $\dot{x} = f(t,x) + G(t,x)\varphi(t,x)$ is uniformly asymptotically stable.

Suppose further that we know a Lyapunov function for ②

that is, we have a continuously differentiable function

$V(t,x)$ that satisfies

$$a_1(\|x\|) \leq V(t,x) \leq a_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\Big(f(t,x) + G(t,x)\varphi(t,x)\Big) \leq -a_3(\|x\|),$$

for all $(t,x) \in [0,+\infty) \times D$, where a_1, a_2, a_3 are class $\mathcal{K}$ functions.

(v) The uncertain term δ satisfies

$$\|\delta(t,x,\varphi(x,t)+v)\| \leq \rho(t,x) + k_0\|v\|, \quad 0 \leq k_0 < 1;$$

for some v to be defined later, $\rho : [0,+\infty) \times D \to \mathbb{R}$ a continuous nonnegative function. The function ρ can be seen as a measure of the uncertainty.

Problem.

Design an additional feedback control v such that the overall control $u = \varphi(t,x) + v$ stabilizes the actual system ① in the presence of the uncertainty.

Remark The design of v is called Lyapunov redesign.

Approach. Consider the closed-loop system of ① as

$$\dot{x} = f(t,x) + G(t,x)\varphi(t,x) + G(t,x)\delta(t,x,\varphi(x,t)+v)$$

③

which is a perturbation of the nominal closed-loop system ②.

We proceed as usual by considering the time derivative of the Lyapunov function $V(t,x)$ of the nominal system ② along the trajectories of ③

[We drop the arguments of functions for convenience]

$$\dot{V} = \underbrace{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}(f + G\varphi)}_{\leq -\alpha_3(\|x\|)} + \frac{\partial V}{\partial x} G(v+\delta) \leq -\alpha_3(\|x\|) + \underbrace{\frac{\partial V}{\partial x}}_{\text{Denote } w^T} G(v+\delta)$$

$$\dot{V} \leq -\alpha_3(\|x\|) + \underbrace{w^T v + w^T \delta}_{\text{choose } v \text{ such that } \leq 0.}$$

Recall $\|\delta(t,x,\varphi(t,x)+u)\|_2 \leq \rho(t,x) + k_0\|u\|_2$, $0 \leq k_0 < 1$

Hence

$$w^T v + w^T \delta \leq w^T v + \|w\|_2 \|\delta\|_2 \leq$$
$$\leq w^T v + \|w\|_2 \left(\rho(t,x) + k_0\|u\|_2\right)$$

Let the control law be of the form $\boxed{v = -n(t,x)\frac{w}{\|w\|_2}}$ * for some nonnegative $n(t,x)$. Then

$$w^T v + w^T \delta \leq -n\|w\|_2 + \rho\|w\|_2 + k_0 n\|w\|_2 =$$
$$= -n(1-k_0)\|w\|_2 + \rho\|w\|_2$$

* Notice however that the control law is in general discontinuous in x !!!

Then, if we choose

$$\underline{n(t,x) \geq \frac{e(t,x)}{1-k_0}} \quad \text{for all } (t,x) \in [0,+\infty) \times D$$

yields $w^T v + w^T \delta \leq -e\|w\|_2 + e\|w\|_2 = 0$

i.e., renders the derivative of $v(t,x)$ along the trajectories of ③ negative definite.

However, the control law $v = -n(t,x) \frac{w}{\|w\|_2}$

is in general a discontinuous function of the state x.

That creates both practical and theoretical challenges.

- Theoretically, we have to investigate the question of existence and uniqueness of solutions more carefully, since the feedback functions are no longer locally Lipschitz in x.
- Theoretically, we also have to consider the definition of the control law to make sure we avoid division by zero.
- Practically, the implementation of such controllers leads to the phenomenon of chattering. (fast switching fluctuations across the switching surface)

To overcome the problem we consider the continuous function

$$v = \begin{cases} -n(t,x) \dfrac{w}{\|w\|_2} & \text{if } n(t,x)\|w\|_2 \geq \epsilon \quad (4.a) \\[2ex] -n^2(t,x) \dfrac{w}{\epsilon} & \text{if } n(t,x)\|w\|_2 < \epsilon \quad (4.b) \end{cases}$$

We have that under (4.a), the derivative $\dot{V}(t,x)$ is negative definite whenever $n(t,x)\|w\|_2 \geq \epsilon$.

We only have to check $\dot{V}$ when $n(t,x)\|w\|_2 < \epsilon$. In this case,

$$\dot{V} \leq -a_3(\|x\|_2) + w^T\left(-n^2 \frac{w}{\epsilon} + \delta\right)$$

$$\leq -a_3(\|x\|_2) - \frac{n^2}{\epsilon}\|w\|_2^2 + \rho\|w\|_2 + k_0\|w\|_2\|u\|_2$$

$$= -a_3(\|x\|_2) - \frac{n^2}{\epsilon}\|w\|_2^2 + \rho\|w\|_2 + k_0\frac{n^2}{\epsilon}\|w\|_2^2$$

$$\leq -a_3(\|x\|_2) + (1+k_0)\underbrace{\left(-\frac{n^2}{\epsilon}\|w\|_2^2 + n\|w\|_2\right)}_{\text{The maximum value of this term is } \frac{\epsilon}{4}\text{, when } n\|w\|_2 = \frac{\epsilon}{2}}$$

Hence, whenever $n(t,x)\|w\|_2 < \epsilon$, we have

$$\dot{V} \leq -a_3(\|x\|_2) + \frac{\epsilon(1-k_0)}{4}$$

On the other hand, we have that whenever $n(t,x)\|w\|_2 \geq \epsilon$, we have $\dot{V} \leq -a_3(\|x\|_2) \leq -a_3(\|x\|_2) + \dfrac{\epsilon(1-k_0)}{4}$.

Therefore, the inequality

$$\boxed{\dot{V} \leq -a_3(\|x\|_2) + \frac{\epsilon(1-k_0)}{4}}$$

holds irrespectively of the value of $n(t,x)\|w\|_2$. Then, application of Theorem 4.18 shows that the solutions of the closed-loop system are uniformly ultimately bounded by a class K function of ϵ.

In fact, Theorem 14.3 summarizes the result and concludes that the ultimate bound $b(\epsilon)$ is a class K function of ϵ. We can observe the following.

- The continuous Lyapunov redesign controller guarantees uniform ultimate boundedness of the solutions
- Since the ultimate bound $b(\epsilon)$ is a class K function of ϵ, it can be made arbitrarily small by choosing ϵ small enough.
- In the limit, as $\epsilon \to 0$, we recover the performance of the discontinuous controller.

In summary: Lyapunov Redesign illustrates the methodological path we can take when addressing the stabilization of systems of the form.

$$\dot{x} = f(t,x) + G(t,x)\left(u + \delta(t,x,u)\right), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^P.$$

Namely, the general approach is to consider the nominal system first, and design a stabilizing feedback controller $u = \varphi(t,x)$. Then, we can consider the design of an additional control law v, such that the control law $u = \varphi(t,x) + v$ renders the solutions of the perturbed system ultimate bounded or GAS (that depends on the form of the disturbance and the choice of v). Some specialized results given certain assumptions on δ, v are provided in Corollary 14.1 (left for optional reading)