Exam Number:________

ROB 501 Exam-II

You can pick any 36 hours between 4:30pm (ET) December 16, 2022 (Friday) and 5:00pm (ET) December 20, 2022 (Tuesday) to solve this exam.

HONOR PLEDGE: Copy (NOW) and SIGN (after the exam is completed): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

S. Rahul Kashyap

SIGNATURE

(Sign **after** the exam is completed)

Swayampakula, Rahul Kashyap

LAST NAME (PRINTED), *FIRST NAME*

RULES:

1. The exam is open book, open lecture handouts and slides, open recitation notes, open HW solutions, open internet (under the communication and usage restrictions mentioned below).
2. If you use MATLAB or any other scientific software to complete some parts of the exam. You are required to submit your script along with your solution in such case.
3. You are not allowed to communicate with anyone other than the Course instructor and the GSIs related to the exam during the entire period. If you have questions, you can post a private Piazza post for the instructors or email `necmiye@umich.edu` with GSIs on cc.
4. You are not allowed to use any online "course helper" sites like Chegg, Course Hero, and Slader, in any part of the exam. You are not allowed to post exam questions on the internet or discuss them online. You are not allowed to use chatGPT or similar large language models.
5. Please do not wait until the last minute to upload your solution to Gradescope and double-check to make sure you uploaded the correct pdf. If you run into problems with Gradescope, email your .pdf file as an attachment to Prof. Ozay as soon as practicable at `necmiye@umich.edu`.

rahulswa.

Answers for Problem 1			
Problem 1(a)	True	$\boxed{\text{False}}$	False
Problem 1(b)	$\boxed{\text{True}}$	False	True
Problem 1(c)	$\mu_Z = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\Sigma_Z = \begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix}$		
Problem 1(d)	$\mu_Y =$ 7 $\text{var}(Y) =$ 10		

Answers for Problem 2			
Problem 2(a)	$(-\infty, -2] \cup \{-1\} \cup [0.1, 5] \cup [7, \infty)$ — same as S		
Problem 2(b)	$\{-1\}$		
Problem 2(c)	$\{-2, -1, 0.1, 5, 7\}$		
Problem 2(d)	True	$\boxed{\text{False}}$	False

Answers for the True/False Part				
	(a)	(b)	(c)	(d)
Problem 3	False	True	True	True
Problem 4	False	True	True	False
Problem 5	False	True	False	True

6. (12 points) The goal of this problem is to derive Minimum Variance Estimator (MVE), but this time, the unknown x is correlated with the measurement noise ϵ. We assume x and ϵ are jointly normal.

Model: $y = Cx + \epsilon, y \in \mathbb{R}^m, x \in \mathbb{R}^n$, and $\epsilon \in \mathbb{R}^m$.

Stochastic assumptions:

$E\{x\} = 0, E\{\epsilon\} = 0$ (means are zero).

$E\left\{\begin{bmatrix} x \\ \epsilon \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix}^T\right\} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{x\epsilon} \\ \Sigma_{\epsilon x} & \Sigma_{\epsilon\epsilon} \end{bmatrix} > 0$ (covariance is positive definite).

Find the minimum variance estimate of x given y.

$\min \left(E(\|\hat{x}-x\|_2^2)\right)$ ST $\hat{x} = Ky$ $\boxed{\text{Defining MVE}}$ (Similar to lecture)

$\Rightarrow$ for each $i \in \{1, 2 \cdots n\}$ $\min \|\hat{x}_i - x_i\|^2$ $\hat{x}_i \in M$ $\downarrow$ Space formed by linear function to $y_1, y_2 \cdots y_m$

$\Rightarrow$ $\hat{x}_i - x_i \perp M$

let $\hat{x}_i = \sum_{j=1}^{m} \alpha_j y_i$ & $G\alpha = \beta$

$= \alpha^T y$

$G_{jk} = \langle y_j, y_k \rangle = E(y_j, y_k)$

$= E\left((C_j x + \epsilon_j)(C_k x + \epsilon_k)^T\right)$

$= E\left(C_j x x^T C_k^T + \epsilon_j x^T C_k^T + C_j x \epsilon_k^T + \epsilon_j \epsilon_k^T\right)$

$= C_j E(x x^T) C_k^T + E(\epsilon x^T) C_k^T + C_j E(x \epsilon_k^T)$

$+ E(\epsilon_j \epsilon_k^T)$

$= C_j \Sigma_{xx} C_k^T + \Sigma_{\epsilon x} C_k^T + C_j \Sigma_{x\epsilon} + \Sigma_{\epsilon\epsilon}$

$\Rightarrow G = C \Sigma_{xx} C^T + \Sigma_{\epsilon x} C^T + C \Sigma_{x\epsilon} + \Sigma_{\epsilon\epsilon}$

Please show your work for question 6.

Similarly $\quad \beta_j = \langle y_j, x_i \rangle = E\{y_j, x_i\}$

$$= E\{(C_j x_j + \varepsilon_j)x_i^T\}$$

$$= C_j E\{x_j x_j\} + E(\varepsilon_j x_i^T)$$

$$\Rightarrow \quad \beta = (C\Sigma_{xx} + \Sigma_{\varepsilon x})$$

$$\alpha = G^{-1}\beta$$

$$\hat{x} = \alpha^T y = \beta^T (G^{-1})^T y$$

$$\boxed{\hat{x} = (C\Sigma_{xx} + \Sigma_{\varepsilon x})^T \left((C\Sigma_{xx}C^T + \Sigma_{\varepsilon x}C^T + C\Sigma_{x\varepsilon} + \Sigma_{\varepsilon\varepsilon})^T\right)^{-1} y}$$

Please show your work for question 7 (copy/paste the scripts as needed).

7)

a)

$$x_{t+1} = Ax_t + Bu_t$$

$$A = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ b_{21} \end{bmatrix}$$

$$\begin{bmatrix} x_1^{t+1} \\ x_2^{t+1} \end{bmatrix} = \begin{bmatrix} 1 & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1^t \\ x_2^t \end{bmatrix} + \begin{bmatrix} 0 \\ b_{21} \end{bmatrix} u_t$$

$$= \begin{bmatrix} x_1^t + a_{12} x_2^t \\ a_{22} x_2^t + b_{21} u_t \end{bmatrix}$$

Subtracting $\begin{bmatrix} x_1^t \\ 0 \end{bmatrix}$ on b.s

$$\Rightarrow \begin{bmatrix} x_1^{t+1} - x_1^t \\ x_2^{t+1} \end{bmatrix} = \begin{bmatrix} a_{12} x_2^t \\ a_{22} x_2^t + b_{21} u_t \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x_1^{t+1} - x_1^t \\ x_2^{t+1} \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} x_2^t & 0 & 0 \\ 0 & x_2^t & u_t \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ b_{21} \end{bmatrix}}_{\theta}$$

$$Z = \begin{bmatrix} x_1^{t+1} - x_1^t \\ x_2^{t+1} \end{bmatrix} \qquad H = \begin{bmatrix} x_2^t & 0 & 0 \\ 0 & x_2^t & u_t \end{bmatrix}$$

7) a) cont.d

Please show your work for question 7 (copy/paste the scripts as needed).

Covariance $\operatorname{cov}(z) = E(zz^T) = \begin{pmatrix} E(pp^T) & E(x_2^{t+1}p^T) \\ E(p\, x_2^{t+1T}) & E(pp^T) \end{pmatrix}$

where $p = x_1^{t+1} - x_1^t$

as

$E(p\, x_2^{t+1T}) = 0$ because its independent of both x_1^t & x_1^{t+1}

$= E(x_1^{t+1} \cancel{x_2^{t+1T}}) - E(x_1^t \cancel{x_2^{t+1T}})$

$= 0 - 0 = 0$

$E(x_2^{t+1} x_2^{t+1}) = 0.01$ $\quad E(pp^T) = E(x_1^{t+1} x_1^{t+1T}) + E(x_1^t x_1^{tT})$
$- E(x_1^{t+1} x_1^{tT}) - E(x_1^t x_1^{t+1T})$

As given w_t are independent of each other

$E(x_1^{t+1} x_1^{tT}) = 0$

$\Rightarrow E(pp^T) = 0.02$ $\quad \Rightarrow \operatorname{cov}(\varepsilon) = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}$

mean: $\mu(p) = \begin{bmatrix} \mu(x_1^{t+1}) - \mu(x_1^t) \\ \mu(x_2^{t+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

7) c) recursive least squares in BLUE

from Batch we know $\hat{x} = (C^T Q^{-1} C)^{-1} C^T Q^{-1} Y$

$$C_n = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \qquad Q = \begin{bmatrix} \Sigma_\varepsilon & 0 & \cdots & 0 \\ \vdots & \Sigma_\varepsilon & 0 & \cdots \\ 0 & \cdots & & \Sigma_\varepsilon \end{bmatrix}$$

n times

$$M_n = \left(\begin{bmatrix} H_1^T & H_2^T & \cdots & H_{n-1}^T & H_n^T \end{bmatrix} \begin{bmatrix} \Sigma_\varepsilon^{-1} & 0 & 0 & \cdots & 0 \\ 0 & \Sigma_\varepsilon^{-1} & 0 & \cdots & \vdots \\ \vdots & & & & \Sigma_\varepsilon^{-1} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix} \right)$$

$$\left(\begin{bmatrix} H_1^T & H_2^T & \cdots & H_{n-1}^T \end{bmatrix} \begin{bmatrix} \Sigma_\varepsilon^{-1} & 0 & \cdots & 0 \\ 0 & \Sigma_\varepsilon^{-1} & \cdots & 0 \\ 0 & & & \\ \vdots & & & \Sigma_\varepsilon^{-1} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_{n-1} \end{bmatrix} + H_n^T \Sigma_\varepsilon^{-1} H_n \right)$$

$$\Rightarrow \quad M_n = M_{n-1} + H_n^T \Sigma_\varepsilon^{-1} H_n$$

$$N_n = \begin{bmatrix} H_1^T & H_2^T & \cdots & H_n^T \end{bmatrix} \begin{bmatrix} \Sigma_\varepsilon^{-1} & 0 & 0 & \cdots & 0 \\ 0 & \Sigma_\varepsilon^{-1} & 0 & \cdots & 0 \\ \vdots & & & & \Sigma_\varepsilon^{-1} \end{bmatrix}$$

$$= N_{n-1} + H_n^T \Sigma_\varepsilon^{-1}$$

$$N_n Y = N_{n-1} Y_{n-1} + H_n^T \Sigma_\varepsilon^{-1} Z_n \equiv M_{n-1} M_{n-1}^{-1} N_{n-1} Y_{n-1} + H_n^T \Sigma_\varepsilon^{-1} Z_n$$

$$\Rightarrow \quad \hat{x}_n = (M_{n-1} + H_n^T \Sigma_\varepsilon^{-1} H_n)^{-1} (M_{n-1} \hat{x}_{n-1} + H_n^T \Sigma_\varepsilon^{-1} Z_n)$$

determined. Then $\mathcal{E}\{x - x\} = 0$.

7) c) cont'd

$$\Rightarrow \hat{x}_n = \hat{x}_{n-1} + \left(M_{n-1} + H_n^T \Sigma_\varepsilon^{-1} H_n\right)^{-1} H_n^T \Sigma_\varepsilon^{-1} \left(y_n - H_n \hat{x}_{n-1}\right)$$

where

$$M_n = M_{n-1} + H_n^T \Sigma_\varepsilon^{-1} H_n$$

$$\Rightarrow \hat{\theta}_{11} = \hat{\theta}_{11} + M_{11} H_{11}^T \Sigma_\varepsilon^{-1} \left(y_{11} - H_{11} \hat{x}_{10}\right)$$

$$y_{11} = \begin{bmatrix} x_{12}(1) - x_{11}(1) \\ x_{12}(2) \end{bmatrix} \qquad \Sigma_\varepsilon = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$H_{11} = \begin{bmatrix} x_{11}(2) & 0 & 0 \\ 0 & x_{11}(2) & u_{11} \end{bmatrix}$$

$$M_{11} = M_{10} + H_{10}^T \Sigma_\varepsilon^{-1} H_{10}$$

8. (18 points) The following are three (3) short answer questions. You do not need to give a formal proof; only give a few short reasons/calculations why something is TRUE or FALSE. **Part (c) is on the next page.**

(a) **(6 Points)** Consider the following set $S = \{x \in \mathbb{R}^2 \mid ||x||_1 + ||x||_2 \leq 2\}$. Then, S is a convex set.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE: $\boxed{\text{True}}$

by convex if $x_1, x_2 \in S$ $\theta_1 x_1 + \theta_2 x_2 \in S$ where $\theta_1 + \theta_2 = 1$

$x_1 \in S \Rightarrow ||x_1||_1 + ||x_1||_2 \leq 2$ ↳ ① $x_2 \in S \Rightarrow ||x_2||_1 + ||x_2||_2 \leq 2$ ↳ ②

θ_1 ① + θ_2 ② & $\theta_1 + \theta_2 = 1$

$\Rightarrow (\theta_1 ||x_1||_1 + \theta_2 ||x_2||_1) + (\theta_1 ||x_1||_2 + \theta_2 ||x_2||_2) \leq (\theta_1 + \theta_2) 2$

WKT $||x+y||_1 \leq ||x||_1 + ||y||_1$, $||x+y||_2 \leq ||x||_2 + ||y||_2$

$\Rightarrow ||\theta_1 x_1 + \theta_2 x_2||_1 + ||\theta_1 x_1 + \theta_2 x_2||_2 \leq (\theta_1 ||x_1||_1 + \theta_2 ||x_2||_1) + (\theta_1 ||x_1||_2 + \theta_2 ||x_2||_2) \leq 2$

$\Rightarrow ||\theta_1 x_1 + \theta_2 x_2||_1 + ||\theta_1 x_1 + \theta_2 x_2||_2 \leq 2$

$\Rightarrow (\theta_1 x_1 + \theta_2 x_2) \in S$ where $\theta_1 + \theta_2 = 1$ $\Rightarrow$ $\boxed{S \text{ is convex}}$

(b) **(6 Points)** Let $(\mathcal{X}, \mathbb{R}, ||\cdot||_{\mathcal{X}})$ be a finite-dimensional normed space and let $S \subset \mathcal{X}$ be nonempty. Let (x_n) be a sequence converging to a point $x^* \in \mathcal{X}$, that is $x_n \to x^*$. If $x_n \in S^\circ$ for all $n \geq 1$, then $x^* \in S$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE: $\boxed{\text{False}}$

$x_n \to x^*$

$\Rightarrow \forall \varepsilon \ \exists N(\varepsilon) < \infty$ s.T $||x_n - x^*|| \leq \varepsilon$ - from def^n of seq

Given $x^* \in S^\circ \Rightarrow$ by above def $d(x, S^\circ) = 0$

if $x \in S^\circ$ then $d(x, S^\circ) = 0$ but can't guarantee other way

So counterexample: let $x_n = 5 - \frac{1}{n}$

& $S = [3, 5)$ $x^* = 5$ but $x^* \notin S$

(c) **(6 Points)** Let $M = AA^T + \epsilon^2 I$ where $A \in \mathbb{R}^{3\times 2}$, $\epsilon \in \mathbb{R}$, and I is the identity matrix of appropriate dimensions. Let $A = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} V^T$ be the singular value decomposition (SVD) of A. Then, the solution of the following problem;

$$X^* = \arg\min \; ||X - M||_2 \quad \text{s.t.} \; \operatorname{rank}(X) = 2$$

is such that $||X^* - M||_2 = \epsilon^2$.

Circle **T** or **F**. Give a few short reasons/calculations why this is TRUE or FALSE:

$\boxed{\text{False}}$

This may not be valid if rank(A) < 2

One counter example can be $\varepsilon = 0$ & $\sigma_1 = \sigma_2 = 0$

in that case $M = 0$ according to above eq^n

rank $(X) = 2$ & $||X^* - M|| = 0$

$\Rightarrow X^* = M$ but Rank$(M) = 0$

$\therefore$ This is false