ROB 501 Homework - 4 Unique Name : rahulswa

1)

To find dimension of space spanned by

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\}$$

Span(s) we need to find maximum no. of elements in any linear independent set of vector.

$S = \{ v_1, v_2, v_3, v_4, v_5 \}$

$\text{Span}(s) = \{ x = \sum_{i=1}^{5} \alpha_i v_i \mid v_i \in S \}$

So set of any elements in x ultimately can be written as linear combination of $\{v_1, v_2, v_3, v_4, v_5\}$

$\therefore \dim(\text{span}(s)) = \dim(s)$

$$\Rightarrow \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + 3\alpha_5 \\ 2\alpha_1 + 8\alpha_3 + \alpha_4 + 3\alpha_5 \\ -\alpha_1 - 4\alpha_3 + \alpha_4 \\ 3\alpha_1 + 2\alpha_2 + 8\alpha_3 + \alpha_4 + 6\alpha_5 \end{bmatrix}$$

lets take only first 2 vectors

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha_1 + \alpha_2 = 0$
$2\alpha_1 = 0$
$\alpha_1 = 0$
$3\alpha_1 + \alpha_2 = 0$

$\Rightarrow \alpha_1 \ \& \ \alpha_2 = 0$

Both are independent

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$ $\qquad$ $3\alpha_1 + 2\alpha_2 + 8\alpha_3 = 0.$

$2\alpha_1 + 8\alpha_3 = 0$

$\alpha_1 + 2\alpha_2 = 0$

$\alpha_1 + 4\alpha_3 = 0$

$\boxed{\alpha_1 = -2\alpha_2}$

$\boxed{\alpha_1 = 4\alpha_3}$

$\Rightarrow \boxed{\alpha_2 = -2\alpha_3}$ $\qquad$ $\Rightarrow 4\alpha_3 - 2\alpha_3 + 2\alpha_3 = 0$

$\boxed{\alpha_3 = 0}$

$\boxed{\alpha_1 = 0}$ $\boxed{\alpha_2 = 0}$

$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix}$ are linearly independent

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \\ 2\alpha_1 + 8\alpha_3 + \alpha_4 \\ -\alpha_1 - 4\alpha_3 + \alpha_4 \\ 3\alpha_1 + 2\alpha_2 + 8\alpha_3 + \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2\alpha_1 + 8\alpha_3 + \alpha_4 = 0 \quad \Rightarrow \boxed{\alpha_4 = 0}$$

$$2\times(-\alpha_1 - 4\alpha_3 + \alpha_4 = 0)$$

if $\alpha_4 = 0$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$ ($\because$ they are linearly independ)

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 8 \\ -4 \\ 8 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + 3\alpha_5 = 0 \quad - (1)$$

$$2\alpha_1 + 8\alpha_3 + \alpha_4 + 3\alpha_5 = 0 \quad - (2)$$

$$-\alpha_1 - 4\alpha_3 + \alpha_4 = 0 \quad \Rightarrow \boxed{\alpha_4 = \alpha_1 + 4\alpha_3} \quad - (3)$$

$$3\alpha_1 + 2\alpha_2 + 8\alpha_3 + \alpha_4 + 6\alpha_5 = 0 \quad - (4)$$

(4) - 2×(1)

$$3\alpha_1 + 2\alpha_2 + 8\alpha_3 + \alpha_4 + 6\alpha_5 = 0$$

$$2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 + 6\alpha_5 = 0$$

$$\alpha_1 + 4\alpha_3 + -\alpha_4 = 0$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_1 + 4\alpha_3 + 3\alpha_5 = 0$$

$$2\alpha_1 + 8\alpha_3 + \alpha_1 + 4\alpha_3 + 3\alpha_5 = 0$$

$$3\alpha_1 + 2\alpha_2 + 8\alpha_3 + \alpha_1 + 4\alpha_3 + 6\alpha_5 = 0$$

$$2\alpha_1 + \alpha_2 + 6\alpha_3 + 3\alpha_5 = 0$$

$$3\alpha_1 + 12\alpha_3 + 3\alpha_5 = 0 \Rightarrow \alpha_1 + 4\alpha_3 + \alpha_5 = 0$$

$$4\alpha_1 + 2\alpha_2 + 12\alpha_3 + 6\alpha_5 = 0 \Rightarrow 2\alpha_1 + \alpha_2 + 6\alpha_3 + 3\alpha_5 = 0$$

$$2\alpha_1 + \alpha_2 + 6\alpha_3 + 3\alpha_5 = 0$$

$$2\alpha_1 + 8\alpha_3 + 2\alpha_5 = 0$$

$$\boxed{\alpha_2 - 2\alpha_3 + \alpha_5 = 0}$$

$\Rightarrow$ there exist ∞ sol^n

$\Rightarrow$ only set of 4 are independend

$\therefore$ $\dim(\text{span}(S)) = 4$

2)

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = v$$

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}$$

$\alpha_1 + \alpha_2 + \alpha_3 = 8$ — (1) $\quad \alpha_1 + 2\alpha_2 + 2\alpha_3 = 7$ — (2)

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 4$ — (3)

(3) - (2)

$$\boxed{\alpha_3 = -3}$$

=>

2 × (1) - (2)

$$2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 16$$

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 7$$

$$\boxed{\alpha_1 = 9}$$

$\alpha_1 + \alpha_2 + \alpha_3 = 8 \Rightarrow 9 + \alpha_2 - 3 = 8$

$$\boxed{\alpha_2 = 2}$$

=> representation is $\begin{bmatrix} 9 \\ 2 \\ -3 \end{bmatrix}$

3) $v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$v = \beta_1 u_{1s} + \beta_2 u_{2s} + \beta_3 u_{3s}$

$(v_{new}) = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ $\quad$ $(v_{old}) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

$[v_{new}] = P[v_{old}]$

$= [P_1 \mid P_2 \mid P_3] [v_{old}]$

$P_1 \rightarrow$ representation of e_i in $\{u_{1s}, u_{2s}, u_{3s}\}$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\alpha_1 + \alpha_2 + \alpha_3 = 1$ $\quad$ $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0.$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$

$\Rightarrow \alpha_3 = 0$ $\quad \Rightarrow \quad$ $\alpha_1 + \alpha_2 = 1$

$\alpha_1 + 2\alpha_2 = 0.$

$\Rightarrow \alpha_2 = -1$ $\quad$ $\alpha_1 = 2$

$P_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

P_2:

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = \alpha_1\begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\2\\2\end{bmatrix} + \alpha_3\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_2+\alpha_3\\ \alpha_1+2\alpha_2+2\alpha_3\\ \alpha_1+2\alpha_2+3\alpha_3\end{bmatrix}$$

$\Rightarrow$ $\alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$ $\Rightarrow$ $\boxed{\alpha_3 = -1}$

$\Rightarrow$ $\alpha_1 + 2\alpha_2 = 3$

$\alpha_1 + \alpha_2 = 1$ $\Rightarrow$ $\boxed{\alpha_2 = 2}$

$\boxed{\alpha_1 = -1}$

$$P_2 = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$$

P_3:

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = \alpha_1\begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2\begin{bmatrix}1\\2\\2\end{bmatrix} + \alpha_3\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_2+\alpha_3\\ \alpha_1+2\alpha_2+2\alpha_3\\ \alpha_1+2\alpha_2+3\alpha_3\end{bmatrix}$$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 1$ — ①

$\alpha_1 + 2\alpha_2 + 2\alpha_3 = 0$ — ②

$\alpha_1 + \alpha_2 + \alpha_3 = 0$. — ③

① - ② $\Rightarrow$ $\boxed{\alpha_3 = 1}$ $\Rightarrow$ $\alpha_1 + \alpha_2 = -1$

$\alpha_1 + 2\alpha_2 = -2$

$-\alpha_2 = 1$

$\boxed{\alpha_2 = -1}$

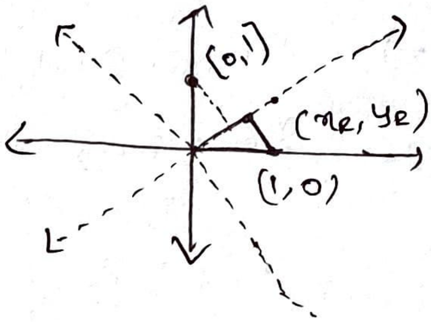
$\Rightarrow$ $\boxed{\alpha_1 = 0}$

$$\Rightarrow P_3 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

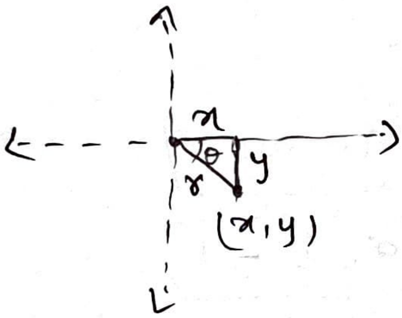
$\Rightarrow$ $P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

4) taking the standard Basis as $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

$[X]_R = P[x]_w$



if we expand diagram

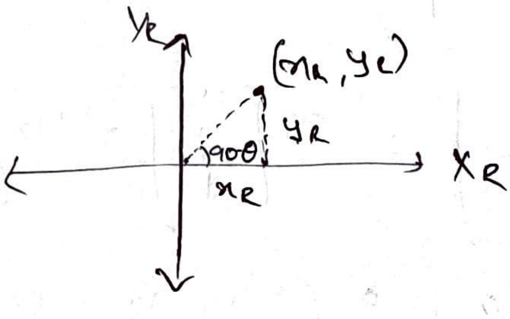


we know that $r = \sqrt{x^2+y^2} = 1$

$y = -r\sin\theta = -\sin\theta$

$x = r\cos\theta = \cos\theta$

similarly for $\begin{bmatrix}0\\1\end{bmatrix}$



$r = \sqrt{x_R^2 + y_R^2} = 1$

$x_R = r\cos(90-\theta) = \sin\theta$

$y_R = r\sin(90-\theta) = \cos\theta$

$\Rightarrow$ $P = [P_1 \mid P_2] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

5) for the set to be basis they should be linearly independent & span the space

$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $M_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $M_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $M_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = 0$

$\Rightarrow \begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{bmatrix} + \begin{bmatrix} \alpha_4 & 0 \\ 0 & -\alpha_4 \end{bmatrix} = 0$

$\begin{bmatrix} \alpha_3+\alpha_4 & \alpha_1-\alpha_2 \\ \alpha_1+\alpha_2 & \alpha_3-\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \alpha_3+\alpha_4 = 0 \quad \alpha_1-\alpha_2 = 0 \quad \alpha_1+\alpha_2 = 0 \quad \alpha_3-\alpha_4 = 0.$

$\Rightarrow \alpha_4 = \alpha_3 \Rightarrow 2\alpha_3 = 0 = 2\alpha_4 \Rightarrow \alpha_3 = \alpha_4 = 0.$

$\alpha_1 = \alpha_2 \Rightarrow 2\alpha_1 = 0 = 2\alpha_2 \Rightarrow \alpha_1 = \alpha_2 = 0.$

$\therefore$ 4 vectors M_1, M_2, M_3, M_4 are independent & they form basis as this is a vector space of dimension = 4

b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4$

$= \alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$+ \alpha_4 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} \alpha_3+\alpha_4 & \alpha_1-\alpha_2 \\ \alpha_1+\alpha_2 & \alpha_3-\alpha_4 \end{bmatrix}$

$\alpha_1 - \alpha_2 = 2 \qquad \alpha_3 - \alpha_4 = 4$

$\alpha_1 + \alpha_2 = 3 \qquad \alpha_3 + \alpha_4 = 1$

$\boxed{\alpha_1 = \frac{5}{2}} \boxed{\alpha_2 = \frac{1}{2}} \qquad \boxed{\alpha_3 = \frac{5}{2}} \boxed{\alpha_4 = \frac{-3}{2}}$

$\Rightarrow \quad [A]_M = \begin{bmatrix} 5/2 & 1/2 \\ 5/2 & -3/2 \end{bmatrix}$

6) a) $S = \{ P_0 = 1, P_1 = x, P_2 = x^2 \}$

$$r(x) = 2 + 3x - x^2 = (-1)P_2 + 3(P_1) + 2(P_0)$$

$$\Rightarrow \quad [r(x)]_S = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

b) $(r(x))_B = \beta_0 q_0 + \beta_1 q_1 + \beta_2 q_2$

$$= \beta_0 (1) + \beta_1 (1 - x) + \beta_2 (x + x^2)$$

$$2 + 3x - x^2 = (\beta_0 + \beta_1) + (\beta_2 - \beta_1) x + \beta_2 x^2$$

$\boxed{\beta_2 = -1}$ $\qquad \beta_2 - \beta_1 = 3$

$\boxed{\beta_1 = \beta_2 - 3 = -4}$

$\beta_0 + \beta_1 = 2 \quad \Rightarrow \quad \boxed{\beta_0 = 6}$

$$[r(x)]_B = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$$

7) a)

$$L(M) = 2(M + M^T)$$

let $N \in X$

$$L(N) = 2(N + N^T)$$

X is vector space so as $M \in X$ & $N \in X$, $M + N \in X$

$\Rightarrow$ $L(M+N) = 2((M+N) + (M+N)^T)$

WKT $(A+B)^T = A^T + B^T$

$\Rightarrow$ $L(M+N) = 2[(M+N) + (M^T + N^T)]$

$= 2[(M+M^T) + (N+N^T)]$

$= 2[M+M^T] + 2[N+N^T]$

$= L(M) + L(N)$

$\Rightarrow$ $\therefore$ By def $L(M+N) = L(M) + L(N)$

& $L(\alpha M) = 2(\alpha M + (\alpha M)^T)$

$(\alpha M)^T = \alpha M^T$

$\Rightarrow$ $L(\alpha M) = 2(\alpha M + \alpha M^T) = \alpha \cdot \{2(M+M^T)\}$

$= \alpha L(M)$

$\therefore \quad L(M+N) = L(M) + L(N)$

& $\quad L(\alpha M) = \alpha L(M)$

$\therefore$ L is linear operator

b) $$L(M) = L(\alpha_{11}E^{11} + \alpha_{12}E^{12} + \alpha_{21}E^{21} + \alpha_{22}E^{22})$$

$$= L(\alpha_{11}E^{11}) + L(\alpha_{12}E^{12}) + L(\alpha_{21}E^{21}) + L(\alpha_{22}E^{22})$$

$$= \alpha_{11}L(E^{11}) + \alpha_{12}L(E^{12}) + \alpha_{21}L(E^{21}) + \alpha_{22}L(E^{22})$$

$$= \left[L_{(E^{11})} \; L_{(E^{12})} \; L_{(E^{21})} \; L_{(E^{22})} \right] \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix}$$

$$= A \begin{bmatrix} \alpha_{1} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix}$$

$L(E^{11}) = 1/2 \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = [\,1$

or) $\quad L(M) = AM$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

$$L(E^{11}) = A E^{11} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{11} & 0 \\ 0 & 0 \end{bmatrix} = 2\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$\cancel{\alpha_{11} = 0}$ $\quad \boxed{\alpha_{11} = 4}$

$$L(E^{12}) = A E^{12} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} 0 & \alpha_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \boxed{\alpha_{12} = 4}$

$$L(E^{21}) = A E^{21} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 2\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} 0 & 0 \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$$

$\Rightarrow \boxed{\alpha_{21} = 4}$

$$L(E^{22}) = A E^{22} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 2\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$\boxed{\alpha_{22} = 4}$

$\Rightarrow A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ where $L(M) = 2(M + M^T)$
$= AM$

8)

a) given $L(x) = Ax$

$L(x) = Ax \quad L(y) = Ay$

$L(x+y) = A(x+y) = Ax + Ay = L(x) + L(y)$

$L(\alpha x) = A(\alpha x) = \alpha(Ax) = \alpha L(x)$

$\therefore$ 'L'(x) is linear operator

let $L(x) = \hat{A} x$

for standard basis in $X = \{x_1, x_2, \dots, x_n\}$

i.e. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

$L(x) = \hat{A} x$

$L(x_1) = \hat{A} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$= [\hat{A}_1 \hat{A}_2 \dots \hat{A}_n] \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [A_1 A_2 \dots A_n] \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$= \hat{A}_1 = A_1$

Similarly $\hat{A}_i = A_i \quad \forall\, i \; 1 < i < n$

$\Rightarrow \quad \boxed{\hat{A} = A}$

b) let new basis

$$B_{new} = \{ e_1, e_2, \cdots e_n \}$$

& let their corresponding eigen values of A are

$$\lambda = \{ \lambda_1, \lambda_2, \cdots \lambda_n \}$$

i.e. $\quad A e_i = \lambda_i e_i \quad \forall \; 1 \le i \le n$

where

$$x_{new} = P x_{old}$$

$$(L(x))_{new} = P (L(x))_{old}$$

$\because$ $L(x) \in$ the same vector space

$$\Rightarrow \hat{A} x_{new} = P (A x_{old})$$

for Basis

$$[\hat{A}_1 \; \hat{A}_2 \cdots \hat{A}_n] \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P(A e_1)$$

$$= P(\lambda_1 e_1)$$

$$= \lambda_1 (P e_1)$$

$x_{new} = P x_{old} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P e_1$

$$\Rightarrow \quad \hat{A}_1 = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly $\hat{A}_2 = \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \quad \hat{A}_n = \lambda_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$$\Rightarrow \quad \hat{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \lambda_3 & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{bmatrix} = \mathrm{Diag}\{\lambda_1, \lambda_2, \cdots \lambda_n\}$$