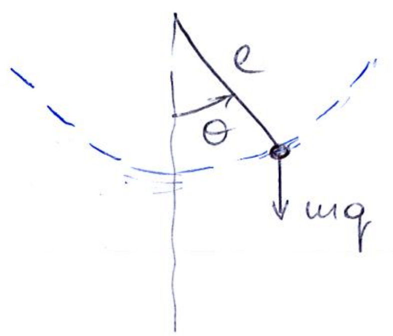
Examples of Nonlinear Systems.

see also Section 1.2 in textbook.

① Pendulum Equation.



Rod of length l and zero mass.
Bob of mass m.
The pendulum swings on the vertical plane.

Frictional force resisting to the motion, assumed to be proportional to the speed of the bob, with coefficient of friction equal to k.

Equation of motion along the tangential direction.
(Newton's Law) $\boxed{ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}}$

We write the equation in state-space form by setting $\theta = x_1$, $\dot{\theta} = x_2$. Then.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

To find the equilibrium points : We set $\dot{x}_1 = 0$, $\dot{x}_2 = 0$ and solve for x_1, x_2. We get:

$$\left.\begin{array}{l} 0 = x_2 \\ 0 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{array}\right\} \Rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = n\pi, \quad n = 0, \pm 1, \pm 2, \dots \end{array}$$

From the physical description of the problem, the pendulum has two equilibrium positions $(0,0)$ and $(\pi, 0)$; other equilibrium points are repetitions of these two positions.

Chapter 2. Second-order systems.

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}\right\}$$

Assume the solution $x(t) = (x_1(t), x_2(t))$, $x(t_0) = x_0$ exists and is unique. Then the locus of $x(t)$ on the $x_1 - x_2$ plane $\forall\, t \geq t_0$ is a curve that passes through x_0.

The right-hand side of the state equation expresses the tangent vector $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t))$ to the solution curve.

The solution curve is often called a trajectory or orbit (from x_0)

The family of all solutions or trajectories is called the phase portrait of the system. We can qualitatively analyze the behavior of second-order systems by using their phase portraits.

- Qualitative Behavior of Linear Systems.

$$\boxed{\dot{x} = Ax} \quad \text{where } A \in \mathbb{R}^{2\times 2}$$

The solution of the system from a given initial state x_0 is

$$x(t) = M \exp(J_r t)\, M^{-1} x_0.$$

where J_r is the real Jordan form of A, and M is real and non-singular such that $M^{-1}AM = J_r$.

Depending on the eigenvalues of A, J_r may be of these forms:

$$\underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{} \qquad \underbrace{\begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}}_{} \qquad \underbrace{\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}}_{}$$

2 real and distinct eigenvalues.

$k = 0, 1$
2 real and equal eigenvalues.

Complex eigenvalues
$\lambda_{1,2} = \alpha \pm j\beta$

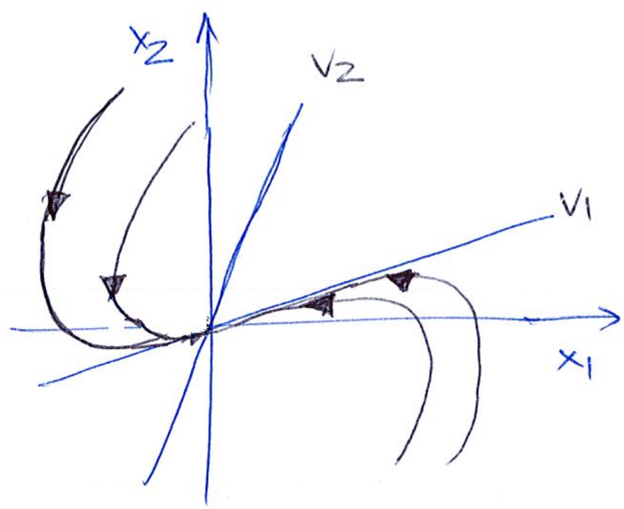
Review in Chapter 2, of the textbook.

Case 1: Both eigenvalues are real, $\lambda_1 \neq \lambda_2 \neq 0$

a) Assume negative eigenvalues, $\lambda_2 < \lambda_1 < 0$.

λ_2: fast eigenvalue

λ_1: slow eigenvalue



Trajectories approach the origin tangent to the slow eigenvector, and are parallel to the fast eigenvector far from the origin.

Stable node.

b) Assume positive eigenvalues, $0 < \lambda_1 < \lambda_2$

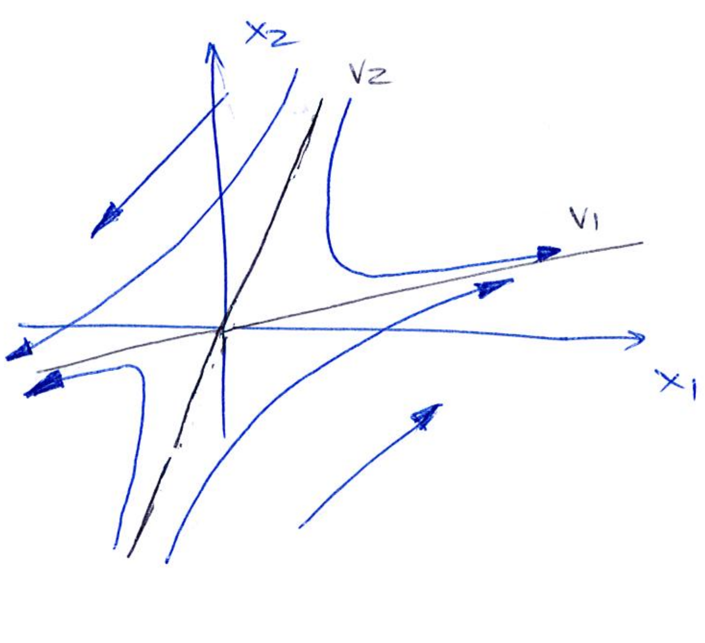
Unstable node.

Reversed trajectory direction compared to previous case.

c) Assume eigenvalues of opposite signs, $\lambda_2 < 0 < \lambda_1$

λ_2: stable eigenvalue

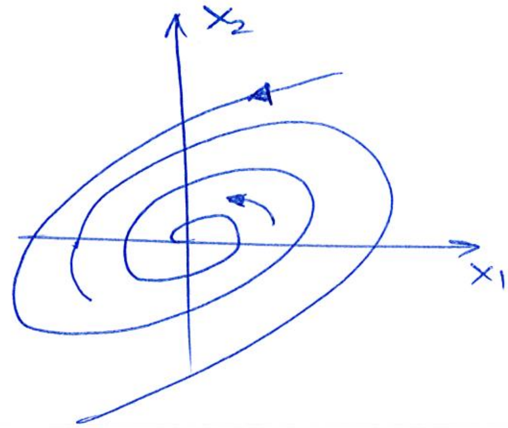
λ_1: unstable eigenvalue.



Saddle point.

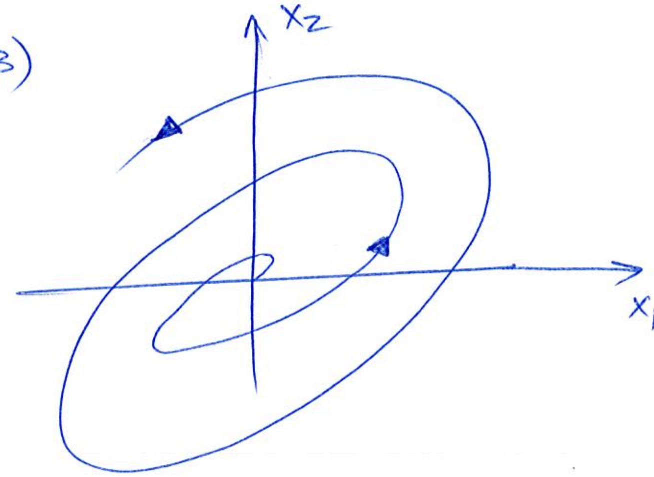
Case 2. Complex eigenvalues $\lambda_{1,2} = a \pm j\beta$.

(a)

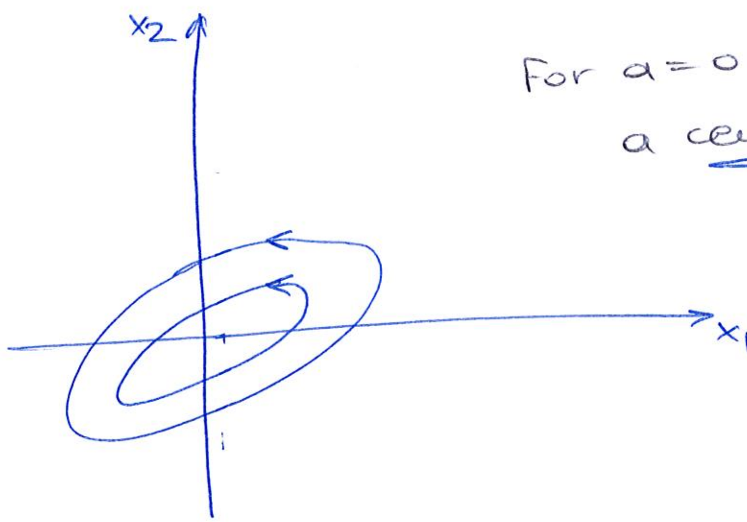


For $a<0$, we have stable focus.

(β)



For $a<0$, we have unstable focus.



For $a=0$, we have a center

Case 3. Non-zero multiple eigenvalues: $\lambda_1 = \lambda_2 = \lambda \neq 0$.

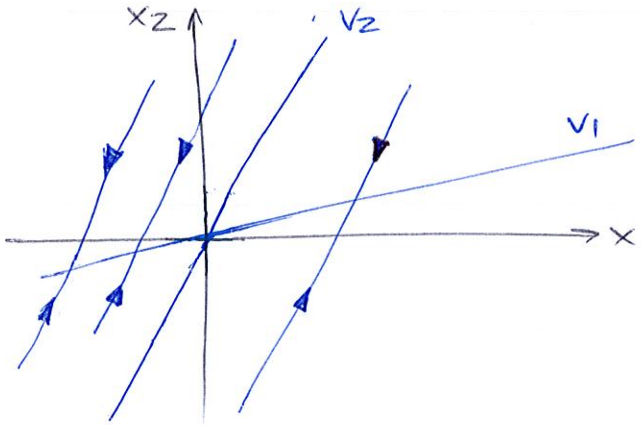
Phase portrait similar to the portrait of a node, but not with the fast-slow asymptotic behavior we noticed earlier.

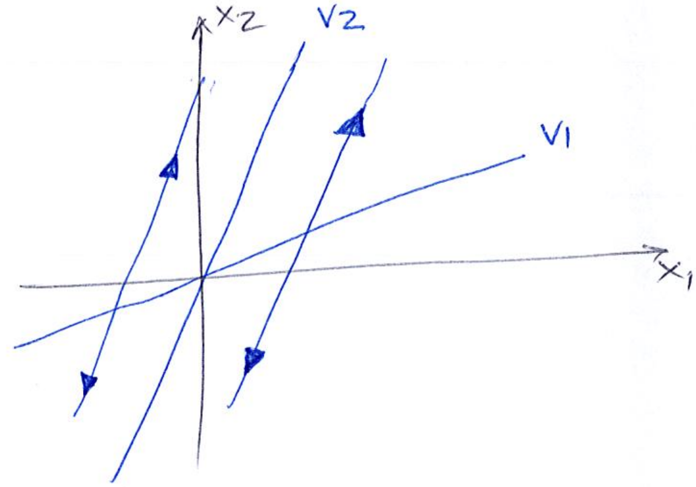
We usually call the case $\lambda<0$ a stable node, and $\lambda>0$ an unstable node.

Case 4. One or both eigenvalues are zero.

In this case the matrix A has a non-trivial null-space; that means, any vector in the null space of A is an equilibrium point for the system.

We also say the system has an equilibrium space instead of equilibrium point.





(a) $\lambda_1 = 0,\ \lambda_2 < 0$ (b) $\lambda_1 = 0,\ \lambda_2 > 0.$

Qualitative Behavior near Equilibrium Points.

Except for some special cases, the qualitative behavior of a nonlinear system near an equilibrium point can be determined via linearization with respect to that point.

Consider $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$ and let $p = (p_1, p_2)$ be an equilibrium point.

Assume f_1, f_2 are continuously differentiable. Writing them as a Taylor series expansion about (p_1, p_2) yields

$$\dot{x}_1 = f_1(p_1, p_2) + \left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}} (x_1 - p_1) + \left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}} + \text{HOT}$$

$$\dot{x}_2 = f_2(p_1, p_2) + \left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}} (x_1 - p_1) + \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}} + \text{HOT.}$$

Since (p_1, p_2) is an equilibrium, we have

$$f_1(p_1, p_2) = 0, \quad f_2(p_1, p_2) = 0.$$

And since we are interested in the trajectories near (p_1, p_2), we define $y_1 = x_1 - p_1$, $y_2 = x_2 - p_2$ to rewrite the system equations as:

$$\begin{cases} \dot{y}_1 = \dot{x}_1 = \underbrace{\left.\dfrac{\partial f_1}{\partial x_1}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}}}_{a_{11}} \cdot y_1 + \underbrace{\left.\dfrac{\partial f_1}{\partial x_2}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}}}_{a_{12}} y_2 + \text{HOT} \\ \dot{y}_2 = \dot{x}_2 = \underbrace{\left.\dfrac{\partial f_2}{\partial x_1}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}}}_{a_{21}} y_1 + \underbrace{\left.\dfrac{\partial f_2}{\partial x_2}\right|_{\substack{x_1 = p_1 \\ x_2 = p_2}}}_{a_{22}} y_2 + \text{HOT} \end{cases}$$

Now, we can restrict our attention to a sufficiently small neighborhood of the equilibrium so that the HOT are negligible, and approximate the nonlinear equations with the linear state equations

$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \end{cases} \Rightarrow \text{in vector form} \quad \dot{y} = Ay, \text{ where}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

evaluated at the equilibrium !!!

The matrix $\left[\dfrac{\partial f}{\partial x}\right]$ is called the Jacobian matrix of $f(x)$, and A is the Jacobian matrix evaluated at the equilibrium $x = p$.

Now, it is true that, if the origin of the linearized system is a stable (resp. unstable) node with distinct eigenvalues, a stable (resp. unstable) focus, or a saddle point, then, in a small neighborhood of the equilibrium point, the trajectories of the nonlinear system will behave like a stable (resp. unstable) node, a stable (resp. unstable) focus, or a saddle point. <u>However</u>, if the linearized system has a center equilibrium (i.e., eigenvalues of the Jacobian A on the imaginary axis), then the behavior of the nonlinear system around the equilibrium point could be quite distinct from that of the linearized system. In fact, in this case, the linearization method around the equilibrium point is inconclusive regarding the type of the equilibrium of the nonlinear system.