ROB 501 Exam-I Solutions

29 October 2019

Problem 1:

(a) False. $n = -1$ is a counterexample.

(b) False. The negation is "there is at least one garden that does not have flowers".

(c) True. First note that $\frac{\epsilon}{2} \le \delta \le \epsilon$ is an AND condition as it states that $\delta \ge \frac{\epsilon}{2}$ AND $\delta \le \epsilon$. Its negation is therefore an OR condition: EITHER $\neg(\delta \ge \frac{\epsilon}{2})$ OR $\neg(\delta \le \epsilon)$. Once you have this the rest of the problem is straightforward: replacing $\neg\, \exists$ with $\forall$ and $\neg\, \forall$ with $\exists$ to obtain the final result

$$\neg(\forall\, \epsilon > 0, \exists \delta > 0 \text{ such that } \frac{\epsilon}{2} \le \delta \le \epsilon) \iff (\exists\, \epsilon > 0 \text{ such that } \forall\, \delta > 0, \text{it is true that either } \delta > \epsilon \text{ or } \delta < \frac{\epsilon}{2})$$

(d) True. Recall that the truth table for $A \implies B$ is

A	B	$A \implies B$
1	1	1
1	0	0
0	1	1
0	0	1

Setting $A = p$ and $B = \neg\, q$ gives the result; indeed

p	q	$\neg q$	$p \implies \neg q$
1	1	0	0
1	0	1	1
0	1	0	1
0	0	1	1

Problem 2:

(a) True. By definition, an eigenvector $v^i \in \mathcal{N}(A - \lambda_i I)$, i.e., $(A - \lambda_i I)v^i = 0$. Moreover, we saw a theorem in lecture stating $\mathcal{R}(B)^\perp = \mathcal{N}(B^T)$ for a real matrix B.

(b) True. From $v = \bar{v}$ we aim to show $\lambda = \bar{\lambda}$. $\overline{Av} = \overline{\lambda v} \implies \bar{A}v = \bar{\lambda}v \implies Av = \lambda v = \bar{\lambda}v \therefore \bar{\lambda} = \lambda$.

(c) True. This is a symmetric matrix, so its eigenvectors are orthogonal. Moreover, this matrix has real eigenvectors so the matrix V is real.

(d) False. The eigenvectors could be complex, so the span would include complex vectors.

Problem 3:

(a) False. If $x \in \mathcal{N}(A)$, x is orthogonal to the rows of A, not the columns of A.

(b) False, $\dim \mathcal{X} \leq 3$. If $\mathcal{X} = \mathbb{R}$, then we could write $\mathcal{X} = \text{span}\{1,2,3\}$ but $\dim \mathbb{R} = 1$ (vectors $\{1,2,3\}$ are linearly dependent).

(c) False. S is not necessarily a subspace, so the direct sum will be missing linear combinations of the elements of S. The correct statement would be $S^\perp \oplus \text{span}\{S\} = \mathcal{X}$.

(d) False. $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$ but not $\mathbb{R}^n$. (On the other hand, $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.)

Problem 4:

(a) True. If A is orthogonal, then $1 = \det(I) = \det\left(A \cdot A^T\right) = \det(A) \cdot \det\left(A^T\right) = \det(A)^2 \Rightarrow \det(A) = \pm 1$. This was posted to Canvas.

(b) False. The zero matrix $0_{n\times n}$ qualifies as positive semi-definite because $x^T 0_{n\times n} x = 0 \geq 0 \; \forall x \in \mathbb{R}^n$, and $\text{tr}(0_{n\times n}) = 0$.

(c) False. This symmetric matrix has a negative diagonal element, and therefore it cannot be positive definite (as discussed in class). The quadratic form $x^T M x$ with $x = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$, $y \neq 0$ arbitrary, is not always positive.

(d) True. First, note this is not a symmetric matrix and thus you cannot immediately use the Schur Complement Theorem. This is, however, a block upper-triangular matrix, and thus its eigenvalues are those of the diagonal blocks $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 10 & -2 \\ -2 & 10 \end{bmatrix}$, as discussed in class. Because these blocks *are* symmetric, you can use the Schur Complement Theorem to easily check that $2 > 0$ and $2 - 1(1/2)1 > 0$, and $10 > 0$ and $10 - (-2)(1/10)(-2) > 0$.

Problem 5:

(a) True. One could check all three axioms:
1. $[u, v] =< u, Pv >= u^T P v = (u^T P v)^T = v^T P u =< v, Pu >= [v, u]$.
2. Linearity in the left argument follows trivially.
3. $[x, x] = x^T P x \geq 0$ for any $x \in \mathbb{R}^n$, and $[x, x] = x^T P x = 0$ if.f. $x = 0$ because $P > 0$.

(b) False. Consider $n = 2$ with $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0$. Clearly $u^T v = 0$ but $u^T P v = 1 \neq 0$, and thus $||u + v||_P^2 = 6$ is not equal to $||u||_P^2 + ||v||_P^2 = 4$. (It would be true for u, v such that $u^T P v = 0$.)

(c) True, stated in lecture. If the vectors $\{v^1, \ldots, v^k\}$ are orthonormal, using the normal equations, $G = I$, so $\alpha = \beta = \begin{bmatrix} < x, v^1 > & \ldots & < x, v^k > \end{bmatrix}^\top$ and $P(x) = \alpha_1 v^1 + \cdots + \alpha_k v^k = \sum_{i=1}^k < x, v^i > v^i$.

(d) False. M is a subspace if and only if $\alpha_0 = 0$. Scalar multiplication and vector addition do not hold if $\alpha_0 \neq 0$: if $x \in M$ and $a \in \mathbb{R}$, ax is not in M since $[3 \quad 5 \quad -2](ax) = a([3 \quad 5 \quad -2]x) = a\alpha_0 \neq \alpha_0$. It is similarly straight-forward to show vector addition does not hold.

Problem 6:

(a) Let A^i denote the ith column of A, then $A^i = [\mathcal{L}(e^i)]_V$.

$$\mathcal{L}(e^2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1 \cdot v^1 - 1 \cdot v^2 + 0 \cdot v^3 \implies A^2 = [\mathcal{L}(e^2)]_V = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

$$\mathcal{L}(e^4) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 0 \cdot v^1 + 1 \cdot v^2 - 1 \cdot v^3 \implies A^4 = [\mathcal{L}(e^4)]_V = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(b) The change of basis matrix from V to $\tilde{V}$ is P. Let P^i denote the ith column of P, then $P^i = [v^i]_{\tilde{V}}$.

$$P^1 = [v^1]_{\tilde{V}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad P^2 = [v^2]_{\tilde{V}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P^3 = [v^3]_{\tilde{V}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, $P = [P^1 \quad P^2 \quad P^3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

Solving for matrix P is sufficient, but you could alternatively take the slightly harder route and solve for $\bar{P} = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, where $\bar{P}^i = [\tilde{v}^i]_V$.

Problem 7:

(a) Noting that y^1, y^2 are linearly independent, we can use the Gram-Schmidt Process to find an orthogonal basis $\{v^1, v^2\}$ for M. This can be normalized to be orthonormal at each step of the process (as we do below), or all at the end.

First define the unit vector $v^1 = y^1/\sqrt{< y^1, y^1 >} = y^1/\sqrt{4} = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$.

Then define the unnormalized vector $\hat{v}^2 = y^2 - \frac{<y^2, v^1>}{<v^1, v^1>} v^1 = y^2 - \frac{-3/2}{1} v^1 = \begin{bmatrix} 3/4 & 1 \\ 1 & 1/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}$.

Finally, normalize this vector to obtain $v^2 = \hat{v}^2/\sqrt{< \hat{v}^2, \hat{v}^2 >} = \hat{v}^2/\sqrt{11/4} = \frac{1}{2\sqrt{11}} \begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}$.

(b) There are two straight-forward ways to solve this problem: 1) use the orthonormal basis from (a) to define the orthogonal projection operator from $\mathcal{X}$ to M (noting that $\hat{x}$ is the orthogonal projection of x onto M), or 2) use the Normal Equations to solve the optimization problem.

Solution 1: The orthogonal projection operator $P(x) :=< x, v^1 > v^1 + < x, v^2 > v^2$, where $< x, v^1 >= 0$ and $< x, v^2 >= 4/\sqrt{11}$. Hence, $\hat{x} = P(x) = \frac{4}{11} \begin{bmatrix} 3/2 & 2 \\ 2 & 1/2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 6 & 8 \\ 8 & 2 \end{bmatrix}$.

Solution 2: To use the normal equations, we define $G_{11} =< y^1, y^1 >= 4$, $G_{12} = G_{21} =< y^1, y^2 >= -3$, $G_{22} =< y^2, y^2 >= 5$, $\beta_1 =< x, y^1 >= 0$, $\beta_2 =< x, y^2 >= 2$.

Then $\alpha = G^{-1}\beta = \frac{1}{\det(G)} \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

Hence, $\hat{x} = \alpha_1 y^1 + \alpha_2 y^2 = \frac{1}{11} \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix} + \frac{1}{11} \begin{bmatrix} 0 & 8 \\ 8 & 8 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 6 & 8 \\ 8 & 2 \end{bmatrix}$.

You could also define the normal equations using the orthonormal basis vectors and you would get the same answer.

Problem 8:

(a) Let's recall the definition of a real inner product:

i) For all $x \in \mathcal{X}$, $<x,x> \geq 0$ and $<x,x>= 0 \iff x = 0$

ii) For $x,y \in \mathcal{X}$, $<x,y>=<y,x>$

iii) For all $\alpha, \beta \in \mathbb{R}$ and $x,y,z \in \mathcal{X}$, $<\alpha x + \beta y, z>= \alpha <x,z> + \beta <y,z>$

All we need is to show a counterexample to one of these properties to show that $<f,g>_\eta := \int_{-2}^{2} f(t)\eta(t)g(t)\;dt$ is not a valid inner product on $(\mathcal{X}, \mathbb{R})$. For example, define

$$f(t) := \begin{cases} 0 & -2 \leq t < 1 \\ (t-1) & 1 \leq t \leq 2 \end{cases}.$$

Then, $f \in \mathcal{X}$ is a continuous function, f is not the zero function, but

$$<f,f>_\eta = \int_{-2}^{2} f(t)\eta(t)f(t)\;dt = \int_{-2}^{1} f(t)\eta(t)f(t)\;dt + \int_{1}^{2} f(t)\eta(t)f(t)\;dt = 0,$$

because the integrand of each integral is identically zero, by the definitions of f and η, respectively. Hence, we have the inner product of a non-zero function with itself being zero, which is not allowed by i) of the definition, giving us a counterexample.

Grading Notes:

- For this proposed inner product, all of the properties hold except $f \neq 0 \iff <f,f> \;\; > \;\; 0$
- If you got this correct, but your function was not continuous, meaning it was not in the given vector space, you earned 4.5 points.
- If you clearly understood that the property $f \neq 0 \iff <f,f>> 0$ fails BUT either you did not provide a specific counterexample or your example was incorrect, then you earned 3.5 points.
- If you understood that a counterexample to one of the properties was needed, but you were working toward a counterexample to one of the other properties, you earned between 2 and 2.5 points, depending on the clarity of your work.
- A common error was to propose a function as a counterexample and then do the integral from -2 to $+1$ instead of -1 to $+1$ and arrive at a wrong conclusion.
- Another common error was to say that $f = 0, \;\; |t| < 1$ and $f = |t|, \;\; |t| > 1$ is a continuous function. There is, however, a jump from zero to 1 at $t = \pm 1$. But hey, in the chaos of an exam, as errors go, it's not a big one!

(b) We use standard induction and define for $n \geq 1$, $P(n)$ to be the statement $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Base Case: We check that $P(1)$ is true: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{1+1}$. Hence, the base case holds.

Induction Step: We show that if $P(n)$ is true for some $n \geq 1$, then it is also true for $n+1$. By the associative property of addition of real numbers, the left-hand side of $P(n+1)$ can be written

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \left(\sum_{k=1}^{n} \frac{1}{k(k+1)}\right) + \left(\frac{1}{(n+1)(n+1+1)}\right) \\
&= \left(\frac{n}{n+1}\right) + \left(\frac{1}{(n+1)(n+1+1)}\right) \text{ where we used } P(n) \text{ is true} \\
&= \left(\frac{n}{n+1}\right)\left(\frac{n+1+1}{n+1+1}\right) + \left(\frac{1}{(n+1)(n+1+1)}\right) \\
&= \left(\frac{n^2+2n}{(n+1)(n+1+1)}\right) + \left(\frac{1}{(n+1)(n+1+1)}\right) \\
&= \left(\frac{n^2+2n+1}{(n+1)(n+1+1)}\right) \\
&= \frac{(n+1)(n+1)}{(n+1)(n+1+1)} \\
&= \frac{(n+1)}{(n+1)+1} \text{ which equals the right-hand side of } P(n+1)
\end{aligned}$$

and therefore, $P(n+1)$ holds. Hence, by the Principle of Induction, we deduce that $P(n)$ is true for all $n \geq 1$.

Grading Notes:

- The absolute key to the problem is to clearly define the property being proved, to establish a base case, and then the induction step.
- A few of you said the base case was $n = 2$ instead of $n = 1$. If the remainder of the proof was rock solid, you earned 9 points.
- If you clearly and correctly delineated the base case and the induction step, but did not complete the algebra, such as not simplifying $\frac{1+n(n+2)}{(n+1)(n+2)} = \frac{n+1}{n+2}$, or not showing why $\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$, **which is what you are trying to show once you substitute in from the induction step**, then you earned between 7 and 8 points, depending on the clarity of your work.
- A clever proof that does not require induction: observe that for $k \geq 1$,

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

and hence we have a telescoping sum:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} = \frac{n}{n+1}
\end{aligned}$$