ROB 501 Exam-II (Fall 2020)

Prof. Robert Gregg
48 Hour Take-Home Exam
Released: 12pm on Sunday, December 13, 2020
Due: 12pm on Tuesday, December 15, 2020

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

SIGNATURE
(Sign **after** the exam is completed)

LAST NAME (PRINTED), *FIRST NAME*

FILL IN YOUR NAME NOW. COPY THE HONOR CODE NOW.

RULES:

1. **NO COLLABORATION OF ANY KIND**
2. OPEN TEXTBOOK, CLASS NOTES, HOMEWORK
3. GOOGLING SOLUTIONS IS CONSIDERED ACADEMIC DISHONESTY, AND MOST PROBLEMS CANNOT BE EASILY FOUND ON THE WEB ANYWAY
4. CALCULATOR/COMPUTER ALLOWED BUT MUST SHOW CALCULATION STEPS FOR FULL CREDIT
5. SUBMIT QUALITY PHOTOS/SCANS TO GRADESCOPE BY DEADLINE (STRICT)

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Problems 1 - 5 (30 points: 5×6)

Instructions. Each part of a question is worth 1.5 points. Submit your answers to questions 1-5 as follows:

1. Download the answer sheet from Canvas (`ROB501 Final Fa2020 TF Answer Sheet.pdf`).
2. Print it, or open it in your favorite PDF viewer app (see Canvas announcement if you need ideas).
3. Clearly mark your answer to each question on the answer sheet.
4. Scan or export your solutions, and upload them to Gradescope.

Do not modify the answer sheet, or attach any extra pages. You do not need to show your work. Answers written directly on the questions below will not be graded.

1. (Real analysis 1) For the following questions, consider a normed space $(\mathcal{X}, ||\cdot||)$.

Circle True or False as appropriate *on the answer sheet* for the following statements:

(a) If a set containing a single point in $\mathcal{X}$ is a closed set, then a set containing two points in $\mathcal{X}$ is also a closed set.

(b) A set that does not contain all of its limit points is open.

(c) If sets S_1 and S_2 are neither closed nor open, it is possible for $S_1 \bigcap S_2$ to be closed or open.

(d) Let $A \subset \mathcal{X}$ be a compact set, and $\{a_n\}$ a sequence in A. If $\{a_{n_i}\}$ is a subsequence of $\{a_n\}$ that converges to some $a^* \in A$, then $\lim\{a_n\} = a^*$.

2. (Real analysis 2)

Circle True or False as appropriate *on the answer sheet* for the following statements:

(a) Consider $\mathbb{Q} \subset (\mathbb{R}, |\cdot|)$, where $|\cdot|$ denotes the absolute value. The number e is a limit point of $\mathbb{Q}$.

(b) Suppose V is a finite-dimensional subspace of normed space $(\mathcal{X}, ||\cdot||)$, and $A \subset V$ is compact. Then A is also complete.

(c) Let $\{x_n\} = \log(n)$. Then $\forall\, \delta > 0$, $\exists\, G \in \mathbb{N}$ s.t. $\forall\, p, q > G$, $|x_p - x_q| < \delta$.

(d) Let $A = [a, b] \subset \mathbb{R}$ be a closed and bounded (compact) interval where $b > a$. Suppose $f : A \longrightarrow \mathbb{R}$ is continuous everywhere except at $x^* \in A$, and f is convex within two subintervals, $[a, x^*)$ and $[x^*, b]$, but not over the entire set A. Then $\exists\, p \in A$ s.t. $f(p) \leq f(x), \forall x \in A$.

3. (Random variables)

Circle True or False as appropriate *on the answer sheet* for the following statements:

(a) Given two jointly distributed random vectors X_1 and X_2, their covariance matrix can be given by $\text{cov}(X_1, X_2) = E\{X_1 X_2^\top\} - E\{X_1\}E\{X_2\}^\top$.

(b) Consider two random vectors, X_1 with mean μ_1 and covariance Σ_1, and X_2 with mean μ_2 and covariance Σ_2. If $E\{(X_1 - \mu_1)(X_2 - \mu_2)^\top\} = 0$, then X_1 and X_2 are independent.

(c) Consider a Gaussian random vector $X \sim \mathcal{N}(\mu_X, \Sigma_X)$. Define $Y = AX + b$. Then, the covariance of Y given X is $\Sigma_{Y|X} = \Sigma_X - \Sigma_X A^\top (A\Sigma_X A^\top)^{-1} \Sigma_X A^\top$.

(d) Consider Gaussian random vector $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ with mean $\mu_x = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$ and covariance $\Sigma_x = \begin{bmatrix} \Sigma_{11} & 0 & 0 \\ 0 & \Sigma_{22} & \Sigma_{23} \\ 0 & \Sigma_{23} & \Sigma_{33} \end{bmatrix}$. Define a new random vector $Y = X_2 + X_3$. Then X_1 and Y are independent.

4. (Estimators) Consider the equation $y_t = Cx_t + \epsilon_t$ where $C \in \mathbb{R}^{n\times m}$. You are given that $E\{\epsilon_t\} = 0$, $E\{\epsilon_t \epsilon_t^\top\} = Q$, $E\{x_t\} = \bar{x}$, and $E\{(x_t - \bar{x})(x_t - \bar{x})^\top\} = P$. Assume that $E\{\epsilon_t x_t^\top\} = 0$, i.e., x_t and ϵ_t are uncorrelated.

Circle True or False as appropriate *on the answer sheet* for the following statements:

(a) Assume that $P = \infty I$, the columns of C are linearly independent, and $Q > 0$. Then $\hat{x}_t = \hat{K} y_t$ is the best linear unbiased estimate of x_t when:

$$\hat{K} = \arg \min_{K \in \mathbb{R}^{m\times n}} \text{tr}(KQK^\top)$$

(b) If $\text{rank}(C) = n$, then the minimum norm solution when the norm is induced by inner product $< x_1, x_2 >= x_1^\top Q^{-1} x_2$ is:

$$\hat{x}_t = \arg \min_{y_t = Cx_t} ||x_t|| = QC^\top (CQC^\top)^{-1} y_t$$

Furthermore, $\hat{x}_t \in \mathcal{R}(C^\top)$.

(c) Assume $Q > 0$ and $P \geq 0$. Then the minimum variance estimate is $\hat{x}_t = \bar{x} + PC^\top (CPC^\top + Q)^{-1} y_t$.

(d) Assume $Q > 0$, $P > 0$ and $\bar{x} = 0$. We seek minimum variance solution $\hat{x}_t = \begin{bmatrix} x_t^{(1)} & \dots & x_t^{(m)} \end{bmatrix}^\top$ such that $\hat{x}_t^{(i)} = \arg \min_{\eta \in M} ||x_t^{(i)} - \eta||$ where $M = \text{span}\{y_t^{(0)}, \dots, y_t^{(n)}\}$. Then, we can use the normal equations and the Gram matrix is $CPC^\top + Q$.

5. (Matrices, Decompositions, Deterministic Estimators)

Circle True or False as appropriate *on the answer sheet* for the following statements:

(a) Let N be a square root of real matrix $M = N^T N$. The singular value decompositions (i.e., $U\Sigma V^T$) of N and M share the same V matrix.

(b) Suppose Q, R are the QR factorization of $A \in \mathbb{R}^{n\times n}$. Denote the sets consisting of the columns of A and Q as $\{a_i\}_{i=1}^n, \{q_i\}_{i=1}^n$, respectively. Then for $\hat{x} \in \mathbb{R}^n$, $d(\hat{x}, \text{span}\{\{a_i\}_{i=1}^n\}) = d(\hat{x}, \text{span}\{\{q_i\}_{i=1}^n\})$.

(c) Consider the subset $L = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid x_2 = px_1 + q \right\}$ of the inner product space $(\mathbb{R}^2, \mathbb{R}, < \cdot, \cdot >)$, where $< \cdot, \cdot >$ is the standard inner product and p, q are fixed real numbers. There exists a unique $\ell \in L$ s.t. $\forall m \in L$, $||\ell|| \leq ||m||$.

(d) Let $A \in \mathbb{R}^{m\times n}$ have rank r. Let σ_r be the smallest (non-zero) singular value of A and denote the corresponding left and right singular vectors as u_r and v_r, respectively. Then $\text{rank}(A - \beta u_r v_r^\top) = r - 1$ for any real $\beta \geq \sigma_r$.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

“I do not know”,

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. <u>If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.</u>

6. (20 points) Consider the robot shown in Figure 1, which moves along the x axis. We want to estimate the scalar position x_t and velocity v_t in the robot state vector $z_t = [x_t, v_t]^T$. The robot is equipped with two sensors: a distance sensor that reads value d_t and an encoder that reads value n_t. It can be controlled by applying a torque τ_t to the wheel.

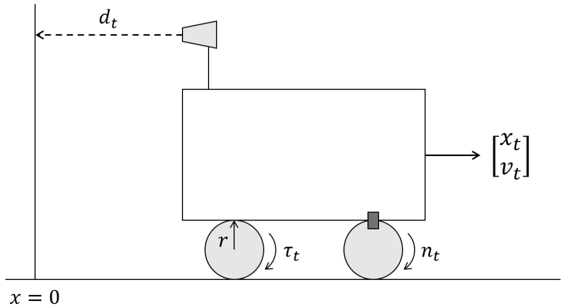


Figure 1: A robot can move along one dimension. It has wheel radius r and mass m.

The distance sensor measures the distance to the wall and produces measurement $y_1 = d_t$. We model the relationship between the sensor output and the robot position as follows:

$$x_t = c_1 d_t + \epsilon_1$$

where c_1 converts the sensor reading to meters. The distance sensor noise is Gaussian distributed with $\epsilon_1 \sim \mathcal{N}(0, q_1)$. The wheel encoder measures the number of "ticks", $y_2 = n_t$, in a time step. The relationship between the number of ticks and the robot velocity over Δt seconds is given as follows:

$$v_t = \frac{2\pi r}{\Delta t N} n_t + \epsilon_2$$

where N is the total number of ticks per wheel encoder rotation, and r is the wheel radius. The encoder sensor noise is Gaussian distributed with $\epsilon_2 \sim \mathcal{N}(0, q_2)$. The following data are given:

$$r = 0.1\,m, \quad m = 10\,kg, \quad N = 60, \quad c_1 = 2 \cdot 10^{-3}, \quad \Delta t = 0.5\,s, \quad q_1 = 3 \cdot 10^4, \quad q_2 = 1 \cdot 10^3.$$

(a) (10 points) The robot does not know its initial position, so it takes three measurements from its two sensors:

$$\left\{ \begin{bmatrix} d_0^{(1)} = 320 \\ n_0^{(1)} = 0 \end{bmatrix}, \begin{bmatrix} d_0^{(2)} = 290 \\ n_0^{(2)} = 1 \end{bmatrix}, \begin{bmatrix} d_0^{(3)} = 270 \\ n_0^{(3)} = 0 \end{bmatrix} \right\}$$

Find the best linear unbiased estimate of the robot state, and give the estimate's covariance P_0. You must define matrices used to calculate your solution; just copying the formula for BLUE will receive zero points.

$\hat{z}_0 = \begin{bmatrix} \hat{x}_0 \\ \hat{v}_0 \end{bmatrix} =$	$P_0 =$

(Show work below)

(b) (10 points) We can drive the robot by sending a torque value to its wheel as a control signal, $u_t = \tau_t$. Given this control signal, the state dynamics are modelled as:

$$x_{t+1} = x_t + (v_t + \frac{\Delta t}{mr}\tau_t)\Delta t + \delta_1$$

$$v_{t+1} = v_t + \frac{\Delta t}{mr}\tau_t + \delta_2$$

The model noise is Gaussian $\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix} \sim \mathcal{N}(0, R)$ where $R = \begin{bmatrix}0.05 & 0.01 \\ 0.01 & 0.08\end{bmatrix}$. Assume $z_0 = \begin{bmatrix}x_0 \\ v_0\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\hat{x}_0 \\ \hat{v}_0\end{bmatrix}, P_0\right)$ with mean and covariance from part (a); if you could not solve part (a) then leave it symbolic.

At time zero, we apply control input $\tau_0 = 0.8$. After Δt, we measure $d_1 = 410$, $n_1 = 18$ from the sensors. Determine the mean, $\hat{z}_{1|1}$, and covariance, $P_{1|1}$, of the state at time $t_1 = 1 \cdot \Delta t$, given the initial control input and the resulting sensor measurement. Again, you must define any matrices used to calculate your solution; just giving a formula will earn zero points.

$\hat{z}_{1\|1} =$	$P_{1\|1} =$

(Show work below)

Please show your work for question 6.

7. (15 points) Let $A = \begin{bmatrix} 1 & -1 \\ b & b \\ 0 & 0 \end{bmatrix}$, for real $b > 1$, be the square root of matrix $M = A^T A$.

(a) (9 points) Calculate the singular value decomposition (SVD) of A, leaving b as a variable.

$\Sigma =$

$U =$

$V =$

(Show your calculations below)

(b) (6 points) Find a rank-1 matrix ΔM of minimum 2-norm such that $\text{rank}(M + \Delta M) = 1$, and calculate its norm $||\Delta M||_2 = \sqrt{\lambda_{\max}((\Delta M)^T \Delta M)}$.

$\Delta M =$	$||\Delta M||_2 =$

(Show work below)

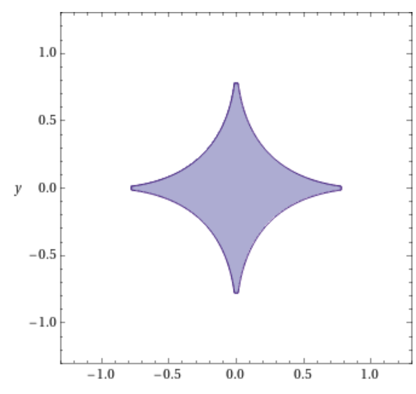
Extra space for question 7.

8. (15 points) The following are three (3) short answer questions. You do not need to give a formal proof; only give a few short reasons/calculations why something is TRUE or FALSE. **Part (c) is on the next page.**

(a) **(5 Points)** Let $(\mathbb{P}^n, \mathbb{R}, ||\cdot||)$ be a normed space, where $\mathbb{P}^n$ is the set of polynomials of degree up to (and including) $n \geq 2$. For $1 \leq m < n$, any Cauchy sequence (x_k) in $\mathbb{P}^m$ has a limit in $\mathbb{P}^m$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

(b) **(5 Points)** A norm exists for vector space $(\mathbb{R}^2, \mathbb{R})$ such that the following set is a ball (of some radius):



Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

(c) **(5 Points)** In normed space $(\mathbb{R}, \mathbb{R}, |\cdot|)$, the intersection of $\bigcap_{n=1}^{\infty}(-\frac{n+1}{n}, \frac{n+1}{n})$ and $\bigcup_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$ is a compact set.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

Extra space for question 8.