Vector spaces

ROB 501

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- **Vector space over a field**
  - **Theorem relating bases and dimension for finite dimensional vector spaces**
  - **Representations of vectors**
  - **Change of basis matrix**
  - **Linear operators (if time)**
    - **Matrix representations of linear operators**

Course Announcements

- Midterm exam is on October 25, from 6:30pm-9pm (in-person exam)
- Final exam is on December 19 (+/- 12 hours). Take home exam. Details (duration, etc.) are being sort out. Make sure you have good internet connection on that date.

Recap

- Last week we defined:
  - Linear combinations
  - Linear independence (of finite and infinite sets)
  - Subspaces
  - Span
  - Basis
  - Dimension

Dimension

The maximal number of elements in any linearly independent set of vector in $(\mathcal{X}, \mathcal{F})$, is called the **dimension** of $(\mathcal{X}, \mathcal{F})$.

Basis

A set of vectors B in $(\mathcal{X}, \mathcal{F})$ is a **basis** if

1. B is linearly independent
2. $span\{B\} = \mathcal{X}$

Theorem: In an $n-$dimensional vector space <u>**ANY**</u> set of n linearly independent vectors is a basis.

Theorem: In an $n-$dimensional vector space **ANY** set of n linearly independent vectors is a basis.

want to show $\{v^1, \ldots, v^n\}$ is a basis.

Proof: Let $(\mathcal{X}, \mathcal{F})$ be $n-$dimensional and let $\{v^1, \ldots, v^n\}$ be a linearly independent set.

To Show: $\forall x \in \mathcal{X}$, $\exists \alpha_1, \ldots, \alpha_n \in \mathcal{F}$ such that $x = \boxed{\alpha_1 v^1 + \ldots + \alpha_n v^n}$, i.e., $\mathcal{X} = \text{span}\{v^1, \ldots, v^n\}$

How: Because $(\mathcal{X}, \mathcal{F})$ is $n-$dimensional, $\boxed{\{x, v^1, \ldots, v^n\}}$ is linearly dependent. (1) Otherwise, the $\dim \mathcal{X} > n$ which it isn't. Hence, $\exists \beta_0, \beta_1, \ldots, \beta_n \in \mathcal{F}$, NOT ALL ZERO, such that $\beta_0 x + \beta_1 v^1 + \ldots + \beta_n v^n = 0$. (2)

(1) by def'n. of dimension

(2) by def'n of being linearly dependent.

Claim: $\beta_0 \neq 0$

Proof of the claim: Assume by contradiction that $\beta_0 = 0$.

Then, 1) At least one of $\beta_1, \ldots, \beta_n$ is non-zero

2) $\beta_1 v^1 + \ldots + \beta_n v^n = 0$. (this is (2))

1) and 2) together imply $\{v^1, \ldots, v^n\}$ is linearly dependent

$\longrightarrow$ contradiction $\therefore$ $\beta_0 \neq 0$

Theorem: In an $n-$dimensional vector space **ANY** set of n linearly independent vectors is a basis.

Proof: Let $(\mathcal{X}, \mathcal{F})$ be $n-$dimensional and let $\{v^1, \dots, v^n\}$ be a linearly independent set.

To Show: $\forall x \in \mathcal{X}$, $\exists \alpha_1, \dots, \alpha_n \in \mathcal{F}$ such that $x = \alpha_1 v^1 + \dots + \alpha_n v^n$, i.e., $\mathcal{X} = \text{span}\{v^1, \dots, v^n\}$

How: Because $(\mathcal{X}, \mathcal{F})$ is $n-$dimensional, $\{x, v^1, \dots, v^n\}$ is linearly dependent. Otherwise, the $\dim \mathcal{X} > n$ which it isn't. Hence, $\exists \beta_0, \beta_1, \dots, \beta_n \in \mathcal{F}$, NOT ALL ZERO, such that $\boxed{\beta_0 x + \beta_1 v^1 + \dots + \beta_n v^n = 0.}$

Claim: $\beta_0 \neq 0$

Proof: Suppose that $\beta_0 = 0$. Then

1. At least one of $\beta_1, \dots, \beta_n$ is non-zero
2. $\beta_1 v^1 + \dots + \beta_n v^n = 0$

1 and 2 above, imply that $\{v^1, \dots, v^n\}$ is linearly dependent, which is a contradiction. Hence $\beta_0 = 0$ cannot hold. Completing the proof, we write

$$\boxed{\beta_0 x = -\beta_1 v^1 - \dots - \beta_n v^n}$$

$$x = \underbrace{\left(\frac{-\beta_1}{\beta_0}\right)}_{\alpha_1} v^1 + \dots + \underbrace{\left(\frac{-\beta_n}{\beta_0}\right)}_{\alpha_n} v^n$$

$$\therefore \alpha_1 = \frac{-\beta_1}{\beta_0}, \dots, \alpha_n = \frac{-\beta_n}{\beta_0}$$

Side question

why $\frac{\beta_i}{\beta_0} \in \mathcal{F}$

$\frac{1}{\beta_0}$ exists (multiplicative inverse)

$\frac{1}{\beta_0} \cdot \beta_i \in \mathcal{F}$ (closedness under multipli.)

□

Proposition Let $(\mathcal{X}, \mathcal{F})$ be a vector space and suppose that $B = \{b^1, b^2, \cdots\}$ is a basis for $(\mathcal{X}, \mathcal{F})$. Let $x \in \mathcal{X}$ and suppose that

$$x = \alpha_1 b^1 + \cdots + \alpha_k b^k \quad (1)$$

and

$$x = \bar{\alpha}_1 b^1 + \cdots + \bar{\alpha}_k b^k \quad (2)$$

Then, $\alpha_i = \bar{\alpha}_i$ for all $1 \leq i \leq k$.

$$0 = \underset{(1)}{x} - \underset{(2)}{x} = (\alpha_1 b^1 + \cdots + \alpha_k b^k) - (\bar{\alpha}_1 b^1 + \cdots + \bar{\alpha}_k b^k)$$

$$0 = (\alpha_1 - \bar{\alpha}_1) b^1 + \cdots + (\alpha_k - \bar{\alpha}_k) b^k$$

Note that $b^1, \ldots, b^k$ are linearly independent (because they are in B and B is a basis).

$$\alpha_1 - \bar{\alpha}_1 = 0, \quad \alpha_k - \bar{\alpha}_k = 0 \;\Big\}\; \alpha_i = \bar{\alpha}_i$$

Proposition Let $(\mathcal{X}, \mathcal{F})$ be a vector space and suppose that $B = \{b^1, b^2, \cdots\}$ is a basis for $(\mathcal{X}, \mathcal{F})$. Let $x \in \mathcal{X}$ and suppose that

$$x = \alpha_1 b^1 + \cdots + \alpha_k b^k$$

and

$$x = \bar{\alpha}_1 b^1 + \cdots + \bar{\alpha}_k b^k$$

Then, $\alpha_i = \bar{\alpha}_i$ for all $1 \leq i \leq k$.

Proof:

$$\begin{aligned} 0 = x - x &= \left(\alpha_1 b^1 + \cdots + \alpha_k b^k\right) - \left(\bar{\alpha}_1 b^1 + \cdots + \bar{\alpha}_k b^k\right) \\ &= (\alpha_1 - \bar{\alpha}_1) b^1 + \cdots + (\alpha_k - \bar{\alpha}_k) b^k \end{aligned}$$

Because $\left\{b^1, \cdots, b^k\right\} \subset B$ implies that $\left\{b^1, \cdots, b^k\right\}$ is linearly independent, we deduce that $\alpha_i - \bar{\alpha}_i = 0$ for all $1 \leq i \leq k$.

Representations of Vectors

Let $(\mathcal{X}, \mathcal{F})$ be an n-dimensional vector space.

Let $V = \{v^1, \ldots, v^n\}$ be a basis. And let $x \in \mathcal{X}$

$x \in \mathcal{X} \longleftrightarrow \underbrace{x = \alpha_1 v^1 + \ldots \alpha_n v^n}_{}$

$$[x]_V = \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n$$

notation for representation of x wrt basis V

unique! (by the previous proposition)

$$[x]_V = \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathcal{F}^n$$

Representations of Vectors

Example: $\mathcal{F} = \mathbb{R}$, $\mathcal{X} = \{2 \times 2$ matrices with real coefficients$\}$

$v = \{v^1, v^2, v^3, v^4\}$

"natural"

Basis 1: $v^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $v^3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $v^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Basis 2: $w^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $w^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $w^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $w^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$w = \{w^1, w^2, w^3, w^4\}$

Consider:

$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$

$[x]_v = ?$ $\qquad$ $[x]_w = ?$

$$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = 5\,v^1 + 3\,v^2 + 1 \cdot v^3 + 4 \cdot v^4$$

$$[x]_V = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^4$$

$$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} = 5\,w^1 + 2\,w^2 + 1\,w^3 + 4\,w^4$$

$$[x]_W = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^4$$

Facts:

1. Addition of vectors in $(\mathcal{X}, \mathcal{F}) \Longleftrightarrow$ Addition of representations in $(\mathcal{F}^n, \mathcal{F})$

$$\forall x, y \in \mathcal{X} \qquad [x+y]_V = [x]_V + [y]_V$$

2. Same for scalar multiplication:

$$\forall x \in \mathcal{X}, \ \forall \alpha \in \mathcal{F} \qquad [\alpha x]_V = \alpha [x]_V$$

3. Once you fix a basis, any n-dimensional vector space $(\mathcal{X}, \mathcal{F})$ "looks like" $(\mathcal{F}^n, \mathcal{F})$

Ex: $\mathcal{X} = \mathbb{P}_3(t) = \{$all polynomials w/ real coefficients w/ degree $\leq 3\}$

$$\mathcal{F} = \mathbb{R}$$

$u = \{\underset{u^1}{1}, \underset{u^2}{t}, \underset{u^3}{t^2}, \underset{u^4}{t^3}\}$ is a basis for $(\mathbb{P}_3(t), \mathbb{R})$

Consider $p_1(t) = t + 2t^2 + 3t^3 \in \mathbb{P}_3(t)$ $\qquad p_1(t) = t + 2t^2 + 3t^3 = 0 \cdot u^1 + 1 \cdot u^2 + 2 \cdot u^3 + 3 \cdot u^4$

$$[p_1]_u = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^4$$

$\bar{u} = \{\overset{\bar{u}^1}{1}, \overset{\bar{u}^2}{t}, \overset{\bar{u}^3}{t + 2t^2}, \overset{\bar{u}^4}{\frac{t^3}{2}}\}$ is also a basis for $(\mathbb{P}_3(t), \mathbb{R})$

$$[p_1]_{\bar{u}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 6 \end{bmatrix} \in \mathbb{R}^4$$

$p_1(t) = t + 2t^2 + 3t^3 = \bar{\alpha}_1 \cdot \bar{u}^1 + \bar{\alpha}_2 \cdot \bar{u}^2 + \bar{\alpha}_3 \cdot \bar{u}^3 + \bar{\alpha}_4 \cdot \bar{u}^4$

Change of Basis Matrix

- We are given a finite dimensional vector space $(\mathcal{X}, \mathcal{F})$ and two bases $\{u\} = \{u^1, u^2, \cdots, u^m\}$ and $\{\bar{u}\} = \{\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^m\}$.

$$\alpha = [x]_u = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \longleftrightarrow x = \alpha_1 u^1 + \cdots + \alpha_m u^m \qquad \bar{\alpha} = [x]_{\bar{u}} = \begin{bmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_m \end{bmatrix} \longleftrightarrow x = \bar{\alpha}_1 \bar{u}^1 + \cdots + \bar{\alpha}_m \bar{u}^m$$

Theorem: There exists an invertible $n \times n$ matrix P with coefficients in $\mathcal{F}$ such that $\forall x \in \mathcal{X}$

$$[x]_{\bar{u}} = P \cdot [x]_u.$$

Moreover, $P = [P_1 \mid P_2 \mid \cdots \mid P_n]$ with $P_i = [u^i]_{\bar{u}}$

Proof: $x = \sum_{i=1}^{n} \alpha_i u^i = \sum_{i=1}^{n} \bar{\alpha}_i \bar{u}^i$

$$\therefore [x]_u = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \doteq \alpha \qquad [x]_{\bar{u}} = \begin{bmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix} \doteq \bar{\alpha}$$

$$\bar{\alpha} = [x]_{\bar{u}} = \left[\sum_{i=1}^{n} \alpha_i u^i\right]_{\bar{u}}$$ by Fact 1 and 2 of representations.

$$= \sum_{i=1}^{n} \alpha_i \underbrace{[u^i]_{\bar{u}}}_{P_i}$$

$$= \sum_{i=1}^{n} \alpha_i P_i = \underbrace{[P_1 | \dots | P_n]}_{P} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\alpha}$$

$\therefore \bar{\alpha} = P \cdot \alpha$, where $P_i = [u^i]_{\bar{u}}$

By the same reasoning, we can find $\bar{P}$ s.t. $\alpha = \bar{P}\bar{\alpha}$, where $\bar{P}_i = [\bar{u}^i]_u$.

Hence, $\bar{\alpha} = P\underbrace{\alpha}_{\bar{P}\cdot\bar{\alpha}} = P\bar{P}\bar{\alpha} \Rightarrow P\bar{P} = I$

Similarly $\alpha = \bar{P}\underbrace{\bar{\alpha}}_{P\alpha} = \bar{P}P\alpha \Rightarrow \bar{P}P = I$

$$\left.\begin{matrix} P\bar{P} = I \\ \bar{P}P = I \end{matrix}\right\} \bar{P} = P^{-1}$$

Example: $\mathcal{X} = \{2 \times 2 \text{ matrices with real coefficients}\}$, $\mathcal{F} = \mathbb{R}$.

$$\boxed{u} = \left\{ \overset{u^1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{u^2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{u^3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{u^4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

$$\bar{u} = \left\{ \underset{\bar{u}^1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{\bar{u}^2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{\bar{u}^3}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \underset{\bar{u}^4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

We have the following relations:

$\bar{\alpha} = P\alpha$, $P_i = [u^i]_{\bar{u}}$, $\quad \alpha = \bar{P}\bar{\alpha}$, $\bar{P}_i = [\bar{u}^i]_u$, $\quad \bar{P}^{-1} = P$, $P^{-1} = \bar{P}$

Typically, compute the easier of P or $\bar{P}$, and compute the other by inversion. For this example, we choose to compute $\bar{P}$

$$\bar{P}_1 = [\bar{u}^1]_u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\bar{u}^1 = 1 \cdot u^1 + 0 \cdot u^2 + 0 \cdot u^3 + 0 \cdot u^4$

$$\bar{P}_2 = [\bar{u}^2]_u = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$\bar{u}^2 = 0 \cdot u^1 + 1 \cdot u^2 + 1 \cdot u^3 + 0 \cdot u^4$

$\vdots$

$$\bar{P}_3 = [\bar{u}^3]_u = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\bar{P}_4 = [\bar{u}^4]_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\bar{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $P = \bar{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

What if we did it the other direction?

$$P_1 = [u^1]_{\bar{u}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{u^1} = 1 \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\bar{u}^1} + 0 \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\bar{u}^2} + 0 \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\bar{u}^3} + 0 \cdot \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\bar{u}^4}$$

$$P_2 = [u^2]_{\bar{u}} = \begin{bmatrix} 0 \\ .5 \\ .5 \\ 0 \end{bmatrix} \leftrightarrow \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{u^2} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_3 = [u^3]_{\bar{u}} = \begin{bmatrix} 0 \\ .5 \\ -.5 \\ 0 \end{bmatrix} \leftrightarrow \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{u^3} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - .5 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_4 = [u^4]_{\bar{u}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\bar{P} = P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise

Representation example from ~~last~~ earlier this lecture:

$$x = \begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix} \qquad [x]_u = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 4 \end{bmatrix} \qquad [x]_{\bar{u}} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

$$[x]_{\bar{u}} = P\,[x]_u$$

$$\underline{[x]_u = \bar{P} \cdot [x]_{\bar{u}}}$$

Check that the P we computed works in relating the two representations (it works for any x)

OFFICE HOURS:

Fact: For any $S \subset X$,

span(S) is a subspace of X.

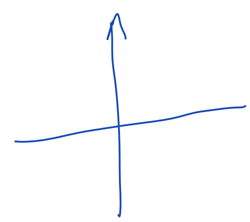
And it is the smallest subspace that contains S.

$$[x]_{\bar{u}} = \left[\sum_{i=1}^{n} \bar{\alpha}_i \, \bar{u}^i \right]_{\bar{u}}$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i \underbrace{\left[\bar{u}^i \right]_{\bar{u}}}$$

$$= \sum_{i=1}^{n} \bar{\alpha}_i \, e^i = \begin{bmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_n \end{bmatrix}$$

$$e^i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i\text{th element}$$



In Q1: $X = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{} \in \mathbb{R}^n$ $\quad x_i \in \mathbb{R}$

$$x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix} \in S_{(c)} \qquad x^{(1)} \cdot x_2^{(1)} = 0$$

$$x^{(2)} \in S_c \qquad x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix} \qquad x_1^{(2)} \cdot x_2^{(2)} = 0$$

$$\alpha x^{(1)} + x^{(2)} = x^{(3)} = \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ \vdots \\ x_n^{(3)} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in S_{(d)}$$ (since $x \in S_{(d)}$, we know $x_1 + \dots + x_n = 0$)

scalar multiplication

$\alpha \in \mathbb{R}$

$$\bar{x} = \alpha x = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$\sum \bar{x}_i = \alpha \sum x_i$
$= 0$

$\Rightarrow \bar{x} \in S_{(d)}$

$x \in S_{(f)}$ iff $Ax = b$

$\bar{x} \in S_{(f)}$ iff $A\bar{x} = b$

does $\alpha x = z$ satisfy $Az = b$?

$A\alpha x = \alpha A x = \alpha b$

is $x + \bar{x} \in S_{(f)}$ meaning

does $x + \bar{x} = y$ satisfy $Ay = b$?

$Ay = A(x + \bar{x}) = Ax + A\bar{x}$

Step 1:

Take $x \in \underbrace{\mathrm{Span}(S_1 \cup S_2)}$, and

show $x \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

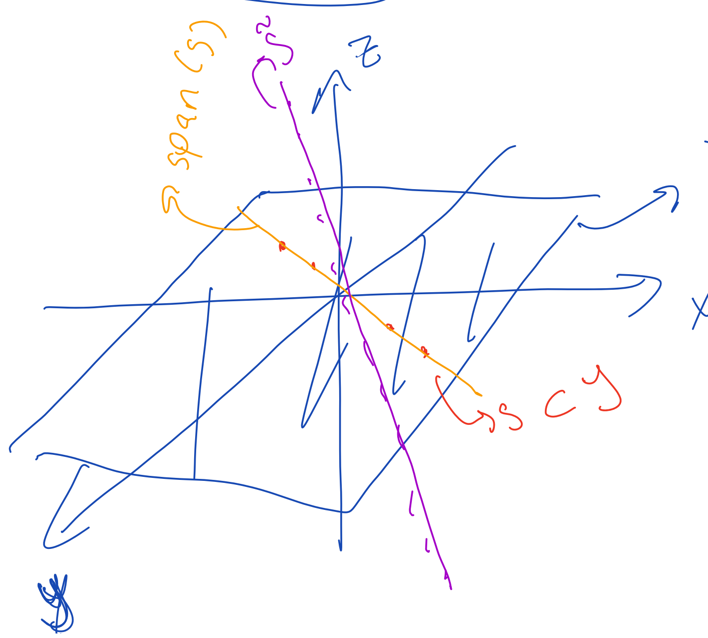
(shows $\mathrm{Span}(S_1 \cup S_2) \subset \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$)

Step 2:

Take $y \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$, and

show $y \in \mathrm{Span}(S_1 \cup S_2)$ (shows $\mathrm{Span}(S_1) + \mathrm{Span}(S_2) \subset \mathrm{Span}(S_1 \cup S_2)$)

1 + 2 $\Longrightarrow$ =



$Y = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; z = 0 \right\}$

want to show

$\mathrm{span}(S) \subset Y$

step 2

Y subspace

($\mathrm{span}(Y) = ?$)

$S \subset Y$

step 1

$\mathrm{span}(S) \subset \mathrm{span}(Y)$