Linear Time-varying Equations.

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \qquad ①$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

We are seeking the <u>form</u> of the solutions of the above LTV system.

Let us recall that for a scalar, Linear Time-Invariant system, $\dot{x}(t) = a\,x(t)$, $x_0 = x(0)$, we have

$$x(t) = e^{at}\,x(0) \qquad ②$$

[Note that: $\frac{d\,x(t)}{dt} = \frac{d}{dt}\left(e^{at}\,x(0)\right) = \underbrace{e^{at}\,x(0)}_{x(t)} \cdot a \Rightarrow$

$$\dot{x}(t) = a\,x(t)$$

hence ② is indeed a solution of ①]

Now, the solution of $\dot{x}(t) = A\,x(t)$, with $x \in \mathbb{R}^n$ is $x(t) = e^{At}\,x(0)$ ③

[Let us verify that:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(e^{At}\,x(0)\right) = e^{At}\cdot x(0)\cdot A =$$
$$= A\cdot e^{At}\,x(0) = A\,x(t)]$$

(Recall that $e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$)

Now let us consider the scalar, time varying equation

(4) $$\dot{x}(t) = a(t)\, x(t)$$

The solution is $$x(t) = e^{\int_0^t a(\tau)\, d\tau}\, x(0)$$ (5)

we can verify this as

$$\frac{d}{dt}\left(e^{\int_0^t a(\tau)\, d\tau}\, x(0)\right) = e^{\int_0^t a(\tau)\, d\tau}\, a(t)\, x(0) =$$

$$= \underbrace{e^{\int_0^t a(\tau)\, d\tau}\, x(0)}_{x(t)}\, a(t)$$

Let us now consider the matrix case for LTV systems.

If we assume that the solution of the system

(6) $$\dot{x}(t) = A(t)\, x(t)$$

is of the form $$x(t) = e^{\int_0^t A(\tau)\, d\tau}\, x(0),$$ (7)

and we further consider the fact that:

(8) $$e^{\int_0^t A(\tau)\, d\tau} = I + \int_0^t A(\tau)\, d\tau + \frac{1}{2!}\left(\int_0^t A(\tau)\, d\tau\right)\left(\int_0^t A(s)\, ds\right) + \cdots$$

then we have that the time derivative of ⑧ reads

$$\frac{d}{dt}x(t) \overset{⑦}{=} \frac{d}{dt}\left(e^{\int_0^t A(\tau)d\tau} x(0)\right) \overset{⑧}{=}$$

$$\left(A(t) + \frac{1}{2!}A(t)\int_0^t A(s)ds + \frac{1}{2!}\left(\int_0^t A(\tau)d\tau\right)A(t) + \cdots\right)x(0)$$

(out of ⑧)

$$\neq A(t)\underbrace{\left(e^{\int_0^t A(\tau)d\tau} x(0)\right)}_{x(t) \text{ out of ⑦}}$$

i.e. ⑦ is not in general a solution of ⑥ !

Hence careful consideration should be put when formalizing the solutions of an LTV system.

For this purpose we will utilize the concept of the state transition matrix.

Let us consider the system ⑥, copied here

$$\dot{x} = A(t)\,x,$$

with $A(t) \in \mathbb{R}^{n \times n}$ being a matrix with continuous functions of t as its entries.

Continuity is a sufficient condition for the existence of a unique solution $x_i(t;t_0)$, $i=1,\dots,n$, from each initial condition $x_i(t_0)$

Now what we can do is arrange n linearly independent solutions as:

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

i.e. in a square matrix X of order n.

Now since every solution x_i satisfies ⑥, we

can write also that

$$\dot{X}(t) = A(t)\,X(t) \qquad ⑨$$

The matrix $X(t)$ is called a fundamental matrix. of the system ⑥.

Example.

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= t\,x_1 \end{aligned} \quad , \quad \text{or} \quad \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x(t)$$

The solution of the first equation is

$$x_1(t) = x_1(0)$$

The solution of the second equation is

$$\dot{x}_2(t) = t\,x_1(t) = t\,x_1(0) \Rightarrow$$

$$x_2(t) = x_2(0) + \int_0^t \tau\,x_1(0)\,d\tau = x_2(0) + x_1(0)\frac{t^2}{2}$$

Then we have

$$x_1(t) = x_1(0)$$
$$x_2(t) = x_2(0) + x_1(0)\frac{t^2}{2}$$

Let us consider the linearly independent initial states (conditions)

$$\begin{cases} x_1(0) = 1 \\ x_2(0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 2 \end{cases}$$

we have.

$$\begin{cases} x_1(t) = 1 \\ x_2(t) = \frac{t^2}{2} \end{cases} \quad \text{and.} \quad \begin{matrix} x_1(t) = 1 \\ x_2(t) = 2 + \frac{t^2}{2} \end{matrix}$$

Then the matrix

$$X(t) = \begin{bmatrix} 1 & 1 \\ \frac{t^2}{2} & 2+\frac{t^2}{2} \end{bmatrix}$$ is a fundamental matrix.

We can now proceed to the definition of the state transition matrix

- Let $X(t)$ be any fundamental matrix of $\dot{x}(t) = A(t)x(t)$.

Then

$$\boxed{\Phi(t, t_0) = X(t)\, X^{-1}(t_0)}$$

is called the <u>state transition matrix</u> of $\dot{x}(t) = A(t)x(t)$

The state transition matrix is the unique solution of

Equation (2.4) in textbook.

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\,\Phi(t, t_0) \\ &\Phi(t_0, t_0) = I. \end{aligned}}$$

with the initial condition

- Important properties of the state transition matrix

$$\Phi(t, t) = I$$

$$\Phi^{-1}(t, t_0) = \left[X(t)\, X^{-1}(t_0)\right]^{-1} = X(t_0)\, X^{-1}(t) = \Phi(t_0, t)$$

$$\Phi(t, t_0) = \Phi(t, t_1)\, \Phi(t_1, t_0)$$

for every t, t_0, t_1.

Example. (work at home) Consider the system of the previous example. Compute the state transition matrix $\Phi(t, t_0)$

Problem. Prove that the solution of the LTV

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \qquad x(t_0) = x_0 \tag{A}$$

is given as

Called "variation of constants" formula in the textbook.

$$\boxed{x(t) = \Phi(t, t_0)\,x_0 + \int_{t_0}^{t} \Phi(t, \tau)\,B(\tau)\,u(\tau)\,d\tau\,,} \tag{B}$$

where $\Phi(t, \tau)$ is the state transition matrix of

Equation (2.6)

$$\dot{x}(t) = A(t)\,x(t)$$

Solution: we will verify that (B) satisfies the initial condition and the state equation.

At $t = t_0$ we have that (B) reads.

$$x(t_0) = \Phi(t_0, t_0)\,x_0 + \int_{t_0}^{t_0} \Phi(t, \tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$= I \cdot x_0 + 0 = x_0.$$

We furthermore have

$$\frac{d}{dt} x(t) = \frac{\partial}{\partial t} \Phi(t, t_0) x_0 + \frac{\partial}{\partial t} \int_{t_0}^{t} \Phi(t,\tau) B(\tau) u(\tau) d\tau =$$

$$= A(t) \Phi(t, t_0) x_0 + \int_{t_0}^{t} \left(\frac{\partial}{\partial t} \Phi(t,\tau) B(\tau) u(\tau) \right) d\tau +$$

$$+ \Phi(t,\tau) B(\tau) u(\tau) \Big|_{\tau = t} =$$

$$= A(t) \Phi(t, t_0) x_0 + \int_{t_0}^{t} A(t) \Phi(t,\tau) B(\tau) u(\tau) d\tau +$$

$$+ \underbrace{\Phi(t,t)}_{I} B(t) u(t) =$$

$$= A(t) \Phi(t, t_0) x_0 + A(t) \int_{t_0}^{t} \Phi(t,\tau) B(\tau) u(\tau) d\tau +$$

$$+ B(t) u(t)$$

$$= A(t) \underbrace{\left(\Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t,\tau) B(\tau) u(\tau) d\tau \right)}_{x(t)} + B(t) u(t)$$

$$= A(t) x(t) + B(t) u(t).$$

The output of the system reads.

Equation (2.7)

$$y(t) = C(t)\,\Phi(t, t_0)\,x_0 + C(t)\int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau \; +$$

$$+ \; D(t)\,u(t)$$

Equation (2.3)

If the input is identically zero: $\boxed{x(t) = \Phi(t, t_0)\,x_0.}$

Some Remarks on the State Transition Matrix

- If the state matrix $A(t)$ has the commutative property, i.e. <u>if</u>

$$A(t)\left(\int_{t_0}^{t} A(\tau)\,d\tau\right) = \left(\int_{t_0}^{t} A(\tau)\,d\tau\right) A(t)$$

for all t, t_0, then the state transition matrix is given as

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)\,d\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)^{k}$$

- <u>Special case</u>: <u>$A(t) = A$ constant</u> then

$$\boxed{\Phi(t,\tau) = e^{A(t-\tau)}, \quad \text{and} \quad \underbrace{x(t) = e^{At}\,x(0)}_{\text{system solution}}}$$

In summary. So far we have.

We consider LTV systems

$$
\begin{cases}
\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) & (2.2)\\
y(t) = C(t)\,x(t) & \\
x(t_0) = x_0 &
\end{cases}
$$

The solution of (2.2) is given out of the "variation of constants" formula (2.6)

$$
x(t) = \underbrace{\Phi(t,t_0)\,x_0}_{\text{free response.}} + \underbrace{\int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau}_{\text{forced response.}}
$$

where $\Phi(t,\tau)$ is the state transition matrix of the system

$$
\begin{cases}
\dot{x}(t) = A(t)\,x(t) & (2.3)\\
x(t_0) = x_0. &
\end{cases}
$$

From (2.6) it readily follows that the solution of (2.3) is given as $x(t) = \Phi(t,t_0)\,x_0.$ (2.4)

Also, from (2.6) we have that the output of (2.2) is

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau \qquad (2.7)$$

- Now, some important properties of the S.T.M.

The S.T.M of (2.2) is the unique solution of

$$\begin{cases} \frac{\partial}{\partial t}\Phi(t,\tau) = A(t)\Phi(t,\tau) \\ \Phi(\tau,\tau) = I_n \text{ (identity matrix)} \end{cases} \qquad (2.5)$$

The S.T.M. is defined for <u>all real</u> <u>t, τ</u>.

Also, it follows that $\forall\ \tau, t, t_1$

$$\left.\begin{array}{l} \Phi^{-1}(t,\tau) = \Phi(\tau,t) \\ \Phi(t,\tau) = \Phi(t,t_1)\Phi(t_1,\tau) \\ \Phi(\tau,\tau) = I. \end{array}\right\}$$

FROM "MODERN CONTROL THEORY", W. L. BROGAN

EXAMPLE 9.1 A system is described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 8 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y(t) = [4 \quad 1]\mathbf{x}(t)$$

The initial conditions are $\mathbf{x}(0) = [1 \quad -4]^T$. Assume that $u(t) = 0$ and analyze this system.

With $\mathbf{A} = \begin{bmatrix}0 & 1\\ 8 & -2\end{bmatrix}$, the eigenvalues are $\lambda_1 = -4, \lambda_2 = 2$.

The eigenvectors are $\xi_1 = [1 \quad -4],^T \xi_2 = [1 \quad 2]^T$.

The modal matrix and its inverse are $\mathbf{M} = \begin{bmatrix}1 & 1\\ -4 & 2\end{bmatrix}, \mathbf{M}^{-1} = \frac{1}{6}\begin{bmatrix}2 & -1\\ 4 & 1\end{bmatrix}$.

Any one of several methods gives

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix}2e^{-4t} + 4e^{2t} & -e^{-4t} + e^{2t}\\ -8e^{-4t} + 8e^{2t} & 4e^{-4t} + 2e^{2t}\end{bmatrix}$$

so the homogeneous solution is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = [e^{-4t} \quad -4e^{-4t}]^T$, and the output is $y(t) = 4e^{-4t} - 4e^{-4t} = 0$ for all t. ■

EXAMPLE 9.2 Modal decomposition is now applied in an attempt to gain insight into the unusual result of Example 9.1.

Since the eigenvalues are distinct, $\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{\Lambda}$ for this system, and Eq. (9.9) becomes

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}-4 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \begin{bmatrix}\frac{1}{6}\\ \frac{5}{6}\end{bmatrix}u$$

The initial conditions are $\mathbf{q}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = [1 \quad 0]^T$. Equation (9.10) becomes $y = [0 \quad 6]\mathbf{q}$. The state vector $\mathbf{x}(t)$ can be written as the sum of two modes,

$$\mathbf{x}(t) = q_1(0)e^{-4t}\begin{bmatrix}1\\ -4\end{bmatrix} + q_2(0)e^{2t}\begin{bmatrix}1\\ 2\end{bmatrix}$$

The particular initial condition selected here has no component along the direction of mode 2, as evidenced by $q_2(0) = 0$. Thus the second mode is not excited, since the input $u(t)$ has been assumed zero. The output of this system consists only of the second mode contribution, as evidenced by $\mathbf{C}_n = [0 \quad 6]$. Mode 1 contributes nothing to the output and mode 2 is not excited, so the output remains identically zero. ■

9.5 THE TIME-VARYING MATRIX CASE

The time-varying homogeneous state equations

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{9.12}$$

are considered first. In order that this qualify as a valid state equation, it is required that there be a *unique* solution for every $\mathbf{x}(t_0) \in \Sigma$. This places some restriction on the kind of time variation allowed on the matrix $\mathbf{A}$. A *sufficient* condition for the existence of unique solutions is to require that all elements $a_{ij}(t)$ of $\mathbf{A}(t)$ be continuous. Weaker conditions may be found in textbooks on differential equations [1, 4].

Since $\dim(\Sigma) = n$, n linearly independent initial vectors $\mathbf{x}_i(t_0)$ can be found, and each one defines a unique solution of Eq. (*9.12*), called $\mathbf{x}_i(t)$, $t \geq t_0$. Define an $n \times n$ matrix $\mathbf{U}(t_0)$ with columns formed by the independent initial condition vectors $\mathbf{x}_i(t_0)$. (A particular set $\mathbf{U}(t_0) = I_n$ is sometimes used, but that restriction is unnecessary.) The n solutions corresponding to these initial conditions are used as the columns in forming an $n \times n$ matrix $\mathbf{U}(t) = [\mathbf{x}_1(t)] \quad \mathbf{x}_2(t) \quad \ldots \quad \mathbf{x}_n(t)]$. Any matrix $\mathbf{U}(t)$ satisfying

$$\dot{\mathbf{U}}(t) = \mathbf{A}(t)\mathbf{U}(t) \tag{9.13}$$

is called a *fundamental solution matrix*, provided that $|\mathbf{U}(t_0)| \neq 0$. Assuming that the fundamental solution matrix is available, the solution to Eq. (*9.12*) with an arbitrary initial condition vector $\mathbf{x}(t_0)$ is

$$\mathbf{x}(t) = \mathbf{U}(t)\mathbf{U}^{-1}(t_0)\mathbf{x}(t_0) \tag{9.14}$$

This is easily verified. Checking initial conditions,

$$\mathbf{x}(t_0) = \mathbf{U}(t_0)\mathbf{U}^{-1}(t_0)\mathbf{x}(t_0) = \mathbf{I}_n\,\mathbf{x}(t_0) = \mathbf{x}(t_0)$$

Checking to see that this solution satisfies the differential equation,

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{U}}(t)\mathbf{U}^{-1}(t_0)\mathbf{x}(t_0) = \mathbf{A}(t)\mathbf{U}(t)\mathbf{U}^{-1}(t_0)\mathbf{x}(t_0) = \mathbf{A}(t)\mathbf{x}(t)$$

Both the initial conditions and the differential equation are satisfied, so this represents the unique solution to the homogeneous problem.

The nonhomogeneous time-varying state equation is solved in an analogous manner to the scalar and constant matrix cases. That is, the equation is reduced to exact differentials so that it can be integrated. Preliminary to this, it is noted that $\mathbf{U}^{-1}(t)$ can be shown to exist for all $t \geq t_0$ and that

$$\mathbf{U}(t)\mathbf{U}^{-1}(t) = \mathbf{I}_n \quad \text{so that} \quad \frac{d}{dt}(\mathbf{U}(t)\mathbf{U}^{-1}(t)) = [\mathbf{0}]$$

or

$$\frac{d\mathbf{U}}{dt}\mathbf{U}^{-1} + \mathbf{U}\frac{d\mathbf{U}^{-1}}{dt} = [\mathbf{0}] \quad \text{or} \quad \frac{d\mathbf{U}^{-1}}{dt} = -\mathbf{U}^{-1}\frac{d\mathbf{U}}{dt}\mathbf{U}^{-1}$$

Therefore,

$$\frac{d\mathbf{U}^{-1}}{dt} = -\mathbf{U}^{-1}(t)\mathbf{A}(t) \tag{9.15}$$

Note that the matrix $\mathbf{K}(t)$ of Sec. 9.3 is an example of $\mathbf{U}^{-1}(t)$. Premultiplying the time-varying version of Eq. (*9.6*) by $\mathbf{U}^{-1}(t)$, postmultiplying Eq. (*9.15*) by $\mathbf{x}(t)$, and adding the results gives

$$\mathbf{U}^{-1}(t)\dot{\mathbf{x}} + \frac{d\mathbf{U}^{-1}}{dt}\mathbf{x}(t) = \mathbf{U}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t)$$

or

$$\frac{d}{dt}[\mathbf{U}^{-1}(t)\mathbf{x}(t)] = \mathbf{U}^{-1}(t)\mathbf{B}(t)\mathbf{u}(t)$$

The nonhomogeneous solution is obtained by integrating both sides from t_0 to t, that is,

$$\mathbf{U}^{-1}(t)\mathbf{x}(t) - \mathbf{U}^{-1}(t_0)\mathbf{x}(t_0) = \int_{t_0}^{t} \mathbf{U}^{-1}(\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau$$

or

$$\mathbf{x}(t) = \mathbf{U}(t)\mathbf{U}^{-1}(t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{U}(t)\mathbf{U}^{-1}(\tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{9.16}$$

The result again takes the form of a term depending on the initial state and a convolution integral involving the input function. In fact, the first term is the same homogeneous solution given by Eq. (*9.14*). This result shows the *form* of the solution, but it may not be immediately useful. It assumes knowledge of the fundamental solution matrix $\mathbf{U}(t)$, and actually finding $\mathbf{U}$ has not yet been addressed.

9.6 THE TRANSITION MATRIX

The preceding results prompt the definition of an important matrix that can be associated with any linear system, namely, the *transition matrix:*

$$\mathbf{\Phi}(t, \tau) \triangleq \mathbf{U}(t)\mathbf{U}^{-1}(\tau) \tag{9.17}$$

This $n \times n$ matrix is a linear transformation or mapping of Σ onto itself. That is, in the absence of any input $\mathbf{u}(t)$, given the state $\mathbf{x}(\tau)$ at any time τ, the state at any other time t is given by the mapping

$$\mathbf{x}(t) = \mathbf{\Phi}(t, \tau)\mathbf{x}(\tau)$$

The mapping of $\mathbf{x}(\tau)$ into itself requires that

$$\mathbf{\Phi}(\tau, \tau) = \mathbf{I}_n \quad \text{for any } \tau \tag{9.18}$$

This is obviously true from Eq. (*9.17*). Differentiating $\mathbf{\Phi}(t, \tau)$ with respect to its first argument t gives

$$\frac{d\mathbf{\Phi}(t, \tau)}{dt} = \frac{d\mathbf{U}(t)}{dt}\mathbf{U}^{-1}(\tau) = \mathbf{A}(t)\mathbf{U}(t)\mathbf{U}^{-1}(\tau)$$

so

$$\frac{d\mathbf{\Phi}(t, \tau)}{dt} = \mathbf{A}(t)\mathbf{\Phi}(t, \tau) \tag{9.19}$$

The set of differential equations (*9.19*), along with the initial condition, Eq. (*9.18*), is often considered as the definition for $\mathbf{\Phi}(t, \tau)$.

Two other important properties of the transition matrix are the semigroup property, mentioned in Chapter 3 while defining state,

$$\mathbf{\Phi}(t_2, t_0) = \mathbf{\Phi}(t_2, t_1)\mathbf{\Phi}(t_1, t_0) \quad \text{for any } t_0, t_1, t_2$$

and the relationship between $\mathbf{\Phi}^{-1}$ and $\mathbf{\Phi}$:

$$\mathbf{\Phi}^{-1}(t, t_0) = \mathbf{\Phi}(t_0, t) \quad \text{for any } t_0, t$$

Both of these propeties are immediately obvious if the definition of Eq. (*9.17*) is considered.

Methods of Computing the Transition Matrix

If the matrix $\mathbf{A}$ is constant, then

$$\mathbf{\Phi}(t, \tau) = e^{(t-\tau)\mathbf{A}} \qquad \text{(Compare Eqs. (9.7) and (9.16).)}$$

Therefore, all the methods of Chapter 8 are applicable for finding $\mathbf{\Phi}$, including

1. $\mathbf{\Phi}(t, 0) = \mathcal{L}^{-1}\{[\mathbf{I}s - \mathbf{A}]^{-1}\}$. $\mathbf{\Phi}(t, \tau)$ is then found by replacing t by $t - \tau$, since $\mathbf{\Phi}(t, \tau) = \mathbf{\Phi}(t - \tau, 0)$ when $\mathbf{A}$ is constant.
2. $\mathbf{\Phi}(t, \tau) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_{n-1}\mathbf{A}^{n-1}$, where $e^{\lambda_i(t-\tau)} = \alpha_0 + \alpha_1\lambda_i + \cdots + \alpha_{n-1}\lambda_i^{n-1}$ and, if some eigenvalues are repeated, derivatives of the above expression with respect to λ must be used.
3. $\mathbf{\Phi}(t, \tau) = \mathbf{M}e^{\mathbf{J}(t-\tau)}\mathbf{M}^{-1}$, where $\mathbf{J}$ is the Jordan form (or the diagonal matrix $\mathbf{\Lambda}$), and $\mathbf{M}$ is the modal matrix.
4. $\mathbf{\Phi}(t, \tau) = \sum_{i=1}^{n} e^{\lambda_i(t-\tau)}\mathbf{Z}_i(\lambda)$, where the $n \times n$ matrices $\mathbf{Z}_i$ are defined in Problem 8.22.
5. $\mathbf{\Phi}(t, \tau) \cong \mathbf{I} + \mathbf{A}(t-\tau) + \frac{1}{2}\mathbf{A}^2(t-\tau)^2 + \frac{1}{3!}\mathbf{A}^3(t-\tau)^3 + \cdots$. This infinite series can be truncated after a finite number of terms to obtain an approximation for the transition matrix. See Problem 9.10 for a more efficient computational form of this series.

A modification of method 1, using signal flow graphs to avoid the matrix inversion, can also be used. Since $\phi_{ij}(s) \triangleq \mathcal{L}\{\phi_{ij}(t, 0)\}$ is the transfer function from the input to the jth integrator to the output of the ith integrator, that is, the ith state variable x_i, Mason's gain rule [5] can be used to write the components $\phi_{ij}(s)$ directly. Inverse Laplace transformations then give the elements of $\mathbf{\Phi}(t, 0)$.

When $\mathbf{A}(t)$ is time-varying, the choices for finding $\mathbf{\Phi}(t, \tau)$ are more restricted:

1. *Computer solution of* $\dot{\mathbf{\Phi}} = \mathbf{A}(t)\mathbf{\Phi}$ *with* $\mathbf{\Phi}(\tau, \tau) = \mathbf{I}$. This is expensive in terms of computer time if the transition matrix is required for all t and τ. It means solving the matrix differential equation many times, using a large set of different τ values as initial times.

2. *Let* $\mathbf{B}(t, \tau) = \int_\tau^t \mathbf{A}(\zeta)\, d\zeta$. Unlike the time-varying scalar case, $\mathbf{\Phi}(t, \tau) \neq e^{\mathbf{B}(t,\tau)}$ unless $\mathbf{B}(t, \tau)$ and $\mathbf{A}(t)$ commute. Unfortunately, they generally do not commute, but two cases for which they do are when $\mathbf{A}$ is constant and when $\mathbf{A}$ is diagonal. Whenever $\mathbf{BA} = \mathbf{AB}$, any method may be used for computing $\mathbf{\Phi}(t, \tau) = e^{\mathbf{B}(t,\tau)}$.

3. *Successive approximations may be used to obtain an approximate transition matrix, as derived in Problem 9.5:*

$$\Phi(t, t_0) = \mathbf{I}_n + \int_{t_0}^{t} \mathbf{A}(\tau_0)\, d\tau_0 + \int_{t_0}^{t} \mathbf{A}(\tau_0) \int_{t_0}^{\tau_0} \mathbf{A}(\tau_1)\, d\tau_1\, d\tau_0$$

$$+ \int_{t_0}^{t} \mathbf{A}(\tau_0) \int_{t_0}^{\tau_0} \mathbf{A}(\tau_1) \int_{t_0}^{\tau_1} \mathbf{A}(\tau_2)\, d\tau_2\, d\tau_1\, d\tau_0 + \cdots$$

4. *In some special cases closed form solutions to the equations may be possible.*

In many cases it is necessary or desirable to select a set of discrete time points, t_k, such that $\mathbf{A}(t)$ can be approximated by a constant matrix over each interval $[t_k, t_{k+1}]$. Then a set of difference equations can be used to describe the state of the system at these discrete times. The approximating difference equation is derived in Sec. 9.8 and Problem 9.10. Solutions of this type of equation are discussed in Sec. 9.9.

9.7 SUMMARY OF CONTINUOUS-TIME LINEAR SYSTEM SOLUTIONS

The most general state space description of a linear system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

The form of the solution for $\mathbf{x}(t)$ has been shown to be

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{9.20}$$

Equation (9.20) is the explicit form of the (linear system) transformation

$$\mathbf{x}(t) = \mathbf{g}(\mathbf{x}(t_0), \mathbf{u}(t), t_0, t)$$

introduced in Chapter 3 when defining state. When the system matrix $\mathbf{A}$ is constant, the transition matrix can always be found in closed form, although it may be tedious to do so for high-order systems. In the time-varying case numerical solutions or approximations must be relied upon. When considering certain questions, it is valuable to know that a solution exists in the stated form, even if it cannot be easily computed.

When the solution for $\mathbf{x}(t)$ is used, the expression for the output becomes

$$\mathbf{y}(t) = \mathbf{C}(t)\Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{C}(t)\Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{D}(t)\mathbf{u}(t)$$

or

$$\mathbf{y}(t) = \mathbf{C}(t)\Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} [\mathbf{C}(t)\Phi(t, \tau)\mathbf{B}(\tau) + \delta(t - \tau)\mathbf{D}(\tau)]\mathbf{u}(\tau)\, d\tau$$

The term inside the integral is an explicit expression for the weighting matrix $\mathbf{W}(t, \tau)$ used in the integral form of the input-output description for the system:

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{W}(t, \tau)\mathbf{u}(\tau)\, d\tau$$