Main Stability Theorem - Examples.

Last time we studied the main theorem for investigation of the stability properties of the equilibrium of $\dot{x} = f(x)$. In summary, the theorem tells us that if we can find a positive definite function $V(x): D \to \mathbb{R}$ *, whose derivative $\dot{V}(x) := \frac{\partial V(x)}{\partial x} \cdot f(x)$ is negative semi-definite, then we can conclude that the equilibrium point is stable in the sense of Lyapunov. If the derivative $\dot{V}(x)$ is negative definite, then the theorem tells us that the equilibrium point is asymptotically stable i.s.L.

* continuously differentiable.

Note: Review the definitions of positive/negative semi-definite functions from the previous lecture

Note: The theorem conditions are sufficient. If we fail to show stability/asymptotic stability with a candidate Lyapunov function $V(x)$, then the theorem does not conclude that the equilibrium point is not stable/asymptotically stable i.s.L. The best we can do in this case is to keep investigating with a different Lyapunov function candidate!

Let us see some illustrative examples.

Example. (4.3 in Khalil.)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin x_1, \quad a>0. \end{cases}$$

(Pendulum without friction)

Consider the equilibrium $(x_1, x_2) = (0, 0)$. We want to investigate its stability properties.

Let us consider the positive definite function

$$\boxed{V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2}$$

Check: Is the function $V(x)$ indeed positive definite?

Well, we have $V(0) = 0$, and $V(x) > 0$, $x \neq 0$.

Hence yes.

Take the time derivative as:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x) = \begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a\sin x_1 \end{bmatrix} = 0.$$

The time derivative is identically zero. From Theorem 4.1 we conclude the origin is stable i.s.L.

In fact, since $\dot{V}(x) :\equiv 0$, we can conclude that the trajectories do not approach the origin. (they get trapped on the level set $V(x(t_0)) = c$, i.e., on the level set of the function they start from. Remember the geometric representation.)

Example. (4.4 in Khalil)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin x_1 - bx_2 \end{cases}, \quad a,b>0$$

(Pendulum with friction)

Similarly we want to investigate the stability properties of $(x_1,x_2)=(0,0)$.

- Let us try with the Lyapunov function candidate of the previous example.

$$\boxed{V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2}$$

- Take the time derivative

$$\dot{V}(x) = \begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a\sin x_1 - bx_2 \end{bmatrix} = -bx_2^2 \leq 0.$$

The time derivative is of non-positive sign.
Caution here: The time derivative $\dot{V}(x)$ is negative semi-definite !!! Why?

Because it is zero for all $(x_1, 0)$, $x_1 \in \mathbb{R}$.

[It would be negative definite if it were zero for $(0,0)$ only!]

From Theorem 4.1, we can conclude that the origin is stable i.s.L.

Important things to notice and summarize here:

- In this example, with the choice of the candidate Lyapunov function $V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$, we were able to conclude stability i.s.L, but not asymptotic stability. However, we suspect (and we can verify from the phase portrait) that the origin of this System is asymptotically stable. There are two ways forward.

  ① La Salle's Theorem / Principle, which we will learn a bit later on, treats such cases where the derivative of the Lyapunov function candidate is only negative semi-definite, and not negative definite.

  ② Keep investigating using a different Lyapunov function candidate.

- Let us for now attempt ②.

Consider $V(x) = a(1-\cos x_1) + \frac{1}{2}x^T P x$,

where $P > 0$ is a positive definite matrix.

(Review the definitions of positive definite matrices from your Linear Systems, and recall that for a 2x2 positive definite matrix we have

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad \boxed{\begin{array}{l} p_{11} > 0, \\ p_{11}p_{22} - p_{12}^2 > 0. \end{array}}$$

Take the time derivative:

$$\dot{V}(x) = a(1-p_{22})\underline{x_2 \sin x_1} - a p_{12} x_1 \sin x_1 + $$
$$+ (p_{11} - p_{12} b)\underline{x_1 x_2} + (p_{12} - p_{22} b) x_2^2.$$

Now, we can proceed as follows.

- We can attempt to pick p_{11}, p_{22}, p_{12} such that $\dot{V}(x)$ is negative definite AND P is positive definite.
- We observe that the coupling terms $x_1 x_2$, $x_2 \sin x_1$ are sign indefinite, in the sense that their signs depend on the signs of x_1; x_2.

We can pick $\boxed{p_{22} = 1}$ and $\boxed{p_{11} = b p_{12}}$

to cancel out the corresponding terms from $\dot{V}(x)$

Then we have: $p_{22} = 1$, $p_{11} = b p_{12}$.

and from Positive definiteness of P: $p_{11} > 0$ and

$p_{11} > p_{12}^2$. $\Rightarrow$

$b p_{12} > p_{12}^2$, $p_{12} > 0$ $\Rightarrow$ $\boxed{0 < p_{12} < b}$ Let $\boxed{p_{12} = \frac{b}{2}}$

We have: $\dot{V}(x) = -\frac{1}{2} a b x_1 \sin x_1 - \frac{1}{2} b x_2^2$

- Now we have $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$.

Hence we can define the domain $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$ and conclude that $\dot{V}(x)$ is negative definite over D.

Hence from Theorem 4.1, the origin is asymptotically stable.

Let us now focus our attention to the following question. First, we note that Theorem 4.1 assumes the existence of a Lyapunov function $V(\ ): D \to \mathbb{R}$, i.e., of a function $V(\cdot)$ defined over a domain D in $\mathbb{R}^n$. Then, if the requirements of the Theorem are met with $\dot{V}(x) < 0$ for $x \neq 0$, we conclude that the origin is asymptotically stable, i.e. that $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that $\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \epsilon$, $\forall t \geq 0$, and $\lim_{t \to \infty} \|x(t, x_0)\| = 0$.

The question we can pose in this case is: How far from the origin can the trajectory be, and still converge to the origin as t tends to $+\infty$? In other words, we are interested in estimating the region of attraction (or domain of attraction, basin, region of asymptotic stability):

$$R_A = \{x_0 \in \mathbb{R}^n \mid \lim_{t \to \infty} x(t, x_0) = 0\}.$$

Important Remark. Finding the exact region of attraction might be exceedingly difficult to compute exactly.

Lyapunov functions can be used to estimate the region of attraction

↓

(or you can read "approximate")

We will see the technique later in class.

Essentially we try to find level sets of Lyapunov functions that are contained in the region of attraction.

Already from the proof of Theorem 4.1, we saw that if $V: D \to \mathbb{R}$ satisfies the conditions of asymptotic stability. (V positive definite, $\dot{V}$ negative definite), and if $\Omega_c = \{x \in D \mid V(x) \leq c\}$ is bounded and contained in D,* then Ω_c is an estimate of the region of attraction as trajectories starting in Ω_c never escape Ω_c and eventually converge to the origin.

* In fact, Ω_c is by definition closed and bounded (i.e, compact) since $\Omega_c = \{x \in \bar{B}_r(0) \mid V(x) \leq \beta\}$, i.e. is a closed set (see Appendix of last lecture) and a bounded set since it is contained in $B_r(0)$.

- The above observation tells us that Ω_c is an estimate of the region of attraction.

<u>However</u>, it might be a very <u>conservative</u> estimate in the sense that it might be much <u>smaller</u> than the actual region of attraction. (recall Ω_c contains points in $B_r(0)$ only).

- We will see techniques on estimating the region of attraction and on enlarging the estimates later in class

In the mean time, we pose the following question:

"Under which conditions will the region of attraction be the <u>whole space</u> $\mathbb{R}^n$?"

- In other words, under which conditions the state trajectory $x(t, x_0)$ will approach the origin as $t \to \infty$, <u>for any initial $x_0 \in \mathbb{R}^n$</u>, no matter how large $\|x\|$ might be? If an asymptotically stable equilibrium at the origin does possess this property, then it is said to be <u>globally asymptotically stable.</u>

- We will seek an answer to this question.

Recall from the proof of Theorem 4.1, that we established asymptotic stability by showing that a ball of radius δ around the origin can be contained in a compact (closed and bounded) set Ω_c.

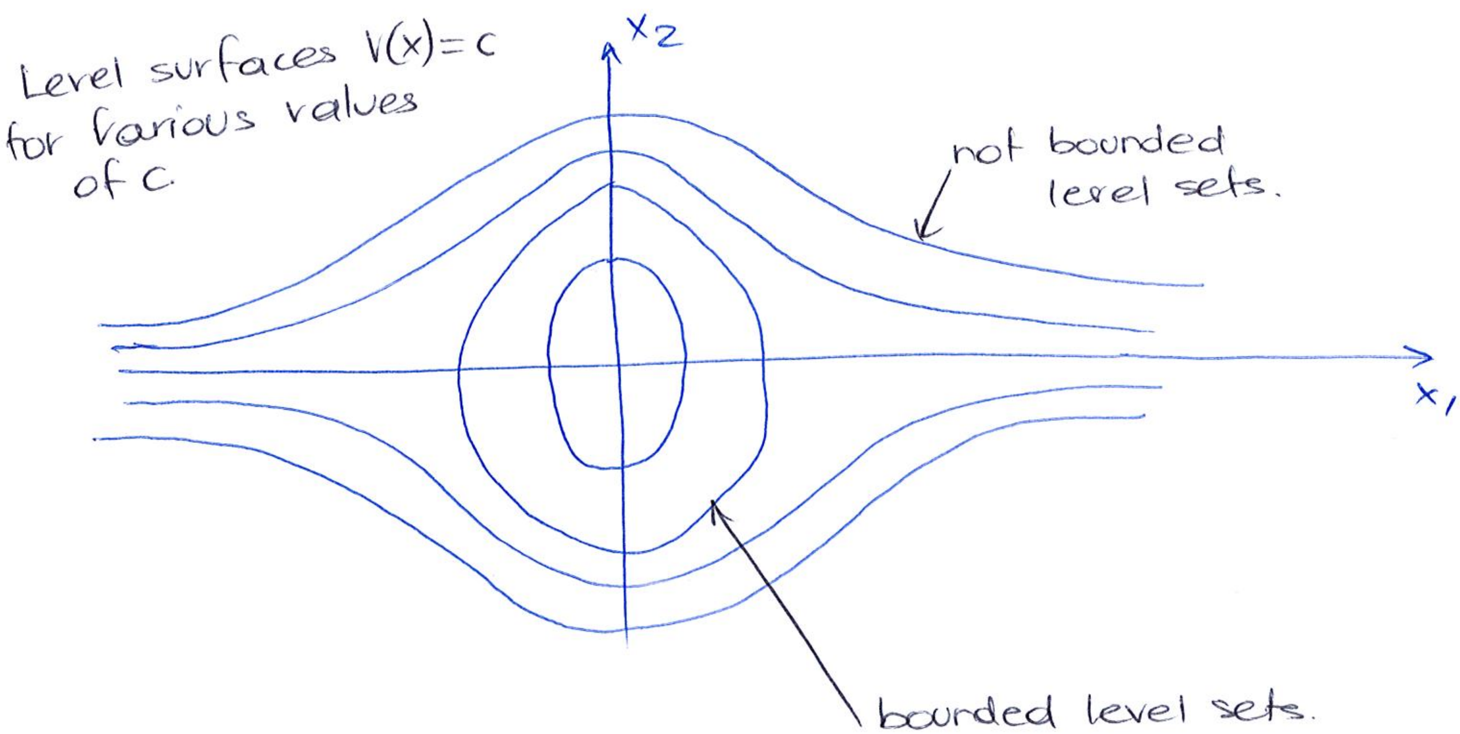
In order to be able to establish that <u>any</u> point $x \in \mathbb{R}^n$ can be contained in the interior of a compact set Ω_c, one obvious observation is that the conditions of the theorem must hold globally, i.e., $D = \mathbb{R}^n$.

However, this condition is not enough.

The problem is that it might be that for large c, the level sets of V might not be bounded sets!

<u>Example</u> $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$

Positive definite function.



In this case, for small enough c, the level surfaces are closed and bounded. That is a consequence of continuity and positive definiteness of V.

However, as c increases, the function $V(x)$ has unbounded level sets. Recall that a set $S \subset \mathbb{R}^n$ is bounded if $\exists k < \infty$ such that $\forall x \in S$, $\|x\| < k$.

Let us find the c for which the set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is bounded.

Let us see how the $\overbrace{\min_{\|x\|=r} V(x)}$ varies as $\underline{r \to \infty}$.

$$l = \lim_{r\to\infty} \underbrace{\inf_{\|x\| \geq r} V(x)} = \lim_{r\to\infty} \underbrace{\min_{\|x\|=r}} V(x) =$$

we want to find the greatest lower bound of $V(x)$ for $\|x\| \geq r$ so that we can fit a set $\Omega_c = \{x \in \mathbb{R}^n / V(x) \leq c\}$ in the interior of $B_r(0)$.

$\|x\| = r$ is a closed and bounded set, hence $V(x)$ attains a minimum value on that set [Weierstrass Thm].

$$= \lim_{r\to\infty} \min_{\|x\|=r} \left[\frac{x_1^2}{1+x_1^2} + x_2^2 \right] = \lim_{\substack{|x_1| \to \infty \\ |x_2| \to 0}} \left[\frac{x_1^2}{1+x_1^2} + x_2^2 \right]$$

$$= \lim_{|x_1| \to \infty} \frac{x_1^2}{1+x_1^2} = 1.$$

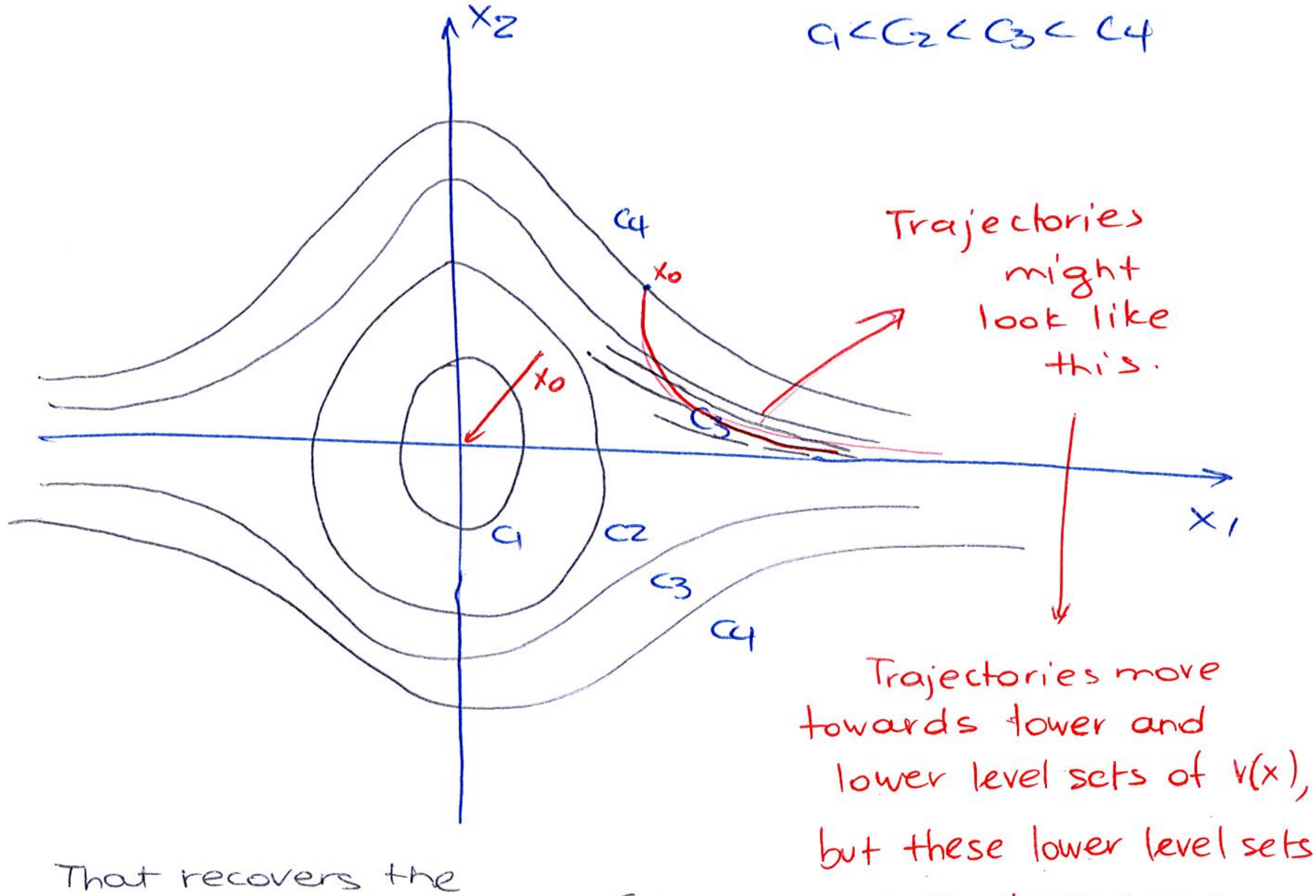
note that this still implies that $\|x\| = r \to \infty$!

Hence we conclude that the level set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is bounded for $c < l = 1$.

Hence, we verified that the level sets of the considered function $V(x)$ are bounded for $c<1$ only.

This means that we can not verify global (i.e, $\forall\, x_0 \in \mathbb{R}^n$) asymptotic stability with such a function.

Intuitively, think that you applied the conditions of Theorem 4.1 with this function, and you indeed had $\dot{V}<0$, $\forall\, x \neq 0$. Then you would conclude that the origin is asymptotically stable for all initial conditions, i.e. globally. Which would be a false result, as you can verify with our geometric representation of Lyapunov stability theorems.



That recovers the "bad" situation we depicted in the proof of Theorem 4.1!

Thus, we have to impose an additional condition, apart from $D = \mathbb{R}^n$. This condition was introduced last time as $V(x)$ being radially unbounded.

Theorem 4.2. Let $x=0$ be an equilibrium point for (4.1). Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

(i) $V(0) = 0$ and $V(x) > 0, \ \forall x \neq 0$.

(ii) $\|x\| \to \infty \Rightarrow V(x) \to \infty$ (radial unboundedness)

(iii) $\dot{V}(x) < 0, \ \forall x \neq 0$.

Then $x=0$ is globally asymptotically stable.

Proof: Given $p \in \mathbb{R}^n$, let $c = V(p)$.

Then from (ii) we have that $\forall c > 0, \exists r > 0$ such that $\|x\| > r \Rightarrow V(x) > c$.

Hence $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is closed and bounded; and can be contained in a ball $B_r(0)$. The rest is similar to the proof of Theorem 4.1.

Remark. Theorem 4.2 is known as Barbashin - Krasovskii Theorem

Lyapunov - Instability.

So far we have studied two main theorems (Theorems 4.1 and 4.2 in our textbook) that can be used to investigate and establish stability and asymptotic stability i.s.L of an equilibrium point.

There are theorems as well that can establish instability of the equilibrium point!

Definition. (Instability) Consider $\dot{x} = f(x)$, a locally Lipschitz continuous ODE on $\mathbb{R}^n$, with equilibrium point x_e.

Then x_e is unstable if $\exists\, \epsilon > 0$ such that, $\forall\, \delta > 0$, $\exists\, x_0 \in B_\delta(x_e)$ and $T > 0$ resulting in

$$\| x(T, x_0) - x_e \| > \epsilon.$$

So the question is, how to characterize instability of the equilibrium? What are the theorems / tools that we can build to establish instability?

Simplest Possible Result. (overkill per Professor Grizzle!) Let $x_e = 0$ be an equilibrium point of the ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let D be an open set containing $x_e = 0$, and suppose that f is locally Lipschitz continuous on D.

→

Suppose there exists a continuously differentiable function $V: D \to \mathbb{R}$ such that

(i) $V(0) = 0$, and $V(x) > 0$ for all $x \in D$, $x \neq 0$.

(ii) $\dot{V}(x) > 0$ for all $x \in D$, $x \neq 0$.

Then the equilibrium point x_e is unstable.

Proof. The key steps are.

(i) Let $\epsilon > 0$ such that $U := \bar{B}_\epsilon(0) \subset D$.

Let $0 < \delta < \epsilon$ be arbitrary, let x_0 be such that $\|x_0\| = \delta$, and let $\phi(t, x_0)$ be a solution from x_0.

Claim: $\exists\, T > 0$ such that $\|\phi(T, x_0)\| > \epsilon$.

i.e, We want to prove that ...

We will pursue proving the claim by contradiction.

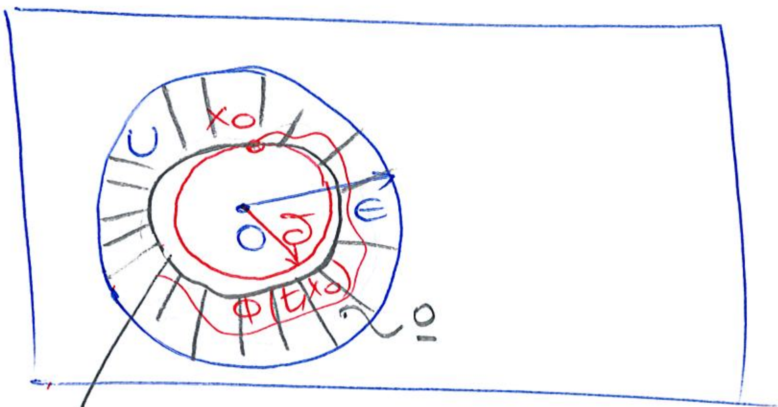
Assume it is not the case that $\exists\, T > 0$ such that $\|\phi(T, x_0)\| > \epsilon$.

That means that $\boxed{\phi(t, x_0) \in U}$, for all $t \geq 0$.

As thus $V(\phi(t, x_0)) \leq \sup\limits_{x \in U} V(x) < \infty$

i.e. $V(\phi(t, x_0))$ is bounded.

② Now we have that $\dot{V} > 0$ (positive definite) which implies that $V(\phi(t,x_0))$ is non-decreasing, and as thus, $V(\phi(t,x_0)) \geq V(x_0)$



③ We furthermore have that the set

$$\Omega = \{x \in U \mid V(x) \geq V(x_0)\}$$

is compact. [the shaded region]

Define $\gamma = \min_{x \in \Omega} \dot{V}(x)$

Then $\gamma > 0$ by the positive definiteness of $\dot{V}$

$U = \bar{B}_\epsilon(0) \subset D$

$\{x \mid V(x) = V(x_0)\}$

So from step ② we have that $\phi(t,x_0)$ evolves outside the set $\{x \mid V(x) = V(x_0)\}$

④ Now by the fundamental Theorem of Calculus.

$$V(\phi(t,x_0)) = V(x_0) + \int_0^T \dot{V}(\phi(\tau,x_0))\,d\tau$$

$$\geq V(x_0) + \gamma t \to \infty \quad \text{as } t \to \infty.$$

which is a contradiction to $V(\phi(t,x_0))$ being bounded!

Hence we prove our initial claim.

Example. Consider the system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2$$

From linear systems tools, you can immediately verify that the origin is unstable.

Let us verify it with our (overkill) result.

Take the positive definite $V(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$

Compute the time derivative

$$\dot{V}(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$$

which is positive definite, hence the origin is unstable.

Question. We may now wonder, if we are to obtain a less restrictive result, what should be weakened? Should we drop V being positive definite, or drop $\dot{V}$ being positive definite?

We investigate the question via some examples.