

ROB 501 Exam-I (Fall 2020)
Prof. Robert Gregg
48 Hour Take-Home Exam
Released: 12pm on Friday, October 23, 2020
Due: 12pm on Sunday, October 25, 2020

HONOR PLEDGE: Copy (NOW) and SIGN (after the exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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(Sign **after** the exam is completed)

LAST NAME (PRINTED), *FIRST NAME*

FILL IN YOUR NAME NOW. COPY THE HONOR CODE NOW.

RULES:

1. **NO COLLABORATION OF ANY KIND**
2. **OPEN TEXTBOOK, CLASS NOTES, HOMEWORK**
3. **GOOGLING SOLUTIONS IS CONSIDERED ACADEMIC DISHONESTY, AND MOST PROBLEMS CANNOT BE EASILY FOUND ON THE WEB ANYWAY**
4. **CALCULATOR/COMPUTER ALLOWED BUT MUST SHOW CALCULATION STEPS FOR FULL CREDIT**
5. **SUBMIT QUALITY PHOTOS/SCANS TO GRADESCOPE BY DEADLINE (STRICT)**

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Problems 1 - 5 (30 points: 5×6)

Instructions. Each part of a question is worth 1.5 points. Submit your answers to questions 1-5 as follows:

1. Download the answer sheet from Canvas (ROB501 Midterm Fa2020 TF Answer Sheet.pdf).
2. Print it, or open it in your favorite PDF viewer app (see Canvas announcement if you need ideas).
3. Clearly mark your answer to each question on the answer sheet.
4. Scan or export your solutions, and upload them to Gradescope.

Do not modify the answer sheet, or attach any extra pages. You do not need to show your work. Answers written directly on the questions below will not be graded.

1. (Logic questions)

Circle True or False as appropriate *on the answer sheet* for the following statements:

- (a) For any statement over the natural numbers, $P(n)$:

$$\left(P(0) \wedge \forall k, P(k) \implies P(k+1) \right) \implies \forall n, P(n)$$

- (b) The negation of “if the leaves are changing, then it is fall,” is “it is not fall, so the leaves are not changing.”

- (c) The negation of the statement

$$\forall x, y \in C \subset \mathcal{X}, \forall 0 \leq \lambda \leq 1, \text{ then } f(\lambda x + (1 - \lambda)y) \leq f(x) + (1 - \lambda)f(y)$$

is:

$$\exists x, y \in C \subset \mathcal{X}, \exists 0 \leq \lambda \leq 1 \text{ s.t. } f(\lambda x + (1 - \lambda)y) > f(x) + (1 - \lambda)f(y)$$

- (d) $\neg(\neg p \vee q) \iff \neg(p \implies q)$.

2. (Change of basis, linear transformations, eigenvalues & eigenvectors)

Circle True or False as appropriate *on the answer sheet* for the following statements:

- (a) Let v be an eigenvector of real symmetric matrix $A \in \mathbb{R}^{n \times n}$. Given non-zero $\alpha \in \mathbb{C}$, then αv is also an eigenvector of A .
- (b) Let $(\mathbb{P}^n, \mathbb{R})$ be the space of polynomials $(p : \mathbb{R} \rightarrow \mathbb{R})$ of degree less than or equal to n over the field \mathbb{R} . Define the linear map $\mathcal{L} : \mathbb{P}^2 \rightarrow \mathbb{P}^3$ as

$$(\mathcal{L}(p))(x) = \int_0^x p(s) ds.$$

Suppose A is the matrix representation of \mathcal{L} with respect to the standard polynomial bases in \mathbb{P}^2 and \mathbb{P}^3 (from lecture). Then A^{-1} represents the linear map $\mathcal{D} : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ given by

$$(\mathcal{D}(p))(x) = \frac{d(p(x))}{dx}.$$

- (c) Consider matrix $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Then, the eigenvalues of matrix $4A + 3I$ are $\{4\lambda_1 + 3, \dots, 4\lambda_n + 3\}$.
- (d) Consider matrix $A \in \mathbb{R}^{n \times n}$ with *distinct* eigenvalues $\{\lambda_1, \dots, \lambda_k\}$, $k < n$, i.e., the other $n - k$ eigenvalues are repeats. Then $\det(A) = \prod_{i=1}^k \lambda_i$.

3. (Inner product spaces, norms, projection theorem)

Circle True or False as appropriate *on the answer sheet* for the following statements:

- (a) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, \langle \cdot, \cdot \rangle)$, where $\langle x, y \rangle = x^T y$ for $x, y \in \mathbb{R}^n$. If $x \perp y$, then $\|x + y\|_1^2 = \|x\|_1^2 + \|y\|_1^2$ where $\|\cdot\|_1$ is the 1-norm.
- (b) Let $(\mathbb{R}^2, \mathbb{R}, \langle \cdot, \cdot \rangle)$ be a 2-dimensional inner product space, and $M, N \subset \mathbb{R}^2$ be two orthogonal subspaces. Then $M^\perp = N$ and $N^\perp = M$.
- (c) Consider the finite-dimensional inner product space $(\mathcal{X}, \mathbb{R}, \langle \cdot, \cdot \rangle)$ and vectors $u_1, u_2, u_3 \in \mathcal{X}$. You are given:

$$\begin{aligned} \langle u_1, u_1 \rangle &= 5 & \langle u_1, u_2 \rangle &= 2 \\ \langle u_2, u_2 \rangle &= 1 & \langle u_1, u_3 \rangle &= 3 \\ \langle u_3, u_3 \rangle &= 9 & \langle u_2, u_3 \rangle &= 0 \end{aligned}$$

Then, for some $v \in \mathcal{X}$:

$$d(v, \text{span}\{u_1, u_2, u_3\}) = \|v - \alpha_1 u_1 - \alpha_2 u_2 - \alpha_3 u_3\|$$

where:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \langle u_1, u_3 \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \langle u_2, u_3 \rangle \\ \langle u_3, u_1 \rangle & \langle u_3, u_2 \rangle & \langle u_3, u_3 \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \langle v, u_3 \rangle \end{bmatrix}$$

- (d) Consider inner product space $(\mathcal{X}, \mathbb{R}, \langle \cdot, \cdot \rangle)$ and let $S \subset \mathcal{X}$ be a subset of \mathcal{X} . Let $U = \text{span}\{S\}$ and $x_1, x_2 \in \mathcal{X}$. If $d(x_1, U) = \|x_1 - u_1\|$ and $d(x_2, U) = \|x_2 - u_2\|$, then

$$d(3x_2 - 2x_1, U) = \|3x_2 - 2x_1 - 3u_2 + 2u_1\|$$

4. (Matrix properties)

Circle True or False as appropriate *on the answer sheet* for the following statements:

- (a) Let $A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} = A^T \in \mathbb{R}^{4 \times 4}$. Then $j - i^2/h > 0$ is a necessary condition for $A > 0$.
- (b) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, \langle \cdot, \cdot \rangle)$ with inner product $\langle y_1, y_2 \rangle = y_1^T S y_2$ for $y_1, y_2 \in \mathbb{R}^n$ where $S = S^T \in \mathbb{R}^{n \times n}$ is a full rank symmetric matrix. Given a matrix $A \in \mathbb{R}^{n \times m}$, where $n > m$, and a vector $y \in \mathbb{R}^n$, $(A^T S A)^{-1} A^T S y \in \mathcal{R}(A)$, where $\mathcal{R}(A)$ is the range of A .
- (c) Let $B \in \mathbb{R}^{n \times n}$ be a (symmetric) positive definite matrix. If $C \in \mathbb{R}^{n \times n}$ is similar to B , then C is (symmetric) positive definite.
- (d) Let $A, B, P \in \mathbb{R}^{n \times n}$, where P is invertible. Suppose $A = P^{-1} B P$. If $\{v^1, \dots, v^k\}$ are the eigenvectors of A , where $1 \leq k \leq n$, then $\{P v^1, \dots, P v^k\}$ are the eigenvectors of B .

5. (Vector spaces and vector fields)

Circle True or False as appropriate *on the answer sheet* for the following statements:

- (a) Let $A, B \subset \mathcal{X}$ be two subsets of vector space $(\mathcal{X}, \mathbb{R})$. Then $A^\perp = \text{span}(B)^\perp \implies A = B$
- (b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a valid basis for both $(\mathbb{R}^2, \mathbb{R})$ and $(\mathbb{C}^2, \mathbb{C})$.
- (c) Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $v^1, v^2, v^3, v^4 \in \mathcal{X}$ be vectors in the vector space. Let $S_1 = \text{span}\{v^1, v^2, v^3\}$ and $S_2 = \text{span}\{v^2, v^3, v^4\}$. If $v^4 \notin S_1$, then $\dim(S_1) = \dim(S_2)$.
- (d) Consider vector space $(\mathbb{R}^n, \mathbb{R})$, $n > 2$. The subset $S = \{x \in \mathbb{R}^n | x_1 = 1, x_2 = 0\}$ is a valid subspace (where x_i is the i^{th} element in vector x).

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you **MUST** show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

“I do not know”,

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.

6. (15 points) (Place your answers in the **boxes** and show your work below.)

Consider the vector spaces $(\mathbb{P}^3, \mathbb{R})$ and $(\mathbb{P}^1, \mathbb{R})$, where \mathbb{P}^n represents the set of polynomials $p : \mathbb{R} \rightarrow \mathbb{R}$ of degree up to (and including) n . Note that \mathbb{P}^n is $n + 1$ dimensional. For the first vector space, you are given the *Bernstein* basis $B = \{(1 - t)^3, 3t(1 - t)^2, 3t^2(1 - t), t^3\}$ (you do **not** need to prove it is a basis, just take it as fact). For the second vector space, you are given the standard basis $E = \{1, t\}$.

Define the linear operator $\mathcal{L} : \mathbb{P}^3 \rightarrow \mathbb{P}^1$ by $(\mathcal{L}(p))(t) = \frac{d^2}{dt^2}p(t), \forall p \in \mathbb{P}^3$.

- (a) (9 points) Find the matrix representation A of $\mathcal{L} : \mathbb{P}^3 \rightarrow \mathbb{P}^1$ with respect to the given bases.

(a) $A =$

- (b) (6 points) Consider the basis $V = \{1, t, t^2/2!, t^3/3!\}$ for $(\mathbb{P}^3, \mathbb{R})$. Let P denote the change of basis matrix from B to V and let \bar{P} denote the change of basis matrix from V to B . **Compute either P or \bar{P} (choose ONE)**. You must clearly indicate which change of basis matrix you are computing. If you mislabel the matrix you will lose half the points.

(b) $P = \begin{bmatrix} & & & \end{bmatrix}$ **OR** $\bar{P} = \begin{bmatrix} & & & \end{bmatrix}$ (only one is required)

Show your steps and reasoning below. No reasoning \implies no points. I do not need to see how you did every algebra calculation, but I do need to see all the steps in the derivation.

Please show your work for question 6.

7. (20 points) Consider the inner product space $(\mathbb{R}^3, \mathbb{R}, \langle \cdot, \cdot \rangle)$ where $\langle x, y \rangle = x^T S y$ for $x, y \in \mathbb{R}^3$ and $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Define $M = \{x \in \mathbb{R}^3 \mid Cx = 0\} \subset \mathbb{R}^3$ for $C = [a, b, 0]$ and non-zero $a, b \in \mathbb{R}$.

(a) (10 points) Find an orthogonal basis for M . It is not necessary to normalize.

(a)

(b) (10 points) Now let $a = 1$, $b = -1$. Find the orthogonal projection of $x = [1, 2, -1]^T$ onto M .

(b) $P(x) =$

Show your steps and reasoning below. No reasoning \implies no points. I do not need to see how you did every algebra calculation, but I do need to see all the steps in the derivation. I suggest you check your calculations using MATLAB by first entering `a = sym('a','real')`, `b = sym('b','real')`. You might find the `simplify` function helpful. That said, this problem is certainly doable without MATLAB.

Please show your work for question 7.

8. (15 points) (Proof Problem) (Done in two parts so that you cannot lose too many points on each part)

- (a) (5 points) Let $x, y, z \in \mathbb{Z}$ with $z \neq 0$. If both x and y are divisible by z , then is $cx + y^d$ divisible by z for all $c, d \in \mathbb{Z}$ with $d \geq 1$? If yes, give the proof; if no, give a counterexample. Be sure to justify why a calculation does or does not give an integer (e.g., “the sum of integers is an integer”).

- (b) (10 points) Show that $\sum_{i=0}^n i(i+2) = \frac{n(n+1)(2n+7)}{6}$ for integers $n > 0$.

Extra space for question 8.