Exam Number:________

ROB 501 Exam-I

From Thursday, October 28, 2021 NOON to Friday, October 29, 2021 11:59pm

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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RULES:

1. The exam is open book, open lecture handouts and slides, open recitation notes, open HW solutions, open internet (under the communication and usage restrictions mentioned below).
2. If you use MATLAB or any other scientific software to complete some parts of the exam. You are required to submit your script along with your solution in such case.
3. You are not allowed to communicate with anyone other than the Course instructor and the GSIs related to the exam during the entire period. If you have questions, you can post a private Piazza post for the instructors or email `necmiye@umich.edu` with GSIs on cc.
4. You are not allowed to use any online "course helper" sites like Chegg, Course Hero, and Slader, in any part of the exam. You are not allowed to post exam questions on the internet or discuss them online.
5. Please do not wait until the last minute to upload your solution to Gradescope and double-check to make sure you uploaded the correct pdf. If you run into problems with Gradescope, email your .pdf file as an attachment to Prof. Ozay as soon as practicable at `necmiye@umich.edu`.

SUBMISSION AND GRADING INSTRUCTIONS:

1. The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly.
2. You must submit your solutions in a single pdf. You will be asked to mark where each solution is.
3. **Honor Code:** The first page of your submitted pdf should include a hand-written and signed honor code (see the first page of this pdf). Without this, your exam will not be graded.
4. **For problems 1-5** Use this page to record your answers. We will NOT grade other pages and we do not care if you make a mistake when copying your answers to this page. Please be careful. If you are submitting handwritten (or word-processed) documents, make sure to make a similar table where you record all your True/False answers. There is no partial credit on these questions. You are welcome to leave some justification but we will not look at them.
5. **For problems 6-7** Record your final answer in the box provided. If you are submitting handwritten (or word-processed) documents, make sure to box or highlight the final result. However, you MUST show your work to get credit. In other words, a correct result with no reasoning or wrong reasoning could lead to no points.
6. **For problems 8a, 8b** These are proof questions. You should show all the steps of your proof carefully.

Answers for the True/False Part				
	(a)	(b)	(c)	(d)
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				

Problems 1 - 5 (30 points: 5×6)

Instructions. For each problem, you should select True or False. **Make sure to record your answers on the second page. Only the second page will be graded!!!**

$\neg(s_1(t) \neq s_2(t)) \equiv \neg[(s_1(t) < s_2(t)) \vee (s_1(t) > s_2(t))]$

$\equiv \neg(s_1(t) < s_2(t)) \wedge \neg(s_1(t) > s_2(t)) \equiv (s_1(t) \geq s_2(t)) \wedge (s_1(t) \leq s_2(t))$

$\equiv s_1(t) = s_2(t)$

1. (Questions on logic and proof methods) Recall that $\wedge$ is 'and', $\vee$ is 'or', and $\neg$ is 'not'. Recall also that the symbol $\Leftrightarrow$ and the written text, "if, and only if", "logically equivalent to", and "have the same truth table", all mean the same thing. For example, in HW, you verified that $\neg(p \wedge q)$ is "logically equivalent to" $(\neg p) \vee (\neg q)$ by proving "they have the same truth table". **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T **F** (a) Negation of "The sky is blue if and only if the grass is green" is "(The sky is not blue and the grass is green) or (The sky is blue and the grass is not green)".

$p \Longleftrightarrow q$ $\quad \neg p \wedge q$ $\quad p \wedge \neg q$

T **F** (b) Let $s_1 : [0, T] \to \mathbb{R}$ and $s_2 : [0, T] \to \mathbb{R}$ be two real valued functions, and let $B \subset \mathbb{R}$. Then,

$$p \quad \boxed{\neg(\forall t \in [0, T],\ \underbrace{(s_1(t) \notin B)}_{a} \wedge \underbrace{(s_2(t) \notin B)}_{b} \wedge \underbrace{(s_1(t) \neq s_2(t))}_{c})} \Leftrightarrow (\exists t \in [0, T],\ (s_1(t) \in B) \vee (s_1(t) = s_2(t)) \vee (s_2(t) \in B)) \quad \vee$$

T **F** (c) You seek to show $\boxed{p \implies q}$ by employing the method of *Proof by Contradiction*. This means that you assume that p is FALSE and q is TRUE, and then seek to deduce a logical statement R that is both TRUE and FALSE.

T **F** (d) The truth table given below is correct for $\neg p$ implies q:

1.b $P \equiv \exists t \in [0,T] \neg(a \wedge b \wedge c)$

$\equiv \exists t \in [0,T]\ (\neg a) \vee (\neg b) \vee \neg(c)$

$\equiv \exists t \in [0,T]\ (s_1(t) \in B) \vee (s_2(t) \in B) \vee (s_1(t) = s_2(t))$

p	q	$\neg p \implies q$	
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	0

$\to p \vee q$

1.a) p: the sky is blue

q: the grass is green

$\neg(p \Leftrightarrow q) = \neg((p \Rightarrow q) \wedge (q \Rightarrow p))$

$\neg((\neg p \vee q) \wedge (\neg q \vee p))$

$(\neg(\neg p \vee q) \vee \neg(\neg q \vee p))$

$(p \wedge \neg q) \vee (q \wedge \neg p)$

1.c. want to show $\boxed{p \Rightarrow q}$ by proof by contradiction

Assume p is true ~~but~~ and q is false.

$(p \Rightarrow q) \equiv (\neg p \vee q)$

By contradiction $\neg(\neg p \vee q) \equiv p \wedge \neg q$

1.d. $(\neg p \Rightarrow q)$

$\equiv (p \vee q)$

2. (Eigenvalues and eigenvectors, linear independence) **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T **F** (a) Let $A \in \mathbb{R}^{n \times n}$. If a nonzero vector v is in the nullspace of A (i.e., $v \in \mathcal{N}(A)$), then v is an eigenvector of A.

T **F** (b) For all $x \in \text{span}\{v^1, v^2, \ldots, v^m\}$ for $m \leq n$ where each v^i is an eigenvector of a $n \times n$ real matrix, there exist unique coefficients $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such $x = \sum_i^m \alpha_i v^i$.

T **F** (c) If matrix A has repeated eigenvalues, then A is always not diagonalizable.

T **F** (d) Let I denote the $n \times n$ identity matrix. For all $x \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}$, if $A = xx^\top + \alpha I$, then $\{x, Ax\}$ must be linearly dependent over $\mathbb{R}$.

2.a. $v \neq 0$ $\quad v \in \mathcal{N}(A) \Rightarrow v \neq 0$ $\quad Av = 0 \Rightarrow v \neq 0$ $\quad Av = 0 \cdot v \Rightarrow$

0 is an eigenvalue of A and v is the corresponding e-vector

2.b. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad v^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad v^2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$x \in \text{span}\{v^1, v^2\}$, there is no unique way of writing $x = \alpha_1 v_1 + \alpha_2 v_2$

$= (\alpha_1 + 2\alpha_2) v_1 + 0 \cdot v_2$

$\therefore v^i$'s can be linearly dependent. For uniqueness we need linear independence

2.c. Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = \lambda_2 = 1$ repeated e-values but A is already diagonal.

Any $v \in \mathbb{R}^2$ is an e-vector of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ b.c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v = 1 \cdot v$

2.d. Let's check $Ax = (xx^\top + \alpha I)x = x(x^\top x) + \alpha x = (\|x\|_2^2 + \alpha)x \to$ scaled version of x $\forall \alpha$

$\Leftrightarrow \{x, bx\}$ are always linearly dependent.

3. (Matrix properties) **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T **F** (a) Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{n\times n}$ be invertible and let A and D be square. Then, A and D are invertible.

p.d.

T **F** (b) Let $M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \in \mathbb{R}^{2k\times 2k}$ be a symmetric positive definite matrix with $A, C \in \mathbb{R}^{k\times k}$. Then $A - BC^{-1}B^\top + C$ is always positive definite.

p.d.

T **F** (c) Suppose P is an $n\times n$ real symmetric positive definite matrix, and Q be an $n \times n$ orthogonal matrix. In the vector space $(\mathbb{R}^n, \mathbb{R})$, $< x, y >= x^\top QPQ^\top y$ satisfies all the conditions of inner product.

T **F** (d) Let A and B be $n\times m$ real matrices.[1] Then, $[A^\top B]_{ij} = (A_i)^\top B_j$.

3.a. Take $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ where $A=1$, $D=0$, M is invertible, D is not invertible.

3.b. Schur complement formula tells us. $C > 0$ and $A - BC^{-1}B^T > 0$. (sum of two p.d. matrices is p.d.

3.c. 1st way: QPQ^T (since $Q^T = Q^{-1}$, this is a similarity transformation and QPQ^T will have the same e-values as P (i.e., they are all real and positive).

$(QPQ^T)^T = (Q^T)^T P^T Q^T = QPQ^T$ symmetric.

3.d. $[A^TB]_{ij}$ is ith row of A^T times jth column of B

2nd way: show symmetry, linearity, non-negativity

ith row of A^T = (ith column of A) transposed.

$n=1$
$m=2$

4. (Inner product spaces, norms, projection theorem) **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

trace(XY) = trace(YX) if X and Y has the same dim

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **T** **F** (a) In $(\mathbb{R}^{n\times n}, \mathbb{R})$, $\rho(A) = |\lambda_{max}(A)|$ is a norm.[2] (spectral radius)

$b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **T** **F** (b) Consider the inner product space $(\mathbb{R}^{n\times n}, \mathbb{R}, < \bullet, \bullet >)$ with inner product defined as $< A, B >:= \text{tr}(A^\top B)$. Let $\mathcal{S} = \{A \in \mathbb{R}^{n\times n} \mid A^\top = A\}$. Then, $\mathcal{S}^\perp = \{A \in \mathbb{R}^{n\times n} \mid A^\top = -A\}$.

$\mathcal{M}$ $\left(\text{trace}(A^TB) = \text{trace}(BA^T)\right)$

$Ax = b$ **T** **F** (c) There exists a finite-dimensional real inner product space $(\mathcal{X}, \mathbb{R}, < \bullet, \bullet >)$ and two vectors $y_1, y_2 \in \mathcal{X}$ such that $< y_1, y_1 >= 1$, $< y_2, y_2 >= 2$, and $< y_1, y_2 >= 3$.

columns of A are linearly indep. $x = \frac{4}{5}$ always $\min \|Ax - b\|$

T **F** (d) Let $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $m > n$, and $nullity(A) = 0$. Then $x = (A^TA)^{-1}A^Tb$ is a unique $\boxed{\text{exact}}$ solution of $Ax = b$.

4.a. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A \neq 0$ ← zero "vector" in $\mathbb{R}^{2\times 2}$ $\quad \rho(A) = 0$ $\quad \rho$ does not satisfy positive definiteness property of norms, it is not a norm.

$\lambda_1 = \lambda_2 = 0$

by def. $\mathcal{M}$

4.b. Let $X \in S$, $\underbrace{Y \in \{A \in \mathbb{R}^{n\times n} \mid A^T = -A\}}_{\mathcal{M}}$

$\langle X, Y \rangle = tr(X^TY) = tr(YX^T) = tr(-Y^TX) = -tr(Y^TX) = -\langle Y, X\rangle$

by def S

$\langle X, Y \rangle = -\langle Y, X \rangle \implies \langle X, Y \rangle = 0 \rightarrow (Y \in \mathcal{M} \implies Y \perp S) \implies \mathcal{M} \perp S$

• We know S is a subspace of $\mathbb{R}^{n\times n}$ and any matrix $Z \in \mathbb{R}^{n\times n}$ $Z = \underbrace{\frac{Z + Z^T}{2}}_{\in S} + \underbrace{\frac{Z - Z^T}{2}}_{\in S^\perp}$

[1]Recall that for any real matrix M, M_i denotes its i-th column and $[M]_{ij}$ denotes its ij-element.

[2]Here $\lambda_{max}(A) \in \mathbb{C}$ denotes the eigenvalue of A with the largest magnitude. Recall also that for a complex number $z \in \mathbb{C}$, $|z|$ denotes its magnitude.

$Z = S \oplus S^\perp$

$\langle \bullet, \bullet \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{F}$

4.C. By Cauchy Schwartz Ineq.: $|\langle u, v\rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$

$3^2 \overset{?}{\leq} 1 \cdot 2 \rightarrow$ Does not hold

4.d. It is not an exact solution if $b \notin R(A)$ (no exact sol'n exists when $b \notin R(A)$) it is the least squares approximation.

5. You are tasked to pick a sensor system that is capable of estimating an unknown quantity $x \in \mathbb{R}^3$. Each sensor i gives a measurement of the form $\boxed{y_i = C_i x}$ (**we assume no noise unless otherwise stated** - these are very expensive sensors :-)), where $C_i \in \mathbb{R}^{1\times 3}$. Here is the list of sensors you must pick from:

5.a. $C_a = \begin{bmatrix} C_1 \\ C_4 \\ C_5 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$C_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$
$$C_2 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}$$
$$C_3 = \begin{bmatrix} 0 & -2 & 0 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}$$
$$C_5 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

rank $(C_a) = 2$

then $y = Cx$ cannot be uniquely solved

$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

no such α_i's $\Downarrow$ $y \notin R(C_c)$

In what follows, if we say k sensors are selected from the above list, we mean the observation model is $y = Cx$ where $C \in \mathbb{R}^{k\times 3}$ and rows of C consist of selected sensors. When we say perfectly solve for the unknown $x \in \mathbb{R}^3$, what we mean is that given y, you can find a $\hat{x}$ and you can guarantee that $\hat{x} = x$.

Answer True or False as appropriate for the following statements. Record your answers on the second page.

T **F** (a) Selecting any set of three sensors from the above list (i.e., $C = \begin{bmatrix} C_i \\ C_j \\ C_k \end{bmatrix}$ with $i \neq j, j \neq k, i \neq k$, and $y = Cx$) is sufficient to perfectly solve for the unknown $x \in \mathbb{R}^3$ given a single measurement $y \in \mathbb{R}^3$.

T **F** (b) The minimum number of sensors that can be selected from the above list so that one can perfectly solve for the unknown $x \in \mathbb{R}^3$ is 2.

need at least 3 lin. indep. measurements

T **F** (c) If we use all of the sensors (i.e., $C_c = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix}$), then there exists some $x \in \mathbb{R}^3$ such that $y = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}$ is a possible measurement we can observe when measuring some $x \in \mathbb{R}^3$ with this C. (is $y \in R(C_c)$?)

$y \notin R(C_c)$

T **F** (d) Assume $C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix}$ as before. However, one and only one of the sensors has an error (i.e., there exists a unique $i^* \in \{1, 2, 3, 4, 5\}$ such that $y_{i^*} = C_{i^*}x + e$ for some error $e \in \mathbb{R}$ and $y_j = C_j x$ for all the remaining $j \in \{1, 2, 3, 4, 5\} \setminus \{i^*\}$). You obtain the measurement $y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then, it is ~~not always~~ possible to tell which sensor failed (i.e., there exists an $x \in \mathbb{R}^3$ that can result in the given measurement y in the existence of a single sensor failure for which it is not possible to tell what i^* is).

This problem at a high-level is about when $y = Cx$ has an exact unique solution, i.e. for existence we need $y \in R(C)$

for uniqueness of the solution we need columns of C to be linearly independent.

4.d. Let C_{-i^*} $\in \mathbb{R}^{4\times3}$ and y_{-i^*} $\in \mathbb{R}^4$ be the measurement matrix and remaining measurement when ignoring senser i^*.

Is there an i^* s.t. $y_{-i^*} \in R(C_{-i^*})$ for $i \neq i^*$ $y_{-i} \notin R(C_{-i})$?

Partial Credit Section of the Exam

For $i^* = 1$, we have $y_{-1}^* \in R(C_{-1})$ and for $i \neq 1$, $y_{-i} \notin R(C_{-i})$.

Hence, if sensor 1 fails, we can tell that it failed.

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it. For example, we proved that the Gram Schmidt Process produces orthogonal vectors. So if you need this fact, simply state it and use it.

Matlab way of checking if $y_{-i} \in R(C_{-i})$ is

$$\mathrm{rank}([C_{-i}\ y_{-i}]) = \begin{cases} \mathrm{rank}([C_{-i}]) \Rightarrow y_{-i} \in R(C_{-i}) \\ > \mathrm{rank}(C_{-i}) \Rightarrow y_{-i} \notin R(C_{-i}) \end{cases}$$

6. (15 points) Let $\mathcal{X}$ be the set of 2×2 matrices with coefficients in $\mathbb{R}$ ($\mathcal{X} = \mathbb{R}^{2\times 2}$). Consider the linear transformation $L : \mathcal{X} \to \mathcal{X}$ given by

4dim 4dim

$$L(M) = \begin{bmatrix} 0.9 & 0.2 \\ -0.1 & 1 \end{bmatrix} M \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 1 \end{bmatrix} - M^\top,$$

We know that

$A \in \mathbb{R}^{4\times 4}$

$$v^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad v^3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v^4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is a basis for $\mathcal{X}$.

(a) (10 points) Find the matrix representation A of this linear transformation with respect to the basis $\{v^1, v^2, v^3, v^4\}$.

$A =$

(b) (5 points) Give the change of basis matrix P from $\{v^1, v^2, v^3, v^4\}$ to $\{\bar{v}^1, \bar{v}^2, \bar{v}^3, \bar{v}^4\}$ where

$$\bar{v}^1 = v^1, \quad \bar{v}^2 = v^1 + v^2, \quad \bar{v}^3 = 3v^1 + v^2 + v^3, \quad \bar{v}^4 = v^1 - v^2 + v^3 - v^4$$

$P =$

Note: You are not asked to show the linear independence of $\{\bar{v}^1, \cdots, \bar{v}^4\}$. And, to be extra clear, you are NOT being asked to find the matrix representation of L in the new basis $\{\bar{v}^1, \cdots, \bar{v}^4\}$....you only need to compute the change of basis matrix. If you need to invert a matrix, you can show it as $[\quad]^{-1}$; you do not need to compute the inverse.

a.) $A = [A_1, A_2, A_3, A_4]$

$$A_1 = [L(v^1)]_v = \left[L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_v = \left[\begin{bmatrix} -0.19 & 0.09 \\ -0.09 & -0.01 \end{bmatrix}\right]_v = \begin{bmatrix} -0.19 \\ -0.01 \\ 0.09 \\ -0.09 \end{bmatrix}$$

$$= -0.19v^1 + (-0.01)v^2 + 0.09v^3 + (-0.09)v^4$$

$$A_2 = [L(v^2)]_v = \left[L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_v = \left[\begin{bmatrix} 0.04 & 0.2 \\ 0.2 & 0 \end{bmatrix}\right]_v = \begin{bmatrix} 0.04 \\ 0 \\ 0.2 \\ 0.2 \end{bmatrix}$$

$$A_3 = [L(v^3)]_v = \left[L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_v = \left[\begin{bmatrix} 0.18 & 0.9 \\ -1.02 & -0.1 \end{bmatrix}\right]_v = \begin{bmatrix} 0.18 \\ -0.1 \\ 0.9 \\ -1.02 \end{bmatrix}$$

Please show your work for question 6.

$$A_4 = \left[L(v^4)\right]_V = \left[L\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)\right]_V = \left[\begin{bmatrix}0.18 & -0.98\\ 0.9 & 0.1\end{bmatrix}\right]_V = \begin{bmatrix}0.18\\ 0.1\\ -0.98\\ 0.9\end{bmatrix}$$

$$A = \begin{bmatrix}-0.19 & 0.04 & 0.18 & 0.18\\ -0.01 & 0 & -0.1 & 0.1\\ 0.09 & 0.2 & 0.9 & -0.98\\ -0.09 & 0.2 & -1.02 & 0.9\end{bmatrix}$$

$\bar{v}^1 = v^1$, $\bar{v}^2 = v^1 + v^2$, $\bar{v}^3 = 3v^1 + v^2 + v^3$, $\bar{v}^4 = v^1 - v^2 + v^3 - v^4$

b. $\bar{v}^1 = v^1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ $\qquad \bar{v}^3 = \begin{bmatrix}3 & 1\\ 0 & 1\end{bmatrix}$

$x = v^1$, $[v^1]_V = e^1$ $\quad P[v^1]_V = p^1$ $\quad [\bar{v}^1]_{\bar V} = p^1$

$x = v^1$, $[v^1]_{\bar V} = P[v^1]_V = P[e^1] = P_1$

$\bar{v}^2 = v^1 + v^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\qquad \bar{v}^4 = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$

to compute P directly, we would need $P = [v]_{\bar V}$

$P_1 = [v^1]_{\bar V}$

$\bar{V} = \{\bar{v}^1, \bar{v}^2, \bar{v}^3, \bar{v}^4\}$ $\forall x$

$[x]_{\bar V} = P[x]_V$ and $[x]_V = \tilde{P}[x]_{\bar V}$

Let $\tilde{P}$ be the change of basis matrix from $\bar{V}$ to V, $\tilde{P} = P^{-1}$

Computing $\tilde{P}$ is easier because it requires representing vectors in V (which is a "simpler" basis). $\tilde{P} = [\tilde{p}_1\ \tilde{p}_2\ \tilde{p}_3\ \tilde{p}_4]$

$\tilde{p}_1 = [\bar{v}^1]_V = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ $\qquad \tilde{p}_3 = \begin{bmatrix}3\\1\\1\\0\end{bmatrix} = [\bar{v}^3]_V$ $\qquad P = \begin{bmatrix}1 & 1 & 3 & 1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}^{-1}$

$\tilde{p}_2 = [\bar{v}^2]_V = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ $\qquad \tilde{p}_4 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = [\bar{v}^4]_V$

$\tilde{S} = \text{span}\left\{ \begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&2\\0&0\end{bmatrix}, \begin{bmatrix}0&3\\0&0\end{bmatrix} \right\}$ $\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}\right\}$ is a basis for $\tilde{S}$.

$\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ is the unique soln.

7. (20 points) Consider the inner product space $(\mathbb{R}^{2\times 2}, \mathbb{R}, <\bullet, \bullet>)$ with inner product defined as $<A, B> := \text{tr}(A^\top B)$.

Define $S = \text{span}\left\{ \underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{s^1}, \underbrace{\begin{bmatrix}1&-1\\0&0\end{bmatrix}}_{s^2}, \underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{s^3}, \underbrace{\begin{bmatrix}0&3\\-1&2\end{bmatrix}}_{s^4} \right\}$.

Record your results on this page. You can use this page or the next page to show your work and/or to state your reasoning. Unsupported answers, even if correct, receive zero credits.

(a) (2 points) Find a basis u for S.

by inspection, we see s^1, s^2, s^3 are linearly independent.

$u = \{s^1, s^2, s^3\}$

$s^4 = 2s^1 - 2s^2 - s^3$

$\Rightarrow s^4$ is linearly dependent on $\{s^1, s^2, s^3\}$

(b) (2 points) Let $W = \begin{bmatrix}2&5\\-1&4\end{bmatrix}$. Find the representation $[W]_u$ of W with respect to the basis you find in the above bullet.

$[W]_u = \begin{bmatrix}4\\-2\\-1\end{bmatrix}$

b. $\begin{bmatrix}2&5\\-1&4\end{bmatrix} = \alpha_1 s^1 + \alpha_2 s^2 + \alpha_3 s^3$

$= 4s^1 - 2s^2 - 1s^3$

(c) (8 points) Find a basis for $S^\perp$, the orthogonal complement of S.

(basis for $S^\perp$) =

c. $\dim(\mathbb{R}^{2\times 2}) = 4$ $\quad \dim(S) = 3$

$\Rightarrow \dim(S^\perp) = 1$

$S \oplus S^\perp = \mathbb{R}^{2\times 2}$

(d) (8 points) Let $Y = \begin{bmatrix}0&1\\1&2\end{bmatrix}$. Find $\hat{X} = \text{argmin}_{X\in S} d(Y, X)$, where the distance d is defined as

$$d(X, Y) = sqrt(\text{tr}((X - Y)^\top (X - Y))). = \sqrt{\langle X-Y, X-Y\rangle}$$

$= \sqrt{\|X-Y\|^2}$

$\hookrightarrow$ this is the norm induced by the inner product

$\hat{X} =$

2nd approach: We know $v = \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$ is a basis for $\mathbb{R}^{2\times 2}$

Then $[s^1]_v = \underbrace{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}_{x_1}$, $[s^2]_v = \underbrace{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}_{x_2}$, $[s^3]_v = \underbrace{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}_{x_3}$, $[s^4]_v = \underbrace{\begin{bmatrix}0\\3\\-1\\2\end{bmatrix}}_{x_4}$

x_1, x_2, x_3 are linearly indep. $\Rightarrow \{s^1, s^2, s^3\}$ forms a basis for S.

Please show your work for question 7.

c. By the dimension, we know we need to find one $u' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S^{\perp}$ $u' \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and it will be a basis.

① $\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, s^1 \right\rangle = 0 \Rightarrow \text{trace}\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$
$= \text{trace}\left(\begin{bmatrix} a & \cdot \\ \cdot & b+d \end{bmatrix}\right) = a+b+d = 0$

② $\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, s^2 \right\rangle = 0 \Rightarrow a - b = 0$

③ $\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, s^3 \right\rangle = 0 \Rightarrow c + b = 0$

from ② $a = b$
from ③ $c = -b = -a$
from ① $d = -2b$

$u' = \begin{bmatrix} b & b \\ -b & -2b \end{bmatrix} = b\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$b \neq 0$

$S^{\perp} = \text{span}\left\{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\right\}$

~~by~~ by adding $\bar{s}^4$, we completed $\{s^1, s^2, s^3\}$ to a basis for $\mathbb{R}^{2\times 2}$.

Second way: - add $\bar{s}^4$ to $\{s^1, s^2, s^3\}$ s.t. $\bar{s}^4$ is linearly indep. of $\{s^1, s^2, s^3\}$. Then apply Gram Schmidt to $\{s^1, s^2, s^3, \bar{s}^4\}$

7.d. $\hat{X} = \underset{X \in S}{\operatorname{argmin}} \|X - Y\|$

where $\{s^1, s^2, s^3\}$ is a basis for the subspace S. We can use normal equations to find a point $\hat{X} \in S$ that is closest to Y.

$$G = \begin{bmatrix} \langle s^1, s^1 \rangle, & \langle s^1, s^2 \rangle, & \langle s^1, s^3 \rangle \\ \langle s^2, s^1 \rangle, & \langle s^2, s^2 \rangle, & \langle s^2, s^3 \rangle \\ \langle s^3, s^1 \rangle, & \langle s^3, s^2 \rangle, & \langle s^3, s^3 \rangle \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\beta = \begin{bmatrix} \langle Y, s^1 \rangle \\ \langle Y, s^2 \rangle \\ \langle Y, s^3 \rangle \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

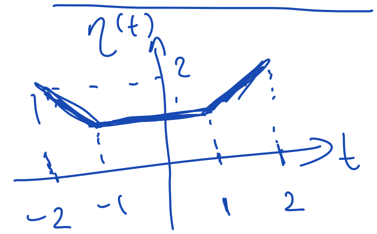
$$G^T \alpha = \beta \qquad \alpha = (G^T)^{-1} \beta = \begin{bmatrix} 0.871 \\ -0.2857 \\ 0.4286 \end{bmatrix}$$

$$\hat{X} = \alpha_1 s^1 + \alpha_2 s^2 + \alpha_3 s^3 = \begin{bmatrix} 0.5714 & 1.5714 \\ 0.4286 & 0.8571 \end{bmatrix}$$

8. (15 points) (Proof Problem) (Done in two parts so that you cannot lose too many points on each part)

(a) (7.5 points) Let $(\mathcal{X}, \mathbb{R})$ be the real vector space of real-valued continuous functions over $[-2, 2]$, that is, $\mathcal{X} = \{f : [-2, 2] \to \mathbb{R},\ f \text{ continuous}\}$. In class we defined an inner product on this vector space by $< f, g >:= \int_{-2}^{2} f(t)g(t)dt$. Suppose we define the function

$$\eta(t) = \begin{cases} |t| & \text{for } |t| > 1 \\ 1 & \text{for } |t| \le 1 \end{cases}.$$



Is $< f, g >_\eta := \int_{-2}^{2} f(t)\eta(t)g(t)\,dt$ a valid inner product on $(\mathcal{X}, \mathbb{R})$? Prove or disprove.

Hint: Note that $< f, g >_\eta = < f, g >$ when $\eta(t) = 1$ for all $t \in [-2, 2]$.

We need to check three conditions:

(a) $\langle f, g\rangle_\eta = \int_{-2}^{2} f(t)\eta(t)g(t)\,dt = \int_{-2}^{2} g(t)\eta(t)f(t)\,dt = \langle g, f\rangle_\eta$ symmetry holds

(b) for $\alpha, \beta \in \mathbb{R}$ $\langle f, \alpha g_1 + \beta g_2\rangle_\eta = \int_{-2}^{2} f(t)\eta(t)[\alpha g_1(t) + \beta g_2(t)]\,dt$ linearity holds

$= \alpha\int_{-2}^{2} f(t)\eta(t)g_1(t)\,dt + \beta\int_{-2}^{2} f(t)\eta(t)g_2(t)\,dt = \alpha\langle f, g_1\rangle + \beta\langle f, g_2\rangle$

(c) $\langle f, f\rangle_\eta = \int_{-2}^{2} f(t)\eta(t)f(t)\,dt = \int_{-2}^{2} f^2(t)\eta(t)\,dt \geq \int_{-2}^{2} f^2(t)\,dt = \langle f, f\rangle > 0$ $f \neq 0$

b.c. $\eta(t) > 1$ $\forall t \in [-2, 2]$ positivity holds

This is an inner product.

(b) (7.5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be an invertible function with inverse f^{-1} (i.e., for all $x, y \in \mathbb{R}$, $f(x) = y \iff x = f^{-1}(y)$). Let $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ be given and define two sets: $S_1 = \{x \mid \exists y \in Y, \exists z \in Z,\ x = f(y) + 5 + z\}$ and $S_2 = \{x \in \mathbb{R} \mid \exists z \in Z,\ f^{-1}(x - 5 - z) \in Y\}$. Show that $S_1 = S_2$. (generally, we need to show $S_1 \subset S_2$ and $S_2 \subset S_1$)

Let $x \in S_1 \iff \exists y \in Y, \exists z \in Z,\ x = f(y) + 5 + z$

$\iff \exists y \in Y, \exists z \in Z \quad y = f^{-1}(x - 5 - z)$

$\iff \exists z \in Z, \exists y \in Y \quad y = f^{-1}(x - 5 - z)$

$\iff \exists z \in Z,\ f^{-1}(x - 5 - z) \in Y$

$\iff x \in S_2$

Please show your work for question 8.