ROB 501 Exam-I (Fall 2020)

Prof. Robert Gregg
48 Hour Take-Home Exam
Released: 12pm on Friday, October 23, 2020
Due: 12pm on Sunday, October 25, 2020

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RULES:

1. **NO COLLABORATION OF ANY KIND**
2. OPEN TEXTBOOK, CLASS NOTES, HOMEWORK
3. GOOGLING SOLUTIONS IS CONSIDERED ACADEMIC DISHONESTY, AND MOST PROBLEMS CANNOT BE EASILY FOUND ON THE WEB ANYWAY
4. CALCULATOR/COMPUTER ALLOWED BUT MUST SHOW CALCULATION STEPS FOR FULL CREDIT
5. SUBMIT QUALITY PHOTOS/SCANS TO GRADESCOPE BY DEADLINE (STRICT)

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Problems 1 - 5 (30 points: 5×6)

Instructions. Each part of a question is worth 1.5 points. Submit your answers to questions 1-5 as follows:

1. Download the answer sheet from Canvas (`ROB501 Midterm Fa2020 TF Answer Sheet.pdf`).
2. Print it, or open it in your favorite PDF viewer app (see Canvas announcement if you need ideas).
3. Clearly mark your answer to each question on the answer sheet.
4. Scan or export your solutions, and upload them to Gradescope.

Do not modify the answer sheet, or attach any extra pages. You do not need to show your work. Answers written directly on the questions below will not be graded.

better way of stating the problem → for statement over $\mathbb{N}_0 = \{0,1,\dots\}$

$[P(0) \wedge (\forall k \in \mathbb{N}_0\ P(k) \Rightarrow P(k+1))] \Rightarrow [\forall n \in \mathbb{N}_0\ P(n)]$

1. (Logic questions)

Circle True or False as appropriate *on the answer sheet* for the following statements:

T

(a) For any statement over the natural numbers, $P(n)$: $\mathbb{N} = \{0,1,\dots\}$

→ Proof by induction

$$\big(P(0)\big) \wedge \big(\forall k, P(k) \implies P(k+1)\big) \implies \boxed{\forall n,\ P(n)}$$

True

F

(b) The negation of "if the leaves are changing, then it is fall," is "it is not fall, so the leaves are not changing."

(c) The negation of the statement

T

$$\neg\big(\forall x, y \in C \subset \mathcal{X},\ \forall 0 \le \lambda \le 1,\ \text{then } f(\lambda x + (1-\lambda)y) \le f(x) + (1-\lambda)f(y)\big)$$

$\exists x, y \in C \subset \mathcal{X}\ \exists \lambda\ 0 \le \lambda \le 1\ \neg\big(f(\lambda x+(1-\lambda)y) \le f(x)+(1-\lambda)f(y)\big)$

is:

~~$p \wedge \neg q$~~

$$\exists\, x, y \in C \subset \mathcal{X},\ \exists\, 0 \le \lambda \le 1 \text{ s.t. } f(\lambda x + (1-\lambda)y) > f(x) + (1-\lambda)f(y)$$

T

(d) $\neg(\neg p \vee q) \iff \neg(p \implies q)$. $\Longleftrightarrow \neg(\neg p \vee q)$

(b) p: leaves changing

q: it is fall

"$p \Rightarrow q$" $\equiv$ "$\neg p \vee q$"

$\neg(p \Rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$

$\underbrace{\neg q \Rightarrow \neg p}_{\text{contrapositive}}$

2. (Change of basis, linear transformations, eigenvalues & eigenvectors)

Circle True or False as appropriate *on the answer sheet* for the following statements:

T

(a) Let v be an eigenvector of real symmetric matrix $A \in \mathbb{R}^{n \times n}$. Given non-zero $\alpha \in \mathbb{C}$, then αv is also an eigenvector of A.

F

(b) Let $(\mathbb{P}^n, \mathbb{R})$ be the space of polynomials $(p : \mathbb{R} \to \mathbb{R})$ of degree less than or equal to n over the field $\mathbb{R}$. Define the linear map $\mathcal{L} : \mathbb{P}^2 \to \mathbb{P}^3$ as

$$(\mathcal{L}(p))(x) = \int_0^x p(s)\mathrm{d}s.$$

Suppose A is the matrix representation of $\mathcal{L}$ with respect to the standard polynomial bases in $\mathbb{P}^2$ and $\mathbb{P}^3$ (from lecture). Then A^{-1} represents the linear map $\mathcal{D} : \mathbb{P}^3 \to \mathbb{P}^2$ given by

$$(\mathcal{D}(p))(x) = \frac{d(p(x))}{dx}.$$

$n \times n$ identity matrix

T

(c) Consider matrix $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Then, the eigenvalues of matrix $4A + 3I$ are $\{4\lambda_1 + 3, \dots, 4\lambda_n + 3\}$.

F

(d) Consider matrix $A \in \mathbb{R}^{n \times n}$ with *distinct* eigenvalues $\{\lambda_1, \dots, \lambda_k\}$, $k < n$, i.e., the other $n-k$ eigenvalues are repeats. Then $\det(A) = \prod_{i=1}^{k} \lambda_i$.

take $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow k=1 \quad \lambda_1 = 2$

$\det(A) = 4 \neq 2$

one counterexample is sufficient to conclude a statement is false (WARNING: an example does not imply the

2.a) A is real symmetric (statement is true)

$v \neq 0$ and $\exists \lambda \in \mathbb{R}$ s.t. $Av = \lambda v$.

given $\alpha \in \mathbb{C}$ want to check if $\tilde{v} = \alpha v$ is an e-vector of A.

$\tilde{v} \neq 0$

true because $v \neq 0$ $\alpha \neq 0$

$$A\tilde{v} = A(\alpha v) = \alpha(Av) = \alpha\lambda v$$
$$= \lambda\alpha \cdot v = \lambda \cdot \tilde{v}$$

$$A\tilde{v} = \lambda\tilde{v}$$

b) $\dim(\mathbb{P}^2) = 3$ $A \in \mathbb{R}^{4\times 3}$

$\dim(\mathbb{P}^3) = 4$

A is not square so A^{-1} is not defined. (can't be representation of anything!)

$\longrightarrow$ False

c) Let $v^1, \dots, v^n$ be the e-vectors corresponding to $\lambda_1, \dots, \lambda_n$

$$(4A + 3I)v^i = 4Av^i + 3v^i = 4\lambda_i v^i + 3v^i$$
$$= (4\lambda_i + 3)v^i$$

d) $\det(A) = \prod_{i=1}^{n} \lambda_i$ (independent of whether λ_i's are repeated or not)

$\|x+y\|_1^2 = \left\|\begin{bmatrix}2\\0\end{bmatrix}\right\|_1^2 = 2^2 = 4 \neq \|x\|_1^2 + \|y\|_1^2 = 4 + 4$

$x = \begin{bmatrix}1\\1\end{bmatrix}$ $y = \begin{bmatrix}1\\-1\end{bmatrix}$

$\langle x,y\rangle_Q = x^TQy$

$\|x\|_Q = \sqrt{x^TQx}$

Q real sym. pos. def.

3. (Inner product spaces, norms, projection theorem)

Circle True or False as appropriate *on the answer sheet* for the following statements:

$\boxed{F}$ → (a) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, <\cdot,\cdot>)$, where $<x,y>= x^Ty$ for $x, y \in \mathbb{R}^n$. If $x \perp y$, then $||x+y||_1^2 = ||x||_1^2 + ||y||_1^2$ where $||\cdot||_1$ is the 1-norm.

$\|x+y\|_Q^2 = \|x\|_Q^2 + \|y\|_Q^2$ ⇐ depends on wrt which inner prod. we define orthogonality with.

$\boxed{T}$ → (b) Let $(\mathbb{R}^2, \mathbb{R}, <\cdot,\cdot>)$ be a 2-dimensional inner product space, and $M, N \subset \mathbb{R}^2$ be two orthogonal 1-dimensional subspaces. Then $M^\perp = N$ and $N^\perp = M$.

$\boxed{F}$ → (c) Consider the finite-dimensional inner product space $(\mathcal{X}, \mathbb{R}, <\cdot,\cdot>)$ and vectors $u_1, u_2, u_3 \in \mathcal{X}$. You are given:

$G = \begin{bmatrix}5 & 2 & 3\\2 & 1 & 0\\3 & 0 & 9\end{bmatrix} \to$ not full rank

$$<u_1,u_1>=5 \quad <u_1,u_2>=2$$
$$<u_2,u_2>=1 \quad <u_1,u_3>=3$$
$$<u_3,u_3>=9 \quad <u_2,u_3>=0$$

$G\alpha = \beta$

$G^T\alpha = \beta$

if G is rank 3, then hence $\alpha = G^{-1}\beta$

Then, for some $v \in \mathcal{X}$:

$$d(v, \text{span}\{u_1,u_2,u_3\}) = ||v - \alpha_1u_1 - \alpha_2u_2 - \alpha_3u_3||$$

where:

Looks like normal equations? →

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}<u_1,u_1> & <u_1,u_2> & <u_1,u_3>\\<u_2,u_1> & <u_2,u_2> & <u_2,u_3>\\<u_3,u_1> & <u_3,u_2> & <u_3,u_3>\end{bmatrix}^{-1} \underbrace{\begin{bmatrix}<v,u_1>\\<v,u_2>\\<v,u_3>\end{bmatrix}}_{\beta}$$

→ not defined

$\boxed{T}$ ← (d) Consider inner product space $(\mathcal{X}, \mathbb{R}, <\cdot,\cdot>)$ and let $S \subset \mathcal{X}$ be a subset of $\mathcal{X}$. Let $U = \text{span}\{S\}$ and $x_1, x_2 \in \mathcal{X}$. If $d(x_1, U) = ||x_1 - u_1||$ and $d(x_2, U) = ||x_2 - u_2||$, then

$$d(3x_2 - 2x_1, U) = ||3x_2 - 2x_1 - 3u_2 + 2u_1||$$

(b) Given: $\dim(M) = 1$, $\dim(N) = 1$, $M \perp N$

We know $\underbrace{\dim(M)}_{1} + \dim(M^\perp) = \underbrace{\dim(\mathbb{R}^2)}_{2} \to \dim(M^\perp) = 1$

First we show $N \subset M^\perp$:
take $n \in N$, since $M \perp N$
$\Rightarrow \langle n, m\rangle = 0 \quad \forall m \in M$
$M^\perp = \{y \mid \langle y, m\rangle = 0 \ \forall m \in M\}$
$\Rightarrow n \in M^\perp \Rightarrow N \subset M^\perp$
$\Rightarrow$ since $\dim(N) = \dim(M^\perp)$ we have $N = M^\perp$

4. (Matrix properties)

Circle True or False as appropriate *on the answer sheet* for the following statements:

$\boxed{T}$ (a) Let $A = \left[\begin{array}{cc|cc}a & b & c & d\\b & e & f & g\\\hline c & f & h & i\\d & g & i & j\end{array}\right] = A^T \in \mathbb{R}^{4\times4}$. Then $j - i^2/h > 0$ is a necessary condition for $A > 0$.

A_1, A_2, A_2^T, A_4

$A > 0 \iff A_4 > 0$ and $A_1 - A_2A_4^{-1}A_2^T > 0$

$A_4 > 0 \iff j > 0$ and $j - i\frac{1}{h}i > 0$, $j - \frac{i^2}{h} > 0$

$\boxed{F}$ (b) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, <\cdot,\cdot>)$ with inner product $<y_1,y_2>= y_1^\top Sy_2$ for $y_1, y_2 \in \mathbb{R}^n$ where $S = S^\top \in \mathbb{R}^{n\times n}$ is a full rank symmetric matrix. Given a matrix $A \in \mathbb{R}^{n\times m}$, where $n > m$, and a vector $y \in \mathbb{R}^n$, $(A^\top SA)^{-1}A^\top Sy \in \mathcal{R}(A)$, where $\mathcal{R}(A)$ is the range of A.

$\exists T \in \mathbb{R}^{n\times n}$ T invertible

$\boxed{F}$ (c) Let $B \in \mathbb{R}^{n\times n}$ be a (symmetric) positive definite matrix. If $C \in \mathbb{R}^{n\times n}$ is similar to B, then C is (symmetric) positive definite.

C is not necessarily symmetric. s.t. $B = T^{-1}CT$

$\boxed{T}$ (d) Let $A, B, P \in \mathbb{R}^{n\times n}$, where P is invertible. Suppose $A = P^{-1}BP$. If $\{v^1, \ldots, v^k\}$ are the eigenvectors of A, where $1 \le k \le n$, then $\{Pv^1, \ldots, Pv^k\}$ are the eigenvectors of B.

3.b alternative: same question with

$\boxed{F}$ Let $(\mathbb{R}^2, \mathbb{R}, <\cdot,\cdot>)$ be a 2-dimensional inner product space, and $M, N \subset \mathbb{R}^2$ be two orthogonal [redacted] subspaces. Then $M^\perp = N$ and $N^\perp = M$.

M = some 1-dim subspace, $N = \{0\}$, $\dim(N) = 0$ } $\dim(M^\perp) = 1 \therefore M^\perp$ cannot be equal to N which is zero dimensional.

3.d this question is about the projection operator P_U on to the subspace U.

And we know projection operator is a linear operator

Given $\left.\begin{array}{l} u_1 = P_U(x_1) \\ u_2 = P_U(x_2) \end{array}\right\}$ $P_U(3x_2 - 2x_1) = 3P_U(x_2) - 2P(x_1)$

$= \underline{3u_2 - 2u_1}$

$d(3x_2 - 2x_1, U) = \| 3x_2 - 2x_1 - 3u_2 + 2u_1 \|$

4b.

$A \in \mathbb{R}^{n \times m}$ $n > m$ $y \in \mathbb{R}^n$

$\underbrace{\underbrace{(A^T S A)^{-1}}_{m \times m} \underbrace{A^T}_{m \times n} \underbrace{S}_{n \times n} \underbrace{y}_{n \times 1}}_{\hat{x}} \in R(A)$

From weighted least squares

$\hat{x} := \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \|y - Ax\|$

$A\hat{x} \in R(A)$

$\left.\begin{array}{l} R(A) \subset \mathbb{R}^n \\ \hat{x} \in \mathbb{R}^m \end{array}\right\}$ False

(c) C will be positive definite by not necessarily symmetric $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

v_1, v_2

$C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is similar to B (reason: C has distinct e-values $1, 2$, therefore it is diagonalizable with $\Lambda = B$.)

$C[v_1\ v_2] = [v_1\ v_2]\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$

$C = [v_1\ v_2]\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}[v_1\ v_2]^{-1}$

5. (Vector spaces and vector fields)

Circle True or False as appropriate *on the answer sheet* for the following statements:

$\boxed{F}$ (a) Let $A, B \subset \mathcal{X}$ be two subsets of vector space $(\mathcal{X}, \mathbb{R})$. Then $A^{\perp} = \text{span}(B)^{\perp} \implies A = B$

False $A = \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ $B = \left\{\begin{bmatrix}2\\2\end{bmatrix}\right\}$

clearly $A \neq B$

but we have $A^{\perp} = \text{span}(B)^{\perp}$

$\boxed{T}$ (b) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a valid basis for both $(\mathbb{R}^2, \mathbb{R})$ and $(\mathbb{C}^2, \mathbb{C})$.

$\boxed{F}$ (c) Let $(\mathcal{X}, \mathcal{F})$ be a vector space and $\overset{=0}{v^1}, \overset{=0}{v^2}, \overset{=0}{v^3}, \overset{\neq 0}{v^4} \in \mathcal{X}$ be vectors in the vector space. Let $S_1 = \text{span}\{v^1, v^2, v^3\}$ and $S_2 = \text{span}\{v^2, v^3, v^4\}$. If $v^4 \notin S_1$, then $\dim(S_1) = \dim(S_2)$.

$\boxed{F}$ (d) Consider vector space $(\mathbb{R}^n, \mathbb{R})$, $n > 2$. The subset $S = \{x \in \mathbb{R}^n | \overset{0}{x_1} = \overset{1}{1}, x_2 = 0\}$ is a valid subspace (where x_i is the i^{th} element in vector x).

not closed under addition, does not contain zero, $0 \in \mathbb{R}^n$

4.d) $A = P^{-1} B P$ $\quad v^1, \dots, v^n$ e-vectors of A

$\{\underbrace{P v^1}_{\tilde{v}^i}, \dots, P v^n\}$ are e-vectors of B.

$A v^i = \lambda_i v^i$

$P^{-1} B P v^i = P^{-1} B \tilde{v}^i = \lambda_i v^i$

$B \tilde{v}^i = \lambda_i \underbrace{P v^i}_{\tilde{v}^i}$

$B \tilde{v}^i = \lambda_i \tilde{v}^i \longrightarrow$ $P v^i$'s are e-vectors of B.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.

6. (15 points) (Place your answers in the **boxes** and show your work below.)

Consider the vector spaces $(\mathbb{P}^3, \mathbb{R})$ and $(\mathbb{P}^1, \mathbb{R})$, where $\mathbb{P}^n$ represents the set of polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree up to (and including) n. Note that $\mathbb{P}^n$ is $n+1$ dimensional. For the first vector space, you are given the *Bernstein* basis $B = \{(1-t)^3, 3t(1-t)^2, 3t^2(1-t), t^3\}$ (you do **not** need to prove it is a basis, just take it as fact). For the second vector space, you are given the standard basis $E = \{1, t\}$.
Define the linear operator $\mathcal{L} : \mathbb{P}^3 \to \mathbb{P}^1$ by $(\mathcal{L}(p))(t) = \frac{d^2}{dt^2} p(t), \forall p \in \mathbb{P}^3$.

(a) (9 points) Find the matrix representation A of $\mathcal{L} : \mathbb{P}^3 \to \mathbb{P}^1$ with respect to the given bases.

(a) $A =$

(b) (6 points) Consider the basis $V = \{1, t, t^2/2!, t^3/3!\}$ for $(\mathbb{P}^3, \mathbb{R})$. Let P denote the change of basis matrix from B to V and let $\bar{P}$ denote the change of basis matrix from V to B. **Compute either P or $\bar{P}$ (choose ONE).** You must clearly indicate which change of basis matrix you are computing. If you mislabel the matrix you will lose half the points.

(b) $P = \begin{bmatrix} \quad \end{bmatrix}$ **OR** $\bar{P} = \begin{bmatrix} \quad \end{bmatrix}$ (only one is required)

Show your steps and reasoning below. No reasoning $\implies$ no points. I do not need to see how you did every algebra calculation, but I do need to see all the steps in the derivation.

Problem 6:

(a) It helps to first expand the basis vectors in $B = \{1 - 3t + 3t^2 - t^3, 0 + 3t - 6t^2 + 3t^3, 0 + 0t + 3t^2 - 3t^3, 0 + 0t + 0t^2 + 1t^3\}$.

Next, define the columns of A using the definition $A_i = [\mathcal{L}(b^i)]_E$:

$$\begin{aligned}
A_1 &= [\mathcal{L}(b^1)]_E = [\frac{d^2}{dt^2}(1 - 3t + 3t^2 - t^3)]_E = [(6 - 6t)]_E = \begin{bmatrix} 6 \\ -6 \end{bmatrix} \\
A_2 &= [\mathcal{L}(b^2)]_E = [\frac{d^2}{dt^2}(0 + 3t - 6t^2 + 3t^3)]_E = [(-12 + 18t)]_E = \begin{bmatrix} -12 \\ 18 \end{bmatrix} \\
A_3 &= [\mathcal{L}(b^3)]_E = [\frac{d^2}{dt^2}(0 + 0t + 3t^2 - 3t^3)]_E = [(6 - 18t)]_E = \begin{bmatrix} 6 \\ -18 \end{bmatrix} \\
A_4 &= [\mathcal{L}(b^4)]_E = [\frac{d^2}{dt^2}(0 + 0t + 0t^2 + 1t^3)]_E = [(0 + 6t)]_E = \begin{bmatrix} 0 \\ 6 \end{bmatrix}
\end{aligned}$$

Hence, $A = \begin{bmatrix} 6 & -12 & 6 & 0 \\ -6 & 18 & -18 & 6 \end{bmatrix}$

(b) Again, stick to the definition of the columns of P:

$$\begin{aligned}
P_1 &= [b^1]_V = [1 - 3t + 3t^2 - t^3]_V = \begin{bmatrix} 1 \\ -3 \\ 6 \\ -6 \end{bmatrix} \\
P_2 &= [b^2]_V = [0 + 3t - 6t^2 + 3t^3]_V = \begin{bmatrix} 0 \\ 3 \\ -12 \\ 18 \end{bmatrix} \\
P_3 &= [b^3]_V = [0 + 0t + 3t^2 - 3t^3]_V = \begin{bmatrix} 0 \\ 0 \\ 6 \\ -18 \end{bmatrix} \\
P_4 &= [b^4]_V = [0 + 0t + 0t^2 + 1t^3]_V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}
\end{aligned}$$

Hence,

$$\begin{aligned}
P &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 6 & -12 & 6 & 0 \\ -6 & 18 & -18 & 6 \end{bmatrix} \\
\bar{P} = P^{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/6 & 0 \\ 1 & 1 & 1/2 & 1/6 \end{bmatrix}.
\end{aligned}$$

7. (20 points) Consider the inner product space $(\mathbb{R}^3, \mathbb{R}, < \cdot, \cdot >)$ where $< x, y >= x^T S y$ for $x, y \in \mathbb{R}^3$ and $S = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Define $M = \{x \in \mathbb{R}^3 \,|\, Cx = 0\} \subset \mathbb{R}^3$ for $C = [a, b, 0]$ and non-zero $a, b \in \mathbb{R}$.

(a) (10 points) Find an orthogonal basis for M. It is not necessary to normalize.

(a)

(b) (10 points) Now let $a = 1$, $b = -1$. Find the orthogonal projection of $x = [1, 2, -1]^\top$ onto M.

(b) $P(x) =$

Show your steps and reasoning below. No reasoning $\implies$ no points. I do not need to see how you did every algebra calculation, but I do need to see all the steps in the derivation. I suggest you check your calculations using MATLAB by first entering `a = sym('a','real'), b = sym('b','real')`. You might find the `simplify` function helpful. That said, this problem is certainly doable without MATLAB.

Problem 7:

(a) As we saw in the homework, M is a subspace. Note that $Cx = ax_1 + bx_2 = 0$ implies $ax_1 = -bx_2$. By inspection, a basis can be chosen as $\{v^1, v^2\} = \left\{ \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, i.e., $M = \text{span}\{v^1, v^2\}$. Note that v^1 was chosen such that $Cv^1 = 0$, and $Cv^2 = 0$ trivially.

These vectors may *look* orthogonal, but they are not with respect to the defined (weighted) inner product, i.e., $< v^1, v^2 > \neq 0$! We must use Gram-Schmidt to find an orthogonal basis $\{u^1, u^2\}$:

$$\begin{aligned} u^1 &= v^1 \quad (1) \\ u^2 &= v^2 - \frac{< v^2, u^1 >}{||u^1||^2} u^1 \\ &= v^2 - \frac{{v^2}^T S u^1}{{u^1}^T S u^1} u^1. \end{aligned}$$

If $v^1 = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$u^1 = \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix} \quad (2)$$

$$\begin{aligned} u^2 &= v^2 - \frac{b}{a^2 + 2b^2} u^1 \\ &= \begin{bmatrix} \frac{-b^2}{a^2+2b^2} \\ \frac{ab}{a^2+2b^2} \\ 1 \end{bmatrix}. \quad (3) \end{aligned}$$

If you reverse the order of the basis $\{v^1, v^2\}$ in G-S, then we instead get the simpler answer

$$u^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

$$u^2 = \begin{bmatrix} b \\ -a \\ -b/2 \end{bmatrix} \quad (5)$$

Again we have shown two possible orthogonal bases for M. Any scaled version of these bases is also valid.

(b) There are two possible approaches: 1) use the definition of orthogonal projection operator after normalizing basis $\{u^1, u^2\}$, or 2) go straight to the normal equations. They are the same thing. We show the second approach here:

$$\begin{aligned} G &= \begin{bmatrix} < u^1, u^1 > & < u^1, u^2 > \\ < u^1, u^2 > & < u^2, u^2 > \end{bmatrix} = \begin{bmatrix} {u^1}^T S u^1 & 0 \\ 0 & {u^2}^T S u^2 \end{bmatrix} \\ \beta &= \begin{bmatrix} < x, u^1 > \\ < x, u^2 > \end{bmatrix} = \begin{bmatrix} x^T S u^1 \\ x^T S u^2 \end{bmatrix}. \end{aligned}$$

Thus, $\alpha = G^{-1}\beta$ and then $\hat{x} = P(x) = \alpha_1 u^1 + \alpha_2 u^2$. This is the best approximation of x in M, i.e., the point in M of minimum distance from x with respect to the inner product space. You could alternatively plug in an orthonormal basis or the linearly independent basis $\{v^1, v^2\}$ into the normal equations and get the same answer $\hat{x}$.

Now let $a = 1$, $b = -1$. For the first choice of basis $\{v^1, v^2\}$ in (a), we have $u^1 = [-1, -1, 0]^T$, $u^2 = [-1/3, -1/3, 1]^T$, and

$$\begin{aligned}
G &= \begin{bmatrix} 3 & 0 \\ 0 & 5/3 \end{bmatrix} \\
\beta &= \begin{bmatrix} -3 \\ -2 \end{bmatrix} \\
\alpha &= \begin{bmatrix} -1 \\ -6/5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1.2 \end{bmatrix} \\
\hat{x} &= P(x) = \begin{bmatrix} 7/5 \\ 7/5 \\ -6/5 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 1.4 \\ -1.2 \end{bmatrix}.
\end{aligned} \tag{6}$$

If we reversed the order of the basis $\{v^1, v^2\}$ in (a), then $u^1 = [0, 0, 1]^T, u^2 = [-1, -1, 0.5]^T$, and

$$\begin{aligned}
G &= \begin{bmatrix} 2 & 0 \\ 0 & 2.5 \end{bmatrix} \\
\beta &= \begin{bmatrix} -1 \\ -3.5 \end{bmatrix} \\
\alpha &= \begin{bmatrix} -0.5 \\ -1.4 \end{bmatrix} \\
\hat{x} &= P(x) = \begin{bmatrix} 1.4 \\ 1.4 \\ -1.2 \end{bmatrix}.
\end{aligned} \tag{7}$$

We ultimately get the same answer regardless of the order of basis $\{v^1, v^2\}$.

8. (15 points) (Proof Problem) (Done in two parts so that you cannot lose too many points on each part)

(a) (5 points) Let $x, y, z \in \mathbb{Z}$ with $z \neq 0$. If both x and y are divisible by z, then is $cx + y^d$ divisible by z for all $c, d \in \mathbb{Z}$ with $d \geq 1$? If yes, give the proof; if no, give a counterexample. Be sure to justify why a calculation does or does not give an integer (e.g., "the sum of integers is an integer").

(b) (10 points) Show that $\sum_{i=0}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}$ for integers $n > 0$.

Problem 8:

(a) Yes. Since z divides x and y, there exist integers m, n such that $x = mz$ and $y = nz$. Then $cx + y^d = cmz + (nz)^d = cmz + n^d z^d = z(cm + n^d z^{d-1})$. Because integer $d \geq 1$, n^d and z^{d-1} are integers, as is cm. Hence, $(cm + n^d z^{d-1})$ is an integer, so the expression $cx + y^d$ is indeed divisible by z.

(b) Proof by Induction:

Begin by labeling $P(n)\ :\ \sum_{i=0}^{n} i(i+2) = n(n+1)(2n+7)/6$

Base case: $P(1)\ :\ LHS = 0 + 1(1+2) = 3$ and $RHS = 1(2)(9)/6 = 3$.

OK if used $P(0)$ as base case, but unnecessary given problem statement $n > 0$.

Induction step: To show: $P(k-1) \Longrightarrow P(k)$ (also fine if you showed $P(k) \Longrightarrow P(k+1)$)

Assume $P(k-1)\ :\ \sum_{i=0}^{k-1} i(i+2) = (k-1)k(2k+5)/6$ holds. Then,

$$\begin{aligned}
\sum_{i=0}^{k} i(i+2) &= \left(\sum_{i=0}^{k-1} i(i+2)\right) + k(k+2) \\
&= \frac{(k-1)k(2k+5)}{6} + k(k+2) \\
&= \frac{(k-1)k(2k+5)}{6} + \frac{6k(k+2)}{6} \\
&= \frac{k[(k-1)(2k+5) + 6(k+2)]}{6} \\
&= \frac{k[2k^2 + 5k - 2k - 5 + 6k + 12]}{6} \\
&= \frac{k[2k^2 + 9k + 7]}{6} \\
&= \frac{k(k+1)(2k+7)}{6}
\end{aligned}$$

Hence, $P(k)$ holds and the induction step is proven. This concludes the proof for $P(n)$.