1) $\dot{x} = \sqrt{x}$ $x(0) = 0$.

$$\frac{dx}{\sqrt{x}} = dt$$

let the cond^n given $\dot{x} = \sqrt{x}$ for $x \geq 0$
satisfies at $t_0 = c$

then $2 \cdot (\sqrt{x(t)} - \sqrt{x(t_0)}) = t - t_0$

$\Rightarrow$ as $t_0 = c$ & $x(t_0) = 0$.

$$\Rightarrow \quad x(t) = \frac{(t-c)^2}{4}$$

& $\dot{x} = \sqrt{x}$ when $x \geq 0 \Rightarrow$ for $t < c$ $x = 0$.
Since $x(0) = 0$.

$$\therefore \quad x_c(t) = \begin{cases} 0 & 0 \leq t < c \\ \frac{(t-c)^2}{4} & c \leq t < \infty \end{cases}$$

$f(x) = \sqrt{x}$ $\quad \dot{f} = \dfrac{1}{2\sqrt{x}}$

$\Rightarrow$ as $x \to 0$ $\quad \dot{f}(x) \to \infty$

$\Rightarrow$ $f(x) = \dot{x} = \sqrt{x}$ is not locally Lipschitz

Even though function is piecewise continuous its not locally Lipschitz

So Theorm 3.1 cannot conclude anything for these systems

The function doesn't violate the theorm but the theorm is not applicable to this function.

2) given $\dot{x}(t) = a(t)x(t)$

& $a(t)$ is continious

if $a(t)$ is continious so does $|a(t)|$ is continious

& from weirstrass Theorm

if $h : S \to \mathbb{R}$ is continious & S is compact
then $\exists s_0 \in S$ such that $\sup_{s \in S} h(s) = h(s_0)$

$\Rightarrow$ let $S = [t_0, T]$
then $\exists t' \in S$ such that $\sup_{t' \in S} |a(t)| = |a(t')|$

$\Rightarrow$ for $t \in [t_0, T]$
$|a(t)| \leq |a(t')|$

$\Rightarrow$ for time $[t_0, t_0 + \delta]$

$\| \dot{x}(t_0) - \dot{x}(t_0 + \delta) \| = \| a(t_0)x(t_0) - a(t_0+\delta)x(t_0+\delta) \|$

as $|a(t')| > |a(t)|$ $t \in [t_0, t_0 + \delta)$

$\Rightarrow \| f(t_0) - f(t_0 + \delta) \| < |a(t')| \, \| x(t_0) - x(t_0 + \delta) \|$

$\Rightarrow$ function is locally Lipschitz

as function is locally Lipschitz &
continuous

The solution exists & unique

So sol^n is

$$\dot{x} = a(t)\, x(t)$$

$$\frac{dx}{x} = a(t)\, dt$$

$$\Rightarrow \log\left(\frac{x}{x_0}\right) = \int_{t_0}^{t} a(\tau)\, d\tau$$

$$\Rightarrow \boxed{x(t) = x_0\, e^{\int_{t_0}^{t} a(\tau)\, d\tau}}$$

3) a) $\frac{d}{dt}(x^T(t)x(t)) = \dot{x}^T(t)x(t) + x^T(t)\dot{x}(t)$

$= 2x^T(t)(\dot{x}(t))$

$= 2x^T(t)f(t,x)$

as we know $|x^Ty| \leq \|x\|_2\|y\|_2$

$\Rightarrow \left|\frac{d}{dt}(x^T(t)x(t))\right| \leq 2\|x^T(t)\|_2\|f(t,x)\|_2$

given $\|f(t,x)\|_2 \leq L\|x\|_2$

$\Rightarrow$

$\left|\frac{d}{dt}(x^T(t))(x(t))\right| \leq 2L\|x^T(t)\|_2\|x(t)\|_2$

$\Rightarrow \left|\frac{d}{dt}\left((x^T(t))(x(t))\right)\right| \leq 2L\|x(t)\|_2^2$

b) let $y^2 = x^T(t)x(t)$.

$\Rightarrow \left|\frac{d}{dt}(y^2)\right| \leq 2Ly^2$

$-2Ly^2 \leq 2y\frac{dy}{dt} \leq 2Ly^2$

as $y^2 > 0$

$$-L \le \frac{1}{y}\frac{dy}{dt} \le L$$

$$\Rightarrow \quad -L\,dt \le \frac{dy}{y} \le L\,dt$$

$$\Rightarrow \quad -L\int_{t_0}^{t} dt \le \int_{y_0}^{y} \frac{dy}{y} \le L\int_{t_0}^{t} dt$$

$$\Rightarrow \quad -L(t-t_0) \le \log\left(\frac{y}{y_0}\right) \le L(t-t_0)$$

By defn $y_0 = \|x_0\|_2 \quad y = \|x(t)\|_2$

$\Rightarrow$

$$e^{-L(t-t_0)} \le \frac{\|x(t)\|_2}{\|x_0\|_2} \le e^{L(t-t_0)}$$

$$\boxed{\|x_0\|_2\, e^{-L(t-t_0)} \le \|x(t)\|_2 \le \|x_0\|_2\, e^{L(t-t_0)}}$$

4) $\dot{x}_1 = -x_1 + \frac{2x_2}{1+x_2^2}$

$-1 < \frac{2x_2}{1+x_2^2} < 1 \quad \forall\, x_2 \in \mathbb{R}$

2) $\dot{x}_1 \le -x_1 + 1$

$\dot{x}_2 \le -x_2 + 1$

comparison lemma $\Rightarrow$ $\dot{u}_1 = -u_1 + 1 \; ; \quad \dot{u}_2 = -u_2 + 1$

$\Rightarrow \frac{du_1}{-u_1+1} = dt \Rightarrow -\log\left(\frac{-u_1(t)+1}{-u_1(0)+1}\right) = t - t_0$

$\Rightarrow (-u_1(t)+1) = (-u_1(0)+1)e^{-t}$

$\Rightarrow u_1(t) = u_1(0)e^{-t} + (1-e^{-t})$

Similarly $u_2(t) = u_2(0)e^{-t} + (1-e^{-t})$

$\|u\|_2 = \sqrt{u_1^2(t) + u_2^2(t)}$

$\le e^{-t}\left(\sqrt{u_1^2(0) + u_2^2(0)}\right) + \sqrt{2(1-e^{-t})^2}$

$\le e^{-t}\|u(0)\|_2 + \sqrt{2}\,(1-e^{-t})$

& as

$x(t) < u(t)$

$\Rightarrow \quad \|x(t)\|_2 \leq \|u(t)\|_2 \quad$ if $x(0) = u(0)$

$\Rightarrow \quad \|x(t)\|_2 \leq e^{-t}\|x(0)\|_2 + \sqrt{2}(1-e^{-t})$

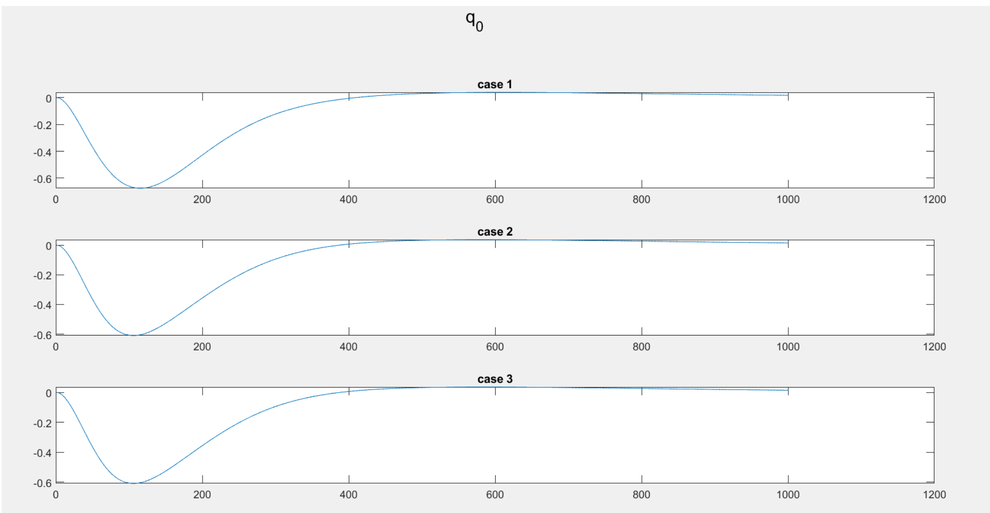
Question 5:

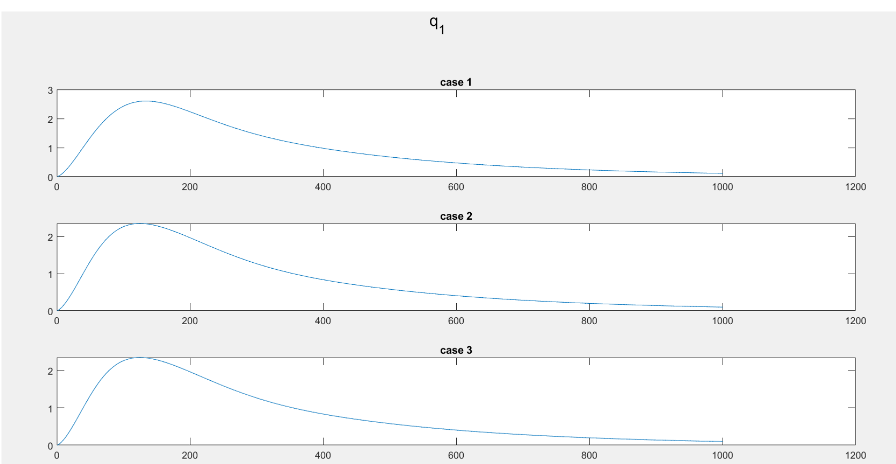
The code was run on three different initial cases(X = [q_0, q_1, q_0_dot, q_1_dot]) . And the trajectories of the q0 and q1 are given as shown below for

Case1: Linear Spong normal system
Case2 : Spong normal
Case 3: Nonlinear system

Initial condition 1:

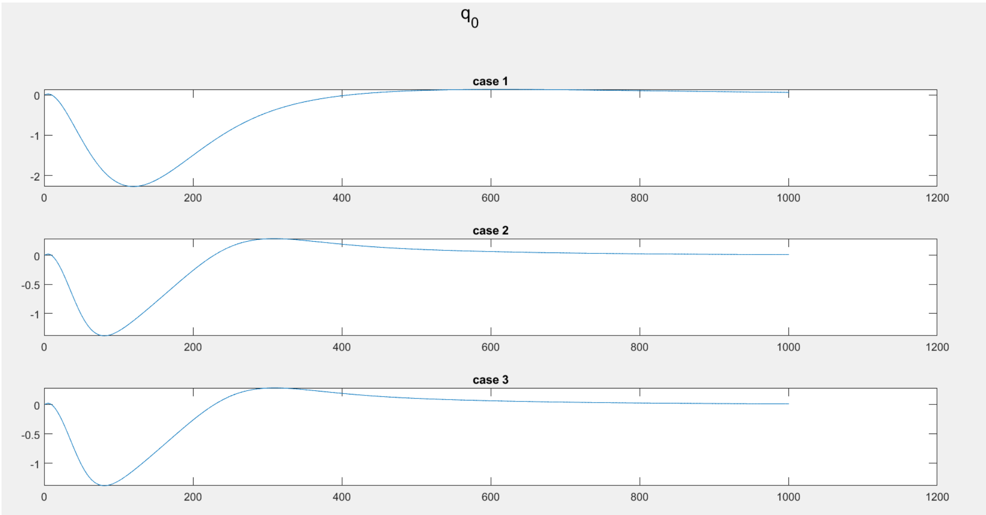
X = [0;0;0;1];

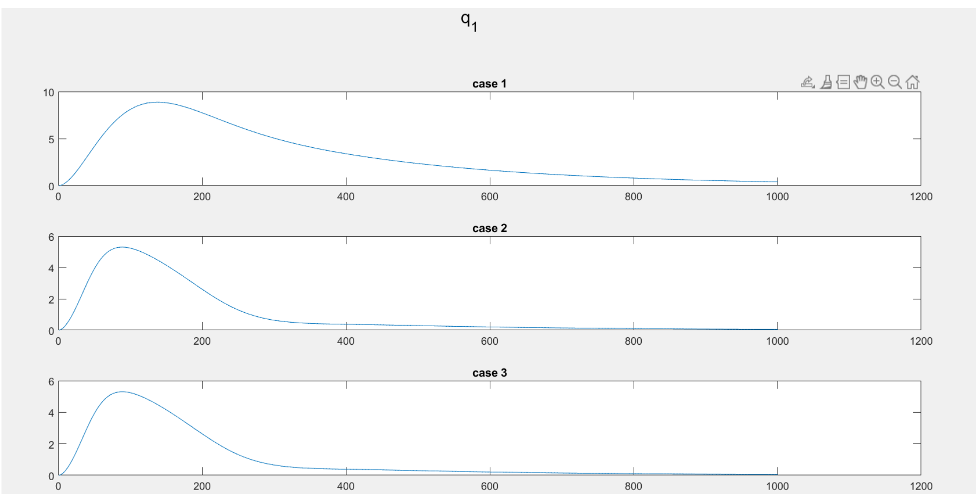




Initial condition 2:

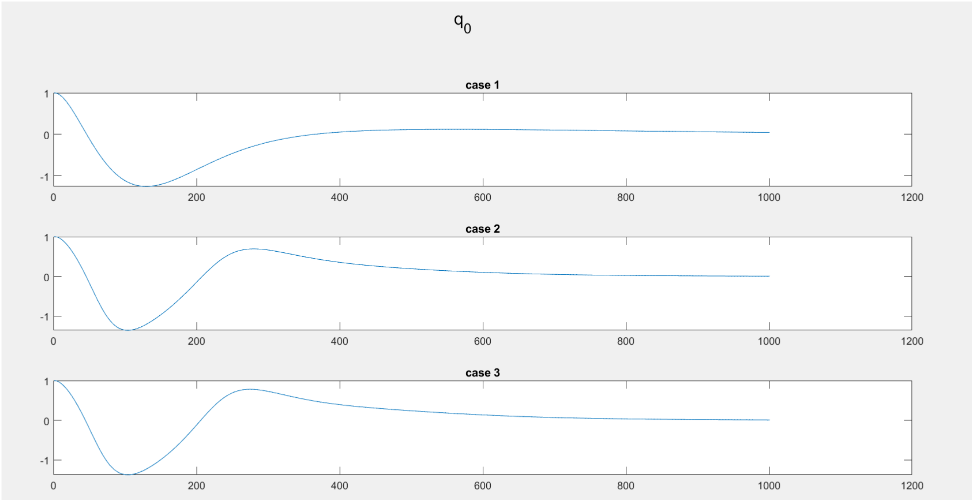
X = [0;0;1;1];

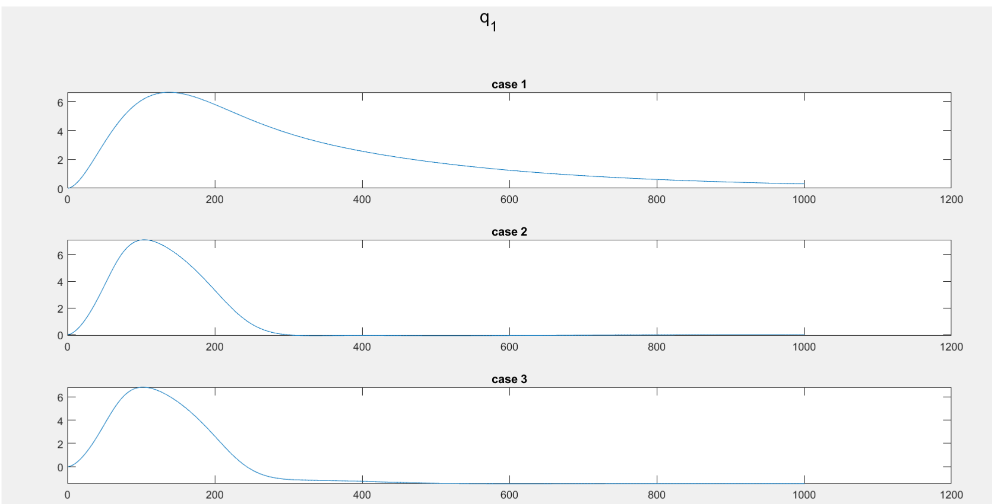




Initial condition 3:

X = [1;0;0;1];





Code:

```
%%values%%

cas = '1';

xx = [0;0;0;1];

% cas = '2';
%
% xx = [0;0;1;1];

% cas = '3';
```

```
%
% xx = [1;0;0;1];

%%
A = [0,(1/16.73),0,-(7.17/16.73),;
    7,0,0,0;
    0,0,0,1;
    0,0,0,0;
    ];
B = [0;0;0;1];
p = [(-1.7+(1j*1.5)),(-1.7-(1j*1.5)),-0.7,-0.35]';
K = place(A,B,p);

sig = (xx(3)*16.73) +(7.17*xx(4));
x0 = [xx(1);sig;xx(2);xx(4)];
% x0 = [0;0;1;1];
% x_new = [0;1;-(7.17/16.73);1;0];
% x_new = [0;0;-(7.17/16.73);1;0];
x_new = [xx(1);xx(2);xx(3);xx(4);sig];
t = 0:0.01:10;

[tP, xP] = ode45(@(t,x)lin_dyn(t,x,A,B,K), t, x0);
[tnP, xnP] = ode45(@(t,x)non_lin(t,x,B,K), t, x0);
[tcP, xcP] = ode45(@(t,x)comp(t,x,K), t, x_new);


fig = figure();
subplot(3,1,1);
plot(xP(:,1));
title("case 1");
hold on;
subplot(3,1,2);
plot(xnP(:,1));
title("case 2");
subplot(3,1,3);
plot(xcP(:,1));
title("case 3");
sgtitle('q_0');
savefig(['./results/q_0_' num2str(cas) '.fig'])

fig = figure();
subplot(3,1,1);
plot(xP(:,3));
title("case 1");
```

```
hold on;
subplot(3,1,2);
plot(xnP(:,3));
title("case 2");
subplot(3,1,3);
plot(xcP(:,2));
title("case 3");
sgtitle('q_1');
savefig(['./results/q_1_' num2str(cas) '.fig'])


function x_dot = lin_dyn(t,x,A,B,K)
x_dot = (A*x) - (B*(K*x));
end

function x_dot = non_lin(t,x,B,K)
v = -K*x;
x_dot = zeros(4,1);
bp = (7*cos(x(1))+9.73);
x_dot(1) = ((x(2)/bp) - (((3.67+(3.5*cos(x(1))))*x(4))/bp));
x_dot(2) = (7*sin(x(1))) - (3.5*sin(x(1))*x_dot(1)*(x_dot(1)+x(4)));
x_dot(3) = x(4);
x_dot(4) = v;
end

function x_dot = comp(t,x,K)

x_dot = zeros(5,1);
y = [x(1);x(5);x(2);x(4)];
aa = 3.67 + (3.5*(cos(x(1))));
a1 = (7*cos(x(1)))+9.73;
D = [a1, aa;
     aa, 3.67];
bb = -3.5*sin(x(1));
C = [bb*x(3),0;
    bb*x(3),0;];
G = [-7*sin(x(1));0];

B= [0;1];

cc = (7*cos(x(1))) + 9.73;
dd = (7+(3.5*x(3)*x(3)));
v = -K*(y);
```

```
u = (bb*x(3)*x(3)) + (sin(x(1))*(aa/cc)*dd) + ((3.67 - ((aa*aa)/cc))*v);

q_dot = [x(3);x(4)];
comp_1 = -inv(D)*((C*q_dot)+G);
comp_2 = inv(D)*B;

x_dot(1) = x(3);
x_dot(2) = x(4);

comp = comp_1 + (comp_2*u);

x_dot(3) = comp(1);
x_dot(4) = comp(2);

x_dot(5) = (7*sin(x(1))) - (3.5*sin(x(1))*x(3)*(x(3)+x(4)));

end
```