Exponential Stability. (of time-invariant and time-varying systems)

In general, there are also other types of stability beyond stability in the sense of Lyapunov. Some of them are:

- Exponential stability
- Stability under permanently acting perturbations
- Stability of non-trivial trajectories
- Orbital stability
- Lagrange stability
- Input - Output stability.

Today, we will focus on exponential stability. We will consider the cases of time-invariant and time-varying systems. The key points to focus on:

- The definition of exponential stability, and how it relates to asymptotic stability i.s.L.
- The characterization of exponential stability via Lyapunov functions.
- For time-invariant systems, the linearization of a system about an equilibrium point provides necessary and sufficient conditions for local exponential stability.

Definition. Let the time-varying system

$$\dot{x} = f(t,x), \text{ where } f : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n.$$

The point $x_e \in \mathbb{R}^n$ is an equilibrium point if

$$f(t, x_e) = 0, \quad \forall\, t \geq 0.$$

Definition. (a) The equilibrium point $x_e = 0$ of $\dot{x} = f(t,x)$ is exponentially stable if,

$\forall\, t_0 \geq 0$, $\exists\, \sigma > 0$, $\gamma > 0$ and $N < \infty$, such that

(i) $\forall\, x_0 \in B_\sigma(0)$, the solution $x(t, t_0, x_0)$ exists on $[t_0, \infty)$ and is unique.

(ii) $\forall\, x_0 \in B_\sigma(0)$ and $t \geq t_0$,

$$\| x(t, t_0, x_0) \| \leq N \, \| x(t_0) \| \, e^{-\gamma (t - t_0)}$$

(b) The equilibrium point $x_e = 0$ is uniformly exponentially stable if σ, γ, N can be chosen independent of t_0.

Remarks

- If $x_e = 0$ is exponentially stable, then there exists $\mu > 0$ such that

$$\lim_{t \to \infty} e^{\mu (t - t_0)} \, x(t, t_0, x_0) = 0.$$

(To verify why, take $0 < \mu < \gamma$ in the definition of exponential stability)

- In general, (uniform) asymptotic stability $\not\Rightarrow$ (uniform) exponential stability.

Indeed, consider the example: $\dot{x} = -x^3$.

The solution is $\boxed{x(t, t_0, x_0) = x_0^2 \sqrt{\frac{1}{1+2(t-t_0)x_0^2}}}$

However, for every $\mu > 0$ and $x_0 \neq 0$, we have:

$$\lim_{t \to \infty} e^{\mu(t-t_0)} |x(t, t_0, x_0)| = \infty$$

[To see why, apply L'Hospital Rule]

Hence, the equilibrium can not be exponentially stable.

- Consider the linear time-invariant system $\dot{x} = Ax$, where A is a real constant matrix. Then the origin is asymptotically stable $\Leftrightarrow$ the origin is exponentially stable.

Indeed, $-\gamma = \max_{1 \leq i \leq n} Re(\lambda_i(A))$

(to be proved later)

With the formal definition of exponential stability at hand, we can proceed now to the characterization of exponential stability by means of Lyapunov functions.

Theorem. (Stated similarly as Theorem 4.10 in the textbook)

Suppose that $x_e = 0$ is an equilibrium of $\dot{x} = f(t,x)$, and that $f : [0,+\infty) \times D \to \mathbb{R}^n$ is locally Lipschitz in x on $[0,+\infty) \times D$, piecewise continuous in t, and that D contains the origin $x_e = 0$.

Then $x_e = 0$ is uniformly exponentially stable if there exists a continuously differentiable function $V : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ and constants $a, \beta, \mu, r > 0$ such that for every $x \in B_r(0)$ and $t \geq 0$,

(i) $a\, x^T x \leq V(t,x) \leq \beta x^T x$

(ii) $\dot{V}(t,x) \leq -\mu V(t,x)$

Proof. Uniform asymptotic stability of the origin is immediate because from (i), $V(t,x)$ is locally positive definite and decrescent, and from (ii), $\dot{V}(t,x)$ is locally negative definite. Hence, all we have to show is that there is an exponential bound on the convergence rate to the origin.

From (ii) we have $\mathring{V}(t,x) \leq -\mu V(t,x)$

From the Comparison Lemma 3.4 in our textbook,

$$\boxed{V(t, x(t, t_0, x_0)) \leq V(t_0, x_0)\, e^{-\mu(t-t_0)}} \quad \text{(iii)}$$

Denote $x(t) = x(t, t_0, x_0)$, then from (i)

$$\alpha \|x(t)\|_2^2 \overset{\text{(i)}}{\leq} V(t, x(t)) \overset{\text{(iii)}}{\leq} V(t_0, x_0)\, e^{-\mu(t-t_0)} \overset{\text{(i)}}{\leq}$$

$$\beta \|x_0\|_2^2\, e^{-\mu(t-t_0)}, \text{ i.e.}$$

$$\alpha \|x(t)\|_2^2 \leq \beta \|x_0\|_2^2\, e^{-\mu(t-t_0)}, \Rightarrow$$

$$\|x(t)\|_2^2 \leq \frac{\beta}{\alpha} \|x_0\|_2^2\, e^{-\mu(t-t_0)} \Rightarrow$$

$$\|x(t)\|_2 \leq \underbrace{\sqrt{\frac{\beta}{\alpha}}}_{N} \|x_0\|\, e^{-\underbrace{\frac{\mu}{2}}_{\gamma}(t-t_0)}$$

which proves that the origin is uniformly exponentially stable.

Our next and final objective is to prove that a time-invariant nonlinear system is exponentially stable if, and only if, its linearization has all of its e-values in the open left-half plane. The following turns out to be the key lemma for proving this fact.

Lemma: Consider the system

$$\dot{x} = Mx + e^{\beta t} g(e^{-\beta t} x) \quad t \geq 0$$

where $\beta > 0$, $g(0) = 0$, and $\frac{\|g(x)\|_2}{\|x\|_2} \xrightarrow[\|x\|_2 \to 0]{} 0$. Suppose that M has at least one e-value in the open right half plane and no e-values on the imaginary axis. Then the origin is unstable.

Proof. Without loses of generality, suppose that $M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$ where $Re\ \lambda(M_{11}) < 0$ and $Re\ \lambda(M_{22}) > 0$. As in a previous proof when proving instability, $\exists P_{11} < 0$ and $P_{22} > 0$ such that $P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$ satisfies

$$M^T P + PM = I$$

Define $V(x) = x^T Px$. Then $\forall \sigma > 0$, $\exists x \in B_\sigma(0)$ such that $V(x) > 0$ [V takes on position values arbitrarily near the origin]. A straightforward computation yields

$$\boxed{\dot{V} = x^T x + 2x^T P e^{\beta t} g(e^{-\beta t} x)}$$

Observations:

a) $\frac{\|g(x)\|_2}{\|x\|_2} \xrightarrow[\|x\|_2 \to 0]{} 0$ means that $\forall \eta > 0, \exists\ \delta > 0$ such that $\forall \bar{x} \in B_\delta(0)$, $\|g(\bar{x})\|_2 \leq \eta \|\bar{x}\|_2$

b) $x \in B_\delta(0) \Rightarrow e^{-\beta t} x \in B_\delta(0)\ \forall t \geq 0$

c) Combining (a) and (b), we have that $\forall \eta > 0, \exists \delta > 0$ such that

$$x \in B_\delta(0) \Rightarrow \|g(e^{-\beta t} x)\|_2 \leq \eta \|e^{-\beta t} x\|_2 = \eta e^{-\beta t} \|x\|_2$$

and thus multipying both sides by $e^{\beta t}$ yields

$$e^{\beta t} \|g(e^{-\beta t} x)\|_2 \leq \eta \|x\|_2$$

d) Using the general fact $|x^T Py| \leq ||x||_2 ||P||_i ||y||_2$, we have

$$|x^T Pe^{\beta t} g(e^{-\beta t} x)| \leq \eta \lambda_{\max}(P) \cdot ||x||_2^2$$

e) If we choose $\eta > 0$ such that

$$\boxed{1 - 2\eta\lambda_{\max}(P) > 0}$$

then,

$$\begin{aligned}
\dot{V}(x) &= x^T x + 2x^T Pe^{\beta t} g(e^{-\beta t} x) \\
&\geq x^T x - 2|x^T Pe^{\beta t} g(e^{-\beta t} x)| \\
&\geq x^T x - 2\eta\lambda_{\max}(P)||x||_2^2 \\
&\geq (1 - 2\eta\lambda_{\max}(P))||x||_2^2
\end{aligned}$$

which shows that $\dot{V}(x)$ is loc. pos def on $B_\delta(0)$. It follows that $x_e = 0$ is unstable.

Exponential Stability and Linearitations

Theorem Consider a time-invariant, continuously differentiable system $\dot{x} = f(x)$ such that $f(0) = 0$. Write $f(x) = Ax + g(x)$ where $\frac{||g(x)||}{||x||} \xrightarrow[||x|| \to 0]{} 0$. Then the $x_e = 0$ is exponential stable $\Leftrightarrow$ A is Hurwitz, that is, if and only if all of the e-values of A have negative real parts.

Proof:

(Sufficiency): Exercise: construct a quadratic Lyapunov function and apply the theorem given at the beginning of this handout.

(Necessity): Suppose the origin is an exponentially stable equilibrium point, that is, $\exists r > 0$, $N < \infty$, and $\alpha > 0$ such that

$$||x(t, t_0, x_0)|| \leq N||x_0||e^{-\alpha(t-t_0)} \quad \forall x \in B_r(0) \quad (*)$$

In particular then, the origin is (unif) asymp. stable and hence the e-values of A are in the closed left-half plane.

To show: e-values of A are in the open left-half plane. We will actually prove that $Re\ \lambda(A) \leq -\alpha$ for $1 \leq i \leq n$, where α is given in (*).

For the moment, let $0 < \beta < \alpha$ be arbitrary. Define $z(t) = e^{\beta t}x(t)$. Then $||z(t)|| = e^{\beta t}||x(t)|| \leq e^{\beta t}Ne^{-\alpha t}||x_0|| \xrightarrow[t\to\infty]{} 0$ because $\beta - \alpha < 0$. Let's find the differential equation satisfied by $z(t)$.

$$\begin{aligned}\frac{d}{dt}z(t) &= \beta e^{\beta t}x(t) + e^{\beta t}\dot{x}(t)\\ &= \beta e^{\beta t}x(t) + e^{\beta t}Ax(t) + e^{\beta t}g(x(t))\\ &= (A + \beta I)z(t) + e^{\beta t}g(e^{-\beta t}z(t))\end{aligned}$$

That is, $z(t)$ is the solution of the time-varying ODE

$$\boxed{\dot{z} = (A + \beta I)z + e^{\beta t}g(e^{-\beta t}z)}$$

Suppose it is not true that all of the e-value of A have real part less than or equal to $-\alpha$. Then then exists some i such that $Re(\lambda_i(A)) > -\alpha$. Choose $0 < \beta < \alpha$ such that $A + \beta I$ has at least one e-value with a positive real part, and no e-value on the imaginary axis. This is always possible because $\lambda_i(A + \beta I) = \lambda_i(A) + \beta$. Hence,

$$\dot{z} = (A + \beta I)z + e^{\beta t}g(e^{-\beta t}z)$$

is unstable by our lemma. But we previously showed that

$$||z(t)|| \leq Ne^{(-\alpha+\beta)t}||z_0|| \xrightarrow[t\to\infty]{} 0$$

which implies that $z_e = 0$ is asymptotically stable.

$\therefore$ We have a contradiction, and thus it could not have been the case that $\exists i$ s.t. $Re(\lambda_i(A)) > -\alpha$, and the result is proven.

Basis for 95% of Control Design

(an exaggeration, but not much of one)

Consider a nonlinear system $\Sigma : \ \dot{x} = f(x,u)$, with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Assume that f is continuously differentiable and $f(0,0) = 0$

Lemma: There exists a continuously differentiable state variable feedback $u = \alpha(x)$, with $\alpha(0) = 0$, rendering the origin an exponentially stable equilibrium point of the closed-loop system $\dot{x} = f(x, \alpha(x))$ if, and only if, the pair (A, B) is stabilizable, where $A = \frac{\partial f}{\partial x}(0,0)$ and $B = \frac{\partial f}{\partial u}(0,0)$.

Proof. Exercise:

Remark: Let $K = \frac{\partial \alpha}{\partial x}(0)$. Then α exponentially stabilizes Σ, if and only if, $Re\lambda_i(A + BK) < 0$