

Homework 7

Problem 2

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

as form of $O\Lambda O^T$

O =

$$\begin{bmatrix} -0.5000 & 0.7071 & 0.5000 \\ -0.7071 & -0.0000 & -0.7071 \\ 0.5000 & 0.7071 & -0.5000 \end{bmatrix}$$

D =

$$\begin{bmatrix} 0.5858 & 0 & 0 \\ 0 & 2.0000 & 0 \\ 0 & 0 & 3.4142 \end{bmatrix}$$

Problem 6

If Matrix A is positive definite or semi-positive definite, then A =

Part a:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$O_a = \begin{bmatrix} -0.9487 & 0.3162 \\ 0.3162 & 0.9487 \end{bmatrix}$$

$$D_a = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & 10 \\ 1 & 3 \end{matrix}$$

The matrix is Semi-Positive Definite by observing the eigenvalues.

$$A_{\text{root}} = \begin{matrix} 0 & 0 \\ 1 & 3 \end{matrix}$$

Part b:

$$B = \begin{bmatrix} 6 & 10 & 11 \\ 10 & 19 & 19 \\ 11 & 19 & 21 \end{bmatrix}$$

$$O_b =$$

$$\begin{matrix} -0.8745 & 0.3270 & 0.3583 \\ -0.0244 & -0.7674 & 0.6407 \\ 0.4845 & 0.5515 & 0.6791 \end{matrix}$$

$$D_b =$$

$$\begin{matrix} 0.1856 & 0 & 0 \\ 0 & 1.0842 & 0 \\ 0 & 0 & 44.7302 \end{matrix}$$

The matrix is Positive Definite by observing the eigenvalues.

$$B_{\text{root}} =$$

$$\begin{matrix} -0.3767 & -0.0105 & 0.2087 \\ 0.3405 & -0.7991 & 0.5743 \\ 2.3963 & 4.2850 & 4.5417 \end{matrix}$$

Part c:

$$C = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$$

O_c =

0.8424	0.4082	0.3516
0.1647	-0.8165	0.5534
-0.5130	0.4082	0.7551

D_c =

-2.9165	0	0
0	0.0000	0
0	0	32.9165

The matrix is **neither** Positive Definite **nor** Semi-Positive Definite by observing the eigenvalues.

Problem 3

$O^T A O$

final_matrix =

-1	0	0
0	2	0
0	0	2

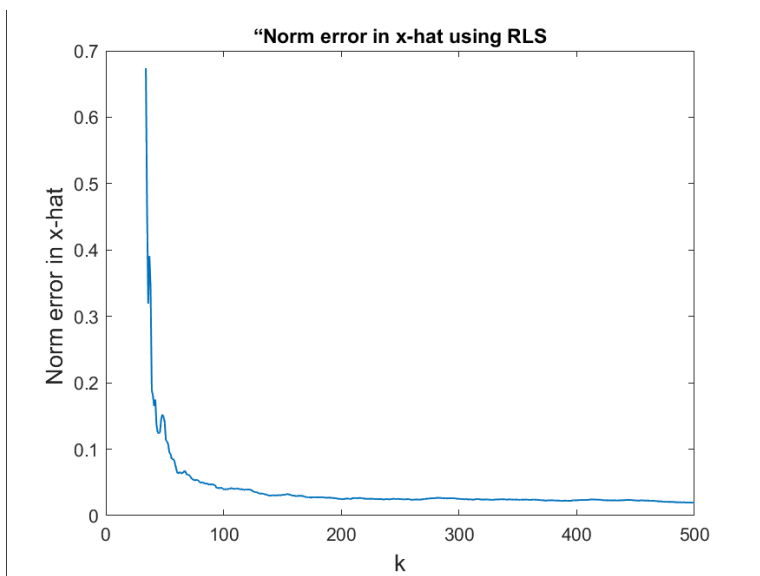
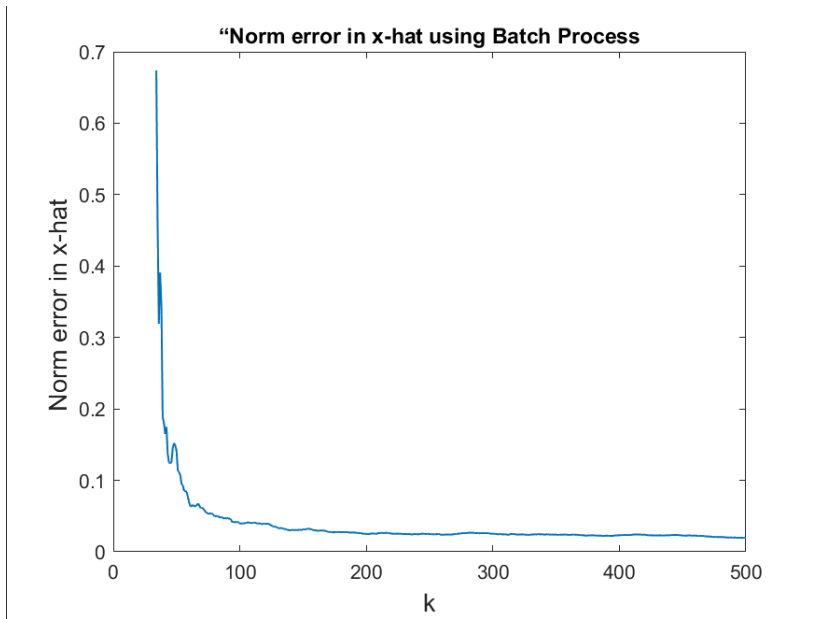
Problem 4

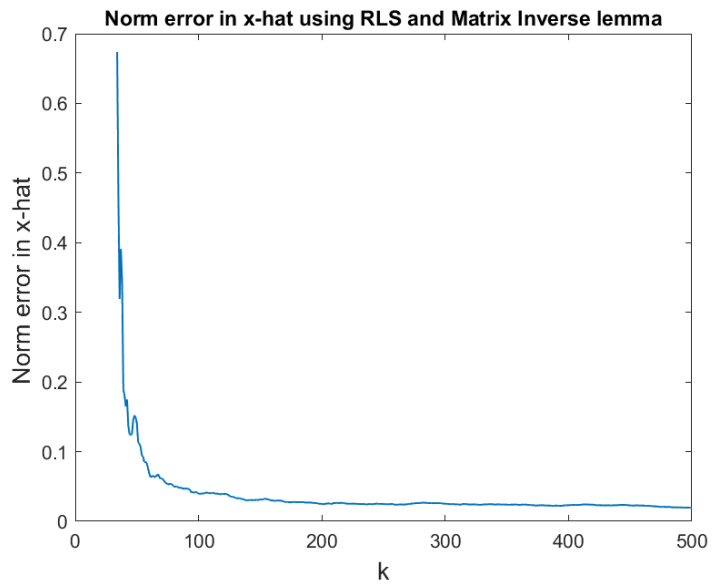
Step (a): n = 34

Total_time for batch process = 3.1442 sec

Total_time for RLS without Matrix Inversion lemma = 0.4463 sec

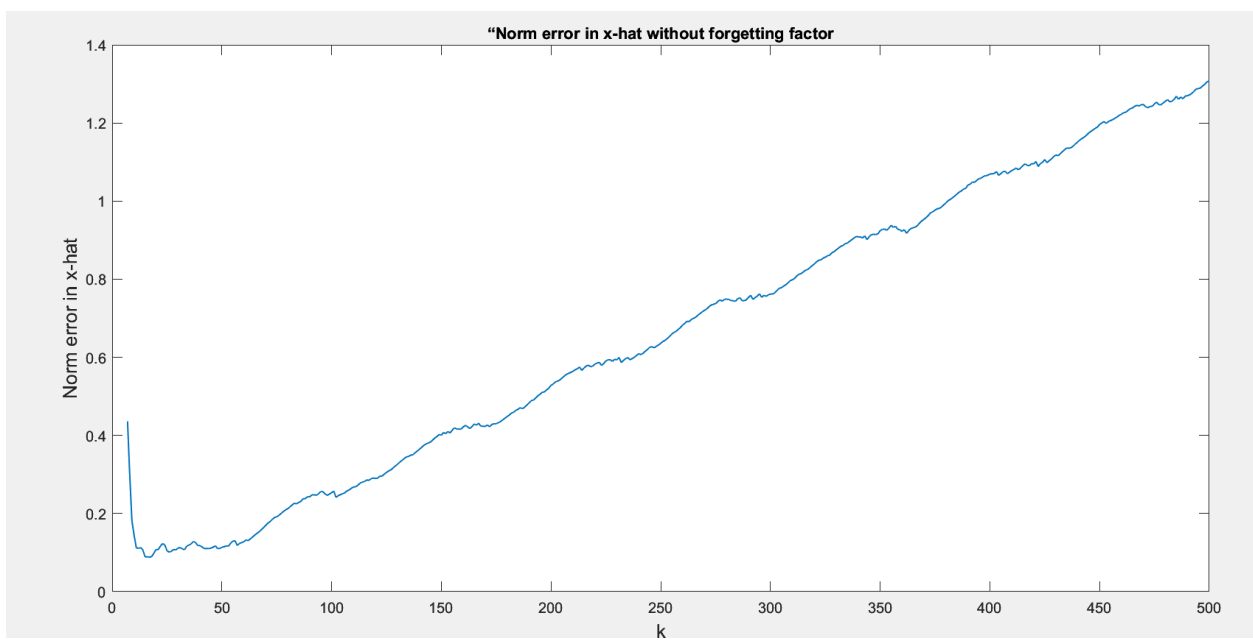
Total_time for RLS with Matrix Inversion lemma = 0.2729 sec

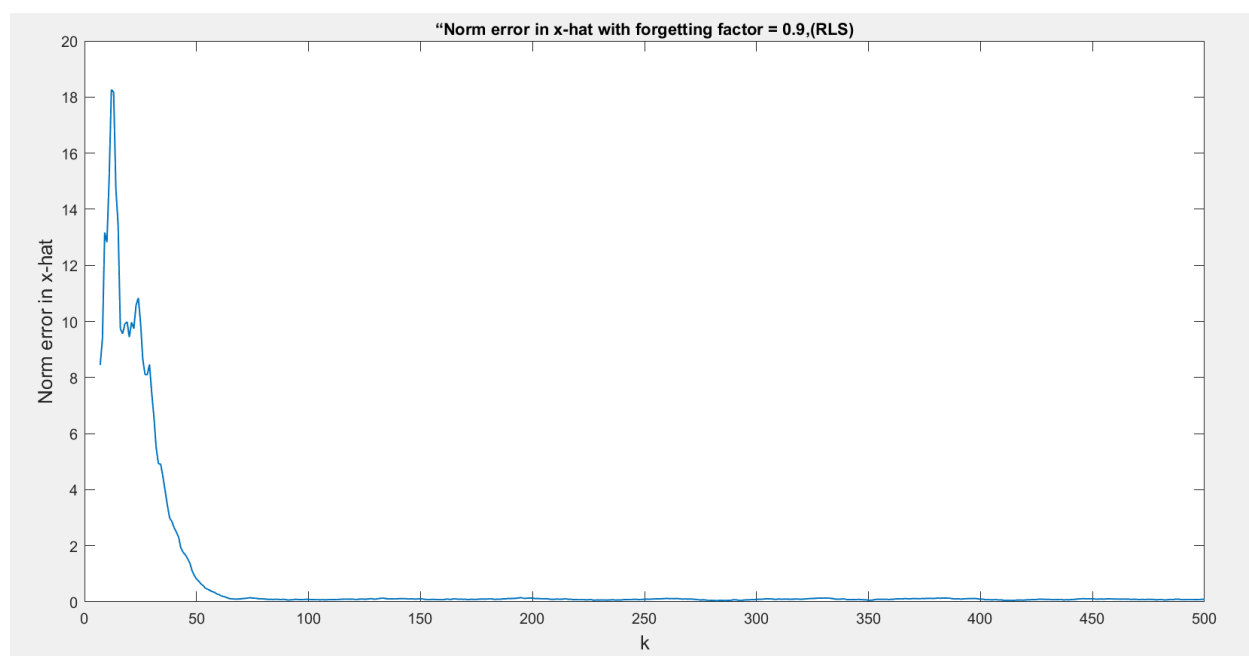
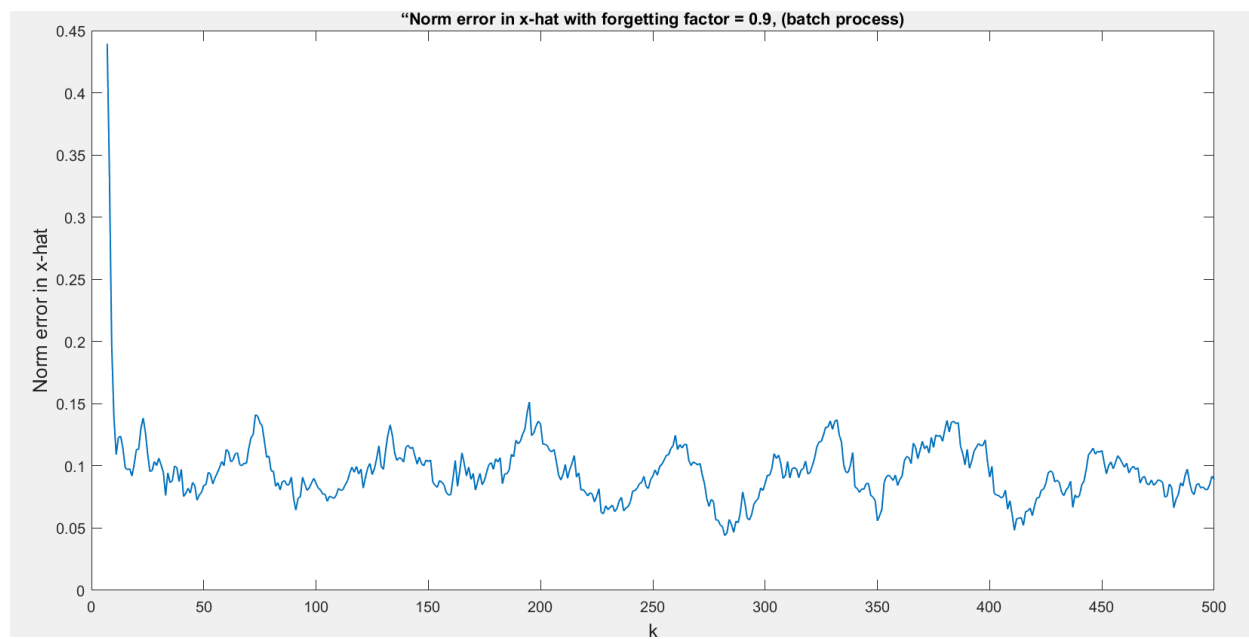




Problem 5:

N value is 7





1) if there are n columns in A

we know that

$$\begin{aligned}\text{Null}(A) &= \dim(\text{nullspace of } A) \\ &= \dim \{ Ax = 0 \text{ where } x \in \mathbb{R}^n \}\end{aligned}$$

now let's say $\text{Rank}(A) = r$ $\& \ r \leq n$

\Rightarrow in row reduce echelon form, there are r rows with leading ones, of $Ax=0$

$\Rightarrow Ax=0$ has $(n-r)$ columns of A without leading zeros

Let x_1, x_2, \dots, x_{n-r} denote solⁿ obtained sequentially by setting each free variable to 1 & remaining free variables be zero

$\therefore x_1, x_2, \dots, x_{n-r}$ will be linearly independent

\therefore solⁿ to $Ax=0$ is $x = \sum_{i=1}^{n-r} \alpha_i x_i$

$\therefore \{x_1, x_2, \dots, x_{n-r}\}$ spans nullspace of (A)

$\Rightarrow \dim \text{nullity}(A) = \dim \{x_1, x_2, \dots, x_{n-r}\}$

as $\{x_1, x_2, \dots, x_{n-r}\}$ are linearly independent

$$\text{nullity}(A) = n - r$$

$\Rightarrow \text{Rank}(A) + \text{nullity}(A) = r + n - r$

$$= n$$

7)

$$a) \quad M = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \approx \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$\text{where } A = 1 \quad B = 3 \quad C = 8$$

$$\text{if } M > 0$$

$$\Leftrightarrow A > 0 \quad \& \quad C - B^T A^{-1} B > 0$$

$$1 > 0 \quad 8 - 3 \cdot 1 \cdot 3 = -1 < 0$$

\therefore The eqⁿ is not satisfied so matrix is not Positive definite

$$b) \quad M = \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 4 & 7 \\ \hline 6 & 7 & 10 \end{array} \right] \approx \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

to check $M > 0$ will check $A > 0$ & $C - B^T A^{-1} B > 0$

$$\text{where } A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \quad C = 10$$

$$\begin{aligned} C - B^T A^{-1} B &= 10 - [6 \ 7] \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \\ &= 10 - \left[36 + \frac{49}{4} \right] < 0 \end{aligned}$$

$\therefore M$ is not Positive definite

7c)

$$M = \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 5 & 7 \\ 6 & 7 & a \end{array} \right] \approx \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \quad C = a$$

if $M > 0$ then $A > 0$ & $C - B^T A^{-1} B > 0$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \approx \begin{bmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{bmatrix} \quad \begin{array}{l} A_1 > 0 \\ 1 > 0 \end{array} \quad \begin{array}{l} C_1 - B_1^T A_1^{-1} B_1 \\ = 5 - 22 = -17 < 0 \end{array}$$

$$\therefore A > 0 \quad A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\begin{aligned} C - B^T A^{-1} B &= a - \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \\ &= a - \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} 16 \\ -5 \end{bmatrix} = a - (96 - 35) \\ &= a - 61 > 0 \end{aligned}$$

$$\Rightarrow \boxed{a > 61}$$

8)

a)

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ 3x_1 + 8x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_1 + 3x_2 + 2x_3 = 1 \quad \text{--- (1)}$$

$$3x_1 + 8x_2 + 4x_3 = 2 \quad \text{--- (2)}$$

$$\text{(2)} - 3 \times \text{(1)} \Rightarrow \begin{array}{rcl} 3x_1 + 8x_2 + 4x_3 & = & 2 \\ \underline{3x_1 + 6x_2 + 4x_3} & & \\ x_2 & = & 0 \end{array}$$

$$\Rightarrow \boxed{x_1 = -2x_2}$$

$$\text{using eqn (1)} \quad -2x_2 + 3x_2 + 2x_3 = 1$$

$$\Rightarrow \boxed{x_3 = \frac{1 - x_2}{2}}$$

as this is the underdetermined system this will have a solⁿ as derived above

but we want solⁿ with min norm

$$\Rightarrow \min(x^T x) \quad \text{given} \quad \begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow x^T x = x_1^2 + x_2^2 + x_3^2$$

$$= (-2x_2)^2 + x_2^2 + \left(\frac{1 - x_2}{2}\right)^2$$

$$= (20x_2^2 + 1 - 2x_2 + x_2)/4 = 1 - 2x_2 + 21x_2^2$$

$$\min(x^T x) = \min\left(\frac{1}{4} (1 - 2x_2 + 21x_2^2)\right)$$

$\Rightarrow \frac{\partial}{\partial x_2} (-RHS) = 0$ will give a solⁿ for it

$$\Rightarrow \frac{1}{4} (-2 + 2 \cdot 21 x_2) = 0$$

$$\Rightarrow \boxed{x_2 = 1/21}$$

$$\boxed{x_2 = 0.0476}$$

$$\Rightarrow x_1 = -2x_2 = ~~-0.0952~~ - 0.0952$$

$$\& x_3 = \frac{1 - x_2}{2} = \frac{1 - 0.0476}{2} = 0.4762$$

b) if inner product defined as

$$\langle x, y \rangle = x^T \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix} y$$

$$\Rightarrow \langle x, x \rangle = x^T \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix} x = 5x_1^2 + 2x_2^2 + 17x_3^2 + 2(x_1x_2 + 9x_1x_3 + x_2x_3)$$

from above eqⁿ

$$5(-2x_2)^2 + 2x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 = 2x_2^2$$

$$+ 9x_1(2x_3) + x_2(2x_3)$$

$$= 20x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 - 9(2x_2)(1-x_2) + x_2(1-x_2)$$

$$= 20x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 - 18x_2 + 18x_2^2 + x_2 - x_2^2$$

$$= 37x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 - 17x_2$$

So $\frac{\partial}{\partial x_2}$ (RHS) will give minima

$$74x_2 + 17(1-x_2) - 17 = 0$$

$$9x_2 = 34$$

$$(x_1, x_2, x_3) = (-0.64, 0.324, 0.337)$$

If a matrix A is positive (or) semi-positive definite
then $A = B^T B$ where B is ^{Square} root of matrix

here after splitting

$$M = O \Lambda O^T = (O \Lambda_{\text{root}}) (\Lambda_{\text{root}} O^T)$$

$$\Rightarrow \boxed{B = \Lambda_{\text{root}} O^T}$$

Λ_{root} is $\sqrt{\text{Square root of diag elements of } \Lambda}$