EECS 562 - Nonlinear Systems and Control
HW #10

Due on Thursday, April 6th, 2023
By 11:59pm, on Canvas

1. (20 points) Consider the nonlinear system and the candidate (local) clf given below:

$$\begin{cases} \dot{x}_1 &= -x_1 - x_2 + \frac{x_1 x_2}{12} \\ \dot{x}_2 &= u \end{cases} = f(x) + g(x)\,u \qquad \begin{aligned} V(x) &= x^T P x \\ P &= \begin{bmatrix} 2 & 1.5 \\ 1.5 & 3 \end{bmatrix} \end{aligned}$$

`>>[M,D]=eig(P)` yields $M = \begin{bmatrix} -0.8112 & -0.5847 \\ 0.5847 & -0.8112 \end{bmatrix}$ and $D = \begin{bmatrix} 0.9189 & 0 \\ 0 & 4.0811 \end{bmatrix}$

   (a) Find an open set about the origin in $\mathbb{R}^2$ for which $\inf_{u \in \mathbb{R}} \dot{V}(x, u) < 0$, $x \neq 0$.

   (b) For this set, find a control law $u = \alpha(x)$ that renders $\dot{V} < 0$ for the closed-loop system.

2. (30 points) Use backstepping to derive a clf and two (2) associated globally asymptotically stabilizing feedbacks for

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned}$$

Make one of your asymptotically stabilizing feedbacks linear and the other nonlinear.

3. (10 points) Simulate any one of the two controllers you computed in Problem 2. Turn in a plot that shows $x_1(t)$, $x_2(t)$, and $x_3(t)$. You do NOT need to provide a copy of your MATLAB code.

4. (20 points) We return now to the infamous inverted pendulum on a cart problem.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-0.25(x_2)^2 \sin(2x_1) + \sin(x_1)}{1 + 0.5\sin^2(x_1)} - \frac{\cos(x_1)}{1 + 0.5\sin^2(x_1)} u \\ x_4 \\ \frac{1.5(x_2)^2 \sin(x_1) - 0.5\sin(2x_1)}{1 + 0.5\sin^2(x_1)} + \frac{3}{1 + 0.5\sin^2(x_1)} u \end{bmatrix} = f(x) + g(x)\,u$$

Recall that from class, we know how to asymptotically stabilize the pendulum alone using a nonlinear feedback, and that this forces the cart to converge to a trajectory of the form $\bar{x}(t) = \begin{bmatrix} 0 \\ 0 \\ p + vt \\ v \end{bmatrix}$. We also have seen how to start near a trajectory of this form, render it asymptotically attractive, and then perturb it slowly in order to "plan" a path back to the origin. We now formally put these two ideas together to get our first "large region of attraction" controller for the inverted pendulum. Let $u = kx$ be as in the project. Define a switched feedback controller by:

$$u = \begin{cases} \frac{k_1 x_1 + k_2 x_2}{\cos(x_1)} & \text{until } \sqrt{(x_1)^2 + (x_2)^2} < \text{threshold} \\ k_1 x_1 + k_2 x_2 + k_3 (x_3 - \xi_1) + k_4 (x_4 - \xi_2) & \text{from then on} \end{cases}$$

where

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ -\frac{\alpha_1 \xi_1}{1+K_1|\xi_1|} - \frac{\alpha_2 \xi_2}{1+K_2|\xi_2|} \end{bmatrix},$$

and at the switching time, T_s, y ou need to initialize properly the $\zeta-$subsystem $\xi_1(T_s) \approx x_3(T_s)$, $\xi_2(T_s) \approx x_4(T_s)$. I will leave it to you to choose a proper threshold (you may wish to choose it on the basis of the region of attraction of the second controller) and values for $K_1 \geq 0$, $K_2 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$.

As in the project, determine the largest initial angle of the pendulum for which you can achieve attractivity to the origin, assuming all other initial conditions of the cart and pendulum are zero. Provide a few plots to support your conclusions. No MATLAB code is required.

Note: This will take some innovation on your part to implement in MATLAB. It can all be done. I've done it, and it works!

Second Note: A better switching strategy may be

$$u = \begin{cases} \frac{k_1 x_1 + k_2 x_2}{\cos(x_1)} & \text{until } V(x_1, x_2, 0, 0) < \text{threshold} \\ k_1 x_1 + k_2 x_2 + k_3(x_3 - \xi_1) + k_4(x_4 - \xi_2) & \text{from then on} \end{cases}$$

where V is a Lyapunov function associated with applying the linear control law to the full nonlinear model of the cart and pendulum. You are not obliged to try both.

5. (20 points) For another controller of the inverted pendulum, try out the following

$$u = \frac{k_1 x_1 + k_2 x_2}{\cos(x_1)} + \frac{k_3 x_3}{1 + K_1|x_3|} + \frac{k_4 x_4}{1 + K_2|x_4|}$$

with

$$k = [19.3000 \quad 22.9750 \quad 1.5900 \quad 5.5250]$$

as in the project, and $K_1 > 0$, $K_2 > 0$ to be chosen by you. Determine the largest initial angle of the pendulum for which you can achieve asymptotic convergence to the origin, assuming all other initial conditions of the cart and pendulum are zero. Turn in a plot that illustrates your result. No MATLAB code is required.

Remark: Since the linearization of the controller about the origin is $u = kx$, the origin of the closed-loop system is (locally) exponentially stable. Simulation will show you that you get a very large region of attraction. Proving that analytically takes some work! If we have time, we will discuss how to go about this in class.

Hints for Problem 1: Find a set about the origin in which $(x \neq 0, L_gV(x) = 0) \Rightarrow L_fV(x) < 0$. Then recall Sontag's formula. While it would be great to find a "large" set about the origin for which the CLF conditions hold, do not spend too much time on that part.

Hints for Problem 2: You can do the backstepping using "first principles", or you can apply the Backstepping Lemma. You do not need to put general coefficients in your feedback, meaning that you can set $c_k = 5$, or whatever when doing your design. You can wait until the last step to introduce a nonlinear term in your feedback solution.

Hints for Problem 4: If you put in a simple `IF-THEN-ELSE STATEMENT`, your controller is likely to "chatter", meaning that it will oscillate about your switching condition. You want to run the the nonlinear controller

$$u = \frac{k_1x_1 + k_2x_2}{\cos(x_1)}$$

`until` the pendulum is nearly upright and has a small angular velocity, and then `switch permanently` to the second controller that steers the cart gently to the origin. I repeat, you want to have a maximum of one switch of controllers during your convergence to the origin. I say maximum of one, because if you start near the origin, you will run immediately the second controller. It will take some time to figure out how to get this to work in MATLAB. It is fine to get hints from the class on PIAZZA, or to seek hints on the WEB for doing this. MATLAB hints will not be considered cheating. If you tune your controller correctly, you will be able to start the pendulum-on-a-cart system with the pendulum upright and at rest, and the cart 100 units away from the origin, and still steer it back to the origin, without knocking over the pendulum.

Hints for Problem 5: You will get this controller to work very quickly. Start out with K_1 and K_2 small and then slowly increase them until you find values you like. The controller is quite nonlinear and illustrates some of the power of our methods in the course. You will enjoy playing with it and seeing just how wonderfully it works. In HW 10, we will design a controller that will swing up the pendulum from a downward initial condition. More fun things to come!