Exam Number:________

ROB 501 Exam-II

You can pick any 36 hours between 4:30pm (ET) December 16, 2022 (Friday) and 5:00pm (ET) December 20, 2022 (Tuesday) to solve this exam.

HONOR PLEDGE: Copy (NOW) and SIGN (after the exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

____
SIGNATURE
(Sign **after** the exam is completed)

________________, ________________
LAST NAME (PRINTED) *FIRST NAME*

RULES:

1. The exam is open book, open lecture handouts and slides, open recitation notes, open HW solutions, open internet (under the communication and usage restrictions mentioned below).
2. If you use MATLAB or any other scientific software to complete some parts of the exam. You are required to submit your script along with your solution in such case.
3. You are not allowed to communicate with anyone other than the Course instructor and the GSIs related to the exam during the entire period. If you have questions, you can post a private Piazza post for the instructors or email `necmiye@umich.edu` with GSIs on cc.
4. You are not allowed to use any online "course helper" sites like Chegg, Course Hero, and Slader, in any part of the exam. You are not allowed to post exam questions on the internet or discuss them online. You are not allowed to use chatGPT or similar large language models.
5. Please do not wait until the last minute to upload your solution to Gradescope and double-check to make sure you uploaded the correct pdf. If you run into problems with Gradescope, email your .pdf file as an attachment to Prof. Ozay as soon as practicable at `necmiye@umich.edu`.

SUBMISSION AND GRADING INSTRUCTIONS:

1. The maximum possible score is 80 (+5 bonus points). To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly.
2. You must submit your solutions in a single pdf. You will be asked to mark where each solution is.
3. **Honor Code:** The first page of your submitted pdf should include a hand-written and signed honor code (see the first page of this pdf). Without this, your exam will not be graded.
4. **For problems 1-5** Use the next page to record your answers. We will NOT grade other pages and we do not care if you make a mistake when copying your answers to the next page. Please be careful. If you are submitting handwritten (or word-processed) documents, make sure to make a similar table where you record all your True/False (and fill in the blanks for 1(c),(d) and 2(a)(b)(c)) answers. There is no partial credit on these questions. You are welcome to leave some justification but we will not look at them.
5. **For problems 6-7-8** Record your final answer in the box whenever one is provided. If you are submitting handwritten (or word-processed) documents, make sure to box or highlight the final result. However, you MUST show your work and submit scripts whenever applicable to get credit . In other words, a correct result with no reasoning or wrong reasoning could lead to no points.

Answers for Problem 1	
Problem 1(a)	True False
Problem 1(b)	True False
Problem 1(c)	$\mu_Z =$ $\Sigma_Z =$
Problem 1(d)	$\mu_Y =$ $\mathrm{var}(Y) =$

Answers for Problem 2	
Problem 2(a)	
Problem 2(b)	
Problem 2(c)	
Problem 2(d)	True False

Answers for the True/False Part				
	(a)	(b)	(c)	(d)
Problem 3				
Problem 4				
Problem 5				

Problems 1 - 5 (30 points: 5 × 6)

Instructions. For each problem, you should select True or False. **Make sure to record your answers on the second page. Only the second page will be graded!!!**

1. (Probability) Consider three random variables X_1, X_2 and X_3. We are given the following information:

- X_1, X_2 and X_3 are jointly normally distributed (i.e., joint Gaussian random variables).
- The marginal distributions are $X_1 \sim \mathcal{N}(1, 1)$, $X_2 \sim \mathcal{N}(3, 9)$, $X_3 \sim \mathcal{N}(-1, 2)$.
- $\text{cov}(X_1, X_2) = 1$, $\text{cov}(X_2, X_3) = 2$.
- The random variables X_1 and X_3 are independent.

For parts (a)-(b), circle True or False as appropriate. For part (c)-(d), give the asked quantities.

(a) The random variables $X_4 = 3X_1 - X_2$ and X_1 are independent. (**True or False**) False

(b) The following two conditional random variables have the same normal distribution: (**True or False**

- X_1 conditioned on $\begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$,
- $(X_1 \mid_{\{X_3=-1\}})$ conditioned on $(X_2 \mid_{\{X_3=-1\}}) = 8$.

True ig

(c) The random vector $Z = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has mean $\mu_Z =$ 1,3 and covariance $\Sigma_Z =$ 1 1 / 1 9.

(d) Define the random variable $X_5 = X_1 + X_2$. Then the conditional random variable $Y = X_5 \mid_{\{X_3=2\}}$ has mean $\mu_Y =$ 7 and variance $\text{var}(Y) =$ 10.

2. (Real Analysis I) Consider the normed space $(\mathbb{R}, \mathbb{R}, |\cdot|)$ (the set of reals equipped with the absolute value as the norm). Consider the set $S \subset \mathbb{R}$ given as $S = (-\infty, -2] \cup \{-1, 0.1, 2\} \cup (0.1, 5] \cup [7, \infty)$. **For parts (a)-(b)-(c), provide the set being asked for (e.g., if you were asked about the interior of $\mathbb{R}$, the answer would be $\mathbb{R}$. Giving the definition of interior will not get you any points.). For part (d) Answer True or False. Record your answers on the second page.**

(a) The closure $\bar{S}$ of S is same as S.

(b) An isolation point of S is a point in S that is not a limit point of S. The set of all isolation points of S is -1.

(c) The boundary ∂S of S is -2 -1 0.1 5 7.

(d) The set S is neither open nor closed. **True or False**.

3. (Real Analysis II) **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) Let C be a compact set in a finite-dimensional normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|_{\mathcal{X}})$ and (x_n) be a Cauchy sequence in C. Let $f : C \to \mathbb{R}$ be a function such that the sequence $(y_n) = (f(x_n))$ is not Cauchy in $(\mathbb{R}, \mathbb{R}, \|\cdot\|)$. Then, for any such f, there exists $x^* \in C$ such that $x^* = \max_{x \in C} f(x)$.

False

T F (b) Consider the following set $M = \{x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \in \mathbb{R}^2 \mid x^{(1)} = 2x^{(2)}\}$ in the normed space $(\mathbb{R}^2, \mathbb{R}, ||\cdot||_2)$. There exists a sequence (x_n) in M that does not have a convergent subsequence. True

T F (c) Let $\mathcal{F} = \{f : [0,1] \to \mathbb{R} \mid \sup_{x \in [0,1]} |f(x)| < \infty\}$ be the set of real-valued bounded functions on $[0,1]$. Define $f_k(x) = \begin{cases} \frac{x}{k} & \text{if } x \in [0, 0.5) \\ \frac{1-x}{k} & \text{otherwise} \end{cases}$. Consider the normed space $(\mathcal{F}, \mathbb{R}, ||\cdot||_\infty)$ with norm $||f||_\infty = \sup_{x \in [0,1]} |f(x)|$. Then, (f_k) is a Cauchy sequence in this space.[1] True

T F (d) Suppose $(\mathcal{X}, \mathbb{R}, \|\cdot\|_{\mathcal{X}})$ is a finite-dimensional normed space and suppose that both $S \subset \mathcal{X}$ and its set complement $\sim S$ are nonempty sets. If $x_0 \in S$ satisfies $\forall \epsilon > 0$, $\exists y \in \sim S$ such that $||x_0 - y|| < \epsilon$, then $x_0 \notin S^\circ$ (where S° is the interior of S).

True

4. (Optimization, SVD, QR factorization) Let $A = U\Sigma V^\top$ be the SVD of a real square matrix A, with $\Sigma = \operatorname{diag}(10, \sigma_2, \sigma_3, 3)$, where $10 \geq \sigma_2 \geq \sigma_3 \geq 3$. For $1 \leq i \leq 4$, let U_i and V_i denote the ith column of U and V respectively. The matrix norm used in this problem is $\|B\| = \sqrt{\lambda_{\max}(B^\top B)}$. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) $\|A\| = 13 + \sigma_2 + \sigma_3$. False

T F (b) There exists an orthogonal matrix Q and an upper-triangular, invertible matrix R such that $A = QR$. True

T F (c) The rank of the matrix $M := U_1 V_1^\top + U_4 V_4^\top$ is larger than or equal to $\|M\|$. True - 99.99

T F (d) The following optimization problems have the same optimum value (i.e., objective function is the same when evaluated at the corresponding optimizing point) when the optimum is finite:

$$\begin{array}{ll} \min_{X \in \mathbb{R}^4} & \max\{c_1^\top X, c_2^\top X\} \\ \text{s.t.} & AX \leq b \end{array} \qquad \begin{array}{ll} \min_{\bar{X} \in \mathbb{R}^5} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{X} \\ \text{s.t.} & \begin{bmatrix} A & \mathbf{0}_{1\times 4} \\ c_2^\top & -1 \\ c_1^\top & -1 \end{bmatrix} \bar{X} \leq \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} \end{array}$$

false

[1] The set $\mathcal{F}$ only includes bounded functions to make sure the given norm is well-defined for all elements of $\mathcal{F}$, hence, the normed space is well-defined. You do not really need to worry about boundedness. You just need to apply the definition of Cauchy sequence for the given norm.

5. (Estimators) Consider a problem with equation $y = Cx + \epsilon$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and both n and m are positive integers. You measure y but ϵ is unknown. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T **F** (a) Suppose $\mathcal{E}\{x\} = 0$, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{x\epsilon^\top\} = 0$, $\mathcal{E}\{xx^\top\} = P$, and $\mathcal{E}\{\epsilon\epsilon^\top\} = Q$. The error of the linear estimator $\hat{x} = PC^\top(CPC^\top + Q)^{-1}y$ has covariance given by $\mathcal{E}\{(\hat{x} - x)(\hat{x} - x)^\top\} = P - PC^T(CPC^\top + Q)^{-1}PC$. False

T **F** (b) Suppose $\mathcal{E}\{x\} = 0$, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{x\epsilon^\top\} = 0$, $\mathcal{E}\{xx^\top\} = P > 0$, and $\mathcal{E}\{\epsilon\epsilon^\top\} = Q > 0$. Then the minimum variance estimator (MVE) of x is given by $\hat{x} = PC^\top(CPC^\top + Q)^{-1}y$ or equivalently $\hat{x} = (C^\top Q^{-1}C + P^{-1})^{-1}C^\top Q^{-1}y$. True

T **F** (c) Suppose $\mathcal{E}\{x\} = 0$, $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{x\epsilon^\top\} = 0$, $\mathcal{E}\{xx^\top\} = I_{n\times n}$, and $\mathcal{E}\{\epsilon\epsilon^\top\} = I_{m\times m}$. Consider the linear estimators $\hat{x}_1 = (C^\top C + I)^{-1}C^\top y$ and $\hat{x}_2 = (C^\top C)^{-1}C^\top y$. Then the variance of the error of $\hat{x}_1$ is greater than for $\hat{x}_2$, i.e, $\mathcal{E}\{(\hat{x}_1 - x)^\top(\hat{x}_1 - x)\} > \mathcal{E}\{(\hat{x}_2 - x)^\top(\hat{x}_2 - x)\}$. False

T **F** (d) Assume x is deterministic, $\mathcal{E}\{\epsilon\} = 0$, and the Best Linear Unbiased Estimate (BLUE) denoted by $\hat{x}$ can be determined. Then $\mathcal{E}\{\hat{x} - x\} = 0$. True

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

“I do not know”,

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it. For example, we proved that real symmetric matrices have real e-values. So if you need this fact, simply state it and use it.

6. (12 points) The goal of this problem is to derive Minimum Variance Estimator (MVE), but this time, the unknown x is correlated with the measurement noise ϵ. We assume x and ϵ are jointly normal.

Model: $y = Cx + \epsilon, y \in \mathbb{R}^m, x \in \mathbb{R}^n,$ and $\epsilon \in \mathbb{R}^m$.

Stochastic assumptions:

$E\{x\} = 0, E\{\epsilon\} = 0$ (means are zero).

$$E\left\{\begin{bmatrix} x \\ \epsilon \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix}^\top\right\} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{x\epsilon} \\ \Sigma_{\epsilon x} & \Sigma_{\epsilon\epsilon} \end{bmatrix} > 0 \text{ (covariance is positive definite).}$$

Find the minimum variance estimate of x given y.

Please show your work for question 6.

7. (20+5 points) In this problem you will work with an unknown discrete-time linear time-invariant system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t\,, \end{aligned}$$

with states $x_t \in \mathbb{R}^2$, control input $u_t \in \mathbb{R}^1$, disturbance $w_t \in \mathbb{R}^1$, observation $y_t \in \mathbb{R}^1$, and noise $v_t \in \mathbb{R}^1$. We will assume that all w_t and v_t are mutually independent with $w_t \sim \mathcal{N}(0, \Sigma_w)$ and $v_t \sim \mathcal{N}(0, \Sigma_v)$.
You will be able to generate data from the system using a provided web app. In your web browser, navigate to the page `https://web.eecs.umich.edu/~necmiye/problem7.html`. The system will be personalized based upon your UMID which must be entered at the top of the page. Your responses to this question will graded upon your system, so please enter your UMID correctly.

1. (15 points) **System Identification**. By analyzing a physical model of the system you know that

$$A = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ b_{21} \end{bmatrix}, \qquad \Sigma_w = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},$$

where a_{12}, a_{22}, and b_{21} are unknown quantities. In order to learn more about the dynamics of the unknown system, i.e, the matrices A and B, you run an experiment in the lab where you can measure the state of the system. This experiment generates data $\{(x_t, u_t) \mid 0 \le t < N\}$ from one trajectory of the system with $N = 1000$ (available from the web app). We will ignore the observation equation $y_t = Cx_t + v_t$ for system identification and assume we can measure x_t directly.

   (a) (3 points) Let the data at time t be (x_t, u_t, x_{t+1}). Let the unknown we want to estimate be $\theta = \begin{bmatrix} a_{12} \\ a_{22} \\ b_{21} \end{bmatrix}$. Write a measurement model $z_t = H_t\theta + \epsilon_t$ that can be used in estimation. Give the expressions for z_t, H_t in terms of the problem variables, as well as the mean and covariance of noise ϵ. (Hint: the *measurements* z_t involve the states x, while the *noises* ϵ involve the disturbances w_t. The matrix H_t should have entries from both the states x_t and the inputs u_t.)

   (b) (5 points) Using the data for $t = 0 : 10$, find the (batch) best linear unbiased estimate (BLUE) $\hat{\theta}_{10}$ and construct the unknown parameters of A and B. You must report the numerical values for A and B, calculated using MATLAB or Python, for instance. Include the script for calculations as well.

   (c) (2 points) Given your estimate $\hat{\theta}_{10}$ from the previous step, give an expression for $\hat{\theta}_{11}$ that is recursively computed after seeing (x_{11}, u_{11}, x_{12}) (i.e., from the new measurement equation $z_{11} = H_{11}\theta + \epsilon_{11}$).

   (d) (5 points) Using the entire data for $t = 0 : 1000$, recursively find the best linear unbiased estimate (BLUE) $\hat{\theta}_t$ (for $t > 10$) and report the final estimates of A and B (based on $\hat{\theta}_{999}$). Include the script for calculations as well.

2. (5+5 points) **State Estimation**. You now wish to estimate the state of the system outside of the lab where you cannot measure the full state directly. You equip the system with a sensor characterized by

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad \Sigma_v = 0.01\,.$$

To minimize your estimation error, you will implement the Kalman filter to dynamically update your estimate using the model you learned in the previous part (7.1.c). Assume the initial state is $x = \begin{bmatrix} 1 & 0.3 \end{bmatrix}^\top$. Using the observations and inputs $\{(u'_t, y'_t) \mid 0 \le t < M\}$ of a new trajectory with $M = 100$ steps (available from the web app).

   (a) (5 points) Give the Kalman gain at time $t = 1$.

   (b) (5 points - bonus) Implement a Kalman filter and calculate the corresponding state estimates $\hat{x}_t$ output by the Kalman filter. Provide two plots: (i) $\hat{x}_t$ vs time (showing two curves one for each state); (ii) $C\hat{x}_t$ vs time together with y'_t vs time (showing two curves on the same plot). Include the script for your Kalman filter implementation.

Hints:

1. Part 7.2 relies upon the model computed in part a. If you did not create a model, you may still complete this part for the following system model:
$$A = \begin{bmatrix} 1 & 0.106 \\ 0 & 1.0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0.0265 \end{bmatrix}.$$
If you use this model, you must generate data for this part using the alternate UMID 00000000 instead of your own in the web app. Be sure to indicate this on your exam.

2. The data for these parts are provided as comma separated values (CSV). See `en.wikipedia.org/wiki/Comma-separated_values` for details. This format can be read and written to by common programming languages.

   In MATLAB, you can load the states for part 1 for example using

   ```
   x = readmatrix("data_1/x.csv");
   ```

   Likewise you can write state estimates for part 2 for example using

   ```
   writematrix(xhat, "data_2/xhat.csv");
   ```

   In Python, you can load the states for part 1 for example using

   ```
   with open("data_1/x.csv", 'r') as f:
       x = [[float(val) for val in row] for row in (csv.reader(f))]
   ```

   Likewise you can write state estimates for part 2 for example using

   ```
   with open('data_2/xhat.csv', 'w') as f:
       writer = csv.writer(f)
       for row in xhat:
           writer.writerow(row)
   ```

   Alternatively if you are using `pandas`, then this can be done with

   ```
   x = pandas.read_csv("data_1/x.csv")
   ```

   and

   ```
   xhat.to_csv("data_2/xhat.csv")
   ```

3. If you have trouble with the webapp, please email the instructors with your UMID and we will send you the CSV file directly.

Please show your work for question 7 (copy/paste the scripts as needed).

Please show your work for question 7 (copy/paste the scripts as needed).

8. (18 points) The following are three (3) short answer questions. You do not need to give a formal proof; only give a few short reasons/calculations why something is TRUE or FALSE. **Part (c) is on the next page.**

(a) **(6 Points)** Consider the following set $S = \{x \in \mathbb{R}^2 \mid ||x||_1 + ||x||_2 \leq 2\}$. Then, S is a convex set.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

True

(b) **(6 Points)** Let $(\mathcal{X}, \mathbb{R}, \|\cdot\|_{\mathcal{X}})$ be a finite-dimensional normed space and let $S \subset \mathcal{X}$ be nonempty. Let (x_n) be a sequence converging to a point $x^* \in \mathcal{X}$, that is $x_n \to x^*$. If $x_n \in S^\circ$ for all $n \geq 1$, then $x^* \in S$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

False`

(c) **(6 Points)** Let $M = AA^T + \epsilon^2 I$ where $A \in \mathbb{R}^{3\times 2}$, $\epsilon \in \mathbb{R}$, and I is the identity matrix of appropriate dimensions. Let $A = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} V^T$ be the singular value decomposition (SVD) of A. Then, the solution of the following problem;

$$\begin{aligned} X^* = \arg\min \quad & ||X - M||_2 \\ \text{s.t.} \quad & \text{rank}(X) = 2 \end{aligned}$$

is such that $||X^* - M||_2 = \epsilon^2$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE: