Sliding Mode Control (Chapter 14.1)

So with Lyapunov Redesign, and its special case. Nonlinear Damping Control, we can address the control design of systems of the form $\dot{x} = f(x) + G(x)\big(u + \delta(t,x,u)\big)$

i.e. where the uncertain term is <u>matched</u> with the control input, i.e, it appears at the exact same place as the control input. A more general case (of matched uncertainty) is:

$$\dot{x} = f(x) + B(x)\Big(G(x)E(x)u + \delta(t,x,u)\Big)$$

where f, B, E are known, $E(x)$ nonsingular, $f(0)=0$. $G(x), \delta(t,x,u)$ are unknown, $G(x)$ diagonal matrix such that $g_i(x) \geq g_0 > 0$. $\forall x \in D$

One can show that there exists a $T: D \to \mathbb{R}^n$ (diffeomorphism) such that $\frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$ so that the system can be rewritten in coordinates $\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x)$, $\eta \in \mathbb{R}^{n-p}$, $\xi \in \mathbb{R}^p$ as

$$\left\{ \begin{aligned} \dot{\eta} &= f_a(\eta, \xi) \\ \dot{\xi} &= f_b(\eta, \xi) + G(x)E(x)u + \delta(t,x,u) \end{aligned} \right\} \text{ often called "regular form"}$$

The Idea is to render the origin $\eta = 0$ G.A.S upon a proper choice of $\xi = \phi(\eta)$, and then make $\xi - \phi(\eta)$ go to zero. We saw how backstepping renders $z = \xi - \phi(\eta)$ G.A.S via proper CLFs.

To solve the same problem, Sliding Mode Control takes a different approach. With SMC is objective is to drive the error $s = \xi - \phi(\eta)$ to zero in finite time (opposed to asymptotically with time that we have done so far with the other techniques) and make it stay zero for all future times.

In other words $s=0$ can be viewed as a sliding surface: SMC forces system trajectories to reach this surface in finite time (reaching phase) and once on the surface, slide along it (without ever after leaving it) towards the origin (asymptotically with time) (sliding phase).

As we will see, sliding mode control is inherently a discontinuous control law, and was introduced in 1977 by V. Utkin as a variable structure system with sliding modes. The paper has been uploaded on Canvas, in case you are interested. From that perspective, SMC can be viewed as an early version of switching control and hybrid control.

SMC was designed to have good robustness properties against matched (and unmatched, yet vanishing) uncertainties. For a more detailed introduction to SMC, interested readers can refer to Slotine and Li "Applied Nonlinear Control." We will study SMC based on a simple example shortly. Yet, we can see the main steps of sliding mode control design based on the general case we have introduced above, i.e. for the system in regular form.

$$\begin{cases} \dot{\eta} = f_a(\eta, \xi) \\ \dot{\xi} = f_b(\eta, \xi) + G(x)E(x)\,u + \delta(t, x, u) \end{cases}$$

The sliding manifold is set as $s = \xi - \phi(n) = 0$. such that when the motion of the system is restricted to $s=0$, then the "reduced order model." $\dot{n} = f_a(n, \phi(n))$ has an asymptotically stable equilibrium at the origin.

Hence the design of $\phi(n)$ is a stabilization problem for $\dot{n} = f_a(n, \xi)$ where ξ is the (virtual) control input. We have seen the same idea before, and we can solve this problem using linearization, feedback linearization, CLF / Sontag and so on. Whatever method we choose, the assumption / requirement to move forward is that we can find a stabilizing, continuously differentiable function $\phi(n)$ such that $\phi(0) = 0$.

Then what sliding mode control does differently compared to other methods we have seen so far is that it forces $s = \xi - \phi(n)$ to be driven to zero in finite time, and remain zero for all future times. The dynamics of s are

$$\dot{s} = f_b(n, \xi) - \frac{\partial \phi}{\partial n} f_a(n, \xi) + G(x) E(x) u + \delta(t, x, u)$$

From here, there are various ways / variations of SMC that we can follow. The more general form for u would be to try to compensate for the uncertain terms $G(x)$ and concurrently make $s=0$ reachable in finite time, and forward invariant thereafter. In fact, note that in the absence of uncertainty, i.e. if δ = 0, G(x) is known, then the control law

$$u = -E^{-1}(x)\, G^{-1}(x) \left[f_b(n, \xi) - \frac{\partial \phi}{\partial n} f_a(n, \xi) \right]$$

results in $\dot{s} = 0$, which means that $s=0$ can be maintained for all future time

On the other hand, if some nominal model / estimate $\hat{G}(x)$ of the uncertain $G(x)$ is available, then the form of the control law can be. $u = -E^{-1}(x)\,\hat{G}^{-1}(x)\left[f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi)\right]$

This is called the "equivalent control" (from the certainty equivalence principle.) Note that this controller maintains (in the absence of uncertainty) the trajectories on the sliding surface $s=0$. In order to make the trajectories reach the sliding surface, another (additional) controller is needed. The overall form of the controller is.

$$u = E^{-1}(x)\left\{-L(x)\left[f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi)\right] + v\right\}$$

where one can set $\boxed{L(x) = \hat{G}^{-1}(x)}$ if it is desired to cancel the estimates of the known terms.

or $\boxed{L(x) = 0}$ otherwise (e.g. when no estimates are available)

Then the question is how to design the control law v. In fact, this is very similar to Lyapunov redesign. Substituting the control law above to the dynamics of s yields for each s_i variable; where $i \in \{1, \cdots, p\}$:

$$\dot{s}_i = g_i(x)\,v_i + \Delta_i(t, x, v),$$

where $\underline{\Delta_i}$ is the i-th component of

$$\Delta(t, x, u) = \delta(t, x, \tilde{u}) + \big(I - G(x)L(x)\big)\left(f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi)\right)$$

where u is the control law above.

and $\underline{g_i}$ is the i-th diagonal element of $G(x)$

Now, the key assumption is that the ratio $\frac{\Delta_i}{g_i}$ satisfies

$$\left| \frac{\Delta_i(t,x,u)}{g_i(x)} \right| \leq \rho(x) + k_0 \|u\|_\infty \qquad \forall (t,x,u) \in [0,\infty) \times D \times \mathbb{R}^p$$

where $\rho(x) \geq 0$ a known continuous function

$k_0 \in [0,1)$ known constant.

Then with the Lyapunov function candidate $V_i = \frac{1}{2} s_i^2$, one can show that the control law

$$\underline{v_i = -\beta(x)\,\mathrm{sgn}(s_i)}, \quad \text{where } \beta(x) \geq \frac{\rho(x)}{1-k_0} + \beta_0,$$

$$\beta_0 > 0$$

ensures that all trajectories starting of the manifold $s=0$ reach it in finite time, and those on the manifold can not leave it.

overall the controller has great robustness to matched uncertainties, since during the sliding phase (reduced-order subsystem $\dot{n} = f_a(n, \phi(n))$) the motion of the system is independent of G, δ, i.e of the uncertain terms.

However: It is notable that the term $\mathrm{sgn}(\cdot)$ is discontinuous.

This can create both theoretical and practical challenges.

Theoretical → existence and uniqueness of solutions has to be established in a framework that does not require locally Lipschitz dynamics.

Practical → the solutions can suffer from chattering due to measurement noise, imperfections in switching devices and delays.

One way to circumvent chattering is to use the continuous approximation

$$\nu_i = -\beta(x)\ \mathrm{sat}\left(\frac{s_i}{\varepsilon}\right),$$

where $\varepsilon > 0$. and sat( ) stands for the saturation function (see later)

Finally, one comment about the case of unmatched uncertainty i.e of systems in the form

$$\boxed{\dot{x} = f(x) + B(x)\Big(G(x)E(x)u + \underbrace{\delta(t,x,u)}_{\text{matched}}\Big) + \underbrace{\delta_1(x)}_{\text{unmatched}}}$$

Then under $\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x)$, $\eta \in \mathbb{R}^{n-p}$, $\xi \in \mathbb{R}^p$ the system is transformed into

$$\begin{aligned} \dot{\eta} &= f_a(\eta,\xi) + \underbrace{\delta_a(\eta,\xi)}_{\text{unmatched}} \\ \dot{\xi} &= f_b(\eta,\xi) + G(x)E(x)u + \underbrace{\delta(t,x,u) + \delta_b(\eta,\xi)}_{\text{matched}} \end{aligned}$$

where $\begin{bmatrix} \delta_a \\ \delta_b \end{bmatrix} = \frac{\partial T}{\partial x} \cdot \delta_1$

In other words, the unmatched disturbance δ_1 contributes a matched component δ_b, and an unmatched component δ_a.

SMC guarantees robustness against matched disturbances, provided that an upper bound is known and the needed control effort can be provided. There is no such guarantee against unmatched disturbances. In some cases, we might be able to robustly stabilize the unmatched uncertainties, but this is not always guaranteed. Much depends on the structure of the reduced-order system. See Examples 14.1 and 14.2 [optional reading]

Sliding Mode Control (Example.)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = h(x) + g(x)u \end{cases}$$

, where $h(x), g(x)$ unknown functions, and $g(x) \geq g_0 > 0$.

Problem: Design a state feedback law to stabilize the origin of the closed-loop system.

Approach: We want to steer the trajectories of the system to a surface $s(x_1, x_2)$ that leads to the origin (passes through) and keep the trajectories /constraint them on this surface until they approach the origin

For example, such a surface could be constructed as

$$s = a_1 x_1 + x_2 = 0 \quad \Rightarrow \quad x_2 = -a_1 x_1.$$

Then on this manifold, we have $\dot{x}_1 = x_2 = -a_1 x_1$

The choice $a_1 > 0$ guarantees that $x_1(t) \to 0$ as $t \to \infty$, and therefore that $x_2(t) \to 0$ as $t \to \infty$ (since $s(t) = 0$ for all $t \geq 0$, remember we assumed we are on this manifold.) So the question becomes: How can we steer the trajectory $x(t)$ to $s = 0$ and maintain it there? This leads to the sliding mode control design. On the manifold the dynamics satisfy

$$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x) u.$$

Take the Lyapunov function candidate

$V(s) = \frac{1}{2}s^2$ and its time derivative

$$\dot{V}(s) = s\left(a_1 x_2 + h(x) + g(x)u\right) = s a_1 x_2 + s h(x) + s g(x) u.$$

$$= s\left(\frac{a_1 x_2 + h(x)}{g(x)}\right) g(x) + s g(x) u.$$

If we can assume a bound $\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq \rho(x), \ \forall x \in \mathbb{R}^2$

then $\dot{V}(s) \leq g(x) s \rho(x) + s g(x) u$

$$\leq g(x)|s|\rho(x) + g(x) s u.$$

Then taking the control law $\boxed{u = -\beta(x)\operatorname{sgn}(s)}$

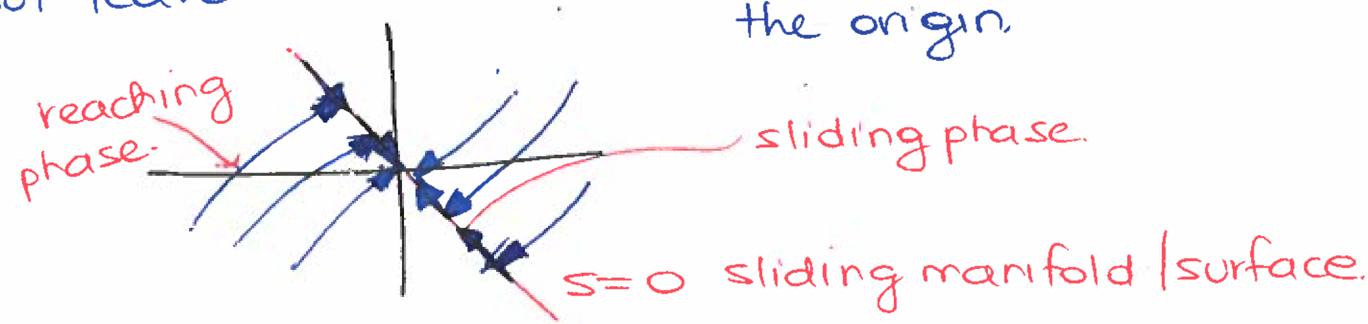
where $\beta(x) \triangleq \rho(x) + \beta_0, \ \beta_0 > 0$, and

$$\operatorname{sgn}(s) = \begin{Bmatrix} 1, & s>0 \\ 0, & s=0 \\ -1, & s<0 \end{Bmatrix}$$

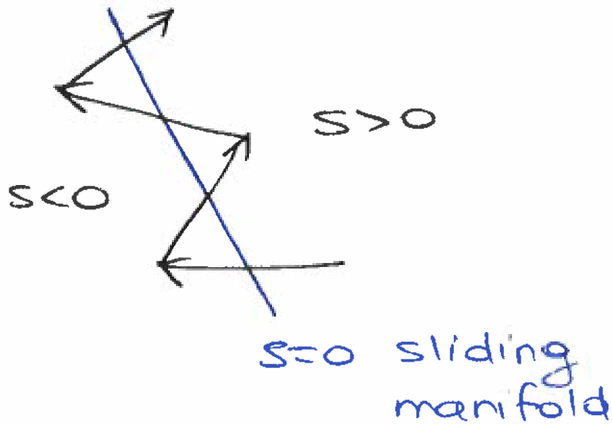
yields $\dot{V}(s) \leq g(x)|s|\rho(x) - g(x)(\rho(x)+\beta_0)|s| =$

$$= -g(x)\beta_0|s| \leq -g_0\beta_0|s|,$$

i.e. the manifold $s=0$ is reached, and in fact it is reached in finite time. (can be seen with the comparison principle) Once $s=0$ is reached, the trajectory can not leave the manifold, but slides on it as it approaches the origin.



SMC suffers from chattering



Due to delays between the time the sign s changes and the control u switches, or due to measurement errors etc., in practice the trajectory does not slide along the $s=0$ manifold, but chatters (zigs-zags) around it.

There are ways to suppress chattering in practice.

One technique is to try to have a continuous control law and the switching control law acting together.

e.g. $$u = -\frac{a_1 x_2 + \hat{h}(x)}{\hat{g}(x)} - \beta(x)\,\mathrm{sgn}(s)$$

where $\hat{h}(x)$, $\hat{g}(x)$ estimates/nominal models of $h(x)$, $g(x)$

Another technique is to substitute the signum function with a high-slope saturation function

$$u = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\varepsilon}\right), \quad \text{where: } \mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y) & \text{if } |y| > 1 \end{cases}$$

$\varepsilon > 0$