EECS 562 - Nonlinear Systems and Control
HW #9

Due by Thursday, March 30th, 2023
by 11:59PM, on Canvas

Preliminary: Recall the normalized model of the inverted pendulum on a cart:

$$\begin{cases} \frac{d^2\phi}{dt^2} &= \frac{-c(\dot{\phi})^2 \sin\phi\cos\phi}{1+c\sin^2\phi} + \frac{\sin\phi}{1+c\sin^2\phi} - \frac{\cos\phi}{1+c\sin^2\phi}\bar{\mu} \\ \frac{d^2\bar{s}}{dt^2} &= \frac{d(\dot{\phi})^2\sin\phi}{1+c\sin^2\phi} - \frac{\cos\phi\sin\phi}{1+c\sin^2\phi} + \frac{b}{1+c\sin^2\phi}\bar{\mu} \end{cases} \tag{1}$$

Choosing the states as $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = \bar{s}$, $x_4 = \dot{\bar{s}}$, letting $u = \bar{\mu}$ and setting $b = 3$, $c = 0.5$ and $d = 1.5$ as in the project, results in

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-0.25(x_2)^2\sin(2x_1)+\sin(x_1)}{1+0.5\sin^2(x_1)} - \frac{\cos(x_1)}{1+0.5\sin^2(x_1)}u \\ x_4 \\ \frac{1.5(x_2)^2\sin(x_1)-0.5\sin(2x_1)}{1+0.5\sin^2(x_1)} + \frac{3}{1+0.5\sin^2(x_1)}u \end{bmatrix} = f(x) + g(x)u \tag{2}$$

1. Show that for every $p \in \mathbb{R}$ and $v \in \mathbb{R}$, $\bar{x}(t) = \begin{bmatrix} 0 \\ 0 \\ p+vt \\ v \end{bmatrix}$ and $\bar{u}(t) = 0$ is a solution of (2). Linearize the system (2) about the trajectory $\bar{x}(t)$, $\bar{u}(t)$. You should find that the linearization is time-invariant and equal to the matrices of your project, so, in particular, you get the same linearization for any values of p and v.

2. Using the state feedback gains of your project, make the trajectory $\bar{x}(t)$, $\bar{u}(t) = 0$ (uniformly) exponentially stable by feeding back $u(t) = k[x(t) - \bar{x}(t)]$ (see Figure 1 below). Perform a simulation that illustrates this for $\bar{x}(t)$ corresponding to $p = 10$ and $v = -0.2$. Choose initial conditions for (2) that are different from
$$\bar{x}_0 = \begin{bmatrix} 0 \\ 0 \\ p \\ v \end{bmatrix}$$
and show that $x(t)$ converges to $\bar{x}(t)$ as $t \longrightarrow \infty$, as long as you choose x_0 sufficiently close to $\bar{x}_0$. [Same $\bar{x}(t)$ and $\bar{u}(t)$ from Prob. 1]

3. What happens if you allow p and v in $\bar{x}(t)$ to vary slowly with time? In particular, let
$$\bar{x}(t) = \begin{bmatrix} 0 \\ 0 \\ 10e^{-\varepsilon t} \\ \frac{d}{dt}(10e^{-\varepsilon t}) \end{bmatrix}$$
for $\varepsilon > 0$ small, and see what happens! We note that $\bar{x}(t)$ is no longer a solution of (1), but it is a "small," slowly -varying, perturbation of a solution of (1). Provide a printout of your simulation. Take your initial condition as $x(0) = \begin{bmatrix} 0 & 0 & 10 & 0 \end{bmatrix}'$.

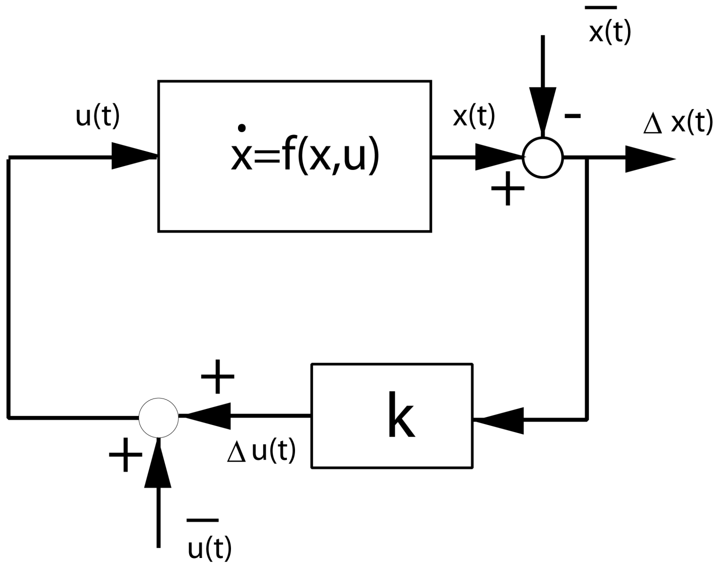

Figure 1: Block diagram of how to implement a linear feedback on the original nonlinear system when the nominal trajectory is not the origin.

4. Prove (with mathematical analysis and not by simulation) that the feedback scheme of Problem 2 does indeed render the trajectory $\bar{x}(t)$ (of Problem 1) uniformly exponentially stable. To do this, translate $\bar{x}(t)$ to the origin via the standard transformation, $z(t) = x(t) - \bar{x}(t)$, and determine the differential equation for $z(t)$. Then apply results from class on uniform exponential stability of the origin. We will analyze why we can "track" slowly varying $p(t)$ and $v(t)$ in a later HW set.

5. Consider the system

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ -\frac{\alpha_1 \xi_1}{1+K_1|\xi_1|} - \frac{\alpha_2 \xi_2}{1+K_2|\xi_2|} \end{bmatrix}$$

(a) Show that for all $K_1 \geq 0$, $K_2 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$, the origin is (locally) exponentially stable. To show exponential stability, you need to linearize the system. It is the NL system differentiable at the origin? Why?

(b) Show that for all $K_1 \geq 0$, $K_2 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ the origin is GAS. To show this try the function

$$V(\xi) = \alpha_1 \int_0^{\xi_1} \frac{x}{1+K_1|x|} dx + \frac{\xi_2^2}{2}$$

(c) Show that for all $\epsilon > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ there exist $K_1 > 0$ and $K_2 > 0$ such that

$$\sup_{(\xi_1,\xi_2)\in\mathbb{R}^2} \left| \frac{\alpha_1 \xi_1}{1+K_1|\xi_1|} + \frac{\alpha_2 \xi_2}{1+K_2|\xi_2|} \right| \leq \varepsilon$$

Remark: Can you see how all of this is fitting together? Suppose that we can find a control law $u = \alpha(x_1, x_2)$ that asymptotically stabilizes the pendulum for large angles, while ignoring the cart (as in the handout on Sontag's feedback controller). Such a feedback will make the pendulum and cart approach a trajectory of the form $\bar{x}(t)$ from Problem 1. From Problem 3, we have the idea that we can "track" a slowly varying version of $\bar{x}(t)$, so it would be nice to create a slowly varying version of $\bar{x}(t)$

that converges to the origin! But this is what Problem 5 shows us how to do...part 5b of that problem gets us convergence to the origin from any initial condition, and part 5c says that we can make the trajectory "uniformly slowly varying." We will put all of this together in an upcoming HW and build a nice controller for the pendulum and cart.

Hints for Problem 1: Plug into the ODE and see if left-hand side equals right-hand side. That is all there is to showing it is a solution.

Hints for Problem 2: We have not spent time in lecture on stabilizing a non-trivial solution of an ODE, hence this problem. It is quite common in practice to implement a scheme as shown in Fig. 1.

Hints for Problem 4: See page 147 of our textbook for the chance of variables being done here. When you have shifted the solution to the origin, you should find yourself in the unusual, but fortunate, situation of $z(t)$ satisfying the same ODE as $x(t)$! In particular, the ODE for $z(t)$ is time-invariant. Therefore, you can prove local uniform exponential stability through linearization or though a quadratic Lyapunov function.

Hints for Problem 5a: Compute the derivative of $\frac{\alpha \xi}{1+K|\xi|}$ for $\xi > 0$ and for $\xi < 0$; then take the limit as $\xi \longrightarrow 0$ and see that you get the same answer in each case.

Hints for Problem 5c: Use the triangle inequality to simplify the problem to two scalar problems of the form $\sup_{x \in \mathbb{R}} \left| \frac{\alpha x}{1+K|x|} \right| \leq \frac{\varepsilon}{2}$