Kinematics

(expressing derivatives of math and physical with respect to possibly moving frames, instead of inertial frames)

one of our final steps before we dive into how accelerometers and gyros work!

Vector differentiation

Given a math vector $r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$

$$\Rightarrow \quad \dot{r} \equiv \frac{d}{dt} r = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix} \qquad \left(\text{where } \dot{(*)} = \frac{d}{dt}(*)\right)$$

$\hookrightarrow$ info about frame that has been used to find r has been omitted

Given a physical vector instead we need to consider a frame such that:

$$\vec{r} = r_1 \hat{i}_A + r_2 \hat{j}_A + r_3 \hat{k}_A$$

$$\Longleftrightarrow \quad \vec{r}\big|_A = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Then, the time derivative of $\vec{r}$ with respect to A frame is defined as:

$$\overset{A\cdot}{\vec{r}} = \dot{r}_1 \hat{i}_A + r_1 \underbrace{\overset{A\cdot}{\hat{i}_A}}_{=0} + \dot{r}_2 \hat{j}_A + r_2 \underbrace{\overset{A\cdot}{\hat{j}_A}}_{=0} + \dot{r}_3 \hat{k}_A + r_3 \underbrace{\overset{A\cdot}{\hat{k}_A}}_{=0}$$

$$= \dot{r}_1 \hat{i}_A + \dot{r}_2 \hat{j}_A + \dot{r}_3 \hat{k}_A$$

$$\Longleftrightarrow \quad \overset{A\cdot}{\vec{r}}\Big|_A = \frac{d}{dt}\left[\vec{r}\big|_A\right] = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}$$

Now ready to express formally acceleration using derivatives of physical and math vectors:

- If $\vec{r}_{y/x}$ is position (of y w.r.t x) then:

  - $\vec{v}_{y/x/A} \triangleq \overset{A\cdot}{\vec{r}}_{y/x}$ is velocity w.r.t A

  - $\vec{a}_{y/x/A} \triangleq \overset{A\cdot}{\vec{v}}_{y/x/A} = \overset{A\cdot\cdot}{\vec{r}}_{y/x}$ is acceleration w.r.t. A

Slash and split holds:

$$\vec{r}_{y/x} = \vec{r}_{y/z} + \vec{r}_{z/x}$$

$$\vec{v}_{y/x/A} = \vec{v}_{y/z/A} + \vec{v}_{z/x/A}$$

$$\vec{a}_{y/x/A} = \vec{a}_{y/z/A} + \vec{a}_{z/x/A}$$

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<u>Frame dot identities</u>

$$\frac{d}{dt}\left[\vec{x}\cdot\vec{y}\right] = \overset{\cdot}{\overbrace{\vec{x}\cdot\vec{y}}} = \overset{A\cdot}{\vec{x}}\cdot\vec{y} + \vec{x}\cdot\overset{A\cdot}{\vec{y}}$$

<u>Definition</u> $\overset{A\cdot}{\overbrace{\vec{x}'}} \stackrel{\Delta}{=} (\overset{A\cdot}{\vec{x}})'$

<u>Definition</u> Let $\vec{M} = \vec{x}\,\vec{y}'$, then

$$\overset{A\cdot}{\vec{M}} \stackrel{\Delta}{=} \overset{A\cdot}{\vec{x}}\,\vec{y}' + \vec{x}\,\overset{A\cdot}{\vec{y}}{}'$$

Fact: $\left(\overset{A\cdot}{\vec{M}}\right)' = \overbrace{\vec{M}'}^{A\cdot}$

Fact $\overbrace{\vec{M}\cdot\vec{z}}^{A\cdot} = \overset{A\cdot}{\vec{M}}\,\vec{z} + \vec{M}\,\overset{A\cdot}{\vec{z}}$

$$\overbrace{\vec{M}\cdot\vec{N}}^{A\cdot} = \overset{A\cdot}{\vec{M}}\,\vec{N} + \vec{M}\,\overset{A\cdot}{\vec{N}}$$

Fact $\overset{A\cdot}{\vec{M}}\Big|_A = \frac{d}{dt}\left[\vec{M}\big|_A\right]$

So far, we established background for expressing <u>linear</u> accelerations. BUT gyros measure <u>angular</u> accelerations.

Super Cross Derivative

$$\overset{A\cdot}{\overbrace{\vec{x}^{\times}}} = \left(\overset{A\cdot}{\vec{x}}\right)^{\times}$$

$$\overset{A\cdot}{\overbrace{\vec{x} \times \vec{y}}} = \overset{A\cdot}{\vec{x}} \times \vec{y} + \vec{x} \times \overset{A\cdot}{\vec{y}}$$

Rotation matrix derivative

__Fact.__ $\overset{A\cdot}{\vec{I}} = \vec{0}$

__Fact__ $\overset{B\cdot}{\vec{R}}_{A|B} = -\vec{R}_{A|B}\,\overset{B\cdot}{\vec{R}}_{B|A} \cdot \vec{R}_{A|B}$

__Proof__

$$\vec{I} = \vec{R}_{A|B} \cdot \vec{R}_{B|A}$$

$$\Rightarrow \vec{0} = \overset{B\cdot}{\vec{R}}_{A|B}\,\vec{R}_{B|A} + \vec{R}_{A|B} \cdot \overset{B\cdot}{\vec{R}}_{B|A}$$

$$\Rightarrow \vec{R}_{A|B} \cdot \overset{B\cdot}{\vec{R}}_{B|A} = -\overset{B\cdot}{\vec{R}}_{A|B} \cdot \vec{R}_{B|A}$$

$\vec{R}_{B|A}\times$

$$\Rightarrow \overset{B\cdot}{\vec{R}}_{B|A} = -\vec{R}_{B|A}\,\overset{B\cdot}{\vec{R}}_{A|B}\,\vec{R}_{B|A}$$

$$\Rightarrow \overset{B\cdot}{\vec{R}}_{A|B} = -\vec{R}_{A|B}\,\overset{B\cdot}{R}_{B|A} \cdot \vec{R}_{A|B}$$

Note $\overset{B\cdot}{\vec{R}}{}'_{B/A} = \overset{B\cdot}{\vec{R}}_{A/B}$.

Fact (non-trivial) $\overset{B\cdot}{\vec{R}}_{B/A} = \vec{R}_{B/A}\,\overset{A\cdot}{\vec{R}}_{B/A}\,\vec{R}_{A/B}$

Note $\overset{B\cdot}{\vec{R}}_{B/A} = 0 \iff \overset{A\cdot}{\vec{R}}_{B/A} = 0$.

Rotating dot identity

Given F_A and F_B, what is the relationship between $\overset{A\cdot}{\vec{x}}$ and $\overset{B\cdot}{\vec{x}}$?

Fact (Transport theorem) $\overset{A\cdot}{\vec{x}} = \overset{B\cdot}{\vec{x}} + R_{A/B}\,\overset{B\cdot}{\vec{R}}_{B/A}\,\vec{x}$.

Note: if $\overset{B\cdot}{\vec{R}}_{B/A} = \overset{A\cdot}{\vec{R}}_{B/A} = 0$ then: $\overset{A\cdot}{\vec{x}} = \overset{B\cdot}{\vec{x}}$

Proof. Define $\vec{y} \doteq \vec{R}_{B/A}\vec{x}$. We use the following Fact:

Fact $\overset{B\cdot}{\vec{y}} = \vec{R}_{B/A} \overset{A\cdot}{\vec{x}}$

Proof

$$\left(\vec{R}_{A/B}\overset{B\cdot}{\vec{y}}\right)\Big|_A = \vec{R}_{A/B}\Big|_A \overset{B\cdot}{\vec{y}}\Big|_A =$$

$$\vec{R}_{A/B}\Big|_A \; \sigma_{A/B} \overset{B\cdot}{\vec{y}}\Big|_B =$$

$$\vec{R}_{A/B}\Big|_A \cdot \vec{R}_{A/B}\Big|_A^T \cdot \overset{B\cdot}{\vec{y}}\Big|_B =$$

$$\overset{B\cdot}{\vec{y}}\Big|_B = \left(\overbrace{\vec{R}_{B|A}\vec{x}}^{B\cdot}\right)\Big|_B = \qquad \overset{B\cdot}{\vec{r}}\Big|_B = \overbrace{\left(\vec{r}|_B\right)}^{\cdot}$$

$$\overbrace{\left(\vec{R}_{B|A}\vec{x}\Big|_B\right)}^{\cdot} = \qquad \vec{z}|_B = \sigma_{B|A}\,\vec{z}|_A$$

$$\overbrace{\Big(\underbrace{\underbrace{\vec{O}_{B|A}}_{\substack{=\vec{R}_{B|A}|_B^T \\ =\vec{R}_{B|A}|_A^T}}\;\vec{R}_{B|A}}_{I}\;\vec{x}|_A\Big)}^{\cdot} = (\dot{\vec{x}}|_A) = \overset{A\cdot}{\vec{x}}\Big|_A$$

Given the blue fact, now we continue with the proof of the orange fact. Blue fact gives:

$$\overset{A\cdot}{\vec{x}} = \vec{R}_{A|B}\,\overset{B\cdot}{\vec{y}}$$

$$= \vec{R}_{A|B}\,\overbrace{\vec{R}_{B|A}\,\vec{x}}^{B\cdot}$$

$$= \vec{R}_{A|B}\Big(\vec{R}_{B|A}\,\overset{B\cdot}{\vec{x}} + \overset{B\cdot}{\vec{R}}_{B|A}\,\vec{x}\Big)$$

$$= \overset{B\cdot}{\vec{a}} + \underbrace{\vec{R}_{A|B}\,\overset{B\cdot}{\vec{R}}_{B|A}}_{\overset{\Delta}{=}\ \vec{\Omega}_{B|A}} \vec{a}$$

<u>Fact</u> $\vec{\Omega}'_{B|A} = -\vec{\Omega}_{B|A}$

$\Rightarrow$ $\vec{\Omega}_{B|A}$ is skew symmetric!

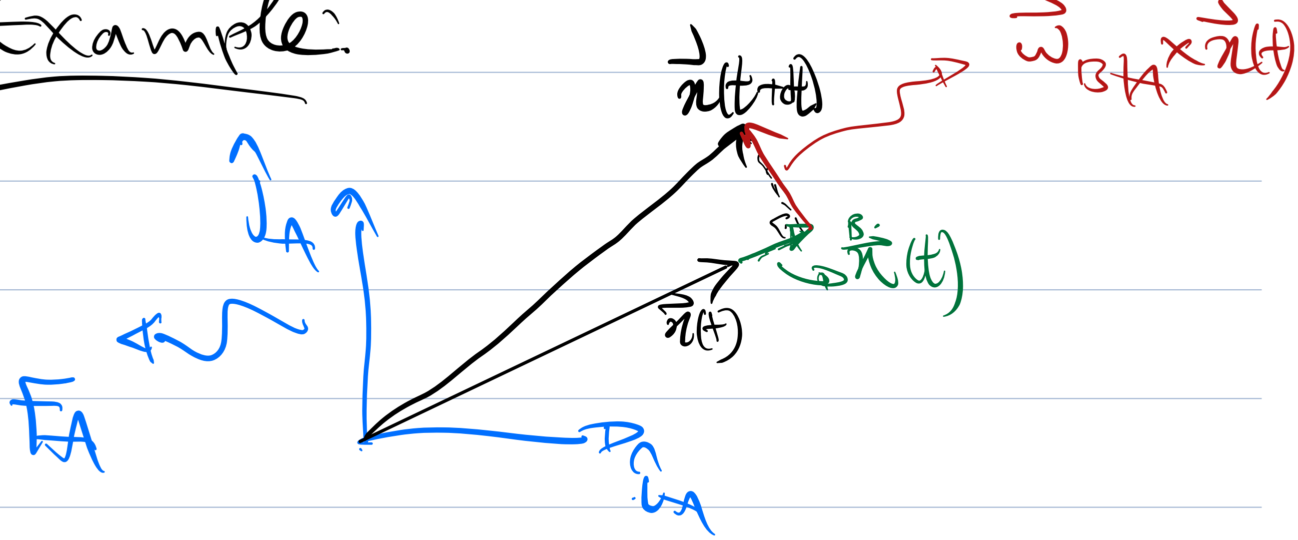
$\Rightarrow$ there exists $\vec{\omega}_{B|A}$ such that

$$\vec{\omega}^{\times}_{B|A} = \vec{\Omega}_{B|A}$$

$\vec{\omega}_{B|A}$ is the angular velocity of F_B relative to F_A.

orange $\Rightarrow$ fact $\quad \overset{A\cdot}{\vec{r}} = \overset{B\cdot}{\vec{r}} + \vec{\omega}_{B|A} \times \vec{r}$

Example:



Assume F_B rotates with $\vec{r}$ such that $\hat{i}_B$ is $\parallel$ with $\vec{r}(t)$ at all times, with angular velocity $\vec{\omega}_{B|A}$.

Fact $\vec{\omega}_{B|A} = -\vec{\omega}_{A|B}$

Fact (slash and split)

$$\vec{\omega}_{C|A} = \vec{\omega}_{C|B} + \vec{\omega}_{B|A}$$

Poisson's Equation

Resolving $\vec{\omega}^{\times}_{B|A}$:

$$\vec{\omega}^{\times}_{B|A}\Big|_B = \vec{R}_{A|B}\Big|_B \cdot \overset{B\cdot}{\vec{R}}_{B|A}\Big|_B$$

$\Theta_{A|B} = \vec{R}_{B|A}|_B$

$\overset{B\cdot}{\vec{M}}|_B = (\dot{\vec{M}}|_B)$

$$= \Theta_{B|A} \cdot \dot{\Theta}_{A|B}$$

$$\Rightarrow \quad \dot{O}_{A|B} = O_{A|B} \cdot \vec{\omega}^{\times}_{B/A}\big|_B$$

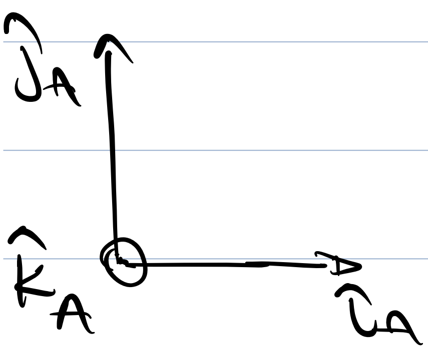
$$\Rightarrow \quad \dot{O}^T_{A|B} = \vec{\omega}^{\times}_{B/A}\big|_B^T \cdot O^T_{A|B}$$

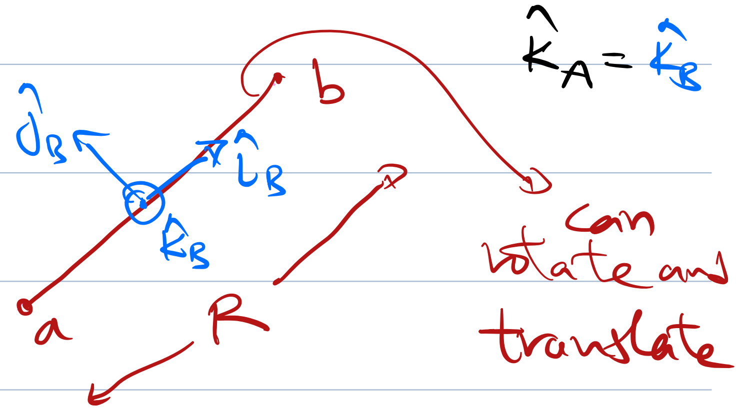
$$\Rightarrow \quad \dot{O}_{B|A} = -\omega^{\times}_{B/A}\big|_B \cdot O_{B|A}$$

(if you know $O_{B|A}(0)$ you integrate above to get $O_{B|A}(t)$)

$\sim$ inertial navigation

Example (Circular motion)





Assume $F_A \xrightarrow[\hat{k}_A]{\theta} F_B \Rightarrow$

$$\vec{\omega}_{B/A} = \dot{\theta}\,\hat{k}_A = \omega\,\hat{k}_A = \omega\,\hat{k}_B$$

$$\Rightarrow \vec{v}_{b/a|A} = \overset{A\cdot}{\vec{r}}_{b/a}$$

$$= \underbrace{\overset{B\cdot}{\vec{r}}_{b/a}}_{=0} + \vec{\omega}_{B/A} \times \vec{r}_{b/a}$$

$$= \omega\,\hat{k}_A \times R\,\hat{\imath}_B$$

$$= \underbrace{\omega R}_{\triangleq\ v}\,\hat{\jmath}_B$$

$$\Rightarrow \vec{a}_{b/a|A} = \overset{A\cdot}{\vec{v}}_{b/a|A}$$

$$= v\,\overset{A\cdot}{\hat{\jmath}}_B + \dot{v}\,\hat{\jmath}_B$$

$$= v\left(\underbrace{\overset{B\cdot}{\hat{j}_B}}_{=0} + \vec{\omega}_{B|A} \times \hat{j}_B \right) + \dot{v}\,\hat{j}_B$$

$$= v \cdot \omega\, \hat{k}_B \times \hat{j}_B + \dot{v}\,\hat{j}_B$$

$$= \underbrace{-\omega^2 R\, \hat{i}_B}_{\text{centripetal acceleration}} + \underbrace{\dot{\omega} R\, \hat{j}_B}_{\text{angular acceleration}}$$

<u>Fact</u> $\overset{A\cdot}{\vec{\omega}_{B|A}} = \overset{B\cdot}{\vec{\omega}_{B|A}}$

<u>Proof</u> $\overset{A\cdot}{\vec{\omega}_{B|A}} = \overset{B\cdot}{\vec{\omega}_{B|A}} + \underbrace{\vec{\omega}_{B|A} \times \vec{\omega}_{B|A}}_{=0}$

Euler angles

$$F_A \xrightarrow[\hat{k}_A]{\psi} F_B \xrightarrow[\hat{j}_B]{\theta} F_C \xrightarrow[\hat{i}_C]{\phi} F_D$$

$$\vec{\omega}_{D|A} = \vec{\omega}_{D|C} + \vec{\omega}_{C|B} + \vec{\omega}_{B|A}$$
$$= \dot{\phi}\,\hat{i}_C + \dot{H}\,\hat{j}_B + \dot{\psi}\,\hat{k}_A$$

Double Transport

$$\vec{a}_{y/x|A} = \underbrace{\vec{a}_{y|x|B}}_{\text{relative acceleration}} + \underbrace{2\vec{\omega}_{B|A} \times \vec{v}_{y|x|B}}_{\text{coriolis acceleration}}$$
$$+ \underbrace{\overset{A\cdot}{\vec{\omega}}_{B|A} \times \vec{r}_{y/x}}_{\text{angular acceleration}} + \underbrace{\vec{\omega}_{B|A} \times (\vec{\omega}_{B|A} \times \vec{r}_{y/x})}_{\text{centripetal acceleration}}$$