

i) a) given $\alpha_0 \in \text{Span}\{y_1, y_2 - y_p\}$ & $\langle m_0, y_i \rangle = c_i$
 $\forall i \in P$

let $\beta \cdot \alpha_1, \{\alpha_i\} \in \text{Span}\{y_1 - y_p\}$

$$\{ \langle \alpha_1, y_i \rangle = c_i \} \quad \{ \langle \alpha_2, y_i \rangle = c_i \}$$

as $\alpha_1 \in \text{Span}\{y_1, y_2 - y_p\}$

$$\alpha_1 = \alpha_1 y_1 + \alpha_2 y_2 + \dots - \alpha_p y_p$$

similarly $\alpha_2 = \beta_1 y_1 + \beta_2 y_2 + \dots - \beta_p y_p$

$$\langle \alpha_1, y_i \rangle = c_i = \alpha_1 \langle y_1, y_i \rangle + \alpha_2 \langle y_2, y_i \rangle + \dots - \alpha_p \langle y_p, y_i \rangle$$

$$\langle \alpha_1, y_p \rangle = c_p = \alpha_1 \langle y_1, y_p \rangle + \alpha_2 \langle y_2, y_p \rangle + \dots - \alpha_p \langle y_p, y_p \rangle$$

$$\Rightarrow \begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 - y_p \rangle \\ \langle y_2, y_1 \rangle & \langle y_2, y_2 - y_p \rangle \\ \vdots & \vdots \\ \langle y_p, y_1 \rangle & \langle y_p, y_2 - y_p \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

$$A = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \quad \text{--- (1)}$$

$\therefore \alpha_0 = \alpha_1 + \alpha_2 + \dots + \alpha_p$

Similarly

$$A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \rightarrow \textcircled{2}$$

$$\text{eq } \textcircled{1} - \textcircled{2}$$

$$A \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \vdots \\ \alpha_p - \beta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

here A is symm & invertible matrx

so

the solⁿ for $A\alpha = 0$ is only

$$\alpha = 0$$

$$\textcircled{2} \quad \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \\ \vdots \\ \alpha_p - \beta_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \underbrace{\alpha_i = \beta_i}_{\textcircled{1}} \Rightarrow \underbrace{\alpha_1 = \alpha_2}_{\textcircled{1}}$$

\Rightarrow Only unique $\alpha_0 \in \text{Span}\{y_1 - y_2\}$ s.t $\alpha_0 \neq 0$

b) $M = (\text{Span}\{y_1, y_2 - y_p\})^\perp$

if $\alpha \in V$

then

$$\alpha = \alpha_0 + m$$

$$\Rightarrow (\alpha - \alpha_0) \in M$$

$$\Rightarrow (\alpha - \alpha_0) \in (\text{Span}\{y_1, y_2 - y_p\})^\perp$$

$$\Rightarrow (\alpha - \alpha_0) \perp \text{Span}\{y_1, -y_p\}$$

if $(\alpha - \alpha_0) \perp \text{Span}\{y_1, y_2 - y_p\}$

$$\Rightarrow (\alpha - \alpha_0) \in (\text{Span}\{y_1, -y_p\})^\perp$$

$$\Rightarrow (\alpha - \alpha_0) = m$$

$$\Rightarrow \alpha = \alpha_0 + m$$

$$\text{where } V = \alpha_0 + M$$

$$\Rightarrow \overline{\alpha_0 + V}$$

$$\therefore (m \in V) \Leftrightarrow (\alpha - \alpha_0) \perp \text{Span}\{y_1, -y_p\}$$

$$\text{where } V = \alpha_0 + M$$

1) c) $\|v^*\| = \inf_{v \in V} \|v\|$

From claim (b)

$$\theta = m_0 + m$$

{ also $\theta = m_0 - m$ from (b) } $m \in M$

$$\Rightarrow \inf \|v\| = \inf \|m_0 + m\| = \inf \|m_0 - m\|$$

bottom of page (not)

$$= d(m_0, M)$$

from projection theorem

unique minimiser $n^* \in M$

$$\|m_0 - n^*\| = d(m_0, M)$$

$$\text{as } m^* \in M \Rightarrow v^* = m_0 - n^*$$

$\Rightarrow m - n^*$ is min norm $\perp M$

$$\therefore v^* = m_0 - n^* \perp M$$

2) using lemma 3

we can say that $v^* = \underset{v \in V}{\operatorname{argmin}} \|v\|$

v^* is unique & $v^* \perp M$

& $v^* \perp M \Rightarrow v^* \in V$

$\Rightarrow v^* \in \text{span}\{y_1, y_2 - y_p\}$

$$\Rightarrow v^* = \sum_{i=1}^p b_i y_i$$

$\sum_i b_i \langle v^*, y_i \rangle = c_i$

2) Similar to expansion lemma 1

$$A \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix}$$

$$A = \begin{bmatrix} \langle y, y_1 \rangle & \langle y, y_2 \rangle & \cdots & \langle y, y_p \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle & \cdots & \langle y_2, y_p \rangle \\ \vdots & \vdots & \ddots & \langle y_p, y_p \rangle \\ \langle y, y_p \rangle & & & \end{bmatrix}$$

3) a) Given

$$\mu = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \quad \Sigma = \text{cov} \left(\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}^T \right)$$
$$= \begin{bmatrix} P & PC^T \\ CP & CPC^T + Q \end{bmatrix}$$

from
conditional dist

$$\mu_{1|2} = \mu_1 + \Sigma_{12} \cdot \Sigma_{22}^{-1} (x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

i) $\mu_{1|2} = \bar{x} + PC^T (CPC^T + Q)^{-1} CP (y - \bar{y})$

$$\Sigma_{1|2} = P - PC^T (CPC^T + Q)^{-1} CP$$

3) b)

For matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Schur complement of D in M is

$$M|D = A - BD^{-1}C$$

$$= P - (Pc^T)(CPG^T + Q)^{-1}eP$$

which is same as $\Sigma_{1/2}$

c)

let $f = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4$

where $f_1 = 1$ $f_2 = t$

$f_3 = t^2$ $f_4 = \sin(\pi t)$

$\Rightarrow \arg \min \|f\| = \min \int (\beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3 + \beta_4 f_4)^2$

$= \min \left([\beta_1, \beta_2, \beta_3, \beta_4] G \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \right)$

$$G = \begin{bmatrix} \langle f_1 f_1 \rangle & \langle f_1 f_2 \rangle & \langle f_1 f_3 \rangle & \langle f_1 f_4 \rangle \\ \langle f_2 f_1 \rangle & \langle f_2 f_2 \rangle & \langle f_2 f_3 \rangle & \langle f_2 f_4 \rangle \\ \langle f_3 f_1 \rangle & \langle f_3 f_2 \rangle & \langle f_3 f_3 \rangle & \langle f_3 f_4 \rangle \\ \langle f_4 f_1 \rangle & \langle f_4 f_2 \rangle & \langle f_4 f_3 \rangle & \langle f_4 f_4 \rangle \end{bmatrix}$$

$$\langle f, t \rangle = 0$$

$$\Rightarrow A \beta = C$$

where

$$A = \begin{bmatrix} \langle f_1 f_1 \rangle & \langle f_1 f_2 \rangle & \langle f_1 f_3 \rangle & \langle f_1 f_4 \rangle \\ \langle f_2 f_1 \rangle & \langle f_2 f_2 \rangle & \langle f_2 f_3 \rangle & \langle f_2 f_4 \rangle \\ \langle f_3 f_1 \rangle & \langle f_3 f_2 \rangle & \langle f_3 f_3 \rangle & \langle f_3 f_4 \rangle \\ \langle f_4 f_1 \rangle & \langle f_4 f_2 \rangle & \langle f_4 f_3 \rangle & \langle f_4 f_4 \rangle \end{bmatrix}$$

$$C = 2$$

$\Rightarrow \beta$ is given by

$$G^T A^T (A G^T A^T)^{-1} C$$

From computing in MATLAB

$$\therefore \beta = \begin{bmatrix} 0 \\ 0.75 \\ 0 \\ 0 \end{bmatrix} \Rightarrow f = 0.75t$$

b) $A = \begin{bmatrix} \langle f_1 f_1 \rangle & \langle f_1 f_2 \rangle & \langle f_1 f_3 \rangle & \langle f_1 f_4 \rangle \\ \langle f_2 f_1 \rangle & \langle f_2 f_2 \rangle & \langle f_2 f_3 \rangle & \langle f_2 f_4 \rangle \\ \langle f_3 f_1 \rangle & \langle f_3 f_2 \rangle & \langle f_3 f_3 \rangle & \langle f_3 f_4 \rangle \\ \langle f_4 f_1 \rangle & \langle f_4 f_2 \rangle & \langle f_4 f_3 \rangle & \langle f_4 f_4 \rangle \end{bmatrix}$

$$C = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$$

From MATLAB

$$\beta = \begin{bmatrix} 0 \\ 1.7688 \\ 0 \\ 4.2677 \end{bmatrix} \Rightarrow f = (1.7688 \cdot t) + (4.2677 \cdot \sin \pi t)$$

5) a) given $\alpha \in \text{ann}(b)$ $\Rightarrow \alpha \perp b$

$$\Rightarrow \alpha = \alpha_{\text{null}(A)^\perp} + \alpha_{\text{null}(A)}$$

$$\Rightarrow A\alpha = b$$

$$A(\alpha_{\text{null}(A)^\perp}) = b$$

If $y \in \text{null}(A^\top)$

$$\Rightarrow y^\top \alpha = 0 \text{ where } \alpha \in \text{Null}(A)$$

$$\Rightarrow \alpha^\top y^\top y = 0$$

$$\Rightarrow y = [\bar{a}_1 \bar{a}_2 \dots \bar{a}_p] \alpha$$

$$\Rightarrow y = A^\top \alpha$$

$$\Rightarrow A(A^\top \alpha) = b$$

$$\Rightarrow \alpha = (A A^\top)^{-1} b$$

$$y = \alpha_{\text{null}(A^\top)} = A^\top \alpha = A^\top (A A^\top)^{-1} b$$

from norm as

$\alpha_{\text{null}(A)}$ & $\alpha_{\text{null}(A)^\perp}$ or for

$$\alpha_{\text{null}(A)^\perp} = 0$$

$$\therefore \hat{x} = A^T (A A^T)^{-1} b$$

b)

$$Ax = b$$

& min norm $\|x\|$

\Rightarrow nothing but

$$\min(x^T x)$$

$$\Theta x = y + \text{null}$$

$$y \in \text{null}(A)^\perp$$

$$x_{\text{null}} \in \text{null}(A)$$

$$\Rightarrow A y = b$$

& from inner product def

$$y^T Q x_{\text{null}} = 0$$

$$\Rightarrow x_{\text{null}}^T Q^T y = 0$$

$$\Rightarrow Q^T y = A^T d \quad Q^T = Q \quad \therefore Q \Delta 0$$

$$\Rightarrow y = Q^{-1} A^T d$$

\Rightarrow

$$A y = b \Rightarrow (A Q^{-1} A^T) d = b$$

$$\Rightarrow d = (A Q^{-1} A^T)^{-1} b$$

$$y = Q^{-1} A^T d$$

$$Y = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b$$

$\{ \quad \alpha_{\text{null}} = 0 \text{ for mandatory}$

non-zero norm of $A^{-1}b$

$$\rightarrow \text{for } y = Q^{-1}A^T(AQ^{-1}A^T)^{-1}b$$

$\|Q^{-1}A^T(AQ^{-1}A^T)^{-1}b\|$

$\rightarrow \text{non-zero norm}$

$\rightarrow \text{non-zero norm}$

$\rightarrow \text{do back-substitution}$

$\rightarrow \text{non-zero norm}$

$$b) Q = [v^1 | v^2]$$

$$\therefore A = QR$$

Q is gram schmidt of columns of A

$$v^1 = \frac{A_1}{\|A_1\|}$$

$$A_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \|A_1\| = \sqrt{1^2 + 3^2 + 5^2} = 5.91$$

$$v^1 = \frac{A_1}{\|A_1\|} = \begin{bmatrix} 0.169 \\ 0.5071 \\ 0.8452 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad v^1 = \begin{bmatrix} 0.169 \\ 0.5071 \\ 0.8452 \end{bmatrix}$$

$$A_2^T v^1 = 7.4374$$

$$\langle A_2, v^1 \rangle v^1 = \begin{bmatrix} 1.2571 \\ 3.7714 \\ 6.2857 \end{bmatrix}$$

$$a_2 = A_2 - \langle A_2, v^1 \rangle v^1$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1.2571 \\ 3.7714 \\ 6.2857 \end{bmatrix} = \begin{bmatrix} 0.7429 \\ 0.2286 \\ -0.2857 \end{bmatrix}$$

$$V^2 = \begin{bmatrix} 0.8971 \\ 0.2760 \\ -0.3450 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.169 & 0.8971 \\ 0.5071 & 0.2760 \\ 0.8452 & -0.3450 \end{bmatrix}$$

$$\frac{3}{4} \frac{x^3}{3} R = [R_1 \quad R_2]$$

~~$$RF = \begin{bmatrix} A_1 v_1 \\ A_2 v_1 \\ A_1 v_2 \end{bmatrix} = \begin{bmatrix} 5.916 \\ 7.4314 \\ 0.8287 \end{bmatrix}$$~~

~~$$R_2 = \begin{bmatrix} (A_2 v_1) \\ (A_2 v_2) \end{bmatrix} = \begin{bmatrix} 7.4314 \\ 0.8287 \end{bmatrix}$$~~

$$R_1 = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 v_1 \\ A_1 v_2 \end{bmatrix} = \begin{bmatrix} 5.916 \\ 0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} (A_2 v_1) \\ (A_2 v_2) \end{bmatrix} = \begin{bmatrix} 7.4314 \\ 0.8287 \end{bmatrix}$$

$$R = \begin{bmatrix} 5.916 & 7.4314 \\ 0 & 0.8287 \end{bmatrix}$$