Exam Number:________

ROB 501 Exam-I

October 25, 2022 (Tuesday) / 6:30pm-9:00pm
A through Li in FXB 1109
Lo through V in FRB 1060
W through Z in FRB 1050

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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S.Rahul.
SIGNATURE
(Sign **after** the exam is completed)

Sivayampakula — *LAST NAME* (PRINTED)   Rahul Kashyap — *FIRST NAME*

RULES:

1. CLOSED TEXT BOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. TWO SHEETS OF NOTE PAPER
6. NO CELLPHONES, SMARTPHONES, TABLETS, LAPTOPS etc.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Answers for the True/False Part				
	(a)	(b)	(c)	(d)
Problem 1	F	T	T	T
Problem 2	T	T	F	T
Problem 3	T	T	T	T
Problem 4	T	F	T	F
Problem 5	F	T	F	T

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Problems 1 - 5 (30 points: 5×6)

Instructions. For each problem, you should select True or False. **Make sure to record your answers on the second page. Only the second page will be graded!!!**

1. (Questions on logic and proof methods) Recall that $\wedge$ is 'and', $\vee$ is 'or', and $\neg$ is 'not'. Recall also that the symbol $\Leftrightarrow$ and the written text, "if, and only if", "logically equivalent to", and "have the same truth table", all mean the same thing. For example, in HW, you verified that $\neg(p \wedge q)$ is "logically equivalent to" $(\neg p) \vee (\neg q)$ by proving "they have the same truth table". **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T **F** (a) The negation of "it is not fall, so it is not raining," is "if it is raining, then it is fall."

T F (b) Let $Y \subset \mathbb{R}$, $Z \subset \mathbb{R}$, $W \subset \mathbb{R}$. Then,

$$(\forall y \in Y, \exists z \in Z, \exists w \in W \text{ s.t. } y + z = w) \Leftrightarrow \neg(\exists y \in Y, \forall z \in Z, y + z \notin W)$$

$\forall w \in W$ $y+z \neq w$

T F (c) For any statement $P(n)$ over natural numbers, the following is true:

$$\Big(P(1) \wedge (\forall k \geq 1, P(k) \Rightarrow P(k+1))\Big) \Leftrightarrow \left(P(1) \wedge \left(\forall k \geq 1, \left(\bigwedge_{j=1}^{k} P(j)\right) \Rightarrow P(k+1)\right)\right)$$

$p \vee \neg q$

T F (d) The truth table given below is correct for $\neg p$ implies $\neg q$:

$\sim p \rightarrow \sim q$

$p \rightarrow q$

$\sim p \vee q$

p	q	$\neg p \implies \neg q$
1	1	1
1	0	1
0	1	0
0	0	1

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} c & b \\ b & a \end{pmatrix}$

2. (Eigenvalues and eigenvectors, linear independence) For square matrix $A \in \mathbb{R}^{n \times n}$, its eigenvalues are $\lambda_i \in \mathbb{C}$, and its corresponding eigenvectors are non-zero $v^i \in \mathbb{C}^n$ s.t. $Av^i = \lambda_i v^i$. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) Let $A \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{C}$. If v is an eigenvector of A, then so is αv.

T F (b) Let $A \in \mathbb{R}^{n \times n}$ and v^1 and v^2 be two eigenvectors of A corresponding to two distinct eigenvalues (i.e., $\lambda_1 \neq \lambda_2$). Define $w^1 = v^1 + v^2$ and $w^2 = v^1 - v^2$. Then $\{w^1, w^2\}$ is always a linearly independent set over $\mathbb{C}$.

T **F** (c) The matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.

$AB x = \lambda x^T$

T F (d) Let A and B be $n \times n$ real symmetric matrices. Then, AB and BA always have the same eigenvalues.

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(AB)^T = (B^T A^T) = BA$

$2a \quad 2b$

$2c \quad 2d$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix}$

$\lambda = 2$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$2a \quad 2a + 2b$

$2c \quad 2c + 2d$

3. (Matrix properties, null space, range, solutions of linear equation systems) When applicable, assume we are working with a real inner product space $(\mathcal{X}, \mathbb{R}, <\bullet, \bullet>)$ where $<x, y> = x^\top y$. **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) Let $A = [A_1 \mid A_2 \mid A_3 \mid A_4] \in \mathbb{R}^{4\times 4}$ and $B = [A_2 \mid A_4 \mid A_3 \mid A_1] \in \mathbb{R}^{4\times 4}$. Then, $\mathcal{N}(A^\top) = (\mathcal{R}(B))^\perp$

T F (b) Let $A \in \mathbb{R}^{m\times n}$ with full row rank (i.e., $rank(A) = m$). Then, $Ax = b$ has an exact (possibly non-unique) solution x for any $b \in \mathbb{R}^m$. $m \le n$

T F (c) Let $A \in \mathbb{R}^{m\times n}$ with full row rank (i.e., $rank(A) = m$). Then, $A^\top Ax = A^\top b$ has an exact (possibly non-unique) solution x for any $b \in \mathbb{R}^m$.

T F (d) Consider the matrix $M = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 5 & 3 \\ 1 & 3 & a \end{bmatrix}$. Then, there exists a value $a \in \mathbb{R}$ such that M is positive definite.

$1/6 \quad 1/3$

$\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$2a - 2c \quad 2b - 2d$

$-2a + 5c \quad -2b + 5d$

$3c = 1 \quad c = 1/3$

$b = d \quad a = \frac{5}{2}c$

$b = d = 1/3 \quad \frac{3}{2}c = 1/2$

$c = 1/3 \quad a = 5/6$

4. (Vector spaces, linear operators) **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) The set $\mathcal{Z} = \{f : \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{\infty} f(x)dx = 0\}$ (i.e., the set of real-valued functions whose integral is zero) is a vector space over the field $\mathbb{R}$.

T F (b) Consider the set $\mathcal{X} = \{f : [-1, 1] \to \mathbb{R}\}$ (i.e., the set of real-valued functions with domain $[-1, 1]$).[1] Now consider the subset $\mathcal{Y} = \left\{ f : [-1, 1] \to \mathbb{R} \mid \exists y \in (-1, 1), \exists a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. } f(x) = \begin{cases} a & \text{if } -1 \le x \le y \\ b & \text{if } y < x \le 1 \end{cases} \right\}$ (i.e., the set of piecewise-constant functions with domain $[-1, 1]$ with at most one discontinuity). Then, $\mathcal{Y}$ is a subspace of $\mathcal{X}$.

T F (c) Consider the set $\mathcal{X} = \{f : [-1, 1] \to \mathbb{R}\}$ (i.e., the set of real-valued functions with domain $[-1, 1]$). Now consider the subset $\mathcal{Y}' = \left\{ f : [-1, 1] \to \mathbb{R} \mid \exists a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. } f(x) = \begin{cases} a & \text{if } -1 \le x \le 0.1 \\ b & \text{if } 0.1 < x \le 1 \end{cases} \right\}$ (i.e., the set of piecewise-constant functions with domain $[-1, 1]$ with at most one discontinuity at 0.1). Then, $\mathcal{Y}'$ is a 2-dimensional subspace of $\mathcal{X}$.

T F (d) Let $L(x) = \min\{x_1, x_2, x_3, x_4, x_5\}$, where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^\top \in \mathbb{R}^5$ (i.e., L is the operator that takes a vector in $\mathbb{R}^5$ and maps it to its minimum element). Then, L is a linear operator from $(\mathbb{R}^5, \mathbb{R})$ to $(\mathbb{R}, \mathbb{R})$.

$\begin{pmatrix} 5/6 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}$

$(1 \quad 3)\begin{pmatrix} 11/6 \\ 4/3 \end{pmatrix} \quad \frac{11}{6}$

[1] In the lecture we saw that $\mathcal{X}$ is a vector space over $\mathbb{R}$.

R S

5. (Projection theorem, least squares) **Answer True or False as appropriate for the following statements. Record your answers on the second page.**

T F (a) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, <\bullet,\bullet>)$, where $<u,v>= u^\top v$. Then, for any subspace S of $\mathbb{R}^n$ and any $y \in \mathbb{R}^n$, the solution $\hat{x} = \text{argmin}_{x\in S}||x-y||_2^2$ always satisfies $\hat{x} \perp y$.

T F (b) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, <\bullet,\bullet>_P)$, where $<u,v>_P= u^\top Pv$ where $P \in \mathbb{R}^{n\times n}$ is symmetric positive definite. Then, for any positive definite matrix P, any subspace S of $\mathbb{R}^n$, any $z \in S$, and any $y \in \mathbb{R}^n$, the solution $\hat{x} = \text{argmin}_{x\in S} <x-y, x-y>_P$ always satisfies $<\hat{x}-y, z>_P = 0$.

T F (c) Consider $A \in \mathbb{R}^{m\times n}$ with $m > n$ and $rank(A) < n$, and $b \in \mathbb{R}^m$. Then, the least squares problem

$$\min_{\alpha\in\mathbb{R}^n} ||A\alpha - b||_2^2$$

always has a unique minimizer α^*.

T F (d) Consider $A \in \mathbb{R}^{m\times n}$ with $m < n$ and $rank(A) = m$, $b \in \mathbb{R}^m$, and $W = N^\top N$ with $N \in \mathbb{R}^{m\times m}$, $rank(N) = m$. Then, $\alpha^* = W^{-1}A^\top(AW^{-1}A^\top)^{-1}b$ satisfies

$$\alpha^* \in \text{argmin}_{\alpha\in\mathbb{R}^n} ||N(A\alpha - b)||_2^2$$

and for any other $\bar{\alpha} \in \text{argmin}_{\alpha\in\mathbb{R}^n}||N(A\alpha-b)||_2^2$, we always have $\boxed{||N\bar{\alpha}||_2 \geq ||N\alpha^*||_2.}$

0

$\begin{bmatrix}1&0&2\\0&0&0\end{bmatrix}$

0 y -

0

$\begin{pmatrix}1&2\\0&0\\0&0\end{pmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$ $\begin{pmatrix}0\\0\\0\end{pmatrix}$

$A\alpha$

$x+2y$

$\hat{y}^\top Q\alpha = 0$

$\hat{\alpha}^\top Q y = 0$

$Qy = A^\top\alpha$

$y = Q^{-1}A^\top\alpha$

$\hat{A}$

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it. For example, we proved that the Gram Schmidt Process produces orthogonal vectors. So if you need this fact, simply state it and use it.

R S

6. (17 points) Let $\mathbb{P}_3(t)$ be the set of polynomials in t with real coefficients with degree less than or equal to 3. Consider the linear operator $L : \mathbb{P}_3(t) \to \mathbb{R}$ given by

$$L(p) = 2p(-1)$$

To be clear this linear operator takes a polynomial $p \in \mathbb{P}_3(t)$, evaluates it at $t = -1$ (i.e., $p(-1) = p(t)|_{t=-1}$) and multiplies the results by 2. You are not asked to show that this is a linear operator, we showed linearity in the lecture. We know that

$$v^1 = 1, \quad v^2 = t, \quad v^3 = t^2, \quad v^4 = t^3$$

is a basis for $\mathbb{P}_3(t)$. Also, $u^1 = 2$ is a basis for $\mathbb{R}$.

(a) (10 points) Find the matrix representation A of the linear transformation L with respect to the basis $\{v^1, v^2, v^3, v^4\}$ for its domain and the basis u^1 for its co-domain.

$A = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$

(b) (7 points) Give the change of basis matrix P from $\{v^1, v^2, v^3, v^4\}$ to $\{\bar{v}^1, \bar{v}^2, \bar{v}^3, \bar{v}^4\}$ where

$$\bar{v}^1 = 1 + t^2, \quad \bar{v}^2 = 1 + t^3, \quad \bar{v}^3 = t - 2t^2, \quad \bar{v}^4 = 1 - t + t^2 - t^3$$

$P = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}^{-1}$

Note: You are not asked to show the linear independence of $\{\bar{v}^1, \cdots, \bar{v}^4\}$. And, to be extra clear, you are NOT being asked to find the matrix representation of L in the new basis $\{\bar{v}^1, \cdots, \bar{v}^4\}$....you only need to compute the change of basis matrix. If you need to invert a matrix, you can show it as $[\quad]^{-1}$; you do not need to compute the inverse.

a) $A = L : [L_1 \; L_2 \; L_3 \; L_4]$

$\beta = L\alpha$, where α are representation of vectors in basis (v) & β are representation of vectors in basis (u)

$\therefore L_i = [L(v_i)]_u$

$\Rightarrow L(v_1) = 2(v_1(-1)) = 2 \cdot (1) = 2$

$[L v(1)]_u = 2/2 = 1$

$\Rightarrow L(v_2) = 2(v_2(-1)) = 2(t)|_{-1} = -2$

$[L(v_2)]_u = -2/2 = -1$

$L(v_3) = 2 v_2(-1) = 2 \cdot 1 \cdot 1 = 2$

$[L(v_3)]_u = 2/2 = 1$

Please show your work for question 6.

$L(v_4) = 2 \cdot v_4(-1) = 2(t^3)_{t=-1} = -2 \quad [L(v_4)]_u = -2/2 = -1$

$\Rightarrow L = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$

$\Rightarrow$ change of basis $P_V^{V'} \to$ from V to V'

$\beta = P\alpha \quad \alpha \to$ representation in V

$P = [P_1 \; P_2 \; P_3 \; P_4] \quad \beta \to$ representation in V'

$\Rightarrow P_i = [V_i]_{V'}$ instead we compute its inverse

$P'_i = [\bar{V}_i]_V \quad P'_1 = [\bar{V}_1]_V = (1+t^2]_V = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$P'_2 = [\bar{V}_2]_V = (1+t^3]_V = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $P'_3 = [\bar{V}_3]_V = [t-2t^2]_V$

$= \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

$P'_4 = (\bar{V}_4]_u = (1-t+t^2-t^3)_u$

$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$\Rightarrow P' = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \Rightarrow P = (P')^{-1}$

$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}^{-1}$

R S

7. (18 points) Assume the possible trajectories of a robot comes from a real inner product space $(\mathcal{X}, \mathbb{R}, < \bullet, \bullet >)$. Assume you have a precomputed trajectory library consisting of two trajectories $x^1, x^2 \in \mathcal{X}$ that are linearly independent vectors in $\mathcal{X}$. Moreover, you are given that

$$< x^1, x^1 >= 3, \quad < x^1, x^2 >= -1, \quad < x^2, x^2 >= 2$$

and you are given a new trajectory $y \in \mathcal{X}$ that satisfies

$$< y, x^1 >= 1, \quad < y, x^2 >= 1$$

Assuming your robot can generate trajectories that are linear combinations of trajectories in your precomputed library, you want to approximate the new trajectory y using such a linear combination.
Record your results on this page. You can use this page or the next page to show your work and/or to state your reasoning. Unsupported answers, even if correct, receive zero credits.

(a) (12 points) Solve

$$\alpha^* = \begin{bmatrix} \alpha_1^* \\ \alpha_2^* \end{bmatrix} = \arg \min_{\alpha_1, \alpha_2 \in \mathbb{R}} ||y - (\alpha_1 x^1 + \alpha_2 x^2)||^2$$

$$\boxed{\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}}$$

(b) (6 points) For another precomputed trajectory library consisting of two other linearly independent vectors $\tilde{x}^1, \tilde{x}^2 \in \mathcal{X}$, suppose you compute

$$< \tilde{x}^1, \tilde{x}^1 >= 2, \quad < \tilde{x}^1, \tilde{x}^2 >= 1, \quad < \tilde{x}^2, \tilde{x}^2 >= 0.5.$$

A friend of yours sees these numbers and immediately says that you must have made a mistake. **Explain why your friend is correct!**

a) let $\hat{x} = \alpha_1^* x^1 + \alpha_2^* x^2$ if $(y - \hat{x})$ is the min distance among all $x = \alpha_1 x^1 + \alpha_2 x^2$

then $(y - \hat{x}) \perp X$ $\Rightarrow \langle (y - \hat{x}), x^1 \rangle = 0$

$\Rightarrow \langle y, x^1 \rangle = \alpha_1 \langle x^1, x^1 \rangle + \alpha_2 \langle x^2, x^1 \rangle$

$\Rightarrow \langle y, x^1 \rangle = \alpha_1 \langle x_1, x_1 \rangle + \alpha_2 \langle x_2, x_1 \rangle$;

$\langle y, x^2 \rangle = \alpha_1 \langle x_1, x_2 \rangle + \alpha_2 \langle x_2, x_2 \rangle$;

Please show your work for question 7.

$1 = 3\alpha_1 - \alpha_2$; $1 = -\alpha_1 + 2\alpha_2$ — ②

$\hookrightarrow$ ①

① + 3②

$$1 = 3\alpha_1 - \alpha_2$$
$$3 = -3\alpha_1 + 6\alpha_2$$
$$4 = 5\alpha_2 \Rightarrow \boxed{\alpha_2 = 4/5}$$

$$3\alpha_1 = 1 + \alpha_2 = 9/5 \Rightarrow \boxed{\alpha_1 = 3/5}$$

The reason for the incorrectness comes from Cauchy-Swartz inequality that

$$\langle x, y\rangle^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$$

here $(\langle \tilde{x}_1, \tilde{x}_2\rangle)^2 = (1)^2 = 1$

$\langle \tilde{x}_1, \tilde{x}_1\rangle = 2$ $\langle \tilde{x}_2, \tilde{x}_2\rangle = 0.5$

$\Rightarrow \langle \tilde{x}_1, \tilde{x}_2\rangle^2 = \langle \tilde{x}_1, \tilde{x}_1\rangle\langle \tilde{x}_2, \tilde{x}_2\rangle$

from the inequality, this equality condition only hol

when $\tilde{x}_1 = \alpha \tilde{x}_2$ $\alpha \in F$

but given $\tilde{x}_1$ & $\tilde{x}_2$ are independent this is not possib

So either of the assumptions are wrong.

R S

8. (15 points) (Proof Problem) (Done in two parts so that you cannot lose too many points on each part)

(a) (7.5 points) Let $||x||_0 = |\{i \mid x_i \neq 0, i = 1, 2, 3\}|$ (i.e., $||x||_0$ counts the number of non-zero entries in $x \in \mathbb{R}^3$).[2] **Show that $|| \cdot ||_0$ is not a valid norm on $(\mathbb{R}^3, \mathbb{R})$.**

For $||x||_0$ needs to be norm, it need to follow Properties

1) $\forall x, \; ||x||_0 \geq 0$ & $||x|| = 0 \iff x = 0$

2) $||x+y||_0 \leq ||x||_0 + ||y||_0$ 3) $||\alpha \cdot x||_0 = |\alpha| \, ||x||_0$.

The above norm fails in property 3 i.e. $||\alpha \cdot x||_0 \neq |\alpha| \, ||x$

if $x = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ where $a, b \neq 0$. $\alpha x = \begin{bmatrix} \alpha a \\ \alpha b \\ 0 \end{bmatrix}$

in this case $||x||_0 = 2$ $||\alpha x||_0 = 2$ $\Rightarrow ||x||_0 \neq \alpha ||x||_0$.

$\therefore$ this is not a norm

(b) (7.5 points) Let $S = \{v^1, v^2, \ldots, v^k\}$ be a linearly independent set. Prove that if $T \subset S$, T is also a linearly independent set.

let $T \in S$ without loss of generality

$T = \{v^1, v^2, \ldots v^p\}$ where $p \leq k$

we will Proove this by contradiction, if T is not linearly independent set

then $\exists \alpha_i \in \mathbb{R}$ Such that atleast one $\alpha_i \neq 0$.

$\Rightarrow \alpha_1 v^1 + \alpha_2 v^2 + \cdots \alpha_p v^p = 0$.

if this is linearly dependent set

$-\alpha_1 v^1 = \alpha_2 v^2 + \cdots \alpha_p v^p$

$v^1 = \left(-\frac{\alpha_2}{\alpha_1}\right) v^2 + \cdots + \left(-\frac{\alpha_p}{\alpha_1}\right) v^p$

---

[2]For a set S, $|S|$ denotes the cardinality of the set and it is equal to the number of elements in the set. For example, $\left\| \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\|_0 = |\{1, 2\}| = 2$ since $\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$ has two non-zero entries or the cardinality of the set of indices that correspond to non-zero entries is two.

Please show your work for question 8.

If we take the same eq^n

$\beta_1 v^1 + \beta_2 v^2 + \cdots \beta_p v^p = 0$

$(\beta_1 v^1 + \beta_2 v^2 + \cdots \beta_p v^p) + (\beta_{p+1} v^{p+1} + \cdots \beta_k v^k) = 0$

where $\beta_{p+1} = \beta_{p+2} \cdots \beta_k = 0$

so $\beta_1 = 1 \quad \beta_2 = \frac{-\alpha_2}{\alpha_1} \quad \cdots \quad \beta_p = \frac{-\alpha_p}{\alpha_1}$

$[1, -\frac{\alpha_2}{\alpha_1} \cdots -\frac{\alpha_p}{\alpha_1}, 0, 0, \cdots 0]$ is one set of

non-zero α's for set $\{v^1, v^2, \cdots v^k\}$

$\Rightarrow \{v^1, v^2, \cdots v^k\}$ is linearly dependent, which is

not true so contradiction is false

$\Rightarrow T = \{v^1, v^2, \cdots v^p\}$ must be set of linearly independ

vectors

R J

Fields and Vector Spaces

Definition 2-1

A field consists of a set, denoted by $\mathscr{F}$, of elements called *scalars* and two operations called addition "+" and multiplication "·"; the two operations are defined over $\mathscr{F}$ such that they satisfy the following conditions:

1. To every pair of elements α and β in $\mathscr{F}$, there correspond an element $\alpha+\beta$ in $\mathscr{F}$ called the *sum* of α and β, and an element $\alpha\cdot\beta$ or $\alpha\beta$ in $\mathscr{F}$, called the *product* of α and β.
2. Addition and multiplication are respectively commutative: For any α, β in $\mathscr{F}$,
$$\alpha+\beta=\beta+\alpha \qquad \alpha\cdot\beta=\beta\cdot\alpha$$
3. Addition and multiplication are respectively associative: For any α, β, γ in $\mathscr{F}$,
$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma) \qquad (\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$$
4. Multiplication is distributive with respect to addition: For any α, β, γ in $\mathscr{F}$,
$$\alpha\cdot(\beta+\gamma)=(\alpha\cdot\beta)+(\alpha\cdot\gamma)$$
5. $\mathscr{F}$ contains an element, denoted by 0, and an element, denoted by 1, such that $\alpha+0=\alpha$, $1\cdot\alpha=\alpha$ for every α in $\mathscr{F}$.
6. To every α in $\mathscr{F}$, there is an element β in $\mathscr{F}$ such that $\alpha+\beta=0$. The element β is called the *additive inverse*.
7. To every α in $\mathscr{F}$ which is not the element 0, there is an element γ in $\mathscr{F}$ such that $\alpha\cdot\gamma=1$. The element γ is called the *multiplicative inverse*. ∎

Chen, Page 8 → 2nd Edition

Definition 2-2

A linear space over a field $\mathscr{F}$, denoted by $(\mathscr{X}, \mathscr{F})$, consists of a set, denoted by $\mathscr{X}$, of elements called *vectors*, a field $\mathscr{F}$, and two operations called *vector addition* and *scalar multiplication*. The two operations are defined over $\mathscr{X}$ and $\mathscr{F}$ such that they satisfy all the following conditions:

1. To every pair of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathscr{X}$, there corresponds a vector $\mathbf{x}_1+\mathbf{x}_2$ in $\mathscr{X}$, called the sum of $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Addition is commutative: For any $\mathbf{x}_1, \mathbf{x}_2$ in $\mathscr{X}$, $\mathbf{x}_1+\mathbf{x}_2=\mathbf{x}_2+\mathbf{x}_1$.
3. Addition is associative: For any $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in $\mathscr{X}$, $(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{x}_3=\mathbf{x}_1+(\mathbf{x}_2+\mathbf{x}_3)$.
4. $\mathscr{X}$ contains a vector, denoted by $\mathbf{0}$, such that $\mathbf{0}+\mathbf{x}=\mathbf{x}$ for every $\mathbf{x}$ in $\mathscr{X}$. The vector $\mathbf{0}$ is called the zero vector or the origin.
5. To every $\mathbf{x}$ in $\mathscr{X}$, there is a vector $\bar{\mathbf{x}}$ in $\mathscr{X}$, such that $\mathbf{x}+\bar{\mathbf{x}}=\mathbf{0}$.
6. To every α in $\mathscr{F}$, and every $\mathbf{x}$ in $\mathscr{X}$, there corresponds a vector $\alpha\mathbf{x}$ in $\mathscr{X}$ called the *scalar product* of α and $\mathbf{x}$.
7. Scalar multiplication is associative: For any α, β in $\mathscr{F}$ and any $\mathbf{x}$ in $\mathscr{X}$, $\alpha(\beta\mathbf{x})=(\alpha\beta)\mathbf{x}$.
8. Scalar multiplication is distributive with respect to vector addition: For any α in $\mathscr{F}$ and any $\mathbf{x}_1, \mathbf{x}_2$ in $\mathscr{X}$, $\alpha(\mathbf{x}_1+\mathbf{x}_2)=\alpha\mathbf{x}_1+\alpha\mathbf{x}_2$.
9. Scalar multiplication is distributive with respect to scalar addition: For any α, β in $\mathscr{F}$ and any $\mathbf{x}$ in $\mathscr{X}$, $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$.
10. For any $\mathbf{x}$ in $\mathscr{X}$, $1\mathbf{x}=\mathbf{x}$, where 1 is the element 1 in $\mathscr{F}$. ∎

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