Pre-projection theorem
Gram-Schmidt Process

ROB 501

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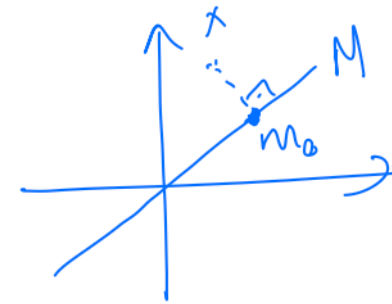
- **Last time: Least squares (high level)**

$$\hat{\alpha} = \underset{\alpha \in \mathbb{R}^2}{\operatorname{argmin}} \, ||Y - A\alpha||^2 \qquad\qquad \hat{\alpha} = (A^T A)^{-1} A^T Y$$

  - **We will build the proof but we need a few new concepts**

Pre-projection Theorem



$(X, \mathbb{R}, \langle \cdot, \cdot \rangle)$ be an inner product space.

Let $M \subset X$ be a subspace, and $x \in X$.

Then,

(a) If $m_0 \in M$ s.t. $\|x - m_0\| \leq \|x - m\|$ $\forall m \in M$,

then m_0 is unique.

(b) A necessary and sufficient condition

$m_0 = \arg\min_{m \in M} d(x, m)$) for $m_0 \in M$ to be a minimizer of

$\min_{m \in M} d(x, m)$ is that the error vector

$(x-m_0)$ is orthagonal to M, i.e. $(x-m_0) \perp M$

Proof: $(\Longrightarrow)$

Claim b) Let $m_0 \in M$. If $\underbrace{\|x-m_0\| = d(x,M)}_{p}$,

then $\underbrace{(x-m_0) \perp M}_{q}$. $\quad p \Rightarrow q$

Contrapositive: $\neg q \Rightarrow \neg p$

$$x-m_0 \not\perp M \Longrightarrow \|x-m_0\| > d(x,M)$$

$$x-m_0 \not\perp M \Rightarrow \exists\, \bar{m} \in M \text{ s.t. } x-m_0 \not\perp \bar{m}$$

$$\| \quad \langle x-m_0, \bar{m} \rangle \neq 0$$

$$\Rightarrow (\bar{m} \text{ is non-zero; ie, } \|\bar{m}\| \neq 0)$$

We can write:

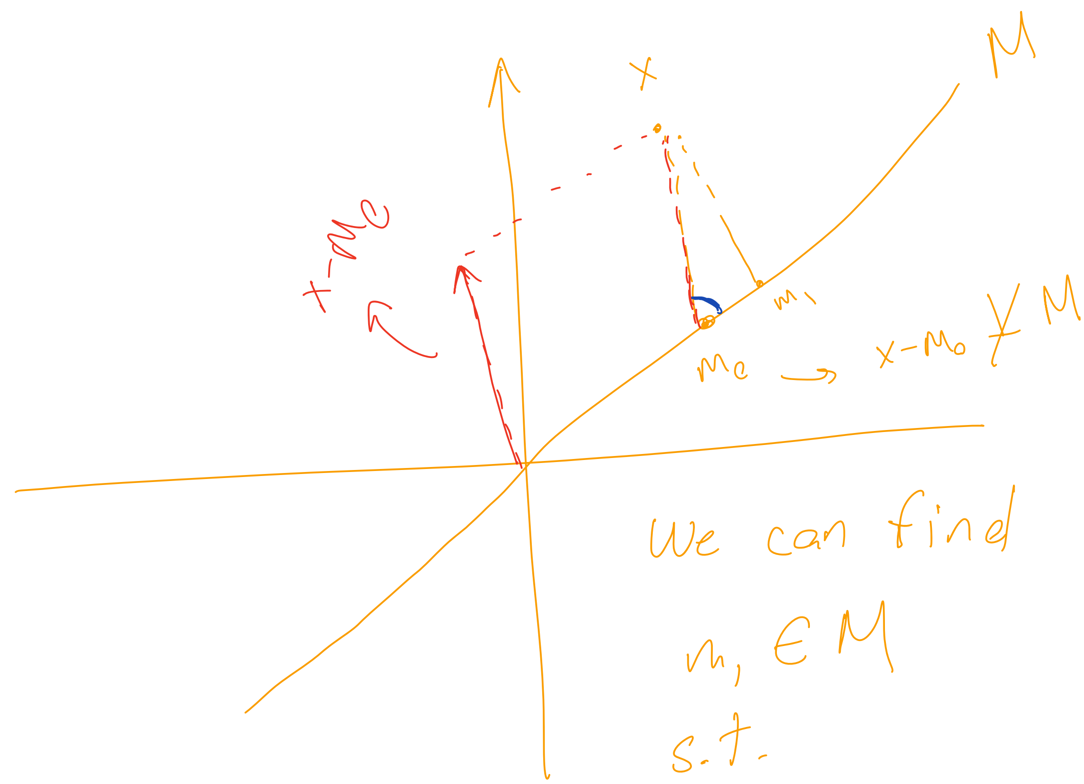
$$\left\langle x-m_0, \frac{\bar{m}}{\|\bar{m}\|} \right\rangle = \underbrace{\frac{1}{\|\bar{m}\|}}_{\neq 0} \underbrace{\langle x-m_0, \bar{m} \rangle}_{\neq 0} \neq 0$$

Thus, without loss of generality, we can assume $\|\bar{m}\| = 1$.

Define: $\beta = \langle x-m_0, \bar{m} \rangle \neq 0$

$$m_1 = m_0 + \beta \bar{m} \in M$$

"To show $\|x-m_1\| < \|x-m_0\|$"



We can find $m_1 \in M$ s.t. $\|x-m_0\| > \|x-m_1\|$

$$\|x-m_1\|^2 = \|x - m_0 - \beta \bar{m}\|^2$$

$$= \langle x - m_0 - \beta \bar{m}, x - m_0 - \beta \bar{m} \rangle$$

$$= \langle x-m_0, x-m_0 \rangle + \langle -\beta \bar{m}, x-m_0 \rangle + \langle x-m_0, -\beta \bar{m} \rangle + \langle -\beta \bar{m}, -\beta \bar{m} \rangle$$

$$= \|x-m_0\|^2 - 2\beta \underbrace{\langle x-m_0, \bar{m} \rangle}_{\beta} + \beta^2$$

$$= \|x-m_0\|^2 - \beta^2 \quad (\text{since } \beta \neq 0,\ \beta^2 > 0)$$

$\Rightarrow$

$\|x-m_1\|^2 < \|x-m_0\|^2$

$\Rightarrow m_0$ cannot be a minimizer.

Other direction for Claim b (+ Claim a)

$(\Leftarrow)$ If $x-m_0 \perp M$, then $\|x-m_0\| = d(x,M)$ (and m_0 is unique).

<u>Proof:</u> Assume $x-m_0 \perp M$ and let $m \in M$ be arbitrary.

$$\|x-m\|^2 = \|\boxed{x-m_0} + \boxed{m_0-m}\|^2$$

Since $(m_0 \in M,\ m \in M) \Rightarrow m_0 - m \in M$

Also $x-m_0 \perp M \Rightarrow (x-m_0) \perp (m_0-m)$

by Pythagorean theorem:

$$\|x-m_0+m_0-m\|^2 = \|x-m_0\|^2 + \|m_0-m\|^2$$

$$\therefore\ d(x,M)^2 = \inf_{m \in M} \|x-m\|^2$$

by our construction

$$= \inf_{m \in M} \|x - m_0\|^2 + \|m_0 - m\|^2$$

$$= \overbrace{\|x-m_0\|^2}^{\geq 0} + \inf_{m \in M} \overbrace{\|m_0 - m\|^2}^{\geq 0}$$

the smallest value of this term is 0, and it is only achieved when $m = m_0$

$\Rightarrow$ m_0 is the unique minimizer.

and for $m = m_0$, we get $d(x, M) = \|x - m_0\|$.

$\boxed{(b')} = \langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle$
$= \alpha_1 \langle x_1, y \rangle$
$+ \alpha_2 \langle x_2, y \rangle$

EECS 560 Handout:[1] Inner Product Spaces

Definition: Let $(X, \mathbb{C})$ be a vector space. A function

$$< \cdot, \cdot >: X \times X \to \mathbb{C}$$

is an **inner product** if

(a) $< x, y > = \overline{< y, x >}$

$\boxed{\text{(b)}}$ $< x, \alpha y_1 + \beta y_2 > = \alpha < x, y_1 > + \beta < x, y_2 >$

(c) $< x, x > \geqslant 0$ for any $x \in X$, and $< x, x > = 0 \Leftrightarrow x = 0$.

In the case of a real vector space $(X, \mathbb{R})$, replace (a) with
(a′): $< x, y > = < y, x >$.

Examples:

(a) $(\mathbb{C}^n, \mathbb{C})$ $\quad < x, y > = \overline{x}^T y = \underbrace{x^*}_{\text{complex conjugate transpose}} y$

(b) $(\mathbb{R}^n, \mathbb{R})$ $\quad < x, y > = x^T y$

(c) $X = C[a, b]$ = space of continuous functions on $[a, b]$

$$< f, g > = \int_a^b f(t) g(t) dt$$

Theorem: [Cauchy-Schwarz Inequality] Suppose that $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$. Let $(X, \mathcal{F}, < \cdot, \cdot >)$ be an **inner product space** (i.e. $(X, \mathcal{F})$ is a vector space and $< \cdot, \cdot >$ is an inner product on X). Then, for all $x, y \in X$,

$$|< x, y >| \leqslant < x, x >^{1/2} \cdot < y, y >^{1/2} .$$

Proof: See Chen, Second Edition, page 59. $\square$

[1] Courtesy of Jessy Grizzle.

Corollary: Let $(X, \mathcal{F}, < \cdot, \cdot >)$ be an inner product space. Then

$$\|x\| := < x, x >^{1/2}$$

is a norm on X.

Proof: The main thing to establish is the triangle inequality:

$$\|x + y\| \leqslant \|x\| + \|y\|.$$

This is equivalent to showing:

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2.$$

Brute force computation:

$$\begin{aligned}
\|x + y\|^2 &= < x + y, x + y > \\
&= < x + y, x > + < x + y, y > \\
&= \overline{< x, x + y >} + \overline{< y, x + y >} \\
&= \overline{< x, x >} + \overline{< x, y >} + \overline{< y, x >} + \overline{< y, y >} \\
&= < x, x > + < y, x > + < x, y > + < y, y > \\
&= \|x\|^2 + \|y\|^2 + < y, x > + < x, y > \\
&\leqslant \|x\|^2 + \|y\|^2 + |< y, x >| + |< x, y >| \\
&\leqslant \|x\|^2 + \|y\|^2 + 2\underbrace{< x, x >^{1/2}}_{\|x\|} \cdot \underbrace{< y, y >^{1/2}}_{\|y\|},
\end{aligned}$$

where the last inequality is from Cauchy-Schwarz Inequality. $\square$

Definition:

(a) Two vectors x and y are **orthogonal** if $< x, y >= 0$. Notation: $x \perp y$.

(b) A **set of vectors** S is **orthogonal** if

$$\forall x, y \in S, \quad x \neq y, \; < x, y >= 0.$$

(c) If in addition $\|x\| = 1 \; \forall x \in S$, S is an *orthonormal set.*

Construct a new set $v = \{v_i \mid i = 1, \dots, n\}$ s.t. v is an orthogonal set.

How to Construct Orthonormal Sets ?

Given

Gram-Schmidt Process: Two steps: orthogonalize, then normalize.

Let $\{y_i \mid i = 1, ..., n\}$ be a *linearly independent* set of vectors. Define a set of vectors $\{v_i \mid i = 1, ..., n\}$ by:

$$v_1 = y_1$$
$$v_2 = y_2 - a_{21} v_1,$$

and **choose** a_{21} so that $< v_1, v_2 >= 0$.

$$\therefore 0 =< v_1, v_2 > =< v_1, y_2 - a_{21} v_1 >$$
$$=< v_1, y_2 > - a_{21} < v_1, v_1 > = 0$$
$$\therefore a_{21} = \frac{< v_1, y_2 >}{\|v_1\|^2}$$

$$a_{21} = \frac{\langle v_1, y_2 \rangle}{\|v_1\|^2}$$

Write $v_3 = y_3 - a_{31} v_1 - a_{32} v_2$

$$\longrightarrow \quad 0 =< v_1, v_3 >=< v_1, y_3 - a_{31} v_1 - a_{32} v_2 >$$
$$=< v_1, y_3 > - a_{31} < v_1, v_1 > - a_{32} \underbrace{\cancel{< v_1, v_2 >}}_{=0}$$

$$y_2 = v_2 + a_{21} v_1$$

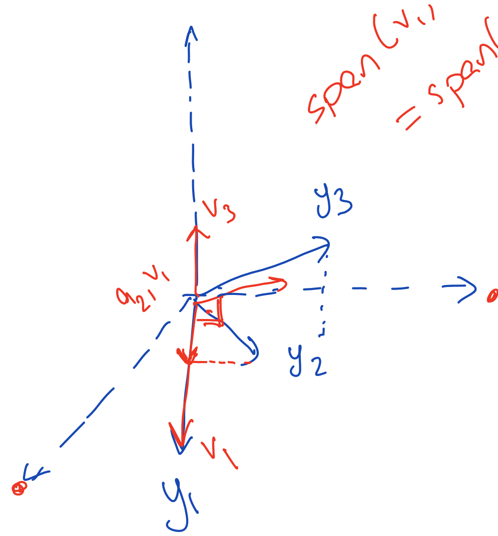
$$\therefore a_{31} = \frac{< v_1, y_3 >}{\|v_1\|^2}$$
$$\longrightarrow \quad 0 =< v_2, v_3 >=< v_2, y_3 - a_{31} v_1 - a_{32} v_2 >$$
$$=< v_2, v_3 >=< v_2, y_3 > - a_{31} \underbrace{\cancel{< v_2, v_1 >}}_{=0} - a_{32} < v_2, v_2 >$$

$$\therefore a_{32} = \frac{< v_2, y_3 >}{\|v_2\|^2}$$

$\mathrm{span}(v_1, v_2) = \mathrm{span}(y_1, y_2)$

In general, one obtains:

$$v_k = y_k - \sum_{j=1}^{k-1} \underbrace{\frac{< v_j, y_k >}{\|v_j\|^2}}_{a_{kj}} \cdot v_j.$$



Now, $\{v_k \mid k = 1, ..., n\}$ is an **orthogonal** set.

Define: $\tilde{v}_i = \dfrac{v_i}{\|v_i\|} \Rightarrow \{\tilde{v}_i \mid i = 1, ..., n\}$ is **orthonormal**.

$$\tilde{v}_i = \frac{v_i}{\|v_i\|}$$

Example: Construct a set of orthonormal vectors from

$$y_1^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad y_2^T = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}, \quad y_3^T = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

The vectors are easily checked to be linearly independent.
Step 1: Let

$$\begin{aligned} v_1 &= y_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \\ v_2 &= y_2 - \frac{< v_1, y_2 >}{< v_1, v_1 >} v_1 = y_2, \quad (\text{because} \quad < v_1, y_2 >= 0) \end{aligned}$$

In this case y_1 and y_2 were already orthogonal, so there was nothing to do. Continuing,

$$\begin{aligned} v_3 &= y_3 - \frac{< v_1, y_3 >}{< v_1, v_1 >} v_1 - \frac{< v_2, y_3 >}{< v_2, v_2 >} v_2 && (1) \\ &= y_3 - \frac{2}{2} v_1 - \frac{4}{6} v_2 && (2) \\ &= \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}^T && (3) \end{aligned}$$

Step 2: Normalize v_i to get $\tilde{v}_i$:

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \tilde{v}_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \tilde{v}_3 = \frac{v_1}{\|v_3\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

$\square$

Problem:[0] Let $\{y_1, \cdots, y_n\}$ be a linearly independent set and let $\{v_1, \cdots, v_n\}$ be the orthogonal set produced by the Gram-Schmidt process. Then, $\forall\ 1 \leqslant l \leqslant n$,

$$\mathrm{span}\{y_1, \cdots, y_l\} = \mathrm{span}\{v_1, \cdots, v_l\}.$$

Solution: Recall

$$v_k = y_k - \sum_{j=1}^{k-1} \frac{< v_j, y_k >}{\|v_j\|^2} \cdot v_j \tag{4}$$

$l = 1$: $v_1 = y_1$ so it is trivially true.

Suppose now it is true for $l = k-1$; will show holds for $l = k$. From (4),

$$y_k = v_k + \sum_{j=1}^{k-1} \frac{< v_j, y_k >}{\|v_j\|^2} \cdot v_j \ \Rightarrow\ \mathrm{span}\{y_1, \cdots, y_k\} \subset \mathrm{span}\{v_1, \cdots, v_k\}.$$

Left to show: $v_k \in \mathrm{span}\{y_1, \cdots, y_k\}$.
By hypothesis,

$$v_j \in \mathrm{span}\{y_1, \cdots, y_{k-1}\} \text{ for all } 1 \leqslant j \leqslant k-1,$$

so

$$\sum_{j=1}^{k-1} \left(\frac{< v_j, y_k >}{\|v_j\|^2} \right) v_j \ \in\ \mathrm{span}\{y_1, \cdots, y_{k-1}\} \ \subset\ \mathrm{span}\{y_1, \cdots, y_k\}.$$

Clearly, $y_k \in \mathrm{span}\{y_1, \cdots, y_k\}$.

$$\therefore v_k = y_k - \sum_{j=1}^{k-1} \left(\frac{< v_j, y_k >}{\|v_j\|^2} \right) v_j \ \in\ \mathrm{span}\{y_1, \cdots, y_k\}$$

because $\mathrm{span}\{y_1, \cdots, y_k\}$ is a subspace. $\square$

Definition: $x \perp y \ \Leftrightarrow\ < x, y >= 0$; $\quad x \perp S \ \Leftrightarrow\ \forall\ y \in S,\ \ < x, y >= 0.$

Problem:[1] Suppose that $x \perp \{y_1, \cdots, y_k\}$ (i.e. $< x, y_i >= 0, 1 \leqslant i \leqslant k$). Then x is $\perp$ to span$\{y_1, \cdots, y_k\}$, i.e, $< x, w >= 0 \; \forall \; w \in$ span$\{y_1, \cdots, y_k\}$.

Solution: $$< x, \sum_{i=1}^{k} \alpha_i y_i >= \sum_{i=1}^{k} \alpha_i < x, y_i >= 0$$ □

$< x, w >$

Problem:[2] Let $\{v_1, \cdots, v_n\}$ be an **orthonormal basis** for a vector space $(X, \mathbb{R})$. Calculate the representation of $x \in X$ with respect to $\{v_1, \cdots, v_n\}$.

Solution: $\{v_1, \cdots, v_n\}$ a basis $\Rightarrow \exists\,!$ coeff. $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ such that

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

Now,

$$\begin{aligned} < v_i, x > &= < v_i, \sum_{k=1}^{n} \alpha_k v_k > \\ &= \sum_{k=1}^{n} \alpha_k \underbrace{< v_i, v_k >}_{\delta_{ik}} \\ &= \alpha_i \end{aligned}$$

$$\therefore \boxed{\alpha_i =< v_i, x >}$$

$$\text{or, } \boxed{x = \sum_{i=1}^{n} < v_i, x > v_i}$$

□

WON'T COVER

Reciprocal Basis Vectors

Problem:[3] Let $(X, \mathbb{R}, < \cdot, \cdot >)$ be an n-dim. inner product space, and let $\{v_1, \cdots, v_n\}$ be a basis for X (*not* necessarily orthonormal). Show that for each $i = 1, 2, \cdots, n$, $\exists\ r_i \in X$ such that

$$< r_i, v_j >= \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \tag{5}$$

Solution: Suffices to prove this for $i = 1$. Apply the Gram-Schmidt process to the linearly independent set $\{v_2, v_3, \cdots, v_n, v_1\}$. Note that v_1 has been permuted to the end. (To compute r_3 for example, you would permute v_3 to the end and apply the same procedure.) The Gram-Schmidt process will produce n-vectors $\{\tilde{v}_2, \tilde{v}_3, \cdots, \tilde{v}_n, \tilde{v}_1\}$. By construction, $< \tilde{v}_1, \tilde{v}_j >= 0$, $j = 2, \cdots, n$, which implies, by Problem 1, that

$$\tilde{v}_1 \perp \operatorname{span}\{\tilde{v}_2, \cdots, \tilde{v}_n\}, \tag{6}$$

and by Problem 0 that

$$\tilde{v}_1 \perp \operatorname{span}\{v_2, \cdots, v_n\}. \tag{7}$$

From the Gram-Schmidt Process,

$$\tilde{v}_1 = v_1 - \sum_{j=2}^{n} \frac{< \tilde{v}_j, v_1 >}{\|\tilde{v}_j\|^2} \tilde{v}_j.$$

Therefore, $v_1 = \tilde{v}_1 + \sum_{j=2}^{n} \frac{< \tilde{v}_j, v_1 >}{\|\tilde{v}_j\|^2} \tilde{v}_j$, and hence,

$$< \tilde{v}_1, v_1 >=< \tilde{v}_1, \tilde{v}_1 > \quad \text{by (6) and (7).}$$

$\therefore$ If we choose $r_1 = \frac{\tilde{v}_1}{\|\tilde{v}_1\|^2}$, we have

$$< r_1, v_j >= \begin{cases} 1 & j = 1 \\ 0 & j \neq 1 \end{cases} \quad \text{as desired.}$$

$\square$

Remarks: (a) $\{r_1, \cdots, r_n\}$ satisfying (5) is called a **reciprocal basis**. (b) In order to find r_k, simply rotate v_k to back as in $\{v_1, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n, v_k\}$ and apply the Gram-Schmidt procedure as above.

WON'T COVER

Problem:[4] Let $(X, \mathbb{R}, < \cdot, \cdot >)$ be an n-dim. inner product space, let $\{v_1, \cdots, v_n\}$ be a basis for X, and let $\{r_1, \cdots, r_n\}$ be the corresponding reciprocal basis. Show that for all $x \in X$,

$$x = \sum_{i=1}^{n} < r_i, x > v_i \tag{8}$$

In other words, determining the representation of x with respect to $\{v_1, \cdots, v_n\}$ can be accomplished by computing inner products!

Solution: Because $\{v_1, \cdots, v_n\}$ is a basis, there exist unique $\alpha_i \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i v_i \tag{9}$$

Hence,

$$< r_i, x > = < r_i, \sum_{k=1}^{n} \alpha_k v_k > \tag{10}$$

$$= \sum_{k=1}^{n} \alpha_k < r_i, v_k > \tag{11}$$

$$= \alpha_i \tag{12}$$

because $< r_i, v_k >$ equals one if $i = k$ and zero otherwise. This proves (8).

Inner Products: Example Computation for $(\mathbb{R}^3, \mathbb{R})$

Given data:

$$< p, q >= p^T q = \sum_{i=1}^{3} p_i q_i$$

$$\{y_1, y_2, y_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Apply Gram-Schmidt to Produce an Orthogonal Basis:

$$v_1 = y_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\|v_1\|^2 = (v_1)^T v_1 = 2;$$

$$v_2 = y_2 - \frac{< v_1, y_2 >}{\|v_1\|^2} v_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$\|v_2\|^2 = 9\frac{1}{2} = \frac{19}{2};$$

$$v_3 = y_3 - \frac{< v_1, y_3 >}{\|v_1\|^2} v_1 - \frac{< v_2, y_3 >}{\|v_2\|^2} v_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{1} \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{3\frac{1}{2}} \frac{1}{\frac{19}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{7}{38} \\ \frac{7}{38} \\ \frac{21}{19} \end{bmatrix} = \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}.$$

Normalize to obtain Orthonormal Basis:

$$\tilde{v}_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\tilde{v}_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{-1}{\sqrt{38}} \\ \frac{1}{\sqrt{38}} \\ 3\sqrt{\frac{2}{19}} \end{bmatrix}$$

$$\tilde{v}_3 = \frac{v_3}{\|v_3\|} = \frac{19}{\sqrt{76}} \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}$$

Obtain Reciprocal Basis:

We seek a basis $\{r_1, r_2, r_3\}$ such that $< r_i, y_j >= \delta_{ij}$.

Step 1: r_1 is found by applying the Gram-Schmidt process to $\{y_2, y_3, y_1\}$.

$$v_1^1 := y_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$v_1^2 := y_3 - \frac{< v_1^1, y_3 >}{\|v_1^1\|^2} v_1^1 = \begin{bmatrix} -\frac{5}{14} \\ \frac{2}{7} \\ -\frac{1}{14} \end{bmatrix}$$

$$v_1^3 := y_1 - \frac{< v_1^1, y_1 >}{\|v_1^1\|^2} v_1^1 - \frac{< v_1^2, y_1 >}{\|v_1^2\|^2} v_1^2 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$r_1 := \frac{v_1^3}{\|v_1^3\|^2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Step 2: r_2 is found by applying the Gram-Schmidt process to $\{y_3, y_1, y_2\}$.

$$v_2^1 := y_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2^2 := y_1 - \frac{< v_1^1, y_1 >}{\|v_1^1\|^2} v_1^1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$v_2^3 := y_2 - \frac{< v_1^1, y_2 >}{\|v_1^1\|^2} v_1^1 - \frac{< v_1^2, y_1 >}{\|v_1^2\|^2} v_1^2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$r_2 := \frac{v_2^3}{\|v_2^3\|^2} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Step 3: r_3 is found by applying the Gram-Schmidt process to $\{y_1, y_2, y_3\}$.

$$v_3^1 := y_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3^2 := y_2 - \frac{< v_1^1, y_2 >}{\|v_1^1\|^2} v_1^1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$$

$$v_3^3 := y_3 - \frac{< v_1^1, y_3 >}{\|v_1^1\|^2} v_1^1 - \frac{< v_1^2, y_3 >}{\|v_1^2\|^2} v_1^2 = \begin{bmatrix} -\frac{6}{19} \\ \frac{6}{19} \\ -\frac{2}{19} \end{bmatrix}$$

$$r_3 := \frac{v_3^3}{\|v_3^3\|^2} = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Inner Products: Example Computation for $(C[0,1],\mathbb{R})$

Given data:

$$C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \;\; \text{continuous}\}, \quad < f,g >= \int_0^1 f(\tau)g(\tau)d\tau$$

$$\{y_1, y_2, y_3\} = \{1, t, t^2\}$$

Apply Gram-Schmidt to Produce an Orthogonal Basis:

$$\begin{aligned}
v_1 &= y_1 = 1 \\
\|v_1\|^2 &= \int_0^1 (1)^2 d\tau = 1;
\end{aligned}$$

$$\begin{aligned}
v_2 &= y_2 - \frac{< v_1, y_2 >}{\|v_1\|^2} v_1 \\
&= t - \underbrace{\int_0^1 1 \cdot \tau d\tau}_{\frac{1}{2}} \cdot \frac{1}{1} \cdot 1 = t - \frac{1}{2} \\
\|v_2\|^2 &= \int_0^1 (\tau - \frac{1}{2})^2 d\tau = \frac{1}{12};
\end{aligned}$$

$$\begin{aligned}
v_3 &= y_3 - \frac{< v_1, y_3 >}{\|v_1\|^2} v_1 - \frac{< v_2, y_3 >}{\|v_2\|^2} v_2 \\
&= t^2 - \underbrace{\int_0^1 1 \cdot \tau^2 d\tau}_{\frac{1}{3}} \cdot \frac{1}{1} \cdot 1 - \underbrace{\int_0^1 (\tau - \frac{1}{2})\tau^2 d\tau}_{\frac{1}{12}} \left(\frac{1}{\frac{1}{12}}\right) \left(t - \frac{1}{2}\right) \\
&= t^2 - \frac{1}{3} - \left(t - \frac{1}{2}\right) \\
&= t^2 - t + \frac{1}{6}.
\end{aligned}$$

Doing Inner products on $C[a,b]$ in MATLAB

```
>> clear *
>> syms t % declare to be a symbolic variable

>> % INT(S,a,b) is the definite integral of S with respect to
   % its symbolic variable from a to b.  a and b are each
   % double or symbolic scalars.

>> y1=1+0*t % Otherwise MATLAB is too dumb to realize
            % that y1 is a trivial function of the symbolic
            % variable t
y1 = 1

>> y2=t;
>> y3=t^2;

% Start the G-S Procedure. Here we assume C[0,1], that is
% C[a,b], with [a,b]=[0,1]

>> v1=y1

v1=1

>> v2=y2-int(v1*y2,0,1)*v1/int(v1^2,0,1)

v2=t-1/2

>> v3=y3-int(v1*y3,0,1)*v1/int(v1^2,0,1)-
int(v2*y3,0,1)*v2/int(v2^2,0,1)

v3=t^2+1/6-t

% Next, normalize to length one

v1_tilde=v1/int(v1^2,0,1)^.5
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v1_tilde=1

>> v2_tilde=v2/int(v2^2,0,1)^.5

v2_tilde=(t-1/2)*12^(1/2)

>> simplify(v2_tilde);

ans=(2*t-1)*3^(1/2)

>> v3_tilde=simplyfy(v3/int(v3^2,0,1)^.5)

v3_tilde=(6*t^2+1-6*t)*5^(1/2)
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A useful lemma

Lemma: Let $(\mathcal{X}, \mathcal{F})$ be an n-dimensional vector space and, for $1 \leq k < n$, let $\{v^1, \ldots, v^k\}$ be a linearly independent set in $\mathcal{X}$. Then, $\exists v^{k+1} \in \mathcal{X}$ such that $\{v^1, \ldots, v^{k+1}\}$ is linearly independent.

Proof: $p \Rightarrow q$ (contrapositive: $\neg q \Rightarrow \neg p$)

p: $1 \leq k < n$, $\{v^1, \ldots, v^k\}$ is lineerly independent

q: $\exists v^{k+1} \in \mathcal{X}$ s.t. $\{v^1, \ldots, v^k, v^{k+1}\}$ is lin. indep.

Assume $\neg q$: $\forall x \in \mathcal{X} \quad x \in \text{span}\{v^1, \ldots, v^k\}$

$\Rightarrow \mathcal{X} \subset \text{span}\{v^1, \ldots, v^k\}$

$n = \dim(\mathcal{X}) \leq \dim(\{v^1, \ldots, v^k\}) \leq k$

$\therefore k \geq n \Rightarrow \neg p$

Completion to a basis:
Proposition: Let $(\mathcal{X}, \mathcal{F})$ be n-dim vect. space. If $\{v^1, \ldots, v^k\}$ is a lin. ind. set $1 \leq k < n$ in $\mathcal{X}$, $\exists \{v^{k+1}, \ldots, v^n\}$ s.t. $\{v^1, \ldots, v^n\}$ is a basis for $\mathcal{X}$.

Orthogonal complement

Consider an inner product space $(\mathcal{X}, \mathcal{F}, \langle \cdot , \cdot \rangle)$, $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$

<u>Def:</u> Suppose $S \subset \mathcal{X}$ is a subset. Then,

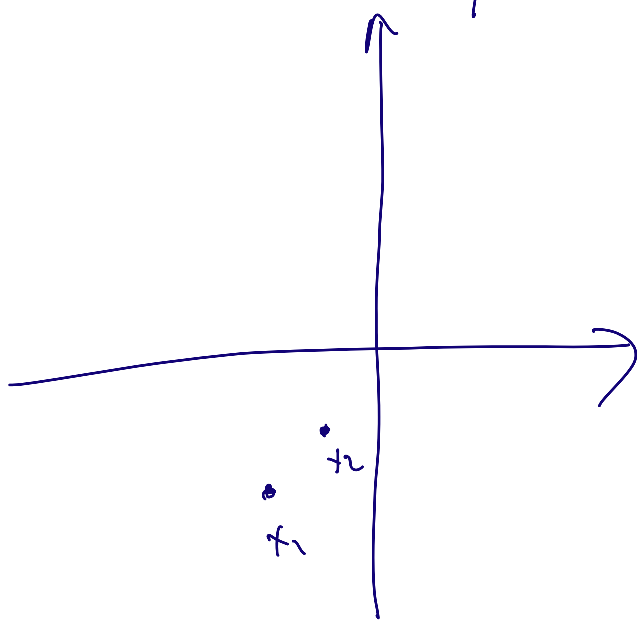
$$S^{\perp} := \{ x \in \mathcal{X} \mid x \perp S \}$$

is the orthagonal complement of S.

<u>Exercise:</u>

1. $S^{\perp}$ is a subspace of $\mathcal{X}$.
2. $S^{\perp} = (\operatorname{span}(S))^{\perp}$

$\mathcal{X} = \mathbb{R}^2$



$S = \{x_1, x_2\}$

$\langle x, y \rangle = \bar{x}^T y$

$\langle y, x \rangle = \bar{y}^T x$

$\overline{\langle y, x \rangle} = \overline{\bar{y}^T x}$
$= y^T \bar{x}$

$x = a\tilde{x}$

$\tilde{x} = x/a$

$\tilde{x} = \dfrac{x}{\|x\|_V}$

$\|x\|_V = a$

$\sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \tilde{x} \neq 0 \\ \|\tilde{x}\|_V = 1}} \dfrac{\|A a\tilde{x}\|_V}{\|a\tilde{x}\|_V} = \dfrac{\cancel{a}\|A\tilde{x}\|_V}{\cancel{a}\|\tilde{x}\|_V = 1}$

$(\mathbb{R}^n, \mathbb{R})$ OFFICE HOUR

$\|\cdot\|_2$

$\|\cdot\|_1$

$\|\cdot\|_p$

$f_1(A) \doteq \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \dfrac{\|Ax\|_1}{\|x\|_1} = \sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_1 = 1}} \|A\tilde{x}\|_1$

$f_2(A) \doteq \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \dfrac{\|Ax\|_2}{\|x\|_2}$

A

$\begin{bmatrix} 4 & -1 \\ -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

→ if $\langle x, y \rangle = x^T A y$ is an inner product

↓

$\langle x, x \rangle^{1/2}$ is a norm

$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$\sup_{\substack{\tilde{x} \in \mathbb{R}^n \\ \|\tilde{x}\|_\infty = 1}} \|A\tilde{x}\|_\infty$

$\tilde{x} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

Ax

$\tilde{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \tilde{x}$