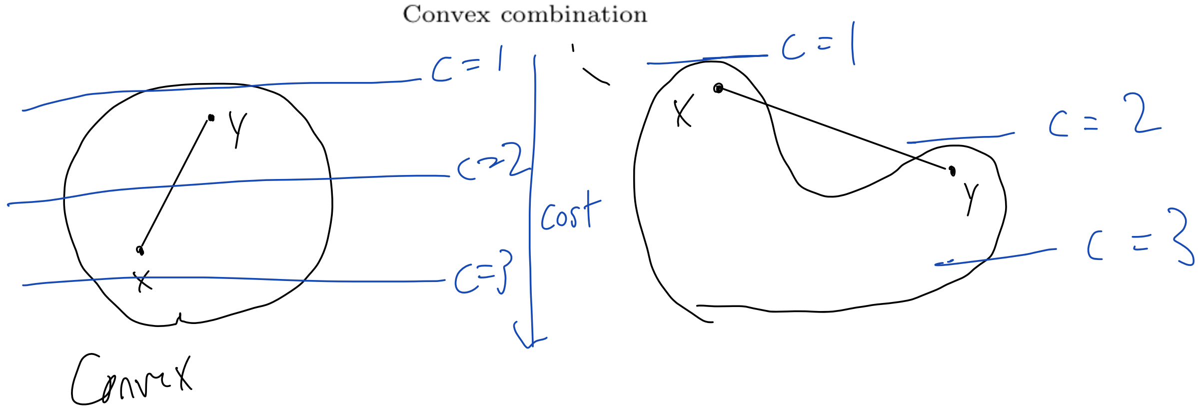
Convex Sets

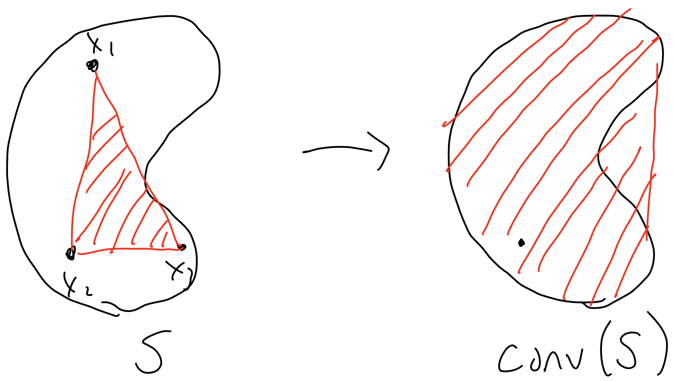
Definition. A set $C \subset \mathcal{X}$ is *convex* if $\forall x, y \in C$ and $\forall \theta \in [0,1]$ it holds that

$$\underbrace{\theta x + (1-\theta) y}_{\text{Convex combination}} \in C\,.$$



Definition. The *convex hull* of a set $S \subset \mathcal{X}$ is the set of convex comb. of S

$$\text{conv}(S) = \{\theta x + (1-\theta) y \mid x, y \in S,\ \theta \in [0,1]\}\,.$$

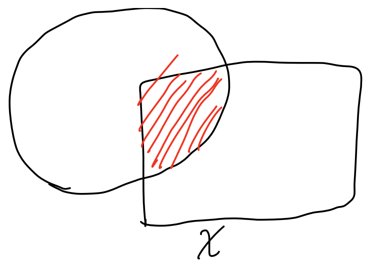


Theorem. The convex hull can be written as

$$\text{conv}(S) = \left\{ \sum_{i=1}^{k} \theta_i x_i \;\middle|\; \forall i \in \{0, \cdots k\}.\ x_i \in S, \quad \theta_i \in [0,1], \quad \sum_{i=1}^{k} \theta_i = 1 \right\}.$$

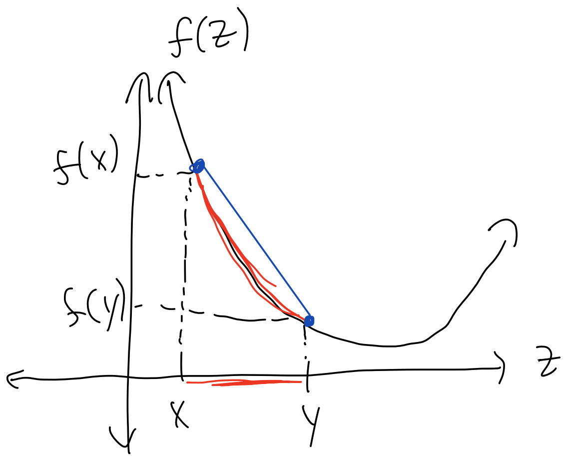
Convex sets are closed under intersection!

Convex Functions

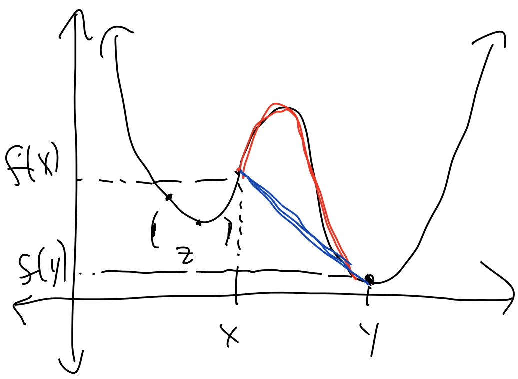


Definition. A function $f : \mathcal{X} \to \mathbb{R}$ is *convex* if $\forall x, y \in$ [redacted] and $\forall \theta \in [0, 1]$,

$$\boxed{\underbrace{f(\theta x + (1-\theta)y)}_{\text{Output of convex comb.}}} \leq \boxed{\underbrace{\theta f(x) + (1-\theta) f(y)}_{\text{Convex comb. of outputs}}}\,.$$



Convex



Not Convex

Examples

$f(x) = a^T x + b$

$f(x) = x^T Q x \qquad Q \succeq 0$

$f(x) = e^x$

$f(x) = |x|^p \qquad p \geq 1$

Convex Optimization

Definition. Let $f : D \to \mathbb{R}$ be a function with $D \subset \mathcal{X}$. A point $x^* \in D$ is said to be

1. a *local minimum* if $\exists \delta > 0$ so $\forall x \in B_\delta(x^*) \cap D,\ f(x^*) \leq f(x)$.
2. a *global minimum* if $\forall x \in D, \quad f(x^*) \leq f(x)$.

Theorem. If $D \subset \mathcal{X}$ and $f : D \to \mathbb{R}$ are convex, then any local minimum of f is a global minimum of f.

Convex Optimization Problem / Convex Program

$$\min f(x) \leftarrow \text{Convex Objective/Cost}$$

$$\text{s.t. } x \in D \leftarrow \text{Convex Constraints (feasible set)}$$

Theorem. If x^*, y^* are local/global minimizers of a convex optimization problem, then so is $\theta x^* + (1-\theta)y^*$ for any $\theta \in [0, 1]$.

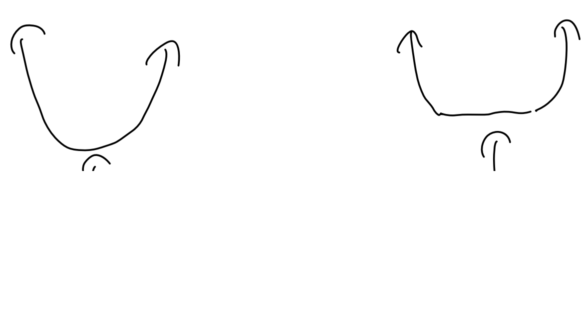
Proof) x^*, y^* are minima $\Rightarrow$ $f(x^*) = f(y^*) = f^*$
$\leq f(\theta x^* + (1-\theta)y^*)$

D is convex, $x^*, y^* \in D \Rightarrow \theta x^* + (1-\theta)y^* \in D$

f is convex $\Rightarrow f(\theta x^* + (1-\theta)y^*) \leq \theta f(x^*) + (1-\theta)f(y^*)$
$= \theta f^* + (1-\theta)f^*$
$= f^*$

$\Rightarrow f(\theta x^* + (1-\theta)y^*) = f^*$

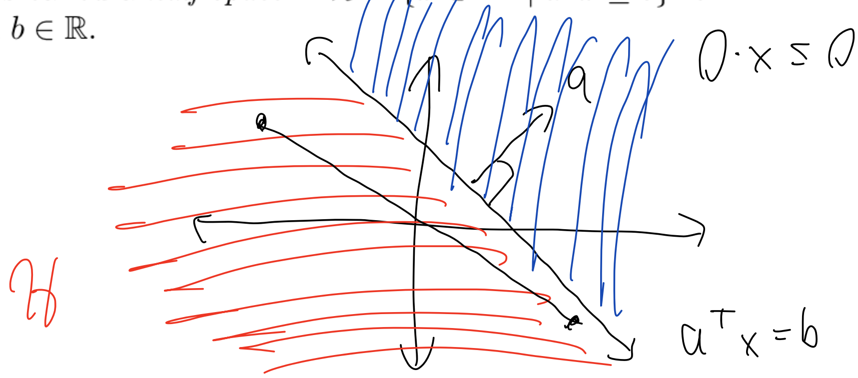
$\square$



unique minimum   non-unique minima

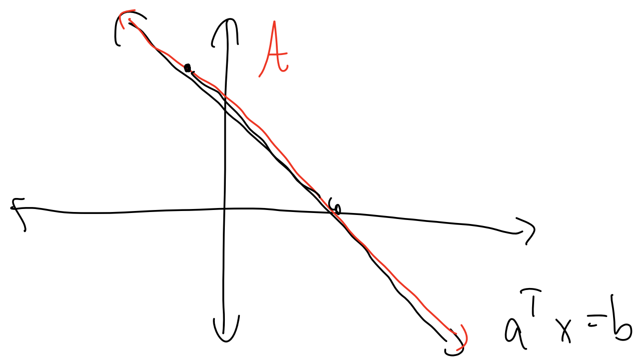
Special Convex Sets

Definition. A set $\mathcal{H} \subset \mathbb{R}^n$ is called a *half-space* if $\mathcal{H} = \{x \in \mathbb{R}^n \mid a^\intercal x \leq b\}$ for some $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}$.



Definition. A set $\mathcal{A} \subset \mathbb{R}^n$ is called an $(n-1)$-dimensional *affine subspace* if $\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\intercal x \underset{=}{\cancel{\leq}} b\}$ for some $a \in \mathbb{R}^n$ with $a \neq 0$ and $b \in \mathbb{R}$.

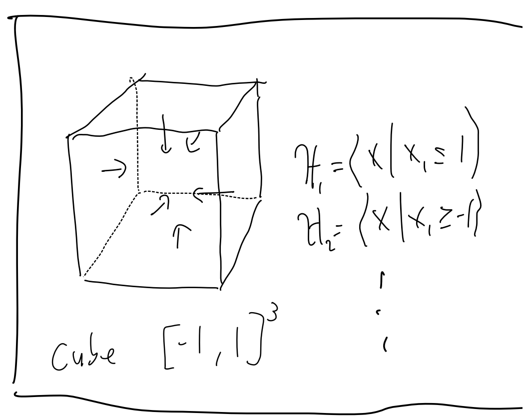
$$\{x \mid a^T x = b\} = \{x \mid a^T x \leq b\} \cap \{x \mid -a^T x \leq -b\}$$



Definition. A set $P \subset \mathbb{R}^n$ is called a *polyhedron* if it is the intersection of a finite number of half-spaces and $(n-1)$-dimensional affine subspaces.

$$P = \left\{ x \;\middle|\; \begin{array}{ccc} \overbrace{a_1^T x \leq b_1}^{H_1} & \cdots & \overbrace{a_n^T x \leq b_n}^{H_n} \\ \underbrace{\tilde{a}_1^T x = \tilde{b}_1}_{A_1} & \cdots & \underbrace{\tilde{a}_m^T x = \tilde{b}_m}_{A_m} \end{array} \right\}$$

$$= \{x \mid Ax \leq b,\ A_{eq} x = b_{eq}\}$$



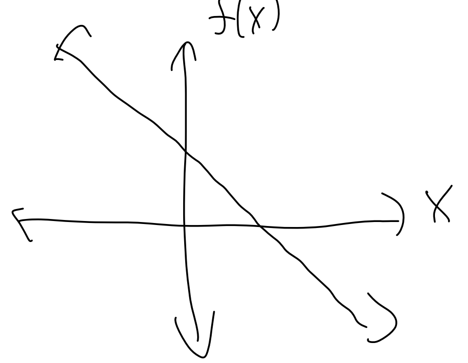
$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad A_{eq} = \begin{bmatrix} \tilde{a}_1^T \\ \vdots \\ \tilde{a}_m^T \end{bmatrix} \quad b_{eq} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{bmatrix}$$

$$\{\underbrace{2x}_{z} \mid A\underbrace{x}_{\frac{z}{2}} \le b \quad A_{eq}\underbrace{x}_{\frac{z}{2}} = b_{eq}\} = \{z \mid \frac{A}{2} z \le b \quad \frac{A_{eq}}{2} z = b_{eq}\}$$

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *affine* if $f(x) = a^\top x + b$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

$\forall x, y \in \mathbb{R}^n \quad \theta \in [0,1]$

$$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y)$$



Definition. The p-norm for $p \in \mathbb{R}$ with $p \geq 1$ is the function $||x||_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.

$||x||_2 \quad ||x||_1$

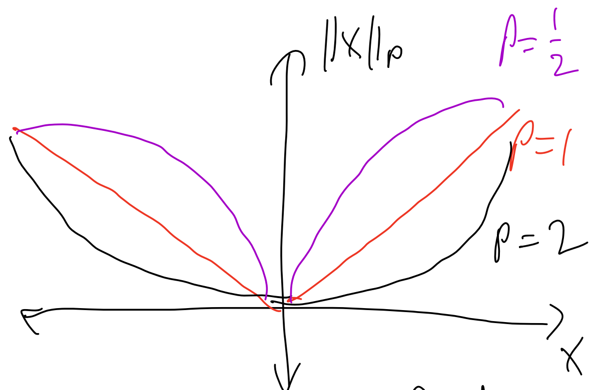
$$||\theta x + (1-\theta)y|| \overset{?}{\le} \theta||x|| + (1-\theta)||y||$$

$$= ||\theta x|| + ||(1-\theta)y||$$

Homogeneity

$||v+u|| \le ||v|| + ||u||$

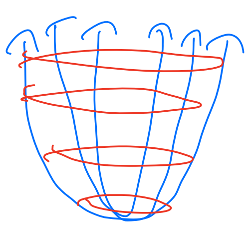
Triangle Inequality



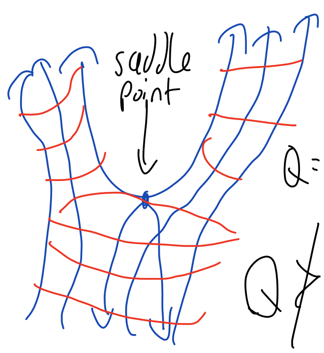
Can drop q

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *quadratic* if $f(x) = x^\top Qx + r^\top x + q$ with $Q \in \mathbb{R}^{n\times n}$, $r \in \mathbb{R}^n$, and $q \in \mathbb{R}$.

$Q \succeq 0 \to f$ is convex



$Q \succ 0$



$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

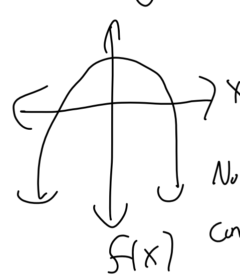
$Q \nsucceq 0$

In $\mathbb{R}^1$

$f(x) = Qx^2 + rx + q$

If $Q < 0$

$\downarrow$



Not convex

$\min f(x)$ vs $\min f(x) + c$ $\longrightarrow$ Same Minimizers

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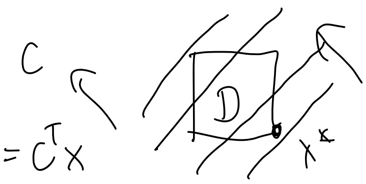
Special Convex Optimization Problems

Linear Programs (LP)

f - affine / linear function $\quad D$ - Polyhedron

$f(x) = c^T x$

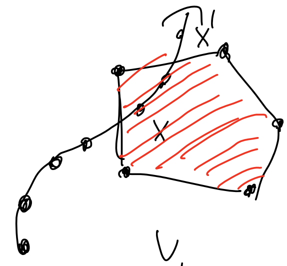


Quadratic Programs (QP)

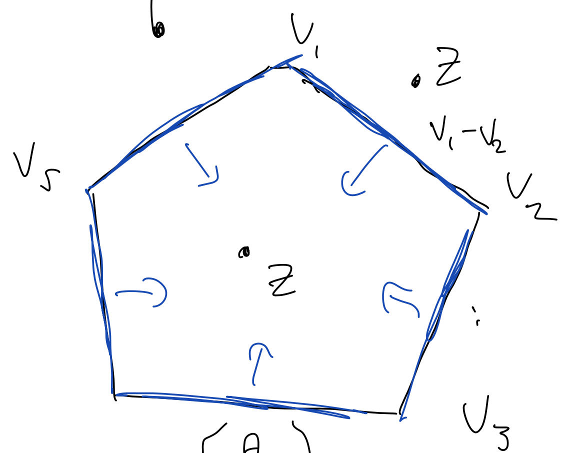
f - Quadratic function $\quad D$ - Polyhedron

$Q \succeq 0$

Motion Planning - Obstacle Avoidance



Problem: Check if $z \in \mathbb{R}^n$ is in $\mathrm{conv}(v_1 \cdots v_k)$.



$$P = \{x \mid A \cdot x \leq b\}$$

$$[1\ 1\ 1\ 1\ 1] \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_5 \end{bmatrix} = 1$$

$$z \in \mathrm{Conv}(\{v_1 \dots v_k\})$$

$$\Longleftrightarrow \quad \underline{\exists \ \theta_1 \dots \theta_k \geq 0}$$

θ - decision variable

$$\sum_{i=1}^{k} \theta_i = 1,$$

$$\sum_{i=1}^{k} \theta_i v_i = z$$

All solutions equally good

$$\text{feasibility} \Leftrightarrow f(\theta) = 0$$

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_k \end{bmatrix} \qquad \underbrace{-I_{k \times k}}_{A} \theta \leq \underbrace{0_{k \times 1}}_{b}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}}_{A_{eq} \in \mathbb{R}^{6 \times 5}} \theta = \underbrace{\begin{bmatrix} 1 \\ z \end{bmatrix}}_{b_{eq}}$$

$$D = \{\theta \mid A\theta \le b,\ A_{eq}\theta = b_{eq}\}$$

LP has solution $\iff D \neq \emptyset \iff z \in \mathrm{conv}(v_1 \cdots v_k)$

LP formulation

See Recitation 13 for a QP formulation of an optimal control problem.

Office Hours

Hyprcube
$[-1,1]^n$

V_4

dim	vertices	facets
2	4	4
3	8	6
4	16	8
⋮		
n	2^n	$2 \cdot n$

hard

Show $\|x\|_2^2 \overset{?}{\le} 2\|x\|_\infty^2$

$\|$ $\|$

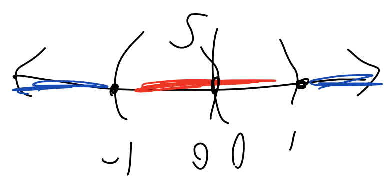
$x_1^2 + x_2^2 \qquad 2\max(x_1^2, x_2^2)$

Either $|x_1| \le |x_2|$ or $|x_1| \ge |x_2|$

$x_1^2 \le x_2^2 \qquad\qquad x_1^2 \ge x_2^2$

$x_1^2 + x_2^2 \le 2x_2^2 \qquad\qquad 2x_1^2 \ge x_1^2 + x_2^2$

$S = (-1, 1)$

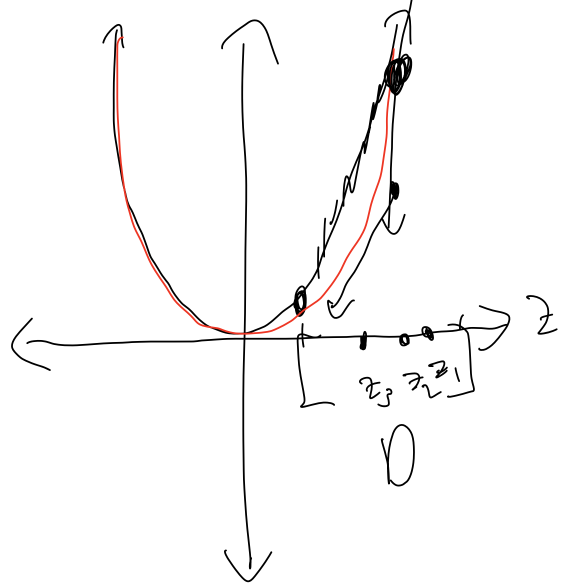


$\bar{S} = [-1, 1]$

$\partial S = \bar{S} \cap \overline{(\sim S)}$

$= \bar{S} \cap (\sim \mathring{S})$

$J(z) = z^T \cdot 1 \cdot z$



$J : \mathbb{R}^n \to \mathbb{R}$

Find $x^* \in \mathbb{R}^n$

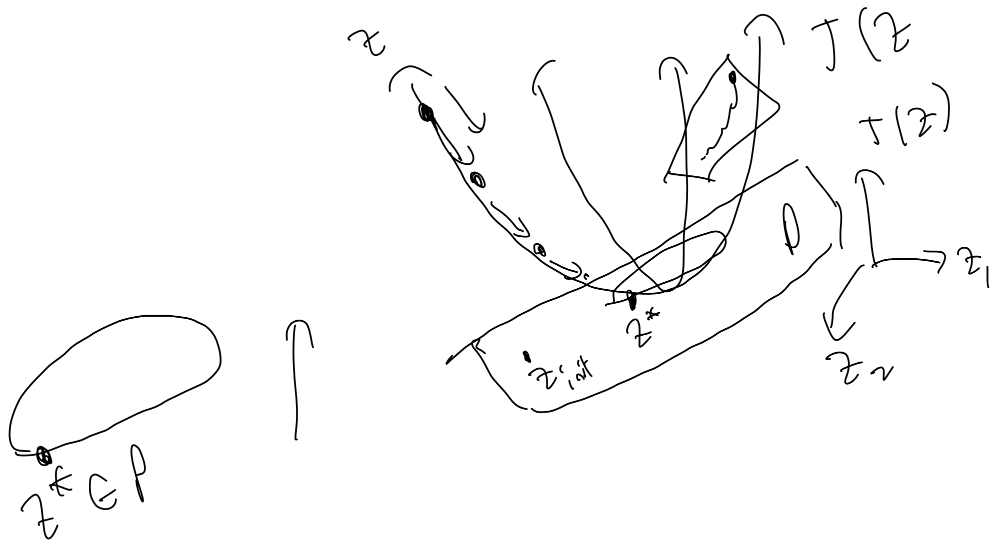
So for all $x \in \mathbb{R}^n$

$J(x^*) \leq J(x)$

$z = [y_0 \; a_0 \; y_1 \; a_1 \; y_2 \; a_2 \; u_0 \; u_1]$

Objective $J(z) \to$ quadratic

Constraints $P \to$ polyhedron



$\cdot x \quad K_1 |||x||| \le ||x|| \le K_2 |||x|||$

$\circ$

0

Assum $|||x||| \le \frac{\varepsilon}{K_2} \xrightarrow{\text{show}} ||x|| \le \varepsilon$

Assum $||x|| < \varepsilon \xrightarrow{\text{show}} |||x||| \le \frac{\varepsilon}{K_1}$

$K_1 |||x||| \le ||x|| \to |||x||| \le \frac{||x||}{K_1} < \frac{\varepsilon}{K_1}$

(x_n) is Cauchy $\leftrightarrow ||x_n - x_m|| \to 0$

as $n, m \to \infty$

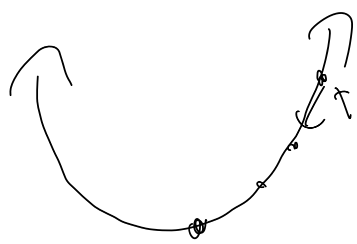
Assume $||x_n - x_m|| \to 0$ as $n, m \to \infty$

Show $|||x_n - x_m||| \to 0$ as $n, m \to \infty$

$\|x\| \le K_2 |||x|||$ $\qquad \tilde{\varepsilon} = \dfrac{\varepsilon}{K_2}$

Pick N so $\forall n, m > N$ $\quad |||x_n - x_m||| < \tilde{\varepsilon}$

$$\begin{aligned}\|x_n - x_m\| &\le K_2 |||x_n - x_m||| \\ &\le K_2 \tilde{\varepsilon} \\ &\le \varepsilon\end{aligned}$$



$x^T Q x + f^T x + q$

$(x - x_0)^T H (x - x_0)$

$ax^2 + bx + c = a(x - r_1)(x - r_2)$

$$\min |x|$$
$$\text{s.t.} \quad ax \le b$$

$$\min s$$
$$\text{s.t.} \quad ax \le b$$
$$x - s \le 0$$
$$s - x \le 0$$

$$|x| = s$$
$$\Downarrow$$
$$x \le s$$
$$x \ge -s$$