EECS 562/AE 551 - Nonlinear Systems and Control
HW #3

Due on Thursday, January 26th, 2023
By 11:59pm, on Canvas

1. (10 points) Consider the scalar differential equation defined for $x \geq 0 : \dot{x} = \sqrt{x}$, $x(0) = 0$. Show that for all $c \geq 0$,

doesn't violate
its giving time shifted val

$$x_c(t) := \begin{cases} 0 & 0 \leq t \leq c \\ \frac{(t-c)^2}{4} & c < t < \infty \end{cases}$$

is a solution. Does this contradict Theorem 3.1 on page 88 of Khalil [2]? Why?

2. (10 points) Consider the scalar time-varying differential equation $\dot{x}(t) = a(t)x(t)$, $x(t_0) = x_0$. Suppose that $a : \left[t_0, \infty\right) \to \mathbb{R}$ is continuous. Prove that

hints help: ok

$$\phi(t) = x_0 e^{\int_{t_0}^{t} a(\tau) d\tau}$$

is the unique solution of the differential equation. That is, (i) prove that the solution exists and that $\phi(t)$ is indeed a solution and (ii) prove that this solution $\phi(t)$ is unique. For (ii), you can apply results from lecture or the textbook.

3. (20 points) Exercise 3.17 on page 107 of Khalil [2]

ok

Remark: Exercise 3.17 shows that the Lipschitz condition prevents the solution from either growing or decaying too quickly; in particular, if $f(x)$ is globally Lipschitz continuous, then the solution can neither escape to infinity in finite time nor converge to an equilibrium point in finite time. The result of this HW problem says that your controller cannot be Lipschitz continuous for finite-time stability, which can be desired in real world applications. Indeed, the controller has to contain something discontinuous, such as a switch, or a non-Lipschitz term, such as $-\text{sgn}(x)\sqrt{|x|}$. You can read [1] for more details.

4. (15 points) Exercise 3.15, page 107 [2].

5. (45 points) Read the supplement to HW 3 dealing with the control of a Segway [attached to this HW set]. Work the problems given in Section 5 of the supplement. Note that this requires MATLAB and simulations, so it will take more time. The purpose of the problem is to give you an opportunity to use some nonlinear control early in the course. Hopefully, you will find that to be fun and worth the time spent on it!

References

[1] V T Haimo. Finite time controllers. *SIAM Journal on Control and Optimization*, 24(4):760–770, 1986.

[2] H.K. Khalil. *Nonlinear Systems.* Prentice Hall, 2002.

EECS 562: Nonlinear Control - Supplement to HW #3

Prof. Grizzle

1 Introduction

Early in the course, we do mostly analysis and very little feedback design. Your project gets you working on how to do feedback design for nonlinear systems on the basis of linearization. The purpose of this problem is do a nonlinear feedback design on the basis of what you know about linear feedback design! If this sounds contradictory, hopefully you will be intrigued and read the following material with great interest. This exercise shows you how nonlinear state variable feedback can be used to obtain a simplified design model on which linear feedback techniques are often more effective than on the original form of the model. You can think of the feedback designed in this exercise as a type of pre-feedback whose purpose is to partially, exactly linearize the model. It is quite effective for a wide class of mechanical models. We will illustrate it on the Segway, which is very similar to the inverted pendulum on a cart.

Remark: This exercise uses facts from mechanics. Don't let that worry you; knowledge of mechanics will not be necessary on the exams. For control applications, on the other hand, it is quite useful. I suggest you skip to that last section, see what you will be asked to do, and then read quickly through the derivation of the method. You can work the problem without understanding completely the derivation.

2 Planar Segway Model

Consider the planar Segway system shown in Figure 1, along with the indicated coordinates, $q = (q_0, q_1) = (\phi, \theta)$, where ϕ is the (absolute) angle of the pendulum with respect to the vertical and θ is the relative angle of the wheel with respect to the pendulum. In both cases, we take clockwise to be positive. The system has one actuator u that acts at the axis connecting the inverted pendulum to the wheel; consequently, θ is the actuated variable and the angle of the pendulum is indirectly regulated[1].

If we assume that there is no slipping between the wheel and the ground, then a model of the system can be easily derived using Lagrangian mechanics. Denote the Lagrangian by $L = K - V$, where K is the kinetic energy and V is the potential energy. The method of Lagrange yields[2]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \begin{cases} 0 & k = 0 \\ u & k = 1 \end{cases}, \tag{1}$$

with u taking values in $\mathbb{R}$.

[1] This is analagous to the inverted pendulum on a cart where cart position is actuated and the angle of the pendulum is indirectly regulated through the position of the cart.

[2] You are not responsible for knowing this method; however, if you plan to work in the area of control system design, it is often convenient to know it when you encounter electromechanical systems.

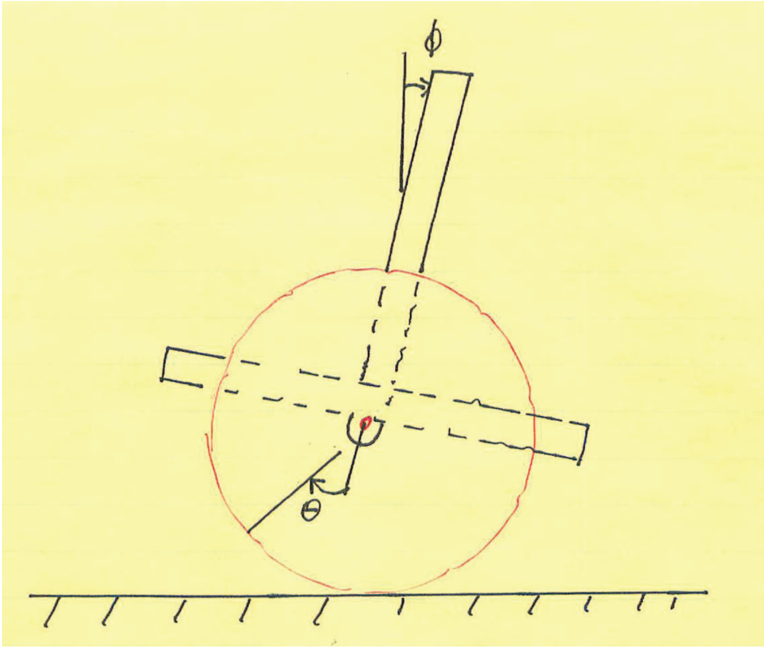

Figure 1: Crude drawing of a planar Segway.

For simple mechanical system of this type, the kinetic energy is quadratic in the velocities, $K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$, with $D(q)$ positive definite[3]. Moreover, the potential energy is independent of the velocities and thus has the form $V = V(q)$. It follows that (1) can be written as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu, \tag{2}$$

where $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The 2×2 matrices D and C and the 2×1 vector G are computed in the symbolic file (available on the course web site)

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symb_model_planar_segway.m
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in case you are curious; they are not really needed for now. All we need to know is that the model can be expressed in the form (2). In this course, you do not need to know how to derive the model.

The standard way to put the system in state variable form is to define $x = (q, \dot{q})$ so that

$$\dot{x} = f(x) + g(x)u, \tag{3}$$

where

$$f(x) = f(q,\dot{q}) = \begin{bmatrix} \dot{q} \\ -D^{-1}(q)\left(C(q,\dot{q})\dot{q} + G(q)\right) \end{bmatrix} \tag{4}$$

and

$$g(x) = g(q) = \begin{bmatrix} 0 \\ D^{-1}(q)B \end{bmatrix}. \tag{5}$$

[3]The fact that D is positive definite follows from facts in mechanics. For us, the thing to note is that because D is positive definite, it is invertible and its diagonal entries are strictly greater than zero.

3 Background on Partial Feedback Linearization of Mechanical Systems

The following is based on [4, 3] and [1, 2, 5]. Here, the general result is specialized (simplified) to a system with two generalized coordinates $q = (q_0, q_1)$ and the special form in (1). You can read the literature for the more general result, if you wish.

Let $F(q, \dot{q}) := C(q, \dot{q})\dot{q} + G(q)$ and recall that we have chosen our generalized coordinates so that they are naturally partitioned into actuated and unactuated parts per $q = (q_0, q_1)$. This induces a decomposition of the model (2)

$$\begin{array}{ccccccc} d_{0,0}\ddot{q}_0 & + & d_{0,1}\ddot{q}_1 & + & F_0 & = & 0 \\ d_{1,0}\ddot{q}_0 & + & d_{1,1}\ddot{q}_1 & + & F_1 & = & u, \end{array} \tag{6}$$

where $d_{i,j} = d_{i,j}(q)$ are entries of the inertia matrix $D(q)$. Define

$$\begin{array}{rcl} \bar{D} & = & d_{1,1} - d_{1,0}d_{0,1}/d_{0,0} \\ \bar{F} & = & F_1 - d_{1,0}F_0/d_{0,0} \\ R & = & -F_0/d_{0,0}, \end{array} \tag{7}$$

and the static state variable feedback

$$u = \bar{D}v + \bar{F}. \tag{8}$$

Using the fact that D is positive definite, it can be shown that this feedback is invertible (i.e., $\bar{D}$ is invertible)[4].

It is straightforward to check that applying (8) to (2) or (6) results in

$$\begin{array}{rcl} \ddot{q}_0 & = & J(q)v + R(q, \dot{q}) \\ \ddot{q}_1 & = & v, \end{array} \tag{9}$$

where $J = -\frac{d_{0,1}}{d_{0,0}}$. This is sometimes called the *Spong normal form*. The signal v is the new control for the model (9). The idea is that you can design a controller for (9) and them implement it on the original system (6) via the preliminary feedback (8). This gives an new way to think about the design of a controller in terms of an "inner-loop outer-loop" paradigm: the inner feedback loop partially linearizes the model, making it more amenable to your outer-loop design, which may be based on linearization, for example[5].

While the form of the model in (9) is quite nice, it is still somewhat complicated because R is quadratic in the velocity and the auxiliary control signal v enters both equations. An even more convenient form of the model is possible by using the notion of conjugate momentum in Lagrangian mechanics. Define

$$\sigma = \frac{\partial L}{\partial \dot{q}_0} = d_{0,0}(q)\dot{q}_0 + d_{0,1}(q)\dot{q}_1. \tag{10}$$

It follows from (1) that in the coordinates $(q_0, \sigma, q_1, \dot{q}_1)$ the model (9) takes the form

$$\begin{array}{rcl} \dot{q}_0 & = & \frac{\sigma}{d_{0,0}(q)} + J(q)\dot{q}_1 \\ \dot{\sigma} & = & \frac{\partial K}{\partial q_0}(q) - \frac{\partial V}{\partial q_0}(q) \\ \ddot{q}_1 & = & v. \end{array} \tag{11}$$

Equation (11) will be called the *modified Spong normal form*.

[4] This is true because $(\det \bar{D})d_{0,0} = \det D$ and $d_{0,0} \neq 0$ (the latter because D is positive definite).

[5] You could try to explore this idea on your project.

4 Specialization to the Segway

With appropriate choice and normalization of parameters, a model of a Segway is given by

$$\begin{bmatrix} 7\cos(q_0)+9.73 & 3.67+3.5\cos(q_0) \\ 3.67+3.5\cos(q_0) & 3.67 \end{bmatrix}\ddot{q} + \begin{bmatrix} -3.5\sin(q_0)\dot{q}_0 & 0 \\ -3.5\sin(q_0)\dot{q}_0 & 0 \end{bmatrix}\dot{q} + \begin{bmatrix} -7\sin(q_0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{12}$$

The modified Spong normal form is then

$$\begin{aligned} \dot{q}_0 &= \tfrac{\sigma}{7\cos(q_0)+9.73} - \tfrac{3.67+3.5\cos(q_0)}{7\cos(q_0)+9.73}\dot{q}_1 \\ \dot{\sigma} &= 7\sin(q_0) - 3.5\sin(q_0)\dot{q}_0(\dot{q}_0+\dot{q}_1) \\ \ddot{q}_1 &= v. \end{aligned} \tag{13}$$

and the pre-feedback is

$$u = -3.5\sin(q_0)\dot{q}_0^2 + \sin(q_0)\left(\frac{3.67+3.5\cos(q_0)}{7\cos(q_0)+9.73}\right)(7+3.5\dot{q}_0^2) + \left[3.67 - \frac{(3.67+3.5\cos(q_0))^2}{7\cos(q_0)+9.73}\right]v. \tag{14}$$

From these two equations you can see what this normal form does: it gives you a simplified design model (13) and pushes most[6] (though not all) of the difficult nonlinearities into the pre-feedback (14). Moreover, this simplification is *exact*; no approximations have been made at this stage. Using techniques developed later in the course, we could directly design a nonlinear controller using this form of the model, (13). Doing so on the original form of the model, (12), would be much more difficult.

Remark: For later use, note that the linearization of (13) about the upright equilibrium point is simply

$$\begin{aligned} \dot{q}_0 &= \tfrac{\sigma}{16.73} - \tfrac{7.17}{16.73}\dot{q}_1 \\ \dot{\sigma} &= 7q_0 \\ \ddot{q}_1 &= v. \end{aligned} \tag{15}$$

5 Problems

1. Design a stabilizing linear state variable feedback for the linearized model (15). Place the e-values of the closed-loop system at $[-1.7+j1.5, -1.7-j1.5, -0.7, -0.35]$ (result of an LQR solution from EECS 565). I am assuming that you will have no trouble writing the model in state variable form.
2. Provide simulations of the closed-loop system consisting of your linear feedback applied to the nonlinear model (13). Just pick two or three initial conditions. Note that your angles are in radians and not degrees when you choose initial conditions. It is optional but probably a good idea to first check your feedback on the closed-loop system consisting of your linear feedback applied to the linearized model (15) because you know it has to be stable.
3. Use (14) to apply your feedback to the Segway model (12). Provide simulations for the same initial conditions used above. You should obtain the same trajectories for ϕ and θ.

Remark: Note that in the last problem, you will have to compute σ from (10) in order to implement your feedback.

[6] It can be shown that this system cannot be exactly linearized through feedback and change of coordinates; partial linearization is all that is possible.

Remark: What have we gained by this procedure? First, we have kept all of the nonlinear terms in (14) without any simplifications. Second, we can see where the really difficult nonlinear term is, namely $\sin(\phi)$ in the derivative of σ: the term $\sin(\phi)$ vanishes when ϕ approaches $\pm\pi/2$, whereas our linear approximation does not capture that phenomenon.

Remark: In case you wish to animate your result, you can compute the horizontal displacement of the vehicle by $x = R(\phi + \theta)$, with $R = 0.5$. You can set the pendulum length to 1.0.

Remark: I have chosen parameters so that the system is pretty easy to control. You should obtain a large "domain of stability" for your closed-loop system.

References

[1] C. Chevallereau, J.W. Grizzle, and C.H. Moog. Nonlinear control of mechanical systems with one degree of underactuation. In *Proc. of the IEEE International Conference on Robotics and Automation, New Orleans, LA*, April 2004.

[2] J. W. Grizzle, C. H. Moog, and C. Chevallereau. Nonlinear control of mechanical systems with an unactuated cyclic variable. *IEEE Transactions on Automatic Control*, 30(5):559–576, May 2005.

[3] M. Reyhanoglu, A. van der Schaft, N.H. McClamroch, and I. Kolmanovsky. Dynamics and control of a class of underactuated mechanical systems. *IEEE Transactions on Automatic Control*, 44(9):1663–1671, 1999.

[4] M.W. Spong. Energy based control of a class of underactuated mechanical systems. In *Proc. of IFAC World Congress, San Francisco, CA*, pages 431–435, 1996.

[5] E. R. Westervelt, J. W. Grizzle, C. Chevallereau, J. H. Choi, and B. Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion.* Taylor & Francis/CRC Press, 2007.

Hints

Hints for Problem 2, part (ii). Prove that for any finite $T \geq t_0$, $\exists c < \infty$ s.t. $\forall t \in [t_0, T]$, $|a(t)| \leq c$. To do this, you need the Weierstrass Theorem (given below). You may use without proof the observation that $a(t)$ continuous implies that $|a(t)|$ is continuous, which, by the way, is true because the composition of continuous functions is also continuous.

Def: A set $S \subset \mathbb{R}^n$ is bounded if $\exists \kappa < \infty$ s.t. $\|x\| \leq \kappa$ for all $x \in S$, that is, $S \subset B_\kappa(0)$ for some finite κ.

Def: A set $S \subset \mathbb{R}^n$ is compact if it is closed and bounded.

Weirstrass Theorem: If $h : S \to \mathbb{R}$ is continuous and S is compact, then $\exists s_0 \in S$ s.t. $\sup_{x \in S} h(x) = h(s_0)$. [You do NOT need to prove this. You can find it in any book on real analysis.]

Remark: It follows that $\exists c < \infty$ s.t. $\sup_{x \in S} h(x) \leq c$ (indeed, just take $c = h(s_0)$)

Hints for Problem 3:, part (a). This is a bit painful, but it shows how one goes about bounding terms in an equation:

(i) The chain rule works for vectors:

$$\begin{aligned} \frac{d}{dt} x^T(t)\, x(t) &= \left(\frac{d}{dt} x(t)\right)^T x(t) + x^T(t) \left(\frac{d}{dt} x(t)\right) \\ &= 2 \left(\frac{d}{dt} x(t)\right)^T x(t) \\ &= 2x^T(t) \left(\frac{d}{dt} x(t)\right) \end{aligned}$$

If you want to prove this (not required): write $x^T(t)x(t)$ as a finite sum $\sum_{k=1}^n (x_k(t))^2$ and compute the derivative with respect to time! It is not difficult, once you write it out.

(ii) Note that $(\|x(t)\|_2)^2 = x^T(t)\, x(t)$

(iii) The following is called the Cauchy-Schwarz inequality: $|x^T y| \leq \|x\|_2 \|y\|_2$ for $x, y \in \mathbb{R}^n$

For part (b), note that from part (a),

$$-2LV(t) \leq \dot{V}(t) \leq 2LV(t).$$

Divide through by $V(t)$, multiply by dt, and integrate. This should get you going:

$$\begin{aligned} \dot{V}(t) &\leq 2LV(t) \\ &\Updownarrow \\ \frac{dV}{V} &\leq 2L dt \\ &\Updownarrow \\ \int_{V_0}^{V(t)} \frac{dV}{V} &\leq \int_{t_0}^{t} 2L d\tau \end{aligned}$$

Do the integrals on each side (very easy), and then exponentiate each side. Then do the other side of the inequality.