1)

a) $f(x_1, x_2, x_3) = 3x_1[2x_2 - x_3^3] + \frac{x_2^4}{3}$

$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1}\left(3x_1\{2x_2 - x_3^3\}\right) + \frac{\partial}{\partial x_1}\left(\frac{x_2^4}{3}\right)$

$\boxed{\frac{\partial f}{\partial x_1} = 3[2x_2 - x_3^3]}$

$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2}\left(3x_1[2x_2]\right) - \frac{\partial}{\partial x_2}\left(3x_1 x_3^3\right)$

$+ \frac{\partial}{\partial x_2}\left(\frac{x_2^4}{3}\right)$

$\boxed{\frac{\partial f}{\partial x_2} = 6x_1 + \frac{4x_2^3}{3}}$

$\frac{\partial f}{\partial x_3} = \frac{\partial}{\partial x_3}\left(-3x_1 x_3^3\right) + \frac{\partial}{\partial x_3}\left(6x_1x_2 + \frac{x_2^4}{3}\right)$

$\boxed{\frac{\partial f}{\partial x_3} = -9x_1 x_3^2}$

$\left.\frac{\partial f}{\partial x}\right|_{x^*} = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3}\right]_{x^*}$

$$\left.\frac{\partial f}{\partial x_1}\right|_{x^*} = 3\left(2(3) - (-1)^3\right) = 21$$

$$\left.\frac{\partial f}{\partial x_2}\right|_{x^*} = 6x_1 + \left.\frac{4x_2^3}{3}\right|_{x^*}$$

$$= 6(1) + \frac{4 \times 3^3}{3} = 42$$

$$\left.\frac{\partial f}{\partial x_3}\right|_{x^*} = -9x_1 x_3^2\Big|_{x^*} = -9 \times 1 \times (-1)^2$$

$$= -9$$

$$\left.\frac{\partial f}{\partial x}\right| = \begin{bmatrix} 21 & 42 & -9 \end{bmatrix}$$

Question 1:

Part b:

Delta value used is 0.01

Jacobian = [
21.00
42.04
-9.03]

Part c:

The delta value is calculated by decreasing the delta value by 1/10th each time. So if the function is linear for *delta1* it will be linear for *delta2* (delta2<delta1) also. And it is observed after delta=0.001. The jacobian values are almost same for delta values less than that.

Jacobian:
[
117.370767151514
-41.3695125586657
-636.444234930384
-3.85196584436720
-11.2049219677841
]

Delta = 0.001

Code

Part b:

```
x = [1, 3, -1];
jac = zeros(3,5);
del = 1;
for j = 1:2
   del = del*1e-1;
   for i = 1:3
```

```
        e = zeros(1,3);
        e(i) = 1;
        x1 = x - del*e;
        x2 = x + del*e;
        jac(i,j) = (part_b(x2) - part_b(x1)).*(1/(2*del));
    end
end

function r = part_b(x)
    r = (3*x(1)*(2*x(2) - (x(3)^3))) + ((x(2)^4)/3);
end
```

Part C:

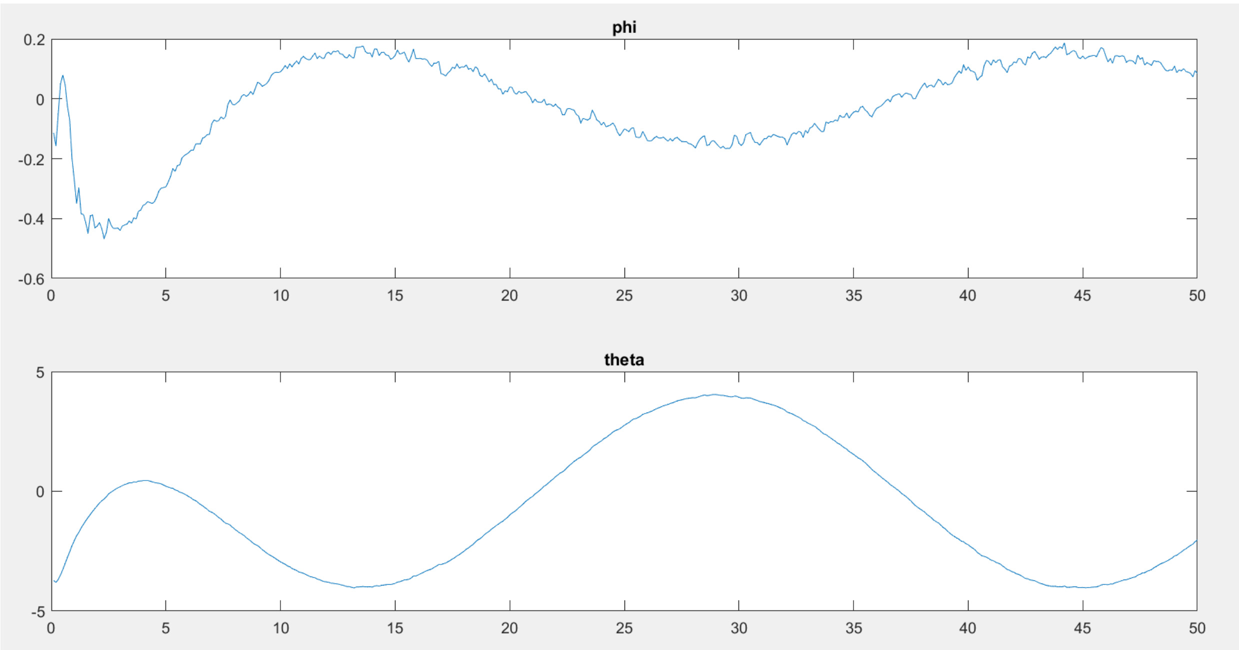
```
x = [1, 3, -1];
jac = zeros(3,5);
del = 1;
for j = 1:2
    del = del*1e-1;
    for i = 1:3
        e = zeros(1,3);
        e(i) = 1;
        x1 = x - del*e;
        x2 = x + del*e;
        jac(i,j) = (part_b(x2) - part_b(x1)).*(1/(2*del));
    end
end

function r = part_b(x)
    r = (3*x(1)*(2*x(2) - (x(3)^3))) + ((x(2)^4)/3);
end
```

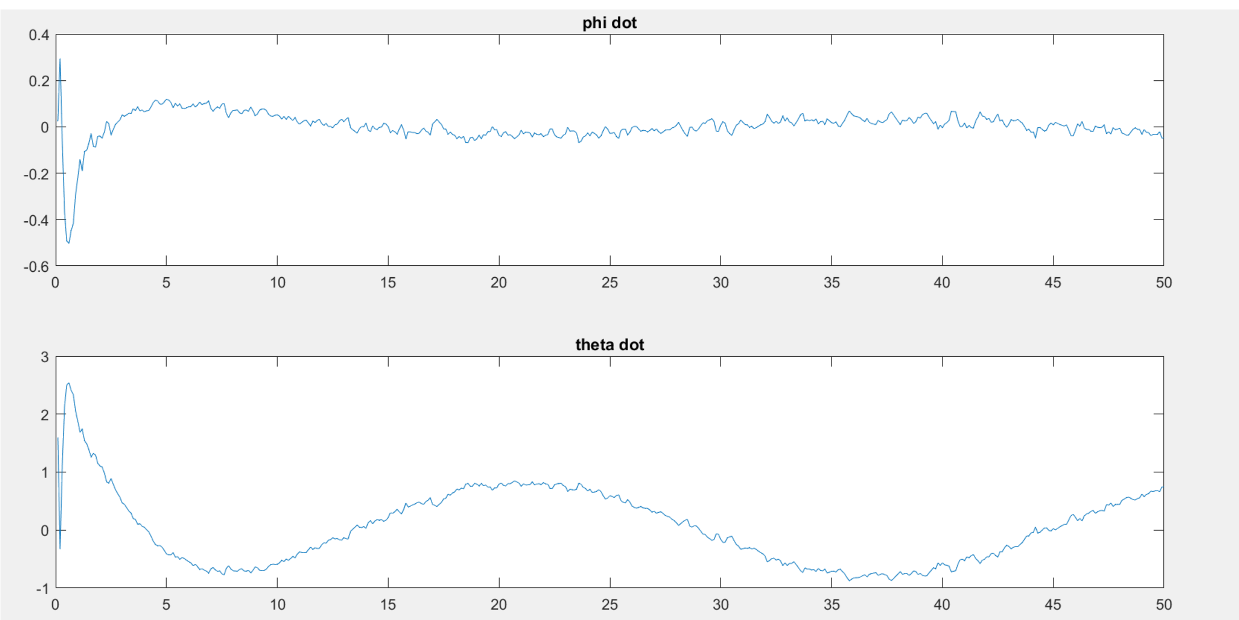
Question 2:

Part b:

Estimates of (phi and theta) vs time:

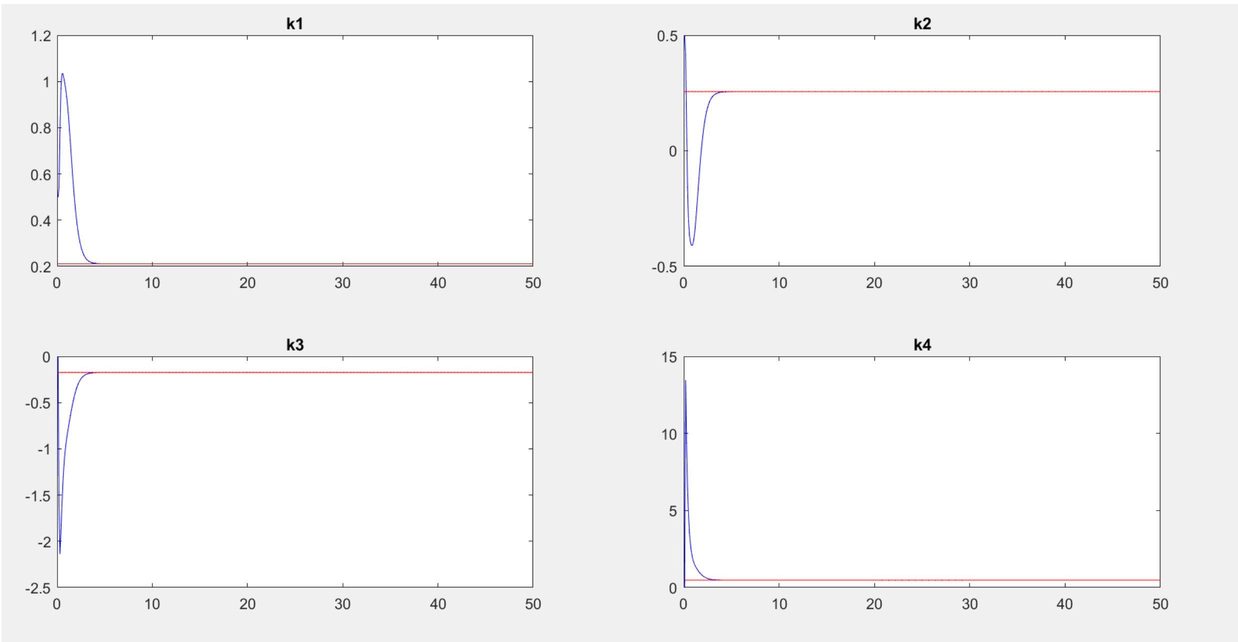


Estimates of derivatives of (phi and theta) w.r.t. time



Elements of K (Kalman gain) vs time:

Blue - Kalman gains calculated w.rt. Time
Red - Steady state Kalman gains using *dlqe()*



Steady state values achieved using:

Using Code :

K =[
 0.2113
 0.2559
-0.1744
 0.4816
]

Using *dlqe()*:

Kss =[
 0.2113
 0.2559
-0.1744
 0.4816
]

Code:

```
load ../HW10data/SegwayData4KF

phi=zeros(N,1);
theta=zeros(N,1);
phi_dot=zeros(N,1);
theta_dot=zeros(N,1);

k1=zeros(N,1);
k2=zeros(N,1);
k3=zeros(N,1);
k4=zeros(N,1);

xk=x0;
pk = P0;

t=zeros(1,N);
for k =1:N
   uk=u(k);
   K = (pk*C')*(inv((C*pk*C')+ Q));
   xkp1=A*xk+B*uk + A*K*(y(k)-(C*xk));
   pkp1= (A*(pk - (K*C*pk))*A') + (G*R*G');
   phi(k)=[1 0 0 0]*xkp1;
   theta(k)=[0 1 0 0]*xkp1;
   phi_dot(k)=[0 0 1 0]*xkp1;
   theta_dot(k)=[0 0 0 1]*xkp1;

   k1(k) = K(1);
   k2(k) = K(2);
   k3(k) = K(3);
   k4(k) = K(4);

   t(k)=k*Ts;
   xk=xkp1;
   pk = pkp1;
end
```

```
[Kss,Pss] = dlqe(A,G,C,R,Q);


fig1 = figure();
subplot(2,1,1);
plot(t, phi);
title("phi");
subplot(2,1,2);
plot(t, theta);
title("theta");


fig2 = figure();
subplot(2,1,1);
plot(t, phi_dot);
title("phi dot");
subplot(2,1,2);
plot(t, theta_dot);
title("theta dot");

fig3 = figure();
subplot(2,2,1);
plot(t, k1, 'b', t,Kss(1)*ones(length(k1)), 'r');
title("k1");
subplot(2,2,2);
plot(t, k2, 'b', t,Kss(2)*ones(length(k2)), 'r');
title("k2");
subplot(2,2,3);
plot(t, k3, 'b', t,Kss(3)*ones(length(k3)), 'r');
title("k3");
subplot(2,2,4);
plot(t, k4, 'b', t,Kss(4)*ones(length(k4)), 'r');
title("k4");
```

Question 3:

The estimated pose of x1 (i.e robot position at t=0.1s) is : 1.799

The mean and variance of x1 is:

> Mean : 1.799
> Variance: 0.0213

Code:

```
B = 0.1;
A = 1;
c = 3*1e8;
C = -2/c;
Q = 1e-18;

x0 = 0.5*randn(1) + 1;
u_hat = 4*randn(1) + 10;
z_meas = 2.2*1e-8;

var_x0 = 0.25;
var_u = 16;

x_est = A*x0 + B*u_hat;
P_10 = (A*var_x0*A') + (B*var_u*B');
z_hat = C*(x_est-5);

K = P_10*C'*(inv(C*P_10*C' + Q));

x1 = x_est + K*(z_meas-z_hat);
P_11 = P_10 - K*C*P_10;

u_last = A*1 + B*10;
```

Question 5:

Delta_A =
```
[
   0.0010   -0.0010    0.0010
  -0.0024    0.0025   -0.0025
   0.0013   -0.0013    0.0013
]
```

A_hat = *A* + *Delta_A* =
```
[
   4.0420    7.0450    3.0150
  10.0426   17.0345    7.0245
  16.0073   27.0037   11.0493
]
```

A =
```
[
   4.0410    7.0460    3.0140
  10.0450   17.0320    7.0270
  16.0060   27.0050   11.0480
]
```

rank(*A_hat*) = 2
rank(*A*) = 3
norm(*delA*) = 0.0051

A_hat is *rank 2 approximation of A*

Code:

```
A = [4.041, 7.046, 3.014;
   10.045, 17.032, 7.027;
   16.006, 27.005, 11.048;];
rank(A)
[U,S,V] = svd(A);
sigma = S(3,3);
ur = U(:,3);
vr = V(:,3);
```

```
delA = sigma*ur*vr';

A_hat = A - delA;

rank(A_hat)
```

4) $x^T \cdot A^T A x$

$A = U \Sigma V^T$

then $A^T A = V \Sigma^T \Sigma V^T$

$x^T A^T A x = x^T V (\Sigma^T \Sigma) V^T x$

let $V^T x = y$ $\quad = y^T (\Sigma^T \Sigma) y$

let $\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ \vdots & \sigma_2 & \cdots & & \\ & \cdots & & & \sigma_n \end{pmatrix}$

$y = [y_1 \; y_2 \; \cdots \; y_n]^T$

$\Rightarrow y^T (\Sigma^T \Sigma) y = \Sigma \sigma_i^2 y_i^2$

given $x^T x = 1$ as $\|v\| = 1$

$x^T V V^T x = 1$

$\Rightarrow y^T y = 1$

$\therefore \min (\Sigma \sigma_i^2 y_i^2) = \min_i (\sigma_i^2)$

$\therefore \Rightarrow y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ (assuming Σ is in decreasing order of eigen values)

2) $y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ ~~ωp~~ $= V^T x$

as collums of V are orthogonal

This canbe possible only when

x = last collum of V

5) The rank 2 approximation for given matrix A can be solved using

$$\Delta A = -\sigma_r u_r v_r^T$$

as given

$\|\Delta A\|$ should have smallest norm

σ_r will be corresponding to least eigen value

$\Rightarrow$ Using SVD

$$A = U D V^T$$

$$U = \begin{bmatrix} -0.215 & 0.9147 & 0.3422 \\ -0.5209 & 0.1890 & -0.8374 \\ -0.8261 & -0.3572 & 0.4358 \end{bmatrix}$$

$$D = \begin{bmatrix} 40.2854 & 0 & 0 \\ 0 & 0.1859 & 0 \\ 0 & 0 & 0.0051 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.4797 & -0.6642 & -0.5734 \\ -0.8116 & 0.0875 & 0.5776 \\ -0.335 & 0.7424 & -0.5810 \end{bmatrix}$$

taking u_3 & v_3 w.r.t eigen value

$\sigma = 0.0051$

$$\Delta A = \begin{pmatrix} 0.001 & -0.001 & 0.001 \\ -0.0024 & 0.0025 & -0.0025 \\ 0.0013 & -0.0013 & 0.0013 \end{pmatrix}$$

$$A + \Delta A = \begin{pmatrix} 4.0420 & 7.0450 & 3.0150 \\ 10.0426 & 17.0346 & 7.0245 \\ 16.0073 & 27.0037 & 11.0493 \end{pmatrix}$$

rank $(A + \Delta A) = 2$

ΔA has least norm $= 0.0051$