Homework - V   uniqID rahulsawa

1) a)

$$A_3 = \begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigen values calulated from characteristic eq^n

$$|A_3 - \lambda I| = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 3-\lambda & 10 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(1-\lambda) = 0$$

$$\boxed{\lambda = 1, 2, 3}$$

$$A_3 x = \lambda x$$

$\lambda = 1$

$$\begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 + 10x_2 \\ 2x_2 + 4x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \quad 3x_1 + 10x_2 = x_1 \quad \Rightarrow \quad 5x_2 = -x_1$$

$$2x_2 + 4x_3 = x_2 \quad \Rightarrow \quad 4x_3 = -x_2$$

let $x_2 = 1$

$$\Rightarrow \quad x_1 = \begin{bmatrix} -5 \\ 1 \\ -1/4 \end{bmatrix}$$

$\lambda = 2$

$$\begin{pmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 + 10x_2 \\ 2x_2 + 4x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$x_3 = 2x_3$

$\Rightarrow \boxed{x_3 = 0}$

$2x_2 + 0 = 2x_2 \Rightarrow x_2 = x_2$

$3x_1 + 10x_2 = 2x_1$

$10x_2 = -x_1$ let $x_1 = 1$

$\Rightarrow \boxed{x_2 = -1/10}$

$$\Rightarrow x_2 = \begin{bmatrix} 1 \\ -1/10 \\ 0 \end{bmatrix}$$

$\lambda = 3$

$$\begin{bmatrix} 3 & 10 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$$

$x_3 = 3x_3$

$\Rightarrow x_3 = 0$

$2x_2 + 4x_3 = 3x_2 \Rightarrow 2x_2 = 3x_2 \Rightarrow x_2 = 0$

$3x_1 + 10x_2 = 3x_1 \Rightarrow x_1 = x_1$ let $\boxed{x_1 = 1}$

$$\Rightarrow x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} -5 \\ 1 \\ -1/4 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1/10 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = \begin{bmatrix} -5a_1 + a_2 + a_3 \\ a_1 - \frac{a_2}{10} \\ -\frac{a_1}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-\frac{a_1}{4} = 0 \Rightarrow \boxed{a_1 = 0}$

$a_1 - \frac{a_2}{10} = 0 \Rightarrow \boxed{a_2 = 0}$

$-5a_1 + a_2 + a_3 = 0 \Rightarrow \boxed{a_3 = 0}$

$\therefore$ linearly independent

b)

$$A_4 = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix} \quad |A_4 - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 1 & 1 \\ 0 & 5-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(5-\lambda)(2-\lambda) = 0$$

$$\boxed{\lambda = 2 \text{ (or) } 5}$$

$A_4 x = \lambda x$

$\lambda = 2$

$$\begin{bmatrix} 5 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} \qquad 2x_3 = 2x_3$$

$5x_1 + x_2 + x_3 = 2x_1$

$5x_2 + 3x_3 = 2x_2$

$\boxed{x_3 = -x_2} \Rightarrow$ $5x_1 = 2x_1$, $\boxed{x_1 = 0}$

$$X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} 5 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ 5x_2 \\ 5x_3 \end{bmatrix}$$

$$5x_1 + x_2 + x_3 = 5x_1$$

$$5x_2 + 3x_3 = 5x_2$$

$$2x_3 = 5x_3 \Rightarrow \boxed{x_3 = 0}$$

$$5x_1 + x_2 = 5x_1 \quad \boxed{x_2 = 0}$$

$$\Rightarrow \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 5 \Rightarrow X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\{X_1\ X_2\ X_3\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

So we can't use this set as basis as vectors are not linear

even if we take $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ it has only 2 vectors

but $\dim(\mathbb{R}^3) = 3$

$\therefore$ it can't span the space

2)

if both A & B are similar then

$B = P^{-1} A P$

$\Rightarrow \det(\lambda I - B) = \det(\lambda I - P^{-1} A P)$

$I = P^{-1} P$

$= \det(\lambda P^{-1} P - P^{-1} A P)$

$= \det(P^{-1} (\lambda I) P - P^{-1} A P)$

$= \det(P^{-1} (\lambda I - A) P)$

$\det(AB) = \det(A) \cdot \det(B)$

$\det(P^{-1}) = (\det(P))^{-1}$

$\rightarrow \det(\lambda I - B) = \det(P^{-1}) \{\det(\lambda I - A)\} \det(P)$

$= \dfrac{\det(\lambda I - A)}{\det(P)} \cdot \det P$

$\boxed{\det(\lambda I - B) = \det(\lambda I - A)}$

as they have same det $\therefore$ same characteristic eq^n

$\rightarrow$ same eigen values for both

3)

let

$$P = [x_1 \mid x_2 \mid x_3]$$

where x_1, x_2, x_3 are eigen vectors ~~values~~ obtained in q1

$$P = \begin{bmatrix} -5 & 1 & 1 \\ 1 & -1/10 & 0 \\ -1/4 & 0 & 0 \end{bmatrix}$$

$$AP = A[x_1 \mid x_2 \mid x_3]$$

$$= [Ax_1 \mid Ax_2 \mid Ax_3]$$

~~for~~ As x_1, x_2, x_3 are eigen vectors $Ax = \lambda x$

$$\Rightarrow AP = [Ax_1 \mid Ax_2 \mid Ax_3]$$

$$= [\lambda_1 x_1 \mid \lambda_2 x_2 \mid \lambda_3 x_3]$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [x_1 \mid x_2 \mid x_3]$$

let

$$\lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Rightarrow AP = \lambda P$$

i) $A = P^{-1} \lambda P$

$$P = \begin{bmatrix} -5 & 1 & 1 \\ 1 & -1/10 & 0 \\ -1/4 & 0 & 0 \end{bmatrix} \qquad \det(P) = \frac{1}{40} \neq 0.$$

$\therefore$ P^{-1} exist

$\therefore$ A & λ are similar matrices where

λ is diagonal matrix of eigen values of A

& P is matrix whose collumns are eigen vectors

6)

$\langle x, y \rangle = x^T \bar{y}$

let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $\bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_n \end{bmatrix}$

$\langle x, y \rangle = x^T \bar{y} = [x_1 \; x_2 \; \cdots \; x_n] \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_n \end{bmatrix}$

$= x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots x_n \bar{y}_n = \sum_{i=1}^{n} x_i \bar{y}_i$

$\langle y, x \rangle = y^T \bar{x} = [y_1 \; y_2 \; \cdots \; y_n] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$

$= y_1 \bar{x}_1 + y_2 \bar{x}_2 + \cdots y_n \bar{x}_n$

$= \sum_{i=1}^{n} \bar{x}_i y_i = \overline{\left(\sum_{i=1}^{n} x_i \bar{y}_i \right)} = \overline{\langle x, y \rangle}$

∴ it follows first requirement

$\langle \alpha x_1 + \alpha x_2, y \rangle$. let $x_1 = [x_{11} \; x_{12} \; \cdots \; x_{1n}]^T$

$x_2 = [x_{21} \; x_{22} \; \cdots \; x_{2n}]^T$

$$\alpha_1 x_1 + \alpha_2 x_2 = \left(\alpha_1 x_{11} + \alpha_2 x_{21},\ \alpha_1 x_{12} + \alpha_2 x_{22} \cdots \alpha_1 x_{1n} + \alpha_2 x_{2n}\right)^T$$

$$\Rightarrow (\alpha_1 x_1 + \alpha_2 x_2)^T \bar{y}$$

$$= [\alpha_1 x_{11} + \alpha_2 x_{21},\ \alpha_1 x_{12} + \alpha_2 x_{22} \cdots \alpha_1 x_{1n} + \alpha_2 x_{2n}] \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} (\alpha_1 x_{1i} \bar{y}_i + \alpha_2 x_{2i} \bar{y}_i) = \alpha_1 \sum_{i=1}^{n} x_{1i} \bar{y}_i + \alpha_2 \sum_{i=1}^{n} x_{2i} \bar{y}_i$$

$$= \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$$

$\therefore$ it satisfies second requirement also.

$$\langle x, x \rangle = [x_1\ x_2 \cdots x_n] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$= x_1 \bar{x}_1 + x_2 \bar{x}_2 \cdots x_n \bar{x}_n = \sum_{i=1}^{n} x_i \bar{x}_i$$

$x_i \bar{x}_i$ is always greater than 0, sum of +ve numbers is greater than 0

$$= \sum_{i=1}^{n} x_i \bar{x}_i \geq 0 \Rightarrow \langle x, x \rangle \geq 0$$

if $\langle x, x \rangle = 0$

then $\sum_{i=1}^{n} x_i \bar{x}_i = 0 \Rightarrow x_i \bar{x}_i = 0 \quad \forall i = 1, 2, \ldots n$

let $x_i = a + jb$

then $(a + jb)(a - jb) = 0$

$a^2 + b^2 = 0 \Rightarrow a = 0 \; b = 0$

$\Rightarrow x_i = 0 \quad \forall \; i = 1, 2, \ldots n$

$\Rightarrow \boxed{x = 0}$

$\therefore \langle x, x \rangle \geq 0 \; \forall \; x \in F$ & $\langle x, x \rangle = 0$ if $x = 0$

$\therefore$ it satisfies all properties, so $\langle x, y \rangle = x^T \bar{y}$ is a inner product

7)

$P_0(x) = 1 \qquad P_1(x) = x$

$P_2(x) = \dfrac{(3x^2-1)}{2} \qquad P_3(x) = \dfrac{(5x^3-3x)}{2}$

$$\langle P_0, P_3 \rangle = \int_{-1}^{1} 1 \cdot \frac{(5x^3-3x)}{2} \cdot dx$$

$$= \frac{1}{2} \int_{-1}^{1} (5x^3 - 3x) \cdot dx$$

$$= \frac{1}{2} \left[\frac{5x^4}{4} - \frac{3x^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{2} [0 - 0] = 0$$

$$\langle P_1, P_2 \rangle = \int_{-1}^{1} x \frac{(3x^2-1)}{2} \cdot dx$$

$$= \frac{1}{2} \int_{-1}^{1} (3x^3 - x) \cdot dx = \frac{1}{2} \left[\frac{3x^4}{4} - \frac{x^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{2} [0 - 0] = 0$$

$\therefore \langle P_0, P_3 \rangle = 0 \; ; \; \langle P_1, P_2 \rangle = 0$

$\therefore$ The given basis is orthogonal basis of P^3

8)

a)

we need to find inverse of $(A+BCD)^{-1}$

$\Rightarrow \quad (A+BCD)X = I \quad \Rightarrow \quad X = (A+BCD)^{-1}$

lets construct this in matrix representation

$$\begin{bmatrix} A & B \\ D & -C^{-1} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$\Rightarrow \quad AX + BY = I \;-①$ multiply 2nd eqn by BC

$DX - C^{-1}Y = 0 \;-②$ then

$$AX + BY = I$$
$$BCDX - BY = 0$$
$$(A+BCD)X = I$$

$\Rightarrow$ solving the above eqn for X is equivalent to find inverse of $(A+BCD)$

so now by eqn ①

$$AX + BY = I$$
$$AX = I - BY$$
$$X = A^{-1}(I - BY)$$

substituting value of X in second

$$\Rightarrow \quad Dx - C^{-1}y = 0$$

$$\Rightarrow \quad D(A^{-1}(I - By)) - C^{-1}y = 0.$$

$$DA^{-1} - DA^{-1}By - C^{-1}y = 0$$

$$\Rightarrow \quad (C^{-1} + DA^{-1}B)\,y = DA^{-1}$$

$$\Rightarrow \quad y = (C^{-1} + DA^{-1}B)^{-1} DA^{-1}$$

$$x = A^{-1}(I - By)$$

$$= A^{-1} - A^{-1}B\left[(C^{-1} + DA^{-1}B)^{-1} DA^{-1}\right]$$

$$\boxed{x = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1} DA^{-1}}$$

$$x = (A + BCD)^{-1}$$

b)

$$A = \mathrm{diag}([0.5, 1, 1, 0.5, 1])$$

$$B = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad C = 0.25 \quad D = B^T = \begin{bmatrix} 3 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$BCD = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot (0.25) \cdot \begin{bmatrix} 3 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 9 & 0 & 6 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$A + BCD = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 9/4 & 0 & 6/4 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 0 \\ 6/4 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 2/4 & 0 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 11/4 & 0 & 6/4 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 0 \\ 6/4 & 0 & 2 & 0 & 1/2 \\ 0 & 0 & 0 & 0.5 & 0 \\ 3/4 & 0 & 2/4 & 0 & 5/4 \end{bmatrix}$$

Calculating inverse above will be tough

So will use

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}$$

BCZ A^{-1} is easy to find & as C is scalar

$(C + DA^{-1}B)^{-1}$ is also easy to find

$$A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$C^{-1} = 4$

$D = [3 \; 0 \; 2 \; 0 \; 1]$

$B = [3 \; 0 \; 2 \; 0 \; 1]^T$

$$DA^{-1}B = [3 \; 0 \; 2 \; 0 \; 1] \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= [3 \; 0 \; 2 \; 0 \; 1] \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 18 + 4 + 1 = 23$$

$$C^{-1} + DA^{-1}B = 4 + 23 = 27 = E$$

$$A^{-1}B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$DA^{-1} = \cancel{\begin{bmatrix} 3 & 0 & 2 & 0 & 1 \end{bmatrix}} \cancel{\begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}} \quad \begin{bmatrix} 3 & 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 2 & 0 & 1 \end{bmatrix}$$

$$A^{-1}B \cdot E^{-1} \cdot DA^{-1} = \frac{1}{27} \left(\begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 2 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{27} \begin{pmatrix} 36 & 0 & 12 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 2 & 0 & 1 \end{pmatrix} = F$$

$$A^{-1} - F = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 36/27 & 0 & 12/27 & 0 & 6/27 \\ 0 & 0 & 0 & 0 & 0 \\ 12/27 & 0 & 4/27 & 0 & 2/27 \\ 0 & 0 & 0 & 0 & 0 \\ 6/27 & 0 & 2/27 & 0 & 1/27 \end{bmatrix}$$

$$D - F = \begin{pmatrix} 18/27 & 0 & -12/27 & 0 & -6/27 \\ 0 & 1 & 0 & 0 & 0 \\ -12/27 & 0 & 23/27 & 0 & -2/27 \\ 0 & 0 & 0 & 2 & 0 \\ -6/27 & 0 & -2/27 & 0 & 26/27 \end{pmatrix}$$

9) $f(x) = (x^T A x)^{1/2}$

$f(x)$ is always greater than or equal to zero.

By defn of positive definite matrix

$x^T A x > 0 \quad \forall x \in \mathbb{R}^n$

& $x^T A x = 0$ if $x = 0$

$\therefore f(x) \geq 0 \quad \forall x \in \mathbb{R}^n$ & $f(x) = 0 \iff x = 0$.

Positive homogeneity: $f(\alpha x) = \|\alpha x\| = \left((\alpha x)^T A (\alpha x)\right)^{1/2}$

$= (\alpha x^T A \alpha x)^{1/2}$

$= (\alpha^2 \cdot x^T A x)^{1/2}$

$= |\alpha| (x^T A x)^{1/2}$

$f(\alpha x) = |\alpha| f(x)$

$\|\alpha x\| = |\alpha| \|x\|$

$\therefore$ it's also satisfied

Triangular inequality $\|x + y\| \leq \|x\| + \|y\|$

$f(x+y) = \left((x+y)^T A (x+y)\right)^{1/2}$

$= \left[(x^T + y^T) \cdot A \cdot (x+y)\right]^{1/2}$

$= \left[(x^T + y^T)\cdot(Ax + Ay)\right]^{1/2} = (x^TAx + y^TAx + x^TAy + y^TAx)^{1/2}$

x^TAy is an Scalar so for scalar it's transpose is itself

$\Rightarrow \quad x^TAy = (x^TAy)^T = y^TA^Tx$ is as A is symmetric

$= y^TAx$

$\Rightarrow \quad (x^TAx + y^TAy + 2y^TAx)^{1/2} = f(x+y)$

& $2y^TAx$, using cholesky factorization

$A = LL^T$

$\Rightarrow 2(y^TLL^Tx)$ let $L^Ty = u$ $\quad L^Tx = v$

$\Rightarrow \quad (u^Tv)$ & $\Rightarrow$ we know that

$(u^Tv)^2 \le (u^Tu)\cdot(v^Tv)$

By Caushy-Schwartz inequality

$\Rightarrow \quad 2(y^TLL^Tx) \le \underbrace{(u^Tu)(v^Tv)}_{\ge 0}$

$\therefore \quad 2(y^TLL^Tx) \ge 0.$

$\Rightarrow$

$$f(x) + f(y) = x^T A x + y^T A y$$

$$\cancel{f(x+y)}^2 = f(x+y)^2 = x^T A x + y^T A y + 2 y^T A x$$

$$(f(x) + f(y))^2 = x^T A x + y^T A y + 2\sqrt{(x^T A x)\cdot(y^T A y)}$$

from Causchy - Swarts we have seen

$$(y^T A x)^2 \leq (u^T u) \cdot (v^T v)$$

$$u = L^T y \qquad v = L^T x$$

$$\Rightarrow (y^T A x) \leq (y^T L L^T y)(x^T L L^T x)$$

$$(y^T A x) \leq (y^T A y) \cdot (x^T A x)$$

$$\Rightarrow (f(x+y))^2 \leq (f(x) + f(y))^2$$

$$\Rightarrow f(x) + f(y) \geq f(x+y)$$

This holds triangular inequality too

$\therefore$ $f(x) = (x^T A x)^{1/2}$ where A is symmetric +ve definite matrix is a form if ~~A~~ $\to 2A$ if A is replaced by $2A$

$f(x) = (x^T(2A)x)^{1/2}$ let $2A = B$

B is +ve definite matrix & symmetric

$\therefore$ $f(x) = (x^T B x)^{1/2}$ will be norm because there is

no change in properties of B

9) b) To check norm condⁿ

$$f_v(A) = \max_{x \in R^n} \frac{\|Ax\|_v}{\|x\|_v} = \sup_{\substack{x \in R^n \\ x \neq 0}} \frac{\|Ax\|_v}{\|x\|_v}$$

i) if $A \neq 0 \Rightarrow$ atleast one collumn will be non zero $\Rightarrow$ let it be a_j

then $$f_v(A) = \max_{\substack{x \in R \\ x \neq 0}} \frac{\|Ax\|_v}{\|x\|_v} \geq \frac{\|\cancel{Ax}\|_v}{\|x\|} = \frac{\|a_j\|_v}{\|x\|_v} \geq 0$$

$\therefore f_v(A) \geq 0$.

if $f_v(A) = 0$ then $\max_{\substack{x \in R \\ x \neq 0}} \|Ax\| = 0 \quad \because \|x\| \neq 0$.

as $\max_{\substack{x \in R \\ x \neq 0}} \|Ax\| = 0$. let $\exists$ a nonzero collumn

$$\Rightarrow \max_{\substack{x \in R \\ x \neq 0}} \|Ax\| = \|a_j x\| = 0$$

$\Rightarrow$ as $x \neq 0$ a_j should be zero.

$$\therefore \boxed{A = 0}$$

ii) $$f_v(\alpha A) = \max_{\substack{x \in R^n \\ x \neq 0}} \frac{\|\alpha Ax\|_v}{\|x\|_v} = \max_{\substack{x \in R^n \\ x \neq 0}} |\alpha| \frac{\|Ax\|_v}{\|x\|_v}$$

$$= |\alpha| f_v(A)$$

iii) $$f_v(A+B) = \max_{\substack{x \in R^n \\ x \neq 0}} \frac{\|(A+B)x\|_v}{\|x\|_v} = \max_{\substack{x \in R^n \\ x \neq 0}} \frac{\|Ax + Bx\|_v}{\|x\|_v}$$

let $Ax = y_1$ $\quad Bx = y_2$

from def of norm of vectors

$$\|y_1 + y_2\| \le \|y_1\| + \|y_2\|$$

$$\Rightarrow \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax + Bx\|_V}{\|x\|_V} \le \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \left(\frac{\|Ax\|_V}{\|x\|_V} + \frac{\|Bx\|_V}{\|x\|_V} \right)$$

$$\max(a+b) = \max(a) + \max(b) \quad \because \boxed{\begin{matrix} a \geq 0 \\ b \geq 0 \end{matrix}}$$

$$\therefore \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax + Bx\|_V}{\|x\|_V} \le \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \left(\frac{\|Ax\|_V}{\|x\|_V} \right) + \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \left(\frac{\|Bx\|_V}{\|x\|_V} \right)$$

$$f_V(A+B) \le f_V(A) + f_V(B)$$

$\therefore$ $f_V(A) = \max\limits_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \dfrac{\|Ax\|_V}{\|x\|_V}$ is a norm

$$f_v(A) = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_v}{\|x\|_v} = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \left\| A \frac{x}{\|x\|_v} \right\|_v$$

$$= \sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_v = 1}} \|A\tilde{x}\|_v .$$

let $A = [a_1 \ a_2 \ --- \ a_n]$

$$f_1(A) = \sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_v = 1}} \|A\tilde{x}\|_1 = \sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_v = 1}} \|x_1 a_1 + x_2 a_2 + \dots x_n a_n\|$$

$$\leq \sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_v = 1}} \left(\|x_1\|_1 \|a_1\| + \|x_2\| \|a_2\| + \dots \|x_n\| \|a_n\| \right)$$

This supremum is bounded by max norm of $\|a_j\|$

$$\Rightarrow \sum_i \|x_i\| \|a_i\| = \|a_j\| \sum_i \|x_i\|$$

where $\|a_j\| = \max_{i = 1 \text{ to } n} (\|a_i\|)$

$$= \sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_1 = 1}} (\underbrace{\|x_1\| + \|x_2\| -- \|x_n\|}_{1}) \|a_j\|$$

$$= \|a_j\|$$

$$\boxed{f_1(A) = \max_{i \in [1,n]} \|a_i\|}$$

where a_i is ith collumn of A

let $A = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix}$

$$f_\infty(A) = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_\infty = 1}} \left(\left\| \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix} x \right\|_\infty \right)$$

$$= \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_\infty = 1}} \begin{bmatrix} b_1^T x \\ b_2^T x \\ \vdots \\ b_n^T x \end{bmatrix} = \sup \left(\max_i \| b_i^T x \| \right)$$

let

$$= \sup \left(\max_i \| b_i^T \| . \| x \| \right)$$

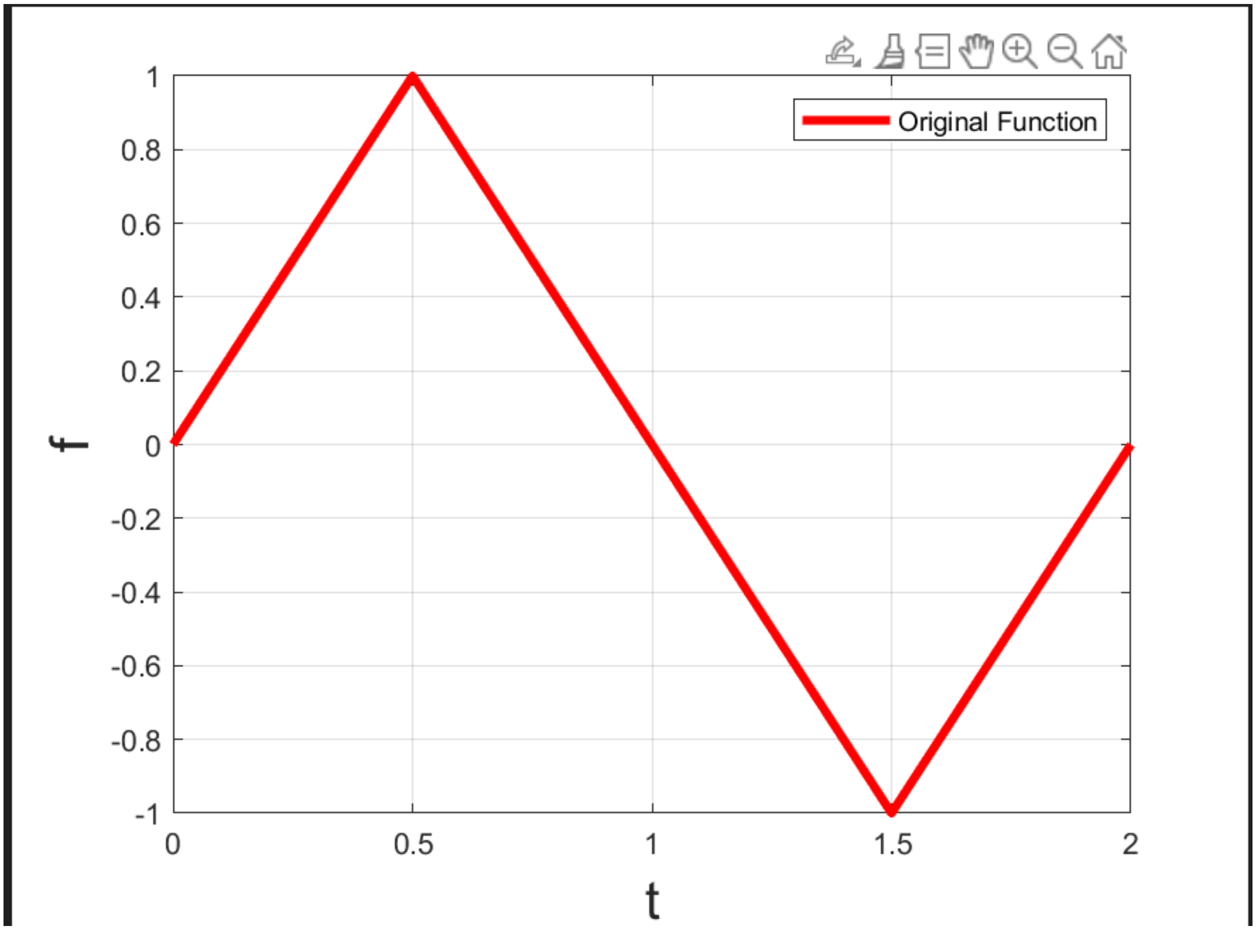
$$= \max_i \| b_i^T \| \sup (\max \| x \|)$$

$$\boxed{f_\infty(A) \;\therefore\; \max_i \| b_i^T \|}$$

where b_i is
ith row of
matrix A

Problem 4

Part a:



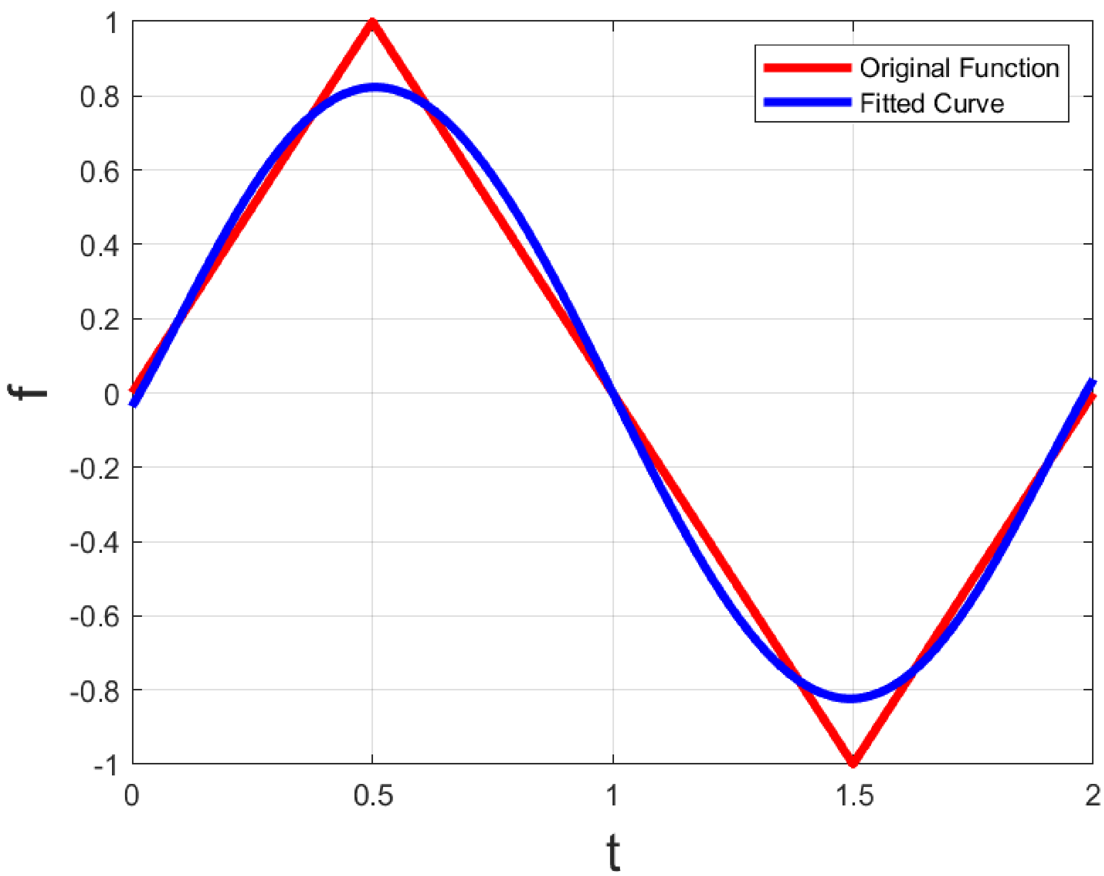
Question 4 Part b:

Coefficients of $\{1, t, t^2, t^3, t^4, t^5\}.$ are

Where ai is coefficient of t^i

a0 = -0.0370426374169580
a1 = 2.21581179193736
a2 = 2.89010976017038
a3 = -11.2271424067268
a4 = 7.69782936506529
a5 = -1.53956587302948

Plot:

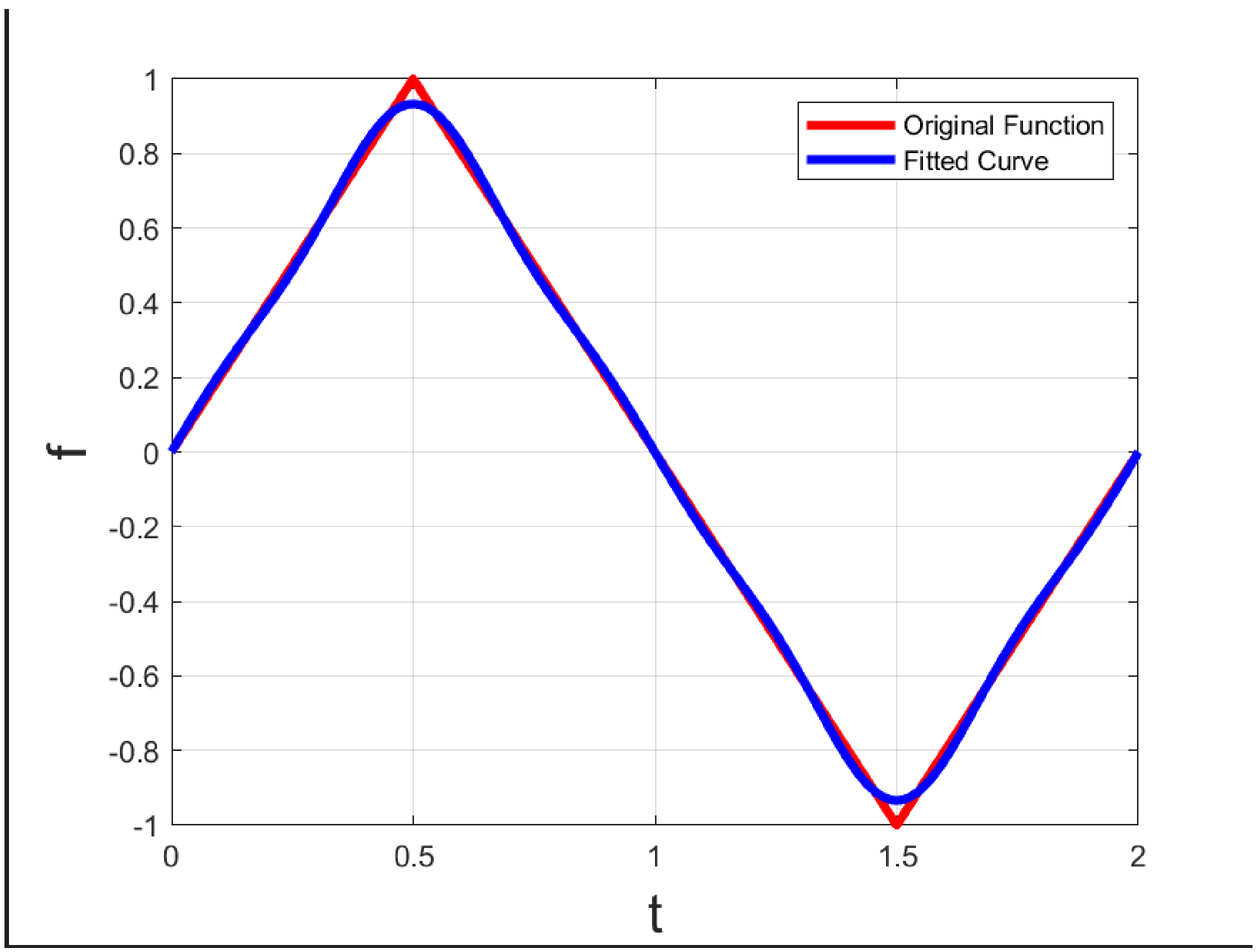


Question 4 part c:

Used function $\{\sin(\pi t), \sin(2\pi t), \cdots, \sin(5\pi t)\}.$

$$||f - \sum_{k=1}^{5} \alpha_k \sin(k\pi t)||^2$$

alpha1 = 0.810636139095366
alpha2 = 3.60590464808585e-17
alpha3 = -0.0901299706346645
alpha4 = 5.01899244788276e-17
alpha5 = 0.0324895277595177



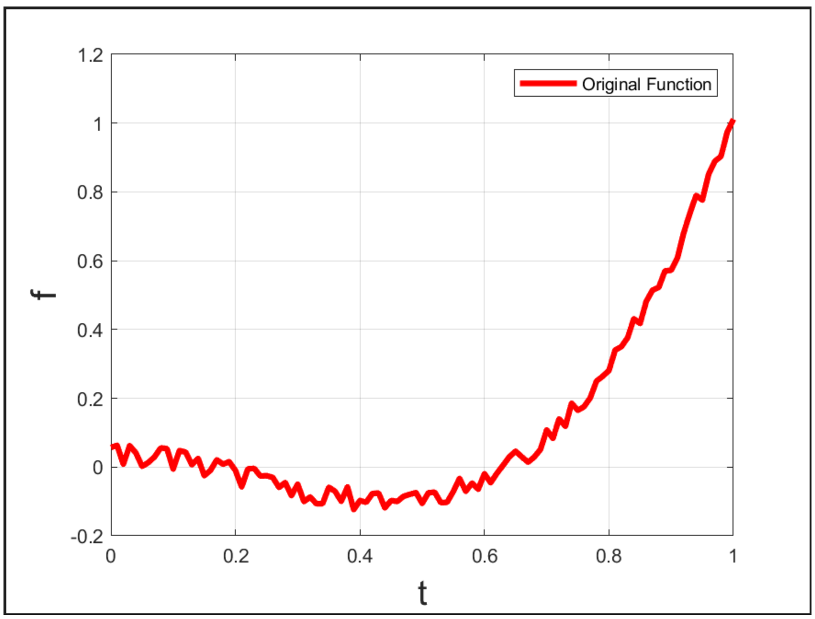
Problem 5:

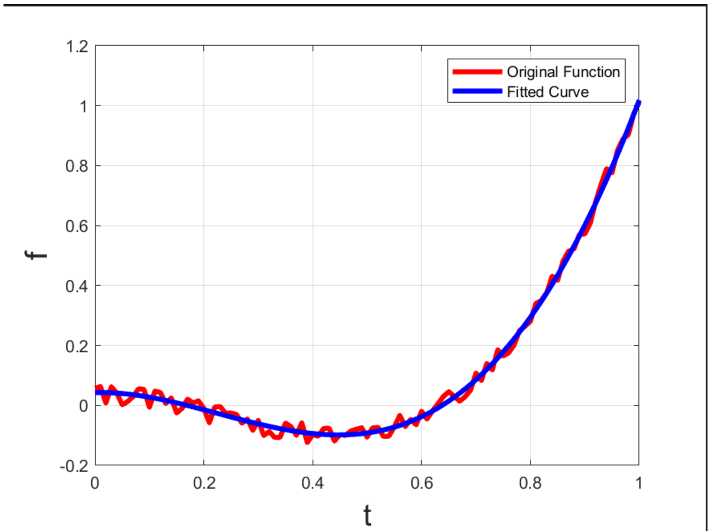
Coefficients of $\{1, t, t^2, t^3, t^4, t^5\}.$ are

Where ai is coefficient of t^i

a0 = 0.0427781164972310
a1 = 0.0504965350306795
a2 = -2.40874043583924
a3 = 3.68499943568390
a4 = -0.232462308825341
a5 = -0.118606010001770

Derivative at t=0.3 is -0.429707351596419





Original Function
Fitted Curve
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
f
0
0.2
0.4
0.6
0.8
1
t