1) i) let $V(x) = x_1^2 + x_2^2$

given $\dot{x}_1 = -x_1 + x_1 x_2$

$\dot{x}_2 = -x_2$

$$\dot{V}(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix}$$

$$= -2x_1^2 + 2x_1^2 x_2 - 2x_2^2$$

if $|x_1| < r \Rightarrow \dot{V}(x) < -2x_1^2 - 2x_2^2 + 2r x_1 x_2$

as $|a||b| \leq \frac{1}{2}(a^2+b^2)$

$\Rightarrow \dot{V}(x) \leq -2x_1^2 - 2x_2^2 + r(x_1^2 + x_2^2)$

$\Rightarrow r < 2$ for $\dot{V}(x) < 0$.

& $\{0\}$ belong to the domain $V(x) > 0$
$\dot{V}(x) < 0$.

& $x = 0$ is eq^m Point

So. the system is locally asymptotically stable at zero.

Global!

$\dot{x}_1 = x_2$

$\Rightarrow x_2 = x_{20} e^{-t}$

$\dot{x}_1 = -x_1 (1 - x_2)$

$\Rightarrow \ln\left(\frac{x_1}{x_{10}}\right) = -\left(t + \underset{>0}{x_{20}}(e^{-t} - 1)\right)$

$\Rightarrow x_1 = x_{10} e^{-ct}$

$\therefore$ as $t \to \infty$ $x_1, x_2 \to 0$

$\therefore$ 0 is globaly asymptotically stable

3)

$V(x) = x_1^2 + x_2^2$ $\qquad$ $\cancel{\dot{x}_1} = \cancel{-x_2} - x_1(1 - x_1^2 - x_2^2)$ $\quad$ $\cancel{\dot{x}_2}$

$\dot{x}_1 = x_2(1 - x_1^2)$

$\dot{x}_2 = -(x_1 + x_2)(1 - x_1^2)$

$$\dot{V}(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_2(1-x_1^2) \\ -(x_1+x_2)(1-x_1^2) \end{bmatrix}$$

$$= 2\left(\cancel{x_1x_2} - \cancel{x_1^3 x_2} + \cancel{x_1^3 x_2} - \cancel{x_1x_2} - x_2^2 + x_2^2 x_1^2\right)$$

$$= 2\left(x_2^2 x_1^2 - x_2^2\right)$$

$\dot{V}(x) < 0 \Rightarrow (x_1^2 - 1) < 0 \Rightarrow |x_1| < 1$

$V(x) > 0$ & $\dot{V}(x) < 0$ & $\{0\} \in$ Domain

Domain $= \{ |x_1| < 1 \}$

$\Rightarrow$ ~~$x \neq 0$~~ & $\dot{x}_1 = 0$ $\quad \dot{x}_2 = 0$ when $x_1, x_2 = 0$

$\therefore$ $x = 0$ is locally asymptotically stable

ξ in the Domain D

$x=0$ is not the only eqm

when $x_1 = 1$ then $\dot{x}_1 = 0$

$\dot{x}_2 = 0$

$\therefore$ $x=0$ is not globaly asymptotically stable

4) $\dot{x}_1 = -x_1 - x_2$

$\dot{x}_2 = 2x_1 - x_2^3$

$\Rightarrow V(x) = x_1^2 + x_2^2$

$\Rightarrow \dot{V}(x) = [2x_1 \quad 2x_2] \begin{bmatrix} -x_1 - x_2 \\ 2x_1 - x_2^3 \end{bmatrix}$

$= 2(-x_1^2 - \cancel{x_1 x_2} + \cancel{x_1 x_2} - x_2^4)$

$\dot{V}(x) < 0$ at x_1, x_2

$\therefore$ as $x \in$ Domain & $\dot{x}_1 = 0$
$\dot{x}_2 = 0$

when $x_1 = 0$ $x_2 = 0$

$\Rightarrow$ $x = 0$ is locally asymptotically stable

&, $\dot{V}(x) < 0$ & $\dot{x} = 0$

$\Rightarrow$ $\dot{x}_1 = 0 \Rightarrow x_1 = -x_2$

$\dot{x}_2 = 0 \Rightarrow 2x_1 - x_2^3 = 0.$

$\Rightarrow 2x_1 + x_1^3 = 0$

$x_1 = 0 \,/\, x_1^2 = -2$

not possible

So $x = 0$ is only eq^m

& as $x \to \infty$ $V(x) \to \infty$

$V(x) = x_1^2 + x_2^2$

$\therefore$ V is radially unbounded

$\therefore$ $x = 0$ is globally asymptotically stable

2) $$V(x) = \frac{(x_1+x_2)^2}{1+(x_1+x_2)^2} + (x_1-x_2)^2$$

a) if $x_1 = 0$

$$\Rightarrow V(x) = \frac{x_2^2}{1+x_2^2} + x_2^2$$

term a : $\frac{x_2^2}{1+x_2^2}$ as $x_2 \to \infty \Rightarrow \frac{2x_2}{2x_2} = 1$

term b : x_2^2 as $x_2 \to \infty \Rightarrow x_2^2 \to \infty$

$\Rightarrow V(x) \to \infty$ as $\|x_2\| \to \infty$ along $x_1 = 0$

Similarly if $x_2 = 0$

$$\Rightarrow V(x) = \frac{x_1^2}{1+x_1^2} + x_1^2$$

term a : $x_1 \to \infty$: $\frac{x_1^2}{1+x_1^2} = \frac{2x_1}{2x_1} = 1$

term b : $x_1 \to \infty$: $x_1^2 \to \infty$.

$\Rightarrow V(x) \to \infty$

$\therefore V(x) \to \infty$ as $\|x\| \to \infty$ along $x_1 = 0$ &

$x_2 = 0$.

b). $V(x) = \frac{(x_1+x_2)^2}{1+(x_1+x_2)^2} + (x_1-x_2)^2$

$V(x)$ is a increasing function w.r.t $\|x\|$

& so. $\inf_{\|x\| \geq k} V(x)$ will be at $\|x\| = k$

$\Rightarrow$ at $\|x\| = k$ $V(x) = \inf V(x)$

$\Rightarrow$ this is possible when $x_1 = \alpha x_2$

let $\alpha = 1$

$\Rightarrow$ $V(x) = \frac{(2x_1)^2}{1+(2x_1)^2} + (0)^2$

as $\|x\| \to \infty \Rightarrow x_1 \to \infty$

$\Rightarrow$ $\lim_{x_1 \to \infty} V(x) = \lim_{x_1 \to \infty} \frac{4x_1^2}{1+4x_1^2}$

$= \lim_{x_1 \to \infty} \frac{8x_1}{8x_1} = 1$

$\therefore$ as $\|x\| \to \infty$ $V(x) \to 1$

$\therefore$ By defn $V(x)$ is not radially Bounded

3) if $w(x)$ is positive definite

let $\alpha(p) := \inf_{p \le \|x\| \le \bar{r}} w(x)$ $\}$ $0 \le \bar{r} < r$

a) $\alpha(0) = \inf_{0 \le \|x\| \le \bar{r}} w(x)$ as $w(x)$ is positive def

min $(w(x)) = 0$ $0 \le \|x\| \le r$

$\Rightarrow$ $\alpha(0) = 0$

for any p $\|x\|$ belongs to compact set $[p, \bar{r}]$

& w is continuous $\Rightarrow \inf_{p \le \|x\| \le \bar{r}} w(x) > 0$

$\Rightarrow$ $\alpha(p) > 0$.

b) $\alpha(\|x\|) = \inf_{\|x\| \le \|y\| \le \bar{r}} w(y) \le w(x)$

$\Rightarrow$ $\alpha(\|x\|) \le w(x)$ & this is irrespective of $\bar{r}$

c) α: non-decreasing & continuous.

$S(p) = \{ x \in \mathbb{R}^n \mid p \le \|x\| \le r \}$

if $p_1 \le p_2 \le r$ $\Rightarrow$ $S(p_2) < S(p_2)$

as W is +ve definite & $S(P_1)$; $S(P_2)$ are compact sets

$$\inf_{x \in S(P_1)} W(x) \leq \inf_{x \in S(P_2)} W(x)$$

$\Rightarrow$ $\alpha(P_1) \leq \alpha(P_2)$ where $P_1 < P_2$

$\therefore$ α is non decreasing

To prove continuity

if $\forall\, \rho\ \exists\, \epsilon$ such that

$$\|P_1 - P_2\| \leq \rho \Rightarrow \|\alpha(P_1) - \alpha(P_2)\| < \epsilon$$

using <u>Lemma 5</u> : Suppose $0 \leq P_1 \leq P_2 < \gamma$

then if $x \in S(P_1)$, $d(x, S(P_2)) \leq P_2 - P_1$

where $d(x, s) := \inf_{y \in S} \|m - y\|$

since $B_r(0)$ is compact set & W is continuous

WTOB $\Rightarrow$ $m_1, m_2 \in B_\delta(0)$

if $\|m_1 - m_2\| < \delta$

$\Rightarrow$ $\|W(m_1) - W(m_2)\| < \frac{\epsilon}{2}$

if $p_1 \le p_2 \le \tilde{r}$

$\Rightarrow \quad p_2 - p_1 < \frac{\delta}{2}$

Since $\alpha(p_1) = \inf_{m \in S(p_1)} w(m)$

$\Rightarrow$ lets choose $m_1 \in S(p_1)$ such that

$$w(m_1) - \alpha(p_1) < \frac{\epsilon}{2}$$

$\Rightarrow$ from lemma 5 we know

$$d(m_1, S(p_2)) \le p_2 - p_1 \le \frac{\delta}{2}$$

$\Rightarrow$ we can choose $m_2 \in S(p_2)$ such that

$$\begin{aligned} w(m_2) - \alpha(p_1) &\ge |w(m_2) - \alpha(p_1)| \\ &\le |w(m_2) - w(m_1)| + |w(m_1) - \alpha(p_2)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Rightarrow$ ~~$\alpha(p_1)$~~ $\Rightarrow \inf_{m \in S(p_2)} w(m) \le w(m_2) \le \alpha(p_1) + \epsilon$

$\Rightarrow \quad \alpha(p_2) - \alpha(p_1) \le \epsilon$

$\Rightarrow \quad \alpha$ is continuous.

if $\exists$ α as shown

$\Rightarrow$ given $W(0) = 0$

& $\alpha(p) > 0 \quad \forall \quad 0 < p < r$

$\Rightarrow$ & $\alpha(\|x\|) \leq W(x)$

as $\alpha(\|x\|) > 0$

$W(x) \geq 0$

& $\dot{\alpha}(\|x\|) \leq \dot{W}(x)$

as α is nondecreasing continuous

$\dot{W}(x) \geq 0$

$\therefore$ W is a positive definite function locally