Discrete-time Kalman Filter Equations for Navigation

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1 Kalman Filter

Consider the system

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + D_{1,k} w_k, \\ y_k &= C_k x_k + D_{2,k} w_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^{l_x}$ is the state, $u_k \in \mathbb{R}^{l_u}$ is the input, $y_k \in \mathbb{R}^{l_y}$ is the measured output, $w_k \in \mathbb{R}^{l_w}$ is the noise, and $A_k, B_k, C_k, D_{1,k}, D_{2,k}$ are real matrices of appropriate dimensions.

Consider the filter

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k, \tag{2}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k (y_{k+1} - C_{k+1} \hat{x}_{k+1|k}). \tag{3}$$

The Kalman filter chooses the gain K_k that minimizes the covariance of the state error. The Kalman gain K_k is thus given by

$$K_k = P_{k+1|k} C_{k+1}^{\mathrm{T}} (C_{k+1} P_{k+1|k} C_{k+1}^{\mathrm{T}} + R_{k+1})^{-1}, \tag{4}$$

where the prior state-error covariance $P_{k+1|k}$ is given by

$$P_{k+1|k} = A_k P_{k|k} A_k^{\mathrm{T}} + Q_k, \tag{5}$$

and the posterior state-error covariance $P_{k+1|k+1}$ is given by

$$P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}. \tag{6}$$

2 Inertial Navigation

In an inertial navigation system, a three-axis accelerometer and a three-axis rate gyro provide the acceleration of the body in the body-fixed frame and the angular velocity of the body relative to an inertial frame. The discretized equations of motion are given by

$$x_{k+1} = A x_k + B u_k, \tag{7}$$

where

$$A = \begin{bmatrix} I_3 & T I_3 \\ 0 & I_3 \end{bmatrix} \in \mathbb{R}^{6\times 6}, \quad B = \begin{bmatrix} T^2/2 I_3 \\ T I_3 \end{bmatrix} \in \mathbb{R}^{6\times 3}, \tag{8}$$

and T is the time step used in the discretization. The input $u_k \in \mathbb{R}^3$ is given by

$$u_k = \chi_k^{\mathrm{T}} a_k - g_{\mathrm{A}}, \tag{9}$$

where $a_k \in \mathbb{R}^3$ is the accelerometer measurement, $\chi_k \triangleq \mathcal{O}_{\mathrm{B/A},k} \in \mathbb{R}^{3\times 3}$ is the orientation matrix, and $g_{\mathrm{A}} \in \mathbb{R}^3$ is the acceleration due to gravity resolved in the inertial frame F_{A}. The orientation matrix is propagated by using the discretized Poisson's equation

$$\chi_{k+1} = \Xi_k \chi_k, \tag{10}$$

where $\Xi_k = e^{-T\omega_k^\times} \in \mathbb{R}^{3\times3}$, T is the sampling rate in seconds, and $\omega_k \in \mathbb{R}^3$ is the rate-gyro measurement.

In an inertial navigation system, the estimate of the position is given by

$$\hat{x}_{k+1} = A\hat{x}_k + B(\hat{\chi}_k^{\mathrm{T}} a_k - g_{\mathrm{A}}), \tag{11}$$

$$\hat{\chi}_{k+1} = \Xi_k \hat{\chi}_k. \tag{12}$$

Note that the Kalman filter is not useful in inertial navigation since the measurements do not include any component of the state x_k and χ_k.

3 GPS-based Navigation

In a GPS-based navigation system, a GPS system provides noisy position measurements. These position measurements can be used to improve the position estimate using the Kalman filter.

The position estimate is given by

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + B(\hat{\chi}_k^{\mathrm{T}} a_k - g_{\mathrm{A}}), \tag{13}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k(y_{k+1} - C_{k+1}\hat{x}_{k+1|k}), \tag{14}$$

$$\hat{\chi}_{k+1} = \Xi_k \hat{\chi}_k, \tag{15}$$

where A and B are given by (8), $C_k \in \mathbb{R}^{3\times6}$ is given by

$$C_k = \begin{cases} \begin{bmatrix} I_3 & 0_3 \end{bmatrix}, & \text{if measurement is available,} \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Note that since accelerometer measurements a_k are noisy , $D_{1,k} = B\hat{\chi}_k^{\mathrm{T}}$ and thus $Q_k = D_{1,k}D_{1,k}^{\mathrm{T}}$.

The Kalman gain is given by

$$K_k = P_{k+1|k}C_{k+1}^{\mathrm{T}}(C_{k+1}P_{k+1|k}C_{k+1}^{\mathrm{T}} + R_{k+1})^{-1}, \tag{17}$$

$$P_{k+1|k} = A_k P_{k|k} A_k^{\mathrm{T}} + Q_k, \tag{18}$$

$$P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}. \tag{19}$$

Note that R_k is defined by the variance of the GPS measurements. However, R_k along with Q_k and $P_{0|0}$ are often used as a hyperparameters to tune the performance of the Kalman filter.