Chapter 3. Fundamental Properties

We highlighted last time via several examples why the fundamental properties of the solutions of ODEs are important to be studied. By "fundamental properties", we refer to existence, uniqueness, continuous dependence on initial conditions, and continuous dependence on parameters. These properties are essential for the state equation $\dot{x} = f(t,x)$ to be a useful model for a physical system.

Example. Consider for instance the pendulum problem. In principle we expect that starting the pendulum from a given initial state at time t_0, will imply that the system will more and its state will be defined in the (at least immediate future) $t > t_0$. Moreover, with a deterministic system, we expect that if we could repeat the experiment exactly, we would get exactly the same motion and the same state at $t > t_0$. For the mathematical model to predict the future state of the system from its current state at t_0, the initial-value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, must have a unique solution. This is the question of existence and uniqueness that is addressed in Chapter 3.1.

We will see that the question of existence and uniqueness can be treated by imposing some constraints on the right-hand side function $f(t,x)$. The key constraint is the so-called Lipschitz condition, whereby $f(t,x)$ satisfies the inequality

$$\boxed{\|f(t,x) - f(t,y)\| \leq L\,\|x-y\|,}$$

for all (t,x) and (t,y) in a neighborhood of (t_0,x_0) for $L<\infty$.

Note that $\|\cdot\|$ denotes any p-norm.

Before proceeding with the mathematical background, let us clarify a bit what we mean by "existence" and "uniqueness" of the solution of a system of ordinary differential equations

Let
$$\boxed{\begin{aligned} &\dot{x}(t) = f(t, x(t)), \quad \forall\, t \geq t_0 \\ &x(t_0) = x_0 \in \mathbb{R}^n \end{aligned}}$$

For this system of ODEs to be a useful representation of a physical system, we want:

⊛ The system to have at least one solution
(Existence of solution)

⊛ Even better, the system to have exactly one solution for sufficiently small values of t
(Local (in terms of time) existence and uniqueness of the solution)

⊛ Even better: The system to have exactly one solution for all $t \in [0, \infty)$!
(Global (in terms of time) existence and uniqueness of the solution!)

Local and Global E&U are the topics of the fundamental Theorems 3.1 and 3.2 in our textbook!

Before proceeding with the mathematical definition of Lipschitz continuity,* let us recall that by the term solution of the

* that plays a key role in E&U of the solution,

initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0 \in \mathbb{R}^n$, over an interval $[t_0, t_1]$, we mean a [continuous function $x: [t_0,t_1] \to \mathbb{R}^n$ such that $\dot{x}(t)$ is defined,]** and $\dot{x}(t) = f(t, x(t))$ for all $t \in [t_0, t_1]$. We will assume that $f(t,x)$ is continuous in x, but only piecewise continuous in t. To make the context concrete, formal definitions will follow.

** essentially, a differentiable function $x: [t_0,t_1] \to \mathbb{R}^n$

Example

Last lecture we studied: $\boxed{\dot{x} = x^{1/3}, \text{ with } x(0) = 0.}$

We saw that $x(t) = \left(\frac{2}{3}t\right)^{3/2}$ and $x(t) = 0$ are both solutions! In noting that the right-hand side of the equation is continuous in x, our first conclusion is that continuity of $f(t,x)$ in x is not sufficient to ensure uniqueness of the solution!!! Extra conditions must be imposed on $f(\cdot,\cdot)$. ■

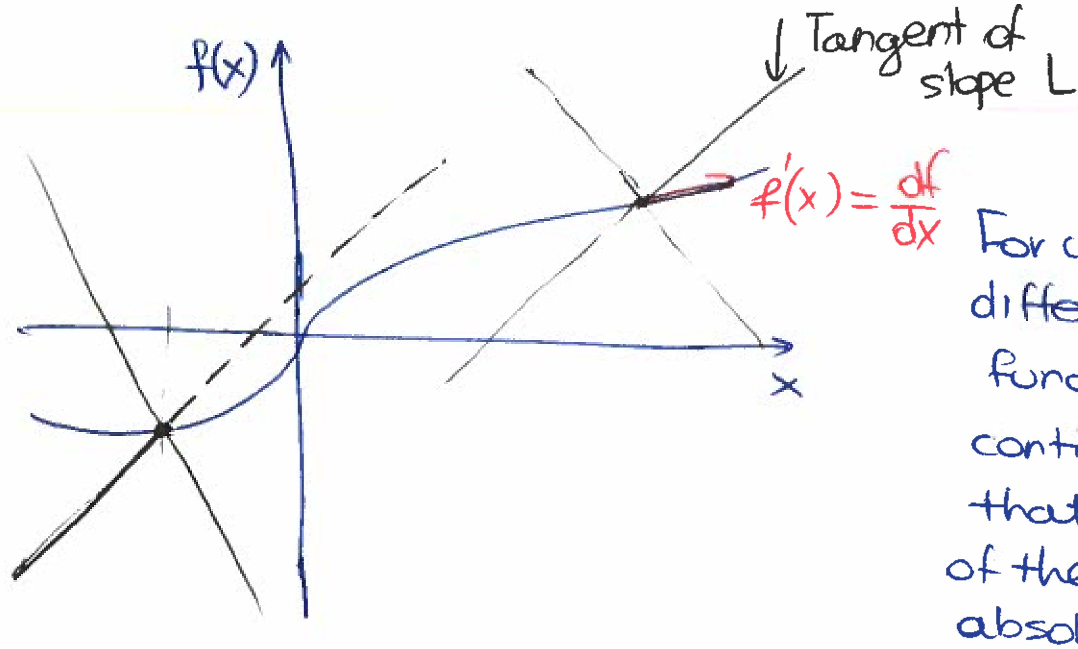
Remark: We noted that this extra condition is the Lipschitz condition $\|f(t,x) - f(t,y)\| \le L\|x-y\|$, $L < \infty$, for all (t,x) and (t,y)

Let us obtain some first insight by considering that $f: \mathbb{R} \to \mathbb{R}$, so that in this case the condition reads:

$$\frac{|f(y)-f(x)|}{|y-x|} \le L.$$

This means, on a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ can not have a slope whose absolute value is greater than L.

Example



For continuously-differentiable functions, Lipschitz continuity means that the derivative of the function in absolute value is bounded by the Lipschitz constant L.

We will come back to the definition of Lipschitz continuity shortly.

Preview of the Main Theorems on Existence and Uniqueness (E&U) of nonlinear ODEs. (Theorems 3.1 and 3.2)

Let us try to summarize in intuitive, but not rigorous (yet) technical terms, the main concepts of the E&U of solutions to the initial-value problem

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0.$$

The first theorem (Theorem 3.1) addresses what we call local existence and uniqueness of the solution $x(t)$ over a time interval $[t_0, t_0+\delta]$, where $\delta > 0$. The Theorem states that if the function $f: \mathbb{R}^n \to \mathbb{R}^n$ is such that

$$\forall\ x, y \in B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}, \ \forall\ x_0 \in \mathbb{R}^n$$

(for all x, y in a ball of radius r in the neighborhood of x_0) (and for all $x_0 \in \mathbb{R}^n$)

it holds that

$$\|f(x)-f(y)\| \leq L\|x-y\| \quad \text{(Lipschitz condition)}$$

then, there exists a $\delta>0$ for which we can guarantee that the solution $x(t)$ exists and is unique over the time interval $[t_0, t_0+\delta]$. But note, we have no control on choosing $\delta>0$! I.e. we can not choose $\delta>0$ arbitrarily large.

- In other words, if the function $f: \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, then the theorem can be used to assert that the solution exists and is unique over some time interval $[t_0, t_0+\delta]$

- You may already have many questions, such as:
  What is a locally Lipschitz function? How do I know that a function is locally Lipschitz? What is the parameter $\delta>0$ that affects the existence and uniqueness interval? Can δ be made arbitrarily large? In other words, can we guarantee global existence and uniqueness, i.e., over the time interval $[t_0, t_1]$, where $t_1 \to \infty$?

- In order to answer those questions in a rigorous manner, we need to first review/establish the proper mathematical framework. In the next part of the lecture notes, as well as in Appendices A and B of the textbook, we summarize the fundamental concepts of Euclidean spaces, normed spaces, open and closed sets, convergence of sequences, which are used towards defining the Lipschitz property, as well as in the proofs of Theorems 3.1 and 3.2.

- Studying the proofs of Theorems 3.1 and 3.2 is left optional however if you do wish to obtain a deeper understanding of the concepts, especially if you are in, or planning to apply to, the PhD program, then I strongly recommend that you read the proofs too.

- The proofs require the notion of a contraction mapping, which we will find in some homework problem as well. So to guide you through the next section "Mathematical Review"

Study the paragraphs ① through ⑫. Some we will review (quickly) in class, but you are expected to review the material also on your own.

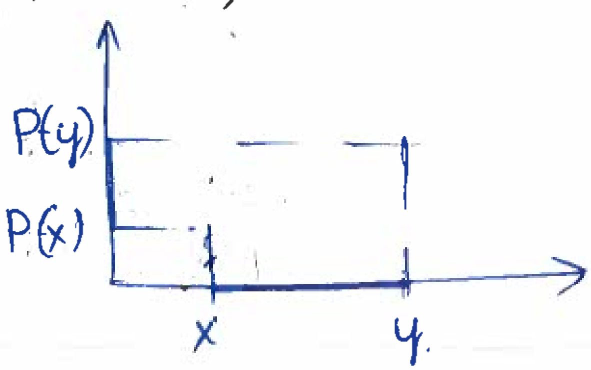
- In class we will cover (briefly) the notion of a contraction mapping, and highlight how it is useful for the E&U theorems. We will also define locally and globally Lipschitz functions and state formally Theorems 3.1 and 3.2.

- The quick summary on contraction mappings is as follows. [Check also in the "Mathematical Review".]

A function $P: \mathcal{X} \to \mathcal{X}$ (where $\mathcal{X}$ is a normed space) is called a contraction if there exists a constant $0 \le c < 1$ such that for all $x, y \in \mathcal{X}$, it holds that

$$\|P(x) - P(y)\| \le c\,\|x-y\|$$

In other words, a contraction mapping maps points closer together.



You can also note that a Lipschitz continuous function with Lipschitz constant $L<1$ is a contraction mapping.

What is nice about contraction mappings is the so-called Contraction Mapping Principle, which says that if $P: \mathcal{X} \to \mathcal{X}$ is a contraction, then (i) there exists a unique point $x^* \in \mathcal{X}$ such that $\underline{P(x^*) = x^*}$ (that is called a fixed point) and (ii) $\forall x_0 \in \mathcal{X}$, the sequence $x_{n+1} = P(x_n)$, $n \geq 0$ is Cauchy and $\lim_{n\to\infty} x_n = x^*$.

Now, the sequence means a collection of points $\{x_0, x_1, \dots, x_s, \dots\}$ indexed by the counting number $s \in \mathbb{Z}^+$ (positive integer). A Cauchy sequence is one whose terms approach closer and closer to each other as the index s increases. So the contraction mapping principle says that from every initial point x_0, applying a contraction mapping iteratively creates a sequence that converges to a unique fixed point.

This is the main idea that the proof of the Theorem 3.1 utilizes: It considers the solution to $\dot{x}(t) = f(t, x(t))$, $x(t_0) = x_0$ given as:

$$x(t) = \underbrace{x_0 + \int_{t_0}^{t} f(s, x(s))\, ds}_{Px(t)}$$

, it views the right hand side as a mapping of the state trajectory $x: [t_0, t_1] \to \mathbb{R}^n$, and applies the contraction mapping principle to identify conditions under which the mapping $Px(t)$ is a contraction.

This is the notation in the textbook, it means $P(x(t))$

This way we obtain that the solution to $x(t) = Px(t)$ exists and is unique – in fact, this is shown for a time interval $[t_0, t_0+\delta] \subset [t_1, t_1]$, where

$$\boxed{\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lr+h}, \frac{\rho}{L}\right\}},$$

where L the Lipschitz constant, r the radius of the ball around x_0, $h = \max_{s\in[t_0,t_1]} \|f(s, x_0)\|$, $\rho < 1$

Theorem 3.1 does not provide a way of arbitrarily choosing $\delta > 0$; the conditions on global existence and uniqueness are given by Theorem 3.2. At this point, review the math background in the next few pages; we resume with the formal definitions of locally Lipschitz and globally Lipschitz functions.

In fact, one last note prior to going into the detailed mathematical review: To continue with the main argument of the proof of Theorem 3.1, which will be useful in one of your homework problems: To prove that the right-hand side of the solution $x(t) = \underbrace{x_0 + \int_{t_0}^{t} f(x(s), s)\, ds}_{Px(t)}$ is a contraction mapping, we apply the contraction mapping principle, i.e, we consider:

$$\| Px(t) - Py(t) \| = \left\| \left[\int_{t_0}^{t} f(s, x(s)) - \int_{t_0}^{t} f(s, y(s)) \right] ds \right\| \leq$$

$$\leq \int_{t_0}^{t} \underbrace{\| f(s, x(s)) - f(s, y(s)) \|}_{\text{from Lipschitz continuity}} ds \leq$$

$$\leq \int_{t_0}^{t} L \| x(s) - y(s) \| ds \leq \int_{t_0}^{t} ds \; L \| x - y \|_c$$

norm on the Banach space where the supremum value of the trajectories over time has been considered:

$$\| x \|_c = \max_{t \in [t_0, t_0 + \delta]} \| x(t) \|$$

Hence we can write the relationship with $\| \cdot \|$ denoting the $\| \cdot \|_c$ norm:

$$\| Px(t) - Py(t) \| \leq \underbrace{(t - t_0)}_{\delta} L \| x - y \| \implies$$

$$\boxed{\| Px(t) - Py(t) \| \leq \delta L \| x - y \|}$$

For your homework problem, start from the above expression, compute L for the given system, and choose a $\delta > 0$ such that Px is a contraction mapping.

Mathematical Review (Textbook: Appendix A and B)

① Euclidean Space

The set of all n-dimensional vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where $x_1, x_2, \dots, x_n$ are real numbers

defines the n-dimensional Euclidean space

denoted as $\mathbb{R}^n$.

② Extended Real Line : $\mathbb{R}_e = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$

where : $x < +\infty$, for all $x \in \mathbb{R}$

$-\infty < x$, for all $x \in \mathbb{R}$

③ Supremum = Least Upper Bound.

Let $S \subset \mathbb{R}$. We say that $a \in \mathbb{R}_e$ is the supremum of S

if ① a is an upper bound : $s \le a$, $\forall s \in S$.

(ii) a is the least bound : If $b \in \mathbb{R}_e$ satisfies

$s \le b \;\; \forall s \in S$, then $a \le b$.

(that means that if b is also an upper bound, then $a \le b$.)

Notation : $a = \sup S$ or $a = \sup\{S\}$

Examples : $S = (0,1)$, then $\sup S = 1$, while the max of S does not exist.

$S = \{x \in \mathbb{R} \mid x \ge 0\}$, then $\sup S = +\infty$.

If $f : V \to \mathbb{R}$, then $\sup\limits_{x \in V} f(x) := \sup\{f(x) \mid x \in V\} = \sup S$

where $S = \{f(x) \mid x \in V\} \subset \mathbb{R}$

Infimum = Greatest Lower Bound.

obtained by reversing all the inequalities used for the supremum.

Maximum: If $S \subset \mathbb{R}$, and $\sup\{S\} \in S$, then $\max\{S\}$ exists, and $\max\{S\} = \sup\{S\}$.

Minimum: Similarly, if $\inf\{S\} \in S$, then $\min\{S\}$ exists and $\min\{S\} = \inf\{S\}$.

④ Norms. Let V be a vector space with field $\mathbb{R}$. Then $\|\cdot\| : V \to [0,\infty)$ is a norm if

(a) $\forall\, x \in V$, $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.

(b) $\forall\, x \in V$, and $a \in \mathbb{R}$, $\|a \cdot x\| = |a| \cdot \|x\|$.

(c) $\forall\, x, y \in V$, $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

Definition. $(V, \|\cdot\|)$ is called a normed space if V is a vector space with field $\mathbb{R}$, and $\|\cdot\|$ is a norm.

Examples.

$V = \mathbb{R}^n$ and
$$\begin{cases} \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \\ \|x\|_1 = \sum_{i=1}^{n} |x_i| \\ \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \end{cases}$$

$V = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$ with the norm defined as $\|f\|_\infty = \max_{a \leq t \leq b} |f(t)|$.

⑤ Induced Norms. Let $A \in \mathbb{R}^{n \times n}$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then the induced norm on A is

$$\|A\|_i = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

Fact: For all $x \in \mathbb{R}^n$, $\|Ax\| \leq \|A\|_i \|x\|$

Example. The only induced norm we will use is the one coming from the 2-norm, $\|\cdot\|_2$. In this case, Appendix A in the textbook shows that.

$$\|A\|_i = \sqrt{\lambda_{max}(A^T A)} = \text{square root of maximum eigenvalue of } A^T A.$$

⑥ Equivalent Norms. Two norms $\|\cdot\|_a : V \to [0, +\infty)$ and $\|\cdot\|_\beta : V \to [0, +\infty)$ are equivalent if there exist positive constants K_1 and K_2 such that, for all $x \in V$,

$$K_1 \|x\|_a \leq \|x\|_\beta \leq K_2 \|x\|_a$$

Remark: It follows from the definition of equivalent norms that

$$\frac{1}{K_2} \|x\|_\beta \leq \|x\|_a \leq \frac{1}{K_1} \|x\|_\beta.$$

Fact: V is finite-dimensional vector space if, and only if, all norms on V are equivalent.

⑦ Open Balls, Open Sets, and Closed Sets.

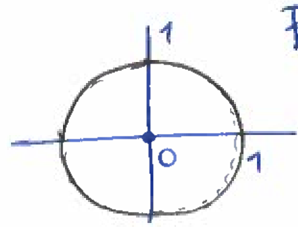
Let $x_0 \in V$, $a \in \mathbb{R}$, with $a > 0$.

1) $B_a(x_0) = \{x \in V \mid \|x - x_0\| < a\}$ is called the open ball of radius a centered at x_0.

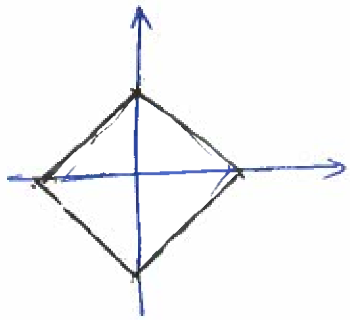
Examples

$B_1(0)$ (unit ball)

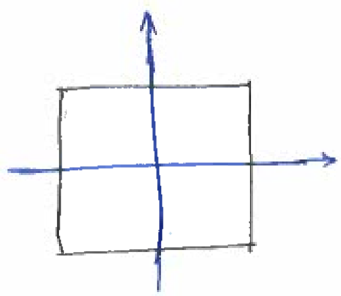
$(\mathbb{R}^2, \|\cdot\|_2)$



$(\mathbb{R}^2, \|\cdot\|_1)$



$(\mathbb{R}^2, \|\cdot\|_\infty)$



Open Sets. A set $S \subset V$ is open if $\forall\, s_0 \in S$, $\exists\, \varepsilon(s_0) > 0$ such that $B_\varepsilon(s_0) \subset S$.

Remark.

The empty set, denoted ϕ, is also said to be open.

Remark → The notation $\varepsilon(s_0)$ simply means that ε can depend on s_0 in the sense that, as you vary the point s_0, you may need a smaller or can use a larger value of ε.

Examples: $S = (0,1) \subset (\mathbb{R}, |\cdot|)$ is open.

$S = [0,1) \subset (\mathbb{R}, |\cdot|)$ is NOT open because $\forall\, \varepsilon > 0$, $B_\varepsilon(0) \not\subset S$.

Fact: Equivalent norms define the same open sets.

Closed Sets: S is closed if $\sim S$ is open, where $\sim S$ denotes the set complement.

Example: $S = [0,1] \subset (\mathbb{R}, |\cdot|)$ is closed because

$$\sim S = (-\infty, 0) \cup (1, +\infty) \text{ is open.}$$

Fact: Arbitrary unions of open sets are open.
Finite intersections of open sets are open

Fact: Arbitrary intersections of closed sets are closed
Finite unions of closed sets are closed.

Example: $S = [0,1)$ is neither open nor closed!

⑧ Convergence of a sequence.

Let $(V, \|\cdot\|)$ be a normed space.

Definition A collection of points in V indexed by the counting numbers (positive integers) is called a sequence.
Common notation is (x_k), (x_n) etc.

Definition A sequence (x_n) converges to a point $\bar{x} \in V$ if $\forall\, \epsilon > 0$, $\exists\, N(\epsilon) < \infty$ such that,

$$\forall\, n \geq N, \quad \|x_n - \bar{x}\| < \epsilon.$$

Remark: $\|x_n - \bar{x}\| < \epsilon \iff x_n \in B_\epsilon(\bar{x})$

Notation: $\lim x_n = \bar{x}$ or $x_n \xrightarrow[n \to \infty]{} \bar{x}$

Examples. let us take $(V, \|\cdot\|) = (\mathbb{R}, |\cdot|)$

Consider $x_n = \frac{1}{n}$ and $\bar{x} = 0$. Then $\lim_{n\to\infty} x_n = \bar{x}$.

Consider $x_n = \left(1 + \frac{1}{n}\right)^n$ and $\bar{x} = e$. Then $\lim_{n\to\infty} x_n = \bar{x}$.

Remark. We will soon use sequences as a means to iteratively construct better and better approximate solutions to an equation, such as an ordinary differential equation, where we want to assert the existence of a solution! To apply the definition we just gave, and test whether a sequence converges or not, we have to know the candidate limit $\bar{x}$.

But, if $\bar{x}$ is in fact what we are trying to construct and prove that exists, then we seem to be in a viscious circle:

we want to prove that $\bar{x}$ exists by proving it is the limit of a convergent sequence, BUT to prove that the sequence converges, we have to know the candidate limit!

emember, this always emains our main goal.

This viscious circle is broken with the notion of a Cauchy sequence.

Definition A sequence (x_n) in V is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N(\epsilon) < \infty$, such that for any pair $m, n \geq N \Rightarrow \|x_n - x_m\| < \epsilon$.

That says, the farther down you go along the sequence, the closer and closer its terms get to one another.

This property can be checked without having a limit in hand!

Fact. Every convergent sequence is Cauchy. But the converse does not always hold.

Definition A normed space $(V, \|\cdot\|)$ is complete if every Cauchy sequence converges (to an element $\bar{x}$ in $(V, \|\cdot\|)$)

Fact: Every finite-dimensional normed space is complete. The infinite-dimensional normed space $(C[a,b], \|\cdot\|_\infty)$ is complete.

9) Fixed Points and Contraction Mappings.

Let $(\mathcal{X}, \|\cdot\|)$ be a normed space and let $P: \mathcal{X} \to \mathcal{X}$ be a mapping.

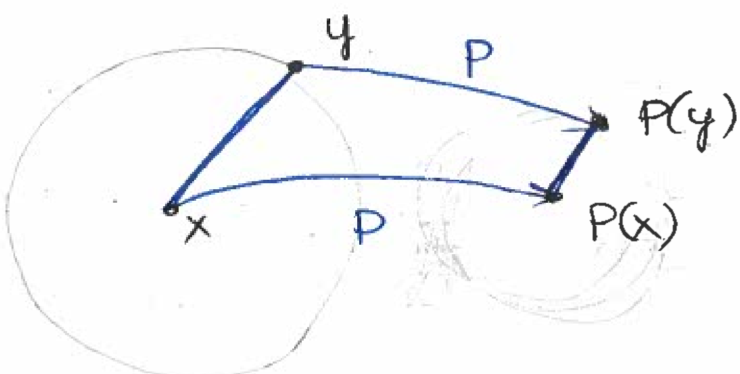
Definition x^* is a fixed point if $P(x^*) = x^*$.

[The mapping $P(\cdot)$ leaves x^* invariant]

Definition $P(\cdot)$ is a contraction if $\exists$ a constant $0 \le c < 1$ such that, $\forall\ x, y \in \mathcal{X}$

$$\| P(x) - P(y) \| \le c \, \| x - y \|$$

In other words, a contraction mapping maps points closer together.



$P(\cdot)$ is a contraction mapping.

$\|P(y) - P(x)\| \le c\,\|x - y\|$, where $0 \le c < 1$.

Contraction Mapping Principle. Let $(\mathcal{X}, \|\cdot\|)$ be a complete normed space, and let $P: \mathcal{X} \to \mathcal{X}$ be a contraction. Then $\exists$ a unique $x^* \in \mathcal{X}$ such that $P(x^*) = x^*$.

Moreover, $\forall\ x_0 \in \mathcal{X}$, the sequence

$$x_{n+1} = P(x_n), \quad n \ge 0$$

is Cauchy, and $\boxed{\lim_{n \to \infty} x_n = x^*}$

What does that mean, and why is it useful to us?

That is a plausible question! See next page. →

Let us see some more intuition behind contraction mappings and fixed points.

Consider an equation of the form $x = P(x)$.

A solution x^* to this equation, i.e., a x^* such that $x^* = P(x^*)$ is said to be a <u>fixed point</u> of the mapping $P(\cdot)$, since $P(\cdot)$ leaves x^* invariant.

A classical idea for finding a fixed point is the successive approximation method: We begin with an initial trial vector x_1 and compute $x_2 = P(x_1)$. Continuing in this matter iteratively we compute successive vectors $x_{n+1} = P(x_n)$.

Now, the contraction mapping principle gives <u>sufficient</u> conditions under which there exists a fixed point x^* of $x = P(x)$, and the sequence (x_n) converges to x^*. [The full proof is in Appendix B of our textbook.] In fact, it is a powerful analysis tool for proving the existence of the solution of an equation of the form $x = T(x)$.

We will later use it for proving the existence of the solution of an ODE with locally Lipschitz right-hand side.

<u>In the mean time, let us wrap-up:</u>

Suppose $P: \mathcal{X} \to \mathcal{X}$ is a contraction mapping. Then:

(a) Fixed points are unique.

(b) $x_{n+1} = P(x_n)$ yields a Cauchy sequence.

Let us see why $\rightarrow$

(a) Fixed points are unique.

To prove that: Suppose that x^* and y^* are both fixed points. Then $x^* = P(x^*)$ and $y^* = P(y^*)$. Hence

$$\|x^* - y^*\| = \|P(x^*) - P(y^*)\| \leq c\,\|x^* - y^*\|$$

This is true if $c = 0$ or $\|x^* - y^*\| = 0$, where

$c = 0$ implies that $\|x^* - y^*\| = 0 \Rightarrow x^* = y^*$.

(b) Then $x_{n+1} = P(x_n)$ yields a Cauchy sequence.

Note: $\|x_{n+1} - x_n\| = \|P(x_n) - P(x_{n-1})\| \leq c\,\|x_n - x_{n-1}\|$

$$\leq c\,\|P(x_{n-1}) - P(x_{n-2})\|$$

$$\leq c^2\,\|x_{n-1} - x_{n-2}\|$$

$$\vdots$$

$$\leq c^{n-1}\,\|x_1 - x_0\| \xrightarrow[n \to \infty]{} 0.$$

Because $\boxed{0 \leq c < 1}$

This is not the complete proof for why the sequence is Cauchy, but it yields why the terms of the sequence as $n \to \infty$ tend to be closer and closer to each other. [The complete proof is in the textbook.]

Example. We consider the normed space $(\mathbb{R}^n, \|\cdot\|_2)$ and seek to solve the linear equation $Ax = b$, where A is an $n\times n$ matrix, b is $n\times 1$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$

(a) Define $P(x) = x + (Ax - b)$

We note that $x^* = P(x^*) \iff x^* = x^* + Ax^* - b$

$$\iff 0 = Ax^* - b$$

$$\iff x^* \text{ solves } Ax = b.$$

(b) When is P a contraction?

$$\|P(x) - P(y)\|_2 = \|x + (Ax - b) - y - Ay + b\|_2 =$$

$$= \|x - y + Ax - Ay\|_2 =$$

$$= \|(I+A)x - (I+A)y\|_2 =$$

$$= \|(I+A)(x-y)\|_2 \leq \|I+A\|_i \, \|x-y\|_2$$

where $\|I+A\|_i$ is the induced norm.

Hence, P is a contraction if

$$c := \|I+A\|_i = \left(\lambda_{max}\,(I+A)^T(I+A)\right)^{1/2} < 1$$

Remark: You can verify convergence in MATLAB. Of course, one would rarely solve a set of linear equations by this method! Here we just illustrated the concept of convergent sequence. Later we will apply the method to proving the existence of solutions to nonlinear differential equations, with a locally Lipschitz continuous right-hand side.

To this end we have to define (local and global) Lipschitz continuity.

⑩ Continuous Functions.

Let $(U, \|\cdot\|)$ and $(W, \|\cdot\|)$ be normed spaces.

Definition The function $h : V \to W$ is continuous at $x_0 \in V$ if $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon, x_0) > 0$ such that

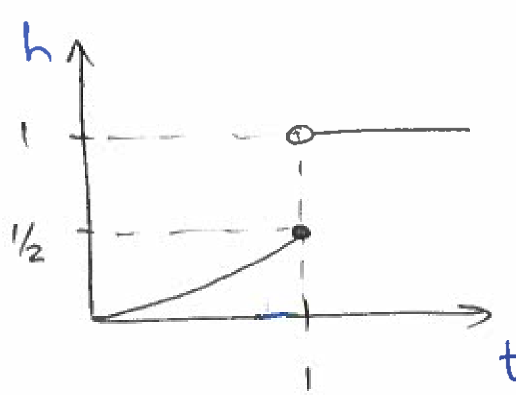
$$\|x - x_0\| < \delta \Rightarrow \|h(x) - h(x_0)\| < \epsilon.$$

Another way to say it: $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon, x_0) > 0$ such that

$$x \in B_\delta(x_0) \Rightarrow h(x) \in B_\epsilon\big(h(x_0)\big).$$

Definition. $h : V \to W$ is continuous if it is continuous at x_0, $\forall\, x_0 \in V$.

Non-example.



The function $h : \mathbb{R} \to \mathbb{R}$ shown above is NOT continuous at $t_0 = 1$ because for $\epsilon = 1/3$, it is not possible to choose $\delta > 0$ such that $|t - t_0| < \delta$ implies that $|h(t) - h(t_0)| < \epsilon$

Indeed, when $\epsilon = \frac{1}{3}$, we have that $\forall \delta > 0$, there exists $t \in B_\delta(t_0)$ that yields $|h(t) - h(t_0)| \geq \frac{1}{3}$.

<u>Lipschitz continuity</u>. $h: V \to W$ is Lipschitz continuous <u>at $x_0 \in V$</u> if $\exists\ r > 0$ and $\boxed{L < \infty}$ such that

$\forall\ x, y \in B_r(x_0)$, $\boxed{\| h(x) - h(y) \| \leq L \|x - y\|}$.

L is called a <u>Lipschitz</u> constant.

$h: V \to W$ is <u>locally</u> Lipschitz continuous if it is Lipschitz continuous at <u>every</u> point of V. (i.e, $\forall x_0 \in V$)

$h: V \to W$ is <u>globally</u> Lipschitz continuous on V if there exists $L < \infty$ such that $\forall\ x, y \in V$,

$$\| h(x) - h(y) \| \leq L \|x - y\|.$$

Remarks and Useful Facts.

(a) The difference between <u>global</u> and <u>local</u> Lipschitz continuity is that with global Lipschitz continuity, there is a single constant L that works everywhere, whereas with local Lipschitz continuity, as you vary x_0 and/or the radius of the open ball $B_r(x_0)$, the Lipschitz constant may have to change.

(b) Lipschitz continuous at x_0 implies continuous at x_0,

indeed, for $\epsilon > 0$, we can select $\delta = \frac{\epsilon}{L}$.

(c) [Easy way to find a Lipschitz constant.]

Suppose that $h: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, and that at some point $x_0 \in \mathbb{R}^n$, there exists $r > 0$ and $L < \infty$ such that, $x \in B_r(x_0) \Rightarrow \| \frac{\partial h}{\partial x}(x) \|_i \leq L$. Then,

$\forall\, x, y \in B_r(x_0)$, $\|h(x) - h(y)\|_2 \leq L \|x - y\|_2$.

Note: $\|\cdot\|_2$ is the 2-norm on $\mathbb{R}^n$ or $\mathbb{R}^m$, whereas $\|\cdot\|_i$ is the induced two-norm on matrices.

(d) If $h: V \to W$ is Lipschitz continuous at $x_0 \in V$ with norm $\|\cdot\|$ on V, and norm $|||\cdot|||$ on W, then it is also Lipschitz continuous at $x_0 \in V$ for any equivalent norms, though the Lipschitz constant will in general be different.

(e) $h: \mathbb{R} \to \mathbb{R}$ defined as $h(x) = \mathrm{sat}(x) = \begin{cases} x, & |x| < 1 \\ 1, & x \geq 1 \\ -1, & x \leq -1 \end{cases}$

is NOT differentiable on $\mathbb{R}$, but it is globally Lipschitz continuous! since

$$|h(x) - h(y)| \leq |x - y|, \quad \forall\, x, y \in \mathbb{R}.$$

⑪ Piecewise continuity

Definition. A function $h: \mathbb{R} \to V$ is piecewise continuous if

(a) For every integer $k > 0$, $h: [-k, k] \to V$ is continuous, except possibly at finite number of points. (This equivalently, reads, any bounded interval contains a finite number of discontinuity points)

(b) At each point of discontinuity t_i, the limits from the left and right both exist and are finite, that is, $\lim_{\delta \downarrow 0} h(t_i + \delta)$ and $\lim_{\delta \uparrow 0} h(t_i + \delta)$ both exist and are finite.

Example. $h(t) = \begin{cases} \sin(t) & t \neq k\pi, \\ k^2\pi^2 & t = k\pi, \end{cases} \quad k = 0, \pm 1, \ldots$

Non-Example $h(t) = \begin{cases} \tan(t), & t \neq k\frac{\pi}{2} \\ 0, & t = k\frac{\pi}{2} \end{cases} \quad k = \pm 1, \pm 2, \ldots$

We note that the limits from left and right are unbounded at points of discontinuity. Hence the function is not piecewise continuous.

Non-example: $h(t) = \begin{cases} 0 & t = 0 \\ \sin\left(\frac{1}{t}\right) & t \neq \frac{2}{k\pi}, \; k = \pm 1, \pm 2, \ldots \\ 1 & t = \frac{2}{k\pi} \end{cases}$

The function is discontinuous at $t_k = \frac{2}{k\pi}$, $k = \pm 1, \pm 2, \ldots$ and the interval $[0,1]$ contains an infinite number of points of discontinuity, hence the function is not piecewise continuous

⑫ Negation of a statement.

Recall the rule from logic that $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$ where the symbol $(\neg)$ denotes negation.

$(\neg q \Rightarrow \neg p)$ is called the contrapositive of $(p \Rightarrow q)$

To apply the contrapositive rule in a proof, we need to be able to form the negation of a property, such as $\neg p$, $\neg q$

Note: You will not need to write proofs on your exams. There will be theory questions, but no written proofs.

Example 1: We let p be the property or statement:

$x_n \xrightarrow[n \to \infty]{} \bar{x}$. $\left\{ \begin{array}{l} \text{The statement is that} \\ \text{a sequence } x_n \text{ converges} \\ \text{to a limit point } \bar{x} \end{array} \right\}$

We want to form its negation.

Write out in math: p: $\forall \epsilon > 0$, $\exists N < \infty$ such that $\forall n \geq N$, $\|x_n - \bar{x}\| < \epsilon$. $\left\{ \begin{array}{l} \text{This is just the} \\ \text{definition of a} \\ \text{convergent sequence.} \end{array} \right\}$

Write out in english: p: For all $\epsilon > 0$, there exists $N < \infty$, such that for all $n \geq N$, $\|x_n - \bar{x}\| < \epsilon$.

Negate in english! : $\neg p$: It is not the case that for all $\epsilon > 0$, there exists $N < \infty$ such that for all $n \geq N$, $\| x_n - \bar{x} \| < \epsilon$.

Negate in english : $\neg p$: For some $\epsilon > 0$, there does not exist any $N < \infty$, such that for all $n \geq N$, $\| x_n - \bar{x} \| < \epsilon$.

Negate in english: $\neg p$: For some $\epsilon > 0$, it is the case that for all $N < \infty$, there is some $n \geq N$, such that $\| x_n - \bar{x} \| > \epsilon$.

Negation in math: $\neg p$: $\exists \, \epsilon > 0$, such that $\forall \, N < \infty$, $\exists \, n \geq N$, such that $\| x_n - \bar{x} \| > \epsilon$.

General Rules for Negation.

(a) The Existential Quantifier $\exists$ is replaced with the Universal Quantifier $\forall$, that is $\exists \to \forall$

(b) The Universal Quantifier $\forall$ is replaced with the Existential Quantifier $\exists$, that is $\forall \to \exists$

Example 2 : We let p be the property: The function h is continuous at x_0.

p: $\forall \epsilon > 0$, $\exists \, \delta > 0$ such that $\forall \, x \in B_\delta(x_0)$, it follows that $\| h(x) - h(x_0) \| < \epsilon$.

$\neg p$: $\exists \, \epsilon > 0$, such that $\forall \, \delta > 0$, $\exists \, x \in B_\delta(x_0)$ such that $\| h(x) - h(x_0) \| > \epsilon$.

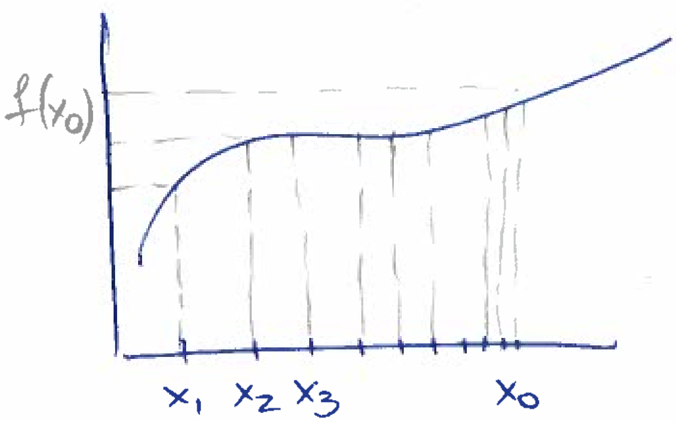
Example 3

When we are given a statement and are asked to prove another statement, it is often useful to write down in math the given statement and take it from there.

For instance, Problem 2 states that you have a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at $x_0 \in \mathbb{R}$ (this is the first given statement) AND also that you have a convergent sequence to $x_0 \in \mathbb{R}$: $\lim_{n\to\infty} x_n = x_0$ (this is a second given statement)

limit point

You are asked to show that the limit point of the sequence $f(x_n)$ will be $f(x_0)$: $\lim_{n\to\infty} f(x_n) = f(x_0)$



Perhaps drawing a figure would help, but it is not necessary.

How to approach such problems? Start with what you know/what you are given.

You know that $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$. This means you can write the definition of a continuous function at a point x_0:

$$\forall \epsilon > 0,\ \exists\ \delta(\epsilon, x_0) \text{ such that } \forall\ x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0))$$

or, simpler: $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$.

(implies that)

You also know that x_n is a convergent sequence to $x_0 \in \mathbb{R}$.

So you can write the definition of a convergent sequence: x_n to a limit point denoted x_0.

$\forall \epsilon > 0$, $\exists N(\epsilon) < \infty$ such that $\forall n \geq N$, it holds that

$$\| x_n - \underbrace{x_0}_{} \| < \epsilon$$

note that x_0 here is the limit point.

Since the definition holds for all $\epsilon > 0$, it as well holds for some $\delta > 0$

Hence we can write $\| x_n - x_0 \| < \delta$, as $n \to \infty$.

From the continuity of $f: \mathbb{R} \to \mathbb{R}$, we then have that $\exists \delta > 0$ such that.

$$\| x_n - x_0 \| < \delta \Rightarrow \| f(x_n) - f(x_0) \| < \epsilon, \quad \forall \epsilon > 0$$

i.e, what we were seeking to prove! The proof is complete.

Example 4. Sometimes it is useful to use the "proof by contradiction" to prove an argument. Consider that you want to prove that a statement/property is true. You start with that ~~negated~~ the property is true, ~~and~~ instead then you derive an illogical/false result. This implies that your original assumption was false, hence the opposite of it is true (i.e what you were seeking to prove)

For example, Problem 3 gives a closed set $S \subset V$, where V is a normed space, and a sequence of points x_n such that $x_n \in S$, $\forall n \geq 1$. The problem also states that the sequence is convergent, i.e, that $\lim_{n\to\infty} x_n = \bar{x}$.

Then, it asks to show that $\underline{\underline{\bar{x} \in S}}$.

We can show the argument utilizing "proof by contradiction". The goal is to show that $\bar{x} \notin S$ leads to a contradiction, i.e, to a false/illogical result according to our assumptions. Let's see how this works.

Let's assume that $\bar{x} \notin S$, i.e that $\bar{x}$ does not belong into the closed set S. Then $\bar{x}$ must belong into the complement set of S, denoted $\sim S$, which <u>must</u> be open (since S is closed). Since $\bar{x} \in \sim S$ and $\sim S$ is an open set, from the definition of an open set we have that for $\bar{x} \in \sim S$, there exists an $\varepsilon(\bar{x}) > 0$ such that $B_\varepsilon(\bar{x}) \subset \sim S$

Then, $B_\varepsilon(\bar{x}) \cap S = \varnothing$, or in other words, $\forall s \in S$, (for all points s in the set S)

$$\|\bar{x} - s\| > \varepsilon$$

We have that $\underline{x_n \in S}$, $\forall n \geq 1$. (this is given by definition of the problem)

Hence $\|x_n - \bar{x}\| \underline{> \varepsilon}$, $\underline{\forall n \geq 1}$. (A)

We also know that the sequence x_n is <u>convergent</u> to $\bar{x}$ (also known by definition) This means that $\|x_n - \bar{x}\| \underline{< \varepsilon}$, $\underline{\forall \varepsilon > 0}$, $\underline{\forall n > N}$, for some $N < \infty$. (B)

Now observe Ⓐ and Ⓑ

Ⓐ says that there exists an $\varepsilon > 0$ such that $\forall n \geq 1$

$$\| x_n - \bar{x} \| > \varepsilon$$

(for all points $x_1, \dots, x_n, \dots$) of the sequence.

Ⓑ says that $\forall \varepsilon > 0$, it must be that, for some $n > N(\varepsilon)$ where $N(\varepsilon) < \infty$,

$$\| x_n - \bar{x} \| < \varepsilon$$

Clearly Ⓐ contradicts Ⓑ since for all elements $x_1, x_2, \dots, x_n, \dots$ of the sequence, Ⓐ implies that the distance of each point of the sequence to $\bar{x}$ can not be made smaller or equal to $\varepsilon > 0$ (the sequence is not convergent)

Hence $\bar{x} \notin S$ was a false statement, i.e. $\bar{x} \in S$.

Existence and Uniqueness of Solutions to ODEs

With our math background in place, we are almost ready to state and prove the existence and uniqueness theorems.

Theorem 3.1 (Local Existence and Uniqueness) Let $f(t,x)$ be piecewise continuous in t, and satisfy the Lipschitz condition

$$\|f(t,x)-f(t,y)\| \leq L\|x-y\|,$$

$\forall\ x,y \in B_r(x_0)=\{x\in\mathbb{R}^n \mid \|x-x_0\|\leq r\}$, $\forall\ t\in[t_0,t_1]$. Then, there exists some $\delta>0$ such that the state equation $\dot{x}=f(t,x)$ with $x(t_0)=x_0$ has a unique solution over $[t_0,t_0+\delta]$.

Remark: Note that we extended the definition of Lipschitz condition to a function $f(t,x)$ instead of $f(x)$ that we saw earlier. See page 89 in your textbook.

Proof. (Appendix C.1) From continuity of $f(\cdot,\ )$, we have that $\dot{x}=f(t,x)$ has at least one solution $x(t)$

A solution $x(t)$ of $\dot{x}=f(t,x)$, $x(t_0)=x_0$, by integration satisfies $\boxed{x(t)=x_0+\int_{t_0}^{t} f(s,x(s))\,ds.}$ Thus we can proceed with investigating the existence and uniqueness of

$$x(t)=x_0+\int_{t_0}^{t} f(s,x(s))\,ds. \qquad (C.2)$$

We view the right-hand side of (C.2) as a mapping of the continuous function

$x(t) : [t_0, t_1] \to \mathbb{R}^n$, and we let it have the form

$$\boxed{x(t) = (Px)(t)} \qquad (C\,3)$$

Now, $(Px)(t)$ is continuous in t, and we can view it as a contraction mapping. More specifically, we have that a solution of $(C\,3)$ is a fixed point of the mapping P that maps x into Px. Then we can use the contraction mapping principle to establish the existence of a fixed point of $(C.\,3)$

To proceed with that we need a Banach space, {read : a complete normed linear space} and a closed set $S \subset \mathcal{X}$ such that the mapping P maps S into S, and is a contraction over S.

Banach space: (Example B.1, p. 654) A complete normed linear space $\mathcal{X}$. Recall that "complete" means that every Cauchy sequence in $\mathcal{X}$ converges to a point in $\mathcal{X}$.

Hence we proceed by (i) defining a Banach space $\mathcal{X}$, i.e, a normed vector space in which every Cauchy sequence converges to a point in $\mathcal{X}$, and (ii) by constructing a closed set $S \subset \mathcal{X}$ and showing that P maps S into S and is a contraction over S.

(i) We consider the set of continuous functions $x : [t_0, t_0+\delta] \to \mathbb{R}^n$ denoted as $C[t_0, t_0+\delta]$ This set forms a vector space on $\mathbb{R}$. [The justification is given in example B.1] For obtaining a normed vector space we have to equip the vector space with a norm. We pick the norm $\|x\|_C = \max\limits_{t\in[t_0, t_0+\delta]} \|x(t)\|$

[why this indeed is a norm is shown in example B.1]

and as thus we have the normed vector space

$$\mathcal{X} = C\,[t_0, t_0+\delta], \quad \text{with norm } \|x\|_C = \max_{t\in[t_0, t_0+\delta]} \|x(t)\|$$

where δ is a positive constant to be chosen.

(ii)

We proceed by defining the set $S = \{x\in \mathcal{X} \mid \|x - x_0\|_C \leq r\}$, where r is the radius of the ball $B_r(x_0) = \{x\in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$

We want to show that P maps S into S. We have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds \Rightarrow x(t) - x_0 = \int_{t_0}^{t} f(s, x(s))\,ds$$

$$\Rightarrow (Px)(t) - x_0 = \int_{t_0}^{t} f(s, x(s))\,ds =$$

$$= \int_{t_0}^{t} \left[f(s, x(s)) - f(s, x_0) + f(s, x_0)\right] ds$$

Jow we can consider that

(a) $f(t, x_0)$ is bounded on $[t_0, t_1]$, since f is piecewise continuous. We denote $h = \max\limits_{t\in[t_0, t_1]} \|f(t, x_0)\|$

(b) $\forall\, x\in S$, $\|x(t) - x_0\|_C = \max \|x(t) - x_0\| \leq r$, $\forall\, t\in[t_0, t_0+\delta]$

and write:

$$\|(Px)(t)-x_0\| \leq \int_{t_0}^{t}\Big[\underbrace{\|f(s,x(s))-f(s,x_0)\|}_{\leq L\|x(s)-x_0\| \text{ from Lipschitz property}} + \overbrace{\|f(s,x_0)\|}^{\leq \max_{s\in[t_0,t_1]}\|f(s,x_0)\|=h}\Big]ds$$

$$\leq \int_{t_0}^{t}\Big[L\underbrace{\|x(s)-x_0\|}_{\leq r}+h\Big]ds \leq \int_{t_0}^{t}(Lr+h)\,ds =$$

$$= (Lr+h)(t-t_0)$$

We can now restrict the choice of δ to satisfy $\boxed{\delta \leq t_1-t_0}$ so that $[t_0, t_0+\delta] \subset [t_0,t_1]$. Then for $t=t_1$ we can write

$\|(Px)(t)-x_0\| \leq (Lr+h)\delta$. Note now that $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$, whereas $\|\cdot\|_C$ is the norm on $\mathcal{X}$. We have

$$\|(Px)(t)-x_0\|_C = \max_{t\in[t_0,t_0+\delta]}\|(Px)(t)-x_0\| \leq (Lr+h)\delta$$

Hence, choosing $\boxed{\delta \leq \frac{r}{Lr+h}}$ ensures that P maps S into S

since $\|\underbrace{(Px)(t)}_{x(t)}-x_0\|_C \leq \frac{r}{\cancel{Lr+h}}\cdot\cancel{(Lr+h)}$, i.e the solution

$x(t)$ exists in the set $S\subset\mathcal{X}$, $\forall\, t\in[t_0,t_0+\delta]$

Now, we want to show that P is a contraction over S.

Let $x, y \in S$ and take

$$\|(Px)(t) - (Py)(t)\| = \left\| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))] ds \right\| \leq$$

$$\leq \int_{t_0}^{t} \|f(s, x(s)) - f(s, y(s))\| ds$$

$$\leq \int_{t_0}^{t} L \underbrace{\|x(s) - y(s)\|}_{\leq \max_{s \in [t_0, t_0+\delta]} \|x(s)-y(s)\|} ds \leq L \|x-y\|_c \underbrace{\int_{t_0}^{t} ds}_{=\delta \text{ for } t = t_0+\delta}$$

$$\leq L \|x-y\|_c \cdot \delta \quad \Rightarrow$$

Thus we have:

$$\|(Px) - (Py)\|_c \leq L\delta \|x-y\|_c \leq \rho \|x-y\|_c, \text{ where } \boxed{\delta \leq \frac{\rho}{L}}$$

Under which condition is this a contraction? We see that, choosing $\rho < 1$ and $\delta \leq \frac{\rho}{L}$, ensures that P is a contraction mapping over S.

Hence, by the contraction mapping theorem, we conclude that if δ is chosen to satisfy

$$\delta \leq \min \left\{ t_1 - t_0, \frac{r}{Lr+h}, \frac{\rho}{L} \right\}, \text{ for } \rho < 1,$$

then $x(t)$ given by (C 2) is unique in S.

To complete the proof, we need to establish uniqueness of the solution of (C 2) in $\mathcal{X}$, i.e., that the solution of (C 2) is unique among all continuous functions $x(t) \in \mathcal{X} = (C[t_0, t_0+\delta], \|\cdot\|_c)$

To establish this, we start with that since $x(t_0) = x_0$, i.e., since the solution $x(t)$ starts in S, and since it is continuous, it must lie in S for some interval of time.

Suppose that $x(t)$ leaves the ball $B_r(x_0)$, (which is by definition the ball of radius r centered at x_0), and that the first time $x(t)$ intersects the boundary of the ball $B_r(x_0)$ is $t_0+\mu$. Then, $\|x(t_0+\mu) - x_0\| = r$. In addition, $\forall\ t \in [t_0, t_0+\mu]$ we can write:

$$\|x(t) - x_0\| \leq \int_{t_0}^{t} \left[\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|\right] ds$$

$$\leq \int_{t_0}^{t} \left[L\,\|x(s) - x_0\| + h\right] ds \leq \int_{t_0}^{t} (Lr+h)\, ds$$

Hence, $r = \|x(t_0+\mu) - x_0\| \leq (Lr+h)(\cancel{t_0}+\mu-\cancel{t_0}) \Rightarrow$

$$\mu \geq \frac{r}{Lr+h} \geq \delta$$

This implies that the time $t_0+\mu$ at which the solution $x(t)$ would leave the ball $B_r(x_0)$ is greater than $t_0+\delta$, or in other words, the solution $x(t)$ cannot leave the ball $B_r(x_0)$ within the time interval $[t_0, t_0+\delta]$, which implies that any solution in $\mathcal{X}$ over the time interval $[t_0, t_0+\delta]$ must lie in S. Consequently, uniqueness of the solution in S implies uniqueness in $\mathcal{X}$. ■

Corollary Suppose $f(t,x)$ is piecewise continuous in t, and there exists $r>0$, $T>t_0$, and $0<L<\infty$ such that, for all $x,y \in B_r(x_0)$ and $t\in[t_0,T]$,

we have $\left\| \frac{\partial f}{\partial x}(t,x) \right\| \leq L$.

Then, there exists $\delta>0$, such that such that the differential equation $\dot{x}=f(t,x(t))$ has exactly one solution over $[t_0, t_0+\delta]$.

Remark: See also Lemmas 3.1, 3.2, 3.3, and as a suggestion, work through the proofs.

Example $f(x) = \begin{bmatrix} -x_1+x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix}$ is continuously differentiable in $\mathbb{R}^2$

Per Lemma 3.2, $f(x)$ is locally Lipschitz on $\mathbb{R}^2$

Let us compute a Lipschitz constant over the convex set
$W = \{x\in\mathbb{R}^2 \mid |x_1|\leq a_1,\ |x_2|\leq a_2\}$

The Jacobian is $\frac{\partial f}{\partial x} = \begin{bmatrix} -1+x_2 & x_1 \\ -x_2 & 1-x_1 \end{bmatrix}$

Use $\|\cdot\|_\infty$ for vectors in $\mathbb{R}^2$ and the induced matrix norm for matrices (see Appendix A), then:

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty = \max\{|-1+x_2|+|x_1|,\ |x_2|+|1-x_1|\}$$

For all points in W, we have
$$|-1+x_2|+|x_1| \leq 1+a_2+a_1$$
$$|x_2|+|1-x_1| \leq a_2+1+a_1$$

Hence $\left\|\frac{\partial f}{\partial x}\right\|_\infty \leq 1+a_1+a_2$, and $L = 1+a_1+a_2$.

Lemma 3.1 Let $f : [t_0, t_1] \times D \to \mathbb{R}^m$ be continuous for some $D \subset \mathbb{R}^r$ open and connected set, also called a domain. Suppose that $\frac{\partial f}{\partial x}$ exists and is continuous on $[t_0, t_1] \times D$. If for some convex subset $W \subset D$ there is a constant $L \geq 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \leq L \quad \text{on } [t_0, t_1] \times W, \text{ then}$$

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\| \quad \forall\, t \in [t_0, t_1],\ \forall\, x, y \in W.$$

- The lemma tells us that a continuously differentiable function (ie a function whose derivative exists and is continuous) with a bounded derivative on a convex set W satisfies the Lipschitz condition on the set $W \subset D$. In fact, the derivative bound is the Lipschitz constant. This leads to the following result.

Lemma 3.2. If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[t_0, t_1] \times D$, for some domain $D \subseteq \mathbb{R}^n$, then f is locally Lipschitz in x on $[t_0, t_1] \times D$.

- An extension to global Lipschitzness is given in lemma 3.3.

Lemma 3.3. If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[t_0, t_1] \times \mathbb{R}^n$ then f is globally Lipschitz in x on $[t_0, t_1] \times \mathbb{R}^n$ if and only if $\frac{\partial f}{\partial x}(t, x)$ is uniformly bounded on $[t_0, t_1] \times \mathbb{R}^n$.

the same bound applies for all $x \in \mathbb{R}^n$.

Theorem 3.2. (Global Existence and Uniqueness) Suppose that $f(t,x)$ is piecewise continuous in t and satisfies

$$\| f(t,x) - f(t,y) \| \le L \, \|x-y\|,$$

$\forall\, x,y \in \mathbb{R}^n$, $\forall\, t \in [t_0, t_1]$. Then, the state equation

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \text{ has}$$

a unique solution over $[t_0, t_1]$.

Proof: See Appendix C. The key point is to show that the constant δ of Theorem 3.1 can be made independent of the initial state x_0.

Examples ① Consider $\dot{x} = 1 + x^2$, $x(t_0) = 0 \in \mathbb{R}$. Check the Lipschitz condition for $-2 < x < 2$.

We have $\left| \frac{\partial f}{\partial x} \right| = |2x| \le 4$

Hence, there exists $\delta > 0$ such that the solution exists and is unique on $[t_0, t_0+\delta]$

② Consider $\dot{x} = \underbrace{Ax(t) + Bu(t)}_{f(t,x)}$, where $t \in [t_0, \infty)$,

$x(t_0) = x_0 \in \mathbb{R}^m$, the input $u : [t_0, t_{\infty}) \to \mathbb{R}^m$ is piecewise continuous, and A, B constant matrices of appropriate dimensions. We check the Lipschitz condition in some norm $\|\cdot\|$ on $\mathbb{R}^n$:

$$\| f(t,x) - f(t,y) \| = \| A(x-y) \| \le \underbrace{\|A\|_i}_{L} \, \|x-y\|$$

Hence solutions exist globally and are unique.

③ Consider $\dot{x} = x^{2/3}$, $x(0) = 0 \in \mathbb{R}$.

Claim $x^{2/3}$ is <u>not</u> Lipschitz continuous at zero.

<u>Intuition behind the claim</u> → Note: This is <u>not</u> a proof, but it gives a hint as to why the function is not Lipschitz continuous.

$$\frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{x^{1/3}}$$

$$\lim_{x \uparrow 0} \frac{1}{x^{1/3}} = -\infty, \quad \lim_{x \downarrow 0} \frac{1}{x^{1/3}} = +\infty.$$

<u>Proof of the claim</u>: Suppose to the contrary that the function is Lipschitz continuous at the origin, i.e, there exist $r > 0$ and $0 < L < \infty$ such that, for all $x, y \in B_r(0)$

$$|x^{2/3} - y^{2/3}| \leq L\,|x - y|$$

Setting $y = 0$, we have that $\forall x \in B_r(0)$

$$|x^{2/3}| \leq L|x|$$

Then for $x \neq 0$, we divide both sides by $|x|$ to obtain $\left|\frac{1}{\sqrt[3]{x}}\right| \leq L$, $x \neq 0$.

However $\lim_{x \to 0} \sqrt[3]{x} = 0$, which implies that $\sup_{0<|x|<r} \left|\frac{1}{\sqrt[3]{x}}\right| = \infty$, which contradicts the assumption that $L < \infty$.

<u>Remark</u>: Since the right-hand side of the ODE is not locally Lipschitz continuous, our theorems on local existence and uniqueness do not apply.

Existence and Uniqueness: Recap of Theorems 3.1 and 3.2 Intro of Theorem 3.3.

What do the Existence and Uniqueness theorems tell us?

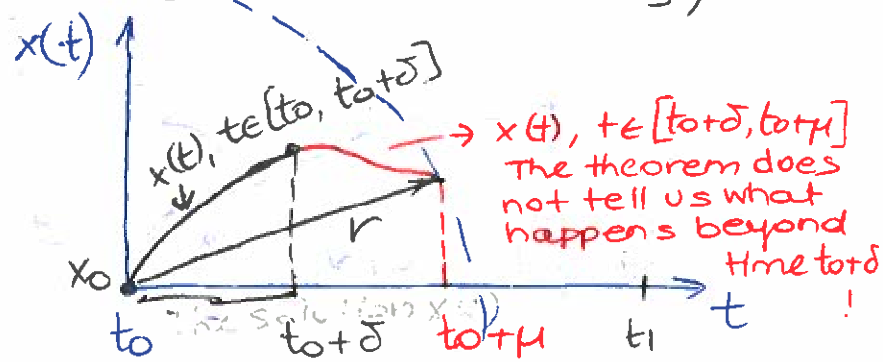
Let us start with the Local Existence and Uniqueness Theorem (Theorem 3.1 in our textbook) The theorem states that.

If the function $f(t,x)$ piecewise continuous in t, and satisfying the local Lipschitz property: $\|f(t,x)-f(t,y)\| \leq L\,\|x-y\|$,

$\forall\ x,y \in B_r(x_0)$, where $B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x-x_0\| \leq r\}$,

$\forall\ t \in [t_0, t_1]$

[that reads, we can find a Lipschitz constant at every point in the ball $B_r(x_0)$]



$f(t,x)$ is locally Lipschitz over $[t_0, t_1]$

Then: We can find a $\delta > 0$, such that the solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$, exists and is unique on the time interval $[t_0, t_0+\delta] \subset [t_0, t_1]$

subset of

To better understand the physical meaning of the mathematical statement, it helps to go over the mathematical proof! See Appendix C.1. (You can skip this) optional reading

In summary, the key points of the proof are:

- We construct the set $S = \{x \in \mathcal{X} \mid \|x - x_0\|_c \leq r\}$, and we prove that: $\underbrace{x(t) = x_0 + \int_{t_0}^{t_0+\delta} f(s, x(s))\,ds}_{Px(t)}$ is a mapping from S to S, for $\left[\delta \leq \dfrac{r}{Lr+h}\right]$, $\forall\ [t \in t_0, t_0+\delta]$

That means, we prove that $x(t) = Px(t)$ exists in the set S over the time interval $[t_0, t_0+\delta]$

[Existence of $x(t)$ in S means, there is at least one solution $x(t) = Px(t)$ in S]

Then we prove that $Px(t)$ is a contraction over S. That further implies that the solution $x(t) = Px(t)$ is unique over S over the time interval $[t_0, t_0+\delta]$

Finally, we prove that the solution of $x(t) = Px(t)$ is unique in $\mathcal{X} = (C(t_0, t_0+\delta), \|\cdot\|_C)$, over the time interval $[t_0, t_0+\delta]$ To do this we prove that the first time $t_0+\mu$ that the trajectory would leave the set S is greater than $t_0+\delta$. That means, for $[t_0, t_0+\delta]$, the unique solution in S is also a unique solution in $\mathcal{X}$.

optional reading.

In summary, if $f(x,t)$ is Lipschitz locally in x on a ball $B_r(x_0)$ over a time interval $[t_0, t_1]$, then we can guarantee that the ODE $\dot{x} = f(t,x)$, $x(t_0) = x_0$, will have a unique solution over $[t_0, t_0+\delta] \subset [t_0, t_1]$.

Hence Theorem 3.1 is a local theorem in the sense that it guarantees existence and uniqueness over a time interval $[t_0, t_0+\delta]$, where δ might be very small. In other words, we have no control over δ, i.e, we can not guarantee existence a uniqueness over a given interval $[t_0, t_1]$.

Some more remarks.

- We saw that Theorem 3.1 is a local theorem in the sense that it guarantees existence and uniqueness over a time interval $[t_0, t_0+\delta]$, where $\delta>0$ might be very small. In other words, we have no control over δ, i.e, we can not guarantee existence and uniqueness over a given interval $[t_0, t_1]$.
- How can we pursue a larger time interval of E&U?

What we can do is to try to extend the existence and uniqueness interval by repeated applications of the local theorem. That is, take $(t_0+\delta)$ as our new initial time, and $x(t_0+\delta)$ as our new initial condition, and try to apply Theorem 3.1 to ensure existence and uniqueness beyond $t_0+\delta$. In fact, if the conditions of the Theorem are satisfied at $(t_0+\delta, x(t_0+\delta))$, then there exists $\delta_2>0$ such that the original state equation has a unique solution over $[t_0+\delta, t_0+\delta+\delta_2]$ passing through $(t_0+\delta, x(t_0+\delta))$. We can piece together the solutions over $[t_0, t_0+\delta]$, $[t_0+\delta, t_0+\delta+\delta_2]$ to establish the existence of a unique solution over $[t_0, t_0+\delta+\delta_2]$

This idea can be repeated to keep extending the solution.
However! In general the interval of the existence of the solution may not be extended indefinitely!

In fact, there is a maximum interval $[t_0, T)$ where the unique solution starting at $(t_0, x(t_0))$ exists.

Also, in general, T may be less than t_1, which means that as $t \to T$, the solution leaves the compact set over which f is locally Lipschitz in x over $[t_0, t_1]$

Example: Consider $\dot{x} = -x^2$, $x(0) = -1$.

$f(x) = -x^2$. $\frac{\partial f}{\partial x} = f'(x) = -2x$. Hence, since the function is continuously differentiable on $\mathbb{R}$,

that means, the function $f(x)$ is differentiable [i.e, the derivative $f'(x)$ exists] and the derivative $f'(x)$ is a continuous function.

it follows that it is locally Lipschitz on every point on $\mathbb{R}$, i.e. $\forall x \in \mathbb{R}$ [Lemma 3.2]

However, it is not globally Lipschitz since the $\frac{\partial f}{\partial x}$ can not be uniformly bounded on $\mathbb{R}$ [Lemma 3.3]

However, we have that $f(x)$ is locally Lipschitz on any compact (closed and bounded) subset of $\mathbb{R}$.

E.g. Consider $-1 \le x \le 1$. Then $\left|\frac{\partial f}{\partial x}\right| \le 2$.

Hence $f(x)$ Lipschitz on $[-1,1] \subset \mathbb{R}$, $\forall t$.

Let us now obtain the solution as $x(t) = \frac{1}{t-1}$.

The solution exists over $[0,1)$. As $t \to 1$, the solution leaves any compact set, e.g. the set $[-1,1]$ over which $f(x)$ is Lipschitz.

Remark: The example above verifies that the condition of $f(t,x)$ being locally Lipschitz over $[t_0, t_1]$ does not guarantee that the solution $x(t)$ will exist and be unique over $[t_0, t_1]$, i.e, for arbitrary time t_1.

The above problem is alleviated by having $f(t,x)$ satisfy the global Lipschitz condition as stated by the Global Existence and Uniqueness Theorem 3.2 (in our textbook and previous lecture)

Remark We should note that the global Lipschitz property is restrictive.

Many physical systems fail to satisfy it, models of
Still those models do have unique global solutions !!! (See Example 3.5)

Remark It is thus useful to have global existence and uniqueness theorems that require the function $f(t,x)$ to be only locally Lipschitz.

Theorem 3.3 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x, $\forall\, t \geq t_0$ and $\forall\, x \in D \subset \mathbb{R}^n$. Let W be a compact subset of D, $x_0 \in W$, and suppose that it is known that every solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$, lies entirely in W. Then there is a unique solution that is defined $\forall\, t \geq t_0$.

Remark: The trick in applying Theorem 3.3 is in checking that every solution lies in W without actually solving the differential equation. Chapter 4 and Lyapunov's method is very valuable in this regard.