Weierstrass Theorem

ROB 501

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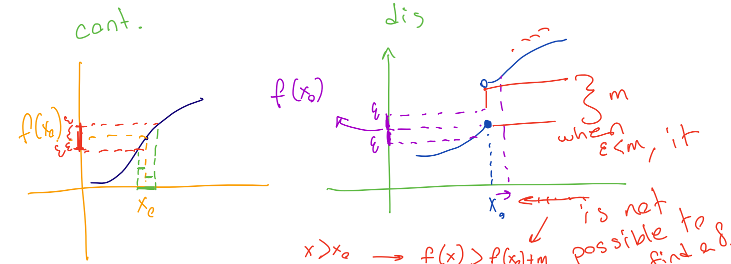
- **Bolzano-Weierstrass Theorem (alternative characterization of compactness in finite dimensional spaces)**
- **Weierstrass Theorem (when does a function guaranteed to have a maximum or minimum in its domain?)**
- **Intro to convexity (if time)**

Last time



- Continuity of a function f at a point x_o
  - Epsilon-delta definition

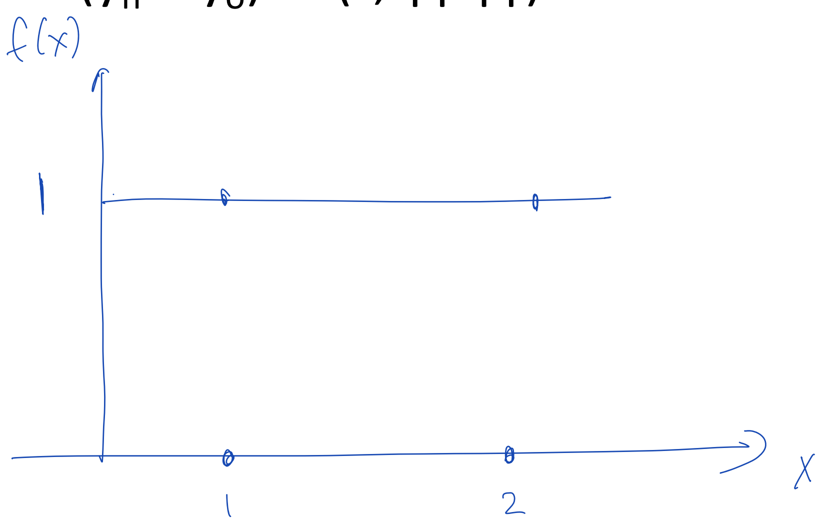
    Alternatively written, continuity of $f : D \to \mathbb{R}$ at $x_0 \in D$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$:

    $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon.$
  - An equivalent characterization in terms of preserving the convergence of sequences, i.e., f is continuous at x_o iff "($x_n \rightarrow x_o$) implies ($f(x_n) \rightarrow f(x_o)$)"
- Question: how about the converse? That is, if
  1. f: X->Y continuous everywhere
  2. $(x_n) \in X$, and $y_n = f(x_n)$
  3. ($y_n \rightarrow y_o$) in (Y, ||.||)

Does there exist a x_o s.t. ($x_n \rightarrow x_o$) in (X, ||.||)?

NO!

$f(x) = 1$



$y_n = f(x_n) = 1$

$(y_n) = 1, 1, \ldots$

converges to 1.

$x_i = \begin{cases} 1 & i \text{ is odd} \\ 2 & i \text{ is even} \end{cases}$

(x_n) is not convergent

Last time

- Continuity of a function f at a point x_o
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    Alternatively written, continuity of $f : D \to R$ at $x_0 \in D$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in D$:

    $$|x - x_0| < \delta \text{ implies } |f(x) - f(x_0)| < \varepsilon.$$

  - An equivalent characterization in terms of preserving the convergence of sequences, i.e., f is continuous at x_o iff "($x_n \to x_o$) implies ($f(x_n) \to f(x_o)$)"

- Compactness

Def: Let $(X, \|\cdot\|)$ be a normed space. Then, $C \subset X$ is compact if every sequence in C has a convergent subsequence with limit in C.

Remark: Often called sequential compactness.

Bolzano-Weierstrass Theorem: (simple characterization of compactness in finite dimensional spaces).

In a finite-dim normed space X, TFAE for a set $C \subset X$:

(a) C is closed and bounded.

(b) C is sequentially compact.

We could alternatively write (a) and (b) as

(a) C is closed and bounded

(b) For every seq. (x_n) in C $(x_n \in C\ \forall n \geq 1)$, $\exists\ x_0 \in C$ and a subsequence (x_{n_i}) of (x_n) s.t. $x_{n_i} \xrightarrow[i \to \infty]{} x_0$. ($C$ is sequentially compact).

<u>Proof:</u> (a) $\Rightarrow$ (b)

<u>Case 1</u>: Suppose (x_n) has only a finite # of distinct elements. Hence, at least one value is repeated an infinite number of times. WLOG, say that is x_5. Then $\exists\ n_1 < n_2 < \ldots < n_k < \ldots$ s.t. $x_{n_i} = x_5$ $i = 1, 2, \ldots$ ((x_{n_i}) is a constant sequence).

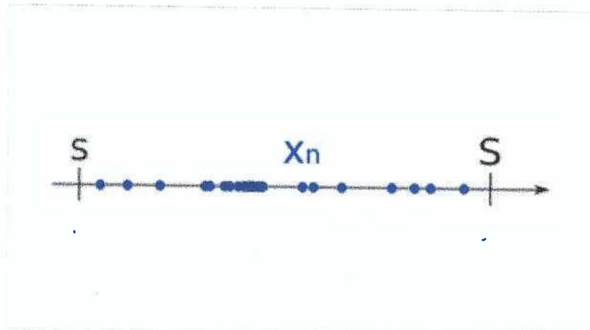
$\therefore x_{n_i} \to x_5$.

<u>Case 2</u>: (x_n) has an ∞ # of distinct elements.

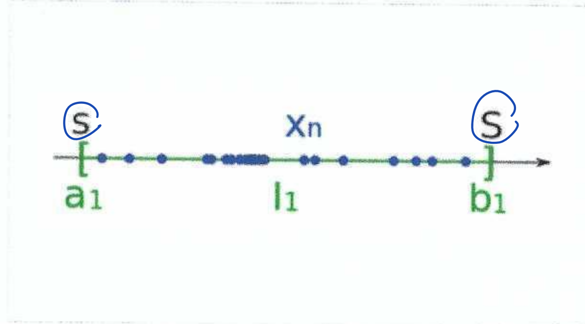
See next page for a Wikipedia proof.

~~Alternative~~ proof [edit]

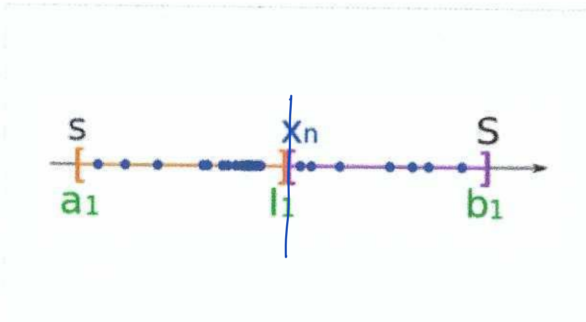
There is also an alternative proof of the Bolzano–Weierstrass theorem using nested intervals. We start with a bounded sequence (x_n):



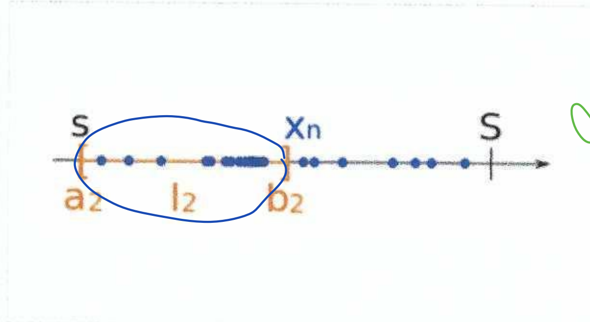
Because $(x_n)_{n\in\mathbb{N}}$ is bounded, this sequence has a lower bound s and an upper bound S.



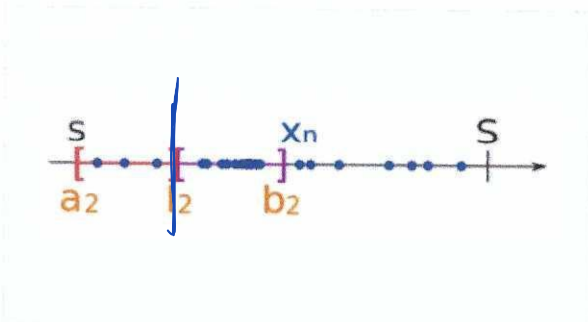
We take $I_1 = [s, S]$ as the first interval for the sequence of nested intervals.



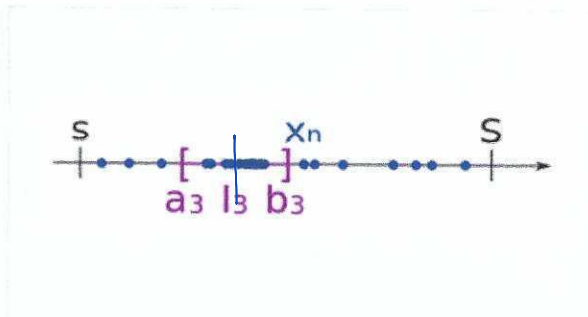
Then we split I_1 at the mid into two equally sized subintervals.



We take this subinterval as the second interval I_2 of the sequence of nested intervals which contains infinitely many members of $(x_n)_{n\in\mathbb{N}}$. Because each sequence has infinitely many members, there must be at least one subinterval which contains infinitely many members.



Then we split I_2 again at the mid into two equally sized subintervals.

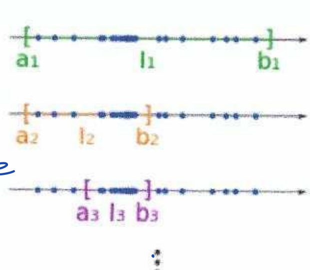


Again we take this subinterval as the third subinterval I_3 of the sequence of nested intervals, which contains infinitely many members of $(x_n)_{n\in\mathbb{N}}$.

x_n

$x_{n_1} = x_1$ →

$x_{n_2} =$ x_i in the orange interval with the smallest index greater than 1.



→ x_{n_3} : x_i in the purple interval with the smallest index greater than n_2.

We continue this process infinitely many times. Thus we get a sequence of nested intervals.

The subsequence, we contstruct as above is a Cauchy sequence (the distance between consequtive points shrinks bc. the size of intervals shrinks) $\rightarrow$ it is convergent (since $\mathbb{R}$ is complete)

What we showed is if C is bounded in a complete space and (x_n) is a seq. in C, then (x_n) has a convergent subsequence.

$(b) \Rightarrow (a)$: We can show $\sim(a) \Rightarrow \sim(b)$

$\sim(a)$: C is either not closed or not bounded

$\sim(b)$: $\exists\ (x_n)$ with no convergent subsequence having limit in C.

Case 1: C is unbounded $\Rightarrow \sim(b)$

Shown in exercise (fact) 2 in the last lecture.

Case 2: C is not closed $\Rightarrow \sim(b)$

Proof of Case 2: C not closed $\Rightarrow$ C does not contain all of its limit points. $\exists\ x_0 \notin C$ and (x_n) with $x_n \in C$ s.t. $x_n \rightarrow x_0$ (x_0 is a limit point of C that is not in C)

Hence, we have a sequence (x_n) for which none of its subsequences have a limit in C (since (x_n) is convergent, all its subsequences converge to x_0, and $x_0 \notin C$).

Hence $(b) \Rightarrow (a)$.

<u>Weierstrass Thm</u>: If C is a compact subset of a normed space $(X, \|\cdot\|)$ and $f: C \to \mathbb{R}$ is continuous at each point of C, then f achieves its extreme values

$$\exists\, x^* \in C \quad \text{s.t.} \quad f(x^*) = \sup_{x \in C} f(x) \quad [\sup = \max]$$

$$\exists\, x_* \in C \quad \text{s.t.} \quad f(x_*) = \inf_{x \in C} f(x) \quad [\inf = \min]$$

<u>Proof</u>:

Claim: $f: C \to \mathbb{R}$ continuous and C compact

$$\Longrightarrow f^* := \sup_{x \in C} f(x) < \infty$$

Proof of the claim:

$p \Rightarrow q \iff (\sim p) \vee q \iff \sim (p \wedge (\sim q))$

p: f cont. and C compact

q: $f^* < \infty$

Suppose $f^* = \infty$. Choose $x_1 \in C$ s.t. $f(x_1) \geq 1$.

By induction, choose $x_{n+1} \in C$ s.t. $f(x_{n+1}) \geq f(x_n) + 1$.

$y_n := f(x_n)$ is a sequence in $\mathbb{R}$ and has no convergent subsequence. (b.c. each consequtive y_i is at least "1" apart by construction).

However, (x_n) is a sequence in C, which is compact. Hence $\exists \tilde{x} \in C$ and a subsequence (x_{n_i}) s.t. $x_{n_i} \to \tilde{x}$. But f is continuous, and thus $f(x_{n_i}) \xrightarrow[i\to\infty]{} f(\tilde{x})$.

Then, (y_{n_i}) is subsequence of (y_n) and $y_{n_i} \to \tilde{y} = f(\tilde{x})$. This is a contradiction!

Hence $(p \wedge \sim q)$ is false!

$\therefore p \Rightarrow q$

Let's go back to the bigger proof knowing $f^* := \sup_{x \in C} f(x) < \infty$.

$\therefore \; \forall n \geq 1, \; \exists x_n \in C$ s.t. $|f(x_n) - f^*| < \frac{1}{n}$

$\therefore \; \underline{y_n} := f(x_n) \to f^*$

Invoke C is compact to choose a point $\tilde{x} \in C$ and a subsequence (x_{n_i}) of (x_n) s.t. $x_{n_i} \xrightarrow[i \to \infty]{} \tilde{x}$.

Question: $f(\tilde{x}) = f^*$?

(y_{n_i}) is a subsequence of (y_n), and $y_n \to f^* \implies y_{n_i} \to f^*$

We know $x_{n_i} \to \tilde{x} \xRightarrow{\text{by continuity of } f.} y_{n_i} \to f(\tilde{x})$

Limits are unique, hence $f^* = f(\tilde{x})$.

Then $x^* = \tilde{x}$ is the point in C that achieves the maximum. $\square$.

The proof for x_* is similar.

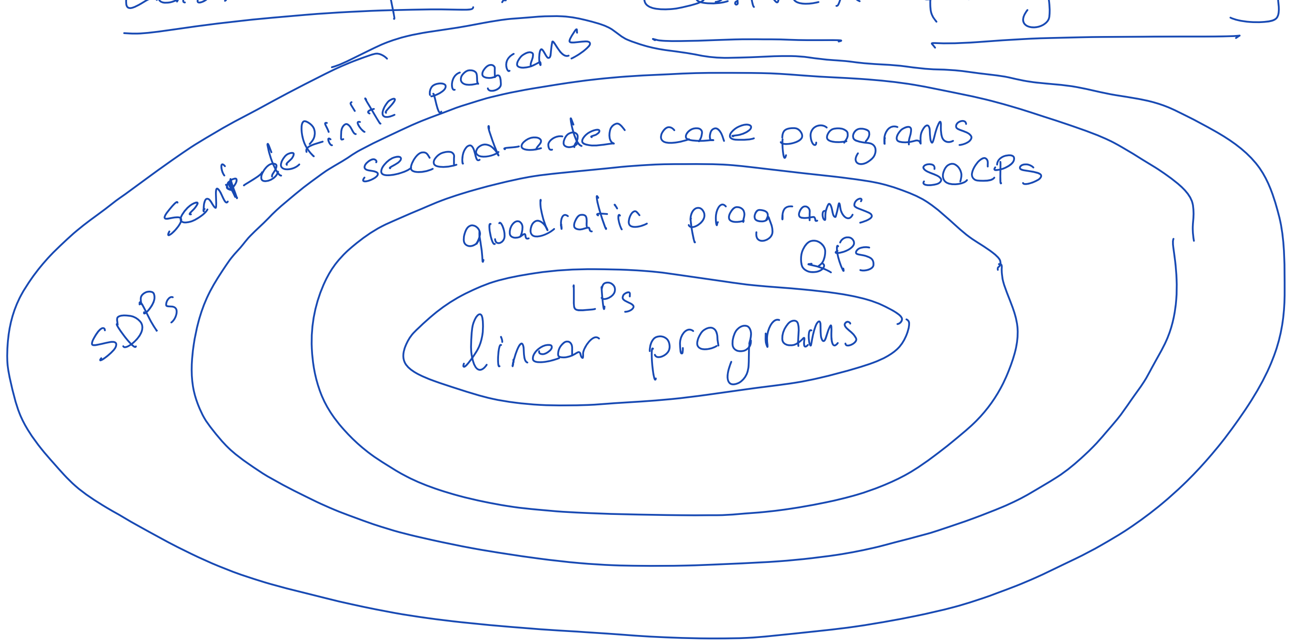
DONE WITH REAL ANALYSIS!

Some follow-up courses: Math 451 (mostly finite dimensional or 1-dimensional)

EECS 600 (infinite dimensional)

and other courses in the math depart

<u>Last topic</u>: <u>Convex programming</u>



- Define convex sets
- Define convex function

$$\min \ \|x\|_1$$

$$\text{s.t.} \quad \underbrace{Hx \le h}\,\Big\}\, k$$

$$\tilde{x} = \begin{bmatrix} x \\ \vdots \\ \vdots \end{bmatrix}$$

$$c^T \tilde{x}$$

$$A\tilde{x} \le b$$

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

$$-\varepsilon_1 \le x_1 \le \varepsilon_1$$

$$\vdots$$

$$-\varepsilon_n \le x_n \le \varepsilon_n$$

$2n$

$$\min \ \varepsilon_1 + \dots + \varepsilon_n$$

$$\tilde{x} = \begin{bmatrix} x \\ \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\min \ [0\ 0 \dots 0\ 1 \dots 1]\ \tilde{x}$$

$$\begin{bmatrix} H & 0 \\ \boxed{} & \end{bmatrix} \tilde{x} \le \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix}$$

$2n+k$

$$x_i = \begin{cases} 1 + \frac{1}{i} & \text{if } i \text{ is odd} \\ 4 - \frac{1}{i} & \text{if } i \text{ is even} \end{cases}$$

Not convergent

$x_1 = 2$

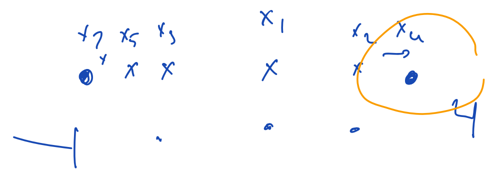
$x_2 = 3.5$

$x_3 = 1.\overline{3}$

$x_4 = 3.75$

$x_5 = 1.2$

$x_6 = 3 - \frac{1}{6}$



for all ε,

find an index $N(\varepsilon)$ that is large enough s.t.

for all indices $i \geq N(\varepsilon)$ $\quad x_i \in B_\varepsilon(x)$

$x_i = \frac{1}{i}$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

for $\varepsilon = 0.5$ $\quad N = 2$ works

$\varepsilon = 0.001$ $\quad N = 1000$ works

Take $x \in B_a(x_0)$ which means $\|x - x_0\| < a$

want to show $x \in \tilde{B}_{\frac{a}{K_1}}(x_0)$.

We know:

$\forall x: K_1 |||x - x_0||| \leq \|x - x_0\|$

if $x \in B_a(x_0)$ we know $\|x - x_0\| < a$,

if $x \in B_a(x_0)$ $\quad K_1 |||x - x_0||| \leq \|x - x_0\| < a$

$K_1 |||x - x_0||| < a$

$|||x - x_0||| < \frac{a}{K_1} \Rightarrow x \in \tilde{B}_{\frac{a}{K_1}}(x_0)$