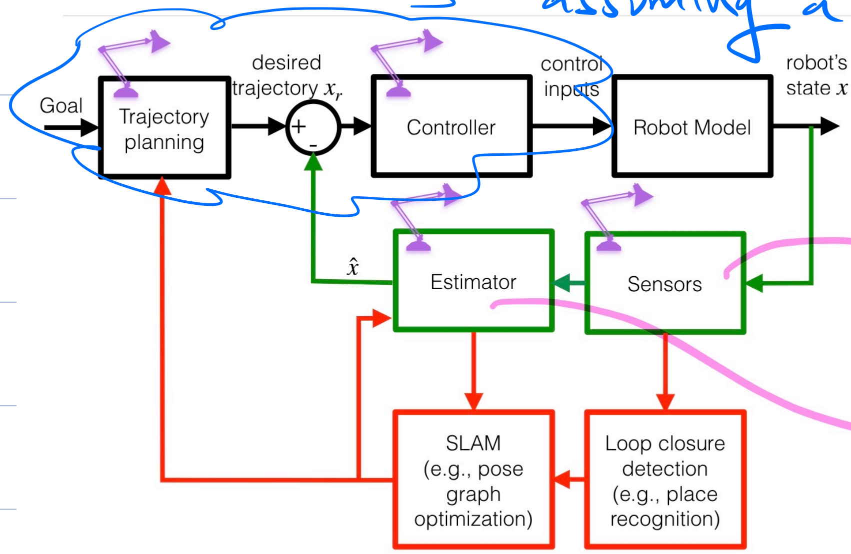
Big Picture

focus from now on, assuming a simplifying robot model (e.g., particle)



accelerometer
gyro

position fixing
(no dynamics
kinematics)

= **focus of ANAV**

Our focus on trajectory planning and control will be from the perspective of pursuit-evasion: a robot (assumed to be a particle) is chasing a target (who, possibly, tries to escape)

Applications

- Missile guidance

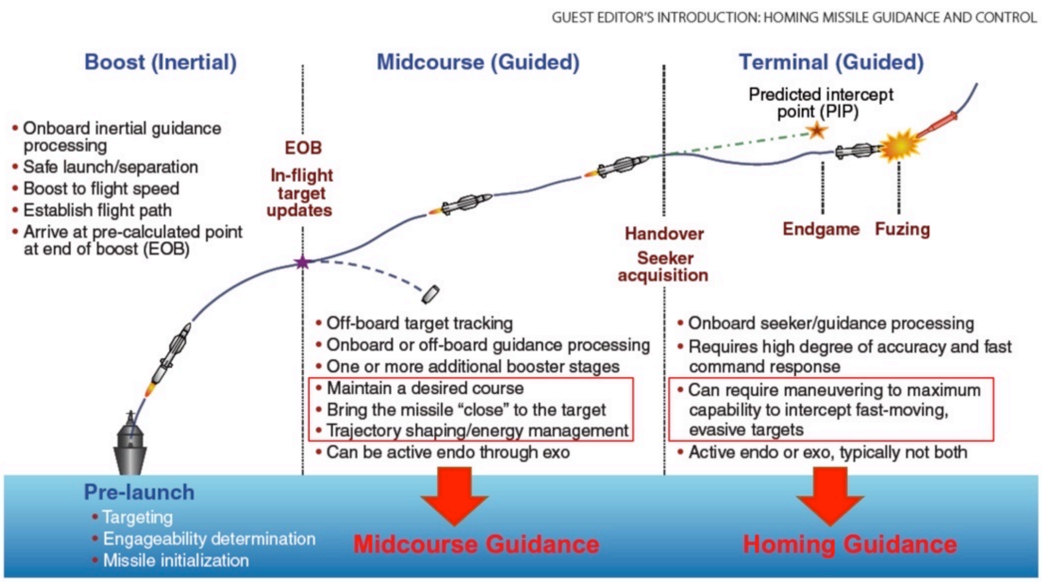
- Rendezvous

- Whatever application requires

a robot come close to, or land on, or collide with, or catch a target.

Missile Guidance

Overview: Phases of Missile Guidance



From Neil F. Palumbo: "Homing Missile Guidance and Control"

our focus

Terminal guidance

The problem of making a pursuer to come close to, or hit, e.t.c. a target.

A type of terminal guidance is homing guidance

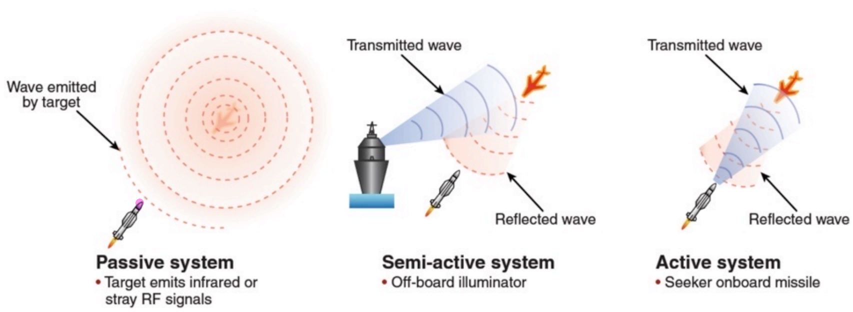
Basic assumption of Homing Guidance

- Pursuer has a velocity of constant norm

Note: The assumption is violated in practice, but helps with the technical analysis.

POLL 1

Overview: Homing Guidance

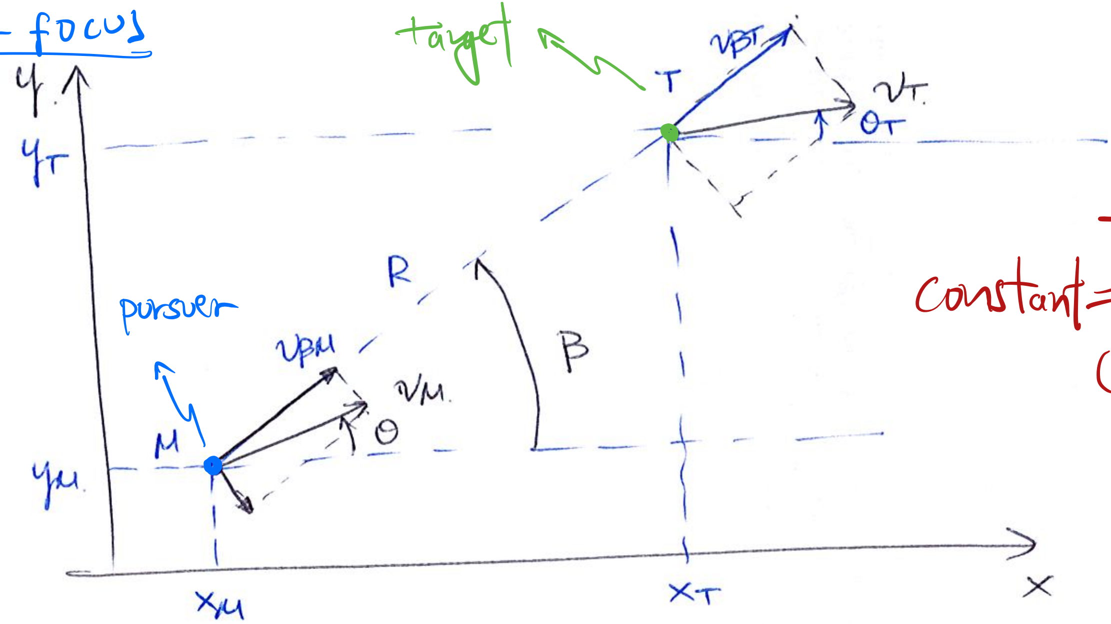


Passive Homing (Seeking): Heat generated by the target is detected and homed on. Also referred to as "heat seeking" or "infrared seeking".

Semi-active Homing (Seeking): Consists of a passive radar receiver on the missile and a separate targeting radar that "illuminates" the target.

Active Homing (Seeking): A radar system on the missile provides a guidance signal. The radar is autonomously kept pointed directly at the target.

2D-focus



Assumption:

constant $= |v_M|, |v_T|$
(constant norm of speeds!)

Homing Problem: For the pursuer (missile) find a guidance law for θ as a function of the line-of-sight β, and its time derivative, $\dot{\beta}$, to cause a collision, or near collision. (R and β are measured)

$$R = \sqrt{(x_T - x_M)^2 + (y_T - y_M)^2}$$

$$\dot{R} = \frac{(x_T - x_M)(\dot{x}_T - \dot{x}_M) + (y_T - y_M)(\dot{y}_T - \dot{y}_M)}{R}$$

$$\dot{R} = \frac{(x_T - x_M)(v_T \cos\theta_T - v_M \cos\theta) + (y_T - y_M)(v_T \sin\theta_T - v_M \sin\theta)}{R}$$

$$\dot{R} = \frac{\cancel{R}\cos\beta\,(v_T \cos\theta_T - v_M \cos\theta) + \cancel{R}\sin\beta\,(v_T \sin\theta_T - v_M \sin\theta)}{\cancel{R}}$$

$$\dot{R} = v_T \cos\beta \cos\theta_T - v_M \cdot \cos\beta \cos\theta + v_T \sin\beta \sin\theta_T - v_M \sin\beta \sin\theta$$

$$\boxed{\dot{R} = v_T \cos(\beta - \theta_T) - v_M \cos(\beta - \theta)} \quad (5.1)$$

(notes written by Prof. Dimitra Panagou)

$$\beta = \arctan\left(\frac{y_T - y_M}{x_T - x_M}\right)$$

$$\dot{\beta} = \frac{1}{1 + \left(\frac{y_T - y_M}{x_T - x_M}\right)^2} \frac{d}{dt}\left(\frac{y_T - y_M}{x_T - x_M}\right) =$$

$$= \frac{\cancel{(x_T - x_M)^2}}{(x_T - x_M)^2 + (y_T - y_M)^2} \frac{(\dot{y}_T - \dot{y}_M)(x_T - x_M) - (y_T - y_M)(\dot{x}_T - \dot{x}_M)}{\cancel{(x_T - x_M)^2}}$$

$$= \frac{(v_T \sin\theta_T - v_M \sin\theta)\cancel{R}\cdot\cos\beta - \cancel{R}\cdot\sin\beta\,(v_T\cos\theta_T - v_M\cos\theta)}{R^{\cancel{2}}}$$

$$= \frac{v_T \sin\theta_T \cos\beta - v_M \sin\theta\cos\beta - v_T \sin\beta\cos\theta_T + v_M \sin\beta\cos\theta}{R}$$

$$= \frac{v_T \sin(\theta_T - \beta) - v_M \sin(\theta - \beta)}{R} \Rightarrow$$

$$\boxed{\dot{\beta} = -\frac{v_T \sin(\beta - \theta_T) - v_M \sin(\beta - \theta)}{R}} \qquad (5.2)$$

Equations (5.1) and (5.2) are the fundamental equations of homing guidance.

Homing Strategies.

① Pursuit Guidance.

The missile is always heading towards the target.

i.e., $\boxed{\theta = \beta}$

controlled → θ; β → measured!

Let us analyze the behavior of the pursuer during pursuit guidance.

For simplicity we set $\theta_T = 0$, $v_T = \text{constant}$ in (5.1), (5.2), i.e, we assume that the target is not maneuvering.

We have

$$\boxed{\begin{aligned} \dot{R} &= v_T \cos\beta - v_M \\ \dot{\beta} &= -\frac{v_T \sin\beta}{R} \end{aligned}}$$

Unless $\beta = 0$, or $\beta = \pi$ (tail-chase or head-on collision course, respectively), we have that $\dot{\beta} \neq 0$.

Consequently, the missile always turns during the engagement, unless the engagement is a tail-chase or head-on collision.

Note! It is advantageous for the pursuer to use a homing strategy that minimizes the requirement to turn.

Why? Because turning requires generating lift, which induces drag, which consumes energy, which makes the pursuer slower.

Let us combine the equations as

$$\frac{\overset{\circ}{R}}{\overset{\circ}{\beta}} = \frac{dR}{d\beta} = \frac{v_T \cos\beta - v_M}{-v_T \cdot \frac{\sin\beta}{R}} = \left(-\operatorname{cotan}\beta + \gamma \operatorname{cosec}\beta\right)R,$$

where we have defined $\boxed{\gamma = \frac{v_M}{v_T}}$ the velocity ratio.

Then we have

$$\frac{dR}{R} = \left(-\operatorname{cotan}\beta + \gamma \operatorname{cosec}\beta\right) d\beta$$

Integrate:

$$\log R = -\log|\sin\beta| + \gamma \log\left|\tan\frac{\beta}{2}\right| + \text{const.}$$

Let us consider for simplicity and without loss of generality that $0 \leq \beta \leq \pi$.

Then $\log R = -\log(\sin\beta) + \gamma \log\left(\tan\frac{\beta}{2}\right) + \text{const.}$

$$\log\left(\frac{R\sin\beta}{\left(\tan\frac{\beta}{2}\right)^\gamma}\right) = \text{const.}$$

This further reads that

$$\frac{R\sin\beta}{\tan\left(\frac{\beta}{2}\right)^\gamma} = \frac{R_0\sin\beta_0}{\tan\left(\frac{\beta_0}{2}\right)^\gamma} = K.$$

where K is a constant, R_0, β_0 and R, β are values for the distance and the line-of-sight at arbitrary time instances.

We also observe the following.

As $R \to 0$, i.e. as the missile approaches the target,

we have $\dfrac{\sin\beta}{\left(\tan\frac{\beta}{2}\right)^\gamma} \to +\infty$, i.e. $\dfrac{\left(\tan\frac{\beta}{2}\right)^\gamma}{\sin\beta} \to 0$.

Since $\sin\beta$ is finite, the possibility of the ratio going to zero can happen if $\beta \to 0$.

$$\Rightarrow \simeq \frac{\beta^{\gamma-1}}{2\gamma} \to 0$$

$$\Rightarrow \beta^{\gamma-1} \to 0$$

$$\Rightarrow \gamma > 1$$

For $\beta \to 0$, we can approximate $\sin\beta \simeq \beta$, $\tan\frac{\beta}{2} \simeq \frac{\beta}{2}$

then $\dfrac{\left(\tan\frac{\beta}{2}\right)^\gamma}{\sin\beta} \simeq \dfrac{\beta^{\gamma-1}}{2^\gamma}$ proportional to $\beta^{\gamma-1}$

Thus for $\beta \to 0$ we need to have $\gamma > 1$,

i.e. $\boxed{v_M > v_T}$! not surprisingly, the missile needs to be faster than the target.

We can now study the turning requirements of the missile.

We have $R = \dfrac{K\left(\tan\frac{\beta}{2}\right)^\gamma}{\sin\beta}$

Then the fundamental homing equation reads

$$\dot{\beta} = -\frac{v_T \sin^2\beta}{K\left(\tan\frac{\beta}{2}\right)^\gamma}$$

Since as the pursuer approaches the target we showed that $\beta \to 0$, we can further

$$\dot{\beta} = - \frac{v_T \sin^2\beta}{K\left(\tan\frac{\beta}{2}\right)^\gamma} \simeq - \frac{v_T\ 2^\gamma}{K} \frac{\beta^2}{\beta^\gamma}$$

which is proportional to $\beta^{2-\gamma}$

Therefore we can conclude that

For $\gamma > 2$ the terminal turning rate $\dot{\beta}$ is infinite

For $\gamma = 2$ 〃 $\dot{\beta}$ is finite

For $\gamma < 2$ 〃 $\dot{\beta}$ is going zero.

We can differentiate once more the above expression to obtain conclusions for the terminal acceleration

In this case we have that

$\ddot{\beta}$ proportional to $\beta^{1-\gamma}\dot{\beta}$, which is proportional to $\beta^{1-\gamma}\beta^{2-\gamma} = \beta^{3-2\gamma}$

Thus for $\gamma > 1{,}5$ the terminal acceleration is infinite

$\gamma = 1{,}5$ 〃 is finite

$\gamma < 1{,}5$ 〃 is zero.

In summary, we must have $1 \le \gamma \le 2$ for a finite turning rate, and $1 \le \gamma \le 1{,}5$ for a finite turning acceleration.

But then the question is, what if we have a pursuer with limitation of the turn rate?

To answer this question we perform an analysis of the pursuer's motion under turn rate limitations.

Let us consider the ideal terminal phase first, where we showed that $\beta \to 0$. We approximate $\cos\beta \simeq 1$, $\sin\beta \simeq \beta$, and simplify the homing equations as.

$$\dot{R} = v_T - v_M$$

$$\dot{\beta} = - \frac{v_T \cdot \beta}{R}$$

We integrate the first equation

$$\frac{dR}{d\tau} = v_T - v_M \Rightarrow \int_t^{t_f} dR = \int_t^{t_f} (v_T - v_M)\, d\tau \Rightarrow$$

$$\cancel{R(t_f)} - R(t) = (v_T - v_M)(t_f - t)$$

we use the boundary condition $R(t_f) = 0$

$$\boxed{R(t) = (v_T - v_M)(t - t_f)}$$

Thus the second equation reads.

$$\dot{\beta} = - \frac{\beta}{\frac{v_T - v_M}{v_T}(t - t_f)} = - \frac{\beta}{(1-\gamma)(t - t_f)}$$

We integrate this equation to obtain

$$\frac{d\beta}{\beta} = \frac{dt}{(\gamma - 1)(t - t_f)} \Rightarrow \boxed{\log\beta = \frac{1}{\gamma - 1} \log(t - t_f) + \text{const.}}$$

which further reads.

$$\log \beta = \log\left((t-t_f)^{\frac{1}{\gamma-1}}\right) + \text{const.}$$

$$\log \frac{\beta}{(t-t_f)^{\frac{1}{\gamma-1}}} = \text{const.} = \log \frac{\beta_0}{(t_0-t_f)^{\frac{1}{\gamma-1}}}$$

where $\beta_0 = \beta(t_0)$ is the line-of-sight at the beginning of the terminal phase (time t_0)

Therefore, under the assumption that $\beta \ll L$, we derived the approximate formula:

$$\boxed{\beta \simeq \beta_0 \left(\frac{t_f - t}{t_f - t_0}\right)^{\frac{1}{\gamma-1}}} \qquad \text{(A)}$$

At this point, you may want to take the first order derivative to verify that this formula captures that

For $\gamma < 2$: we get $\dot{\beta} \to 0$ as $t \to t_f$

For $\gamma > 2$: we get $\dot{\beta} \to -\infty$ as $t \to t_f$.

Similarly for the second-order derivative.

Let us write the formulas just for the sake of completeness

$$\dot{\beta} = \beta_0 \frac{1}{(1-\gamma)(t_f - t_0)} \left(\frac{t_f - t}{t_f - t_0}\right)^{\frac{2-\gamma}{\gamma-1}} \qquad \text{(B.)}$$

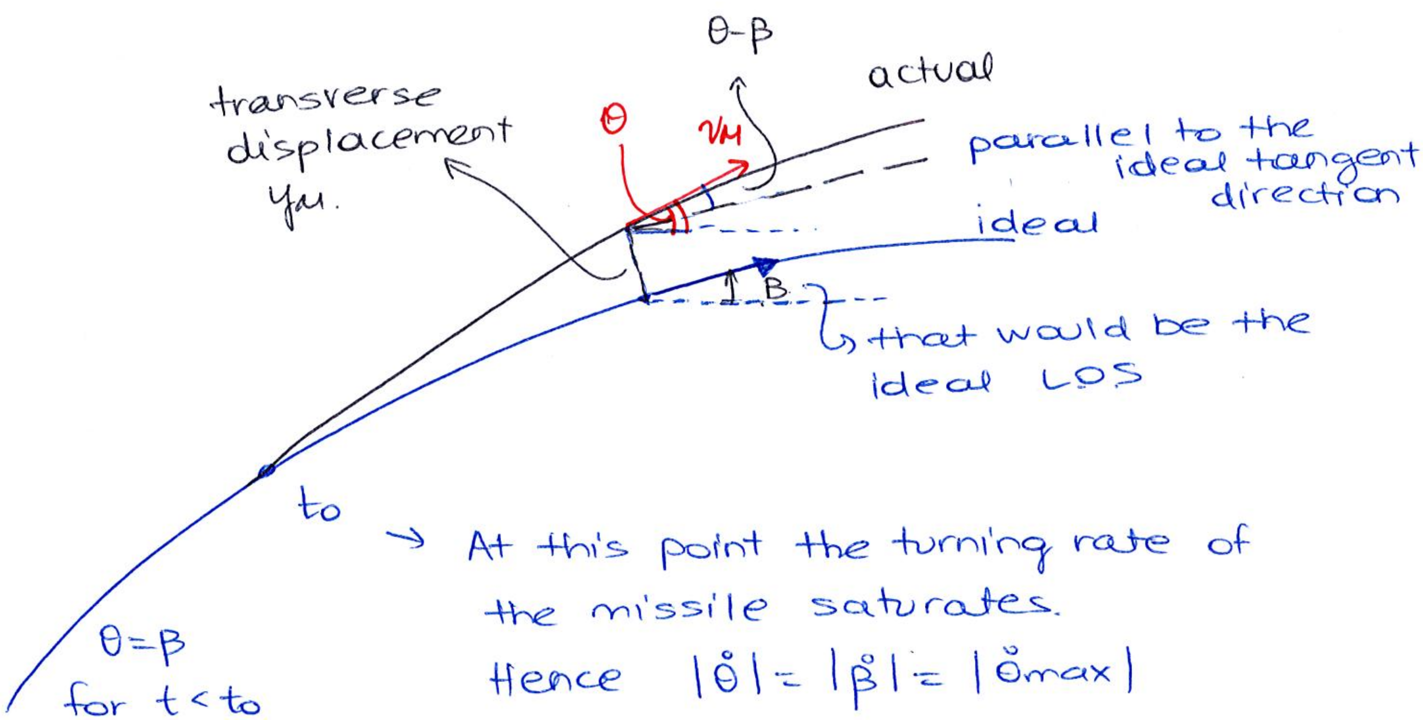
And.

$$\ddot{\beta} = \frac{\beta_0 (2-\gamma)}{(\gamma-1)^2 (t_f - t_o)^2} \left(\frac{t_f - t}{t_f - t_o} \right)^{\frac{3-2\gamma}{\gamma-1}}$$

Recall that these formulas are valid approximations when $\beta \ll 1$.

Let us now proceed with the miss distance analysis for the case that the pursuer is subject to a limitation on its turn rate as

$|\dot{\theta}| \leq |\dot{\theta}_{max}|$, where $|\dot{\theta}_{max}|$ is given



→ At this point the turning rate of the missile saturates.
Hence $|\dot{\theta}| = |\dot{\beta}| = |\dot{\theta}_{max}|$

For $|\theta-\beta| \ll 1$, we can approximate

$$\boxed{\dot{y}_M = v_M (\theta - \beta)} \qquad (c)$$

We also have

$$\mathring{\theta} = -|\mathring{\theta}_{max}| = \text{constant}.$$

We integrate this equation to obtain

$$\boxed{\theta = \beta_0 - |\mathring{\theta}_{max}|(t - t_0)} \quad (D)$$

where we used the boundary condition $\theta(t_0) = \beta_0$.

Let us plug (D) and (A) into (C). to obtain

$$\mathring{y}_M = U_M \left(\beta_0 - |\mathring{\theta}_{max}|(t - t_0) - \beta_0 \left(1 - \frac{t - t_0}{t_f - t_0}\right)^{\frac{1}{\gamma - 1}} \right)$$

Integrating this equation with the boundary condition $y_M(t_0) = 0$. yields the following expression for the transverse distance.

$$y_M = U_M \left(\beta_0 (t - t_0) - |\mathring{\theta}_{max}| \frac{(t - t_0)^2}{2} - \frac{\gamma - 1}{\gamma} \beta_0 (t - t_f) \left(1 - \frac{t - t_0}{t_f - t_0}\right)^{\frac{1}{\gamma - 1}} - \frac{\gamma - 1}{\gamma} \beta_0 (t_f - t_0) \right) \quad (E)$$

The miss distance is defined as $M \simeq y_M(t_f)$ Thus

$$\boxed{M = y(t_f) = U_M (t_f - t_0) \left(\frac{\beta_0}{\gamma} - |\mathring{\theta}_{max}| \frac{t_f - t_0}{2} \right)} \quad (F)$$

Let us now denote R_0 the distance at time t_0

Then we have $R_0 = (v_T - v_M)(t_0 - t_f) = (v_M - v_T)(t_f - t_0)$

$$\boxed{t_f - t_0 = \frac{R_0}{v_M - v_T} = \frac{R_0}{v_T(\gamma - 1)} = \frac{\gamma R_0}{v_M(\gamma - 1)}} \quad (G)$$

Also near intercept

$$\dot{\beta}(t_0) = \dot{\beta}_0 = -|\dot{\theta}_{max}| \overset{(B)}{=} \frac{\beta_0}{(1-\gamma)(t_f - t_0)}$$

Hence

$$\boxed{-\frac{|\dot{\theta}_{max}|(t_f - t_0)}{2} = \frac{\beta_0}{2(1-\gamma)}} \quad (H)$$

We can now plug (G) and (H) into (F) to obtain

$$M = \frac{\gamma R_0}{\gamma - 1}\left(\frac{\beta_0}{\gamma} + \frac{\beta_0}{2(1-\gamma)}\right) = \frac{\gamma R_0}{\gamma - 1} \, \frac{2\beta_0(1-\gamma) + \beta_0\gamma}{2\gamma(1-\gamma)}$$

$$= \frac{\gamma R_0}{\gamma - 1} \, \frac{2\beta_0 - 2\beta_0\gamma + \beta_0\gamma}{2\gamma(1-\gamma)} = \frac{\gamma R_0(2\beta_0 - \beta_0\gamma)}{2\gamma(1-\gamma)(\gamma-1)} = \frac{R_0\beta_0(2-\gamma)}{2(1-\gamma)(\gamma-1)}$$

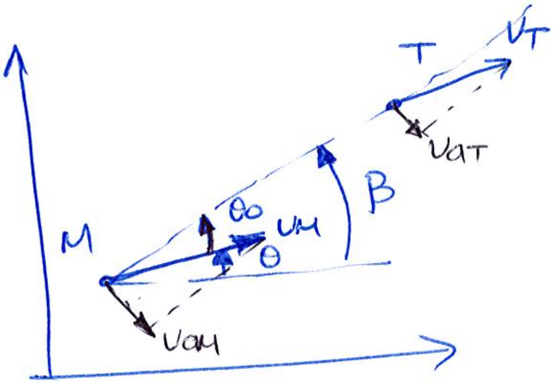
$$\boxed{M = \frac{R_0\beta_0(\gamma - 2)}{2(\gamma - 1)^2}, \quad \text{where } \beta_0 \ll 1}$$

Note that as $\gamma \to \infty$, $M \to 0$. This implies that if $\gamma \gg 1$, then pursuit guidance is still quite effective despite the miss due to turn rate limitation.

$\boxed{\text{Fixed - Lead Guidance}}$

The missile always "leads ahead the target" by a fixed lead angle θ_0

$$\boxed{\theta = \beta - \theta_0}$$



For a given nominal β_0 and a nonmaneuvering target, one can find a specific fixed lead angle θ_0 that leads to a nonmaneuvering collision course (for the missile)

as denoted in the figure

However, if β_0 changes, then θ_0 changes, hence the missile should maneuver during the operation

$\boxed{\text{Constant Bearing Guidance}}$

We require that the line-of-sight remains constant

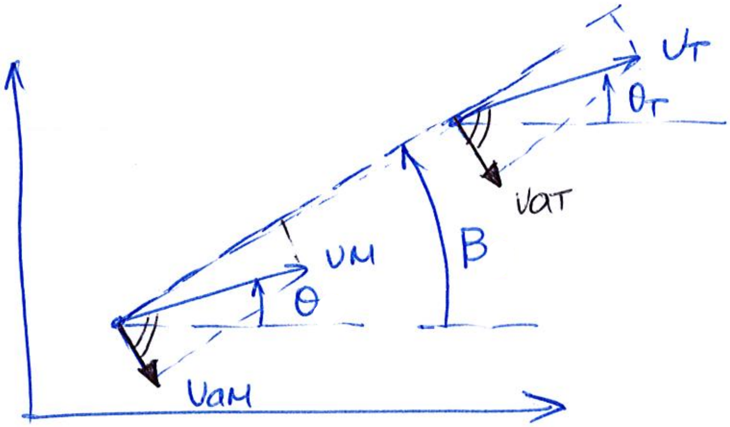
$$\boxed{\dot{\beta} = 0}$$

This is achieved if $\boxed{V_{aM} = V_{aT}.}$

This implies that $\boxed{\dot{V}_{aM} = \dot{V}_{aT},}$ i.e. the missile must react instantaneously to accelerations of the target.

to match the component of vertical velocity and acceleration orthogonal to the L.O.S.

Hard to Implement in practice.



Now with reference to the figure above
[note that θ_0 denotes initial orientation !]

the condition yielding a nonmaneuvering interception (assuming a non-maneuvering target) reads.

$$v_{aM} = v_{aT} \Rightarrow$$

$$\sin(\beta-\theta) \cdot v_M = \sin(\beta-\theta_T)\, v_T \Rightarrow$$

$$\boxed{\frac{v_T}{\sin(\beta-\theta)} = \frac{v_M}{\sin(\beta-\theta_T)}}$$

Hence in the presence of launch error in θ and a maneuvering target, the idea is to let the missile turn appropriately.

This leads to the guidance method called **Proportional Navigation**.

Note! it is NOT an estimation method!

Proportional Navigation

The guidance law is defined as

$$\boxed{\dot{\theta} = \lambda \dot{\beta}}$$

Integrating this relation yields $\theta = \lambda\beta + C_0$

integration constant.

This indicates that the previously studied guidance methods are particular cases of proportional navigation.

For $\lambda = 1$ and $C_0 = 0$ $\Rightarrow$ we recover pure pursuit. guidance

For $\lambda = 1$ and $C_0 \neq 0$ $\Rightarrow$ we recover fixed-lead guidance

For $\lambda \to \infty$, we have $\dot{\beta} \to 0$ $\Rightarrow$ we recover (approximately) constant bearing guidance.

We derived earlier that the initial conditions yielding non-maneuvering (i.e. constant bearing) interception of a non-maneuvering target. read.

$$\boxed{\sin(\beta_0 - \theta_0) = \frac{\sin(\beta_0 - \theta_T)}{\gamma}}$$

In the sequel we analyze the performance of PN in the presence of launch errors, target maneuvers, noisy measurements, and the effect of the navigation constant.

The Method of Adjoints - Chapter 2.4 of the textbook.

In principle, we can use the variation of constants formula (2.7) from Chapter 2, and the definition of the state transition matrix, to compute the output of a given linear dynamic system at a given final time., However, this requires a prohibitive amount of computations.

Example: Let us consider the LTV

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t)$$
$$x(t_0) = 0 . \qquad t \in [t_0, t_f]$$

Then the output of the system at time t_f is

$$y(t_f, t_0, x_0, u) = y(t_f) \overset{(2.7)}{\underset{x(t_0)=0}{=}} = \int_{t_0}^{t_f} \underbrace{C(t_f)\Phi(t_f,\tau)B(\tau)}_{\text{impulse response matrix.}}u(\tau)\,d\tau$$

$$= \int_{t_0}^{t_f} G(t_f,\tau)\,u(\tau)\,d\tau$$

Thus, for evaluating $y(t_f)$, we need to know $\Phi(t_f,\tau)$ for all $\tau \in [t_0, t_f]$

However, this implies that at each time instance τ, one needs to integrate the differential equations involved in the definition of the STM.

Fortunateley, the method of adjoints overcomes this limitation.

Q. How do we formulate the method of adjoints?

A. Given the LTV

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$
$$x(t_0) = x_0, \quad t \in [t_0, t_f]$$

we define the adjoint system as.

$$\dot{p}(t) = -A^T(t)p(t)$$

What is the usefulness of this system? To answer this, let us consider that

$$\frac{d}{dt}(p^T x) = \dot{p}^T x + p^T \dot{x} = -\cancel{p^T A x} + \cancel{p^T A x} + p^T B u$$
$$= p^T B u.$$

Integrating between t_0, t_f yields.

$$p^T(t_f)\,x(t_f) = p^T(t_0)\,x(t_0) + \int_{t_0}^{t_f} p^T(\tau)B(\tau)u(\tau)\,d\tau \quad (A)$$

Now, we can notice that if we choose the final value of the adjoint vector as

(B) $\boxed{p(t_f) = C^T(t_f)}$ then

we have

$$y(t_f) = C(t_f)\,x(t_f) \overset{(B)}{=} p^T(t_f)\,x(t_f) \overset{(A)}{=}$$

$$= p^T(t_0)\,x(t_0) + \int_{t_0}^{t_f} p^T(\tau)\,B(\tau)\,u(\tau)\,d\tau \qquad (C)$$

This way we have expressed the seeked output $y(t_f)$ in terms of the time history of the adjoint vector $p(\tau)$, $\tau \in [t_0, t_f]$

In fact, from (C) we note that we now need:

- one backward integration to obtain the time history of the adjoint vector $p^T(t)$
- one forward integration to obtain $y(t_f)$

Example. Consider the first-order linear system

$$\dot{x}(t) = t\,x(t) + u(t)$$
$$y(t) = x(t)$$
$$x(t_0) = x_0.$$

The adjoint system is $\dot{p}(t) = -t\,p(t)$

We choose the value of the adjoint vector at final time t_f as $p(t_f) = C^T(t_f) = 1$

We now solve for $p(t)$ "backwards", i.e. we have

$$\dot{p}(t) = -t\,p(t) \Rightarrow \frac{dp(t)}{p(t)} = -t\,dt$$

$$\int_{p(t_f)}^{p(t)} \frac{dp}{p} = \int_{t_f}^{t} -\tau\,d\tau \Rightarrow \ln p(t) - \cancel{\ln p(t_f)}^{\,0} =$$

$$= -\frac{t^2}{2} + \frac{t_f^2}{2} \Rightarrow$$

$$\boxed{p(t) = e^{\frac{t_f^2 - t^2}{2}}}$$

Hence we now perform the "forward" integration to obtain

$$y(t_f) = p^T(t_0)\,x(t_0) + \int_{t_0}^{t_f} p^T(\tau)\,B(\tau)\,u(\tau)\,d\tau$$