Proportional navigation (continuation...)

Recap

Proportional navigation has the advantage that doesn't require knowledge of velocity of target! Instead, it's required for:

- Constant Bearing Pursuit ($\dot{\beta}=0$):

$$\frac{v_T}{\sin(\beta-\theta)} = \frac{v_M}{\sin(\beta-\theta_T)}$$

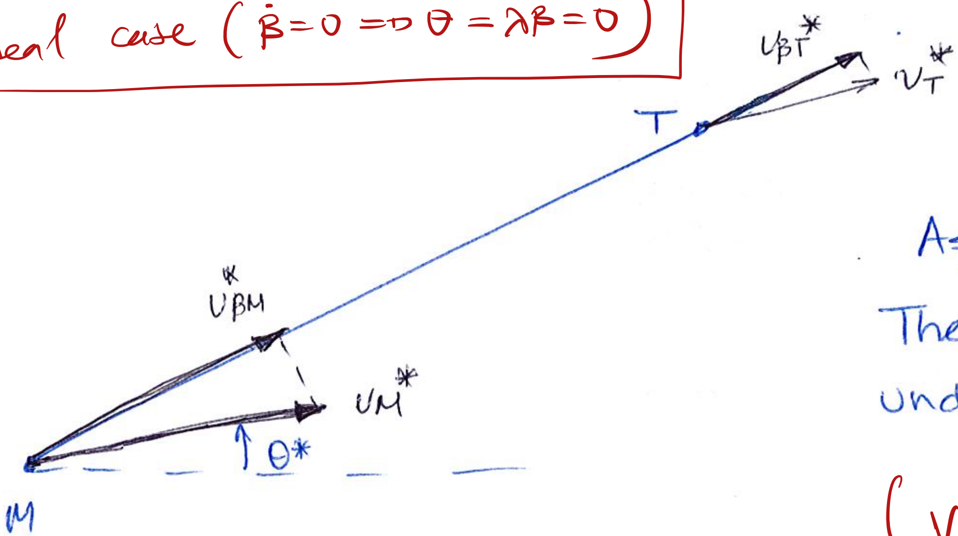
Linearized Proportional Navigation.

We start with a constant bearing reference (ideal) trajectory, and we linearize around it.

We focus on the lateral displacements of the missile and the target wrt the nominal (constant-bearing) L.O.S.

We are primarily interested in modeling how the motions of the target cause (induce) motions of the missile.

$\boxed{\text{ideal case } (\dot{\beta} = 0 \Rightarrow \dot{\theta} = \lambda \dot{\beta} = 0)}$



Assumption

The missile is moving under constant velocity.

$(V_{aM}^* = V_{aT}^*)$

We have $\dot{R}^* = v_{\beta T}^* - v_{\beta M}^* = \text{const.}$ (0)

We integrate with boundary condition $t(0) = 0$.

$$\boxed{R^*(t) = R_0 - |v_{\beta T}^* - v_{\beta M}^*|\, t}$$

We obtain $$\boxed{t_f^* = \frac{R_0}{|v_{\beta T}^* - v_{\beta M}^*|}}$$

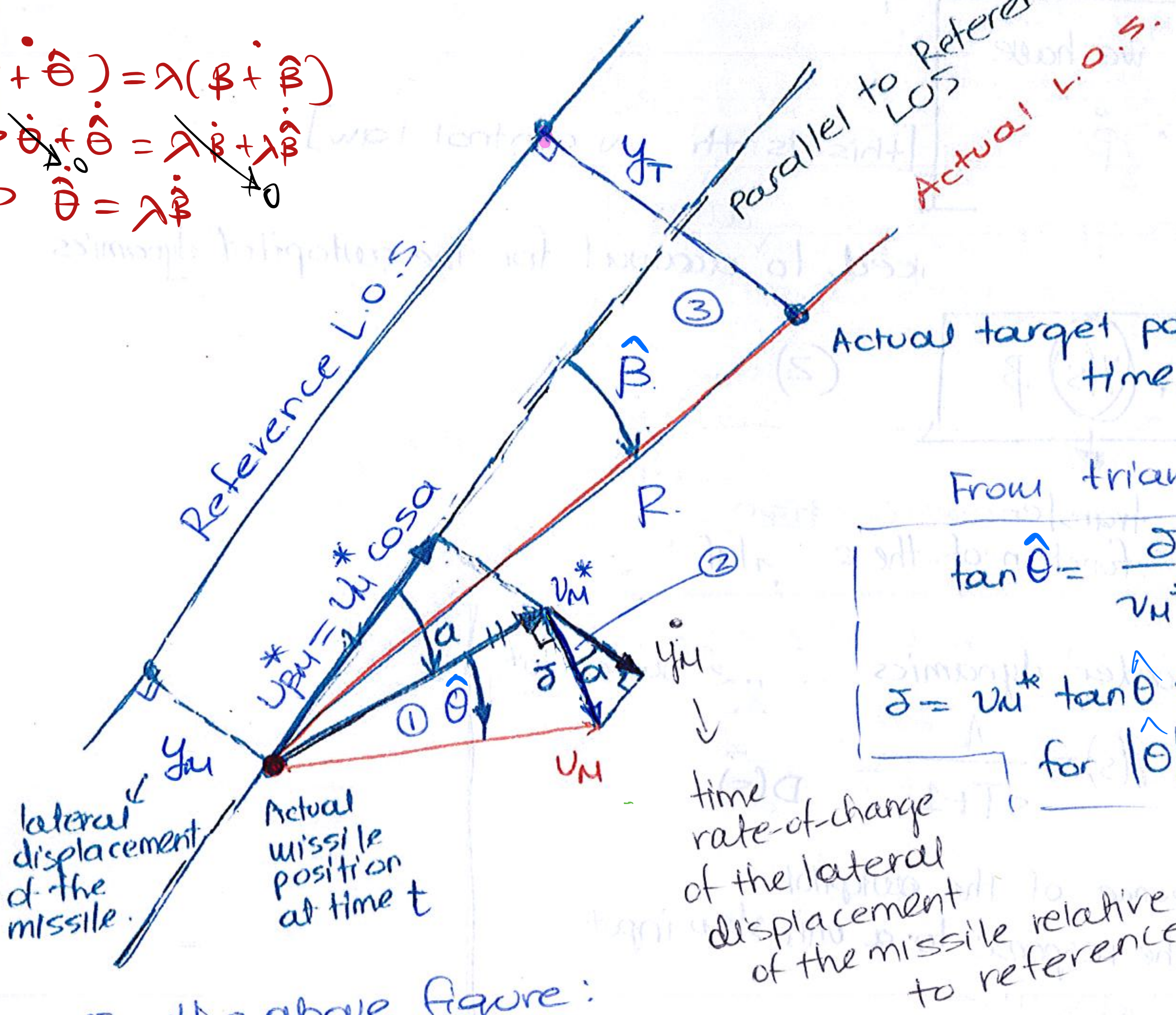
Reference time-to-impact. I.e. time-to-impact under constant bearing guidance.

$\boxed{\text{now we have a small pertrubation } \hat{\beta} \text{ from } \dot{\beta}=0}$

$$(\theta+\dot{\hat{\theta}}) = \lambda(\beta+\dot{\hat{\beta}})$$

$$\Rightarrow \cancel{\dot{\theta}}^{0} + \dot{\hat{\theta}} = \cancel{\lambda\dot{\beta}}^{0} + \lambda\dot{\hat{\beta}}$$

$$\Rightarrow \dot{\hat{\theta}} = \lambda\dot{\hat{\beta}}$$



From triangle ①

$$\tan\hat{\theta} = \frac{\delta}{v_M^*}$$

$$\delta = v_M^* \tan\hat{\theta} \simeq v_M^* \hat{\theta} \quad \text{for } |\hat{\theta}| \ll 1$$

In the above figure:

$\hat{\beta}$ is the angle between reference and actual L.O.S. $\quad |\beta| \ll 1$

$\hat{\theta}$ is the angle between reference and actual missile velocity. $\quad |\theta| \ll 1$

Our goal is to obtain an expression for $y_T(t_f^*) - y_M(t_f^*)$.

We assume we measure this!

From triangle ②

$$\dot{y}_M = \delta\cos a \Rightarrow$$

$$\dot{y}_M \overset{①}{=} \hat{\theta}\, v_M^* \cos a \Rightarrow$$

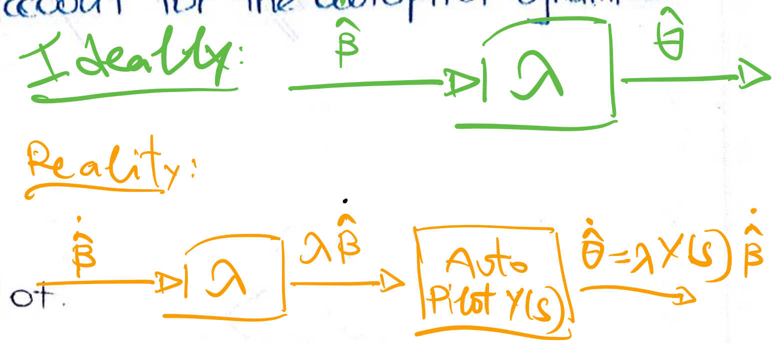
$$\dot{y}_M = \hat{\theta}\, v_{PM}^*$$

Thus

$$\boxed{\hat{\theta} = \frac{\dot{y}_M}{v_{PM}^*}} \quad ①$$

Now, $\boxed{\text{Ideally, we have } \quad \dot{\hat{\theta}} = \lambda \dot{\hat{\beta}}}$ [this is the PN control law]

But, in practice we need to also account for the autopilot dynamics.



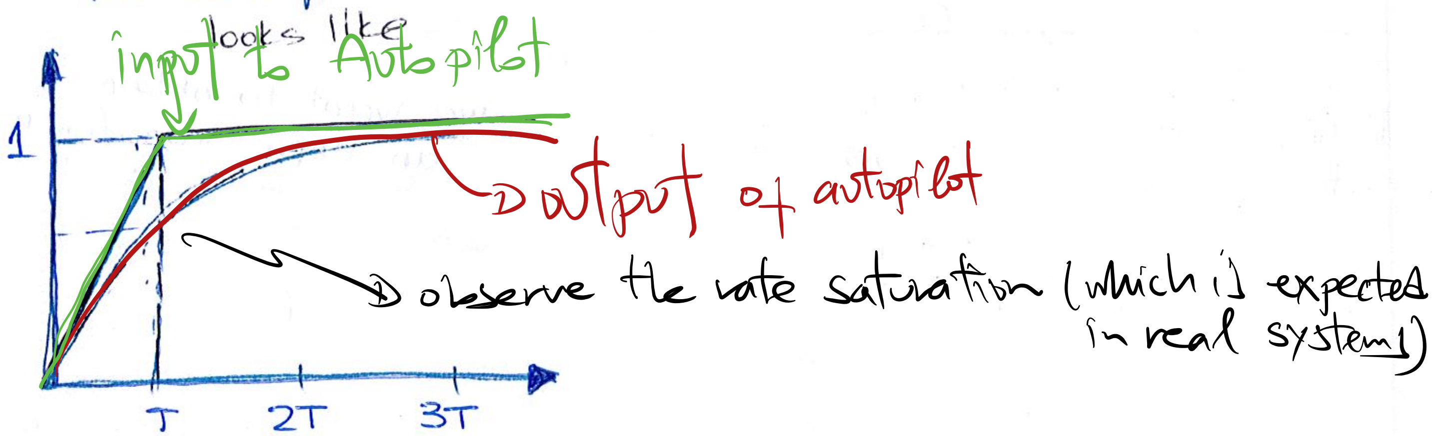
Thus: $\boxed{\dot{\hat{\theta}} = \lambda\, Y(s)\, \dot{\hat{\beta}}}$ (2)

$Y(s)$ → transfer function of the autopilot.

Eg. First-order dynamics of the autopilot:

$$Y(s) = \frac{1}{sT+1} = \frac{N(s)}{D(s)}$$

The step response of the autopilot
i.e. the response to a unit step input
looks like



Then, integrating (2) with boundary conditions $\hat{\theta}(0) = 0$, $\hat{\beta}(0) = 0$

yields. $\boxed{\hat{\theta} = \lambda\, Y(s)\, \hat{\beta}}$ (3)

Now from triangle 3 in the figure. we have

$$\boxed{\hat{\beta} = \frac{y_T - y_M}{R}} \quad (|\hat{\beta}| << 1)$$

Also, we can assume $R \simeq R^*$, since $|\hat{\beta}| << 1$

Then: $$\boxed{\hat{\beta} = \frac{y_T - y_M}{R^*}} \quad (4)$$

It follows that (3), after plugging in (1),(4), reads:

$$\boxed{\frac{\dot{y}_M}{v_{BM}^*} = \lambda\, Y(s) \left(\frac{y_T - y_M}{R^*(t)}\right)} \quad (5)$$

which further reads:

$$\boxed{D(s)\,\dot{y}_M = v_{BM}^* \lambda N(s) \left(\frac{y_T - y_M}{R^*(t)}\right)} \quad (6)$$

For $R^*(t_f^*) = 0$, from eq.(0) we have:

$$\boxed{R^*(t) = (t_f^* - t)\,|v_{BT}^* - v_{BM}^*|} \quad (7)$$

Then (6) reads:

$$\boxed{D(s)\,\dot{y}_M = \underbrace{\boxed{\frac{v_{BM}^* \lambda}{|v_{BT}^* - v_{BM}^*|}}}_{\Lambda} N(s) \left(\frac{y_T - y_M}{t_f^* - t}\right)}$$

We ended up with a LTV system!

input: y_T

output: y_M.

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Effective Navigation Constant.

So we ended up with the LTV system

$$D(s)\,\dot{y}_M = \Lambda N(s) \left(\frac{y_T - y_M}{t_f^* - t} \right),$$

where :

$$\Lambda = \frac{v_{BM}^* \lambda}{|v_{BT}^* - v_{BM}^*|}$$ is called the effective navigation constant;

y_T is the input to the system.

y_M is the output of the system.

This system describes the resulting displacement y_M of the missile wrt the reference (constant-bearing) line-of-sight due to displacement y_T of the target.

The miss distance can be approximated as

$$\boxed{M = y_T(t_f^*) - y_M(t_f^*)}$$

We can use the method of adjoints (chapter 2.4) to evaluate $y_M(t_f^*)$

Eg. Evaluate $y_M(t_f^*)$ using the method of adjoints and under the consideration of first order dynamics for the autopilot. $N(s) = 1$, $D(s) = sT + 1$. We have:

$$(sT+1)\,\dot{y}_M = \frac{\Lambda}{t_f^* - t}(y_T - y_M) \Rightarrow$$

$$s\,\dot{y}_M + \frac{1}{T}\dot{y}_M = \frac{\Lambda}{(t_f^* - t)T}(y_T - y_M) \Rightarrow$$

$$\ddot{y}_M + \frac{1}{T}\dot{y}_M = \frac{\Lambda}{(t_f^* - t)T}(y_T - y_M)$$

Let us write the system in state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{\Lambda}{(t_f^* - t)} & -\dfrac{1}{T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\Lambda}{(t_f^* - t) T} \end{bmatrix} y_T$$

$$y_M = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{where} \quad \boxed{\begin{aligned} x_1 &= y_M \\ x_2 &= \dot{y}_M \end{aligned}}$$

We can apply the method of adjoints to evaluate $y_M(t_f^*)$

Let us define the adjoint system

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\Lambda}{(t_f^* - t)} \\ -1 & \dfrac{1}{T} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

and the boundary condition $\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}(t_f^*) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Out of the method of adjoints (chapter 2.4) we have

$$y_M(t_f^*) = p^T(t_f^*)\, x(t_f^*) = p^T(0)\, \cancel{x(0)} + \int_0^{t_f^*} p^T(\tau) B(\tau) u(\tau)\, d\tau$$

zero since we have assumed zero initial conditions.
$y_M(0) = x_1(0) = 0, \; \dot{y}_M(0) = x_2(0) = 0.$

Hence
$$\boxed{y_M(t_f^*) = \int_0^{t_f^*} p^T(\tau) B(\tau) u(\tau)\, d\tau = \int_0^{t_f^*} \frac{\Lambda\, p_2(\tau)\, y_T(\tau)}{T\,(t_f^* - \tau)}\, d\tau}$$

We can also relate the output and the input through the impulse response $G(t,\tau)$

$$y_M(t_f^*) = \int_0^{t_f^*} G(t_f^*, \tau)\, y_T(\tau)\, d\tau$$

why this name?
Because if $y_T(t) = \delta(t-\bar{\tau})$
$y_M(t) = \int_0^t G(t,\tau)\, y_T(\tau)\, d\tau = G(t,\bar{\tau})$
$\Rightarrow G(t,\bar{\tau})$ is the system output at time t when the input y_T at time $\bar{\tau}$ is an impulse!

Thus we can write:

$$\boxed{\frac{\Lambda}{T}\,\frac{P_2(\tau)\,\cancel{y_T(\tau)}}{t_f^* - \tau} = \underbrace{G(t_f^*, \tau)}_{\text{Impulse Response}}\,\cancel{y_T(\tau)}}$$

Remark:

Note:
Recall that the impulse response matrix $G_{ij}(t,\tau)$ represents the response of the i-th output at time t due to an impulse applied at time τ in the j-th input

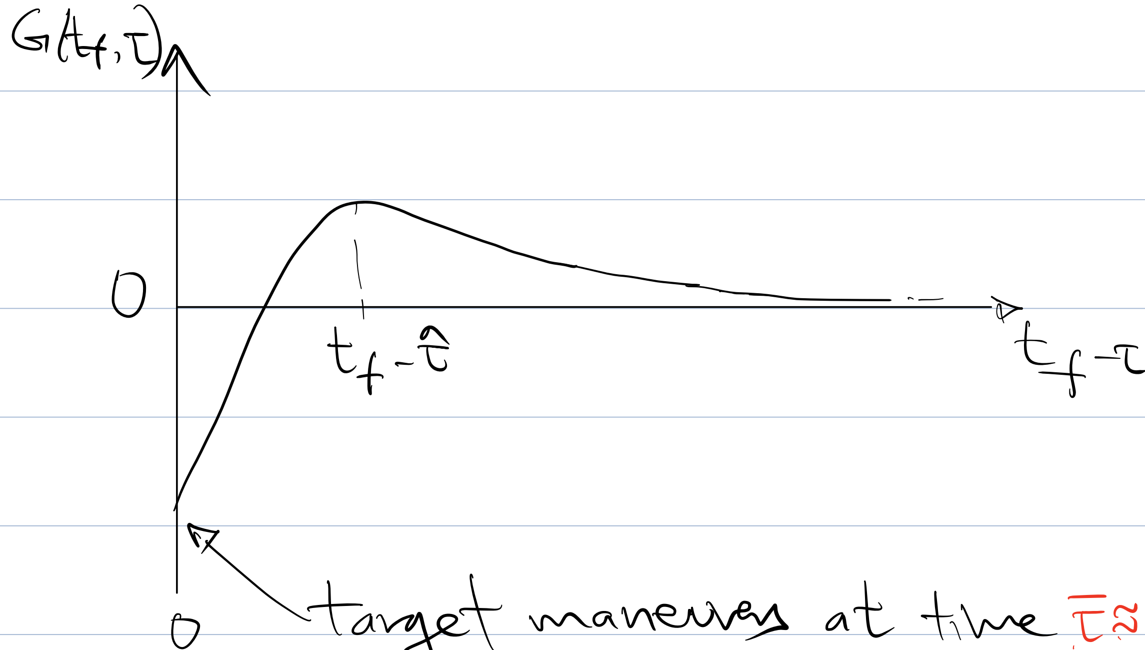
which reads:

$$\boxed{G(t_f^*, \tau) = \frac{\Lambda}{T}\,\frac{P_2(\tau)}{t_f^* - \tau}}$$

[see also example on page 135, Figure 5.8]

Now, since $y_M(t_f^*) = \int_0^{t_f^*} G(t_f^*, \tau)\, y_T(\tau)\, d\tau$, we can conclude that:
An impulsive maneuver of the target just before impact, i.e. at $(t_f^*)^-$ will result in larger miss distance

When $|\tau - t_f^*| >> T$, the miss distance tends to zero
i.e, the autopilot has time to react to the maneuver. of the target.



target maneuvers at time $\bar{\tau} \approx t_f^*$
(impulse maneuver at $\bar{\tau}$)

$$\Rightarrow y_M(t_f) = \int_0^{t_f} G(t_f,\tau)\, \underbrace{y_T(\tau)}_{= \delta(\tau-\bar{\tau})} d\tau$$

$$= G(t_f,\bar{\tau})$$

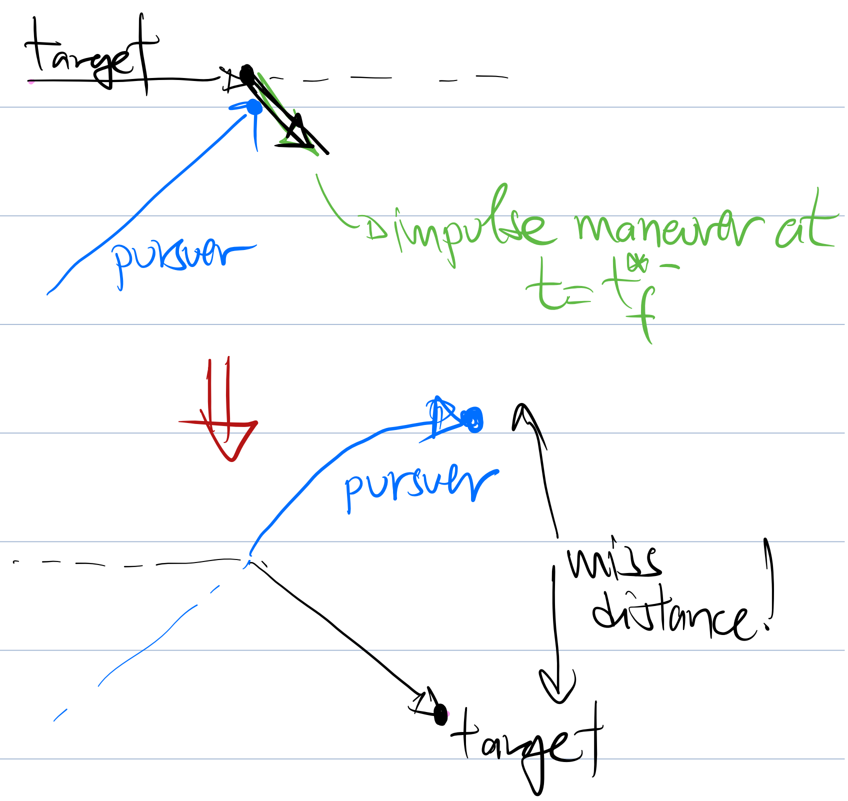
$\Rightarrow$ miss distance is maximal when

$t_f - \bar{\tau} \to 0$, in which case:

$$y_T(t_f^*) - G(t_f^*,\bar{\tau})$$

$$= \underbrace{y_T(t_f^*)}_{\substack{\text{and that's} \\ \text{positive, since} \\ \text{a positive impulse} \\ \text{was assumed}}} + \left| G(t_f^*, \bar{\tau}) \right|$$

<u>Example 1</u> $\left(t = \bar{\tau} = t_f^{*-}\right)$



Example 2 ($t = \hat{\tau}$)

$\hat{\tau}$ seems also a good (but not as good) time to maneuver (note: $\hat{\tau} < \bar{\tau}$, i.e., $\hat{\tau}$ corresponds to earlier maneuver, much before t_f in comparison to $\bar{\tau} \simeq t_f^*$)