Instability. (Continued)

Last time we investigated a first "overkill" (i.e, conservative) result on how to establish the instability of an equilibrium point.

The importance of this result is twofold. First, it verifies our physical intuition that, if the time derivative of a candidate Lyapunov function is positive on a domain D that contains the equilibrium point, then this implies that the system trajectories must move along higher level sets of the function V, i.e, deviate from the equilibrium point and not be bounded within a ball of arbitrarily small radius ϵ. Second, the proof gives us a nice practice example on how to construct an argument based on the δ-ϵ formulation (similarly to the stability proof of Theorem 4.1).

Before we proceed let us define some notation.

Definition. Let $S \subset \mathbb{R}^n$. The closure of S, denoted $\bar{S}$, is the smallest closed set in $\mathbb{R}^n$ that contains S.

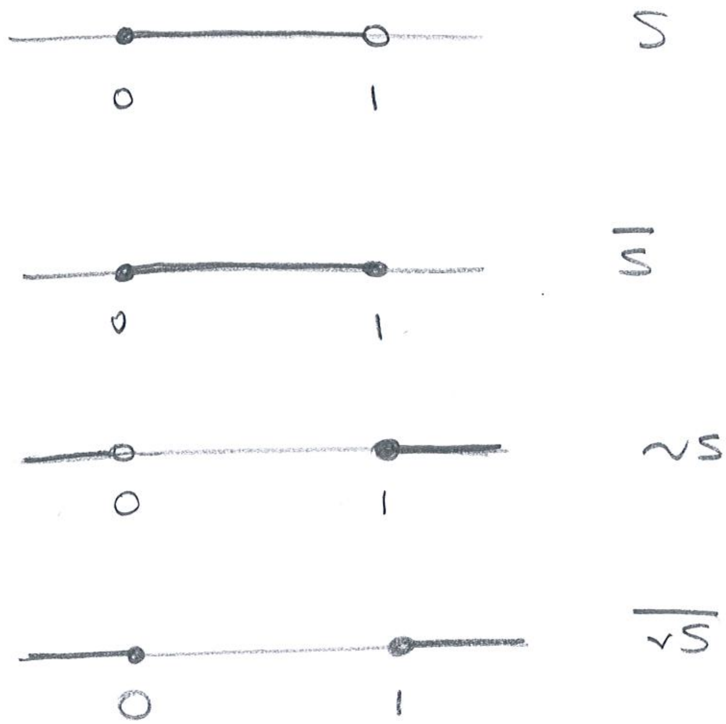
Examples (a) $S = [0,1)$. Then $\bar{S} = [0,1]$

(b) $S = B_r(0)$. Then $\bar{S} = \bar{B}_r(0) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$

Definition The boundary of S, denoted ∂S, is defined as $\partial S = \overline{S} \cap \overline{(\sim S)}$

Examples. (a) $S = [0,1)$. Then $\partial S = \{0\} \cup \{1\}$

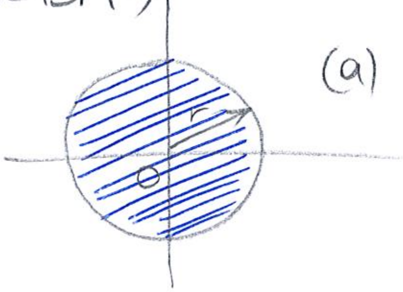
Check that graphically...

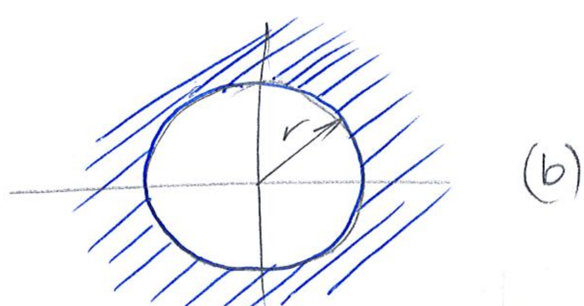


The intersection of these two sets is the set $\{0\} \cup \{1\}$

(b) $S = B_r(0)$, then $\partial S = \{x \in \mathbb{R}^n \mid \|x\| = r\}$

$S = B_r(0)$,

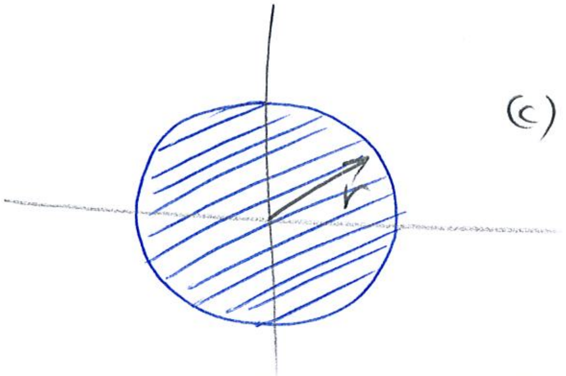


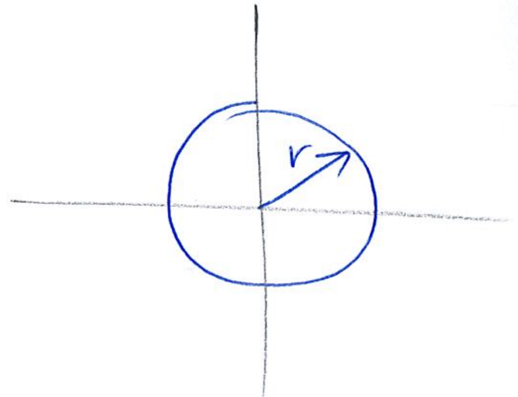


Open ball is the shaded region the circular disk of radius r without the circle of radius r

Complement of the open ball $B_r(0)$, $(\sim S) = (\sim B_r(0))$ is everything outside the circle of radius r. This set coincides with its closure, $\overline{(\sim S)}$

(c)





The closure of the open ball, $\bar{B}_r(0)$ is the closed ball of radius r (i.e, the circular disk including the circle of radius r)

Now, the intersection of sets in figures (b) and (c), i.e, the boundary of the open ball $B_r(0)$, denoted $\partial B_r(0) = \partial S$, is the Circle of radius r.

(c) $S = \bar{B}_r(0)$, then $\partial S = \{x \in \mathbb{R}^n \mid \|x\| = r\}$.

Definition Let $S \subset \mathbb{R}^n$. The interior of S, denoted $\mathring{S}$, is the largest open set contained in S.

Examples. (a) $S = [0,1)$, then $\mathring{S} = (0,1)$

(b) $S = \bar{B}_r(0)$, then $\mathring{S} = B_r(0)$.

Fact: $x_0 \in \mathring{S} \iff \exists\ \epsilon > 0$ such that $B_\epsilon(x_0) \subset S$.

We can now proceed with examples on instability and more instability results. $\longrightarrow$

Example. Consider the system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2$$

From linear systems tools, you can immediately verify that the origin is unstable.

Let us verify it with our (overkill) result.

Take the positive definite $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

Compute the time derivative

$$\dot{V}(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$$

which is positive definite, hence the origin is unstable.

Question. We may now wonder, if we are to obtain a less restrictive result, what should be weakened? Should we drop V being positive definite, or drop $\dot{V}$ being positive definite?

We investigate the question via some examples.

Less trivial example. (Trial / Take 1)

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 \end{cases}$$

from linear systems theory, you can easily verify that the origin is unstable.

Let us build a case where things "do not work".

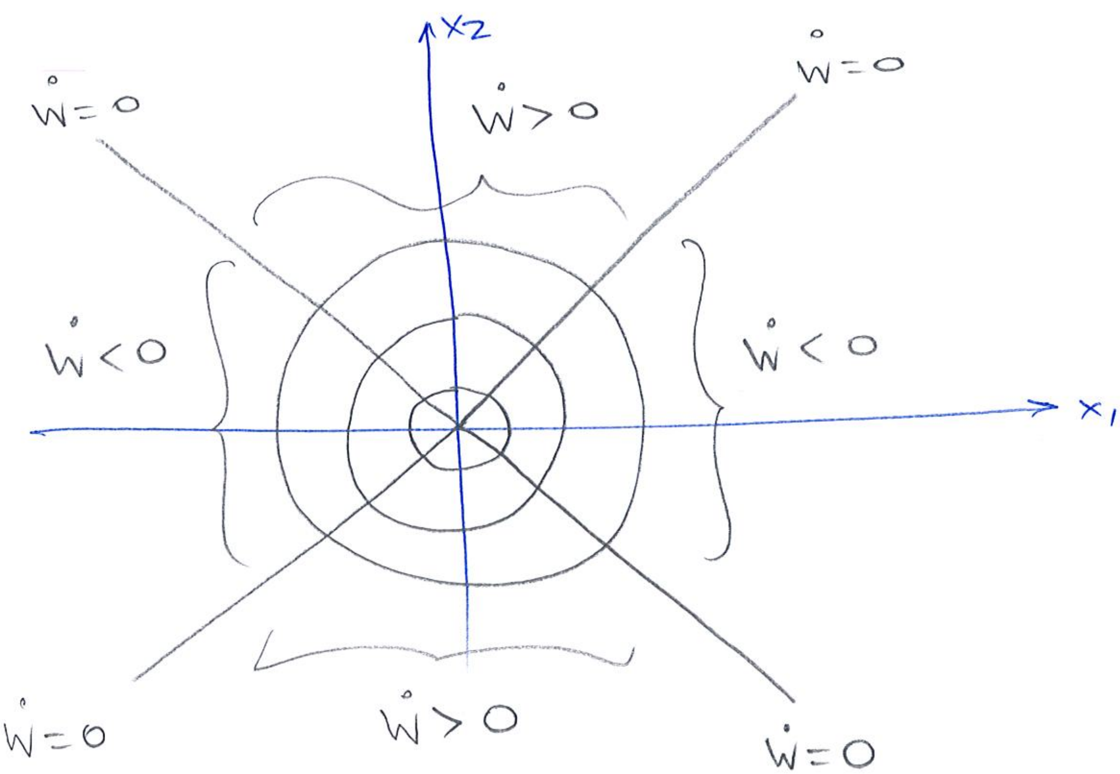
Consider $W(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$, positive definite.

Compute the time derivative as

$$\dot{W}(x_1, x_2) = [x_1 \quad x_2] \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = -x_1^2 + x_2^2$$

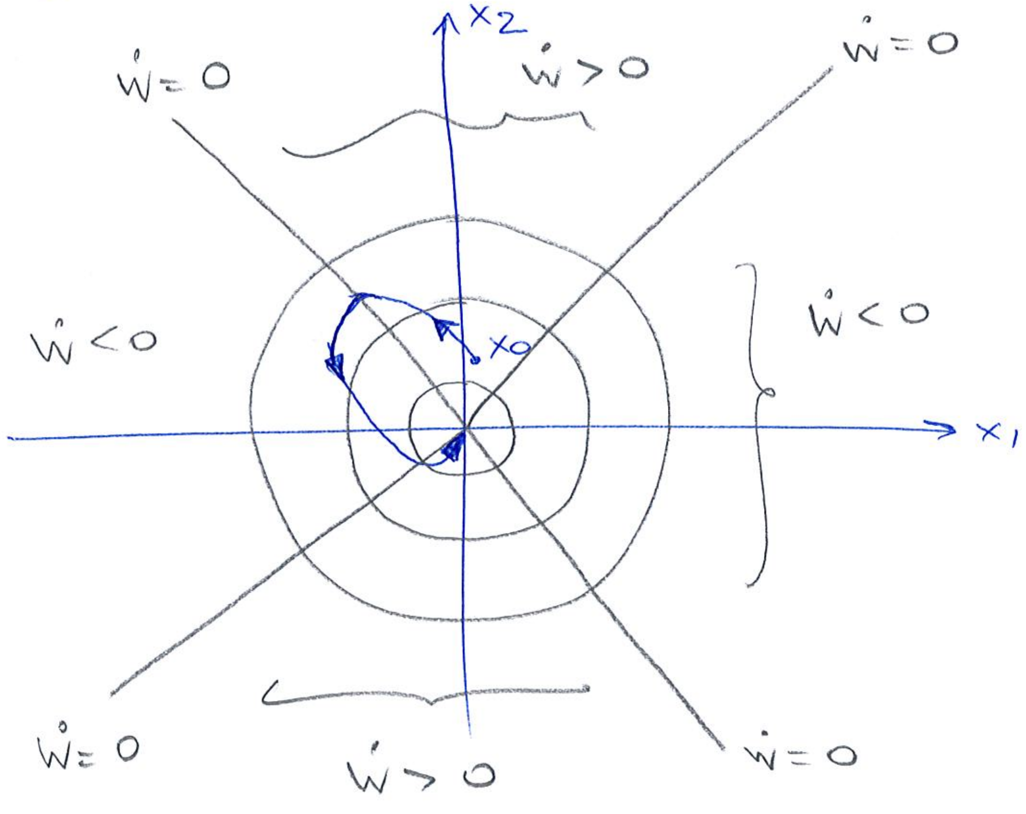
We note that $\dot{W}(x_1, x_2) = 0$ when $|x_1| = |x_2|$, and that $\dot{W}(x_1, x_2)$ can take both negative and positive values.

(in that case, we call the function <u>indefinite</u>)



The above plot shows the sign of $\dot{W}(x_1, x_2)$ for the various values of x_1, x_2.

Then, one could consider conceptually a trajectory that starts out at a region where $\dot{W} > 0$, moves through increasing level sets where $\dot{W} = 0$, and enters the region $\dot{W} < 0$, at which point, it could move through decreasing level sets, and approach the origin.



However! The origin is unstable! Where is the catch?

Given ANY $\epsilon > 0$, we can not be sure that we can find a $\delta(\epsilon) > 0$, such that $\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \epsilon$, $\forall\, t \geq 0$.

Fact: If $W(x)$ is positive definite and $\dot{W}(x)$ is indefinite, then NOTHING can be concluded on the basis of $W(x)$.

Less Trivial Example (Trial / Take 2).

Same system

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2. \end{cases}$$

Let us build a case where things "do work"

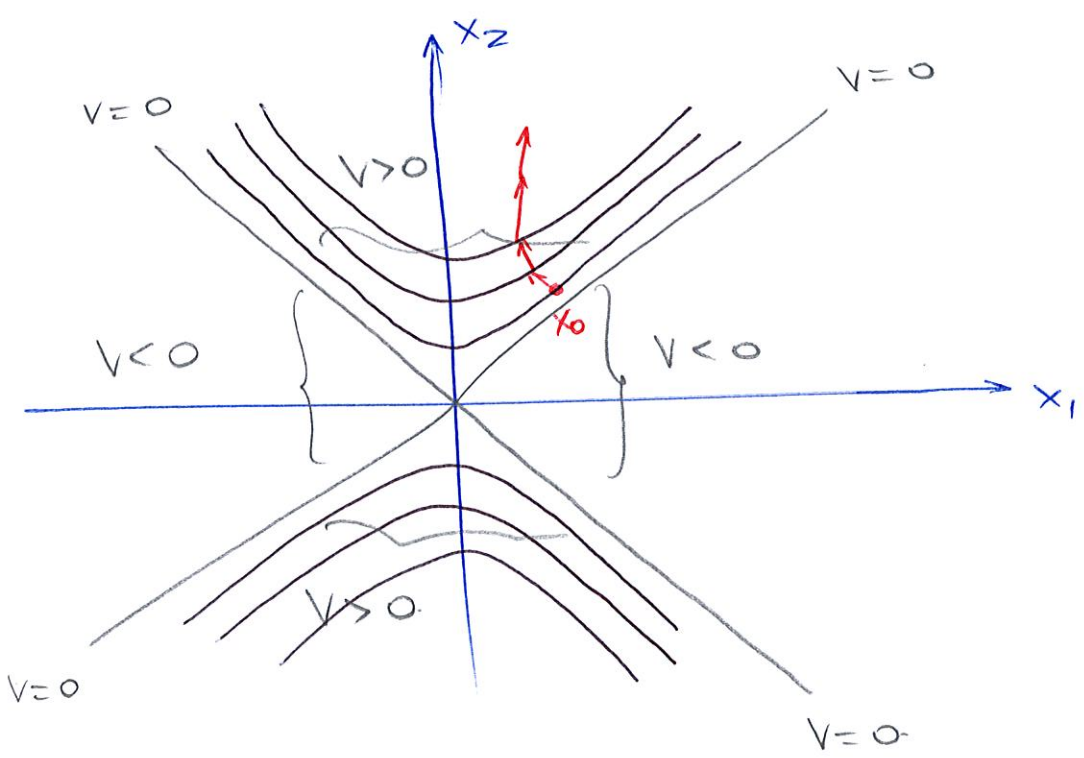
Consider $\boxed{V(x_1, x_2) = -\frac{1}{2} x_1^2 + \frac{1}{2} x_2^2},$ NOTE: Clearly INDEFINITE !!!

And let us take its time derivative

$$\dot{V}(x_1, x_2) = \begin{bmatrix} -x_1 & x_2 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2,$$

which is positive definite.

Noting that $V(x_1, x_2) = 0$ when $|x_1| = |x_2|$, we have



Now we can observe the following.

If a trajectory starts in a region where both $V > 0$ and $\dot{V} > 0$, then it must move along increasing level sets.

If the region is bounded by the sets where $V = 0$, then the trajectory must move outward away from the origin, showing instability!

Remark. These ideas underlie the primary instability theorem from our textbook (given next), and an improved version (given after the book's version).

Bottom line: If $V(x)$ is indefinite, and $\dot{V}(x)$ is positive definite, then it may be possible to conclude something on the basis of $V(x)$.

We will see later that a weaker condition is even possible.

Theorem 4.3 (from our textbook, page 125)

Let $x_e = 0$ be an equilibrium point of the ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let D be an open set containing $x_e = 0$, and suppose f is locally Lipschitz on D.

Suppose also there exists a continuously differentiable function $V: D \to \mathbb{R}^n$ such that

$\rightarrow$

(a) $V(0) = 0$ and

(b) For every $\delta > 0$, $\exists\, x_0 \in D_1$, $\|x_0\| < \delta$ such that $V(x_0) > 0$.

Choose $r > 0$ such that $B_r(0) \subset D_1$ and define

$$U = \{ x \in B_r(0) \mid V(x) > 0 \}.$$

Furthermore, suppose that

(c) $\dot{V}(x) > 0$ for all $x \in U$.

Then the equilibrium point x_e is unstable.

Remark. Condition (b) is assuring that the origin is on the boundary of the set U, that is, $0 \in \partial U$. That means that no matter how small $\delta > 0$ is selected, you can always find initial conditions x_0 which will be forced "far" (at least distance ϵ) away from the origin.

Theorem (Improvement over Theorem 4.3 in the book)

Let $x_e = 0$ be an equilibrium point of the ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let D be an open set containing $x_e = 0$ and suppose f is locally Lipschitz on D.

Suppose also there exists a continuously differentiable function $V: D \to \mathbb{R}$ such that

$\longrightarrow$

(a) $V(0) = 0$

and in addition suppose there exists an open set $U \subset D$ such that

(b) $0 \in \partial U$

(c) $\dot{V}(x) > 0$ for all $x \in U$

(d) $\forall \delta > 0$, $\exists x_0 \in U$ such that $V(x_0) > 0$ and $\|x_0\| < \delta$.

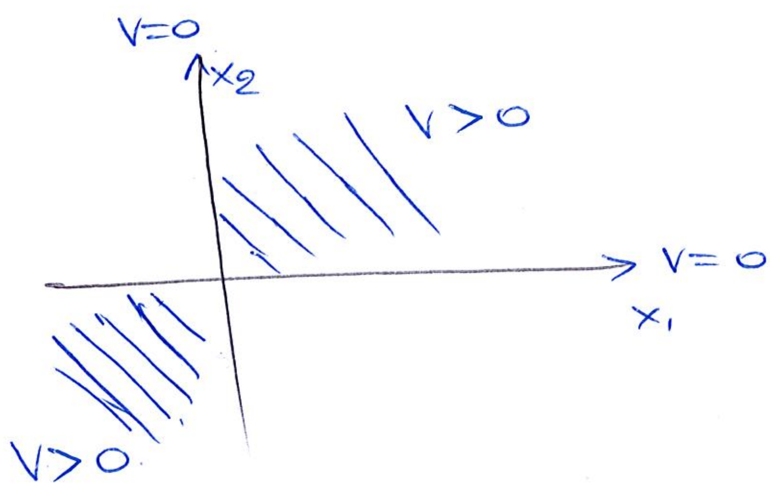
(e) $\exists r > 0$ such that, $x \in \partial U$ and $\|x\| < r \Rightarrow V(x) = 0$.

Then the equilibrium point is unstable.

Example.

$$\dot{x}_1 = 3x_1 + x_2 + x_1^2$$

$$\dot{x}_2 = x_1 - x_2 + x_1 x_2$$



Consider the function

$$\boxed{V(x_1, x_2) = x_1 x_2}$$

Take the time derivative.

$$\dot{V}(x) = x_2(3x_1 + x_2 + x_1^2) +$$

$$+ x_1(x_1 - x_2 + x_1 x_2) =$$

$$= 3x_1x_2 + x_2^2 + x_1^2 x_2 + x_1^2 - x_1x_2 + x_1^2 x_2 =$$

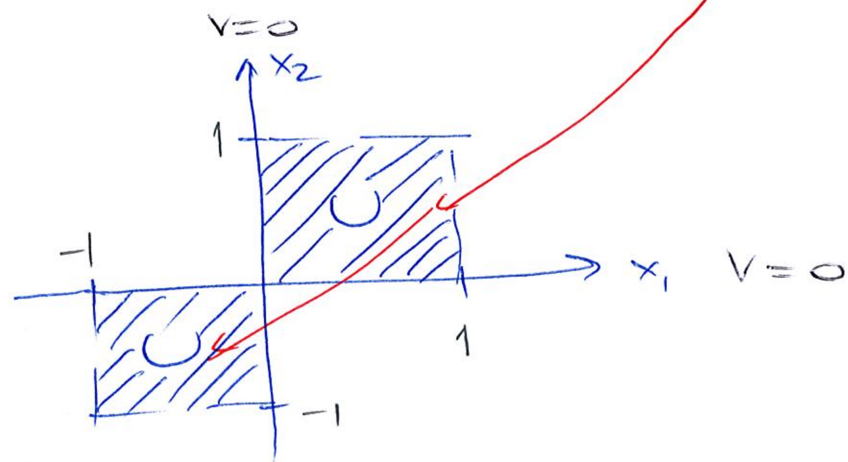
$$= x_1^2 + x_2^2 + 2x_1^2 x_2 + 2x_1 x_2 =$$

$$= x_1^2 + x_2^2 + 2x_1 x_2(x_1 + 1)$$

Solution 1

Choose $\|\cdot\| = \|\cdot\|_\infty$. Let $r=1$.

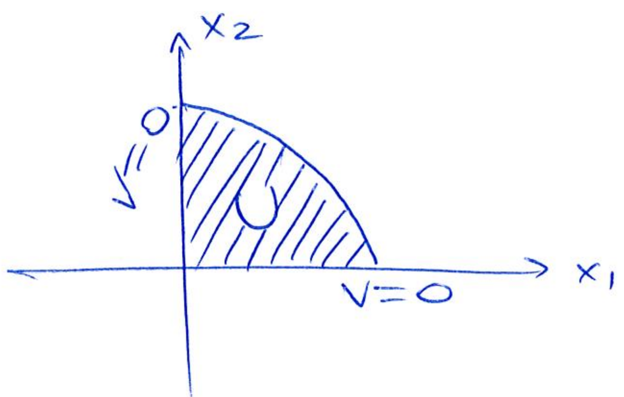
Define $U = \{x \in B_r(0) \mid V(x) > 0\}$



Then $\dot{V}(x) > 0$, $\forall\, x \in U$. From Theorem 4.3, the origin $x_e = 0$ is unstable.

Solution 2.

$\|\cdot\|$ is any norm. Pick $0 < r < \infty$ arbitrarily.



Define $U = \{x \in B_r(0) \mid x_1 > 0,\ x_2 > 0\}$

Then we have.

① $0 \in \partial U$

② $\dot{V}(x) > 0$, $\forall\, x \in U$

③ $\forall\, \delta > 0$, $\exists\, x_0 \in U$, $\|x_0\| < \delta$, such that $V(x_0) > 0$

④ $x \in \partial U$, $\|x\| < r \Rightarrow V(x) = 0$.

⑤ $V(0) = 0$.

Then ~~per~~ the "improved version" of Theorem 43, the origin is unstable.

Some more terminology and sketches regarding the "Improved version" of Theorem 4.3 available in our notes.

Q. What are the different, more relaxed hypotheses of the "improvement version" relative to Theorem 4.3?

A. The improvement version does not require $\dot{V}(x)$ to be positive definite on $U = \{ x \in \bar{B}_r(0) \mid V(x) > 0 \}$ for any $r > 0$. *

Notice the requirements of the "improvement theorem"

- Suppose there exists a continuously differentiable function $V: D \to \mathbb{R}$ such that

(a) $V(0) = 0$, and in addition suppose there exists an open set $U \subset D$ such that

(b) $0 \in \partial U$

(c) $\dot{V}(x) > 0$ for all $x \in U$

(d) $\forall \delta > 0, \exists x_0 \in U$ such that $V(x_0) > 0$ and $\|x_0\| < \delta$

(e) $\exists r > 0$ such that, $x \in \partial U$ and $\|x\| < r \Rightarrow V(x) = 0$.

- Conclusion: If all above assumptions are met, then the origin is unstable.

* Essentially, the "improvement theorem" constructs the set U in a more relaxed way that can still help us conclude instability. See the following example.

Some more terminology and sketches regarding Theorem 4.3 (page 125 in our textbook)

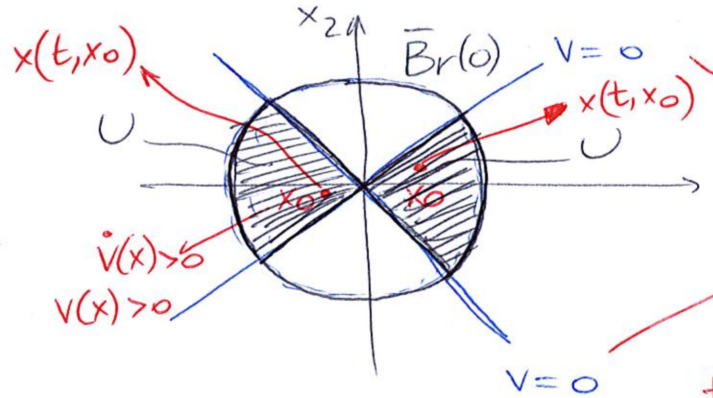
- Let $V: D \to \mathbb{R}$ continuously differentiable on a domain $D \subset \mathbb{R}^n$ that contains the origin

- Suppose $V(0) = 0$

- Suppose also there is a point x_0 arbitrarily close to the origin $x=0$, i.e, contained with a ball $B_\delta(0)$ for any $\delta > 0$, $\|x_0\| < \delta$, such that $V(x_0) > 0$.

- Choose $r > 0$ such that the ball $\bar{B}_r(0) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ is contained in D, and let $U = \{x \in \bar{B}_r(0) \mid V(x) > 0\}$.

Example: Take the indefinite function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$



Trajectories starting in U can not escape through $V=0$ because $\dot{V}(x) > 0$ in U. Hence trajectories must escape through $\{\|x\|=r\} \subset U$.

In the region U, we have that $V(x) > 0$. By construction, the boundary ∂U of the set U is the surface $V(x) = 0$ and the sphere $\|x\| = r$.

- Suppose $\dot{V}(x) > 0$ in U.

- Conclusion: Then the origin is unstable.

Remark: Since we can find an x_0 arbitrarily close to the origin, $\|x_0\| < \delta$, from which trajectories can not be confined within any $\epsilon > 0$, the origin is unstable ISL.

Example. Let $V(x) = x_1^4 - x_2^2$

$\dot{V}(x) = -4x_1^5$

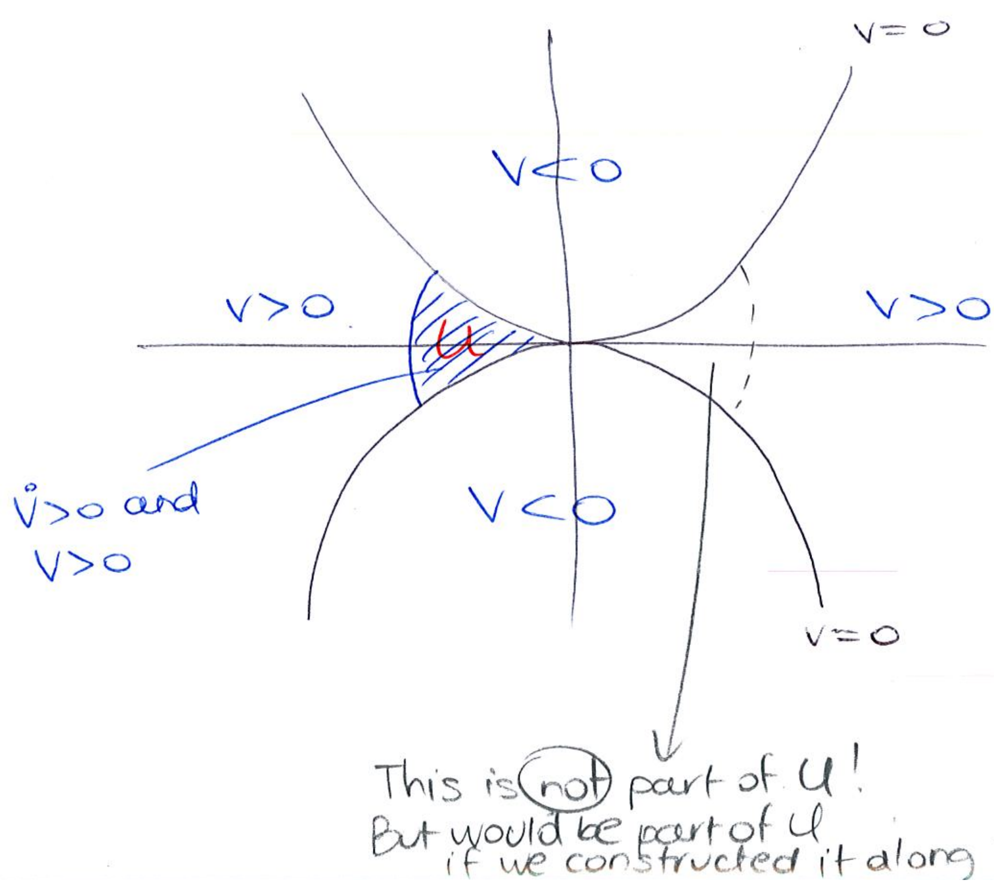
Remark 1: $V(x)$ is not positive definite, hence not be used to assess stability.

Remark 2: $V(x)$ takes on positive values near the origin, hence it might be able to be used in proving instability.

We have $\{x \in \mathbb{R}^n \mid \dot{V}(x) > 0\} = \{x \in \mathbb{R}^n \mid x_1 < 0\}$.

Next, we have to sketch where $V=0$, $V>0$, $V<0$.

$\{x \in \mathbb{R}^2 \mid V(x) = 0\} = \{x \in \mathbb{R}^2 \mid |x_2| = x_1^2\}$



Let $r=1$ and consider the ball $B_1(0)$.

Then we consider the set U as shown in the shaded region,

$U = \{x \in B_1(0) \mid V(x) > 0, x_1 < 0\}$

Now for this $\mathcal{U}$ it is easy to verify that

(a) $V(0) = 0$

(b) $0 \in \partial \mathcal{U}$

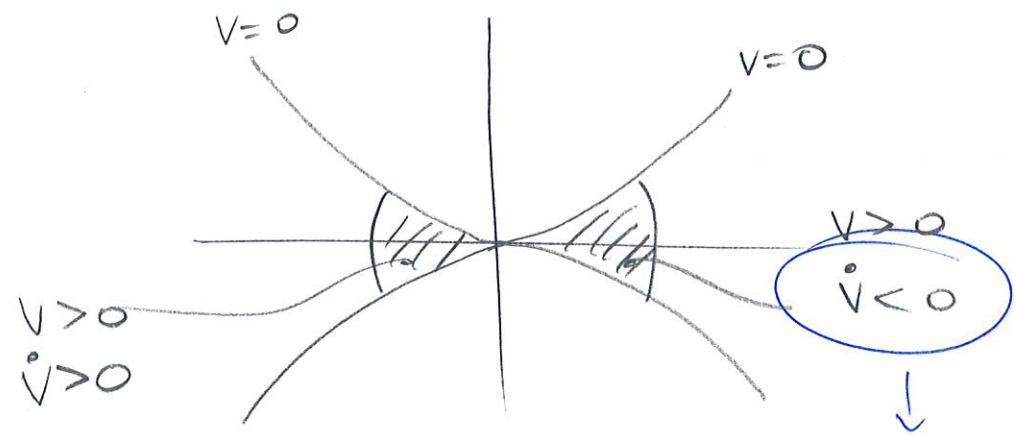
(c) $\dot{V}(x) > 0 \quad \forall\, x \in \mathcal{U}$

(d) $\forall\, \delta > 0 \quad \exists\, x_0 \in \mathcal{U}$ such that $\|x_0\| < \delta$ and $V(x_0) > 0$.

(e) $\forall\, x \in \partial\mathcal{U},\ \|x\| < 1, \Rightarrow V(x) \equiv 0$.

Hence, by the "improved theorem", the origin <u>is unstable</u>.

<u>Remark.</u> If we constructed the set $\mathcal{U}$ per the requirements of Theorem 4.3, we would have.



Hence $\dot{V}(x)$ is <u>not</u> positive definite in U, i.e, $\dot{V}(x)$ is not > 0 $\forall\, x \in U$!

Hence the hypotheses of Theorem 4.3 are not met in this case, and as thus Theorem 4.3 can not be applied in this case !!!

Remark. Let M be an arbitrary, real, $n \times n$ matrix.

Write M as $M = \dfrac{M+M^T}{2} + \dfrac{M-M^T}{2}$.

Claim: (i) $x^T (M - M^T) x = 0$, for all $x \in \mathbb{R}^n$

(ii) $\dfrac{M+M^T}{2}$ is symmetric and is called the symmetric part of M.

(iii) $\dfrac{M-M^T}{2}$ is skew-symmetric and is called the skew-symmetric part of M.

Proof. : We prove only the first claim; the rest are left as an exercise for the reader.

We note that $x^T(M - M^T)x$ is a scalar, hence equal to its transpose. Therefore, for any $x \in \mathbb{R}^n$ we have

$$x^T(M-M^T)x = x^T(M^T - M)x = -x^T(M-M^T)x$$

which proves the result, since the only real number equal to its negative is zero.

Remark 2: From the above we conclude that

$$x^T M x = x^T \left(\frac{M+M^T}{2}\right) x.$$

As thus, symmetry is always assumed as part of the definition of positive and negative (semi-) definite matrices.

Positive Definite Matrices and the Lyapunov Equation.

We have already seen examples from our textbook where the use of positive definite matrices in the definition of our Lyapunov function candidates was useful in establishing the time derivative of the function negative definite (and hence in being able to conclude asymptotic stability)

Let us summarize/review the properties of positive definite matrices and see their usefulness in finding Lyapunov functions for linear systems.

<u>Definition</u> Let P be a real, $n \times n$, symmetric ($P^T = P$) matrix.

(i) P is <u>positive semi-definite</u> if $x^T P x \geq 0$, for all $x \in \mathbb{R}^n$

Notation: $P \geq 0$.

(ii) P is <u>positive definite</u> if $x^T P x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.

Notation: $P > 0$.

(iii) P is <u>negative definite</u> if $x^T P x < 0$, for all $x \in \mathbb{R}^n$, $x \neq 0$.

Notation: $P < 0$

(iv) P is <u>negative semi-definite</u> if $x^T P x \leq 0$, for all $x \in \mathbb{R}^n$

Notation: $P \leq 0$.

Remark !!! $P > 0$ does not mean that each of the entries of P are positive !!!