Vector spaces

ROB 501

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- **Vector space over a field**
  - **Span**
  - **Basis**
  - **Dimension**
  - **Representation of vectors (if time)**

Recap: linear combination $(\mathcal{X}, \mathcal{F})$

A linear combination is any **finite** sum of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

where $\alpha_i \in \mathcal{F}$, $x_i \in \mathcal{X}$, $1 \leq i \leq n$, n arbitrary $n \geq 1$

$\sum_{i=1}^{\infty} \alpha_i x_i$ is NOT a linear comb.

Recap: linear independence

A finite set of vectors $x_1, \ldots, x_k \in \mathcal{X}$ is **linearly dependent** if there exist scalars $\alpha_1, \ldots, \alpha_k \in \mathcal{F}$, NOT ALL ZERO, such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k = 0 \in \mathcal{X}$.

zero vector

"at least one α_i is non-zero"

Otherwise, the set is linearly independent.

Exercise:

<u>Ex</u> $\mathcal{X} = \mathbb{R}^{2\times 3}$, $\mathcal{F} = \mathbb{R}$

$A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 6 & 0 & 6 \end{bmatrix}$

linearly independent? YES

$$\alpha_1 \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 & 1 & 0 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6 eqns:

$$\left\{ \begin{array}{l} \alpha_1 + 4\alpha_2 = 0 \\ 3\alpha_1 + 6\alpha_2 = 0 \\ 0\cdot\alpha_1 + 1\cdot\alpha_2 = 0 \\ \vdots \end{array} \right\}$$

the only solution is

$\alpha_1 = \alpha_2 = 0$

$\Rightarrow$ A_1 and A_2 are linearly independent.

Ex $\mathcal{X} = \mathbb{R}^{2\times 3}$, $\mathcal{F} = \mathbb{R}$

$A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ $A_3 = \begin{bmatrix} -2 & 0 & -8 \\ -6 & 2 & -4 \end{bmatrix}$

linearly independent? NO.

$$\alpha_1 \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} -2 & 0 & -8 \\ -6 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\left.\alpha_1 = 2 \quad \alpha_3 = 1\right\}$ equality is satisfied

Therefore, A_1 and A_3 are linearly dependent.

Ex: $\mathcal{X} = \mathbb{P}(t)$, $\mathcal{F} = \mathbb{R}$

$p_1(t) = t^2 + t$ $p_2(t) = 2t$ $p_3(t) = t^2 + 5t$

Is $\{p_1, p_2, p_3\}$ linearly independent? NO

$$\alpha_1 (t^2 + t) + \alpha_2 (2t) + \alpha_3 (t^2 + 5t) \equiv 0$$

$\alpha_1 = 1$ $\alpha_2 = 2$ $\alpha_3 = -1$ satisfies $\rightarrow$ linearly dependent

Linear independence of arbitrary (not necessarily finite) sets

An arbitrary set of vectors $\mathcal{S} \subset \mathcal{X}$ is **linearly independent** if every finite subset is linearly independent.

polynomials in t with real coefficients

Ex: $(\mathcal{X}, \mathcal{F}) = (\mathbb{P}(t), \mathbb{R})$

$S = \{1, t, t^2, t^3, \dots\}$ = monomials w/ unit coefficient.

Claim: S is linearly independent.

"Sketch" of the proof:

Let's take a (finite) arbitrary

$p : \mathbb{R} \to \mathbb{R}$

lin. combination of elements of S.

↳ $\cdot\ p(t) = \boxed{\alpha_0 \cdot 1 + \alpha_1 t + \dots + \alpha_k t^k}$ ← linear comb.

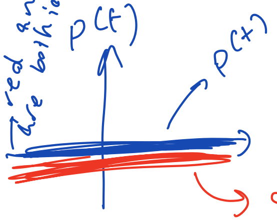
zero polynomial

To show

$$\boxed{p(t) \equiv 0 \iff \alpha_0 = \alpha_1 = \dots = \alpha_k = 0}$$

Want to show

red and blue are both identically zero



$\frac{d\,p(t)}{dt}$

↳ $\boxed{\forall t \in \mathbb{R} \quad p(t) = 0}$ ← Given the defn. of a zero polynomial

$\Updownarrow$

$p(t) \equiv 0$ ↳ this is the defn of zero polynomial.

Observe: $p(t)\big|_{t=0} = p(0) = 0 \iff \alpha_0 = 0 \in \mathbb{R} = \mathbb{F}$

$\in P(t) = X$

"$p(t) \equiv 0$" $\Rightarrow \frac{d}{dt} p(t) \equiv 0$

$$\frac{d}{dt} p(t) = \alpha_1 + 2\alpha_2 t + \dots + k\alpha_k t^k \equiv 0$$

$$\frac{d}{dt} p(0) = \alpha_1 = 0$$

"proceed by induction"

Proof: **Correct Proof by Induction:** Let $k \geq 0$, and define the property $\mathcal{P}(k)$ by

$$\mathcal{P}(k): \text{The set } \{1, t, \ldots, t^k\} \text{ is linearly independent}$$

Base Case: $\mathcal{P}(0)$ is true; that is, the set $\{1\}$ is linearly independent. (You can work this out at home).

Induction Step: For $k \geq 0$, we assume that $\mathcal{P}(k)$ is true and we must show that $\mathcal{P}(k+1)$ is true, that is,

$$\{1, t, \ldots, t^{k+1}\} \text{ is linearly independent}$$

Assume $p_{k+1}(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_{k+1} t^{k+1} = 0$, the zero polynomial, and hence, is zero for all t. Then,

$$0 = \frac{d^{k+1} p_{k+1}}{dt^{k+1}}\Big|_{t=0} = (k+1)!\, a_{k+1}$$

and hence $a_{k+1} = 0$. It follows that

$$p_{k+1}(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_k t^k = 0.$$

$P(k): \{1, t, \ldots, t^k\}$ lin. indep

By the induction step, this implies that

$$a_0 = 0, a_1 = 0, \ldots, a_k = 0,$$

and thus we are done.

□

Span

Let $\boxed{\mathcal{S} \subset \mathcal{X}}$ be a subset of the $(\mathcal{X}, \mathcal{F})$.
The **span** of $\mathcal{S}$ is the set of all linear combinations of elements of $\mathcal{S}$:

$$span\{\mathcal{S}\} = \{\, x \in \mathcal{X} \,|\, \exists k < \infty,\, x^1, \ldots, x^k \in \mathcal{S},\, \alpha_1, \ldots, \alpha_k \in \mathcal{F},$$
$$x = \underbrace{\alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_k x^k}\}$$

Facts: for any $S \subset X$, span(S) is always a subspace of X!

$\Longleftrightarrow$ span(S) is closed under vector addition and scalar multiplication.

Ex 1) $\underline{S} = \underbrace{\left\{ \overset{s_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \overset{s_2}{\begin{bmatrix} 3 \\ 6 \end{bmatrix}} \right\}}$ $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$

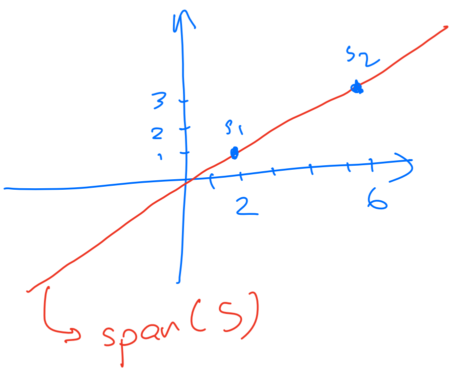
Exercise: What is $\text{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}\right)$?

$$\text{span}(S) = \underline{\left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} \middle| \beta \in \mathbb{R} \right\}}$$

↳ this is the set of all $x = \alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\forall\, \alpha_1, \alpha_2 \in \mathcal{F}$

$S = \{s_1, s_2\}$



Remark: Both in Ex 1 and 2, S is not a subspace of the corresponding $\mathcal{X}$, but span(S) is a subspace of $\mathcal{X}$.

Ex 2) $\mathcal{F} = \mathbb{R}$, $\mathcal{X} = \{ f : \mathbb{R} \to \mathbb{R} \}$

$S = \{1, t, t^2, \dots\}$ → (monomials w/ unit coefficient)

$\text{span}(S) = P(t)$ → set of all polynomials

$$= \{ \alpha_0 + \alpha_1 t + \dots + \alpha_k t^k \mid k \in \mathbb{N},\ \alpha_0, \dots, \alpha_k \in \mathbb{R} \}$$

Question: Is $e^t \in \text{span}(S)$? (for Ex 2)

NO! b.c. no (finite) linear combination can represent e^t.

Basis

A set of vectors B in $(\mathcal{X}, \mathcal{F})$ is a **basis** if

1. B is linearly independent (over $\mathcal{F}$)
2. $span\,\{B\} = \mathcal{X}$ (over $\mathcal{F}$)

Ex: 1) Natural Basis Vectors:

$$\left.\begin{array}{l} (\mathcal{F}^n, \mathcal{F}) \\ \text{e.g. } (\mathbb{R}^n, \mathbb{R}) \\ (\mathbb{C}^n, \mathbb{C}) \\ (\mathbb{Q}^n, \mathbb{Q}) \end{array}\right\} \quad e^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e^n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$B = \{e^1, e^2, \ldots, e^n\}$ is a basis for $(\mathcal{F}^n, \mathcal{F})$

$\Rightarrow$ 2) $(\mathbb{C}^n, \mathbb{R}) \longrightarrow B = \{e^1, e^2, \dots, e^n, je^1, je^2, \dots, je^n\}$

B is lin. indepent over $\mathbb{R}$
$\underbrace{\text{span}(B)}_{\text{when picking the coefs from } \mathbb{R}} = \mathbb{C}^n$
} B is a basis for $(\mathbb{C}^n, \mathbb{R})$

Important Note: $B = \{e^1, e^2, \dots, e^n, je^1, je^2, \dots, je^n\}$ is not a basis for $(\mathbb{C}^n, \mathbb{C})$. bc it is not a linearly independent set when coef. are picked from $\mathbb{C}$ (when doing linear combinations)

3) $(\mathbb{P}(t), \mathbb{R}) \rightarrow B = \{1, t, t^2, \dots\}$ (monomials w/ unit coefs) is a basis for $(\mathbb{P}(t), \mathbb{R})$

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Simplified version of 2). Consider $(\mathbb{C}, \mathbb{R})$. A basis B for $(\mathbb{C}, \mathbb{R})$ is $B = \{1, j\}$. Why?

- $\alpha_0 \cdot 1 + \alpha_1 \cdot j = 0$ with $\alpha_0, \alpha_1 \in \mathbb{R} \Rightarrow \alpha_0 = \alpha_1 = 0$
  $\longrightarrow$ B linearly indep.
- $\text{span}(B) = \{\alpha_0 + \alpha_1 j \mid \alpha_0, \alpha_1 \in \mathbb{R}\} = \mathbb{C}$ ✓

Consider $(\mathbb{C}, \underline{\mathbb{C}})$, $B = \{1, j\}$ is not a basis for $(\mathbb{C}, \mathbb{C})$. Why

- $\alpha_0 \cdot 1 + \alpha_1 \cdot j = 0$ with $\alpha_0, \alpha_1 \in \mathbb{C}$ can be achieved by $\alpha_0 = 1$ $\alpha_1 = j$ (not all zero) $\Rightarrow$ B is linearly <u>dependent</u>.

Dimension

The maximal number of elements in any linearly independent set of vector in $(\mathcal{X}, \mathcal{F})$, is called the **dimension** of $(\mathcal{X}, \mathcal{F})$.

Equivalent Defn.

Let $n \geq 1$ be a finite integer.

$(\mathcal{X}, \mathcal{F})$ has dimension n if

a) $\exists$ a set of n vectors in $\mathcal{X}$ that are linearly independent.

b) Every set with $n+1$ vectors is linearly dependent.

Def: $(X, \mathbb{F})$ is infinite dimensional if $\forall n \geq 1$, $\exists$ set of n linearly independent vectors.

Ex: 1. $\dim(\mathbb{F}^n, \mathbb{F}) = n$

2. $\dim(\mathbb{C}^n, \mathbb{R}) = 2n$

3. $\dim(\mathbb{P}(t), \mathbb{R}) = \infty$

not in any ROB 501 Exam

4. $\dim(\mathbb{R}, \mathbb{Q}) = \infty$

there is a nice proof based on primes.

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Proof: Let $(\mathcal{X}, \mathcal{F})$ be $n-$dimensional and let $\{v^1, \ldots, v^n\}$ be a linearly independent set.

To Show: $\forall x \in \mathcal{X}$, $\exists \alpha_1, \ldots, \alpha_n \in \mathcal{F}$ such that $x = \alpha_1 v^1 + \ldots + \alpha_n v^n$, i.e., $\mathcal{X} = \text{span}\{v^1, \ldots, v^n\}$

How: Because $(\mathcal{X}, \mathcal{F})$ is $n-$dimensional, $\{x, v^1, \ldots, v^n\}$ is linearly dependent. Otherwise, the $\dim \mathcal{X} > n$ which it isn't. Hence, $\exists \beta_0, \beta_1, \ldots, \beta_n \in \mathcal{F}$, NOT ALL ZERO, such that $\beta_0 x + \beta_1 v^1 + \ldots + \beta_n v^n = 0$.

Theorem: In an $n-$ dimensional vector space **ANY** set of n linearly independent vectors is a basis.

Proof: Let $(\mathcal{X}, \mathcal{F})$ be $n-$dimensional and let $\{v^1, \dots, v^n\}$ be a linearly independent set.

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Claim: $\beta_0 \neq 0$

Proof: Suppose that $\beta_0 = 0$. Then

1. At least one of $\beta_1, \dots, \beta_n$ is non-zero
2. $\beta_1 v^1 + \dots + \beta_n v^n = 0$

1 and 2 above, imply that $\{v^1, \dots, v^n\}$ is linearly dependent, which is a contradiction. Hence $\beta_0 = 0$ cannot hold. Completing the proof, we write

$$\beta_0 x = -\beta_1 v^1 - \dots - \beta_n v^n$$

$$x = \left(\frac{-\beta_1}{\beta_0}\right) v^1 - \dots - \left(\frac{-\beta_n}{\beta_0}\right) v^n$$

$$\therefore \alpha_1 = \frac{-\beta_1}{\beta_0}, \dots, \alpha_n = \frac{-\beta_n}{\beta_0}$$

$\square$

Proposition Let $(\mathcal{X}, \mathcal{F})$ be a vector space and suppose that $B = \{b^1, b^2, \cdots\}$ is a basis for $(\mathcal{X}, \mathcal{F})$. Let $x \in \mathcal{X}$ and suppose that

$$x = \alpha_1 b^1 + \cdots + \alpha_k b^k$$

and

$$x = \bar{\alpha}_1 b^1 + \cdots + \bar{\alpha}_k b^k$$

Then, $\alpha_i = \bar{\alpha}_i$ for all $1 \leq i \leq k$.

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Then, $\alpha_i = \bar{\alpha}_i$ for all $1 \leq i \leq k$.

Proof:

$$\begin{aligned} 0 = x - x &= \left(\alpha_1 b^1 + \cdots + \alpha_k b^k\right) - \left(\bar{\alpha}_1 b^1 + \cdots + \bar{\alpha}_k b^k\right) \\ &= (\alpha_1 - \bar{\alpha}_1) b^1 + \cdots + (\alpha_k - \bar{\alpha}_k) b^k \end{aligned}$$

Because $\left\{b^1, \cdots, b^k\right\} \subset B$ implies that $\left\{b^1, \cdots, b^k\right\}$ is linearly independent, we deduce that $\alpha_i - \bar{\alpha}_i = 0$ for all $1 \leq i \leq k$.

Representations of Vectors

Example: $\mathcal{F} = \mathbb{R}$, $\mathcal{X} = \{2 \times 2 \text{ matrices with real coefficients}\}$

"natural"

Basis 1: $v^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $v^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $v^3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $v^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Basis 2: $w^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $w^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $w^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $w^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$