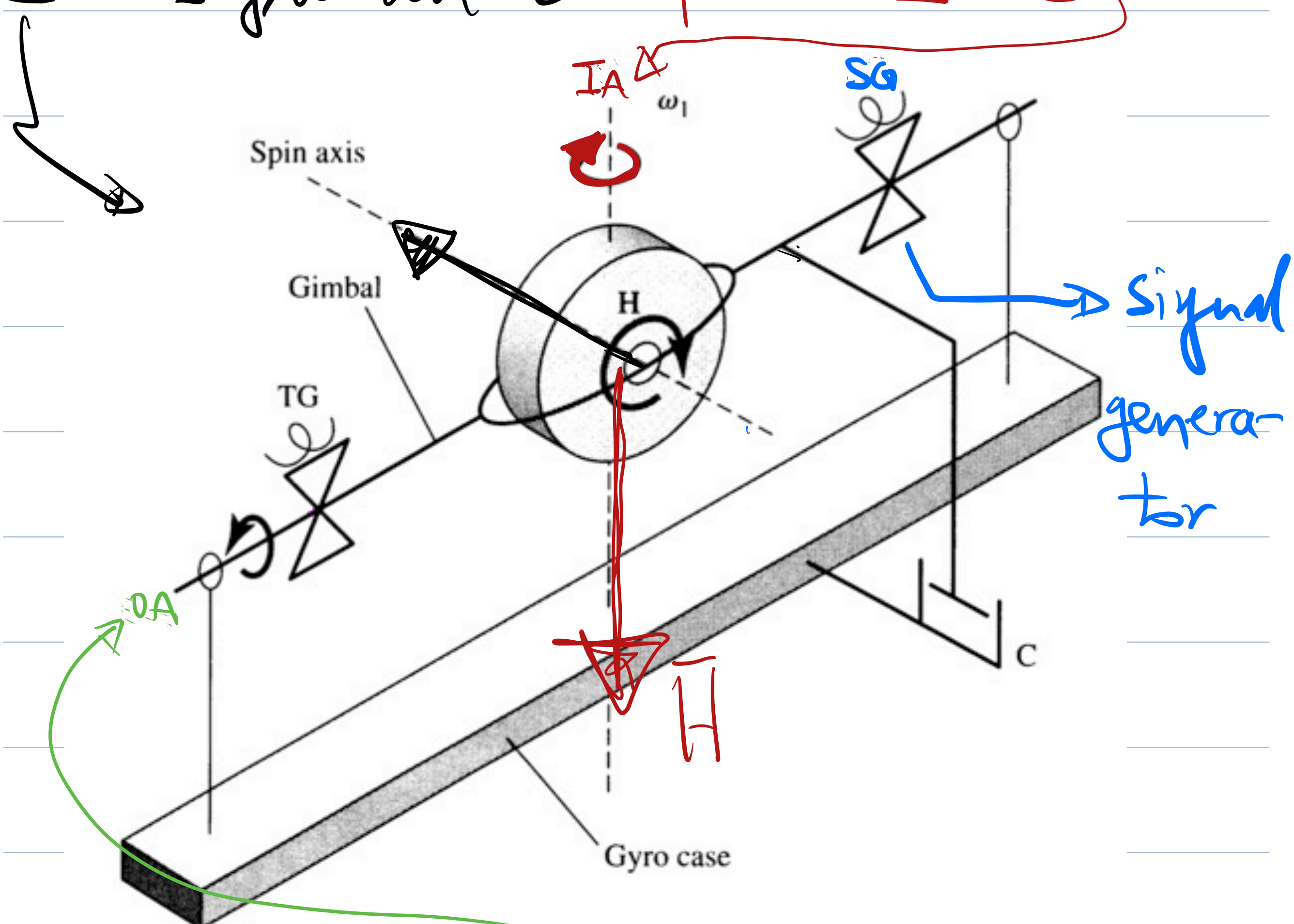
Gyros

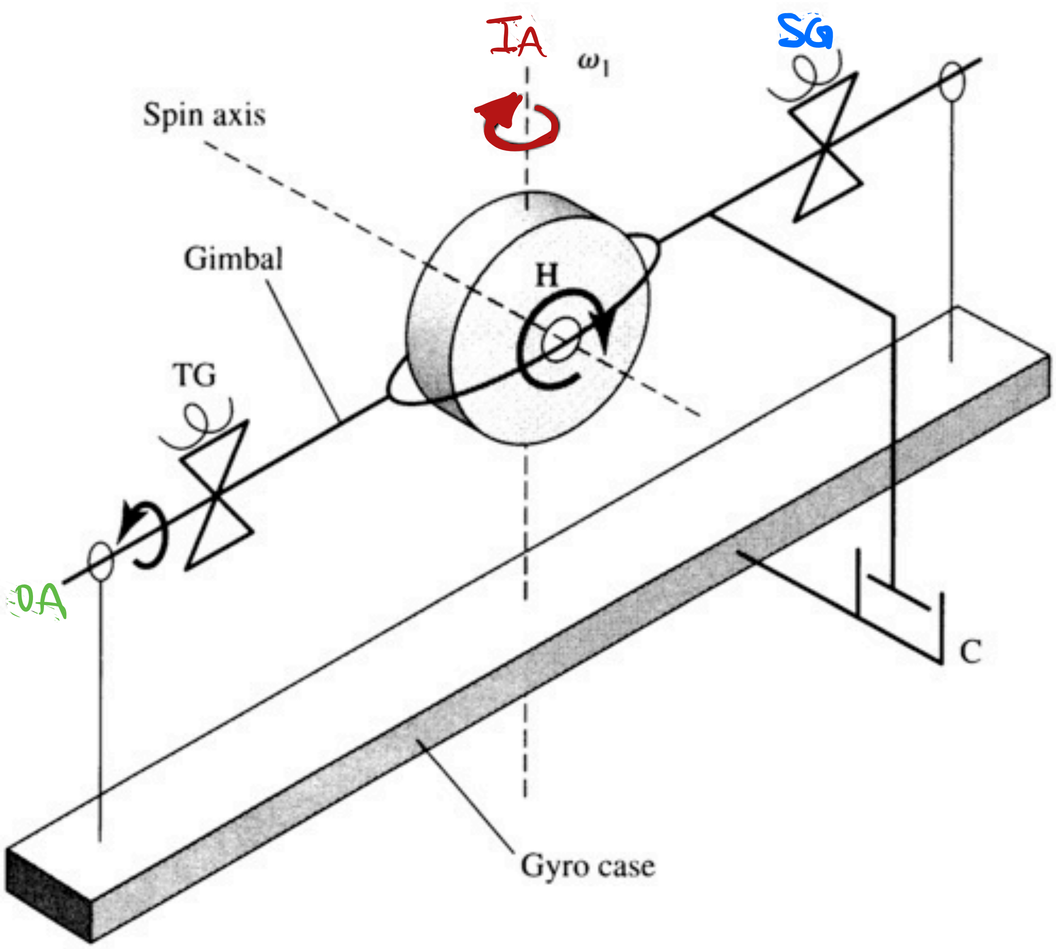
1-axis gyro with input axis the



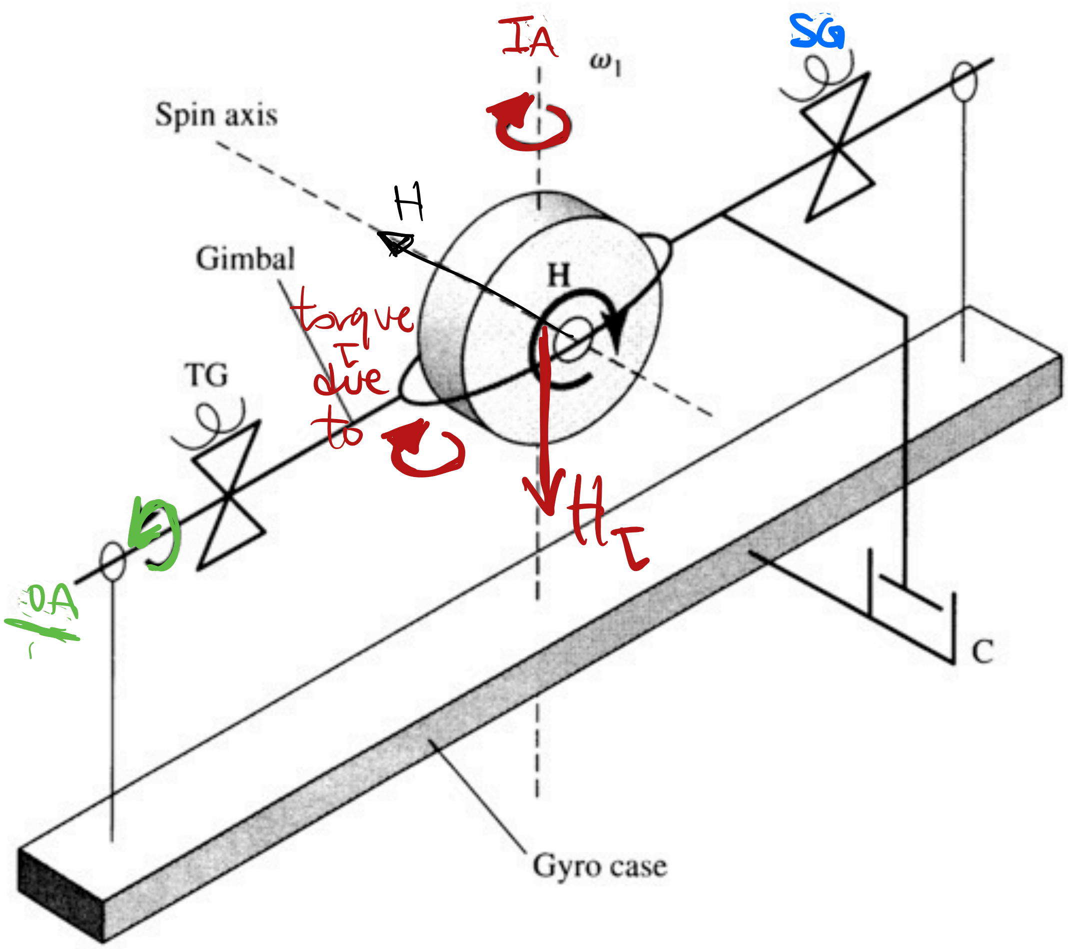
and output axis the. The gyro wheel is spinning around the horizontal axis "spin axis".

Assume the gyro is mounted on a car, turning left.

Then, the gimbal turns ↺ by the IA axis,



The motion induces torques τ to the wheel, resulting to angular momentum H_τ, perpendicular to H due to the spinning of the wheel.



H_T causes the wheel to precess, with the wheel turning around OA B (see first video on last page of Lecture notes no. 7+8:

https://youtu.be/ekzwbt3hu2k
)

POLL 1.

Then, the OA axis is equipped with devices that detect this motion, turn it to electrical signal, which is amplified and "outputed" as sensor measurement (see SG on figure).

In reality, we don't allow the wheel to move along the IA, in the sense that once a desire to move is detected then it's cancelled out by a feedback controller (see torque generator TG in the Fig.) That's similar to the accelerometer discussion we made during the previous lecture.

Let's now do some math:

What Does a Rate Gyro Measure?

Let $\mathrm{F_A}$ be an inertial frame, and let $\mathrm{F_G}$ be a frame whose axes are aligned with the measurement axes of a 3-axis rate gyro mounted at the point p on the rigid body $\mathcal{B}$. The 3-axis rate gyro measures the spin rate around each of its axes. To make this precise, let $\vec{\omega}_{\mathrm{G/A}}$ be the angular velocity of $\mathrm{F_G}$ relative to $\mathrm{F_A}$. Then the rate-gyro measurements are the components of $\vec{\omega}_{\mathrm{G/A}}$ resolved in $\mathrm{F_G}$, which is written as

$$\omega(t) \triangleq \vec{\omega}_{\mathrm{G/A}}(t)\Big|_{\mathrm{G}}. \tag{27}$$

Note that (27) makes no reference to the physical location p of the 3-axis rate gyro on $\mathcal{B}$, and thus the measurements of the 3-axis rate gyro are independent of its location. Letting $\mathrm{F_B}$ be a body-fixed frame, it follows that $\vec{\omega}_{\mathrm{G/B}} = 0$, and thus $\vec{\omega}_{\mathrm{G/A}} = \vec{\omega}_{\mathrm{G/B}} + \vec{\omega}_{\mathrm{B/A}} = \vec{\omega}_{\mathrm{B/A}}$.

For convenience, assume that $\mathrm{F_G} = \mathrm{F_B}$, and consider the single-axis rate gyro shown in Figure 3, which measures the component of the angular velocity of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ along $\hat{\imath}_{\mathrm{B}}$. The frame $\mathrm{F_B}$ is attached to the rigid body $\mathcal{B}$, and the frame $\mathrm{F_C}$ is attached to the gimbal $\mathcal{C}$. The rotor $\mathcal{D}$ is mounted on $\mathcal{C}$, and the frame $\mathrm{F_D}$ is attached to $\mathcal{D}$ such that $\hat{k}_{\mathrm{D}}$ is aligned with $\hat{k}_{\mathrm{C}}$. The axis $\hat{\jmath}_{\mathrm{C}}$, which is aligned with $\hat{\jmath}_{\mathrm{B}}$, is the measurement axis of the rate gyro. These frames are related by the Euler-angle rotations

$$\mathrm{F_A} \xrightarrow[1]{\Psi} \mathrm{F_B} \xrightarrow[2]{\Theta} \mathrm{F_C} \xrightarrow[3]{\Phi} \mathrm{F_D}, \tag{28}$$

where the Euler angles Ψ, Θ, and Φ correspond to rotations around $\hat{\imath}_{\mathrm{A}}$, $\hat{\jmath}_{\mathrm{B}}$, and $\hat{k}_{\mathrm{C}}$, respectively. Note that Θ is the gimbal angle.

Henceforth, assume that the angular velocity of $\mathcal{B}$ relative to $\mathrm{F_A}$ is aligned with $\hat{\imath}_{\mathrm{B}}$, and thus $\mathcal{B}$ rotates relative to $\mathrm{F_A}$ about $\hat{\imath}_{\mathrm{B}}$. It thus follows that

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\Psi}\hat{\imath}_{\mathrm{B}} = \omega_{\mathrm{in}}\hat{\imath}_{\mathrm{B}}, \tag{29}$$

$$\vec{\omega}_{\mathrm{C/B}} = \dot{\Theta}\hat{\jmath}_{\mathrm{C}}, \tag{30}$$

$$\vec{\omega}_{\mathrm{D/C}} = \dot{\Phi}\hat{k}_{\mathrm{D}} = \omega\hat{k}_{\mathrm{D}}. \tag{31}$$

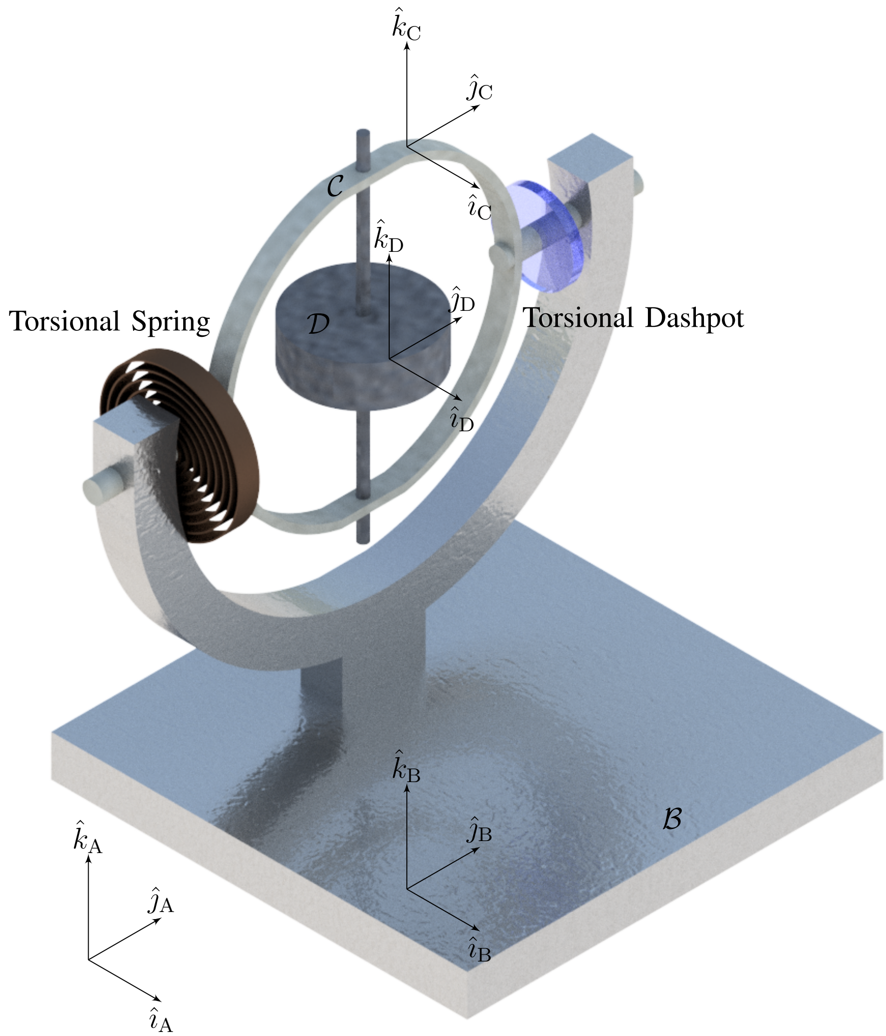


Figure 3: Rate-gyro setup. $\mathrm{F_A}$ is an inertial frame, $\mathrm{F_B}$ is attached to the rigid body $\mathcal{B}$, $\mathrm{F_C}$ is attached to the gimbal $\mathcal{C}$, and $\mathrm{F_D}$ is attached to the rotor $\mathcal{D}$.

Therefore,

$$\begin{aligned}
\vec{\omega}_{\mathrm{D/A}} &= \vec{\omega}_{\mathrm{D/C}} + \vec{\omega}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}} \\
&= \omega \hat{k}_{\mathrm{D}} + \dot{\Theta}\hat{\jmath}_{\mathrm{C}} + \omega_{\mathrm{in}}\hat{\imath}_{\mathrm{B}} \\
&= \omega \hat{k}_{\mathrm{D}} + \dot{\Theta}[(\sin\Phi)\hat{\imath}_{\mathrm{D}} + (\cos\Phi)\hat{\jmath}_{\mathrm{D}}] + \omega_{\mathrm{in}}[(\cos\Theta)\hat{\imath}_{\mathrm{C}} + (\sin\Theta)\hat{k}_{\mathrm{C}}] \\
&= \omega \hat{k}_{\mathrm{D}} + \dot{\Theta}[(\sin\Phi)\hat{\imath}_{\mathrm{D}} + (\cos\Phi)\hat{\jmath}_{\mathrm{D}}] + \omega_{\mathrm{in}}[\cos\Theta[(\cos\Phi)\hat{\imath}_{\mathrm{D}} - (\sin\Phi)\hat{\jmath}_{\mathrm{D}}] + (\sin\Theta)\hat{k}_{\mathrm{D}}] \\
&= [\dot{\Theta}\sin\Phi + \omega_{\mathrm{in}}(\cos\Theta)\cos\Phi]\hat{\imath}_{\mathrm{D}} + [\dot{\Theta}\cos\Phi - \omega_{\mathrm{in}}(\cos\Theta)\sin\Phi]\hat{\jmath}_{\mathrm{D}} + [\omega + \omega_{\mathrm{in}}\sin\Theta]\hat{k}_{\mathrm{D}}.
\end{aligned} \tag{32}$$

Resolving (32) in $\mathrm{F_D}$ yields

$$\vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}} = \begin{bmatrix} (\cos\Theta)\cos\Phi & \sin\Phi & 0 \\ -(\cos\Theta)\sin\Phi & \cos\Phi & 0 \\ \sin\Theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_{\mathrm{in}} \\ \dot{\Theta} \\ \omega \end{bmatrix}. \tag{33}$$

Next, let the moment of inertia of $\mathcal{D}$ relative to its center of mass $\mathrm{cm}_{\mathcal{D}}$ be resolved in $\mathrm{F_D}$ as

$$\vec{J}_{\mathcal{D}/\mathrm{cm}_{\mathcal{D}}}\Big|_{\mathrm{D}} = \begin{bmatrix} J_{\mathcal{D},11} & 0 & 0 \\ 0 & J_{\mathcal{D},22} & 0 \\ 0 & 0 & J_{\mathcal{D},33} \end{bmatrix}. \tag{34}$$

Then, Euler's equation has the form

$$\vec{J}_{\mathcal{D}/\mathrm{cm}_{\mathcal{D}}} \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}} + \vec{\omega}_{\mathrm{D/A}} \times \vec{J}_{\mathcal{D}/\mathrm{cm}_{\mathcal{D}}} \vec{\omega}_{\mathrm{D/A}} = \vec{M}_{\mathcal{D}}, \tag{35}$$

where

$$\begin{aligned} \vec{J}_{\mathcal{D}/\mathrm{cm}_{\mathcal{D}}} \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}} = {} & J_{\mathcal{D},11}[\ddot{\Theta}\sin\Phi + \dot{\Theta}\omega\cos\Phi + \dot{\omega}_{\mathrm{in}}(\cos\Theta)\cos\Phi - \omega_{\mathrm{in}}\dot{\Theta}(\sin\Theta)\cos\Phi \\ & - \omega_{\mathrm{in}}\omega(\cos\Theta)\sin\Phi]\hat{\imath}_{\mathrm{D}} + J_{\mathcal{D},22}[\ddot{\Theta}\cos\Phi - \dot{\Theta}\omega\sin\Phi - \dot{\omega}_{\mathrm{in}}(\cos\Theta)\sin\Phi \\ & + \omega_{\mathrm{in}}\dot{\Theta}(\sin\Theta)\sin\Phi - \omega_{\mathrm{in}}\omega(\cos\Theta)\cos\Phi]\hat{\jmath}_{\mathrm{D}} + J_{\mathcal{D},33}[\dot{\omega} + \dot{\omega}_{\mathrm{in}}\sin\Theta + \omega_{\mathrm{in}}\dot{\Theta}\cos\Theta]\hat{k}_{\mathrm{D}}, \end{aligned} \tag{36}$$

$$\begin{aligned} \vec{\omega}_{\mathrm{D/A}} \times \vec{J}_{\mathcal{D}/\mathrm{cm}_{\mathcal{D}}} \vec{\omega}_{\mathrm{D/A}} = {} & [(\omega + \omega_{\mathrm{in}}\sin\Theta)(\dot{\Theta}\cos\Phi - \omega_{\mathrm{in}}(\cos\Theta)\sin\Phi)(J_{\mathcal{D},33} - J_{\mathcal{D},22})]\hat{\imath}_{\mathrm{D}} \\ & + [(\omega + \omega_{\mathrm{in}}\sin\Theta)(\dot{\Theta}\sin\Phi + \omega_{\mathrm{in}}(\cos\Theta)\cos\Phi)(J_{\mathcal{D},11} - J_{\mathcal{D},33})]\hat{\jmath}_{\mathrm{D}} \\ & + [(\dot{\Theta}\cos\Phi - \omega_{\mathrm{in}}(\cos\Theta)\sin\Phi)(\dot{\Theta}\sin\Phi + \omega_{\mathrm{in}}(\cos\Theta)\cos\Phi)(J_{\mathcal{D},22} - J_{\mathcal{D},11})]\hat{k}_{\mathrm{D}}. \end{aligned} \tag{37}$$

A motor mounted on $\mathcal{C}$ is used to keep the rotor spin rate ω constant. Letting $\vec{M}_{\mathrm{r}/\mathcal{D}/\mathcal{C}} = M_{\mathcal{D},1}\hat{\imath}_{\mathrm{D}} + M_{\mathcal{D},2}\hat{\jmath}_{\mathrm{D}}$ denote the reaction torque on $\mathcal{D}$ due to $\mathcal{C}$, and letting $\vec{M}_{\mathrm{m}} = \tau_{\mathrm{m}}\hat{k}_{\mathrm{D}}$ denote the torque applied to $\mathcal{D}$ by the motor, it follows that the total torque on $\mathcal{D}$ is given by

$$\begin{aligned} \vec{M}_{\mathcal{D}} &= \vec{M}_{\mathrm{r}/\mathcal{D}/\mathcal{C}} + \vec{M}_{\mathrm{m}} \\ &= M_{\mathcal{D},1}\hat{\imath}_{\mathrm{D}} + M_{\mathcal{D},2}\hat{\jmath}_{\mathrm{D}} + \tau_{\mathrm{m}}\hat{k}_{\mathrm{D}}. \end{aligned} \tag{38}$$

Using $J_{\mathcal{D},22} = J_{\mathcal{D},11}$ and substituting (36), (37), and (38) into (35) yields

$$J_{\mathcal{D},33}[\dot{\omega} + \dot{\omega}_{\rm in} \sin \Theta + \omega_{\rm in}(\cos \Theta)\dot{\Theta}] = \tau_{\rm m}, \tag{39}$$

$$\begin{aligned} &J_{\mathcal{D},11}[\ddot{\Theta} \sin \Phi + \dot{\Theta}\omega \cos \Phi + \dot{\omega}_{\rm in}(\cos \Theta) \cos \Phi - \omega_{\rm in}\dot{\Theta}(\sin \Theta) \cos \Phi - \omega_{\rm in}\omega(\cos \Theta) \sin \Phi] \\ &\quad +(\omega + \omega_{\rm in} \sin \Theta)(\dot{\Theta} \cos \Phi - \omega_{\rm in}(\cos \Theta) \sin \Phi)(J_{\mathcal{D},33} - J_{\mathcal{D},22}) = M_{\mathcal{D},1}, \end{aligned} \tag{40}$$

$$\begin{aligned} &J_{\mathcal{D},22}[\ddot{\Theta} \cos \Phi - \dot{\Theta}\omega \sin \Phi - \dot{\omega}_{\rm in}(\cos \Theta) \sin \Phi + \omega_{\rm in}\dot{\Theta}(\sin \Theta) \sin \Phi - \omega_{\rm in}\omega(\cos \Theta) \cos \Phi] \\ &\quad +(\omega + \omega_{\rm in} \sin \Theta)(\dot{\Theta} \sin \Phi + \omega_{\rm in}(\cos \Theta) \cos \Phi)(J_{\mathcal{D},11} - J_{\mathcal{D},33}) = M_{\mathcal{D},2}. \end{aligned} \tag{41}$$

The rotor spin rate ω is assumed to be kept approximately constant by setting $\tau_{\rm m} = J_{\mathcal{D},33}(\dot{\omega}_{\rm in} \sin \Theta + \omega_{\rm in}(\cos \Theta)\dot{\Theta})$.

Next, it follows from (29) and (30) that

$$\begin{aligned} \vec{\omega}_{\rm C/A} &= \vec{\omega}_{\rm C/B} + \vec{\omega}_{\rm B/A} \\ &= \dot{\Theta}\hat{\jmath}_{\rm C} + \omega_{\rm in}\hat{\imath}_{\rm B} \\ &= \omega_{\rm in}(\cos \Theta)\hat{\imath}_{\rm C} + \dot{\Theta}\hat{\jmath}_{\rm C} + \omega_{\rm in}(\sin \Theta)\hat{k}_{\rm C}. \end{aligned} \tag{42}$$

Furthermore, let the moment of inertia of $\mathcal{C}$ relative to its center of mass ${\rm cm}_{\mathcal{C}}$ be resolved in ${\rm F_C}$ as

$$\vec{J}_{\mathcal{C}/{\rm cm}_{\mathcal{C}}}\Big|_{\rm C} = \begin{bmatrix} J_{\mathcal{C},11} & 0 & 0 \\ 0 & J_{\mathcal{C},22} & 0 \\ 0 & 0 & J_{\mathcal{C},33} \end{bmatrix}. \tag{43}$$

Then, Euler's equation has the form

$$\vec{J}_{\mathcal{C}/{\rm cm}_{\mathcal{C}}} \overset{\rm C\bullet}{\vec{\omega}}_{\rm C/A} + \vec{\omega}_{\rm C/A} \times \vec{J}_{\mathcal{C}/{\rm cm}_{\mathcal{C}}}\vec{\omega}_{\rm C/A} = \vec{M}_{\mathcal{C}}, \tag{44}$$

where

$$\vec{J}_{\mathcal{C}/{\rm cm}_{\mathcal{C}}} \overset{\rm C\bullet}{\vec{\omega}}_{\rm C/A} = J_{\mathcal{C},11}[\dot{\omega}_{\rm in} \cos \Theta - \omega_{\rm in}\dot{\Theta} \sin \Theta]\hat{\imath}_{\rm C} + J_{\mathcal{C},22}\ddot{\Theta}\hat{\jmath}_{\rm C} + J_{\mathcal{C},33}[\dot{\omega}_{\rm in} \sin \Theta + \omega_{\rm in}\dot{\Theta} \cos \Theta]\hat{k}_{\rm C}, \tag{45}$$

$$\begin{aligned} \vec{\omega}_{\rm C/A} \times \vec{J}_{\mathcal{C}/{\rm cm}_{\mathcal{C}}}\vec{\omega}_{\rm C/A} &= [(J_{\mathcal{C},33} - J_{\mathcal{C},22})\dot{\Theta}\omega_{\rm in} \sin \Theta]\hat{\imath}_{\rm C} + [(J_{\mathcal{C},11} - J_{\mathcal{C},33})\omega_{\rm in}^2 \cos \Theta \sin \Theta]\hat{\jmath}_{\rm C} \\ &\quad + [(J_{\mathcal{C},22} - J_{\mathcal{C},11})\omega_{\rm in}\dot{\Theta} \cos \Theta]\hat{k}_{\rm C}, \end{aligned} \tag{46}$$

and, letting $\vec{M}_{\mathrm{r}/\mathcal{C}/\mathcal{B}} = M_{\mathcal{C},1}\hat{\imath}_{\mathrm{C}} + M_{\mathcal{C},3}\hat{k}_{\mathrm{C}}$ denote the reaction torque on $\mathcal{C}$ due to $\mathcal{B}$, the total torque on $\mathcal{C}$ is given by

$$\begin{aligned}\vec{M}_{\mathcal{C}} &= \vec{M}_{\mathrm{r}/\mathcal{C}/\mathcal{D}} + \vec{M}_{\mathrm{r}/\mathcal{C}/\mathcal{B}} - (k\Theta + c\dot{\Theta})\hat{\jmath}_{\mathrm{C}} - \tau_{\mathrm{m}}\hat{k}_{\mathrm{D}} \\ &= -M_{\mathcal{D},1}\hat{\imath}_{\mathrm{D}} - M_{\mathcal{D},2}\hat{\jmath}_{\mathrm{D}} + M_{\mathcal{C},1}\hat{\imath}_{\mathrm{C}} + M_{\mathcal{C},3}\hat{k}_{\mathrm{C}} - (k\Theta + c\dot{\Theta})\hat{\jmath}_{\mathrm{C}} - \tau_{\mathrm{m}}\hat{k}_{\mathrm{C}} \\ &= [-M_{\mathcal{D},1}\cos\Phi + M_{\mathcal{D},2}\sin\Phi + M_{\mathcal{C},1}]\hat{\imath}_{\mathrm{C}} + [-M_{\mathcal{D},1}\sin\Phi - M_{\mathcal{D},2}\cos\Phi - k\Theta - c\dot{\Theta}]\hat{\jmath}_{\mathrm{C}} \\ &\quad + (M_{\mathcal{C},3} - \tau_{\mathrm{m}})\hat{k}_{\mathrm{C}},\end{aligned} \tag{47}$$

where $k > 0$ is the stiffness of the torsional spring and $c > 0$ is the damping coefficient of the torsional dashpot. Substituting (45), (46), and (47) into Euler's equation (44) and equating the coefficients of $\hat{\jmath}_{\mathrm{C}}$ yields

$$J_{\mathcal{C},22}\ddot{\Theta} + (J_{\mathcal{C},11} - J_{\mathcal{C},33})\omega_{\mathrm{in}}^2(\cos\Theta)\sin\Theta = -k\Theta - c\dot{\Theta} - M_{\mathcal{D},1}\sin\Phi - M_{\mathcal{D},2}\cos\Phi. \tag{48}$$

Finally, substituting $M_{\mathcal{D},1}$ given by (40) and $M_{\mathcal{D},2}$ given by (41) into (48) yields

$$\bar{J}_{22}\ddot{\Theta} + c\dot{\Theta} + k\Theta = (\bar{J}_{31}\omega_{\mathrm{in}}\sin\Theta + J_{\mathcal{D},33}\omega)\omega_{\mathrm{in}}\cos\Theta, \tag{49}$$

where

$$\bar{J}_{22} \triangleq J_{\mathcal{C},22} + J_{\mathcal{D},22}, \quad \bar{J}_{31} \triangleq J_{\mathcal{C},33} - J_{\mathcal{C},11} + J_{\mathcal{D},33} - J_{\mathcal{D},11}. \tag{50}$$

Next, defining $x \triangleq [\Theta\ \dot{\Theta}]^{\mathrm{T}}$ and $u \triangleq \omega_{\mathrm{in}}$, (49) can be written as

$$\dot{x} = f(x, u), \tag{51}$$

where

$$f(x,u) \triangleq \begin{bmatrix} x_2 \\ \frac{1}{\bar{J}_{22}}\left(\bar{J}_{31}(\sin x_1)(\cos x_1)u^2 + J_{\mathcal{D},33}\omega(\cos x_1)u - cx_2 - kx_1\right) \end{bmatrix}. \tag{52}$$

Letting $(\bar{x}, \bar{u}) = ([\bar{\Theta}\ 0]^{\mathrm{T}}, \bar{\omega}_{\mathrm{in}})$ denote an equilibrium of (51), where $\bar{\omega}_{\mathrm{in}}$ represents an arbitrary constant rate of rotation, the linearized dynamics in a neighborhood of $(\bar{x}, \bar{u})$ are given by

$$\dot{\xi} = A\xi + Bv, \tag{53}$$

where

$$A \triangleq \begin{bmatrix} 0 & 1 \\ \frac{1}{\bar{J}_{22}}\left(\bar{J}_{31}\bar{u}^2\cos 2\bar{x}_1 - J_{\mathcal{D},33}\omega\bar{u}\sin\bar{x}_1 - k\right) & \frac{-c}{\bar{J}_{22}} \end{bmatrix}, \tag{54}$$

$$B \triangleq \begin{bmatrix} 0 \\ \frac{1}{\bar{J}_{22}}\left(\bar{J}_{31}(\sin\bar{x}_1)(\cos\bar{x}_1)2\bar{u} + J_{\mathcal{D},33}\omega(\cos\bar{x}_1)\right) \end{bmatrix}. \tag{55}$$

In particular, for $\bar{\Theta} = 0$ and $\bar{\omega}_{\rm in} = 0$, it follows that

$$A \triangleq \begin{bmatrix} 0 & 1 \\ \dfrac{-k}{\bar{J}_{22}} & \dfrac{-c}{\bar{J}_{22}} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ \dfrac{J_{\mathcal{D},33}\omega}{\bar{J}_{22}} \end{bmatrix}. \tag{56}$$

Defining $C \triangleq [1\ 0]$, the transfer function of the linearized dynamics is given by

$$G(s) = C(sI - A)^{-1}B = \frac{J_{\mathcal{D},33}\omega}{\bar{J}_{22}s^2 + cs + k}. \tag{57}$$

Hence, for small $\omega_{\rm in}$, the Laplace transform of the gimbal angle Θ is given approximately by the gyro transfer function [34, eqn. (3.15)]

$$\hat{\Theta}(s) = \frac{J_{\mathcal{D},33}\omega}{\bar{J}_{22}s^2 + cs + k}\hat{\omega}_{\rm in}(s), \tag{58}$$

and thus the gimbal angle satisfies

$$\Theta(t) = -\frac{\bar{J}_{22}}{k}\ddot{\Theta}(t) - \frac{c}{k}\dot{\Theta}(t) + \frac{J_{\mathcal{D},33}\omega}{k}\omega_{\rm in}(t). \tag{59}$$

Since (59) represents a damped oscillator, it follows that, when $\dot{\Theta}(t) \approx 0$ and $\ddot{\Theta}(t) \approx 0$, the rate-gyro measurement is

$$\boxed{\omega_{\rm in}(t) = \frac{k}{J_{\mathcal{D},33}\omega}\Theta(t).} \tag{60}$$

In the case where $\bar{J}_{22} = 0$ and $k = 0$, the rate-gyro in fact provides an estimate of the angle Ψ instead of $\omega_{\rm in} = \dot{\Psi}$. This case is discussed in the sidebar "What Is an Integrating Gyro?".

Having a 3-axis gyro (composed of 3 1-axis, orthogonal gyros), we can estimate attitude in time:

measured given

Poisson's eq.

$$\dot{\theta}_{B/A}(t) = -\omega^{\times}_{B/A}\big|_B(t)\,\theta_{B/A}(t), \quad \theta_{B/A}(0)$$

linear time-varying ODE (ordinary diff. equation)

It can be written in vector form:

$$\dot{x}(t) = A(t)\,x(t),$$

where $x(t) \in \mathbb{R}^9$, and, then,
nine

$$x(t) = \Phi(t,0)\,x(0),$$

where, for all $t \geq 0, \tau \geq 0$,

$$\frac{\partial}{\partial t} \Phi(t,\tau) = A(t)\,\Phi(t,\tau),$$

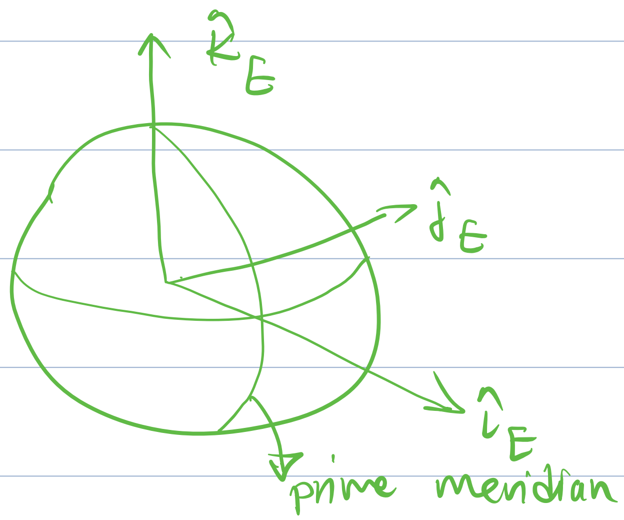
$$\Phi(t,t) = I$$

[$\Phi \in \mathbb{R}^{9\times 9}$ is the "state transition matrix]

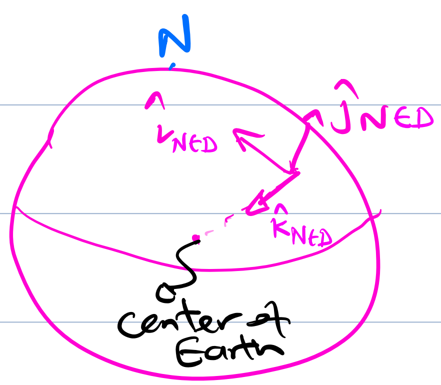
Using MATLAB, we can solve Poisson's equation numeracally. (ode45 function)

Computing position

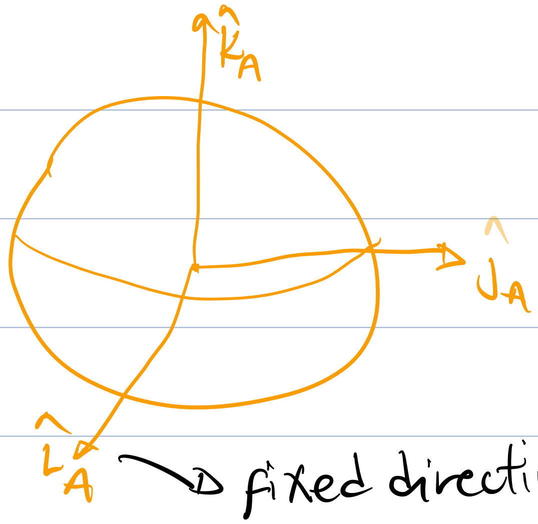
Let's consider on-Earth motions.
3 frames become relevant:



Earth-centered frame.



North-East-Down (NED) frame



Earth-centered inertial frame F_A

$\hat{i}_A \rightarrow$ fixed directions to stars (doesn't move with Earth)

Now:

$$r(t) \triangleq \vec{r}_{y/w}(t)\big|_E$$

$$v(t) \triangleq \vec{v}_{y/w/A}(t)\big|_E$$

$$a(t) \triangleq \vec{a}_{y/w/A}(t)\big|_E .$$

The accelerometer measures:

$$a_{meas}(t) = \vec{a}_{y/w/A}\big|_B + \vec{g}\big|_B$$

$$= \mathcal{O}_{B|E}(t)\, a(t) + \mathcal{O}_{B/NED}(t) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$$\Rightarrow \quad a(t) = \mathcal{O}_{E|B}(t)\, a_{meas}(t) - \underbrace{\mathcal{O}_{E|B}\, \mathcal{O}_{B|NED}}_{\mathcal{O}_{E|NED}} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$$\Rightarrow r(t) = \int_0^t \int_0^\tau \mathcal{O}_{E|B}(s)\, a_{meas}(s)\, ds\, d\tau$$

$$- \int_0^t \int_0^\tau \mathcal{O}_{E|NED}(s)\, ds\, d\tau \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + t\, v(0) + r(0)$$

Here:

$$\mathcal{O}_{E|B} = \underbrace{\mathcal{O}_{E|A}}_{\text{Known}} \cdot \underbrace{\mathcal{O}_{A|B}}_{\text{from Poisson's Eq.}}$$