Introduction to proof techniques

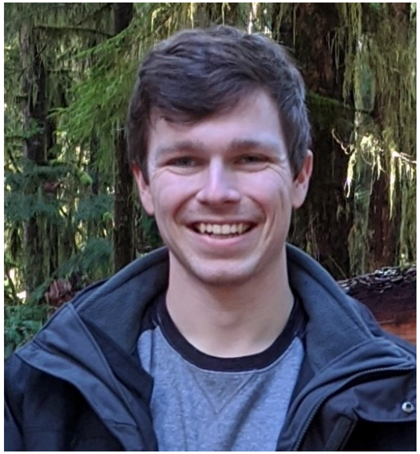
ROB 501

Course administration

- Instructor: Necmiye Ozay

  (Pronunciation: n-edge-me-ye o-zay)
- Office hours: MW, 10:30-11:30am, in front of CHRYS 220
- Lectures: MW, 9:00-10:20am, CHRYS 220 (recordings available after lecture)

Course administration

- Discussion: W 6:30-7:30pm, Th 4:30-5:30, FRB 1060
- GSIs: Andrew Wintenberg and Ishank Juneja
  - office hours: mixture of in-person and zoom
  - times: fill in the poll to choose best office hour times
- Course website: Canvas (we will also use Piazza!)

TODOs

- Resources on Canvas
- HW #1 will be posted today and due on Wednesday (9/7) 3pm
- You are expected to submit HW #1 even if you register late, etc.
- COVID policies:
  - Follow the guidelines at https://campusblueprint.umich.edu/

Course administration

- 15% Homework (~10 homework sets)
  - Each problem will be graded over 3pnts
  - Late HW policy: two grace periods of two days each (use it if you are sick, busy, etc.)
  - Two lowest hw scores will be dropped while computing your hw average
  - Mostly posted on ~~Monday~~ Wednesday and due on the following ~~Monday~~ Wednesday
- Exams (dates TBA soon):
  - One midterm exam (evening) and a final exam (likely take-home over a period of 24-hours)
  - 40%-45%: Lower exam score accounts for 40% of your grade and the other accounts for 45%

Please read the syllabus carefully for class policies!!!

Topics (and tentative schedule)

- Proof techniques (2 Lectures)
- Abstract linear algebra fundamentals (5 Lectures)
- Least squares problems, projection theorem,and the normal equations (3-4Lectures)
- Some advanced properties of matrices (2 to 3 Lectures)
- Filtering I: (2 Lectures)
- Review of random variables and joint probability distributions (1 to 2 Lectures)
- Filtering II: Kalman filter and extensions (2 to 3 Lectures)
- Introduction to real analysis in Rn (4 to 6 Lectures)
- Convex sets and functions (1 Lecture)
- Users tour of optimization (1 to 2 Lectures)

Warnings about the course (It is assumed that you have read this statement and accept it): We do lots of proofs and no realistic examples. This is a theory course on mathematical methods. If you are seeking practical knowledge about robots or mechanical systems, this is not your course. We cover linear algebra, and thus if you have had EECS 560 = AERO 550 = ME 564 (Linear Systems), you will find that part of the material repetitive and boring (about 2.5 weeks of material). This cannot be helped because the course is designed for first-year students who would not have had the Linear Systems course. Besides this, I think it is a great course! Your math skills will increase greatly, which is what the course is all about.

Notation:

$\mathbb{Z}_0 = \{0, 1, 2, \ldots\}$

$= \mathbb{Z}_+$

$\mathbb{N}$ = $\mathbb{N} = \{1, 2, 3, \cdots\}$ Natural numbers or counting numbers

$\mathbb{Z}$ $\mathbb{Z} = \mathcal{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ Integers or whole numbers

$\mathbb{Q}$ $\mathbb{Q} = \left\{\frac{m}{q} | m, q \in \mathbb{Z}, q \neq 0, \text{no common factors (reduce all fractions)}\right\}$ Rational numbers

$\mathbb{R}$ $\mathbb{R}$ = Real numbers

$\mathbb{C}$ $\mathbb{C} = \{\alpha + j\beta \mid \alpha, \beta \in \mathbb{R}, j^2 = -1\}$ Complex numbers

$\forall$ $\forall$ means "for every", "for all", "for each".

$\exists$ $\exists$ means "for some", "there exist(s)", "there is/are", "for at least one".

$\in$ means "element of" as in "$x \in A$" (x is an element of the set A)

$\in$

Ex 1: English: Every non-~~zero~~ real number has a multiplicative inverse.

Math: $\forall x \in \mathbb{R}\ x \neq 0, \exists y \in \mathbb{R}$ $\underbrace{\text{s.t.}}_{\text{such that}}$ $xy = 1$.

*Important: The choice of y depends on x. (x comes before y in the statement)

Ex 2: English: Every real number x can be arbitrarily closely approximated by a rational number.

Math:

Option 1: $\forall x \in \mathbb{R}\ \forall \varepsilon \in \mathbb{R}, \varepsilon > 0, \quad \exists q \in \mathbb{Q}$ s.t. $|x-q| < \varepsilon$.

depends on x and ε (n)

Equivalent statement (option 2):
$\forall x \in \mathbb{R}$ and $\forall n \in \mathbb{N}, \exists q \in \mathbb{Q}$ s.t. $|x-q| < \frac{1}{n}$

acts as epsilon in option 1.

$x = \pi \quad n = 10 \Rightarrow$ one possibility for q is $q = 3.1$

$n = 100 \Rightarrow$ one possibility for q is $q = 3.14$

Logic Notation:

$\sim$ means "not". In books, and some of our handouts, you see $\neg$.

$\neg$: negation (not)

$\wedge$: and

$\vee$: or

$p \Rightarrow q$ means "if p is true, then q is true.". (implication)

$\boxed{p \iff q}$ means "p is true if and only if q is true".

$p \iff q$ is logically equivalent to:

(a) $p \Rightarrow q$ and

(b) $q \Rightarrow p$.

$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

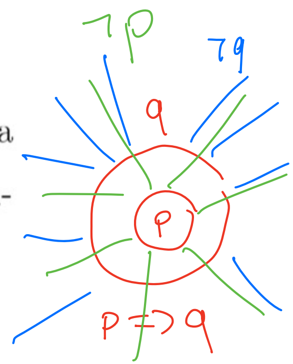
The contrapositive of $p \Rightarrow q$ is $(\sim q) \Rightarrow (\sim p)$ (logically equivalent). $(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$

The converse of $p \Rightarrow q$ is $q \Rightarrow p$.

Relation: $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$

However, in general, $(p \Rightarrow q)$ DOES NOT IMPLY $(q \Rightarrow p)$, and vice-versa

$\square$ = Q.E.D. (Latin:"quod erat demonstrandum" = "thus it was demonstrated")



$\Rightarrow$ we can read this as "p implies q".

- $(p \Rightarrow q) \equiv (\neg p \vee q)$ (1)
- $(\neg q \Rightarrow \neg p) \overset{\text{by (1)}}{\equiv} (q \vee \neg p) \equiv (\neg p \vee q) \overset{\text{by (1)}}{\equiv} (p \Rightarrow q)$

commutativity of $\vee$

Review of proof techniques:

Direct proof: We derive a result by applying the rules of logic to the given assumptions, definitions, and known theorems.

Remark: In HW, "show" $\equiv$ prove

Ex:

Def: An integer n is even if $n = 2k$ for some integer k, and odd otherwise.

Remark: (In definitions, "if" = "if and only if" otherwise they are distinct.)

Proposition: The sum of two odd integers is even.

Proof [Direct]: Let a and b be two odd integers. Hence, $\exists$ two integers k_1 and k_2 s.t. $a = 2k_1 + 1$, $b = 2k_2 + 1$.

Then $a + b = 2k_1 + 1 + 2k_2 + 1 = 2\overbrace{\underbrace{(k_1 + k_2 + 1)}_{\in \mathbb{Z}}}^{=k \in \mathbb{Z}}$

$\Rightarrow a + b$ is even. $\square$

↳ QED (end of proof)

Define $k := k_1 + k_2 + 1$, since integers are closed under summation and $k_1, k_2, 1 \in \mathbb{Z}$, then $k \in \mathbb{Z}$. So, $a + b = 2k$, $k \in \mathbb{Z} \Rightarrow a + b$ is even.

Proof by contrapositive: To establish $p \Rightarrow q$, we show instead $\neg q \Rightarrow \neg p$.

Ex: Proposition: Let n be an integer. If n^2 is even, then n is even.

Proof: p: "n^2 is even"
q: "n is even" } Want to show $p \Rightarrow q$.

$\neg p$: "n^2 is odd"
$\neg q$: "n is odd" } We will show $\underset{\neg q}{\text{"}n \text{ is odd"}} \Rightarrow \underset{\neg p}{\text{"}n^2 \text{ is odd"}}$ (which by contrapositive, equivalent to showing $p \Rightarrow q$)

$\neg q$: "n is odd" $\overset{\text{def.}}{\Longleftrightarrow} (\exists k \in \mathbb{Z} \text{ s.t. } n = 2k+1)$

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\underbrace{(2k^2+2k)}_{\in \mathbb{Z}} + 1$$

$\Rightarrow$ "n^2 is odd": $\neg p$.

This shows $\neg q \Rightarrow \neg p \equiv p \Rightarrow q$ $\square$.

Proof by Exhaustion: Reduce the proof to a finite # of cases and then check every one of them.

Ex: Four color problem (Google it!)

Office Hours

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \Leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

$\neg$, $\wedge$ required operators

$\vee$

$\Rightarrow$: $(p \Rightarrow q) \equiv (\neg p \vee q)$

$\Leftrightarrow$: $(p \Leftrightarrow q) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

XOR

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1