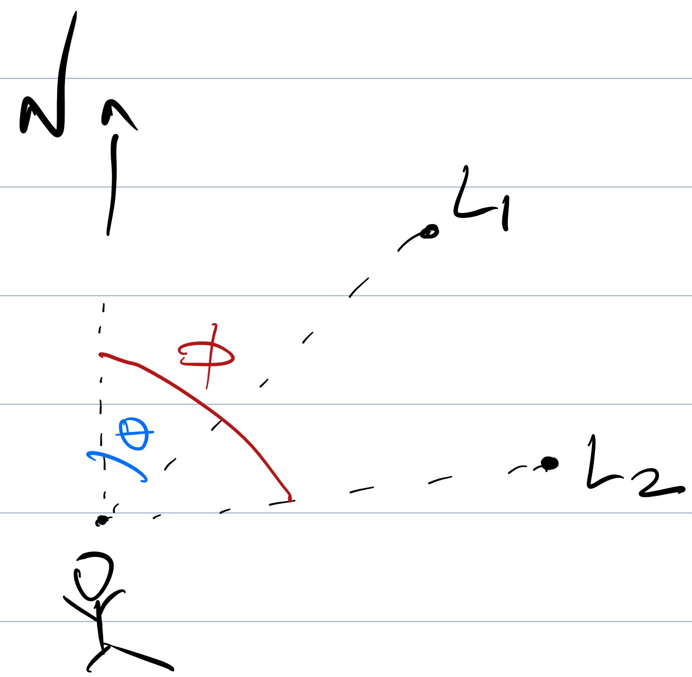
Position fixing with physical vectors

Our goal is to determine equations for position-fixing problems using various types of measurements.

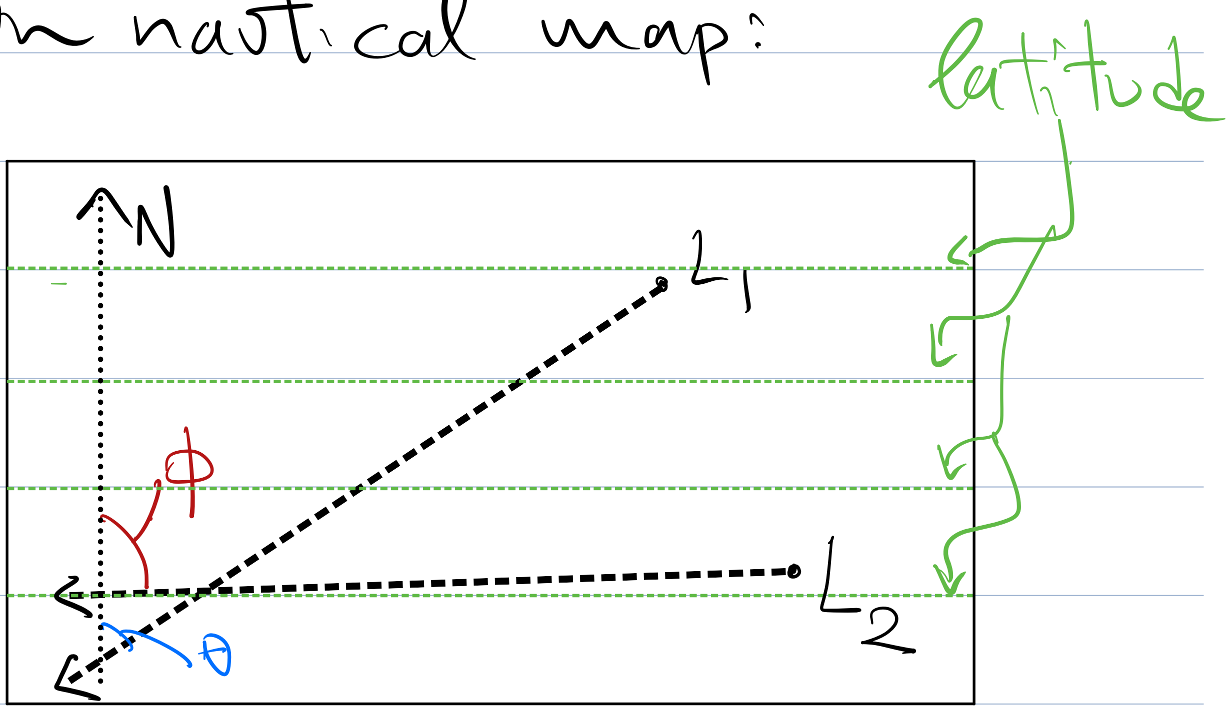
So far, 3 types of measurements:

1) Range
2) Bearing
3) Subtended angle

2 Bearing measurements in 2D



On nautical map:

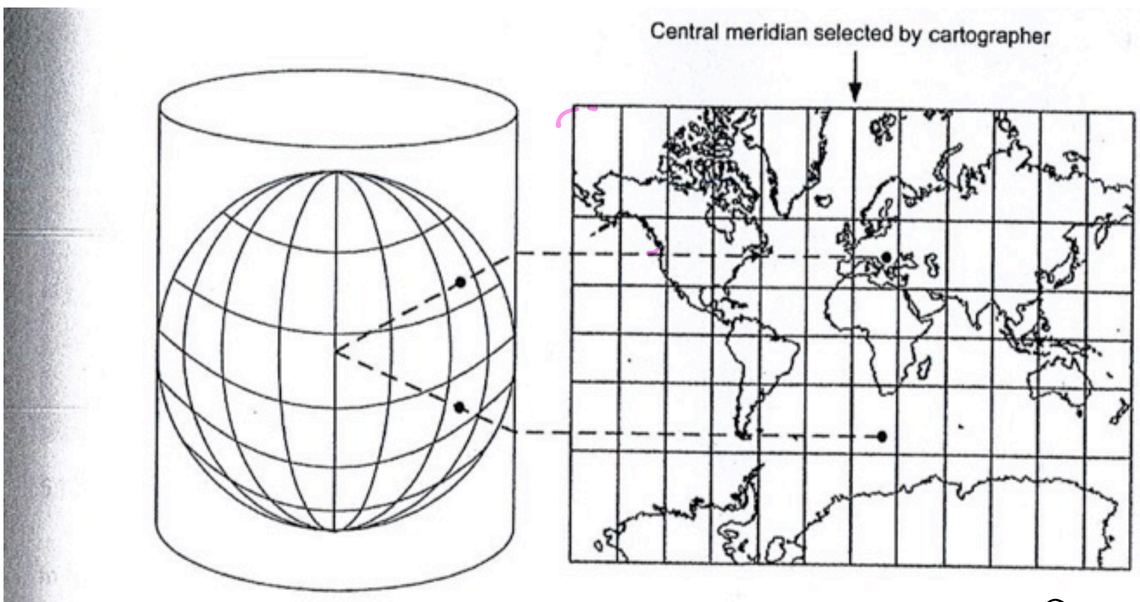


2 bearing measurements

Φ and θ

$\Rightarrow$ position fix at the intersection of 2 rays.

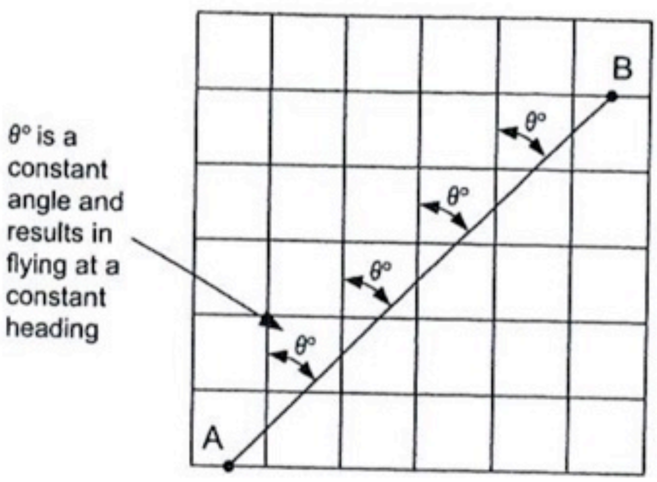
But what a straight line means on a nautical map?



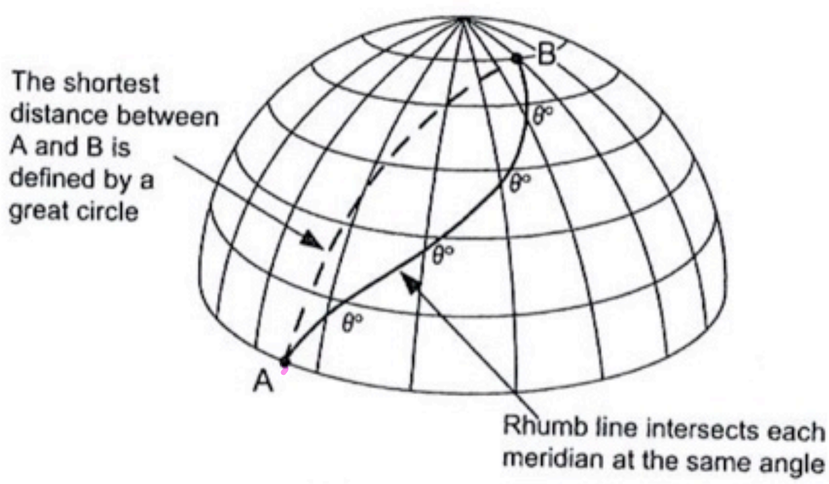
5.9 Mercator projection (for constructing a map)

[In Fig. 5.9: "meridian" = "longitude"]

A straight line on a "Mercator" map, corresponds to a curved line on actual Earth:



(a) Local meridians and the rhumb line

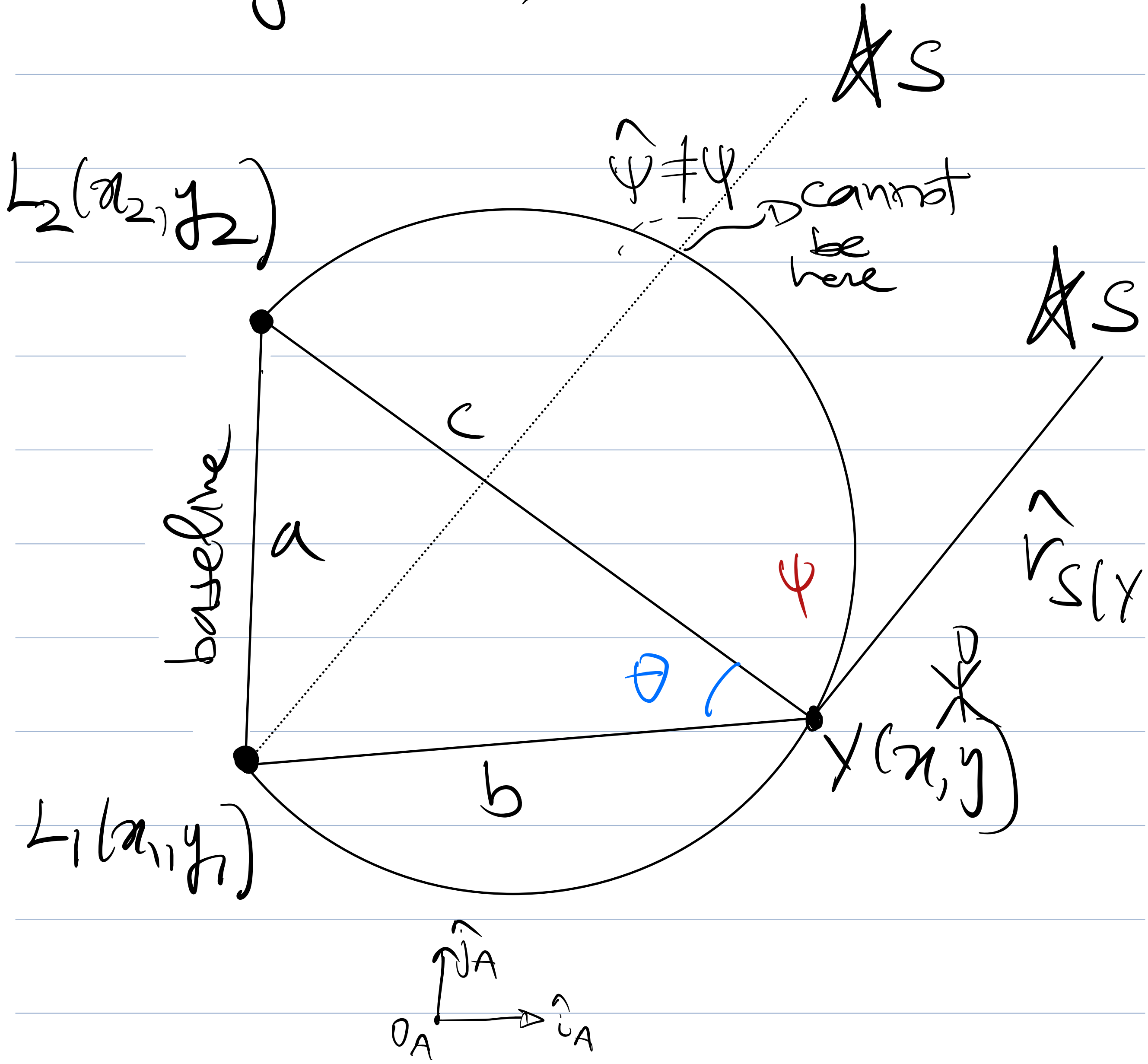


(b) Great circle and the rhumb line

5.4 Rhumb line

2D position fixing with 1 subtended angle and 1 bearing

2 lighthouses and 1 star:



Data uniquely fixes $Y=(x,y)$!

- $\vec{r}_{L_2|L_1} = \vec{r}_{L_2|Y} + \vec{r}_{Y|L_1}$ (E1)
- $a = |\vec{r}_{L_2|L_1}|$ Known baseline
- $\hat{r}_{S|Y}$ Known direction to star
- Frame $F_A = [\hat{i}_A \ \hat{j}_A \ \hat{k}_A]$

Let's find some useful equations:

$$(E1) \overset{\vec{r}_{Y|L_1} \cdot}{\Longrightarrow}$$

$$\vec{r}_{Y|L_1} \cdot \vec{r}_{L_2|L_1} = \vec{r}_{Y|L_1} \cdot \vec{r}_{L_2|Y} + \vec{r}_{Y|L_1} \cdot \vec{r}_{Y|L_1}$$

$$= -\vec{r}_{L_1|Y} \cdot \vec{r}_{L_2|Y} + |\vec{r}_{Y|L_1}|^2$$

$$= -b|\vec{r}_{L_2|Y}| \cos\theta + b^2 \quad (*)$$

where $\quad b \triangleq |\vec{r}_{Y|L_1}|$

Let's resolve vectors in F_A:

$$\vec{r}_{Y|L_1}\big|_A = \vec{r}_{Y/O_A}\big|_A - \vec{r}_{L_1/O_A}\big|_A$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} =$$

$$= \begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix}$$

$$(*) \Rightarrow \begin{bmatrix} x-x_1 \\ y-y_1 \end{bmatrix}^T \begin{bmatrix} x_2-x_1 \\ y_2-y_1 \end{bmatrix}$$

$$= - \left\| \begin{bmatrix} x-x_1 \\ y-y_1 \end{bmatrix} \right\| \cdot \left\| \begin{bmatrix} x-x_2 \\ y-y_2 \end{bmatrix} \right\| \cos\theta$$

$$+ \left\| \begin{bmatrix} x-x_1 \\ y-y_1 \end{bmatrix} \right\|^2 \quad (E2)$$

(E2) is 1 equation with 2 unknowns.

$\Rightarrow$ we need 1 more equation

Let:

$$\hat{r}_{S/Y}\Big|_A = \begin{bmatrix} x_s \\ y_s \end{bmatrix} \quad (x_s^2 + y_s^2 = 1)$$

Then,

$$\vec{r}_{L_2/Y} \cdot \hat{r}_{S/Y} = |\vec{r}_{L_2/Y}| \cos\psi$$

$$\Rightarrow \begin{bmatrix} x_2 - x \\ y_2 - y \end{bmatrix}^T \begin{bmatrix} x_s \\ y_s \end{bmatrix} = \left\| \begin{bmatrix} x - x_2 \\ y - y_2 \end{bmatrix} \right\| \cdot \cos\psi \quad (E3)$$

(E3) is 1 equation with 2 unknowns.

$\Rightarrow$

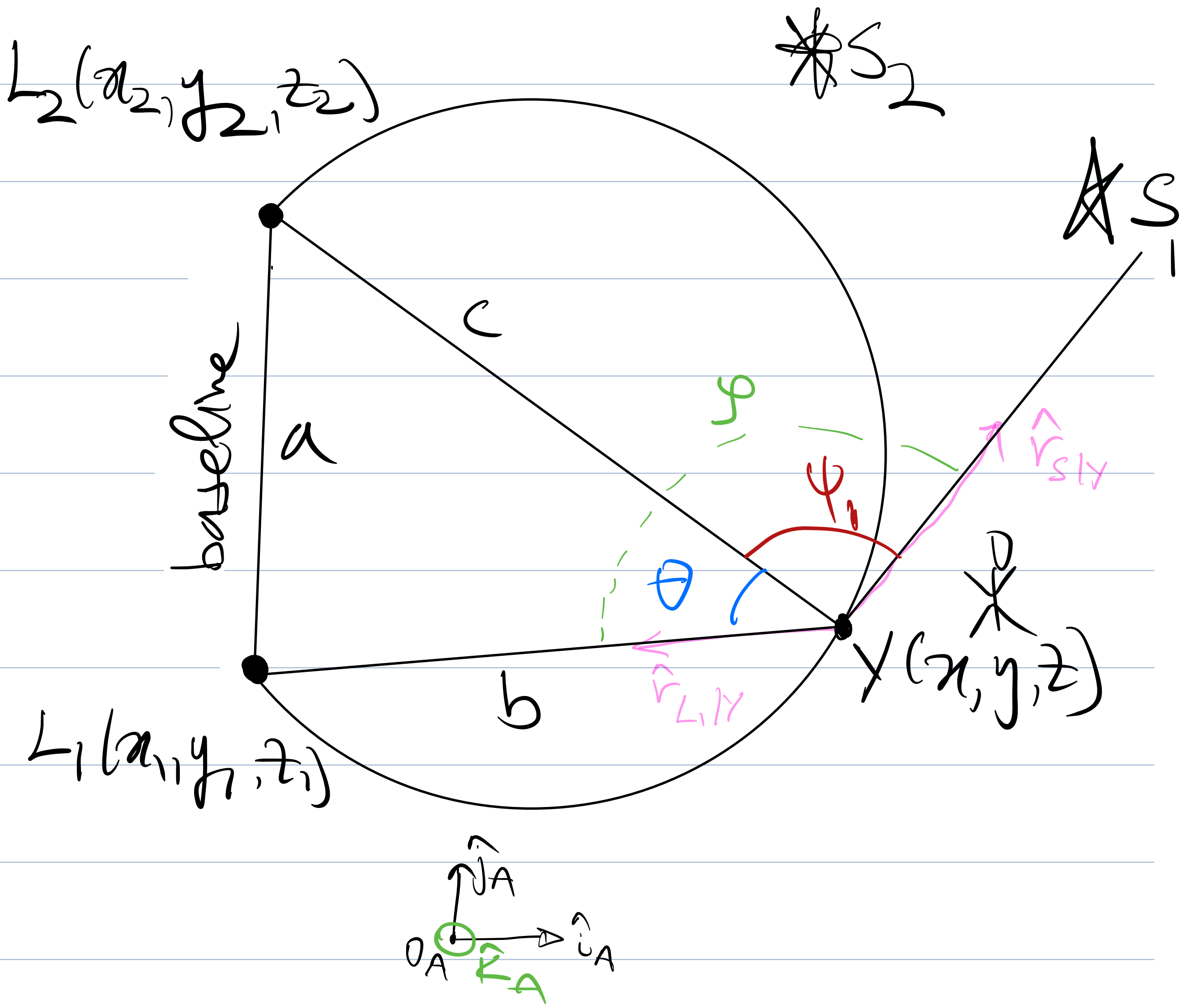
Use (numerical) optimization to solve (E2)-(E3).

Note: So far, we have <u>not</u> ever actually used the knowledge of the baseline a (not even in the previous lectures!), where

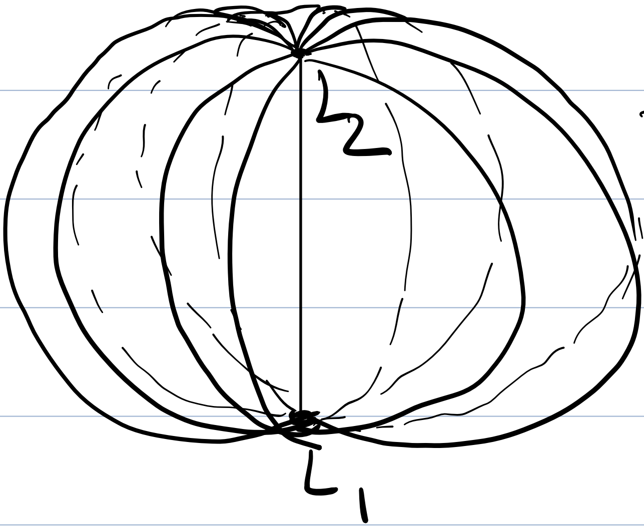
$$a = (L_1 L_2) = \left\| \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \right\|$$

3D position fixing with 1 subtended angle and 1 bearing

Similar to before, but now L_1, L_2, Y, S NOT necessarily on same plane

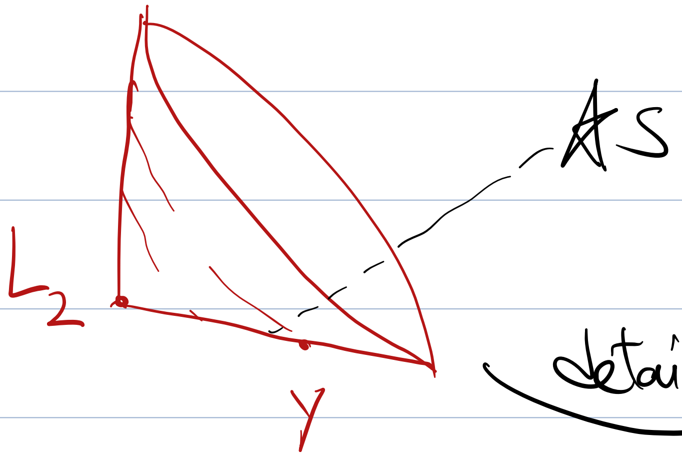


- θ determines a rotated circular arc fix:

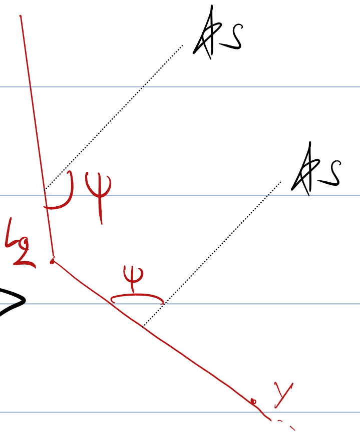


like a pumpkin

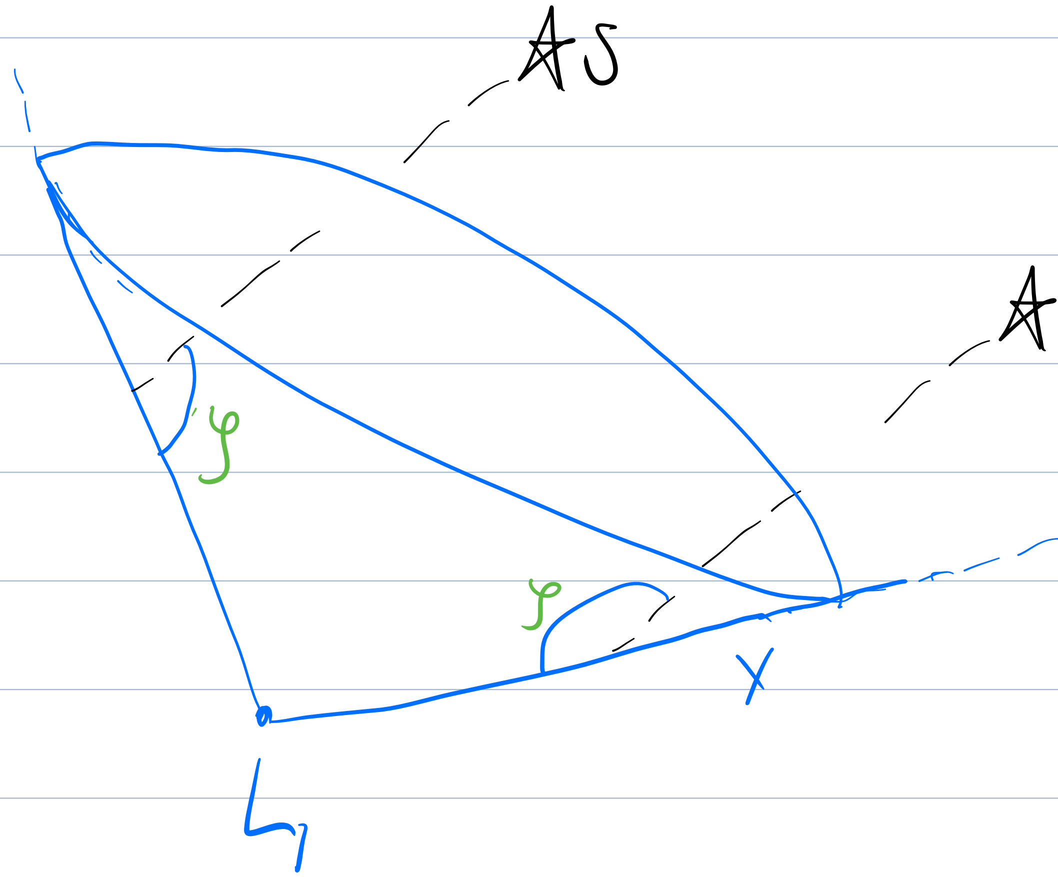
- ψ determines a conical fix



detail



• φ determines another conical fix, where $\varphi \doteq \operatorname{acos} \hat{r}_{S|Y} \cdot \hat{r}_{L_1|Y}$



The intersection of the 3 fixes seems to be at least 2 points:

- The two cones intersect on 2 rays.
- Then, the intersection of the 2 rays with the "pumpkin" is 2 points.

$\Rightarrow$ ambiguous.

<u>Idea</u>: Choose <u>2nd star</u> not in L_1, L_2, S plane

3D equations for position fixes with 2 stars and a subtended angle

Similarly to the 2D case:

$$\vec{r}_{L_2|L_1} = \vec{r}_{L_2|Y} + \vec{r}_{Y|L_1}$$

$$\vec{r}_{Y|L_1} \cdot$$
$\Rightarrow$

$$\vec{r}_{Y|L_1} \cdot \vec{r}_{L_2|L_1} = \vec{r}_{Y|L_1} \cdot \vec{r}_{L_2|Y} + \vec{r}_{Y|L_1} \cdot \vec{r}_{Y|L_1}$$

$$= -\vec{r}_{L_1|Y} \cdot \vec{r}_{L_2|Y} + |\vec{r}_{Y|L_1}|^2$$

$b \triangleq |\vec{r}_{Y|L_1}|$

$$= -b|\vec{r}_{L_2|Y}| \cos\theta + b^2 \quad (*)$$

Resolve vectors in F_A:

$$\vec{r}_{Y|L_1}\Big|_A = \vec{r}_{Y|O_A}\Big|_A - \vec{r}_{L_1|O_A}\Big|_A$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$= \begin{bmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{bmatrix}$$

$$(*) \Rightarrow \begin{bmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{bmatrix}^T \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$$

$$= - \left\| \begin{bmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{bmatrix} \right\| \cdot \left\| \begin{bmatrix} x - x_2 \\ y - y_2 \\ z - z_2 \end{bmatrix} \right\| \cos\theta +$$

$$+ \left\| \begin{bmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{bmatrix} \right\|^2 \quad (E4)$$

(E4) is 1 equation with 3 unknowns $\Rightarrow$ need 2 more eqs.

Star 1

$$\hat{r}_{S_1/Y}\Big|_A = \begin{bmatrix} x_{S_1} \\ y_{S_1} \\ z_{S_1} \end{bmatrix}, \text{ where } x_{S_1}^2 + y_{S_1}^2 + z_{S_1}^2 = 1.$$

Then,

$$\vec{r}_{L_2/Y} \cdot \hat{r}_{S_1/Y} = |\vec{r}_{L_2/Y}| \cos\psi_1$$

$$\Rightarrow \begin{bmatrix} x_2 - x \\ y_2 - y \\ z_2 - z \end{bmatrix}^T \begin{bmatrix} x_{S_1} \\ y_{S_1} \\ z_{S_1} \end{bmatrix} = \left\| \begin{bmatrix} x - x_2 \\ y - y_2 \\ z - z_2 \end{bmatrix} \right\| \cos\psi_1 \qquad \text{(E5)}$$

That's another equation with 3 unknowns.

Need 1 more equation:

Star 2

Similarly to above:

$$\begin{bmatrix} x_2 - x \\ y_2 - y \\ z_2 - z \end{bmatrix}^T \begin{bmatrix} x_{s_2} \\ y_{s_2} \\ z_{s_2} \end{bmatrix} = \left\| \begin{bmatrix} x - x_2 \\ y - y_2 \\ z - z_2 \end{bmatrix} \right\| \cos\psi_2 \qquad \text{(E6)}$$

where $x_{s_2}^2 + y_{s_2}^2 + z_{s_2}^2 = 1$.

$\Rightarrow$ solve (E4)–(E6) for x, y, z.