Real Analysis

ROB 501

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- Open and closed sets (wrap up)
- Sequences

Announcements

- No new problem set this week.
- No lecture on Wednesday
- No recitation this week.

$(X, \mathbb{R}, \|\cdot\|)$ Review of last lecture

Let (X, ||.||) be a normed space. Let P⊂X.

- p∈P is an **interior point of P** if there exist an ε>0 s.t. $B_\varepsilon(p)$⊂P (the open epsilon ball around p is contained in P)
- Interior of P: P^o={set of all interior points} = {x ∈X | d(x,~P)>0} P^o
- x∈X is a **closure point of P** if for all ε>0 s.t. $B_\varepsilon(x)$∩P≠∅ (the distance of x to the set P is zero)
- Closure of P: $\bar{P}$={set of all closure points} = {x ∈X | d(x,P)=0} $\bar{P}$ $\{x \in X \mid d(x, \sim P) = 0\}$

Def: 1) P is an **open set** if $P^o = P$. **Def:** 2)P is a **closed set** if $P = \bar{P}$.

In general, we have: $P^o \subset P \subset \bar{P}$.

Ex: ① Is $P = [0,1)$ a closed set? Not closed because $1 \notin P$, $d(1, P) = 0$,

$\Rightarrow 1 \in \overline{P} \Rightarrow P \neq \overline{P}$.

$\overline{P} = [0,1]$ $\quad P^{o} = (0,1)$

② Is $P = \mathbb{Q}$ (set of rational numbers) closed?

$x = \sqrt{2} \notin P$ but $d(\sqrt{2}, P) = 0 \Rightarrow \overset{P=}{\mathbb{Q}}$ is not closed

(different proof of not being closed)

Fact: $\overline{\mathbb{Q}} = \mathbb{R} \Rightarrow \overline{\mathbb{Q}} \neq \mathbb{Q} \Rightarrow \mathbb{Q}$ not closed!

(but also $\mathbb{Q}$ not open: $\mathbb{Q} \neq \mathbb{Q}^{o}$)

Theorem: P is open iff ~P is closed

<u>Theorem</u>: Let $(\mathcal{X}, \|\cdot\|)$ be a normed space, and $P \subset \mathcal{X}$ a subset. Then P is open iff $\sim P$ is closed.

- P open $\iff \sim P$ is closed.
- P closed $\iff \sim P$ is open

<u>Proof</u>: P open $\iff P^\circ = P$

$$\iff P = \{x \in \mathcal{X} \mid d(x, \sim P) > 0\}$$

$$\iff P = \{x \in \mathcal{X} \mid \exists \varepsilon > 0,\ B_\varepsilon(x) \cap \sim P = \emptyset\}$$

$$\iff P = \sim\{x \in \mathcal{X} \mid \underbrace{\forall \varepsilon > 0,\ B_\varepsilon(x) \cap \sim P \neq \emptyset}_{d(x, \sim P) = 0}\}$$

$$\iff P = \sim\{x \in \mathcal{X} \mid d(x, \sim P) = 0\}$$

$$\iff \sim P = \{x \in \mathcal{X} \mid d(x, \sim P) = 0\} \iff \sim P = \overline{(\sim P)} \iff \sim P \text{ is closed.}$$

Are there sets that are both open and closed? Yes, they are called CLOPEN sets. X is both open and closed. Ø is both open and closed (by convention).

$X = \mathbb{R}^n$

Some facts

- Arbitrary unions of open sets are open
- **Finite** intersections of open sets are open
- Arbitrary intersections of closed sets are closed
- **Finite** unions of closed sets are closed

Ex1: A countably infinite intersection of open sets that is not open.

$A_1 = (-2, 2)$

$A_2 = (-\frac{3}{2}, \frac{3}{2})$

$\forall n \geq 1$, define $a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

Consider $A_n = (-a_n, a_n) \Rightarrow$ all A_n are open

$$\bigcap_{n=1}^{\infty} A_n = \lim_{k \to \infty} \bigcap_{n=1}^{k} A_n$$

$$[-1,1] \subset (-a_n, a_n) \quad \forall n \geq 1 \Rightarrow [-1,1] \subset \bigcap_{n=1}^{\infty} A_n$$

We note that $|x| > 1$ (this defines $\sim[-1,1]$), $\exists k < \infty$

s.t. $\frac{k+1}{k} < |x| \Rightarrow x \notin A_k = (-a_k, a_k)$

$$\Rightarrow x \notin \bigcap_{n=1}^{\infty} A_n \quad \left(\sim[-1,1] \cap \bigcap_{n=1}^{\infty} A_n = \phi\right)$$

$$\therefore \underbrace{\bigcap_{n=1}^{\infty} A_n}_{\text{all open}} = \underbrace{[-1,1]}_{\text{closed}}$$

Exercise: Consider $B_n = (0, a_n)$ (defined as above), is $\bigcap_{n=1}^{\infty} B_n$ open or closed or both or neither?

Ex 2: Infinite union of closed sets that are not closed.

Ex 2a) $A_n = \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$

$A_1 = \{1\}$

$A_2 = \left[\frac{1}{2}, \frac{3}{2}\right]$

$A_3 = \left[\frac{1}{3}, \frac{5}{6}\right]$

$\vdots$

$$\underbrace{\bigcup_{n=1}^{\infty} A_n}_{\text{all closed}} = \underbrace{(0, 2)}_{\text{open}}$$

Ex 2b) Let $S \subset \mathcal{X}$ be any set.

$\bigcup_{x \in S} \{x\} = S$ ↪ anything

↓ all closed

(Singleton set $\{x\}$ for $x \in \mathbb{R}$ is closed bc. $B_\varepsilon(x) \cap \{x\} \neq \emptyset$ $\Rightarrow$ x is a closure pnt. of $\{x\}$ $\Rightarrow$ $\overline{\{x\}} = \{x\}$ $\Rightarrow$ $\{x\}$ is closed.)

Def: Boundary of a set, $\partial P = \overline{P} \cap \overline{(\sim P)}$

Exercise: Prove that $\partial P = \overline{P} \setminus P^o$.

Note: $\partial \mathcal{X} = \emptyset$. (e.g. $\mathbb{R}^n$ has no boundary $\partial \mathbb{R}^n = \emptyset$)

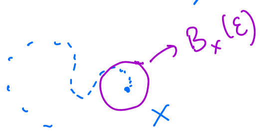
SEQUENCES:

Given $(\mathcal{X}, \|\cdot\|)$ a normed space.

Def: A set of vectors (x_n) indexed by counting numbers is called a sequence.

– sometimes we denote sequences by $\{x_n\}$ or $\{x_n\}_{n=1}^{\infty}$.

Def: A sequence converges to a point $x \in \mathcal{X}$ if $\forall \varepsilon$, $\exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N$, $\|x - x_n\| < \varepsilon$.



Notation: $x = \lim_{n \to \infty} x_n$

or $x_n \xrightarrow[n \to \infty]{} x$ or $x_n \to x$

Proposition: If $x_n \to x$ and $x_n \to y$, then $x = y$.
(limits of sequences are unique)

Proof: Idea: $\underbrace{\|x-y\|}_{\text{show this is } 0} = \|x - x_n + x_n - y\|$

$\leq \|x-x_n\| + \|y-x_n\|$

will show $\|x-x_n\| \to 0$, $\|y-x_n\| \to 0$

Let $\varepsilon > 0$ be given. Because $x_n \to x$, $\exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N$, $\|x_n - x\| < \frac{\varepsilon}{2}$

Because $x_n \to y$, $\exists M(\varepsilon) < \infty$ s.t. $\forall m \geq M$, $\|x_m - y\| < \frac{\varepsilon}{2}$.

Let $L = \max(M, N) < \infty$, then $\forall \ell \geq L$

$\|x-y\| \leq \underbrace{\|x - x_\ell\|}_{< \frac{\varepsilon}{2}} + \underbrace{\|y - x_\ell\|}_{< \frac{\varepsilon}{2}} < \varepsilon$ (we showed that $\|x-y\| < \varepsilon$ for all $\varepsilon > 0$.)

$\therefore \|x-y\| = 0 \implies x - y = 0 \implies x = y$. $\square$.

Def: Let $P \subset X$ and $x \in X$. Then, x is <u>limit point</u> of P if $\exists$ a sequence (x_n) satisfying:

a) $\forall n \geq 1, \; x_n \in P \setminus \{x\}$

b) $x_n \longrightarrow x$

<u>Proposition</u>: x is a limit point of $P \iff x \in \overline{P \setminus \{x\}}$

<u>Proof</u>:

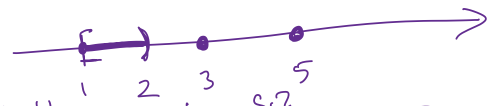
($\Longrightarrow$) If x is a limit point of P, $\exists$ a sequence (x_n) s.t. $\forall n \geq 1$, $x_n \in P \setminus \{x\}$ and $x_n \to x$.

$\therefore \forall \varepsilon > 0, \; \exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N \; \|x_n - x\| < \varepsilon$.

$\Longrightarrow d(x, P \setminus \{x\}) = 0$

$\Longrightarrow x \in \overline{P \setminus \{x\}}$

Ex: $P = [1,2) \cup \{3,5\}$



Let's consider $\{1\}$.

$\overline{P \setminus \{1\}} = [1,2] \cup \{3,5\}$

$1 \in$

↳ limit point

We can consider the sequence

$x_n = 1 + \frac{1}{2n} \in P \setminus \{1\} \; \forall n$

$x_n \longrightarrow 1$

—o—

Let consider $\{3\}$:

3 is not a limit point

$3 \notin \overline{P \setminus \{3\}} = [1,2] \cup \{5\}$

3 is called an isolation point.

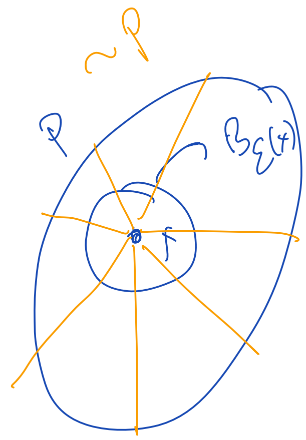
—o—

The set of limit points of P is $[1,2]$

($\Leftarrow$) Next lecture

inf dx

OFFICE HOURS



$x \in P^\circ$ if $\exists \varepsilon$ s.t. $B_\varepsilon(x) \subset P$.

$\iff \exists \varepsilon$ s.t. $B_\varepsilon(x) \cap \sim P = \emptyset$

$\iff d(x, \sim P) > \varepsilon$

$d(x, \sim P) > 0$

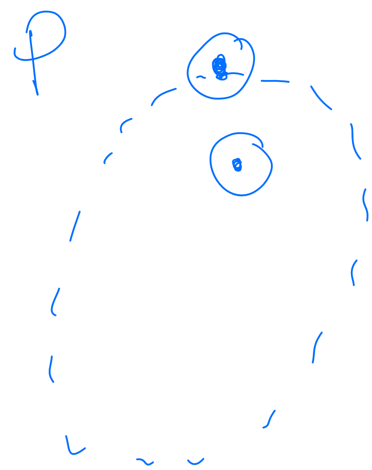
$X = \mathbb{R}$

$P = [1, 2]$ $\qquad \sim P = (-\infty, 1) \cup (2, \infty)$

$d(1, \sim P) = 0 \implies 1 \notin P^\circ$

$\sim P = \mathbb{R} \setminus P \qquad P^\circ = (1, 2)$

$x \in \overline{P}$ if $\forall\, \varepsilon > 0$, $B_\varepsilon(x) \cap P \neq \emptyset$



$x_n = 1 + \frac{2}{n}$ $\quad (x_n \to 1)$

Claim: x_n converges to 1.

$\varepsilon = 0.1$, can you find an $\underline{N}$ s.t. $\forall n \geq N$
$\implies \| 1 - x_n \| < 0.1$

can you find an N s.t. $\forall n \geq N$

$\cancel{1} - \cancel{1} + \frac{2}{n} < 0.1$

$n \geq 21$

$N(0.1) = 21$

$\varepsilon = 0.01$, $\cancel{1} - \cancel{1} + \frac{2}{n} < 0.01$

$n \geq 201$ $N(0.01) = 201$

Def: A sequence converges to a point $x \in \mathcal{X}$ if $\forall \varepsilon$, $\exists N(\varepsilon) < \infty$ s.t. $\forall n \geq N$, $\|x - x_n\| < \varepsilon$.

$y_k - \hat{y}_{k|k-1}$ $\hat{y}_{k|k-1} = \hat{C}\hat{x}_{k|k-1}$

$y = C x_k + v_k$

$z = C x_k + K + v_k$

$y = \underline{z - K}$

sensor model

augmented sensor model

$y = C x_k + v_k$

- z_k → comes from the device

$\tilde{y}_k = y_k$

$\hat{z}_{k|k-1}$

$\hat{y}_{k|k-1} = C x_{k|k-1}$