Exam Number:________

ROB 501 Exam-I
[DO NOT OPEN UNTIL TOLD TO DO SO]

Tuesday, October 29, 2019, 6:10 PM–8:00 PM

Rooms (First letter of last name): (A-E) in EECS 1303; (F-L) in EECS 1311; and (M-Z) in EECS 1500.

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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FILL IN YOUR NAME NOW. COPY THE HONOR CODE NOW. DO NOT COUNT PAGES.

DO NOT OPEN THE EXAM BOOKLET UNTIL TOLD TO DO SO.

RULES:

1. CLOSED TEXTBOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. 1 SHEET OF NOTE PAPER (Front and Back), US Letter Size.
6. NO CALCULATORS, CELL PHONES, HEADSETS, nor DIGITAL DEVICES of any KIND.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

Problems 1 - 5 (30 points: 5 × 6)

Instructions. For each problem, there is at least one answer that is true (i.e., **T**) and one answer that is false (i.e., **F**). *You will receive no credit for your responses if you either mark all of the answers as true or all as false*, **because then we assume you are guessing**. Otherwise, each part of a question is worth 1.5 points.

1. (Questions on logic and proof methods) Recall that $\wedge$ is 'and', $\vee$ is 'or', and $\neg$ is 'not'. Recall also that the symbol $\Leftrightarrow$ and the written text, "if, and only if", "logically equivalent to", and "have the same truth table", all mean the same thing. For example, in HW, you verified that $\neg(p \wedge q)$ is "logically equivalent to" $(\neg p) \vee (\neg q)$ by proving "they have the same truth table". **Circle True or False as appropriate for the following statements:**

T **F** (a) Let n be an integer. If n^2 is positive, then so is n.

T **F** (b) The negation of "all gardens have flowers" is "some gardens have trees".

T **F** (c) $\neg(\forall\ \epsilon > 0, \exists \delta > 0 \text{ such that } \frac{\epsilon}{2} \leq \delta \leq \epsilon) \iff (\exists\ \epsilon > 0 \text{ such that } \forall\ \delta > 0, \text{it is true that either } \delta > \epsilon \text{ or } \delta < \frac{\epsilon}{2})$

T **F** (d) The truth table given below is correct for $p \implies \neg q$

p	q	$p \implies \neg q$
1	1	0
1	0	1
0	1	1
0	0	1

2. (Eigenvalues and eigenvectors) For square matrix $A \in \mathbb{R}^{n\times n}$, its eigenvalues are $\lambda_i \in \mathbb{C}$, and its corresponding eigenvectors are non-zero $v^i \in \mathbb{C}^n$ s.t. $Av^i = \lambda_i v^i$. When applicable, assume we are working with an inner product space where $< x, y >= x^T \bar{y}$. **Circle True or False as appropriate for the following statements:**

T **F** (a) Assuming λ_i and v^i are real, then $v^i \in (\mathcal{R}(A^T - \lambda_i I))^\perp$, where $\mathcal{R}$ stands for range space.

T **F** (b) $v^i \in \mathbb{R}^n \implies \lambda_i \in \mathbb{R}$.

T **F** (c) If $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 3 & 11 \\ -3 & 11 & 5 \end{bmatrix}$ and $V = \begin{bmatrix} \frac{v^1}{||v^1||} & \frac{v^2}{||v^2||} & \frac{v^3}{||v^3||} \end{bmatrix}$, then $V^\top V = I$.

T **F** (d) $\text{span}\{v^1, v^2, \ldots, v^m\} = \mathbb{R}^m$ for $m \leq n$.

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Anything written here will not be graded. You can use as scratch paper.

3. (Span, null space, range space) Assume we are working with a real inner product space $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ where $< x, y >= x^T y$. **Circle True or False as appropriate for the following statements:**

T **F** (a) For $A \in \mathbb{R}^{m \times n}$, $x \in \mathcal{N}(A) \implies x \perp A_i$, $1 \le i \le n$, where A_i is the i-th column of A.

T **F** (b) Let $\mathcal{X}$ be finite-dimensional. If $\mathcal{X} = \text{span}\{u^1, u^2, u^3\}$ for vectors u^1, u^2, u^3, then $\dim \mathcal{X} = 3$.

T **F** (c) Let $S \subset \mathcal{X}$ be a nonempty subset of $\mathcal{X}$, then $S \oplus \text{span}\{S\}^\perp = \mathcal{X}$.

T **F** (d) Given $A \in \mathbb{R}^{m \times n}$, $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^n$.

4. (Matrices) **Circle True or False as appropriate for the following statements:**

T **F** (a) The determinant of any orthogonal matrix is $+1$ or -1.

T **F** (b) The trace of any positive semi-definite matrix is strictly positive.

T **F** (c) The matrix $M = \begin{bmatrix} 1 & 9 & 2 \\ 9 & -3 & 5 \\ 2 & 5 & 10 \end{bmatrix}$ is positive definite.

T **F** (d) The eigenvalues of matrix $M = \begin{bmatrix} 2 & 1 & -1 & 7 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 10 & -2 \\ 0 & 0 & -2 & 10 \end{bmatrix}$ are positive.

5. (Vector spaces, inner products, and norms) **Circle True or False as appropriate for the following statements:**

T **F** (a) Consider the inner product space $(\mathbb{R}^n, \mathbb{R}, < \cdot, \cdot >)$ where $< u, v >= u^T v$ for $u, v \in \mathbb{R}^n$. Then for any positive definite matrix $P \in \mathbb{R}^{n \times n}$, the binary operation $[u, v] :=< u, Pv >$ is also an inner product.

T **F** (b) The normed space $(\mathbb{R}^n, \mathbb{R}, || \cdot ||_P)$ with norm $||x||_P := \sqrt{x^T P x}$, for some positive definite matrix $P \in \mathbb{R}^{n \times n}$, satisfies $||u + v||_P^2 = ||u||_P^2 + ||v||_P^2$ for $u, v \in \mathbb{R}^n$ such that $u^T v = 0$.

T **F** (c) Consider the inner product space $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ and let $P : \mathcal{X} \to M$ be the orthogonal projection of $x \in \mathcal{X}$ onto subspace M. If $\{v^1, \ldots, v^k\}$ is an orthonormal basis for M, then $P(x) = \sum_{i=1}^{k} < x, v^i > v^i$.

T **F** (d) Consider the vector space $(\mathbb{R}^3, \mathbb{R})$ and let $\alpha_0 \in \mathbb{R}$ be fixed. Then $M = \{x \in \mathbb{R}^3 \mid [3 \quad 5 \quad -2]x = \alpha_0\}$ is a subspace if, and only if, $\alpha_0 \neq 0$.

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Anything written here will not be graded. You can use as scratch paper.

Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. <u>If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.</u> For example, we proved that the Projection Theorem. So if you need results associated with it, simply state them and use them.

6. (15 points) (Place your answers in the **boxes** and show your work below.)
Consider the vector spaces $(\mathbb{R}^{3\times 2}, \mathbb{R})$ and $(\mathbb{R}^3, \mathbb{R})$. You are given bases for each of them, respectively, as follows:

$$E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } V = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{1}$$

Define the linear operator $\mathcal{L} : \mathbb{R}^{3\times 2} \to \mathbb{R}^3$ by: for $x \in \mathbb{R}^{3\times 2}$, $\mathcal{L}(x) = xW$, where $W = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

(a) (9 points) Find the <u>2nd and 4th columns</u> of the matrix representation A of $\mathcal{L} : \mathbb{R}^{3\times 2} \to \mathbb{R}^3$ with respect to the given bases.

(a) $A_2 = \begin{bmatrix} \quad \end{bmatrix}$, $A_4 = \begin{bmatrix} \quad \end{bmatrix}$

(b) (6 points) Consider the natural basis $\tilde{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $(\mathbb{R}^3, \mathbb{R})$. Let P denote the change of basis matrix from V to $\tilde{V}$ and let $\bar{P}$ denote the change of basis matrix from $\tilde{V}$ to V. **Compute either P or $\bar{P}$ (choose ONE).** You must clearly indicate which change of basis matrix you are computing. If you mislabel the matrix you will lose half the points.

(b) $P = \begin{bmatrix} \quad \end{bmatrix}$ **OR** $\bar{P} = \begin{bmatrix} \quad \end{bmatrix}$ (only one is required)

Show your calculations and reasoning below. No reasoning $\Longrightarrow$ no points.

Please show your work for question 6.

7. (20 points) Consider the real vector space of 2×2 real symmetric matrices, $\mathcal{X} := \{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{ij} \in \mathbb{R}, \ \ a_{12} = a_{21} \}$, with matrix addition and the product of a (real) scalar and a matrix defined in the usual manners. We define an inner product on $(\mathcal{X}, \mathbb{R})$ by, for $A, B \in \mathcal{X}$

$$\langle A, B \rangle := \left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right\rangle := a_{11}b_{11} + 2a_{12}b_{12} + 3a_{22}b_{22}.$$

(You are given that this is a valid inner product; you do not have to check it yourself.) Define $M = \text{span}\{y^1, y^2\}$ for $y^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $y^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) (10 points) Find an <u>orthonormal</u> basis, $\{v^1, v^2\}$, for M.

(a) $v^1 = \begin{bmatrix} & \\ & \end{bmatrix}$, $v^2 = \begin{bmatrix} & \\ & \end{bmatrix}$

(b) (10 points) Solve $\hat{x} = \text{argmin}\ d(x, M)$ for $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) $\hat{x} = \begin{bmatrix} & \\ & \end{bmatrix}$

Show your calculations below. Note: If you encounter a square root that does not simplify, leave it as a scalar multiple. It is fine to leave part (b) as a linear combination of at most two matrices if they are cumbersome to combine, as long as you specify clearly all of the constants.

Please show your work for question 7.

8. (15 points) (Proof Problem) (Done in two parts so that you cannot lose too many points on each part)

(a) (5 points) Let $(\mathcal{X}, \mathbb{R})$ be the real vector space of real-valued continuous functions over $[-2, 2]$, that is, $\mathcal{X} = \{f : [-2, 2] \to \mathbb{R},\ f \text{ continuous}\}$. In class we defined an inner product on this vector space by $< f, g >:= \int\limits_{-2}^{2} f(t)g(t)dt$.

Suppose we define the continuous function

$$\eta(t) = \begin{cases} 0 & \text{for } |t| > 1 \\ 1 - |t| & \text{for } |t| \leq 1 \end{cases}.$$

Show that $< f, g >_\eta := \int\limits_{-2}^{2} f(t)\eta(t)g(t)\ dt$ **is not a valid inner product on** $(\mathcal{X}, \mathbb{R})$**.**

(b) (10 points) For all $n \geq 1$, prove that $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Extra space for question 8.

(Scratch Paper)

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(If you write anything here, be sure to indicate to which problem it applies.)