Exam Number:________

Date is Monday, December 16, 2019, 4:10 PM–6:00 PM

Prof. Robert Gregg

Chesebrough Auditorium (Chrysler Center)

HONOR PLEDGE: Copy (NOW) and SIGN (**after the exam is completed**): I have neither given nor received aid on this exam, nor have I observed a violation of the Engineering Honor Code.

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DO NOT OPEN THE EXAM BOOKLET UNTIL TOLD TO DO SO.

RULES:

1. CLOSED TEXTBOOK
2. CLOSED CLASS NOTES
3. CLOSED HOMEWORK
4. CLOSED HANDOUTS
5. 2 SHEETS OF NOTE PAPER (Front and Back), US Letter Size.
6. NO CALCULATORS, CELL PHONES, HEADSETS, nor DIGITAL DEVICES of any KIND.

The maximum possible score is 80. To maximize your own score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly on this booklet.

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Problems 1 - 5 (30 points: 5×6)

Instructions. Each part of a question is worth 1.5 points.

1. Let $A \in \mathbb{R}^{m \times n}$ be a real matrix. Let the SVD of the matrix be $A = U\Sigma V^\top$, where the columns of V and U are the right and left singular vectors of A, respectively. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$ denote the singular values of A, where $p = \min(m, n)$. Let the smallest non-zero singular value of A be $\sigma_r > 0$, such that $1 \leq r \leq p$. **Circle True or False as appropriate for the following statements:**

T **F** (a) Suppose $A = A^\top > 0$. Then $\Sigma = \text{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i are the eigenvalues of A, the columns of U are the eigenvectors of A, and $V = U$.

T **F** (b) If $m \geq n$, then $\exists$ an orthogonal matrix Q and an upper-triangular, invertible matrix R such that $A = QR$.

T **F** (c) For any matrix E with norm $\|E\| = \sigma_r$, $\text{rank}(A - E) = r$.

T **F** (d) Suppose $m = n$ and $r = p$. Let $\mathcal{M}$ be the set of all singular matrices, $\mathcal{M} = \{M \in \mathbb{R}^{n \times n} \mid \text{rank}(M) < n\}$. Then, for matrix norm $\|B\|_2 = \sqrt{\lambda_{\max}(B^\top B)}$,

$$\sigma_r = \inf_{M \in \mathcal{M}} \|A - M\|_2.$$

2. Consider the normed space $(\mathbb{R}, \mathbb{R}, |\cdot|)$. **Circle True or False as appropriate for the following statements:**

T **F** (a) The sequence $x_n = \sqrt{n}$, $n \geq 1$, is Cauchy since as $n \to \infty$, $|\sqrt{n+1} - \sqrt{n}| \to 0$.

T **F** (b) Let $S \in \mathbb{R}$ be a non-empty open subset. Suppose (x_k) is a sequence in S which converges to $x^* \in S$, i.e., as $k \to \infty$, $x_k \to x^* \in S$. Then $\exists\ \epsilon > 0$ and $N < \infty$ such that for $n \geq N$, $d(x_n, \sim S) \geq \epsilon$.

T **F** (c) Consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. The interior of $\mathbb{Q}$ is non-empty, i.e., $\mathring{\mathbb{Q}} \neq \emptyset$.

T **F** (d) If $S \subset \mathbb{R}$ is closed and bounded, then S is both complete and compact.

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3. Consider a finite-dimensional normed space $(\mathcal{X}, \mathbb{R}, \|\cdot\|)$. **Circle True or False as appropriate for the following statements:**

T **F** (a) Suppose $S \subset \mathcal{X}$ is a complete subset. Consider function $T : S \to S$ such that $\forall x, y \in S$, $\|T(x) - T(y)\| \leq 0.5\|x - y\|$. Define the sequence $x_n = T(x_{n-1})$ starting at x_0. For any $x_0 \in S$, $x_n \to T(x^*)$ and x^* is unique.

T **F** (b) Given $x_0 \in \mathcal{X}$, suppose that $\exists\, \epsilon > 0$ such that $\forall\, \delta > 0$, $\exists\, x \in B_\delta(x_0)$ such that $f(x) \notin B_\epsilon(f(x_0))$. Then for *any* sequence $(x_n) \in \mathcal{X}$ where $x_n \to x_0$, $f(x_n)$ does not converge to $f(x_0)$.

T **F** (c) If $S \subset \mathcal{X}$ is a non-empty closed and bounded subset, then there exists $x^* \in S$ such that $\|x^*\| = \inf_{x \in S} \|x\|$.

T **F** (d) Suppose every sequence (x_n) in $C \subset \mathcal{X}$ has a subsequence which converges to a limit also in C. Then, given function $f : C \to \mathbb{R}$, $\exists\, x^* \in C$ such that $f(x^*) = \sup_{x \in C} f(x)$.

4. Consider the equation $y = Cx + \epsilon$, where $C \in \mathbb{R}^{n \times m}$. You can infer the dimensions of x, y and ϵ. **Circle True or False as appropriate for the following statements:**

T **F** (a) Suppose $\mathcal{E}\{x\} = 0$ and x is uniformly distributed such that $-\infty < x < \infty$. In other words, $\mathcal{E}\{xx^\top\} = P \to \infty I$. Furthermore, suppose $\mathcal{E}\{\epsilon\} = 0$ and $\mathcal{E}\{\epsilon\epsilon^\top\} = Q > 0$ and C is full rank. Then the minimum variance estimate is $\hat{x} = (C^\top Q^{-1} C)^{-1} C^\top Q^{-1} y$.

T **F** (b) Suppose $E\{x\} = 0$, $E\{\epsilon\} = 0$, $E\{x\epsilon^\top\} = 0$, $E\{xx^\top\} = P \geq 0$, $E\{\epsilon\epsilon^\top\} = Q \geq 0$ and $C^\top PC + Q \geq 0$. Then the Minimum Variance Estimate (MVE) can be determined.

T **F** (c) Suppose we can form the minimum variance estimate of x. Then the estimate minimizes the mean squared error between $\hat{x}$ and the true value x, i.e., $E\{\|\hat{x} - x\|^2\}$.

T **F** (d) Suppose $\mathcal{E}\{\epsilon\} = 0$, $\mathcal{E}\{\epsilon\epsilon^\top\} = Q > 0$ and the rows of C are linearly independent. Then a Best Linear Unbiased Estimate (BLUE) of x can be determined.

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5. Consider three random variables X_1, X_2 and X_3 and the random vector $Z = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}^\top$. We are given the following information:

- X_1, X_2 and X_3 are jointly normally distributed.
- $\mathcal{E}\{Z\} = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix}^\top$.
- $\operatorname{cov}(Z, Z) = \Sigma_Z = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

Circle True or False as appropriate for the following statements:

T **F** (a) The eigenvalues of matrix Σ_Z are real and non-negative, $\lambda_i \geq 0$.

T **F** (b) X_1 and X_2 are uncorrelated but are not necessarily independent.

T **F** (c) The covariance of X_3 conditioned on X_1 is $\Sigma_{3|1} = 2$.

T **F** (d) The distribution of random variable X_1 conditioned on X_2 and X_3, or $X_{1|X_2=x_2, X_3=x_3}$, is the same as the distribution of $X_{1|X_3=x_3}$ conditioned on $X_{2|X_3=x_3}$.

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Partial Credit Section of the Exam

For the next problems, partial credit is awarded and you MUST show your work. Unsupported answers, even if correct, receive zero credit. In other words, right answer, wrong reason or no reason could lead to no points. If you come to me and ask whether you have written enough, my answer will be,

"I do not know",

because answering "yes" or "no" would be unfair to everyone else. If you show the steps you followed in deriving your answer, you'll probably be fine. <u>If something was explicitly derived in lecture, handouts or homework, you do not have to re-derive it. You can state it as a known fact and then use it.</u> For example, we proved that real symmetric matrices have real e-values. So if you need this fact, simply state it and use it.

6. (15 points) (Place your answers in the boxes and show your work below.) We are trying to estimate the velocity and acceleration of a vehicle from wheel velocity measurements. Consider its discrete-time dynamics

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}}_{A} x_k + \underbrace{\begin{bmatrix} 0.5 \\ 0 \end{bmatrix}}_{G} w_k$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x_k + v_k,$$

where w_k and v_k are (scalar) zero mean white Gaussian (i.e., normal) noise processes with constant covariances, satisfying all of the standard assumptions in our Kalman Filter handout. The collection of measurements at time k is denoted $Y_k = (y_k, y_{k-1}, \ldots, y_0)$.

At time $k = 5$ the Kalman filter has the data $\hat{x}_{6|5} := \mathcal{E}\{x_6|Y_5\} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and $P_{5|5} := \mathcal{E}\{(x_5 - \hat{x}_{5|5})(x_5 - \hat{x}_{5|5})^T|Y_5\} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$, and for all $k \geq 0$, $R = R_k = \text{cov}\{w_k\} = 3$ and $Q = Q_k = \text{cov}\{v_k\} = 4$.

Determine:

(a) (3 points) The value of $\hat{x}_{5|5}$.

(b) (8 points) The Kalman gain K_6 time $k = 6$.

(c) (4 points) $\mathcal{E}\left\{ x_6|Y_5 \;\middle|\; y_6|Y_5 \right\}$ given measured value $y_6 = 1.5$.

$\hat{x}_{5\|5} =$	$K_6 =$	$\mathcal{E}\left\{ x_6\|Y_5 \;\middle\|\; y_6\|Y_5 \right\} =$

Please show your work for question 6.

7. (20 points) Consider three ZERO MEAN, jointly normal, random variables X_1, X_2, X_3 with covariance matrix

$$\Sigma = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 5 & 1 \\ 4 & 1 & 3 \end{bmatrix}.$$

The above data is used for parts (a) and (b) of the problem.

(a) (8 points) Find the covariance of the random vector $Y = \begin{bmatrix} X_1 - X_2 \\ X_1 + X_2 \end{bmatrix}$

$\Sigma_Y =$

(Show your calculations below)

(b) (12 points) We build a real inner product space $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$ out of the three random variables X_1, X_2,X_3, just as we did when working on the minimum variance estimator. Specifically, we define $\mathcal{X} := \text{span}\{X_1, X_2, X_3\}$ and the inner product by, for $Z_1, Z_2 \in \mathcal{X}$,

$$< Z_1, Z_2 >:= \mathcal{E}\{Z_1 Z_2\}.$$

Use the normal equations to compute the orthogonal projection of $Z := X_1 - X_2$ on $\mathcal{Y} := \text{span}\{X_1 + X_2, X_3\}$. Equivalently, use the normal equations to solve

$$X^* := \arg\min_{Y \in \mathcal{Y}} \; ||Z - Y||^2,$$

where, just to be extra clear, $|| \cdot ||$ is the norm induced by the inner product. *Note: you have all the data you need to compute any required inner product and obtain the corresponding real number!*

$X^* =$

(Show work below)

Please show your work for question 7.

8. (15 points) The following are three (3) short answer questions. You do not need to give a formal proof; only give a few short reasons/calculations why something is TRUE or FALSE. **Part (c) is on the next page.**

(a) **(5 Points)** Suppose $Ax = b$ is overdetermined with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) < n$. Let $A = QR$ be the QR factorization of A. Then $A^T A\hat{x} = A^T b \iff R\hat{x} = Q^T b$ for least squares solution $\hat{x}$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

(b) **(5 Points)** Consider an n-dimensional, real inner product space $(\mathcal{X}, \mathbb{R}, < \cdot, \cdot >)$. Let $\{y^1, \dots, y^k\}$ be linearly independent vectors in $\mathcal{X}$ with $1 < k < n$, and $M := \text{span}\{y^1, \dots, y^k\}$. Consider the standard norm induced by the inner product. For $x_0 \in \mathcal{X}$,

$$\min_{z \in M^\perp} ||x_0 - z||$$

has a unique solution $x^* = x_0 - (\alpha_1 y^1 + \dots + \alpha_k y^k)$ where α satisfies the normal equations $G\alpha = \beta$.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE:

(c) **(5 Points)** The intersection of two non-convex sets is non-convex.

Circle **T or F**. Give a few short reasons/calculations why this is TRUE or FALSE: