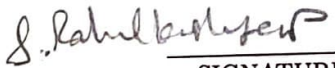


Exam Number: \_\_\_\_\_

## ROB 501 Exam-II

You can pick any 36 hours between 4:30pm (ET) December 16, 2022 (Friday) and 5:00pm (ET) December 20, 2022 (Tuesday) to solve this exam.

**HONOR PLEDGE:** Copy (NOW) and SIGN (after the exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

  
SIGNATURE

(Sign after the exam is completed)

Swayam Pakula  
LAST NAME (PRINTED)

Rahul Kashyap  
FIRST NAME

### RULES:

1. The exam is open book, open lecture handouts and slides, open recitation notes, open HW solutions, open internet (under the communication and usage restrictions mentioned below).
2. If you use MATLAB or any other scientific software to complete some parts of the exam. You are required to submit your script along with your solution in such case.
3. You are not allowed to communicate with anyone other than the Course instructor and the GSIs related to the exam during the entire period. If you have questions, you can post a private Piazza post for the instructors or email [necmiye@umich.edu](mailto:necmiye@umich.edu) with GSIs on cc.
4. You are not allowed to use any online "course helper" sites like Chegg, Course Hero, and Slader, in any part of the exam. You are not allowed to post exam questions on the internet or discuss them online. You are not allowed to use chatGPT or similar large language models.
5. Please do not wait until the last minute to upload your solution to Gradescope and double-check to make sure you uploaded the correct pdf. If you run into problems with Gradescope, email your .pdf file as an attachment to Prof. Ozay as soon as practicable at [necmiye@umich.edu](mailto:necmiye@umich.edu).

Answers for Problem 1			
Problem 1(a)	True	False	False
Problem 1(b)	True	False	True
Problem 1(c)	$\mu_Z = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \Sigma_Z = \begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix}$		
Problem 1(d)	$\mu_Y = 7 \quad \text{var}(Y) = 10$		

Answers for Problem 2	
Problem 2(a)	$(-\infty, -2] \cup \{-1\} \cup [0.1, 5] \cup [7, \infty)$ - same as S
Problem 2(b)	$\{-1\}$
Problem 2(c)	$\{-2, -1, 0.1, 5, 7\}$
Problem 2(d)	True    False    False

Answers for the True/False Part				
	(a)	(b)	(c)	(d)
Problem 3	False	True	True	True
Problem 4	False	True	True	False
Problem 5	False	True	False	True

6. (12 points) The goal of this problem is to derive Minimum Variance Estimator (MVE), but this time, the unknown  $x$  is correlated with the measurement noise  $\epsilon$ . We assume  $x$  and  $\epsilon$  are jointly normal.

Model:  $y = Cx + \epsilon$ ,  $y \in \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$ , and  $\epsilon \in \mathbb{R}^m$ .

Stochastic assumptions:

$E\{x\} = 0$ ,  $E\{\epsilon\} = 0$  (means are zero).

$$E \left\{ \begin{bmatrix} x \\ \epsilon \end{bmatrix} \begin{bmatrix} x \\ \epsilon \end{bmatrix}^T \right\} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{x\epsilon} \\ \Sigma_{\epsilon x} & \Sigma_{\epsilon\epsilon} \end{bmatrix} > 0 \text{ (covariance is positive definite).}$$

Find the minimum variance estimate of  $x$  given  $y$ .

$$\min (E(\|\hat{x} - x\|^2)) \quad \text{s.t.} \quad \hat{x} = Ky \quad \boxed{\text{Defining MVE}} \quad (\text{similar to lectured})$$

$$\Rightarrow \text{for each } i \in \{1, 2, \dots, n\} \quad \min \|\hat{x}_i - x_i\|^2 \quad \hat{x}_i \in M$$

$\downarrow$   
 Space formed by linear function to  $y_1, y_2, \dots, y_m$

$$\Rightarrow \hat{x}_i - x_i \perp M$$

$$\text{let } \hat{x}_i = \sum_{j=1}^m \alpha_j y_j \quad \text{f} \quad G\alpha = \beta$$

$$= x^T y$$

$$G_{jk} = \langle y_j, y_k \rangle = E(y_j, y_k)$$

$$= E((C_j x + \epsilon_j)(C_k x + \epsilon_k)^T)$$

$$= E(C_j x x^T C_k^T + \epsilon_j x^T C_k^T + C_j x \epsilon_k^T + \epsilon_j \epsilon_k^T)$$

$$= C_j E(x x^T) C_k^T + E(\epsilon_j x^T) C_k^T + C_j E(x \epsilon_k^T) + E(\epsilon_j \epsilon_k^T)$$

$$= C_j \Sigma_{xx} C_k^T + \Sigma_{\epsilon x} C_k^T + C_j \Sigma_{x\epsilon} + \Sigma_{\epsilon\epsilon}$$

$$\Rightarrow G = C \Sigma_{xx} C^T + \Sigma_{\epsilon x} C^T + C \Sigma_{x\epsilon} + \Sigma_{\epsilon\epsilon}$$

Please show your work for question 6.

Similarly 
$$\begin{aligned} \beta_j &= \langle y_j, x_j \rangle = E \{ y_j, x_j \} \\ &= E \{ (c_j x_j + \epsilon_j) x_j^T \} \\ &= c_j E \{ x_j x_j^T \} + E \{ \epsilon_j x_j^T \} \end{aligned}$$

$$\Rightarrow B = (C \Sigma_{xx} + \Sigma_{\epsilon x}) \quad \alpha = G^{-1} B$$

$$\hat{\alpha} = \alpha^T y = B^T (G^{-1})^T y$$

$$\boxed{\hat{\alpha} = (C \Sigma_{xx} + \Sigma_{\epsilon x})^T ((C \Sigma_{xx} C^T + \Sigma_{\epsilon x} C^T + C \Sigma_{x\epsilon} + \Sigma_{\epsilon\epsilon})^T)^{-1} y}$$

Please show your work for question 7 (copy/paste the scripts as needed).

7)  
a)

$$x_{t+1} = Ax_t + Bu_t$$

$$A = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ b_{21} \end{bmatrix}$$

$$\begin{bmatrix} x_1^{t+1} \\ x_2^{t+1} \end{bmatrix} = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1^t \\ x_2^t \end{bmatrix} + \begin{bmatrix} 0 \\ b_{21} \end{bmatrix} u_t$$

$$= \begin{bmatrix} x_1^t + a_{12} x_2^t \\ a_{22} x_2^t + b_{21} u_t \end{bmatrix}$$

subtracting  $\begin{bmatrix} x_1^t \\ 0 \end{bmatrix}$  on b.s

$$\Rightarrow \begin{bmatrix} x_1^{t+1} - x_1^t \\ x_2^{t+1} \end{bmatrix} = \begin{bmatrix} a_{12} x_2^t \\ a_{22} x_2^t + b_{21} u_t \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_1^{t+1} - x_1^t \\ x_2^{t+1} \end{bmatrix}}_Z = \underbrace{\begin{bmatrix} x_2^t & 0 & 0 \\ 0 & x_2^t & u_t \end{bmatrix}}_H \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ b_{21} \end{bmatrix}}_\theta$$

$$Z = \begin{bmatrix} x_1^{t+1} - x_1^t \\ x_2^{t+1} \end{bmatrix}$$

$$H = \begin{bmatrix} x_2^t & 0 & 0 \\ 0 & x_2^t & u_t \end{bmatrix}$$



7) a) cont.d

Please show your work for question 7 (copy/paste the scripts as needed).

$$\text{Covariance } \text{COV}(Z) = E(ZZ^T) = \begin{pmatrix} E(PP^T) & E(n_1^{t+1}P^T) \\ E(P^T n_2^{t+1}) & E(PP^T) \end{pmatrix}$$

as where  $P = n_1^{t+1} - n_1^t$

$$\begin{aligned} E(P n_2^{t+1}) &= 0 \quad \text{because it's independent of both } n_1^t \text{ \& } n_1^{t+1} \\ &= E(n_1^{t+1} n_2^{t+1}) - E(n_1^t n_2^{t+1}) \\ &= 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned} E(n_1^{t+1} n_2^{t+1}) &= 0.01 \quad E(PP^T) = E(n_1^{t+1} n_1^{t+1T}) + E(n_1^t n_1^{tT}) \\ &\quad - E(n_1^{t+1} n_1^t) - E(n_1^t n_1^{t+1T}) \end{aligned}$$

As given  $w_t$  are independent of each other,

$$E(n_1^{t+1} n_1^{tT}) = 0$$

$$\Rightarrow E(PP^T) = 0.02 \quad \Rightarrow \text{COV}(Z) = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}$$

mean  $\mu(P) = \begin{bmatrix} \mu(x_1^{t+1}) - \mu(x_1^t) \\ \mu(x_2^{t+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

7) c) recursive least squares in BLUE

from  $\hat{x} = (C^T Q^{-1} C)^{-1} C^T Q^{-1} y$

Batch  
we know

$$C_n = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \quad Q = \begin{bmatrix} \Sigma_E & 0 & \dots & 0 \\ 0 & \Sigma_E & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \Sigma_E \end{bmatrix}$$

$n+1$

Port 1

$$M_n = \left( \begin{bmatrix} H_1^T & H_2^T & \dots & H_n^T \end{bmatrix} \begin{bmatrix} \Sigma_E^{-1} & 0 & 0 & \dots & 0 \\ 0 & \Sigma_E^{-1} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \Sigma_E^{-1} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \right)$$

$$\left( \begin{bmatrix} H_1^T & H_2^T & \dots & H_{n-1}^T \end{bmatrix} \begin{bmatrix} \Sigma_E^{-1} & 0 & \dots & 0 \\ 0 & \Sigma_E^{-1} & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \Sigma_E^{-1} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_{n-1} \end{bmatrix} + H_n^T \Sigma_E^{-1} H_n \right)$$

$$\Rightarrow M_n = M_{n-1} + H_n^T \Sigma_E^{-1} H_n$$

$$N_n = \begin{bmatrix} H_1^T & H_2^T & \dots & H_n^T \end{bmatrix} \begin{bmatrix} \Sigma_E^{-1} & 0 & 0 & \dots & 0 \\ 0 & \Sigma_E^{-1} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \Sigma_E^{-1} \end{bmatrix}$$

$$= N_{n-1} + H_n^T \Sigma_E^{-1}$$

$$N_n y = N_{n-1} y_{n-1} + H_n^T \Sigma_E^{-1} z_n = M_{n-1} M_{n-1}^{-1} N_{n-1} y_{n-1} + H_n^T \Sigma_E^{-1} z_n$$

$$\Rightarrow \hat{x}_n = (M_{n-1} + H_n^T \Sigma_E^{-1} H_n)^{-1} (M_{n-1} \hat{x}_{n-1} + H_n^T \Sigma_E^{-1} z_n)$$

7) c) contd

$$\Rightarrow \hat{x}_{n-1} = \hat{x}_{n-1} + (M_{n-1} + H_n^T \Sigma_E^{-1} H_n)^{-1} H_n^T \Sigma_E^{-1} (y_n - H_n \hat{x}_{n-1})$$

where

$$M_n = M_{n-1} + H_n^T \Sigma_E^{-1} H_n$$

$$\Rightarrow \hat{\theta}_{11} = \hat{\theta}_{11} + M_{11} H_{11}^T \Sigma_E^{-1} (y_{11} - H_{11} \hat{x}_{10})$$

$$y_{11} = \begin{bmatrix} x_{12}(1) - x_{11}(1) \\ x_{12}(2) \end{bmatrix} \quad \Sigma_E = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$H_{11} = \begin{bmatrix} x_{11}(2) & 0 & 0 \\ 0 & x_{11}(2) & u_{11} \end{bmatrix}$$

$$M_{11} = M_{10} + H_{10}^T \Sigma_E^{-1} H_{10}$$



8. (18 points) The following are three (3) short answer questions. You do not need to give a formal proof; only give a few short reasons/calculations why something is TRUE or FALSE. Part (c) is on the next page.

(a) (6 Points) Consider the following set  $S = \{x \in \mathbb{R}^2 \mid \|x\|_1 + \|x\|_2 \leq 2\}$ . Then,  $S$  is a convex set.

Circle T or F. Give a few short reasons/calculations why this is TRUE or FALSE:

**True**

by convex if  $x_1, x_2 \in S$   $\theta_1 x_1 + \theta_2 x_2 \in S$  where  $\theta_1 + \theta_2 = 1$

$$x_1 \in S \Rightarrow \|x_1\|_1 + \|x_1\|_2 \leq 2 \quad x_2 \in S \Rightarrow \|x_2\|_1 + \|x_2\|_2 \leq 2$$

$$\theta_1 \textcircled{1} + \theta_2 \textcircled{2} \text{ \& } \theta_1 + \theta_2 = 1 \quad \hookrightarrow \textcircled{1}$$

$$\hookrightarrow \textcircled{2}$$

$$\Rightarrow (\theta_1 \|x_1\|_1 + \theta_2 \|x_2\|_1) +$$

$$(\theta_1 \|x_1\|_2 + \theta_2 \|x_2\|_2) \leq$$

$$\text{wkt } \|x+y\|_1 \leq \|x\|_1 + \|y\|_1, \quad \|x+y\|_2 \leq \|x\|_2 + \|y\|_2 \quad (\theta_1 + \theta_2) 2$$

$$\Rightarrow \| \theta_1 x_1 + \theta_2 x_2 \|_1 + \| \theta_1 x_1 + \theta_2 x_2 \|_2 \leq (\theta_1 \|x_1\|_1 + \theta_2 \|x_2\|_1) + (\theta_1 \|x_1\|_2 + \theta_2 \|x_2\|_2) \leq 2$$

$$\Rightarrow \| \theta_1 x_1 + \theta_2 x_2 \|_1 + \| \theta_1 x_1 + \theta_2 x_2 \|_2 \leq 2$$

$$\Rightarrow (\theta_1 x_1 + \theta_2 x_2) \in S \text{ where } \theta_1 + \theta_2 = 1 \Rightarrow \boxed{S \text{ is convex}}$$

(b) (6 Points) Let  $(X, \mathbb{R}, \|\cdot\|_X)$  be a finite-dimensional normed space and let  $S \subset X$  be nonempty. Let  $(x_n)$  be a sequence converging to a point  $x^* \in X$ , that is  $x_n \rightarrow x^*$ . If  $x_n \in S^\circ$  for all  $n \geq 1$ , then  $x^* \in S$ .

Circle T or F. Give a few short reasons/calculations why this is TRUE or FALSE:

**False**

$$x_n \rightarrow x^*$$

$$\Rightarrow \forall \epsilon \exists N(\epsilon) < \infty \text{ s.t. } \|x_n - x^*\| \leq \epsilon \text{ - from def of seq}$$

$$\text{Given } x^* \in S^\circ \Rightarrow \text{by above def } d(x^*, S^\circ) = 0$$

if  $x \in S$  then  $d(x, S^\circ) = 0$  but can't guarantee other way

$$\text{So counterexample: let } x_n = 5 - \frac{1}{n}$$

$$\text{ \& } S = [3, 5) \quad x^* = 5 \text{ but } x^* \notin S$$

(c) (6 Points) Let  $M = AA^T + \epsilon^2 I$  where  $A \in \mathbb{R}^{3 \times 2}$ ,  $\epsilon \in \mathbb{R}$ , and  $I$  is the identity matrix of appropriate dimensions. Let

$A = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} V^T$  be the singular value decomposition (SVD) of  $A$ . Then, the solution of the following problem;

$$X^* = \arg \min_{\text{s.t. rank}(X) = 2} \|X - M\|_2$$

is such that  $\|X^* - M\|_2 = \epsilon^2$ .

Circle T or F. Give a few short reasons/calculations why this is TRUE or FALSE:

**False**

This may not be valid if  $\text{rank}(A) < 2$

one counter example can be  $\epsilon = 0$  &  $\sigma_1 = \sigma_2 = 0$

in that case  $M = 0$  according to above eq<sup>n</sup>

$$\text{rank}(X) = 2 \quad \& \quad \|X^* - M\| = 0$$

$$\Rightarrow X^* = M \quad \text{but Rank}(M) = 0$$

$\therefore$  This is false

## Question 7.1

### Results Part b:

Using batch BLUE for  $t = 0:10$

**Estimated value of theta :**  $\theta_{\text{est}} = [0.134; 0.794; -0.366]$

i.e.  $a_{12} = 0.134$ ,  $a_{22} = 0.794$ ,  $b_{21} = -0.366$

so ,  $A = \begin{bmatrix} 1, & 0.134; \\ & 0, & 0.794; \end{bmatrix}$ ;  
 $B = \begin{bmatrix} & 0; \\ & -0.366 \end{bmatrix}$

Using **batch BLUE** for  $t = 0:1000$

**Estimated value of theta :**  $\theta_{\text{est}} = [0.1502; 0.9984; 0.0303]$

i.e.  $a_{12} = 0.1502$ ,  $a_{22} = 0.9984$ ,  $b_{21} = 0.0303$

so ,  $A = \begin{bmatrix} 1, & 0.1502; \\ & 0, & 0.9984; \end{bmatrix}$ ;  
 $B = \begin{bmatrix} & 0; \\ & 0.0303 \end{bmatrix}$

### Results Part d:

Using **Recursive BLUE** for  $t = 0:1000$

**Estimated value of theta :**  $\theta_{\text{est}} = [0.1507; 0.9983; 0.0307]$

i.e.  $a_{12} = 0.1507$ ,  $a_{22} = 0.9983$ ,  $b_{21} = 0.0304$

so ,  $A = \begin{bmatrix} 1, & 0.1507; \\ & 0, & 0.9983; \end{bmatrix}$ ;  
 $B = \begin{bmatrix} & 0; \\ & 0.0304 \end{bmatrix}$

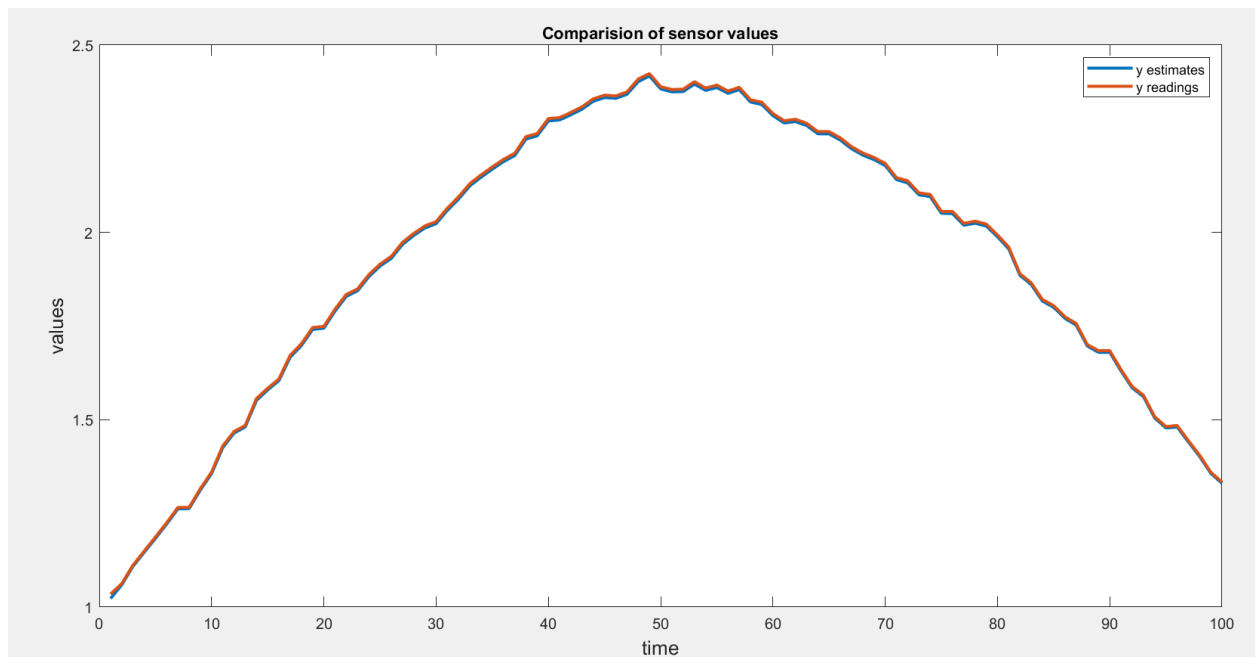
## Question 7.2:

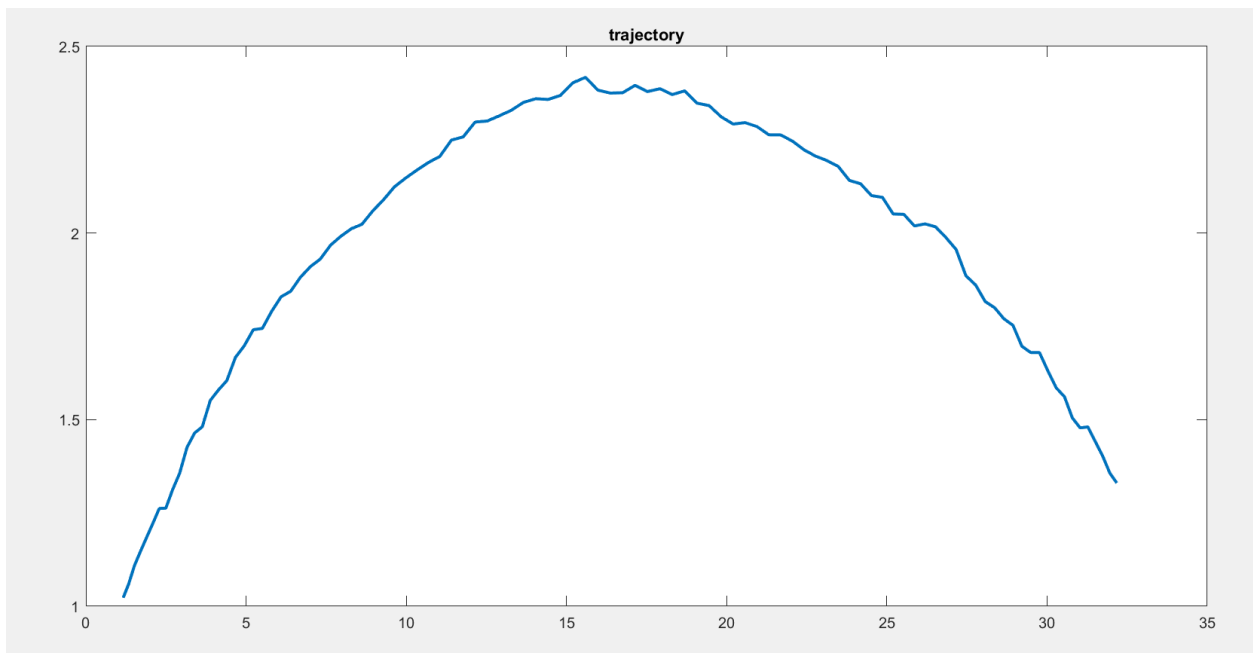
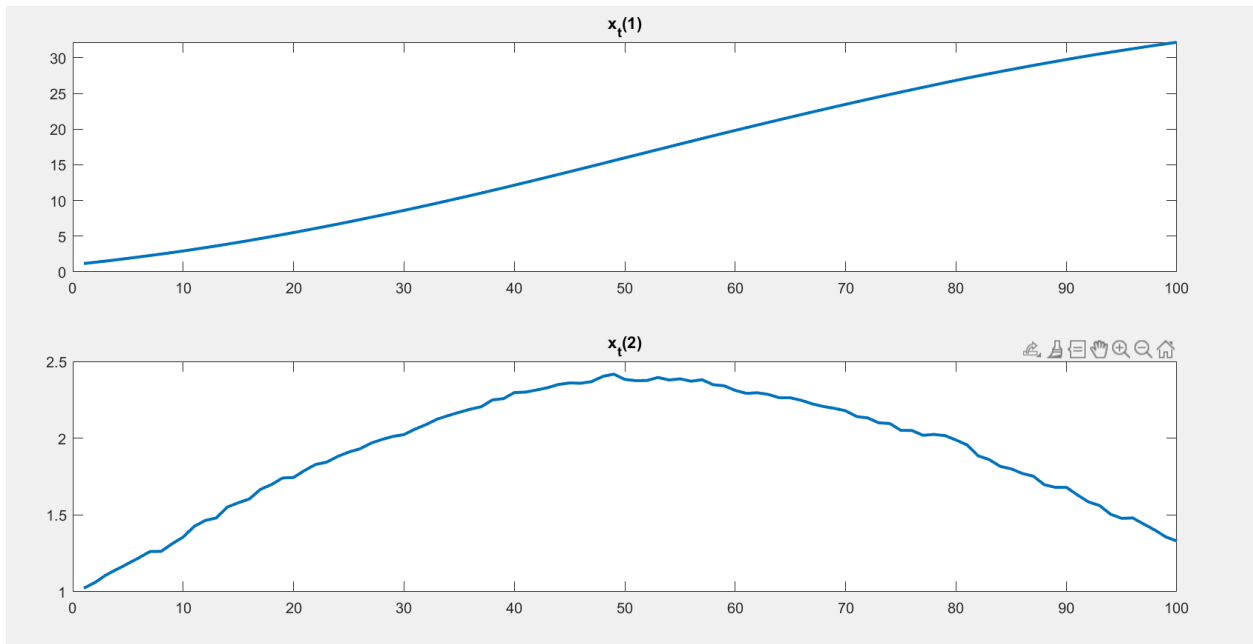
### Results Part a:

When the **Initial P matrix** is chosen as **Identity matrix: eye(2)**

Kalman gain for step 1:  $K = \begin{bmatrix} 0.1661; \\ 0.9843 \end{bmatrix}$ ;

### Results Part b:





### Code Question 7.1.b:

```
x = readmatrix("./data_1/x.csv" );
u = readmatrix("./data_1/u.csv" );

H = [];
z = [];
Q = [];

for i = 1:10

    H_add = [x(i,2),0,0;
             0, x(i,2), u(i)];
    z_add = [x(i+1,1)-x(i,1);x(i+1,2)];
    Q_add = [0.02,0;
             0,0.01;];

    H_new = [H;
             H_add];
    z_new = [z;z_add];
    if(~isempty(Q))
        Q_new = [Q,zeros(length(Q(:,1)),2);
                 zeros(2,length(Q(1,:))),Q_add];
    else
        Q_new = Q_add;
    end

    H = H_new;
    Q = Q_new;
    z = z_new;
end

Q_in = inv(Q);
est_mat = inv((H')*Q_in*H)*(H')*Q_in;
theta_est = est_mat*z;

x_new = validate_dyn(theta_est, x(80,:)', u(80));
```



### Code Question 7.1.d:

```
x = readmatrix("./data_1/x.csv" );
u = readmatrix("./data_1/u.csv" );

Q = zeros(3,3);

A_i = [];
for i = 1:999
    H_i = [x(i,2),0,0;
           0, x(i,2), u(i)];
    A_i = [A_i;H_i];
    z_i = [x(i+1,1)-x(i,1);x(i+1,2)];
    s_i = [0.02,0;
           0,0.01];
    if(i>=10)
        C_k_new = H_i;
        S_k_new = inv(s_i);

        Q_new = Q + C_k_new*S_k_new*C_k_new;
        K_new = inv(Q_new)*C_k_new*S_k_new;

        x_new = x_old + K_new*(z_i-H_i*x_old);

        x_old = x_new;
        Q = Q_new;
    else
        addthing = H_i*s_i*H_i;
        x_old = zeros(3,1);
        Q = Q + addthing;
    end
end
```

## Code Question 7.2::

```
y = readmatrix("./data_2/y.csv" );
u = readmatrix("./data_2/u.csv" );

x_hat = zeros(100,2);
y_hat = zeros(100,1);

E_k = [];
% theta_est = [0.150;0.998;0.030];
theta_est = [0.134;0.794;-0.366];
x_new = [1;0.3];
P_new = [1,0;
         0, 1];

% [0.134194742312479;0.794913610698950;-0.366540891243778]
% K gain [0.148806565381450;0.990059681671246]
for i = 1:100
    A = [1,theta_est(1);
         0, theta_est(2)];
    C_i = [0,1];
    Q = 0.01;
    y_i = y(i);

    x_old = validate_dyn(theta_est,x_new,u(i));
    P_old = A*P_new*A';

    K = (P_old*C_i')*inv((C_i*P_old*C_i')+Q);

    x_new = x_old + K*(y_i - C_i*x_old);

    y_hat(i) = C_i*x_new;

    x_hat(i,1) = x_new(1);
    x_hat(i,2) = x_new(2);

    if(i==1)
        initial_k_gain = K;
    end
end

end

fig = figure('visible','on');
plot(y_hat,'LineWidth',2.0);
hold on
plot(y,'LineWidth',2.0);
hold off
legend('y estimates','y readings');
```

```

ylabel('values','FontSize',13)
xlabel('time','FontSize',13)
title("Comparision of sensor values")
saveas(fig, "compare_y_value.png");

```

```

fig = figure('visible','on');
plot(x_hat(:,1),x_hat(:,2),'LineWidth',2.0);
title("trajectory")
saveas(fig, "traj.png");

```

```

fig = figure('visible','on');
subplot(2,1,1);
plot(x_hat(:,1),'LineWidth',2.0);
subplot(2,1,2);
plot(x_hat(:,2),'LineWidth',2.0);
saveas(fig, "traj_comp.png");

```

## Helper function for 7.1 and 7.2

```

function x_new = validate_dyn(theta,x,u)
A = [1, theta(1);
     0, theta(2)];
B = [0;
     theta(3)];
x_new = A*x + B*u;
end

```