2)

a) for jointly random variables (X, Y, Z)

$$\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

we need $P(X, Y \mid Z = z)$

:) $[X, Y]$ is one random variable & Z is another

WKT

$$\mu_{1|2} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (\alpha_2 - \mu_2)$$

$$\Sigma_{12} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

$$\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Sigma = \left[\begin{array}{cc|c} 2 & 2 & 1 \\ 2 & 4 & 2 \\ \hline 1 & 2 & 2 \end{array}\right]$$

$$\mu_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mu_2 = 0 \quad \Sigma_{11} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \quad \Sigma_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Sigma_{21} = [1 \;\; 2] \quad \Sigma_{22} = 2$$

$$\mu_{1|2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot (1/2) \cdot (z - 0) =$$

$$\boxed{\mu_{12} = \begin{bmatrix} -1 + ((z-1)/2) \\ (z-1) \end{bmatrix}}$$

$$\Sigma_{1|2} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot [1/2 \;\; 1] = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\boxed{\Sigma_{1|2} = \begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix}}$$

b) ~~Ref~~ x so from above

$P(X, Y | Z=z)$ ⇒ 2 random variables are $X|Z=z$ & $Y|Z=z$

⇒ $\mu = \begin{bmatrix} -1 + (z-1)/2 \\ (z-1) \end{bmatrix}$ $\Sigma = \begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix}$

to find $P((X|Z=z) | (Y=y|Z=z))$

$\boxed{\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)}$ $\boxed{\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}$

$\mu_1 = -1 + (z-1)/2$; $\mu_2 = (z-1)$ $\Sigma_{11} = 3/2$ $\Sigma_{12} = 1$ $\Sigma_{22} = 2$

$\Sigma_{22}^{-1} = 1/2$

⇒ $\mu_{1|2} = (-1 + (z-1)/2) + 1 \cdot \frac{1}{2}(y - (z-1))$

$\boxed{\mu_{1|2} = -1 + y/2}$

$\Sigma_{1|2} = (3/2) - 1 \cdot (1/2) \cdot 1 = 1$

$\boxed{\Sigma_{1|2} = 1}$

c) To find $P(X \mid Y=y, Z=z)$ let take random variables X, $[Y, Z]$

$$\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Sigma = \left[\begin{array}{c|cc} 2 & 1 & 2 \\ \hline 1 & 4 & 2 \\ 2 & 2 & 2 \end{array}\right]$$

$\mu_1 = -1 \quad \mu_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Sigma_{11} = 2 \quad \Sigma_{12} = [2 \;\; 1]$

$$\Sigma_{22} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \quad \Sigma_{12}^{-1} = \begin{bmatrix} 0.5 & \\ & \end{bmatrix}$$

$$\Sigma_{22}^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$\Sigma_{12} \cdot \Sigma_{22}^{-1} = [2 \;\; 1] \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$= [0.5 \;\; 0]$$

$$\mu_{1|2} = -1 + [0.5 \;\; 0] \begin{bmatrix} y-0 \\ z-1 \end{bmatrix}$$

$$\boxed{\mu_{1|2} = -1 + y/2}$$

$$\Sigma_{1|2} = 2 - [0.5 \;\; 0] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1$$

$$\boxed{\Sigma_{1|2} = 1}$$

d) Mean & variance of guassian distribution in both (b) & (c) are same i.e. Both distributions are equivalent

3) a)

wkT

for $\vec{\alpha}_k$ $\quad G_k^T \alpha_k = \beta_k$

So similarly for $k+1$

$$G_{k+1}^T \alpha_{k+1} = \beta_{k+1}$$

as its said that y_{k+1} is independent of

span$\{y_1, y_2 \cdots y_k\}$

$$G_{k+1}^T = \begin{bmatrix} G_k^T & 0 \\ 0 & \langle y_{k+1}, y_{k+1} \rangle \end{bmatrix}$$

$\beta_i = \langle \alpha, y_i \rangle$ as y_{k+1} don't have any component

$$\beta = \begin{bmatrix} \beta_k \\ \langle \alpha, y_{k+1} \rangle \end{bmatrix}$$

$$\alpha_{k+1} = (G_{k+1}^T)^{-1} \beta_{k+1} = \begin{bmatrix} G_k^{-1} & 0 \\ 0 & \frac{1}{\langle y_{k+1}, y_{k+1} \rangle} \end{bmatrix} \begin{bmatrix} \beta_k \\ \langle \alpha, y_{k+1} \rangle \end{bmatrix}$$

$$\begin{bmatrix} \alpha_k \\ \alpha \end{bmatrix} = \begin{bmatrix} G_k^{-1} \beta_k \\ \dfrac{\langle \alpha, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle} \end{bmatrix}$$

$$\hat{\alpha}_{k+1} = \hat{\alpha}_k + \alpha y_{k+1}$$

$$\alpha_{k+1} = \underbrace{G_k^{-1} \beta_k}_{\alpha_k} + \underbrace{\frac{\langle \alpha, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle}}_{\beta_q} y_{k+1}$$

$$= \alpha_k + \alpha y_{k+1}$$

$\therefore$ $\exists$ some scalar α where

$$\alpha_{k+1} = \alpha_k + \alpha y_{k+1}$$

b)

$$\hat{y}_{k+1|k} = \underset{m \in M}{\arg\min} \| y_{k+1} - m \|$$

$(\hat{y}_{k+1|k} - y_{k+1}) \perp M_k$ → from projection theorem

for given $\hat{y}_{k+1|k}$ to be belong to M_k

$$\Rightarrow M_{k+1} = M_k \oplus \text{span}\{ y_{k+1} \}$$

$$= M_k \oplus \text{span}\{ y_{k+1} - v \} \quad \text{where } v \in M_k$$

& $M_k \perp (y_{k+1} - \hat{y}_{k+1|k})$ $\because \hat{y}_{k+1|k} \in M_k$

$$\Rightarrow \boxed{\bar{y}_{k+1} = y_{k+1} - \hat{y}_{k+1|k}}$$

using result from a

$$\boxed{\hat{x}_{k+1} = \hat{x}_k + \frac{\langle x, \bar{y}_{k+1} \rangle}{\langle \bar{y}_{k+1}, \bar{y}_{k+1} \rangle} \bar{y}_{k+1}}$$

Homework solutions

Question 1:

The below are solutions obtained from code where

x_sol is the estimated solution for x;
cov is the covariance of the estimate;

Part a:

cov =

```
    4.0000   -2.7500
   -2.7500    2.0000
```

x_sol =

```
    0.6194
    0.4591
```

Part b

cov =

```
    0.0679   -0.0260
   -0.0260    0.1129
```

x_sol =

```
   -1.4303
    1.8791
```

Part c

cov =

```
    0.0487    0.0054
    0.0054    0.0618
```

x_sol =

-1.2201
1.5368

Question 4

The below are solutions obtained from code where

x_sol is the estimated solution for x;
covariance is the covariance of the estimate;

Part a:

covariance =

0.1938 -0.0812
-0.0812 0.1188

x_sol =

0.4504
0.4963

Part b

covariance =

0.0545 -0.0105
-0.0105 0.0828

x_sol =

-1.0134
1.2402

Part c

covariance =

```
    0.0437    0.0072
    0.0072    0.0538
```

x_sol =

```
   -1.0296
    1.2667
```

Question 5

Part a:

The below are solutions obtained from code where
x_sol_wls is the estimated solution for x from WLS;

x_sol_wls =

```
   -1.3169
    1.4368
```

Part b:

The below are solutions obtained from code where
x_sol_blue is the estimated solution for x from BLUE;
cov_blue is the covariance of the estimated solution;

cov_blue =

```
    0.0317   -0.0079
   -0.0079    0.0198
```

x_sol_blue =

-1.3169
1.4368

Part c:

The below are solutions obtained from code where
x_sol_less is the estimated solution for x from MVE with P= 100I;
covariance_less is the covariance of the estimated solution with P= 100I;

covariance_less =

0.0317 -0.0079
-0.0079 0.0198

x_sol_less =

-1.3163
1.4365

The below are solutions obtained from code where
x_sol_more is the estimated solution for x from MVE with P= 1e6*I;
covariance_more is the covariance of the estimated solution with P= 1e6*I;

covariance_more =

0.0327 -0.0086
-0.0070 0.0182

x_sol_more =

-1.3169
1.4368

Part d:

All the solutions are close to each other. Having the *e* variance of identity matrix is close to performing WLS and the variance of the model is also not affecting much because of this.

Question 6

The below are solutions obtained from code where
x_sol is the estimated solution for x from MVE;
covariance is the covariance of the estimated solution;

covariance =

0.0437	0.0072
0.0072	0.0538

x_sol =

-0.8836
1.0802