1)

if there are n collumns in A

we know that

$$\text{null}(A) = \dim(\text{nullspace of } A)$$
$$= \dim(\{Ax = 0 \text{ where } x \in \mathbb{R}^n\})$$

now lets say $\text{Rank}(A) = r$ & $r \leq n$

$\Rightarrow$ in row reduse echelon form of $Ax=0$, there are r rows with leading ones ;

$\Rightarrow$ $Ax = 0$ has $(n-r)$ collums of A without leading zeroes

Let $x_1, x_2 \cdots x_{n-r}$ denote sol^n obtained sequentially by setting each free varriable to 1 & remaining free variables be zero

$\therefore$ $x_1, x_2 \cdots x_{n-r}$ will be linearly independent

$\therefore$ sol^n to $Ax=0$ is $x = \sum_{i=1}^{n-r} \alpha_i x_i$

$\Rightarrow$ $\{x_1, x_2 \cdots x_{n-r}\}$ spans nullspace of(A)

$\Rightarrow$ & nullity $(A) = \dim(\{x_1, x_2 \cdots x_{n-r}\})$

as $\{x_1, x_2 \cdots x_{n-r}\}$ are linearly independent

nullity $(A) = n - r$

$\Rightarrow$ $\text{Ran}(A) + \text{nullity}(A) = r + n - r$
$= n$

7)

a) $$M = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \approx \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where $A = 1$ $\quad B = 3$ $\quad C = 8$

if $M > 0$

$\Leftrightarrow$ $A > 0$ & $C - B^T A^{-1} B > 0$

$1 > 0$ $\quad 8 - 3.1.3 = -1 < 0$

$\therefore$ The eqn is not satisfied so matrix is not Positive definte

b) $$M = \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 4 & 7 \\ \hline 6 & 7 & 10 \end{array}\right] \approx \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

to check $M > 0$ will check $A > 0$ & $C - B^T A^{-1} B > 0$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$ $\quad C = 10$

$$C - B^T A B = 10 - \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= 10 - \left[36 + \frac{49}{4}\right] < 0$$

$\therefore$ M is not Positive definite

7)
c)

$$M = \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 5 & 7 \\ \hline 6 & 7 & a \end{array}\right] \approx \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \quad C = a$$

if $M > 0$ then $A > 0$ & $C - B^T A^{-1} B > 0$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \approx \begin{bmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{bmatrix} \quad \begin{array}{l} A_1 > 0 \\ 1 > 0 \end{array} \quad \begin{array}{l} C_1 - B_1^T A_1^{-1} B \\ = 5 - 22 = 1 > 0 \end{array}$$

$\therefore$ $A > 0$. $\quad A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$$C - B^T A^{-1} B = a - \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

$$= a - \begin{bmatrix} 6 & 7 \end{bmatrix} \begin{bmatrix} 16 \\ -5 \end{bmatrix} = a - (96 - 35)$$

$$= a - 61 > 0$$

$$\Rightarrow \boxed{a > 61}$$

8)

a) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ 3x_1 + 8x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_1 + 3x_2 + 2x_3 = 1 \quad - (1)$

$3x_1 + 8x_2 + 4x_3 = 2 \quad - (2)$

$(2) - 2 \times (1) \Rightarrow$

$$\begin{array}{r} 3x_1 + 8x_2 + 4x_3 = 2 \\ 2x_1 + 6x_2 + 4x_3 = 2 \\ \hline x_1 + 2x_2 = 0 \end{array}$$

$\Rightarrow \boxed{x_1 = -2x_2}$

using eq^n (1) $\quad -2x_2 + 3x_2 + 2x_3 = 1$

$\Rightarrow \boxed{x_3 = \frac{1 - x_2}{2}}$

as this is the undermined system this will have ∞ sol^n as desired above

but we want sol^n with min norm

$\Rightarrow \min(x^T x)$ given $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 8 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow x^T x = x_1^2 + x_2^2 + x_3^2$

$= (-2x_2)^2 + x_2^2 + \left(\frac{1 - x_2}{4}\right)^2$

$= (20x_2^2 + 1 - 2x_2 + x_2)/4 = 1 - 2x_2 + 21x_2^2$

$\min(x^T x) = \min\left(\frac{1}{4}\left(1 - 2x_2 + 21x_2^2\right)\right)$

$\Rightarrow \frac{\partial}{\partial x_i}(\text{RHS}) = 0$ will give a solⁿ for it.

$\Rightarrow \frac{1}{4}(-2 + 2 \cdot 21 x_2) = 0$

$\Rightarrow \boxed{x_2 = 1/21}$ $\boxed{x_2 = 0.0476}$

$\Rightarrow x_1 = -2x_2 = \cancel{-0.952}\ -0.0952$

& $x_3 = \frac{1 - x_2}{2} = \frac{1 - 0.0476}{2} = 0.4762$

b) if inner product defined as

$$\langle x, y \rangle = x^T \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix} y$$

$$\Rightarrow \langle x, x \rangle = x^T \begin{bmatrix} 5 & 1 & 9 \\ 1 & 2 & 1 \\ 9 & 1 & 17 \end{bmatrix} x = 5x_1^2 + 2x_2^2 + 17x_3^2 + 2(x_1x_2 + 9x_1x_3 + x_2x_3)$$

from above eqⁿ

$$5(-2x_2)^2 + \cancel{2x_2^2} + 17\left(\frac{1-x_2}{2}\right)^2 - \cancel{2x_2^2}$$

$$+ 9x_1(2x_3) + x_2(2x_3)$$

$$= 20x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 - 9(2x_2)(1-x_2) + x_2(1-x_2)$$

$$= 20x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 - 18x_2 + 18x_2^2 + x_2 - x_2^2$$

$$= 37x_2^2 + 17\left(\frac{1-x_2}{2}\right)^2 - 17x_2$$

So $\frac{\partial}{\partial x_2}$ (RHS) will give minima

$$76x_2 - 17(1-x_2) - 17 = 0$$

$$93x_2 = 34$$

$$(x_1, x_2, x_3] = [-0.64 \quad 0.324 \quad 0.337)$$

If a matrix A is positive (or) semi-positive definite then $A = B^T B$ where B is square root of matrix

here After splitting

$$M = O \Lambda O^T = (O \Lambda_{root}) (\Lambda_{root} O^T)$$

$$\Rightarrow \boxed{B = \Lambda_{root} O^T}$$

Λ_{root} is : $\sqrt{}$ square root of diag elements of Λ