2)

a) for jointly random variables (X, Y, Z)

$$\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

we need $P(X, Y \mid Z = z)$

=) $[X, Y]$ is one random variable & Z is another

WKT

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

$$\Sigma_{12} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\mu_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mu_2 = 0 \quad \Sigma_{11} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \quad \Sigma_{12} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Sigma_{21} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \Sigma_{22} = 2$$

$$\mu_{1|2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot (1/2) \cdot (z - 0) =$$

$$\boxed{\mu_{12} = \begin{bmatrix} -1 + ((z-1)/2) \\ (z-1) \end{bmatrix}}$$

$$\Sigma_{1|2} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\boxed{\Sigma_{1|2} = \begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix}}$$

b) ~~Ref~~ so from above

$P(X, Y | Z=2) \Rightarrow$ 2 random variables are $X | Z=2$ & $Y | Z=2$

$$\Rightarrow \mu = \begin{bmatrix} -1 + (z-1)/2 \\ (z-1) \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 3/2 & 1 \\ 1 & 2 \end{bmatrix}$$

to find $P((X|z=2) | (Y=y|z=2))$

$$\boxed{\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)} \quad \boxed{\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}$$

$\mu_1 = -1 + (z-1)/2$; $\mu_2 = (z-1)$ $\Sigma_{11} = 3/2$ $\Sigma_{12} = 1$ $\Sigma_{22} = 2$

$\Sigma_{22}^{-1} = 1/2$

$$\Rightarrow \mu_{1|2} = (-1 + (z-1)/2) + 1 \cdot \frac{1}{2}(y - (z-1))$$

$$\boxed{\mu_{1|2} = -1 + y/2}$$

$$\Sigma_{1|2} = (3/2) - 1 \cdot (1/2) \cdot 1 = 1$$

$$\boxed{\Sigma_{1|2} = 1}$$

c) To find $P(X \mid Y=y, Z=2)$ lets take random variables X, $[Y, Z]$

$$\mu = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \Sigma = \left[\begin{array}{c|cc} 2 & 2 & 1 \\ \hline 2 & 4 & 2 \\ 1 & 2 & 2 \end{array}\right]$$

$\mu_1 = -1 \quad \mu_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Sigma_{11} = 2 \quad \Sigma_{12} = \begin{bmatrix} 2 & 1 \end{bmatrix}$

$$\Sigma_{22} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\Sigma_{22}^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$\Sigma_{12} \cdot \Sigma_{22}^{-1} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 \end{bmatrix}$$

$$\mu_{1|2} = -1 + \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} y-0 \\ z-1 \end{bmatrix}$$

$$\boxed{\mu_{1|2} = -1 + y/2}$$

$$\Sigma_{1|2} = 2 - \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1$$

$$\boxed{\Sigma_{1|2} = 1}$$

d) Mean & variance of guassian distoubution in both (b) & (c) are same i.e. Both distoibutions are equivalent

3) a)

WKT

for $\hat{\alpha}_k$, $G_k^T \alpha_k = \beta_k$

So similarly for $k+1$

$$G_{k+1}^T \alpha_{k+1} = \beta_{k+1}$$

as its said that y_{k+1} is independent of $\text{span}\{y_1, y_2 \ldots y_k\}$

$$G_{k+1}^T = \begin{bmatrix} G_k^T & 0 \\ 0 & \langle y_{k+1}, y_{k+1} \rangle \end{bmatrix}$$

$\beta_i = \langle \alpha, y_i \rangle$ as y_{k+1} dont have any component

$$\beta = \begin{bmatrix} \beta_k \\ \langle \alpha, y_{k+1} \rangle \end{bmatrix}$$

$$\alpha_{k+1} = (G_{k+1}^T)^{-1} \beta_{k+1} = \begin{bmatrix} G_k^{-1} & 0 \\ 0 & \frac{1}{y_{k+1}, y_{k+1}} \end{bmatrix} \begin{bmatrix} \beta_k \\ \langle \alpha, y_{k+1} \rangle \end{bmatrix}$$

$$\begin{bmatrix} \alpha_k \\ \alpha \end{bmatrix} = \begin{bmatrix} G_k^{-1} \beta_k \\ \dfrac{\langle \alpha, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle} \end{bmatrix}$$

$$\hat{x}_{k+1} = \cancel{\hat{x}_\beta} + \cancel{\ } \alpha y_{k+1}$$

$$x_{k+1} = \underbrace{G_k^{-1} \beta_k}_{x_k} + \underbrace{\frac{\langle x, y_{k+1} \rangle}{\langle y_{k+1}, y_{k+1} \rangle}}_{\beta \alpha} y_{k+1}$$

$$= x_k + \alpha y_{k+1}$$

$\therefore$ $\exists$ some scalar α where

$$x_{k+1} = x_k + \alpha y_{k+1}$$

b)

$$\hat{y}_{k+1|k} = \underset{m \in M}{\arg\min} \| y_{k+1} - m \|$$

$(\hat{y}_{k+1|k} - y_{k+1}) \perp M_k$ → from projection theorm

for given $\hat{y}_{k+1|k}$ to be belong to M_k

$\Rightarrow$ $M_{k+1} = M_k \oplus \text{span}\{ y_{k+1} \}$

$= M_k \oplus \text{span}\{ y_{k+1} - v \}$ where $v \in M_k$

& $M_k \perp (y_{k+1} - y_{k+1|k})$ ∵ $y_{k+1|k} \in M_k$

$$\Rightarrow \boxed{\bar{y}_{k+1} = y_{k+1} - \hat{y}_{k+1|k}}$$

using result from a

$$\boxed{\hat{x}_{k+1} = \hat{x}_k + \frac{\langle x, \bar{y}_{k+1} \rangle}{\langle \bar{y}_{k+1}, \bar{y}_{k+1} \rangle} \bar{y}_{k+1}}$$