1)

$\dot{x} = a x$

$\Rightarrow \frac{dx}{x(t)} = a \cdot dt$

$\Rightarrow x = \phi(t, t_0)\, x_0.$

$\phi(t, t_0) = e^{\int_{t_0}^{t} a(t) \cdot dt}$

if asymptotically stable, is $V(t, x_t) \underset{t \to \infty}{\to} 0$

$\Rightarrow x(t) \to 0$ as $t \to \infty$ $\forall \|x(t_0)\| < c$, $c > 0.$

$\Rightarrow$ ~~[illegible]~~ $\Rightarrow x = \left(e^{\int_{t_0}^{t} a(t) \cdot dt}\right) x_0$

if $x(t) \to 0$ as $t \to \infty.$

$\Rightarrow \lim_{t \to \infty} \left(e^{\int_{t_0}^{t} a(t)\, dt}\, x_0\right) \to 0$

$\Rightarrow e^{\int_{t_0}^{t} a(t) \cdot dt} \to 0$

$\Rightarrow$ as $t \to \infty$ $x(t) \to 0.$

$\therefore$ $x(t)$ only takes $0$ as $t \to \infty$

$\therefore$ it is globally asymptotically stable

⇒ If its globally asymptotically stable

⇒ $x(t) \to 0$ as $t \to \infty$

$\forall\ \|x_0\| < c \qquad c > 0$

→ this also satisfies asymptotic conditions

So, its asymptotically stable too

2)
$$\dot{x} = \frac{-x}{1+t}$$

$$\Rightarrow \frac{dx}{x} = \frac{-dt}{1+t}$$

$$\Rightarrow \log\left(\frac{x}{x(t_0)}\right) = -\int_{t_0}^{t} \frac{1}{1+\tau}\, d\tau$$

$$= \log\left(\frac{1+t_0}{1+t}\right)$$

$$\Rightarrow x(t) = x(t_0)\left(\frac{1+t_0}{1+t}\right)$$

as $t \to \infty$ $x(t) \to 0$

$\therefore$ system is asymptotically stable

& for any given $x(t_0)$ & $t_0$ $x(t) \to 0$

as $t \to \infty$

$\therefore$ As it doesn't depend on initial conditions,

the $x=0$ is uniformly asymptotically stable

3)

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 + (x_1^4 - 2)x_2$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$V = x^T P x$$

$$= 3x_1^2 - 2x_1x_2 + x_2^2$$

$$\dot{V} = 2(3x_1\dot{x}_1 - x_1\dot{x}_2 - \dot{x}_1x_2 + x_2\dot{x}_2)$$

$$= 2(x_2^2 - \cancel{3x_1x_2} - x_1^2 + \cancel{2x_1x_2}$$
$$- x_1^5x_2 + \cancel{x_1x_2} + x_2^2x_1^4 - 2x_2^2)$$

$$= 2(-x_1^2 - x_2^2 - x_1^5x_2 + x_2^2x_1^4)$$

$$= 2(-x_1^2(1 - x_1^3x_2) - x_2^2(1 - x_1^4))$$

let $a = 1 - x_1^3x_2$ $\quad b = 1 - x_1^4$

$$\dot{V} = -2(ax_1^2 + bx_2^2)$$

if $\|x\| < 1 \Rightarrow a > 0$
$b > 0$

$\Rightarrow \quad (a \cdot m_i^2 + b m_i^2) > 0$

$\Rightarrow \quad \dot{V} < 0$

$\therefore$ the domain $D = B_{r_1}(0)$

$\Rightarrow \quad \lambda_{min} r^2 < V < \lambda_{max} r^2$

$\Rightarrow \quad \cancel{\lambda_{max}} \quad c^* = \lambda_{min} r^2$

$\Rightarrow \quad c^* = \lambda_{min} = 0.5858$

for mor conservative sol^n

$$r^* = \sqrt{\frac{c^*}{\lambda_{max}}} = \sqrt{\frac{0.5858}{3.414}}$$

we can say ROE as

$\cancel{ROE} \quad \{ x \in B_{r^*}(0) \} \quad (or) \quad \{ x : (V < c^*) \}$.

4) $\dot{x}_1 = x_2$

$\dot{x}_2 = -x_2 - (x_1 - x_1^3)$

$$V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} (y - y^3)\,dy$$

$$\Rightarrow \cancel{\dot{V}(x)} \quad V(x) = \frac{x_2^2}{2} + \frac{x_1^2}{2} - \frac{1}{4}x_1^4$$

$$\dot{V}(x) = x_2\dot{x}_2 + x_1\dot{x}_1 - x_1^3\dot{x}_1$$

$$= x_2\left(-x_2 - (x_1 - x_1^3)\right)$$

$$+ x_1x_2 - x_1^3x_2$$

$\boxed{\dot{V}(x) = -x_2^2}$ $\Rightarrow$ $\dot{V}(x)$ is -ve semi definite

$\dot{V}(x) = 0$ when $x_2 = 0$ & $\forall x_1 \in \mathbb{R}$

According to lasalle's theorm

~~if~~

the system will remain in the invariant set

so. instead of searching for domain where $\dot{V}(x) < 0$ we find a Domain $D$ where $\{0\}$ is the largest invariant set

$\Rightarrow$ if we Define $B_r(0)$ where $r = 1$ then it will be a domain where largest invariant set is $\{0\}$

So inside this $r < 1$

if we take $V(x)$,

$$V(x) < \frac{1}{2}(x_1^2 + x_2^2)$$

as $V(x) = \frac{1}{2}(x_1^2 + x_2^2) - \frac{x_1^4}{4}$.

& as $r < 1 \Rightarrow |x_1| < 1$

$\Rightarrow$ $x_1^4 < x_2^2$

$$\Rightarrow \frac{m_1^2+m_2^2}{2} - \frac{m_1^2}{4} < V(m)$$

$$\Rightarrow V(m) > \frac{m_1^2}{4} + \frac{m_2^2}{2}$$

so to find $c^*$

$$\Rightarrow c^* = \min_{\|m\|=r} V$$

as here $\partial D = \partial B_r(0)$ $r=1$

$$\Rightarrow r=1$$

$$\Rightarrow c^* = \min_{\|m\|=1} V = \min_{\|m\|=1} \frac{m_1^2}{4} + \frac{m_2^2}{2}$$

$$= \min_{\|m\|=1} \left(\frac{m_1^2+m_2^2}{4}\right) + \left(\frac{m_2^2}{4}\right)$$

$$= \frac{1}{4} + \min_{\|m_2\|=1} \frac{m_2^2}{4}$$

$$\boxed{c^* = \frac{1}{4}}$$

$\therefore$ ROC

can be defined as:

$$\Omega_c = \{ x_0 \in D \mid \text{where } V(x) < c^* \}$$

where $\boxed{c^* = 1/4}$

5) for eq$^{m}$ points

$$\dot{x} = \begin{bmatrix} x_2 \\ -2\sin(x_1) - 0.5x_2 + 1.3 \end{bmatrix}$$

$\dot{x} = 0 \Rightarrow x_2 = 0$

$-2\sin x_1 - 0.5x_2 + 1.3 = 0.$

$-2\sin x_1 + 1.3 = 0.$

$\Rightarrow \sin(x_1) = 1.3/2$

$x_1 = n\pi \pm 0.707$

where $n$ is integer

if $\begin{cases} x_1 = 2n\pi \pm 0.707 \\ x_2 = 0. \end{cases}$

6)

$\Rightarrow e = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

By observing jacobian calculated in matlab

we can say these points are stable

& for $x_1 = (2n+1)\pi + 0.707$

$x_2 = 0$ These are unstable

b) $\therefore \quad x_s = \begin{bmatrix} 0.707 \\ 0 \end{bmatrix}$ in the interval

$x_1 \in [-2.5, 3.5]$

$$V(x) = \frac{1}{2}(x_2 - x_2^s)^2 - 2(\cos(x_1) - \cos(x_1^s)) - 1.3(x_1 - x_1^s)$$

$$\dot{V}(x) = (x_2 - x_2^s)\,\dot{x}_2 + 2\sin x_1\,\dot{x}_1 - 1.3\,\dot{x}_1$$

$$= x_2(-2\sin x_1 - 0.5x_2 + 1.3) + 2\sin x_1\, x_2 - 1.3x_2 = -0.5x_2^2$$

$\Rightarrow \quad \dot{V}(x) = -0.5x_2^2$

is negative semi-definite

But if we define our domain

$$D = \{x \in D \mid -2.5 \leq x_1 < 3.5\}$$

$\Rightarrow$ The only invariant set is $\{x^s\}$

So ~~as~~ similar to prev question we can claim the ROA

## Question 5:

**Part b:**

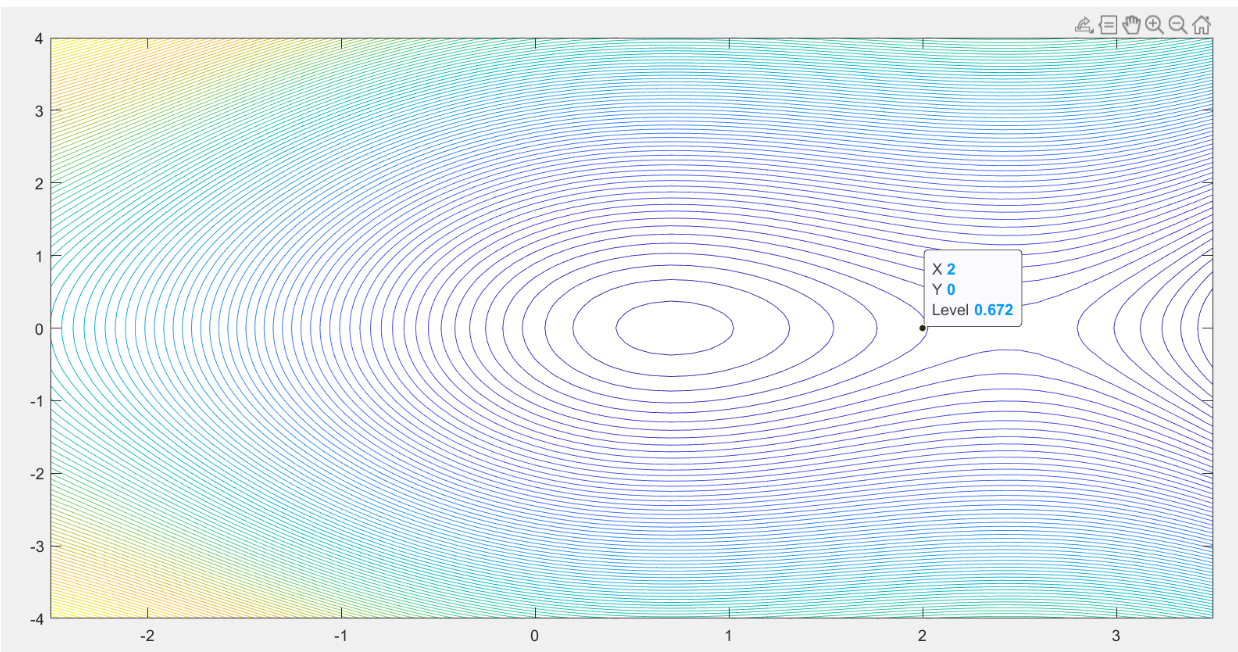
From the plot of the Lyapunov functions we can see that, the in defined region the contour

V(x) < 0.672 ; can be safely assumed as  the region of attraction

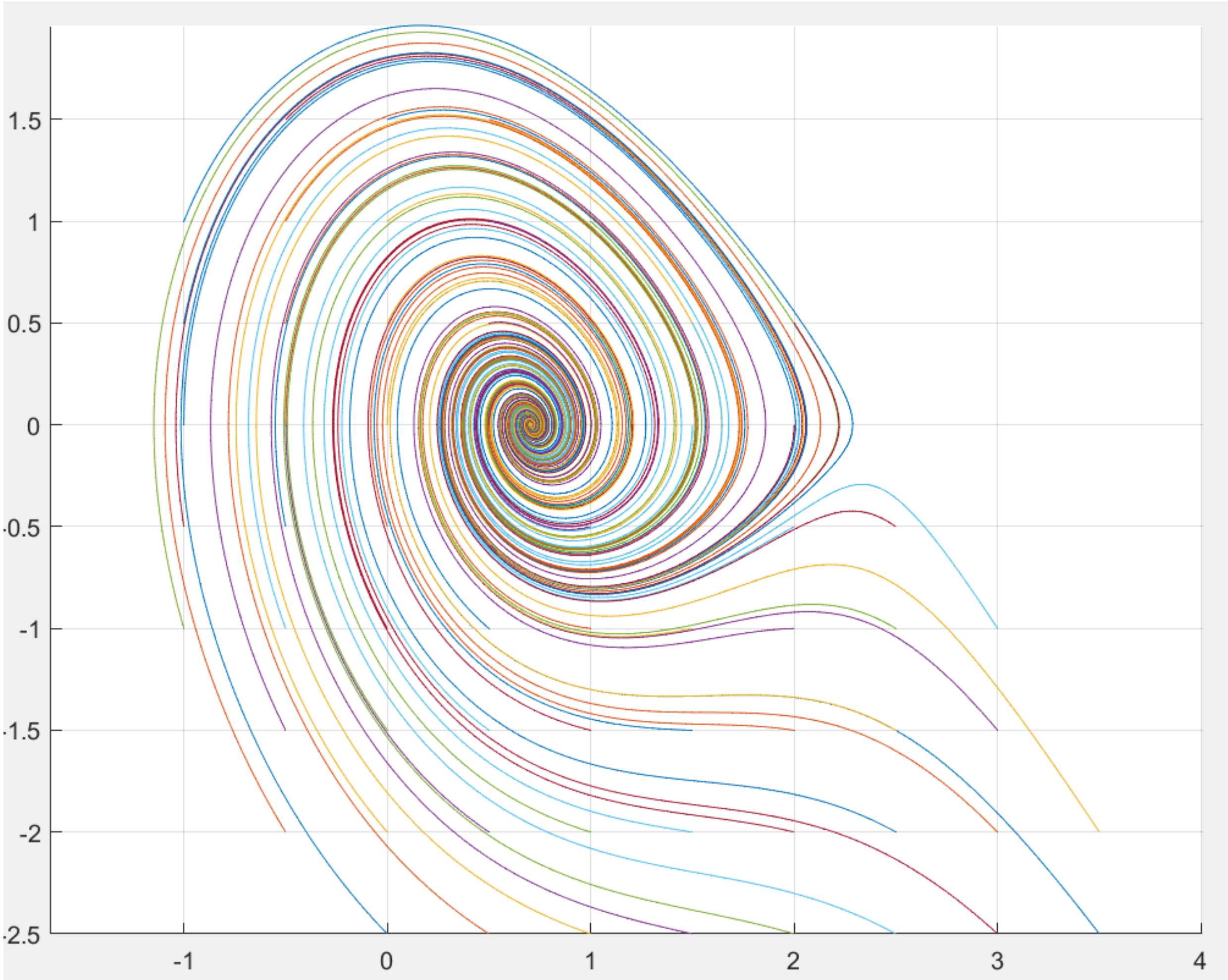
For **part C,** the **maximum value of lyapunov** obtained from **numerical analysis is 0.672**

This is because as we saw that

- the derivative is negative semi definite over the whole domain
- [0.707,0] is the only point in invariant set
- And v(x) =0.672 is the maximum compact set found in the domain through the plot




But if we observe the actual region of attraction its different as shown below:




**Code:**

```
%% Sub section 1
epsilon = 0.2;
samp = 10;

t = 0:0.01:30;

optPos = odeset('RelTol', 1e-6, 'AbsTol',1e-6);

x = -2.5:0.5:3.5;
y = -2.5:0.5:3.5;
[X,Y] = meshgrid(x,y);
```

```
ind = size(X);

fig = figure();
for i=1:ind(1)
    for j=1:ind(2)

        x0 = [X(i,j);Y(i,j)];
        [tP, x_new] = ode45(@(t,x)dyn(t,x), t, x0, optPos);
        if((abs(x_new(end,1)-0.707)<0.1) && (abs(x_new(end,2))<0.1))
            hold on
            plot(x_new(:,1),x_new(:,2))
        end
        % quiver(x_new(1:samp:end,1),x_new(1:samp:end,2))

    end
end
hold off
axis equal
grid on



function x_dot = dyn(t,x)
x_dot = [x(2); (-2*sin(x(1)))- (0.5*x(2)) + 1.3];
end
```

—-----------------------------------------------------------------

```
x = -2.5:0.1:3.5;
y = -4:0.1:4;
[X,Y] = meshgrid(x,y);

xs = 0.707;
ys = 0;

X_dot = Y;

F = ((0.5*(Y.*Y) -2*(cos(X)-cos(xs)) - 1.3*(X-xs)));

G = (((Y.*(-2*sin(X)- 0.5*Y + 1.3)) +2*(sin(X).*Y) - 1.3*(Y)));

R = sqrt((X-xs).^2 + (Y-ys).^2);
```

```
figure();
contour(X,Y,F,100);
```