A summary of (Integrator) Backstepping

see also 2nd half of lecture notes.

- Nonlinear control design method
  - Systematic (i.e, constructive) method
  - Recursive procedure
  - Appropriate for stabilization, tracking, robust control problems

- Key idea: Break down the control problem of the full system into a sequence of control problems for simpler systems.

- Avoids cancellation of useful nonlinearities (unlike feedback linearization)

Approach (in a nutshell)

$$(\Sigma)\begin{cases} \dot{x} = f(x) + g(x)\,\xi & (1.1), \\ \dot{\xi} = u & (1.2) \end{cases}$$

where the state vector is $\begin{bmatrix} x \\ \xi \end{bmatrix} \in \mathbb{R}^{n+1}$

- We consider the state variable ξ as a virtual control input.

- Find a stabilizing feedback control law →

$\boxed{\xi = \phi(x)}$ for eq. (1.1) so that the origin $x = 0$

$\overset{\circ}{x} = f(x) + g(x)\phi(x)$ is asymptotically stable.

- Add and subtract the term $g(x)\phi(x)$ at the right-hand side of (1.1) to obtain

$$\begin{cases} \overset{\circ}{x} = f(x) + g(x)\phi(x) + g(x)\left(\xi - \phi(x)\right) & (2.1) \\ \overset{\circ}{\xi} = u. & (2.2) \end{cases}$$

- Consider the change of variables $\boxed{z = \xi - \phi(x)}$ to further rewrite the system as

$$\begin{cases} \overset{\circ}{x} = f(x) + g(x)\phi(x) + g(x)z & (3.1) \\ \overset{\circ}{z} = \underbrace{u - \overset{\circ}{\phi}(x)}_{v} & (3.2) \end{cases}$$

- The form (3.1-3.2) is similar to the original one (1.1-1.2), with one key difference:

The origin $x = 0$ of system (3.1) is now asymptotically stable when the virtual input z is zero !!!

- Hence the control / stabilization problem reduces to designing the control input v so that it stabilizes the origin of the overall system.

Application Example.

Let
$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u \end{cases}$$

Design a stabilizing state feedback control law that renders the origin $(x_1, x_2) = (0,0)$ (globally) asymptotically stable.

- Approach. We will apply backstepping.

  Note $f(x) = x_1^2$

  $g(x) = 1$

- Let x_2 be treated as a virtual control input for the first equation.

We want to design a control law $x_2 = \phi(x_1)$ to stabilize x_1 to the origin.

Let $\boxed{x_2 = -x_1^2 - x_1}$ the "virtual control law."

Take
$$V(x_1) = \frac{1}{2} x_1^2 \Rightarrow$$

$$\dot{V}(x_1) = x_1(x_1^2 + x_2) = x_1(x_1^2 - x_1^2 - x_1) = -x_1^2 < 0$$

which reads, $x_1 = 0$ is asymptotically stable.

- Now rewrite the system as

$$\dot{x}_1 = x_1^2 + x_2 - (-x_1^2 - x_1) + \underbrace{(-x_1^2 - x_1)}_{g(x)\phi(x)} =$$
$$= -x_1 + \underbrace{x_2 - (-x_1^2 - x_1)}_{\xi - \phi(x)}$$
$$\dot{x}_2 = u.$$

- Let $z = \xi - \phi(x) = x_2 + x_1^2 + x_1$

Then $\dot{z} = \dot{x}_2 - \frac{\partial \phi(x_1)}{\partial x_1} \dot{x}_1 = u + (2x_1 + 1)(-x_1 + z)$

Then we have.

$$\begin{cases} \dot{x}_1 = -x_1 + z \\ \dot{z} = \underbrace{u + (2x_1+1)(-x_1+z)}_{v} \end{cases}$$

- Now we design v to stabilize the overall system.

$$V_1(x_1, z) = V(x_1) + \frac{1}{2}z^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$$

$$\dot{V}_1(x_1, z) = x_1\dot{x}_1 + z\dot{z} = x_1(-x_1+z) + z\dot{z} =$$

$$= -x_1^2 + x_1 z + z v =$$

$$= -x_1^2 + z(x_1+v)$$

- Choose $\boxed{v = -x_1 + kz, \quad k>0}$ then

$$\dot{V}_1(x_1, z) = -x_1^2 - kz^2 < 0$$

Hence the control law

$\boxed{u = v - (2x_1+1)(-x_1+z)}$ where $v = -x_1 - kz$, $k>0$

renders the origin of the overall system G.A.S.

$$\boxed{u = -x_1 - k(x_2 + x_1^2 + x_1) - (2x_1+1)(x_2+x_1^2), \quad k>0}$$

Another view of the Application Example.

[Eas~~y~~ier way of performing the coordinate transformation]

Let

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = u \end{cases}, \quad \text{where } f(x) = x_1^2, \; g(x) = 1.$$

① Define the new variable $\boxed{\xi_1 = x_1}$

Then $\dot{\xi}_1 = \dot{x}_1 = x_1^2 + x_2 = -\xi_1 + \underbrace{\xi_1 + x_1^2}_{a_1(\xi_1)} + x_2.$

② Define now the variable $\boxed{\xi_2 = x_2 + a_1(\xi_1)}$

so that the first equation is rewritten

$$\dot{\xi}_1 = -\xi_1 + \xi_2.$$

and the second equation reads

$$\dot{\xi}_2 = \dot{x}_2 + \frac{\partial a_1(\xi_1)}{\partial \xi_1} \dot{\xi}_1$$

$$= u + (1 + 2\xi_1)(-\xi_1 + \xi_2) \Rightarrow$$

$$\dot{\xi}_2 = -\xi_2 + u + (1+2\xi_1)(-\xi_1+\xi_2) + \xi_2$$

③ In summary, we have

$$\begin{cases} \dot{\xi}_1 = -\xi_1 + \xi_2 \\ \dot{\xi}_2 = -\xi_2 + u + (1+2\xi_1)(-\xi_1+\xi_2) + \xi_2. \end{cases}$$

④ So now under the feedback control law

$$u = -(1+2\xi_1)(-\xi_1+\xi_2) - \xi_2 =$$

$$= -3x_1^3 - 2x_1^2 - 2x_1x_2 - 2x_1 - x_1$$

We have that the closed-loop system is rewritten as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},$$

and since the state matrix is Hurwitz, we have.

$$\lim_{t\to\infty} \xi_1(t) = 0 \Rightarrow x_1(t) \xrightarrow[t\to\infty]{} 0$$

$$\lim_{t\to\infty} \xi_2(t) = 0 \Rightarrow x_2(t) \xrightarrow[t\to\infty]{} 0.$$

Recursive Feedback Design - Backstepping

From our textbook → Chapter 14.3, page 589.

Today we will learn a powerful control design technique, called "backstepping". We will consider, similarly to last time, a single-input, control affine nonlinear system of the form

$$\boxed{\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R},}$$

where $f(0) = 0$, and f and g are locally Lipschitz continuous.

Let us suppose that there exists a continuously differentiable, feedback control law $\boxed{u = a(x), \ a(0) = 0,}$ and a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$, such that

$$\forall x \in \mathbb{R}^n, \quad L_f V(x) + [L_g V(x)]\, a(x) \leq -W(x),$$

where $W(x)$ is positive semi-definite.

[Notation] We denote $L_f V(x) = \frac{\partial V}{\partial x}(x) \cdot f(x)$

and $L_g V(x) = \frac{\partial V}{\partial x}(x) \cdot g(x)$.

What can we conclude under the stated assumptions?

Well, we have that the origin $x=0$ is a stable equilibrium of the closed-loop system

$$\dot{x} = f(x) + g(x)\,a(x),$$

we also have that all solutions exist on $[0, +\infty)$ and are globally bounded, and that

$$\lim_{t \to \infty} W(x(t, x_0)) = 0.$$

[Concept question: Does the above condition imply that the solution $x(t, x_0) \xrightarrow[t\to\infty]{} 0$?]

In addition, we have that <u>if</u> the only solution of

$$\dot{x} = f(x) + g(x)\,a(x)$$

which can stay entirely (identically) in the set $Z := \{x \in \mathbb{R}^n \mid W(x) = 0\}$ is the trivial trajectory $x(t) \equiv 0$, then the origin is globally asymptotically stable (GAS)

Obviously, we have that if $W(x)$ is positive definite, then the origin is globally asymptotically stable. (GAS). [Why?]

Under these assumptions, we will state the following "integrator backstepping" technique, which at the same time guides the control design* in the case of the considered systems.

* that is, the finding of $u = a(x)$.

Integrator Backstepping

Suppose the above assumptions hold for the system

$$\boxed{(\Sigma) \quad \dot{x} = f(x) + g(x)u.}$$

The name "integrator backstepping" technique originates from the idea that we can consider $\xi = a(x)$ as a "virtual" control input, and cast the control design of stabilizing the origin $x=0$ of the system (Σ) to a problem of stabilizing the control input u to the "virtual" control input $a(x)$. To this end, we augment the original system with an integrator as

$$\boxed{(\Sigma_a) \quad \begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u. \end{aligned}}$$

Now for the system (Σ_a), we consider

$$V_a(x, \xi) = V(x) + \frac{1}{2}\left(\xi - a(x)\right)^2,$$ where $V(x)$ is positive definite and radially unbounded.

Then, along the system trajectories,

$$\dot{V}_a(x,\xi) = \dot{V}(x) + \big(\xi - a(x)\big)\big(\dot{\xi} - \dot{a}(x)\big),$$

where $\dot{a}(x) = \frac{\partial a}{\partial x}(x)\,\big(f(x) + g(x)\,\xi\big)$

$\dot{\xi} = u$

$\dot{V}(x) = \frac{\partial V}{\partial x}(x)\big(f(x) + g(x)\,\xi\big)$

all from system dynamics.

Denote (for compactness):

$$\dot{a}(x) = \frac{\partial a}{\partial x}(x)\,\big(f(x) + g(x)\,\xi\big) := L_{f+g\xi}\, a(x)$$

Then, under all the previously stated conditions, we have that the control law $u = q_a(x,\xi)$, given as:

$$\boxed{u = -c\big(\xi - a(x)\big) + L_{f+g\xi}\, a(x) - L_g V(x), \quad c>0} \qquad (C.1)$$

renders the origin $x_e = 0$, $\xi_e = 0$ GAS.

Proof: Let us consider the error variable

$$z = \xi - a(x)$$

and rewrite the system (Σ_a) in the $\begin{bmatrix} x \\ z \end{bmatrix}$ coordinates $\rightarrow$

$$(\Sigma_\beta)\begin{cases} \dot{x} = f(x) + g(x)\,(a(x)+z) \\ \dot{z} = \dot{\xi} - \dot{a} = u - \dfrac{\partial a}{\partial x}(x)\,\dot{x} \\ \quad = u - \dfrac{\partial a}{\partial x}(x)\left\{f(x)+g(x)\,(a(x)+z)\right\} \end{cases}$$

Differentiating $V_a(x,z) = V(x) + \frac{1}{2}z^2$

along the trajectories of $(\Sigma_\beta.)$ yields.

$$\dot{V}_a(x,z) = \frac{\partial V}{\partial x}(x)\left\{f(x)+g(x)\left[a(x)+z\right]\right\} +$$

$$+ z\left(u - \frac{\partial a}{\partial x}(x)\left\{f(x)+g(x)\,(a(x)+z)\right\}\right) =$$

$$= \underbrace{\frac{\partial V}{\partial x}(x)\left\{f(x)+g(x)\,a(x)\right\}}_{\substack{\text{from the assumption} \\ \leq -W(x)}} + z\left(u - \frac{\partial a}{\partial x}(x)\left\{f(x)+g(x)\,(a(x)+z)\right\} + \frac{\partial V}{\partial x}(x)\,g(x)\right)$$

$$\leq -W(x) + z\Big(u - \underbrace{\frac{\partial a}{\partial x}(x)\left\{f(x)+g(x)(a(x)+z)\right\}}_{\text{denoted } L_{f+g\xi}\, a(x)} + \underbrace{\frac{\partial V}{\partial x}(x)\,g(x)}_{\substack{\text{denoted} \\ L_g V(x)}}\Big)$$

Then, under the control law $\boxed{C.1}$

we get: $\boxed{\dot{V}_a(x,z) \leq -W(x) - cz^2}$

Hence, we verified that the control law $u = q_a(x,z)$ given by (C.1) is one choice of control that renders $\mathring{V}_a(x,z)$ negative semi-definite.

In addition, if $W(x)$ is positive definite, then

$\mathring{V}_a$ is negative definite.

Note The control law given by (C.1) is just one of many controls that renders $\mathring{V}_a$ negative semi definite.

Note If W is positive definite, we can use Sontag's formula as well to obtain that $\mathring{V}_a$ is negative definite.

Post-lecture note : Today in class we started from the first part of the lecture (in a nutshell) and the example.

Suggestion: Read the example in parallel to the generic formulation (second part of lecture notes) and check old handouts for typed notes!

Strict Feedback Systems

Definition A system that can be expressed in the form

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)\xi_1 \\
\dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \\
\dot{\xi}_2 &= f_2(x,\xi_1,\xi_2) + g_2(x,\xi_1,\xi_2)\xi_3 \\
&\vdots \\
\dot{\xi}_{k-1} &= f_{k-1}(x,\xi_1,\xi_2,\ldots,\xi_{k-1}) + g_{k-1}(x,\xi_1,\xi_2,\ldots,\xi_{k-1})\xi_k \\
\dot{\xi}_k &= f_k(x,\xi_1,\xi_2,\ldots,\xi_{k-1},\xi_k) + g_k(x,\xi_1,\xi_2,\ldots,\xi_{k-1},\xi_k)u,
\end{aligned}
$$

where $x \in I\!R^n$, $u \in I\!R$, $\xi_i \in I\!R$, $f_0(0) = 0$, $f_i(0,\ldots,0) = 0$, $1 \le i \le k$, **and** the x-subsystem satisfies Assumption A.1, is called a strict feedback system.

For such system, backstepping proceeds just as we have done in previous examples:

- We know that $\xi_1 = \alpha_0(x)$ renders the origin of

$$\dot{x} = f_0(x) + g_0(x)\alpha_0(x)$$

  globally asymptotically stable (GAS).

- We propose $V_1(x,\xi) = V_0(x) + \frac{1}{2}\left[\xi_1 - \alpha_0(x)\right]^2$ as a CLF for

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)\left[\xi_1 - \alpha_0(x) + \alpha_0(x)\right] \\
\dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 \\
&\Updownarrow \\
\dot{x} &= f_0(x) + g_0(x)\alpha_0(x) + g_0(x)\left[\xi_1 - \alpha_0(x)\right] \\
\dot{\xi}_1 &= f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2
\end{aligned}
$$

  and we seek $\xi_2 = \alpha_1(x,\xi_1)$ to make $\dot{V}_1$ negative definite.

- Doing the calculation we obtain

$$
\begin{aligned}
\dot{V}(x,\xi_1) =& \frac{\partial V_0(x)}{\partial x}\dot{x} + [\xi_1 - \alpha_0(x)]\left[\dot{\xi}_1 - \dot{\alpha}_0(x)\right] \\
=& \frac{\partial V_0(x)}{\partial x}\left\{f_0(x) + g_0(x)\alpha_0(x) + g_0(x)\left[\xi_1 - \alpha_0(x)\right]\right\} \\
& + [\xi_1 - \alpha_0(x)]\left[f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 - \frac{\partial \alpha_0(x)}{\partial x}\Big(f_0(x) + g_0(x)\xi_1\Big)\right] \\
\leq & -W(x) + [\xi_1 - \alpha_0(x)] \times \\
& \left[\frac{\partial V_0(x)}{\partial x}g_0(x) + f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 - \frac{\partial \alpha_0(x)}{\partial x}\Big(f_0(x) + g_0(x)\xi_1\Big)\right]
\end{aligned}
$$

- If $g_1(x,\xi_1) \neq 0$ for all $x \in I\!R^n$, $\xi_1 \in I\!R$, then we can solve for ξ_2 via

$$
\left[\frac{\partial V_0(x)}{\partial x}g_0(x) + f_1(x,\xi_1) + g_1(x,\xi_1)\xi_2 - \frac{\partial \alpha_0(x)}{\partial x}\Big(f_0(x) + g_0(x)\xi_1\Big)\right] = -c_1[\xi_1 - \alpha_0(x)],
$$

with $c_1 > 0$. This yields

$$
\begin{aligned}
\xi_2 =& \alpha_1(x,\xi_1) \\
=& \frac{-c_1\left[\xi_1 - \alpha_0(x)\right] - L_{g_0}V_0(x) - f_1(x,\xi_1) + L_{f_0}\alpha_0(x) + \xi_1 L_{g_0}\alpha_0(x)}{g_1(x,\xi_1)}
\end{aligned}
$$

and

$$
\dot{V}_1(x,\xi_1) \leq -W(x) - c_1\left[\xi_1 - \alpha_0(x)\right]^2
$$

- If $g_1(x,\xi_1)$ vanishes at some points, then there may or may not be a feedback that renders $\dot{V}_1$ negative definite or negative semi-definite. The conclusion will depend on the particular example at hand.

- Assuming you were successful at this stage, you then proceed by induction. The condition $g_i(x,\xi_1,\ldots,\xi_i) \neq 0$ for all $x,\xi_1,\ldots,\xi_i$ will guarantee the existence of functions $\alpha_2(x,\xi_1,\xi_2),\ldots,\alpha_k(x,\xi_1,\ldots,\xi_k)$ such that

$$
V_k(x,\xi_1,\ldots,\xi_k) = V_0(x) + \frac{1}{2}\left[\xi_1 - \alpha_0(x)\right]^2 + \frac{1}{2}\sum_{i=2}^{k}\left[\xi_i - \alpha_{i-1}(x,\xi_1,\ldots,\xi_{i-1})\right]^2
$$

is a CLF for the overall system.

EECS 562 Handout: Grizzle
Recursive Feedback Design
(Backstepping)

Remark: See Chapter 14.3, page 589, of our textbook, Nonlinear Systems, H. Khalil, Third Edition.

Assumption A.1 Consider the system

$$\dot{x} = f(x) + g(x)u \qquad\qquad (\Sigma)$$

where $f(0) = 0$, f and g are locally Lipschitz continuous, $x \in I\!R^n$, and $u \in I\!R$. We suppose that there exists a continuously differentiable feedback control law

$$u = \alpha(x), \quad \alpha(0) = 0$$

and a continuously differentiable, positive definite, radially unbounded function $V : I\!R^n \to I\!R$ such that, for all $x \in I\!R^n$,

$$L_f V(x) + [L_g V(x)]\,\alpha(x) \leq -W(x),$$

where W is positive $\underline{\text{semi-definite}}$.

Remark: Under Assumption A.1, the origin is a stable equilibrium of the closed-loop system

$$\dot{x} = f(x) + g(x)\alpha(x),$$

all solutions exist on $[0, \infty)$ and are globally bounded, and

$$\lim_{t\to\infty} W(x(t, x_0)) = 0.$$

Moreover, if the only solution of $\dot{x} = f(x) + g(x)\alpha(x)$ that lies entirely in

$$Z = \{x \mid W(x) = 0\}$$

is $x(t) \equiv 0$, then the origin is GAS. If W is positive definite, then the equilibrium is GAS.

Integrator Backstepping Lemma Suppose that Assumption A.1 holds for

$$\dot{x} = f(x) + g(x)u \qquad (\Sigma)$$

and consider the above system augmented with an integrator:

$$\begin{array}{rcl} \dot{x} & = & f(x) + g(x)\xi \\ \dot{\xi} & = & u \qquad\qquad (\Sigma_a) \end{array}$$

(i) If $W(x)$ is positive definite, then

$$V_a(x,\xi) = V(x) + \frac{1}{2}\left[\xi - \alpha(x)\right]^2 \qquad (*)$$

is a clf for Σ_a; that is, there exists a feedback $u = \alpha_a(x,\xi)$ which renders the equilibrium $x_e = 0,\ \xi_e = 0$ GAS. Moreover, one such feedback is

$$u = -\,c\,(\xi - \alpha(x)) + L_{f+g\xi}\,\alpha(x) - L_g V(x), \quad c > 0. \quad (**)$$

$$= -\,c\,(\xi - \alpha(x)) + \frac{\partial \alpha(x)}{\partial x}\big(f(x) + g(x)\xi\big) - \frac{\partial V(x)}{\partial x} g(x)$$

Note: $L_{f+g\xi}\,\alpha(x) = \dfrac{\partial \alpha(x)}{\partial x}\big(f(x) + g(x)\xi\big)$ and $L_g V(x) = \dfrac{\partial V(x)}{\partial x} g(x)$.

(ii) If $W(x)$ is only positive semi-definite, then there exists a feedback control which renders

$$\dot{V}_a(x,\xi) \leq -W_a(x,\xi) \leq 0$$

and such that $W_a(x,\xi) > 0$ whenever $W(x) > 0$ or $\xi \neq \alpha(x)$. This control law provides stability i.s.L., global boundedness, and convergence of the state $(x(t),\xi(t))$ of (Σ_a) to the largest invariant set M_a contained in

$$Z_a = \left\{ \begin{bmatrix} x \\ \xi \end{bmatrix} \in I\!\!R^{n+1} \mid W(x) = 0,\ \xi = \alpha(x) \right\}.$$

Proof: Introduce the error variable

$$z = \xi - \alpha(x)$$

and rewrite Σ_a in the $\begin{bmatrix} x \\ z \end{bmatrix}$ coordinates:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)[\alpha(x) + z] \\
\dot{z} &= \dot{\xi} - \frac{\partial \alpha}{\partial x}(x)\dot{x} \\
&= u - \frac{\partial \alpha}{\partial x}(x)\{f(x) + g(x)[\alpha(x) + z]\}
\end{aligned} \qquad (***)$$

Differentiating $V_a(x, z) = V(x) + \frac{1}{2}z^2$ along the solutions of (***) yields

$$\begin{aligned}
\dot{V}_a(x, z) =& \frac{\partial V}{\partial x}(x)\{f(x) + g(x)[\alpha(x) + z]\} \\
&+ z\{u - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\langle \alpha(x) + z \rangle]\} \\
=& \frac{\partial V(x)}{\partial x}\{f(x) + g(x)\alpha(x)\} \\
&+ z\{u - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\langle \alpha(x) + z \rangle] + \frac{\partial V}{\partial x}(x)g(x)\} \\
\leq& - W(x) + z\underbrace{\{u - \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\langle \alpha(x) + z \rangle] + \frac{\partial V}{\partial x}(x)g(x)\}}_{-cz \quad , \quad c>0}
\end{aligned}$$

$$\Rightarrow \boxed{u = -cz + \tfrac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\langle \alpha(x) + z \rangle] - \tfrac{\partial V}{\partial x}(x)g(x)} \qquad (****)$$

is one choice of control that renders $\dot{V}_a$ negative semi-definite. Indeed,

$$\dot{V}_a(x, z) = -W(x) - cz^2,$$

so that if $W(x)$ is positive definite, then $\dot{V}_a$ is negative definite.

Note: u given by (****) is just one of many controls that renders $\dot{V}_a$ neg semi-def. If $W > 0$, one can use Sontag's formula as well to obtain $\dot{V}_a < 0$