Region of Attraction

Often, it is not enough to merely establish that a system has an asymptotically stable equilibrium.

It is also important to know how far from the equilibrium can the system trajectories start from, and still converge to the equilibrium. In other words,

Let $\dot{x} = f(x)$, $f: D \to \mathbb{R}^n$ locally Lipschitz, and let the origin $x=0$ be the equilibrium. D is a domain $D \subset \mathbb{R}^n$ that contains the origin. We are interested in finding the set

$$\boxed{R_A = \left\{ x_0 \in D \;\middle|\; x(t, x_0) \text{ is defined } \forall\, t \geq 0, \text{ and } \lim_{t\to\infty} x(t, x_0) = 0. \right\}}$$

called, the region of attraction R_A.

We have indeed seen this definition earlier in class.

Remark: We already know that if the conditions of Theorem 4.2 are met, then the origin is globally asymptotically stable. That means, the region of attraction is the entire $\mathbb{R}^n$.

Problem: We want to employ Lyapunov's Theorem 4.1 to obtain estimates of the region of attraction. By estimate we mean a set $\Omega \subset R_A$ such that any trajectory starting in Ω approaches the origin as $t \to \infty$.

Assumptions From Theorem 4.1, we have.

(i) $D \subset \mathbb{R}^n$ an open set containing the origin $x=0$.

(ii) $\dot{x} = f(x)$ locally Lipschitz on D

(iii) $f(0)=0$, i.e. the origin is the equilibrium

(iv) $V: D \to \mathbb{R}$ continuously differentiable, and positive definite on D.

(v) $\dot{V}: D \to \mathbb{R}$ negative definite on D.

Caution! Given that $V(x)$ positive definite on D, and $\dot{V}(x)$ negative definite on D, we might want to jump into the conclusion that D is an estimate of R_A. However this conjecture is not true!

The textbook gives a counter-example

Example where D is not an estimate of RoA!

[Example 8.8] Let

$$\boxed{\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \frac{1}{3}x_1^3 - x_2.\end{aligned}}$$

A Lyapunov function is chosen as

$$V(x) = \frac{1}{2}x^T \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}}_{P>0} x + \int_0^{x_1} \left(y - \frac{1}{3}y^3\right) dy.$$

Remark The Lyapunov candidate function and the system, ~~are~~ generalizations of the pendulum and mass-spring-damper examples; the first term corresponding to "kinetic" energy and the second term to "potential" energy

The time derivative reads $\boxed{\dot{V}(x) = -\frac{1}{2}x_1^2\left(1-\frac{1}{3}x_1^2\right) - \frac{1}{2}x_2^2}$

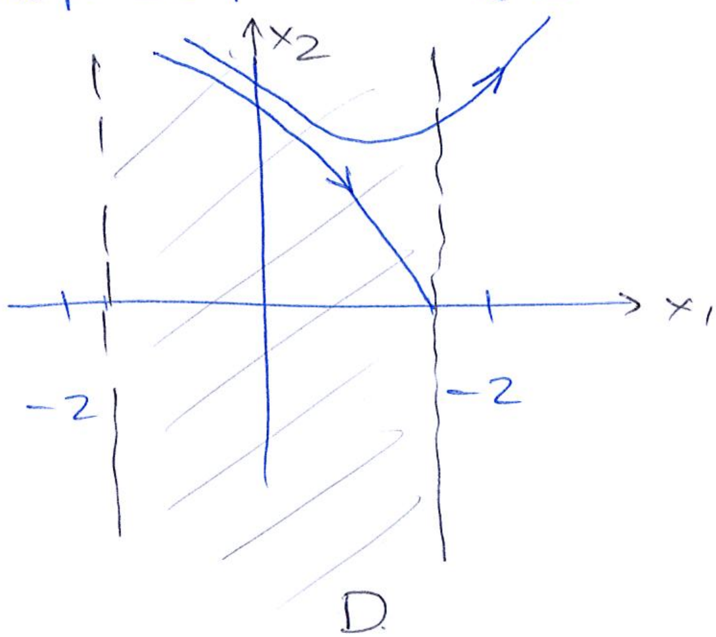
Then, if we define the domain $D = \{x \in \mathbb{R}^2 \mid -\sqrt{3} < x_1 < \sqrt{3}\}$

we have that $V(x)$ is positive definite on D, and

$\dot{V}(x)$ is negative definite on D. Theorem 4.1 concludes

that the origin is asymptotically stable.

However!!! the domain D is not a subset of the region of attraction R_A. This can be verified by inspection of the phase portrait. (Figure 8.3)



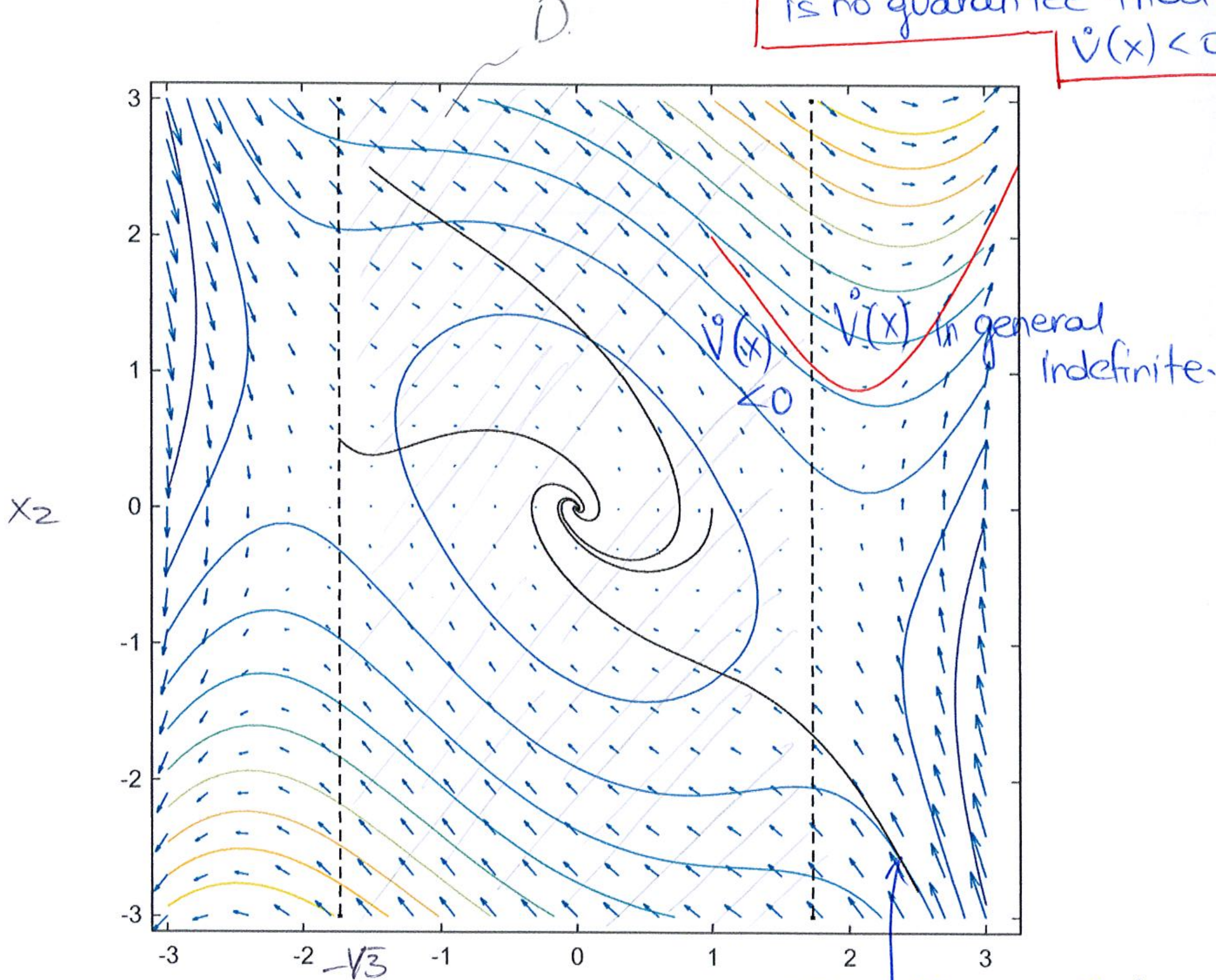
There are trajectories starting in D that will not stay in D, and hence will not approach the origin.

Where is the catch? As we have mentioned earlier in class, the catch lies in the fact that the level sets of V are not necessarily compact for any $c \in \mathbb{R}^+$!

The phase portrait of the system in example 8.8 with the level surfaces of the Lyapunov function used.

We verify that the domain $D = \{x \in \mathbb{R}^2 \mid |x_1| < \sqrt{3}\}$ is <u>not</u> a subset of the region of attraction. For instance, the red trajectory starts in D, but escapes D.

Beyond that point there is no guarantee that $\dot{V}(x) < 0$.



Also, worth to Note that this trajectory does approach the origin, despite starting outside D. What are you suspecting for $\dot{V}(x)$ here?

<u>Remark</u>

In general, the problem is addressed by trying to estimate the region of attraction via <u>compact</u> level sets of $V(x)$, that in addition are positively invariant, so that every trajectory starting in the set stays for ever in the set.

Hence, Problem Objective: We seek $c>0$ such that

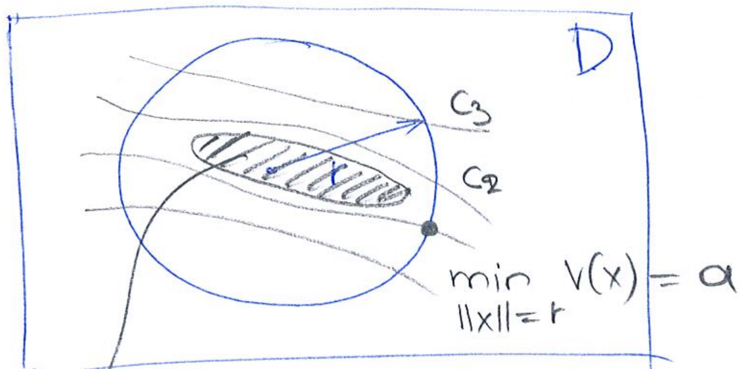
$$\Omega_c = \{x \in D \mid V(x) \leq c\} \text{ is compact}$$

Then, from Theorem 4.1, we have that

$$\forall\, x_0 \in \Omega_c,\ \lim_{t \to \infty} x(t, x_0) = 0.$$

Hence, in other words, $\Omega_c \subset R_A$.

How to pick compact level sets?



Ω_c is compact

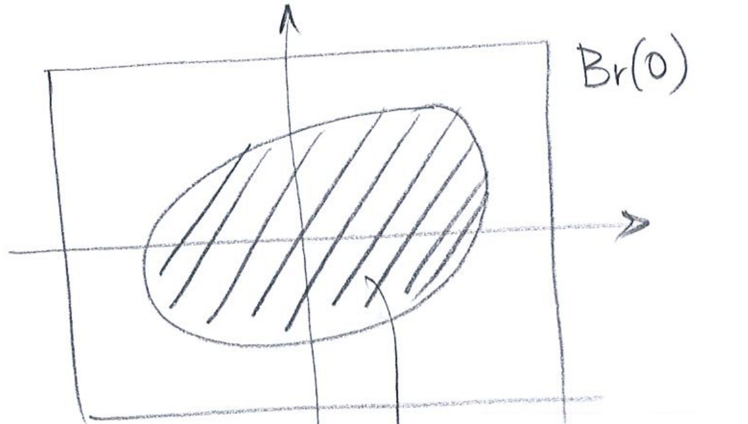
We construct

$$\Omega_c = \{V(x) \leq c\}$$

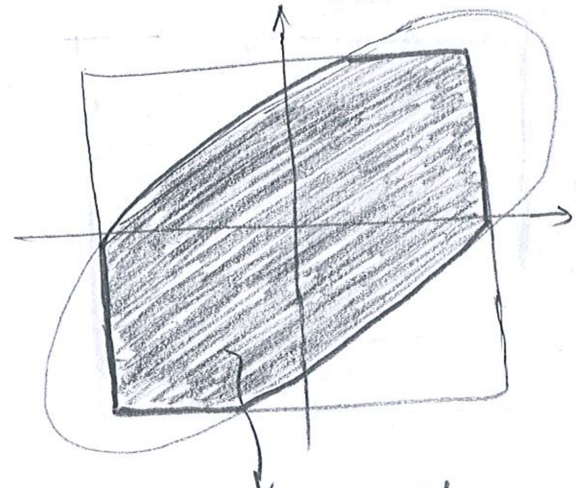
where $c < a = \min_{\|x\|=r} V(x)$.

Then this set is closed because it contains its limit points, and bounded because it is contained in $B_r(0)$.

In general, given any norm on $\mathbb{R}^n$, you can think of compact sets as those that can be contained in $B_r(0)$, such that $\partial\Omega \cap \partial B_r(0) = \emptyset$



Ω_c is compact



NOT compact!

Approach. We will investigate the estimation of the region of attraction using quadratic Lyapunov functions, i.e, functions of the form

$$\boxed{V(x) = x^T P x} \,, \quad P = P^T \text{ (symmetric), and real.}$$

Fact: If P is symmetric and real, then its eigenvalues are real.

Fact: $\lambda_{min}\, x^T x \leq x^T P x \leq \lambda_{max}\, x^T x$, where

λ_{min} and λ_{max} are the minimum and maximum eigenvalues of P, respectively.

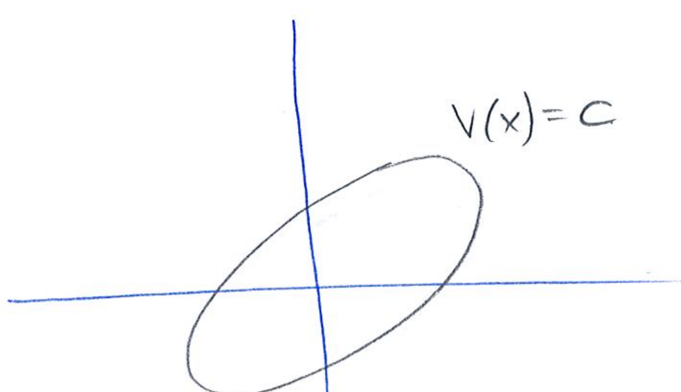
Assumption P is positive definite, $P > 0$.

Then $\lambda_{min} > 0$.

Take $c > 0$ and let us investigate the geometry of the level sets of

$V(x) = x^T P x$.

1



The level surfaces

$\Omega_c := \{x \mid V(x) = c\} =$
$= \{x \mid x^T P x = c\}$

are ellipses. The major axis is aligned with the eigenvector of λ_{min}.

② Now let us consider the level sets

$$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\} = \{x \in \mathbb{R}^n \mid x^T P x \leq c\}$$

For this set to be contained in a ball $B_r(0) := D$, it suffices to pick

$$\boxed{c < \min_{\|x\|=r} (x^T P x) = \lambda_{min} r^2}$$

So this equivalently reads:

ⓘ For a given r, the largest $c > 0$ such that

$\Omega_c = \{x \in \mathbb{R}^n \mid x^T P x \leq c\} \subset \bar{B}_r(0)$ is $c^* = \lambda_{min} r^2$

[Geometrically, this characterizes the largest ellipse that can be contained in a given circle]

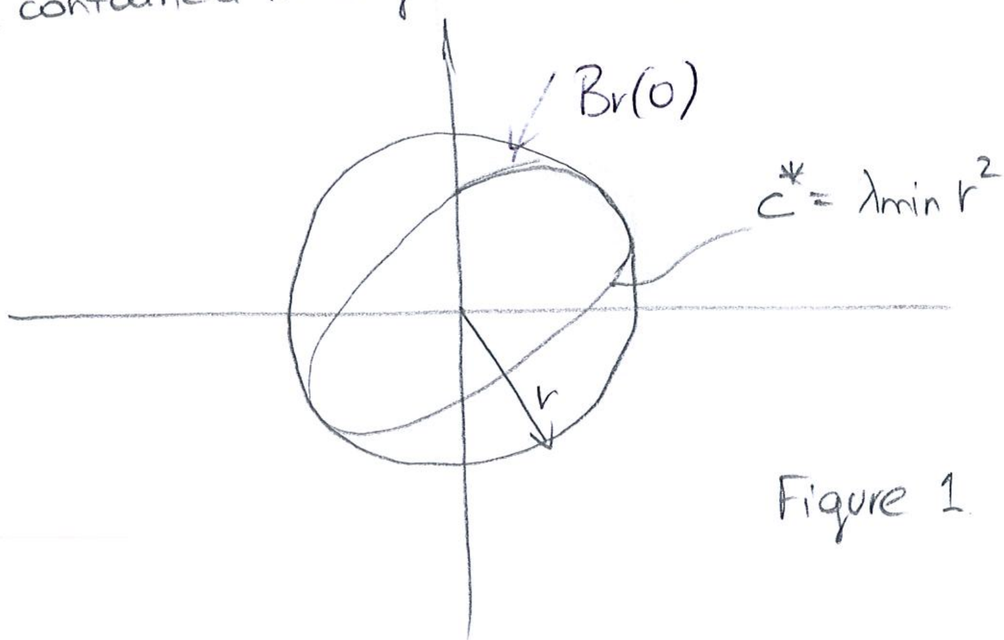


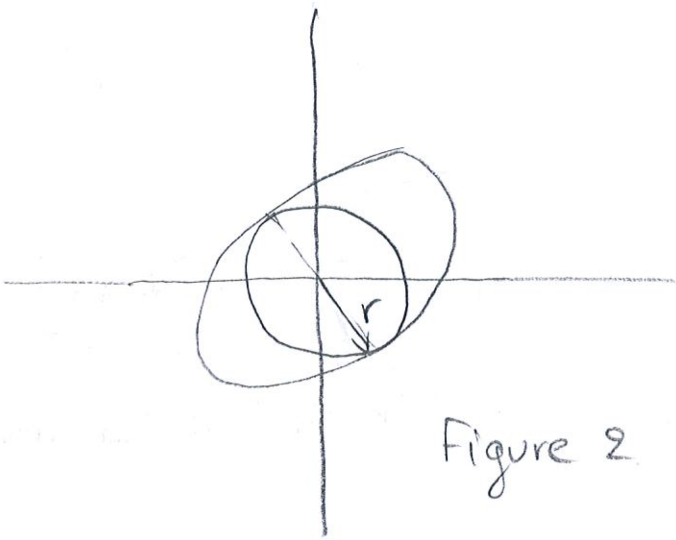
Figure 1

ⓘⓘ Equivalently, for given $c > 0$, the smallest $r > 0$ such that $\Omega_c \subset \bar{B}_r(0)$ is $r^* = \sqrt{\frac{c}{\lambda_{min}}}$

[Geometrically, this characterizes the smallest circle that can contain a given ellipse].

Both (i) and (ii) refer to Figure 1.

(3) We might also want to define a ball $\bar{B}_r(0)$ contained in the level set $\Omega_c = \{x \in \mathbb{R}^n \mid x^T P x \leq c\}$.



For this it suffices to pick

$$c > \max_{\|x\|=r} (x^T P x) = $$
$$= \lambda_{max} r^2$$

Figure 2

(i) That means, for given $r > 0$, the smallest $c > 0$ such that $\bar{B}_r(0) \subset \Omega_c = \{x \in \mathbb{R}^n \mid x^T P x \leq c\}$ is $c_* = \lambda_{max} r^2$

[Geometrically, that characterizes the smallest ellipse that contains a given circle]

(ii) We can also view it as, for given $c > 0$, the largest $r > 0$ such that $\bar{B}_r(0) \subset \Omega_c = \{x \in \mathbb{R}^n \mid x^T P x \leq c\}$ is $r^* = \sqrt{\dfrac{c}{\lambda_{max}}}$

[Geometrically, this is the largest circle contained in a given ellipse]

Both (i) and (ii) correspond to Figure 2.

~~Remark In fact, we know that~~

We will now see how we can use these bounds to estimate the region of attraction through an example.

Recall the Lyapunov equation $A^T P + PA = -Q$, $P > 0$, $Q > 0$ and what it implies for a system whose matrix A is Hurwitz.

Example.

$$\left.\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - (x_1^2 + 1)\, x_2 \end{aligned}\right\} = f(x)$$

We are asked to determine if the origin is asymptotically stable, and if so, to give an estimate of the R_A

Step 1) We resort to linearization method to draw conclusions for the stability of the origin.

$$A = \left.\frac{\partial f}{\partial x}\right|_0 = \left.\begin{bmatrix} 0 & -1 \\ 1 - x_2 \cdot 2x_1 & -(x_1^2+1) \end{bmatrix}\right|_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Compute the eigenvalues as $\det(\lambda I - A) = 0 \Rightarrow$

$\lambda(\lambda+1) + 1 = 0 \Rightarrow \lambda^2 + \lambda + 1 = 0 \Rightarrow$ asymptotically stable

Hence the origin of the nonlinear system is asympt. stable. So now we need a Lyapunov function. for the nonlinear system.

We consider the Lyapunov equation for $Q = I$. We solve

$$A^T P + PA = -I \Rightarrow P = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} > 0$$ [easy to verify through principal minors.]

So we take the Lyapunov function candidate for the nonlinear system as

$$V(x) = x^T P x = \frac{3}{2} x_1^2 - x_1 x_2 + x_2^2 > 0, \quad \forall x \neq 0.$$

The time derivative reads

$$\underline{\dot{V}(x) = -x_1^2 (1 - x_1 x_2) - x_2^2 (1 + 2x_1^2)}$$

We have that $\dot{V}(x) < 0$ if $\underline{1 - x_1 x_2 > 0}$, and $x \neq 0$

We notice that $x_1 x_2 < 1 \Rightarrow |x_1 x_2| < 1$.

Now we also have that $|x_1 x_2| \leq \frac{1}{2} \|x\|_2^2$ (recall HW 2)

Hence if $\frac{1}{2} \|x\|_2^2 < 1 \Rightarrow \underline{\|x\|_2 < \sqrt{2}}$, we also have $|x_1 x_2| < 1$.

Thus, we conclude that $\dot{V}(x)$ is negative definite on

$D = B_{\sqrt{2}}(0)$ in the euclidean norm.

So the problem now reads: Find $c > 0$ such that

$$\Omega_c = \{ x \in \bar{B}_{\sqrt{2}}(0) \mid V(x) \leq c \} \subset B_{\sqrt{2}}(0)$$

In other words, find the largest ellipse contained in

$$\{ x \mid x^T x \leq (\sqrt{2})^2 \}$$

We know that the major axis of the seeked ellipse is aligned with the eigenvector corresponding to $\lambda_{min}(P)$.

Matlab can give us the eigenvalues and eigenvectors

$$[V, D] = \mathrm{eig}(P)$$

In this example we get.

$$V = \begin{bmatrix} -0{,}85 & -0{,}53 \\ 0{,}53 & -0.85 \end{bmatrix} \qquad D = \begin{bmatrix} 1.81 & 0 \\ 0 & 0.691 \end{bmatrix}$$

1) So the <u>largest</u> ellipse contained in $\{x \mid x^T x \leq (\sqrt{2})^2\}$ is the level surface $\{x \in B_{\sqrt{2}}(0) \mid x^T P x = c^*\}$, where

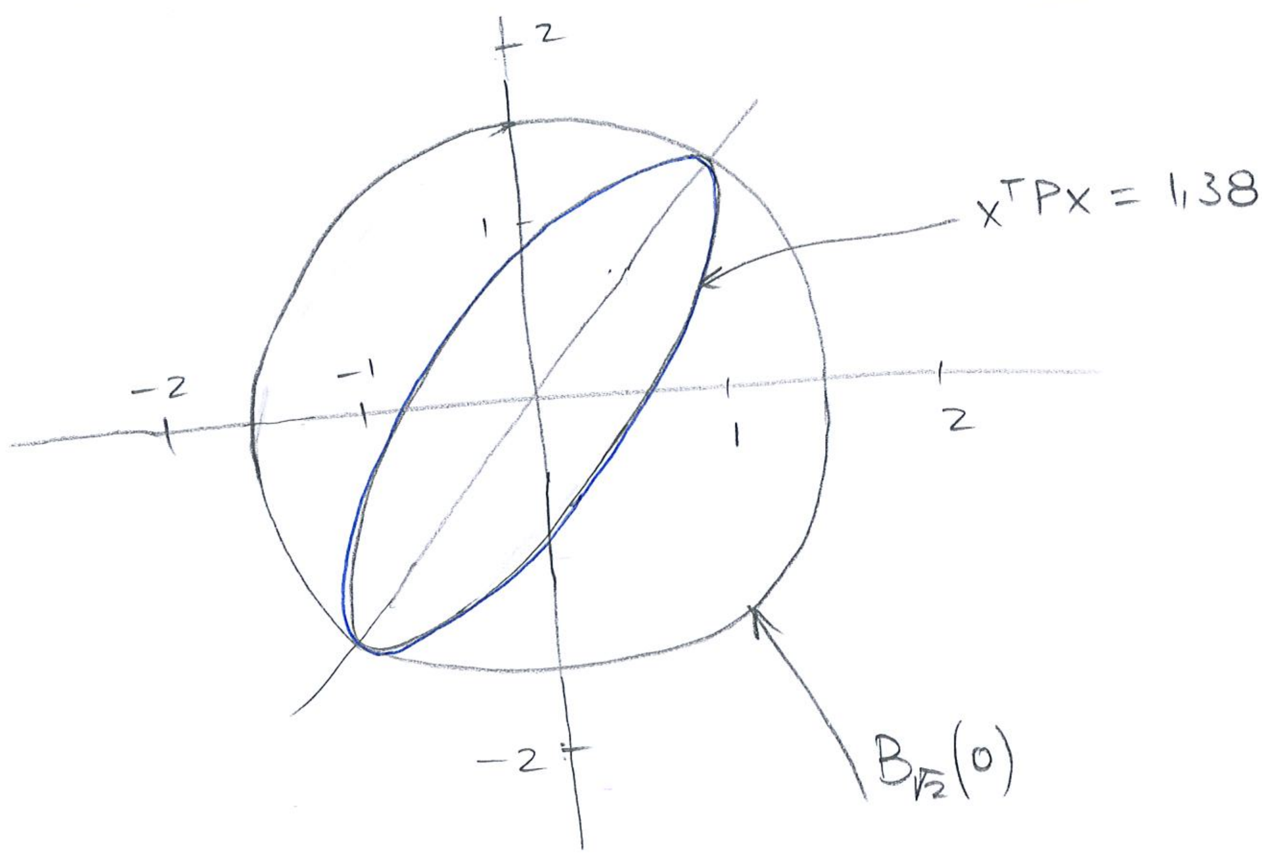
$$\boxed{c^* = r^2 \cdot \lambda_{min} = (\sqrt{2})^2 \cdot 0{,}691 = 1{,}38.}$$

2) That means, $\forall\ 0 < c < c^*$, $\Omega_c = \{x \mid x^T P x = c\} \subset B_r(0)$.

3) Hence Ω_c is compact, and $x \in \Omega_c, x \neq 0$ implies that

$$V(x) > 0, \quad \dot{V}(x) < 0$$

Thus, Ω_c is an estimate of the <u>region of attraction</u>!



Remark. A more conservative estimate is the largest ball contained in Ω_c.

The radius $r^* = \sqrt{\dfrac{c^*}{\lambda_{max}}} = \sqrt{\dfrac{1{,}38}{1{,}81}} \simeq 0{,}873$

Alternative solution. We can attempt to enlarge our estimate of the region of attraction

Notice that we had $\dot{V}(x) = -x_1^2 + x_1^2 x_1 x_2 - x_2^2(1+2x_1^2)$

$$= -[x_1 \; x_2] \underbrace{\begin{bmatrix} 1 & -\frac{x_1^2}{2} \\ -\frac{x_1^2}{2} & 1+2x_1^2 \end{bmatrix}}_{M(x)} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then if $M(x) > 0$, we have $\dot{V}(x) < 0$.

For $M(x) > 0$, $\iff$ $1 + 2x_1^2 - \dfrac{x_1^4}{4} > 0 \iff$

$$-x_1^4 + 8x_1^2 + 4 > 0.$$

We consider $\mu = x_1^2$, then the solutions of $\boxed{-\mu^2 + 8\mu + 4 = 0}$

are $\mu_{1,2} = 4 \pm 2\sqrt{5} \Rightarrow \mu_1 \simeq -0{,}4721$, $\mu_2 \cong 8{,}4721$.

Hence, we take $x_1^2 = 8{,}4721 \iff M(x) = 0$, which implies that at $x_1 = \pm\sqrt{8.4721}$ are points were $M(x)$ is no longer positive definite.

We conclude that $\boxed{\dot{V}(x) < 0 \text{ for } x_1^2 < 8.472, \ x \neq 0.}$

We now note that $\|x\|_2 < \underbrace{\sqrt{8.472}}_{2.91} \Rightarrow x_1^2 < 8.472$

Hence we pick the ball $D = B_{2.91}(0)$ in Euclidean norm.

Then we have $\begin{cases} V: D \to \mathbb{R} \text{ such that } V(x) > 0, \ x \neq 0, \ x \in D. \\ \dot{V}: D \to \mathbb{R} \text{ such that } \dot{V}(x) < 0, \ x \neq 0, \ x \in D. \end{cases}$

We now can approximate the region of <u>attraction</u> as the largest ellipse contained in $\bar{B}_{2.91}(0) = \{x \mid x^T x \leq 2.91^2\}$

We have:

$c^* = \lambda_{min} r^2 = 0.691 \cdot 2.91^2 = 5.85$

$$\boxed{\text{Thus, } \forall \ 0 < c < c^* = 5.85 \text{ we have } \Omega_c = \{x \mid x^T P x \leq c\} \subset B_{2.91}(0)}$$

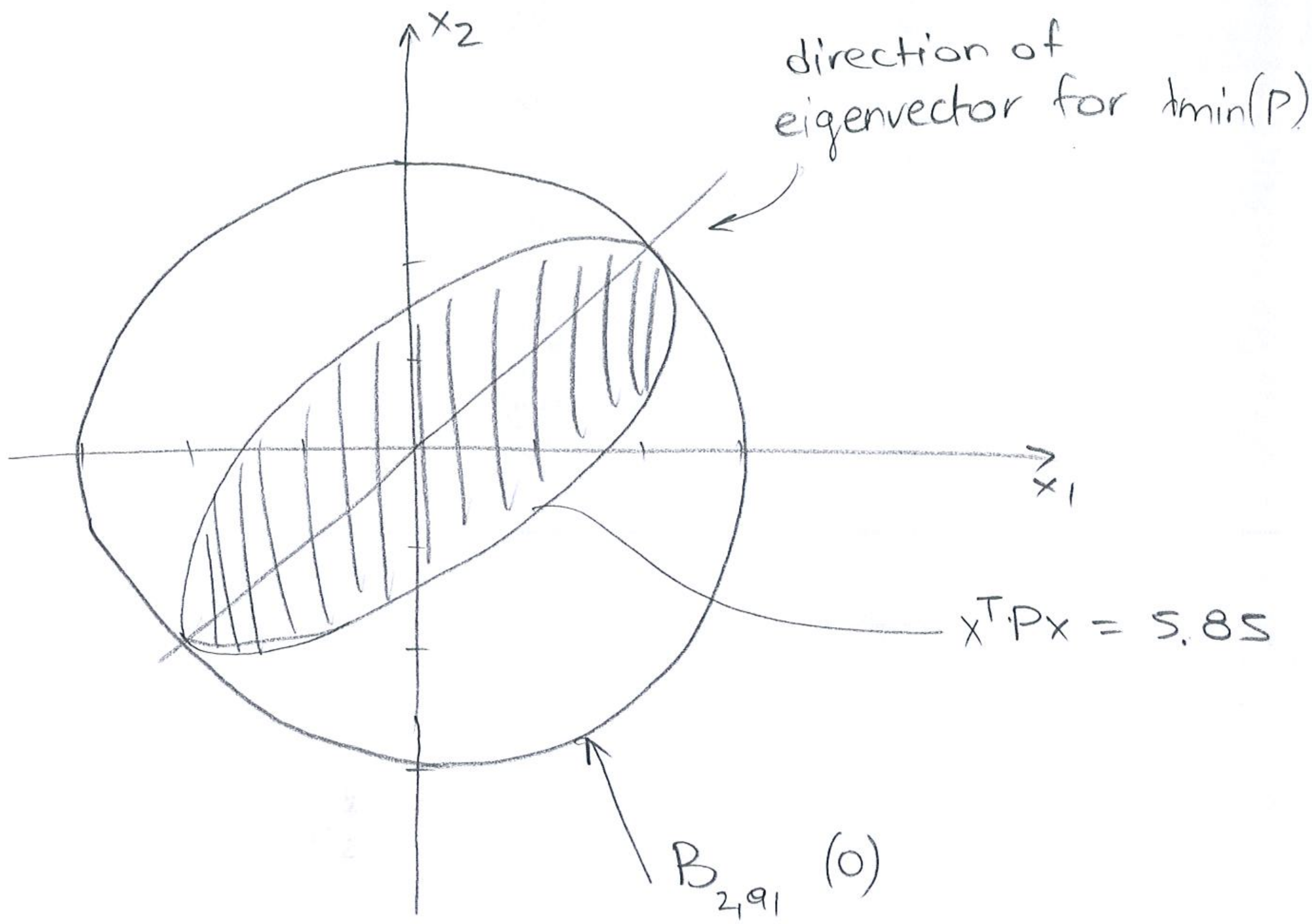
$\Rightarrow \Omega_c$ is compact; in addition for all $x \in \Omega_c$ we have $\dot{V}(x) < 0, \ V(x) > 0, \ x \neq 0$.

$\Rightarrow \Omega_c$ is an <u>estimate</u> of the region of attraction.

<u>Remark</u>. The estimate we obtained here is much bigger than the previous estimate.

The improvement came from finding a larger set where $\dot{V}(x) < 0$.

How does it look like?



Remark The fact that we obtained different results in the two approaches is that we only obtain estimates of the region of attraction.

The RoA of the equilibrium point is the same in both cases, but in the first case we found a smaller region where $\dot{V} < 0$, compared to the second one. If we tried with a different Lyapunov function (i.e, different Q which would give a different P out of the Lyapunov equation) we would get a different result as well.