Sequences, completeness, and contraction mapping theorem

ROB 501

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- Sequences
  - Cauchy sequences
- Complete spaces and complete sets
- Newton Raphson Algorithm
- Contraction mapping

From last lecture

Definition of convergence and limit points:

<u>Def</u>: A sequence <u>converges</u> to a point $x \in \mathcal{X}$ if $\forall \varepsilon$, $\exists \, \underline{N}(\varepsilon) < \infty$ s.t. $\forall n \geq N$, $\|x - x_n\| < \varepsilon$.

Notation: $x = \lim_{n \to \infty} x_n$

or $x_n \xrightarrow[n \to \infty]{} x$ or $x_n \to x$

<u>Def</u>: Let $P \subset \mathcal{X}$ and $x \in \mathcal{X}$. Then, x is <u>limit point</u> of P if $\exists$ a sequence (x_n) satisfying:

a) $\forall n \geq 1$, $x_n \in P \setminus \{x\}$

b) $x_n \to x$

Proposition: x is a limit point of $P \iff x \in \overline{P\setminus\{x\}}$

closure of $P\setminus\{x\}$

Proof:

($\Longrightarrow$) If x is a limit point of P, $\exists$ a sequence (x_n) s.t. $\forall n \geq 1$, $x_n \in P\setminus\{x\}$ and $x_n \to x$.

$\therefore \forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N$ $\|x_n - x\| < \varepsilon$.

$\Rightarrow d(x, P\setminus\{x\}) = 0$

$\Longrightarrow x \in \overline{P\setminus\{x\}}$

Ex: $P = [1,2) \cup \{3,5\}$

Let's consider $\{1\}$.

$\overline{P\setminus\{1\}} = [1,2] \cup \{3,5\}$

$1 \in$

↳ limit point

We can consider the sequence

$x_n = 1 + \frac{1}{2n} \in P\setminus\{1\}$ $\forall n$

$x_n \to 1$

— o —

Let consider $\{3\}$:

3 is not a limit point

$3 \notin \overline{P\setminus\{3\}} = [1,2] \cup \{5\}$

3 is called an isolation point.

— o —

The set of limit points of P is $[1,2]$

is 2 a limit point of P?

Proof 1: Yes. Here is a sequence

$x_n = 2 - \frac{1}{2n}$

$x_n \in P\setminus\{x\}$ $\forall n \geq 1$

$x_n \to 2$

Alternative proof (using the proposition):

$P\setminus\{2\} = P = [1,2) \cup \{3,5\}$

$\overline{P\setminus\{2\}} = [1,2] \cup \{3,5\}$

$2 \in \overline{P\setminus\{2\}}$ ✓

Ex 2: $P = (1,5)$

The set of limit points of $P'' = \overline{P}$

$(\Leftarrow)$ Suppose $x \in \overline{P \setminus \{x\}}$, then $d(x, P\setminus\{x\}) = 0$.

Hence, $\forall n < \infty$ $\exists x_n \in P\setminus\{x\}$ s.t. $\|x - x_n\| < \frac{1}{n}$.

(We construct a sequence x_n by taking

$x_n \in \left(B_{\frac{1}{n}}(x) \cap (P\setminus\{x\}) \right)$

To argue x_n is well-defined for all n, we need to show $B_{\frac{1}{n}}(x) \cap (P\setminus\{x\}) \neq \emptyset$ but this is true because $x \in \overline{P\setminus\{x\}}$.)

$\therefore x_n \longrightarrow x$ and $x_n \in P\setminus\{x\}$

$\Longrightarrow x$ is a limit point of P.

<u>Corollary</u>: P is closed $\Longleftrightarrow$ P contains all of its limit points

The main drawback of the notion of converging sequences is that you have to know a priori the limit x to prove/check $\|x_n - x\| \to 0$ (i.e. $x_n \to x$).

<u>Def:</u> A sequence (x_n) is Cauchy if, $\forall \varepsilon > 0$, $\exists N(\varepsilon) < \infty$ s.t. $\forall n, m \geq N$, $\|x_n - x_m\| < \varepsilon$

* Cauchy definition only depends on the elements of the sequence.

<u>Notation:</u> $\|x_n - x_m\| \xrightarrow[n,m\to\infty]{} 0$

<u>Proposition:</u> If $x_n \to x$, then (x_n) is Cauchy.

<u>Proof idea:</u> $\|x_n - x_m\| = \|x_n - x + x - x_m\|$

$$\leq \underbrace{\|x_n - x\|}_{< \frac{\varepsilon}{2}} + \underbrace{\|x_m - x\|}_{< \frac{\varepsilon}{2}} \to 0$$

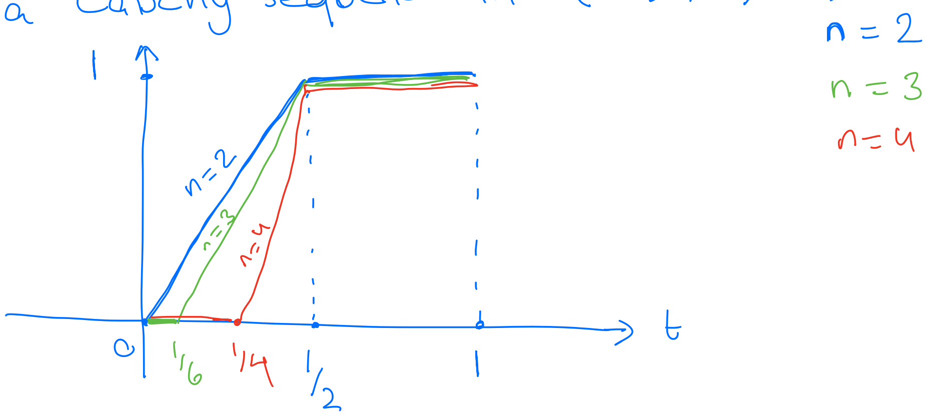
<u>Question:</u> Do all Cauchy sequences have limits?

Unfortunately not!

<u>Ex:</u> $\mathcal{X} = \{ f : [0,1] \to \mathbb{R} \mid f \text{ continuous} \} = C[0,1]$ continuous.

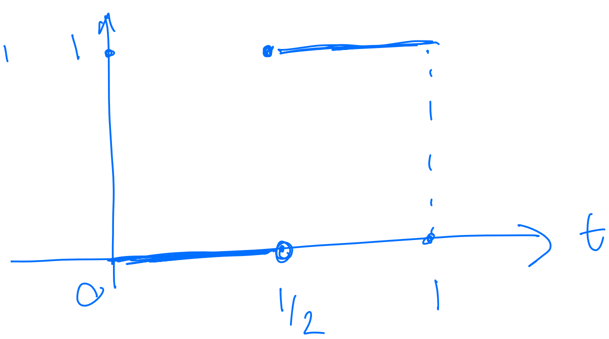
$$f_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ 1 + n\cdot\left(t - \frac{1}{2}\right) & \text{if } \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \text{if } t \ge \frac{1}{2} \end{cases}$$

f_n is a Cauchy sequence in $(C[0,1], \|\cdot\|_1)$



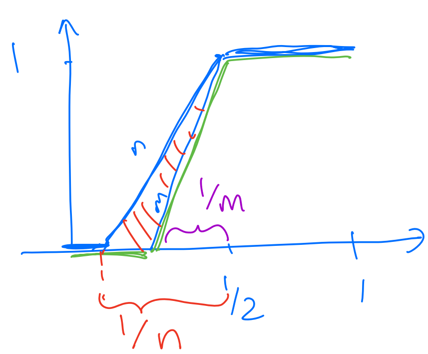
$\lim_{n\to\infty} f_n$ $\Downarrow$ step func $\notin C[0,1]$

w.r.t. $\|\cdot\|_1$



1. Why f_n is Cauchy in $(C[0,1], \|\cdot\|_1)$?

$$\|f_n - f_m\|_1 = \int_{t=0}^{1} |f_n(t) - f_m(t)|\,dt$$



$$\|f_n - f_m\|_1 = \frac{\left(\frac{1}{n} - \frac{1}{m}\right)\cdot 1}{2} = \frac{m-n}{2m\cdot n} \xrightarrow[m,n\to\infty]{} 0$$

$\longrightarrow$ f_n is Cauchy in $(C[0,1], \|\cdot\|_1)$.

but does not have a limit in $C[0,1]$.

<u>Prop:</u> Let $(X, \|\cdot\|)$ be a normed space. If X is <u>finite-dimensional</u>, then every Cauchy sequence has a limit in X.

<u>Def:</u> A normed space is <u>complete</u> if every Cauchy sequence has a limit in the given normed space. A complete normed space is called a <u>Banach space</u>.

<u>Ex:</u> $(C[0,1], \|\cdot\|_1)$ is <u>NOT</u> a Banach space

(i.e. <u>not</u> complete.)

Bc. we just saw a Cauchy sequence in this space without limit in it.

<u>Ex:</u> $(C[0,1], \|\cdot\|_\infty)$ is Banach space (i.e. complete).

(recall $\|f\|_\infty = \sup_{t\in[0,1]} f(t)$)

Our earlier example:

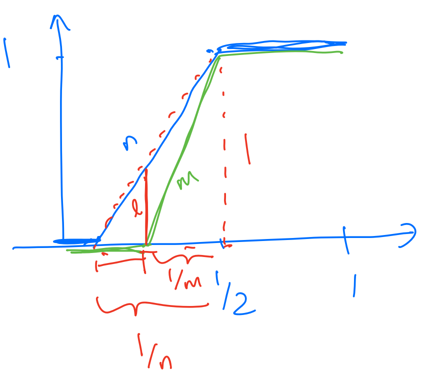
$$f_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ 1 + n\cdot(t - \frac{1}{2}) & \text{if } \frac{1}{2} - \frac{1}{n} \le t \le \frac{1}{2} \\ 1 & \text{if } t \ge \frac{1}{2} \end{cases}$$

f_n is not a Cauchy sequence in $(C[0,1], \|\cdot\|_\infty)$!

To why

$$l = \|f_n - f_m\|_\infty = \sup_{t\in[0,1]} |f_n(t) - f_m(t)| = \frac{\frac{m-n}{nm}}{\frac{1}{n}} = \frac{m-n}{m}$$

assume WLOG $m > n$



from the red triangle $\frac{l}{1} = \frac{\frac{1}{n} - \frac{1}{m}}{\frac{1}{n}}$

$\Rightarrow$ for any N if we pick $m = 2n$

$\Rightarrow \|f_n - f_m\|_\infty = \frac{2n-n}{2n} = \frac{1}{2}$

So, the distance cannot be made arbitrarily small (for $\varepsilon < \frac{1}{2}$, it does not work)

<u>Def:</u> (complete sets). Let $(X, \|\cdot\|)$ be a normed space. $S \subset X$ is <u>complete</u> if every Cauchy sequence constructed from elements of S has a limit in S.

<u>Prop:</u> Let $(\mathcal{X}, \|\cdot\|)$ be a normed space.

(a) If $S \subset \mathcal{X}$ is complete, then S is closed.

(b) If $(\mathcal{X}, \|\cdot\|)$ is complete and $S \subset \mathcal{X}$ is closed, the S is complete.

(c) All finite-dimensional subspaces of $\mathcal{X}$ are complete.

ROB 501 Handout: Grizzle

$h(x) = y$

Newton Raphson Algorithm

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, and satisfy

$$\boxed{\det\left(\frac{\partial h}{\partial x}(x)\right) \neq 0 \quad \forall x \in \mathbb{R}^n}$$

Problem: For $y \in \mathbb{R}^n$ fixed, find a solution of $\boxed{y = h(x)}$; i.e, find $x^* \in \mathbb{R}^n$ s.t. $y = h(x^*)$. We note that this is equivalent to $h(x^*) - y = 0$. In other words, we are looking for a root of the equation $h(x) - y = 0$,

Approach: Find a convergent sequence $x_k \to x^*$ such that

$$\boxed{\lim_{k\to\infty} h(x_k) - y = h(x^*) - y = 0}$$

that is, $x^* = \lim_{k\to\infty} x_k$ is a root of $h(x) - y = 0$

Idea: Write $x_{k+1} = \boxed{x_k + \Delta x_k}$. We want

$$h(x_{k+1}) - y = \underbrace{h(\overbrace{x_k + \Delta x_k}^{x_{k+1}}) - y \approx 0.}$$

What should Δx_k look like ?

$h(x_k + \Delta_k) \approx h(x_k) + \frac{\partial}{\partial x} h(x_k) \Delta x_k + \text{H.O.T.}$

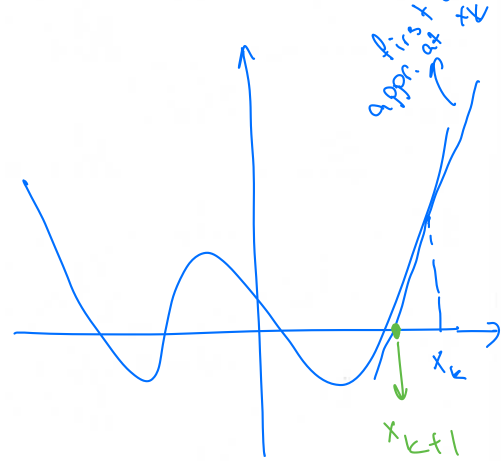
first order appr. at x_k

Apply Taylor's Theorem, to get

$$\boxed{h(x_k) + \frac{\partial h}{\partial x}(x_k)\Delta x_k} - y \approx 0$$

$$\therefore \frac{\partial h}{\partial x}(x_k)\Delta x_k \approx -(h(x_k) - y)$$

$$\Delta x_k \approx -\left(\frac{\partial h}{\partial x}(x_k)\right)^{-1}(h(x_k) - y)$$



Recalling that $x_{k+1} = x_k + \Delta x_k$, we arrive at Newton's Algorithm,

$$\boxed{x_{k+1} = x_k - \left(\frac{\partial h}{\partial x}(x_k)\right)^{-1}(h(x_k) - y)}$$

In practice, the change in x_k given by $\Delta x_k = -\left(\frac{\partial h}{\partial x}(x_k)\right)^{-1}(h(x_k) - y)$ is often too large. Hence, one uses the so-called <u>Damped Newton Algorithm</u>

$$\boxed{x_{k+1} = x_k - \epsilon\left(\frac{\partial h}{\partial x}(x_k)\right)^{-1}(h(x_k) - y)}$$

where $\epsilon > 0$ provides step size control!

<u>**Remark:**</u> Looking ahead to our discussion of contraction mappings, let's rewrite the algorithm as the iteration of a mapping $x_{k+1} = P(x_k)$

$$\boxed{P(x) := x - \epsilon\left(\frac{\partial h}{\partial x}(x)\right)^{-1}(h(x) - y)}$$

A solution of $h(x) - y$ is a <u>fixed point</u> of $P(x)$. Indeed,

$$\begin{gathered}
x^* = P(x^*) \\
\Updownarrow \\
x^* = x^* - \epsilon\left(\frac{\partial h}{\partial x}(x^*)\right)^{-1}(h(x^*) - y) \\
\Updownarrow \\
0 = -\epsilon\left(\frac{\partial h}{\partial x}(x^*)\right)^{-1}(h(x^*) - y) \\
\Updownarrow \\
0 = (h(x^*) - y).
\end{gathered}$$

It can be shown that P is a <u>local contraction</u> on an open ball around a solution of $h(x) - y = 0$.

Example Find the solution to the coupled NL equations

$$0 = h(x) = \begin{pmatrix} x_1 + 2x_2 - x_1(x_1 + 4x_2) - x_2(4x_1 + 10x_2) + 3 \\ 3x_1 + 4x_2 - x_1(x_1 + 4x_2) - x_2(4x_1 + 10x_2) + 4 \\ \sin(x_3)^7 + \frac{\cos(x_1)}{2} \\ {x_4}^3 - 2{x_2}^2 \sin(x_1) \end{pmatrix}$$

Initial Guess: $x_0 = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 10 \end{bmatrix}$

We do 16 iterations of Newton's Algorithm (a nonlinear root finding algorithm) and we obtain:

$$x^* = \begin{pmatrix} -2.25957308738366677539068499960 \\ 1.75957308738366677539068499960 \\ 189.5095410061333978330549312824 \\ -1.68458069860197189523093013800 \end{pmatrix}$$

And the error is:

$$h(x^*) = \begin{bmatrix} 3.6734198 \times 10^{-39} \\ 2.9387359 \times 10^{-39} \\ 1.2765134 \times 10^{-38} \\ -2.5915832 \times 10^{-32} \end{bmatrix}$$

Rob 501 Handout: Grizzle
A Useful Cauchy Sequence in $(\mathbb{R}, |\cdot|)$

Proposition Let $0 \leq c < 1$ and let (a_n) be a sequence of real numbers satisfying, $\forall\, n \geq 1$,

$$|a_{n+1} - a_n| \leq c|a_n - a_{n-1}|.$$

"contracting"

Then (a_n) is Cauchy in $(\mathbb{R}, |\cdot|)$.

Proof:

Claim 1: $\forall\, n \geq 1$, $|a_{n+1} - a_n| \leq c^n |a_1 - a_0|$.

Proof: First observe that

$$|a_3 - a_2| \leq c\underbrace{|a_2 - a_1|}_{\leq c \cdot |a_1 - a_0|} \leq c^2 |a_1 - a_0|.$$

Then complete the proof by induction.

Claim 2: $\forall\, n \geq 1, k \geq 1$, $|a_{n+k} - a_n| \leq \frac{c^n}{1-c}|a_1 - a_0|$.

Proof:

$$\begin{aligned}
|a_{n+k} - a_n| &\leq |a_{n+k} - a_{n+k-1} + a_{n+k-1} - a_{n+k-2} + \cdots a_{n+1} - a_n| \\
&\leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots |a_{n+1} - a_n| \\
&\leq c^{n+k-1}|a_1 - a_0| + c^{n+k-2}|a_1 - a_0| + \cdots + c^n |a_1 - a_0| \\
&\leq c^n \Big(\sum_{i=0}^{k-1} c^i \Big) \; |a_1 - a_0| \\
&\leq c^n \Big(\sum_{i=0}^{\infty} c^i \Big) \; |a_1 - a_0| \\
&\leq c^n \Big(\frac{1}{1-c} \Big) \; |a_1 - a_0| \\
&\leq \frac{c^n}{1-c} \; |a_1 - a_0|
\end{aligned}$$

Claim 3: (a_n) is Cauchy

Proof: Consider m and n. WLOG, suppose $m \geq n$. If $m = n$, then $|a_m - a_n| = 0$. Thus assume $m = n + k$ for some $k \geq 1$. Then

$$|a_m - a_n| = |a_{n+k} - a_n| \leq \frac{c^n}{1-c}|a_1 - a_0| \xrightarrow[n\to\infty, m\to\infty]{0},$$

and thus it is Cauchy.

Remark: Because WLOG we could assume $m \geq n$, from $n \to \infty$, we have both $n \to \infty$ and $m \to \infty$.