Vector spaces

ROB 501

Necmiye Ozay

- **Vector space over a field**
  - **More examples of vector spaces**
  - **Subspaces**
  - **Linear independence**
  - **Basis (if time)**

Course announcements

- Next lecture is also over Zoom (we will be in-person next week)
- Recitation sessions are in-person

Vector spaces

A linear (vector) space over a field $\mathcal{F}$, denoted by $(\mathcal{X}, \mathcal{F})$ consists of a set $\mathcal{X}$ of vectors, a field $\mathcal{F}$, and two operations *vector addition* and *scalar multiplication* such that

$x_1, x_2 \in \mathcal{X}$ $\quad x_1 + x_2$ $\qquad \alpha \in \mathcal{F}, \; x \in \mathcal{X}$ $\quad \alpha x$

related to vector addition

1. $\mathcal{X}$ is closed under vector addition
2. Vector addition is commutative
3. Vector addition is associative
4. $\mathcal{X}$ contains a zero vector $\mathbf{0}$ (origin of the vector space) with $\boxed{\mathbf{0} + x = x \text{ for all } \mathbf{x} \in \mathcal{X}}$ additive identity for vector addition
5. Each element of $\mathcal{X}$ has an additive inverse

related to scalar multiplication

6. $\mathcal{X}$ closed under scalar multiplication for any $\alpha \in \mathcal{F}$ $\quad \alpha x \in \mathcal{X}$
7. Scalar multiplication is associative
8. Scalar multiplication is distributive over vector addition
9. Scalar multiplication is distributive over scalar addition
10. For any $\mathbf{x} \in \mathcal{X}$, $1\mathbf{x} = \mathbf{x}$ where 1 $(\in \mathcal{F})$ is the multiplicative identity in $\mathcal{F}$

$(\mathbb{R}, \mathbb{C})$ is not a vector space. fails Axiom 6

"$\mathcal{X}$" $5 \in \mathbb{R}$ $\quad i \in \mathbb{C}$ "$\mathcal{F}$"

$5 \cdot i \notin \mathbb{R}$

Question from last time

Some questions: (you can prove)

1. For fields, the element $\underline{0}$ & $\underline{1}$ are unique

2. For vector spaces, the <u>origin</u> ($\mathbf{0}$) is unique.

Proof of 2: (Proof by contradiction)

Let $\mathbf{0}_1, \mathbf{0}_2 \in X$ be origins ("zero vectors") of the vector space X and $\underline{\mathbf{0}_1 \neq \mathbf{0}_2}$.

If 0_1 is a zero vector, by Axiom 4, we have:

$$0_1 + x = x \quad \forall x \in X. \text{ In particular, } 0_2 \in X,$$

therefore $\underline{0_1 + 0_2} = \boxed{0_2}$. (1)

If 0_2 is a zero vector, by Axiom 4, we have:

$$0_2 + x = x \quad \forall x \in X. \text{ In particular, } 0_1 \in X,$$

therefore $0_2 + 0_1 = 0_1$. (2)

by commutativity of vector addition $\quad \underline{0_1 + 0_2} = \boxed{0_1}$ (3)

from (1) and (3), $\underline{0_1 = 0_2}$ (because left hand sides are equal).

$\Rightarrow$ Contradiction. $0 = 0_1 = 0_2$ unique vector.

Examples

- Pick $\mathcal{X} \doteq \mathcal{F}$, $\mathcal{F} \doteq \mathcal{F}$. $(\mathcal{F}, \mathcal{F})$ is a vector space. That is, every field forms a vector space over itself. Eg. $(\mathbb{R}, \mathbb{R})$, $(\mathbb{C}, \mathbb{C})$, $(\mathbb{R}(s), \mathbb{R}(s))$

  Pick $\mathcal{X} \doteq \mathbb{C}$, $\mathcal{F} \doteq \mathbb{R}$. $(\mathbb{C}, \mathbb{R})$ is a vector space. That is, the set of complex numbers forms a vector space over reals.
  The set of real numbers is **not** a vector space over complex numbers. $(\mathbb{R}, \mathbb{C})$ is not a vector space!

- Pick $\mathcal{X} \doteq \{f : [a, b] \to \mathbb{R}\}$, $\mathcal{F} \doteq \mathbb{R}$. $(\mathcal{X}, \mathbb{R})$ is a vector space. That is, the set of functions with domain $[a, b]$ and that take values in reals (range is a subset of reals) forms a vector space over reals.

With the usual definition of addition and scalar multiplication.

Examples

Pick $\mathcal{X} \doteq \mathcal{F}^n$, $\mathcal{F} \doteq \mathcal{F}$. $(\mathcal{F}^n, \mathcal{F})$ is a vector space. That is, the set of n-tuples of elements in $\mathcal{F}$, written as columns, forms a vector space over $\mathcal{F}$.

$$\mathcal{F}^n = \left\{ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} |\alpha_i \in \mathcal{F},\, i = 1, \ldots, n \right\}$$

$(\mathcal{F}^n, \mathcal{F})$ is a vector space.

Define vector addition by: $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix}$

and scalar times vector multiplication by: $\alpha \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha\beta_1 \\ \vdots \\ \alpha\beta_n \end{bmatrix}$

$(\mathbb{R}^n, \mathbb{R})$, $(\mathbb{C}^n, \mathbb{C})$, etc.

Subspace

Remark on notation: For sets A and B,

$A \subset B$ means $\forall a \in A$, $a \in B$ (A is a subset of B)

$A = B$ means $A \subset B$ and $B \subset A$.

In ROB 501, we have no notation like $A \subseteq B$, $A \subsetneq B$

Defn. Let $(\mathcal{X}, \mathcal{F})$ be vector space and $\mathcal{Y} \subset \mathcal{X}$. $\mathcal{Y}$ is subspace of $\mathcal{X}$ (or of $(\mathcal{X}, \mathcal{F})$) if $(\mathcal{Y}, \mathcal{F})$ is a vector space when using the vector addition and scalar multiplication operations from $(\mathcal{X}, \mathcal{F})$.

<u>Remark</u>: In principle, must check $\mathcal{Y}$ satisfies all 10 axioms to verify it is a subspace.

Questions to think about:

- Is $(X, \mathcal{F})$ a subspace of $(\mathcal{X}, \mathcal{F})$?

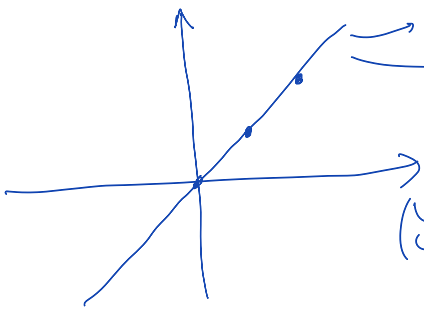
  Yes. $X \subset \mathcal{X}$ and $(X, \mathcal{F})$ is vector space.

- Let $\mathbf{0}$ be the zero vector of $(\mathcal{X}, \mathcal{F})$, and consider $\mathcal{Y} = \{\mathbf{0}\}$. Is $(\mathcal{Y}, \mathcal{F})$ a subspace of $(\mathcal{X}, \mathcal{F})$? $\mathcal{Y} \subset \mathcal{X}$ is $(\mathcal{Y}, \mathcal{F})$ a vector space

  Yes $(\mathcal{Y}, \mathcal{F})$ is a vector space and a subspace of $(X, \mathcal{F})$

- Let $Y \subset X$ and let $\overset{\text{origin}}{0} \notin Y$. Can Y be subspace of $(X, \mathcal{F})$? No, because Y cannot be a vector space without an additive inverse (= w/out an origin).

- Ex of $(\mathbb{R}^2, \mathbb{R})$ subspaces of $(\mathbb{R}^2, \mathbb{R})$



Y is $\{$a line passing through the origin$\}$

$(Y, \mathbb{R})$ is a subspace

<u>Proposition</u>: The following are equivalent (TFAE) for a vector space $(X, \mathcal{F})$ and $Y \subset X$:

a) Y is a subspace of X by 10 axioms.

b) $\forall v^1, v^2 \in Y$, $v^1 + v^2 \in Y$ → closed under addition

and

$\forall \alpha \in \mathcal{F}$, $\forall v \in Y$ $\alpha v \in Y$ → closed scalar multiplication

c) $\forall \alpha \in \mathcal{F}$ and $\forall v^1, v^2 \in Y$, $\alpha v^1 + v^2 \in Y$

→ d) $\forall\, \alpha_1, \alpha_2 \in \mathcal{F}$ and $\forall\, v^1, v^2 \in \mathcal{Y}$, $\underbrace{\alpha_1 v^1 + \alpha_2 v^2}\in \mathcal{Y}$

Examples of subspaces:

1) $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$

$$\mathcal{Y} = \left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} \;\middle|\; \beta \in \mathbb{R} \right\}$$

$$\underbrace{\begin{bmatrix} \beta_1 \\ 2\beta_1 \end{bmatrix}}_{v^1} + \underbrace{\begin{bmatrix} \beta_2 \\ 2\beta_2 \end{bmatrix}}_{v^2} = \begin{bmatrix} \beta_1 + \beta_2 \\ 2(\beta_1 + \beta_2) \end{bmatrix} \in \mathcal{Y}$$ → is closed under vector addition.

$\alpha \in \mathbb{R}$ $$\alpha \cdot \underbrace{\begin{bmatrix} \beta \\ 2\beta \end{bmatrix}}_{v} = \begin{bmatrix} (\alpha\cdot\beta) \\ 2(\alpha\cdot\beta) \end{bmatrix} \in \mathcal{Y}$$ → is closed under scalar multiplication

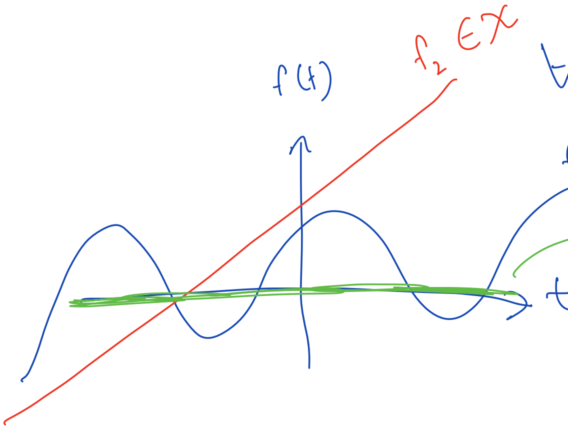
$\therefore$ By b) $\mathcal{Y}$ is a vector space.

2) $\mathcal{F} = \mathbb{R}$, $\underline{\mathcal{X}} = \{ f : \mathbb{R} \to \mathbb{R} \}$ We showed that $(\mathcal{X}, \mathcal{F})$ is a vector space.

$\mathcal{Y} := \mathbb{P}(t) := \{$ polynomials in t with real coefficients $\}$

if $p \in \mathbb{P}(t)$, $p(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_n t^k$
$\alpha_i \in \mathbb{R}$ $k \in \mathbb{N}$

$\mathcal{Y}$ is a subspace because $\forall\, p_1, p_2 \in \mathbb{P}(t)$
$p_1 + p_2 \in \mathbb{P}(t)$



$\forall \alpha \in \mathbb{R} \ \forall p \in \mathbb{P}(t) \quad \alpha \cdot p \in \mathbb{P}(t)$

constant zero function $(\mathbf{0})$ is the origin of $\mathcal{X} \ni \mathbf{0}$

and $y_{\mathbf{0}}$ $(k=0 \ \ \alpha_0 = 0)$

$\bar{\mathcal{Y}} \subset \mathcal{X}$ $\quad \bar{\mathcal{Y}} := \{ f : \mathbb{R} \to \mathbb{R} \mid f$ is differentiable and $\frac{df}{dt} \equiv \mathbf{0} \}$

↳ constant zero function.

$\bar{\mathcal{Y}}$ is subspace.

<u>Non-example</u>: $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$

$\mathcal{Y} = \left\{ \begin{bmatrix} \beta \\ 2\beta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid \beta \in \mathbb{R} \right\}$ <u>not</u> a subspace $\quad \mathbf{0} \notin \mathcal{Y}$

$\mathbf{0} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

— a —

Linear combinations and linear independence.

Let $(\mathcal{X}, \mathcal{F})$ be a vector space.

<u>Defn</u>: A <u>linear combination</u> is any <u>finite</u> sum of the form:

$$\alpha_1 v^1 + \alpha_2 v^2 + \dots + \alpha_n v^n$$

where $n \geq 1$ (and finite), $\alpha_i \in \mathcal{F}$, $v^i \in \mathcal{X}$, $1 \leq i \leq n$.

<u>Note</u>: $\sum_{i=1}^{\infty} \alpha_i v^i$ is <u>not</u> a linear combination.

$\qquad \overset{\triangle}{=} \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i v^i$ (and to define limits, we need more structure than the 10 axioms of vector space. We need real analysis (later in the semester!).

<u>Mind set</u>: Consider $(\mathbb{R}^n, \mathbb{R})$. Let A be an $n \times m$ real matrix and $x \in \mathbb{R}^m$.

Then Ax is a linear combination of column's of A:

$$A = [A_1 | A_2 | \dots | A_m] \qquad x \in \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$Ax = \underbrace{x_1}_{\text{scalars}} \overbrace{A_1}^{\text{vectors}} + \dots + \underbrace{x_m}_{\text{scalars}} \overbrace{A_m}^{\text{vectors}}$$

Linear independence

<u>Def:</u> A finite set of vectors $\{v^1, \ldots, v^k\} \subset \mathcal{X}$ is <u>linearly dependent</u> if $\exists\, \alpha_1, \ldots, \alpha_k \in \mathcal{F}$ not all zero s.t.

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_k v^k = \mathbf{0} \in \mathcal{X} \qquad (*)$$

Otherwise, $\{v^1, \ldots, v^k\}$ is linearly <u>independent</u>.

($\Leftrightarrow$ only sol'n of $(*)$ is the trivial one $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$.)

Ex: $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R})$

$v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v^2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ linearly independent?

Let $\alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $(*)$

$\left.\begin{aligned} \alpha_1 + 4\alpha_2 &= 0 \\ \alpha_1 &= 0 \end{aligned}\right\} \Rightarrow$ $\alpha_1 = 0 = \alpha_2$ is the only sol'n to $(*)$.

$\Rightarrow$ v^1 and v^2 are linearly independent.

Exercise:

Ex $\mathcal{X} = \mathbb{R}^{2\times 3}$, $\mathcal{F} = \mathbb{R}$

$A_1 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 4 & 1 & 0 \\ 6 & 0 & 6 \end{bmatrix}$

linearly independent?

OFFICE HOURS

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

equation (*) in terms of x, M, λ

→ the set of "x, λ" pairs that satisfy eq. (*) are "......"'s of <u>M</u>

and

"....."'s

Proof: **Correct Proof by Induction:** Let $k \geq 0$, and define the property $\mathcal{P}(k)$ by

$$\mathcal{P}(k) : \text{The set } \{1, t, \ldots, t^k\} \text{ is linearly independent}$$

Base Case: $\mathcal{P}(0)$ is true; that is, the set $\{1\}$ is linearly independent. (You can work this out at home).

Induction Step: For $k \geq 0$, we assume that $\mathcal{P}(k)$ is true and we must show that $\mathcal{P}(k+1)$ is true, that is,

$$\{1, t, \ldots, t^{k+1}\} \text{ is linearly independent}$$

Assume $p_{k+1}(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_{k+1} t^{k+1} = 0$, the zero polynomial, and hence, is zero for all t. Then,

$$0 = \frac{d^{k+1} p_{k+1}}{dt^{k+1}}\Big|_{t=0} = (k+1)!\; a_{k+1}$$

and hence $a_{k+1} = 0$. It follows that

$$p_{k+1}(t) := \alpha_0 + \alpha_1 t + \cdots + \alpha_k t^k = 0.$$

$\mathcal{P}(k) : \{1, t, \ldots, t^k\}$ lin. indep

By the induction step, this implies that

$$a_0 = 0, a_1 = 0, \ldots, a_k = 0,$$

and thus we are done.

□