Symmetric matrices

ROB 501

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- **Ax=b, underdetermined case and minimum norm solution**
- **Symmetric matrices**
- **Quadratic forms**

$b \in \mathbb{R}^m$, $\underset{\text{full rank}}{A} \in \mathbb{R}^{m \times n}$, seeking $x \in \mathbb{R}^n$

Case 3: $n > m$ (underdetermined) $\underbrace{A = [\quad\quad]}_{\text{wide matrix}}$

↳ less equations than unknowns

↳ many solutions (if A full rank)

* Recall $x \in \mathbb{R}^n$ and $\mathbb{R}^n = R(A^T) \oplus N(A)$

Idea: decompose $\hat{x}$ into a component in $R(A^T)$ and another in $N(A)$

$$\hat{x} = \underset{\in R(A^T)}{\hat{x}_{R(A^T)}} + \underset{\in N(A)}{\hat{x}_{N(A)}}$$

$$b = A\hat{x} = \underbrace{A(\hat{x}_{R(A^T)} + \hat{x}_{N(A)})}_{A\hat{x}_{R(A^T)} = b} = b$$

Choose $\hat{x} = \hat{x}_{R(A^T)} \in R(A^T) \implies \hat{x} = A^T\alpha$ ($\hat{x}$ is a lin. comb of cols of A^T)

Then, $A\hat{x} = AA^T\alpha = b$

$$\implies \alpha = \underbrace{(AA^T)^{-1}}_{\text{invertible bc. } A \text{ full rank}} b$$

$$\underline{\hat{x} = A^T(AA^T)^{-1}b}$$

This is the sol'n with minimum norm!

Proof: Take an arbitrary sol'n $\tilde{x}$ s.t. $A\tilde{x} = b$

$$\|\tilde{x}\|^2 = \|\tilde{x}_{R(A^T)} + \tilde{x}_{N(A)}\|^2 = \|\tilde{x}_{R(A^T)}\|^2 + \underset{\geq 0}{\|\tilde{x}_{N(A)}\|^2}$$

$$\tilde{x}_{R(A^T)} = \underline{\hat{x}}$$

Note that $\tilde{x}_{R(A^T)} = \hat{x} \implies \|\tilde{x}\|^2 = \|\hat{x}\|^2 + \underset{\geq 0}{\|\tilde{x}_{N(A)}\|^2}$

$\Rightarrow \hat{x}$ has minimum norm among all sol'n.
$\quad \overset{\shortparallel}{A^T (AA^T)^{-1} b}$

Symmetric Real Matrices

$y = Cx$
$e = y - Cx$

Def. An $n \times n$ matrix A is symmetric if $A = A^T$.
(since A is assumed real, $A = \bar{A}$)

Claim 1: If A is real and symmetric, then $\forall x, y \in \mathbb{C}^n$

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

Proof: $\langle Ax, y \rangle = (Ax)^T \bar{y} = x^T A^T \bar{y} \overset{A \text{ sym.}}{=} x^T A \bar{y}$

$\langle x, Ay \rangle = x^T \overline{(Ay)} = x^T \bar{A} \bar{y} = x^T A \bar{y}$ (A real)

Claim 2: E-values of real symmetric matrices are real.

Proof: Want to show:
if $Av = \lambda v$, $v \neq 0$, then $\lambda = \bar{\lambda}$.

Apply claim 1 by $x = y = v$:

$$\langle Av, v \rangle = \langle v, Av \rangle$$

$$\langle \lambda v, v \rangle = \langle v, \lambda v \rangle$$

$$\lambda \langle v, v \rangle = \bar{\lambda} \langle v, v \rangle$$

$$\lambda \|v\|^2 = \bar{\lambda} \|v\|^2 \quad (v \neq 0 \Rightarrow \|v\|^2 \neq 0)$$

$$\Rightarrow \lambda = \bar{\lambda} \quad \therefore \lambda \text{ is real!}$$

Remark: $(A - \lambda I) v = 0 \rightarrow$ we can assume e-vectors are real as well.

Claim 3: E-vectors of symmetric real matrices are orthogonal.

Proof: Consider case where $\lambda_1, \dots, \lambda_n$ distinct (general case in HW)

$\lambda_1 \neq \lambda_2$, $v^1, v^2 \neq 0$, s.t. $Av_1 = \lambda_1 v^1$, $Av_2 = \lambda_2 v^2$

Want to show $\langle v^1, v^2 \rangle = 0$.

Apply claim 1, with $x = v^1$ and $y = v^2$:

$$\langle Av^1, v^2 \rangle = \langle v^1, Av^2 \rangle$$

$$\langle \lambda_1 v^1, v^2 \rangle = \langle v^1, \lambda_2 v^2 \rangle$$

$$\lambda_1 \langle v^1, v^2 \rangle = \bar{\lambda}_2 \langle v^1, v^2 \rangle$$

$\parallel$ by claim 2: $\lambda_2 = \bar{\lambda}_2$

$$\lambda_2 \langle v^1, v^2 \rangle$$

$$\lambda_1 \langle v^1, v^2 \rangle - \lambda_2 \langle v^1, v^2 \rangle = 0$$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle v^1, v^2 \rangle = 0 \implies \langle v^1, v^2 \rangle = 0 \implies v^1 \perp v^2.$$

bc. λ_1, λ_2 distinct.

Def: An $n \times n$ real matrix Q is orthogonal if $Q^T Q = \underline{I}$

$Q = [Q_1 | \dots | Q_n]$ $\quad [Q^T Q]_{ij} = \langle Q_i, Q_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$\rightarrow$ Columns of Q are an orthonormal basis of $\mathbb{R}^n$.

$$Q^T = Q^{-1}$$

Claim 4: If A is real symmetric, then $\exists$ an orthogonal matrix Q s.t.

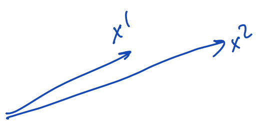
$$Q^T A Q = \Lambda = \mathrm{diag}(d_1, \ldots, d_n)$$

Proof: Just pick $Q = [v^1 | \ldots | v^n]$ where v^i's are e-vectors of A. (and WLOG we can pick $\|v^i\| = 1$)

* Why orthogonal matrices are interesting?

1. $$\|Qx\|_2^2 = x^T \underbrace{Q^T Q}_{I} x = x^T x = \|x\|_2^2$$ (preserve euclidean distance)

2. Numerical robustness



x^1 and x^2 are numerically close to being dependent as angle $\longrightarrow 0$

but orthogonal matrices are numerically stable, easy to invert, etc.

Consider the following $[x^1\ x^2]$ $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 100 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}}_{\substack{\det = 1 \\ \mathrm{rank} = 2}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 10^{-2} & 0 \end{bmatrix}}_{\text{very ``small'' perturbation}} = \underbrace{\begin{bmatrix} 1 & 100 \\ 10^{-2} & 1 \end{bmatrix}}_{\substack{\det = 0 \\ \text{singular!}}}$$

this does not happen w/ orthogonal matrices!

__Observation:__ Let $A \in \mathbb{R}^{m \times n}$, A^TA and AA^T are symmetric.

__Claim 5:__ E-values of A^TA and AA^T are non-negative.

(If λ_i is an e-value of A^TA (or AA^T), $\lambda_i \geq 0 \; \forall i$)

__Proof:__ Let v be an e-vector of A^TA, $v \neq 0$.

$A^TAv = \lambda v$, where $\lambda \in \mathbb{R}$ is e-value.

$$v^T(A^TA)v = \lambda v^T v$$

$$\underbrace{\|Av\|_2^2}_{\geq 0} = \lambda \underbrace{\|v\|_2^2}_{\geq 0} \implies \lambda \geq 0$$

Quadratic Forms

We will use these in optimization (estimation) problems where some measurements are "more uncertain" than others.

Let M be $n \times n$ real matrix. Let $x \in \mathbb{R}^n$

$$x^T M x = x^T \Big[\underbrace{\frac{M+M^T}{2}}_{\text{symmetric part of } M} + \underbrace{\frac{M-M^T}{2}}_{\text{skew-symmetric part of } M:\ \left(\frac{M-M^T}{2}\right)^T = -\left(\frac{M-M^T}{2}\right)} \Big] x$$

$K^T = -K$: K is skew symmetric

$$= \underbrace{x^T \left(\frac{M+M^T}{2}\right) x}_{\text{scalar}} + \underbrace{\cancel{x^T \left(\frac{M-M^T}{2}\right) x}}_{\text{scalar}} \to 0$$

$$\frac{x^T M x - \overbrace{x^T M^T x}^{x^T M x}}{2} = 0$$

$\Rightarrow$ Symmetric M is the most general quadratic to optimize

<u>Def</u>: A real symmetric matrix P is positive definite if $\forall x \in \mathbb{R}^n, x \neq 0$ $x^T P x > 0$

OFFICE HOURS

$\hat{x}$

$\parallel$

$\tilde{x} = \underbrace{\tilde{x}_{R(A^T)}} + \tilde{x}_{N(A)}$

$\tilde{x}_{R(A^T)} = A^T\tilde{\alpha}$

$\parallel$

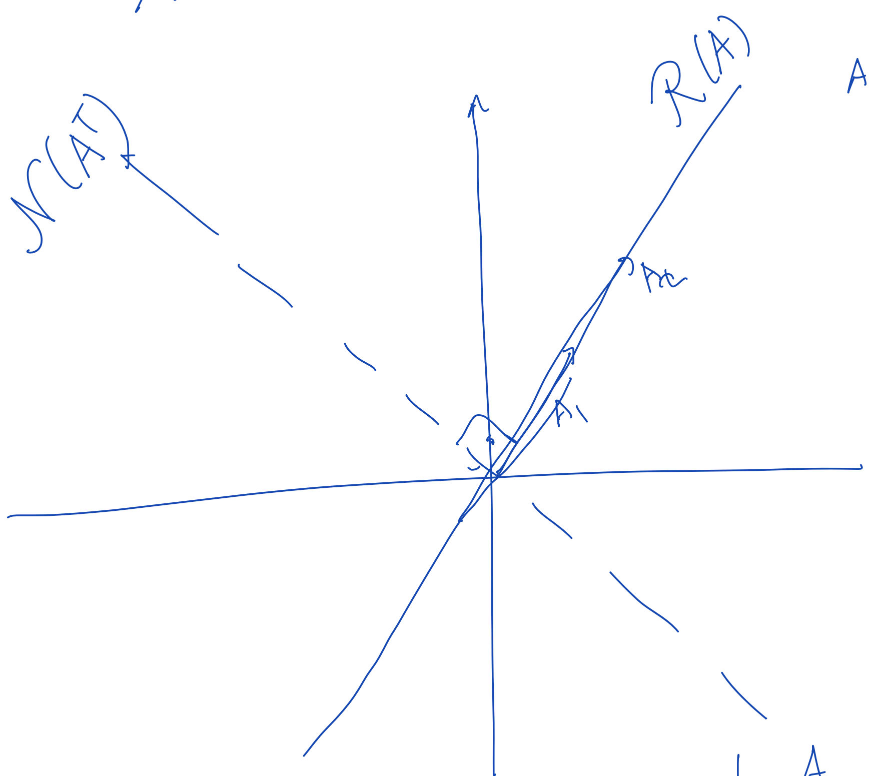
$A^T(AA^T)^{-1}b$

$b = A\tilde{x} = A\tilde{x}_{R(A^T)}$

$b = \underbrace{AA^T}\tilde{\alpha}$

$\tilde{\alpha} = (AA^T)^{-1}b$

$A \in \mathbb{R}^{m \times n}$



$$A = \begin{array}{c} A_1 \; A_2 \; \dots \; A_n \\ \left[\; \right] \end{array}$$

$A^T = \left[\; \right]$

$A \in \mathbb{R}^{2\times 2}$

$A = [A_1, A_2]$

assume

$A_2 = \alpha A_1$

$x \perp A_1$

$x \perp A_2$

$A^T = \begin{bmatrix} A_1^T \\ A_2^T \end{bmatrix}$

$A_1^T x = 0$

$A_2^T x = 0$

$N(A^T) = \{x \mid A^T x = 0\}$

$$\|x+y\|_{\$} = \|x\|_{\$} + \|y\|_{\$} \Rightarrow \frac{x = \alpha y}{\{x, y\} \text{ lin dep.}}$$

$$a \Rightarrow b$$

$$\neg b \Rightarrow \neg a$$

$$\|a+b\| = \|a\| + \|b\|$$

$$a = \alpha b$$

$$\frac{x - m_1}{2} = \alpha \frac{x - m_2}{2}$$

$$\|x+y\| = \|x\| + \|y\| \Rightarrow \exists \alpha \geq 0 \text{ s.t. } x = \alpha y$$

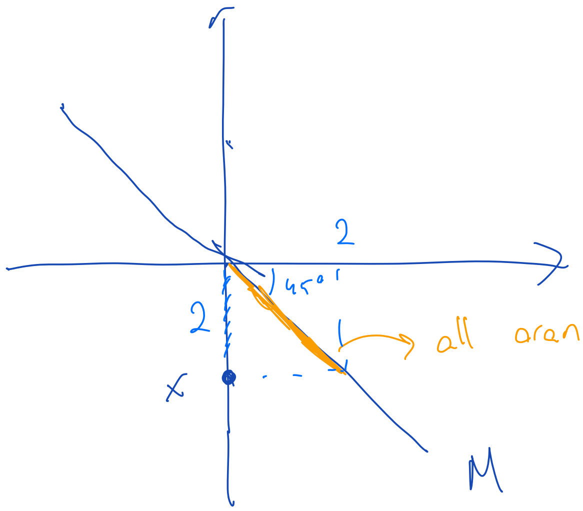
$$\exists \alpha \geq 0 \text{ s.t. } x = \alpha y \Rightarrow \|x+y\| = \|x\| + \|y\|$$

$$\|x+y\| = \|\alpha y + y\| = (\alpha+1)\|y\| = \alpha\|y\| + \|y\|$$
$$= \|\alpha y\| + \|y\|$$
$$= \|x\| + \|y\|$$

if you find x, y s.t. $\|x+y\| = \|x\| + \|y\|$

but $x \neq \alpha y \rightarrow$ then

$\star$ does not hold.



all orange points have the same $\|\cdot\|_1$ distance to x since $\|\cdot\|_1$ is not a strict norm

$\inf\limits_{m \in M} \|x - m\|$