Introduction to Real Analysis

ROB 501

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- **(SVD quick wrap up)**
- **Open balls and closed balls**
- **Open and closed sets**

unitarily invariant norms in $\mathbb{R}^{m\times n}$ are s.t. for any orthogonal metrix Q $\|A\| = \|QA\|$

Singular Value Decomposition

SVD Theorem: Any $m \times n$ real matrix A can be factored as

assume $m \le n$

$$A = U\Sigma V^\top$$

where

$$A = U\Sigma V^\top = \underbrace{\sum_{i=1}^m \sigma_i u_i v_i^\top}$$

$U = m \times m$ orthogonal matrix

$V = n \times n$ orthogonal matrix

$\Sigma = m \times n$ rectangular diagonal matrix

rank(A) = # of non-zero singular values

$E = A - \hat{A}$

$\min_{\hat{A}} \|A - \hat{A}\|$ s.t. $\text{rank}(\hat{A}) = k < r$

and $\text{diag}(\Sigma) = [\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_p]$ satisfies $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ where $p = \min(m,\ n)$. Moreover, the columns of U are eigenvectors of $AA^\top$, the columns of V are eigenvectors of $A^\top A$, and the $(\sigma_i)^2$ are eigenvalues of both $AA^\top$ and $A^\top A$.

$AA^T = U\Sigma V^T V \Sigma^T U^T = U\Sigma\Sigma^T U^T$

$A^T A = V\Sigma^T \Sigma V^T$

$\|A - \hat{A}\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$

$\|A - \hat{A}\|_F = \sqrt{\sum_i\sum_j a_{ij}^2}$

$E = A - \hat{A}$

For spectral norm

Fact: Suppose that $\text{rank}(A) = r$, so that σ_r is the smallest non-zero singular value. Then

Best low rank approximation for any unitarily invariant matrix norm, e.g., Frobenius norm, matrix 2-norm (aka spectral norm).

(i) if an $n \times m$ matrix E satisfies $||E|| < \sigma_r$, then $\text{rank}(A + E) \geq r$.

(ii) there exists E with $||E|| = \sigma_r$ and $\text{rank}(A + E) < r$.

(iii) In fact, for $E = -\sigma_r u_r v_r^\top$, $\text{rank}(A + E) = r - 1$.

(iv) Moreover, for $E = \underbrace{-\sigma_r u_r v_r^\top - \sigma_{r-1} u_{r-1} v_{r-1}^\top}$, $\text{rank}(A + E) = r - 2$.

given rank(A) = r

$\sigma_r = \min_E \|E\|$ s.t. $\text{rank}(A+E) = r-1$

$A + E = U\Sigma_{r-1}V^T$

set $\sigma_r = 0$ in Σ

Corollary: Suppose A is square and invertible. Then σ_r measures the distance from A to the nearest singular matrix.

$$\left.\begin{aligned} A &= \sigma_1 u_1 v_1^T + \overbrace{\cdots + \sigma_m u_m v_m^T} \\ A + E &= \sigma_1 u_1 v_1^T + \cdots + \sigma_{m-1} u_{m-1} v_{m-1}^T \end{aligned}\right\} E = -\sigma_m u_m v_m^T$$

$$\|A - \hat{A}\|_2 \doteq \max_{\|x\|_2 = 1} \|Ax\|_2 \doteq \sup_x \frac{\|Ax\|_2}{\|x\|_2}$$

Intro to Real Analysis

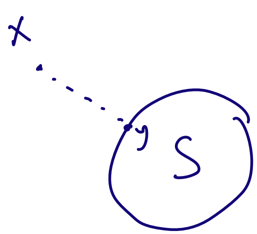
Let $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$ be a normed space. We will use $\mathbb{F} = \mathbb{R}$ in this course, so will simply write $(\mathcal{X}, \|\cdot\|)$.

Recall: a) $\forall x, y \in \mathcal{X}$, $d(x,y) = \|x-y\|$

b) $\forall x \in \mathcal{X}$ and $\forall S \subset \mathcal{X}$

$$d(x, S) := \inf_{y \in S} d(x,y) = \inf_{y \in S} \|x-y\|$$

Remarks: 1) $d(x, S) = 0 \iff \forall \varepsilon > 0,\ \exists y \in S$ s.t. $\|x-y\| < \varepsilon$



($d(x, S) = 0$ does not imply $x \in S$, e.g. take $x = 1$, $S = (0,1)$)

$x \notin S$, $d(x,S) = 0$

(2) $d(x,S) > 0 \iff \exists \varepsilon > 0, \ \forall y \in S \ \ \|x-y\| \geq \varepsilon$

Open and closed sets

Def: Let $x_0 \in \mathcal{X}$, and $a \in \mathbb{R}$, $a > 0$. Then, the open ball of radius a about x_0 is the set

$$B_a(x_0) = \{ x \in \mathcal{X} \mid \|x - x_0\| < a \}$$

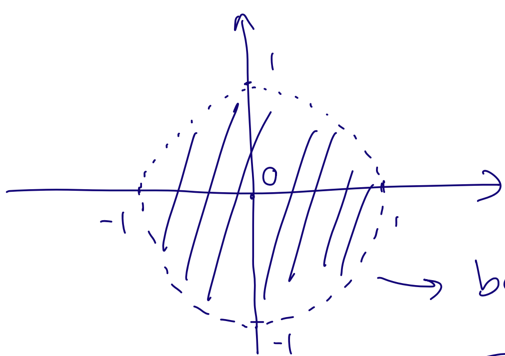
- The actual shape Depends on the norm.
- If we use "$\|x-x_0\| \leq a$" in the def., we get a closed ball.

$B_a^c(x)$

$\overline{B}_a(x)$

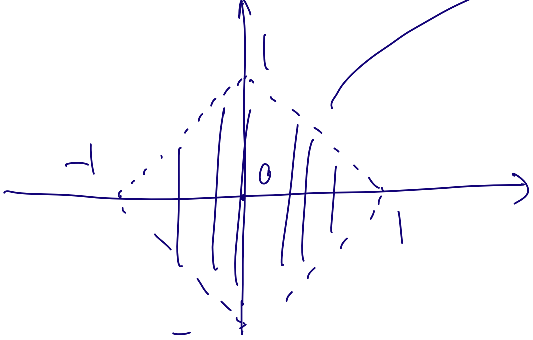
Examples: $B_1(0)$ in $(\mathbb{R}^2, \|\cdot\|)$

$(\mathbb{R}^2, \|\cdot\|_2)$

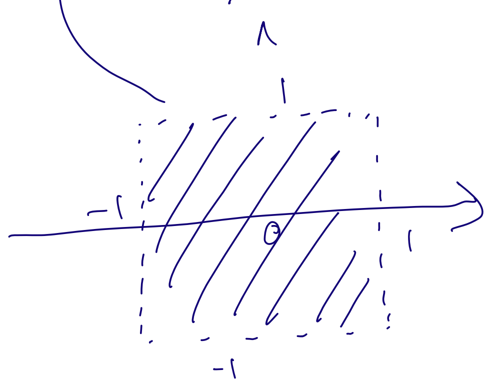


boundary is not included

$(\mathbb{R}^2, \|\cdot\|_1)$

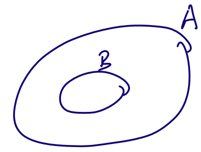


$(\mathbb{R}^2, \|\cdot\|_\infty)$

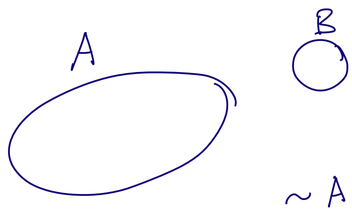


Recall: $B \subset A \iff B \cap (\sim A) = \emptyset$

$\sim A = A^c$



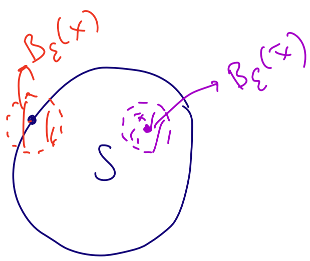
$B \subset (\sim A) \iff \underline{B \cap A = \emptyset}$



Important observations:

(a) $d(x, S) = 0 \iff \forall \varepsilon > 0, \exists y \in S, \ \|x - y\| < \varepsilon$

$\iff \forall \varepsilon > 0, \ B_\varepsilon(x) \cap S \neq \emptyset$



$\left(\neq \emptyset \iff \exists y \text{ s.t. } y \in B_\varepsilon(x) \text{ and } y \in S \right)$

$\iff \exists \varepsilon > 0, \ \forall y \in S \ \ \|x - y\| \geq \varepsilon$

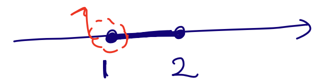
(d) $d(x, S) > 0 \iff \exists \varepsilon > 0, \ B_\varepsilon(x) \cap S = \emptyset$

$\iff \exists \varepsilon > 0 \ \ B_\varepsilon(x) \subset (\sim S)$

Def Let $P \subset X$ be a subset

(a) $p \in P$ is an <u>interior point</u> of P if $\exists \varepsilon > 0 \ \ B_\varepsilon(p) \subset P$.

(Ex: 1) Take $P_1 = [1, 2]$

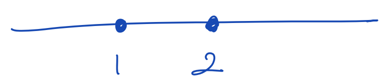


$1 \in P_1$ but is <u>not</u> an interior point. (any open ball around 1 intersects $\sim P$)!

$1.0001 \in P_1$ is an interior point.

b.c. take $\varepsilon = 0.00005 \implies B_\varepsilon(1.0001) \subset P_1$).

Ex 2) $P_2 = \{1, 2\}$



P_2 does not have any interior points.

<u>(b)</u> $P^\circ = \{ p \in P \mid p$ is an interior point of $P \}$ is called the <u>interior</u> of P.

Ex (con't) 1) $P_1^\circ = (1, 2)$

2) $P_2^\circ = \emptyset$

<u>Remark</u>: $P^\circ = \{ p \in P \mid \exists \varepsilon > 0, B_\varepsilon(p) \subset P \}$

$= \{ p \in P \mid d(p, \sim P) > 0 \}$

$= \{ x \in X \mid d(x, \sim P) > 0 \}$

b.c.: if $x \in \sim P$, then $d(x, \sim P) = 0$ (hence we are not adding anything new)

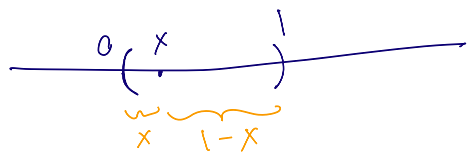
$$\boxed{P^\circ = \{ x \in \mathcal{X} \mid d(x, \sim P) > 0 \}}$$ ← most useful defn. of interior.

<u>Def.</u> P is open if $P = P^\circ$

$\therefore$ P is open $\iff$ $P = \{ x \in \mathcal{X} \mid d(x, \sim P) > 0 \}$

<u>Ex:</u> $(\mathcal{X} = \mathbb{R},\ \|\cdot\| = |\cdot|)$

Is $P = (0,1)$ open?



<u>Method 1 to show P is open:</u>

Take $x \in P \iff 0 < x < 1$

let $\varepsilon = \min\left\{ \frac{x}{2}, \frac{1-x}{2} \right\}$

$\therefore$ $B_\varepsilon(x) \subset P \implies x \in P^\circ$

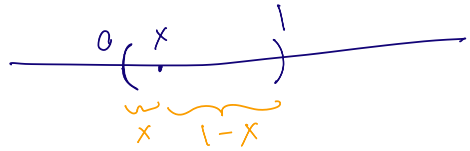
(since this is true for all $x \in P \implies P = P^\circ$)

$\implies P \subset P^\circ$ (by defn. of P° we have $P^\circ \subset P$)

$\therefore$ P is open.

Method 2 to show P is open:

$\cdot\ P = (0,1)$, $\sim P = (-\infty, 0] \cup [1, \infty)$



$x \in P \quad d(x, (-\infty, 0]) = x > 0$

$\qquad\quad d(x, [1, \infty)) = 1 - x > 0$

$\therefore\ d(x, \sim P) = \min\{x, 1-x\} > 0$

$\therefore\ x \in P \iff d(x, \sim P) > 0 \qquad \therefore P$ is open

Def: $P \subset X$ a subset

(a) x is a closure point of P if

$\forall \varepsilon > 0$, $\exists y \in P$, $\|x - y\| < \varepsilon$ (i.e. $d(x, P) = 0$)

(b) The closure of P, denoted $\overline{P}$, is

$\overline{P} := \{x \in X \mid x \text{ is a closure point of } P\}$

$= \{x \in X \mid d(x, P) = 0\}$

Def: P is closed if $P = \overline{P}$.

Ex: (1) Is $P = [0,1)$ a closed set? Not closed because $1 \notin P$, $d(1,P) = 0$,

$\Rightarrow 1 \in \overline{P} \Rightarrow P \neq \overline{P}$.

$\overline{P} = [0,1]$

(2) Is $P = \mathbb{Q}$ (set of rational numbers) closed?

Exercises

Some facts:

- Arbitrary unions of open sets are open
- **Finite** intersections of open sets are open
- Arbitrary intersections of closed sets are closed
- **Finite** unions of closed sets are closed

$$v = z\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$ where $z \notin \mathbb{R}$

$z \in \mathbb{C}$

$$v = \begin{bmatrix} z \\ \vdots \\ z \end{bmatrix}$$

$$Av = \lambda v$$

$$A \cancel{z} \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{} = \lambda \cancel{z} \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{}$$

$\alpha = \bar{z}$

$$\tilde{v} = \bar{z} v = \begin{bmatrix} |z|^2 \\ \vdots \\ |z|^2 \end{bmatrix}$$

$A\tilde{v} =$

exactness $\Longleftrightarrow$ representation of b as a linear comb. of columns of A $\Longleftrightarrow$ $b \in R(A)$

$$\begin{bmatrix} \langle 1, t \rangle \\ \langle t, t \rangle \\ \langle t^2, t^2 \rangle \\ \langle \sin \pi t, \sin \pi t \rangle \end{bmatrix} \cdot [\langle 1, t \rangle, \langle t, t \rangle, \dots \langle \ \rangle]$$

$$y_2 y_3 \int_0^2 dt$$

$$2 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} [y_1 \quad y_2 \quad y_3 \quad y_4] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} 2(\alpha_1 y_1 + \dots \alpha_4 y_4)$$

$M = A B$

$\mathrm{rank}(M) \leq \min(\mathrm{rank}(A), \mathrm{rank}(B))$

$\mathrm{rank}(B) = \mathrm{rank}(B^T)$

$\dim(R(A)) = m$

$$m \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$R(A) \subseteq \mathbb{R}^m$

$b \notin \mathrm{range}(A)$

$\mathrm{rank}(A) = \dim(R(A))$

Assume dep.

$\Leftrightarrow \exists\, \alpha_1, \alpha_2$ at least one non-zero

s.t.

$$\alpha_1 (v^1 + v^2) + \alpha_2 (v^1 - v_2) = 0$$

$$(\alpha_1 + \alpha_2) \underline{v}^1 + (\alpha_1 - \alpha_2) \underline{v}_2 = 0$$

need to show

$\alpha_1+\alpha_2$ or $\alpha_1-\alpha_2$ is non-zero

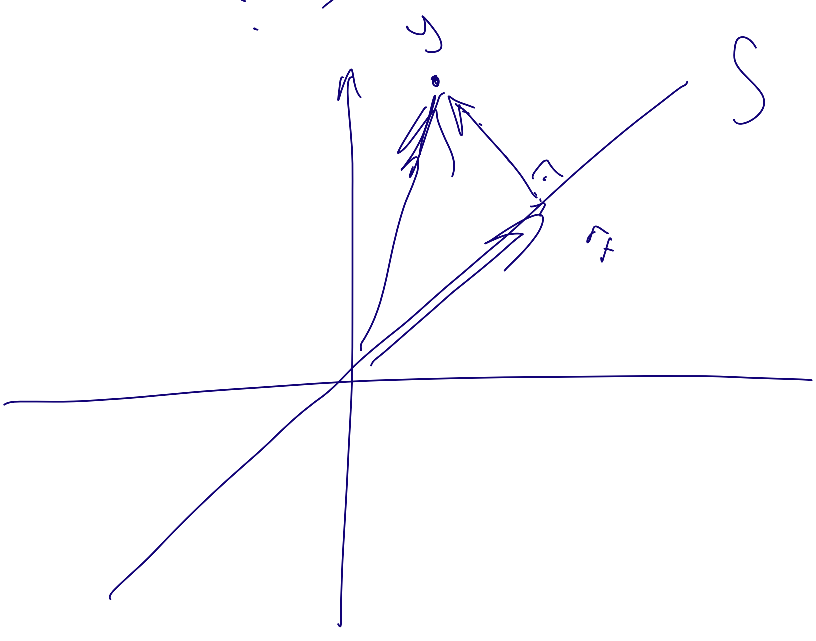
let's assume zero

$\alpha_1+\alpha_2=\alpha_1-\alpha_2=0$

$\alpha_1=0, \ \alpha_2=0$

$\Rightarrow$ $\alpha_1+\alpha_2$ or $\alpha_1-\alpha_2$ is non-zero

$\Rightarrow$ v's are lin. dep.



$(\tilde{x}-y)\perp S$