1)

a) in interior of set ~~all interior pot.~~

its collection of all interior points of set

$\therefore \quad S^{o} = (-2,3) \cup (3,4) \cup (7,\infty)$

b) limit points of S

$= \{-2, 3, 4, 7\}$

c) $\partial S = S - \bar{S}$

$\bar{S} = S^{o} \cup \{\text{limit points of } S\}$

$\Rightarrow \partial S = \{5\}$

2)

a) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

Solving for $n=2$ case

$$\|x\|_1 = |x_1| + |x_2|$$
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$(\|x\|_1)^2 = (|x_1|)^2 + (|x_2|)^2 + 2|x_1||x_2|$$
$$= \|x\|_2 + 2|x_1||x_2|$$

$$(\|x\|_1)^2 \ge (\|x\|_2)^2 \quad \because |x_1| \,\&\, |x_2| \ge 0$$

as $\|x\|_1$ & $\|x_2\|$ are +ve

$$\|x\|_1 \ge \|x\|_2$$

$$\|x\|_1 \le \sqrt{2}\,\|x\|_2$$

S.O.B.S.

$$(|x_1| + |x_2|)^2 \le 2(|x_1|^2 + |x_2|^2)$$
$$|x_1|^2 + |x_2|^2 + 2|x_1||x_2| \le 2|x_1|^2 + 2|x_2|^2$$

$$\Rightarrow |x_1|^2 + |x_2|^2 - 2|x_1||x_2| \ge 0$$
$$(|x_1| - |x_2|)^2 \ge 0$$
which is true

$\therefore$

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{2}\,\|x\|_2$$

b) let ~~$\max\{|x_1|, |x_2|\}$~~ x_1

$$\Rightarrow \|x\|_\infty = |x_1|$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|_2^2 = x_1^2 + x_2^2 = \|x\|_\infty^2 + x_2^2$$

$$\Rightarrow \|x\|_2^2 \geq \|x\|_\infty^2$$

$$2\|x\|_\infty^2 = 2|x_1|^2$$

$$= |x_1|^2 + |x_1|^2$$

as $|x_1| \geq |x_2|$

$$\Rightarrow \; = |x_1|^2 + |x_1|^2 \geq |x_1|^2 + |x_2|^2$$

$$2\|x\|_\infty^2 \geq \|x\|_2^2$$

$$\Rightarrow \|x\|_2 \leq \sqrt{2}\,\|x\|_\infty$$

$$\Rightarrow \|x\|_\infty \leq \|x\|_2 \leq \sqrt{2}\,\|x\|_\infty$$

c) $\|x\|_1 = |x_1| + |x_2|$

as $|x_1| > |x_2|$

$\Rightarrow \quad |x_1| + |x_1| \geq |x_1| + |x_2|$

$\Rightarrow \quad 2\|x\|_\infty \geq \|x\|_1$

$\|x\|_\infty = |x_1|$

$\Rightarrow \|x\|_1 = \|x\|_\infty + |x_2|$

$\Rightarrow \|x\|_1 > \|x\|_\infty$

$\Rightarrow \|x\|_\infty \leq \|x\|_1 \leq 2\|x\|_\infty$

3) a) if $x \in B_a(x_0)$ then

$$\|x - x_0\| \leq a$$

& $x \in \tilde{B}_b(x_0)$

$$|||x - x_0||| \leq b$$

from defⁿ of equivalent norms

$$K_1 |||x - x_0||| \leq \|x - x_0\| \leq K_2 |||x - x_0|||$$

if $x \in \tilde{B}_{\frac{a}{K_2}}(x_0)$

$$\Rightarrow \quad |||x - x_0||| \leq \frac{a}{K_2}$$

$$K_2 |||x - x_0||| \leq a$$

WKT $\|x - x_0\| \leq |||x - x_0||| \cdot K_2$

$$\Rightarrow \quad \|x - x_0\| \leq K_2 |||x - x_0||| \leq a$$

$\Rightarrow$ every element that belongs to $\tilde{B}_{\frac{a}{K_2}}(x_0)$ belongs to $B_a(x_0)$

$\therefore$ ~~$B_a(x_0)$~~ But the reverse is not guarenteed

so. $\tilde{B}_{\frac{a}{K_2}}(x_0) \subset B_a(x_0)$

Similarly if $x \in B_a(x_0)$

$$\|x - x_0\| \leq a$$

$$k_1 |||x - x_0||| \leq \|x - x_0\| \leq a$$

$$\Rightarrow |||x - x_0||| \leq \frac{a}{k_1}$$

$\Rightarrow x \in \tilde{B}_{\frac{a}{k_1}}(x_0)$ & reverse is not guaranteed

$$B_a(x_0) \subset \tilde{B}_{\frac{a}{k_1}}(x_0)$$

$$\therefore \quad \tilde{B}_{\frac{a}{k_2}}(x_0) \subset B_a(x_0) \subset \tilde{B}_{\frac{a}{k_1}}(x_0)$$

3) b) if a set is open then $P = P^\circ$

$\Rightarrow P = \{x \in X \mid d(x, \sim P) > 0\}$

let d be distance induced by norm $\|\cdot\|$

& $\tilde{d}$ be distance induced by norm $|||\cdot|||$

$$\Rightarrow k_1 \tilde{d}(x,y) \leq d(x,y) \leq k_2 \tilde{d}(x,y)$$

if P is open set in norm $\|\cdot\|$
then for x let

$$d(x, \sim P) = \epsilon \quad \text{where } \epsilon > 0$$

$\Rightarrow$

$$\tilde{d}(x, \sim P) > \frac{d(x, \sim P)}{k_2} > \frac{\epsilon}{k_2}$$

$$\Rightarrow \tilde{d}(x, \sim P) > 0$$

$\Rightarrow$ P is open set in norm $|||\cdot|||$

similarly if P is open in $|||\cdot|||$

$$\Rightarrow \tilde{d}(x, \sim P) > 0 \Rightarrow \tilde{d}(x, \sim P) = \epsilon \quad \epsilon > 0$$

$\Rightarrow$ as $d(x, \sim P) > k_1 \tilde{d}(x, \sim P) \geq k_1 \epsilon$

$$\Rightarrow d(x, \sim P) > k_1 \epsilon$$

$\Rightarrow d(x, \sim p) > 0$

$\therefore$ p is open in $\|\cdot\|$ if it is open in $|||\cdot|||$

$\therefore$ p is open in $(X, \mathbb{R}, \|\cdot\|)$ if & only if open in $(X, \mathbb{R}, |||\cdot|||)$

3) c) given a cauchy sequence

$\forall \epsilon > 0 \quad \exists N(\epsilon) < \infty$ S.T $\forall n, m \geq N$

$\|x_n - x_m\| < \epsilon$

as we know from, norm def^n equi

$$k_1 |||x_n - x_m||| \leq \|x_n - x_m\|$$

$$k_1 |||x_n - x_m||| \leq \|x_n - x_m\| < \epsilon$$

$$|||x_n - x_m||| \leq \frac{\epsilon}{k_1}$$

Let $\frac{\epsilon}{k_1}$ be new Epsilon (ϵ')

$\Rightarrow \forall \epsilon' > 0$ (as $\epsilon > 0$ $\frac{\epsilon}{k_1} > 0$) $\exists N(\epsilon') < \infty$

$\forall n, m \geq N$

Such that
$$|||x_n - x_m||| \geq \epsilon'$$
$\therefore$ This is a Cauchy sequence in norm $|||\cdot|||$

Similarly if a seq is Cauchy in $|||\cdot|||$

$\Rightarrow$ $\forall\ \epsilon > 0$ $\exists\ N(\epsilon) < \infty$ s.t. $\forall\ n, m \geq N$
$$|||x_n - x_m||| \leq \epsilon$$

from eq norm def
$$\|x_n - x_m\| \leq k_2\ |||x_n - x_m|||$$
$$\|x_n - x_m\| \leq k_2\ |||x_n - x_m||| \leq k_2 \epsilon$$

$$\Rightarrow \|x_n - x_m\| \leq k_2 \epsilon$$
now $k_2\epsilon$ will be new epsilon to form Cauchy seq

$\therefore$ if a seq is Cauchy in $|||\cdot|||$ then its Cauchy in $\|\cdot\|$

$\therefore$ ~~if a~~ seq is Cauchy if and only if its Cauchy in its equivalent norms

4) if x seq is converging

$\forall N < \infty \quad \exists\, n \geq N \quad$ s.t $\quad \|y_n - y_0\| < \epsilon$

if x_n is converging to x_0

$$\|x_n - x_0\| \leq \delta$$

as f is continuous $\quad \exists\, \epsilon$ for δ such that

$$\|f(x_n) - f(x_0)\| < \epsilon$$

$$\Rightarrow \|y_n - y_0\| < \epsilon$$

$\Rightarrow$ y_n converges to y_0

$$\Rightarrow \lim_{n \to \infty} y_n = y_0$$

$$\lim_{n \to \infty} f(x_n) = f(x_0)$$

now reg discontinuity

if $f(x)$ is discontinuous at x_0
then ~~if~~ $\exists \epsilon$ for some δ

$$\|f(x) - f(x_0)\| \geq \epsilon$$

$$\|x - x_0\| \leq \delta.$$

a for a sequence to converge

$$\| y_n - y_0 \| \leq \epsilon$$

$\forall N < \infty \ \exists n \geq N$

here as $n \to \infty$ $x_n \to x_0$

let $\delta = \frac{1}{n}$

$\Rightarrow \quad \| x_n - x_0 \| \leq \frac{1}{n}$

by def^n of discontinuity

$\exists \ \epsilon$ for δ s.t

$$\| f(x_n) - f(x_0) \| \geq \epsilon$$
$$x_n - x_0 \leq \delta$$

$\Rightarrow$ let $f(x_n) = y_n$

$\Rightarrow \quad \| y_n - y_0 \| \geq \epsilon$

$\Rightarrow$ seq $y_n = f(x_n)$ doesn't converge to $f(x_0)$

6)

a) True

$\cancel{f(x)} = \lambda e^{-\lambda x}$ $\lambda = 0.001$

$f(x)$

$$P(X = x) = \int_0^x \lambda e^{-\lambda x} \cdot dx$$

for 1 year

$$P(X = 365) = \left[\frac{\lambda e^{-\lambda x}}{-\lambda}\right]_0^{365}$$

$$= 1 - e^{-0.365}$$

$$P(X = 365) = 0.31$$

$$P(X = x (x < 365)) < P(x = 365)$$

$\therefore$ failing less than 1 year will be less than 0.31

6) b) False (if $\mu \neq 0$)

as per def^n of unbiased estimator

$$E(\hat{x}) = x$$

so :- $E(\hat{x}) = E(ky)$

$= E(k(cx + \epsilon))$

$= E(kcx + k\epsilon)$

$=$ if $kc = 1$

$\Rightarrow = x + E(k\epsilon) = x + k\,E(\epsilon)$

$E(\hat{x}) = x$ if $E(\epsilon) = 0$ else non zero

$\therefore$ it cannot be guaranteed as unbiased estimator

6) c) False. if random variables are independent they are co-related but not otherway

$X_1 = X$ $\quad X_2 = \sqrt{2}\,X^2$ ~~where~~ can be this a counter example for this case