1) a) given $x_0 \in \text{span}\{y_1, y_2 \ldots y_p\}$ & $\langle x_0, y_i \rangle = c_i$ $\forall 1 \leq i \leq p$

let $\exists$ x_1 & $x_2 \in \text{span}\{y_1 \ldots y_p\}$

& $\langle x_1, y_i \rangle = c_i$ & $\langle x_2, y_i \rangle = c_i$

as $x_1 \in \text{span}\{y_1, y_2 \ldots y_p\}$

$x_1 = \alpha_1 y_1 + \alpha_2 y_2 + \ldots \alpha_p y_p$

similarly $x_2 = \beta_1 y_1 + \beta_2 y_2 + \ldots \beta_p y_p$

$\langle x_1, y_1 \rangle = c_1 = \alpha_1 \langle y_1, y_1 \rangle + \alpha_2 \langle y_1, y_2 \rangle + \ldots \alpha_p \langle y_1, y_p \rangle$

$\vdots$

$\langle x_1, y_p \rangle = c_p = \alpha_1 \langle y_1, y_p \rangle + \alpha_2 \langle y_2, y_p \rangle + \ldots \alpha_p \langle y_p, y_p \rangle$

$$\Rightarrow \underbrace{\begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle & \ldots & \langle y_1, y_p \rangle \\ \langle y_2, y_1 \rangle & \langle y_2, y_2 \rangle & \ldots & \langle y_2, y_p \rangle \\ \vdots & & & \\ \langle y_p, y_1 \rangle & \langle y_p, y_2 \rangle & \ldots & \langle y_p, y_p \rangle \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \quad \text{--- (1)}$$

Similarly

$$A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \quad — (2)$$

eq^n (1) - (2)

$$A \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \vdots \\ \alpha_p - \beta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

here A is symm & invertible matrix

So

the sol^n for $Ax = 0$ is only

$$x = 0$$

$$\Rightarrow \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \vdots \\ \alpha_p - \beta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow \boxed{\alpha_i = \beta_i}$ $\quad \Rightarrow \boxed{x_1 = x_2}$

$\Rightarrow \exists$ only unique $x_0 \in \operatorname{span}\{y_1 — y_p\}$ s.t. $\langle x_0, y_i \rangle = c_i$

b) $M = (\text{span}\{y_1, y_2 - y_p\})^{\perp}$

if $x \in V$

then

$x = x_0 + M$

$\Rightarrow (x - x_0) = M$

$\Rightarrow (x - x_0) \in (\text{span}\{y_1, y_2 - y_p\})^{\perp}$

$\Rightarrow (x - x_0) \perp \text{span}\{y_1, - y_p\}$

if $(x - x_0) \perp \text{span}\{y_1, y_2 - y_p\}$

$\Rightarrow (x - x_0) \in (\text{span}\{y_1 - y_p\})^{\perp}$

$\Rightarrow (x - x_0) = M$

$\Rightarrow x = x_0 + M$

WKT $V = x_0 + M$

$\Rightarrow \boxed{x \in V}$

$\therefore (x \in V) \Leftrightarrow (x - x_0) \perp \text{span}\{y_1, - y_p\}$

where $V = x_0 + M$

1) c) $\|v^*\| = \inf_{v \in V} \|v\|$

From claim (b)

$V = x_0 + M$

& also $V = x_0 - M$ $\because m$ & $-m \in M$

$\Rightarrow \inf \|v\| = \inf \|x_0 + m\| = \inf \|x_0 - m\|$

$= d(x_0, M)$

$\Rightarrow$ from projection theorm

$\exists$ unique minimiser $x^* \in M$

such that

$\|x_0 - x^*\| = d(x_0, M)$

as $x^* \in M \Rightarrow v^* = x_0 - x^*$

$\Rightarrow$ $x - x^*$ is min norm & $\perp$ to M

$\therefore v^* = x_0 - x^* \perp M$

2) using lemma 3

we can say that $v^* = \underset{v \in V}{\arg\min} \|v\|$

v^* is unique & $v^* \perp M$

& $v^* \perp M \Rightarrow v^* \in V$

$\Rightarrow v^* \in \text{span}\{y_1, y_2 \ldots y_p\}$

$\Rightarrow v^* = \sum_{i=1}^{p} \beta_i y_i$

& if $\langle v^*, y_i \rangle = c_i$

$\Rightarrow$ similar to expansion in lemma 1

$$A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

$$A = \begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle & \cdots & \langle y_1, y_p \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle & \cdots & \langle y_2, y_p \rangle \\ \vdots & \vdots & & \\ \langle y_1, y_p \rangle & & & \langle y_p, y_p \rangle \end{bmatrix}$$

3) a) Given

$$\mu = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \qquad \Sigma = \mathrm{Cov}\left(\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

$$= \begin{bmatrix} P & PC^T \\ CP & CPC^T + Q \end{bmatrix}$$

from conditional dis

$$\mu_{1|2} = \mu_1 + \Sigma_{12} \cdot \Sigma_{22}^{-1} (x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

2)

$$\mu_{1|2} = \bar{x} + PC^T (CPC^T + Q)^{-1} CP (y - \bar{y})$$

$$\Sigma_{1|2} = P - PC^T (CPC^T + Q)^{-1} CP$$

3) b) For matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Schur complement of D in M is

$$M/D = A - BD^{-1}C$$

$$= P - (PC^T)(CPC^T + Q)^{-1} CP$$

which is same as $\Sigma_{1/2}$

4)

let $f = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4$

where $f_1 = 1$ $\quad f_2 = t$

$f_3 = t^2$ $\quad f_4 = \sin(\pi t)$

$$\Rightarrow \arg\min \|f\| = \min \int_0^2 (\beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4)^2$$

$$= \min \left([\beta_1 \ \beta_2 \ \beta_3 \ \beta_4] \, G \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} \right)$$

$$G = \begin{bmatrix} \langle f_1 f_1 \rangle & \langle f_1 f_2 \rangle & \langle f_1 f_3 \rangle & \langle f_1 f_4 \rangle \\ \langle f_1, f_2 \rangle & \langle f_2 f_2 \rangle & \langle f_2 f_3 \rangle & \langle f_2 f_4 \rangle \\ \langle f_3 f_1 \rangle & \langle f_3 f_2 \rangle & \langle f_3 f_3 \rangle & \langle f_3 f_4 \rangle \\ \langle f_4 f_1 \rangle & \langle f_4 f_2 \rangle & \langle f_4 f_3 \rangle & \langle f_4 f_4 \rangle \end{bmatrix}$$

$\langle f, t \rangle = 0$

4

$\Rightarrow$ $A\beta = C$

where

$A = [\langle f_1 f_2 \rangle \; \langle f_2 f_2 \rangle \; \langle f_2 f_3 \rangle \; \langle f_2 f_4 \rangle]$

$C = 2$

i) β is given by

$$G^{-1} A^T (A G^{-1} A^T)^{-1} C$$

from computing in matlab

$$\therefore \beta = \begin{bmatrix} 0 \\ 0.75 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{f = 0.75t}$$

b) $A = \begin{bmatrix} \langle f_2 f_1 \rangle & \langle f_2 f_2 \rangle & \langle f_2 f_3 \rangle & \langle f_2 f_4 \rangle \\ \langle f_4 f_1 \rangle & \langle f_4 f_2 \rangle & \langle f_4 f_3 \rangle & \langle f_4 f_4 \rangle \end{bmatrix}$

$C = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$

from matlab

$$\beta = \begin{bmatrix} 0 \\ 1.7688 \\ 0 \\ 4.2677 \end{bmatrix} \Rightarrow f = (1.7688 \cdot t) + (4.2677 \cdot \sin \pi t)$$

5) a) given $\hat{x} : \underset{Ax=b}{\operatorname{argmin}} \|x\|$

$\Rightarrow$ $x = x_{Null(A)^\perp} + x_{Null(A)}$

$\Rightarrow$ $Ax = b$

$A(x_{Null\,A^\perp}) = b$

if $y \in (Null\,A^\perp)$

$\Rightarrow$ $y^T x = 0$ where $x \in Null(A)$

$\Rightarrow$ $x^T y = 0$.

$\Rightarrow$ $y : [\bar{A}_1 \; \bar{A}_2 \; --- \; \bar{A}_p] \alpha$

$\Rightarrow$ $y = A^T \alpha$

$\Rightarrow$ $A(A^T \alpha) = b$

$\Rightarrow$ $\alpha = (AA^T)^{-1} b$

& $y = x_{Null\,A^\perp} = A^T \alpha = A^T (AA^T)^{-1} b$

for min norm as

$x_{Null(A)}$ & $x_{Null(A)^\perp}$ are $\perp$ar

$x_{(Null(A))^\perp} = 0$.

$\therefore \quad x = A^T (A A^T)^{-1} b$

b) $\quad Ax = b \quad$ & min norm $\|x\|$
where $\|x\| = x^T Q x$
$\Rightarrow$ nothing but
min $(x^T Q x)$

$x = y + x_{null} \qquad y \in Null(A)^{\perp}$
$x_{null} \in Null(A)$

$\Rightarrow \quad Ay = b$

& from inner Product def

$y^T Q x_{null} = 0$

$\Rightarrow x_{null}^T Q^T y = 0$

$\Rightarrow \quad Q^T y = A^T \alpha \qquad Q^T = Q \quad \because Q \succ 0$

$\Rightarrow \quad y = Q^{-1} A^T \alpha$

$\Rightarrow$

$Ay = b \Rightarrow (A Q^{-1} A^T) \alpha = b$

$\Rightarrow \quad \alpha = (A Q^{-1} A^T)^{-1} b$

$y = Q^{-1} A^T \alpha$

$$\boxed{y = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b}$$

ξ $x_{null} = 0$ for maintaining min norm

$$\therefore \boxed{x = y = Q^{-1}A^T (AQ^{-1}A^T)^{-1} b}$$

6) $Q = [v^1 | v^2]$

$A = QR$

Q is graam schmit of collumns of A

$$v^1 = \frac{A_1}{\|A_1\|}$$

$$A_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \|A_1\| = \sqrt{1^2+3^2+5^2} = 5.91$$

$$v^1 = \frac{A_1}{\|A_1\|} = \begin{bmatrix} 0.169 \\ 0.5071 \\ 0.8452 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad v^1 = \begin{bmatrix} 0.169 \\ 0.5071 \\ 0.8452 \end{bmatrix}$$

$A_2^T v^1 = 7.4374$

$$\langle A_2, v^1 \rangle v^1 = \begin{bmatrix} 1.2571 \\ 3.7714 \\ 6.2857 \end{bmatrix}$$

$$a_2 = A_2 - \langle A_2 v^1 \rangle v^1$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1.2571 \\ 3.7714 \\ 6.2877 \end{bmatrix} = \begin{bmatrix} 0.7429 \\ 0.2286 \\ -0.2857 \end{bmatrix}$$

$$v^2 = \begin{bmatrix} 0.8971 \\ 0.2760 \\ -0.3450 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.169 & 0.8971 \\ 0.5071 & 0.2760 \\ 0.8452 & -0.3450 \end{bmatrix}$$

$\frac{3}{4}$ $\frac{t^3}{3}$

$$R = \begin{bmatrix} R_1 & R_2 \end{bmatrix}$$

$$\cancel{R_1 = \begin{bmatrix} \langle A_1, v_1 \rangle \\ \langle A_i, v_2 \rangle \end{bmatrix} = \begin{bmatrix} 5.916 \\ 0 \end{bmatrix}}$$

$$\cancel{R_2 = \begin{bmatrix} \langle A_2, v_1 \rangle \\ \langle A_2, v_2 \rangle \end{bmatrix} = \begin{bmatrix} 7.4314 \\ 0.8284 \end{bmatrix}}$$

$$R_1 = \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} \langle A_1, v_1 \rangle \\ \langle A_1, v_2 \rangle \end{bmatrix} = \begin{bmatrix} 5.916 \\ 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \langle A_2, v_1 \rangle \\ \langle A_2, v_2 \rangle \end{bmatrix} = \begin{bmatrix} 7.4314 \\ 0.8287 \end{bmatrix}$$

$$R = \begin{bmatrix} 5.916 & 7.4314 \\ 0 & 0.8287 \end{bmatrix}$$

Q6 Code:

```
A = [1,2;
    3,4;
    5,6;
    ];
[Q1, R1] = qr(A,0)
[Q2, R2] = qr(A)
```

Matlab results:

Q1 and R1 are results from *economy* operation & Q2, R2 are results from *QR() MATLAB*.

The economy operation is close to what is done by hand but differs in the signs. Another function makes Q a square matrix rather than R as square matrix.

```
Q1 =

  -0.1690    0.8971
  -0.5071    0.2760
  -0.8452   -0.3450


R1 =

  -5.9161   -7.4374
        0    0.8281


Q2 =

  -0.1690    0.8971    0.4082
  -0.5071    0.2760   -0.8165
  -0.8452   -0.3450    0.4082


R2 =
```

```
  -5.9161   -7.4374
        0    0.8281
        0         0
```

—---

Code Q4:

```
f1 = @(t) 1;
f2 = @(t) t;
f3 = @(t) t.^2;
f4 = @(t) sin(pi*t);

f11 = @(t) 1;
f12 = @(t) t;
f13 = @(t) t.^2;
f14 = @(t) sin(pi*t);
f22 = @(t) t.^2;
f23 = @(t) t.^3;
f24 = @(t) t.*sin(pi*t);
f33 = @(t) t.^4;
f34 = @(t) (t.^2).*sin(pi*t);
f44 = @(t) (sin(pi*t)).*sin(pi*t);
li =2;
%% for part a

a11 = 2;
a12 = integral(f12,0,li);
a13 = integral(f13,0,li);
a14 = integral(f14,0,li);
a22 = integral(f22,0,li);
a23 = integral(f23,0,li);
a24 = integral(f24,0,li);
a33 = integral(f33,0,li);
a34 = integral(f34,0,li);
a44 = integral(f44,0,li);
```

```
G = [
    a11,a12,a13,a14;
    a12,a22,a23,a24;
    a13,a23,a33,a34;
    a14,a24,a34,a44;
    ];

A = [a12,a22,a23,a24;];
b = 2;
P = inv(G)*A';
x1 = P*(inv(A*P))*b;


A = [a12,a22,a23,a24;
    a14,a24,a34,a44;];
b = [2;pi];
P = inv(G)*A';
x2 = P*(inv(A*P))*b;
```