AEROSP 584 - Navigation and Guidance: From Perception to Control

Unmute Start Video

Lectures start at 10:30am EST

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Lecture 13

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To ask questions:

Raise Hand or Chat

Optimal Predictor

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_k = Cx_k + D_2 w_k$$

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + K_k(y_k - C\hat{x}_k)$$

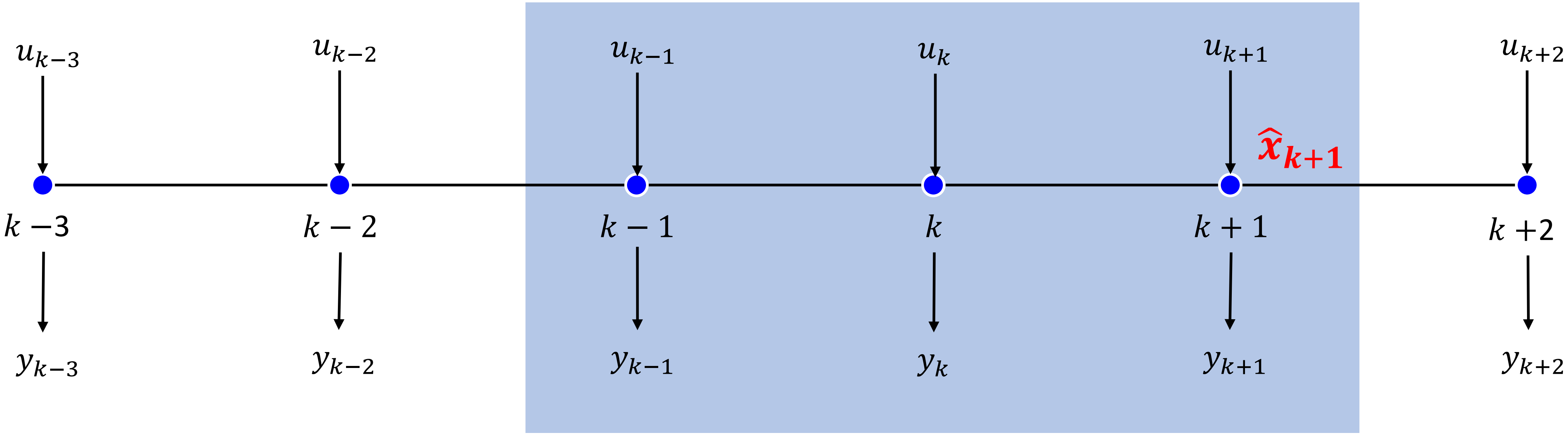
$$K_k = \left(AP_kC^\mathrm{T} + S\right)\left(CP_kC^\mathrm{T} + R\right)^{-1}$$

$$P_{k+1} = AP_kA^\mathrm{T} + Q - K_k\left(AP_kC^\mathrm{T} + S\right)^\mathrm{T}$$

State Estimators

Predictor

Filter



Filter

$$x_{k+1} = x_k + u_k + w_k$$
$$y_k = x_k + v_k$$

- Use y_{k+1} to estimate x_{k+1}

$$\hat{x}_{k+1} = \hat{x}_k + u_k + K(y_{k+1} - \hat{y}_{k+1})$$

- Use the dynamics of the system to produce a pre-estimate of $\hat{x}_{k+1}$

$$\hat{x}_{k+1} = \hat{x}_k + u_k + K(x_{k+1} + v_{k+1} - ??)$$

$$e_{k+1} = (1 - K)e_k + (K - 1)w_k + Kv_{k+1}$$

Filter vs Predictor

$$x_{k+1} = x_k + u_k + w_k$$
$$y_k = x_k + v_k$$

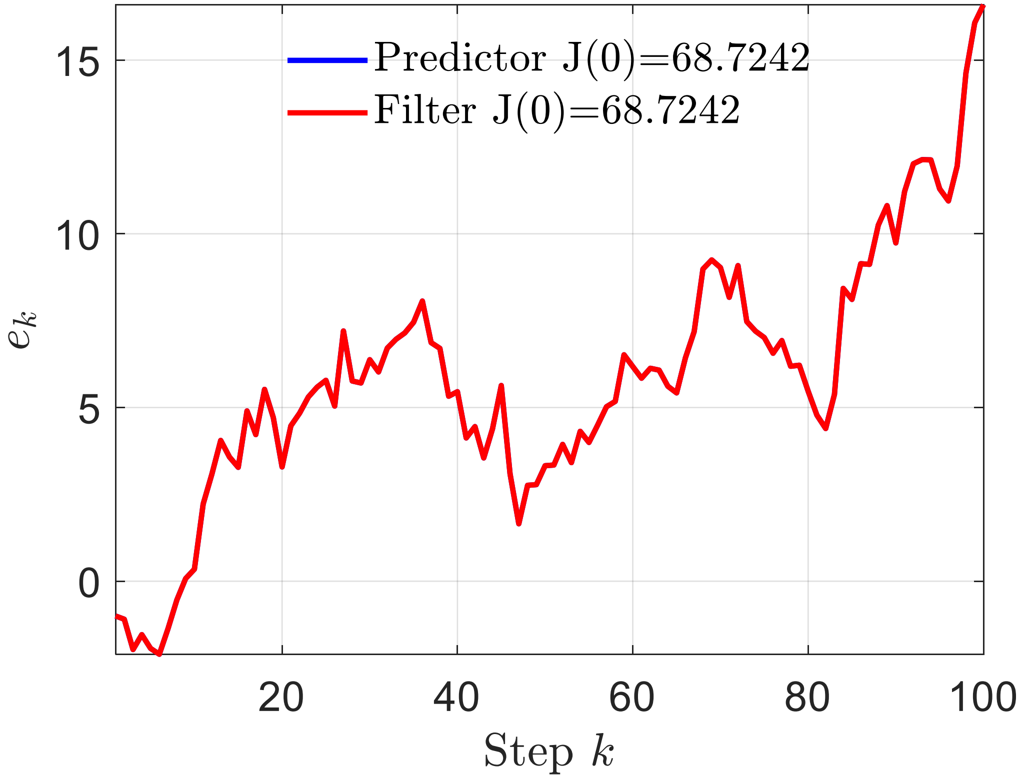
- Filter $e_{k+1} = (1-K)e_k + (K-1)w_k + Kv_{k+1}$
- Predictor $e_{k+1} = (1-K)e_k - w_k + Kv_k$

Run for 100 steps with

$$x_0 = 1, \hat{x}_0 = 0 \Rightarrow e_0 = \text{-}1$$

$$v_k, w_k \sim \mathcal{N}(0,1)$$

$$J(K) = \sum_{i=0}^{100} e_i^2$$



$$e_{k+1} = e_k - w_k$$
$$e_{k+1} = e_k - w_k$$

Filter vs Predictor

$$x_{k+1} = x_k + u_k + w_k$$
$$y_k = x_k + v_k$$

- Filter $e_{k+1} = (1-K)e_k + (K-1)w_k + Kv_{k+1}$
- Predictor $e_{k+1} = (1-K)e_k - w_k + Kv_k$

Run for 100 steps with

$$x_0 = 1, \hat{x}_0 = 0 \Rightarrow e_0 = \text{-}1$$

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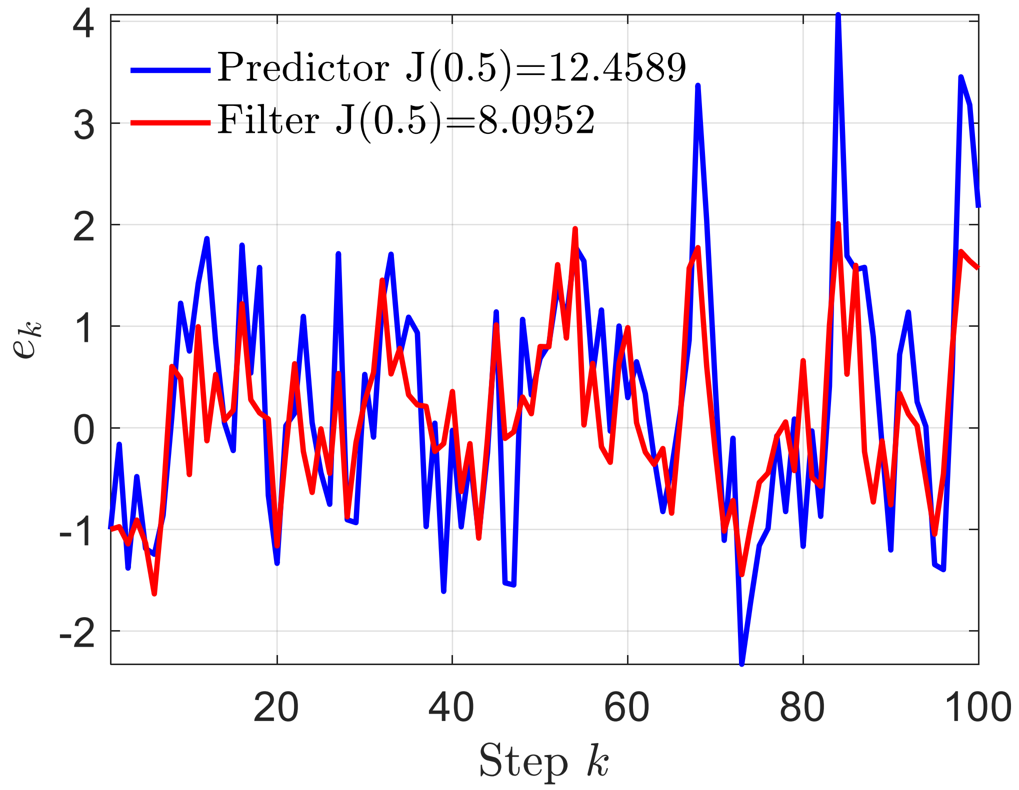


$$e_{k+1} = 0.5e_k + 0.5v_{k+1} - 0.5w_k$$
$$e_{k+1} = 0.5e_k + 0.5v_k - w_k$$

Filter vs Predictor

$$x_{k+1} = x_k + u_k + w_k$$
$$y_k = x_k + v_k$$

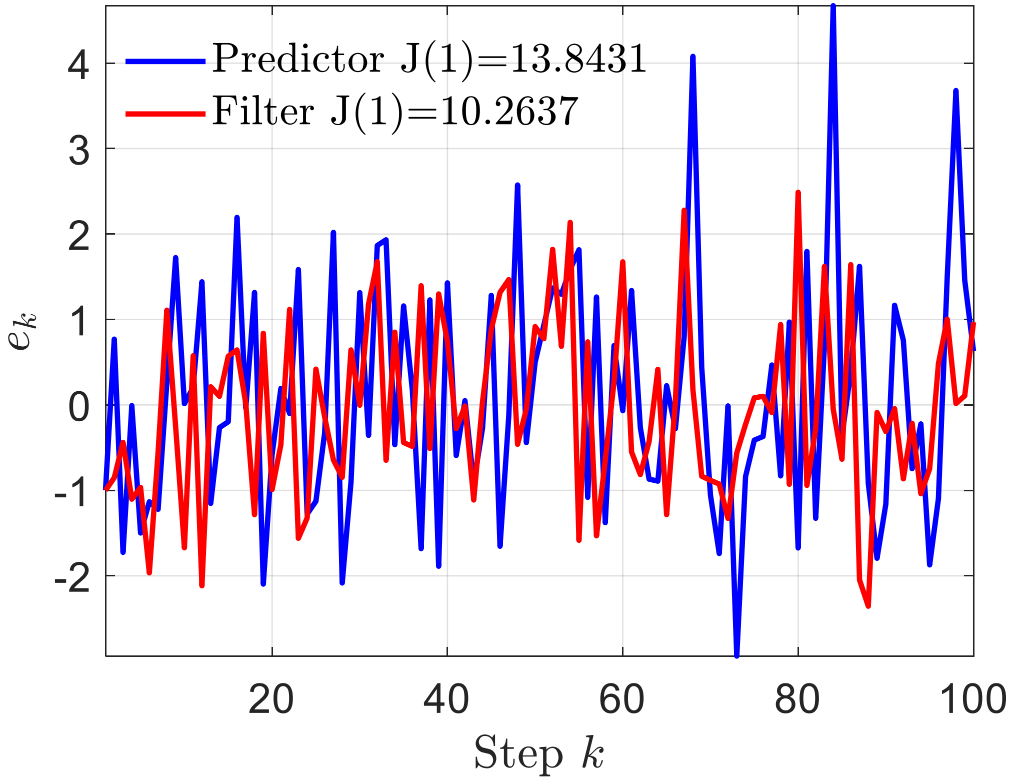
- Filter $e_{k+1} = (1-K)e_k + (K-1)w_k + Kv_{k+1}$
- Predictor $e_{k+1} = (1-K)e_k - w_k + Kv_k$

Run for 100 steps with

$$x_0 = 1, \hat{x}_0 = 0 \Rightarrow e_0 = \text{-}1$$

$$v_k, w_k \sim \mathcal{N}(0,1)$$

$$J(K) = \sum_{i=0}^{100} e_i^2$$



$$e_{k+1} = v_{k+1}$$
$$e_{k+1} = v_k - w_k$$

Optimal Filter

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_k = Cx_k + D_2 w_k$$

Assumptions

- A, B, C, D_1, D_2 known
- u_k known
- $w_k \sim \mathcal{N}(0, I), \quad \mathbb{E}[x_0] = \bar{x}$
- $\mathrm{Cov}[x_0, w_k] = 0$

Observations

- Since w_k is a RV, x_k and y_k are random vectors
- Since w_k does not affect x_k, $\mathrm{Cov}[x_k, w_k] = 0$

Two Step Filter

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_{k+1} = Cx_{k+1} + D_2 w_{k+1}$$

- Break the estimator in two steps

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$

  - $\hat{x}_{k+1|k}$ prior estimate of x_{k+1}
  - Physics update of the state estimate

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

  - $\hat{x}_{k+1|k+1}$ posterior estimate of x_{k+1}
  - Assimilation of the measurement

- We optimize K

Optimal Filter

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_{k+1} = Cx_{k+1} + D_2 w_{k+1}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

- Define
  - $e_{k+1|k} \triangleq \hat{x}_{k+1|k} - x_{k+1}$ prior error
  - $e_{k+1|k+1} \triangleq \hat{x}_{k+1|k+1} - x_{k+1}$ posterior error

  - $P_{k+1|k} \triangleq \mathrm{Cov}\left[e_{k+1|k}\right]$ prior error covariance
  - $P_{k+1|k+1} \triangleq \mathrm{Cov}\left[e_{k+1|k+1}\right]$ posterior error covariance

- Optimal Filter chooses K that minimizes the covariance of the posterior error

$$K_k = \min_{\hat{K}\in\mathbb{R}^{l_x \times l_y}} \operatorname{trace} P_{k+1|k+1}$$

Optimal Filter – Error Dynamics

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_{k+1} = Cx_{k+1} + D_2 w_{k+1}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

- $e_{k+1|k} =$

- $e_{k+1|k+1} =$

Optimal Filter – Error Dynamics

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_{k+1} = Cx_{k+1} + D_2 w_{k+1}$$

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

- $$\begin{aligned} e_{k+1|k} &= A\hat{x}_{k|k} + Bu_k - Ax_k - Bu_k - D_1 w_k \\ &= Ae_{k|k} - D_1 w_k \end{aligned}$$

- $$\begin{aligned} e_{k+1|k+1} &= \hat{x}_{k+1|k} + K\left(y_{k+1} - C\hat{x}_{k+1|k}\right) - x_{k+1} \\ &= e_{k+1|k} + K\left(Cx_{k+1} + D_2 w_{k+1} - C\hat{x}_{k+1|k}\right) \\ &= e_{k+1|k} + K\left(D_2 w_{k+1} - Ce_{k+1|k}\right) \\ &= (I - KC)e_{k+1|k} + KD_2 w_{k+1} \end{aligned}$$

Optimal Filter – Covariance Matrices

$$P_{k+1|k} \triangleq \mathrm{Cov}\left[e_{k+1|k}\right] = \mathbb{E}\left[\left(e_{k+1|k} - \mathbb{E}\left[e_{k+1|k}\right]\right)\left(e_{k+1|k} - \mathbb{E}\left[e_{k+1|k}\right]\right)^{\mathrm{T}}\right]$$

- We would like $\mathbb{E}\left[e_{k+1|k}\right] = 0$.

- The prior state error satisfies
$$e_{k+1|k} = Ae_{k|k} - D_1 w_k$$
$$\mathbb{E}\left[e_{k+1|k}\right] = \mathbb{E}\left[Ae_{k|k} - D_1 w_k\right] = A\mathbb{E}\left[e_{k|k}\right]$$
- If $\mathbb{E}\left[e_{k|k}\right] = 0$, then $\mathbb{E}\left[e_{k+1|k}\right] = 0$.

- The posterior state error satisfies
$$e_{k+1|k+1} = (I - KC)e_{k+1|k} + KD_2 w_{k+1}$$
$$\mathbb{E}\left[e_{k+1|k+1}\right] = \mathbb{E}\left[(I - KC)e_{k+1|k} + KD_2 w_{k+1}\right] = (I - KC)\mathbb{E}\left[e_{k+1|k}\right]$$
- If $\mathbb{E}\left[e_{k+1|k}\right] = 0$, then $\mathbb{E}\left[e_{k+1|k+1}\right] = 0$.

- If $\mathbb{E}\left[e_{0|0}\right] = 0$, then, $\mathbb{E}\left[e_{1|0}\right] = \mathbb{E}\left[e_{1|1}\right] = \cdots = 0$.

$$\mathbb{E}\left[e_{0|0}\right] = \mathbb{E}\left[\hat{x}_{0|0} - x_0\right] = \mathbb{E}\left[\hat{x}_{0|0}\right] - \mathbb{E}[x_0] = \mathbb{E}\left[\hat{x}_{0|0}\right] - \bar{x}$$

- If $\mathbb{E}\left[\hat{x}_{0|0}\right] = \bar{x}$. Then, $\mathbb{E}\left[e_{k+1|k}\right] = 0$ and $\mathbb{E}\left[e_{k+1|k+1}\right] = 0$

Optimal Filter – Covariance Matrices

$$P_{k+1|k} = \mathbb{E}\left[e_{k+1|k} e_{k+1|k}^{\mathrm{T}}\right]$$
$$P_{k+1|k+1} = \mathbb{E}\left[e_{k+1|k+1} e_{k+1|k+1}^{\mathrm{T}}\right]$$

- The prior state error satisfies $e_{k+1|k} = Ae_{k|k} - D_1 w_k$

$$P_{k+1|k} = \mathbb{E}\left[\left(Ae_{k|k} - D_1 w_k\right)\left(Ae_{k|k} - D_1 w_k\right)^{\mathrm{T}}\right]$$

- The posterior state error satisfies $e_{k+1|k+1} = (I - KC)e_{k+1|k} + KD_2 w_{k+1}$

$$P_{k+1|k+1} = \mathbb{E}\left[\left((I - KC)e_{k+1|k} + KD_2 w_{k+1}\right)\left((I - KC)e_{k+1|k} + KD_2 w_{k+1}\right)^{\mathrm{T}}\right]$$

Optimal Filter – Covariance Matrices

$$P_{k+1|k} = \mathbb{E}\left[e_{k+1|k} e_{k+1|k}^{\mathrm{T}}\right]$$
$$P_{k+1|k+1} = \mathbb{E}\left[e_{k+1|k+1} e_{k+1|k+1}^{\mathrm{T}}\right]$$

- The prior state error satisfies $e_{k+1|k} = Ae_{k|k} - D_1 w_k$

$$\begin{aligned} P_{k+1|k} &= \mathbb{E}\left[\left(Ae_{k|k} - D_1 w_k\right)\left(Ae_{k|k} - D_1 w_k\right)^{\mathrm{T}}\right] \\ &= AP_{k|k}A^{\mathrm{T}} + Q \end{aligned}$$

- The posterior state error satisfies $e_{k+1|k+1} = (I - KC)e_{k+1|k} + KD_2 w_{k+1}$

$$\begin{aligned} P_{k+1|k+1} &= \mathbb{E}\left[\left((I - KC)e_{k+1|k} + KD_2 w_{k+1}\right)\left((I - KC)e_{k+1|k} + KD_2 w_{k+1}\right)^{\mathrm{T}}\right] \\ &= P_{k+1|k} - KCP_{k+1|k} - P_{k+1|k}C^{\mathrm{T}}K^{\mathrm{T}} + K\left(CP_{k+1|k}C^{\mathrm{T}} + R\right)K^{\mathrm{T}} \end{aligned}$$

Optimal Filter – Minimizing Gain

$$P_{k+1|k+1} = P_{k+1|k} - KCP_{k+1|k} - P_{k+1|k}C^{\mathrm{T}}K^{\mathrm{T}} + K\left(CP_{k+1|k}C^{\mathrm{T}} + R\right)K^{\mathrm{T}}$$

$$K_k = \min_{\hat{K}\in\mathbb{R}^{l_x\times l_y}} \operatorname{trace} P_{k+1|k+1}$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\operatorname{tr}(XA) = A^{\mathrm{T}}$$
$$\frac{\mathrm{d}}{\mathrm{d}X}\operatorname{tr}\left(XAX^{\mathrm{T}}\right) = XA^{\mathrm{T}} + XA$$
$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

$$\frac{\mathrm{d}}{\mathrm{d}K}\operatorname{trace} P_{k+1|k+1} = -P_{k+1|k}C^{\mathrm{T}} + K\left(CP_{k+1|k}C^{\mathrm{T}} + R\right)$$

$$K_k = P_{k+1|k}C^{\mathrm{T}}\left(CP_{k+1|k}C^{\mathrm{T}} + R\right)^{-1}$$

Optimal Filter

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

$$P_{k+1|k} = AP_{k|k}A^{\mathrm{T}} + Q$$

$$K_k = P_{k+1|k}C^{\mathrm{T}}\left(CP_{k+1|k}C^{\mathrm{T}} + R\right)^{-1}$$

$$P_{k+1|k+1} = P_{k+1|k} - K_kCP_{k+1|k}$$

Optimal Filter - Simple Example

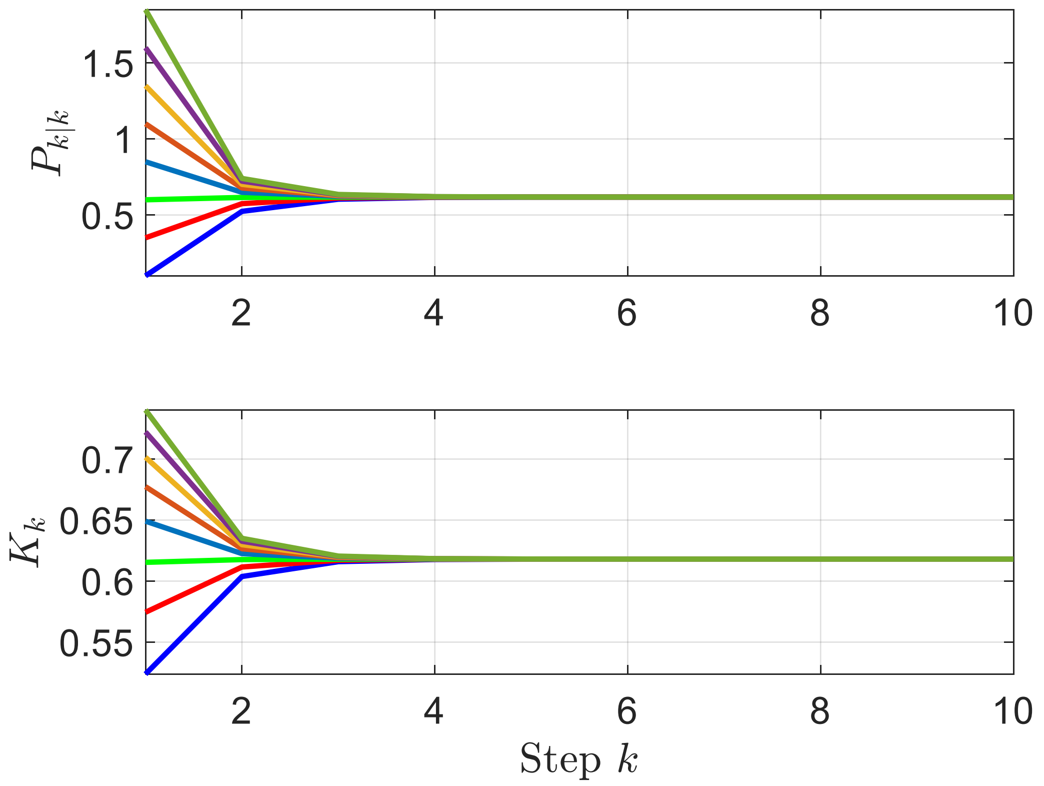
$$x_{k+1} = x_k + u_k + [1 \quad 0]w_k$$
$$y_k = x_k + [0 \quad 1]w_k$$

$$\hat{x}_{k+1|k} = \hat{x}_{k|k} + u_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - \hat{x}_{k+1|k}\right)$$

$$P_{k+1|k} = P_{k|k} + 1, K_k = \frac{P_{k+1|k}}{P_{k+1|k} + 1}, P_{k+1|k+1} = \frac{P_{k+1|k}}{P_{k|k+1} + 1}$$

- If $P_{k+1|k+1}$ converges, then $P_{k+1|k+1} - P_{k|k} \to 0$

$$P_{\infty|\infty} = \frac{1}{2}\left(-1 + \sqrt{5}\right) \approx 0.6180$$
$$K_{\infty} = \frac{1}{2}\left(-1 + \sqrt{5}\right) \approx 0.6180$$



Does OF Improve Accuracy?

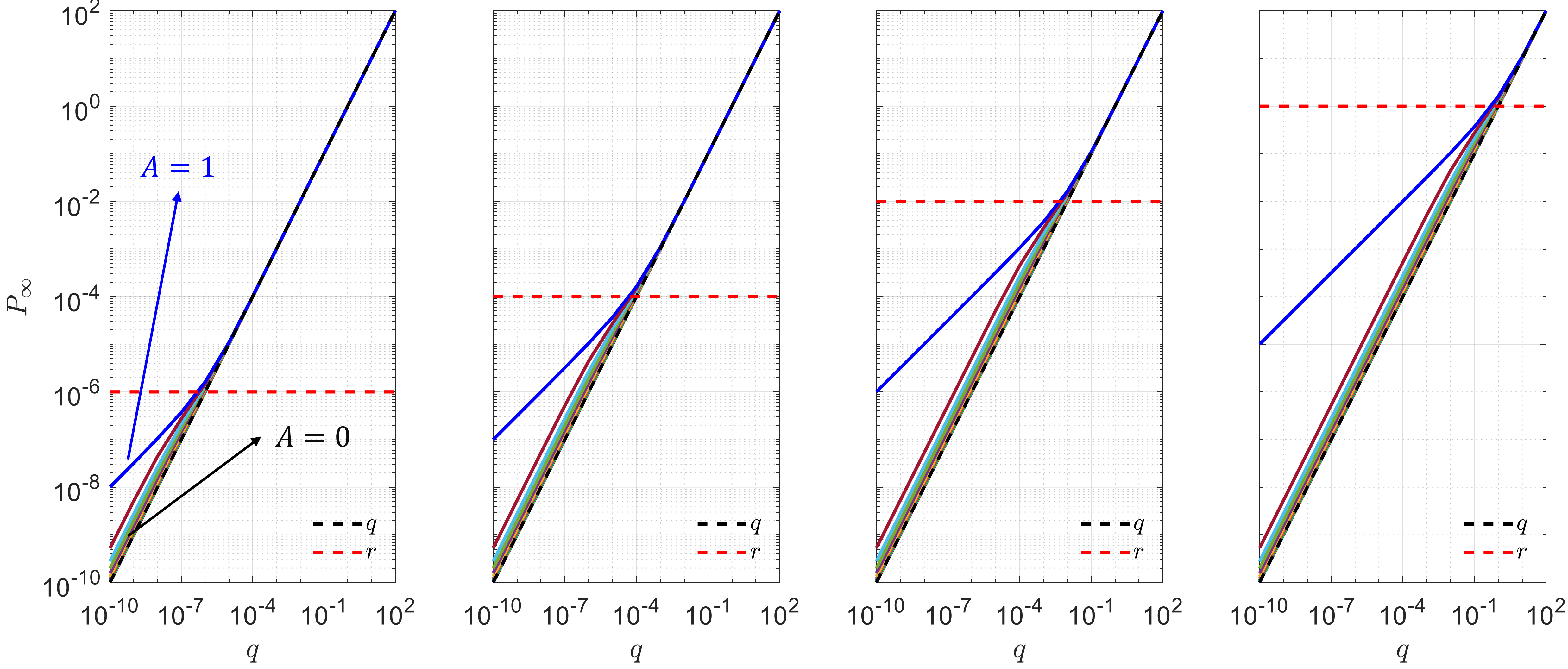
$$x_{k+1} = a x_k + b u_k + [\sqrt{q} \quad 0] w_k$$
$$y_k = x_k + [0 \quad \sqrt{r}] w_k$$

$$\hat{x}_{k+1|k} = a \hat{x}_{k|k} + b u_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k \left(y_{k+1} - \hat{x}_{k+1|k} \right)$$

- $P_{k+1|k} = A P_{k|k} A^{\mathrm{T}} + Q$
- $K_k = P_{k+1|k} C^{\mathrm{T}} \left(C P_{k+1|k} C^{\mathrm{T}} + R \right)^{-1}$
- $P_{k+1|k+1} = P_{k+1|k} - K_k C P_{k+1|k}$

- $P_{k+1|k} = a^2 P_{k|k} + q$
- $K_k = \frac{P_{k+1|k}}{P_{k+1|k} + r}$
- $P_{k+1|k+1} = \frac{P_{k+1|k}\, r}{P_{k+1|k} + r}$

Does OP Improve Accuracy?



- If $q > r$, then $P_\infty > q > r$
- If $r > q$, then $r > P_\infty > q$

Does OF Improve Accuracy?

$$P_{k+1|k+1} = \frac{\left(a^2 P_{k|k} + q\right) r}{a^2 P_{k|k} + q + r}$$

- Assume that $P_{k|k}$ converges to $P_{\infty|\infty}$

$$a^2 P_{\infty|\infty}^2 + P_{\infty|\infty}\left(q + r\left(1 - a^2\right)\right) - q\, r = 0$$

- $a = 0 \Rightarrow P_{\infty|\infty} = \frac{qr}{q+r} = \left(\frac{1}{q} + \frac{1}{r}\right)^{-1}$
- $a \neq 0 \Rightarrow$

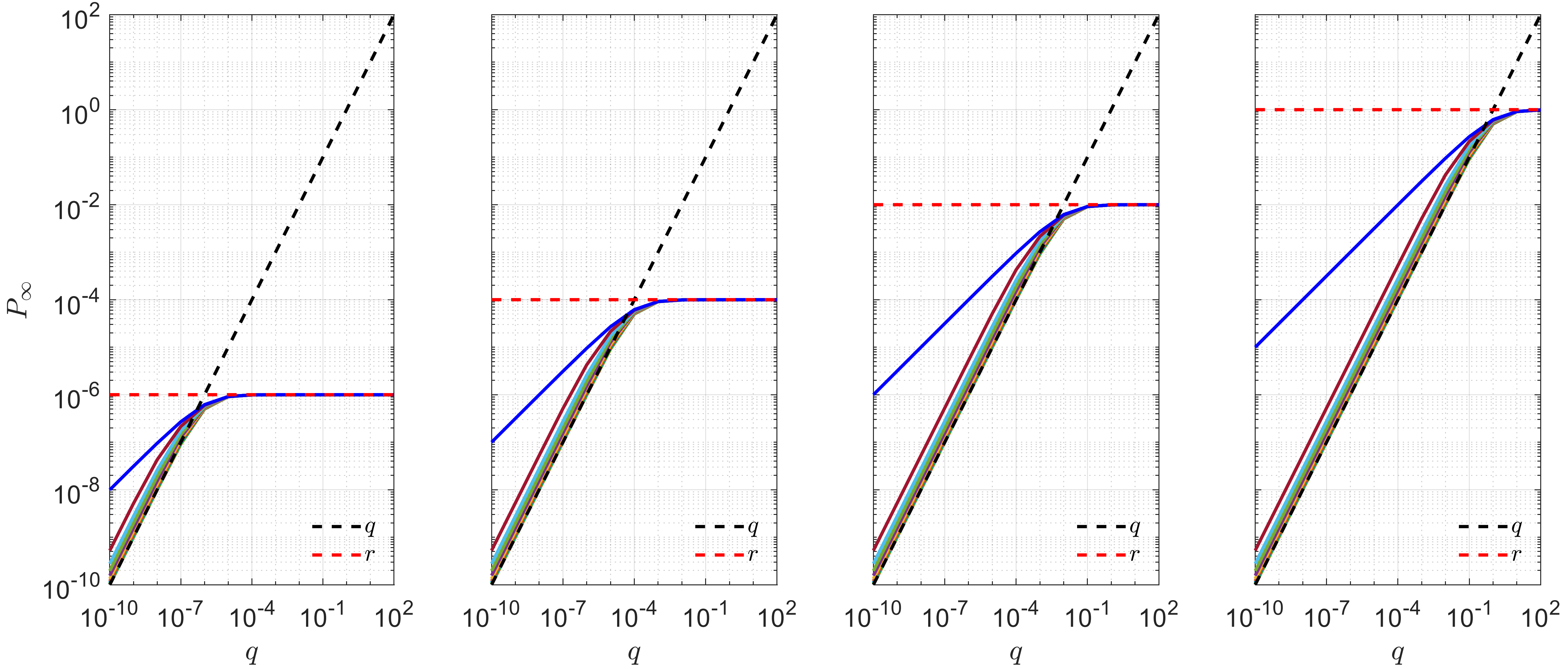
$$P_{\infty|\infty} = \frac{1}{2a^2}\left(-\left(q + r\left(1 - a^2\right)\right) + \sqrt{\left(q + r\left(1 - a^2\right)\right)^2 + 4a^2 qr}\right)$$

Does OF Improve Accuracy?

$$P_{\infty|\infty} = \frac{1}{2a^2}\left(-(q + r(1 - a^2)) + \sqrt{(q + r(1 - a^2))^2 + 4a^2qr}\right)$$

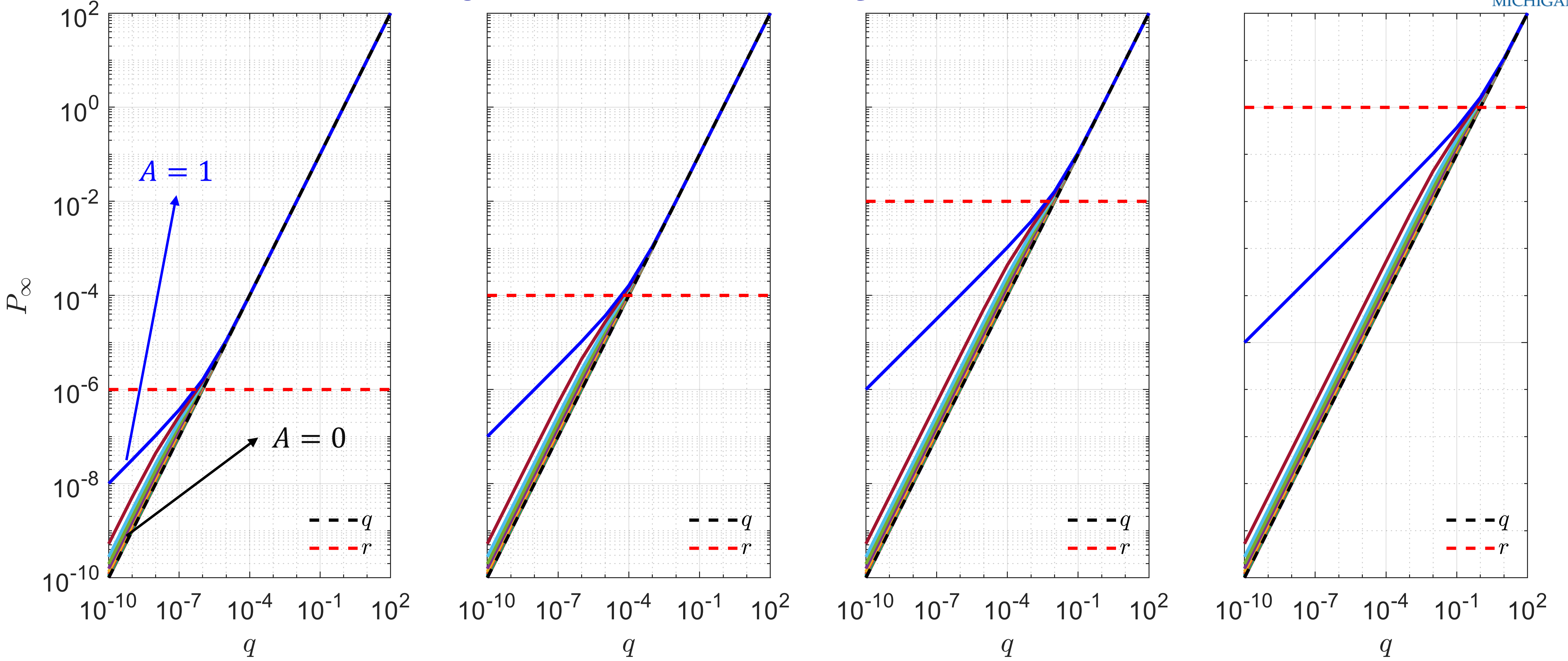
Case	$P_{\infty\|\infty}$	Observation
$q = 0$	0	If the model is good, no matter how bad the sensor is, the estimator state converges to the true state (if $\|a\| \leq 1$)
$r = 0$	0	If the sensor is perfect, no matter how bad the model is, the filter state converges to the true state

Does OF Improve Accuracy?



$$P_{\infty|\infty} < r$$

Does OSE Improve Accuracy?



- If $q > r$, then $P_\infty > q > r$
- If $r > q$, then $r > P_\infty > q$

Is OF better than OP?

$$P_{k+1|k+1} = \frac{a^2 P_{k|k} r + qr}{a^2 P_{k|k} + q + r}$$

- Let $a = 0$. Then,

$$P_{k+1|k+1} = \frac{qr}{q + r}$$

$$P_{k+1} = \frac{a^2 P_k r + qr + qP_k}{P_k + r}$$

- Let $a = 0$. Then,

$$P_{k+1} = q$$

Optimal Filter for LTV System

$$x_{k+1} = A_k x_k + B_k u_k + D_{1,k} w_k$$
$$y_k = C_k x_k + D_{2,k} w_k$$

- $\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k$
- $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k \left(y_{k+1} - C_{k+1} \hat{x}_{k+1|k} \right)$

- $P_{k+1|k} = A_k P_{k|k} A_k^{\mathrm{T}} + Q_k$
- $K_k = P_{k+1|k} C_{k+1}^{\mathrm{T}} \left(C_{k+1} P_{k+1|k} C_{k+1}^{\mathrm{T}} + R_{k+1} \right)^{-1}$
- $P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}$

Summary

- Using the measurement y_{k+1}, a filter provides a better estimate of the state than the predictor

- In the next lectures (on applications),
  - We will formulate the navigation problem as a state estimation problem
  - Can the Kalman Filter help in inertial navigation?