1) $y_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \quad y_3 = \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix}$

$$\boxed{K_1 = y_1}$$

$$K_2 = y_2 - \frac{\langle y_1, y_2 \rangle}{\|y_1\|^2} y_1$$

$$\langle y_1, y_2 \rangle = (-2 \cdot (0) + 4 - 1)$$
$$= 3$$

$$\|y_1\|^2 = (-2)^2 + (1)^2 + (1)^2 = 6.$$

$$K_2 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} - \frac{3}{6} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\cancel{\boxed{K_2 = \begin{bmatrix} 2/6 \\ 23/6 \\ -7/6 \end{bmatrix}}} \qquad \boxed{K_2 = \begin{bmatrix} 1 \\ 7/2 \\ -3/2 \end{bmatrix}}$$

$$K_3 = y_3 - \left(\frac{\langle y_1, y_3\rangle}{\langle y_1, y_1\rangle} \cdot y_1\right) - \left(\frac{\langle y_2\, y_3\rangle}{\langle y_2\, y_2\rangle} y_2\right)$$

$$\langle y_1, y_3\rangle = -2 \cdot 4 - 4 + 6 = -6$$

$$\langle y_2\, y_3\rangle = 0 \cdot 4 - 4 \cdot 4 - 1 \cdot 6 = -22$$

$$\langle y_1, y_1\rangle = 6$$

$$\langle y_2, y_2\rangle = 0 + 16 + 1 = 17$$

$$y_3 - \left(\frac{\langle y_1\, y_3\rangle}{\langle y_1, y_1\rangle} y_1\right) = y_3 + y_1 = \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} = p'$$

$$p' - \frac{\langle y_2\, y_3\rangle}{\langle y_2\, y_2\rangle} y_2 = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + \frac{22}{17}\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 37/17 \\ 97/17 \end{bmatrix}$$

Q4) $x = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

$y_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$; $y_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

let $\hat{x}$ be the sol^n

then

$\langle (x - \hat{x}), y_i \rangle = 0$

$\Rightarrow \langle x, y_i \rangle = \langle \hat{x}, y_i \rangle$ let $\hat{x} = \alpha_1 y_1 + \alpha_2 y_2$

$\Rightarrow \alpha_i \langle y_i, y_i \rangle = \langle x, y_i \rangle$

$\langle x, y_1 \rangle = \text{trace}(x^T y_1)$

$= \text{tr}\left(\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right)$

$= \text{tr}\left(\begin{bmatrix} 4 & 0 \\ -1 & 0 \end{bmatrix} \right) = 4$

$\langle x, y_2 \rangle = \text{trace}(x^T y_2)$

$= \text{trace}\left(\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$

$= \text{tr}\left(\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \right) = 1$

$$\langle y_1, y_1\rangle = tr\left(\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}\right) = 5$$

$$\langle y_2, y_2\rangle = tr\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\right) = 4$$

$$\alpha_1 = \frac{\langle x, y_1\rangle}{\langle y_1, y_1\rangle} = \frac{4}{5} \qquad \alpha_2 = \frac{\langle x, y_2\rangle}{\langle y_2, y_2\rangle} = \frac{1}{4}$$

$$\hat{x} = \alpha_1 y_1 + \alpha_2 y_2$$

$$= \frac{4}{5}\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

$$\boxed{\hat{x} = \begin{bmatrix}21/20 & 1/4\\ 37/20 & 1/4\end{bmatrix}}$$

8) $\hat{x} = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3$

$x = e^t$

$y_1 = 1 \qquad y_2 = t \qquad y_3 = \frac{1}{2}(3t^2 - 1)$

$\langle x - \hat{x}, y_i \rangle = 0$

$\langle \hat{x}, y_i \rangle = \langle x, y_i \rangle \qquad \hat{x} = \sum \alpha_i y_i$

c) $\alpha_i = \dfrac{\langle x, y_i \rangle}{\langle y_i, y_i \rangle}$

$\langle x, y_1 \rangle = \int_{-1}^{1} e^t \cdot dt = [e^t]_{-1}^{1} = \frac{e^2 - 1}{e} = e - e^{-1}$

$\langle x, y_2 \rangle = \int_{-1}^{1} te^t \cdot dt = [te^t - e^t]_{-1}^{1}$

$= [e^1 - e^1] - [-e^{-1} - e^{-1}]$

$= 2e^{-1}$

$\langle x, y_3 \rangle = \frac{1}{2}\left[\int_{-1}^{1} e^t \cdot 3t^2 - \int_{-1}^{1} e^t\right]$

$= \frac{1}{2}\left[3\int_{-1}^{1} t^2 e^t - \int_{-1}^{1} e^t\right]$

$$\langle x, y_3 \rangle = \frac{1}{2}\left[3\left[t^2e^t - 2te^t + 2e^t \right] - e^t \right]_{-1}^{1}$$

$$= \frac{1}{2}\left[3t^2e^t - 6te^t + 5e^t \right]_{-1}^{1}$$

$$= \frac{1}{2}\left[(3e - 6e + 5e) - (3e^{-1} + 6e^{-1} + 5e^{-1}) \right]$$

$$= \frac{1}{2}\left[2e - 14e^{-1} \right] = e - 7e^{-1}$$

$$\langle y_1, y_1 \rangle = \int_{-1}^{1} 1.1.dt = 2$$

$$\langle y_2, y_2 \rangle = \int_{-1}^{1} t.t.dt = \left[\frac{t^3}{3} \right]_{-1}^{1} = 2/3$$

$$\langle y_3, y_3 \rangle = \int_{-1}^{1} \left(\frac{3t^2 - 1}{2} \right)^2 = \cancel{\left[\frac{t^3}{2} \right]_{-1}^{1} - \left[\frac{t}{2} \right]_{-1}^{1}}$$

$$= \cancel{[illegible]}\ 0.$$

$$= \frac{2}{4} \int_0^1 (3t^2 - 1)^2 = \frac{1}{2} \int_0^1 (9t^4 - 6t^2 + 1)$$

$$= \frac{1}{2}\left(\frac{9}{5} - \frac{6}{3} + 1 \right) = 0.4$$

$$\alpha_1 = \frac{e - e^{-1}}{2}$$

$$\alpha_2 = \frac{2e^{-1}}{(2/3)} = 3e^{-1}$$

$$\alpha_3 = \frac{10}{4}\left(e - 7e^{-1}\right) = \frac{5}{2}\left(e - 7e^{-1}\right)$$

$$\hat{x} = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3$$

wher $y_1 = 1$

$y_2 = t$

$$y_3 = \frac{(3t^2 - 1)}{2}$$

5)

$$\delta = \inf_{y \in M} \|x - y\| \leq \left\| \frac{x - m_1}{2} + \frac{x - m_2}{2} \right\|$$

let m_1 & m_2 be 2 sol^n

where

$$\|x - m_1\| = \delta \quad \& \quad \|x - m_2\| = \delta$$

$$\Rightarrow \left\| \frac{x - m_1}{2} + \frac{x - m_2}{2} \right\| \leq \left\| \frac{x - m_1}{2} \right\| + \left\| \frac{x - m_2}{2} \right\|$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2}$$

$$\leq \delta$$

$$\Rightarrow \delta \leq \left\| \frac{x - m_1}{2} + \frac{x - m_2}{2} \right\| \leq \delta$$

$$\Rightarrow \left\| \frac{x - m_1}{2} + \frac{x - m_2}{2} \right\| = \delta$$

$$= \left\| \frac{x - m_1}{2} \right\| + \left\| \frac{x - m_2}{2} \right\|$$

$\Rightarrow$ by def^n

$$(x - m_1) = \alpha (x - m_2)$$

taking norm

$$\|x - m_1\| = \|\alpha\| \, \|x - m_2\|$$

$$\Rightarrow \|\alpha\| = 1 \quad \Rightarrow \alpha = \pm 1$$

$$\Rightarrow x - m_1 = 1\,(x - m_2) \quad \Rightarrow m_1 = m_2$$

$$(x - m_1) = -(x - m_2) \Rightarrow m_2 = 2x - m_1$$

$\Rightarrow$ ~~ok~~ $\Rightarrow 2x - m_1 \in M$

& distance b/w both m & x is ~~not~~ zero

$\therefore$ only sol^n is $\boxed{m_1 = m_2}$

6)

a) $\|x\| = |x_1| + |x_2|$

So let $x = \begin{bmatrix}1\\0\end{bmatrix}$ $y = \begin{bmatrix}0\\1\end{bmatrix}$

$\|x+y\|_1 = \left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|_1 = 2$

$\|x\|_1 + \|y\|_1 = 1 + 1 = 2$

but $\boxed{x \neq \alpha y}$

$\therefore$

c) $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

$x = \begin{bmatrix}1\\0\end{bmatrix}$ $y = \begin{bmatrix}1\\1\end{bmatrix}$

$\|x+y\|_\infty = \left\|\begin{bmatrix}2\\1\end{bmatrix}\right\|_\infty = 2$

$\|x\|_\infty = \left\|\begin{bmatrix}1\\0\end{bmatrix}\right\|_\infty = 1$

$\|y\|_\infty = \left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|_\infty = 1$

$\|x+y\|_\infty = \|x\|_\infty + \|y\|_\infty$ but $\boxed{x \neq \alpha y}$