2) given $\frac{d(x(t))}{dt} = f(x(t))$

& $y(\tau) = x(\tau/a)$

$$\frac{d\,y(\tau)}{d\tau} = \frac{d}{d\tau}(x(\tau/a))$$

applying $\tau/a = t$

$$= \frac{d\,(x(t))}{dt}\frac{dt}{d\tau}$$

$$= \frac{1}{a} f(x(t))$$

$$= \frac{1}{a} f(x(\tau/a))$$

$$\boxed{\frac{d\,y(\tau)}{d\tau} = \frac{1}{a} f(y(\tau))}$$

b) applying $\tau = -t$

$\Rightarrow \frac{1}{a} = -1$

$\Rightarrow \frac{d\,y(\tau)}{d\,\tau} = (-1) \cdot f(y(\tau))$

$$\boxed{\frac{d\,y(\tau)}{d\,\tau} = -f(y(\tau))}$$

c) given $\frac{d\,x(t)}{d\,t} = 3\,x(t)$

$\Rightarrow f(x(t)) = 3\,x(t)$

$\Rightarrow \frac{d\,y(\tau)}{d\,\tau} = -3\,y(\tau)$

4) a) $\dot{x} = x^4$

$$\frac{dx}{x^4} = dt$$

$$\left[-\frac{4}{x^3}\right]_{x(0)}^{x(t)} = t$$

$$1 - \frac{1}{x^3} = t/4$$

$$x^3 = 1/(1 - t/4)$$

$$\boxed{x(t) = (1/(1 - t/4))^{1/3}}$$ will go to ∞ at $t = 4$

4) b) $\dot{x} = x^4 \sin x$

This sequence will always be inside a compact set W around the equilibrium point

where $W = \left(\theta - \frac{\pi}{2}, \theta + \pi/2\right)$

where $\theta \to$ equilibrium point $= n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$

The sequence won't escape the set

So. It doesn't have finite time escape.

And from Theorm 3.3 it will converge to ~~[illegible]~~ equilibrium point

5) a) Yes, the function is locally Lipschitz continuous

& the Lipschitz constant can be given by

$$\frac{\| f(x) - f(y) \|}{\| x - y \|} \leq L$$

& L can be given by $\max(f'(z))$ where $z \in (x, y)$

if $x \in (x_1, x_2)$

$$f'(x) = 2x + 3x^2 (\text{sign}(x))$$

$$\max_{x \in (x_1, x_2)} |f'(x)| = \begin{cases} |2x_2 + 3x_2^2 \,\text{sign}(x_2)| & \text{if } |x_2| > |x_1| \\ |2x_1 + 3x_1^2 \,\text{sign}(x_1)| & |x_1| > |x_2| \end{cases}$$

b) This is not Lipschitz continuous

because the function is not piecewise continuous

& it has discontinuity at

$x = 0$ $\quad f(x) = \begin{cases} \cos x & x \geq 0 \\ -\cos x & x < 0 \end{cases}$

This can also be verified

if we take $x = \delta$ $y = -\delta$

where $\delta \to 0$

then $\cos x = \cos\delta \approx 1$

$\cos y = \cos(-\delta) \approx 1$

$f(x) \approx 1$ $\quad f_y \approx -1$

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \approx \frac{2}{2\|\delta\|} \text{ as } \delta \to 0.$$

The value shoots to ∞.

$\therefore$ not Lipschitz continuous

c) ~~A~~ Yes, It is locally Lipschitz

& similar to Part (a)

$$L = \max_{x \in (x_1, x_2)} (|f'(x)|)$$

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

∴ ~~kee~~ = ~~[illegible]~~

& $f'(x)$ has range from $(-1, 1)$

∴ $L = 1 \quad \forall x \in D$

∴ We can also say that this is globally Lipschitz

d) No, This is not locally Lipschitz

Because the function is discontinuous

at $x = -1, 1$

$$f(x) = \begin{cases} x^3 & |x| \leq 1 \\ x & x > 1 \\ -x & x < 1 \end{cases}$$

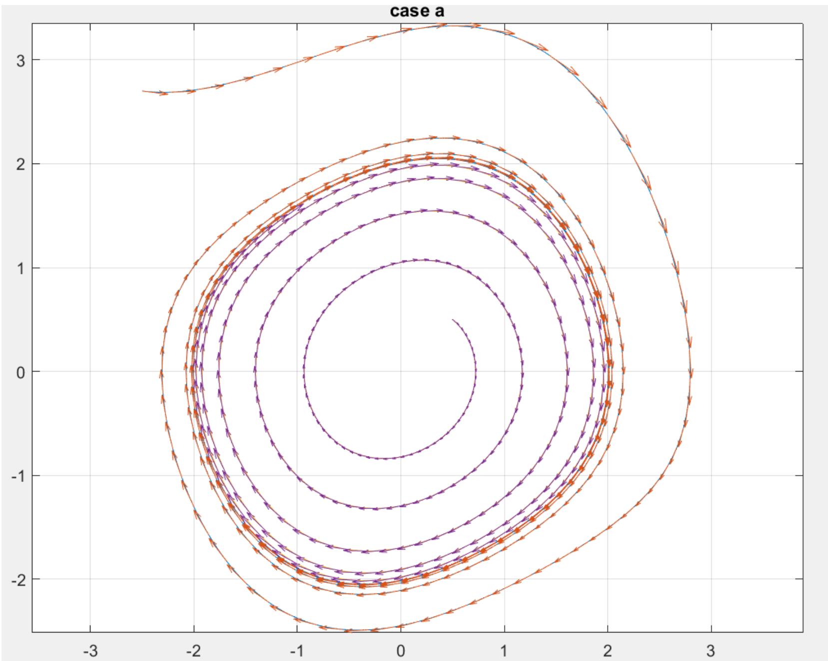
$\therefore$ at $\lim_{x \to 1^-} f(x) = -1$

$\lim_{x \to 1^+} f(x) = 1$

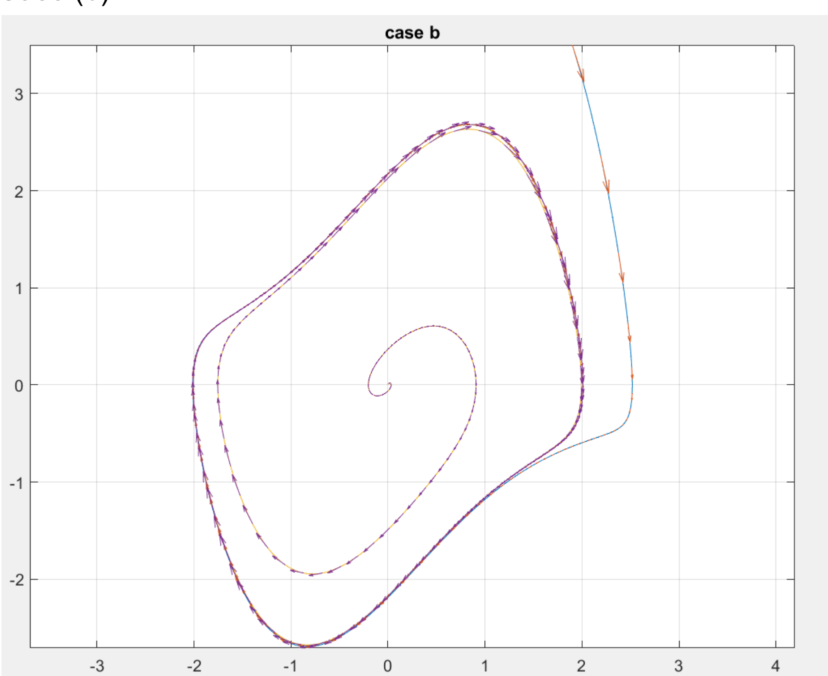
Homework 1

Question 1:

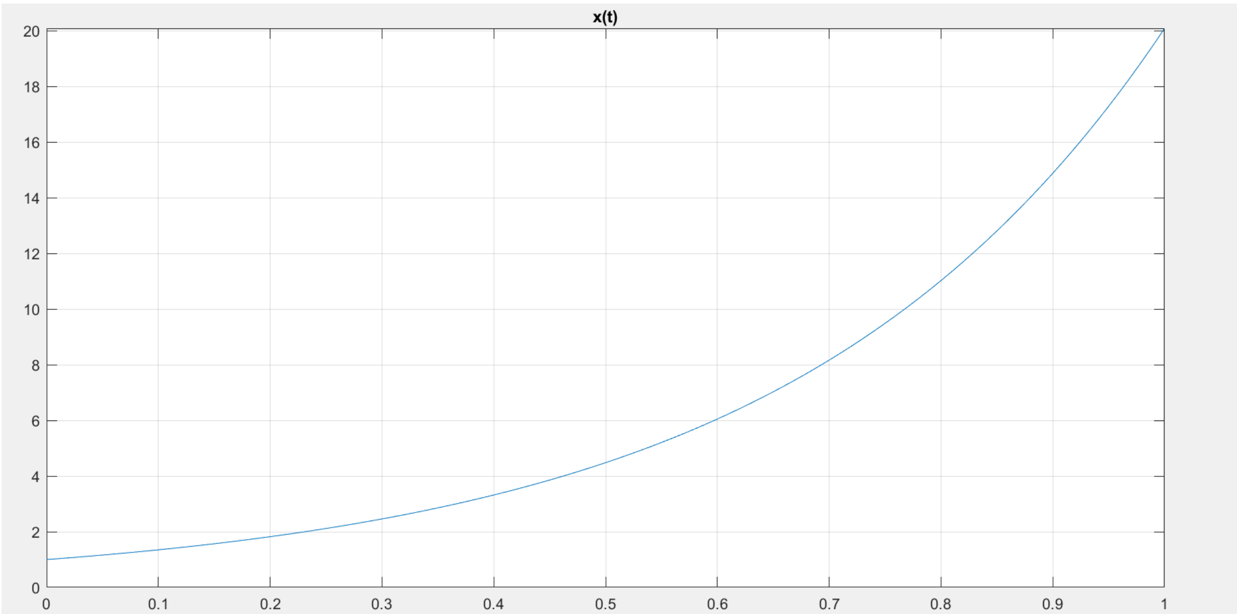
Case (a):

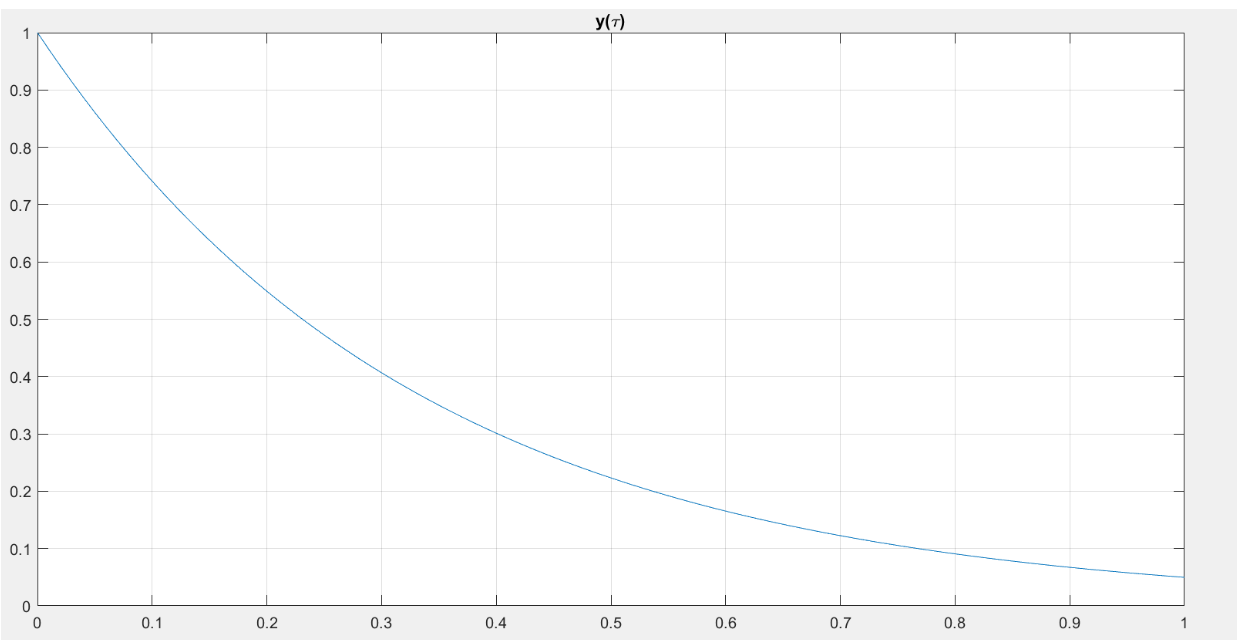


Case (b):



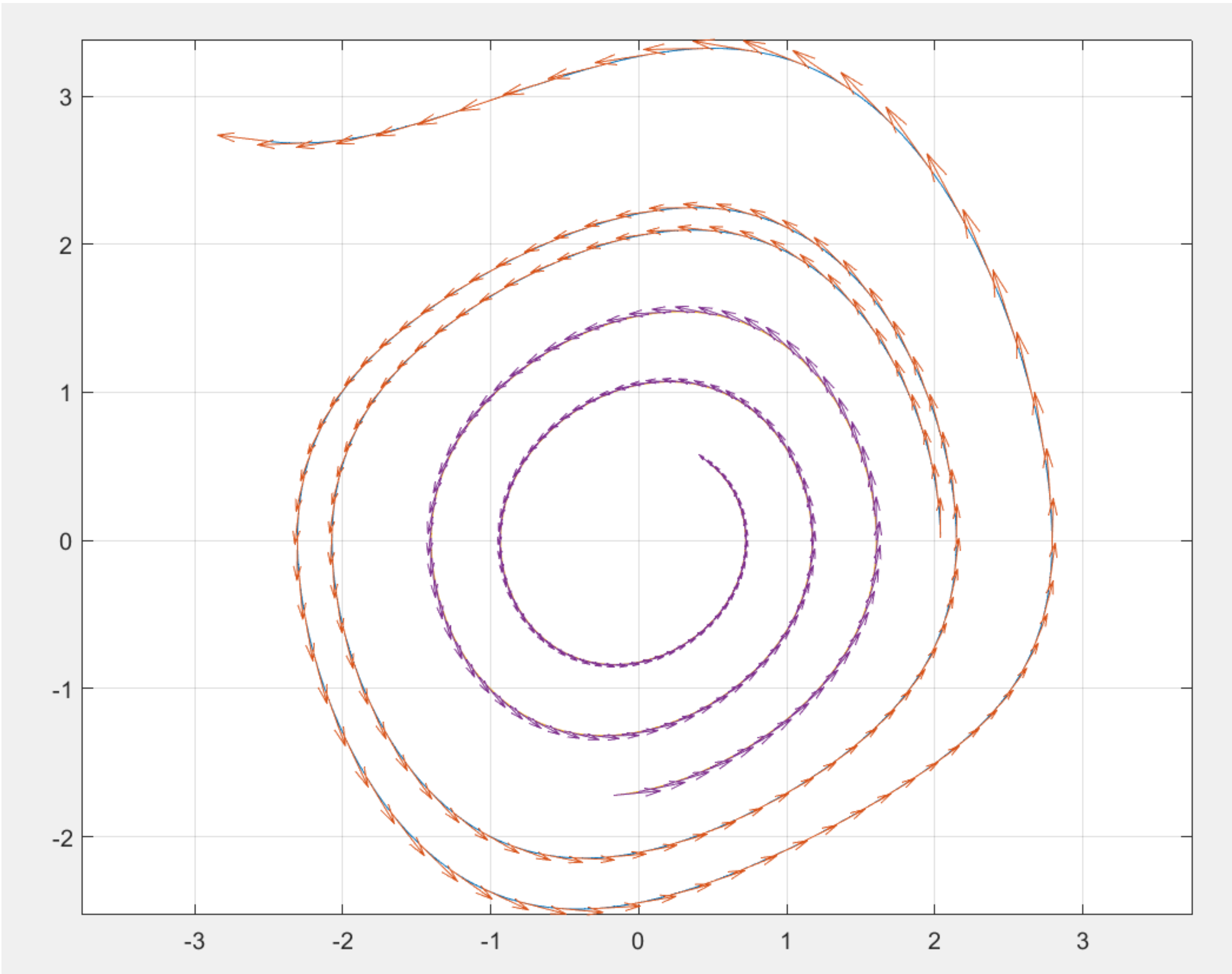
Question 2:


x(t)
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


y(τ)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

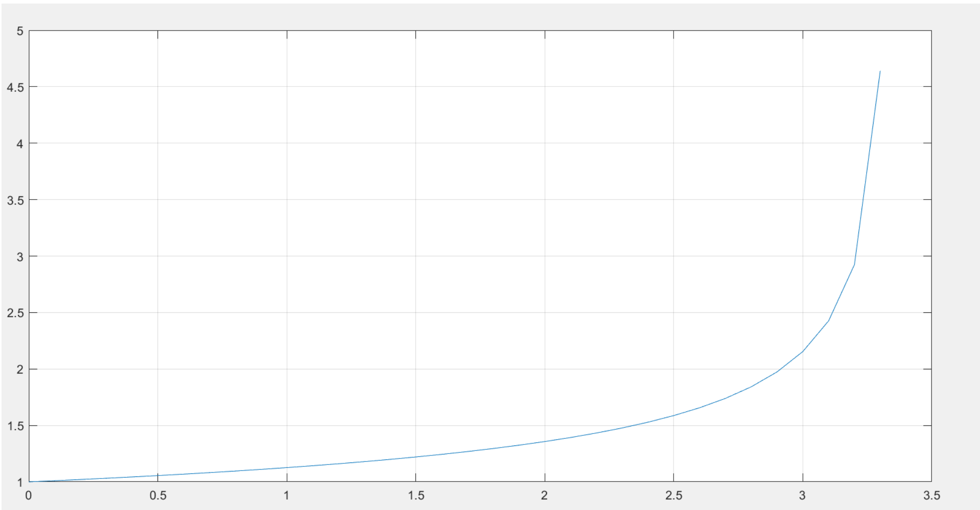
Question 3:

The plot is similar to question q with the direction reversed in the opposite direction. This is because the derivative is negative of the one in question 1.



Question 4:
Case a:

The ode stops near **t = 4**



Case b:

ODE approaching next nearest equilibrium point