# Physical vectors

GEOMETRY, KINEMATICS, STATICS, AND DYNAMICS

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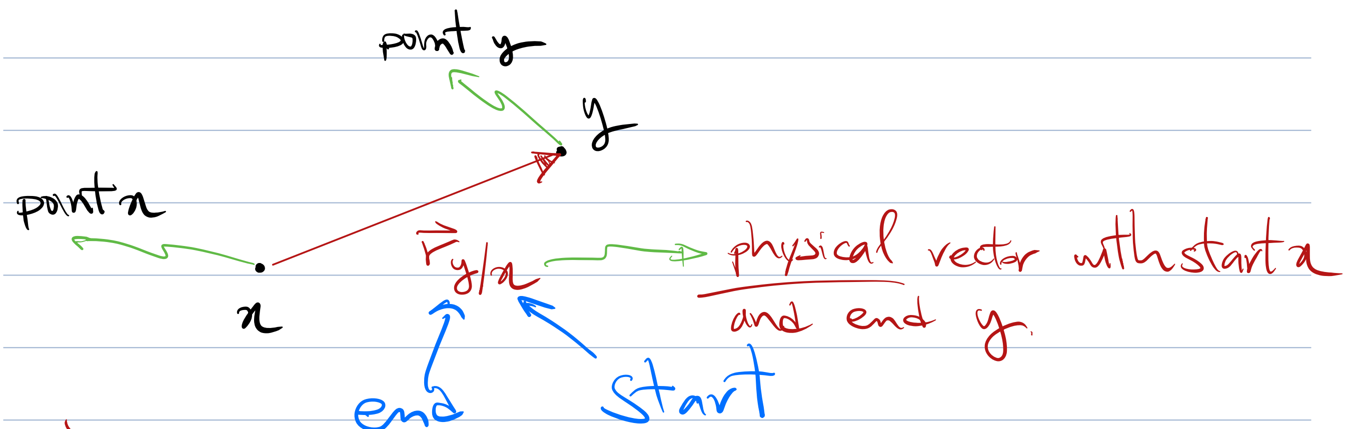
Ch. 1, 2, 4, 5

Let's start from the basics:

- Consider points $x$ and $y$:



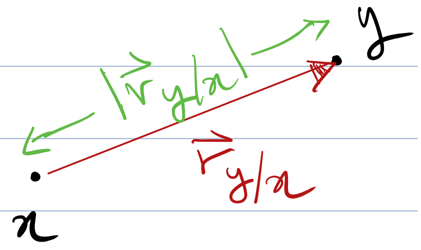

$\vec{r}_{y/x}$ is the position of $y$ relative to $x$.

→ it represents something physical
(but it's an imaginary object)
($\vec{r}_{y/x}$ has no physical location)
(it literally isn't anywhere)

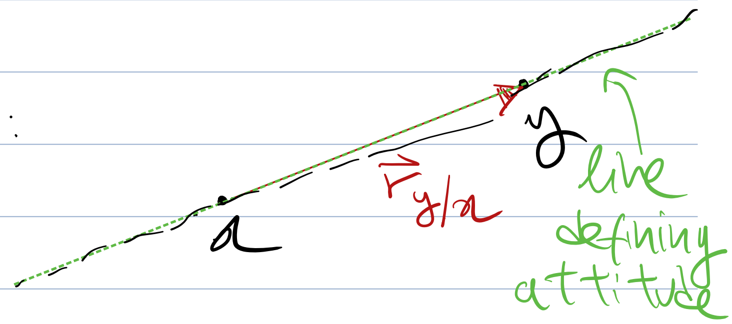
## Properties

- $\vec{r}_{y/x}$ has 3 attributes.

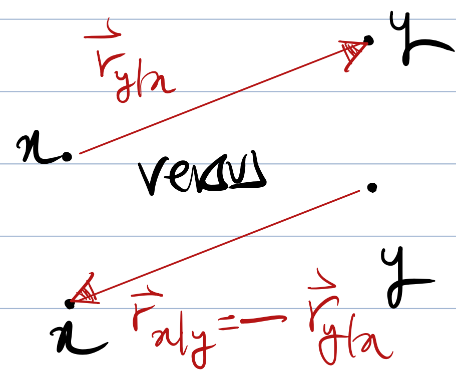
1) length of $\vec{r}_{y/x} \triangleq |\vec{r}_{y/x}|$ :
(also called "magnitude")




2) Attitude :




3) Orientation :

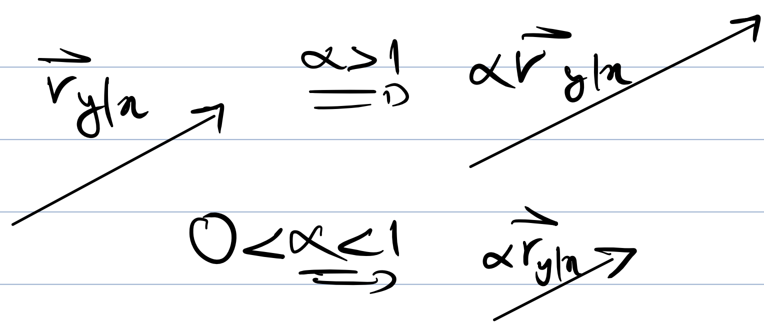



(2) and (3) constitute together <u>direction</u>.

<u>Note 1</u>: To define a <u>physical</u> vector, the notions of frame and coordinates are NOT needed. To define a <u>mathematical</u> vector, then they are needed.

# Scaling Vectors

Consider $\vec{r}_{y|x}$ and $\alpha$ to be a real number.




Generally,

$$|\alpha \vec{r}_{y|x}| = \underbrace{|\alpha|}_{\hookrightarrow \text{ absolute value of } \alpha} |\vec{r}_{y|x}|$$

Hence:

$$|\vec{r}_{x|y}| = |-\vec{r}_{y|x}| = |\vec{r}_{y|x}|$$

If $x$ and $y$ are colocates (same point), then $\vec{r}_{y|x} = \vec{r}_{y|y} = \vec{0}$ (zero physical vector)

Evidently, $|\vec{0}| = 0$.

## Unit vectors

Consider $|\vec{r}_{y|x}| \neq 0$. Then, define the vector:

$$\hat{r}_{y|x} \triangleq \frac{1}{|\vec{r}_{y|x}|} \vec{r}_{y|x} \quad (E1)$$

Then,

1) $\hat{r}_{y|x}$ has unit length:

$$|\hat{r}_{y|x}| = 1.$$

2) $\hat{r}_{y|x}$ is dimensionless

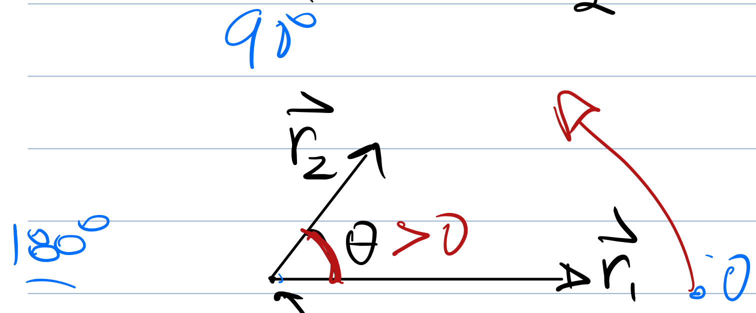
From (E1), $\vec{r}_{y|x} = \underbrace{|\vec{r}_{y|x}|}_{\text{length}} \cdot \underbrace{\hat{r}_{y|x}}_{\text{direction}}$

Note 2: To define a physical vector, we do NOT need a starting and ending point! Only length and direction!

POLL 1

# Physical dot product

Let $\vec{r}_1$ and $\vec{r}_2$ be physical vectors:




imagine their tails are joined

Then, their physical dot product is:

$$\vec{r}_1 \cdot \vec{r}_2 \triangleq |\vec{r}_1||\vec{r}_2| \cos\theta$$

If $\vec{r}_1 \neq \vec{0}$, $\vec{r}_2 \neq \vec{0}$, then:

$$\cos\theta = \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|}$$

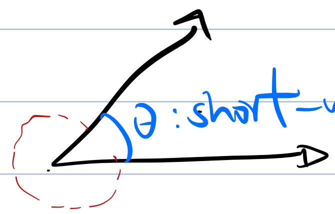
$$\Rightarrow \quad \theta = \operatorname{acos} \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|}$$

Constraining $\theta \in (0, \pi)$ $\Rightarrow$ $\theta$ is the short-way angle

open

This is a convention in the definition of acos. It's needed since $\cos(-\theta) = \cos(\theta)$. (hence, we contraint $\theta$ such that $0 < \theta < \pi$, to avoid ambiguity)

## Short-way versus long-way angles

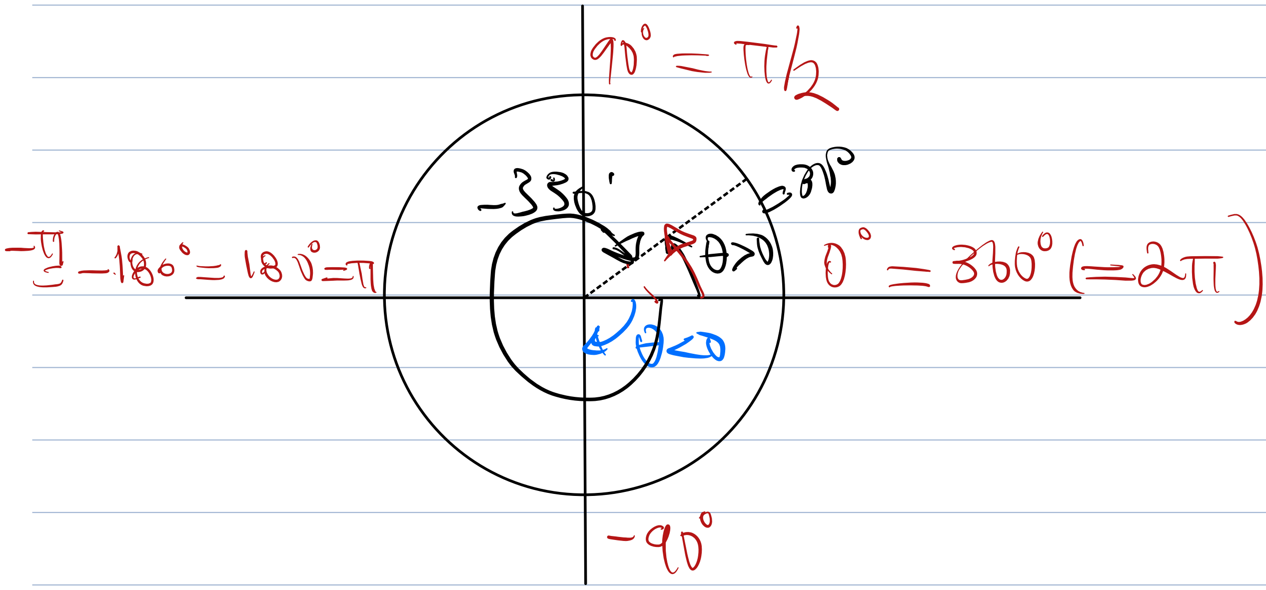


$\theta$: short-way angle $(0 \leq \theta \leq \pi)$

$\psi$: long-way angle $(\pi < \psi < 2\pi)$

We can use negative angles for long-way angles, since $\cos(\pi + \theta) = \cos(-\theta)$ for $\theta \in (0, \pi)$

# Wrapped angles

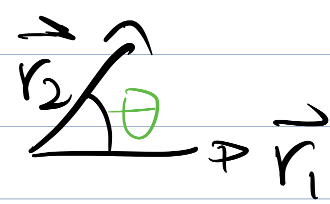



Angles "wrap" every $2\pi$.

## Conventions:

- Wrapped angles are in $\boxed{(-\pi, \pi]}$ open! closed!
- Short-way angles are in $[0, \pi]$ $(\theta \geq 0)$

$$\theta_{\vec{r}_1/\vec{r}_2} = \theta_{\vec{r}_2/\vec{r}_1} = \theta$$

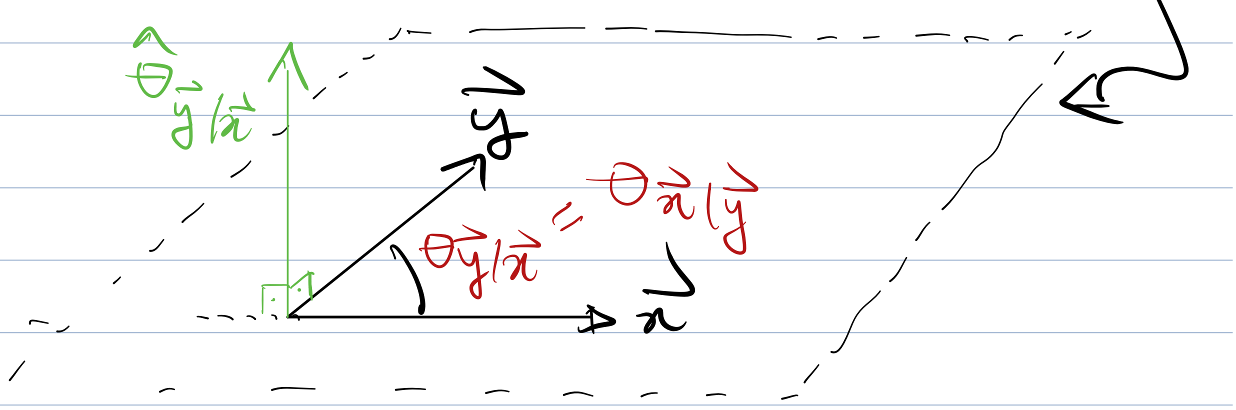



subscript order irrelevant by definition

# Unit angle vector

Assume $\vec{y}$ and $\vec{x}$ are non-zero and non parallel.

Then, $\vec{y}$ and $\vec{x}$ define a plane:




Unit vector $\hat{\theta}_{\vec{y}/\vec{x}}$ is the direction perpendicular to both $\vec{x}$ and $\vec{y}$ that obeys the right-hand rule:

Curl your finger of your right hand from $\vec{x}$ to $\vec{y}$ along short angle. Then, thumb defines $\hat{\theta}_{\vec{y}/\vec{x}}$ direction.

Similarly to notation for $\vec{r}_{y/x}$,

to from

for $\hat{\theta}_{\vec{y}/\vec{x}}$ it is:

$$\hat{\theta}_{\vec{y}\,|\,\vec{x}}$$

to from

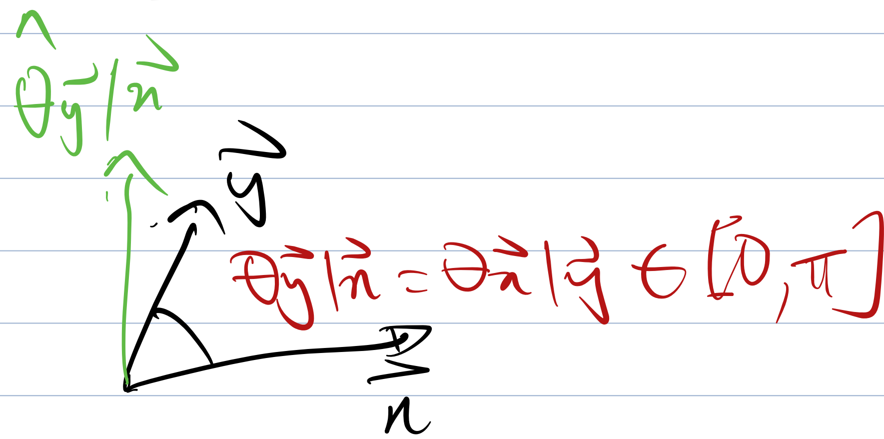
In contrast to $\theta_{\vec{y}/\vec{x}}$ (short-way angle), order of $\vec{y}$ and $\vec{x}$ in subscript in $\hat{\theta}_{\vec{y}/\vec{x}}$ <u>matters</u>.

POLL 2.

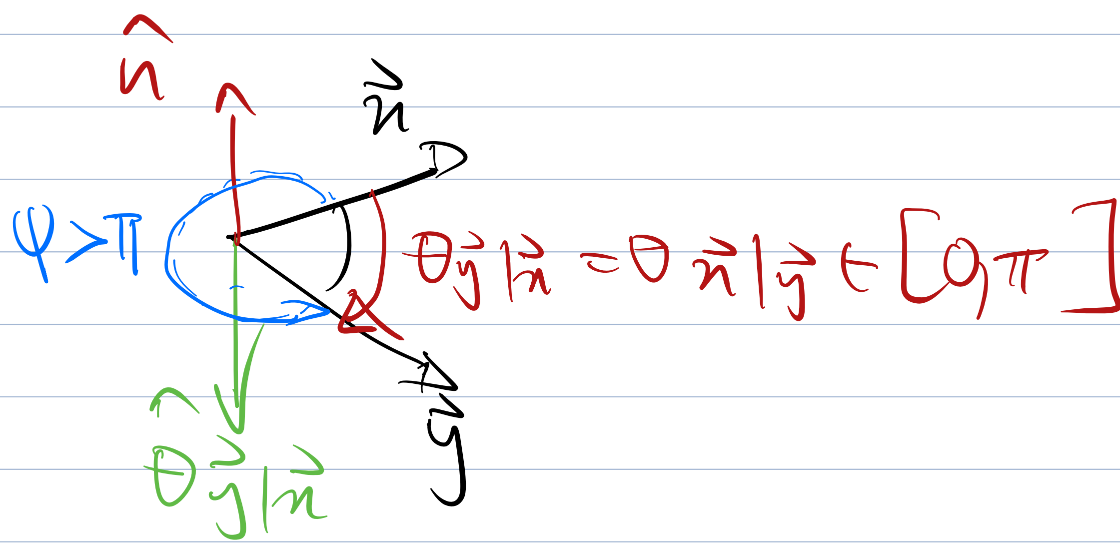
# Physical cross product

$$\vec{x} \times \vec{y} \hat{=} |\vec{x}|\,|\vec{y}|\,(\sin \underbrace{\theta_{\vec{y}/\vec{x}}}_{0 \le \theta_{\vec{y}/\vec{x}} \le \pi})\, \hat{\theta}_{\vec{y}/\vec{x}} \quad \text{(E2)}$$

## Example 1




## Example 2



In example 2, notice that if we would follow the right-hand rule (curling fingers of right hand) from $\vec{x}$ to $\vec{y}$ along the $\psi$ angle (long-way), then it holds:

$$\vec{x} \times \vec{y} = |\vec{x}| \cdot |\vec{y}| (\sin\psi)\, \hat{n} \quad \text{(E3)}$$

where $\hat{n}$ is direction of thumb (see figure of example 2).

(E2) and (E3) are equivalent because:

$$(\sin\psi)\,\hat{n} = \left[\sin(2\pi - \theta_{\vec{y}|\vec{x}})\right]\hat{n}$$

$$= -(\sin\theta_{\vec{y}|\vec{x}})\,\hat{n}$$

$\psi + \theta_{\vec{y}|\vec{x}} = 2\pi$

$$= (\sin\theta_{\vec{y}|\vec{x}})\underbrace{(-\hat{n})}_{=\hat{\theta}_{\vec{y}|\vec{x}}}$$

Let's generalize:

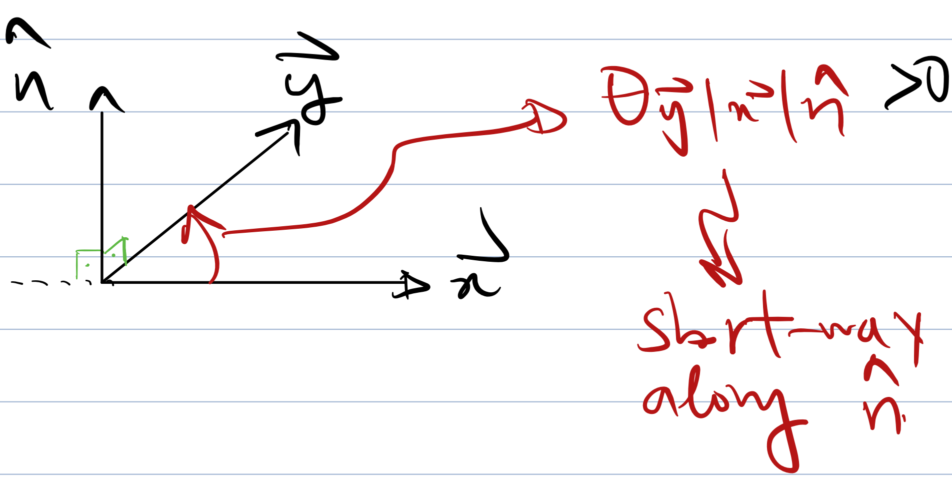
## Directed angles

Assume $\vec{x}, \vec{y}$ both non-zero and non-parallel.

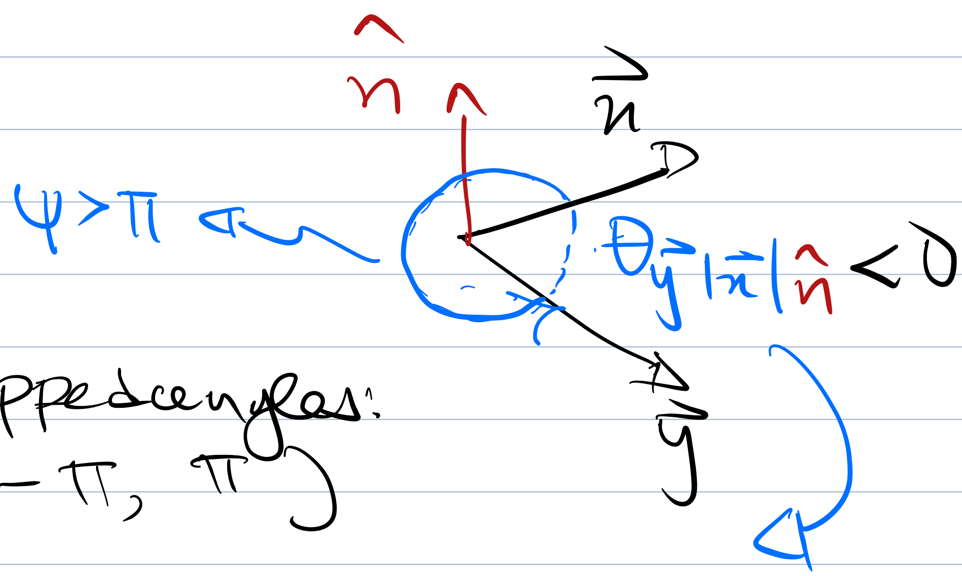
Also, assume unit $\hat{n}$, right to $\vec{x}, \vec{y}$

Then, $\theta_{\vec{y}|\vec{x}|\hat{n}}$ is the angle from $\vec{x}$ to $\vec{y}$ determined by right-hand rule with thumb along $\hat{n}$.

Example 3:



Example 4 (Example 2 revisited):




wrapped angles:
$(-\pi, \pi]$

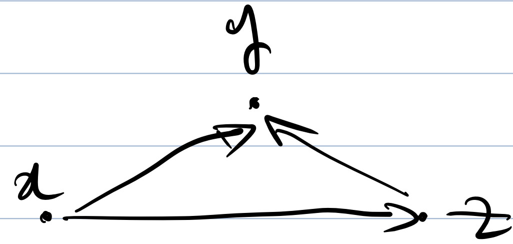
Long-way angle $\psi$ is represented by negative short-way $\theta_{\vec{y}|\vec{x}|\hat{n}}$.

$\Downarrow$

All directed angles are wrapped.

## Back to vectors

### Slash and split




$$\vec{r}_{y|x} = \vec{r}_{y|z} + \vec{r}_{z|x}$$

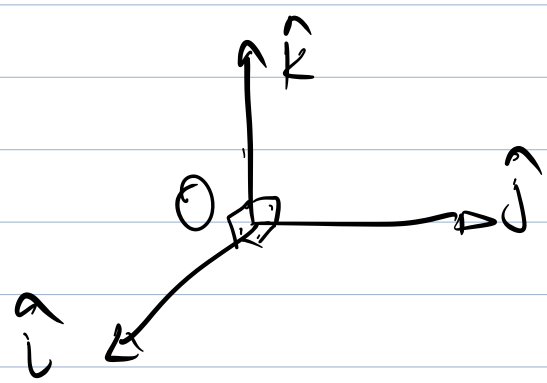
### Triangle inequality

$$\leq |\vec{r}_{y|x}| \leq |\vec{r}_{y|z}| + |\vec{r}_{z|x}|$$

# Frames

Assume 3 mutually orthogonal (unit) vectors:


For convenience draw them "from" a point:




This is a <u>frame</u> F; $\hat{i}$, $\hat{j}$, $\hat{k}$ are its axes.

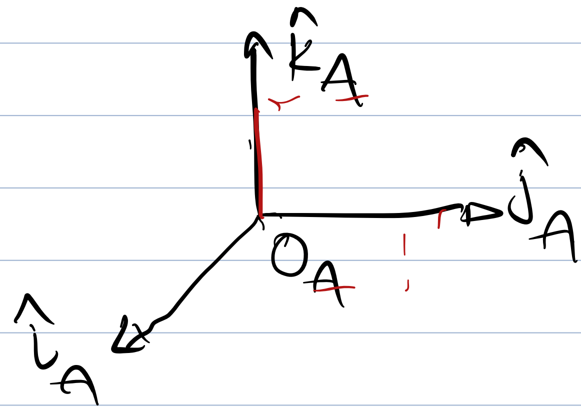
$$F = [\hat{i}, \hat{j}, \hat{k}]$$

- Since $\hat{i} \times \hat{j} = \hat{k}$, it's a right-handed frame

# Origin of frame

A frame has no physical location.

But we can assign an <u>origin</u> to a frame:

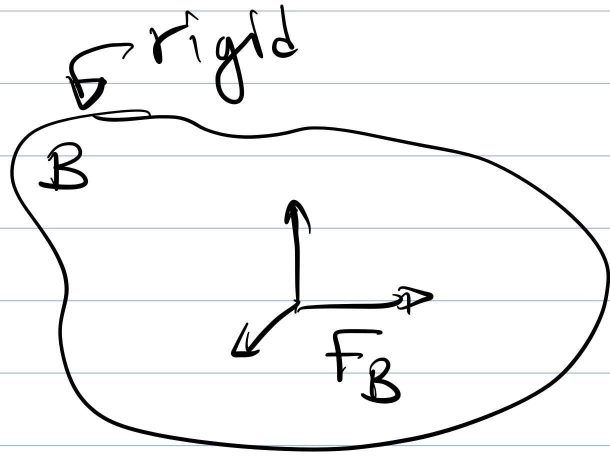



$$F_A = [\hat{i}_A, \hat{j}_A, \hat{k}_A]$$

$O_A$ = origin of $F_A$

- $O_A$ is <u>any</u> convenient reference point.
- $O_A$ can translate; $F_A$ cannot translate ($F_A$ has no location)

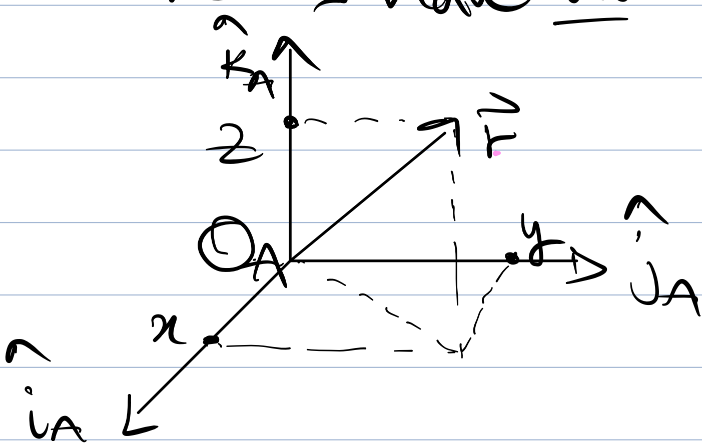
A rigid body B may have a <u>body-fixed</u> frame:




B may translate and rotate.
But $F_B$ does NOT translate; it only rotates according to how B rotates.

<u>Coordinates</u>

Physical vectors have <u>no</u> components.



Project $\vec{r}$ onto each frame axis

Then, given $x, y, z$, we <u>resolve</u> $\vec{r}$ in $F_A$ to get:

$$\boxed{\vec{r}\big|_A \triangleq \begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

$x, y, z$ are $\vec{r}$'s cordinates when $\vec{r}$ is expressed in $F_A$:

$$\boxed{\vec{r} = x\hat{i}_A + y\hat{j}_A + z\hat{k}_A.}$$

- $\vec{r}$ is a <u>physical</u> vector
- $\vec{r}\big|_A$ is a <u>math</u> vector

It is: $|\vec{r}| \underset{\text{norm}}{\overset{\text{if Euclidean}}{=}} \|\vec{r}|_A\|$ $\sqrt{x^2+y^2+z^2}$