GEOMETRY, KINEMATICS, STATICS, AND DYNAMICS

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Contents

Chapter One
Introduction

The principles of kinematics and dynamics presented in this book are consistent with the many available books on these subjects. However, the presentation differs from other books in crucial ways. In particular, we define concepts and properties of idealized objects with extreme care in order to provide a precise foundation for the key results, which are derived and proved in a rigorous, mathematical style. This approach is intended to add clarity to the basic ideas of kinematics and dynamics, which are often obscure in traditional textbooks. We also find that a careful treatment of the concepts, notation, and terminology of kinematics and dynamics is helpful for solving problems, where intuition and insight can be transferred to precise and unambiguous language and notation.

1.1 Points, Particles, and Bodies

A *point* has zero size and zero mass. A *particle* has zero size and nonnegative mass. A point can thus be viewed as a particle with zero mass. Points and particles have positions *relative* to other points and particles. Two points, two particles, or a point and a particle are *colocated* if they are located at the same place. Two particles are in contact if and only if they are colocated. A *reference point* (such as the origin of a frame) is a point relative to which the positions of other points and particles are determined.

Points and particles can translate relative to other points and particles. Translational motion includes velocity and acceleration. For example, the point or particle x has a position relative to the point or particle y. Likewise, the point or particle x has a velocity and acceleration *relative* to the point or particle y and with *respect* to frame $\mathrm{F_A}$. Unlike bodies, points and particles cannot rotate.

A *body* is a finite collection of particles connected by massless links, linear and rotary springs, dashpots, and inerters, and joints. A single particle is a body, as is a pair of particles that are either freely moving or rigidly connected by a massless link. A *sub-body* is a subset of a body. A body consisting of at least two noncolocated particles is a *rigid body* if, at all distinct times t and t', the orientations of the body are related by a physical rotation matrix. A particle is thus not a rigid body. If a body is rigid, then the distances between every pair of particles are constant. Note that, if a body is modified to match its reflection in a mirror, then the distances between every pair of particles do not change. If, however, the reflected image of the body cannot be rotated to match the original body, then the body is not rigid.

Each particle in a body may be subject to *internal forces*, which arise from the interactions among sub-bodies. An *external force* is a force on a body that is not due to interactions among sub-bodies. A force on a particle or a body produces a *moment relative to a reference point.* A collection of forces applied to a body is *balanced* if the net force is zero. A moment that arises from a collection of balanced forces is a *torque*.

Most of the results in this book are derived for bodies that consist of a finite number of particles.

	Translation	Rotation
Geometry and Kinematics	Point	Frame
Dynamics	Particle	Body

Table 1.1.1: Conceptual roadmap for kinematics and dynamics. For translational geometry and kinematics, mass is irrelevant, and thus a particle is effectively a point. Furthermore, for rotational geometry and kinematics, mass distribution is irrelevant, and thus a body is viewed as a frame. Points and particles cannot rotate, and thus rotational geometry, kinematics, and dynamics apply only to frames and bodies.

A *continuum rigid body* is a rigid shape in two or three dimensions characterized by its density rather than discrete particles. Results derived for rigid bodies can be directly applied to continuum rigid bodies.

The role of points, particles, frames, and bodies in kinematics and dynamics is summarized in Table 1.1.1.

In bodies comprised of particles connected by massless links, the internal forces on pairs of interconnected particles due to their mutual interaction are equal in magnitude, opposite in direction, and parallel with the line that passes through the particles. Bodies can be mechanically coupled through mechanical coupling due to linear and rotary springs, dashpots, and inerters. Bodies can also interact due to field forces. Newton's law of gravity, which involves only attractive forces directed along the line passing through the particles, satisfies this assumption, as does Coulomb's law for electric charges, where the forces may be either repulsive or attractive depending on whether the electric charges are the same or opposite. A rigid body consisting of interconnected permanent magnets does not fit into this framework since the internal forces that attract or repel the magnets follow curved field lines.

A rigid body is *degenerate* if its particles lie along a single line. A massless link is thus a degenerate continuum body. A rigid body that contains three particles in a triangle is not degenerate. Since nonzero forces on a massless body give rise to infinite acceleration, the net force on such a body is taken to be zero. Likewise, when moments or torques on a degenerate body give rise to infinite angular acceleration, these moments or torques are taken to be zero. These conventions are convenient for analyzing internal forces in bodies involving massless links, for example, the simple pendulum. A rigid body that has at least three points that form a triangle can possess a body-fixed frame.

An *unforced particle* is a particle that has no force applied to it. A *massive particle* is a particle with infinite mass. Since a massive particle is unaffected by all forces, every massive particle is effectively an unforced particle. A *massive body* is a rigid body that has at least three massive particles that form a triangle. A massive body is unaffected by all forces, moments, and torques. Consequently, every particle in a massive body is unaffected by forces, and thus every particle in a massive body is unforced.

An inertial frame is a frame with respect to which Newton's first and second laws hold. An unforced particle moves along a straight line with respect to an inertial frame and has zero inertial acceleration. A frame that has zero angular velocity relative to an inertial frame is also an inertial frame.

A massive body is *inertially nonrotating* if its angular velocity relative to an inertial frame is zero; otherwise it is *inertially rotating*. Consequently, every body-fixed frame associated with an inertially nonrotating massive body $\mathcal{B}$ is an inertial frame, and every particle in $\mathcal{B}$ is unforced. Every point that is fixed in an inertially nonrotating massive body has zero inertial acceleration. The Earth is not a massive body since it is affected by central gravity from the Sun and other planets. In addition, the Earth is rotating relative to inertial frames. However, the assumption that the Earth is an inertially nonrotating massive body is useful in many applications. Walls, ceilings, floors, and the ground are conceptual examples of inertially nonrotating massive bodies to which mechanical systems can be attached for the study of dynamics.

The study of mechanics may include either time or force. The various branches of mechanics are outlined in Table 1.1.2.

	Space	Motion	Force
Geometry	Yes	No	No
Kinematics	Yes	Yes	No
Statics	Yes	No	Yes
Dynamics	Yes	Yes	Yes

Table 1.1.2: Definitions of the various branches of mechanics.

1.2 Newton's Third Law and Reciprocal Forces

Rigid bodies that interact through joints are *articulated*. A *prismatic joint* allows translational motion along a line, curve, or surface. Friction may or may not be present in these mechanisms. The reaction force in a frictionless prismatic joint is zero in the direction of translation. A *rotary joint* allows rotation around one or more directions. The reaction torque in a frictionless rotary joint is zero along the axes of rotation. Terminology for joints having 1, 2, or 3 degrees of freedom is summarized in Table 1.2.1. A *welded joint*, which is a rigid connection having zero degrees of freedom, supports reaction forces and reaction torques in all directions.

	1DOF	2DOF	3DOF
Revolute	Pin (Hinge)	Dual Pin (Universal Joint)	Ball Joint (Triple Gimbal)
Prismatic	Sleeve	Dual Sleeve (x-y Stage)	x-y-z Stage
Combined		Slotted pin Collar	Dual Sleeve-Pin Collar-Pin

Table 1.2.1: Terminology for rotary and prismatic joints having 1, 2, or 3 degrees of freedom. A slotted pin is a groove within which a pin translates. A collar is a ring that slides along a shaft while rotating. Alternative names appear in parentheses. The two joints in the $(3,2)$ entry are different, and likewise for the two joints in the $(3,3)$ entry.

A mechanical system consists of particles and rigid bodies (either discrete or continuum, massive or not massive) that interact through contact, mechanical coupling due to linear and rotary springs, dashpots, and inerters as well as joints and field forces. Contact may occur through rolling, sliding, and joints with or without friction. We consider five types of joints. A *pin* allows rotation around a single axis and no translation. A *sleeve* allows translation along a single direction but no rotation. A *collar* is a combination of a pin and a sleeve that allows rotation around a single axis and translation along a single direction, where the axis of rotation and the direction of translation are parallel. A *slotted pin* is a combination of a pin and a sleeve that allows rotation around a single axis and translation along a single direction, where the axis of rotation is perpendicular to the direction of translation. A *universal joint* allows rotation around two axes, whereas a *ball joint* allows rotation around three axes.

The dynamics of a particle depend on the net force acting on the particle. Likewise, the dynamics of a rigid body depend on the net force, moment, and torque acting on the body. These observations provide the ability to analyze the dynamics of the particles and rigid bodies within a body in terms of a *free-body diagram* for each rigid body.

A body may consist of a collection of particles and rigid bodies that interact with each other in various ways. Contact between rigid bodies may involve one or more contact points in each rigid body. Contact may occur as a collision or through a rotary joint, or it may involve time-varying contact points, as in the case of a prismatic joint, sliding (relative translation without relative rotation), or rolling (relative translation and relative rotation with or without slipping). Interaction involving time-dependent contact points can occur with or without friction. .

Newton's third law applies to reaction forces arising from contact between bodies. Interaction forces also arise from mechanical coupling and field forces. Mechanical coupling between particles and rigid bodies can occur through linear and rotary springs, dashpots, and inerters. Field forces include central gravity, electrostatics, and magnetism. An extensions of Newton's third law applies in these cases as well, except that the reciprocal forces arising from magnetic forces are not parallel with the line passing through the magnetic dipoles. Newton's third law does not hold in the case of electrodynamics; for details, see [4, pp. 349–351].

1.3 Physical Vectors and Frames

The notion of location is *relative*; in other words, the location of a point or particle is meaningful only when given in terms of other points and particles. An analogous statement applies to motion. We do not ascribe meaning to the word "stationary"; in fact, the location of a point or particle can be "fixed" only relative to other points and particles. Consequently, the position, velocity, and acceleration of a point or particle are meaningful only when used in a relative sense. Analogous statements apply to bodies under translation and rotation.

This book develops kinematics and dynamics in terms of 12 types of physical vectors. A *physical vector* has a magnitude (which may be dimensional or dimensionless) and direction, but it has no physical location. For example, although the points x and y have locations relative to each other, the physical position vector $\vec{r}_{x/y}$ has no physical location. Likewise, although the force vector $\vec{f}$ can represent a force applied to a particle or body, the physical force vector $\vec{f}$ has no physical location. Consequently, the net force on a particle or the center of mass of a rigid body can be determined by summing individual forces by plotting them tip-to-tail.

Two nonzero physical vectors are *parallel* if one is a scalar multiple of the other, and are *codirectional* if one is a positive multiple of the other.

The purpose of a *frame*, which is a set of three linearly independent (and usually mutually perpendicular) physical vectors, is to define directions in three-dimensional space. For example, the frame $\mathrm{F_A}$ is represented by the row vectrix $\mathrm{F_A} = [\hat{\imath}_\mathrm{A}\ \hat{\jmath}_\mathrm{A}\ \hat{k}_\mathrm{A}]$ and the column vectrix $\mathcal{F}_\mathrm{A} = \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}$. Since physical vectors have no physical location, a frame has no location. Since a frame has no physical location, it is meaningless to refer to its velocity and acceleration. This conception of a frame, which is a unique feature of this book, stresses its role in defining direction as distinct from location. For a rigid body $\mathcal{B}$, a body-fixed frame $\mathrm{F_B}$ is a frame whose axes have fixed directions in $\mathcal{B}$. The frame $\mathrm{F_B}$ thus rotates as $\mathcal{B}$ rotates.

It is traditional but not necessary to associate with a frame a reference point designated as the *origin* of the frame. Like any other point, the origin of a frame has position, velocity, and acceleration relative to other points, and it can be used to define the relative position, velocity, and acceleration of other points. The traditional notion of the "acceleration of a frame" refers to the motion of its origin rather than the axes of the frame, which have no physical location. A frame need not be assigned an origin, however, although we often do this for convenience. The choice of a reference point, is arbitrary; however, for a body-fixed frame, it is convenient to choose a point that is fixed in the body, and, for an inertial frame, it is convenient to choose a point that has zero inertial acceleration.

It is not meaningful to say that a point is "fixed in a frame," although it is meaningful to say that a point is fixed in a rigid body. A point p may be fixed relative to a rigid body but not part of the rigid body by viewing p as rigidly attached to the rigid body by means of a massless link. It is meaningful to say that a direction is fixed in a rigid body and with respect to a frame in the sense that the direction of the vector depends on the orientation of the frame. We almost exclusively consider only *standard frames*, which are orthogonal and right-handed with dimensionless, unit-length axes. Frames that are nonstandard are considered only in Section 2.19.

Velocity and acceleration depend on the frame with *respect* to which changes are observed. Hence, derivatives of physical vectors are defined only with respect to frames; these derivatives are called *frame derivatives*.

An *unforced particle* is a particle that has no force (that is, zero net force) applied to it. The motion of an unforced particle is thus determined by its initial position and velocity. An *inertial frame* is a frame that has the property that the relative acceleration with respect to the frame of every pair of unforced particles is zero. This is Newton's first law. Like all frames, an inertial frame has no location, and thus the velocity and acceleration of an inertial frame are meaningless. There are an infinite number of inertial frames, and the relative angular velocity of each pair of inertial frames is zero. We do not recognize the notion of an "absolute" frame.

1.4 Remarks on Notation

The notation in this book differs from other books on dynamics. With modest effort, the reader will find that this notation is extremely helpful for understanding the principles of kinematics and dynamics and for solving problems. Some of the features of this notation are described below.

First, a half arrow over a symbol such as $\vec{r}_{x/y}$, where x and y are points or particles, emphasizes the fact that $\vec{r}_{x/y}$ denotes a physical vector, which is not *resolved* in a frame. Derivations and calculations can be carried out to the greatest possible extent without resolving physical vectors in a specific frame. At a later stage, every vector in the equation can be resolved in any frame of interest to obtain mathematical vectors, which are column vectors with numerical or symbolic components. This approach facilitates the physical interpretation of the components of mathematical vectors.

The notation used in this book strives to be 1) independent of context, 2) explicit, and 3) unambiguous. The meaning of each symbol can be determined by its appearance alone without the need for additional verbiage, commentary, or explanation. This interpretation is facilitated by subscripts. For example, $\vec{r}_{x/y}$ denotes the position of the point or particle x relative to the point or particle y. The notation $\vec{f}_y$ denotes a generic force applied to the point or particle y.

A physical vector can be multiplied by a real scalar, as in $3\vec{f}$ or $-6\vec{f}$. The zero physical vector is denoted by $\vec{0}$, and therefore $\vec{r}_{x/x} = \vec{0}$; typically, the harpoon is omitted. A physical vector such as $\vec{r}_{x/y}(t)$ can be a function of time, although the time argument is usually omitted. For a nonzero physical vector $\vec{x}$, the notation $\hat{x}$ represents a dimensionless, unit-length physical vector whose direction is the same as the direction of $\vec{x}$.

When rate is involved, an additional subscript is included to denote the frame used for the frame derivative as in, for example,

$$\vec{v}_{x/y/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{x/y}, \tag{1.4.1}$$

which denotes the velocity of the point or particle x relative to the point or particle y with respect to the frame $\mathrm{F_A}$. Frame derivatives are denoted by $\overset{\mathrm{A}\bullet}{\vec{r}}_{x/y}$, $\overset{\mathrm{B}\bullet}{\vec{r}}_{x/y}$, $\overset{\mathrm{C}\bullet}{\vec{r}}_{x/y}$, and so forth. Likewise,

$$\vec{a}_{x/y/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{v}}_{x/y/\mathrm{A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{x/y} \tag{1.4.2}$$

denotes the acceleration of x relative to y with respect to the frame $\mathrm{F_A}$.

The subscripts on these vectors are convenient for the following reason. If x, y, and z are points or particles, then the physical position, velocity, and acceleration vectors satisfy the *slash and split* identities

$$\vec{r}_{z/x} = \vec{r}_{z/y} + \vec{r}_{y/x}, \tag{1.4.3}$$

$$\vec{v}_{z/x/\mathrm{A}} = \vec{v}_{z/y/\mathrm{A}} + \vec{v}_{y/x/\mathrm{A}}, \tag{1.4.4}$$

$$\vec{a}_{z/x/\mathrm{A}} = \vec{a}_{z/y/\mathrm{A}} + \vec{a}_{y/x/\mathrm{A}}. \tag{1.4.5}$$

Furthermore,

$$\vec{r}_{x/y} = -\vec{r}_{y/x}, \tag{1.4.6}$$

$$\vec{v}_{x/y/\mathrm{A}} = -\vec{v}_{y/x/\mathrm{A}}, \tag{1.4.7}$$

$$\vec{a}_{x/y/\mathrm{A}} = -\vec{a}_{y/x/\mathrm{A}}. \tag{1.4.8}$$

Note, however, that slash and split does not hold for unit vectors. In particular, $\hat{r}_{z/x} = \hat{r}_{z/y} + \hat{r}_{y/x}$ is correct if and only if x, y, z are the vertices of an equilateral triangle.

A physical matrix $\vec{M}$ can be viewed as a 3×3 matrix that is not resolved in a frame. A physical matrix is traditionally called a dyad or a second-order tensor. Physical rotation matrices and physical inertia matrices are physical matrices that play key roles in kinematics and dynamics.

We use only a single font for all symbols without the need for bold letters. This style allows easy presentation on a whiteboard without the need for underscores and undertildes. We also avoid the use of superscripts, both pre and post.

1.5 Resolving Physical Vectors

A physical vector has no components, and thus it is distinct from a mathematical column vector of the form $[1 \;\; -6 \;\; 3]^{\mathsf{T}}$. However, any physical vector can be resolved in any frame. For example, the velocity vector $\vec{v}_{y/x/\mathrm{A}}$ can be resolved in $\mathrm{F_B}$ by writing

$$\vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}}. \tag{1.5.1}$$

The resolved vector can also be represented by

$$v_{y/x/\mathrm{A}|\mathrm{B}} = \vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}}. \tag{1.5.2}$$

1.6 Types of Physical Vectors

A *physical vector* (as distinct from a mathematical vector, which is a column of numbers) is an abstract quantity having a *tip* and a *tail* and thus magnitude and direction. A physical vector is not a physical object, and thus it is not located anywhere, although we can envision its tail located at an arbitrary location for convenience. A physical vector is denoted with a half arrow or hat over the symbol denoting the physical quantity. For example, $\vec{f}$ is a physical vector representing a force applied to a particle in a body, while $\vec{r}_{x/y}$ is the physical vector representing the position of the point x *relative* to the point y. We may envision the tip of $\vec{r}_{x/y}$ at x and its tail at y. However, the physical vector $\vec{r}_{x/y}$ has no physical location. The magnitude of $\vec{x}$ is denoted by $|\vec{x}|$.

A physical vector may have dimensions or it may be dimensionless. A frame consists of three unit, dimensionless physical vectors that are mutually orthogonal. The motion of a point, particle, or body relative to another point, particle, or body is determined with *respect* to a frame. Differentiation with respect to a frame is discussed in Chapter 4.

Statics, kinematics, and dynamics are based on 12 types of physical vectors, namely:

i) Dimensionless. A dimensionless physical vector has no physical units associated with it. A unit dimensionless physical vector is written as $\hat{\imath}$. Three mutually orthogonal unit dimensionless physical vectors comprise a frame. The unit dimensionless physical vector that points in the direction of the nonzero physical vector $\vec{x}$ is denoted by $\hat{x}$. Hence, $\vec{x} = |\vec{x}|\hat{x}$. If $\vec{x} = 0$, then $\hat{x}$ is not defined.

ii) Unit angle vector. The unit angle vector of the physical vector $\vec{y}$ relative to the physical vector $\vec{x}$, where $\vec{y}$ and $\vec{x}$ are nonzero and not parallel, is written as $\hat{\theta}_{\vec{y}/\vec{x}}$. The direction of $\hat{\theta}_{\vec{y}/\vec{x}}$ is given by the right hand rule with the fingers curled from $\vec{x}$ to $\vec{y}$ through the short-way angle $\theta_{\vec{y}/\vec{x}} = \theta_{\vec{x}/\vec{y}}$ between $\vec{x}$ and $\vec{y}$. Hence, $\hat{\theta}_{\vec{x}/\vec{y}} = -\hat{\theta}_{\vec{y}/\vec{x}}$. The notation $\vec{\theta}_{\vec{y}/\vec{x}}$ is not defined or used.

iii) Position. The position of the point y relative to the point x is written as $\vec{r}_{y/x}$.

iv) Velocity. The velocity of the point y relative to the point x with respect to the frame $\mathrm{F_A}$ is written as $\vec{v}_{y/x/\mathrm{A}}$. Each component of $\vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{C}}$ is a speed.

v) Acceleration. The acceleration of the point y relative to the point x with respect to the frame $\mathrm{F_A}$ is written as $\vec{a}_{y/x/\mathrm{A}}$.

vi) Angular velocity. The angular velocity of the frame $\mathrm{F_B}$ relative to the frame $\mathrm{F_A}$ is written as $\vec{\omega}_{\mathrm{B/A}}$. Each component of $\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{C}}$ is a spin rate.

vii) Angular acceleration. The angular acceleration of the frame $\mathrm{F_B}$ relative to the frame $\mathrm{F_A}$ with respect to the frame $\mathrm{F_C}$ is written as $\vec{\alpha}_{\mathrm{B/A/C}}$.

viii) Momentum. The momentum of the particle y relative to the point x with respect to the frame $\mathrm{F_A}$ is written as $\vec{p}_{y/x/\mathrm{A}}$. The momentum of the body $\mathcal{B}$ relative to the point x with respect to the frame $\mathrm{F_A}$ is written as $\vec{p}_{\mathcal{B}/x/\mathrm{A}}$.

ix) Force. A force $\vec{f}$ can be applied to a point or a particle, where a point is viewed as a particle with zero mass. We allow a force to be applied to a point as long as neither infinite acceleration nor infinite angular acceleration can occur. For example, a force can be applied to a point along a massless link in a body. A force on a particle in a body can be either an external force or an internal reaction force. Reaction forces are forces due to the interaction between particles; these forces may or may not involve contact. Reaction forces due to the interaction with a massive body are viewed as external forces. The force on a particle or body due to gravity can be either uniform, that is, a function of mass, or central, that is, a function of mass and position.

x) Angular momentum. The angular momentum of the particle x relative to the point w with respect to the frame $\mathrm{F_A}$ is written as $\vec{H}_{x/w/\mathrm{A}}$. The angular momentum of the body $\mathcal{B}$ relative to the point w with respect to the frame $\mathrm{F_A}$ is written as $\vec{H}_{\mathcal{B}/w/\mathrm{A}}$.

xi) Moment. A moment can be applied to either a particle or a body. A moment on the particle x relative to the point y is written as $\vec{M}_{x/y}$. A moment on the body $\mathcal{B}$ relative to the point y is written as $\vec{M}_{\mathcal{B}/y}$. A moment can be applied to a trivial rigid body as long as infinite angular acceleration cannot occur.

xii) Torque. A torque on a body can be either an external torque or an internal reaction torque. A torque on the body $\mathcal{B}$ is written as $\vec{T}_{\mathcal{B}}$. A nonzero torque can be applied to an arbitrary rigid body as long as infinite angular acceleration cannot occur.

Position, velocity, acceleration, momentum, and force can be projected onto a direction $\hat{n}$; the resulting vector is the position, velocity, acceleration, momentum, and force *along* $\hat{n}$. Angular velocity, angular acceleration, angular momentum, moment, and torque can be projected onto a direction $\hat{n}$; the resulting vector is the angular velocity, angular acceleration, angular momentum, moment, and torque *around* $\hat{n}$.

Energy is a scalar quantity that is associated with a particle or a body. Potential energy can be defined in terms of a spring or a uniform or central gravitational vector field. Kinetic energy is defined in terms of velocity with respect to an inertial frame.

1.7 Mechanical Systems

We apply the techniques of kinematics and dynamics to various types of mechanical systems. These systems may be one-dimensional (linear), two-dimensional (planar), or three-dimensional (spatial), depending on whether the motion occurs along a line, in a plane, or in three-dimensional space. The systems may involve various joints, they may involve one or more rigid bodies, they may include the effect of gravity, they may include springs and dashpots (either linear or torsional), and they may involve rolling (with or without slipping), sliding (with or without friction), and collisions. For convenience, we refer to these examples by the following terminology:

Pendulum. A pendulum is a planar or spatial mechanical system connected to a massive body by means of a rotary joint. Gravity is usually present. Springs and dashpots can be included, as well as multiple rigid bodies.

MCK system. Rigid bodies, springs, and dashpots can be connected to form a planar or spatial mechanical system.

Gimbal. A gimbal is a spatial mechanical system with multiple rotary joints. Springs and dashpots can be included, as well as a spinning payload.

Shaft. A shaft is a three-dimensional mechanical system consisting of a rotating rigid body connected to a massive body by means of a rotary joint. Additional rigid bodies may be connected to the shaft by means of rotary or prismatic joints.

Linkage. A linkage is a planar or spatial device involving multiple rigid bodies connected by rotary or prismatic joints. Springs and dashpots can be included, as well as rolling disks.

Rolling body. A disk, sphere, and cone can roll over a surface that is either flat or curved.

Spinning top. A top is a spinning body connected to a massive body by means of a ball joint.

The above classification is not precise and is for convenience only. For example, a pendulum can be viewed as a linkage, a gimbal can be viewed as a type of a shaft, and rolling bodies can be combined with other types of mechanical systems.

1.8 Classification of Forces and Moments

Forces that do not entail a loss of energy are called *conservative forces*, while forces that give rise to a loss of energy are called *dissipative forces*.

Table 1.8.1 classifies reaction and non-reaction forces and moments in terms of energy conservation and dissipation. Reaction forces due to elastic collisions, rolling without slipping, sliding without friction, and pivoting, as well as springs and inerters are conservative. Forces due to friction (except rolling without slipping), inelastic collisions, and dashpots are dissipative. When two bodies are in contact, the reaction force may be either tangential or normal. The reaction force between two bodies that are in contact is frictionless if the tangential contact force is zero. Reaction forces may be associated with the Coriolis, angular-acceleration, and centripetal components of acceleration. Angular-acceleration and centripetal reaction forces involve zero relative motion, whereas a Coriolis contact force involves nonzero relative motion (such as a particle sliding with friction within a groove on a rotating platform).

	Conservative	Dissipative
Reaction forces	Elastic impact Joint constraint Frictionless sliding Rolling without slipping Frictionless slipping Frictionless pivoting Spring Inerter Central gravity Electrostatic force magnetic force	Inelastic impact Sliding with friction Slipping with friction Dashpot
Reaction torques	Joint constraint Rotational spring Rotational inerter	Pivoting with friction Rotational dashpot
Non-reaction forces and moments	Uniform gravity	Control Disturbance External forces

Table 1.8.1: Classification of reaction and non-reaction forces, moments, and torques. Dissipative forces, moments, and torques entail a loss of energy, whereas energy is conserved by conservative forces, moments, and torques. Control and disturbance forces and moments can increase or decrease energy. Contact forces include normal and tangential reaction forces due to collision, rolling (with or without slipping), sliding (with or without friction), and pivoting. Mechanical coupling forces may be due to springs, dashpots, and inerters. Field forces include gravity and electromagnetic forces.

Chapter Two
Geometry

2.1 Angle and Dot Product

An angle θ confined to $(-\pi, \pi]$ is a *wrapped angle*; otherwise, $\theta \in \mathbb{R}$ is an *unwrapped angle*. The wrapped angle $\theta \in [0, \pi]$ between two physical vectors is the *short-way angle* between the vectors.

Since θ and $\theta + 2n\pi$, where n is an integer, represent the same angle, wrapped angles suffice to represent all possible angles between a pair of physical vectors. However, sums and differences of wrapped angles can violate this constraint, and thus are unwrapped. To facilitate addition and subtraction of wrapped angles without using unwrapped angles, we extend the notion of equality for angles. In particular, for $a, b \in \mathbb{R}$, the notation $a \equiv b$ means that $a - b$ is an integer multiple of 2π.

Fact 2.1.1. Let $a, b, c, d \in \mathbb{R}$. Then, the following statements hold:

i) $a \equiv b$ if and only if $a - b \equiv 0$.

ii) $a \equiv b$ if and only if $-a \equiv -b$.

iii) If $a \equiv b$ and n is an integer, then $na \equiv nb$.

iv) Let n and m be integers such that $n + m$ is even. Then, $a \equiv n\pi$ if and only if $a \equiv m\pi$.

v) $a \equiv -a$ if and only if there exists an integer n such that $a = n\pi$.

vi) The following statements are equivalent:

a) $a \equiv -a \equiv 0$.

b) There exists an even integer n such that $a = n\pi$.

c) $a \equiv 0$.

vii) The following statements are equivalent:

a) $a \equiv -a \equiv \pi$.

b) There exists an odd integer n such that $a = n\pi$.

c) $a \equiv \pi$.

viii) If $a \equiv b$ and $c \equiv d$, then $a + c \equiv b + d$.

Let $\theta_{\vec{x}/\vec{y}} = \theta_{\vec{y}/\vec{x}} \in [0, \pi]$ denote the short-way angle between the physical vectors $\vec{x}$ and $\vec{y}$. If either $\vec{x}$ or $\vec{y}$ is the zero physical vector $\vec{0}$ (also written as just 0), then $\theta_{\vec{x}/\vec{y}} = \theta_{\vec{y}/\vec{x}} = 0$. The *length* of the physical vector $\vec{x}$ is denoted by $|\vec{x}|$. The *dot product* $\vec{x} \cdot \vec{y}$ of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \cdot \vec{y} \triangleq |\vec{x}||\vec{y}| \cos \theta_{\vec{x}/\vec{y}}. \tag{2.1.1}$$

Since $\cos \theta_{\vec{x}/\vec{x}} = \cos 0 = 1$, it follows that

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}. \tag{2.1.2}$$

Furthermore,

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} \tag{2.1.3}$$

and

$$|\vec{x} \cdot \vec{y}| = |\vec{x}|\,|\vec{y}|\,|\cos \theta_{\vec{x}/\vec{y}}|. \tag{2.1.4}$$

If $\vec{x}$ and $\vec{y}$ are nonzero, then the unit vectors $\hat{x}$ and $\hat{y}$ satisfy

$$\hat{x} \cdot \hat{y} = \cos \theta_{\vec{x}/\vec{y}}, \tag{2.1.5}$$

and thus

$$\theta_{\vec{x}/\vec{y}} = \operatorname{acos} \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|\,|\vec{y}|} = \operatorname{acos} \hat{x} \cdot \hat{y} \in [0, \pi]. \tag{2.1.6}$$

Note that the range of the function acos is $[0, \pi]$.

Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors. Then $\vec{x}$ and $\vec{y}$ are *mutually orthogonal* if $\vec{x} \cdot \vec{y} = 0$, that is, if $\theta_{\vec{y}/\vec{x}} = \pi/2$. Equivalently, we say that $\vec{x}$ is *orthogonal to* $\vec{y}$, and $\vec{y}$ is *orthogonal to* $\vec{x}$. Furthermore, $\vec{x}$ and $\vec{y}$ are *parallel* if either $\theta_{\vec{x}/\vec{y}} = 0$ or $\theta_{\vec{x}/\vec{y}} = \pi$. Equivalently, we say that $\vec{x}$ is *parallel with* $\vec{y}$, and $\vec{y}$ is *parallel with* $\vec{x}$. Note that $\vec{x}$ and $\vec{y}$ are parallel if and only if

$$|\vec{x} \cdot \vec{y}| = |\vec{x}|\,|\vec{y}|. \tag{2.1.7}$$

Finally, $\vec{x}$ and $\vec{y}$ are *codirectional* if one is a positive multiple of the other, that is, if $\hat{x} = \hat{y}$.

We define

$$\vec{x}'\vec{y} \triangleq \vec{x} \cdot \vec{y}, \tag{2.1.8}$$

where the *physical covector* $\vec{x}'$ operates on the physical vector $\vec{y}$ to produce the real scalar $\vec{x} \cdot \vec{y}$. Note that $\vec{x}'\vec{y} = \vec{y}'\vec{x}$. The physical covector $\vec{x}'$ is the *coform* of the physical vector $\vec{x}$. We define $(\vec{x}')' \triangleq \vec{x}$.

Fact 2.1.2. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then, $\vec{x} = \vec{y}$ if and only if $\vec{x}' = \vec{y}'$.

The following identity will be useful.

Fact 2.1.3. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$(\vec{x} + \vec{y})'(\vec{x} + \vec{y}) = \vec{x}'\vec{x} + \vec{y}'\vec{y} + 2\vec{x}'\vec{y}, \tag{2.1.9}$$

that is,

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 + 2\vec{x}'\vec{y}. \tag{2.1.10}$$

Proof. Applying the cosine rule to the triangle with sides $\vec{y}$, $-\vec{x}$, and $\vec{y}+\vec{x}$ yields

$$
\begin{aligned}
(\vec{y}+\vec{x})'(\vec{y}+\vec{x}) &= |\vec{y}+\vec{x}|^2 \\
&= |\vec{y}|^2 + |-\vec{x}|^2 - 2|\vec{y}||-\vec{x}|\cos\theta_{(-\vec{x})/\vec{y}} \\
&= |\vec{y}|^2 + |\vec{x}|^2 - 2|\vec{y}||\vec{x}|\cos(\pi - \theta_{\vec{x}/\vec{y}}) \\
&= |\vec{y}|^2 + |\vec{x}|^2 + 2|\vec{y}||\vec{x}|\cos\theta_{\vec{x}/\vec{y}} \\
&= \vec{x}'\vec{x} + \vec{y}'\vec{y} + 2\vec{y}'\vec{x}. \qquad \square
\end{aligned}
$$

The set of physical covectors corresponding to a set $\mathcal{V}$ of physical vectors is denoted by $\mathcal{V}'$. Each physical covector can be associated with a hyperplane in the space of physical vectors, that is, a plane that is translated away from the origin. Specifically, for the physical vector $\vec{x}$, define

$$\mathcal{H}(\vec{x}) \triangleq \{\vec{y} \in \mathcal{V} \colon \vec{x}'\vec{y} = 1\}. \tag{2.1.11}$$

To show that $\mathcal{H}(\vec{x})$ is a hyperplane, let the physical vector $\vec{y}_0$ satisfy $\vec{x}'\vec{y}_0 = 1$. Then,

$$\mathcal{H}(\vec{x}) = \vec{y}_0 + \{\vec{y} \in \mathcal{V} \colon \vec{x}'\vec{y} = 0\}. \tag{2.1.12}$$

2.2 Unit Angle Vector and Cross Product

Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors that are not parallel so that $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$. The *unit angle vector* $\hat{\theta}_{\vec{y}/\vec{x}}$ of $\vec{y}$ relative to $\vec{x}$ is the unit dimensionless physical vector orthogonal to both $\vec{x}$ and $\vec{y}$ whose direction is determined by the right hand rule with the fingers curled from $\vec{x}$ to $\vec{y}$ through the positive short-way angle $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$ between $\vec{x}$ and $\vec{y}$. See Figure 2.2.1.

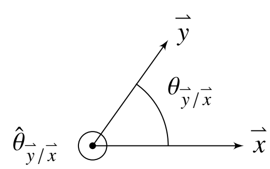


Figure 2.2.1: The unit angle vector $\hat{\theta}_{\vec{y}/\vec{x}}$ of $\vec{y}$ relative to $\vec{x}$ is the dimensionless physical vector whose direction is determined by the direction of the right-hand thumb with the fingers curled from $\vec{x}$ to $\vec{y}$ through the positive short-way angle $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$. For the example shown, $\hat{\theta}_{\vec{y}/\vec{x}}$ points out of the page. Note that $\theta_{\vec{y}/\vec{x}} = \theta_{\vec{x}/\vec{y}}$ and $\hat{\theta}_{\vec{x}/\vec{y}} = -\hat{\theta}_{\vec{y}/\vec{x}}$.

Let $\vec{x}$ and $\vec{y}$ be physical vectors. If $\theta_{\vec{y}/\vec{x}}$ is either 0 or π, then the *cross product* $\vec{x}\times\vec{y}$ is defined to be the zero physical vector. On the other hand, if $\vec{x}$ and $\vec{y}$ are nonzero and not parallel, then the *cross product* of $\vec{x}$ and $\vec{y}$ is defined as

$$\vec{x}\times\vec{y} \triangleq |\vec{x}||\vec{y}|(\sin\theta_{\vec{y}/\vec{x}})\hat{\theta}_{\vec{y}/\vec{x}}. \tag{2.2.1}$$

Note that, since $\theta_{\vec{y}/\vec{x}} \in (0,\pi)$, it follows that $\sin\theta_{\vec{y}/\vec{x}} > 0$. Therefore,

$$\hat{x}\times\hat{y} = (\sin\theta_{\vec{y}/\vec{x}})\hat{\theta}_{\vec{y}/\vec{x}}, \tag{2.2.2}$$

$$|\vec{x} \times \vec{y}| = |\vec{x}|\,|\vec{y}| \sin \theta_{\vec{y}/\vec{x}}, \tag{2.2.3}$$

$$\hat{\theta}_{\vec{y}/\vec{x}} = \frac{1}{|\vec{x} \times \vec{y}|} \vec{x} \times \vec{y} = \frac{1}{|\vec{x}|\,|\vec{y}| \sin \theta_{\vec{y}/\vec{x}}} \vec{x} \times \vec{y} = \frac{1}{\sin \theta_{\vec{y}/\vec{x}}} \hat{x} \times \hat{y}, \tag{2.2.4}$$

$$\vec{y} \times \vec{x} = -(\vec{x} \times \vec{y}) = (-\vec{x}) \times \vec{y} = \vec{x} \times (-\vec{y}), \tag{2.2.5}$$

$$\vec{x} \times \vec{x} = 0. \tag{2.2.6}$$

2.3 Directed Angles

Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors, and let $\hat{n}$ be a unit dimensionless physical vector that is orthogonal to both $\vec{x}$ and $\vec{y}$; that is, either $\hat{n} = \hat{\theta}_{\vec{y}/\vec{x}}$ or $\hat{n} = -\hat{\theta}_{\vec{y}/\vec{x}}$. The *directed angle* $\theta_{\vec{y}/\vec{x}/\hat{n}}$ *from* $\vec{x}$ to $\vec{y}$ *around* $\hat{n}$ is defined by

$$\theta_{\vec{y}/\vec{x}/\hat{n}} \triangleq \begin{cases} 0, & \text{if } \theta_{\vec{y}/\vec{x}} = 0, \\ \theta_{\vec{y}/\vec{x}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in (0,\pi) \text{ and } \hat{n} = \hat{\theta}_{\vec{y}/\vec{x}}, \\ -\theta_{\vec{y}/\vec{x}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in (0,\pi) \text{ and } \hat{n} = -\hat{\theta}_{\vec{y}/\vec{x}}, \\ \pi, & \text{if } \theta_{\vec{y}/\vec{x}} = \pi. \end{cases} \tag{2.3.1}$$

In the first and last cases, $\vec{x}$ and $\vec{y}$ are parallel. In the second case, the directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is positive; in the third case, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is negative. Therefore, $\theta_{\vec{y}/\vec{x}/\hat{n}} \in (-\pi,\pi]$, and thus $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is a wrapped angle. Figure 2.3.1 shows that, in the second case, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is the angle from $\vec{x}$ to $\vec{y}$, as determined by the right-hand rule with the thumb pointing in the direction of $\hat{n}$. Figure 2.3.1 also shows that, in the third case, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is the angle from $\vec{x}$ to $\vec{y}$, as determined by the left-hand rule with the thumb pointing in the direction of $\hat{n}$. Note that

$$\theta_{\vec{x}/\vec{y}/\hat{n}} = \begin{cases} -\theta_{\vec{y}/\vec{x}/\hat{n}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in [0,\pi), \\ \pi, & \text{if } \theta_{\vec{y}/\vec{x}} = \pi, \end{cases} \tag{2.3.2}$$

$$\theta_{\vec{y}/\vec{x}/-\hat{n}} = \begin{cases} -\theta_{\vec{y}/\vec{x}/\hat{n}}, & \text{if } \theta_{\vec{y}/\vec{x}} \in [0,\pi), \\ \pi, & \text{if } \theta_{\vec{y}/\vec{x}} = \pi. \end{cases} \tag{2.3.3}$$

Hence,

$$\theta_{\vec{x}/\vec{y}/-\hat{n}} = \theta_{\vec{y}/\vec{x}/\hat{n}}. \tag{2.3.4}$$

Note that all directed angles are elements of $(-\pi,\pi]$, and thus are wrapped. In addition, since the magnitude of a directed angle is not larger than π, a directed angle can be viewed as a *signed short-way angle*, that is, a short-way angle that may be positive or negative, as shown in Figure 2.3.1.

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be nonzero physical vectors that are linearly dependent, and let $\hat{n}$ denote a unit vector that is orthogonal to $\vec{x}$, $\vec{y}$, and $\vec{z}$. Then,

$$\theta_{\vec{z}/\vec{x}/\hat{n}} \equiv \theta_{\vec{z}/\vec{y}/\hat{n}} + \theta_{\vec{y}/\vec{x}/\hat{n}}. \tag{2.3.5}$$

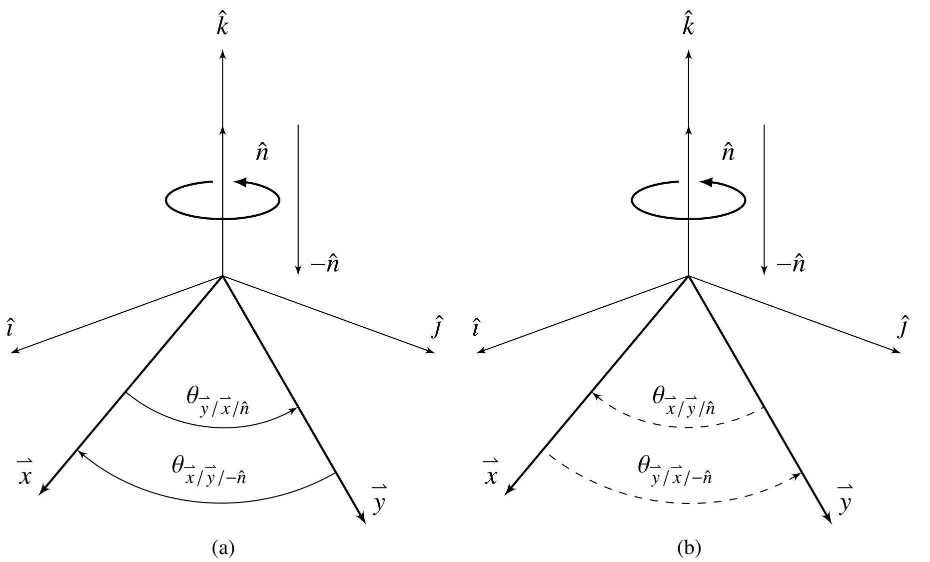


Figure 2.3.1: Directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}}$ from $\vec{x}$ to $\vec{y}$ around $\hat{n}$. The value of $\theta_{\vec{y}/\vec{x}/\hat{n}}$ in (a) is determined by the curled fingers of the right hand when the right-hand thumb points in the direction of $\hat{n}$. As shown, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is a positive directed angle, as indicated by the solid curved arrow. The directed angle $\theta_{\vec{x}/\vec{y}/-\hat{n}}$ from $\vec{y}$ to $\vec{x}$ around $-\hat{n}$ is also shown, and it can be seen that $\theta_{\vec{x}/\vec{y}/-\hat{n}} = \theta_{\vec{y}/\vec{x}/\hat{n}} > 0$. In (b), the value of $\theta_{\vec{x}/\vec{y}/\hat{n}}$ is determined by the curled fingers of the *left* hand when the left-hand thumb points in the direction of $\hat{n}$. As shown, $\theta_{\vec{x}/\vec{y}/\hat{n}}$ is a negative directed angle, and thus a signed short-way angle, as indicated by the dashed curved arrow. The directed angle $\theta_{\vec{y}/\vec{x}/-\hat{n}}$ from $\vec{y}$ to $\vec{x}$ around $-\hat{n}$ is also shown, and it can be seen that $\theta_{\vec{y}/\vec{x}/-\hat{n}} = \theta_{\vec{x}/\vec{y}/\hat{n}} < 0$.

This shows that, accounting for wrapping, slash and split holds for directed angles. For details, see Fact 2.20.1.

The directed angle $\theta_{\vec{y}/\vec{x}/\hat{n}} \in (-\pi, \pi]$ can be understood in the following way. Define a frame $\mathrm{F} = [\hat{\imath}\ \hat{\jmath}\ \hat{k}]$ such that $\hat{\imath} = \hat{x}$, $\vec{y}$ lies in the $\hat{\imath}$-$\hat{\jmath}$ plane, and $\hat{k} = \hat{n}$. Furthermore, write $\vec{y} = y_1\hat{\imath} + y_2\hat{\jmath}$. Next, we view $\hat{\imath}$ and $\hat{\jmath}$ as defining a complex plane, where $\hat{\imath}$ points the direction of the positive real axis, and $\hat{\jmath}$ points in the direction of the positive imaginary axis. Then, the vector $\vec{y}$ can be viewed as the position of the complex number $y_1 + y_2\jmath$ relative to the origin. With this construction, $\theta_{\vec{y}/\vec{x}/\hat{n}}$ is the angle of the complex number $y_1 + y_2\jmath$ in the complex plane with the usual convention that clockwise rotations correspond to increasing angles with zero radians assigned to points on the positive real axis. Hence,

$$\tan\theta_{\vec{y}/\vec{x}/\hat{n}} = \frac{y_2}{y_1}. \tag{2.3.6}$$

Furthermore,

$$\theta_{\vec{y}/\vec{x}/\hat{n}} = \operatorname{atan2}(y_2, y_1), \tag{2.3.7}$$

where atan2 is the four-quadrant inverse of the tangent function, that is,

$$\operatorname{atan2}(y_2, y_1) = \begin{cases} 0, & y_2 = y_1 = 0, \\ \tan^{-1} \frac{y_2}{y_1}, & y_1 > 0, \\ -\pi/2, & y_2 < 0,\ y_1 = 0, \\ \pi/2, & y_2 > 0,\ y_1 = 0, \\ -\pi + \tan^{-1} \frac{y_2}{y_1}, & y_2 < 0,\ y_1 < 0, \\ \pi + \tan^{-1} \frac{y_2}{y_1}, & y_2 \geq 0,\ y_1 < 0. \end{cases} \tag{2.3.8}$$

Note that the range of the function $\tan^{-1}$ is $(-\pi/2, \pi/2)$, whereas the range of the function atan2 is $(-\pi, \pi]$. Equivalently,

$$\operatorname{atan2}(y_2, y_1) = \begin{cases} 0, & y_2 = y_1 = 0, \\ 2\tan^{-1} \frac{y_2}{\sqrt{y_1^2 + y_2^2} + y_1}, & \sqrt{y_1^2 + y_2^2} + y_1 > 0, \\ \pi, & y_2 = 0,\ y_1 < 0. \end{cases} \tag{2.3.9}$$

2.4 Frames

A frame is a collection of three unit dimensionless physical vectors, called *axes*, that are pairwise mutually orthogonal. Since each frame vector is a physical vector, the notion of the "location" of the frame is meaningless. In addition, since a frame has no location, it cannot translate, and thus a frame has no velocity or acceleration.

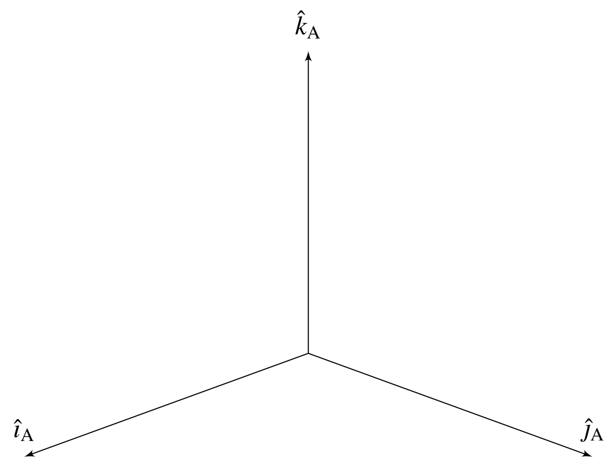


Figure 2.4.1: Right-handed frame F_A with pairwise mutually orthogonal axes $\hat{\imath}_A, \hat{\jmath}_A, \hat{k}_A$.

Nevertheless, it is often useful to associate a *reference point* with a frame. When we do this, we call the reference point the *origin of the frame*, and we draw the frame as if all of the axes were located at the reference point, which may have nonzero velocity and acceleration. Hence, the notion of a "translating frame" refers to the motion of the origin of the frame but not the axes that comprise the frame. A frame has no position, velocity, or acceleration since physical vectors have no location and thus cannot translate.

Let $\mathrm{F_A}$ be a frame with axes $\hat{\imath}_\mathrm{A}, \hat{\jmath}_\mathrm{A}, \hat{k}_\mathrm{A}$. Since these axes are pairwise mutually orthogonal, it follows that

$$\hat{\imath}_\mathrm{A} \cdot \hat{\imath}_\mathrm{A} = \hat{\jmath}_\mathrm{A} \cdot \hat{\jmath}_\mathrm{A} = \hat{k}_\mathrm{A} \cdot \hat{k}_\mathrm{A} = 1, \tag{2.4.1}$$

$$\hat{\imath}_\mathrm{A} \cdot \hat{\jmath}_\mathrm{A} = \hat{\jmath}_\mathrm{A} \cdot \hat{k}_\mathrm{A} = \hat{k}_\mathrm{A} \cdot \hat{\imath}_\mathrm{A} = 0. \tag{2.4.2}$$

The frame $\mathrm{F_A}$ is *right handed* if the labeling of the axes conforms to

$$\hat{\imath}_\mathrm{A} \times \hat{\jmath}_\mathrm{A} = \hat{k}_\mathrm{A}.$$

Consequently,

$$\hat{\jmath}_\mathrm{A} \times \hat{k}_\mathrm{A} = \hat{\imath}_\mathrm{A},$$

$$\hat{k}_\mathrm{A} \times \hat{\imath}_\mathrm{A} = \hat{\jmath}_\mathrm{A}.$$

See Figure 2.4.1. All orthogonal frames in this book are right handed.

For convenience, we write

$$\mathrm{F_A} = \begin{bmatrix} \hat{\imath}_\mathrm{A} & \hat{\jmath}_\mathrm{A} & \hat{k}_\mathrm{A} \end{bmatrix}, \tag{2.4.3}$$

which is a *row vectrix*, as well as its transpose

$$\mathcal{F}_\mathrm{A} \triangleq \mathrm{F}_\mathrm{A}^\mathsf{T} = \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}, \tag{2.4.4}$$

which is a *column vectrix*. Furthermore, we define the *coframe*

$$\mathrm{F}'_\mathrm{A} \triangleq \begin{bmatrix} \hat{\imath}'_\mathrm{A} & \hat{\jmath}'_\mathrm{A} & \hat{k}'_\mathrm{A} \end{bmatrix}, \tag{2.4.5}$$

whose transpose is given by

$$\mathrm{F}'^\mathsf{T}_\mathrm{A} = \mathcal{F}'_\mathrm{A} = \mathrm{F}^{\mathsf{T}\prime}_\mathrm{A} = \begin{bmatrix} \hat{\imath}'_\mathrm{A} \\ \hat{\jmath}'_\mathrm{A} \\ \hat{k}'_\mathrm{A} \end{bmatrix}. \tag{2.4.6}$$

The coframe F'_A is a *row covectrix*, and its transpose $\mathcal{F}'_\mathrm{A}$ is a *column covectrix*. The axes $\hat{\imath}'_\mathrm{A}, \hat{\jmath}'_\mathrm{A}, \hat{k}'_\mathrm{A}$ of the coframe can be viewed as a basis for the space $\mathcal{V}'$ of covectors. Note that the components of a vectrix are physical vectors, whereas the components of a covectrix are physical covectors. More generally, let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors. Then,

$$\begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}, \quad \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}, \tag{2.4.7}$$

$$\begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}' = \begin{bmatrix} \vec{x}' & \vec{y}' & \vec{z}' \end{bmatrix}, \quad \begin{bmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{bmatrix}' = \begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}, \tag{2.4.8}$$

$$\begin{bmatrix} \vec{x}' & \vec{y}' & \vec{z}' \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}, \quad \begin{bmatrix} \vec{x}' \\ \vec{y}' \\ \vec{z}' \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \vec{x}' & \vec{y}' & \vec{z}' \end{bmatrix}. \tag{2.4.9}$$

Vectrices and covectrices are multiplied according to the rules

$$\begin{bmatrix} \vec{x}_1' & \vec{y}_1' & \vec{z}_1' \end{bmatrix} \begin{bmatrix} \vec{x}_2 \\ \vec{y}_2 \\ \vec{z}_2 \end{bmatrix} = \vec{x}_1'\vec{x}_2 + \vec{y}_1'\vec{y}_2 + \vec{z}_1'\vec{z}_2, \tag{2.4.10}$$

$$\begin{bmatrix} \vec{x}_1' \\ \vec{y}_1' \\ \vec{z}_1' \end{bmatrix} \begin{bmatrix} \vec{x}_2 & \vec{y}_2 & \vec{z}_2 \end{bmatrix} = \begin{bmatrix} \vec{x}_1'\vec{x}_2 & \vec{x}_1'\vec{y}_2 & \vec{x}_1'\vec{z}_2 \\ \vec{y}_1'\vec{x}_2 & \vec{y}_1'\vec{y}_2 & \vec{y}_1'\vec{z}_2 \\ \vec{z}_1'\vec{x}_2 & \vec{z}_1'\vec{y}_2 & \vec{z}_1'\vec{z}_2 \end{bmatrix}, \tag{2.4.11}$$

$$\begin{bmatrix} \vec{x}_1 & \vec{y}_1 & \vec{z}_1 \end{bmatrix} \begin{bmatrix} \vec{x}_2' \\ \vec{y}_2' \\ \vec{z}_2' \end{bmatrix} = \vec{x}_1\vec{x}_2' + \vec{y}_1\vec{y}_2' + \vec{z}_1\vec{z}_2'. \tag{2.4.12}$$

In particular,

$$\mathrm{F}_{\mathrm{A}}'\mathcal{F}_{\mathrm{A}} = 3, \quad \mathcal{F}_{\mathrm{A}}'\mathrm{F}_{\mathrm{A}} = I_3, \quad \mathrm{F}_{\mathrm{A}}\mathcal{F}_{\mathrm{A}}' = \vec{I}, \tag{2.4.13}$$

where I_3 is the 3×3 identity matrix and $\vec{I}$ is the physical identity matrix defined in Section 2.8

Let F_{A} be a frame, and let $\vec{x}$ be a physical vector. Then, $\vec{x}\big|_{\mathrm{A}}$ is the physical vector $\vec{x}$ *resolved* in F_{A}. In fact, $\vec{x}\big|_{\mathrm{A}}$ is the *mathematical vector* defined by

$$\vec{x}\Big|_{\mathrm{A}} \triangleq \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \cdot \vec{x} \\ \hat{\jmath}_{\mathrm{A}} \cdot \vec{x} \\ \hat{k}_{\mathrm{A}} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{2.4.14}$$

where x_1, x_2, and x_3 are the *components* of the physical vector $\vec{x}$ resolved in F_{A}. Every physical vector is uniquely specified by resolving it in a frame. In particular, $\vec{x}$ can be reconstructed from $\vec{x}\big|_{\mathrm{A}}$ by means of

$$\vec{x} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}} & \hat{\jmath}_{\mathrm{A}} & \hat{k}_{\mathrm{A}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}. \tag{2.4.15}$$

In other words,

$$\vec{x} = \mathrm{F}_{\mathrm{A}}\left(\vec{x}\Big|_{\mathrm{A}}\right) = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}} \mathcal{F}_{\mathrm{A}}. \tag{2.4.16}$$

A shorthand notation for $\vec{x}\big|_{\mathrm{A}}$ is given by

$$x_{|\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}}. \tag{2.4.17}$$

Fact 2.4.1. Let F_{A} be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{x} = \vec{y} \tag{2.4.18}$$

if and only if

$$\vec{x}\Big|_{\mathrm{A}} = \vec{y}\Big|_{\mathrm{A}}. \tag{2.4.19}$$

The physical covector $\vec{x}'$ is resolved according to

$$\vec{x}'\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}, \tag{2.4.20}$$

and vectrices and covectrices are resolved as

$$\begin{bmatrix}\vec{x} & \vec{y} & \vec{z}\end{bmatrix}\Big|_{\mathrm{A}} = \begin{bmatrix}\vec{x}\Big|_{\mathrm{A}} & \vec{y}\Big|_{\mathrm{A}} & \vec{z}\Big|_{\mathrm{A}}\end{bmatrix}, \tag{2.4.21}$$

$$\begin{bmatrix}\vec{x}'\\ \vec{y}'\\ \vec{z}'\end{bmatrix}\Bigg|_{\mathrm{A}} = \begin{bmatrix}\vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\\ \vec{y}\Big|_{\mathrm{A}}^{\mathsf{T}}\\ \vec{z}\Big|_{\mathrm{A}}^{\mathsf{T}}\end{bmatrix}. \tag{2.4.22}$$

In particular,

$$\mathrm{F}_{\mathrm{A}}|_{\mathrm{A}} = \mathcal{F}'_{\mathrm{A}}\Big|_{\mathrm{A}} = \mathrm{F}_{\mathrm{A}}^{\mathsf{T}\prime}\Big|_{\mathrm{A}} = I_3. \tag{2.4.23}$$

However, $\mathrm{F}_{\mathrm{A}}^{\mathsf{T}}\Big|_{\mathrm{A}}$ and $\mathrm{F}'_{\mathrm{A}}\Big|_{\mathrm{A}}$ are not defined.

Let F_{A} be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors, where

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \quad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}. \tag{2.4.24}$$

Then, using the geometric fact that $x_1\hat{\imath}_{\mathrm{A}}$ and $x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}$ are mutually orthogonal, Fact 2.1.3 implies that

$$\begin{aligned}|\vec{x}|^2 &= |x_1\hat{\imath}_{\mathrm{A}}|^2 + |x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}|^2 + 2(x_1\hat{\imath}_{\mathrm{A}})'(x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}})\\ &= x_1^2 + |x_2\hat{\jmath}_{\mathrm{A}}|^2 + |x_3\hat{k}_{\mathrm{A}}|^2 + 2(x_2\hat{\jmath}_{\mathrm{A}})'x_3\hat{k}_{\mathrm{A}}\\ &= x_1^2 + x_2^2 + x_3^2.\end{aligned} \tag{2.4.25}$$

Therefore,

$$\begin{aligned}\vec{x}\cdot\vec{y} &= (x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}})\cdot(y_1\hat{\imath}_{\mathrm{A}} + y_2\hat{\jmath}_{\mathrm{A}} + y_3\hat{k}_{\mathrm{A}})\\ &= \tfrac{1}{2}[(\vec{x}+\vec{y})'(\vec{x}+\vec{y}) - \vec{x}'\vec{x} - \vec{y}'\vec{y}]\\ &= \tfrac{1}{2}[(x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 - (x_1^2+x_2^2+x_3^2) - (y_1^2+y_2^2+y_3^2)]\\ &= x_1y_1 + x_2y_2 + x_3y_3\\ &= \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\vec{y}\Big|_{\mathrm{A}}.\end{aligned} \tag{2.4.26}$$

Note that

$$\vec{x}'\vec{y} = \vec{x}\cdot\vec{y} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\vec{y}\Big|_{\mathrm{A}} = \vec{x}'\Big|_{\mathrm{A}}\vec{y}\Big|_{\mathrm{A}}. \tag{2.4.27}$$

For $x = [x_1 \ \ x_2 \ \ x_3]^\mathsf{T} \in \mathbb{R}^3$, define

$$\|x\| \triangleq \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{2.4.28}$$

It thus follows that

$$|\vec{x}| = \sqrt{\vec{x}'\vec{x}} = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{\vec{x}\big|_\mathrm{A}^\mathsf{T}\,\vec{x}\big|_\mathrm{A}} = \left\|\vec{x}\big|_\mathrm{A}\right\|. \tag{2.4.29}$$

Let $x, y \in \mathbb{R}^3$, let $\mathrm{F_A}$ be a frame, and define $\vec{x} \triangleq \mathrm{F_A}x$ and $\vec{y} \triangleq \mathrm{F_A}y$. Then, the cross product of x and y is defined by

$$x \times y = \vec{x}\big|_\mathrm{A} \times \vec{y}\big|_\mathrm{A} \triangleq (\vec{x}\times\vec{y})\big|_\mathrm{A}. \tag{2.4.30}$$

Therefore,

$$\begin{aligned}
\vec{x}\big|_\mathrm{A} \times \vec{y}\big|_\mathrm{A} &= (\vec{x}\times\vec{y})\big|_\mathrm{A}\\
&= [(x_1\hat{\imath}_\mathrm{A} + x_2\hat{\jmath}_\mathrm{A} + x_3\hat{k}_\mathrm{A}) \times (y_1\hat{\imath}_\mathrm{A} + y_2\hat{\jmath}_\mathrm{A} + y_3\hat{k}_\mathrm{A})]\big|_\mathrm{A}\\
&= [(x_2y_3 - x_3y_2)\hat{\imath}_\mathrm{A} - (x_1y_3 - x_3y_1)\hat{\jmath}_\mathrm{A} + (x_1y_2 - x_2y_1)\hat{k}_\mathrm{A}]\big|_\mathrm{A}\\
&= \begin{bmatrix} x_2y_3 - x_3y_2\\ x_3y_1 - x_1y_3\\ x_1y_2 - x_2y_1 \end{bmatrix}\\
&= \begin{bmatrix} 0 & -x_3 & x_2\\ x_3 & 0 & -x_1\\ -x_2 & x_1 & 0 \end{bmatrix}\begin{bmatrix} y_1\\ y_2\\ y_3 \end{bmatrix}.
\end{aligned} \tag{2.4.31}$$

Defining the cross-product matrix

$$\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix}^\times \triangleq \begin{bmatrix} 0 & -x_3 & x_2\\ x_3 & 0 & -x_1\\ -x_2 & x_1 & 0 \end{bmatrix}, \tag{2.4.32}$$

which is a 3×3 skew-symmetric matrix, it follows that

$$\vec{x}\big|_\mathrm{A} \times \vec{y}\big|_\mathrm{A} = \vec{x}\big|_\mathrm{A}^\times\,\vec{y}\big|_\mathrm{A}. \tag{2.4.33}$$

Finally, we have the formal identity

$$\vec{x}\times\vec{y} = \det\begin{bmatrix} \hat{\imath}_\mathrm{A} & \hat{\jmath}_\mathrm{A} & \hat{k}_\mathrm{A}\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3 \end{bmatrix}. \tag{2.4.34}$$

Fact 2.4.2. Let $\vec{x}, \vec{y}, \vec{z}$ be physical vectors. Then,

$$\vec{x}\times(\vec{y}\times\vec{z}) = (\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z}, \tag{2.4.35}$$

$$(\vec{x}\times\vec{y})\times\vec{z} = (\vec{x}\cdot\vec{z})\vec{y} - (\vec{y}\cdot\vec{z})\vec{x}, \tag{2.4.36}$$

$$(\vec{x}\times\vec{y})\cdot\vec{z} = \vec{x}\cdot(\vec{y}\times\vec{z}). \tag{2.4.37}$$

Furthermore, let $\mathrm{F_A}$ be a frame. Then,

$$(\vec{x}\times\vec{y})\cdot\vec{z} = \left(\vec{x}\big|_\mathrm{A} \times \vec{y}\big|_\mathrm{A}\right)^\mathsf{T}\vec{z}\big|_\mathrm{A} = \det\left[\vec{x}\big|_\mathrm{A} \quad \vec{y}\big|_\mathrm{A} \quad \vec{z}\big|_\mathrm{A}\right]. \tag{2.4.38}$$

If a frame rotates according to the rotation of a rigid body, then the frame is a *body-fixed frame*. (Since a frame has no physical location, this terminology should not be interpreted literally.) The axes of a body-fixed frame can thus be painted on a rigid body. The origin of a body-fixed frame is usually taken to be a point in the body. The orientation of a rigid body is thus specified by the orientation of a body-fixed frame.

The physical vectors $\vec{x}, \vec{y}, \vec{z}$ are *linearly independent* if the only the real numbers α, β, γ that satisfy

$$\alpha\vec{x} + \beta\vec{y} + \gamma\vec{z} = \vec{0} \tag{2.4.39}$$

are $\alpha = \beta = \gamma = 0$. Note that, if $\vec{x}, \vec{y}, \vec{z}$ are linear independent, then they are nonzero. Furthermore, $\vec{x}, \vec{y}, \vec{z}$ are *linearly dependent* if they are not linearly independent; intuitively, this is equivalent to saying that $\vec{x}, \vec{y}, \vec{z}$ lie in a single plane when their tails are placed at a single point.

Now, let $\mathrm{F_A}$ be a frame. Then, it can be seen that the physical vectors $\vec{x}, \vec{y}, \vec{z}$ are linearly independent if and only if the only the real numbers α, β, γ that satisfy

$$\alpha\vec{x}\Big|_{\mathrm{A}} + \beta\vec{y}\Big|_{\mathrm{A}} + \gamma\vec{z}\Big|_{\mathrm{A}} = \vec{0} \tag{2.4.40}$$

are $\alpha = \beta = \gamma = 0$.

Fact 2.4.3. The physical vectors $\vec{x}, \vec{y}, \vec{z}$ are linearly independent if and only if

$$\vec{x} \cdot (\vec{y} \times \vec{z}) \neq 0. \tag{2.4.41}$$

Proof. Let $\mathrm{F_A}$ be a frame. Then, it follows from (2.4.38) that (2.4.41) is equivalent to

$$\det\left[\vec{x}\Big|_{\mathrm{A}} \quad \vec{y}\Big|_{\mathrm{A}} \quad \vec{z}\Big|_{\mathrm{A}}\right] \neq 0,$$

which is equivalent to the fact that the mathematical vectors $\vec{x}\Big|_{\mathrm{A}}, \vec{y}\Big|_{\mathrm{A}}, \vec{z}\Big|_{\mathrm{A}}$ are linearly independent, and thus the physical vectors $\vec{x}, \vec{y}, \vec{z}$, are linearly independent. □

Fact 2.4.4. The physical vectors $\vec{x}, \vec{y}, \vec{z}$ are linearly independent if and only if, for every physical vector $\vec{w}$, there exist unique real numbers α, β, γ such that

$$\vec{w} = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}. \tag{2.4.42}$$

2.5 Position Vectors, Lines, and Planes

Let x and y be points. Then, the *position of* y *relative to* x is denoted by $\vec{r}_{y/x}$. Note that $\vec{r}_{x/y} = -\vec{r}_{y/x}$. Although the points x and y have physical locations, the position vector $\vec{r}_{y/x}$ has no physical location. This property of physical vectors is evident in the following fundamental property of three-dimensional space.

Fact 2.5.1. Let x, y, z be points. Then, there exists a point w such that $\vec{r}_{y/x} = \vec{r}_{w/z}$.

If, in addition, z is a point, then vector addition yields the "slash and split" identity

$$\vec{r}_{y/x} = \vec{r}_{y/z} + \vec{r}_{z/x}. \tag{2.5.1}$$

This identity can be resolved in $\mathrm{F_A}$ by writing

$$\vec{r}_{y/x}\Big|_{\mathrm{A}} = \vec{r}_{y/z}\Big|_{\mathrm{A}} + \vec{r}_{z/x}\Big|_{\mathrm{A}}. \tag{2.5.2}$$

Equivalently, we can write

$$r_{y/x|\mathrm{A}} = r_{y/z|\mathrm{A}} + r_{z/x|\mathrm{A}}. \tag{2.5.3}$$

Let x be a point, and let $\hat{n}$ be a unit dimensionless physical vector. Then, the *line through x parallel to $\hat{n}$* is the set of points defined by

$$\mathcal{L}(x,\hat{n}) \triangleq \{y \in \mathbb{R}^3\colon \text{there exists } \alpha \in \mathbb{R} \text{ such that } \vec{r}_{y/x} = \alpha\hat{n}\}. \tag{2.5.4}$$

Now, let $\hat{n}_1$ and $\hat{n}_2$ be linearly independent unit dimensionless physical vectors. Then, the *plane through x parallel to $\hat{n}$* is the set of points defined by

$$\mathcal{P}(x,\hat{n}_1,\hat{n}_2) \triangleq \{y \in \mathbb{R}^3\colon \text{there exist } \alpha,\beta \in \mathbb{R} \text{ such that } \vec{r}_{y/x} = \alpha\hat{n}_1 + \beta\hat{n}_2\}. \tag{2.5.5}$$

Just as a point has a physical location, so do lines and planes. For points x and y, $[x,y]$ denotes the line segment connecting x and y.

2.6 Physical Matrices

Let $\vec{x}_1,\ldots,\vec{x}_n$ and $\vec{y}_1,\ldots,\vec{y}_n$ be physical vectors. Then,

$$\vec{M} \triangleq \sum_{i=1}^{n} \vec{x}_i\vec{y}_i^{\,\prime} \tag{2.6.1}$$

is a *physical matrix*. A physical matrix is a second-order component-free tensor. The zero physical matrix is denoted by $\vec{0}$ or just 0. Physical matrices operate on physical vectors according to the rules given below.

Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors, and define

$$\vec{M} \triangleq \vec{x}\vec{y}^{\,\prime}. \tag{2.6.2}$$

Then, $\vec{M}$ satisfies the multiplication rules

$$\vec{M}\vec{z} = (\vec{x}\vec{y}^{\,\prime})\vec{z} \triangleq \vec{x}\vec{y}\cdot\vec{z} = (\vec{y}\cdot\vec{z})\vec{x}, \tag{2.6.3}$$

$$\vec{z}^{\,\prime}\vec{M} = \vec{z}^{\,\prime}(\vec{x}\vec{y}^{\,\prime}) = (\vec{z}\cdot\vec{x})\vec{y}^{\,\prime}. \tag{2.6.4}$$

Let $\vec{w}$ and $\vec{v}$ be physical vectors, and define

$$\vec{N} \triangleq \vec{w}\vec{v}^{\,\prime}. \tag{2.6.5}$$

Then,

$$\vec{M}\vec{N} = \vec{M}\vec{w}\vec{v}^{\,\prime} = \vec{x}(\vec{y}\cdot\vec{w})\vec{v}^{\,\prime} = (\vec{y}\cdot\vec{w})\vec{x}\vec{v}^{\,\prime}, \tag{2.6.6}$$

$$\vec{M}\vec{N}\vec{z} = (\vec{x}\vec{y}^{\,\prime})(\vec{w}\vec{v}^{\,\prime})\vec{z} = \vec{x}(\vec{y}\cdot\vec{w})(\vec{v}\cdot\vec{z}) = (\vec{y}\cdot\vec{w})(\vec{v}\cdot\vec{z})\vec{x}. \tag{2.6.7}$$

Let $\vec{x}$ and $\vec{y}$ be physical vectors, and define

$$\vec{M} \triangleq \vec{x}\vec{y}'. \tag{2.6.8}$$

Then, the *coform* $\vec{M}'$ of $\vec{M}$ is defined by

$$\vec{M}' \triangleq \vec{y}\vec{x}', \tag{2.6.9}$$

which is also a physical matrix. Furthermore, let $\vec{N}$ and $\vec{L}$ be physical matrices. Then,

$$(\vec{N} + \vec{L})' = \vec{N}' + \vec{L}', \tag{2.6.10}$$

$$(\vec{N}\vec{L})' = \vec{L}'\vec{N}'. \tag{2.6.11}$$

Finally, if $\vec{z}$ is a physical vector, then

$$(\vec{M}\vec{z})' = \vec{z}'\vec{M}'. \tag{2.6.12}$$

Let $\vec{M}$ be a physical matrix, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{M}\vec{x}\vec{y}' = \vec{M}(\vec{x}\vec{y}') = (\vec{M}\vec{x})\vec{y}'. \tag{2.6.13}$$

The physical matrix $\vec{M}$ is *symmetric* if $\vec{M}' = \vec{M}$ and *skew symmetric* if $\vec{M}' = -\vec{M}$.

Fact 2.6.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and define

$$\vec{M} \triangleq \vec{x}\vec{y}' - \vec{y}\vec{x}'. \tag{2.6.14}$$

Then, $\vec{M}$ is skew symmetric.

Let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\mathrm{F_A}$ be a frame. Then, we define

$$(\vec{x}\vec{y}')\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}} \vec{y}\Big|_{\mathrm{A}}^{\mathsf{T}}. \tag{2.6.15}$$

Note that $(\vec{x}\vec{y}')\Big|_{\mathrm{A}}$ is a 3×3 matrix whose rank is 1 if and only if $\vec{x}$ and $\vec{y}$ are nonzero, and whose rank is 0 if and only if either $\vec{x}$ or $\vec{y}$ is zero. Furthermore, if $\vec{w}$ and $\vec{z}$ are physical vectors, then

$$(\vec{x}\vec{y}' + \vec{w}\vec{z}')\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}} \vec{y}\Big|_{\mathrm{A}}^{\mathsf{T}} + \vec{w}\Big|_{\mathrm{A}} \vec{z}\Big|_{\mathrm{A}}^{\mathsf{T}}. \tag{2.6.16}$$

Fact 2.6.2. Let $\vec{M}$ be a physical matrix, let $\vec{z}$ be a physical vector, and let $\mathrm{F_A}$ be a frame. Then,

$$(\vec{M}\vec{z})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} \vec{z}\Big|_{\mathrm{A}}. \tag{2.6.17}$$

Proof. Assuming that $\vec{M}$ has the form (2.6.1),

$$(\vec{M}\vec{z})\Big|_{\mathrm{A}} = \sum_{i=1}^{n}(\vec{y}_i \cdot \vec{z})\vec{x}_i\Bigg|_{\mathrm{A}} = \sum_{i=1}^{n} \vec{y}_i\Big|_{\mathrm{A}}^{\mathsf{T}} \vec{z}\Big|_{\mathrm{A}} \vec{x}_i\Big|_{\mathrm{A}} = \sum_{i=1}^{n} \vec{x}_i\Big|_{\mathrm{A}} \vec{y}_i\Big|_{\mathrm{A}}^{\mathsf{T}} \vec{z}\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} \vec{z}\Big|_{\mathrm{A}}. \qquad \square$$

The following result is analogous to Fact 2.4.1.

Fact 2.6.3. Let $\vec{M}$ and $\vec{N}$ be physical matrices. Then,

$$\vec{M} = \vec{N} \tag{2.6.18}$$

if and only if

$$\vec{M}\Big|_{\mathrm{A}} = \vec{N}\Big|_{\mathrm{A}}. \tag{2.6.19}$$

Fact 2.6.4. Let $\vec{M}$ and $\vec{N}$ be physical matrices. Then,

$$\vec{M} = \vec{N} \tag{2.6.20}$$

if and only if, for all physical vectors $\vec{x}$,

$$\vec{M}\vec{x} = \vec{N}\vec{x}. \tag{2.6.21}$$

Fact 2.6.5. Let $\mathrm{F_A}$ be a frame, let $\vec{M}$ and $\vec{N}$ be physical matrices, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{M}'\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}}^{\mathsf{T}}, \tag{2.6.22}$$

$$(\vec{M}+\vec{N})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} + \vec{N}\Big|_{\mathrm{A}}, \tag{2.6.23}$$

$$(\vec{M}\vec{N})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}}\,\vec{N}\Big|_{\mathrm{A}}, \tag{2.6.24}$$

$$(\vec{M}\vec{x})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}}\,\vec{x}\Big|_{\mathrm{A}}, \tag{2.6.25}$$

$$(\vec{x}'\vec{M})\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\,\vec{M}\Big|_{\mathrm{A}}, \tag{2.6.26}$$

$$\vec{x}'\vec{M}\vec{y} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\,\vec{M}\Big|_{\mathrm{A}}\,\vec{y}\Big|_{\mathrm{A}}, \tag{2.6.27}$$

$$\vec{M}\Big|_{\mathrm{A}} = \begin{bmatrix} \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} & \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} & \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} & \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} & \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} & \hat{k}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} & \hat{k}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}} \end{bmatrix}. \tag{2.6.28}$$

It can be seen that the coform of a physical vector or a physical matrix is analogous to the transpose of a mathematical vector or a mathematical matrix.

The following definition concerns eigenvalues and eigenvectors of physical matrices.

Definition 2.6.6. Let $\vec{M}$ be a physical matrix, let $\vec{x}$ be a nonzero dimensionless physical vector, let λ be a complex number, and assume that

$$\vec{M}\vec{x} = \lambda\vec{x}. \tag{2.6.29}$$

Then, λ is an *eigenvalue* of $\vec{M}$, and $\vec{x}$ is an *eigenvector* of $\vec{M}$ associated with λ.

The following result shows that the eigenvalues and eigenvectors of a physical matrix correspond to the eigenvalues and eigenvectors of 3×3 matrices.

Fact 2.6.7. Let $\vec{M}$ be a physical matrix, let λ be an eigenvalue of $\vec{M}$, let $\vec{x}$ be a eigenvector of $\vec{M}$ associated with λ, and let $\mathrm{F_A}$ be a frame. Then, λ is an eigenvalue of $\vec{M}\Big|_{\mathrm{A}}$, and $\vec{x}\Big|_{\mathrm{A}}$ is an eigenvector of $\vec{M}\Big|_{\mathrm{A}}$ associated with λ.

2.7 Physical Projector Matrices

Let $\vec{y}$ be a nonzero physical vector. Then, the *physical projector matrix* $\vec{P}_{\vec{y}}$ *onto the line spanned by* $\vec{y}$ is defined by

$$\vec{P}_{\vec{y}} \triangleq \frac{1}{|\vec{y}|^2}\vec{y}\vec{y}', \tag{2.7.1}$$

and the *physical projector matrix* $\vec{P}_{\vec{y}}$ *onto a plane orthogonal to* $\vec{y}$ is defined by

$$\vec{P}_{\vec{y}\perp} \triangleq \vec{I} - \vec{P}_{\vec{y}}. \tag{2.7.2}$$

Note that

$$\vec{P}_{\vec{y}}^{\,2} = \vec{P}_{\vec{y}}, \tag{2.7.3}$$

$$\vec{P}_{\vec{y}\perp}^{\,2} = \vec{P}_{\vec{y}\perp}. \tag{2.7.4}$$

If $\vec{y}$ has unit length, then

$$\vec{P}_{\hat{y}} = \hat{y}\hat{y}', \tag{2.7.5}$$

$$\vec{P}_{\hat{y}\perp} = \vec{I} - \hat{y}\hat{y}'. \tag{2.7.6}$$

Let $\vec{y}$ be a nonzero physical vector, and let $\vec{x}$ be a physical vector. Then, the *projection of* $\vec{x}$ *onto the line spanned by* $\vec{y}$ is given by

$$\vec{P}_{\vec{y}}\vec{x} = \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y}, \tag{2.7.7}$$

and the *projection of* $\vec{x}$ *onto the plane that is orthogonal to* $\vec{y}$ is given by

$$\vec{P}_{\vec{y}\perp}\vec{x} = (\vec{I} - \vec{P}_{\vec{y}})\vec{x} = \vec{x} - \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y}. \tag{2.7.8}$$

Note that $\vec{P}_{\vec{y}}\vec{y} = \vec{y}$ and $\vec{P}_{\vec{y}\perp}\vec{y} = \vec{0}$.

Fact 2.7.1. Let $\vec{y}$ be a nonzero physical vector, and let $\vec{x}$ be a physical vector. Then,

$$|\vec{P}_{\vec{y}}\vec{x}| = \frac{|\vec{x}\cdot\vec{y}|}{|\vec{y}|} = |\vec{x}||\cos\theta_{\vec{y}/\vec{x}}|. \tag{2.7.9}$$

Figure 2.7.1 illustrates the physical projector matrix.

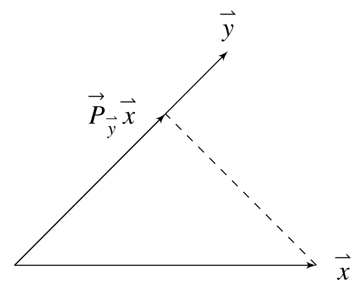


Figure 2.7.1: The projection $\vec{P}_{\vec{y}}\vec{x}$ of $\vec{x}$ onto $\vec{y}$.

Now, let $\vec{y}$ and $\vec{z}$ be nonzero physical vectors that are mutually orthogonal. Then, the *physical projector matrix* $\vec{P}_{\vec{y},\vec{z}}$ *onto the plane spanned by* $\vec{y}$ *and* $\vec{z}$ is defined by

$$\vec{P}_{\vec{y},\vec{z}} \triangleq \vec{P}_{\vec{y}} + \vec{P}_{\vec{z}}. \tag{2.7.10}$$

For each physical vector $\vec{x}$, the *projection of* $\vec{x}$ *onto the plane spanned by* $\vec{y}$ *and* $\vec{z}$ is the physical vector $\vec{P}_{\vec{y},\vec{z}}\vec{x}$ given by

$$\vec{P}_{\vec{y},\vec{z}}\vec{x} = \vec{P}_{\vec{y}}\vec{x} + \vec{P}_{\vec{z}}\vec{x} = \frac{\vec{x}\cdot\vec{y}}{|\vec{y}|^2}\vec{y} + \frac{\vec{x}\cdot\vec{z}}{|\vec{z}|^2}\vec{z}. \tag{2.7.11}$$

If $\vec{y}$ and $\vec{z}$ have unit length, then

$$\vec{P}_{\hat{y},\hat{z}} = \hat{y}\hat{y}' + \hat{z}\hat{z}'. \tag{2.7.12}$$

Finally, if $\vec{y}$ and $\vec{z}$ are not mutually orthogonal, then $\vec{y} - P_{\vec{z}}\vec{y}$ and $\vec{y}$ are mutually orthogonal, and we define

$$\vec{P}_{\vec{y},\vec{z}} \triangleq \vec{P}_{\vec{y}-\vec{P}_{\vec{z}}\vec{y}} + \vec{P}_{\vec{z}}. \tag{2.7.13}$$

Problem 2.24.7 shows that this definition does not depend on the order of $\vec{y}$ and $\vec{z}$.

2.8 Physical Rotation Matrices

Let $\mathrm{F_A}$ be a frame. Then, the *physical identity matrix* $\vec{I}$ is defined by

$$\vec{I} \triangleq \hat{\imath}_\mathrm{A}\hat{\imath}_\mathrm{A}' + \hat{\jmath}_\mathrm{A}\hat{\jmath}_\mathrm{A}' + \hat{k}_\mathrm{A}\hat{k}_\mathrm{A}' = \mathrm{F_A}\mathcal{F}_\mathrm{A}'. \tag{2.8.1}$$

The following result shows that $\vec{I}$ is independent of the choice of frame in (2.8.1). Let I_3 denote the 3×3 identity matrix, and let e_i denote the ith column of I_3.

Fact 2.8.1. Let $\mathrm{F_A}$ be a frame, and define $\vec{I}$ by (2.8.1). Then, for all physical vectors $\vec{x}$,

$$\vec{I}\vec{x} = \vec{x}, \tag{2.8.2}$$

and, for all physical covectors $\vec{x}'$,

$$\vec{x}'\vec{I} = \vec{x}'. \tag{2.8.3}$$

Now, let $\mathrm{F_B}$ be a frame. Then,

$$\vec{I}\Big|_{\mathrm{B}} = I_3. \tag{2.8.4}$$

Proof. Writing $\vec{x} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}$, it follows that

$$\vec{I}\vec{x} = (\hat{\imath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\hat{k}'_{\mathrm{A}})(x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}) = \vec{x}.$$

Consequently,

$$\vec{I}\Big|_{\mathrm{B}}\,\vec{x}\Big|_{\mathrm{B}} = \vec{x}\Big|_{\mathrm{B}}.$$

Therefore,

$$\begin{aligned}\vec{I}\Big|_{\mathrm{B}} &= \vec{I}\Big|_{\mathrm{B}} I_3 = \begin{bmatrix}\vec{I}\Big|_{\mathrm{B}} e_1 & \vec{I}\Big|_{\mathrm{B}} e_2 & \vec{I}\Big|_{\mathrm{B}} e_3\end{bmatrix} = \begin{bmatrix}(\vec{I}\hat{\imath}_{\mathrm{B}})\Big|_{\mathrm{B}} & (\vec{I}\hat{\jmath}_{\mathrm{B}})\Big|_{\mathrm{B}} & (\vec{I}\hat{k}_{\mathrm{B}})\Big|_{\mathrm{B}}\end{bmatrix}\\ &= \begin{bmatrix}\hat{\imath}_{\mathrm{B}}|_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{B}} & \hat{k}_{\mathrm{B}}|_{\mathrm{B}}\end{bmatrix} = \begin{bmatrix}e_1 & e_2 & e_3\end{bmatrix} = I_3.\end{aligned}$$

□

Let $\vec{M}$ and $\vec{N}$ be physical matrices. If $\vec{M}\vec{N} = \vec{I}$, then we define

$$\vec{M}^{-1} \triangleq \vec{N}. \tag{2.8.5}$$

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, the *physical rotation matrix* $\vec{R}_{\mathrm{B/A}}$ is defined by

$$\vec{R}_{\mathrm{B/A}} \triangleq \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{A}}. \tag{2.8.6}$$

Note that

$$\vec{R}_{\mathrm{B/A}} = \begin{bmatrix}\hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}}\end{bmatrix}\begin{bmatrix}\hat{\imath}'_{\mathrm{A}}\\ \hat{\jmath}'_{\mathrm{A}}\\ \hat{k}'_{\mathrm{A}}\end{bmatrix} = \mathrm{F_B}\mathcal{F}'_{\mathrm{A}}, \tag{2.8.7}$$

$$\vec{R}_{\mathrm{A/A}} = \mathrm{F_A}\mathcal{F}'_{\mathrm{A}} = \vec{I}. \tag{2.8.8}$$

The physical matrix $\vec{R}$ is a *physical rotation matrix* if there exist frames $\mathrm{F_A}$ and $\mathrm{F_B}$ such that $\vec{R} = \vec{R}_{\mathrm{B/A}}$. The following result shows that $\vec{R}_{\mathrm{B/A}}$ rotates $\mathrm{F_A}$ to $\mathrm{F_B}$.

Fact 2.8.2. Let F_A and F_B be frames. Then,

$$\hat{\imath}_B = \vec{R}_{B/A}\hat{\imath}_A, \tag{2.8.9}$$

$$\hat{\jmath}_B = \vec{R}_{B/A}\hat{\jmath}_A, \tag{2.8.10}$$

$$\hat{k}_B = \vec{R}_{B/A}\hat{k}_A. \tag{2.8.11}$$

Furthermore,

$$\vec{R}_{B/A} = \vec{R}'_{A/B} \tag{2.8.12}$$

$$\vec{R}_{B/A}\vec{R}_{A/B} = \vec{I}, \tag{2.8.13}$$

$$\vec{R}_{B/A} = \vec{R}^{-1}_{A/B} = \vec{R}'_{A/B}. \tag{2.8.14}$$

We thus have

$$\begin{aligned} F_B &= \begin{bmatrix} \hat{\imath}_B & \hat{\jmath}_B & \hat{k}_B \end{bmatrix} \\ &= \begin{bmatrix} \vec{R}_{B/A}\hat{\imath}_A & \vec{R}_{B/A}\hat{\jmath}_A & \vec{R}_{B/A}\hat{k}_A \end{bmatrix} \\ &= \vec{R}_{B/A}\begin{bmatrix} \hat{\imath}_A & \hat{\jmath}_A & \hat{k}_A \end{bmatrix} \\ &= \vec{R}_{B/A}F_A. \end{aligned} \tag{2.8.15}$$

Since $\vec{R}^{-1}_{A/B} = \vec{R}'_{A/B}$, it follows that $\vec{R}_{A/B}$ is an *orthogonal physical matrix*.

Fact 2.8.3. Let F_A and F_B be frames. Then, there exists a unique physical rotation matrix $\vec{R}$ such that $F_B = \vec{R}F_A$. In particular, $\vec{R} = \vec{R}_{B/A}$.

The following result concerns the rotation of a physical position vector.

Fact 2.8.4. Let x and y be points, and let $\vec{R}$ be a physical rotation matrix. Then, there exists a point z such that $\vec{r}_{z/x} = \vec{R}\vec{r}_{y/x}$.

Proof. Since $\vec{R}\vec{r}_{y/x}$ is a position vector, the result follows from Fact 2.5.1.

In Fact 2.8.4, note that the position vectors $\vec{r}_{y/x}$ and $\vec{r}_{z/x}$ have the same tail, namely, the point x. Consequently, the physical rotation matrix can be viewed as rotating the physical position vector $\vec{r}_{y/x}$ around the point x so that the tip moves from y to z while the tail remains at x.

2.9 Physical Cross Product Matrix

Let $\vec{x}$ be a physical vector. Then, for all physical vectors $\vec{y}$, the *physical cross product matrix* $\vec{M} \triangleq \vec{x}^{\times}$ is defined by

$$\vec{M}\vec{y} = \vec{x}^{\times}\vec{y} \triangleq \vec{x} \times \vec{y}. \tag{2.9.1}$$

Fact 2.9.1. Let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ be a frame. Then,

$$\vec{x}^{\times}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times} = \begin{bmatrix} 0 & -\hat{k}_{\mathrm{A}}\cdot\vec{x} & \hat{\jmath}_{\mathrm{A}}\cdot\vec{x} \\ \hat{k}_{\mathrm{A}}\cdot\vec{x} & 0 & -\hat{\imath}_{\mathrm{A}}\cdot\vec{x} \\ -\hat{\jmath}_{\mathrm{A}}\cdot\vec{x} & \hat{\imath}_{\mathrm{A}}\cdot\vec{x} & 0 \end{bmatrix}, \tag{2.9.2}$$

$$\vec{x}^{\times} = (\hat{\imath}_{\mathrm{A}}\cdot\vec{x})(\hat{k}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}}) + (\hat{\jmath}_{\mathrm{A}}\cdot\vec{x})(\hat{\imath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}}) + (\hat{k}_{\mathrm{A}}\cdot\vec{x})(\hat{\jmath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}}). \tag{2.9.3}$$

Proof. Let $\vec{y}$ be a physical vector. We thus have

$$\vec{x}^{\times}\Big|_{\mathrm{A}}\vec{y}\Big|_{\mathrm{A}} = (\vec{x}^{\times}\vec{y})\Big|_{\mathrm{A}} = (\vec{x}\times\vec{y})\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}\times\vec{y}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times}\vec{y}\Big|_{\mathrm{A}}.$$

It thus follows from Fact 2.6.4 that $\vec{x}^{\times}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{A}}^{\times}$. The second equality in (2.9.2) follows from (2.4.32). Finally, resolving the right-hand side of (2.9.3) yields the matrix in (2.9.2), and thus the second statement follows from Fact 2.6.3. □

Defining

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x}\Big|_{\mathrm{A}}, \tag{2.9.4}$$

it follows from (2.9.2) that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{\times} = \vec{x}\Big|_{\mathrm{A}}^{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{2.9.5}$$

Fact 2.9.2. Let $\vec{x}$ be a physical vector. Then,

$$\vec{x}^{\times\prime} = -\vec{x}^{\times}, \tag{2.9.6}$$

$$\vec{x}^{\times}\vec{x} = 0, \tag{2.9.7}$$

$$\vec{x}'\vec{x}^{\times} = 0, \tag{2.9.8}$$

$$\vec{x}^{\times 2} = \vec{x}\vec{x}' - |\vec{x}|^2\vec{I}, \tag{2.9.9}$$

$$(\vec{I} + \vec{x}^{\times})^{-1} = \frac{1}{1+|\vec{x}|^2}(\vec{I} + \vec{x}\vec{x}' - \vec{x}^{\times}) \tag{2.9.10}$$

$$= \vec{I} + \frac{1}{1+|\vec{x}|^2}(\vec{x}^{\times 2} - \vec{x}^{\times}). \tag{2.9.11}$$

Now, let $\mathrm{F_A}$ be a frame, and define $x \triangleq \vec{x}\Big|_{\mathrm{A}}$. Then,

$$x^{\times\mathsf{T}} = -x^{\times}, \tag{2.9.12}$$

$$x^{\times}x = 0, \tag{2.9.13}$$

$$x^{\mathsf{T}}x^{\times} = 0, \tag{2.9.14}$$

$$x^{\times 2} = xx^{\mathsf{T}} - x^{\mathsf{T}}xI_3, \tag{2.9.15}$$

$$(I_3 + x^\times)^{-1} = \frac{1}{1+\|x\|^2}(I_3 + xx^\mathsf{T} - x^\times) \tag{2.9.16}$$

$$= I_3 + \frac{1}{1+\|x\|^2}(x^{\times 2} - x^\times). \tag{2.9.17}$$

Equation (2.9.6) shows that the physical cross product matrix $\vec{x}^\times$ is skew symmetric. The following result provides the converse result, namely, that if the physical matrix $\vec{M}$ is skew symmetric, then it must be a physical cross product matrix.

Fact 2.9.3. Let $\vec{M}$ be a physical matrix, and assume that $\vec{M}$ is skew symmetric. Then, there exists a physical vector $\vec{x}$ such that $\vec{M} = \vec{x}^\times$.

Proof. Let $\mathrm{F_A}$ be a frame, and define $M \triangleq \vec{M}\big|_\mathrm{A}$. Furthermore, define $\vec{x} = -M_{(2,3)}\hat{\imath}_\mathrm{A} + M_{(1,3)}\hat{\jmath}_\mathrm{A} - M_{(1,2)}\hat{k}_\mathrm{A}$. Then, $\vec{M}\big|_\mathrm{A} = \vec{x}^\times\big|_\mathrm{A}$, and thus $\vec{M} = \vec{x}^\times$. □

Fact 2.9.4. Let $\vec{x}$ be a physical vector, let α and β be real numbers, and assume that either $\alpha \neq 0$ or $\beta|\vec{x}|^2 \neq 1$. Then,

$$(\vec{I} + \alpha\vec{x}^\times + \beta\vec{x}^{\times 2})^{-1} = \vec{I} - \frac{\alpha}{\alpha^2|\vec{x}|^2 + (\beta|\vec{x}|^2 - 1)^2}\vec{x}^\times + \frac{\alpha^2 + \beta^2|\vec{x}|^2 - \beta}{\alpha^2|\vec{x}|^2 + (\beta|\vec{x}|^2 - 1)^2}\vec{x}^{\times 2}. \tag{2.9.18}$$

Now, let $\mathrm{F_A}$ be a frame, and define $x \triangleq \vec{x}\big|_\mathrm{A}$. Then,

$$(I_3 + \alpha x^\times + \beta x^{\times 2})^{-1} = I_3 - \frac{\alpha}{\alpha^2\|x\|^2 + (\beta\|x\|^2 - 1)^2}x^\times + \frac{\alpha^2 + \beta^2\|x\|^2 - \beta}{\alpha^2\|x\|^2 + (\beta\|x\|^2 - 1)^2}x^{\times 2}. \tag{2.9.19}$$

Fact 2.9.5. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$(\vec{x} \times \vec{y})' = -\vec{y}'\vec{x}^\times, \tag{2.9.20}$$

$$(\vec{x} \times \vec{y})^\times = \vec{y}\vec{x}' - \vec{x}\vec{y}', \tag{2.9.21}$$

$$\vec{x}^\times\vec{y}^\times = \vec{y}\vec{x}' - (\vec{y}'\vec{x})\vec{I}. \tag{2.9.22}$$

Now, let $\mathrm{F_A}$ be a frame, and define $x \triangleq \vec{x}\big|_\mathrm{A}$ and $y \triangleq \vec{y}\big|_\mathrm{A}$. Then,

$$(x \times y)^\mathsf{T} = -y^\mathsf{T}x^\times, \tag{2.9.23}$$

$$(x \times y)^\times = yx^\mathsf{T} - xy^\mathsf{T}, \tag{2.9.24}$$

$$x^\times y^\times = yx^\mathsf{T} - y^\mathsf{T}xI_3. \tag{2.9.25}$$

Proof. To prove (2.9.20), note that (2.9.6) implies that

$$(\vec{x} \times \vec{y})' = (\vec{x}^\times\vec{y})' = \vec{y}'\vec{x}^{\times\prime} = -\vec{y}'\vec{x}^\times.$$

Next, to prove (2.9.21) let $\vec{z}$ be a physical vector. Then, Fact 2.4.2 implies that

$$(\vec{x} \times \vec{y})^{\times}\vec{z} = (\vec{x} \times \vec{y}) \times \vec{z} = (\vec{x}\,'\vec{z})\vec{y} - (\vec{y}\,'\vec{z})\vec{x} = (\vec{y}\,\vec{x}\,' - \vec{x}\,\vec{y}\,')\vec{z}.$$

Finally, to prove (2.9.22), let $\vec{z}$ be a physical vector. Then, Fact 2.4.2 implies that

$$\vec{x}^{\times}\vec{y}^{\times}\vec{z} = \vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x}\,'\vec{z})\vec{y} - (\vec{x}\,'\vec{y})\vec{z} = [\vec{y}\,\vec{x}\,' - (\vec{y}\,'\vec{x})\vec{I}\,]\vec{z}. \qquad \square$$

For the 3×3 matrix M, the trace of M, which is denoted by $\operatorname{tr} M$, is the sum of the diagonal entries of M. The following result uses the *Frobenius norm* $\|\cdot\|_{\mathrm{F}}$ of a 3×3 real matrix M defined by

$$\|M\|_{\mathrm{F}} \triangleq \sqrt{\operatorname{tr} MM^{\mathsf{T}}}. \tag{2.9.26}$$

Fact 2.9.6. Let $\mathcal{S}$ be a parallelogram with vertices a, b, c, d so that $\vec{r}_{b/a} = \vec{r}_{d/c}$ and $\vec{r}_{c/a} = \vec{r}_{d/b}$, and let $\theta \in (0, \pi)$ be the angle between $\vec{r}_{b/a}$ and $\vec{r}_{c/a}$. Then,

$$\operatorname{area}(\mathcal{S}) = |\vec{r}_{b/a}|\,|\vec{r}_{c/a}| \sin\theta = |\vec{r}_{b/a} \times \vec{r}_{c/a}| = |(\vec{r}_{c/a}\vec{r}\,'_{b/a} - \vec{r}_{b/a}\vec{r}\,'_{c/a})^{-\times}|. \tag{2.9.27}$$

Now, define $x \triangleq \vec{r}_{b/a}\big|_{\mathrm{A}}$ and $y \triangleq \vec{r}_{c/a}\big|_{\mathrm{A}}$. Then,

$$\operatorname{area}(\mathcal{S}) = \|x\|\|y\| \sin\theta = \|x \times y\| = \|(x \times y)^{\times}\|_{\mathrm{F}} = \|(yx^{\mathsf{T}} - xy^{\mathsf{T}})^{-\times}\| = \|yx^{\mathsf{T}} - xy^{\mathsf{T}}\|_{\mathrm{F}}. \tag{2.9.28}$$

Fact 2.9.6 shows that the cross product $\vec{r}_{b/a} \times \vec{r}_{c/a}$ can be viewed as a *directed area*, and likewise for the physical matrix $\vec{r}_{b/a}\vec{r}\,'_{c/a} - \vec{r}_{c/a}\vec{r}\,'_{b/a}$. It will be shown in Chapter 3 that $\vec{r}_{b/a} \wedge \vec{r}\,'_{c/a} = \vec{r}_{b/a} \otimes \vec{r}\,'_{c/a} - \vec{r}_{c/a} \otimes \vec{r}\,'_{b/a} = \vec{r}_{b/a}\vec{r}\,'_{c/a} - \vec{r}_{c/a}\vec{r}\,'_{b/a}$, where $\vec{r}_{b/a} \wedge \vec{r}\,'_{c/a}$ is a bivector

Fact 2.9.7. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors, and let $\mathrm{F_A}$ be a frame. Then,

$$(\vec{x} \times \vec{y})'\vec{z} = \vec{x}\,'(\vec{y} \times \vec{z}) = \det\begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix}. \tag{2.9.29}$$

Proof. Note that

$$(\vec{x} \times \vec{y})'\vec{z} = -(\vec{y} \times \vec{x})'\vec{z} = -(\vec{y}^{\times}\vec{x})'\vec{z} = -\vec{x}\,'\vec{y}^{\times\prime}\vec{z} = \vec{x}\,'\vec{y}^{\times}\vec{z} = \vec{x}\,'(\vec{y} \times \vec{z}).$$

Finally, note that

$$(\vec{x} \times \vec{y})'\vec{z} = \left(\vec{x}\big|_{\mathrm{A}} \times \vec{y}\big|_{\mathrm{A}}\right)^{\mathsf{T}} \vec{z}\big|_{\mathrm{A}} = \det\begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix}. \qquad \square$$

Fact 2.9.8. Let $\vec{x}$ be a physical vector, and let $\vec{R}$ be a physical rotation matrix. Then,

$$(\vec{R}\vec{x})^{\times} = \vec{R}\vec{x}^{\times}\vec{R}\,'. \tag{2.9.30}$$

Proof. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames such that $\vec{R} = \vec{R}_{\mathrm{B/A}}$, and write $\vec{x}\big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Using (2.9.3) we have

$$\vec{R}_{\mathrm{B/A}}\vec{x}^{\times}\vec{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{B/A}}[x_1(\hat{k}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}}) + x_2(\hat{\imath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}}) + x_3(\hat{\jmath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}})]\vec{R}_{\mathrm{A/B}}$$

$$= x_1(\hat{k}_{\rm B}\hat{\jmath}'_{\rm B} - \hat{\jmath}_{\rm B}\hat{k}'_{\rm B}) + x_2(\hat{\imath}_{\rm B}\hat{k}'_{\rm B} - \hat{k}_{\rm B}\hat{\imath}'_{\rm B}) + x_3(\hat{\jmath}_{\rm B}\hat{\imath}'_{\rm B} - \hat{\imath}_{\rm B}\hat{\jmath}'_{\rm B})$$
$$= x_1\hat{\imath}_{\rm B}^\times + x_2\hat{\jmath}_{\rm B}^\times + x_3\hat{k}_{\rm B}^\times = (x_1\hat{\imath}_{\rm B} + x_2\hat{\jmath}_{\rm B} + x_3\hat{k}_{\rm B})^\times = (\vec{R}_{\rm B/A}\vec{x})^\times. \qquad \square$$

Fact 2.9.9. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\vec{R}$ be a physical rotation matrix. Then,

$$\vec{R}(\vec{x}\times\vec{y}) = (\vec{R}\vec{x})\times(\vec{R}\vec{y}). \tag{2.9.31}$$

Now, let $\rm F_A$ be a frame and define $x \triangleq \vec{x}\big|_{\rm A}$, $y \triangleq \vec{y}\big|_{\rm A}$, and $\mathcal{R} \triangleq \vec{R}\big|_{\rm A}$. Then,

$$\mathcal{R}(x\times y) = (\mathcal{R}x)\times(\mathcal{R}y). \tag{2.9.32}$$

Proof. Using (2.9.30) it follows that

$$\vec{R}(\vec{x}\times\vec{y}) = \vec{R}\vec{x}^\times\vec{y} = \vec{R}\vec{x}^\times\vec{R}'\vec{R}\vec{y} = (\vec{R}\vec{x})^\times\vec{R}\vec{y} = (\vec{R}\vec{x})\times(\vec{R}\vec{y}). \qquad \square$$

2.10 Rotation and Orientation Matrices

The following result is needed for the subsequent development.

Fact 2.10.1. Let $\rm F_A$ and $\rm F_B$ be frames. Then,

$$\vec{R}_{\rm B/A}\Big|_{\rm B} = \vec{R}_{\rm B/A}\Big|_{\rm A} = \begin{bmatrix} \hat{\imath}_{\rm A}\cdot\hat{\imath}_{\rm B} & \hat{\imath}_{\rm A}\cdot\hat{\jmath}_{\rm B} & \hat{\imath}_{\rm A}\cdot\hat{k}_{\rm B} \\ \hat{\jmath}_{\rm A}\cdot\hat{\imath}_{\rm B} & \hat{\jmath}_{\rm A}\cdot\hat{\jmath}_{\rm B} & \hat{\jmath}_{\rm A}\cdot\hat{k}_{\rm B} \\ \hat{k}_{\rm A}\cdot\hat{\imath}_{\rm B} & \hat{k}_{\rm A}\cdot\hat{\jmath}_{\rm B} & \hat{k}_{\rm A}\cdot\hat{k}_{\rm B} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\rm B}|_{\rm A} & \hat{\jmath}_{\rm B}|_{\rm A} & \hat{k}_{\rm B}|_{\rm A} \end{bmatrix} = {\rm F_B}|_{\rm A}. \tag{2.10.1}$$

Proof. Note that

$$\vec{R}_{\rm B/A}\Big|_{\rm B} = e_1\hat{\imath}_{\rm A}|_{\rm B}^{\rm T} + e_2\,\hat{\jmath}_{\rm A}|_{\rm B}^{\rm T} + e_3\hat{k}_{\rm A}|_{\rm B}^{\rm T} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}\begin{bmatrix} \hat{\imath}_{\rm A}|_{\rm B}^{\rm T} \\ \hat{\jmath}_{\rm A}|_{\rm B}^{\rm T} \\ \hat{k}_{\rm A}|_{\rm B}^{\rm T} \end{bmatrix}$$
$$= \begin{bmatrix} \hat{\imath}_{\rm A}|_{\rm B}^{\rm T} \\ \hat{\jmath}_{\rm A}|_{\rm B}^{\rm T} \\ \hat{k}_{\rm A}|_{\rm B}^{\rm T} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\rm A}|_{\rm B}^{\rm T} \\ \hat{\jmath}_{\rm A}|_{\rm B}^{\rm T} \\ \hat{k}_{\rm A}|_{\rm B}^{\rm T} \end{bmatrix}\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$
$$= \begin{bmatrix} \hat{\imath}_{\rm A}|_{\rm B}^{\rm T}e_1 & \hat{\imath}_{\rm A}|_{\rm B}^{\rm T}e_2 & \hat{\imath}_{\rm A}|_{\rm B}^{\rm T}e_3 \\ \hat{\jmath}_{\rm A}|_{\rm B}^{\rm T}e_1 & \hat{\jmath}_{\rm A}|_{\rm B}^{\rm T}e_2 & \hat{\jmath}_{\rm A}|_{\rm B}^{\rm T}e_3 \\ \hat{k}_{\rm A}|_{\rm B}^{\rm T}e_1 & \hat{k}_{\rm A}|_{\rm B}^{\rm T}e_2 & \hat{k}_{\rm A}|_{\rm B}^{\rm T}e_3 \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\rm A}\cdot\hat{\imath}_{\rm B} & \hat{\imath}_{\rm A}\cdot\hat{\jmath}_{\rm B} & \hat{\imath}_{\rm A}\cdot\hat{k}_{\rm B} \\ \hat{\jmath}_{\rm A}\cdot\hat{\imath}_{\rm B} & \hat{\jmath}_{\rm A}\cdot\hat{\jmath}_{\rm B} & \hat{\jmath}_{\rm A}\cdot\hat{k}_{\rm B} \\ \hat{k}_{\rm A}\cdot\hat{\imath}_{\rm B} & \hat{k}_{\rm A}\cdot\hat{\jmath}_{\rm B} & \hat{k}_{\rm A}\cdot\hat{k}_{\rm B} \end{bmatrix}$$
$$= \begin{bmatrix} \hat{\imath}_{\rm B}|_{\rm A} & \hat{\jmath}_{\rm B}|_{\rm A} & \hat{k}_{\rm B}|_{\rm A} \end{bmatrix} = {\rm F_B}|_{\rm A} = \begin{bmatrix} \hat{\imath}_{\rm B}|_{\rm A} & \hat{\jmath}_{\rm B}|_{\rm A} & \hat{k}_{\rm B}|_{\rm A} \end{bmatrix}\begin{bmatrix} e_1^{\rm T} \\ e_2^{\rm T} \\ e_3^{\rm T} \end{bmatrix}$$
$$= \hat{\imath}_{\rm B}|_{\rm A}\,e_1^{\rm T} + \hat{\jmath}_{\rm B}|_{\rm A}\,e_2^{\rm T} + \hat{k}_{\rm B}|_{\rm A}e_3^{\rm T} = \vec{R}_{\rm B/A}\Big|_{\rm A}. \qquad \square$$

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define the *rotation matrix from* $\mathrm{F_A}$ *to* $\mathrm{F_B}$ to be the 3×3 matrix

$$\mathcal{R}_{\mathrm{B/A}} \triangleq R_{\mathrm{B/A|B}} = R_{\mathrm{B/A|A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \mathrm{F_B}|_{\mathrm{A}}\,. \tag{2.10.2}$$

Furthermore, define the *orientation matrix of* $\mathrm{F_A}$ *relative* to $\mathrm{F_B}$ to be the 3×3 matrix

$$\mathcal{O}_{\mathrm{A/B}} \triangleq \mathcal{R}_{\mathrm{B/A}}. \tag{2.10.3}$$

Hence,

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = R_{\mathrm{A/B|A}} = R_{\mathrm{A/B|B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} = \mathrm{F_A}|_{\mathrm{B}}\,. \tag{2.10.4}$$

Note that the distinction between the orientation matrix $\mathcal{O}_{\mathrm{B/A}}$ and the rotation matrix $\mathcal{R}_{\mathrm{A/B}}$ resides in the order of subscripts. The subsequent development shows that, when working with resolved vectors, $\mathcal{O}_{\mathrm{B/A}}$ is far more natural and convenient than $\mathcal{R}_{\mathrm{A/B}}$.

Fact 2.10.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{\mathsf{T}} = \mathcal{O}_{\mathrm{A/B}}^{\mathsf{T}}, \tag{2.10.5}$$

$$\mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{-1}, \tag{2.10.6}$$

$$\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{B/A}}^{-1}. \tag{2.10.7}$$

Therefore,

$$\mathcal{R}_{\mathrm{B/A}}^{\mathsf{T}} = \mathcal{R}_{\mathrm{B/A}}^{-1}, \tag{2.10.8}$$

$$\mathcal{O}_{\mathrm{B/A}}^{\mathsf{T}} = \mathcal{O}_{\mathrm{B/A}}^{-1}. \tag{2.10.9}$$

Proof. Note that

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = \vec{R}'_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathsf{T}} = \mathcal{R}_{\mathrm{B/A}}^{\mathsf{T}} = \mathcal{O}_{\mathrm{A/B}}^{\mathsf{T}}.$$

Next, since $\vec{I} = \vec{R}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/A}}$, it follows from (2.10.1) that

$$I_3 = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}}\,\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}}\,\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\mathcal{R}_{\mathrm{B/A}}.$$

Hence, $\mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{-1}$. □

It follows from (2.10.8) and (2.10.9) that $\mathcal{R}_{\mathrm{A/B}}$ and $\mathcal{O}_{\mathrm{A/B}}$ are orthogonal matrices.

Fact 2.10.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{O}_{\mathrm{A/B}} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}}\cdot\hat{\imath}_{\mathrm{B}} & \hat{\imath}_{\mathrm{A}}\cdot\hat{\jmath}_{\mathrm{B}} & \hat{\imath}_{\mathrm{A}}\cdot\hat{k}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{A}}\cdot\hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{A}}\cdot\hat{\jmath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{A}}\cdot\hat{k}_{\mathrm{B}} \\ \hat{k}_{\mathrm{A}}\cdot\hat{\imath}_{\mathrm{B}} & \hat{k}_{\mathrm{A}}\cdot\hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{A}}\cdot\hat{k}_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}}|_{\mathrm{A}} & \hat{k}_{\mathrm{B}}|_{\mathrm{A}} \end{bmatrix} = \mathrm{F_B}|_{\mathrm{A}}\,. \tag{2.10.10}$$

We can write $\mathcal{O}_{\mathrm{A/B}}$ in terms of row and column vectrices as

$$\mathcal{O}_{\mathrm{A/B}} = \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix} \cdot \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} \hat{\imath}'_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} = \mathcal{F}'_{\mathrm{A}}\mathrm{F_B}. \tag{2.10.11}$$

The following result shows that the entries of $\mathcal{O}_{\mathrm{A/B}}$ are the cosines of the angles between pairs of vectors in frames $\mathrm{F_A}$ and $\mathrm{F_B}$. Consequently, $\mathcal{O}_{\mathrm{A/B}}$ is a *direction cosine matrix*.

Fact 2.10.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{O}_{\mathrm{A/B}} = \begin{bmatrix} \cos\theta_{\hat{\imath}_\mathrm{A}/\hat{\imath}_\mathrm{B}} & \cos\theta_{\hat{\imath}_\mathrm{A}/\hat{\jmath}_\mathrm{B}} & \cos\theta_{\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{B}} \\ \cos\theta_{\hat{\jmath}_\mathrm{A}/\hat{\imath}_\mathrm{B}} & \cos\theta_{\hat{\jmath}_\mathrm{A}/\hat{\jmath}_\mathrm{B}} & \cos\theta_{\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{B}} \\ \cos\theta_{\hat{k}_\mathrm{A}/\hat{\imath}_\mathrm{B}} & \cos\theta_{\hat{k}_\mathrm{A}/\hat{\jmath}_\mathrm{B}} & \cos\theta_{\hat{k}_\mathrm{A}/\hat{k}_\mathrm{B}} \end{bmatrix}. \tag{2.10.12}$$

The following result relates column vectrices that represent different frames. In particular, (2.10.13) shows that the column vectrix $\mathcal{F}_\mathrm{B}$ is the product of the orientation matrix $\mathcal{O}_{\mathrm{B/A}}$ and the the column vectrix $\mathcal{F}_\mathrm{A}$. The multiplication is performed in the same way as standard matrix-vector multiplication by viewing the components of $\mathcal{F}_\mathrm{A}$ as scalars.

Fact 2.10.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\begin{bmatrix} \hat{\imath}_\mathrm{B} \\ \hat{\jmath}_\mathrm{B} \\ \hat{k}_\mathrm{B} \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}, \tag{2.10.13}$$

where

$$\mathcal{O}_{\mathrm{B/A}} = \begin{bmatrix} \hat{\imath}_\mathrm{B}\cdot\hat{\imath}_\mathrm{A} & \hat{\imath}_\mathrm{B}\cdot\hat{\jmath}_\mathrm{A} & \hat{\imath}_\mathrm{B}\cdot\hat{k}_\mathrm{A} \\ \hat{\jmath}_\mathrm{B}\cdot\hat{\imath}_\mathrm{A} & \hat{\jmath}_\mathrm{B}\cdot\hat{\jmath}_\mathrm{A} & \hat{\jmath}_\mathrm{B}\cdot\hat{k}_\mathrm{A} \\ \hat{k}_\mathrm{B}\cdot\hat{\imath}_\mathrm{A} & \hat{k}_\mathrm{B}\cdot\hat{\jmath}_\mathrm{A} & \hat{k}_\mathrm{B}\cdot\hat{k}_\mathrm{A} \end{bmatrix}. \tag{2.10.14}$$

Proof. Note that

$$\begin{aligned}
\begin{bmatrix} \hat{\imath}_\mathrm{B} \\ \hat{\jmath}_\mathrm{B} \\ \hat{k}_\mathrm{B} \end{bmatrix} &= \begin{bmatrix} \vec{I}\,\hat{\imath}_\mathrm{B} \\ \vec{I}\,\hat{\jmath}_\mathrm{B} \\ \vec{I}\,\hat{k}_\mathrm{B} \end{bmatrix} = \begin{bmatrix} (\hat{\imath}_\mathrm{A}\hat{\imath}'_\mathrm{A} + \hat{\jmath}_\mathrm{A}\hat{\jmath}'_\mathrm{A} + \hat{k}_\mathrm{A}\hat{k}'_\mathrm{A})\hat{\imath}_\mathrm{B} \\ (\hat{\imath}_\mathrm{A}\hat{\imath}'_\mathrm{A} + \hat{\jmath}_\mathrm{A}\hat{\jmath}'_\mathrm{A} + \hat{k}_\mathrm{A}\hat{k}'_\mathrm{A})\hat{\jmath}_\mathrm{B} \\ (\hat{\imath}_\mathrm{A}\hat{\imath}'_\mathrm{A} + \hat{\jmath}_\mathrm{A}\hat{\jmath}'_\mathrm{A} + \hat{k}_\mathrm{A}\hat{k}'_\mathrm{A})\hat{k}_\mathrm{B} \end{bmatrix} \\
&= \begin{bmatrix} \hat{\imath}'_\mathrm{A}\hat{\imath}_\mathrm{B} & \hat{\jmath}'_\mathrm{A}\hat{\imath}_\mathrm{B} & \hat{k}'_\mathrm{A}\hat{\imath}_\mathrm{B} \\ \hat{\imath}'_\mathrm{A}\hat{\jmath}_\mathrm{B} & \hat{\jmath}'_\mathrm{A}\hat{\jmath}_\mathrm{B} & \hat{k}'_\mathrm{A}\hat{\jmath}_\mathrm{B} \\ \hat{\imath}'_\mathrm{A}\hat{k}_\mathrm{B} & \hat{\jmath}'_\mathrm{A}\hat{k}_\mathrm{B} & \hat{k}'_\mathrm{A}\hat{k}_\mathrm{B} \end{bmatrix} \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix} \\
&= \begin{bmatrix} \hat{\imath}_\mathrm{B}\cdot\hat{\imath}_\mathrm{A} & \hat{\imath}_\mathrm{B}\cdot\hat{\jmath}_\mathrm{A} & \hat{\imath}_\mathrm{B}\cdot\hat{k}_\mathrm{A} \\ \hat{\jmath}_\mathrm{B}\cdot\hat{\imath}_\mathrm{A} & \hat{\jmath}_\mathrm{B}\cdot\hat{\jmath}_\mathrm{A} & \hat{\jmath}_\mathrm{B}\cdot\hat{k}_\mathrm{A} \\ \hat{k}_\mathrm{B}\cdot\hat{\imath}_\mathrm{A} & \hat{k}_\mathrm{B}\cdot\hat{\jmath}_\mathrm{A} & \hat{k}_\mathrm{B}\cdot\hat{k}_\mathrm{A} \end{bmatrix} \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix} \\
&= \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}_\mathrm{A} \\ \hat{\jmath}_\mathrm{A} \\ \hat{k}_\mathrm{A} \end{bmatrix}.
\end{aligned}$$

□

Note that (2.10.13) can be written as

$$\mathcal{F}_\mathrm{B} = \mathcal{O}_{\mathrm{B/A}}\mathcal{F}_\mathrm{A} = \mathcal{R}_{\mathrm{A/B}}\mathcal{F}_\mathrm{A}. \tag{2.10.15}$$

Therefore,

$$\mathrm{F_B} = \mathrm{F_A}\mathcal{O}_{\mathrm{A/B}} = \mathrm{F_A}\mathcal{R}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A}. \tag{2.10.16}$$

Note the "commuting" property in the last equality in (2.10.16), which implies that

$$\mathrm{F_A} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A}\mathcal{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{A/B}}\mathrm{F_A}\mathcal{R}_{\mathrm{B/A}}. \tag{2.10.17}$$

Resolving (2.10.16) in $\mathrm{F_A}$ yields

$$\mathcal{O}_{\mathrm{A/B}} = \mathrm{F_B}\big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}, \tag{2.10.18}$$

whereas resolving (2.10.16) in $\mathrm{F_B}$ yields

$$I_3 = \mathcal{O}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{B/A}}. \tag{2.10.19}$$

To directly show the equality between the second and fourth terms in (2.10.16), note that

$$\vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \mathrm{F_B}\mathcal{F}'_{\mathrm{A}}\mathrm{F_A} = \mathrm{F_B}I_3 = \mathrm{F_B} = \vec{I}\,\mathrm{F_B} = \mathrm{F_A}\mathcal{F}'_{\mathrm{A}}\mathrm{F_B} = \mathrm{F_A}\mathcal{O}_{\mathrm{A/B}}. \tag{2.10.20}$$

Note that the last three equalities in (2.10.20) show that

$$\mathcal{F}_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{F}_{\mathrm{A}}, \tag{2.10.21}$$

which is (2.10.13). Finally, it follows from (2.10.20) and (2.4.13) that

$$\vec{R}_{\mathrm{B/A}} = \mathrm{F_A}\mathcal{O}_{\mathrm{A/B}}\mathcal{F}'_{\mathrm{A}} = \mathrm{F_A}\mathcal{R}_{\mathrm{B/A}}\mathcal{F}'_{\mathrm{A}}. \tag{2.10.22}$$

To relate (2.10.13) to (2.10.10), note that it follows from (2.10.13) that

$$\begin{bmatrix} \vec{\imath}'_{\mathrm{B}} \\ \vec{\jmath}'_{\mathrm{B}} \\ \hat{k}'_{\mathrm{B}} \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \vec{\imath}'_{\mathrm{A}} \\ \vec{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix}, \tag{2.10.23}$$

that is,

$$\mathcal{F}'_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\mathcal{F}'_{\mathrm{A}}. \tag{2.10.24}$$

Now, taking the transpose of (2.10.24) and multiplying on the right by $\mathcal{O}_{\mathrm{B/A}}$ yields

$$\mathrm{F}'_{\mathrm{A}} = \mathrm{F}'_{\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}, \tag{2.10.25}$$

that is,

$$\begin{bmatrix} \vec{\imath}'_{\mathrm{A}} & \vec{\jmath}'_{\mathrm{A}} & \hat{k}'_{\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \vec{\imath}'_{\mathrm{B}} & \vec{\jmath}'_{\mathrm{B}} & \hat{k}'_{\mathrm{B}} \end{bmatrix}\mathcal{O}_{\mathrm{B/A}}. \tag{2.10.26}$$

Furthermore, resolving (2.10.23) in $\mathrm{F_A}$ yields

$$\begin{bmatrix} \hat{\imath}_{\mathrm{B}}\big|_{\mathrm{A}}^{\mathsf{T}} \\ \hat{\jmath}_{\mathrm{B}}\big|_{\mathrm{A}}^{\mathsf{T}} \\ \hat{k}_{\mathrm{B}}\big|_{\mathrm{A}}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \vec{\imath}'_{\mathrm{B}}\big|_{\mathrm{A}} \\ \vec{\jmath}'_{\mathrm{B}}\big|_{\mathrm{A}} \\ \hat{k}'_{\mathrm{B}}\big|_{\mathrm{A}} \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}}, \tag{2.10.27}$$

which implies that

$$\mathcal{O}_{\mathrm{A/B}} = \begin{bmatrix} \hat{\imath}_{\mathrm{B}}\big|_{\mathrm{A}} & \hat{\jmath}_{\mathrm{B}}\big|_{\mathrm{A}} & \hat{k}_{\mathrm{B}}\big|_{\mathrm{A}} \end{bmatrix} = \mathrm{F_B}\big|_{\mathrm{A}}. \tag{2.10.28}$$

The following identities are useful.

Fact 2.10.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{I} = \begin{bmatrix} \hat{\imath}_\mathrm{B} & \hat{\jmath}_\mathrm{B} & \hat{k}_\mathrm{B} \end{bmatrix} \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}'_\mathrm{A} \\ \hat{\jmath}'_\mathrm{A} \\ \hat{k}'_\mathrm{A} \end{bmatrix}, \tag{2.10.29}$$

$$\vec{R}_{\mathrm{B/A}} = \begin{bmatrix} \hat{\imath}_\mathrm{B} & \hat{\jmath}_\mathrm{B} & \hat{k}_\mathrm{B} \end{bmatrix} \mathcal{O}_{\mathrm{A/B}} \begin{bmatrix} \hat{\imath}'_\mathrm{B} \\ \hat{\jmath}'_\mathrm{B} \\ \hat{k}'_\mathrm{B} \end{bmatrix}. \tag{2.10.30}$$

Proof. It follows from (2.10.23) that

$$\begin{bmatrix} \hat{\imath}_\mathrm{B} & \hat{\jmath}_\mathrm{B} & \hat{k}_\mathrm{B} \end{bmatrix} \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}'_\mathrm{A} \\ \hat{\jmath}'_\mathrm{A} \\ \hat{k}'_\mathrm{A} \end{bmatrix} = \begin{bmatrix} \hat{\imath}_\mathrm{B} & \hat{\jmath}_\mathrm{B} & \hat{k}_\mathrm{B} \end{bmatrix} \begin{bmatrix} \hat{\imath}'_\mathrm{B} \\ \hat{\jmath}'_\mathrm{B} \\ \hat{k}'_\mathrm{B} \end{bmatrix} = \hat{\imath}_\mathrm{B}\hat{\imath}'_\mathrm{B} + \hat{\jmath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{k}'_\mathrm{B} = \vec{I}. \qquad \square$$

The following result is especially useful.

Fact 2.10.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_\mathrm{B} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_\mathrm{A}, \tag{2.10.31}$$

$$\vec{x}\Big|_\mathrm{B} = \mathcal{R}_{\mathrm{A/B}}\ \vec{x}\Big|_\mathrm{A}. \tag{2.10.32}$$

Proof. Note that

$$\vec{x}\Big|_\mathrm{B} = I_3\ \vec{x}\Big|_\mathrm{B} = \mathcal{F}'_\mathrm{B}\mathrm{F_B}\ \vec{x}\Big|_\mathrm{B} = \mathcal{F}'_\mathrm{B}\mathrm{F_A}\ \vec{x}\Big|_\mathrm{A} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_\mathrm{A}. \qquad \square$$

Fact 2.10.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\vec{x}$ be a physical vector, and let $\vec{y} = \vec{R}_{\mathrm{B/A}}\vec{x}$. Then,

$$\vec{y}\Big|_\mathrm{A} = \mathcal{R}_{\mathrm{B/A}}\ \vec{x}\Big|_\mathrm{A} = \mathcal{R}^2_{\mathrm{B/A}}\ \vec{x}\Big|_\mathrm{B}, \tag{2.10.33}$$

$$\vec{y}\Big|_\mathrm{B} = \mathcal{R}_{\mathrm{B/A}}\ \vec{x}\Big|_\mathrm{B} = \vec{x}\Big|_\mathrm{A}. \tag{2.10.34}$$

The following result shows that the orthogonal matrix $\mathcal{O}_{\mathrm{A/B}}$ is proper, and thus represents the result of a physical rotation.

Fact 2.10.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\det \mathcal{O}_{\mathrm{B/A}} = 1. \tag{2.10.35}$$

Proof. Using Fact 2.9.7, it follows that

$$\det \mathcal{O}_{\mathrm{B/A}} = \det \begin{bmatrix} \hat{\imath}_\mathrm{A}|_\mathrm{B} & \hat{\jmath}_\mathrm{A}|_\mathrm{B} & \hat{k}_\mathrm{A}|_\mathrm{B} \end{bmatrix} = (\hat{\imath}_\mathrm{A}|_\mathrm{B} \times \hat{\jmath}_\mathrm{A}|_\mathrm{B})^\mathsf{T}\ \hat{k}_\mathrm{A}\Big|_\mathrm{B} = \hat{k}_\mathrm{A}\Big|_\mathrm{B}^\mathsf{T}\ \hat{k}_\mathrm{A}\Big|_\mathrm{B} = 1. \qquad \square$$

Example 2.10.10. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames such that

$$\hat{\imath}_\mathrm{B} = -\hat{k}_\mathrm{A},\ \hat{\jmath}_\mathrm{B} = \hat{\jmath}_\mathrm{A},\ \hat{k}_\mathrm{B} = \hat{\imath}_\mathrm{A}. \tag{2.10.36}$$

Therefore, $\vec{R}_{\mathrm{B/A}}$ rotates $\mathrm{F_A}$ by $\pi/2$ rad according to the right hand rule around $\hat{\jmath}_{\mathrm{A}}$. Furthermore,

$$\mathcal{O}_{\mathrm{B/A}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{2.10.37}$$

which satisfies (2.10.35). Finally,

$$\begin{bmatrix} \hat{\imath}_{\mathrm{B}} & \hat{\jmath}_{\mathrm{B}} & \hat{k}_{\mathrm{B}} \end{bmatrix} \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}'_{\mathrm{A}} \\ \hat{\jmath}'_{\mathrm{A}} \\ \hat{k}'_{\mathrm{A}} \end{bmatrix} = -\hat{\imath}_{\mathrm{B}}\hat{k}'_{\mathrm{A}} + \hat{k}_{\mathrm{B}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{A}} = \hat{\imath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} = \vec{I},$$

which confirms (2.10.29).

Fact 2.10.11. Let $\vec{M}$ be a physical matrix. Then,

$$\vec{M}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{M}\Big|_{\mathrm{A}}\ \mathcal{O}_{\mathrm{A/B}}. \tag{2.10.38}$$

Proof. Write

$$\vec{M} = \sum_{i=1}^{n} \vec{x}_i \vec{y}_i{}'.$$

We thus have

$$\begin{aligned} \vec{M}\Big|_{\mathrm{B}} &= \sum_{i=1}^{n} \vec{x}_i\Big|_{\mathrm{B}}\ \vec{y}_i\Big|_{\mathrm{B}}^{\mathsf{T}} = \sum_{i=1}^{n} \mathcal{O}_{\mathrm{B/A}}\ \vec{x}_i\Big|_{\mathrm{A}} \left(\mathcal{O}_{\mathrm{B/A}}\ \vec{y}_i\Big|_{\mathrm{A}}\right)^{\mathsf{T}} \\ &= \mathcal{O}_{\mathrm{B/A}} \sum_{i=1}^{n} \vec{x}_i\Big|_{\mathrm{A}}\ \vec{y}_i\Big|_{\mathrm{A}}^{\mathsf{T}}\ \mathcal{O}_{\mathrm{B/A}}^{\mathsf{T}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{M}\Big|_{\mathrm{A}}\ \mathcal{O}_{\mathrm{A/B}}. \end{aligned}$$

□

Fact 2.10.12. Let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$(\mathcal{R}_{\mathrm{B/A}}x)^{\times} = \mathcal{R}_{\mathrm{B/A}}x^{\times}\mathcal{R}_{\mathrm{A/B}}, \tag{2.10.39}$$

$$(\mathcal{O}_{\mathrm{B/A}}x)^{\times} = \mathcal{O}_{\mathrm{B/A}}x^{\times}\mathcal{O}_{\mathrm{A/B}}. \tag{2.10.40}$$

Proof. The result follows from Fact 2.9.8. □

Fact 2.10.13. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{O}_{\mathrm{B/A}}\left(\vec{x}\Big|_{\mathrm{A}} \times \vec{y}\Big|_{\mathrm{A}}\right) = \vec{x}\Big|_{\mathrm{B}} \times \vec{y}\Big|_{\mathrm{B}}. \tag{2.10.41}$$

Proof. The result follows from Fact 2.9.9. □

Fact 2.10.14. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Then,

$$\mathcal{O}_{\mathrm{A/B}}\ \vec{x}^{\times}\Big|_{\mathrm{B}} = \vec{x}^{\times}\Big|_{\mathrm{A}}\ \mathcal{O}_{\mathrm{A/B}}. \tag{2.10.42}$$

That is,

$$\mathcal{O}_{\mathrm{A/B}}\begin{bmatrix} 0 & -\hat{k}_{\mathrm{B}}\cdot\vec{x} & \hat{\jmath}_{\mathrm{B}}\cdot\vec{x} \\ \hat{k}_{\mathrm{B}}\cdot\vec{x} & 0 & -\hat{\imath}_{\mathrm{B}}\cdot\vec{x} \\ -\hat{\jmath}_{\mathrm{B}}\cdot\vec{x} & \hat{\imath}_{\mathrm{B}}\cdot\vec{x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\hat{k}_{\mathrm{A}}\cdot\vec{x} & \hat{\jmath}_{\mathrm{A}}\cdot\vec{x} \\ \hat{k}_{\mathrm{A}}\cdot\vec{x} & 0 & -\hat{\imath}_{\mathrm{A}}\cdot\vec{x} \\ -\hat{\jmath}_{\mathrm{A}}\cdot\vec{x} & \hat{\imath}_{\mathrm{A}}\cdot\vec{x} & 0 \end{bmatrix}\mathcal{O}_{\mathrm{A/B}}. \tag{2.10.43}$$

Fact 2.10.15. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\vec{R}_{\mathrm{C/A}} = \vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}}. \tag{2.10.44}$$

Furthermore,

$$\mathcal{O}_{\mathrm{C/A}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}, \tag{2.10.45}$$

$$\mathcal{R}_{\mathrm{C/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}}. \tag{2.10.46}$$

Proof. The first equality follows directly from the definition of the physical rotation matrix. Next, using (2.10.44), (2.6.24), and (2.10.38), we have

$$\mathcal{O}_{\mathrm{C/A}} = \vec{R}_{\mathrm{A/C}}\Big|_{\mathrm{C}} = \left(\vec{R}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/C}}\right)\Big|_{\mathrm{C}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{C}}\vec{R}_{\mathrm{B/C}}\Big|_{\mathrm{C}} = \mathcal{O}_{\mathrm{C/B}}\,\vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}}\mathcal{O}_{\mathrm{B/C}}\mathcal{O}_{\mathrm{C/B}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}. \quad \square$$

For four frames we have the following immediate extension.

Fact 2.10.16. Let $\mathrm{F_A}$, $\mathrm{F_B}$, $\mathrm{F_C}$, and $\mathrm{F_D}$ be frames. Then,

$$\vec{R}_{\mathrm{D/A}} = \vec{R}_{\mathrm{D/C}}\vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}}. \tag{2.10.47}$$

Furthermore,

$$\mathcal{O}_{\mathrm{D/A}} = \mathcal{O}_{\mathrm{D/C}}\mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}, \tag{2.10.48}$$

$$\mathcal{R}_{\mathrm{D/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}}\mathcal{R}_{\mathrm{D/C}}. \tag{2.10.49}$$

For the physical matrix $\vec{M}$, we define

$$\operatorname{tr}\vec{M} \triangleq \operatorname{tr}\vec{M}\Big|_{\mathrm{A}}, \tag{2.10.50}$$

where $\mathrm{F_A}$ is an arbitrary frame. This definition is independent of the choice of frame since, if $\mathrm{F_B}$ is also a frame, then

$$\operatorname{tr}\vec{M}\Big|_{\mathrm{A}} = \operatorname{tr}\left(\mathcal{O}_{\mathrm{A/B}}\vec{M}\Big|_{\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}\right) = \operatorname{tr}\left(\mathcal{O}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}\vec{M}\Big|_{\mathrm{B}}\right) = \operatorname{tr}\vec{M}\Big|_{\mathrm{B}}. \tag{2.10.51}$$

Note that

$$\operatorname{tr}\vec{M} = \hat{\imath}'_{\mathrm{A}}\vec{M}\hat{\imath}_{\mathrm{A}} + \hat{\jmath}'_{\mathrm{A}}\vec{M}\hat{\jmath}_{\mathrm{A}} + \hat{k}'_{\mathrm{A}}\vec{M}\hat{k}_{\mathrm{A}}. \tag{2.10.52}$$

Likewise, we define

$$\det\vec{M} \triangleq \det\vec{M}\Big|_{\mathrm{A}}, \tag{2.10.53}$$

which is also independent of the choice of frame.

Fact 2.10.17. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\operatorname{tr} \vec{x}^{\times} = 0, \tag{2.10.54}$$

$$\operatorname{tr} \vec{x}\vec{y}' = \vec{y}'\vec{x}, \tag{2.10.55}$$

$$\operatorname{tr} \vec{x}^{\times}\vec{y}^{\times} = -2\vec{x}'\vec{y}. \tag{2.10.56}$$

The following result shows that the trace of a physical rotation matrix lies in the range $[-1, 3]$.

Fact 2.10.18. Let $\vec{R}$ be a physical rotation matrix. Then,

$$-1 \le \operatorname{tr} \vec{R} \le 3. \tag{2.10.57}$$

Furthermore, $\operatorname{tr} \vec{R} = 3$ if and only if $\vec{R} = \vec{I}$.

Proof. Let $\mathrm{F_A}$ be a frame, and define $\mathcal{R} \triangleq \vec{R}\Big|_{\mathrm{A}}$. Then, it follows from Problem 2.24.15 that the eigenvalues of $\mathcal{R}$ are 1, λ, and $\overline{\lambda}$, where $|\lambda| = |\overline{\lambda}| = 1$. Therefore, $-2 \le \lambda + \overline{\lambda} \le 2$, and thus $-1 \le \lambda + \overline{\lambda} + 1 = \operatorname{tr} \vec{R} \le 3$. Therefore, $\operatorname{tr} \mathcal{R} = \operatorname{tr} \vec{R}\Big|_{\mathrm{A}} = \operatorname{tr} \vec{R} = 3$ if and only if $\mathcal{R} = I_3$, which is the case if and only if $\vec{R} = \vec{I}$. □

The following result is the *Cayley-Hamilton* theorem for physical matrices. This result shows that every physical matrix satisfies a polynomial of degree 3.

Fact 2.10.19. Let $\vec{M}$ be a physical matrix. Then, $\vec{M}$ satisfies

$$\vec{M}^3 - (\operatorname{tr} \vec{M})\vec{M}^2 + \tfrac{1}{2}[(\operatorname{tr} \vec{M})^2 - \operatorname{tr} \vec{M}^2]\vec{M} - (\det \vec{M})\vec{I} = 0. \tag{2.10.58}$$

In addition,

$$\det \vec{M} = \tfrac{1}{3} \operatorname{tr} \vec{M}^3 - \tfrac{1}{2}(\operatorname{tr} \vec{M}) \operatorname{tr} \vec{M}^2 + \tfrac{1}{6}(\operatorname{tr} \vec{M})^3. \tag{2.10.59}$$

Finally, if $\vec{M}$ is nonsingular, then

$$\operatorname{tr} \vec{M}^{-1} = \frac{(\operatorname{tr} \vec{M})^2 - \operatorname{tr} \vec{M}^2}{2 \det \vec{M}}. \tag{2.10.60}$$

Proof. See [1, p. 283] or [5, p. 87]. □

2.11 Eigenaxis Rotations and Rodrigues's Formula

Let $\hat{n}$ be a unit dimensionless physical vector, let $\theta \in (-\pi, \pi]$, and define the physical matrix

$$\vec{R}_{\hat{n}}(\theta) \triangleq (\cos\theta)\vec{I} + (1 - \cos\theta)\hat{n}\hat{n}' + (\sin\theta)\hat{n}^{\times}, \tag{2.11.1}$$

which is *Rodrigues's formula*. Equivalently,

$$\vec{R}_{\hat{n}}(\theta) = \hat{n}\hat{n}' + (\cos\theta)(\vec{I} - \hat{n}\hat{n}') + (\sin\theta)\hat{n}^{\times}. \tag{2.11.2}$$

Using (2.9.9), an equivalent form of (2.11.1) is given by

$$\vec{R}_{\hat{n}}(\theta) = \vec{I} + (1 - \cos\theta)\hat{n}^{\times 2} + (\sin\theta)\hat{n}^{\times}. \tag{2.11.3}$$

The notation "$\vec{n}$" is not needed or used.

Resolving (2.11.1), (2.11.2), and (2.11.3) in $\mathrm{F_A}$ yields

$$\mathcal{R}_n(\theta) = (\cos\theta)I_3 + (1 - \cos\theta)nn^{\mathsf{T}} + (\sin\theta)n^{\times} \tag{2.11.4}$$

$$= nn^{\mathsf{T}} + (\cos\theta)(I_3 - nn^{\mathsf{T}}) + (\sin\theta)n^{\times} \tag{2.11.5}$$

$$= I_3 + (1 - \cos\theta)n^{\times 2} + (\sin\theta)n^{\times}, \tag{2.11.6}$$

where

$$\mathcal{R}_n(\theta) \triangleq \vec{R}_{\hat{n}}(\theta)\Big|_{\mathrm{A}}, \tag{2.11.7}$$

$$n \triangleq \hat{n}|_{\mathrm{A}}. \tag{2.11.8}$$

Fact 2.11.1. Let $\hat{n}$ be a unit dimensionless physical vector, and let $\theta \in (-\pi, \pi]$. Then,

$$\operatorname{tr}\vec{R}_{\hat{n}}(\theta) = \operatorname{tr}\mathcal{R}_n(\theta) = 1 + 2\cos\theta. \tag{2.11.9}$$

Proof. Using Fact 2.10.17, it follows from (2.11.3) that

$$\begin{aligned}\operatorname{tr}\vec{R}_{\hat{n}}(\theta) &= \operatorname{tr}\vec{I} + \operatorname{tr}(1 - \cos\theta)\hat{n}^{\times 2} + \operatorname{tr}(\sin\theta)\hat{n}^{\times}\\ &= 3 + (1 - \cos\theta)(-2)\\ &= 1 + 2\cos\theta.\end{aligned}$$

□

Now, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames and assume that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. Then, we write

$$\mathrm{F_A} \xrightarrow[\hat{n}]{\theta} \mathrm{F_B}, \tag{2.11.10}$$

which is equivalent to

$$\mathrm{F_B} \xrightarrow[\hat{n}]{-\theta} \mathrm{F_A}, \tag{2.11.11}$$

$$\mathrm{F_B} \xrightarrow[-\hat{n}]{\theta} \mathrm{F_A}, \tag{2.11.12}$$

$$\mathrm{F_A} \xrightarrow[-\hat{n}]{-\theta} \mathrm{F_B}. \tag{2.11.13}$$

This angle is the *eigenangle* of $\vec{R}_{\hat{n}}(\theta)$, while the unit dimensionless physical $\hat{n}$ is the *eigenaxis* of $\vec{R}_{\hat{n}}(\theta)$. The following result shows that $\vec{R}_{\hat{n}}(\theta)$ is the physical rotation matrix that rotates vectors around the eigenaxis $\hat{n}$ through the eigenangle θ.

Fact 2.11.2. Let $\hat{n}$ be a unit dimensionless physical vector, and let $\theta \in (-\pi, \pi]$. Then, $\vec{R}_{\hat{n}}(\theta)$ is a physical rotation matrix. Furthermore, let $\vec{x}$ be a nonzero physical vector, and let $\vec{x}_{\perp} \triangleq \vec{P}_{\hat{n}\perp}\vec{x}$ be the component of $\vec{x}$ that is orthogonal to $\hat{n}$. Then, the physical vector $\vec{y} = \vec{R}_{\hat{n}}(\theta)\vec{x}$ is obtained by rotating $\vec{x}$ according to the right hand rule around $\hat{n}$ by the angle θ, which is the directed angle

$$\theta = \theta_{\vec{R}_{\hat{n}}(\theta)\vec{x}_\perp/\vec{x}_\perp/\hat{n}}, \tag{2.11.14}$$

In particular,

$$\vec{R}_{\hat{n}}(\theta)\hat{n} = \hat{n}. \tag{2.11.15}$$

Furthermore,

$$\vec{R}_{\hat{n}}(-\theta) = \vec{R}_{-\hat{n}}(\theta) = \vec{R}'_{\hat{n}}(\theta), \tag{2.11.16}$$

$$\vec{R}_{\hat{n}}(\theta) = \vec{R}_{-\hat{n}}(-\theta) = \vec{R}'_{\hat{n}}(-\theta). \tag{2.11.17}$$

In addition,

$$\cos\theta = \tfrac{1}{2}(\operatorname{tr}\vec{R}_{\hat{n}}(\theta) - 1), \tag{2.11.18}$$

$$(\sin\theta)\hat{n}^\times = \tfrac{1}{2}\left(\vec{R}_{\hat{n}}(\theta) - \vec{R}'_{\hat{n}}(\theta)\right). \tag{2.11.19}$$

Furthermore, if $\theta \neq 0$ and $\theta \neq \pi$, then

$$\hat{n}^\times = \frac{1}{2\sin\theta}\left(\vec{R}_{\hat{n}}(\theta) - \vec{R}'_{\hat{n}}(\theta)\right). \tag{2.11.20}$$

Finally, if $\theta \neq 0$, $\theta \neq \pi$, and $\vec{x}$ and $\hat{n}$ are orthogonal, then

$$\hat{n} = \begin{cases} \hat{\theta}_{\vec{R}_{\hat{n}}(\theta)\vec{x}/\vec{x}}, & \theta > 0, \\ -\hat{\theta}_{\vec{R}_{\hat{n}}(\theta)\vec{x}/\vec{x}}, & \theta < 0. \end{cases} \tag{2.11.21}$$

Proof. Using (2.9.9) we have

$$\begin{aligned}
\vec{R}_{\hat{n}}(\theta)\vec{R}'_{\hat{n}}(\theta) &= (\cos\theta)^2\vec{I} + 2(\cos\theta)(1-\cos\theta)\hat{n}\hat{n}' - (\cos\theta)(\sin\theta)\hat{n}^\times \\
&\quad + (\cos\theta)(\sin\theta)\hat{n}^\times + (1-\cos\theta)^2\hat{n}\hat{n}' - (\sin\theta)^2(\hat{n}^\times)^2 \\
&= (\cos\theta)^2\vec{I} + (1-\cos^2\theta)\hat{n}\hat{n}' - (\sin\theta)^2(\hat{n}\hat{n}' - \vec{I}) \\
&= (\cos^2\theta + \sin^2\theta)\vec{I} + (1 - \cos^2\theta - \sin^2\theta)\hat{n}\hat{n}' \\
&= \vec{I}.
\end{aligned}$$

To prove that $\vec{R}_{\hat{n}}(\theta)$ is proper, let $\mathrm{F_A}$ be such that $\hat{n} = \hat{\imath}_{\mathrm{A}}$. Then,

$$\det \vec{R}_{\hat{n}}(\theta)\Big|_{\mathrm{A}} = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = 1.$$

Next, to demonstrate the effect of applying $\vec{R}_{\hat{n}}(\theta)$ to $\vec{x}$, we write $\vec{x} = x_{\mathrm{par}}\hat{n} + \vec{x}_\perp$, where $x_{\mathrm{par}} = \hat{n}'\vec{x}$ and thus $\hat{n}'\vec{x}_\perp = 0$. We then have

$$\vec{R}_{\hat{n}}(\theta)\vec{x} = (\cos\theta)\vec{x} + x_{\mathrm{par}}(1-\cos\theta)\hat{n} + (\sin\theta)\hat{n}\times\vec{x}_\perp$$

$$= x_{\rm par}(\cos\theta)\hat{n} + (\cos\theta)\vec{x}_\perp + x_{\rm par}(1-\cos\theta)\hat{n} + (\sin\theta)\hat{n}\times\vec{x}_\perp$$
$$= x_{\rm par}\hat{n} + [(\cos\theta)\vec{x}_\perp + (\sin\theta)\hat{n}\times\vec{x}_\perp],$$

which shows that $\vec{R}_{\hat{n}}(\theta)$ rotates $\vec{x}$ according to the right hand rule around $\hat{n}$ by the angle θ.

Finally, (2.11.18) and (2.11.19) follow from (2.11.1). □

The following result uses an eigenaxis rotation to obtain an identity between a pair of nonzero physical vectors.

Fact 2.11.3. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors, assume that $\vec{x}$ and $\vec{y}$ are not parallel, let $\hat{n}$ be a unit dimensionless physical vector that is orthogonal to both $\vec{x}$ and $\vec{y}$, and define $\theta \triangleq \theta_{\vec{y}/\vec{x}/\hat{n}}$. Then,

$$\vec{y} = \frac{|\vec{y}|}{|\vec{x}|}\vec{R}_{\hat{n}}(\theta)\vec{x} \tag{2.11.22}$$
$$= \frac{|\vec{y}|}{|\vec{x}|}\Big[(\cos\theta)\vec{x} + (\sin\theta)\hat{n}\times\vec{x}\Big]. \tag{2.11.23}$$

Proof. First note that $\hat{n} = [(\operatorname{sign}\theta)/|\vec{x}\times\vec{y}|](\vec{x}\times\vec{y})$. It thus follows from (2.11.1), (2.4.36), and (2.1.1) that

$$\begin{aligned}
\vec{R}_{\hat{n}}(\theta)\vec{x} &= [(\cos\theta)\vec{I} + (1-\cos\theta)\hat{n}\hat{n}' + (\sin\theta)\hat{n}^\times]\vec{x}\\
&= (\cos\theta)\vec{x} + (1-\cos\theta)\hat{n}\hat{n}'\vec{x} + (\sin\theta)\hat{n}\times\vec{x}\\
&= (\cos\theta)\vec{x} + (\sin\theta)\hat{n}\times\vec{x}\\
&= (\cos\theta)\vec{x} + \frac{(\operatorname{sign}\theta)\sin\theta}{|\vec{x}\times\vec{y}|}(\vec{x}\times\vec{y})\times\vec{x}\\
&= (\cos\theta)\vec{x} + \frac{(\operatorname{sign}\theta)\sin\theta}{|\vec{x}\times\vec{y}|}[(\vec{x}\cdot\vec{x})\vec{y} - (\vec{x}\cdot\vec{y})\vec{x}]\\
&= (\cos\theta)\vec{x} + \frac{(\operatorname{sign}\theta)\sin\theta}{|\vec{x}||\vec{y}||\sin\theta|}[|\vec{x}|^2\vec{y} - |\vec{x}||\vec{y}|(\cos\theta)\vec{x}]\\
&= (\cos\theta)\vec{x} + \frac{1}{|\vec{x}||\vec{y}|}[|\vec{x}|^2\vec{y} - |\vec{x}||\vec{y}|(\cos\theta)\vec{x}]\\
&= \frac{|\vec{x}|}{|\vec{y}|}\vec{y}.
\end{aligned}$$
□

Note that (2.11.23) can be written as

$$\vec{y} = \frac{|\vec{y}|}{|\vec{x}|}\vec{M}\vec{x}, \tag{2.11.24}$$

where

$$\vec{M} \triangleq (\cos\theta)\vec{I} + (\sin\theta)\hat{\theta}^\times_{\vec{y}/\vec{x}}. \tag{2.11.25}$$

However, $\vec{M}$ does not necessarily satisfy $\vec{M}\vec{M}' = \vec{I}$, and thus is not necessarily a physical rotation matrix.

The following result shows that the eigenaxis vector $\hat{n}$ has the same components when resolved in both frames. Consequently, when the two frames coincide with the same body-fixed frame for a rigid body before and after rotation, the vector $\hat{n}$ can be viewed as body-fixed.

Fact 2.11.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\hat{n}$ be a unit dimensionless physical vector, let $\theta \in (-\pi, \pi]$, and assume that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. Then,

$$\hat{n}|_{\mathrm{B}} = \hat{n}|_{\mathrm{A}}. \tag{2.11.26}$$

Proof. Note that

$$\hat{n}|_{\mathrm{B}} = (\vec{R}_{\mathrm{B/A}}\hat{n})\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \hat{n}|_{\mathrm{B}} = \mathcal{O}_{\mathrm{A/B}}\, \hat{n}|_{\mathrm{B}} = \hat{n}|_{\mathrm{A}}. \qquad \square$$

Each pair of frames $\mathrm{F_A}$ and $\mathrm{F_B}$ is related by a unique physical rotation matrix, namely, $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A}$. The following result is *Euler's theorem*, which states that every physical rotation matrix can be expressed in the eigenaxis/eigenangle form (2.11.1). In particular, this result provides explicit expressions for the eigenaxis and eigenangle for a given physical rotation matrix.

We consider three cases separately. In the first case, the rotation is through an eigenangle of 0 rad, and thus the eigenaxis is arbitrary. In the second case, the frames are related by a rotation through an eigenangle of π rad, and the eigenaxis can be chosen in two distinct ways. In the last case, there are two distinct eigenangles in the range $(-\pi, \pi)$ with corresponding the eigenaxes. These cases are distinguished by the trace inequalities given by Fact 2.10.18.

We first consider a rotation through 0 rad. This case corresponds to the condition $\operatorname{tr} \vec{R}_{\mathrm{B/A}} = 3$, which occurs when the upper bound is attained in (2.10.57).

Fact 2.11.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that

$$\operatorname{tr} \vec{R}_{\mathrm{B/A}} = 3. \tag{2.11.27}$$

Then, for every unit dimensionless physical vector $\hat{n}$, $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(0) = \vec{I}$.

We next consider a rotation through an eigenangle of π rad. This case corresponds to the condition $\operatorname{tr} \vec{R}_{\mathrm{B/A}} = -1$, which occurs when the lower bound is attained in (2.10.57). In this case there is one eigenangle and two distinct eigenaxes.

Fact 2.11.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that

$$\operatorname{tr} \vec{R}_{\mathrm{B/A}} = -1. \tag{2.11.28}$$

Then, there exist exactly two representations of $\vec{R}_{\mathrm{B/A}}$ of the form (2.11.1). In particular, let $\hat{n}_{\mathrm{B/A}}$ be a unit dimensionless physical vector satisfying

$$\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} = \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} + \vec{I}\right) \tag{2.11.29}$$

or, equivalently,

$$\hat{n}_{\mathrm{B/A}}^{\times 2} = \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} - \vec{I}\right). \tag{2.11.30}$$

Therefore,

$$n_{\mathrm{B/A}} n_{\mathrm{B/A}}^{\mathsf{T}} = \tfrac{1}{2}(\mathcal{R}_{\mathrm{B/A}} + I_3), \tag{2.11.31}$$

$$n_{\mathrm{B/A}}^{\times 2} = \tfrac{1}{2}(\mathcal{R}_{\mathrm{B/A}} - I_3). \tag{2.11.32}$$

Then, $\vec{R}_{\mathrm{B/A}}$ has the two representations

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\pi) = \vec{R}_{-\hat{n}_{\mathrm{B/A}}}(\pi). \tag{2.11.33}$$

Furthermore,

$$\mathcal{R}_{\mathrm{B/A}} = -I_3 + 2 n_{\mathrm{B/A}} n_{\mathrm{B/A}}^{\mathsf{T}} \tag{2.11.34}$$

$$= I_3 + 2 n_{\mathrm{B/A}}^{\times 2}, \tag{2.11.35}$$

where

$$n_{\mathrm{B/A}} \triangleq \hat{n}_{\mathrm{B/A}}\big|_{\mathrm{B}} = \hat{n}_{\mathrm{B/A}}\big|_{\mathrm{A}}. \tag{2.11.36}$$

Finally, the eigenvalues of $\mathcal{R}_{\mathrm{B/A}}$ are 1, −1, and −1,

$$\mathcal{R}_{\mathrm{B/A}} n_{\mathrm{B/A}} = n_{\mathrm{B/A}}, \tag{2.11.37}$$

and, if $m \in \mathbb{R}^3$ satisfies $m^{\mathsf{T}} n_{\mathrm{B/A}} = 0$, then

$$\mathcal{R}_{\mathrm{B/A}} m = -m. \tag{2.11.38}$$

It follows from (2.11.29) that

$$|\hat{n}_{\mathrm{B/A}}|^2 = \hat{n}'_{\mathrm{B/A}} \hat{n}_{\mathrm{B/A}} = \operatorname{tr} \hat{n}_{\mathrm{B/A}} \hat{n}'_{\mathrm{B/A}} = \tfrac{1}{2} \operatorname{tr}\left(\vec{R}_{\mathrm{B/A}} + \vec{I}\right) = \tfrac{1}{2}(-1+3) = 1.$$

Therefore, $\hat{n}_{\mathrm{B/A}}$ satisfying (2.11.29) is a unit vector.

In Fact 2.11.6 the unit vector $n_{\mathrm{B/A}}$ satisfies (2.11.31). Consequently, the matrix $\frac{1}{2}(\mathcal{R}_{\mathrm{B/A}} + I_3)$ is positive semidefinite and has rank 1. Thus, there exist exactly two vectors $x \in \mathbb{R}^3$ that satisfy $xx' = \frac{1}{2}(\mathcal{R}_{\mathrm{B/A}} + I_3)$, namely, $x = n_{\mathrm{B/A}}$ and $x = -n_{\mathrm{B/A}}$.

In the last case, the eigenangle is assumed to be neither 0 rad nor π rad. This condition is equivalent to strict inequality in the lower and upper bounds in (2.11.39). In this case, there are two distinct eigenangles and two distinct eigenaxes.

Fact 2.11.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that

$$-1 < \operatorname{tr} \vec{R}_{\mathrm{B/A}} < 3. \tag{2.11.39}$$

Then, there exist exactly two representations of $\vec{R}_{\mathrm{B/A}}$ of the form (2.11.1). In particular, let $\theta_{\mathrm{B/A}} \in (0, \pi)$ satisfy

$$\cos \theta_{\mathrm{B/A}} = \tfrac{1}{2}(\operatorname{tr} \vec{R}_{\mathrm{B/A}} - 1), \tag{2.11.40}$$

and let $\hat{n}_{\mathrm{B/A}}$ be the unit dimensionless physical vector satisfying

$$\hat{n}_{\mathrm{B/A}}^{\times} = \frac{1}{2\sin\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right). \tag{2.11.41}$$

Then, the eigenvalues of $\vec{R}_{\mathrm{B/A}}$ are $1, \lambda, \overline{\lambda}$, where $\lambda \triangleq \cos\theta_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\jmath$. Furthermore,

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) = \vec{R}_{-\hat{n}_{\mathrm{B/A}}}(-\theta_{\mathrm{B/A}}), \tag{2.11.42}$$

$$\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.11.43}$$

In terms of $\mathcal{R}_{\mathrm{B/A}}$,

$$\cos\theta_{\mathrm{B/A}} = \tfrac{1}{2}(\operatorname{tr}\mathcal{R}_{\mathrm{B/A}} - 1), \tag{2.11.44}$$

$$n_{\mathrm{B/A}}^{\times} = \frac{1}{2\sin\theta_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}}), \tag{2.11.45}$$

$$\mathcal{R}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}})I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathsf{T}} + (\sin\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} \tag{2.11.46}$$

$$= I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times 2} + (\sin\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times}, \tag{2.11.47}$$

$$\mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}} = n_{\mathrm{B/A}}, \tag{2.11.48}$$

where

$$n_{\mathrm{B/A}} \triangleq \hat{n}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \hat{n}_{\mathrm{B/A}}\Big|_{\mathrm{A}}. \tag{2.11.49}$$

Finally, let $m = m_1 + m_2\jmath \in \mathbb{C}^3$, where $m_1, m_2 \in \mathbb{R}^3$ satisfy $n_{\mathrm{B/A}}^{\mathsf{T}}m_1 = n_{\mathrm{B/A}}^{\mathsf{T}}m_2 = m_1^{\mathsf{T}}m_2 = 0$. Then,

$$\mathcal{R}_{\mathrm{B/A}}m = [\cos\theta_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\jmath]m. \tag{2.11.50}$$

Proof. Since $\vec{R}_{\mathrm{B/A}}$ is a physical rotation matrix, it follows that one of its eigenvalues is 1. It follows from (2.11.39) that the remaining eigenvalues of $\vec{R}_{\mathrm{B/A}}$ are not real. Furthermore, it follows from (2.11.40) that $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 1 + 2\cos\theta_{\mathrm{B/A}}$, and thus the eigenvalues of $\vec{R}_{\mathrm{B/A}}$ are $1, \lambda, \overline{\lambda}$, where λ satisfies $|\lambda| = 1$ and λ is neither 1 nor -1.

To show that $\hat{n}_{\mathrm{B/A}}$ given by (2.11.41) is a unit vector, note that

$$\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - |\hat{n}_{\mathrm{B/A}}|^2\vec{I} = \hat{n}_{\mathrm{B/A}}^{\times 2} = \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right)^2 = \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}^2_{\mathrm{B/A}} - 2\vec{I} + \vec{R}'^2_{\mathrm{B/A}}\right).$$

Taking the trace yields

$$-2|\hat{n}_{\mathrm{B/A}}|^2 = \frac{\operatorname{tr}\vec{R}^2_{\mathrm{B/A}} - 3}{2\sin^2\theta_{\mathrm{B/A}}}.$$

Hence

$$|\hat{n}_{\mathrm{B/A}}|^2 = \frac{3 - \operatorname{tr}\vec{R}^2_{\mathrm{B/A}}}{4\sin^2\theta_{\mathrm{B/A}}} = \frac{3 - (1 + \lambda^2 + \overline{\lambda}^2)}{4\sin^2\theta_{\mathrm{B/A}}} = \frac{1 + \sin^2\theta_{\mathrm{B/A}} - \cos^2\theta_{\mathrm{B/A}}}{2\sin^2\theta_{\mathrm{B/A}}} = 1.$$

Next, note that

$$\begin{aligned}
\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} &= \hat{n}^{\times 2}_{\mathrm{B/A}} + \vec{I} \\
&= \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right)^2 + \vec{I} \\
&= \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}^2_{\mathrm{B/A}} - 2\vec{I} + \vec{R}'^2_{\mathrm{B/A}}\right) + \vec{I} \\
&= \frac{1}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) + \vec{I} - \frac{1}{2\sin^2\theta_{\mathrm{B/A}}}\vec{I}.
\end{aligned}$$

Therefore, using (2.11.40), (2.11.41), and Problem 2.24.17 it follows that

$$\begin{aligned}
&\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) \\
&= (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \\
&= (\cos\theta_{\mathrm{B/A}})\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{4\sin^2\theta_{\mathrm{B/A}}}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) + (1-\cos\theta_{\mathrm{B/A}})\vec{I} - \frac{1-\cos\theta_{\mathrm{B/A}}}{2\sin^2\theta_{\mathrm{B/A}}}\vec{I} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \\
&= \vec{I} + \frac{1}{4(1+\cos\theta_{\mathrm{B/A}})}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) - \frac{1}{2(1+\cos\theta_{\mathrm{B/A}})}\vec{I} + \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right) \\
&= \frac{1+2\cos\theta_{\mathrm{B/A}}}{2(1+\cos\theta_{\mathrm{B/A}})}\vec{I} + \frac{1}{4(1+\cos\theta_{\mathrm{B/A}})}\left(\vec{R}^2_{\mathrm{B/A}} + \vec{R}'^2_{\mathrm{B/A}}\right) + \frac{1}{2}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right) = \vec{R}_{\mathrm{B/A}}. \qquad \square
\end{aligned}$$

Note that, in Fact 2.11.7, $\theta_{\mathrm{B/A}}$ is defined such that $\theta_{\mathrm{B/A}} \in (0, \pi)$. In fact,

$$\theta_{\mathrm{B/A}} = \operatorname{acos}[\tfrac{1}{2}(\operatorname{tr}\vec{R}_{\mathrm{B/A}} - 1)]. \tag{2.11.51}$$

Furthermore, since Rodrigues's formula (2.11.1) is defined for $\theta \in (-\pi, \pi]$, within the context of Fact 2.11.5, we define $\theta_{\mathrm{B/A}} = 0$, while, within the context of Fact 2.11.6, we define $\theta_{\mathrm{B/A}} = \pi$. Consequently, in all cases, (2.11.51) is valid, and the notation $\theta_{\mathrm{B/A}}$ denotes an element of $[0, \pi]$, despite the fact that θ in Rodrigues's formula (2.11.1) may be an element of $(-\pi, \pi]$. Since, by definition, $\theta_{\mathrm{B/A}} \in [0, \pi]$ in all cases, it follows that

$$\theta_{\mathrm{B/A}} = \theta_{\mathrm{A/B}}. \tag{2.11.52}$$

If $\theta_{\mathrm{B/A}} \in (0, \pi)$, then the corresponding eigenaxis $\hat{n}_{\mathrm{B/A}}$ is unique. Therefore, if $\theta_{\mathrm{B/A}} \in (0, \pi)$, then

$$\hat{n}_{\mathrm{B/A}} = -\hat{n}_{\mathrm{A/B}}. \tag{2.11.53}$$

However, in the case $\theta_{\mathrm{B/A}} = 0$, the eigenaxis $\hat{n}_{\mathrm{B/A}}$ is arbitrary, and thus (2.11.53) is not meaningful. Furthermore, in the case $\theta_{\mathrm{A/B}} = \pi$, there exist exactly two choices of $\hat{n}_{\mathrm{B/A}}$, which are related by the factor -1. Therefore, the notation $\hat{n}_{\mathrm{B/A}}$ is ambiguous, and thus (2.11.53) is not meaningful.

The following result states Euler's theorem for the three cases considered in Fact 2.11.5, Fact 2.11.6, and Fact 2.11.7.

Fact 2.11.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, there exist a unit dimensionless physical vector $\hat{n}$ and $\theta \in (-\pi, \pi]$ such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. In addition,

$$\hat{n}\big|_{\mathrm{B}} = \hat{n}\big|_{\mathrm{A}}. \tag{2.11.54}$$

Proof. Since $\vec{R}_{\mathrm{B/A}}$ is a physical rotation matrix, there exists a unit dimensionless physical vector $\hat{n}$ such that $\hat{n} = \vec{R}_{\mathrm{B/A}}\hat{n}$. If $\vec{R}_{\mathrm{B/A}} = \vec{I}$, then the result holds with $\theta = 0$. Now assume that $\vec{R}_{\mathrm{B/A}} \neq \vec{I}$. Since $\vec{R}_{\mathrm{B/A}}$ has exactly one eigenvalue equal to 1 and since $\hat{n}$ is a unit dimensionless eigenvector of $\vec{R}_{\mathrm{B/A}}$, it follows that $\hat{n}$ must be equal to either $\hat{n}_{\mathrm{B/A}}$ or $-\hat{n}_{\mathrm{B/A}}$. In both cases, there exists $\theta \in (-\pi, \pi]$ such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. In particular, if $\hat{n} = \hat{n}_{\mathrm{B/A}}$, then $\theta = \theta_{\mathrm{B/A}}$, whereas, if $\hat{n} = -\hat{n}_{\mathrm{B/A}}$, then $\theta = \theta_{\mathrm{B/A}} - 2\pi$. □

The following result considers the reverse rotation in all three cases. This is a restatement of (2.11.1), (2.11.2), and (2.11.3) with $\theta = \theta_{\mathrm{B/A}}$ and $\hat{n} = \hat{n}_{\mathrm{B/A}}$ using (2.11.16), (2.11.52), and (2.11.53).

Fact 2.11.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in (0, \pi)$. Then,

$$\vec{R}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}, \tag{2.11.55}$$

$$\vec{R}_{\mathrm{A/B}} = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}, \tag{2.11.56}$$

$$\vec{R}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\cos\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}, \tag{2.11.57}$$

$$\vec{R}_{\mathrm{A/B}} = \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\cos\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) - (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}, \tag{2.11.58}$$

$$\vec{R}_{\mathrm{B/A}} = \vec{I} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}^{\times 2}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}, \tag{2.11.59}$$

$$\vec{R}_{\mathrm{A/B}} = \vec{I} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}^{\times 2}_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}. \tag{2.11.60}$$

The following result assigns a physical rotation matrix to a pair of physical vectors of the same length. This physical rotation matrix is defined in terms of an eigenaxis rotation, which is unique if and only if the eigenangle is neither 0 rad nor π rad.

Fact 2.11.10. Let $\vec{x}$ and $\vec{y}$ be physical vectors, and assume that $|\vec{x}| = |\vec{y}| \neq 0$. Then, there exists a physical rotation matrix $\vec{R}$ such that $\vec{y} = \vec{R}\vec{x}$. Now, let $\vec{R}$ be a physical rotation matrix such that $\vec{y} = \vec{R}\vec{x}$. Then, the following statements hold:

i) If either $\theta_{\hat{y}/\hat{x}} = 0$ or $\theta_{\hat{y}/\hat{x}} = \pi$, then, for every unit dimensionless physical vector $\hat{n}$ such that $\hat{n}'\vec{x} = 0$, it follows that $\vec{R} = \vec{R}_{\hat{n}}(\theta_{\hat{y}/\hat{x}})$.

ii) If $\theta_{\hat{y}/\hat{x}} \in (0, \pi)$, then there exists a unique unit dimensionless physical vector $\hat{n}$ such that $\hat{n}'\vec{x} = \hat{n}'\vec{y} = 0$ and $\vec{R} = \vec{R}_{\hat{n}}(\theta_{\hat{y}/\hat{x}})$. In particular, $\hat{n} = \hat{\theta}_{\hat{y}/\hat{x}}$. In addition, $\vec{R} = \vec{R}_{-\hat{n}}(-\theta_{\hat{y}/\hat{x}})$.

The following result replaces $\hat{n}^{\times}_{\mathrm{B/A}}$ in (2.11.1) by a difference of physical matrices

Fact 2.11.11. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\hat{v}_{\mathrm{B/A}}$ and $\hat{w}_{\mathrm{B/A}}$ satisfy $\hat{n}_{\mathrm{B/A}} = \hat{v}_{\mathrm{B/A}} \times \hat{w}_{\mathrm{B/A}}$. Then,

$$\vec{R}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})(\hat{w}_{\mathrm{B/A}}\hat{v}'_{\mathrm{B/A}} - \hat{v}_{\mathrm{B/A}}\hat{w}'_{\mathrm{B/A}}). \tag{2.11.61}$$

Proof. The result follows from (2.9.21). □

The following result determines the eigenaxis rotation arising from a pair of eigenaxis rotations.

Fact 2.11.12. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\cos\tfrac{1}{2}\theta_{\mathrm{C/A}} = (\cos\tfrac{1}{2}\theta_{\mathrm{C/B}})(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}}) - (\sin\tfrac{1}{2}\theta_{\mathrm{C/B}})(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}'_{\mathrm{C/B}}\hat{n}_{\mathrm{B/A}}, \tag{2.11.62}$$

$$\begin{aligned}\hat{n}_{\mathrm{C/A}} &= (\csc\tfrac{1}{2}\theta_{\mathrm{C/B}})[(\sin\tfrac{1}{2}\theta_{\mathrm{C/B}})(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{C/B}} + (\cos\tfrac{1}{2}\theta_{\mathrm{C/B}})(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\\ &\quad + (\sin\tfrac{1}{2}\theta_{\mathrm{C/B}})(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{C/B}}\times\hat{n}_{\mathrm{B/A}})]\\ &= \frac{\cot\tfrac{1}{2}\theta_{\mathrm{C/A}}}{1-\hat{n}'_{\mathrm{C/B}}\hat{n}_{\mathrm{B/A}}(\tan\tfrac{1}{2}\theta_{\mathrm{C/B}})\tan\tfrac{1}{2}\theta_{\mathrm{B/A}}}[(\tan\tfrac{1}{2}\theta_{\mathrm{C/B}})\hat{n}_{\mathrm{C/B}} + (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\\ &\quad + (\tan\tfrac{1}{2}\theta_{\mathrm{C/B}})(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})(\hat{n}'_{\mathrm{C/B}}\hat{n}_{\mathrm{B/A}})].\end{aligned} \tag{2.11.63}$$

2.12 Rotating a Rigid Body around a Point

Since a frame has no location, rotating a frame affects only the orientation of the frame. Rotating a body, however, concerns a physical object, which has an orientation as well as a location in space. Consequently, rotating a body does not fully determine the resulting location of the body; additional information is needed to specify the effect of the rotation on the location of the body. Although time plays no role in this chapter, we assume that the body is rigid in order to emphasize that the shape of the body is unchanged by the rotation.

In order to discuss the effect of rotating a rigid body, we specify a point that is fixed in the body and that remains spatially invariant despite the rotation. For example, assume that the rigid body $\mathcal{B}$ is a cube, let the point x be one of the vertices of $\mathcal{B}$, let $\vec{R}$ be a physical rotation matrix, let $\mathcal{B}'$ denote the body $\mathcal{B}$ rotated by $\vec{R}$, and let x' denote the point on $\mathcal{B}'$ corresponding to x on $\mathcal{B}$. We then assume that the rotation of $\mathcal{B}$ occurs so that x is spatially invariant in the sense that $\vec{r}_{x'/x} = \vec{0}$. In this case, we say that $\mathcal{B}$ *is rotated by* $\vec{R}$ *around* x. To be more precise, let $\mathcal{B} = \{y_1, \ldots, y_l\}$, where $y_1, \ldots, y_l$ are particles comprising $\mathcal{B}$, and let $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$, where $y'_1, \ldots, y'_l$ are the corresponding particles in $\mathcal{B}'$ after rotation. Note that, since mass plays no role here, the particles $y_1, \ldots, y_l$ can be viewed as points in space that define $\mathcal{B}$, while the particles $y'_1, \ldots, y'_l$ in $\mathcal{B}'$ can be viewed as the points in $\mathcal{B}'$ that correspond to $y_1, \ldots, y_l$ in $\mathcal{B}$. Since $\mathcal{B}$ is rotated by $\vec{R}$ around x, it follows that every particles in $\mathcal{B}$ is rotated by $\vec{R}$ around x, and thus, for all $i = 1, \ldots, l$, $\vec{r}_{y'_i/x} = \vec{R}\vec{r}_{y_i/x}$.

Fact 2.8.4 shows that a physical position vector can be rotated around its tail to obtain a physical position vector with a different tip but the same tail. Using this idea, a body $\mathcal{B}$ can be rotated around a point x that is not necessarily contained in $\mathcal{B}$. In particular, if x is not a point in $\mathcal{B}$, then we can view x as connected to $\mathcal{B}$ by means of a rigid massless link. With this extension, $\mathcal{B}$ can be rotated around x whether or not x is a point in $\mathcal{B}$.

Definition 2.12.1. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ be a rigid body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$ be a rigid body with particles $y'_1, \ldots, y'_l$ whose masses are $m_1, \ldots, m_l$, respectively. Then, $\mathcal{B}$ and $\mathcal{B}'$ are *identical* if there exists a physical rotation matrix $\vec{R}$ such that, for all $i, j = 1, \ldots, l$, $\vec{r}_{y'_i/y'_j} = \vec{R}\vec{r}_{y_i/y_j}$; $\mathcal{B}$ and $\mathcal{B}'$ have the *same orientation* if, for all $i, j = 1, \ldots, l$, $\vec{r}_{y'_i/y'_j} = \vec{r}_{y_i/y_j}$; and $\mathcal{B}$ and $\mathcal{B}'$ are *colocated* if, for all $i = 1, \ldots, l$, $\vec{r}_{y'_i/y_i} = \vec{0}$.

Note that, if $\mathcal{B}$ and $\mathcal{B}'$ have the same orientation, then they are identical with $\vec{R} = \vec{I}$. Furthermore, if $\mathcal{B}$ and $\mathcal{B}'$ are colocated, then, for all $i, j = 1, \ldots, l$,

$$\vec{r}_{y_i/y_j} = \vec{r}_{y_i/y'_i} + \vec{r}_{y'_i/y'_j} + \vec{r}_{y'_j/y_j} = \vec{r}_{y'_i/y'_j}, \tag{2.12.1}$$

and thus $\mathcal{B}$ and $\mathcal{B}'$ have the same orientation.

If $\mathcal{B}$ and $\mathcal{B}'$ are identical, then the points y_i and y'_i are *corresponding particles.* The following result shows that, if two bodies are identical, then the distances between the particles in all corresponding pairs of particles are equal.

Fact 2.12.2. Let $\mathcal{B}$ and $\mathcal{B}'$ be identical rigid bodies. Then, for all $i, j = 1, \ldots, l$, $|\vec{r}_{y_i/y_j}| = |\vec{r}_{y'_i/y'_j}|$

Proof. Note that, for all $i, j \in \{1, \ldots, l\}$,

$$|\vec{r}_{y'_i/y'_j}| = |\vec{r}_{y'_i/x} - \vec{r}_{y'_j/x}| = |\vec{R}(\vec{r}_{y_i/x} - \vec{r}_{y_j/x})| = |\vec{r}_{y_i/x} - \vec{r}_{y_j/x}| = |\vec{r}_{y_i/y_j}|. \qquad \square$$

The converse of Fact 2.12.2 is not true. Consider, for example, rigid bodies $\mathcal{B}$ and $\mathcal{B}'$, which are mirror images of each other. In this case, distances between corresponding particles are equal, but the bodies are not necessarily related by a physical rotation matrix.

As an alternative interpretation, let $\mathcal{B}$ be a rigid body, let x be a point, and rotate $\mathcal{B}$ around the point x by means of the physical rotation matrix $\vec{R}$. It follows from Rodrigues's formula that there exists a unit dimensionless physical vector $\hat{n}$ and an eigenangle θ such that $\vec{R} = \vec{R}_{\hat{n}}(\theta)$. The following result shows that rotating $\mathcal{B}$ around the point x by means of $\vec{R}$ is equivalent to rotating $\mathcal{B}$ around the line $\mathcal{L}(x, \hat{n})$ by means of $\vec{R}_{\hat{n}}(\theta)$. Recall that $\mathcal{L}(x, \hat{n})$ is the line parallel to $\hat{n}$ and passing through x.

Fact 2.12.3. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ be a rigid body, let $\vec{R}_{\hat{n}}(\theta)$ be a physical rotation matrix, let x be a point, and let $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$ denote $\mathcal{B}$ rotated by $\vec{R}_{\hat{n}}(\theta)$ around x. Then, for all $i = 1, \ldots, l$, $\hat{n}'\vec{r}_{y'_i/y_i} = 0$ and $\theta_{\vec{r}_{y'_i/x}/\vec{r}_{y_i/x}} = \theta$.

Proof. Let $i \in \{1, \ldots, l\}$. Note that

$$\hat{n}'\vec{r}_{y'_i/y_i} = \hat{n}'(\vec{r}_{y'_i/x} + \vec{r}_{x/y_i}) = \hat{n}'(\vec{R}_{\hat{n}}(\theta)\vec{r}_{y_i/x} - \vec{r}_{y_i/x}) = \hat{n}'(\vec{R}_{\hat{n}}(\theta) - \vec{I})\vec{r}_{y_i/x} = 0. \qquad \square$$

Prove theta result. Need figure to illustrate this

2.13 Euler Angles, Euler Rotation Matrices, and Euler Orientation Matrices

Rotating one frame to yield another frame can be achieved through a sequence of three eigenaxis rotations, where each eigenaxis is chosen to be an axis of either the initial frame or the frame resulting from the preceding rotation. Consequently, every physical rotation matrix can be expressed as the product of three physical rotation matrices, where each physical rotation matrix is an eigenaxis rotation represented by Rodrigues's formula. The three rotations involve a total of four frames, namely, the initial and final frames as well as two intermediate frames. Each eigenaxis rotation is an *Euler rotation*, and the directed angle that defines the rotation is an *Euler angle*. Consequently, the orientation of the final frame relative to the initial frame can be expressed as the product of three orientation matrices.

There are twelve different Euler-rotation sequences, for example, 1-2-3, where the first rotation is around the $\hat{\imath}$ axis of the initial frame, the second rotation is around the $\hat{\jmath}$ axis of the second frame, and the third rotation is around the $\hat{k}$ axis of the third frame. The 3-2-1 and 3-1-3 Euler-rotation sequences are the most frequently used. Although there are 12 Euler-rotation sequences that transform a given frame into another given frame, by renaming the frame axes it can be seen that there are only two distinct sequences; these can be represented by the Euler-rotation sequences 3-2-1 and 3-1-3. Each of the remaining ten Euler-rotation sequences is equivalent to one of these two Euler-rotation sequences under a renaming of the frame axes.

As an example of an Euler-rotation sequence, assume that the frame $\mathrm{F_B}$ is obtained by rotating the frame $\mathrm{F_A}$ around the eigenaxis $\hat{k}_\mathrm{A}$. Then,

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}}) = \vec{R}_{\hat{k}_\mathrm{A}}(\theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}).$$

Hence, using the notation of (2.11.10),

$$\mathrm{F_A} \xrightarrow[\hat{k}_\mathrm{A}]{\theta} \mathrm{F_B}, \tag{2.13.1}$$

and, since $\hat{k}_\mathrm{B} = \hat{k}_\mathrm{A}$,

$$\mathrm{F_A} \xrightarrow[\hat{k}_\mathrm{B}]{\theta} \mathrm{F_B}, \tag{2.13.2}$$

where θ is the name of the eigenangle, that is, $\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}} = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}$. For convenience, we write

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}. \tag{2.13.3}$$

Note that (2.13.1) is equivalent to

$$\mathrm{F_A} \xrightarrow[-\hat{k}_\mathrm{A}]{-\theta} \mathrm{F_B}, \tag{2.13.4}$$

which we write as

$$\mathrm{F_A} \xrightarrow[-3]{-\theta} \mathrm{F_B}. \tag{2.13.5}$$

Equivalently,

$$\mathrm{F_B} \xrightarrow[3]{-\theta} \mathrm{F_A}, \tag{2.13.6}$$

$$\mathrm{F_B} \xrightarrow[-3]{\theta} \mathrm{F_A}. \tag{2.13.7}$$

If $\theta = \pi$, then $-\theta$ must be replaced by π in the above expressions. See (2.3.2).

The following result considers (2.11.1) when the eigenaxis is a frame axis.

Fact 2.13.1. Let $\mathrm{F_A}$ be a frame, and let $\theta \in (-\pi, \pi]$. If $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{\imath}_\mathrm{A}}(\theta)\mathrm{F_A}$, then

$$\mathcal{O}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\Big|_\mathrm{A}^\mathsf{T} = \vec{R}_{\hat{\imath}_\mathrm{A}}(\theta)\Big|_\mathrm{A}^\mathsf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{2.13.8}$$

$$\theta = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{\imath}_\mathrm{A}} = \theta_{\hat{k}_\mathrm{B}/\hat{k}_\mathrm{A}/\hat{\imath}_\mathrm{A}}. \tag{2.13.9}$$

If $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\mathrm{F_A}$, then

$$\mathcal{O}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathsf{T}} = \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathsf{T}} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.13.10}$$

$$\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{\jmath}_\mathrm{A}} = \theta_{\hat{k}_\mathrm{B}/\hat{k}_\mathrm{A}/\hat{\jmath}_\mathrm{A}}. \tag{2.13.11}$$

If $\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\mathrm{F_A}$, then

$$\mathcal{O}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathsf{T}} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathsf{T}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.13.12}$$

$$\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}} = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}. \tag{2.13.13}$$

For convenience we define the *Euler rotation matrices*

$$\mathcal{R}_1(\theta) \triangleq \vec{R}_{\hat{\imath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \tag{2.13.14}$$

$$\mathcal{R}_2(\theta) \triangleq \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.13.15}$$

$$\mathcal{R}_3(\theta) \triangleq \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.13.16}$$

and the *Euler orientation matrices*

$$\mathcal{O}_1(\theta) \triangleq \mathcal{R}_1^{\mathsf{T}}(\theta) = \vec{R}_{\hat{\imath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{2.13.17}$$

$$\mathcal{O}_2(\theta) \triangleq \mathcal{R}_2^{\mathsf{T}}(\theta) = \vec{R}_{\hat{\jmath}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathsf{T}} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{2.13.18}$$

$$\mathcal{O}_3(\theta) \triangleq \mathcal{R}_3^{\mathsf{T}}(\theta) = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)\Big|_{\mathrm{A}}^{\mathsf{T}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.13.19}$$

Note that, for all angles θ, ϕ,

$$\mathcal{O}_1(\theta)\mathcal{O}_1(\phi) = \mathcal{O}_1(\theta + \phi), \tag{2.13.20}$$

and likewise for $\mathcal{O}_2$ and $\mathcal{O}_3$.

For a 3-2-1 rotation sequence, the Euler rotations and Euler angles are denoted by

$$\mathrm{F_A} \xrightarrow[3]{\Psi} \mathrm{F_B} \xrightarrow[2]{\Theta} \mathrm{F_C} \xrightarrow[1]{\Phi} \mathrm{F_D}. \tag{2.13.21}$$

In aircraft kinematics and dynamics, the 3-2-1 Euler angles Ψ, Θ, and Φ refer to *azimuth*, *elevation*, and *bank*, respectively. Thus,

$$\mathrm{F_B} = \vec{R}_{\mathrm{B/A}}\mathrm{F_A} = \vec{R}_{\hat{k}_\mathrm{A}}(\Psi)\mathrm{F_A}, \tag{2.13.22}$$

$$\mathrm{F_C} = \vec{R}_{\mathrm{C/B}}\mathrm{F_B} = \vec{R}_{\hat{\jmath}_\mathrm{B}}(\Theta)\mathrm{F_B}, \tag{2.13.23}$$

$$\mathrm{F_D} = \vec{R}_{\mathrm{D/C}}\mathrm{F_C} = \vec{R}_{\hat{\imath}_\mathrm{C}}(\Phi)\mathrm{F_C}, \tag{2.13.24}$$

where

$$\Psi = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}} = \theta_{\hat{\jmath}_\mathrm{B}/\hat{\jmath}_\mathrm{A}/\hat{k}_\mathrm{A}}, \tag{2.13.25}$$

$$\Theta = \theta_{\hat{\imath}_\mathrm{C}/\hat{\imath}_\mathrm{B}/\hat{\jmath}_\mathrm{B}} = \theta_{\hat{k}_\mathrm{C}/\hat{k}_\mathrm{B}/\hat{\jmath}_\mathrm{B}}, \tag{2.13.26}$$

$$\Phi = \theta_{\hat{\jmath}_\mathrm{D}/\hat{\jmath}_\mathrm{C}/\hat{\imath}_\mathrm{C}} = \theta_{\hat{k}_\mathrm{D}/\hat{k}_\mathrm{C}/\hat{\imath}_\mathrm{C}}. \tag{2.13.27}$$

Hence,

$$\vec{R}_{\mathrm{D/A}} = \vec{R}_{\mathrm{D/C}}\vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{\imath}_\mathrm{C}}(\Phi)\vec{R}_{\hat{\jmath}_\mathrm{B}}(\Theta)\vec{R}_{\hat{k}_\mathrm{A}}(\Psi). \tag{2.13.28}$$

Each component of a 3-2-1 rotation sequence can be interpreted as either a rotation or an orientation. The matrix

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{\mathsf{T}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathsf{T}} = \vec{R}_{\hat{k}_\mathrm{A}}(\Psi)\Big|_{\mathrm{A}}^{\mathsf{T}} = \mathcal{O}_3(\Psi) = \mathcal{R}_3(\Psi)^{\mathsf{T}} \tag{2.13.29}$$

gives the orientation of $\mathrm{F_B}$ with respect to $\mathrm{F_A}$ as a function of the eigenangle Ψ, which is measured from $\hat{\imath}_\mathrm{A}$ to $\hat{\imath}_\mathrm{B}$ or from $\hat{\jmath}_\mathrm{A}$ to $\hat{\jmath}_\mathrm{B}$ as shown in Figure 2.13.1. Likewise, as illustrated in Figure 2.13.2 and Figure 2.13.3,

$$\mathcal{O}_{\mathrm{C/B}} = \mathcal{R}_{\mathrm{B/C}} = \mathcal{R}_{\mathrm{C/B}}^{\mathsf{T}} = \vec{R}_{\mathrm{C/B}}\Big|_{\mathrm{B}}^{\mathsf{T}} = \vec{R}_{\hat{\jmath}_\mathrm{B}}(\Theta)\Big|_{\mathrm{B}}^{\mathsf{T}} = \mathcal{O}_2(\Theta) = \mathcal{R}_2(\Theta)^{\mathsf{T}}, \tag{2.13.30}$$

$$\mathcal{O}_{\mathrm{D/C}} = \mathcal{R}_{\mathrm{C/D}} = \mathcal{R}_{\mathrm{D/C}}^{\mathsf{T}} = \vec{R}_{\mathrm{D/C}}\Big|_{\mathrm{C}}^{\mathsf{T}} = \vec{R}_{\hat{\imath}_\mathrm{C}}(\Phi)\Big|_{\mathrm{C}}^{\mathsf{T}} = \mathcal{O}_1(\Phi) = \mathcal{R}_1(\Phi)^{\mathsf{T}}. \tag{2.13.31}$$

Using Fact 2.10.16 to combine the 3-2-1 rotation sequence yields the product of Euler orientation matrices given by

$$\mathcal{O}_{\mathrm{D/A}} = \mathcal{O}_{\mathrm{D/C}}\mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi). \tag{2.13.32}$$

Similarly, in terms of Euler rotation matrices,

$$\mathcal{R}_{\mathrm{D/A}} = \mathcal{R}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{C/B}}\mathcal{R}_{\mathrm{D/C}} = \mathcal{R}_3(\Psi)\mathcal{R}_2(\Theta)\mathcal{R}_1(\Phi). \tag{2.13.33}$$

For a 3-1-3 rotation sequence, the Euler rotations and Euler angles are denoted by

$$\mathrm{F_A} \xrightarrow[3]{\Phi} \mathrm{F_B} \xrightarrow[1]{\Theta} \mathrm{F_C} \xrightarrow[3]{\Psi} \mathrm{F_D}. \tag{2.13.34}$$

The orientation matrices for the 3-1-3 sequence are thus given by

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{B/A}}^{\mathsf{T}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}^{\mathsf{T}} = \vec{R}_{\hat{k}_\mathrm{A}}(\Phi)\Big|_{\mathrm{A}}^{\mathsf{T}} = \mathcal{O}_3(\Phi) = \mathcal{R}_3(\Phi)^{\mathsf{T}}, \tag{2.13.35}$$

$$\mathcal{O}_{\mathrm{C/B}} = \mathcal{R}_{\mathrm{B/C}} = \mathcal{R}_{\mathrm{C/B}}^{\mathsf{T}} = \vec{R}_{\mathrm{C/B}}\Big|_{\mathrm{B}}^{\mathsf{T}} = \vec{R}_{\hat{\imath}_\mathrm{B}}(\Theta)\Big|_{\mathrm{B}}^{\mathsf{T}} = \mathcal{O}_1(\Theta) = \mathcal{R}_1(\Theta)^{\mathsf{T}}, \tag{2.13.36}$$

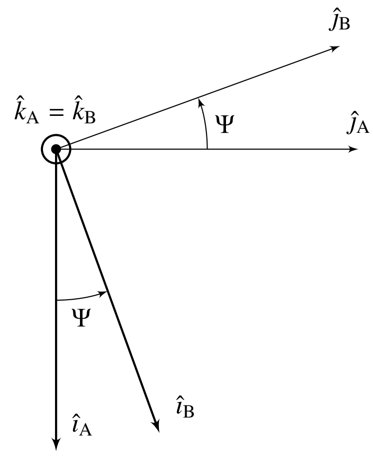


Figure 2.13.1: Rotation from F_A to F_B.

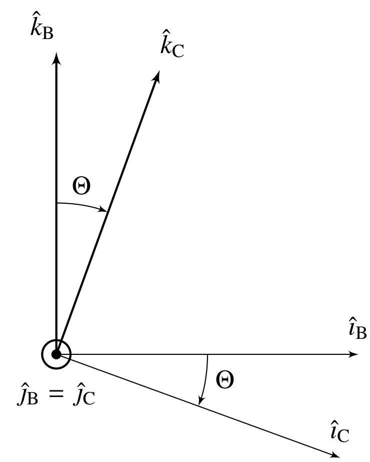


Figure 2.13.2: Rotation from F_B to F_C.

$$\mathcal{O}_{D/C} = \mathcal{R}_{C/D} = \mathcal{R}_{D/C}^{\mathsf{T}} = \vec{R}_{D/C}\Big|_{C}^{\mathsf{T}} = \vec{R}_{\hat{k}_C}(\Psi)\Big|_{C}^{\mathsf{T}} = \mathcal{O}_3(\Psi) = \mathcal{R}_3(\Psi)^{\mathsf{T}}. \tag{2.13.37}$$

Using Fact 2.10.15 to combine the 3-1-3 rotation sequence yields the product of Euler orientation matrices

$$\mathcal{O}_{D/A} = \mathcal{O}_{D/C}\mathcal{O}_{C/B}\mathcal{O}_{B/A} = \mathcal{O}_3(\Psi)\mathcal{O}_1(\Theta)\mathcal{O}_3(\Phi). \tag{2.13.38}$$

Similarly, in terms of Euler rotation matrices,

$$\mathcal{R}_{D/A} = \mathcal{R}_{B/A}\mathcal{R}_{C/B}\mathcal{R}_{D/C} = \mathcal{R}_3(\Phi)\mathcal{R}_1(\Theta)\mathcal{R}_3(\Psi). \tag{2.13.39}$$

For spacecraft attitude dynamics, the 3-1-3 Euler angles Φ, Θ, and Ψ refer to *precession*, *nutation*, and *spin*, respectively. For satellite orbital dynamics, the 3-1-3 Euler angles Ω, i, and ω refer to *right ascension of the ascending node*, *inclination*, and *argument of periapsis*, respectively. In

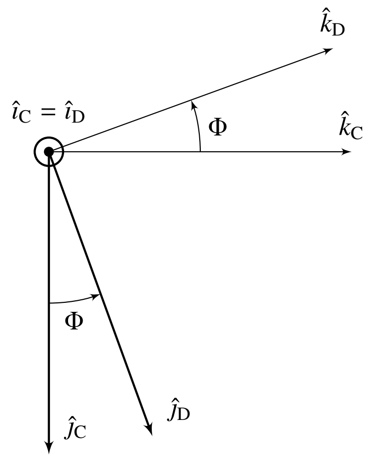


Figure 2.13.3: Rotation from F_C to F_D.

this case,

$$\mathcal{O}_{D/A} = \mathcal{O}_{D/C}\mathcal{O}_{C/B}\mathcal{O}_{B/A} = \mathcal{O}_3(\omega)\mathcal{O}_1(i)\mathcal{O}_3(\Omega). \tag{2.13.40}$$

2.14 Orientation Matrices and Euler Angles

Consider the 3-2-1 Euler angle rotation sequence

$$\mathrm{F_A} \xrightarrow[3]{\Psi} \mathrm{F_B} \xrightarrow[2]{\Theta} \mathrm{F_C} \xrightarrow[1]{\Phi} \mathrm{F_D}, \tag{2.14.1}$$

where Ψ, Θ, and Φ are the azimuth, elevation, and bank Euler angles, respectively. The corresponding orientation matrices are given by

$$\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_3(\Psi) \triangleq \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.14.2}$$

$$\mathcal{O}_{\mathrm{C/B}} = \mathcal{O}_2(\Theta) \triangleq \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix}, \tag{2.14.3}$$

$$\mathcal{O}_{\mathrm{D/C}} = \mathcal{O}_1(\Phi) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix}. \tag{2.14.4}$$

Combining these orientation matrices yields

$$\begin{aligned} \mathcal{O}_{\mathrm{D/A}} &= \mathcal{O}_{\mathrm{D/C}}\mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}} \\ &= \begin{bmatrix} (\cos\Theta)\cos\Psi & (\cos\Theta)\sin\Psi & -\sin\Theta \\ (\sin\Phi)(\sin\Theta)\cos\Psi - (\cos\Phi)\sin\Psi & (\sin\Phi)(\sin\Theta)\sin\Psi + (\cos\Phi)\cos\Psi & (\sin\Phi)\cos\Theta \\ (\cos\Phi)(\sin\Theta)\cos\Psi + (\sin\Phi)\sin\Psi & (\cos\Phi)(\sin\Theta)\sin\Psi - (\sin\Phi)\cos\Psi & (\cos\Phi)\cos\Theta \end{bmatrix}. \end{aligned} \tag{2.14.5}$$

Given an orientation matrix $\mathcal{O}_{\mathrm{D/A}}$, the Euler angles Ψ, Θ, and Φ are computed as shown below. Note that we want $\Psi, \Theta, \Phi \in (-\pi, \pi]$. First, note that

$$\mathcal{O}_{\mathrm{D/A}}(1,3) = -\sin\Theta. \tag{2.14.6}$$

Using the fact that $\sin(\pi - x) = \sin x$, it follows that two values of Θ that satisfy (2.14.6) are

$$\Theta_1 = -\sin^{-1}(\mathcal{O}_{\mathrm{D/A}}(1,3)), \tag{2.14.7}$$

$$\Theta_2 = \pi - \Theta_1. \tag{2.14.8}$$

Note that $\Theta_1 \in [-\pi/2, \pi/2]$ and thus $\Theta_2 \in [\pi/2, 3\pi/2]$. The range of elevation angle Θ_2 can be fixed by redefining it as

$$\Theta_2 = \begin{cases} \pi - \Theta_1, & \Theta_1 \in [0, \pi/2], \\ -\pi - \Theta_1, & \Theta_1 \in [-\pi/2, 0). \end{cases} \tag{2.14.9}$$

Next, note that

$$\mathcal{O}_{\mathrm{D/A}}(1,2) = (\cos\Theta)\sin\Psi, \tag{2.14.10}$$

$$\mathcal{O}_{\mathrm{D/A}}(1,1) = (\cos\Theta)\cos\Psi. \tag{2.14.11}$$

Assuming $\Theta \neq \pm\pi/2$ and using the function $\mathrm{atan2} : [-1,1] \times [-1,1] \to (-\pi, \pi]$, the two possible values of azimuth are given by

$$\Psi_1 = \mathrm{atan2}\left(\frac{\mathcal{O}_{\mathrm{D/A}}(1,2)}{\cos\Theta_1}, \frac{\mathcal{O}_{\mathrm{D/A}}(1,1)}{\cos\Theta_1}\right), \tag{2.14.12}$$

$$\Psi_2 = \operatorname{atan2}\left(\frac{\mathcal{O}_{\mathrm{D/A}}(1,2)}{\cos\Theta_2}, \frac{\mathcal{O}_{\mathrm{D/A}}(1,1)}{\cos\Theta_2}\right). \tag{2.14.13}$$

Note that $\Psi_1, \Psi_2 \in (-\pi, \pi]$.

Next, note that

$$\mathcal{O}_{\mathrm{D/A}}(2,3) = (\cos\Theta)\sin\Phi, \tag{2.14.14}$$

$$\mathcal{O}_{\mathrm{D/A}}(3,3) = (\cos\Theta)\cos\Phi. \tag{2.14.15}$$

Assuming $\Theta \neq \pm\pi/2$, the two possible values of bank Euler angle are given by

$$\Phi_1 = \operatorname{atan2}\left(\frac{\mathcal{O}_{\mathrm{D/A}}(2,3)}{\cos\Theta_1}, \frac{\mathcal{O}_{\mathrm{D/A}}(3,3)}{\cos\Theta_1}\right), \tag{2.14.16}$$

$$\Phi_2 = \operatorname{atan2}\left(\frac{\mathcal{O}_{\mathrm{D/A}}(2,3)}{\cos\Theta_2}, \frac{\mathcal{O}_{\mathrm{D/A}}(3,3)}{\cos\Theta_2}\right). \tag{2.14.17}$$

Note that $\Phi_1, \Phi_2 \in (-\pi, \pi]$.

Finally, we consider the case where $\Theta = \pm\pi/2$. First, consider $\Theta = \pi/2$. Note that

$$\mathcal{O}_{\mathrm{D/A}}(2,1) = (\sin\Phi)\cos\Psi - (\cos\Phi)\sin\Psi = \sin(\Phi - \Psi), \tag{2.14.18}$$

$$\mathcal{O}_{\mathrm{D/A}}(2,2) = (\sin\Phi)\sin\Psi + (\cos\Phi)\cos\Psi = \cos(\Phi - \Psi), \tag{2.14.19}$$

and thus

$$\Phi - \Psi = \operatorname{atan2}\left(\mathcal{O}_{\mathrm{D/A}}(2,1), \mathcal{O}_{\mathrm{D/A}}(2,2)\right) \tag{2.14.20}$$

Next, consider $\Theta = -\pi/2$. Note that

$$\mathcal{O}_{\mathrm{D/A}}(2,1) = -(\sin\Phi)\cos\Psi - (\cos\Phi)\sin\Psi = -\sin(\Phi + \Psi), \tag{2.14.21}$$

$$\mathcal{O}_{\mathrm{D/A}}(2,2) = -(\sin\Phi)\sin\Psi + (\cos\Phi)\cos\Psi = -\cos(\Phi + \Psi), \tag{2.14.22}$$

and thus

$$\Phi + \Psi = \operatorname{atan2}\left(-\mathcal{O}_{\mathrm{D/A}}(2,1), -\mathcal{O}_{\mathrm{D/A}}(2,2)\right). \tag{2.14.23}$$

2.15 Products of Euler Orientation Matrices†

Recall that, for $a, b \in \mathbb{R}$, the notation $a \equiv b$ means that $a - b$ is an integer multiple of 2π.

Fact 2.15.1. The following statements hold:

i) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $a \equiv 0$ if and only if $\mathcal{O}_i(a) = I$.

ii) Let $a \in \mathbb{R}$. Then, the following statements are equivalent:
a) $a \equiv \pi$.
b) $\mathcal{O}_1(a) = \operatorname{diag}(1, -1, -1)$.
c) $\mathcal{O}_2(a) = \operatorname{diag}(-1, 1, -1)$.
d) $\mathcal{O}_3(a) = \operatorname{diag}(-1, -1, 1)$.

iii) Let $i, j, k \in \{1,2,3\}$ be distinct. Then, $\mathcal{O}_i(\pi)\mathcal{O}_j(\pi)\mathcal{O}_k(\pi) = I_3$ and $\mathcal{O}_i(\pi) = \mathcal{O}_j(\pi)\mathcal{O}_k(\pi)$.

iv) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, the following statements are equivalent:
a) Either $a \equiv 0$ or $a \equiv \pi$.

b) $\mathcal{O}_i(a)$ is symmetric.
c) $\mathcal{O}_i(a)$ is diagonal.

v) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $\mathcal{O}_i(a)e_i = e_i$, where e_i is the ith column of I_3.

vi) Let $a \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $\mathcal{O}_i(-a) = \mathcal{O}_i(a)^{-1} = \mathcal{O}_i(a)^{\mathsf{T}}$.

vii) Let $a \in \mathbb{R}$, and let $i, j \in \{1,2,3\}$ be distinct. Then, $\mathcal{O}_i(\pi)\mathcal{O}_j(a)\mathcal{O}_i(\pi) = \mathcal{O}_j(-a)$.

viii) Let $a \in \mathbb{R}$, and let $i, j, k \in \{1,2,3\}$ be distinct. Then,

$$\mathcal{O}_i(\pm\pi/2)\mathcal{O}_j(a)\mathcal{O}_i(\mp\pi/2) = \begin{cases} \mathcal{O}_k(\pm a), & (i,j) \in \{(1,3),(2,1),(3,2)\}, \\ \mathcal{O}_k(\mp a), & (i,j) \in \{(1,2),(2,3),(3,1)\}. \end{cases} \tag{2.15.1}$$

ix) Let $a, b \in \mathbb{R}$ and $i \in \{1,2,3\}$. Then, $\mathcal{O}_i(a)\mathcal{O}_i(b) = \mathcal{O}_i(a+b)$.

x) Let $a, b \in \mathbb{R}$, and let $i, j \in \{1,2,3\}$ be distinct. Then, the following statements are equivalent:
a) $a \equiv b \equiv 0$.
b) $\mathcal{O}_i(a) = \mathcal{O}_j(b)$.
c) $\mathcal{O}_i(a)\mathcal{O}_j(b) = I$.

The entries of $\mathcal{O} \in \mathbb{R}^{3\times 3}$ are written as

$$\mathcal{O} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{2.15.2}$$

Note that, if $\mathcal{O}$ is an orientation matrix, then, for all $i, j \in \{1,2,3\}$, $|r_{ij}| \le 1$.

3-2-1 factorizations of an orientation matrix are considered in two cases. When $|r_{13}| < 1$, b can assume two distinct values, and a and c are uniquely determined by b. When $|r_{13}| = 1$, b is unique and a and c can assume infinitely many values. It is helpful to note that

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) = \begin{bmatrix} \mathrm{C}b\mathrm{C}c & \mathrm{C}b\mathrm{S}c & -\mathrm{S}b \\ \mathrm{C}c\mathrm{S}a\mathrm{S}b - \mathrm{C}a\mathrm{S}c & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}b\mathrm{S}c & \mathrm{C}b\mathrm{S}a \\ \mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b & \mathrm{C}a\mathrm{S}b\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{C}a\mathrm{C}b \end{bmatrix}. \tag{2.15.3}$$

The following result shows that, when $|r_{13}| < 1$, the entries r_{21}, r_{22}, r_{31}, and r_{32} of an orientation matrix $\mathcal{O}$ are uniquely determined by the remaining entries r_{11}, r_{12}, r_{13}, r_{23}, and r_{33}.

Fact 2.15.2. Let $\mathcal{O} \in \mathbb{R}^{3\times 3}$, and assume that $r_{11}^2 + r_{12}^2 > 0$ and $r_{11}^2 + r_{12}^2 + r_{13}^2 = r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$. Then, $\mathcal{O}$ is an orientation matrix if and only if

$$r_{21} = -\frac{r_{11}r_{13}r_{23} + r_{12}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{22} = \frac{r_{11}r_{33} - r_{12}r_{13}r_{23}}{r_{11}^2 + r_{12}^2}, \tag{2.15.4}$$

$$r_{31} = \frac{r_{12}r_{23} - r_{11}r_{13}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{32} = -\frac{r_{12}r_{13}r_{33} + r_{11}r_{23}}{r_{11}^2 + r_{12}^2}. \tag{2.15.5}$$

Proof. To prove sufficiency, it can be shown that, with (2.15.4) and (2.15.5), $\mathcal{O}$ satisfies $\mathcal{O}\mathcal{O}^{\mathsf{T}} = I$ and $\det \mathcal{O} = 1$. To prove necessity, note that $\mathcal{O}\mathcal{O}^{\mathsf{T}} = I$ yields five equalities involving r_{21}, r_{22}, r_{31}, and r_{32}, namely,

$$r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} = 0, \tag{2.15.6}$$

$$r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} = 0, \tag{2.15.7}$$

$$r_{21}^2 + r_{22}^2 + r_{23}^2 = 1, \tag{2.15.8}$$

$$r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} = 0, \tag{2.15.9}$$

$$r_{31}^2 + r_{32}^2 + r_{33}^2 = 1. \tag{2.15.10}$$

Since $r_{11}^2 + r_{12}^2 > 0$, it follows that either $r_{11} \neq 0$ or $r_{12} \neq 0$. The case where $r_{12} \neq 0$ is considered first. In this case, (2.15.6) implies that

$$r_{22} = -(r_{11}r_{21} + r_{13}r_{23})/r_{12}, \tag{2.15.11}$$

which, using (2.15.8) yields

$$(r_{11}^2 + r_{12}^2)r_{21}^2 + 2r_{11}r_{23}r_{13}r_{21} + r_{23}^2(r_{13}^2 + r_{12}^2) - r_{12}^2 = 0. \tag{2.15.12}$$

Solving (2.15.12) for r_{21} yields

$$\begin{aligned} r_{21} &= \frac{-r_{11}r_{23}r_{13} \pm \sqrt{r_{11}^2 r_{23}^2 r_{13}^2 - (r_{11}^2 + r_{12}^2)(r_{23}^2(r_{13}^2 + r_{12}^2) - r_{12}^2)}}{r_{11}^2 + r_{12}^2} \\ &= \frac{-r_{11}r_{23}r_{13} \pm r_{12}r_{33}}{r_{11}^2 + r_{12}^2}. \end{aligned} \tag{2.15.13}$$

Substituting r_{21} given by (2.15.13) into (2.15.11) yields

$$r_{22} = \frac{\mp r_{11}r_{33} - r_{23}r_{13}r_{12}}{r_{11}^2 + r_{12}^2}. \tag{2.15.14}$$

The same expressions (2.15.13) and (2.15.14) are obtained when $r_{11} \neq 0$. Note that (2.15.13) and (2.15.14) imply that r_{21} and r_{22} are given by either (2.15.4) or

$$r_{21} = -\frac{r_{11}r_{13}r_{23} - r_{12}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{22} = -\frac{r_{11}r_{33} + r_{12}r_{13}r_{23}}{r_{11}^2 + r_{12}^2}. \tag{2.15.15}$$

Following a similar procedure using (2.15.7) and (2.15.10), it follows that r_{31} and r_{32} are given by either (2.15.5) or

$$r_{31} = -\frac{r_{12}r_{23} + r_{11}r_{13}r_{33}}{r_{11}^2 + r_{12}^2}, \quad r_{32} = \frac{r_{11}r_{23} - r_{12}r_{13}r_{33}}{r_{11}^2 + r_{12}^2}. \tag{2.15.16}$$

It thus follows from (2.15.4), (2.15.15), (2.15.5), and (2.15.16) that r_{21}, r_{22}, r_{31}, and r_{32} are given by either *i*) (2.15.4) and (2.15.5), *ii*) (2.15.4) and (2.15.16), *iii*) (2.15.15) and (2.15.5), or *iv*) (2.15.15) and (2.15.16). In case *i*), $\mathcal{O}\mathcal{O}^\mathsf{T} = I$ and $\det \mathcal{O} = 1$. In case *ii*), $\mathcal{O}\mathcal{O}^\mathsf{T} \neq I$ and $\det \mathcal{O} = -1$. In particular, the $(2,3)$ and $(3,2)$ entries of $\mathcal{O}\mathcal{O}^\mathsf{T}$ are not zero. In case *iii*), $\mathcal{O}^\mathsf{T} \neq I$ and $\det \mathcal{O} = 1$. In particular, the $(2,3)$ and $(3,2)$ entries of $\mathcal{O}\mathcal{O}^\mathsf{T}$ are not zero. In case *iv*), $\mathcal{O}\mathcal{O}^\mathsf{T} = I$ and $\det \mathcal{O} = -1$. Therefore, cases *ii*), *iii*), and *iv*) are spurious, and thus r_{21}, r_{22}, r_{31}, and r_{32} satisfy (2.15.4) and (2.15.5). □

The following result characterizes the feasible Euler angles having the property that a 3-2-1 product of Euler rotation matrices is equal to a given rotation matrix. This result consider the case where $|r_{13}| < 1$; the case where $|r_{13}| = 1$ is considered by Fact 2.15.5.

Fact 2.15.3. Let $\mathcal{O}$ be an orientation matrix, and assume that $|r_{13}| < 1$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O} = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) \tag{2.15.17}$$

if and only if either

$$b \equiv -\operatorname{asin} r_{13} \tag{2.15.18}$$

or

$$b \equiv \operatorname{asin} r_{13} + \pi \tag{2.15.19}$$

and

$$a \equiv \operatorname{atan2}\left(\frac{r_{23}}{\cos b}, \frac{r_{33}}{\cos b}\right), \quad c \equiv \operatorname{atan2}\left(\frac{r_{12}}{\cos b}, \frac{r_{11}}{\cos b}\right). \tag{2.15.20}$$

Proof. To prove necessity, note that it follows from the $(1,3)$ entry of (2.15.3) that b satisfies either (2.15.18) or (2.15.19). Furthermore, it follows from the $(1,1)$, $(1,2)$, $(2,3)$, and $(3,3)$ entries of (2.15.3) that a and c are given by (2.15.20).

To prove sufficiency note that it follows from (2.15.18) that the $(1,3)$ entry of (2.15.3) is $-\sin b = \sin \operatorname{asin} r_{13} = r_{13}$. Likewise, it follows from (2.15.19) that the $(1,3)$ entry of (2.15.3) is $-\sin b = -\sin(\operatorname{asin} r_{13} + \pi) = r_{13}$. Furthermore, it follows from (2.15.20) that the $(1,1)$ entry of (2.15.3) is given by $(\cos b)\cos c = (\cos b)r_{11}/(\cos b) = r_{11}$ and likewise for the $(1,2)$, $(2,3)$, and $(3,3)$ entries of (2.15.3). Next, since $\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)$ is an orientation matrix, Fact 2.15.2 implies that its $(2,1)$, $(2,2)$, $(3,1)$, and $(3,2)$ entries are determined by its $(1,1)$, $(1,2)$, $(1,3)$, $(2,3)$, and $(3,3)$ entries. In particular, the $(2,1)$ entry of $\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)$ is given by

$$\begin{aligned}(\cos c)(\sin a)\sin b - (\cos a)\sin c &= \frac{r_{11}}{\cos b}\frac{r_{23}}{\cos b}\sin b - \frac{r_{33}}{\cos b}\frac{r_{12}}{\cos b} \\ &= \frac{r_{11}r_{23}\sin b - r_{33}r_{12}}{\cos^2 b} \\ &= \frac{-r_{11}r_{23}r_{13} - r_{33}r_{12}}{1 - r_{13}^2} \\ &= r_{21},\end{aligned}$$

where the last equality follows from (2.15.4) of Fact 2.15.2. Similar calculations show that the $(2,2)$, $(3,1)$, and $(3,2)$ entries of $\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)$ are given by r_{22}, r_{31}, and r_{32}, respectively. Consequently, (2.15.17) is satisfied. □

In the special case where $r_{13} = 0$, it follows from Theorem 1 that $a, b, c \in \mathbb{R}$ satisfy (2.15.17) if and only if either $b \equiv 0$ or $b \equiv \pi$ and

$$a \equiv \operatorname{atan2}(r_{23}, r_{33}), \quad c \equiv \operatorname{atan2}(r_{12}, r_{11}). \tag{2.15.21}$$

Setting $\mathcal{O} = \mathcal{O}_2(d)$ in Fact 2.15.3 yields the following result.

Fact 2.15.4. Let $d \in \mathbb{R}$, and assume that $d \not\equiv \pi/2$ and $d \not\equiv -\pi/2$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c). \tag{2.15.22}$$

if and only if either *i*) $a \equiv c \equiv 0$ and $b \equiv d$ or *ii*) $a \equiv c \equiv \pi$ and $b \equiv \pi - d$.

The following result extends Fact 2.15.3 to the case where $|r_{13}| = 1$, that is, $r_{11} = r_{12} = 0$. The proof is omitted.

Fact 2.15.5. Let $\mathcal{O}$ be an orientation matrix, and assume that $|r_{13}| = 1$. Then, $a, b, c \in \mathbb{R}$ satisfy (2.15.17) if and only if either

$$b \equiv \pi/2, \quad a \equiv c + \operatorname{atan2}(r_{21}, r_{22}) \tag{2.15.23}$$

or

$$b \equiv -\pi/2, \quad a \equiv -c + \mathrm{atan2}(-r_{21}, r_{22}). \tag{2.15.24}$$

When $|r_{13}| = 1$, it follows that $r_{31} = -r_{13}r_{22}$ and $r_{32} = r_{13}r_{21}$. Therefore, in Fact 2.15.5, (2.15.23) and (2.15.24) can be replaced, respectively, by

$$b \equiv \pi/2, \quad a \equiv c + \mathrm{atan2}(-r_{32}, r_{31}), \tag{2.15.25}$$

$$b \equiv -\pi/2, \quad a \equiv -c + \mathrm{atan2}(-r_{32}, -r_{31}). \tag{2.15.26}$$

Setting $\mathcal{O} = \mathcal{O}_2(d)$ in Fact 2.15.5 yields the following result.

Fact 2.15.6. Let $d \in \mathbb{R}$, and assume that either $d \equiv \pi/2$ or $d \equiv -\pi/2$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c). \tag{2.15.27}$$

if and only if either *i)* $a \equiv c$ and $b \equiv d \equiv \pi/2$ or *ii)* $a \equiv -c$ and $b \equiv d \equiv -\pi/2$.

1-2-1 factorizations of an orientation matrix $\mathcal{O}$ are considered in two cases. When $|r_{11}| < 1$, b can assume two distinct values and a and c are uniquely determined by b. When $|r_{11}| = 1$, b is unique and a and c can assume infinitely many values. It is helpful to note that

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c) = \begin{bmatrix} \mathrm{C}b & \mathrm{S}b\mathrm{S}c & -\mathrm{S}b\mathrm{C}c \\ \mathrm{S}a\mathrm{S}b & \mathrm{C}a\mathrm{C}c - \mathrm{C}b\mathrm{S}a\mathrm{S}c & \mathrm{C}a\mathrm{S}c + \mathrm{C}b\mathrm{C}c\mathrm{S}a \\ \mathrm{C}a\mathrm{S}b & -\mathrm{C}c\mathrm{S}a - \mathrm{C}a\mathrm{C}b\mathrm{S}c & \mathrm{C}a\mathrm{C}b\mathrm{C}c - \mathrm{S}a\mathrm{S}c \end{bmatrix}. \tag{2.15.28}$$

The following result characterizes the feasible Euler angles having the property that a 1-2-1 product of Euler rotation matrices is equal to an given rotation matrix. This result consider the case where $|r_{11}| < 1$; the case where $|r_{11}| = 1$ is considered by Fact 2.15.8.

Fact 2.15.7. Let $\mathcal{O}$ be an orientation matrix, and assume that $|r_{11}| < 1$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O} = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c). \tag{2.15.29}$$

if and only if either

$$b \equiv \mathrm{acos}\, r_{11} \tag{2.15.30}$$

or

$$b \equiv -\mathrm{acos}\, r_{11} \tag{2.15.31}$$

and

$$a \equiv \mathrm{atan2}\left(\frac{r_{21}}{\sin b}, \frac{r_{31}}{\sin b}\right), \quad c \equiv \mathrm{atan2}\left(\frac{r_{12}}{\sin b}, \frac{-r_{13}}{\sin b}\right). \tag{2.15.32}$$

The following result extends Fact 2.15.7 to the case where $|r_{11}| = 1$.

Fact 2.15.8. Let $\mathcal{O}$ be an orientation matrix, and assume that $|r_{11}| = 1$. Then, $a, b, c \in \mathbb{R}$ satisfy

$$\mathcal{O} = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c). \tag{2.15.33}$$

if and only if either

$$b \equiv 0, \quad a \equiv -c + \mathrm{atan2}(r_{23}, r_{22}) \tag{2.15.34}$$

or

$$b \equiv \pi, \quad a \equiv c + \operatorname{atan2}(r_{23}, r_{22}). \tag{2.15.35}$$

When $|r_{11}| = 1$, it follows that $r_{32} = -r_{11}r_{23}$ and $r_{33} = r_{11}r_{22}$. Therefore, in Fact 2.15.8, (2.15.34) and (2.15.35) can be replaced, respectively, by

$$b \equiv 0, \quad a \equiv -c + \operatorname{atan2}(-r_{32}, r_{33}), \tag{2.15.36}$$

$$b \equiv \pi, \quad a \equiv c + \operatorname{atan2}(r_{32}, -r_{33}). \tag{2.15.37}$$

The following result considers the case where a 1-2-1 product of Euler rotation matrices is equal to the identity matrix.

Fact 2.15.9. Let $a, b, c \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c) = I \tag{2.15.38}$$

if and only if $b \equiv 0$ and $a \equiv -c$.

Proof. Rewriting (2.15.38) as

$$\mathcal{O}_1(a + c) = \mathcal{O}_2(-b), \tag{2.15.39}$$

x) of Fact 2.15.1 implies that (2.15.39) holds if and only if $a + c \equiv b \equiv 0$. □

The following result considers the case where a 3-2-1 product of Euler rotation matrices is equal to the identity matrix.

Fact 2.15.10. Let $a, b, c \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) = I \tag{2.15.40}$$

if and only if either $a \equiv b \equiv c \equiv 0$ or $a \equiv b \equiv c \equiv \pi$.

Proof. Sufficiency is immediate. To prove necessity, note that, by using (2.15.3) to rewrite (2.15.40) as

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}c & \mathrm{C}b\mathrm{S}c & -\mathrm{S}b \\ \mathrm{C}c\mathrm{S}a\mathrm{S}b - \mathrm{C}a\mathrm{S}c & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}b\mathrm{S}c & \mathrm{C}b\mathrm{S}a \\ \mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b & \mathrm{C}a\mathrm{S}b\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{C}a\mathrm{C}b \end{bmatrix} = I,$$

it follows from the (1,1) entry that $\mathrm{C}b \neq 0$, and thus from the (1,2), (1,3), and (2,3) entries that $\mathrm{S}a = \mathrm{S}b = \mathrm{S}c = 0$. Hence, it follows from the (2,2) and (3,3) entries that $\mathrm{C}a\mathrm{C}c = \mathrm{C}a\mathrm{C}b = 1$, and thus either $\mathrm{C}a = \mathrm{C}b = \mathrm{C}c = 1$ or $\mathrm{C}a = \mathrm{C}b = \mathrm{C}c = -1$. Hence, either $a \equiv b \equiv c \equiv 0$ or $a \equiv b \equiv c \equiv \pi$. □

The following result considers the case where a 2-3-2-1 product of Euler orientation matrices is equal to the identity matrix.

Fact 2.15.11. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)\mathcal{O}_2(d) = I \tag{2.15.41}$$

if and only if either *i*) $b \equiv -d \equiv \pi/2$ and $a \equiv c$, *ii*) $b \equiv -d \equiv -\pi/2$ and $a \equiv -c$, *iii*) $a \equiv c \equiv 0$ and $b \equiv -d$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv d + \pi$.

Proof. Sufficiency is immediate. To prove necessity, note that (2.15.41) implies

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}c\mathrm{C}d - \mathrm{S}b\mathrm{S}d & \mathrm{C}b\mathrm{S}c & -\mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{C}c\mathrm{S}d \\ \mathrm{C}b\mathrm{S}a\mathrm{S}d - \mathrm{C}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a\mathrm{S}b) & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}b\mathrm{S}c & \mathrm{S}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a\mathrm{S}b) + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ \mathrm{C}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b) + \mathrm{C}a\mathrm{C}b\mathrm{S}d & \mathrm{C}a\mathrm{S}b\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{C}a\mathrm{C}b\mathrm{C}d - \mathrm{S}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b) \end{bmatrix} = I,$$

Since $\mathrm{C}b\mathrm{S}c = 0$, it follows that either $\mathrm{C}b = 0$ or $\mathrm{S}c = 0$. Therefore, either *i*) $b \equiv \pi/2$, *ii*) $b \equiv -\pi/2$, *iii*) $c \equiv 0$, or *iv*) $c \equiv \pi$.

Case *i*)**:** $b \equiv \pi/2$. In this case,

$$\begin{bmatrix} -\mathrm{S}d & 0 & -\mathrm{C}d \\ -\mathrm{C}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}c & \mathrm{S}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ \mathrm{C}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c) & \mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & -\mathrm{S}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I,$$

Since $\mathrm{S}d = -1$ and $\mathrm{C}d = 0$, it follows that $d \equiv -\pi/2$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}c & -(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) \\ 0 & \mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a-c) & \sin(a-c) \\ 0 & -\sin(a-c) & \cos(a-c) \end{bmatrix} = I.$$

Hence, $a - c \equiv 0$.

Case *ii*)**:** $b \equiv -\pi/2$. In this case,

$$\begin{bmatrix} \mathrm{S}d & 0 & \mathrm{C}d \\ -\mathrm{C}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a) & \mathrm{C}a\mathrm{C}c - \mathrm{S}a\mathrm{S}c & \mathrm{S}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a) \\ \mathrm{C}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) & -\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & -\mathrm{S}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I,$$

Since $\mathrm{S}d = 1$ and $\mathrm{C}d = 0$, it follows that $d \equiv \pi/2$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{C}a\mathrm{C}c - \mathrm{S}a\mathrm{S}c & \mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a \\ 0 & -\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & -(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a+c) & \sin(a+c) \\ 0 & -\sin(a+c) & \cos(a+c) \end{bmatrix} = I.$$

Hence, $a + c \equiv 0$.

Case *iii*)**:** $c \equiv 0$. In this case,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{S}d \\ \mathrm{C}b\mathrm{S}a\mathrm{S}d + \mathrm{C}d\mathrm{S}a\mathrm{S}b) & \mathrm{C}a & -\mathrm{S}d\mathrm{S}a\mathrm{S}b + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ \mathrm{C}d\mathrm{C}a\mathrm{S}b + \mathrm{C}a\mathrm{C}b\mathrm{S}d & -\mathrm{S}a & \mathrm{C}a\mathrm{C}b\mathrm{C}d - \mathrm{S}d\mathrm{C}a\mathrm{S}b \end{bmatrix} = I,$$

Since $\mathrm{C}a = 1$ and $\mathrm{S}a = 0$, it follows that $a \equiv 0$. Hence,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{S}d \\ 0 & 1 & 0 \\ \mathrm{C}d\mathrm{S}b + \mathrm{C}b\mathrm{S}d & 0 & \mathrm{C}b\mathrm{C}d - \mathrm{S}d\mathrm{S}b \end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix}\cos(b+d) & 0 & -\sin(b+d)\\ 0 & 1 & 0\\ \sin(b+d) & 0 & \cos(b+d)\end{bmatrix} = I,$$

Hence, $b + d \equiv 0$.

Case *iv*)**:** $c \equiv \pi$. In this case,

$$\begin{bmatrix}-\mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b + \mathrm{C}b\mathrm{S}d\\ \mathrm{C}b\mathrm{S}a\mathrm{S}d - \mathrm{C}d\mathrm{S}a\mathrm{S}b & -\mathrm{C}a & \mathrm{S}d\mathrm{S}a\mathrm{S}b + \mathrm{C}b\mathrm{C}d\mathrm{S}a\\ -\mathrm{C}d\mathrm{C}a\mathrm{S}b + \mathrm{C}a\mathrm{C}b\mathrm{S}d & \mathrm{S}a & \mathrm{C}a\mathrm{C}b\mathrm{C}d + \mathrm{S}d\mathrm{C}a\mathrm{S}b\end{bmatrix} = I,$$

Since $\mathrm{C}a = -1$ and $\mathrm{S}a = 0$, it follows that $a \equiv \pi$. Hence,

$$\begin{bmatrix}-\mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b + \mathrm{C}b\mathrm{S}d\\ 0 & 1 & 0\\ \mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{S}d & 0 & -\mathrm{C}b\mathrm{C}d - \mathrm{S}d\mathrm{S}b\end{bmatrix} = I,$$

which can be written as

$$\begin{bmatrix}-\cos(b-d) & 0 & -\sin(b-d)\\ 0 & 1 & 0\\ \sin(b-d) & 0 & -\cos(b-d)\end{bmatrix} = I,$$

Hence, $b - d \equiv \pi$. □

The following result considers the case where a 3-2-1 product of Euler orientation matrices is equal to a 2-axis Euler orientation matrix.

Fact 2.15.12. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c). \tag{2.15.42}$$

if and only if either *i*) $a \equiv c$ and $b \equiv d \equiv \pi/2$, *ii*) $a \equiv -c$ and $b \equiv d \equiv -\pi/2$, *iii*) $a \equiv c \equiv 0$ and $b \equiv d$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv \pi - d$.

The following result considers the case where a 2-1-2-1 product of Euler orientation matrices is equal to the identity matrix.

Fact 2.15.13. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c)\mathcal{O}_2(d) = I \tag{2.15.43}$$

if and only if either *i*) $b \equiv d \equiv 0$ and $a \equiv -c$, *ii*) $a \equiv c \equiv 0$ and $b \equiv -d$, *iii*) $b \equiv d \equiv \pi$ and $a \equiv c$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv d$.

Proof. Sufficiency is immediate. To prove necessity, note that (2.15.43) implies

$$\begin{bmatrix}\mathrm{C}b\mathrm{C}d - \mathrm{C}c\mathrm{S}b\mathrm{S}d & \mathrm{S}b\mathrm{S}c & -\mathrm{C}b\mathrm{S}d - \mathrm{C}c\mathrm{C}d\mathrm{S}b\\ \mathrm{S}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}b\mathrm{C}c\mathrm{S}a) + \mathrm{C}d\mathrm{S}a\mathrm{S}b & \mathrm{C}a\mathrm{C}c - \mathrm{C}b\mathrm{S}a\mathrm{S}c & \mathrm{C}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}b\mathrm{C}c\mathrm{S}a) - \mathrm{S}a\mathrm{S}b\mathrm{S}d\\ \mathrm{C}a\mathrm{C}d\mathrm{S}b - \mathrm{S}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}b\mathrm{C}c) & -\mathrm{C}c\mathrm{S}a - \mathrm{C}a\mathrm{C}b\mathrm{S}c & -\mathrm{C}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}b\mathrm{C}c) - \mathrm{C}a\mathrm{S}b\mathrm{S}d\end{bmatrix} = I.$$

Since $\mathrm{S}b\mathrm{S}c = 0$, it follows that either *i*) $b \equiv 0$, *ii*) $c \equiv 0$, *iii*) $b \equiv \pi$, or *iv*) $c \equiv \pi$.

Case *i*)**:** $b \equiv 0$. In this case,

$$\begin{bmatrix} \mathrm{C}d & 0 & -\mathrm{S}d \\ \mathrm{S}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a) & \mathrm{C}a\mathrm{C}c - \mathrm{S}a\mathrm{S}c & \mathrm{C}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a) \\ -\mathrm{S}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) & -\mathrm{C}c\mathrm{S}a - \mathrm{C}a\mathrm{S}c & -\mathrm{C}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I.$$

Since $\mathrm{C}d = 1$ and $\mathrm{S}d = 0$, it follows that $d \equiv 0$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{C}a\mathrm{C}c - \mathrm{S}a\mathrm{S}c & \mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a \\ 0 & -\mathrm{C}c\mathrm{S}a - \mathrm{C}a\mathrm{S}c & -\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a+c) & \sin(a+c) \\ 0 & -\sin(a+c) & \cos(a+c) \end{bmatrix} = I.$$

Hence, $a + c \equiv 0$.

Case *ii*)**:** $c \equiv 0$. In this case,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}b\mathrm{S}d - \mathrm{C}d\mathrm{S}b \\ \mathrm{S}d\mathrm{C}b\mathrm{S}a + \mathrm{C}d\mathrm{S}a\mathrm{S}b & \mathrm{C}a & \mathrm{C}d\mathrm{C}b\mathrm{S}a - \mathrm{S}a\mathrm{S}b\mathrm{S}d \\ \mathrm{C}a\mathrm{C}d\mathrm{S}b + \mathrm{S}d\mathrm{C}a\mathrm{C}b & -\mathrm{S}a & \mathrm{C}d\mathrm{C}a\mathrm{C}b - \mathrm{C}a\mathrm{S}b\mathrm{S}d \end{bmatrix} = I.$$

Since $\mathrm{C}a = 1$ and $\mathrm{S}a = 0$, it follows that $a = 0$. Hence,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}b\mathrm{S}d - \mathrm{C}d\mathrm{S}b \\ 0 & 1 & 0 \\ \mathrm{C}d\mathrm{S}b + \mathrm{S}d\mathrm{C}b & 0 & \mathrm{C}d\mathrm{C}b - \mathrm{S}b\mathrm{S}d \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} \cos(b+d) & 0 & -\sin(b+d) \\ 0 & 1 & 0 \\ \sin(b+d) & 0 & \cos(b+d) \end{bmatrix} = I.$$

Hence, $b + d \equiv 0$.

Case *iii*)**:** $b \equiv \pi$. In this case,

$$\begin{bmatrix} -\mathrm{C}d & 0 & \mathrm{S}d \\ \mathrm{S}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}c & \mathrm{C}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) \\ -\mathrm{S}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c) & -\mathrm{C}c\mathrm{S}a + \mathrm{C}a\mathrm{S}c & -\mathrm{C}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I.$$

Since $\mathrm{C}d = -1$ and $\mathrm{S}d = 0$, it follows that $d \equiv \pi$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}c & -(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) \\ 0 & -\mathrm{C}c\mathrm{S}a + \mathrm{C}a\mathrm{S}c & \mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a-c) & \sin(a-c) \\ 0 & -\sin(a-c) & \cos(a-c) \end{bmatrix} = I.$$

Hence, $a \equiv c$.

Case *iv*)**:** $c \equiv \pi$. In this case,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d + \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}b\mathrm{S}d + \mathrm{C}d\mathrm{S}b \\ -\mathrm{S}d\mathrm{C}b\mathrm{S}a + \mathrm{C}d\mathrm{S}a\mathrm{S}b & -\mathrm{C}a & -\mathrm{C}d\mathrm{C}b\mathrm{S}a - \mathrm{S}a\mathrm{S}b\mathrm{S}d \\ \mathrm{C}a\mathrm{C}d\mathrm{S}b - \mathrm{S}d\mathrm{C}a\mathrm{C}b & \mathrm{S}a & -\mathrm{C}d\mathrm{C}a\mathrm{C}b - \mathrm{C}a\mathrm{S}b\mathrm{S}d \end{bmatrix} = I.$$

Since $\mathrm{C}a = -1$ and $\mathrm{S}a = 0$, it follows that $a = \pi$. Hence,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d + \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}b\mathrm{S}d + \mathrm{C}d\mathrm{S}b \\ 0 & 1 & 0 \\ -\mathrm{C}d\mathrm{S}b + \mathrm{S}d\mathrm{C}b & 0 & \mathrm{C}d\mathrm{C}b + \mathrm{S}b\mathrm{S}d \end{bmatrix} = I,$$

which can be rewritten as

$$\begin{bmatrix} \cos(d-b) & 0 & -\sin(d-b) \\ 0 & 1 & 0 \\ \sin(d-b) & 0 & \cos(d-b) \end{bmatrix} = I.$$

Hence, $d - b \equiv 0$. □

The following result considers the case where a 1-2-1 product of Euler orientation matrices is equal to a 2-axis Euler orientation matrix.

Fact 2.15.14. Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c). \tag{2.15.44}$$

if and only if either *i*) $b \equiv -d \equiv 0$ and $a \equiv -c$, *ii*) $a \equiv c \equiv 0$ and $b \equiv d$, *iii*) $b \equiv -d \equiv \pi$ and $a \equiv c$, or *iv*) $a \equiv c \equiv \pi$ and $b \equiv -d$.

Fact 2.15.13 yields the following result on commuting Euler orientation matrices.

Fact 2.15.15. Let $a, b \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b) = \mathcal{O}_2(b)\mathcal{O}_1(a) \tag{2.15.45}$$

if and only if either $a \equiv 0$, $b \equiv 0$, or $a \equiv b \equiv \pi$.

These results show that there is one permutationally distinct product of two Euler rotation matrices whose product is the identity; one permutationally distinct product of three Euler rotation matrices whose product is the identity; and two permutationally distinct products of four Euler rotation matrices whose product is the identity.

Products of five or more Euler rotation matrices can be considered. In addition to axis relabeling, by choosing an alternative starting point, sequence cycling can be disregarded in the sense that the sequences 2-1-2-1-3 and 1-2-1-3-2 are identical. Furthermore, by multiplying each angle by -1, sequence reversal can be disregarded in the sense that 2-1-2-1-3 and 3-1-2-1-2 are identical. Consequently, in characterizing all closed rotation sequences of a given length, a pair of sequences can be viewed as identical if one sequence can be obtained from the other by axis relabeling (with axis reflection to retain right-handedness), sequence cycling, and sequence reversal. It thus follows that there is one permutationally distinct product of five Euler rotation matrices whose product is the identity, namely, 1-2-1-2-3; there are four permutationally distinct products of six Euler rotation matrices whose product is the identity, namely, 1-2-1-2-1-2, 1-2-1-2-1-3, 1-2-1-3-2-3, and 1-2-3-1-2-3; and there are three permutationally distinct products of seven Euler rotation matrices whose product is the identity, namely, 1-2-1-2-1-2-3, 1-2-1-2-3-1-3, and 1-2-1-3-2-1-3.

2.16 Exponential Representation of Rotation Matrices and the Eigenaxis Angle Vector†

An alternative way to express a physical rotation matrix is in terms of the exponential of a skew-symmetric physical matrix. For the physical matrix $\vec{M}$ define the *exponential* of $\vec{M}$ by

$$\exp(\vec{M}) \triangleq \vec{I} + \vec{M} + \tfrac{1}{2}\vec{M}^2 + \tfrac{1}{3!}\vec{M}^3 + \cdots. \tag{2.16.1}$$

Therefore, for each frame $\mathrm{F_A}$,

$$\exp(\vec{M})\Big|_{\mathrm{A}} = e^M = I_3 + M + \tfrac{1}{2}M^2 + \tfrac{1}{3!}M^3 + \cdots, \tag{2.16.2}$$

where $M \triangleq \vec{M}\Big|_{\mathrm{A}}$.

The following result provides an exponential matrix representation of the physical rotation matrix in Fact 2.11.10.

Fact 2.16.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors, assume that $|\vec{x}| = |\vec{y}| \neq 0$, and assume that $\vec{x}$ and $\vec{y}$ are not parallel. Then,

$$\vec{R}_{\hat{\theta}_{\vec{y}/\vec{x}}}(\theta_{\vec{y}/\vec{x}}) = \exp\left(\theta_{\vec{y}/\vec{x}}\hat{\theta}^{\times}_{\vec{y}/\vec{x}}\right) = \exp\left(\frac{\theta_{\vec{y}/\vec{x}}}{\sin\theta_{\vec{y}/\vec{x}}}(\hat{x}\times\hat{y})^{\times}\right). \tag{2.16.3}$$

Furthermore,

$$\vec{y} = \exp\left(\theta_{\vec{y}/\vec{x}}\hat{\theta}^{\times}_{\vec{y}/\vec{x}}\right)\vec{x}. \tag{2.16.4}$$

Proof. The result follows from Fact 11.11.6 in [1]. The second equality in (2.16.3) follows from (2.2.2). □

Recall that, if $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 3$, then $\theta_{\mathrm{B/A}} = 0$ and $\hat{n}_{\mathrm{B/A}}$ is an arbitrary unit dimensionless physical vector. Furthermore, if $\operatorname{tr}\vec{R}_{\mathrm{B/A}} \in (-1,3)$, then $\theta_{\mathrm{B/A}} = \operatorname{acos}[\tfrac{1}{2}(\operatorname{tr}\vec{R}_{\mathrm{B/A}} - 1)]$ and $\hat{n}^{\times}_{\mathrm{B/A}} = \frac{1}{2\sin\theta_{\mathrm{B/A}}}(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}})$, which is uniquely defined. Finally, if $\operatorname{tr}\vec{R}_{\mathrm{B/A}} = -1$, then $\theta_{\mathrm{B/A}} = \pi$ and $\hat{n}_{\mathrm{B/A}}$ satisfies $\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} = \tfrac{1}{2}(\vec{R}_{\mathrm{B/A}} + \vec{I})$. In this case, there are two eigenaxes in the sense that, if $\hat{n}_{\mathrm{B/A}}$ is an eigenaxis then so is $-\hat{n}_{\mathrm{B/A}}$.

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. If $\theta_{\mathrm{B/A}} \in [0,\pi)$, then we define the *eigenaxis angle vector* by

$$\vec{\Theta}_{\mathrm{B/A}} \triangleq \begin{cases} \vec{0}, & \theta_{\mathrm{B/A}} = 0, \\ \theta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}, & \theta_{\mathrm{B/A}} \in (0,\pi). \end{cases} \tag{2.16.5}$$

Therefore, for all $\theta_{\mathrm{B/A}} \in [0,\pi)$, it follows that

$$\vec{\Theta}_{\mathrm{A/B}} = -\vec{\Theta}_{\mathrm{B/A}}, \tag{2.16.6}$$

$$\hat{\Theta}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.16.7}$$

Finally, define

$$\Theta_{\mathrm{B/A}} \triangleq \vec{\Theta}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{\Theta}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \theta_{\mathrm{B/A}}n_{\mathrm{B/A}}. \tag{2.16.8}$$

If $\theta_{\mathrm{B/A}} = \pi$, then $\vec{R}_{\mathrm{B/A}}$ has two eigenaxes related by -1, and, therefore, $\hat{n}_{\mathrm{B/A}}$ is not uniquely defined. Consequently, $\vec{\Theta}_{\mathrm{B/A}}$ is also not uniquely defined. In this case, we follow the convention that $\hat{n}_{\mathrm{B/A}}$ represents one of two possible eigenaxes, and that $\vec{\Theta}_{\mathrm{B/A}}$ represents one of two possible eigenaxis angle vectors.

In terms of $\theta_{\mathrm{B/A}}$ and $\hat{n}_{\mathrm{B/A}}$, (2.11.1), (2.11.2), and (2.11.3) can be written as

$$\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \tag{2.16.9}$$

$$= \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\cos\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \tag{2.16.10}$$

$$= \vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}^{\times 2}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}. \tag{2.16.11}$$

The next result expresses $\vec{R}_{\mathrm{B/A}}$ in terms of $\vec{\Theta}_{\mathrm{B/A}}$.

Fact 2.16.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in (0,\pi)$. Then,

$$\vec{R}_{\mathrm{B/A}} = \exp\left(\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \tag{2.16.12}$$

$$= (\cos\theta_{\mathrm{B/A}})\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} \tag{2.16.13}$$

$$= \frac{1}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\cos\theta_{\mathrm{B/A}})\left(\vec{I} - \frac{1}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}}\right) + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} \tag{2.16.14}$$

$$= \vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}} + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} \tag{2.16.15}$$

$$= \vec{I} + \frac{\sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{1}{2}\left(\frac{\sin\frac{1}{2}\theta_{\mathrm{B/A}}}{\frac{1}{2}\theta_{\mathrm{B/A}}}\right)^2\vec{\Theta}^{\times 2}_{\mathrm{B/A}}. \tag{2.16.16}$$

Proof. The result follows from Fact 2.16.1 by setting $\vec{\theta}^{\times}_{\vec{y}/\vec{x}} = \vec{\Theta}_{\mathrm{B/A}}$, that is, by setting $\theta_{\vec{y}/\vec{x}} = \theta_{\mathrm{B/A}}$ and $\hat{\theta}_{\vec{y}/\vec{x}} = \hat{n}_{\mathrm{B/A}}$. □

Combining (2.11.1), (2.11.45), and (2.16.12) yields the identities

$$\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) \tag{2.16.17}$$

$$= (\cos\theta_{\mathrm{B/A}})\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \tag{2.16.18}$$

$$= \exp\left(\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \tag{2.16.19}$$

$$= \exp\left(\theta\vec{n}^{\times}_{\mathrm{B/A}}\right) \tag{2.16.20}$$

$$= \exp\left[\frac{\theta_{\mathrm{B/A}}}{2\sin\theta_{\mathrm{B/A}}}\left(\vec{R}_{\mathrm{B/A}} - \vec{R}'_{\mathrm{B/A}}\right)\right], \tag{2.16.21}$$

where (2.16.20) and (2.16.21) are valid for the case $\theta_{\mathrm{B/A}} \in (0,\pi)$. It follows from (2.16.21)–(2.16.21) that

$$\mathcal{R}_{\mathrm{B/A}} = \mathcal{R}_{n_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) \tag{2.16.22}$$

$$= (\cos\theta_{\mathrm{B/A}})I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathsf{T}} + (\sin\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} \tag{2.16.23}$$

$$= \exp\left(\Theta_{\mathrm{B/A}}^{\times}\right) \tag{2.16.24}$$

$$= \exp\left(\theta n_{\mathrm{B/A}}^{\times}\right) \tag{2.16.25}$$

$$= \exp\left[\frac{\theta_{\mathrm{B/A}}}{2\sin\theta_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{B/A}}^{\mathsf{T}})\right], \tag{2.16.26}$$

where (2.16.25) and (2.16.26) are valid for the case $\theta_{\mathrm{B/A}} \in (0,\pi)$.

The following result expresses $\vec{R}_{\mathrm{C/A}} = \vec{R}_{\mathrm{C/B}}\vec{R}_{\mathrm{B/A}}$ in terms of eigenaxis angle vectors.

Fact 2.16.3. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\exp\left(\vec{\Theta}_{\mathrm{C/A}}^{\times}\right) = \exp\left(\vec{\Theta}_{\mathrm{C/B}}^{\times}\right)\exp\left(\vec{\Theta}_{\mathrm{B/A}}^{\times}\right). \tag{2.16.27}$$

The following result gives an alternative exponential representation of a physical rotation matrix.

Fact 2.16.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $\vec{\Theta}_{\mathrm{B/A}}$ as in Fact 2.16.2. Furthermore, let $\vec{v}_{\mathrm{B/A}}$ and $\vec{w}_{\mathrm{B/A}}$ satisfy $\vec{\Theta}_{\mathrm{B/A}} = \vec{v}_{\mathrm{B/A}} \times \vec{w}_{\mathrm{B/A}}$. Then,

$$\vec{R}_{\mathrm{B/A}} = \exp(\vec{w}_{\mathrm{B/A}}\vec{v}_{\mathrm{B/A}}' - \vec{v}_{\mathrm{B/A}}\vec{w}_{\mathrm{B/A}}'). \tag{2.16.28}$$

Proof. It follows from (2.16.12) and (2.9.21) that

$$\vec{R}_{\mathrm{B/A}} = \exp\left(\vec{\Theta}_{\mathrm{B/A}}^{\times}\right) = \exp[(\vec{v}_{\mathrm{B/A}} \times \vec{w}_{\mathrm{B/A}})^{\times}] = \exp(\vec{w}_{\mathrm{B/A}}\vec{v}_{\mathrm{B/A}}' - \vec{v}_{\mathrm{B/A}}\vec{w}_{\mathrm{B/A}}'). \qquad \square$$

In Fact 2.16.4, the vectors $\vec{v}_{\mathrm{B/A}}$ and $\vec{w}_{\mathrm{B/A}}$ define a plane that is orthogonal to $\vec{\Theta}_{\mathrm{B/A}}$. This plane and the length of the cross product of $\vec{v}_{\mathrm{B/A}}$ and $\vec{w}_{\mathrm{B/A}}$ characterize the physical rotation matrix. An analogous idea is given by the concept of a rotor in Chapter 3.

2.17 Euler Parameters

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames with eigenangle $\theta_{\mathrm{B/A}} \in [0,\pi]$ and eigenaxis $n_{\mathrm{B/A}}$. Then, rewriting (2.11.40) as

$$\cos\theta_{\mathrm{B/A}} = 2\cos^2 \tfrac{1}{2}\theta_{\mathrm{B/A}} - 1 = \tfrac{1}{2}(\operatorname{tr}\mathcal{R}_{\mathrm{B/A}} - 1), \tag{2.17.1}$$

we define

$$a \triangleq \cos\tfrac{1}{2}\theta_{\mathrm{B/A}} = \tfrac{1}{2}\sqrt{1 + \operatorname{tr}\mathcal{R}_{\mathrm{B/A}}}, \tag{2.17.2}$$

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} \triangleq (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}. \tag{2.17.3}$$

Note that $a \geq 0$, and that $a = 1$ if and only if $\theta_{\mathrm{B/A}} = 0$ rad, whereas $a = 0$ if and only if $\theta_{\mathrm{B/A}} = \pi$ rad. Furthermore, $\theta_{\mathrm{B/A}} = 0$ rad if and only if

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = 0, \tag{2.17.4}$$

whereas $\theta_{\mathrm{B/A}} = \pi$ rad if and only if

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = n_{\mathrm{B/A}}. \tag{2.17.5}$$

Now, assume that $\theta_{\mathrm{B/A}} \in [0, \pi)$, so that $a > 0$. Then, it follows from (2.11.45) and the identity $\sin\theta_{\mathrm{B/A}} = 2(\sin\frac{1}{2}\theta_{\mathrm{B/A}})\cos\frac{1}{2}\theta_{\mathrm{B/A}}$ that

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix}^{\times} = (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} = \frac{\sin\frac{1}{2}\theta_{\mathrm{B/A}}}{2\sin\theta_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}}) = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}}). \tag{2.17.6}$$

Therefore,

$$\begin{bmatrix} b \\ c \\ d \end{bmatrix} = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}})^{-\times}, \tag{2.17.7}$$

that is,

$$b = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}(3,2)} - \mathcal{R}_{\mathrm{B/A}(2,3)}), \tag{2.17.8}$$

$$c = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}(1,3)} - \mathcal{R}_{\mathrm{B/A}(3,1)}), \tag{2.17.9}$$

$$d = \frac{1}{4a}(\mathcal{R}_{\mathrm{B/A}(2,1)} - \mathcal{R}_{\mathrm{B/A}(1,2)}). \tag{2.17.10}$$

Fact 2.17.1. Define a, b, c, d by (2.17.2) and (2.17.3). Then,

$$a^2 + b^2 + c^2 + d^2 = 1. \tag{2.17.11}$$

Proof. Note that

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= a^2 + \begin{bmatrix} b \\ c \\ d \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} b \\ c \\ d \end{bmatrix} \\ &= \cos^2\tfrac{1}{2}\theta_{\mathrm{B/A}} + (\sin^2\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\mathsf{T}}n_{\mathrm{B/A}} \\ &= \cos^2\tfrac{1}{2}\theta_{\mathrm{B/A}} + \sin^2\tfrac{1}{2}\theta_{\mathrm{B/A}} = 1. \end{aligned}$$

□

The case $\theta_{\mathrm{B/A}} = \pi$ must be handled separately.

Fact 2.17.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, assume that $\theta_{\mathrm{B/A}} = \pi$, and define a, b, c, d by (2.17.2) and (2.17.3). Then,

$$\mathcal{R}_{\mathrm{B/A}} = \begin{bmatrix} 2b^2-1 & 2bc & 2bd \\ 2bc & 2c^2-1 & 2cd \\ 2bd & 2cd & 2d^2-1 \end{bmatrix}. \tag{2.17.12}$$

Proof. It follows from (2.11.35) that

$$\mathcal{R}_{\mathrm{B/A}} = -I_3 + 2n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathsf{T}} = -I_3 + 2\begin{bmatrix} b \\ c \\ d \end{bmatrix}\begin{bmatrix} b & c & d \end{bmatrix}. \qquad \square$$

Fact 2.17.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, assume that $\theta_{\mathrm{B/A}} \in [0, \pi)$, and define a, b, c, d by (2.17.2) and (2.17.3). Then,

$$\mathcal{R}_{\mathrm{B/A}} = \begin{bmatrix} 2a^2+2b^2-1 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & 2a^2+2c^2-1 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & 2a^2+2d^2-1 \end{bmatrix} \tag{2.17.13}$$

$$= \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix}. \tag{2.17.14}$$

Proof. It follows from (2.11.47) that

$$\begin{aligned} \mathcal{R}_{\mathrm{B/A}} &= (\cos\theta_{\mathrm{B/A}})I_3 + (1-\cos\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathsf{T}} + (\sin\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} \\ &= (2\cos^2\tfrac{1}{2}\theta_{\mathrm{B/A}} - 1)I_3 + 2(\sin^2\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\mathsf{T}} + 2(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}^{\times} \\ &= (2a^2-1)I_3 + 2\begin{bmatrix} b \\ c \\ d \end{bmatrix}\begin{bmatrix} b & c & d \end{bmatrix} + 2a\begin{bmatrix} b \\ c \\ d \end{bmatrix}^{\times}. \end{aligned} \qquad \square$$

Next, in analogy with the eigenaxis angle vector defined in (2.16.5), we define the *Euler vector*

$$\vec{\varepsilon}_{\mathrm{B/A}} \triangleq (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}. \tag{2.17.15}$$

This vector is uniquely defined for all $\theta_{\mathrm{B/A}} \in [0, \pi)$. It follows from (2.11.52) and (2.11.53) that

$$\vec{\varepsilon}_{\mathrm{B/A}} = -\vec{\varepsilon}_{\mathrm{A/B}}. \tag{2.17.16}$$

In the case $\theta_{\mathrm{B/A}} = \pi$, there are two possible choices of the eigenaxis $\hat{n}_{\mathrm{A/B}}$, and thus two possible choices of $\vec{\varepsilon}_{\mathrm{B/A}}$; these choices are related by the factor -1. Despite this ambiguity, we write $\vec{\varepsilon}_{\mathrm{A/B}} = -\vec{\varepsilon}_{\mathrm{B/A}}$ in all cases. Note that, if $\theta_{\mathrm{B/A}} \in (0, \pi]$, then

$$\hat{\varepsilon}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.17.17}$$

Next, define

$$\eta_{\mathrm{B/A}} \triangleq a, \tag{2.17.18}$$

$$\varepsilon_{\mathrm{B/A}} \triangleq \vec{\varepsilon}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{\varepsilon}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}} = \begin{bmatrix} b \\ c \\ d \end{bmatrix}. \tag{2.17.19}$$

Then, the *Euler parameter vector* $q_{\mathrm{B/A}}$ of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ is defined by

$$q_{\mathrm{B/A}} \triangleq \begin{bmatrix} \eta_{\mathrm{B/A}} \\ \varepsilon_{\mathrm{B/A}} \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\theta_{\mathrm{B/A}} \\ (\sin\frac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}. \tag{2.17.20}$$

Note that $\eta_{\mathrm{B/A}} \ge 0$, and that $\eta_{\mathrm{B/A}} = 0$ if and only if $\theta_{\mathrm{B/A}} = \pi$ rad. Furthermore, if $\theta_{\mathrm{B/A}} = 0$, then

$$q_{\mathrm{B/A}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.17.21}$$

whereas, if $\theta_{\mathrm{B/A}} = \pi$, then

$$q_{\mathrm{B/A}} = \begin{bmatrix} 0 \\ n_{\mathrm{B/A}} \end{bmatrix}. \tag{2.17.22}$$

The Euler parameter vector $q_{\mathrm{B/A}}$ provides a representation of the rotation matrix $\mathcal{R}_{\mathrm{B/A}}$. The components a, b, c, d of $q_{\mathrm{B/A}}$ are the *Euler parameters*. It follows from (2.11.52) and (2.11.53) that

$$\eta_{\mathrm{B/A}} = \eta_{\mathrm{A/B}}, \tag{2.17.23}$$

$$\varepsilon_{\mathrm{B/A}} = -\varepsilon_{\mathrm{A/B}}. \tag{2.17.24}$$

The following result shows that $q_{\mathrm{B/A}}$ is an element of the unit sphere in $\mathbb{R}^4$.

Fact 2.17.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\eta^2_{\mathrm{B/A}} + \vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}} = 1, \tag{2.17.25}$$

and thus

$$\eta^2_{\mathrm{B/A}} + \varepsilon^{\mathsf{T}}_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}} = 1. \tag{2.17.26}$$

If $\theta_{\mathrm{B/A}} \in (-2\pi, 2\pi]$ so that $\frac{1}{2}\theta_{\mathrm{B/A}} \in (-\pi, \pi]$, then $q_{\mathrm{B/A}}$ and $-q_{\mathrm{B/A}}$ both represent $\mathcal{R}_{\mathrm{B/A}}$, and thus (2.17.12) and (2.17.14) are two-to-one. With this increased range of $\theta_{\mathrm{B/A}}$, the values of $q_{\mathrm{B/A}}$ are in one-to-one correspondence with each point on the unit sphere in $\mathbb{R}^4$. However, unless stated otherwise, the eigenangle $\theta_{\mathrm{B/A}}$ is assumed to be an element of $[0, \pi]$.

Fact 2.17.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{R}_{\mathrm{B/A}} = (2\eta^2_{\mathrm{B/A}} - 1)\vec{I} + 2\eta_{\mathrm{B/A}}\vec{\varepsilon}^{\,\times}_{\mathrm{B/A}} + 2\vec{\varepsilon}_{\mathrm{B/A}}\vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}} \tag{2.17.27}$$

$$= \vec{I} + 2\eta_{\mathrm{B/A}}\vec{\varepsilon}^{\,\times}_{\mathrm{B/A}} + 2\vec{\varepsilon}^{\,\times 2}_{\mathrm{B/A}}. \tag{2.17.28}$$

Proof. The result follows from Fact 2.11.9 using the identities

$$2(\cos^2\tfrac{1}{2}\theta) - 1 = \cos\theta, \quad 2\cos\tfrac{1}{2}\theta = 1 - \cos\theta, \quad 2(\cos\tfrac{1}{2}\theta)\sin\tfrac{1}{2}\theta = \sin\theta. \qquad \square$$

Resolving (2.17.27) and (2.17.28) yields the following result.

Fact 2.17.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\mathcal{R}_{\mathrm{B/A}} = (2\eta_{\mathrm{B/A}}^2 - 1)I_3 + 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\mathsf{T}} \tag{2.17.29}$$

$$= I_3 + 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}^{\times 2}. \tag{2.17.30}$$

Furthermore,

$$\mathcal{O}_{\mathrm{B/A}} = (2\eta_{\mathrm{B/A}}^2 - 1)I_3 - 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\mathsf{T}} \tag{2.17.31}$$

$$= I_3 - 2\eta_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A}}^{\times} + 2\varepsilon_{\mathrm{B/A}}^{\times 2}. \tag{2.17.32}$$

The following result determines the Euler-parameter vector for an orientation matrix rising from a pair of orientatioin matrices expressed in terms of Euler parameters. In effect, this result provides an expression for the product of Euler-parameter vectors, corresponding to the identity $\mathcal{O}_{\mathrm{C/A}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}$.

For the next result, define

$$\mathcal{Q}(q) \triangleq \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, \tag{2.17.33}$$

where $q \triangleq [a\ \ b\ \ c\ \ d]^{\mathsf{T}}$.

Fact 2.17.7. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$q_{\mathrm{C/A}} = \begin{bmatrix} \eta_{\mathrm{C/A}} \\ \varepsilon_{\mathrm{C/A}} \end{bmatrix} = \begin{bmatrix} \eta_{\mathrm{C/B}}\eta_{\mathrm{B/A}} - \varepsilon_{\mathrm{C/B}}^{\mathsf{T}}\varepsilon_{\mathrm{B/A}} \\ \eta_{\mathrm{B/A}}\varepsilon_{\mathrm{C/B}} + \eta_{\mathrm{C/B}}\varepsilon_{\mathrm{B/A}} + \varepsilon_{\mathrm{C/B}} \times \varepsilon_{\mathrm{B/A}} \end{bmatrix} = \mathcal{Q}(q_{\mathrm{C/B}})q_{\mathrm{B/A}}. \tag{2.17.34}$$

Proof. Define

$$\mathcal{R}_3 \triangleq \mathcal{R}_{\mathrm{C/A}}, \quad \mathcal{R}_2 \triangleq \mathcal{R}_{\mathrm{C/B}}, \quad \mathcal{R}_1 \triangleq \mathcal{R}_{\mathrm{B/A}},$$

$$q_3 \triangleq \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} \triangleq q_{\mathrm{C/A}}, \quad q_2 \triangleq \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \triangleq q_{\mathrm{C/B}}, \quad q_1 \triangleq \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \triangleq q_{\mathrm{B/A}}.$$

Then, for $i = 1, 2, 3$, it follows from (2.17.30) that

$$\mathcal{R}_i = (2a_i^2 - 1)I_3 + 2\begin{bmatrix} b_i \\ c_i \\ d_i \end{bmatrix}\begin{bmatrix} b_i & c_i & d_i \end{bmatrix} + 2a_i\begin{bmatrix} b_i \\ c_i \\ d_i \end{bmatrix}^{\times}.$$

Hence,

$$a_3 = \tfrac{1}{2}\sqrt{1 + \operatorname{tr}\mathcal{R}_3} = \tfrac{1}{2}\sqrt{1 + \operatorname{tr}\mathcal{R}_1\mathcal{R}_2} = a_2a_1 - \begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix}^{\mathsf{T}}\begin{bmatrix} b_1 \\ c_1 \\ d_1 \end{bmatrix},$$

which confirms the expression for $\eta_{\mathrm{C/A}}$ in (2.17.34). Furthermore,

$$\begin{bmatrix} b_3 \\ c_3 \\ d_3 \end{bmatrix} = \frac{1}{4a_3}(\mathcal{R}_3 - \mathcal{R}_3^{\mathsf{T}}) = \frac{1}{4a_3}(\mathcal{R}_1\mathcal{R}_2 - \mathcal{R}_2^{\mathsf{T}}\mathcal{R}_1^{\mathsf{T}}) = a_1 \begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix} + a_2 \begin{bmatrix} b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix} \times \begin{bmatrix} b_1 \\ c_1 \\ d_1 \end{bmatrix},$$

which confirms the expression for $\varepsilon_{\mathrm{C/A}}$ in (2.17.34). The last expression for $q_{\mathrm{C/A}}$ can be confirmed directly. □

2.18 Quaternions†

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}, \tag{2.18.1}$$

and define the *quaternion numbers*

$$\mathbb{H} \triangleq \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \colon a, b, c, d \in \mathbb{R}\}. \tag{2.18.2}$$

Furthermore, for all $a, b, c, d \in \mathbb{R}$, define

$$\mathbf{q} \triangleq a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \tag{2.18.3}$$

$$\overline{\mathbf{q}} \triangleq a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}, \tag{2.18.4}$$

$$|\mathbf{q}| \triangleq \sqrt{\mathbf{q}\overline{\mathbf{q}}} = \sqrt{a^2 + b^2 + c^2 + d^2} = |\overline{\mathbf{q}}|. \tag{2.18.5}$$

Then,

$$\mathbf{q}I_4 = U\mathcal{Q}(\mathbf{q})U, \tag{2.18.6}$$

where

$$\mathcal{Q}(\mathbf{q}) \triangleq \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, \quad U \triangleq \frac{1}{2}\begin{bmatrix} 1 & \mathbf{i} & \mathbf{j} & k \\ -\mathbf{i} & 1 & \mathbf{k} & -\mathbf{j} \\ -\mathbf{j} & -\mathbf{k} & 1 & \mathbf{i} \\ -\mathbf{k} & \mathbf{j} & -\mathbf{i} & 1 \end{bmatrix}. \tag{2.18.7}$$

Note that $\mathcal{Q}(\mathbf{q})$ has the same form as $\mathcal{Q}(q)$ given by (2.17.33), and that $U^2 = I_4$. In addition,

$$\det \mathcal{Q}(\mathbf{q}) = (a^2 + b^2 + c^2 + d^2)^2. \tag{2.18.8}$$

Therefore, if $|\mathbf{q}| = 1$, then $\mathcal{Q}(\mathbf{q})$ is orthogonal and $\det \mathcal{Q}(\mathbf{q}) = 1$.

For $i = 1, 2$, let $a_i, b_i, c_i, d_i \in \mathbb{R}$ and define $\mathbf{q}_i \triangleq a_i + b_i\mathbf{i} + c_i\mathbf{j} + d_i\mathbf{k}$ and $v_i \triangleq [b_i \;\; c_i \;\; d_i]^{\mathsf{T}}$. In addition, define $\mathbf{q}_3 \triangleq \mathbf{q}_2\mathbf{q}_1 = a_3 + b_3\mathbf{i} + c_3\mathbf{j} + d_3\mathbf{k}$. Then,

$$\mathcal{Q}(\mathbf{q}_3) = \mathcal{Q}(\mathbf{q}_2)\mathcal{Q}(\mathbf{q}_1), \quad \overline{\mathbf{q}_3} = \overline{\mathbf{q}_2}\,\overline{\mathbf{q}_1}, \tag{2.18.9}$$

$$|\mathbf{q}_3| = |\mathbf{q}_2\mathbf{q}_1| = |\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1\overline{\mathbf{q}_2}| = |\overline{\mathbf{q}_1}\mathbf{q}_2| = |\overline{\mathbf{q}_1}\,\overline{\mathbf{q}_2}| = |\mathbf{q}_1||\mathbf{q}_2|. \tag{2.18.10}$$

Furthermore, it follows from (2.17.34) that

$$\begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \mathcal{Q}(\mathbf{q}_2)\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} a_2a_1 - v_2^{\mathsf{T}}v_1 \\ a_1v_2 + a_2v_1 + v_2 \times v_1 \end{bmatrix}. \tag{2.18.11}$$

This expression shows that the unit quaternions satisfy the multiplicative property (2.17.34) for the Euler parameters. Consequently, the unit quaternions $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ provide a representation of the Euler parameters.

The quaternions can be represented in various ways using real and complex matrices. In particular, the complex number $\jmath$ can be represented by the real 2×2 skew-symmetric matrix

$$J_2 \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.18.12}$$

which satisfies $J_2^2 = -I_2$. Furthermore, define the complex 2×2 τ-*matrices*

$$\tau_1 \triangleq \begin{bmatrix} 0 & -\jmath \\ -\jmath & 0 \end{bmatrix}, \quad \tau_2 \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tau_3 \triangleq \begin{bmatrix} -\jmath & 0 \\ 0 & \jmath \end{bmatrix}, \tag{2.18.13}$$

which satisfy

$$\tau_1^2 = \tau_2^2 = \tau_3^2 = -I_2, \tag{2.18.14}$$

$$\tau_1\tau_2 = -\tau_2\tau_1 = \tau_3, \quad \tau_2\tau_3 = -\tau_3\tau_2 = \tau_1, \quad \tau_3\tau_1 = -\tau_1\tau_3 = \tau_2, \tag{2.18.15}$$

$$\tau_1\tau_2\tau_3 = \tau_2\tau_3\tau_1 = \tau_3\tau_1\tau_2 = -I_2, \tag{2.18.16}$$

as well as the real 4×4 f-*matrices*

$$f_1 \triangleq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad f_2 \triangleq \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad f_3 \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{2.18.17}$$

which satisfy

$$f_1^2 = f_2^2 = f_3^2 = -I_4, \tag{2.18.18}$$

$$f_1f_2 = -f_2f_1 = f_3, \quad f_2f_3 = -f_3f_2 = f_1, \quad f_3f_1 = -f_1f_3 = f_2, \tag{2.18.19}$$

$$f_1f_2f_3 = f_2f_3f_1 = f_3f_1f_2 = -I_4. \tag{2.18.20}$$

Equivalent multiplication tables for complex scalars, 2×2 real matrices, τ-matrices, and f-matrices are given in Table 2.18.1.

	1	$\jmath$
1	1	$\jmath$
$\jmath$	$\jmath$	-1

(a)

	I_2	J_2
I_2	I_2	J_2
J_2	J_2	$-I_2$

(b)

	I_2	τ_1
I_2	I_2	τ_1
τ_1	τ_1	$-I_2$

(c)

	I_4	f_1
I_4	I_4	f_1
f_1	f_1	$-I_4$

(d)

Table 2.18.1: Equivalent multiplication tables for (a) complex scalars, (b) 2×2 real matrices, (c) τ-matrices, and (d) f-matrices.

	1	**i**	**j**	**k**
1	1	**i**	**j**	**k**
i	**i**	-1	**k**	−**j**
j	**j**	−**k**	−1	**i**
k	**k**	**j**	−**i**	−1

(a)

	I_2	τ_1	τ_2	τ_3
I_2	I_2	τ_1	τ_2	τ_3
τ_1	τ_1	$-I_2$	τ_3	$-\tau_2$
τ_2	τ_2	$-\tau_3$	$-I_2$	τ_1
τ_3	τ_3	τ_2	$-\tau_1$	$-I_2$

(b)

	I_4	f_1	f_2	f_3
I_4	I_4	f_1	f_2	f_3
f_1	f_1	$-I_4$	f_3	$-f_2$
f_2	f_2	$-f_3$	$-I_4$	f_1
f_3	f_3	f_2	$-f_1$	$-I_4$

(c)

Table 2.18.2: Equivalent multiplication tables for (a) complex scalars, (b) 2×2 real matrices, (c) τ-matrices, and (d) f-matrices.

2.19 Gibbs Parameters†

An alternative representation of rotation matrices is given in terms of the Gibbs vector. This representation is valid for all rotations except for the case where the eigenangle is π rad.

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in [0,\pi)$. In analogy with the Euler vector defined in (2.17.15) and the eigenaxis angle vector defined in (2.16.5), we define the *Gibbs vector*

$$\vec{g}_{\mathrm{B/A}} \triangleq (\tan \tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}. \tag{2.19.1}$$

If $\theta_{\mathrm{B/A}} = 0$, then $\vec{g}_{\mathrm{A/B}} = 0$. If $\theta_{\mathrm{B/A}} \in (0,\pi)$, then $\theta_{\mathrm{B/A}} = \theta_{\mathrm{A/B}}$ and $\hat{n}_{\mathrm{B/A}} = -\hat{n}_{\mathrm{A/B}}$, and thus $\vec{g}_{\mathrm{A/B}} = -\vec{g}_{\mathrm{B/A}}$. In the case $\theta_{\mathrm{B/A}} = \pi$, $\vec{g}_{\mathrm{B/A}}$ is not defined. Note that, if $\theta_{\mathrm{B/A}} \in (0,\pi)$, then

$$\hat{g}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}. \tag{2.19.2}$$

Finally, define the *Gibbs parameter vector* by

$$g_{\mathrm{B/A}} \triangleq \vec{g}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{g}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = (\tan \tfrac{1}{2}\theta_{\mathrm{B/A}})n_{\mathrm{B/A}}. \tag{2.19.3}$$

Hence,

$$g_{\mathrm{B/A}} = -g_{\mathrm{A/B}}. \tag{2.19.4}$$

Next, define the physical matrix

$$\vec{R}_{\vec{g}_{\mathrm{B/A}}} \triangleq \frac{1}{1+\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}}[(1-\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}})\vec{I} + 2\vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + 2\vec{g}^{\times}_{\mathrm{B/A}}]. \tag{2.19.5}$$

The following result shows that $\vec{R}_{\vec{g}_{\mathrm{B/A}}}$ is the physical rotation matrix whose eigenaxis is $\hat{n}_{\mathrm{B/A}}$ and eigenangle is $\theta_{\mathrm{B/A}}$.

Fact 2.19.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \in [0,\pi)$. Then,

$$\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) = \vec{R}_{\vec{g}_{\mathrm{B/A}}}. \tag{2.19.6}$$

Furthermore,

$$\vec{R}_{\vec{g}_{\mathrm{B/A}}} = (\vec{I} - \vec{g}^{\times}_{\mathrm{B/A}})^{-1}(\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}}) \tag{2.19.7}$$

$$= (\vec{I} + \vec{g}^{\times}_{\mathrm{B/A}})(\vec{I} - \vec{g}^{\times}_{\mathrm{B/A}})^{-1}. \tag{2.19.8}$$

Proof. To prove (2.19.6) note that it follows from (2.11.1) and the identities

$$\cos\theta = \frac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}, \quad 1-\cos\theta = \frac{2\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}, \quad \sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

that

$$\begin{aligned}
\vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) &= (\cos\theta)\vec{I} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} \\
&= \frac{1-\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}{1+\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}\vec{I} + \frac{2\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}{1+\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} + \frac{2\tan\frac{1}{2}\theta_{\mathrm{B/A}}}{1+\tan^2\frac{1}{2}\theta_{\mathrm{B/A}}}\hat{n}^{\times}_{\mathrm{B/A}} \\
&= \frac{1}{1+\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}}[(1-\vec{g}'_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}})\vec{I} + 2\vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + 2\vec{g}^{\times}_{\mathrm{B/A}}] = \vec{R}_{\vec{g}_{\mathrm{B/A}}}.
\end{aligned}$$

Next, to prove (2.19.7) note that

$$
\begin{aligned}
(\vec{I} - \vec{g}_{\mathrm{B/A}}^{\times})^{-1}(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times}) &= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}'\vec{g}_{\mathrm{B/A}}}(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}' + \vec{g}_{\mathrm{B/A}}^{\times})(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times})\\
&= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}'\vec{g}_{\mathrm{B/A}}}[\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}' + 2\vec{g}_{\mathrm{B/A}}^{\times} + \vec{g}_{\mathrm{B/A}}^{\times 2}]\\
&= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}'\vec{g}_{\mathrm{B/A}}}[\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}' + 2\vec{g}_{\mathrm{B/A}}^{\times} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}' - \vec{g}_{\mathrm{B/A}}'\vec{g}_{\mathrm{B/A}}\vec{I}]\\
&= \frac{1}{1 + \vec{g}_{\mathrm{B/A}}'\vec{g}_{\mathrm{B/A}}}[(1 - \vec{g}_{\mathrm{B/A}}'\vec{g}_{\mathrm{B/A}})\vec{I} + 2\vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}' + 2\vec{g}_{\mathrm{B/A}}^{\times}] = \vec{R}_{\vec{g}_{\mathrm{B/A}}}. \qquad \square
\end{aligned}
$$

The following result relates the Gibbs parameter vector to the Euler parameter vector $q_{\mathrm{B/A}}$ defined by (2.17.20).

Fact 2.19.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \neq \pi$. Then, $\eta_{\mathrm{B/A}} \neq 0$. Furthermore,

$$g_{\mathrm{B/A}} = \frac{1}{\eta_{\mathrm{B/A}}}\varepsilon_{\mathrm{B/A}}. \tag{2.19.9}$$

Finally,

$$g_{\mathrm{B/A}} = \frac{1}{1 + \operatorname{tr}\mathcal{R}_{\mathrm{B/A}}}(\mathcal{R}_{\mathrm{B/A}} - \mathcal{R}_{\mathrm{A/B}})^{-\times}. \tag{2.19.10}$$

The following result determines the Gibbs parameter vector for a rotation arising from a pair of rotation matrices expressed in terms of Gibbs parameter vectors. In effect, this result provides an expression for the product of Gibbs-parameter vectors, corresponding to the identity $\mathcal{O}_{\mathrm{C/A}} = \mathcal{O}_{\mathrm{C/B}}\mathcal{O}_{\mathrm{B/A}}$.

Fact 2.19.3. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames, and assume that neither of the eigenangles $\theta_{\mathrm{C/A}}$, $\theta_{\mathrm{C/B}}$, nor $\theta_{\mathrm{B/A}}$ is equal to π rad. Then,

$$\vec{g}_{\mathrm{C/A}} = \frac{1}{1 - \vec{g}_{\mathrm{C/B}}'\vec{g}_{\mathrm{B/A}}}(\vec{g}_{\mathrm{C/B}} + \vec{g}_{\mathrm{B/A}} - \vec{g}_{\mathrm{C/B}} \times \vec{g}_{\mathrm{B/A}}). \tag{2.19.11}$$

Table 2.19.1 summarizes the existence and uniqueness of the eigenaxis, eigenaxis angle vector, Euler vector, and Gibbs vector in terms of the eigenangle.

$\operatorname{tr}\vec{R}_{\mathrm{B/A}}$	$\theta_{\mathrm{B/A}}$	$\hat{n}_{\mathrm{B/A}}$	$\vec{\Theta}_{\mathrm{B/A}} = \theta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$	$\vec{\varepsilon}_{\mathrm{B/A}} = (\sin\frac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}$	$\vec{g}_{\mathrm{B/A}} = (\tan\frac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}$
$\operatorname{tr}\vec{R}_{\mathrm{B/A}} = -1$	$\theta_{\mathrm{B/A}} = \pi$	Two choices $\pm\hat{n}_{\mathrm{B/A}}$	Two choices $\vec{\Theta}_{\mathrm{B/A}} = \pm\pi\hat{n}_{\mathrm{B/A}}$	Unique $\vec{\varepsilon}_{\mathrm{B/A}} = \hat{n}_{\mathrm{B/A}}$	Not Defined
$-1 < \operatorname{tr}\vec{R}_{\mathrm{B/A}} < 3$	$\theta_{\mathrm{B/A}} \in (0, \pi)$	Unique	Unique	Unique	Unique
$\operatorname{tr}\vec{R}_{\mathrm{B/A}} = 3$	$\theta_{\mathrm{B/A}} = 0$	Arbitrary	Unique $\vec{\Theta}_{\mathrm{B/A}} = 0$	Unique $\vec{\varepsilon}_{\mathrm{B/A}} = 0$	Unique $\vec{g}_{\mathrm{B/A}} = 0$

Table 2.19.1: Existence and uniqueness of the eigenaxis, eigenaxis angle vector, Euler vector, and Gibbs vector in terms of the eigenangle.

2.20 Additivity of Angle Vectors†

The following result concerns the additivity of angles for linearly independent physical vectors.

Fact 2.20.1. Assume that the nonzero physical vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly dependent, and let $\hat{n}$ denote a unit vector that is orthogonal to $\vec{x}$, $\vec{y}$, and $\vec{z}$. Then, there exists $i \in \{-1, 0, 1\}$ such that

$$\theta_{\vec{z}/\vec{x}/\hat{n}} = \theta_{\vec{z}/\vec{y}/\hat{n}} + \theta_{\vec{y}/\vec{x}/\hat{n}} \pm 2i\pi. \tag{2.20.1}$$

In addition,

$$\vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{x}/\hat{n}}) = \vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{y}/\hat{n}})\vec{R}_{\hat{n}}(\theta_{\vec{y}/\vec{x}/\hat{n}}). \tag{2.20.2}$$

Now, assume that $\theta_{\vec{z}/\vec{x}/\hat{n}}, \theta_{\vec{z}/\vec{y}/\hat{n}}, \theta_{\vec{y}/\vec{x}/\hat{n}} \in (0, \pi)$, $\theta_{\vec{z}/\vec{y}/\hat{n}} \le \theta_{\vec{z}/\vec{x}/\hat{n}}$, and $\theta_{\vec{y}/\vec{x}/\hat{n}} \le \theta_{\vec{z}/\vec{x}/\hat{n}}$. Then,

$$\theta_{\vec{z}/\vec{x}} = \theta_{\vec{z}/\vec{y}} + \theta_{\vec{y}/\vec{x}}, \tag{2.20.3}$$

$$\vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{x}}) = \vec{R}_{\hat{n}}(\theta_{\vec{z}/\vec{y}})\vec{R}_{\hat{n}}(\theta_{\vec{y}/\vec{x}}). \tag{2.20.4}$$

The case corresponding to (2.20.3) is illustrated in Figure 2.20.1.

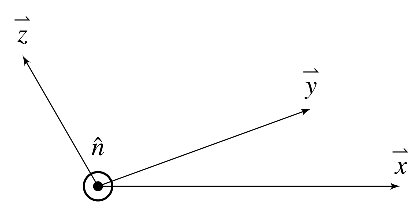


Figure 2.20.1: Angle additivity.

Fact 2.20.2. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be unit dimensionless physical vectors. Then,

$$|\hat{x} \times \hat{y}|^2 \cos^2 \theta_{\hat{x}\times\hat{y}/\hat{z}} = 1 + 2(\cos\theta_{\hat{x}/\hat{y}})(\cos\theta_{\hat{y}/\hat{z}})(\cos\theta_{\hat{z}/\hat{x}}) - \cos^2\theta_{\hat{x}/\hat{y}} - \cos^2\theta_{\hat{y}/\hat{z}} - \cos^2\theta_{\hat{z}/\hat{x}}. \tag{2.20.5}$$

Proof. Using Fact 2.9.7 we have

$$|\hat{x} \times \hat{y}|^2 \cos^2 \theta_{\hat{x}\times\hat{y}/\hat{z}} = [(\hat{x}\times\hat{y})'\hat{z}]^2 = \det \begin{bmatrix} \vec{x}\big|_{\mathrm{A}}^{\mathsf{T}} \\ \vec{y}\big|_{\mathrm{A}}^{\mathsf{T}} \\ \vec{z}\big|_{\mathrm{A}}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & \hat{x}'\hat{y} & \hat{x}'\hat{z} \\ \hat{x}'\hat{y} & 1 & \hat{y}'\hat{z} \\ \hat{x}'\hat{z} & \hat{y}'\hat{z} & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & \cos\theta_{\hat{x}/\hat{y}} & \cos\theta_{\hat{x}/\hat{z}} \\ \cos\theta_{\hat{x}/\hat{y}} & 1 & \cos\theta_{\hat{y}/\hat{z}} \\ \cos\theta_{\hat{x}/\hat{z}} & \cos\theta_{\hat{y}/\hat{z}} & 1 \end{bmatrix}. \qquad \square$$

The following result shows that angle vectors are additive if and only if both angles lie in the same plane.

Fact 2.20.3. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be nonzero physical vectors, no two of which are parallel. Then, the following statements are equivalent:

i) $\hat{x}$, $\hat{y}$, and $\hat{z}$ are linearly dependent.

ii) $\hat{\theta}_{\hat{z}/\hat{x}}$, $\hat{\theta}_{\hat{x}/\hat{y}}$, and $\hat{\theta}_{\hat{y}/\hat{z}}$ are parallel.

iii) At least two of the vectors $\hat{\theta}_{\hat{z}/\hat{x}}$, $\hat{\theta}_{\hat{x}/\hat{y}}$, and $\hat{\theta}_{\hat{y}/\hat{z}}$ are parallel.

iv) Either $\theta_{\hat{x}/\hat{y}} + \theta_{\hat{y}/\hat{z}} + \theta_{\hat{z}/\hat{x}} = 2\pi$, $\theta_{\hat{z}/\hat{x}} = \theta_{\hat{z}/\hat{y}} + \theta_{\hat{y}/\hat{x}}$, $\theta_{\hat{y}/\hat{z}} = \theta_{\hat{y}/\hat{x}} + \theta_{\hat{x}/\hat{z}}$, or $\theta_{\hat{z}/\hat{y}} = \theta_{\hat{z}/\hat{x}} + \theta_{\hat{x}/\hat{y}}$.

v) $\cos\theta_{\hat{x}\times\hat{y}/\hat{z}} = 0$.

vi) $\cos^2\theta_{\hat{x}/\hat{y}} + \cos^2\theta_{\hat{y}/\hat{z}} + \cos^2\theta_{\hat{z}/\hat{x}} = 1 + 2(\cos\theta_{\hat{x}/\hat{y}})(\cos\theta_{\hat{y}/\hat{z}})(\cos\theta_{\hat{z}/\hat{x}})$.

vii) $\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}})\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})$.

The following result considers the case where the vectors are not necessarily linearly dependent.

Fact 2.20.4. Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be unit dimensionless physical vectors, and assume that $\hat{x}$ and $\hat{y}$ are not parallel and $\hat{y}$ and $\hat{z}$ are not parallel. Then,

$$\begin{aligned}
\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) - \vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}})\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})&\\
= (\sin\theta_{\hat{z}/\hat{y}})(\sin\theta_{\hat{y}/\hat{x}})(\hat{\theta}_{\hat{z}/\hat{y}} \times \hat{\theta}_{\hat{y}/\hat{x}})^{\times}&\\
+ (\sin\theta_{\hat{z}/\hat{y}})(1-\cos\theta_{\hat{y}/\hat{x}})(\hat{\theta}^{\times}_{\hat{z}/\hat{y}}\hat{\theta}_{\hat{y}/\hat{x}}\hat{\theta}'_{\hat{y}/\hat{x}} - \hat{\theta}_{\hat{y}/\hat{x}}\hat{\theta}'_{\hat{y}/\hat{x}}\hat{\theta}^{\times}_{\hat{z}/\hat{y}})&\\
+ (\sin\theta_{\hat{y}/\hat{x}})(1-\cos\theta_{\hat{z}/\hat{y}})(\hat{\theta}_{\hat{z}/\hat{y}}\hat{\theta}'_{\hat{z}/\hat{y}}\hat{\theta}^{\times}_{\hat{y}/\hat{x}} - \hat{\theta}^{\times}_{\hat{y}/\hat{x}}\hat{\theta}_{\hat{z}/\hat{y}}\hat{\theta}'_{\hat{z}/\hat{y}})&\\
+ (1-\cos\theta_{\hat{z}/\hat{y}})(1-\cos\theta_{\hat{y}/\hat{x}})\hat{\theta}'_{\hat{z}/\hat{y}}\hat{\theta}_{\hat{y}/\hat{x}}(\hat{\theta}_{\hat{z}/\hat{y}}\hat{\theta}'_{\hat{y}/\hat{x}} - \hat{\theta}_{\hat{y}/\hat{x}}\hat{\theta}'_{\hat{z}/\hat{y}}).&
\end{aligned} \tag{2.20.6}$$

Finally,

$$\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}})\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}})\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) \tag{2.20.7}$$

if and only if at least one of the following conditions holds:

i) $\hat{\theta}_{\hat{z}/\hat{y}}$ and $\hat{\theta}_{\hat{y}/\hat{x}}$ are parallel.

ii) Either $\vec{R}_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) = \vec{I}$ or $\vec{R}_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{I}$.

iii) $\vec{R}^2_{\hat{\theta}_{\hat{z}/\hat{y}}}(\theta_{\hat{z}/\hat{y}}) = \vec{I}$, $\vec{R}^2_{\hat{\theta}_{\hat{y}/\hat{x}}}(\theta_{\hat{y}/\hat{x}}) = \vec{I}$, and $\hat{\theta}'_{\hat{y}/\hat{x}}\hat{\theta}_{\hat{z}/\hat{y}} = 0$.

Proof. See [1, p. 211]. □

2.21 Nonstandard Frames and Reciprocal Frames†

Thus far, and throughout this book, all frames are assumed to be *standard frames*, which are orthogonal, right-handed frames with dimensionless, unit-length axes. In this section we develop properties of frames that consist of three linearly independent axes. These frames arise naturally in many applications.

A *nonstandard frame* is a collection of three linearly independent dimensionless physical vectors. Note that "nonstandard" means "not necessarily standard." Letting $\mathrm{F_A}$ be a nonstandard frame, we denote its dimensionless axes by $\vec{a}_1, \vec{a}_2, \vec{a}_3$. These vectors are not necessarily mutually orthogonal and need not have unit length. Furthermore, it follows from Fact 2.4.4 that, for each physical vector $\vec{x}$, there exist unique real numbers x_1, x_2, x_3 such that

$$\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3. \tag{2.21.1}$$

We thus write

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{2.21.2}$$

Fact 2.21.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be nonstandard frames with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{b}_1, \vec{b}_2, \vec{b}_3$, respectively. Then

$$\begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}, \tag{2.21.3}$$

where $\mathcal{O}_{\mathrm{B/A}} \in \mathbb{R}^{3\times 3}$ is defined by

$$\mathcal{O}_{\mathrm{B/A}} \triangleq \frac{1}{\alpha} \begin{bmatrix} \vec{b}_1 \cdot (\vec{a}_2 \times \vec{a}_3) & \vec{b}_1 \cdot (\vec{a}_3 \times \vec{a}_1) & \vec{b}_1 \cdot (\vec{a}_1 \times \vec{a}_2) \\ \vec{b}_2 \cdot (\vec{a}_2 \times \vec{a}_3) & \vec{b}_2 \cdot (\vec{a}_3 \times \vec{a}_1) & \vec{b}_2 \cdot (\vec{a}_1 \times \vec{a}_2) \\ \vec{b}_3 \cdot (\vec{a}_2 \times \vec{a}_3) & \vec{b}_3 \cdot (\vec{a}_3 \times \vec{a}_1) & \vec{b}_3 \cdot (\vec{a}_1 \times \vec{a}_2) \end{bmatrix}, \tag{2.21.4}$$

where

$$\alpha \triangleq \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3). \tag{2.21.5}$$

Furthermore, $\mathcal{O}_{\mathrm{B/A}}$ is nonsingular, and

$$\mathcal{O}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{B/A}}^{-1}. \tag{2.21.6}$$

Now, let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{A/B}}^{\mathsf{T}}\ \vec{x}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}}^{-\mathsf{T}}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.21.7}$$

Finally, if $\mathrm{F_A}$ and $\mathrm{F_B}$ are standard frames, then $\mathcal{O}_{\mathrm{B/A}}$ is a rotation matrix and

$$\vec{x}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.21.8}$$

Proof. To verify (2.21.3), note that the first equation is given by

$$\alpha\vec{b}_1 = \vec{b}_1 \cdot (\vec{a}_2 \times \vec{a}_3)\vec{a}_1 + \vec{b}_1 \cdot (\vec{a}_3 \times \vec{a}_1)\vec{a}_2 + \vec{b}_1 \cdot (\vec{a}_1 \times \vec{a}_2)\vec{a}_3.$$

Resolving each vector in this equation yields *xlvii*) in [1, p. 386]. □

Let $\mathrm{F_A}$ be a nonstandard frame with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Now, define the dimensionless physical vectors

$$\vec{a}_{[1]} \triangleq \frac{1}{\alpha}\vec{a}_2 \times \vec{a}_3, \tag{2.21.9}$$

$$\vec{a}_{[2]} \triangleq \frac{1}{\alpha}\vec{a}_3 \times \vec{a}_1, \tag{2.21.10}$$

$$\vec{a}_{[3]} \triangleq \frac{1}{\alpha}\vec{a}_1 \times \vec{a}_2, \tag{2.21.11}$$

where α is defined by (2.21.5).

Fact 2.21.2. Let F_A be a nonstandard frame with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Then, the physical vectors $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$ are linearly independent.

The physical vectors $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$ define the *reciprocal frame* $F_{[A]}$. Using these definitions, (2.21.4) can be written as

$$\mathcal{O}_{[A]/A} = \begin{bmatrix} \vec{b}_1 \cdot \vec{a}_{[1]} & \vec{b}_1 \cdot \vec{a}_{[2]} & \vec{b}_1 \cdot \vec{a}_{[3]} \\ \vec{b}_2 \cdot \vec{a}_{[1]} & \vec{b}_2 \cdot \vec{a}_{[2]} & \vec{b}_2 \cdot \vec{a}_{[3]} \\ \vec{b}_3 \cdot \vec{a}_{[1]} & \vec{b}_3 \cdot \vec{a}_{[2]} & \vec{b}_3 \cdot \vec{a}_{[3]} \end{bmatrix}. \tag{2.21.12}$$

Fact 2.21.3. Let F_A be a nonstandard frame with axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$, and let $F_{[A]}$ denote the reciprocal frame with axes $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$. Then, for all $i, j = 1, 2, 3$,

$$\vec{a}_{[i]} \cdot \vec{a}_j = \delta_{i,j}. \tag{2.21.13}$$

Hence,

$$\begin{bmatrix} \vec{a}_{[1]} \\ \vec{a}_{[2]} \\ \vec{a}_{[3]} \end{bmatrix} = \mathcal{O}_{[A]/A} \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}, \tag{2.21.14}$$

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \mathcal{O}_{A/[A]} \begin{bmatrix} \vec{a}_{[1]} \\ \vec{a}_{[2]} \\ \vec{a}_{[3]} \end{bmatrix}, \tag{2.21.15}$$

where

$$\mathcal{O}_{[A]/A} = \begin{bmatrix} \vec{a}_{[1]} \cdot \vec{a}_{[1]} & \vec{a}_{[1]} \cdot \vec{a}_{[2]} & \vec{a}_{[1]} \cdot \vec{a}_{[3]} \\ \vec{a}_{[2]} \cdot \vec{a}_{[1]} & \vec{a}_{[2]} \cdot \vec{a}_{[2]} & \vec{a}_{[2]} \cdot \vec{a}_{[3]} \\ \vec{a}_{[3]} \cdot \vec{a}_{[1]} & \vec{a}_{[3]} \cdot \vec{a}_{[2]} & \vec{a}_{[3]} \cdot \vec{a}_{[3]} \end{bmatrix}, \tag{2.21.16}$$

$$\mathcal{O}_{A/[A]} = \mathcal{O}_{[A]/A}^{-1} = \begin{bmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \vec{a}_1 \cdot \vec{a}_3 \\ \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \vec{a}_2 \cdot \vec{a}_3 \\ \vec{a}_3 \cdot \vec{a}_1 & \vec{a}_3 \cdot \vec{a}_2 & \vec{a}_3 \cdot \vec{a}_3 \end{bmatrix}. \tag{2.21.17}$$

Now, let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_{[1]} \cdot \vec{x} \\ \vec{a}_{[2]} \cdot \vec{x} \\ \vec{a}_{[3]} \cdot \vec{x} \end{bmatrix}, \tag{2.21.18}$$

$$\vec{x}\Big|_{[A]} = \begin{bmatrix} x_{[1]} \\ x_{[2]} \\ x_{[3]} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vec{a}_2 \cdot \vec{x} \\ \vec{a}_3 \cdot \vec{x} \end{bmatrix}. \tag{2.21.19}$$

Furthermore,

$$\vec{x}\Big|_{[A]} = \mathcal{O}_{A/[A]}\ \vec{x}\Big|_{A}, \tag{2.21.20}$$

$$\vec{x}\Big|_{A} = \mathcal{O}_{[A]/A}\ \vec{x}\Big|_{[A]}. \tag{2.21.21}$$

Finally, if F_A is a standard frame, then $F_A = F_{[A]}$.

By rewriting (2.21.13) as

$$\vec{a}'_{[i]}\vec{a}_j = \delta_{i,j}, \tag{2.21.22}$$

the covectors $\vec{a}'_{[1]}, \vec{a}'_{[2]}, \vec{a}'_{[3]}$ corresponding to the axes $\vec{a}_{[1]}, \vec{a}_{[2]}, \vec{a}_{[3]}$ of the reciprocal frame can be viewed as the axes of the *co-reciprocal frame* $F_{[A]'}$, which is a basis for the space $\mathcal{V}'$ of covectors. If F_A is a standard frame, then $F_{[A]} = F_A$, and thus $F_{[A]'} = F_{A'}$.

Using a frame and its reciprocal, it is possible to resolve a physical vector $\vec{x}$ in two ways, namely,

$$\vec{x} = \sum_{i=1}^{3} x_i\vec{a}_i = \sum_{i=1}^{3} x_{[i]}\vec{a}_{[i]}, \tag{2.21.23}$$

where

$$x_i = \vec{a}'_{[i]}\vec{x}, \quad x_{[i]} = \vec{a}'_i\vec{x}. \tag{2.21.24}$$

As a side remark, (2.21.23) and (2.21.24) are written traditionally as

$$\vec{x} = \sum_{i=1}^{3} x^i\vec{a}_i = \sum_{i=1}^{3} x_i\vec{a}^{\,i}, \tag{2.21.25}$$

where

$$x^i = \vec{a}^{\,i} \cdot \vec{x}, \quad x_i = \vec{a}_i \cdot \vec{x}. \tag{2.21.26}$$

Note that $\mathcal{O}_{A/[A]}$ and $\mathcal{O}_{[A]/A}$ are symmetric, and therefore the transpose in (2.21.7) does not appear in (2.21.20) and (2.21.21).

The following result shows that the reciprocal frame is the unique nonstandard frame satisfying (2.21.13).

Fact 2.21.4. Let F_A and F_B be nonstandard frames such that, for all $i, j = 1, 2, 3$, $\vec{b}_i \cdot \vec{a}_j = \delta_{i,j}$. Then, $F_B = F_{[A]}$.

Fact 2.21.5. Let F_A and F_B be nonstandard frames, and let $\vec{x}$ be a physical vector. Then,

$$\vec{x}\Big|_{[B]} = \mathcal{O}_{B/A}\ \vec{x}\Big|_{[A]} = \mathcal{O}^{\mathsf{T}}_{[A]/[B]}\ \vec{x}\Big|_{[A]}. \tag{2.21.27}$$

Furthermore,

$$\mathcal{O}_{B/A} = \mathcal{O}_{B/[B]}\mathcal{O}^{\mathsf{T}}_{A/B}\mathcal{O}_{[A]/A}. \tag{2.21.28}$$

Proof. Note that

$$\vec{x}\Big|_{[\mathrm{B}]} = \begin{bmatrix} \vec{x}\cdot\vec{b}_1 \\ \vec{x}\cdot\vec{b}_2 \\ \vec{x}\cdot\vec{b}_3 \end{bmatrix} = \begin{bmatrix} \vec{x}\cdot \mathrm{row}_1(\mathcal{O}_{\mathrm{B/A}})\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \\ \vec{x}\cdot \mathrm{row}_2(\mathcal{O}_{\mathrm{B/A}})\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \\ \vec{x}\cdot \mathrm{row}_3(\mathcal{O}_{\mathrm{B/A}})\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathrm{row}_1(\mathcal{O}_{\mathrm{B/A}}) \\ \mathrm{row}_2(\mathcal{O}_{\mathrm{B/A}}) \\ \mathrm{row}_3(\mathcal{O}_{\mathrm{B/A}}) \end{bmatrix}\begin{bmatrix} \vec{x}\cdot\vec{a}_1 \\ \vec{x}\cdot\vec{a}_2 \\ \vec{x}\cdot\vec{a}_3 \end{bmatrix} = \mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_{[\mathrm{A}]}.$$

Next, using (2.21.7) and (2.21.20) it follows that

$$\mathcal{O}_{\mathrm{A/B}}^{\mathsf{T}}\ \vec{x}\Big|_{\mathrm{A}} = \vec{x}\Big|_{\mathrm{B}} = \mathcal{O}_{[\mathrm{B}]/\mathrm{B}}\ \vec{x}\Big|_{[\mathrm{B}]} = \mathcal{O}_{[\mathrm{B}]/\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}\ \vec{x}\Big|_{[\mathrm{A}]} = \mathcal{O}_{[\mathrm{B}]/\mathrm{B}}\mathcal{O}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A}/[\mathrm{A}]}\ \vec{x}\Big|_{\mathrm{A}}. \qquad \square$$

The components of $\vec{x}\Big|_{[\mathrm{A}]}$ are the *reciprocal components* of $\vec{x}$.

The components of $\vec{x}\Big|_{\mathrm{A}}$ are the *contravariant components* of $\vec{x}$, whereas the components of $\vec{x}\Big|_{[\mathrm{A}]}$ are the *covariant components* of $\vec{x}$. The word contravariant reflects the reversal in the formulas

$$\begin{bmatrix}\vec{b}_1\\ \vec{b}_2\\ \vec{b}_3\end{bmatrix} = \mathcal{O}_{\mathrm{B/A}}\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix} \tag{2.21.29}$$

and

$$\vec{x}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}}^{\mathsf{T}}\ \vec{x}\Big|_{\mathrm{B}}. \tag{2.21.30}$$

Fact 2.21.6. Let $\mathrm{F_A}$ be a nonstandard frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{x}\cdot\vec{y} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\ \vec{y}\Big|_{[\mathrm{A}]} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\ \mathcal{O}_{\mathrm{A}/[\mathrm{A}]}\ \vec{y}\Big|_{\mathrm{A}}. \tag{2.21.31}$$

In particular,

$$|\vec{x}|^2 = \vec{x}\cdot\vec{x} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\ \vec{x}\Big|_{[\mathrm{A}]} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}\ \mathcal{O}_{\mathrm{A}/[\mathrm{A}]}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.21.32}$$

Proof. Let

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\quad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}.$$

Then,

$$\vec{x}\cdot\vec{y} = (x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3)\cdot(y_1\vec{a}_{[1]} + y_2\vec{a}_{[2]} + y_3\vec{a}_{[3]})$$

$$= x_1y_1 + x_2y_2 + x_3y_3 = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}} \vec{y}\Big|_{[\mathrm{A}]} = \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}} \mathcal{O}_{\mathrm{A}/[\mathrm{A}]} \vec{y}\Big|_{\mathrm{A}}. \qquad \square$$

It follows from (2.21.32) that, if $\mathrm{F_A}$ is nonstandard, then $|\vec{x}|$ and $\left\|\vec{x}\Big|_{\mathrm{A}}\right\|$ may be different. For example, assume that $\vec{a}_1 = \hat{\imath}_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}$, $\vec{a}_2 = \hat{\jmath}_{\mathrm{A}}$, and $\vec{a}_3 = \hat{k}_{\mathrm{A}}$, and let $\vec{x} = \hat{\imath}_{\mathrm{A}}$. Then, $|\vec{x}| = 1$, but $\left\|\vec{x}\Big|_{\mathrm{A}}\right\| = \sqrt{3}$.

The following result follows from (2.21.32).

Fact 2.21.7. Let $\mathrm{F_A}$ be a nonstandard frame. Then, $\mathcal{O}_{\mathrm{A}/[\mathrm{A}]}$ is positive semidefinite.

Let $\mathrm{F_A}$ be a nonstandard frame. As in the case of orthogonal frames, define the row vector

$$\vec{x}'\Big|_{\mathrm{A}} \triangleq \vec{x}\Big|_{\mathrm{A}}^{\mathsf{T}}. \tag{2.21.33}$$

Furthermore, for the physical matrix

$$\vec{M} = \sum_{i=1}^{r} \vec{x}_i \vec{y}'_i, \tag{2.21.34}$$

we define

$$\vec{M}\Big|_{\mathrm{A}} \triangleq \sum_{i=1}^{r} \vec{x}_i\Big|_{\mathrm{A}} \vec{y}_i\Big|_{\mathrm{A}}^{\mathsf{T}}. \tag{2.21.35}$$

Fact 2.21.8. Let $\mathrm{F_A}$ be a nonstandard frame, and let $\vec{M}$ be a physical matrix. Then,

$$\vec{M}\Big|_{\mathrm{A}} = \begin{bmatrix} \vec{a}'_{[1]}\vec{M}\vec{a}_{[1]} & \vec{a}'_{[1]}\vec{M}\vec{a}_{[2]} & \vec{a}'_{[1]}\vec{M}\vec{a}_{[3]} \\ \vec{a}'_{[2]}\vec{M}\vec{a}_{[1]} & \vec{a}'_{[2]}\vec{M}\vec{a}_{[2]} & \vec{a}'_{[2]}\vec{M}\vec{a}_{[3]} \\ \vec{a}'_{[3]}\vec{M}\vec{a}_{[1]} & \vec{a}'_{[3]}\vec{M}\vec{a}_{[2]} & \vec{a}'_{[3]}\vec{M}\vec{a}_{[3]} \end{bmatrix}. \tag{2.21.36}$$

Furthermore,

$$\vec{M} = \sum_{i,j=1}^{3} \left(\vec{a}'_{[i]}\vec{M}\vec{a}_{[j]}\right)\vec{a}_i\vec{a}'_j. \tag{2.21.37}$$

Proof. For simplicity, let $\vec{M} = \vec{x}\vec{y}'$, and write

$$\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3, \quad \vec{y} = y_1\vec{a}_1 + y_2\vec{a}_2 + y_3\vec{a}_3,$$

so that

$$\vec{M} = \sum_{i,j=1}^{3} x_i y_j \vec{a}_i \vec{a}'_j. \tag{2.21.38}$$

Hence,

$$\vec{M}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3. \end{bmatrix}.$$

It follows from (2.21.38) that, for all $i, j = 1,2,3$, $\vec{a}'_{[i]}\vec{M}\vec{a}_{[j]} = x_i y_j$, which yields (2.21.36) and (2.21.37). □

Defining

$$M_{ij} \triangleq \vec{a}'_{[i]}\vec{M}\vec{a}_{[j]}, \tag{2.21.39}$$

which, by (2.21.18), is the i, j entry of $\vec{M}\Big|_{\mathrm{A}}$, (2.21.37) can be written as

$$\vec{M} = \sum_{i,j=1}^{3} M_{ij}\vec{a}_i\vec{a}'_j. \tag{2.21.40}$$

Similarly, defining

$$M_{[i][j]} \triangleq \vec{a}'_i\vec{M}\vec{a}_j, \tag{2.21.41}$$

which, by (2.21.19), is the i, j entry of $\vec{M}\Big|_{[\mathrm{A}]}$, we have

$$\vec{M} = \sum_{i,j=1}^{3} M_{[i][j]}\vec{a}_{[i]}\vec{a}'_{[j]}. \tag{2.21.42}$$

Likewise, defining

$$M_{i[j]} \triangleq \vec{a}'_{[i]}\vec{M}\vec{a}_j, \quad M_{[i]j} \triangleq \vec{a}'_i\vec{M}\vec{a}_{[j]}, \tag{2.21.43}$$

it follows that

$$\vec{M} = \sum_{i,j=1}^{3} M_{i[j]}\vec{a}_i\vec{a}'_{[j]} = \sum_{i,j=1}^{3} M_{[i]j}\vec{a}_{[i]}\vec{a}'_j. \tag{2.21.44}$$

Consequently, by using a frame and the associated coframe, reciprocal, and co-reciprocal frames, a physical matrix $\vec{M}$ can be resolved in four ways, namely,

$$\vec{M} = \sum_{i,j=1}^{3} M_{ij}\vec{a}_i\vec{a}'_j = \sum_{i,j=1}^{3} M_{[i][j]}\vec{a}_{[i]}\vec{a}'_{[j]} = \sum_{i,j=1}^{3} M_{i[j]}\vec{a}_i\vec{a}'_{[j]} = \sum_{i,j=1}^{3} M_{[i]j}\vec{a}_{[i]}\vec{a}'_j. \tag{2.21.45}$$

Fact 2.21.9. Let $\vec{M}$ be a physical matrix, let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ be a nonstandard frame. Then,

$$(\vec{M}\vec{x})\Big|_{\mathrm{A}} = \vec{M}\Big|_{\mathrm{A}} \mathcal{O}_{\mathrm{A}/[\mathrm{A}]}\ \vec{x}\Big|_{\mathrm{A}}. \tag{2.21.46}$$

Fact 2.21.9 implies that

$$\vec{I}\Big|_{\mathrm{A}} = \mathcal{O}_{[\mathrm{A}]/\mathrm{A}}, \quad \vec{I}\Big|_{[\mathrm{A}]} = \mathcal{O}_{\mathrm{A}/[\mathrm{A}]}. \tag{2.21.47}$$

The following result extends (2.10.11) to nonstandard frames.

Fact 2.21.10. Let $\vec{M}$ be a physical matrix, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be nonstandard frames. Then,

$$\vec{M}\Big|_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}^{-\mathsf{T}}\, \vec{M}\Big|_{\mathrm{A}} \mathcal{O}_{\mathrm{A/B}}. \tag{2.21.48}$$

Fact 2.21.11. Let $\vec{M}$ be a physical matrix, and let $\mathrm{F_A}$ be a nonstandard frame. Then,

$$\vec{M}\Big|_{[\mathrm{A}]} = \mathcal{O}_{\mathrm{A/[A]}}\, \vec{M}\Big|_{\mathrm{A}} \mathcal{O}_{\mathrm{A/[A]}}. \tag{2.21.49}$$

Fact 2.21.12. Let $\vec{M}$ be a physical matrix, let $\mathrm{F_A}$ be a nonstandard frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors satisfying $\vec{y} = \vec{M}\vec{x}$. Then,

$$\vec{y}\Big|_{[\mathrm{A}]} = \mathcal{O}_{\mathrm{A/[A]}}\, \vec{M}\Big|_{\mathrm{A}} \mathcal{O}_{\mathrm{A/[A]}}\, \vec{x}\Big|_{[\mathrm{A}]}. \tag{2.21.50}$$

Consider the physical matrix

$$\vec{M} = \sum_{i=1}^{r} \vec{x}_i \vec{y}_i', \tag{2.21.51}$$

and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be nonstandard frames. Then, we define

$$\vec{M}\Big|_{\mathrm{A,B}} \triangleq \sum_{i=1}^{r} \vec{x}_i\Big|_{\mathrm{A}} \vec{y}_i\Big|_{\mathrm{B}}^{\mathsf{T}}. \tag{2.21.52}$$

Fact 2.21.13. Let $\vec{M}$ be a physical matrix, let $\vec{x}$ be a physical vector, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be nonstandard frames. Then,

$$(\vec{M}\vec{x})\Big|_{\mathrm{B}} = \vec{M}\Big|_{\mathrm{B,A}} \vec{x}\Big|_{[\mathrm{A}]}. \tag{2.21.53}$$

It follows from Fact 2.21.13 that

$$\vec{I}\Big|_{[\mathrm{A}]/\mathrm{A}} = \vec{I}\Big|_{\mathrm{A}/[\mathrm{A}]} = I_3. \tag{2.21.54}$$

Fact 2.21.14. Let $\vec{M}$ be a physical matrix, and let $\mathrm{F_A}$ be a nonstandard frame. Then,

$$\vec{M}\Big|_{[\mathrm{A}],\mathrm{A}} = \mathcal{O}_{\mathrm{A/[A]}}\, \vec{M}\Big|_{\mathrm{A}}, \tag{2.21.55}$$

$$\vec{M}\Big|_{\mathrm{A},[\mathrm{A}]} = \vec{M}\Big|_{\mathrm{A}} \mathcal{O}_{\mathrm{A/[A]}}. \tag{2.21.56}$$

2.22 Partial Derivatives and Gradients

Let $f\colon \mathbb{R}^n \to \mathbb{R}$, let $i \in \{1,\ldots,n\}$, and define the function $g_i\colon \mathbb{R} \to \mathbb{R}$ by $g_i(x_i) \triangleq f(x_1,\ldots,x_{i-1}, x_i, x_{i+1},\ldots,x_n)$. The *partial derivative of f with respect to x_i* is the function $g_i'\colon \mathbb{R} \to \mathbb{R}$. The notation $\partial_{x_i} f(x)$, where $x = (x_1,\ldots,x_n)$, represents $g_i'(x_i)$. The *gradient* of f is the function $\partial_x f\colon \mathbb{R}^n \to \mathbb{R}^{1\times n}$ defined by

$$\partial_x f(x) \triangleq [\partial_{x_1} f(x) \ \cdots \ \partial_{x_n} f(x)]. \tag{2.22.1}$$

Next, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, where $f = [f_1 \ \cdots \ f_n]^\mathsf{T}$. Then, the derivative $f'\colon \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is the matrix-valued function whose (i, j) entry is the partial derivative $\partial_{x_j} f_i$. The function f' is the *Jacobian* of f. Note that the ith row of $f'(x)$ is $\partial_x f_i(x)$.

Next, let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ so that f is a real-valued function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then, for all $y \in \mathbb{R}^m$, the function $\partial_x f(\cdot, y)\colon \mathbb{R}^n \to \mathbb{R}^{1\times n}$ is the *gradient* of f *with respect to* x, and $\partial_x f(x, y)|_{x=\overline{x}}$ represents the gradient of f with respect to x evaluated at $(\overline{x}, y)$. Using x to denote the first argument of f, it follows that $\partial_x f(\overline{x}, y)$ denotes $\partial_x f(x, y)|_{x=\overline{x}}$. Defining the function $g\colon \mathbb{R}^n \to \mathbb{R}^{1\times n}$ by $g(x) \stackrel{\triangle}{=} \partial_x f(x, y)$, it follows that $\partial_x f(\overline{x}, y) = g'(\overline{x})$.

As a special case, let $f\colon \mathbb{R}^3 \to \mathbb{R}$, where the argument of f is denoted by $r = [x\ y\ z]^\mathsf{T} \in \mathbb{R}^3$. Then, the gradient of f evaluated at $r \in \mathbb{R}^3$ is the row vector

$$\partial_r f(r) = [\partial_x f(r) \ \ \partial_y f(r) \ \ \partial_z f(r)] \in \mathbb{R}^{1\times 3}, \tag{2.22.2}$$

where $\partial_x f(r)$ is the partial derivative of f with respect to x evaluated at r.

Now, let f denote a mapping from physical vectors to real numbers, that is, for each physical vector $\vec{r}$, let $f(\vec{r}) \in \mathbb{R}$. Furthermore, for the frame $\mathrm{F_A}$, define $f_\mathrm{A}\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$f_\mathrm{A}(r) \stackrel{\triangle}{=} f(\mathrm{F_A} r). \tag{2.22.3}$$

Defining $r \stackrel{\triangle}{=} r_{|\mathrm{A}} = \vec{r}\Big|_\mathrm{A} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, it follows from (2.22.3) that

$$f(\vec{r}) = f(\mathrm{F_A} r_{|\mathrm{A}}) = f(\mathrm{F_A} r) = f_\mathrm{A}(r) = f_\mathrm{A}(r_{|\mathrm{A}}). \tag{2.22.4}$$

Furthermore, if $\mathrm{F_B}$ is also a frame, then

$$f_\mathrm{A}(r_{|\mathrm{A}}) = f(\mathrm{F_A} r_{|\mathrm{A}}) = f(\mathrm{F_B} \mathcal{O}_{\mathrm{B/A}} r_{|\mathrm{A}}) = f_\mathrm{B}(\mathcal{O}_{\mathrm{B/A}} r_{|\mathrm{A}}) = f_\mathrm{B}(r_{|\mathrm{B}}). \tag{2.22.5}$$

It thus follows from the chain rule that

$$\partial_{r_{|\mathrm{A}}} f_\mathrm{A}(r_{|\mathrm{A}}) = \partial_{r_{|\mathrm{A}}} f_\mathrm{B}(\mathcal{O}_{\mathrm{B/A}} r_{|\mathrm{A}}) = \partial_{\mathcal{O}_{\mathrm{B/A}} r_{|\mathrm{A}}} f_\mathrm{B}(\mathcal{O}_{\mathrm{B/A}} r_{|\mathrm{A}}) \mathcal{O}_{\mathrm{B/A}} = \partial_{r_{|\mathrm{B}}} f_\mathrm{B}(r_{|\mathrm{B}}) \mathcal{O}_{\mathrm{B/A}}. \tag{2.22.6}$$

Next, define the *physical gradient* $\vec{\partial} f(\vec{r})$ of f at $\vec{r}$ by

$$\vec{\partial} f(\vec{r}) \stackrel{\triangle}{=} \partial_{r_{|\mathrm{A}}} f_\mathrm{A}(r_{|\mathrm{A}}) \mathrm{F}_\mathrm{A}'^\mathsf{T}. \tag{2.22.7}$$

Hence,

$$\vec{\partial} f(\vec{r}) = \partial_{\overline{x}} f_\mathrm{A}(r_{|\mathrm{A}}) \hat{\imath}_\mathrm{A}' + \partial_{\overline{y}} f_\mathrm{A}(r_{|\mathrm{A}}) \hat{\jmath}_\mathrm{A}' + \partial_{\overline{z}} f_\mathrm{A}(r_{|\mathrm{A}}) \hat{k}_\mathrm{A}', \tag{2.22.8}$$

and thus

$$\vec{\partial} f(\vec{r})\Big|_\mathrm{A} = \partial_{r_{|\mathrm{A}}} f_\mathrm{A}(r_{|\mathrm{A}}) = \begin{bmatrix} \partial_{\overline{x}} f_\mathrm{A}(r_{|\mathrm{A}}) & \partial_{\overline{y}} f_\mathrm{A}(r_{|\mathrm{A}}) & \partial_{\overline{z}} f_\mathrm{A}(r_{|\mathrm{A}}) \end{bmatrix}. \tag{2.22.9}$$

Note that the physical gradient is a covector.

Fact 2.22.1. $\vec{\partial} f(\vec{r})$ is independent of the choice of the frame $\mathrm{F_A}$ used in (2.22.7).

Proof. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $r_{|\mathrm{A}} \stackrel{\triangle}{=} \vec{r}\Big|_\mathrm{A}$ and $r_{|\mathrm{B}} \stackrel{\triangle}{=} \vec{r}\Big|_\mathrm{B}$. Then, it follows from

(2.10.25) and (2.22.6) that

$$\begin{aligned}\partial_{r_{|\mathrm{A}}} f_{\mathrm{A}}(r_{|\mathrm{A}})\mathrm{F}_{\mathrm{A}}'^{\mathsf{T}} &= \partial_{r_{|\mathrm{B}}} f_{\mathrm{B}}(r_{|\mathrm{B}})\mathcal{O}_{\mathrm{B/A}}\mathrm{F}_{\mathrm{A}}'^{\mathsf{T}}\\ &= \partial_{r_{|\mathrm{B}}} f_{\mathrm{B}}(r_{|\mathrm{B}})\mathrm{F}_{\mathrm{B}}'^{\mathsf{T}}.\end{aligned} \qquad \square$$

For a scalar-valued function f whose domain is physical vectors $\vec{r}$, $\vec{\partial}_{\vec{r}} f(\vec{r})\big|_{\vec{r}=\vec{r}_{y/z}}$ denotes the physical gradient with respect to $\vec{r}$ evaluated at $\vec{r}_{y/z}$. If $\vec{r}_{y/z}$ is nonzero, then

$$\vec{\partial}_{\vec{r}}|\vec{r}|\Big|_{\vec{r}=\vec{r}_{y/z}} = \frac{1}{|\vec{r}_{y/z}|}\vec{r}'_{y/z} = \hat{r}'_{y/z}, \tag{2.22.10}$$

and, for all $n \ge 1$,

$$\vec{\partial}_{\vec{r}} \frac{1}{|\vec{r}|^n}\Big|_{\vec{r}=\vec{r}_{y/z}} = -\frac{n}{|\vec{r}_{z/y}|^{n+1}}\vec{\partial}_{\vec{r}}|\vec{r}|\Big|_{\vec{r}=\vec{r}_{y/z}} = -\frac{n}{|\vec{r}_{z/y}|^{n+2}}\vec{r}'_{z/y}. \tag{2.22.11}$$

Furthermore, for all covectors $\vec{v}'$,

$$\vec{\partial}_{\vec{r}}(\vec{v}'\vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}} = \vec{v}'. \tag{2.22.12}$$

The operator $\vec{\nabla}$ is the vector version of the covector physical gradient $\vec{\partial}$; that is, $\vec{\partial} = \vec{\nabla}'$. The divergence and curl of the vector field $\vec{M}$ are thus given by

$$\vec{\nabla}_{\vec{r}} \cdot \vec{M}(\vec{r}) = \operatorname{tr} \vec{\partial}_{\vec{r}}\vec{M}(\vec{r}), \tag{2.22.13}$$

$$\vec{\nabla}_{\vec{r}} \times \vec{M}(\vec{r}) = \vec{\nabla}^{\times}_{\vec{r}}\vec{M}(\vec{r}). \tag{2.22.14}$$

Finally, we note the Jacobian

$$\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{r} = \vec{I}. \tag{2.22.15}$$

The following result will be useful.

Fact 2.22.2. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors that are not parallel. Then,

$$\vec{\partial}_{\theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}}(\vec{x}'\vec{y}) = -\vec{x} \times \vec{y}. \tag{2.22.16}$$

Proof. Note that

$$\begin{aligned}\vec{\partial}_{\theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}}\vec{x}'\vec{y} &= \vec{\partial}_{\theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}}|\vec{x}||\vec{y}|\cos\theta\\ &= \vec{\partial}_{\theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}}|\vec{x}||\vec{y}|\cos(\theta\hat{\theta}'_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}})\\ &= |\vec{x}||\vec{y}|\vec{\partial}_{\theta_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}}}\cos(\theta\hat{\theta}'_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}})\end{aligned}$$

$$= |\vec{x}||\vec{y}|[-\sin(\theta\hat{\theta}'_{\vec{y}/\vec{x}}\hat{\theta}_{\vec{y}/\vec{x}})]\hat{\theta}_{\vec{y}/\vec{x}}$$

$$= -|\vec{x}||\vec{y}|(\sin\theta)\hat{\theta}_{\vec{y}/\vec{x}}$$

$$= -\vec{x}\times\vec{y}. \qquad \diamond$$

2.23 Examples

Example 2.23.1. Consider the 3-bar linkage shown from above in Figure 2.23.1 with links of lengths $\ell_1 = 3$, $\ell_2 = 4$, and $\ell_3 = 2$, and pin joints a, b, and c labeled as shown. The links are initially configured as shown, lying in a horizontal plane, that is, the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. The pins at joints a and c are vertical, that is, parallel with $\hat{k}_{\mathrm{A}}$, and the pin at joint b is horizontal, that is, lying in the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. The rotation angles at joints a, b, and c are ψ, θ, and ϕ, respectively, where a positive value of θ indicates that joint c moves in the negative $\hat{k}_{\mathrm{A}}$ direction. The joints are rotated such that $\psi = 30$ deg, $\theta = -20$ deg, and $\phi = 45$ deg. In terms of the frame F_{A}, determine the position of the tip d of the linkage relative to a after these rotations.

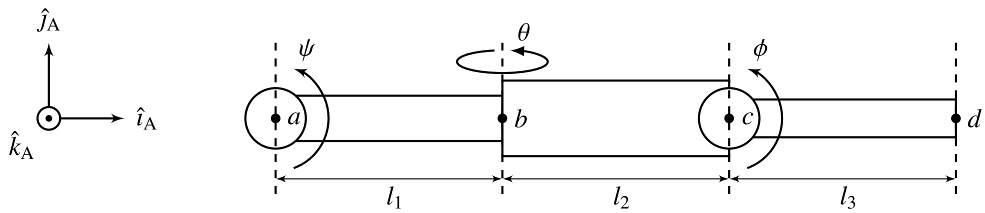


Figure 2.23.1: Example 2.23.1. Three-bar linkage. The orientation of the linkage as shown corresponds to $\psi = \theta = \phi = 0$.

Solution. Define the sequence of rotations

$$\mathrm{F}_{\mathrm{A}} \xrightarrow[3]{\psi} \mathrm{F}_{\mathrm{B}} \xrightarrow[2]{\theta} \mathrm{F}_{\mathrm{C}} \xrightarrow[3]{\phi} \mathrm{F}_{\mathrm{D}}.$$

The position of d relative to a is given by

$$\begin{aligned}\vec{r}_{d/a} &= \vec{r}_{d/c} + \vec{r}_{c/b} + \vec{r}_{b/a}\\ &= \ell_3\hat{\imath}_{\mathrm{D}} + \ell_2\hat{\imath}_{\mathrm{C}} + \ell_1\hat{\imath}_{\mathrm{B}}.\end{aligned}$$

Resolving this equation in F_{A} yields

$$\begin{aligned}\vec{r}_{d/a}\Big|_{\mathrm{A}} &= \ell_3\ \hat{\imath}_{\mathrm{D}}|_{\mathrm{A}} + \ell_2\ \hat{\imath}_{\mathrm{C}}|_{\mathrm{A}} + \ell_1\ \hat{\imath}_{\mathrm{B}}|_{\mathrm{A}}\\ &= \ell_3\mathcal{O}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/C}}\mathcal{O}_{\mathrm{C/D}}\ \hat{\imath}_{\mathrm{D}}|_{\mathrm{D}} + \ell_2\mathcal{O}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/C}}\ \hat{\imath}_{\mathrm{C}}|_{\mathrm{C}} + \ell_1\mathcal{O}_{\mathrm{A/B}}\ \hat{\imath}_{\mathrm{B}}|_{\mathrm{B}}\\ &= [\ell_3\mathcal{O}_3^{\mathsf{T}}(\psi)\mathcal{O}_2^{\mathsf{T}}(\theta)\mathcal{O}_3^{\mathsf{T}}(\phi) + \ell_2\mathcal{O}_3^{\mathsf{T}}(\psi)\mathcal{O}_2^{\mathsf{T}}(\theta) + \ell_1\mathcal{O}_3^{\mathsf{T}}(\psi)]\begin{bmatrix}1\\0\\0\end{bmatrix}\\ &= \mathcal{O}_3^{\mathsf{T}}(\psi)[\ell_3\mathcal{O}_2^{\mathsf{T}}(\theta)\mathcal{O}_3^{\mathsf{T}}(\phi) + \ell_2\mathcal{O}_2^{\mathsf{T}}(\theta) + \ell_1 I_3]\begin{bmatrix}1\\0\\0\end{bmatrix}\end{aligned}$$

$$= \mathcal{O}_3^{\mathsf{T}}(\psi)\left(\ell_3\mathcal{O}_2^{\mathsf{T}}(\theta)\begin{bmatrix}\cos\phi\\ \sin\phi\\ 0\end{bmatrix} + \ell_2\begin{bmatrix}\cos\theta\\ 0\\ -\sin\theta\end{bmatrix} + \ell_1\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)$$

$$= \begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\ell_3(\cos\theta)\cos\phi + \ell_2\cos\theta + \ell_1\\ \ell_3\sin\phi\\ -\ell_3(\sin\theta)\cos\phi - \ell_2\sin\theta\end{bmatrix}$$

$$= \begin{bmatrix}\ell_3(\cos\psi)(\cos\theta)\cos\phi + \ell_2(\cos\psi)\cos\theta + \ell_1\cos\psi - \ell_3(\sin\psi)\sin\phi\\ \ell_3(\sin\psi)(\cos\theta)\cos\phi + \ell_2(\sin\psi)\cos\theta + \ell_1\sin\psi + \ell_3(\cos\psi)\sin\phi\\ -\ell_3(\sin\theta)\cos\phi - \ell_2\sin\theta\end{bmatrix}.$$

Evaluating this expression at $\psi = 30$ deg, $\theta = -20$ deg, and $\phi = 45$ deg yields

$$\vec{r}_{d/a}\Big|_{\mathrm{A}} = \begin{bmatrix}6.29\\ 5.27\\ 1.85\end{bmatrix}. \qquad \diamond$$

2.24 Theoretical Problems

Problem 2.24.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Show that the following statements are equivalent:

i) $\theta_{\vec{y}/\vec{x}} = \pi/2$.

ii) $\vec{x} \cdot \vec{y} = 0$.

iii) $|\vec{x} \times \vec{y}| = |\vec{x}||\vec{y}|$.

Now, assume that $\vec{x}$ and $\vec{y}$ are nonzero. Then, show that the following statements are equivalent:

iv) $\theta_{\vec{y}/\vec{x}} = \pi$.

v) $|\hat{x} \cdot \hat{y}| = 1$.

vi) $\vec{x} \times \vec{y} = 0$.

vii) Either $\hat{x} = \hat{y}$ or $\hat{x} = -\hat{y}$.

Finally, assume that $\vec{x}$ and $\vec{y}$ are nonzero and not parallel, and let $\vec{z}$ be a physical vector. Show that, if $\vec{z} \times \vec{x} = 0$ and $\vec{z} \times \vec{y} = 0$, then $\vec{z} = 0$.

Problem 2.24.2. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors. Determine the length of the projection of $\vec{x}$ onto the line through $\vec{y}$ in terms of $|\vec{x} \cdot \vec{y}|$. Likewise, determine the length of the projection of $\vec{y}$ onto the line through $\vec{x}$. Separately consider the cases where $\theta_{\vec{x}/\vec{y}} \in [0, \pi/2]$ and $\theta_{\vec{x}/\vec{y}} \in [\pi/2, \pi]$.

Problem 2.24.3. Define $\vec{x} \triangleq 3\hat{\imath}_{\mathrm{A}} - 4\hat{\jmath}_{\mathrm{A}}$, $\vec{y} \triangleq -\hat{\imath}_{\mathrm{A}} + 5\hat{\jmath}_{\mathrm{A}} - 2\hat{k}_{\mathrm{A}}$, $\vec{M} \triangleq \vec{x}\vec{y}'$, and $\vec{N} \triangleq \vec{y}\vec{x}'$. Then, do the following:

i) Resolve $\vec{M}$, $\vec{N}$, $\vec{M}\vec{N}$, and $\vec{M}\vec{x}$ in F_{A}.

ii) Use your answer to *i)* to confirm that $(\vec{M}\vec{N})|_{\mathrm{A}} = \vec{M}|_{\mathrm{A}}\vec{N}|_{\mathrm{A}}$ and $(\vec{M}\vec{x})|_{\mathrm{A}} = \vec{M}|_{\mathrm{A}}\vec{x}|_{\mathrm{A}}$.

(Note: In order to confirm $(\overset{\rightharpoonup}{M}\overset{\rightharpoonup}{N})|_{\mathrm{A}} = \overset{\rightharpoonup}{M}|_{\mathrm{A}}\overset{\rightharpoonup}{N}|_{\mathrm{A}}$, first determine $(\overset{\rightharpoonup}{M}\overset{\rightharpoonup}{N})|_{\mathrm{A}}$ by resolving the product $\overset{\rightharpoonup}{M}\overset{\rightharpoonup}{N}$ you obtained in part *i*) in $\mathrm{F_A}$. Then, resolve $\overset{\rightharpoonup}{M}$ and $\overset{\rightharpoonup}{N}$ separately in $\mathrm{F_A}$ and compute the matrix product $\overset{\rightharpoonup}{M}|_{\mathrm{A}}\overset{\rightharpoonup}{N}|_{\mathrm{A}}$.)

Problem 2.24.4. Let $\overset{\rightharpoonup}{M}$ and $\overset{\rightharpoonup}{N}$ be physical matrices, and let $\mathrm{F_A}$ be a frame. Show that if $\overset{\rightharpoonup}{M}\hat{\imath}_{\mathrm{A}} = \overset{\rightharpoonup}{N}\hat{\imath}_{\mathrm{A}}$, $\overset{\rightharpoonup}{M}\hat{\jmath}_{\mathrm{A}} = \overset{\rightharpoonup}{N}\hat{\jmath}_{\mathrm{A}}$, and $\overset{\rightharpoonup}{M}\hat{k}_{\mathrm{A}} = \overset{\rightharpoonup}{N}\hat{k}_{\mathrm{A}}$, then $\overset{\rightharpoonup}{M} = \overset{\rightharpoonup}{N}$.

Problem 2.24.5. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors lying in the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. Show that

$$x_1y_2 - x_2y_1 = |\vec{x}|\,|\vec{y}|\sin\theta_{\vec{y}/\vec{x}/\hat{k}_{\mathrm{A}}},$$

where

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \qquad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}.$$

Problem 2.24.6. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors. Show that

$$\vec{x} \times (\vec{y} \times \vec{z}) + \vec{y} \times (\vec{z} \times \vec{x}) + \vec{z} \times (\vec{x} \times \vec{y}) = 0.$$

This is *Jacobi's identity*.

Problem 2.24.7. Let $\vec{y}$ and $\vec{z}$ be nonzero physical vectors that are not parallel. Show that

$$\overset{\rightharpoonup}{P}_{\vec{y}-P_{\vec{z}}\vec{y}} + \overset{\rightharpoonup}{P}_{\vec{z}} = \frac{1}{|\vec{z}|^2|\vec{y}|^2 - (\vec{y}\,'\vec{z})^2}\left[|\vec{z}|^2\vec{y}\,\vec{y}\,' - \vec{y}\,'\vec{z}(\vec{y}\,\vec{z}\,' + \vec{z}\,\vec{y}\,') + |\vec{y}|^2\vec{z}\,\vec{z}\,'\right],$$

and thus

$$\overset{\rightharpoonup}{P}_{\vec{y}-P_{\vec{z}}\vec{y}} + \overset{\rightharpoonup}{P}_{\vec{z}} = \overset{\rightharpoonup}{P}_{\vec{z}-\overset{\rightharpoonup}{P}_{\vec{y}}\vec{z}} + \overset{\rightharpoonup}{P}_{\vec{y}}.$$

Problem 2.24.8. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be linearly independent physical vectors, and define

$$\begin{aligned} \vec{u} &\triangleq \vec{x}, \\ \vec{v} &\triangleq (\overset{\rightharpoonup}{I} - \overset{\rightharpoonup}{P}_{\vec{x}})\vec{y}, \\ \vec{w} &\triangleq (\overset{\rightharpoonup}{I} - \overset{\rightharpoonup}{P}_{\vec{x}} - \overset{\rightharpoonup}{P}_{\vec{v}})\vec{z}. \end{aligned}$$

Show that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal. (Remark: This is *Gram-Schmidt orthogonalization*.)

Problem 2.24.9. Let $\mathrm{F_A}$ be a frame, and let the frame $\mathrm{F_B}$ be obtained by rotating $\mathrm{F_A}$ according to the right hand rule around the axis $\hat{k}_{\mathrm{A}}$ by the angle $\theta_{\mathrm{B/A}} = \pi/2$. Use (2.8.6) to determine $\overset{\rightharpoonup}{R}_{\mathrm{B/A}}$ and $\mathcal{R}_{\mathrm{B/A}}$ by computing both $\overset{\rightharpoonup}{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}$ and $\overset{\rightharpoonup}{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$. Next, use (2.11.44) to determine $\theta_{\mathrm{B/A}}$ and (2.11.45) to determine $n_{\mathrm{B/A}}$. Finally, verify (2.11.47) and (2.11.48).

Problem 2.24.10. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Express $\mathcal{O}_{\mathrm{C/A}}$ in terms of a product of Euler orientation matrices for the following cases:

i) $\mathrm{F_B} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\psi)\mathrm{F_A}$ and $\mathrm{F_C} = \vec{R}_{\hat{\jmath}_{\mathrm{B}}}(\theta)\mathrm{F_B}$.

ii) $\mathrm{F_B} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\psi)\mathrm{F_A}$ and $\mathrm{F_C} = \vec{R}_{\hat{\jmath}_{\mathrm{A}}}(\theta)\mathrm{F_B}$.

Problem 2.24.11. Let $\mathrm{F_A}$ be a frame, and let $S \in \mathbb{R}^{3\times3}$ be a rotation matrix. Show that there exists a frame $\mathrm{F_B}$ such that $\vec{R}_{\mathrm{A/B}}\big|_{\mathrm{B}} = S$.

Problem 2.24.12. Let $\vec{x}$ be a physical vector, and let $\vec{R}$ be a physical rotation matrix. Use Fact 2.9.8 to show that $\vec{R}\vec{x} = \vec{x}$ if and only if $\vec{R}\vec{x}^{\times} = \vec{x}^{\times}\vec{R}$.

Problem 2.24.13. Let $\hat{n}$ be a unit dimensionless physical vector, let $\theta \in (-\pi, \pi]$, and let $\vec{S}$ be a physical rotation matrix. Show that

$$\vec{R}_{\vec{S}\hat{n}}(\theta) = \vec{S}\vec{R}_{\hat{n}}(\theta)\vec{S}'.$$

Problem 2.24.14. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $\theta_{\mathrm{B/A}}$ and $n_{\mathrm{B/A}}$ as in Fact 2.11.7. Then, show that

$$\begin{aligned}
\mathcal{O}_{\mathrm{A/B}}n_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}} &= n_{\mathrm{B/A}},\\
\mathcal{O}_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\times} &= n_{\mathrm{B/A}}^{\times}\mathcal{O}_{\mathrm{B/A}},\\
\mathcal{R}_{\mathrm{B/A}}n_{\mathrm{B/A}}^{\times} &= n_{\mathrm{B/A}}^{\times}\mathcal{R}_{\mathrm{B/A}}.
\end{aligned}$$

Problem 2.24.15. Let M be an $n \times n$ orthogonal matrix, that is, a nonsingular matrix that satisfies $M^{\mathsf{T}} = M^{-1}$. Show that $|\det M| = 1$ and that the absolute value of every eigenvalue of M is 1. Now, assume that M is a rotation matrix, that is, a real 3×3 orthogonal matrix whose determinant is 1. Show that the eigenvalues of M are given by $\{1, \lambda, \overline{\lambda}\}$, where λ is a complex (possibly real) number whose absolute value is 1. Next, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\hat{n}$ and $\theta \in (-\pi, \pi]$ be such that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}}(\theta)$. Show that $\hat{n}|_{\mathrm{A}}$ is an eigenvector of $\mathcal{R}_{\mathrm{B/A}}$ corresponding to the eigenvalue 1, and show that $\lambda = (\cos\theta) + (\sin\theta)\jmath$. (Hint: Let $v \in \mathbb{C}^n$ be an eigenvector of M associated with the eigenvalue λ, and note that $v^*M^{\mathsf{T}}Mv = \lambda v^*M^{\mathsf{T}}v$.)

Problem 2.24.16. Let R be a rotation matrix. Show that

$$\begin{aligned}
(\operatorname{tr} R)^2 &= \operatorname{tr} R^2 + 2\operatorname{tr} R,\\
(\operatorname{tr} R)^3 + 2\operatorname{tr} R^3 &= 3(\operatorname{tr} R)(\operatorname{tr} R^2) + 6.
\end{aligned}$$

Problem 2.24.17. Let R be a rotation matrix. Show that

$$R^3 - (\operatorname{tr} R)R^2 + (\operatorname{tr} R)R - I_3 = 0.$$

Now, define $c \triangleq \frac{1}{2}(\operatorname{tr} R - 1)$. Show that

$$R^4 - (2 + 2c)R^3 + (2 + 4c)R^2 - (2 + 2c)R + I_3 = 0.$$

Furthermore, if $c \neq -1$, then show that

$$\frac{1+2c}{2(1+c)}I_3 + \frac{1}{4(1+c)}(R^2 + R^{2\mathsf{T}}) + \tfrac{1}{2}(R - R^{\mathsf{T}}) = R.$$

(Hint: For the first equality, use the Cayley-Hamilton theorem. For the second equality, use Problem 2.24.11 to express c in terms of the eigenvalues of R, and use an orthogonal transformation to diagonalize R.)

Problem 2.24.18. Let $\vec{R} = \vec{R}_{\hat{n}}(\theta)$ be a physical rotation matrix, where $\hat{n}$ is a unit dimensionless physical vector and $\theta \in [0, \pi]$. Show that the following statements are equivalent:

i) $\vec{R}$ is symmetric.

ii) Either $\operatorname{tr} \vec{R} = -1$ or $\operatorname{tr} \vec{R} = 3$.

iii) Either $\theta = 0$ or $\theta = \pi$.

Problem 2.24.19. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that

$$\vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{B/A}}\Big|_{\mathrm{A}}$$

in two different ways, namely:

i) By resolving $\vec{R}_{\mathrm{B/A}}$ in both $\mathrm{F_A}$ and $\mathrm{F_B}$ and using (2.10.31).

ii) By using Euler's theorem to express $\vec{R}_{\mathrm{B/A}}$ in terms of Rodrigues's formula.

Problem 2.24.20. Let b, c, d be real numbers, and define the pure quaternion $\mathbf{q} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Show that

$$e^{\frac{1}{2}\mathbf{q}} = \cos\tfrac{1}{2}|\mathbf{q}| + \frac{\sin\frac{1}{2}|\mathbf{q}|}{|\mathbf{q}|}\mathbf{q}$$

and thus

$$|e^{\frac{1}{2}\mathbf{q}}| = 1.$$

(Remark: Define $(\sin 0)/0 \triangleq 1$ when $b = c = d = 0$.) (Remark: See [7, p. 71].)

Problem 2.24.21. Let a, b, c, d be real numbers such that $\sqrt{a^2 + b^2 + c^2 + d^2} = 1$, and define the unit quaternion $\mathbf{q} \triangleq a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Furthermore, let $\mathrm{F_A}$ be a frame, let $\vec{x}$ be a physical vector, and let

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Finally, define the pure quaternion

$$\mathbf{x} \triangleq x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

Then, show that

$$\mathbf{y} \triangleq y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} \triangleq \mathbf{q}\mathbf{x}\mathbf{q}^{-1}$$

is a pure quaternion and that the physical vector

$$\vec{y} \triangleq y_1\hat{\imath}_{\mathrm{A}} + y_2\hat{\jmath}_{\mathrm{A}} + y_3\hat{k}_{\mathrm{A}}$$

satisfies

$$\vec{y} = \vec{R}\vec{x},$$

where $\vec{R}$ is the physical rotation matrix

$$\vec{R}\Big|_{\mathrm{A}} \triangleq \begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 \end{bmatrix}.$$

(Hint: Note that $\mathbf{q}^{-1} = \overline{\mathbf{q}} \triangleq a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.) (Remark: This result shows that the two-sided transformation $\mathbf{y} = \mathbf{q}\mathbf{x}\mathbf{q}^{-1}$, where $\mathbf{q}$ is a unit quaternion, represents a physical rotation matrix. Conversely, since $\mathbf{y} = (-\mathbf{q})\mathbf{x}(-\mathbf{q})^{-1}$, this result also shows that every physical rotation matrix that is not the identity can be represented by two distinct unit quaternions, namely, $\mathbf{q}$ and $-\mathbf{q}$.)

2.25 Applied Problems

Problem 2.25.1. Consider the box shown in Figure 2.25.1 with side lengths and vertices as shown. The box is rotated 30 degrees clockwise around its edge *ab* as viewed from *a* to *b*, that is, 30 degrees by the right-hand rule around $\hat{r}_{b/a}$. Next, the box is rotated 45 degrees counterclockwise around its edge *ae* as viewed from *a* to *e*. In terms of the frame $\mathrm{F_A}$, determine the position of *g* relative to *a* after both rotations.

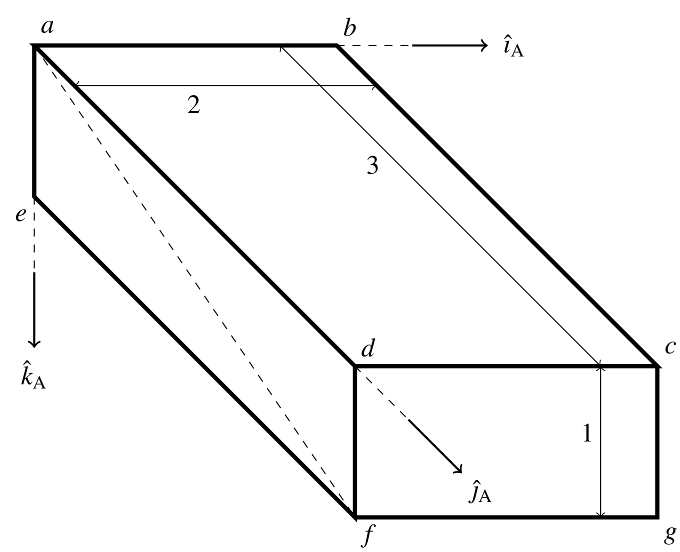


Figure 2.25.1: Box for Problem 2.25.1 and Problem 2.25.2.

Problem 2.25.2. Consider the box shown in Figure 2.25.1 with side lengths and vertices as shown. The box is rotated 60 degrees counterclockwise around the diagonal af as viewed from *a* to *f*, that is, -60 degrees by the right-hand rule around $\hat{r}_{f/a}$. In terms of the frame $\mathrm{F_A}$, determine the position of *g* relative to *a* after the rotation.

Problem 2.25.3. Consider the bent wire *abc* shown in Figure 2.25.2. This wire consists of two straight segments of length ℓ_1 and ℓ_2, and lies in the $\hat{\imath}_{\mathrm{A}}$-$\hat{k}_{\mathrm{A}}$ plane. The angle θ describes how much the wire is bent at the point b. The bent wire is rotated counterclockwise around the line passing through the points a and c (as seen looking from a to c) by the angle $\phi > 0$. Determine the distance from the original position of the point b to its final position after the rotation. Check your solution by specializing it to the case $\theta = 0$.

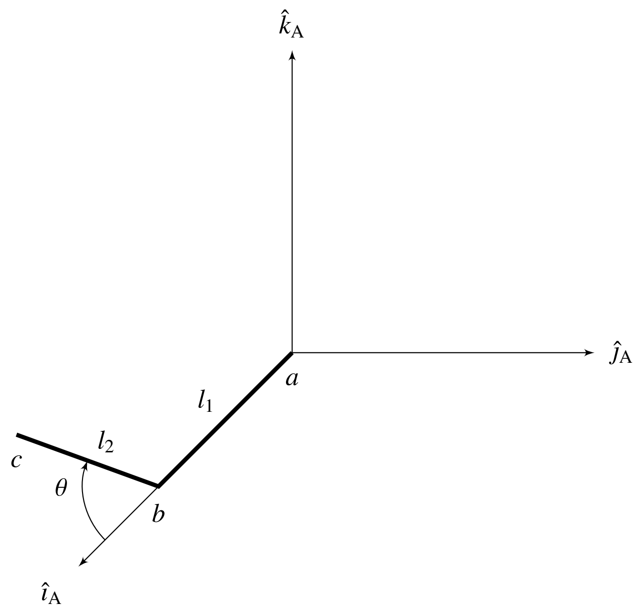


Figure 2.25.2: Bent wire for Problem 2.25.3.

Problem 2.25.4. Consider a 3-gimbal mechanism that emulates 3-2-1 Euler angles, that is, the outermost gimbal rotates around the $\hat{k}$-axis of the support frame, the intermediate axis rotates around the $\hat{\jmath}$-axis of a frame attached to the outermost gimbal, and the innermost gimbal rotates around the $\hat{\imath}$-axis of the intermediate gimbal. Physically, it can be seen that rotating each gimbal relative to its support can be performed in an arbitrary order without changing the final configuration. Prove this mathematically, that is, show that if the outermost gimbal is rotated first, followed by the intermediate gimbal, and, then, finally, by the innermost gimbal, then the configuration that results is the same as the configuration that results from rotating the gimbals in the reverse order. To do this, resolve two products of physical rotation matrices in the support frame.

Symbol	Definition
x	Point or particle x
$\mathcal{B}$	Body $\mathcal{B}$
$\vec{x}$	Physical vector
$\lvert\vec{x}\rvert$	Magnitude of the physical vector $\vec{x}$
$\hat{\imath}, \hat{\jmath}, \hat{k}$	Unit dimensionless physical vectors
$\theta_{\vec{y}/\vec{x}}$	Angle in $[0, \pi]$ between $\vec{y}$ and $\vec{x}$
$\vec{\theta}_{\vec{y}/\vec{x}}$	Angle vector of $\vec{y}$ relative to $\vec{x}$
$\vec{r}_{y/x}$	Position of y relative to x
$\theta_{\vec{y}/\vec{x}/\hat{n}}$	Directed angle in $(-\pi, \pi]$ from $\vec{x}$ to $\vec{y}$ around $\hat{n}$
$\vec{\theta}_{\vec{y}/\vec{x}/\hat{n}}$	Directed angle vector of $\vec{y}$ relative to $\vec{x}$ around $\hat{n}$
$\vec{r}_{y/x}$	Position of y relative to x
$\mathrm{F_A}$	Frame A written as a row vectrix
$\mathcal{F}_\mathrm{A}$	Frame A written as a column vectrix
$\vec{R}_\mathrm{B/A}$	Physical rotation matrix that rotates $\mathrm{F_A}$ to $\mathrm{F_B}$
$\mathcal{R}_\mathrm{B/A}$	Rotation matrix from $\mathrm{F_A}$ to $\mathrm{F_B}$
$\mathcal{O}_\mathrm{B/A}$	Orientation matrix of $\mathrm{F_B}$ relative to $\mathrm{F_A}$
$q_\mathrm{B/A}$	Euler parameter vector of $\mathrm{F_B}$ relative to $\mathrm{F_A}$
3-2-1 rotation: Ψ, Θ, Φ	Azimuth, elevation, bank Euler angles
3-1-3 rotation: Φ, Θ, Ψ	Precession, nutation, spin Euler angles
3-1-3 rotation: Ω, i, ω	Right ascension of the ascending node, inclination, argument of periapsis Euler angles

Table 2.25.1: Notation for Chapter 2.

Chapter Three
Tensors[†]

3.1 Tensors

A *tensor* is a real-valued function of physical covectors and physical vectors that is multilinear, that is, linear in each argument separately. For example, if $\vec{w}$ is a physical vector and $\vec{z}'$ is a physical covector, then the function $\vec{T}\colon \mathcal{V}' \times \mathcal{V} \mapsto \mathbb{R}$ defined by

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{x}'\vec{w})(\vec{z}'\vec{y}) \tag{3.1.1}$$

is a tensor, where $\mathcal{V}$ denotes the set of physical vectors and $\mathcal{V}'$ denotes the set of physical covectors. Note that the value of $\vec{T}(\vec{x}', \vec{y})$ is a product of inner products. In this case we write

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{w} \otimes \vec{z}')(\vec{x}', \vec{y}), \tag{3.1.2}$$

where the tensor $\vec{T}$ is represented by the tensor product notation

$$\vec{T} = \vec{w} \otimes \vec{z}'. \tag{3.1.3}$$

The physical vector $\vec{w}$ and the physical covector $\vec{z}'$ are the factors of $\vec{T}$, whereas the physical covector $\vec{x}'$ and the physical vector $\vec{y}$ are the arguments of $\vec{T}$.

The tensor $\vec{T}$ defined by (3.1.1) can be viewed as a physical matrix. To see this, define $\vec{M} = \vec{x}\vec{y}'$. Then, for every physical vector $\vec{z}$ and physical covector $\vec{w}'$,

$$\vec{w}'\vec{M}\vec{z} = \vec{w}'\vec{x}\vec{y}'\vec{z} = (\vec{x}'\vec{w})(\vec{z}'\vec{y}) = (w \otimes \vec{z}')(\vec{x}', \vec{y}) = \vec{T}(\vec{w}', \vec{z}). \tag{3.1.4}$$

We thus identify $\vec{x}\vec{y}'$ with $\vec{x} \otimes \vec{y}'$.

Like physical matrices, tensors of the form (3.1.2) can be added. For example, for physical vectors $\vec{w}_1, \vec{w}_2$ and physical covectors $\vec{z}'_1, \vec{z}'_2$, we can define the tensor

$$\vec{T} = \vec{w}_1 \otimes \vec{z}'_1 + \vec{w}_2 \otimes \vec{z}'_2, \tag{3.1.5}$$

which satisfies

$$\begin{aligned}\vec{T}(\vec{x}', \vec{y}) &= (\vec{w}_1 \otimes \vec{z}'_1)(\vec{x}', \vec{y}) + (\vec{w}_2 \otimes \vec{z}'_2)(\vec{x}', \vec{y}) \\ &= (\vec{x}'\vec{w}_1)(\vec{z}'_1\vec{y}) + (\vec{x}'\vec{w}_2)(\vec{z}'_2\vec{y}).\end{aligned} \tag{3.1.6}$$

As another example, the function $\vec{T}\colon \mathcal{V}\times\mathcal{V}\mapsto\mathbb{R}$ defined by

$$\vec{T}(\vec{x},\vec{y}) = \vec{x}\,'\vec{y} \tag{3.1.7}$$

is a tensor. To see this, note that

$$\begin{aligned}(\hat{\imath}'_{\rm A}\otimes\hat{\imath}'_{\rm A}+\hat{\jmath}'_{\rm A}\otimes\hat{\jmath}'_{\rm A}+\hat{k}'_{\rm A}\otimes\hat{k}'_{\rm A})(\vec{x},\vec{y}) &= (\hat{\imath}'_{\rm A}\vec{x})(\hat{\imath}'_{\rm A}\vec{y})+(\hat{\jmath}'_{\rm A}\vec{x})(\hat{\jmath}'_{\rm A}\vec{y})+(\hat{k}'_{\rm A}\vec{x})(\hat{k}'_{\rm A}\vec{y})\\ &= \vec{x}\,'\vec{y}.\end{aligned} \tag{3.1.8}$$

Hence,

$$\vec{T} = \hat{\imath}'_{\rm A}\otimes\hat{\imath}'_{\rm A}+\hat{\jmath}'_{\rm A}\otimes\hat{\jmath}'_{\rm A}+\hat{k}'_{\rm A}\otimes\hat{k}'_{\rm A}. \tag{3.1.9}$$

Alternatively, the function $\vec{T}\colon \mathcal{V}'\times\mathcal{V}\mapsto\mathbb{R}$ defined by

$$\vec{T}(\vec{x}\,',\vec{y}) = \vec{x}\,'\vec{y} \tag{3.1.10}$$

is a tensor since

$$\begin{aligned}(\hat{\imath}_{\rm A}\otimes\hat{\imath}'_{\rm A}+\hat{\jmath}_{\rm A}\otimes\hat{\jmath}'_{\rm A}+\hat{k}_{\rm A}\otimes\hat{k}'_{\rm A})(\vec{x}\,',\vec{y}) &= (\vec{x}\,'\hat{\imath}_{\rm A})(\hat{\imath}'_{\rm A}\vec{y})+(\vec{x}\,'\hat{\jmath}_{\rm A})(\hat{\jmath}'_{\rm A}\vec{y})+(\vec{x}\,'\hat{k}_{\rm A})(\hat{k}'_{\rm A}\vec{y})\\ &= (\hat{\imath}'_{\rm A}\vec{x})(\hat{\imath}'_{\rm A}\vec{y})+(\hat{\jmath}'_{\rm A}\vec{x})(\hat{\jmath}'_{\rm A}\vec{y})+(\hat{k}'_{\rm A}\vec{x})(\hat{k}'_{\rm A}\vec{y})\\ &= \vec{x}\,'\vec{y}.\end{aligned} \tag{3.1.11}$$

Hence,

$$\vec{T} = \hat{\imath}_{\rm A}\otimes\hat{\imath}'_{\rm A}+\hat{\jmath}_{\rm A}\otimes\hat{\jmath}'_{\rm A}+\hat{k}_{\rm A}\otimes\hat{k}'_{\rm A}. \tag{3.1.12}$$

The *type* of a tensor $\vec{T}$ is denoted by (p,q), where p is the number of physical vectors multiplied together (*contravariant factors*) to form $\vec{T}$, while q is the number of physical covectors that are multiplied together (*covariant factors*) to form $\vec{T}$. Equivalently, p is the number of physical covectors (*contravariant arguments*) that $\vec{T}$ operates on, and q is the number of physical vectors (*covariant* arguments) that $\vec{T}$ operates on.

The *order* of a tensor of type (p,q) is $p+q$. Therefore, a tensor of type $(2,0)$, $(1,1)$, and $(0,2)$ is a second-order tensor. A scalar is a zeroth-order tensor.

If the factors of $\vec{T}$ include both physical vectors and physical covectors, then the physical vectors appear first, followed by the physical covectors. In this case, the arguments of $\vec{T}$ include both physical covectors and physical vectors, and thus the physical covectors are listed first followed by the physical vectors. Hence, if $\vec{T}=\vec{w}\otimes\vec{z}\,'$, then we write $\vec{T}(\vec{x}\,',\vec{y})$. The set of tensors of type (p,q) is denoted by $\mathcal{T}_{(p,q)}$, and thus, for $\vec{T}\in\mathcal{T}_{(p,q)}$, we write $\vec{T}\colon \mathcal{V}'^{\,p}\times\mathcal{V}^q\mapsto\mathbb{R}$.

Given p physical vectors and q physical covectors, it is possible to construct a tensor of type (p,q). Specifically, given physical vectors $\vec{w}_1,\ldots,\vec{w}_p$ and physical covectors $\vec{z}\,'_1,\ldots,\vec{z}\,'_q$, we can construct the tensor $\vec{T}$ of type (p,q) given by

$$\vec{T} = \vec{w}_1\otimes\cdots\otimes\vec{w}_p\otimes\vec{z}\,'_1\otimes\cdots\otimes\vec{z}\,'_q. \tag{3.1.13}$$

Then, $\vec{T}$ operates on the physical covectors $\vec{x}'_1, \ldots, \vec{x}'_p$ and the physical vectors $\vec{y}_1, \ldots, \vec{y}_q$ according to

$$\begin{aligned}\vec{T}(\vec{x}'_1, \ldots, \vec{x}'_p, \vec{y}_1, \ldots, \vec{y}_q) &= (\vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q)(\vec{x}'_1, \ldots, \vec{x}'_p, \vec{y}_1, \ldots, \vec{y}_q) \\ &= (\vec{x}'_1 \vec{w}_1) \cdots (\vec{x}'_p \vec{w}_p)(\vec{z}'_1 \vec{y}_1) \cdots (\vec{z}'_q \vec{y}_q). \qquad (3.1.14)\end{aligned}$$

Note that the value of $\vec{T}$ is the product of $p + q$ inner products. A sum of tensors of type (p, q) is also a tensor of type (p, q). That is, if $\vec{T}_1 \in \mathfrak{T}_{(p,q)}$ and $\vec{T}_2 \in \mathfrak{T}_{(p,q)}$, then $\vec{T}_1 + \vec{T}_2 \in \mathfrak{T}_{(p,q)}$.

Let $\vec{T} \in \mathfrak{T}_{(p,q)}$. Then, $\vec{T}$ is *wide* if $p < q$, *square* if $p = q$, and tall if $p > q$. Note that $\vec{T}$ is square if and only if $\vec{T}$ has an equal number of vector and covector factors.

The *coform* of the tensor $\vec{T}$ given by (3.1.13) is the (q, p) tensor

$$\vec{T}' = \vec{z}_1 \otimes \cdots \otimes \vec{z}_q \otimes \vec{w}'_1 \otimes \cdots \otimes \vec{w}'_p. \qquad (3.1.15)$$

For example,

$$(\vec{w}_1 \otimes \vec{z}'_1)' = \vec{z}_1 \otimes \vec{w}'_1, \qquad (3.1.16)$$

$$(\vec{w}_1 \otimes \vec{w}_2)' = \vec{w}'_1 \otimes \vec{w}'_2. \qquad (3.1.17)$$

If $\vec{T}_1$ and $\vec{T}_2$ are tensors of the same type, then

$$(\vec{T}_1 + \vec{T}_2)' = \vec{T}'_1 + \vec{T}'_2. \qquad (3.1.18)$$

Let $\vec{z}'$ be a physical covector. Then,

$$\vec{T} = \vec{z}' \qquad (3.1.19)$$

is a first-order tensor of type $(0, 1)$. Furthermore, for each physical vector $\vec{x}$, $\vec{T}(\vec{x})$ is given by

$$\vec{T}(\vec{x}) = \vec{z}'\vec{x}. \qquad (3.1.20)$$

Similarly, let $\vec{w}$ be a physical vector. Then,

$$\vec{T} = \vec{w} \qquad (3.1.21)$$

is a first-order tensor of type $(1, 0)$. Furthermore, for each physical covector $\vec{x}'$, $\vec{T}(\vec{x}')$ is given by

$$\vec{T}(\vec{x}') = \vec{x}'\vec{w}. \qquad (3.1.22)$$

The second-order tensor of type (1,1) given by $\vec{T} = \vec{w} \otimes \vec{z}'$ operates on the pair $(\vec{x}', \vec{y})$ to yield the real number

$$\vec{T}(\vec{x}', \vec{y}) = (\vec{w} \otimes \vec{z}')(\vec{x}', \vec{y}) = (\vec{x}'\vec{w})(\vec{z}'\vec{y}) = \vec{x}'\vec{M}\vec{y}, \qquad (3.1.23)$$

where $\vec{M}$ is the physical matrix $\vec{M} = \vec{w}\vec{z}'$. For example, letting $\mathrm{F_A}$ be a frame, it follows that

$$\vec{T} = \hat{\imath}_\mathrm{A} \otimes \hat{\imath}'_\mathrm{A} + \hat{\jmath}_\mathrm{A} \otimes \hat{\jmath}'_\mathrm{A} + \hat{k}_\mathrm{A} \otimes \hat{k}'_\mathrm{A} = \hat{\imath}_\mathrm{A}\hat{\imath}'_\mathrm{A} + \hat{\jmath}_\mathrm{A}\hat{\jmath}'_\mathrm{A} + \hat{k}_\mathrm{A}\hat{k}'_\mathrm{A} = \vec{I}. \tag{3.1.24}$$

Therefore,

$$\vec{T}(\vec{x}', \vec{y}) = \vec{x}'\vec{I}\vec{y} = \vec{x}'\vec{y}. \tag{3.1.25}$$

In the case of a tensor of type (1,1) it is convenient to omit "⊗" and recognize that every tensor of type $(1, 1)$ is a physical matrix, and vice versa. We thus write $\vec{T} = \vec{w} \otimes \vec{z}' = \vec{w}\vec{z}'$.

As a final example, let $\vec{z}'_1$ and $\vec{z}'_2$ be physical covectors. Then, the second-order tensor

$$\vec{T} = \vec{z}'_1 \otimes \vec{z}'_2 \tag{3.1.26}$$

is of type $(0, 2)$. In particular,

$$\vec{T}(\vec{y}_1, \vec{y}_2) = (\vec{z}'_1 \otimes \vec{z}'_2)(\vec{y}_1, \vec{y}_2) = (\vec{z}'_1\vec{y}_1)(\vec{z}'_2\vec{y}_2). \tag{3.1.27}$$

3.2 Tensor Contraction and Tensor Multiplication

Let

$$\vec{T} = \vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_q \tag{3.2.1}$$

be a tensor of type (p, q), let $1 \le i \le p$ and $1 \le j \le q$. Then, a single *contraction* of $\vec{T}$ yields the tensor of type $(p-1, q-1)$ given by

$$\vec{T}_{(i,j)} = (\vec{z}'_j\vec{w}_i)\vec{w}_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+1} \cdots \otimes \vec{w}_p \otimes \vec{z}'_1 \otimes \cdots \otimes z_{j-1} \otimes z_{j+1} \otimes \cdots \otimes \vec{z}'_q. \tag{3.2.2}$$

Note that $\vec{w}_i$ and $\vec{z}_j$ are removed from the tensor product and now appear as a scalar factor in an inner product. Likewise, if $1 \le i \le p$ and $1 \le k \le p$ are distinct and $1 \le j \le q$ and $1 \le l \le q$ are distinct, then a *double contraction* of $\vec{T}$ yields the tensor of type $(p-2, q-2)$ given by

$$\vec{T}_{(i,j),(k,l)} = (\vec{z}'_j\vec{w}_i)(\vec{z}'_l\vec{w}_k)\vec{T}_1, \tag{3.2.3}$$

where $\vec{T}_1$ is identical to $\vec{T}$ except that $\vec{w}_i, \vec{w}_k, \vec{z}_j, \vec{z}_l$ are removed. More generally, application of an r-contraction to a (p, q) tensor, where $r \le \min\{p, q\}$, yields a tensor of type $(p-r, q-r)$. Each contraction thus reduces the order of a tensor by 2.

As an extreme case of contraction, assume that $\vec{T} \in \mathcal{T}_{(p,p)}$, and thus $\vec{T}$ is square. Then, applying p contractions to $\vec{T}$ yields a scalar, which is a product of p inner products. The p contractions constitute a *total contraction* of $\vec{T}$. Note that there are $p!$ different total contractions of $\vec{T}$. A *partial contraction* of $\vec{T} \in \mathcal{T}_{(p,q)}$ is a contraction that is not a total contraction. Note that, if $\vec{T}$ is not square, then $\vec{T}$ does not have a total contraction.

Next, to define tensor multiplication, let

$$\vec{T}_1 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_{p_1} \otimes \vec{z}'_1 \otimes \cdots \otimes \vec{z}'_{q_1} \tag{3.2.4}$$

be a tensor of type (p_1, q_1), and let

$$\vec{T}_2 = \vec{w}_{p_1+1} \otimes \cdots \otimes \vec{w}_{p_1+p_2} \otimes \vec{z}\,'_{q_1+1} \otimes \cdots \otimes \vec{z}\,'_{q_1+q_2} \tag{3.2.5}$$

be a tensor of type (p_2, q_2). Then, the *tensor product* $\vec{T}_1 \otimes \vec{T}_2$ of $\vec{T}_1$ and $\vec{T}_2$ is the tensor of type $(p_1 + p_2, q_1 + q_2)$ given by

$$\vec{T}_1 \otimes \vec{T}_2 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_{p_1+p_2} \otimes \vec{z}\,'_1 \otimes \cdots \otimes \vec{z}\,'_{q_1+q_2}. \tag{3.2.6}$$

Next, consider the tensor of type (p, q) given by

$$\vec{T}_1 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}\,'_1 \otimes \cdots \otimes \vec{z}\,'_q \tag{3.2.7}$$

and evaluate $\vec{T}_1$ at the arguments $z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}$. This yields the real number

$$\begin{aligned} T_1(z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}) &= (\vec{w}_1 \otimes \cdots \otimes \vec{w}_p \otimes \vec{z}\,'_1 \otimes \cdots \otimes \vec{z}\,'_q)(z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}) \\ &= (\vec{z}\,'_{q+1}\vec{w}_1) \cdots (\vec{z}\,'_{p+q}\vec{w}_p)(\vec{z}\,'_1\vec{w}_{p+1}) \cdots (\vec{z}\,'_q\vec{w}_{p+q}). \end{aligned} \tag{3.2.8}$$

Now, using the arguments $z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}$, construct the tensor

$$\vec{T}_2 \triangleq \vec{w}_{p+1} \otimes \cdots \otimes \vec{w}_{p+q} \otimes \vec{z}\,'_{q+1} \otimes \cdots \otimes \vec{z}\,'_{q+p}. \tag{3.2.9}$$

Then, the tensor product $\vec{T}_3 \triangleq \vec{T}_1 \otimes \vec{T}_2$ is given by

$$T_3 \triangleq \vec{T}_1 \otimes \vec{T}_2 = \vec{w}_1 \otimes \cdots \otimes \vec{w}_{p+q} \otimes \vec{z}\,'_1 \otimes \cdots \otimes \vec{z}\,'_{p+q}. \tag{3.2.10}$$

Note that $\vec{T}_3$ is a square tensor of type $(p + q, p + q)$. Furthermore, a total contraction of $\vec{T}_3$ is given by

$$\begin{aligned} (\vec{T}_3)_{(1,q+1),\ldots,(p,q+p),(p+1,1),\ldots,(p+q,q)} &= (\vec{z}\,'_{q+1}\vec{w}_1) \cdots (\vec{z}\,'_{p+q}\vec{w}_p)(\vec{z}\,'_1\vec{w}_{p+1}) \cdots (\vec{z}\,'_q\vec{w}_{p+q}) \\ &= T_1(z'_{q+1}, \ldots, z'_{q+p}, w_{p+1}, \ldots, w_{p+q}). \end{aligned} \tag{3.2.11}$$

Note that the value of this total contraction is precisely the value of $\vec{T}_1$ evaluated at the arguments $(\vec{z}\,'_{q+1}, \ldots, \vec{z}\,'_{q+p}, \vec{w}_{p+1}, \ldots, \vec{w}_{p+q})$. It thus follows that the evaluation of $\vec{T}_1$ at given arguments can be expressed as the total contraction of the tensor product of $\vec{T}_1$ and a tensor $\vec{T}_2$ whose factors are the arguments of $\vec{T}_1$. Conversely, every total contraction of the product of two tensors $\vec{T}_1$ and $\vec{T}_2$ can be expressed as the evaluation of $\vec{T}_1$ at arguments that are the factors of $\vec{T}_2$.

To illustrate this connection, consider the physical matrices $\vec{M} \triangleq \vec{x}\vec{y}\,' = \vec{x} \otimes \vec{y}\,'$ and $\vec{N} \triangleq \vec{w}\vec{z}\,' = \vec{w} \otimes \vec{z}\,'$. Then, a total contraction of the tensor product

$$\vec{T} \triangleq \vec{M} \otimes \vec{N} = (\vec{x}\vec{y}\,') \otimes (\vec{w}\vec{z}\,') = (\vec{x} \otimes \vec{y}\,') \otimes (\vec{w} \otimes \vec{z}\,') = \vec{x} \otimes \vec{w} \otimes \vec{y}\,' \otimes \vec{z}\,' \tag{3.2.12}$$

is given by

$$(\vec{T})_{(2,1),(1,4)} = (\vec{M} \otimes \vec{N})_{(2,1),(1,4)} = (\vec{y}\,'\vec{w})(\vec{z}\,'\vec{x}), \tag{3.2.13}$$

which is one of two possible total contractions of $\vec{T}$. On the other hand,

$$\vec{M}(\vec{z}', \vec{w}) = (\vec{z}'\vec{x})(\vec{y}'\vec{w}) = (\vec{M} \otimes \vec{N})_{(2,1),(1,4)}. \tag{3.2.14}$$

Consequently, evaluating the second-order tensor $\vec{M}$ at the arguments $\vec{z}', \vec{w}$ is equivalent to taking the tensor product $\vec{M}$ and $\vec{N}$ and then forming a total contraction of the product. Note, however, that the physical matrix $\vec{P} = \vec{M}\vec{N} = (\vec{x}\vec{y}')(\vec{w}\vec{z}') = (\vec{y}'\vec{w})\vec{x}\vec{z}'$ is not a scalar and thus is not a total contraction. In fact, $\vec{P}$ can be viewed as a partial contraction of $\vec{M} \otimes \vec{N}$, as discussed in the next section.

3.3 Partial Tensor Evaluation and the Contracted Tensor Product

In the previous section, we showed that evaluating a tensor at given arguments is equivalent to multiplying the tensor by another tensor whose factors are the arguments and then forming a total contraction of the product. In this case, if the first tensor is of type (p, q), then the second tensor is of type (q, p). In this section we generalize this idea by introducing partial evaluation of a tensor. In this case, the number of covectors and vectors comprising the arguments may be less than the order of the tensor. We then show that partial evaluation is equivalent to constructing a tensor from the given arguments, forming the tensor product, and then partially contracting the product. Both operations–partial evaluation versus tensor multiplication followed by partial tensor contraction—yield a tensor. In the case where the evaluation is performed with the maximum number of arguments, the result is equivalent to a total contraction, as discussed in the previous section. Partial contraction of the tensor product allows us to extend the notion of the tensor product to obtain a tensor of lower order than the tensor product defined above. For example, with this extension, the product of a fourth-order tensor and a second-order tensor may be a tensor of order 6, 4, or 2.

To illustrate the main idea, let $\vec{T}_1$ be the fifth-order tensor of type $(3, 2)$ given by

$$\vec{T}_1 = \vec{w}_1 \otimes \vec{w}_2 \otimes \vec{w}_3 \otimes z_1' \otimes z_2'. \tag{3.3.1}$$

Although $\vec{T}_1$ has five arguments, we partially evaluate $\vec{T}_1$ at the arguments $\vec{x}', \vec{y}_1, \vec{y}_2$ to obtain the third-order tensor $\vec{T}_3$ of type $(1, 2)$ given by

$$\vec{T}_3 \triangleq \vec{T}_1(\vec{x}', \vec{y}_1, \vec{y}_2; 3, 2, 1) = (\vec{x}'\vec{w}_3)(\vec{z}'_2\vec{y}_1)(\vec{z}'_1\vec{y}_2)\vec{w}_1 \otimes \vec{w}_2. \tag{3.3.2}$$

Note that $(3, 2, 1)$ associates the first argument, which is a covector, with the third vector factor in $\vec{T}_1$; the first vector argument with the second covector factor in $\vec{T}_1$; and the second vector argument with the first covector factor in $\vec{T}_1$.

We can arrive at partial evaluation from a different direction. To do this, define the $(2, 1)$ tensor

$$\vec{T}_2 \triangleq \vec{y}_1 \otimes \vec{y}_2 \otimes \vec{x}' \tag{3.3.3}$$

and multiply it by $\vec{T}_1$ given by (3.3.1) to obtain the $(5, 3)$ tensor

$$\vec{T}_1 \otimes \vec{T}_2 = \vec{w}_1 \otimes \vec{w}_2 \otimes \vec{w}_3 \otimes \vec{y}_1 \otimes \vec{y}_2 \otimes z_1' \otimes z_2' \otimes \vec{x}'. \tag{3.3.4}$$

It thus follows that

$$\begin{aligned}(\vec{T}_1 \otimes \vec{T}_2)_{(3,3),(4,2),(5,1)} &= (\vec{w}_1 \otimes \vec{w}_2 \otimes \vec{w}_3 \otimes \vec{y}_1 \otimes \vec{y}_2 \otimes z_1' \otimes z_2' \otimes \vec{x}')_{(3,3),(4,2),(5,1)} \\ &= (\vec{x}'\vec{w}_3)(\vec{z}_2'\vec{y}_1)(\vec{z}_1'\vec{y}_2)\vec{w}_1 \otimes \vec{w}_2 \\ &= \vec{T}_1(\vec{x}', \vec{y}_1, \vec{y}_2; 3, 2, 1) \\ &= \vec{T}_3, \end{aligned} \tag{3.3.5}$$

where $\vec{T}_3$ is defined by (3.3.2). Note that $\vec{T}_3$ arises in two different but equivalent ways, namely, from the partial evaluation $\vec{T}_1(\vec{x}', \vec{y}_1, \vec{y}_2; 3, 2, 1)$ of $\vec{T}_1$ as well as from the *contracted tensor product* $(\vec{T}_1 \otimes \vec{T}_2)_{(3,3),(4,2),(5,1)}$, both of which produce a tensor of lower rank than the tensor product $\vec{T}_1 \otimes \vec{T}_2$.

As another example, we compare "standard" multiplication and tensor multiplication of the physical matrix $\vec{M} = \vec{x} \otimes \vec{y}' = \vec{x}\vec{y}'$, which is a tensor of type $(1, 1)$, with the physical vector $\vec{z}$, which is a tensor of type $(1, 0)$. In this case, the tensor product of $\vec{M}$ and z is the $(2, 1)$ tensor

$$\vec{T} \triangleq \vec{M} \otimes \vec{z} = \vec{x} \otimes \vec{z} \otimes \vec{y}'. \tag{3.3.6}$$

On the other hand, standard multiplication yields

$$\vec{M}\vec{z} = \vec{x}\vec{y}'\vec{z} = (\vec{y}'\vec{z})\vec{x} = \vec{M}(\vec{z}; 1) = (\vec{x} \otimes \vec{y}')(\vec{z}; 1) = (\vec{x} \otimes \vec{z} \otimes \vec{y}')_{(2,1)} = (M \otimes \vec{z})_{(2,1)}. \tag{3.3.7}$$

Next, consider the physical matrices $\vec{M} = \vec{w}\vec{z}'$ and $\vec{N} = \vec{x}\vec{y}'$. Their product can be written as the contracted tensor product

$$\vec{M}\vec{N} = (\vec{w} \otimes \vec{x} \otimes \vec{z}' \otimes \vec{y}')_{(2,1)} = (\vec{z}'\vec{x})\vec{w}\vec{y}'. \tag{3.3.8}$$

As another example, consider the *metric tensor*, which is the $(0, 2)$ tensor

$$\vec{G} = \hat{\imath}_{\mathrm{A}}' \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}}' \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}}' \otimes \hat{k}_{\mathrm{A}}'. \tag{3.3.9}$$

Letting $\vec{x} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}$, it follows that

$$\begin{aligned}(\vec{G} \otimes \vec{x})_{(1,1)} &= ((\hat{\imath}_{\mathrm{A}}' \otimes \hat{\imath}_{\mathrm{A}}' + \hat{\jmath}_{\mathrm{A}}' \otimes \hat{\jmath}_{\mathrm{A}}' + \hat{k}_{\mathrm{A}}' \otimes \hat{k}_{\mathrm{A}}') \otimes (x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}))_{(1,1)} \\ &= x_1\hat{\imath}_{\mathrm{A}}' + x_2\hat{\jmath}_{\mathrm{A}}' + x_3\hat{k}_{\mathrm{A}}' \\ &= \vec{G}(\vec{x}; 1) \\ &= \vec{x}'. \end{aligned} \tag{3.3.10}$$

The metric tensor thus maps the vector $\vec{x}$ to the corresponding covector $\vec{x}'$. Likewise, the *cometric tensor* $\vec{G}' = \hat{\imath}_{\mathrm{A}} \otimes \hat{\imath}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}} \otimes \hat{\jmath}_{\mathrm{A}} + \hat{k}_{\mathrm{A}} \otimes \hat{k}_{\mathrm{A}}$, which is of type $(2, 0)$, converts the covector $\vec{x}'$ to the corresponding vector $\vec{x}$; that is, $(\vec{G}' \otimes \vec{x}')_{(1,1)} = \vec{x}$. Consequently,

$$(\vec{G}' \otimes (\vec{G} \otimes \vec{x})_{(1,1)})_{(1,1)} = (\vec{G}' \otimes \vec{x}')_{(1,1)} = \vec{x}, \tag{3.3.11}$$

$$(\vec{G} \otimes (\vec{G}' \otimes \vec{x}')_{(1,1)})_{(1,1)} = (\vec{G} \otimes \vec{x})_{(1,1)} = \vec{x}'. \tag{3.3.12}$$

Furthermore,

$$(\vec{G}' \otimes \vec{G})_{(1,1)} = (\vec{G} \otimes \vec{G}')_{(1,1)} = \vec{U}. \tag{3.3.13}$$

As another example, consider the identity (2.9.3) for the physical cross product matrix, that is,

$$(\hat{k}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}})\hat{\imath}'_{\mathrm{A}}\vec{x} + (\hat{\imath}_{\mathrm{A}}\hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}})\hat{\jmath}'_{\mathrm{A}}\vec{x} + (\hat{\jmath}_{\mathrm{A}}\hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}}\hat{\jmath}'_{\mathrm{A}})\hat{k}'_{\mathrm{A}}\vec{x} = \vec{x}^{\times}. \tag{3.3.14}$$

Defining the $(1,2)$ tensor

$$\vec{T} = (\hat{k}_{\mathrm{A}} \otimes \hat{\jmath}'_{\mathrm{A}} - \hat{\jmath}_{\mathrm{A}} \otimes \hat{k}'_{\mathrm{A}}) \otimes \hat{\imath}'_{\mathrm{A}} + (\hat{\imath}_{\mathrm{A}} \otimes \hat{k}'_{\mathrm{A}} - \hat{k}_{\mathrm{A}} \otimes \hat{\imath}'_{\mathrm{A}}) \otimes \hat{\jmath}'_{\mathrm{A}} + (\hat{\jmath}_{\mathrm{A}} \otimes \hat{\imath}'_{\mathrm{A}} - \hat{\imath}_{\mathrm{A}} \otimes \hat{\jmath}'_{\mathrm{A}}) \otimes \hat{k}'_{\mathrm{A}}, \tag{3.3.15}$$

it follows that

$$(\vec{T} \otimes \vec{x})_{(2,2)} = \vec{T}(\vec{x}; 2) = \vec{x}^{\times}. \tag{3.3.16}$$

A compact expression for the tensor (3.3.15) is given by

$$\vec{T} = -\sum_{i,j,k=1}^{n} \epsilon_{ijk}\hat{e}_i \otimes \hat{e}'_j \otimes e'_k, \tag{3.3.17}$$

where

$$\epsilon_{ijk} \triangleq \begin{cases} 1, & ijk \in \{123, 231, 312\}, \\ -1, & ijk \in \{321, 132, 213\}, \\ 0, & (i-j)(j-k)(k-i) = 0. \end{cases} \tag{3.3.18}$$

As another example, define the fourth-order tensor $\vec{U}_4 \triangleq \vec{U} \otimes \vec{U}$ of type $(2,2)$, and let $\vec{M} = \sum_{i,j=1}^{3} m_{ij}\hat{e}_i \otimes \hat{e}'_j$ be a physical matrix. Then,

$$\vec{U}_4 = \left(\sum_{i=1}^{3} \hat{e}_i \otimes \hat{e}'_i\right) \otimes \left(\sum_{j=1}^{3} \hat{e}_j \otimes \hat{e}'_j\right) = \sum_{i,j=1}^{3} \left(\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}'_i \otimes \hat{e}'_j\right). \tag{3.3.19}$$

Therefore,

$$\begin{aligned} \vec{U}_4 \otimes \vec{M} &= \sum_{i,j=1}^{3} \left(\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}'_i \otimes \hat{e}'_j\right) \otimes \sum_{k,l=1}^{3} m_{kl}\hat{e}_k \otimes \hat{e}_l \\ &= \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} m_{kl} \left(\hat{e}_i \otimes \hat{e}_j \otimes \hat{e}_k \otimes \hat{e}'_i \otimes \hat{e}'_j \otimes e'_l\right) \end{aligned} \tag{3.3.20}$$

and thus

$$(\vec{U}_4 \otimes \vec{M})_{(2,3),(3,1)} = \sum_{i,j=1}^{3} m_{ij}\hat{e}_i \otimes \hat{e}'_j = \vec{M}. \tag{3.3.21}$$

Hence, with the contraction $(2,3),(3,1)$, $\vec{U}_4$ can be viewed as the identity tensor on the $\mathcal{T}_{(1,1)}$. Alternatively, note that

$$(\vec{U}_4 \otimes \vec{M})_{(2,3),(3,2)} = \sum_{i,j=1}^{3} m_{jj}\hat{e}_i \otimes \hat{e}'_i = (\operatorname{tr} \vec{M})\vec{U}. \tag{3.3.22}$$

3.4 Stress, Strain, and Elasticity Tensors

Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the axes of an orthogonal frame, and consider stress and strain tensors, which are (1, 1) tensors, that is, physical matrices, of the form

$$\vec{\sigma} = \sum_{i,j=1}^{3} \sigma_{ij}\hat{e}_i\hat{e}_j', \tag{3.4.1}$$

$$\vec{\varepsilon} = \sum_{i,j=1}^{3} \varepsilon_{ij}\hat{e}_i\hat{e}_j'. \tag{3.4.2}$$

Next, define the *stiffness tensor* $\vec{\mathcal{K}}$, which is a fourth-order tensor of type (2, 2) of the form

$$\vec{\mathcal{K}} = \sum_{k,l,m,n=1}^{3} c_{klmn}\hat{e}_k \otimes \hat{e}_l \otimes \hat{e}_m' \otimes \hat{e}_n'.$$

The tensor product of $\vec{\mathcal{K}}$ and $\vec{\varepsilon}$ is given by

$$\vec{\mathcal{K}} \otimes \vec{\varepsilon} = \sum_{k,l,m,n=1}^{3} \sum_{i,j=1}^{3} c_{klmn}\varepsilon_{ij}(\hat{e}_k \otimes \hat{e}_l \otimes \hat{e}_i \otimes \hat{e}_m' \otimes \hat{e}_n' \otimes \hat{e}_j'). \tag{3.4.3}$$

Then, *Hooke's law* is the constitutive law given by the contracted tensor product

$$\vec{\sigma} = (\vec{\mathcal{K}} \otimes \vec{\varepsilon})_{(2,3),(3,1)}. \tag{3.4.4}$$

Therefore,

$$\begin{aligned}
\vec{\sigma} &= \sum_{k,l,m,n=1}^{3} \sum_{i,j=1}^{3} c_{klmn}\varepsilon_{ij}(\hat{e}_k \otimes \hat{e}_l \otimes \hat{e}_i \otimes \hat{e}_m' \otimes \hat{e}_n' \otimes \hat{e}_j')_{(2,3),(3,1)} \\
&= \sum_{k,l,m,n=1}^{3} c_{klmn} \sum_{i,j=1}^{3} \varepsilon_{ij}(\hat{e}_j'\hat{e}_l)(\hat{e}_m'\hat{e}_i)(\hat{e}_k \otimes \hat{e}_n') \\
&= \sum_{k,l,m,n=1}^{3} c_{klmn}\varepsilon_{ml}(\hat{e}_k \otimes \hat{e}_n') \\
&= \sum_{i,j=1}^{3} \sigma_{ij}\hat{e}_i \otimes \hat{e}_j',
\end{aligned}$$

where, for all $i, j = 1, 2, 3$,

$$\sigma_{ij} \triangleq \sum_{k,l=1}^{3} c_{iklj}\varepsilon_{lk}. \tag{3.4.5}$$

For physical reasons, the strain and stress tensors are symmetric, that is, $\vec{\sigma}' = \vec{\sigma}$ and $\vec{\varepsilon}' = \vec{\varepsilon}$. Therefore, for all $i, j = 1, 2, 3$, $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$. Since $\vec{\sigma}$ is symmetric, it follows that, for all $i, j = 1, 2, 3$, $c_{iklj} = c_{jkli}$. Consequently, $\mathcal{K}$ can be characterized by at most $3^4 - 3 \cdot 9 = 81 - 27 = 54$ parameters. Furthermore, since $\vec{\varepsilon}$ is symmetric, it follows that, for all $i, j = 1, 2, 3$, $c_{iklj} = c_{ilkj}$. Consequently, $\mathcal{K}$ can be characterized by at most $54 - 3 \cdot 6 = 54 - 18 = 36$ parameters. Alternatively, note that the symmetry of $\vec{\sigma}$ and $\vec{\varepsilon}$ implies that each of these tensors can be represented by a vector with six components. Consequently, (3.4.5) can be represented by

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123}\\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223}\\ c_{3311} & c_{3322} & c_{3333} & c_{3312} & c_{3313} & c_{3323}\\ c_{1211} & c_{1222} & c_{1233} & c_{1212} & c_{1213} & c_{1223}\\ c_{1311} & c_{1322} & c_{1333} & c_{1312} & c_{1313} & c_{1323}\\ c_{2311} & c_{2322} & c_{2333} & c_{2312} & c_{2313} & c_{2323}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{bmatrix}, \tag{3.4.6}$$

where the matrix in (3.4.6) has 36 entries.

Since the strain energy $\mathcal{E}$ is given by

$$\mathcal{E} = \frac{1}{2}\sum_{i,j,k,l=1}^{3} c_{ikjl}\varepsilon_{ij}\varepsilon_{kl}, \tag{3.4.7}$$

it follows that, for all $i, j, k, l = 1, 2, 3$, $c_{ijkl} = c_{klij}$. The 6×6 matrix in (3.4.6) is thus characterized by $6 + 5 + 4 + 3 + 2 + 1 = 21$ (rather than 36) constants, and (3.4.6) can be written as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1113} & c_{1123}\\ c_{1122} & c_{2222} & c_{2233} & c_{2212} & c_{2213} & c_{2223}\\ c_{1133} & c_{2233} & c_{3333} & c_{3312} & c_{3313} & c_{3323}\\ c_{1112} & c_{2212} & c_{3312} & c_{1212} & c_{1213} & c_{1223}\\ c_{1113} & c_{1322} & c_{3313} & c_{1213} & c_{1313} & c_{1323}\\ c_{1123} & c_{2322} & c_{3323} & c_{1223} & c_{1323} & c_{2323}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{bmatrix}. \tag{3.4.8}$$

For an isotropic material it can be shown that (3.4.8) can be characterized by 9 (rather than 21) constants, and (3.4.8) can be written as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0\\ c_{1122} & c_{2222} & c_{2233} & 0 & 0 & 0\\ c_{1133} & c_{2233} & c_{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{1212} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{1313} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{2323}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{bmatrix}. \tag{3.4.9}$$

The 12 nonzero entries in (3.4.9) can be parameterized by two constants. Specifically,

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{bmatrix}, \tag{3.4.10}$$

where the positive numbers μ and λ are the *Lame constants*. In tensor form, it follows from (3.4.10) that Hooke's law for isotropic materials has the form

$$\vec{\sigma} = 2\mu\vec{\varepsilon} + \lambda(\mathrm{tr}\,\vec{\varepsilon})\vec{I}, \tag{3.4.11}$$

which defines the fourth-order stiffness tensor $\vec{\mathcal{K}}$. In particular, it follows from (3.3.21) and (3.3.22) that

$$\begin{aligned}\vec{\sigma} &= (\vec{\mathcal{K}} \otimes \vec{\varepsilon})_{(2,3),(3,1)} \\ &= 2\mu\vec{\varepsilon} + \lambda(\operatorname{tr}\vec{\varepsilon})\vec{I} \\ &= 2\mu(\vec{U}_4 \otimes \vec{\varepsilon})_{(2,3),(3,1)} + \lambda(\vec{U}_4 \otimes \vec{\varepsilon})_{(2,3),(3,2)}.\end{aligned} \tag{3.4.12}$$

The inverse of the stiffness tensor is the *compliance tensor* $\vec{\mathcal{C}}$, which satisfies

$$\vec{\varepsilon} = (\vec{\mathcal{C}} \otimes \vec{\sigma})_{(2,3),(3,1)}. \tag{3.4.13}$$

The compliance tensor is represented by the relation

$$\vec{\varepsilon} = \frac{1+\nu}{E}\vec{\sigma} - \frac{\nu}{E}(\operatorname{tr}\vec{\sigma})\vec{I}, \tag{3.4.14}$$

where ν is *Poisson's ratio* and E is *Young's modulus*. In terms of μ and λ, the parameters ν and E are given by

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \tag{3.4.15}$$

Conversely, μ and λ are given in terms of ν and E by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{3.4.16}$$

3.5 Kronecker Algebra

For matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, we consider the Kronecker product $A \otimes B$. For details, see [1]. Note that "$\otimes$" is the same notation used for the tensor product. However, no confusion can occur since the tensor product is used for physical vectors and physical covectors, whereas the Kronecker product is used only for math vectors and math matrices. In this section we show that the Kronecker product can be used to resolve tensors.

For $A \in \mathbb{F}^{n\times m}$ define the *vec* operator as

$$\operatorname{vec} A \triangleq \begin{bmatrix}\operatorname{col}_1(A) \\ \vdots \\ \operatorname{col}_m(A)\end{bmatrix} \in \mathbb{F}^{nm}, \tag{3.5.1}$$

which is the column vector of size $nm \times 1$ obtained by stacking the columns of A. We recover A from $\operatorname{vec} A$ by writing

$$A = \operatorname{vec}^{-1}(\operatorname{vec} A). \tag{3.5.2}$$

Note that, if $x \in \mathbb{F}^n$, then $\operatorname{vec} x = \operatorname{vec} x^{\mathsf{T}} = x$.

Fact 3.5.1. Let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{m\times n}$. Then,

$$\operatorname{tr} AB = (\operatorname{vec} A^{\mathsf{T}})^{\mathsf{T}} \operatorname{vec} B = (\operatorname{vec} B^{\mathsf{T}})^{\mathsf{T}} \operatorname{vec} A. \tag{3.5.3}$$

Next, we introduce the Kronecker product.

Definition 3.5.2. Let $A \in \mathbb{F}^{n\times m}$ and $B \in \mathbb{F}^{l\times k}$. Then, the *Kronecker product* $A \otimes B \in \mathbb{F}^{nl\times mk}$ of A and B is the partitioned matrix

$$A \otimes B \triangleq \begin{bmatrix} A_{(1,1)}B & A_{(1,2)}B & \cdots & A_{(1,m)}B \\ \vdots & \vdots & \therefore & \vdots \\ A_{(n,1)}B & A_{(n,2)}B & \cdots & A_{(n,m)}B \end{bmatrix}. \tag{3.5.4}$$

Unlike matrix multiplication, the Kronecker product $A \otimes B$ does not entail a restriction on either the size of A or the size of B.

The following results are immediate consequences of the definition of the Kronecker product.

Fact 3.5.3. Let $\alpha \in \mathbb{F}$, $A \in \mathbb{F}^{n\times m}$, and $B \in \mathbb{F}^{l\times k}$. Then,

$$\alpha \otimes A = A \otimes \alpha = \alpha A, \quad A \otimes (\alpha B) = (\alpha A) \otimes B = \alpha (A \otimes B), \tag{3.5.5}$$

$$\overline{A \otimes B} = \overline{A} \otimes \overline{B}, \quad (A \otimes B)^{\mathsf{T}} = A^{\mathsf{T}} \otimes B^{\mathsf{T}}, \quad (A \otimes B)^{*} = A^{*} \otimes B^{*}. \tag{3.5.6}$$

Fact 3.5.4. Let $A, B \in \mathbb{F}^{n\times m}$ and $C \in \mathbb{F}^{l\times k}$. Then,

$$(A + B) \otimes C = A \otimes C + B \otimes C, \tag{3.5.7}$$

$$C \otimes (A + B) = C \otimes A + C \otimes B. \tag{3.5.8}$$

The next result shows that the Kronecker product is associative.

Fact 3.5.5. Let $A \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{l\times k}$, and $C \in \mathbb{F}^{p\times q}$. Then,

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C. \tag{3.5.9}$$

We thus write $A \otimes B \otimes C$ for $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$.

The next result shows how matrix multiplication interacts with the Kronecker product.

Fact 3.5.6. Let $A \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{l\times k}$, $C \in \mathbb{F}^{m\times q}$, and $D \in \mathbb{F}^{k\times p}$. Then,

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \tag{3.5.10}$$

Next, we consider the inverse of a Kronecker product.

Fact 3.5.7. Assume that $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$ are nonsingular. Then, $A \otimes B$ is nonsingular, and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{3.5.11}$$

Fact 3.5.8. Let $x \in \mathbb{F}^{n}$ and $y \in \mathbb{F}^{m}$. Then,

$$xy^{\mathsf{T}} = x \otimes y^{\mathsf{T}} = y^{\mathsf{T}} \otimes x, \tag{3.5.12}$$

$$\operatorname{vec} xy^{\mathsf{T}} = y \otimes x. \tag{3.5.13}$$

Fact 3.5.9. Let $A \in \mathbb{F}^{n\times m}$, $B \in \mathbb{F}^{m\times l}$, and $C \in \mathbb{F}^{l\times k}$. Then,

$$\operatorname{vec}(ABC) = (C^{\mathsf{T}} \otimes A)\operatorname{vec} B. \tag{3.5.14}$$

3.6 Composing Tensors

Let $\vec{T} \in \mathfrak{T}_{(p,q)}$ be given by

$$\vec{T} = \sum_{i=1}^{r} \vec{w}_{1i} \otimes \cdots \otimes \vec{w}_{pi} \otimes \vec{z}^{\,\prime}_{1i} \otimes \cdots \otimes \vec{z}^{\,\prime}_{qi}, \tag{3.6.1}$$

and let $\mathrm{F_A}$ be a frame. Then, we define

$$\vec{T}\Big|_{\mathrm{A}} \triangleq \sum_{i=1}^{r} \vec{w}_{1i}\Big|_{\mathrm{A}} \otimes \cdots \otimes \vec{w}_{pi}\Big|_{\mathrm{A}} \otimes \vec{z}_{1i}\Big|_{\mathrm{A}}^{\mathsf{T}} \otimes \cdots \otimes \vec{z}_{qi}\Big|_{\mathrm{A}}^{\mathsf{T}}. \tag{3.6.2}$$

Therefore, $\vec{T}\Big|_{\mathrm{A}} \in \mathbb{R}^{3^p \times 3^q}$.

The following result shows that the Kronecker product representation of tensors is compatible with tensor multiplication.

Fact 3.6.1. Let $\vec{T}_1 \in \mathfrak{T}_{(p_1,q_1)}$ and $\vec{T}_2 \in \mathfrak{T}_{(p_2,q_2)}$, and define $\vec{T}_3 \triangleq \vec{T}_1 \otimes \vec{T}_2 \in \mathfrak{T}_{(p_1+p_2,q_1+q_2)}$. Then,

$$\vec{T}_3\Big|_{\mathrm{A}} = \vec{T}_1\Big|_{\mathrm{A}} \otimes \vec{T}_2\Big|_{\mathrm{A}}. \tag{3.6.3}$$

Finally, the Kronecker product representation of tensors is also compatible with restricted tensor multiplication. To show this, we need to introduce the *restricted Kronecker product*. Let $\vec{T}_1 \in \mathfrak{T}_{(p_1,q_1)}$ and $\vec{T}_2 \in \mathfrak{T}_{(p_2,q_2)}$, and define

$$T_1 \triangleq \vec{T}_1\Big|_{\mathrm{A}} = \sum \alpha_{j_1,\ldots,j_p,k_1,\ldots,k_q} e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{k_1}^{\mathsf{T}} \otimes \cdots \otimes e_{k_q}^{\mathsf{T}} \tag{3.6.4}$$

and

$$T_2 \triangleq \vec{T}_2\Big|_{\mathrm{A}} = \sum \beta_{j_1,\ldots,j_p,k_1,\ldots,k_q} e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{k_1}^{\mathsf{T}} \otimes \cdots \otimes e_{k_q}^{\mathsf{T}}. \tag{3.6.5}$$

where $e_i \triangleq \hat{e}_i|_{\mathrm{A}}$ for $i = 1, 2, 3$, and the summation is over all indices in the range $1, 2, 3$. Then, for example,

$$T_1 \otimes_{(j_p,k_q)} T_2 \triangleq \sum \alpha_{j_1,\ldots,j_p,k_1,\ldots,k_{q-1},j_p} \beta_{j_1,\ldots,j_p,k_1,\ldots,k_{q-1},j_p} e_{j_1} \otimes \cdots \otimes e_{j_{p-1}} \otimes e_{k_1}^{\mathsf{T}} \otimes \cdots \otimes e_{k_{q-1}}^{\mathsf{T}}. \tag{3.6.6}$$

3.7 Alternating Tensors and the Wedge Product

An *alternating tensor* (also called a *skew-symmetric tensor*) is a contravariant or covariant tensor whose sign changes when two of its arguments are interchanged. The set of alternating covariant tensors in $\mathfrak{T}_{(0,q)}$ is denoted by $\hat{\mathfrak{T}}_{(0,q)}$, while the set of alternating contravariant tensors in $\mathfrak{T}_{(p,0)}$ is denoted by $\hat{\mathfrak{T}}_{(p,0)}$. By definition, $\hat{\mathfrak{T}}_{(0,1)} = \mathfrak{T}_{(0,1)}$ and $\hat{\mathfrak{T}}_{(1,0)} = \mathfrak{T}_{(1,0)}$. The wedge product is used to construct alternating tensors.

3.7.1 Bivectors

Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then, the *wedge product* (also called the *exterior product*) $\vec{x} \wedge \vec{y}$, of $\vec{x}$ and $\vec{y}$ is the $(2,0)$ tensor defined by

$$\vec{x} \wedge \vec{y} \triangleq \vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x}. \tag{3.7.1}$$

The wedge product satisfies the properties

$$\vec{x} \wedge \vec{x} = 0, \tag{3.7.2}$$

$$\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x}, \tag{3.7.3}$$

$$(\alpha\vec{x}) \wedge \vec{y} = \alpha(\vec{x} \wedge \vec{y}), \tag{3.7.4}$$

$$\vec{x} \wedge (\alpha\vec{y}) = \alpha(\vec{x} \wedge \vec{y}), \tag{3.7.5}$$

for all real numbers α. The identity (3.7.3) shows that $\vec{x} \wedge \vec{y}$ is an alternating tensor. The contravariant alternating tensor $\vec{x} \wedge \vec{y}$ is called a *bivector*.

The identity (3.7.1) is analogous to the physical cross-product matrix identity given by (2.9.21). In particular, note that

$$(\vec{x} \times \vec{y})^{\times} = \vec{y}\vec{x}' - \vec{x}\vec{y}' = \vec{y} \otimes \vec{x}' - \vec{x} \otimes \vec{y}', \tag{3.7.6}$$

which is a $(1,1)$ tensor.

Fact 3.7.1. Let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\begin{aligned}(\vec{x} \wedge \vec{y})\Big|_{\mathrm{A}} &= \operatorname{vec}\left[(\vec{x} \times \vec{y})^{\times}\Big|_{\mathrm{A}}\right] = \operatorname{vec}\left[(\vec{x} \times \vec{y})\Big|_{\mathrm{A}}^{\times}\right] \\ &= \left\|(\vec{x} \times \vec{y})^{\times}\Big|_{\mathrm{A}}\right\|_{\mathrm{F}} = \left\|(\vec{x} \times \vec{y})\Big|_{\mathrm{A}}^{\times}\right\|_{\mathrm{F}} \\ &= \operatorname{vec}\left[(\vec{x}\vec{y}' - \vec{y}\vec{x}')\Big|_{\mathrm{A}}\right] = \left\|(\vec{x}\vec{y}' - \vec{y}\vec{x}')\Big|_{\mathrm{A}}\right\|_{\mathrm{F}}.\end{aligned} \tag{3.7.7}$$

Now, define $x \triangleq \vec{x}\Big|_{\mathrm{A}}$ and $y \triangleq \vec{y}\Big|_{\mathrm{A}}$. Then,

$$\begin{aligned}x \otimes y - y \otimes x &= \operatorname{vec}(y \otimes x^{\mathsf{T}} - x \otimes y^{\mathsf{T}}) = \operatorname{vec}(yx^{\mathsf{T}} - xy^{\mathsf{T}}) \\ &= \operatorname{vec}[(x \times y)^{\times}] = \|(x \times y)^{\times}\|_{\mathrm{F}}.\end{aligned} \tag{3.7.8}$$

Proof. It follows from [1, p. 682] that $y \otimes x = \operatorname{vec}(x \otimes y^{\mathsf{T}}) = \operatorname{vec} xy^{\mathsf{T}}$, which proves the first and second equalities in (3.7.8). Next, it follows from (2.9.25) that $yx^{\mathsf{T}} - xy^{\mathsf{T}} = (x \times y)^{\times}$, which proves the third equality in (3.7.8). Finally, (3.7.7) is a restatement of the equality between the first and last terms in (3.7.8). □

As noted after Fact 2.9.6, the cross product $\vec{x} \times \vec{y}$ can be viewed as the directed area of a parallelogram. It thus follows from (3.7.7) that the bivector $\vec{x} \wedge \vec{y}$ can be viewed in the same way.

Fact 3.7.2. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors lying in the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. Then

$$\vec{x} \wedge \vec{y} = |\vec{x}||\vec{y}| \sin\theta_{\vec{y}/\vec{x}/\hat{k}_{\mathrm{A}}} \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}.$$

Proof. Let

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \qquad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}.$$

Then, it follows from Problem 2.24.5 that

$$\begin{aligned} \vec{x} \wedge \vec{y} &= (x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}}) \wedge (y_1\hat{\imath}_{\mathrm{A}} + y_2\hat{\jmath}_{\mathrm{A}}) \\ &= (x_1y_2 - x_2y_1)\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \\ &= |\vec{x}||\vec{y}| \sin\theta_{\vec{y}/\vec{x}/\hat{k}_{\mathrm{A}}} \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}. \end{aligned}$$

□

The bivector $\vec{x} \wedge \vec{y}$ can be visualized as a planar region. This region can have the form of a parallelogram constructed by sweeping $\vec{y}$ along $\vec{x}$. The sides of the parallelogram are thus $\vec{x}$, $\vec{y}$, $-\vec{x}$, and $-\vec{y}$, and the magnitude of the bivector $\vec{x} \wedge \vec{y}$ is defined to be the area of the parallelogram, that is,

$$|\vec{x} \wedge \vec{y}| = |\vec{x}||\vec{y}| \sin\theta_{\vec{x}/\vec{y}} |\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}| = |\vec{x}||\vec{y}| \sin\theta_{\vec{x}/\vec{y}}, \tag{3.7.9}$$

where the area of the bivector $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$ is the area of the unit square, namely, 1. Note that

$$|\vec{x} \wedge \vec{y}| = |\vec{x} \times \vec{y}|. \tag{3.7.10}$$

Therefore, the magnitude of $\vec{x} \wedge \vec{y}$ is the area of the bivector, its attitude is given by the plane within which the region lies, and its orientation is given by the direction determined by the right hand rule when $\vec{x}$ is rotated to $\vec{y}$, that is, the direction of $\vec{x} \times \vec{y}$. The shape of a bivector need not be a parallelogram, however; for example, it may be ellipsoidal. If the bivector is visualized as a square, then the length of each side is $\sqrt{|\vec{x}||\vec{y}| \sin\theta_{\vec{x}/\vec{y}}}$.

3.7.2 Trivectors

The tensor $\vec{T} = \vec{x} \wedge \vec{y}$ has the property that its sign changes when its arguments are interchanged. In particular, note that, if $\vec{u}'$ and $\vec{v}'$ are physical covectors, then

$$\begin{aligned} \vec{T}(\vec{u}', \vec{v}') &= (\vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x})(\vec{u}', \vec{v}') \\ &= \vec{u}'\vec{x}\vec{v}'\vec{y} - \vec{u}'\vec{y}\vec{v}'\vec{x} \\ &= -\vec{T}(\vec{v}', \vec{u}'). \end{aligned}$$

The wedge product can be applied to more than two physical vectors. For example, let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors. Then, $\vec{T} = \vec{x} \wedge \vec{y} \wedge \vec{z}$ is the $(3,0)$ tensor defined by

$$\vec{x} \wedge \vec{y} \wedge \vec{z} \triangleq \vec{x} \otimes \vec{y} \otimes \vec{z} + \vec{y} \otimes \vec{z} \otimes \vec{x} + \vec{z} \otimes \vec{x} \otimes \vec{y} - \vec{x} \otimes \vec{z} \otimes \vec{y} - \vec{y} \otimes \vec{x} \otimes \vec{z} - \vec{z} \otimes \vec{y} \otimes \vec{x}. \tag{3.7.11}$$

Note that

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = -\vec{y} \wedge \vec{x} \wedge \vec{z} = \vec{y} \wedge \vec{z} \wedge \vec{x} = -\vec{z} \wedge \vec{y} \wedge \vec{x} = \vec{z} \wedge \vec{x} \wedge \vec{y} = -\vec{x} \wedge \vec{z} \wedge \vec{y}. \tag{3.7.12}$$

Consequently, if $\vec{u}'$, $\vec{v}'$, and $\vec{w}'$ are physical covectors, then

$$\vec{T}(\vec{u}', \vec{v}', \vec{w}') = -\vec{T}(\vec{v}', \vec{u}', \vec{w}') = \vec{T}(\vec{v}', \vec{w}', \vec{u}') = -\vec{T}(\vec{w}', \vec{v}', \vec{u}') = \vec{T}(\vec{w}', \vec{u}', \vec{v}') = -\vec{T}(\vec{u}', \vec{w}', \vec{v}'). \tag{3.7.13}$$

Let $\vec{T}$ denote a $(p, 0)$ tensor, let $\vec{w}'_1, \ldots, \vec{w}'_p$ be physical covectors, and let σ denote a permutation of the integers $1, \ldots, p$. Then, we define the *σ-permutation* $\vec{T}_\sigma$ of $\vec{T}$ by

$$\vec{T}_\sigma(\vec{w}'_1, \ldots, \vec{w}'_p) \triangleq \vec{T}(\vec{w}'_{\sigma(1)}, \ldots, \vec{w}'_{\sigma(p)}). \tag{3.7.14}$$

The parity $\mathrm{sign}(\sigma)$ of σ is either 1 or -1 depending on whether the number of transpositions of $\sigma(1), \ldots, \sigma(p)$ needed to reach $1, \ldots, p$ is even or odd, respectively. It is useful to note that, if $\vec{T} = \vec{x}_1 \otimes \cdots \otimes \vec{x}_p$, then $\vec{T}_\sigma = \vec{x}_{\sigma(1)} \otimes \cdots \otimes \vec{x}_{\sigma(p)}$.

The $(p, 0)$ tensor $\vec{T}$ is an *alternating tensor* if, for every permutation σ of the integers $1, \ldots, p$,

$$\vec{T}_\sigma = \mathrm{sign}(\sigma)\vec{T}. \tag{3.7.15}$$

Consequently, if $\vec{T}$ is an alternating tensor, then, for all physical covectors $\vec{w}'_1, \ldots, \vec{w}'_p$, it follows that

$$\vec{T}(\vec{w}'_{\sigma(1)}, \ldots, \vec{w}'_{\sigma(p)}) = \mathrm{sign}(\sigma)\vec{T}(\vec{w}'_1, \ldots, \vec{w}'_p). \tag{3.7.16}$$

Hence the sign of $\vec{T}(\vec{w}'_1, \ldots, \vec{w}'_p)$ changes whenever any two of its arguments are interchanged. As a special case of this definition, every physical vector is an alternating tensor.

The wedge product can be used to construct alternating tensors from each pair of alternating tensors. Let $\vec{T}_1$ and $\vec{T}_2$ be alternating tensors of orders $(p_1, 0)$ and $(p_2, 0)$, respectively. Then,

$$\vec{T}_1 \wedge \vec{T}_2 \triangleq \frac{1}{p_1!p_2!} \sum \mathrm{sign}(\sigma)(\vec{T}_1 \otimes \vec{T}_2)_\sigma, \tag{3.7.17}$$

where the summation is taken over all permutations σ of $1, \ldots, p_1 + p_2$.

To illustrate (3.7.17), let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be physical vectors, and define $\vec{T}_1 = \vec{x}$ and $\vec{T}_2 = \vec{y} \wedge \vec{z}$. Then,

$$\begin{aligned}
\vec{x} \wedge (\vec{y} \wedge \vec{z}) &= \vec{T}_1 \wedge \vec{T}_2 \\
&= \frac{1}{2} \sum \mathrm{sign}(\sigma)[\vec{x} \otimes (\vec{y} \otimes \vec{z} - \vec{z} \otimes \vec{y})]_\sigma \\
&= \frac{1}{2} \sum \mathrm{sign}(\sigma)[\vec{x} \otimes \vec{y} \otimes \vec{z} - \vec{x} \otimes \vec{z} \otimes \vec{y}]_\sigma \\
&= \frac{1}{2}[\vec{x} \otimes \vec{y} \otimes \vec{z} - \vec{x} \otimes \vec{z} \otimes \vec{y} - (\vec{y} \otimes \vec{x} \otimes \vec{z} - \vec{z} \otimes \vec{x} \otimes \vec{y}) \\
&\quad - (\vec{z} \otimes \vec{y} \otimes \vec{x} - \vec{y} \otimes \vec{z} \otimes \vec{x}) - (\vec{x} \otimes \vec{z} \otimes \vec{y} - \vec{x} \otimes \vec{y} \otimes \vec{z}) \\
&\quad + \vec{y} \otimes \vec{z} \otimes \vec{x} - \vec{z} \otimes \vec{y} \otimes \vec{x} + \vec{z} \otimes \vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x} \otimes \vec{z}] \\
&= \vec{x} \wedge \vec{y} \wedge \vec{z}.
\end{aligned}$$

Hence the wedge product is associative for physical vectors, and we can write $\vec{x} \wedge \vec{y} \wedge \vec{z}$ without ambiguity. However, the cross product is not associative. Likewise, for alternating tensors $\vec{T}_1$, $\vec{T}_2$, and $\vec{T}_3$ of orders $(p_1, 0)$, $(p_2, 0)$, and $(p_3, 0)$, respectively, it follows that

$$\vec{T}_1 \wedge (\vec{T}_2 \wedge \vec{T}_3) = (\vec{T}_1 \wedge \vec{T}_2) \wedge \vec{T}_3. \tag{3.7.18}$$

The wedge product is thus associative for tensors, and we can write $\vec{T}_1 \wedge \vec{T}_2 \wedge \vec{T}_3$. In fact,

$$\vec{T}_1 \wedge \vec{T}_2 \wedge \vec{T}_3 = \frac{1}{p_1!p_2!p_3!} \sum \mathrm{sign}(\sigma)(\vec{T}_1 \otimes \vec{T}_2 \otimes \vec{T}_3)_\sigma. \tag{3.7.19}$$

For details, see [2, p. 260], [3, p. 278].

The following observation is useful.

Fact 3.7.3. Let $\vec{x}$ and $\vec{y}$ be nonzero physical vectors. Then, $\vec{x}$ and $\vec{y}$ are colinear if and only if $\vec{x} \wedge \vec{y} = 0$.

The following result shows that wedge products of three physical vectors are closely related to determinants.

Fact 3.7.4. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be nonzero physical vectors, and let $\mathrm{F_A}$ be a frame. Then

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = \det\begin{bmatrix} \vec{x}\big|_\mathrm{A} & \vec{y}\big|_\mathrm{A} & \vec{z}\big|_\mathrm{A} \end{bmatrix} \hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A} \wedge \hat{k}_\mathrm{A} \tag{3.7.20}$$

Consequently, $\vec{x}$, $\vec{y}$, and $\vec{z}$ are linearly dependent if and only if $\vec{x} \wedge \vec{y} \wedge \vec{z} = 0$.

Note that (3.7.20) can be written as

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = (\vec{x} \times \vec{y})' \vec{z} (\hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A} \wedge \hat{k}_\mathrm{A}). \tag{3.7.21}$$

Fact 3.7.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then

$$\hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A} \wedge \hat{k}_\mathrm{A} = \hat{\imath}_\mathrm{B} \wedge \hat{\jmath}_\mathrm{B} \wedge \hat{k}_\mathrm{B}. \tag{3.7.22}$$

Proof. Let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be such that $(\vec{x} \times \vec{y})' \vec{z} \neq 0$. Then, it follows from (3.7.21) that $\hat{\imath}_\mathrm{A} \wedge \hat{\jmath}_\mathrm{A} \wedge \hat{k}_\mathrm{A} = \frac{1}{(\vec{x}\times\vec{y})'\vec{z}} \vec{x} \wedge \vec{y} \wedge \vec{z}$. Likewise, $\hat{\imath}_\mathrm{B} \wedge \hat{\jmath}_\mathrm{B} \wedge \hat{k}_\mathrm{B} = \frac{1}{(\vec{x}\times\vec{y})'\vec{z}} \vec{x} \wedge \vec{y} \wedge \vec{z}$. Hence (3.7.22) is satisfied. □

The quantity $\vec{x} \wedge \vec{y} \wedge \vec{z}$ is called a *trivector*. The trivector $\vec{x} \wedge \vec{y} \wedge \vec{z}$ can be visualized as a parallelepiped three of whose edges are $\vec{x}$, $\vec{y}$, $\vec{z}$. This parallelepiped is constructed by sweeping the bivector $\vec{x} \wedge \vec{y}$ along $\vec{z}$. The magnitude of $\vec{x} \wedge \vec{y} \wedge \vec{z}$ is given by its volume

$$|\vec{x} \wedge \vec{y} \wedge \vec{z}| \triangleq \left|\det\begin{bmatrix} \vec{x}\big|_\mathrm{A} & \vec{y}\big|_\mathrm{A} & \vec{z}\big|_\mathrm{A} \end{bmatrix}\right|, \tag{3.7.23}$$

while its orientation is given by the sign of the determinant in (3.7.23).

Let $x, y, z \in \mathbb{R}^3$. Then, we define

$$x \wedge y \wedge z \triangleq x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y - x \otimes z \otimes y - y \otimes x \otimes z - z \otimes y \otimes x. \tag{3.7.24}$$

Fact 3.7.6. Let $x, y, z \in \mathbb{R}^3$, and let e_1, e_2, e_3 denote the columns of I_3. Then,

$$x \wedge y \wedge z = \det\,[x \;\; y \;\; z](e_1 \wedge e_2 \wedge e_3). \tag{3.7.25}$$

Therefore,

$$|\det[x\ \ y\ \ z]| = \frac{1}{\sqrt{6}}\|x \wedge y \wedge z\|. \tag{3.7.26}$$

3.7.3 Bicovectors, Tricovectors, and Forms

We define the *physical bicovector*

$$\vec{x}' \wedge \vec{y}' \triangleq \vec{x}' \otimes \vec{y}' - \vec{y}' \otimes \vec{x}'. \tag{3.7.27}$$

Note that

$$\vec{x}' \wedge \vec{y}' = (\vec{y} \otimes \vec{x} - \vec{x} \otimes \vec{y})' = (\vec{y} \wedge \vec{x})' = -(\vec{x} \wedge \vec{y})'. \tag{3.7.28}$$

Likewise, we define the *physical tricovector*

$$\begin{aligned}\vec{x}' \wedge \vec{y}' \wedge \vec{z}' \\ \triangleq \vec{x}' \otimes \vec{y}' \otimes \vec{z}' + \vec{y}' \otimes \vec{z}' \otimes \vec{x}' + \vec{z}' \otimes \vec{x}' \otimes \vec{y}' - \vec{z}' \otimes \vec{y}' \otimes \vec{x}' - \vec{x}' \otimes \vec{z}' \otimes \vec{y}' - \vec{y}' \otimes \vec{x}' \otimes \vec{z}'\end{aligned} \tag{3.7.29}$$

Note that

$$\begin{aligned}\vec{x}' \wedge \vec{y}' \wedge \vec{z}' &= (\vec{z} \otimes \vec{y} \otimes \vec{x} + \vec{x} \otimes \vec{z} \otimes \vec{y} + \vec{y} \otimes \vec{x} \otimes \vec{z} - \vec{x} \otimes \vec{y} \otimes \vec{z} - \vec{y} \otimes \vec{z} \otimes \vec{x} - \vec{z} \otimes \vec{x} \otimes \vec{y})' \\ &= -(\vec{x} \otimes \vec{y} \otimes \vec{z} + \vec{y} \otimes \vec{z} \otimes \vec{x} + \vec{z} \otimes \vec{x} \otimes \vec{y} - \vec{z} \otimes \vec{y} \otimes \vec{x} - \vec{x} \otimes \vec{z} \otimes \vec{y} - \vec{y} \otimes \vec{x} \otimes \vec{z})' \\ &= -(\vec{x} \wedge \vec{y} \wedge \vec{z})'.\end{aligned} \tag{3.7.30}$$

A *0-form* $\vec{\varphi}$ is a real-valued function $\vec{\varphi}\colon\ \mathcal{V} \to \mathbb{R}$. A 0-form is also called a *scalar field*.

A *1-form* $\vec{\varphi}$ is a mapping from the physical vectors to the set $\hat{\mathfrak{T}}_{(0,1)}$ of alternating covariant tensors, that is, $\vec{\varphi}\colon\ \mathcal{V} \to \hat{\mathfrak{T}}_{(0,1)}$. In particular, if $\vec{\varphi}$ is a 1-form, then, given the frame $\mathrm{F_A}$, there exist real-valued functions $\varphi_1, \varphi_2, \varphi_3$ on $\mathcal{V}$ such that

$$\vec{\varphi}(\vec{x}) = \varphi_1(\vec{x})\hat{\imath}'_{\mathrm{A}} + \varphi_2(\vec{x})\hat{\jmath}'_{\mathrm{A}} + \varphi_3(\vec{x})\hat{k}'_{\mathrm{A}}. \tag{3.7.31}$$

Note that every physical covector is a 1-form. A 1-form is also called a *covector field*.

A *2-form* $\vec{\varphi}$ is a mapping from the physical vectors to the set $\hat{\mathfrak{T}}_{(0,2)}$ of alternating covariant tensors, that is, $\vec{\varphi}\colon\ \mathcal{V} \to \hat{\mathfrak{T}}_{(0,2)}$. In particular, if $\vec{\varphi}$ is a 2-form, then, given the frame $\mathrm{F_A}$, there exist real-valued functions $\varphi_1, \varphi_2, \varphi_3$ on $\mathcal{V}$ such that

$$\vec{\varphi}(\vec{x}) = \varphi_1(\vec{x})\hat{\imath}'_{\mathrm{A}} \wedge \hat{\jmath}'_{\mathrm{A}} + \varphi_2(\vec{x})\hat{\jmath}'_{\mathrm{A}} \wedge \hat{k}'_{\mathrm{A}} + \varphi_3(\vec{x})\hat{k}'_{\mathrm{A}} \wedge \hat{\imath}'_{\mathrm{A}}. \tag{3.7.32}$$

Note that every physical bicovector is a 2-form. A 2-form is also called a *bicovector field*.

Finally, a *3-form* $\vec{\varphi}$ is a mapping from the physical vectors to the set $\hat{\mathfrak{T}}_{(0,3)}$ of alternating covariant tensors, that is, $\vec{\varphi}\colon\ \mathcal{V} \to \hat{\mathfrak{T}}_{(0,3)}$. In particular, if $\vec{\varphi}$ is a 3-form, then, given the frame $\mathrm{F_A}$, there exists a real-valued function φ_1 on $\mathcal{V}$ such that

$$\vec{\varphi}(\vec{x}) = \varphi_1(\vec{x})\hat{\imath}'_{\mathrm{A}} \wedge \hat{\jmath}'_{\mathrm{A}} \wedge \hat{k}'_{\mathrm{A}}. \tag{3.7.33}$$

Note that every physical tricovector is a 3-form. A 3-form is also called a *tricovector field*.

3.8 Multivectors

The real scalars α, vectors $\vec{x}$, bivectors $\vec{x} \wedge \vec{y}$, and trivectors $\vec{x} \wedge \vec{y} \wedge \vec{z}$ can be used to represent 0-, 1-, 2-, and 3-dimensional objects in 3-dimensional space. Building on the basis vectors $\hat{\imath}_{\mathrm{A}}$, $\hat{\jmath}_{\mathrm{A}}$, $\hat{k}_{\mathrm{A}}$ of the frame $\mathrm{F_A}$, we can view 3-dimensional space as an 8-dimensional vector space spanned by $1, \hat{\imath}_{\mathrm{A}}, \hat{\jmath}_{\mathrm{A}}, \hat{k}_{\mathrm{A}}, \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}, \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}, \hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$, and $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$. The elements of this space are *multivectors* $\vec{\vec{S}}$, which have the form

$$\vec{\vec{S}} = \alpha + \beta\hat{\imath}_{\mathrm{A}} + \gamma\hat{\jmath}_{\mathrm{A}} + \delta\hat{k}_{\mathrm{A}} + \varepsilon\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} + \phi\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} + \psi\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}} + \rho\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}, \tag{3.8.1}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \psi, \rho$ are real numbers. The trivector

$$\vec{\mathfrak{I}} \triangleq \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$$

is called the *pseudoscalar*. Fact 3.7.5 shows that $\vec{\mathfrak{I}}$ is independent of the choice of frame. In terms of $\vec{\mathfrak{I}}$, (3.7.20) can be rewritten as

$$\vec{x} \wedge \vec{y} \wedge \vec{z} = \det\begin{bmatrix} \vec{x}\big|_{\mathrm{A}} & \vec{y}\big|_{\mathrm{A}} & \vec{z}\big|_{\mathrm{A}} \end{bmatrix} \vec{\mathfrak{I}}. \tag{3.8.2}$$

Multivectors can be multiplied by introducing a suitable multiplication operation. For example, the *geometric product* of the vectors $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x}\vec{y} \triangleq \vec{x} \cdot \vec{y} + \vec{x} \wedge \vec{y}, \tag{3.8.3}$$

which can be rewritten as

$$\vec{x}\vec{y} = \vec{x} \cdot \vec{y} + \vec{x} \otimes \vec{y} - \vec{y} \otimes \vec{x}. \tag{3.8.4}$$

Note that $\vec{x}\vec{y}$ is a multivector since it is a linear combination of a scalar and a bivector. In particular,

$$\vec{x}^{\,2} \triangleq \vec{x}\vec{x} = \vec{x} \cdot \vec{x} \tag{3.8.5}$$

is a scalar.

Since $\vec{y} \wedge \vec{x} = -\vec{x} \wedge \vec{y}$, it follows that

$$\vec{y}\vec{x} = \vec{x} \cdot \vec{y} - \vec{x} \wedge \vec{y}. \tag{3.8.6}$$

Hence,

$$\vec{x}\vec{y} + \vec{y}\vec{x} = 2\vec{x} \cdot \vec{y}, \tag{3.8.7}$$

$$\vec{x}\vec{y} - \vec{y}\vec{x} = 2\vec{x} \wedge \vec{y}. \tag{3.8.8}$$

Therefore,

$$\vec{x} \cdot \vec{y} = \tfrac{1}{2}(\vec{x}\vec{y} + \vec{y}\vec{x}), \tag{3.8.9}$$

$$\vec{x} \wedge \vec{y} = \tfrac{1}{2}(\vec{x}\vec{y} - \vec{y}\vec{x}). \tag{3.8.10}$$

Note that, if $\vec{x}$ and $\vec{y}$ are mutually orthogonal, then

$$\vec{x}\vec{y} = -\vec{x}\vec{y} = \vec{x} \wedge \vec{y}. \tag{3.8.11}$$

The geometric product can be applied to an arbitrary collection of multivectors. This product is associative, that is, $\overset{\twoheadrightarrow}{S}_1(\overset{\twoheadrightarrow}{S}_2\overset{\twoheadrightarrow}{S}_3) = (\overset{\twoheadrightarrow}{S}_1\overset{\twoheadrightarrow}{S}_2)\overset{\twoheadrightarrow}{S}_3$, and thus can be written as $\overset{\twoheadrightarrow}{S}_1\overset{\twoheadrightarrow}{S}_2\overset{\twoheadrightarrow}{S}_3$. In particular, the geometric product of three physical vectors is given by

$$\vec{x}\vec{y}\vec{z} = (\vec{y}\cdot\vec{z})\vec{x} - (\vec{x}\cdot\vec{z})\vec{y} + (\vec{x}\cdot\vec{y})\vec{z} + \vec{x}\wedge\vec{y}\wedge\vec{z}. \tag{3.8.12}$$

We thus have the identities

$$\vec{x}\vec{y}\vec{x} = 2(\vec{x}\cdot\vec{y})\vec{x} - (\vec{x}\cdot\vec{x})^2\vec{y}, \tag{3.8.13}$$

$$\vec{x}\vec{y}\vec{z} + \vec{y}\vec{x}\vec{z} = 2(\vec{x}\cdot\vec{y})\vec{z}, \tag{3.8.14}$$

$$\vec{x}\vec{y}\vec{z} + \vec{z}\vec{y}\vec{x} = 2[(\vec{y}\cdot\vec{z})\vec{x} - (\vec{x}\cdot\vec{z})\vec{y} + (\vec{x}\cdot\vec{y})\vec{z}], \tag{3.8.15}$$

$$\vec{x}\vec{y}\vec{z} - \vec{y}\vec{z}\vec{x} = 2[(\vec{x}\cdot\vec{y})\vec{z} - (\vec{x}\cdot\vec{z})\vec{y}]. \tag{3.8.16}$$

Furthermore,

$$\begin{aligned} \vec{x}(\vec{y}\wedge\vec{z}) &= \tfrac{1}{2}(\vec{x}\vec{y}\vec{z} - \vec{x}\vec{z}\vec{y}) && (3.8.17)\\ &= (\vec{x}\cdot\vec{y})\vec{z} - (\vec{x}\cdot\vec{z})\vec{y} + \tfrac{1}{2}(\vec{z}\vec{x}\vec{y} - \vec{y}\vec{x}\vec{z}), && (3.8.18) \end{aligned}$$

$$\begin{aligned} (\vec{x}\wedge\vec{y})\vec{z} &= \tfrac{1}{2}(\vec{x}\vec{y}\vec{z} - \vec{y}\vec{x}\vec{z}) && (3.8.19)\\ &= (\vec{y}\cdot\vec{z})\vec{x} - (\vec{x}\cdot\vec{z})\vec{y} + \tfrac{1}{2}(\vec{y}\vec{z}\vec{x} - \vec{x}\vec{z}\vec{y}), && (3.8.20) \end{aligned}$$

$$\vec{x}(\vec{y}\wedge\vec{z}) - (\vec{y}\wedge\vec{z})\vec{x} = 2(\vec{x}\cdot\vec{y})\vec{z} - 2(\vec{x}\cdot\vec{z})\vec{y}, \tag{3.8.21}$$

$$\begin{aligned} \vec{x}\wedge\vec{y}\wedge\vec{z} &= \tfrac{1}{2}[\vec{x}(\vec{y}\wedge\vec{z}) + (\vec{y}\wedge\vec{z})\vec{x}] && (3.8.22)\\ &= \tfrac{1}{2}(\vec{x}\vec{y}\vec{z} - \vec{z}\vec{y}\vec{x}) && (3.8.23)\\ &= \tfrac{1}{4}(\vec{x}\vec{y}\vec{z} + \vec{y}\vec{z}\vec{x} - \vec{x}\vec{z}\vec{y} - \vec{z}\vec{y}\vec{x}) && (3.8.24)\\ &= \tfrac{1}{6}(\vec{x}\vec{y}\vec{z} + \vec{y}\vec{z}\vec{x} + \vec{z}\vec{x}\vec{y} - \vec{x}\vec{z}\vec{y} - \vec{y}\vec{x}\vec{z} - \vec{z}\vec{y}\vec{z}). && (3.8.25) \end{aligned}$$

For comparison, recall that (3.7.11) states that

$$\vec{x}\wedge\vec{y}\wedge\vec{z} = \vec{x}\otimes\vec{y}\otimes\vec{z} + \vec{y}\otimes\vec{z}\otimes\vec{x} + \vec{z}\otimes\vec{x}\otimes\vec{y} - \vec{x}\otimes\vec{z}\otimes\vec{y} - \vec{y}\otimes\vec{x}\otimes\vec{z} - \vec{z}\otimes\vec{y}\otimes\vec{x}. \tag{3.8.26}$$

Moreover,

$$\vec{\mathfrak{I}}^2 = -1. \tag{3.8.27}$$

For the case of four physical vectors, we have

$$\vec{x}\wedge\vec{y}\wedge\vec{z}\wedge\vec{w} = 0, \tag{3.8.28}$$

$$\begin{aligned} \vec{x}\vec{y}\vec{z}\vec{w} = {}& (\vec{x}\cdot\vec{y})(\vec{z}\cdot\vec{w}) - (\vec{x}\cdot\vec{z})(\vec{y}\cdot\vec{w}) + (\vec{x}\cdot\vec{w})(\vec{y}\cdot\vec{z})\\ &+ (\vec{z}\cdot\vec{w})\vec{x}\wedge\vec{y} - (\vec{y}\cdot\vec{w})\vec{x}\wedge\vec{z} + (\vec{y}\cdot\vec{z})\vec{x}\wedge\vec{w}\\ &+ (\vec{x}\cdot\vec{w})\vec{y}\wedge\vec{z} - (\vec{x}\cdot\vec{z})\vec{y}\wedge\vec{w} + (\vec{x}\cdot\vec{y})\vec{z}\wedge\vec{w}. \end{aligned} \tag{3.8.29}$$

Hence,

$$\vec{x}\vec{y}\vec{x}\vec{y} = 2(\vec{x}\cdot\vec{y})^2 - (\vec{x}\cdot\vec{x})\vec{y}\cdot\vec{y} + 2(\vec{x}\cdot\vec{y})\vec{x}\wedge\vec{y}. \tag{3.8.30}$$

Furthermore,

$$(\vec{x}\wedge\vec{y})(\vec{z}\wedge\vec{w}) = (\vec{x}\cdot\vec{y} - \vec{x}\vec{y})(\vec{z}\cdot\vec{w} - \vec{z}\vec{w}). \tag{3.8.31}$$

In particular,

$$(\vec{x}\wedge\vec{y})^2 = (\vec{x}\cdot\vec{y})^2 - (\vec{x}\cdot\vec{x})\vec{y}\cdot\vec{y}, \tag{3.8.32}$$

which is a scalar. Finally, for each frame $\mathrm{F_A}$,

$$\vec{\mathfrak{I}}\vec{x} = (\hat{\imath}_\mathrm{A}\cdot\vec{x})\hat{\jmath}_\mathrm{A}\wedge\hat{k}_\mathrm{A} + (\hat{\jmath}_\mathrm{A}\cdot\vec{x})\hat{k}_\mathrm{A}\wedge\hat{\imath}_\mathrm{A} + (\hat{k}_\mathrm{A}\cdot\vec{x})\hat{\imath}_\mathrm{A}\wedge\hat{\jmath}_\mathrm{A}. \tag{3.8.33}$$

The following equalities connect the geometric product with the cross product.

Fact 3.8.1. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then,

$$\vec{x}\times\vec{y} = -\vec{\mathfrak{I}}(\vec{x}\wedge\vec{y}), \tag{3.8.34}$$

$$\vec{x}\wedge\vec{y} = \vec{\mathfrak{I}}(\vec{x}\times\vec{y}). \tag{3.8.35}$$

Furthermore,

$$\vec{x}\wedge\vec{y} = \begin{bmatrix}\hat{\imath}_\mathrm{A}\wedge\hat{\jmath}_\mathrm{A} & \hat{\jmath}_\mathrm{A}\wedge\hat{k}_\mathrm{A} & \hat{k}_\mathrm{A}\wedge\hat{\imath}_\mathrm{A}\end{bmatrix}(\vec{x}\times\vec{y})\Big|_\mathrm{A}. \tag{3.8.36}$$

Moreover,

$$\vec{x}\times\vec{y} = \tfrac{1}{2}(\vec{y}\vec{\mathfrak{I}}\vec{x} - \vec{\mathfrak{I}}\vec{x}\vec{y}). \tag{3.8.37}$$

Finally, let $\vec{z}$ be a physical vector. Then

$$\vec{x}\cdot(\vec{y}\times\vec{z}) = \det\begin{bmatrix}\vec{x}\Big|_\mathrm{A} & \vec{y}\Big|_\mathrm{A} & \vec{z}\Big|_\mathrm{A}\end{bmatrix} = -(\vec{x}\wedge\vec{y}\wedge\vec{z})\vec{\mathfrak{I}}, \tag{3.8.38}$$

$$\vec{x}\times(\vec{y}\times\vec{z}) = \tfrac{1}{2}[(\vec{y}\wedge\vec{z})\vec{x} - \vec{x}(\vec{y}\wedge\vec{z})]. \tag{3.8.39}$$

Proof. Note that

$$\begin{aligned}
-\vec{\mathfrak{I}}(\vec{x}\wedge\vec{y}) &= -\vec{\mathfrak{I}}[(x_1\hat{\imath}_\mathrm{A} + x_2\hat{\jmath}_\mathrm{A} + x_3\hat{k}_\mathrm{A})\wedge(y_1\hat{\imath}_\mathrm{A} + y_2\hat{\jmath}_\mathrm{A} + y_3\hat{k}_\mathrm{A})]\\
&= -\vec{\mathfrak{I}}[(x_1y_2 - y_1x_2)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A} + (x_1y_3 - y_1x_3)\hat{\imath}_\mathrm{A}\hat{k}_\mathrm{A} + (x_2y_3 - y_2x_3)\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}]\\
&= (y_1x_2 - x_1y_2)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A} + (y_1x_3 - x_1y_3)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\hat{\imath}_\mathrm{A}\hat{k}_\mathrm{A} + (y_2x_3 - x_2y_3)\hat{\imath}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\hat{\jmath}_\mathrm{A}\hat{k}_\mathrm{A}\\
&= -(y_1x_2 - x_1y_2)\hat{k}_\mathrm{A} + (y_1x_3 - x_1y_3)\hat{\jmath}_\mathrm{A} - (y_2x_3 - x_2y_3)\hat{\imath}_\mathrm{A}\\
&= (x_2y_3 - y_2x_3)\hat{\imath}_\mathrm{A} + (y_1x_3 - x_1y_3)\hat{\jmath}_\mathrm{A} + (x_1y_2 - y_1x_2)\hat{k}_\mathrm{A}\\
&= \vec{x}\times\vec{y}.
\end{aligned}$$

□

The matrix in (3.8.36) is called a *bivectrix*.

Let $\mathrm{F_A}$ be a frame, and consider the multivector $\overset{\twoheadrightarrow}{S}$

$$\overset{\twoheadrightarrow}{S} = \alpha + \beta \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}, \tag{3.8.40}$$

where $\overset{\twoheadrightarrow}{S}$ is a linear combination of a scalar and a bivector. Since $\hat{\imath}_{\mathrm{A}} \cdot \hat{\jmath}_{\mathrm{A}} = 0$, it follows that $\hat{\imath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}} = \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} = -\hat{\jmath}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}} = -\hat{\jmath}_{\mathrm{A}} \hat{\imath}_{\mathrm{A}}$. Therefore, since $\hat{\imath}_{\mathrm{A}} \hat{\imath}_{\mathrm{A}} = \hat{\jmath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}} = 1$, it follows that

$$(\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}})^2 = (\hat{\imath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}})^2 = \hat{\imath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}} \hat{\imath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}} = -\hat{\imath}_{\mathrm{A}} \hat{\imath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}} \hat{\jmath}_{\mathrm{A}} = -1. \tag{3.8.41}$$

Therefore, $\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$ behaves like $\jmath = \sqrt{-1}$. It is therefore useful to define the conjugate $\overline{\overset{\twoheadrightarrow}{S}}$ of $\overset{\twoheadrightarrow}{S}$ as

$$\overline{\overset{\twoheadrightarrow}{S}} = \alpha - \beta \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}. \tag{3.8.42}$$

Therefore,

$$\overline{\overset{\twoheadrightarrow}{S}}\overset{\twoheadrightarrow}{S} = \overset{\twoheadrightarrow}{S}\overline{\overset{\twoheadrightarrow}{S}} = \alpha^2 + \beta^2. \tag{3.8.43}$$

More generally, the conjugate of $\overset{\twoheadrightarrow}{S}$ defined by (3.8.1) is given by

$$\overline{\overset{\twoheadrightarrow}{S}} \triangleq \alpha - \beta \hat{\imath}_{\mathrm{A}} - \gamma \hat{\jmath}_{\mathrm{A}} - \delta \hat{k}_{\mathrm{A}} - \varepsilon \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} - \phi \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}} - \psi \hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}} - \rho \hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}. \tag{3.8.44}$$

As in the case of the quaternions, the basis multivectors can be represented by various objects, such as complex and real matrices. It is convenient to define the complex 2×2 *σ-matrices*

$$\sigma_1 = \jmath\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \jmath\tau_2 = \begin{bmatrix} 0 & -\jmath \\ \jmath & 0 \end{bmatrix}, \quad \sigma_3 = \jmath\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.8.45}$$

which satisfy

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2, \tag{3.8.46}$$

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \jmath\sigma_3 = -\tau_3 = \begin{bmatrix} \jmath & 0 \\ 0 & -\jmath \end{bmatrix}, \tag{3.8.47}$$

$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = \jmath\sigma_1 = -\tau_1 = \begin{bmatrix} 0 & \jmath \\ \jmath & 0 \end{bmatrix}, \tag{3.8.48}$$

$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = \jmath\sigma_2 = -\tau_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.8.49}$$

$$\sigma_1\sigma_2\sigma_3 = \sigma_2\sigma_3\sigma_1 = \sigma_3\sigma_1\sigma_2 = \jmath I_2, \tag{3.8.50}$$

as well as the real 4×4 g-matrices

$$g_1 \triangleq \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad g_2 \triangleq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad g_3 \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{3.8.51}$$

which satisfy

$$g_1^2 = g_2^2 = g_3^2 = I_4, \tag{3.8.52}$$

$$g_1g_2 = -g_2g_1 = -f_3, \quad g_2g_3 = -g_3g_2 = -f_1, \quad g_3g_1 = -g_1g_3 = -f_2, \tag{3.8.53}$$

$$g_4 \triangleq g_1 g_2 g_3 = g_2 g_3 g_1 = g_3 g_1 g_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{3.8.54}$$

Multiplication tables involving two multivectors are given in Table 3.8.1 along with an equivalent table involving two complex scalars. These tables are equivalent to the tables in Figure 2.18.1.

A multiplication table involving three bivectors is given in Table 3.8.2 along with an equivalent multiplication table involving the quaternions.

A multiplication table involving four multivectors is given in Table 3.8.3 along with equivalent multiplication tables involving 2×2 complex matrices and 4×4 real matrices.

	1	$\hat{\imath}_A \wedge \hat{\jmath}_A$
1	1	$\hat{\imath}_A \wedge \hat{\jmath}_A$
$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$	-1

(a)

	1	$\hat{\imath}_A \wedge \hat{\jmath}_A \wedge \hat{k}_A$
1	1	$\hat{\imath}_A \wedge \hat{\jmath}_A \wedge \hat{k}_A$
$\hat{\imath}_A \wedge \hat{\jmath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A \wedge \hat{k}_A$	-1

(b)

	1	$\jmath$
1	1	$\jmath$
$\jmath$	$\jmath$	-1

(c)

Table 3.8.1: Equivalent multiplication tables for (a), (b) two multivectors, and (c) complex scalars. These tables are equivalent to the tables in Figure 2.18.1.

	1	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{k}_A \wedge \hat{\imath}_A$
1	1	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{k}_A \wedge \hat{\imath}_A$
$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$	-1	$\hat{k}_A \wedge \hat{\imath}_A$	$-\hat{\imath}_A \wedge \hat{k}_A$
$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{k}_A \wedge \hat{\imath}_A$	-1	$-\hat{\imath}_A \wedge \hat{\jmath}_A$
$\hat{k}_A \wedge \hat{\imath}_A$	$\hat{k}_A \wedge \hat{\imath}_A$	$-\hat{\jmath}_A \wedge \hat{k}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$	-1

(a)

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

(b)

Table 3.8.2: Equivalent multiplication tables involving (a) bivectors and (b) quaternions. Additional equivalent multiplication tables are given in Figure 3.8.3.

	1	$\hat{\imath}_A$	$\hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$
1	1	$\hat{\imath}_A$	$\hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$
$\hat{\imath}_A$	$\hat{\imath}_A$	1	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\jmath}_A$
$\hat{\jmath}_A$	$\hat{\jmath}_A$	$-\hat{\imath}_A \wedge \hat{\jmath}_A$	1	$-\hat{\imath}_A$
$\hat{\imath}_A \wedge \hat{\jmath}_A$	$\hat{\imath}_A \wedge \hat{\jmath}_A$	$-\hat{\jmath}_A$	$\hat{\imath}_A$	-1

(a)

	I_2	σ_1	σ_2	$-\tau_3$
I_2	I_2	σ_1	σ_2	$-\tau_3$
σ_1	σ_1	I_2	$-\tau_3$	σ_2
σ_2	σ_2	τ_3	I_2	$-\sigma_1$
$-\tau_3$	$-\tau_3$	$-\sigma_2$	σ_1	$-I_2$

(b)

	I_4	g_1	g_2	$-f_3$
I_4	I_4	g_1	g_2	$-f_3$
g_1	g_1	I_4	$-f_3$	g_2
g_2	g_2	f_3	I_4	$-g_1$
$-f_3$	$-f_3$	$-g_2$	g_1	$-I_4$

(c)

Table 3.8.3: Equivalent multiplication tables involving (a) multivectors, (b) σ- and τ-matrices, and (c) g- and f-matrices.

	1	$\hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\vec{\mathcal{I}}$
1	1	$\hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\vec{\mathcal{I}}$
$\hat{\imath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}}$	1	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\vec{\mathcal{I}}$	$-\hat{k}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$
$\hat{\jmath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	1	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\vec{\mathcal{I}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$
$\hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	1	$\vec{\mathcal{I}}$	$-\hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$
$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}}$	$\vec{\mathcal{I}}$	-1	$-\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$-\hat{k}_{\mathrm{A}}$
$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\vec{\mathcal{I}}$	$-\hat{k}_{\mathrm{A}}$	$\hat{\jmath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	-1	$-\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}}$
$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}}$	$\vec{\mathcal{I}}$	$-\hat{\imath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	-1	$-\hat{\jmath}_{\mathrm{A}}$
$\vec{\mathcal{I}}$	$\vec{\mathcal{I}}$	$\hat{\jmath}_{\mathrm{A}} \wedge \hat{k}_{\mathrm{A}}$	$\hat{k}_{\mathrm{A}} \wedge \hat{\imath}_{\mathrm{A}}$	$\hat{\imath}_{\mathrm{A}} \wedge \hat{\jmath}_{\mathrm{A}}$	$-\hat{k}_{\mathrm{A}}$	$-\hat{\imath}_{\mathrm{A}}$	$-\hat{\jmath}_{\mathrm{A}}$	-1

Table 3.8.4: Multiplication table involving multivectors. Additional equivalent multiplication tables are given in Figure 3.8.5.

	I_2	σ_1	σ_2	σ_3	$-\tau_3$	$-\tau_1$	$-\tau_2$	$\jmath I_2$
I_2	I_2	σ_1	σ_2	σ_3	$-\tau_3$	$-\tau_1$	$-\tau_2$	$\jmath I_2$
σ_1	σ_1	I_2	$-\tau_3$	τ_2	σ_2	$\jmath I_2$	$-\sigma_3$	$-\tau_1$
σ_2	σ_2	τ_3	I_2	$-\tau_1$	$-\sigma_1$	$\hat{k}_{\mathrm{A}}$	$\jmath I_2$	$-\tau_2$
σ_3	σ_3	$-\tau_2$	τ_3	I_2	$\jmath I_2$	$-\sigma_2$	σ_1	$-\tau_3$
$-\tau_3$	$-\tau_3$	$-\sigma_2$	σ_1	$\jmath I_2$	$-I_2$	τ_2	$-\tau_1$	$-\sigma_3$
$-\tau_1$	$-\tau_1$	$\jmath I_2$	$-\sigma_3$	σ_2	$-\tau_2$	$-I_2$	τ_3	$-\sigma_1$
$-\tau_2$	$-\tau_2$	σ_3	$\jmath I_2$	$-\sigma_1$	τ_1	$-\tau_3$	$-I_2$	$-\sigma_2$
$\jmath I_2$	$\jmath I_2$	$-\tau_1$	$-\tau_2$	$-\tau_3$	$-\sigma_3$	$-\sigma_1$	$-\sigma_2$	$-I_2$

(a)

	I_4	g_1	g_2	g_3	$-f_3$	$-f_1$	$-f_2$	g_4
I_4	I_4	g_1	g_2	g_3	$-f_3$	$-f_1$	$-f_2$	g_4
g_1	g_1	I_4	$-f_3$	f_2	g_2	g_4	$-g_3$	$-f_1$
g_2	g_2	f_3	I_4	$-f_1$	$-g_1$	$\hat{k}_{\mathrm{A}}$	g_4	$-f_2$
g_3	g_3	$-f_2$	f_3	I_4	g_4	$-g_2$	g_1	$-f_3$
$-f_3$	$-f_3$	$-g_2$	g_1	g_4	$-I_4$	f_2	$-f_1$	$-g_3$
$-f_1$	$-f_1$	g_4	$-g_3$	g_2	$-f_2$	$-I_4$	f_3	$-g_1$
$-f_2$	$-f_2$	g_3	g_4	$-g_1$	f_1	$-f_3$	$-I_4$	$-g_2$
g_4	g_4	$-f_1$	$-f_2$	$-f_3$	$-g_3$	$-g_1$	$-g_2$	$-I_4$

(b)

Table 3.8.5: Equivalent multiplication tables involving (a) σ- and τ-matrices and (b) g- and f-matrices. These multiplication tables are equivalent to the multivector multiplication table in Figure 3.8.4.

3.9 Rotations and Reflections

Let $\hat{v}$ and $\hat{w}$ be orthogonal vectors, and let θ denote an angle. Then, the *rotor* $\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)$ is the multivector

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta) \triangleq \cos\theta/2 - (\sin\theta/2)(\hat{v}\wedge\hat{w}), \tag{3.9.1}$$

which is a linear combination of a scalar and a bivector. Therefore,

$$\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta) = \cos\theta/2 + (\sin\theta/2)(\hat{v}\wedge\hat{w}). \tag{3.9.2}$$

The following result shows that the rotor $\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)$ rotates each vector by the angle θ around the direction orthogonal to the plane spanned by $\hat{v}$ and $\hat{w}$ and in the direction determined by the orientation of the bivector $\hat{v}\wedge\hat{w}$.

Since $(\hat{v}\wedge\hat{w})^2 = -1$, (3.9.1) can be written as

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta) = \exp[-(\hat{v}\wedge\hat{w})\theta/2]. \tag{3.9.3}$$

In other words, (3.9.1) is analogous to Euler's formula $e^{\jmath\theta} = \cos\theta + \jmath\sin\theta$. Furthermore, note that

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta) = \overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta)\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta) = 1. \tag{3.9.4}$$

Fact 3.9.1. Let θ be an angle, let $\hat{v}$ and $\hat{w}$ denote orthogonal vectors, and let $\hat{n} = \hat{v}\times\hat{w}$. Then, for every physical vector $\vec{x}$,

$$\vec{R}_{\hat{n}}(\theta)\vec{x} = \overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta). \tag{3.9.5}$$

Proof. Note that

$$\begin{aligned}
&\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{w}}(\theta)\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\hat{v}\wedge\hat{w}}(\theta)\\
&\quad= [\cos\theta/2 - (\sin\theta/2)(\hat{v}\wedge\hat{w})]\vec{x}[\cos\theta/2 + (\sin\theta/2)(\hat{v}\wedge\hat{w})] \qquad (3.9.6)\\
&\quad= [\cos\theta/2 - (\sin\theta/2)\hat{v}\hat{w}]\vec{x}[\cos\theta/2 + (\sin\theta/2)\hat{v}\hat{w}]\\
&\quad= (\cos^2\theta/2)\vec{x} + \tfrac{1}{2}(\sin\theta)(\hat{x}\hat{v}\hat{w} - \hat{v}\hat{w}\hat{x}) + (\sin^2\theta/2)\hat{v}\hat{w}\hat{x}\hat{w}\hat{v}\\
&\quad= (\cos^2\theta/2)\vec{x} + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}](\sin^2\theta/2)[\vec{x} - 2(\hat{v}\cdot\hat{x})\hat{v} - 2(\hat{w}\cdot\hat{x})\hat{w}]\\
&\quad= \vec{x} + [(\cos\theta) - 1][(\hat{v}\cdot\hat{x})\hat{v} + (\hat{w}\cdot\hat{x})\hat{w}] + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)(\hat{w}\otimes\hat{v}' - \hat{v}\otimes\hat{w}')\otimes_{(2,1)}[(\hat{v}\cdot\vec{x})\hat{w} - (\hat{w}\cdot\vec{x})\hat{v}] + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)(\hat{w}\otimes\hat{v}' - \hat{v}\otimes\hat{w}')^2\otimes_{(2,1)}\vec{x} + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)(\hat{v}\times\hat{w})^{\times 2}\otimes_{(2,1)}\vec{x} + (\sin\theta)[(\hat{v}\cdot\hat{x})\hat{w} - (\hat{w}\cdot\hat{x})\hat{v}]\\
&\quad= \vec{x} + (1-\cos\theta)\hat{n}^{\times 2}\otimes_{(2,1)}\vec{x} + (\sin\theta)(\hat{w}\otimes\hat{v}' - \hat{v}\otimes\hat{w}')\otimes_{(2,1)}\vec{x}\\
&\quad= \vec{x} + (1-\cos\theta)\hat{n}^{\times 2}\otimes_{(2,1)}\vec{x} + (\sin\theta)\hat{n}^{\times}\otimes_{(2,1)}\vec{x}\\
&\quad= \vec{R}_{\hat{n}}(\theta)\otimes_{(2,1)}\vec{x}
\end{aligned}$$

$$= \vec{R}_{\hat{n}}(\theta)\vec{x}. \qquad \square$$

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, we let $\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}$ denote the rotor that corresponds to $\vec{R}_{\mathrm{B/A}}$ in the sense that, for every physical vector $\vec{x}$,

$$\vec{R}_{\mathrm{B/A}}\vec{x} = \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}. \tag{3.9.7}$$

Fact 3.9.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ and $\vec{y}$ be physical vectors. Then

$$\vec{x}\cdot(\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}) = (\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{x}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\cdot\vec{y} \tag{3.9.8}$$

and

$$\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}(\vec{x}\wedge\vec{y})\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}} = (\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{x}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\wedge(\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{y}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}). \tag{3.9.9}$$

Proof. To prove (3.9.8) note that

$$\vec{x}\cdot(\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}) = \vec{x}\cdot(\vec{R}_{\mathrm{B/A}}\vec{y}) = (\vec{R}'_{\mathrm{B/A}}\vec{x})\cdot\vec{y} = (\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{x}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\cdot\vec{y}.$$

Next, to prove (3.9.9) note that

$$\begin{aligned}
\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}(\vec{x}\wedge\vec{y})\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}} &= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}(\vec{x}\vec{y}-\vec{x}\cdot\vec{y})\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\\
&= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\vec{y}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}} - \vec{x}\cdot\vec{y}\\
&= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}} - \vec{x}\cdot\vec{y}\\
&= \overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{x}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}\vec{y}\,\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}} - (\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{x}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\cdot(\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{y}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\\
&= (\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{x}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}})\wedge(\overline{\overset{\twoheadrightarrow}{R}}_{\mathrm{B/A}}\vec{y}\,\overset{\twoheadrightarrow}{R}_{\mathrm{B/A}}). \qquad \square
\end{aligned}$$

Let $\hat{n}$ and $\vec{x}$ be physical vectors. Then, $\vec{x}$ can be written as the sum

$$\vec{x} = \vec{x}_{\mathrm{perp},\hat{n}} + \vec{x}_{\mathrm{par},\hat{n}}, \tag{3.9.10}$$

where $\vec{x}_{\mathrm{perp},\hat{n}}$ is the component of $\vec{x}$ in the plane orthogonal to $\hat{n}$ and $\vec{x}_{\mathrm{par},\hat{n}}$ is the component of $\vec{x}$ in the direction of $\hat{n}$. Furthermore, the reflection of $\vec{x}$ in the plane orthogonal to $\hat{n}$ is defined by

$$\vec{x}_{\mathrm{refl}} \triangleq \vec{x}_{\mathrm{perp},\hat{n}} - \vec{x}_{\mathrm{par},\hat{n}}. \tag{3.9.11}$$

Fact 3.9.3. Let $\hat{n}$ be a physical vector and let $\vec{x}$ be a physical vector. Then,

$$\begin{aligned}
\vec{x}_{\mathrm{perp},\hat{n}} &= \hat{n}(\hat{n}\wedge\vec{x}) &(3.9.12)\\
&= (\hat{n}\cdot\vec{x})\hat{n} - \hat{n}\vec{x}\hat{n} &(3.9.13)\\
&= \tfrac{1}{2}(\vec{x} - \hat{n}\vec{x}\hat{n}) &(3.9.14)\\
&= \vec{x} - (\hat{n}\cdot\vec{x})\hat{n}, &(3.9.15)
\end{aligned}$$

$$\vec{x}_{\mathrm{par},\hat{n}} = (\hat{n} \cdot \vec{x})\hat{n}, \tag{3.9.16}$$

$$\begin{aligned} \vec{x}_{\mathrm{refl},\hat{n}} &= -\hat{n}\vec{x}\hat{n} && (3.9.17)\\ &= \vec{x} - 2(\hat{n} \cdot \vec{x})\hat{n}. && (3.9.18) \end{aligned}$$

Proof. Since $\hat{n}^2 = 1$, it follows that

$$\vec{x} = \hat{n}^2\vec{x} = \hat{n}(\hat{n} \wedge \vec{x} + \hat{n} \cdot \vec{x}) = \vec{x}_{\mathrm{perp},\hat{n}} + \vec{x}_{\mathrm{par},\hat{n}}. \tag{3.9.19}$$

Furthermore, using (3.8.17) and (3.8.19) we have

$$\begin{aligned} \vec{x}_{\mathrm{refl},\hat{n}} &= \hat{n}(\hat{n} \wedge \vec{x}) - (\hat{n} \cdot \vec{x})\hat{n} = -(\hat{n} \wedge \vec{x})\hat{n} - (\hat{n} \cdot \vec{x})\hat{n} = -\hat{n}\vec{x}\hat{n}\\ &= \hat{n}(\hat{n}\vec{x} - \hat{n} \cdot \vec{x}) - (\hat{n} \cdot \vec{x})\hat{n} = \hat{n}^2\vec{x} - 2(\hat{n} \cdot \vec{x})\hat{n} = \vec{x} - 2(\hat{n} \cdot \vec{x})\hat{n}. \qquad \square \end{aligned}$$

The following result shows that the rotation of the vector $\vec{x}$ around the normal to the plane spanned by two physical vectors $\hat{v}$ and $\hat{w}$ through twice the angle between $\hat{v}$ and $\hat{w}$ is equivalent to reflecting $\vec{x}$ successively in the planes orthogonal to $\hat{v}$ and $\hat{w}$. Note that $\hat{v}$ and $\hat{w}$ are not necessarily orthogonal. This result is illustrated in [9, pp. 133–137].

Fact 3.9.4. Let $\hat{v}$ and $\hat{w}$ be physical vectors that are not colinear, and define $\hat{n} \triangleq \frac{1}{|\hat{v}\times\hat{w}|}\hat{v} \times \hat{w}$ and $\hat{z} \triangleq \hat{n} \times \hat{v}$. Then

$$\overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{z}}(2\theta_{\hat{v}/\hat{w}}) = \hat{w}\hat{v}. \tag{3.9.20}$$

Consequently, for every physical vector $\vec{x}$,

$$\vec{R}_{\hat{n}}(2\theta_{\hat{v}/\hat{w}})\vec{x} = \hat{w}\hat{v}\vec{x}\hat{v}\hat{w}. \tag{3.9.21}$$

Proof. Note that

$$(\sin\theta_{\hat{v}/\hat{w}})\hat{z} = \hat{w} - (\cos\theta_{\hat{v}/\hat{w}})\hat{v}.$$

Therefore,

$$\begin{aligned} \overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{z}}(2\theta_{\hat{v}/\hat{w}}) &= \cos\theta_{\hat{v}/\hat{w}} - (\sin\theta_{\hat{v}/\hat{w}})(\hat{v} \wedge \hat{z})\\ &= \cos\theta_{\hat{v}/\hat{w}} - \hat{v} \wedge \hat{w}\\ &= \cos\theta_{\hat{v}/\hat{w}} + \hat{w} \wedge \hat{v}\\ &= \cos\theta_{\hat{v}/\hat{w}} + \hat{w}\hat{v} - \hat{w} \cdot \hat{v}\\ &= \hat{w}\hat{v}. \qquad \square \end{aligned}$$

Note that (3.9.20) implies

$$\begin{aligned} \overset{\twoheadrightarrow}{R}_{\hat{v}\wedge\hat{z}}(2\theta_{\hat{v}/\hat{w}}) &= \hat{w}\hat{v}\\ &= \exp[-(\hat{v} \wedge \hat{w})\theta_{\hat{v}/\hat{w}}/|\hat{v} \times \hat{w}|]\\ &= \exp[-(\hat{v} \wedge \hat{w})\theta_{\hat{v}/\hat{w}}/\sin\theta_{\hat{v}/\hat{w}}]. && (3.9.22) \end{aligned}$$

3.10 Problems

Problem 3.10.1. Let $\vec{T}_1, \vec{T}_2, \vec{T}_3 \in \mathcal{T}_{(p,p)}$, and let $\mathcal{S} = \{(i_1, j_1), \ldots, (i_p, j_p)\}$, where $i_1, \ldots, i_p$ are distinct integers in $\{1, \ldots, 2p\}$ and $j_1, \ldots, j_p$ are distinct integers in $\{1, \ldots, 2p\}$. Show that

$$((\vec{T}_1 \otimes \vec{T}_2)_{\mathcal{S}} \otimes \vec{T}_3)_{\mathcal{S}} = (\vec{T}_1 \otimes (\vec{T}_2 \otimes \vec{T}_3)_{\mathcal{S}})_{\mathcal{S}}.$$

Problem 3.10.2. Let $\vec{x}$, $\vec{y}$, $\vec{z}$, and $\vec{w}$ be physical vectors. Show that

$$(\vec{x} \wedge \vec{y})'(\vec{z}, \vec{w}) = (\vec{x} \times \vec{y}) \cdot (\vec{z} \times \vec{w}) = (\vec{x} \cdot \vec{z})(\vec{y} \cdot \vec{w}) - (\vec{y} \cdot \vec{z})(\vec{x} \cdot \vec{y}).$$

Problem 3.10.3. Let $\vec{T}_1$ and $\vec{T}_2$ be tensors of order $(p_1, 0)$ and $(p_2, 0)$, respectively. Show that

$$\vec{T}_1 \wedge \vec{T}_2 = (-1)^{p_1 p_2} \vec{T}_2 \wedge \vec{T}_1.$$

Problem 3.10.4. Let $\vec{w}'_1$, $\vec{w}'_2$, $\vec{w}'_3$ be physical covectors, and let $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ be physical vectors. Show that

$$(\vec{w}'_1 \wedge \vec{w}'_1 \wedge \vec{w}'_1)(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \det \begin{bmatrix} \vec{w}'_1 \vec{x}_1 & \vec{w}'_1 \vec{x}_2 & \vec{w}'_1 \vec{x}_3 \\ \vec{w}'_2 \vec{x}_1 & \vec{w}'_2 \vec{x}_2 & \vec{w}'_2 \vec{x}_3 \\ \vec{w}'_3 \vec{x}_1 & \vec{w}'_3 \vec{x}_2 & \vec{w}'_3 \vec{x}_3 \end{bmatrix}.$$

Problem 3.10.5. Let x, y be real numbers, let θ be an angle, and define u, v by

$$u + v\jmath = e^{\theta\jmath}(x + y\jmath) = [\cos\theta + (\sin\theta)\jmath](x + y\jmath).$$

Show that

$$u\mathbf{i} + v\mathbf{j} = e^{\frac{1}{2}\theta\mathbf{k}}(x\mathbf{i} + y\mathbf{j})e^{-\frac{1}{2}\theta\mathbf{k}} = [(\cos\tfrac{1}{2}\theta) + (\sin\tfrac{1}{2}\theta)\mathbf{k}](x\mathbf{i} + y\mathbf{j})[(\cos\tfrac{1}{2}\theta) - (\sin\tfrac{1}{2}\theta)\mathbf{k}].$$

Chapter Four
Kinematics

4.1 Frame Derivatives

Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ be a physical vector expressed as

$$\vec{x} = x_1\hat{\imath}_{\mathrm{A}} + x_2\hat{\jmath}_{\mathrm{A}} + x_3\hat{k}_{\mathrm{A}}. \tag{4.1.1}$$

We can thus write

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{4.1.2}$$

Let $\mathrm{F_B}$ be a frame. Then, the *derivative of* $\vec{x}$ *with respect to* $\mathrm{F_B}$ is defined by

$$\overset{\mathrm{B}\bullet}{\vec{x}} \triangleq \dot{x}_1\hat{\imath}_{\mathrm{A}} + x_1\overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}} + \dot{x}_2\hat{\jmath}_{\mathrm{A}} + x_2\overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}} + \dot{x}_3\hat{k}_{\mathrm{A}} + x_3\overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}}. \tag{4.1.3}$$

In particular,

$$\overset{\mathrm{A}\bullet}{\vec{x}} = \dot{x}_1\hat{\imath}_{\mathrm{A}} + x_1\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{A}}} + \dot{x}_2\hat{\jmath}_{\mathrm{A}} + x_2\overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{A}}} + \dot{x}_3\hat{k}_{\mathrm{A}} + x_3\overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{A}}}. \tag{4.1.4}$$

Since the axes of $\mathrm{F_A}$ are constant with respect to $\mathrm{F_A}$, it follows that $\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{A}}} = \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{A}}} = \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{A}}} = 0$. Therefore, (4.1.4) can be written as

$$\overset{\mathrm{A}\bullet}{\vec{x}} \triangleq \dot{x}_1\hat{\imath}_{\mathrm{A}} + \dot{x}_2\hat{\jmath}_{\mathrm{A}} + \dot{x}_3\hat{k}_{\mathrm{A}}. \tag{4.1.5}$$

Resolving $\overset{\mathrm{A}\bullet}{\vec{x}}$ in $\mathrm{F_A}$ yields

$$\overset{\mathrm{A}\bullet}{\vec{x}}\Big|_{\mathrm{A}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}, \tag{4.1.6}$$

that is,

$$\overset{\mathrm{A}\bullet}{\vec{x}}\Big|_{\mathrm{A}} = \overset{\bullet}{\overbrace{\vec{x}\Big|_{\mathrm{A}}}}. \tag{4.1.7}$$

If, for all time, $\overset{\mathrm{A}\bullet}{\vec{x}}\Big|_{\mathrm{A}} = 0$, then the components of $\overset{\mathrm{A}\bullet}{\vec{x}}\Big|_{\mathrm{A}}$ are constant; these components are *constants of the motion with respect to* $\mathrm{F_A}$.

Let x and y be points, and let $\mathrm{F_A}$ be a frame. Then, the *velocity of y relative to x with respect to* $\mathrm{F_A}$ is defined by

$$\vec{v}_{y/x/\mathrm{A}} \triangleq \overset{\mathrm{A}\bullet}{\vec{r}}_{y/x}, \tag{4.1.8}$$

and the *acceleration of y relative to x with respect to* $\mathrm{F_A}$ is defined by

$$\vec{a}_{y/x/\mathrm{A}} \triangleq \overset{\mathrm{A}\bullet}{\vec{v}}_{y/x/\mathrm{A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/x}. \tag{4.1.9}$$

Note that, if

$$\vec{r}_{y/x}\Big|_{\mathrm{A}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{4.1.10}$$

then

$$\vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \dot r_1 \\ \dot r_2 \\ \dot r_3 \end{bmatrix}, \quad \vec{a}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \ddot r_1 \\ \ddot r_2 \\ \ddot r_3 \end{bmatrix}. \tag{4.1.11}$$

More generally, if $\mathrm{F_B}$ is a frame, then each component of $\vec{v}_{y/x/\mathrm{A}}\big|_{\mathrm{B}}$ is a *speed*. Note that a speed may be positive, negative, or zero.

Let x and y be points, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, we define the notation

$$r_{y/x|\mathrm{B}} \triangleq \vec{r}_{y/x}\Big|_{\mathrm{B}}. \tag{4.1.12}$$

Note that

$$r_{y/x|\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}\, \vec{r}_{y/x}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}} r_{y/x|\mathrm{A}}. \tag{4.1.13}$$

Next, define

$$v_{y/x/\mathrm{A|B}} \triangleq \vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{y/x}\Bigg|_{\mathrm{B}}. \tag{4.1.14}$$

Note that

$$v_{y/x/\mathrm{A|B}} = \mathcal{O}_{\mathrm{B/A}}\, \vec{v}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}} v_{y/x/\mathrm{A|A}} = \mathcal{O}_{\mathrm{B/A}} \dot r_{y/x|\mathrm{A}}. \tag{4.1.15}$$

Furthermore, define

$$a_{y/x/\mathrm{A|B}} \triangleq \vec{a}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}} = \overset{\mathrm{A}\bullet}{\vec{v}}_{y/x|\mathrm{A}}\Bigg|_{\mathrm{B}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/x}\Bigg|_{\mathrm{B}}. \tag{4.1.16}$$

Note that

$$a_{y/x/\mathrm{A|B}} = \mathcal{O}_{\mathrm{B/A}}\, \vec{a}_{y/x/\mathrm{A}}\Big|_{\mathrm{A}} = \mathcal{O}_{\mathrm{B/A}} a_{y/x/\mathrm{A|A}} = \mathcal{O}_{\mathrm{B/A}} \dot v_{y/x/\mathrm{A|A}} = \mathcal{O}_{\mathrm{B/A}} \ddot r_{y/x|\mathrm{A}}. \tag{4.1.17}$$

The *mixed acceleration* of y relative to x with respect to frames $(\mathrm{F_A}, \mathrm{F_B})$ is defined by

$$\vec{a}_{y/x/\mathrm{A/B}} \triangleq \overset{\mathrm{B}\bullet}{\vec{v}}_{y/x/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{r}}}_{y/x}. \tag{4.1.18}$$

As a special case, $\vec{a}_{y/x/\mathrm{A}/\mathrm{A}} = \vec{a}_{y/x/\mathrm{A}}$. Now let $\mathrm{F_C}$ be a frame. Then, the mixed acceleration can be resolved as

$$a_{y/x/\mathrm{A}/\mathrm{B}|\mathrm{C}} \triangleq \vec{a}_{y/x/\mathrm{A}/\mathrm{B}}\Big|_{\mathrm{C}}. \tag{4.1.19}$$

As a special case, $a_{y/x/\mathrm{A}/\mathrm{A}|\mathrm{C}} = a_{y/x/\mathrm{A}|\mathrm{C}}$. Note that

$$\begin{aligned}\vec{a}_{y/x/\mathrm{A}/\mathrm{B}|\mathrm{C}} = \mathcal{O}_{\mathrm{C}/\mathrm{B}}\, \vec{a}_{y/x/\mathrm{A}/\mathrm{B}}\Big|_{\mathrm{B}} &= \mathcal{O}_{\mathrm{C}/\mathrm{B}}\, \overset{\mathrm{B}\bullet}{\vec{v}}_{y/x/\mathrm{A}}\Big|_{\mathrm{B}} \\ &= \mathcal{O}_{\mathrm{C}/\mathrm{B}}\dot{v}_{y/x/\mathrm{A}|\mathrm{B}} = \mathcal{O}_{\mathrm{C}/\mathrm{B}}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{O}_{\mathrm{B}/\mathrm{A}}\dot{r}_{y/x|\mathrm{A}}) \\ &= \mathcal{O}_{\mathrm{C}/\mathrm{B}}(\dot{\mathcal{O}}_{\mathrm{B}/\mathrm{A}}\dot{r}_{y/x|\mathrm{A}} + \mathcal{O}_{\mathrm{B}/\mathrm{A}}\ddot{r}_{y/x|\mathrm{A}}) \\ &= \mathcal{O}_{\mathrm{C}/\mathrm{B}}\dot{\mathcal{O}}_{\mathrm{B}/\mathrm{A}}\dot{r}_{y/x|\mathrm{A}} + \mathcal{O}_{\mathrm{C}/\mathrm{A}}\ddot{r}_{y/x|\mathrm{A}}.\end{aligned} \tag{4.1.20}$$

If $\mathrm{F_B} = \mathrm{F_A}$, then (4.1.20) yields (4.1.17).

Fact 4.1.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let x, y, and z be points. Then,

$$\begin{aligned}\vec{v}_{z/x/\mathrm{A}} &= \vec{v}_{z/y/\mathrm{A}} + \vec{v}_{y/x/\mathrm{A}}, \\ \vec{a}_{z/x/\mathrm{A}} &= \vec{a}_{z/y/\mathrm{A}} + \vec{a}_{y/x/\mathrm{A}}, \\ \vec{a}_{z/x/\mathrm{A}/\mathrm{B}} &= \vec{a}_{z/y/\mathrm{A}/\mathrm{B}} + \vec{a}_{y/x/\mathrm{A}/\mathrm{B}}.\end{aligned} \tag{4.1.21}$$

Fact 4.1.1 is based on the assumptions of Newtonian mechanics and thus is not valid within the context of relativity. For a particle of light in a medium such as vacuum or water, the speed of light c is independent of the velocity of all other bodies. Therefore, if x denotes a particle of light, w denotes a particle, and $\mathrm{F_A}$ is a frame, then

$$|\vec{v}_{x/w/\mathrm{A}}| = c. \tag{4.1.22}$$

Therefore, (4.1.21) does not hold for light particles.

The following result shows that the velocity of y relative to x with respect to $\mathrm{F_A}$ is zero if and only if the position of y relative to x is constant with respect to $\mathrm{F_A}$. Likewise, the acceleration of y relative to x with respect to $\mathrm{F_A}$ is zero if and only if y moves in a straight line at constant speed relative to x and with respect to $\mathrm{F_A}$.

Fact 4.1.2. Let $\mathrm{F_A}$ be a frame, let y and x be points, and let $t_1 < t_2$. Then, the following statements hold:

i) $\vec{v}_{y/x/\mathrm{A}}(t) = 0$ for all $t \in [t_1, t_2]$ if and only if there exist real numbers $\alpha_1, \alpha_2, \alpha_3$ such that, for all $t \in [t_1, t_2]$,

$$\vec{r}_{y/x}(t)\Big|_{\mathrm{A}} = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}. \tag{4.1.23}$$

ii) $\vec{a}_{y/x/\mathrm{A}}(t) = 0$ for all $t \in [t_1, t_2]$ if and only if there exist real numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, such that, for all $t \in [t_1, t_2]$,

$$\vec{r}_{y/x}(t)\Big|_{\mathrm{A}} = \begin{bmatrix}\alpha_1 t + \beta_1\\ \alpha_2 t + \beta_2\\ \alpha_3 t + \beta_3\end{bmatrix}. \tag{4.1.24}$$

Fact 4.1.3. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}(t)$ and $\vec{y}(t)$ be physical vectors. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{x}'\vec{y}) = \overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\vec{y} + \vec{x}'\,\overset{\mathrm{A}\bullet}{\vec{y}}. \tag{4.1.25}$$

In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{x}'\vec{x}) = 2\,\overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\vec{x}. \tag{4.1.26}$$

Proof. Write

$$\vec{x}\Big|_{\mathrm{A}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{y}\Big|_{\mathrm{A}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\vec{x}'\vec{y}) &= \frac{\mathrm{d}}{\mathrm{d}t}(x_1y_1 + x_2y_2 + x_3y_3) \\
&= \dot{x}_1y_1 + \dot{x}_2y_2 + \dot{x}_3y_3 + x_1\dot{y}_1 + x_2\dot{y}_2 + x_3\dot{y}_3 \\
&= \overbrace{\vec{x}\Big|_{\mathrm{A}}}^{\cdot}{}^{\mathrm{T}}\,\vec{y}\Big|_{\mathrm{A}} + \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}\,\overbrace{\vec{y}\Big|_{\mathrm{A}}}^{\cdot} \\
&= \overset{\mathrm{A}\bullet}{\vec{x}}\Big|_{\mathrm{A}}^{\mathrm{T}}\,\vec{y}\Big|_{\mathrm{A}} + \vec{x}\Big|_{\mathrm{A}}^{\mathrm{T}}\,\overset{\mathrm{A}\bullet}{\vec{y}}\Big|_{\mathrm{A}} \\
&= \overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\vec{y} + \vec{x}'\,\overset{\mathrm{A}\bullet}{\vec{y}}. \qquad \square
\end{aligned}$$

Fact 4.1.4. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}(t)$ be a physical vector that is nonzero for all $t \in [t_1, t_2]$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}| = \frac{1}{|\vec{x}|}\overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\vec{x} = \overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\hat{x}. \tag{4.1.27}$$

Furthermore, if $|\vec{x}(t)|$ is constant, then $\vec{x}(t)$ and $\overset{\mathrm{A}\bullet}{\vec{x}}(t)$ are mutually orthogonal. In particular, $\hat{x}(t)$ and $\overset{\mathrm{A}\bullet}{\hat{x}}(t)$ are mutually orthogonal.

Fact 4.1.5. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}(t)$ be a physical vector that is nonzero for all $t \in [t_1, t_2]$. Then,

$$\overset{\mathrm{A}\bullet}{\vec{x}} = |\vec{x}|\,\overset{\mathrm{A}\bullet}{\hat{x}} + \left(\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}|\right)\hat{x} = |\vec{x}|\,\overset{\mathrm{A}\bullet}{\hat{x}} + \overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\hat{x}\hat{x}. \tag{4.1.28}$$

Therefore,

$$\overset{\mathrm{A}\bullet}{\hat{x}} = \frac{1}{|\vec{x}|}\overset{\mathrm{A}\bullet}{\vec{x}} - \frac{1}{|\vec{x}|}\overset{\mathrm{A}\bullet}{\vec{x}}{}'\,\hat{x}\hat{x}. \tag{4.1.29}$$

Proof. Differentiating $\vec{x} = |\vec{x}|\hat{x}$ with respect to $\mathrm{F_A}$ and using (4.1.27) yields (4.1.28). $\square$

Fact 4.1.4 and Fact 4.1.5 show that the frame derivative of a physical vector $\vec{x}$ can be decomposed as the sum of two terms, namely, $|\vec{x}|\overset{\mathrm{A}\bullet}{\hat{x}}$ and $\left(\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}|\right)\hat{x}$. Since $\hat{x}$ has unit length, it follows that $\overset{\mathrm{A}\bullet}{\hat{x}}$ is orthogonal to $\hat{x}$, and thus $|\vec{x}|\overset{\mathrm{A}\bullet}{\hat{x}}$ is orthogonal to $\vec{x}$. On the other hand, $\left(\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}|\right)\hat{x}$ is parallel with $\vec{x}$. Hence, $|\vec{x}|\overset{\mathrm{A}\bullet}{\hat{x}}$ and $\left(\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}|\right)\hat{x}$ are mutually orthogonal.

Let $\mathrm{F_A}$ be a frame, let $\vec{x}(t)$ and $\vec{y}(t)$ be physical vectors, and let $\vec{M} = \vec{x}\vec{y}'$. Then, we define

$$\overset{\mathrm{A}\bullet}{\vec{y}'} \triangleq \overset{\mathrm{A}\bullet\prime}{\vec{y}}, \tag{4.1.30}$$

$$\overset{\mathrm{A}\bullet}{\vec{M}} \triangleq \overset{\mathrm{A}\bullet}{\vec{x}}\,\vec{y}' + \vec{x}\,\overset{\mathrm{A}\bullet\prime}{\vec{y}}. \tag{4.1.31}$$

Fact 4.1.6. Let $\mathrm{F_A}$ be a frame, let $\vec{M}(t)$ and $\vec{N}(t)$ be physical matrices, and let $\vec{x}(t)$ be a physical vector. Then,

$$\overbrace{\vec{M}\vec{x}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{M}}\,\vec{x} + \vec{M}\,\overset{\mathrm{A}\bullet}{\vec{x}}, \tag{4.1.32}$$

$$\overbrace{\vec{M}\vec{N}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{M}}\,\vec{N} + \vec{M}\,\overset{\mathrm{A}\bullet}{\vec{N}}. \tag{4.1.33}$$

Furthermore,

$$\overset{\mathrm{A}\bullet}{\vec{M}}\Big|_{\mathrm{A}} = \overbrace{\vec{M}\Big|_{\mathrm{A}}}^{\cdot}. \tag{4.1.34}$$

Proof. For convenience, assume that $\vec{M} = \vec{y}\vec{z}'$. Using Fact 4.1.3, it follows that

$$\begin{aligned}
\overbrace{\vec{M}\vec{x}}^{\mathrm{A}\bullet} &= \overbrace{(\vec{y}\vec{z}')\vec{x}}^{\mathrm{A}\bullet}\\
&= \overbrace{(\vec{z}'\vec{x})\vec{y}}^{\mathrm{A}\bullet}\\
&= \overbrace{(\vec{z}'\vec{x})}^{\cdot}\,\vec{y} + (\vec{z}'\vec{x})\,\overset{\mathrm{A}\bullet}{\vec{y}}\\
&= (\overset{\mathrm{A}\bullet\prime}{\vec{z}}\,\vec{x})\vec{y} + (\vec{z}'\,\overset{\mathrm{A}\bullet}{\vec{x}})\vec{y} + (\vec{z}'\vec{x})\,\overset{\mathrm{A}\bullet}{\vec{y}}\\
&= (\vec{y}\,\overset{\mathrm{A}\bullet\prime}{\vec{z}})\vec{x} + (\overset{\mathrm{A}\bullet}{\vec{y}}\,\vec{z}')\vec{x} + (\vec{y}\vec{z}')\,\overset{\mathrm{A}\bullet}{\vec{x}}\\
&= \overset{\mathrm{A}\bullet}{\vec{M}}\,\vec{x} + \vec{M}\,\overset{\mathrm{A}\bullet}{\vec{x}}.
\end{aligned}$$

Next, let $\vec{N} = \vec{w}\vec{u}{}'$ and note that

$$\begin{aligned}
\overbrace{\vec{M}\vec{N}}^{\mathrm{A}\bullet} &= \overbrace{(\vec{y}\vec{z}{}')(\vec{w}\vec{u}{}')}^{\mathrm{A}\bullet}\\
&= \overbrace{\vec{z}{}'\vec{w}(\vec{y}\vec{u}{}')}^{\mathrm{A}\bullet}\\
&= \vec{z}{}'\vec{w}\,\overbrace{(\vec{y}\vec{u}{}')}^{\mathrm{A}\bullet} + \overbrace{\vec{z}{}'\vec{w}}^{\cdot}\,(\vec{y}\vec{u}{}')\\
&= \vec{z}{}'\vec{w}(\overset{\mathrm{A}\bullet}{\vec{y}}\,\vec{u}{}' + \vec{y}\,\overset{\mathrm{A}\bullet\prime}{\vec{u}}) + (\overset{\mathrm{A}\bullet\prime}{\vec{z}}\,\vec{w} + \vec{z}{}'\,\overset{\mathrm{A}\bullet}{\vec{w}})\vec{y}\vec{u}{}'\\
&= (\overset{\mathrm{A}\bullet}{\vec{y}}\,\vec{z}{}' + \vec{y}\,\overset{\mathrm{A}\bullet\prime}{\vec{z}})\vec{w}\vec{u}{}' + \vec{y}\vec{z}{}'(\overset{\mathrm{A}\bullet}{\vec{w}}\,\vec{u}{}' + \vec{w}\,\overset{\mathrm{A}\bullet\prime}{\vec{u}})\\
&= \overset{\mathrm{A}\bullet}{\vec{M}}\,\vec{N} + \vec{M}\,\overset{\mathrm{A}\bullet}{\vec{N}}.
\end{aligned}$$

□

Fact 4.1.7. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}(t)$ be a physical vector. Then,

$$\overbrace{\vec{x}^{\times}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{x}}{}^{\times}, \tag{4.1.35}$$

$$\overset{\mathrm{A}\bullet}{\vec{x}} \times \vec{x} = -\vec{x} \times \overset{\mathrm{A}\bullet}{\vec{x}}, \tag{4.1.36}$$

$$(\vec{x} \times \overset{\mathrm{A}\bullet}{\vec{x}})^{\times} = \overset{\mathrm{A}\bullet}{\vec{x}}\,\vec{x}{}' - \vec{x}\,\overset{\mathrm{A}\bullet\prime}{\vec{x}}. \tag{4.1.37}$$

If, in addition, $\vec{y}$ is a physical vector, then

$$\overbrace{\vec{x} \times \vec{y}}^{\mathrm{A}\bullet} = \overset{\mathrm{A}\bullet}{\vec{x}} \times \vec{y} + \vec{x} \times \overset{\mathrm{A}\bullet}{\vec{y}}. \tag{4.1.38}$$

Fact 4.1.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{I}} = 0, \tag{4.1.39}$$

$$\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}}\,\hat{\imath}_{\mathrm{B}}' + \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}}\,\hat{\jmath}_{\mathrm{B}}' + \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}}\,\hat{k}_{\mathrm{B}}' = -(\hat{\imath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}_{\mathrm{B}}} + \hat{\jmath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}_{\mathrm{B}}} + \hat{k}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{k}_{\mathrm{B}}}). \tag{4.1.40}$$

Fact 4.1.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} = -\vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}}, \tag{4.1.41}$$

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = -\vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}}. \tag{4.1.42}$$

Proof. Differentiating $\vec{R}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/A}} = \vec{I}$ and using (4.1.33) yields

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}\,\vec{R}_{\mathrm{B/A}} + \vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = 0,$$

which implies (4.1.41). □

Fact 4.1.10. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}} = -\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}\,\vec{R}_{\mathrm{B/A}} = -\vec{R}_{\mathrm{B/A}}\,\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}}. \tag{4.1.43}$$

Proof. Note that

$$\begin{aligned}\left.\left(\vec{R}_{\mathrm{A/B}}\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\right)\right|_{\mathrm{B}} &= \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}}\,\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}\mathcal{R}_{\mathrm{A/B}}\\ &= \mathcal{O}_{\mathrm{B/A}}\,\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\,\mathcal{O}_{\mathrm{A/B}}\,\vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{B}}\,\vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} = \left.\left(\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}}\right)\right|_{\mathrm{B}}.\end{aligned}$$ □

Fact 4.1.11. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet\prime}{\vec{R}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}. \tag{4.1.44}$$

Proof. Note that

$$\begin{aligned}\overset{\mathrm{B}\bullet\prime}{\vec{R}}_{\mathrm{B/A}} &= \left(\overbrace{(\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{A}})}^{\mathrm{B}\bullet}\right)' = \left(\hat{\imath}_{\mathrm{B}}\overset{\mathrm{B}\bullet\prime}{\hat{\imath}_{\mathrm{A}}} + \hat{\jmath}_{\mathrm{B}}\overset{\mathrm{B}\bullet\prime}{\hat{\jmath}_{\mathrm{A}}} + \hat{k}_{\mathrm{B}}\overset{\mathrm{B}\bullet\prime}{\hat{k}_{\mathrm{A}}}\right)'\\ &= \overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}}\hat{\imath}'_{\mathrm{B}} + \overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}}\hat{\jmath}'_{\mathrm{B}} + \overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}}\hat{k}'_{\mathrm{B}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}.\end{aligned}$$ □

4.2 The Mixed-Dot Identity and the Physical Angular Velocity Matrix

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Then, the frame derivatives $\overset{\mathrm{A}\bullet}{\vec{x}}$ and $\overset{\mathrm{B}\bullet}{\vec{x}}$ may be different when $\mathrm{F_A}$ and $\mathrm{F_B}$ are rotating relative to each other. The following result, called the *rotating-dot identity*, will be useful for deriving a relationship between $\overset{\mathrm{A}\bullet}{\vec{x}}$ and $\overset{\mathrm{B}\bullet}{\vec{x}}$.

Fact 4.2.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}(t)$ be a physical vector. Then,

$$\overbrace{\vec{R}_{\mathrm{B/A}}\vec{x}}^{\mathrm{B}\bullet} = \vec{R}_{\mathrm{B/A}}\,\overset{\mathrm{A}\bullet}{\vec{x}}\,. \tag{4.2.1}$$

Proof. Defining $\vec{y} \triangleq \vec{R}_{\mathrm{B/A}}\vec{x}$, note that

$$\left(\vec{R}_{\mathrm{A/B}}\overset{\mathrm{B}\bullet}{\vec{y}}\right)\Bigg|_{\mathrm{A}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{A}}\overset{\mathrm{B}\bullet}{\vec{y}}\Bigg|_{\mathrm{A}} = \mathcal{R}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{A/B}}\overset{\mathrm{B}\bullet}{\vec{y}}\Bigg|_{\mathrm{B}} = \overbrace{\vec{R}_{\mathrm{B/A}}\vec{x}}^{\mathrm{B}\bullet}\Bigg|_{\mathrm{B}} = \overbrace{\vec{R}_{\mathrm{B/A}}\vec{x}\Big|_{\mathrm{B}}}^{\cdot}$$

$$= \overbrace{\mathcal{O}_{\mathrm{B/A}}\vec{R}_{\mathrm{B/A}}\vec{x}\Big|_{\mathrm{A}}}^{\cdot} = \overbrace{\mathcal{O}_{\mathrm{B/A}}\mathcal{R}_{\mathrm{B/A}}\vec{x}\Big|_{\mathrm{A}}}^{\cdot} = \overbrace{\vec{x}\Big|_{\mathrm{A}}}^{\cdot} = \overset{\mathrm{A}\bullet}{\vec{x}}\Bigg|_{\mathrm{A}}. \qquad \square$$

The following result used the rotating-dot identity to obtain the *mixed-dot identity*. This result shows how derivatives with respect to different frames are related.

Fact 4.2.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}(t)$ be a physical vector. Then,

$$\overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{B}\bullet}{\vec{x}} + \vec{R}_{\mathrm{A/B}}\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\vec{x}. \tag{4.2.2}$$

Proof. Using (4.2.1) we have

$$\begin{aligned}\overset{\mathrm{A}\bullet}{\vec{x}} &= \vec{R}_{\mathrm{A/B}}\overbrace{\vec{R}_{\mathrm{B/A}}\vec{x}}^{\mathrm{B}\bullet}\\ &= \vec{R}_{\mathrm{A/B}}\left(\vec{R}_{\mathrm{B/A}}\overset{\mathrm{B}\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\vec{x}\right)\\ &= \overset{\mathrm{B}\bullet}{\vec{x}} + \vec{R}_{\mathrm{A/B}}\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\vec{x}. \qquad \square\end{aligned}$$

Since $\overset{\mathrm{A}\bullet}{\vec{x}}$ and $\overset{\mathrm{B}\bullet}{\vec{x}}$ differ due to the relative rotation of $\mathrm{F_A}$ and $\mathrm{F_B}$, we have the following definition.

Definition 4.2.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, the *physical angular velocity matrix* of $\mathrm{F_B}$ *relative* to $\mathrm{F_A}$ is defined by

$$\vec{\Omega}_{\mathrm{B/A}} \triangleq \vec{R}_{\mathrm{A/B}}\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}. \tag{4.2.3}$$

It follows from (4.1.41) that

$$\vec{\Omega}_{\mathrm{B/A}} = -\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/A}}. \tag{4.2.4}$$

Furthermore, (4.2.2) can be written as

$$\overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\Omega}_{\mathrm{B/A}}\vec{x}. \tag{4.2.5}$$

The following result shows that $\vec{\Omega}_{\mathrm{B/A}}$ is skew symmetric.

Fact 4.2.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{\Omega}'_{\mathrm{B/A}} = -\vec{\Omega}_{\mathrm{B/A}}. \tag{4.2.6}$$

Proof. Using (4.1.44) and (4.2.4), it follows that

$$\vec{\Omega}'_{\mathrm{B/A}} = \left(\vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\right)' = \overset{\mathrm{B}\bullet\prime}{\vec{R}}_{\mathrm{B/A}} \vec{R}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} \vec{R}_{\mathrm{B/A}} = -\vec{\Omega}_{\mathrm{B/A}}. \qquad \square$$

Fact 4.2.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\vec{\Omega}_{\mathrm{B/A}}, \tag{4.2.7}$$

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} = -\vec{\Omega}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}}. \tag{4.2.8}$$

Proof. (4.2.7) follows from (4.2.3), and (4.2.8) follows from (4.2.4). $\square$

Fact 4.2.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{\Omega}_{\mathrm{A/B}} = \hat{\imath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}_{\mathrm{B}}} + \hat{\jmath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}_{\mathrm{B}}} + \hat{k}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{k}_{\mathrm{B}}}, \tag{4.2.9}$$

$$\vec{\Omega}_{\mathrm{A/B}} = -(\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}}\hat{\imath}'_{\mathrm{B}} + \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}}\hat{\jmath}'_{\mathrm{B}} + \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}}\hat{k}'_{\mathrm{B}}), \tag{4.2.10}$$

$$\vec{\Omega}_{\mathrm{B/A}} = \hat{\imath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\imath}_{\mathrm{A}}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\jmath}_{\mathrm{A}}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{k}_{\mathrm{A}}}, \tag{4.2.11}$$

$$\vec{\Omega}_{\mathrm{B/A}} = -(\overset{\mathrm{B}\bullet}{\hat{\imath}_{\mathrm{A}}}\hat{\imath}'_{\mathrm{A}} + \overset{\mathrm{B}\bullet}{\hat{\jmath}_{\mathrm{A}}}\hat{\jmath}'_{\mathrm{A}} + \overset{\mathrm{B}\bullet}{\hat{k}_{\mathrm{A}}}\hat{k}'_{\mathrm{A}}). \tag{4.2.12}$$

Proof. Note that

$$\begin{aligned}
\vec{\Omega}_{\mathrm{A/B}} &= \vec{R}_{\mathrm{B/A}}\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} \\
&= (\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{A}} + \hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{A}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{A}})(\hat{\imath}_{\mathrm{A}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}_{\mathrm{B}}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}_{\mathrm{B}}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{A}\bullet\prime}{\hat{k}_{\mathrm{B}}}) \\
&= \hat{\imath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\imath}_{\mathrm{B}}} + \hat{\jmath}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}_{\mathrm{B}}} + \hat{k}_{\mathrm{B}}\overset{\mathrm{A}\bullet\prime}{\hat{k}_{\mathrm{B}}},
\end{aligned}$$

which proves (4.2.9). Using (4.1.40) yields (4.2.10). $\square$

Fact 4.2.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} = \vec{\Omega}_{\mathrm{B/A}}\hat{\imath}_{\mathrm{B}}, \tag{4.2.13}$$

$$\overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} = \vec{\Omega}_{\mathrm{B/A}}\hat{\jmath}_{\mathrm{B}}, \tag{4.2.14}$$

$$\overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}} = \vec{\Omega}_{\mathrm{B/A}}\hat{k}_{\mathrm{B}}. \tag{4.2.15}$$

Proof. Using (4.2.11) and Fact 4.2.8, we have

$$
\begin{aligned}
\vec{\Omega}_{\mathrm{B/A}}\hat\imath_{\mathrm{B}} &= \Big(\hat\imath_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm{A}}} + \hat\jmath_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat\jmath_{\mathrm{A}}} + \hat k_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat k_{\mathrm{A}}}\Big)\hat\imath_{\mathrm{B}}\\
&= \Big(\overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}\Big)\hat\imath_{\mathrm{A}} + \Big(\overset{\mathrm{B}\bullet\prime}{\hat\jmath_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}\Big)\hat\jmath_{\mathrm{A}} + \Big(\overset{\mathrm{B}\bullet\prime}{\hat k_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}\Big)\hat k_{\mathrm{A}}\\
&= \Big(\hat\imath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}}\Big)\hat\imath_{\mathrm{A}} + \Big(\hat\jmath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}}\Big)\hat\jmath_{\mathrm{A}} + \Big(\hat k'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}}\Big)\hat k_{\mathrm{A}}\\
&= (\hat\imath_{\mathrm{A}}\hat\imath'_{\mathrm{A}})\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} + (\hat\jmath_{\mathrm{A}}\hat\jmath'_{\mathrm{A}})\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} + (\hat k_{\mathrm{A}}\hat k'_{\mathrm{A}})\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}}\\
&= \vec{I}\,\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} = \overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}}. \qquad\square
\end{aligned}
$$

Fact 4.2.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\hat\imath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}, \quad \hat\imath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\jmath_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm{A}}}\,\hat\jmath_{\mathrm{B}}, \quad \hat\imath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat k_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm{A}}}\,\hat k_{\mathrm{B}}, \tag{4.2.16}$$

$$\hat\jmath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat\jmath_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}, \quad \hat\jmath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\jmath_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat\jmath_{\mathrm{A}}}\,\hat\jmath_{\mathrm{B}}, \quad \hat\jmath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat k_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat\jmath_{\mathrm{A}}}\,\hat k_{\mathrm{B}}, \tag{4.2.17}$$

$$\hat k'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat k_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}, \quad \hat k'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\jmath_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat k_{\mathrm{A}}}\,\hat\jmath_{\mathrm{B}}, \quad \hat k'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat k_{\mathrm{B}}} = \overset{\mathrm{B}\bullet\prime}{\hat k_{\mathrm{A}}}\,\hat k_{\mathrm{B}}. \tag{4.2.18}$$

Proof. Note that $\hat\imath'_{\mathrm{A}}\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}} = \frac{\mathrm{d}}{\mathrm{d}t}(\hat\imath'_{\mathrm{A}}\hat\imath_{\mathrm{B}}) = \overset{\mathrm{B}\bullet\prime}{\hat\imath_{\mathrm{A}}}\,\hat\imath_{\mathrm{B}}$. $\square$

Fact 4.2.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\vec{\Omega}_{\mathrm{A/B}} = -\vec{\Omega}_{\mathrm{B/A}}, \tag{4.2.19}$$

$$\vec{\Omega}_{\mathrm{A/B}} = \vec{\Omega}{}'_{\mathrm{B/A}}. \tag{4.2.20}$$

Proof. (4.2.19) follows from the equality of the first, second, and fourth terms in (4.1.43). Alternatively, Fact 4.2.6 and (4.1.40) imply that

$$
\begin{aligned}
\vec{\Omega}_{\mathrm{A/B}} &= -\Big(\overset{\mathrm{A}\bullet}{\hat\imath_{\mathrm{B}}}\,\hat\imath'_{\mathrm{B}} + \overset{\mathrm{A}\bullet}{\hat\jmath_{\mathrm{B}}}\,\hat\jmath'_{\mathrm{B}} + \overset{\mathrm{A}\bullet}{\hat k_{\mathrm{B}}}\,\hat k'_{\mathrm{B}}\Big)\\
&= -\Big[\Big(\vec{\Omega}_{\mathrm{B/A}}\hat\imath_{\mathrm{B}}\Big)\hat\imath'_{\mathrm{B}} + \Big(\vec{\Omega}_{\mathrm{B/A}}\hat\jmath_{\mathrm{B}}\Big)\hat\jmath'_{\mathrm{B}} + \Big(\vec{\Omega}_{\mathrm{B/A}}\hat k_{\mathrm{B}}\Big)\hat k'_{\mathrm{B}}\Big]\\
&= -\vec{\Omega}_{\mathrm{B/A}}\vec{I} = -\vec{\Omega}_{\mathrm{B/A}}. \qquad\square
\end{aligned}
$$

It follows from (4.2.19) that

$$\vec{\Omega}_{\mathrm{B/A}} = -\vec{R}_{\mathrm{B/A}}\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}}, \tag{4.2.21}$$

$$\vec{\Omega}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}}. \tag{4.2.22}$$

Fact 4.2.10. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define

$$\Omega_{\mathrm{B/A|B}} \triangleq \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}. \tag{4.2.23}$$

Then,

$$\Omega_{\mathrm{B/A|B}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}} = -\dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}, \tag{4.2.24}$$

$$\Omega_{\mathrm{B/A|B}} = -\Omega_{\mathrm{B/A|B}}^{\mathsf{T}}. \tag{4.2.25}$$

Furthermore,

$$\Omega_{\mathrm{A/B|A}} \triangleq \vec{\Omega}_{\mathrm{A/B}}\Big|_{\mathrm{A}} = -\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = -\mathcal{O}_{\mathrm{A/B}}\Omega_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/A}} = -\dot{\mathcal{O}}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/A}}. \tag{4.2.26}$$

Proof. It follows from (4.2.3) that

$$\Omega_{\mathrm{B/A|B}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}},$$

which proves (4.2.24). Next, it follows from (4.2.6) that

$$\Omega_{\mathrm{B/A|B}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = -\vec{\Omega}'_{\mathrm{B/A}}\Big|_{\mathrm{B}} = -\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}^{\mathsf{T}} = -\Omega_{\mathrm{B/A|B}}^{\mathsf{T}},$$

which proves (4.2.25). Finally, (4.2.26) follows from (4.2.19). □

Note that (4.2.26) shows that, except in special cases, $\Omega_{\mathrm{B/A|B}} \neq -\Omega_{\mathrm{A/B}}$.

4.3 The Physical Angular Velocity Vector and Poisson's Equation

To understand the meaning of $\vec{\Omega}_{\mathrm{B/A}}$, let $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$ so that $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)$ on a time interval, where $\theta = \theta_{\hat{\imath}_\mathrm{B}/\hat{\imath}_\mathrm{A}/\hat{k}_\mathrm{A}}$. Therefore,

$$\vec{\Omega}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_\mathrm{A}}(-\theta) \overset{\mathrm{B}\bullet}{\vec{R}}_{\hat{k}_\mathrm{A}}(\theta). \tag{4.3.1}$$

It thus follows that

$$\vec{\Omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_\mathrm{A}^{\times}. \tag{4.3.2}$$

To see this in terms of components, we resolve (4.3.1) to obtain

$$\Omega_{\mathrm{B/A|B}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \vec{R}_{\mathrm{A/B}}\Big|_{\mathrm{B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}}\Big|_{\mathrm{B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}}. \tag{4.3.3}$$

Since $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_\mathrm{A}}(\theta)$ represents a rotation around $\hat{k}_\mathrm{A}$ through the angle θ, it follows that $\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_3(\theta)$. Hence,

$$\Omega_{\mathrm{B/A|B}} = \mathcal{O}_3(\theta)\dot{\mathcal{O}}_3^{\mathsf{T}}(\theta) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}^{\times}. \tag{4.3.4}$$

Consequently, $\Omega_{\mathrm{B/A|B}}$ is associated with a physical vector that is codirectional with the axis of rotation and whose length represents the rate of rotation around that axis.

The next result shows that the physical angular velocity matrix $\vec{\Omega}_{\mathrm{B/A}}$ can be written as a physical cross product matrix. This physical cross product matrix is written as $\vec{\omega}_{\mathrm{B/A}}^{\times}$, where the physical vector $\vec{\omega}_{\mathrm{B/A}}$ is the *angular velocity of* $\mathrm{F_B}$ *relative to* $\mathrm{F_A}$.

Fact 4.3.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, there exists a physical vector $\vec{\omega}_{\mathrm{B/A}}$ such that

$$\vec{\Omega}_{\mathrm{B/A}} = \vec{\omega}_{\mathrm{B/A}}^{\times}. \tag{4.3.5}$$

Furthermore,

$$\Omega_{\mathrm{B/A|B}} = \omega_{\mathrm{B/A|B}}^{\times} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{4.3.6}$$

where

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \triangleq \omega_{\mathrm{B/A|B}} = \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}. \tag{4.3.7}$$

In addition,

$$\vec{\omega}_{\mathrm{A/B}} = -\vec{\omega}_{\mathrm{B/A}}. \tag{4.3.8}$$

Finally,

$$\omega_{\mathrm{B/A|B}} = -\mathcal{O}_{\mathrm{B/A}}\omega_{\mathrm{A/B|A}} = -\mathcal{R}_{\mathrm{A/B}}\omega_{\mathrm{A/B|A}}, \tag{4.3.9}$$

$$\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \omega_{\mathrm{B/A|A}} = -\omega_{\mathrm{A/B|A}} = \mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{B/A|B}}. \tag{4.3.10}$$

Proof. It follows from Fact 4.2.4 that $\vec{\Omega}_{\mathrm{B/A}}$ is skew symmetric. Fact 2.9.3 thus implies that there exists a physical vector $\vec{\omega}_{\mathrm{B/A}}$ satisfying (4.3.5). Next, (4.3.8) follows from (4.2.19). Finally, (4.3.9) follows from (4.2.26). □

The components ω_1, ω_2, and ω_3 of $\vec{\omega}_{\mathrm{B/A}}$ resolved in an arbitrary frame are *spin rates.* Note that a spin rate may be positive, negative, or zero.

It follows from (4.2.3), (4.2.4), (4.2.21), and (4.2.22) that

$$\vec{\omega}_{\mathrm{B/A}}^{\times} = -\vec{R}_{\mathrm{B/A}} \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} \vec{R}_{\mathrm{A/B}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = -\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} \vec{R}_{\mathrm{B/A}}. \tag{4.3.11}$$

Assume, for example, $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)$ on a time interval so that $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$, where $\theta = \theta_{\hat{\imath}_{\mathrm{B}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}$. Then, it follows from (4.3.2) that

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_{\mathrm{A}} = \dot{\theta}_{\hat{\imath}_{\mathrm{B}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}\hat{k}_{\mathrm{A}}. \tag{4.3.12}$$

For the case where $\theta_{\mathrm{B/A}}$ and $\hat{n}_{\mathrm{B/A}}$ are not necessarily constant during the rotation, the relationship between $\vec{\omega}_{\mathrm{B/A}}$ and $\theta_{\mathrm{B/A}}$ and $\hat{n}_{\mathrm{B/A}}$ and their derivatives is given by (4.7.18) and (4.7.19).

Note that (4.3.9) shows that $\omega_{\mathrm{B/A|B}} = -\omega_{\mathrm{A/B|A}}$ may not be true. The following result gives conditions under which $\omega_{\mathrm{B/A|B}} = -\omega_{\mathrm{A/B|A}}$.

Fact 4.3.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time, the following statements are equivalent:

i) $\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$.

ii) $\Omega_{\mathrm{A/B|A}} = -\Omega_{\mathrm{B/A|B}}$.

iii) $\dot{\mathcal{O}}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}}$.

iv) $\dot{\mathcal{O}}_{\mathrm{A/B}}\mathcal{O}_{\mathrm{B/A}} = -\dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}}$.

v) $\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$.

vi) $\omega_{\mathrm{A/B|A}} = -\omega_{\mathrm{B/A|B}}$.

vii) $\mathcal{O}_{\mathrm{B/A}}\omega_{\mathrm{A/B|A}} = \omega_{\mathrm{A/B|A}}$.

viii) $\mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{A/B|A}} = \omega_{\mathrm{A/B|A}}$.

Fact 4.3.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time, the following statements are equivalent:

i) $n_{\mathrm{B/A}} \times \omega_{\mathrm{B/A|B}} = 0$.

ii) $n_{\mathrm{B/A}} \times \omega_{\mathrm{A/B|A}} = 0$.

Fact 4.3.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time, the following statements are equivalent:

i) $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} = 0$.

ii) $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} = 0$.

iii) $\dot{\omega}_{\mathrm{B/A|B}} = 0$.

iv) $\dot{\omega}_{\mathrm{A/B|A}} = 0$.

Proof. To prove that *iii*) implies *iv*), note that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} = \overset{\cdot}{\overbrace{\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}}} = \dot{\omega}_{\mathrm{B/A|B}} = 0.$$

Hence, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, and thus $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$. Therefore,

$$\dot{\omega}_{\mathrm{A/B}} = \overset{\cdot}{\overbrace{\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}}} = \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{A}} = 0. \qquad \square$$

Using (4.3.5) we can rewrite Fact 4.2.7 as follows.

Fact 4.3.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} = \vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}}, \tag{4.3.13}$$

$$\overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} = \vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_{\mathrm{B}}, \tag{4.3.14}$$

$$\overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} = \vec{\omega}_{\mathrm{B/A}} \times \hat{k}_{\mathrm{B}}. \tag{4.3.15}$$

The following result gives *Poisson's* equation (4.3.19).

Fact 4.3.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{B/A}}\vec{\omega}^{\times}_{\mathrm{B/A}}, \tag{4.3.16}$$

$$\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}\omega^{\times}_{\mathrm{B/A|B}}. \tag{4.3.17}$$

Furthermore,

$$\dot{\mathcal{O}}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{A/B}}\omega^{\times}_{\mathrm{B/A|B}}, \tag{4.3.18}$$

and thus

$$\dot{\mathcal{O}}_{\mathrm{B/A}} = -\omega^{\times}_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/A}}. \tag{4.3.19}$$

Hence,

$$\omega^{\times}_{\mathrm{B/A|B}} = \mathcal{O}_{\mathrm{B/A}}\dot{\mathcal{O}}_{\mathrm{A/B}} = -\dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}} = \mathcal{R}_{\mathrm{A/B}}\dot{\mathcal{R}}_{\mathrm{B/A}} = -\dot{\mathcal{R}}_{\mathrm{A/B}}\mathcal{R}_{\mathrm{B/A}}. \tag{4.3.20}$$

Proof. Resolving (4.2.7) in $\mathrm{F_B}$ yields (4.3.17). □

The following result provides a vectrix version of Poisson's equation.

Fact 4.3.7. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} \end{bmatrix} = -\omega^{\times}_{\mathrm{B/A|B}} \begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix}. \tag{4.3.21}$$

Proof. It follows from (2.10.13) and (4.3.20) that

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} \\ \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}} \end{bmatrix} = \dot{\mathcal{O}}_{\mathrm{B/A}} \begin{bmatrix} \hat{\imath}_{\mathrm{A}} \\ \hat{\jmath}_{\mathrm{A}} \\ \hat{k}_{\mathrm{A}} \end{bmatrix} = \dot{\mathcal{O}}_{\mathrm{B/A}}\mathcal{O}_{\mathrm{A/B}} \begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix} = -\omega^{\times}_{\mathrm{B/A|B}} \begin{bmatrix} \hat{\imath}_{\mathrm{B}} \\ \hat{\jmath}_{\mathrm{B}} \\ \hat{k}_{\mathrm{B}} \end{bmatrix}.$$ □

Resolving (4.3.21) in $\mathrm{F_A}$ yields (4.3.17). Furthermore, (4.3.21) can be written as

$$\overset{\mathrm{A}\bullet}{\mathcal{F}}_{\mathrm{B}} = -\omega^{\times}_{\mathrm{B/A|B}}\mathcal{F}_{\mathrm{B}}. \tag{4.3.22}$$

4.4 The Transport Theorem

The following result is the *transport theorem*. This result expresses the difference between $\overset{\text{A}\bullet}{\vec{x}}$ and $\overset{\text{B}\bullet}{\vec{x}}$ in terms of $\vec{\omega}_{\text{B/A}}$ and $\vec{x}$. Note the "ABBA" pattern.

Fact 4.4.1. Let $\vec{x}$ be a physical vector, and let F_A and F_B be frames. Then,

$$\overset{\text{A}\bullet}{\vec{x}} = \overset{\text{B}\bullet}{\vec{x}} + \vec{\omega}_{\text{B/A}} \times \vec{x}. \tag{4.4.1}$$

Proof. The result follows from (4.3.5) and (4.2.5). □

The following result is an immediate consequence of the transport theorem.

Fact 4.4.2. Let F_A and F_B be frames. Then,

$$\overset{\text{A}\bullet}{\vec{\omega}}_{\text{B/A}} = \overset{\text{B}\bullet}{\vec{\omega}}_{\text{B/A}}. \tag{4.4.2}$$

Proof.

$$\overset{\text{A}\bullet}{\vec{\omega}}_{\text{B/A}} = \overset{\text{B}\bullet}{\vec{\omega}}_{\text{B/A}} + \vec{\omega}_{\text{B/A}} \times \vec{\omega}_{\text{B/A}} = \overset{\text{B}\bullet}{\vec{\omega}}_{\text{B/A}}.$$ □

The *angular acceleration of* F_B *relative to* F_A is defined by

$$\vec{\alpha}_{\text{B/A}} \triangleq \overset{\text{A}\bullet}{\vec{\omega}}_{\text{B/A}} = \overset{\text{B}\bullet}{\vec{\omega}}_{\text{B/A}}. \tag{4.4.3}$$

It thus follows from (4.3.8) that

$$\vec{\alpha}_{\text{A/B}} = -\vec{\alpha}_{\text{B/A}}. \tag{4.4.4}$$

Now, let C be a frame, Then, the *mixed angular acceleration* is defined by

$$\vec{\alpha}_{\text{B/A/C}} \triangleq \overset{\text{C}\bullet}{\vec{\omega}}_{\text{B/A}}. \tag{4.4.5}$$

Note that $\vec{\alpha}_{\text{B/A/A}} = \vec{\alpha}_{\text{B/A/B}} = \vec{\alpha}_{\text{B/A}}$. Furthermore, let F_D be a frame. Then, the angular acceleration is resolved as

$$\alpha_{\text{B/A|D}} \triangleq \vec{\alpha}_{\text{B/A}}\Big|_{\text{D}}. \tag{4.4.6}$$

Note that

$$\alpha_{\text{B/A|D}} = \overset{\text{A}\bullet}{\vec{\omega}}_{\text{B/A}}\Big|_{\text{D}} = \mathcal{O}_{\text{D/A}}\dot{\omega}_{\text{B/A|A}} = \overset{\text{B}\bullet}{\vec{\omega}}_{\text{B/A}}\Big|_{\text{D}} = \mathcal{O}_{\text{D/B}}\dot{\omega}_{\text{B/A|B}}. \tag{4.4.7}$$

Finally, the mixed angular acceleration is resolved as

$$\alpha_{\text{B/A/C|D}} \triangleq \vec{\alpha}_{\text{B/A/C}}\Big|_{\text{D}} \triangleq \overset{\text{C}\bullet}{\vec{\omega}}_{\text{B/A}}\Big|_{\text{D}}. \tag{4.4.8}$$

Note that

$$\alpha_{\mathrm{B/A/C|D}} = \mathcal{O}_{\mathrm{D/C}}\ \vec{\alpha}_{\mathrm{B/A/C}}\Big|_{\mathrm{C}} = \mathcal{O}_{\mathrm{D/C}}\ \overset{\mathrm{C}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{C}} = \mathcal{O}_{\mathrm{D/C}}\dot{\omega}_{\mathrm{B/A|C}}. \tag{4.4.9}$$

The following result is an immediate consequence of the transport theorem.

Fact 4.4.3. Let $\mathrm{F_A}$ be a frame, let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let x, y, z be points, and assume that y and z are fixed in $\mathcal{B}$. Then,

$$\vec{v}_{z/x/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{z/y} + \vec{v}_{y/x/\mathrm{A}}. \tag{4.4.10}$$

The following result is based on Figure 4.4.1.

Fact 4.4.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames with origins $\mathrm{o_A}$ and $\mathrm{o_B}$, respectively, and let x be a point. Then,

$$\vec{v}_{x/\mathrm{o_A}/\mathrm{A}} = \vec{v}_{x/\mathrm{o_A}/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{o_A}}, \tag{4.4.11}$$

$$\vec{v}_{x/\mathrm{o_A}/\mathrm{A}} = \vec{v}_{x/\mathrm{o_B}/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{o_B}} + \vec{v}_{\mathrm{o_B}/\mathrm{o_A}/\mathrm{A}}. \tag{4.4.12}$$

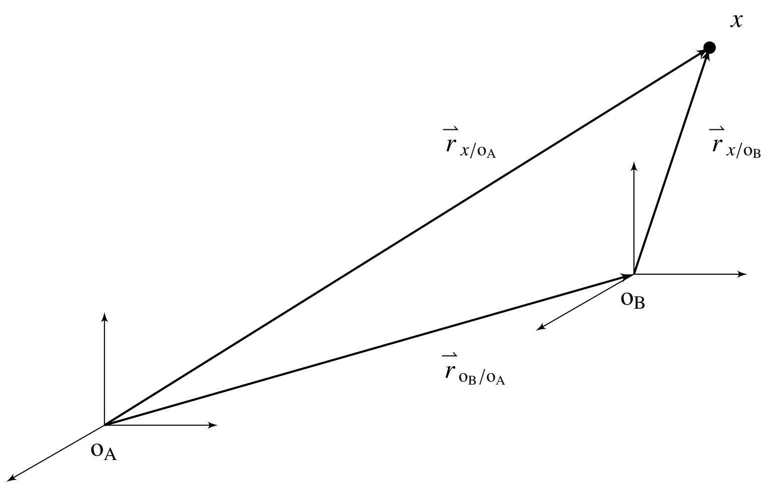


Figure 4.4.1: Geometry for relative motion.

Fact 4.4.5. Let $\vec{M}$ be a physical matrix, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{M}} = \overset{\mathrm{B}\bullet}{\vec{M}} + \vec{\Omega}_{\mathrm{B/A}}\vec{M} - \vec{M}\vec{\Omega}_{\mathrm{B/A}}. \tag{4.4.13}$$

Proof. For convenience, let $\vec{M} = \vec{x}\,\vec{y}\,'$. Then,

$$\begin{aligned}\overset{\mathrm{A}\bullet}{\vec{M}} &= \overset{\mathrm{A}\bullet}{\vec{x}}\ \vec{y}\,' + \vec{x}\,\overset{\mathrm{A}\bullet}{\vec{y}}{}'\\ &= (\overset{\mathrm{B}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{x})\vec{y}\,' + \vec{x}(\overset{\mathrm{B}\bullet}{\vec{y}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{y})'\end{aligned}$$

$$
\begin{aligned}
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet\prime}{\vec{y}} + (\vec{\omega}_{\mathrm{B/A}} \times \vec{x})\vec{y}\,' + \vec{x}(\vec{\omega}_{\mathrm{B/A}} \times \vec{y})' \\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet\prime}{\vec{y}} + (\vec{\omega}_{\mathrm{B/A}}^{\times}\vec{x})\vec{y}\,' + \vec{x}(\vec{\omega}_{\mathrm{B/A}}^{\times}\vec{y})' \\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet\prime}{\vec{y}} + \vec{\omega}_{\mathrm{B/A}}^{\times}\vec{x}\vec{y}\,' + \vec{x}\vec{y}\,'\vec{\omega}_{\mathrm{B/A}}^{\times\prime} \\
&= \overset{\mathrm{B}\bullet}{\vec{x}}\,\vec{y}\,' + \vec{x}\,\overset{\mathrm{B}\bullet\prime}{\vec{y}} + \vec{\omega}_{\mathrm{B/A}}^{\times}\vec{x}\vec{y}\,' - \vec{x}\vec{y}\,'\vec{\omega}_{\mathrm{B/A}}^{\times} \\
&= \overset{\mathrm{B}\bullet}{\vec{M}} + \vec{\Omega}_{\mathrm{B/A}}\vec{M} - \vec{M}\vec{\Omega}_{\mathrm{B/A}}. \qquad \square
\end{aligned}
$$

Fact 4.4.6. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} + \vec{\Omega}_{\mathrm{B/A}}\vec{R}_{\mathrm{B/A}} - \vec{R}_{\mathrm{B/A}}\vec{\Omega}_{\mathrm{B/A}}, \tag{4.4.14}$$

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = \overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}} + \vec{\Omega}_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}} - \vec{R}_{\mathrm{A/B}}\vec{\Omega}_{\mathrm{B/A}}. \tag{4.4.15}$$

4.5 Summation of Angular Velocities and Angular Accelerations

The following result shows that slash and split holds for the angular velocity vector.

Fact 4.5.1. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\vec{\Omega}_{\mathrm{C/A}} = \vec{\Omega}_{\mathrm{C/B}} + \vec{\Omega}_{\mathrm{B/A}}, \tag{4.5.1}$$

$$\vec{\omega}_{\mathrm{C/A}} = \vec{\omega}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}}. \tag{4.5.2}$$

Furthermore,

$$\Omega_{\mathrm{C/A|C}} = \Omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}}\Omega_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/C}}, \tag{4.5.3}$$

$$\omega_{\mathrm{C/A|C}} = \omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}}\omega_{\mathrm{B/A|B}}. \tag{4.5.4}$$

Proof. Using the transport theorem and (4.2.19) yields

$$
\begin{aligned}
\vec{\Omega}_{\mathrm{C/A}} &= \hat{\imath}_{\mathrm{A}}\overset{\mathrm{C}\bullet\prime}{\hat{\imath}}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{C}\bullet\prime}{\hat{\jmath}}_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{C}\bullet\prime}{\hat{k}}_{\mathrm{A}} \\
&= \hat{\imath}_{\mathrm{A}}\left(\overset{\mathrm{B}\bullet}{\hat{\imath}}_{\mathrm{A}} + \vec{\Omega}_{\mathrm{B/C}}\hat{\imath}_{\mathrm{A}}\right)' + \hat{\jmath}_{\mathrm{A}}\left(\overset{\mathrm{B}\bullet}{\hat{\jmath}}_{\mathrm{A}} + \vec{\Omega}_{\mathrm{B/C}}\hat{\jmath}_{\mathrm{A}}\right)' + \hat{k}_{\mathrm{A}}\left(\overset{\mathrm{B}\bullet}{\hat{k}}_{\mathrm{A}} + \vec{\Omega}_{\mathrm{B/C}}\hat{k}_{\mathrm{A}}\right)' \\
&= \hat{\imath}_{\mathrm{A}}(\vec{\Omega}_{\mathrm{B/C}}\hat{\imath}_{\mathrm{A}})' + \hat{\jmath}_{\mathrm{A}}\left(\vec{\Omega}_{\mathrm{B/C}}\hat{\jmath}_{\mathrm{A}}\right)' + \hat{k}_{\mathrm{A}}\left(\vec{\Omega}_{\mathrm{B/C}}\hat{k}_{\mathrm{A}}\right)' + \hat{\imath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\imath}}_{\mathrm{A}} + \hat{\jmath}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{\jmath}}_{\mathrm{A}} + \hat{k}_{\mathrm{A}}\overset{\mathrm{B}\bullet\prime}{\hat{k}}_{\mathrm{A}} \\
&= \hat{\imath}_{\mathrm{A}}\hat{\imath}_{\mathrm{A}}'\vec{\Omega}_{\mathrm{C/B}} + \hat{\jmath}_{\mathrm{A}}\hat{\jmath}_{\mathrm{A}}'\vec{\Omega}_{\mathrm{C/B}} + \hat{k}_{\mathrm{A}}\hat{k}_{\mathrm{A}}'\vec{\Omega}_{\mathrm{C/B}} + \vec{\Omega}_{\mathrm{B/A}} \\
&= \vec{\Omega}_{\mathrm{C/B}} + \vec{\Omega}_{\mathrm{B/A}}.
\end{aligned}
$$

Finally, to prove (4.5.3), note that

$$\Omega_{\mathrm{C/A|C}} = \vec{\Omega}_{\mathrm{C/A}}\Big|_{\mathrm{C}}$$

$$
\begin{aligned}
&= \Omega_{\mathrm{C/B|C}} + \vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{C}} \\
&= \Omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}}\,\vec{\Omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}\,\mathcal{O}_{\mathrm{B/C}} \\
&= \Omega_{\mathrm{C/B|C}} + \mathcal{O}_{\mathrm{C/B}}\Omega_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/C}}. \qquad \square
\end{aligned}
$$

The following result shows that slash and split does not hold for the angular acceleration vector.

Fact 4.5.2. Let $\mathrm{F_A}$, $\mathrm{F_B}$, and $\mathrm{F_C}$ be frames. Then,

$$\vec{\alpha}_{\mathrm{C/A}} = \vec{\alpha}_{\mathrm{C/B}} + \vec{\alpha}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{C/B}} \times \vec{\omega}_{\mathrm{B/A}}. \tag{4.5.5}$$

Fact 4.5.3. Let $\mathrm{F_A}$, $\mathrm{F_B}$, $\mathrm{F_C}$, and $\mathrm{F_D}$ be frames. Then,

$$
\begin{aligned}
\vec{\alpha}_{\mathrm{D/A}} &= \vec{\alpha}_{\mathrm{D/C}} + \vec{\alpha}_{\mathrm{C/B}} + \vec{\alpha}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{D/C}} \times \vec{\omega}_{\mathrm{C/B}} - \vec{\omega}_{\mathrm{D/B}} \times \vec{\omega}_{\mathrm{B/A}} && (4.5.6)\\
&= \vec{\alpha}_{\mathrm{D/C}} + \vec{\alpha}_{\mathrm{C/B}} + \vec{\alpha}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{D/C}} \times \vec{\omega}_{\mathrm{C/A}} - \vec{\omega}_{\mathrm{C/B}} \times \vec{\omega}_{\mathrm{B/A}}. && (4.5.7)
\end{aligned}
$$

4.6 The Double Transport Theorem

Applying the transport theorem twice yields the *double transport theorem*. This result expresses the difference between $\overset{\mathrm{A\bullet\bullet}}{\vec{x}}$ and $\overset{\mathrm{B\bullet\bullet}}{\vec{x}}$ in terms of $\vec{x}$, $\overset{\mathrm{B\bullet}}{\vec{x}}$, $\vec{\omega}_{\mathrm{B/A}}$, and $\overset{\mathrm{B\bullet}}{\vec{\omega}}_{\mathrm{B/A}}$.

Fact 4.6.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Then,

$$\overset{\mathrm{A\bullet\bullet}}{\vec{x}} = \overset{\mathrm{B\bullet\bullet}}{\vec{x}} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B\bullet}}{\vec{x}} + \overset{\mathrm{B\bullet}}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}). \tag{4.6.1}$$

Proof. Note that

$$
\begin{aligned}
\overset{\mathrm{A\bullet\bullet}}{\vec{x}} &= \overset{\mathrm{A\bullet}}{\overset{\mathrm{B\bullet}}{\vec{x}}} + \overbrace{\vec{\omega}_{\mathrm{B/A}} \times \vec{x}}^{\mathrm{A\bullet}} \\
&= \overset{\mathrm{B\bullet\bullet}}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B\bullet}}{\vec{x}} + \overset{\mathrm{B\bullet}}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{A\bullet}}{\vec{x}} \\
&= \overset{\mathrm{B\bullet\bullet}}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B\bullet}}{\vec{x}} + \overset{\mathrm{B\bullet}}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times (\overset{\mathrm{B\bullet}}{\vec{x}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{x}) \\
&= \overset{\mathrm{B\bullet\bullet}}{\vec{x}} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B\bullet}}{\vec{x}} + \overset{\mathrm{B\bullet}}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}). \qquad \square
\end{aligned}
$$

If $\vec{x}$ is the position vector $\vec{r}_{x/w}$, then (4.6.1) can be written as

$$\overset{\mathrm{A\bullet\bullet}}{\vec{r}}_{x/w} = \overset{\mathrm{B\bullet\bullet}}{\vec{r}}_{x/w} + \underbrace{2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B\bullet}}{\vec{r}}_{x/w}}_{\text{Coriolis acceleration}} + \underbrace{\overset{\mathrm{B\bullet}}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{x/w}}_{\substack{\text{angular-acceleration}\\ \text{acceleration}}} + \underbrace{\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/w})}_{\text{centripetal acceleration}}, \tag{4.6.2}$$

which can be rewritten as

$$\vec{a}_{x/w/\mathrm{A}} = \vec{a}_{x/w/\mathrm{B}} + \underbrace{2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/w/\mathrm{B}}}_{\text{Coriolis acceleration}} + \underbrace{\vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/w}}_{\substack{\text{angular-acceleration}\\ \text{acceleration}}} + \underbrace{\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/w})}_{\text{centripetal acceleration}}, \tag{4.6.3}$$

or, more succinctly,

$$\vec{a}_{x/w/\mathrm{A}} = \vec{a}_{x/w/\mathrm{B}} + \underbrace{2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/w/\mathrm{B}}}_{\vec{a}_{\mathrm{Cor}}} + \underbrace{\vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/w}}_{\vec{a}_{\mathrm{aa}}} + \underbrace{\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/w})}_{\vec{a}_{\mathrm{cp}}}. \tag{4.6.4}$$

Equation (4.6.3) shows how accelerations with respect to different frames differ, specifically,

$$\vec{a}_{x/w/\mathrm{A}} - \vec{a}_{x/w/\mathrm{B}} = \underbrace{2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/w/\mathrm{B}}}_{\vec{a}_{\mathrm{Cor}}} + \underbrace{\vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/w}}_{\vec{a}_{\mathrm{aa}}} + \underbrace{\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/w})}_{\vec{a}_{\mathrm{cp}}}. \tag{4.6.5}$$

Furthermore, note that

$$\begin{aligned}
\vec{a}_{x/w/\mathrm{B}} &= \vec{a}_{x/w/\mathrm{A}} - 2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/w/\mathrm{B}} - \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/w} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/w})\\
&= \vec{a}_{x/w/\mathrm{A}} + 2\vec{\omega}_{\mathrm{A/B}} \times (\vec{v}_{x/w/\mathrm{A}} + \vec{\omega}_{\mathrm{A/B}} \times \vec{r}_{x/w}) + \vec{\alpha}_{\mathrm{A/B}} \times \vec{r}_{x/w} - \vec{\omega}_{\mathrm{A/B}} \times (\vec{\omega}_{\mathrm{A/B}} \times \vec{r}_{x/w})\\
&= \vec{a}_{x/w/\mathrm{A}} + 2\vec{\omega}_{\mathrm{A/B}} \times \vec{v}_{x/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{A/B}} \times \vec{r}_{x/w} + \vec{\omega}_{\mathrm{A/B}} \times (\vec{\omega}_{\mathrm{A/B}} \times \vec{r}_{x/w}),
\end{aligned} \tag{4.6.6}$$

which shows that the frames $\mathrm{F_A}$ and $\mathrm{F_B}$ are interchangeable in (4.6.3), as expected.

It is often the case in practice that the Coriolis acceleration, angular-acceleration acceleration, and centripetal acceleration involve sums of terms, many of which are common but have opposite signs and thus cancel. Therefore, unless it is of interest to determine these terms separately, it is more efficient to compute $\vec{a}_{x/w/\mathrm{A}}$ directly than to compute each term in (4.6.4).

The following result extends (4.6.3) to the case where $\mathrm{F_A}$ and $\mathrm{F_B}$ have possibly different origins, as illustrated in Figure 4.4.1.

Fact 4.6.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames with origins $\mathrm{o_A}$ and $\mathrm{o_B}$, respectively, and let x be a point. Then,

$$\begin{aligned}
\vec{a}_{x/\mathrm{o_A}/\mathrm{A}} = \vec{a}_{x/\mathrm{o_B}/\mathrm{B}} &+ 2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/\mathrm{o_B}/\mathrm{B}} + \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{o_B}}\\
&+ \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{o_B}}) + \vec{a}_{\mathrm{o_B}/\mathrm{o_A}/\mathrm{A}}.
\end{aligned} \tag{4.6.7}$$

Using (4.4.11), it follows that (4.6.7) can be written as

$$\begin{aligned}
\vec{a}_{x/\mathrm{o_A}/\mathrm{A}} = \vec{a}_{x/\mathrm{o_B}/\mathrm{B}} &+ 2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{x/\mathrm{o_B}/\mathrm{B}} + \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{x/\mathrm{o_B}}\\
&+ \vec{\omega}_{\mathrm{B/A}} \times (\vec{v}_{x/\mathrm{o_B}/\mathrm{A}} - \vec{v}_{x/\mathrm{o_B}/\mathrm{B}}) + \vec{a}_{\mathrm{o_B}/\mathrm{o_A}/\mathrm{A}}.
\end{aligned} \tag{4.6.8}$$

Assume that $\vec{\omega}_{\mathrm{B/A}}$ is orthogonal to $\vec{r}_{x/\mathrm{o_B}}$. Then, the centripetal acceleration is given by $-|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{r}_{x/\mathrm{o_B}}$, and thus the direction of the centripetal acceleration is opposite to the direction of $\vec{r}_{x/\mathrm{o_B}}$. This case occurs when a point moves in a circle.

4.7 The Angular Velocity Vector and Eigenaxis Derivative†

In this section we relate the angular velocity vector to the derivative of the eigenaxis and eigenangle, which appear in Rodrigues's formula.

Fact 4.7.1. Let $\mathrm{F_A}$ be a frame, and let $\hat{n}$ be a unit dimensionless physical vector. Then, for all time,

$$\overset{\mathrm{A}\bullet\prime}{\hat{n}}\,\hat{n} = \hat{n}'\,\overset{\mathrm{A}\bullet}{\hat{n}} = 0, \tag{4.7.1}$$

$$\hat{n}'\,\overset{\mathrm{A}\bullet\times}{\hat{n}} = -\overset{\mathrm{A}\bullet\prime}{\hat{n}}\,\hat{n}^{\times}, \tag{4.7.2}$$

$$\hat{n}'(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}) = 0, \tag{4.7.3}$$

$$\overset{\mathrm{A}\bullet\prime}{\hat{n}}\,(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}) = 0, \tag{4.7.4}$$

$$\hat{n}\times(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}) = -\overset{\mathrm{A}\bullet}{\hat{n}}, \tag{4.7.5}$$

$$\hat{n}\,\overset{\mathrm{A}\bullet\prime}{\hat{n}}\,\hat{n}^{\times} + \hat{n}^{\times}\,\overset{\mathrm{A}\bullet}{\hat{n}}\,\hat{n}' = -\overset{\mathrm{A}\bullet\times}{\hat{n}}, \tag{4.7.6}$$

$$\overset{\mathrm{A}\bullet\times}{\hat{n}}\,\hat{n}^{\times} = \hat{n}\,\overset{\mathrm{A}\bullet\prime}{\hat{n}}, \tag{4.7.7}$$

$$\hat{n}^{\times}\,\overset{\mathrm{A}\bullet\times}{\hat{n}} = \overset{\mathrm{A}\bullet}{\hat{n}}\,\hat{n}', \tag{4.7.8}$$

$$(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}})^{\times} = \overset{\mathrm{A}\bullet}{\hat{n}}\,\hat{n}' - \hat{n}\,\overset{\mathrm{A}\bullet\prime}{\hat{n}}, \tag{4.7.9}$$

$$|\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}| = |\overset{\mathrm{A}\bullet}{\hat{n}}|, \tag{4.7.10}$$

$$\overset{\mathrm{A}\bullet\bullet\prime}{\hat{n}}\,\hat{n} + \overset{\mathrm{A}\bullet\prime}{\hat{n}}\,\overset{\mathrm{A}\bullet}{\hat{n}} = 0. \tag{4.7.11}$$

Furthermore, at each instant of time, the following conditions are equivalent:

i) $\overset{\mathrm{A}\bullet}{\hat{n}} = 0$.

ii) $\overset{\mathrm{A}\bullet}{\hat{n}}\,\hat{n}' + \hat{n}\,\overset{\mathrm{A}\bullet\prime}{\hat{n}} = 0$.

iii) $\hat{n}\times(\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}}) = 0$.

iv) $\hat{n}\,\overset{\mathrm{A}\bullet\prime}{\hat{n}}\,\hat{n}^{\times} + \hat{n}^{\times}\,\overset{\mathrm{A}\bullet}{\hat{n}}\,\hat{n}' = 0$.

v) $\hat{n}\times\overset{\mathrm{A}\bullet}{\hat{n}} = 0$.

Finally, if, for all time, $\overset{\mathrm{A}\bullet\bullet}{\hat{n}} = 0$, then, for all time, $\overset{\mathrm{A}\bullet}{\hat{n}} = 0$.

Proof. The proof is left to the reader. □

For the following result note that

$$\cot\tfrac{1}{2}\theta_{\mathrm{B/A}} = \frac{\sin\theta_{\mathrm{B/A}}}{1-\cos\theta_{\mathrm{B/A}}} = \frac{1+\cos\theta_{\mathrm{B/A}}}{\sin\theta_{\mathrm{B/A}}}. \tag{4.7.12}$$

Fact 4.7.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.13}$$

$$= (\cos\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.14}$$

$$= \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}}, \tag{4.7.15}$$

$$\begin{aligned}\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} = {} & [(\sin\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - \vec{I}) + (\cos\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\dot{\theta}_{\mathrm{B/A}} \\ & + (1-\cos\theta_{\mathrm{B/A}})(\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}\, \hat{n}'_{\mathrm{B/A}} + \hat{n}_{\mathrm{B/A}}\, \overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{A}\bullet\times}{\hat{n}}_{\mathrm{B/A}},\end{aligned} \tag{4.7.16}$$

$$\begin{aligned}\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{B/A}} = {} & [(\sin\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - \vec{I}) + (\cos\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\dot{\theta}_{\mathrm{B/A}} \\ & + (1-\cos\theta_{\mathrm{B/A}})(\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}\, \hat{n}'_{\mathrm{B/A}} + \hat{n}_{\mathrm{B/A}}\, \overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}) + (\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{B}\bullet\times}{\hat{n}}_{\mathrm{B/A}},\end{aligned} \tag{4.7.17}$$

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.18}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}, \tag{4.7.19}$$

$$|\vec{\omega}_{\mathrm{B/A}}| = \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(1-\cos\theta_{\mathrm{B/A}})\,|\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2} \tag{4.7.20}$$

$$= \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(1-\cos\theta_{\mathrm{B/A}})\,|\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2}, \tag{4.7.21}$$

$$\dot{\theta}_{\mathrm{B/A}} = \hat{n}'_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}, \tag{4.7.22}$$

$$\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \vec{\omega}_{\mathrm{B/A}} = (\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}} \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.23}$$

$$= -(\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{A}\bullet\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \hat{n}_{\mathrm{B/A}}, \tag{4.7.24}$$

$$\overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \vec{\omega}_{\mathrm{B/A}} = (\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.25}$$

$$= -(\sin\theta_{\mathrm{B/A}})\, \overset{\mathrm{B}\bullet\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \hat{n}_{\mathrm{B/A}}. \tag{4.7.26}$$

Finally, if $\theta_{\mathrm{B/A}} \neq 0$, then

$$|\vec{\omega}_{\mathrm{B/A}}| = \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\, \overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \vec{\omega}_{\mathrm{B/A}}} \tag{4.7.27}$$

$$= \sqrt{\dot{\theta}^2_{\mathrm{B/A}} + 2(\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\, \overset{\mathrm{B}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \vec{\omega}_{\mathrm{B/A}}}, \tag{4.7.28}$$

$$\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \tfrac{1}{2}[(\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) - \hat{n}^{\times}_{\mathrm{B/A}}]\vec{\omega}_{\mathrm{B/A}} \tag{4.7.29}$$

$$= \tfrac{1}{2}[\dot{\theta}_{\mathrm{B/A}}(\cot \tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\cot \tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}}], \tag{4.7.30}$$

$$\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} = \tfrac{1}{2}[(\cot \tfrac{1}{2}\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}}) + \hat{n}^{\times}_{\mathrm{B/A}}]\vec{\omega}_{\mathrm{B/A}} \tag{4.7.31}$$

$$= \tfrac{1}{2}[\dot{\theta}_{\mathrm{B/A}}(\cot \tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\cot \tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}}]. \tag{4.7.32}$$

Proof. Using Fact 2.11.9 and Fact 4.7.1, recalling that

$$\vec{R}_{\mathrm{A/B}} = (\cos\theta_{\mathrm{B/A}})\vec{I} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}} - (\sin\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}},$$

it follows that

$$\vec{\omega}^{\times}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}}\,\vec{R}_{\mathrm{A/B}}$$
$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}^{\times}_{\mathrm{B/A}} + (1 - \cos\theta_{\mathrm{B/A}})(\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}})^{\times} + (\sin\theta_{\mathrm{B/A}})\,\overset{\mathrm{A}\bullet}{\hat{n}}{}^{\times}_{\mathrm{B/A}}.$$

Next, sufficiency in *i*) is immediate. To prove necessity, assume that $\vec{\omega}_{\mathrm{B/A}} = 0$. Then, $\dot{\theta}_{\mathrm{B/A}} = \hat{n}'_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}} = 0$. Furthermore, note that $(\sin\theta_{\mathrm{B/A}})|\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}|^2 = \overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}} = 0$. Therefore, either $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$ or $\theta_{\mathrm{B/A}} = 0$ or $\theta_{\mathrm{B/A}} = \pi$. In the case that $\theta_{\mathrm{B/A}} = \pi$, it follows that $\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$. Finally, substituting (4.7.18) into the right hand side of (4.7.29) yields $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}$.

To derive (4.7.29), multiply (4.7.18) by $\hat{n}^{\times}_{\mathrm{B/A}}$, which yields

$$\hat{n}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = (\cos\theta_{\mathrm{B/A}} - 1)[\vec{I} - (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\,\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}.$$

Hence,

$$\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \frac{1}{\cos\theta_{\mathrm{B/A}} - 1}[\vec{I} - (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]^{-1}(\hat{n}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}}).$$

Now, using (2.9.10) yields (4.7.29). □

Equation (4.7.18) gives an expression for $\vec{\omega}_{\mathrm{B/A}}$ in terms of the eigenaxis $\hat{n}_{\mathrm{B/A}}$ and its derivative. If the eigenaxis is constant, then $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$, which includes an Euler rotation as a special case.

Fact 4.7.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then, at each instant of time the following statements are equivalent:

i) $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}$.

ii) $\vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}} = 0$.

iii) $\omega_{\mathrm{B/A}} \times n_{\mathrm{B/A}} = 0$.

iv) $\omega_{\mathrm{A/B}} \times n_{\mathrm{B/A}} = 0$.

v) $\mathcal{O}_{\mathrm{A/B}}\dot{n}_{\mathrm{B/A}} = \dot{n}_{\mathrm{B/A}}$.

vi) $\mathcal{O}_{\mathrm{B/A}}\dot{n}_{\mathrm{B/A}} = \dot{n}_{\mathrm{B/A}}$.

Furthermore, at each instant of time the following statements hold:

vii) If *i*)–*vi*) are satisfied and $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \neq 0$, then $\theta_{\mathrm{B/A}} = 0$.

viii) If $\theta_{\mathrm{B/A}} = 0$, then *i*)–*vi*) are satisfied.

ix) If *i*)–*vi*) are satisfied and $\theta_{\mathrm{B/A}} \neq 0$, then $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

x) The following conditions are equivalent:

a) $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

b) $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

xi) The following conditions are equivalent:

a) Either $\theta_{\mathrm{B/A}} = 0$ or $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

b) $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}$.

xii) If $\theta_{\mathrm{B/A}} = 0$ and $\dot{\theta}_{\mathrm{B/A}} = 0$, then $\vec{\omega}_{\mathrm{B/A}} = 0$.

xiii) The following conditions are equivalent:

a) $\vec{\omega}_{\mathrm{B/A}} = 0$.

b) $\dot{\theta}_{\mathrm{B/A}} = 0$ and either $\theta_{\mathrm{B/A}} = 0$ or $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

c) $\dot{\theta}_{\mathrm{B/A}} = 0$ and either $\theta_{\mathrm{B/A}} = 0$ or $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

xiv) The following conditions are equivalent:

a) $\vec{\omega}_{\mathrm{B/A}} \neq 0$.

b) Either $\dot{\theta}_{\mathrm{B/A}} \neq 0$ or both $\theta_{\mathrm{B/A}} \neq 0$ and $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \neq 0$.

c) Either $\dot{\theta}_{\mathrm{B/A}} \neq 0$ or both $\theta_{\mathrm{B/A}} \neq 0$ and $\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \neq 0$.

xv) If $\vec{\omega}_{\mathrm{B/A}} \neq 0$, then either $\theta_{\mathrm{B/A}} \neq 0$ or $\dot{\theta}_{\mathrm{B/A}} \neq 0$.

Fact 4.7.4. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and, for all time, assume that $\omega_{\mathrm{B/A|B}} \neq 0$ and $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$. Then, for all time, the following conditions are equivalent:

i) $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

ii) $\vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}} = 0$.

Proof. The result follows from Problem 4.17.3. □

For small angles $\theta_{\mathrm{B/A}}$, we have

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + [\tfrac{1}{2}\theta_{\mathrm{B/A}}^2 + O(\theta_{\mathrm{B/A}}^4)]\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + [\theta_{\mathrm{B/A}} + O(\theta_{\mathrm{B/A}}^3)]\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.7.33}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - [\tfrac{1}{2}\theta_{\mathrm{B/A}}^2 + O(\theta_{\mathrm{B/A}}^4)]\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + [\theta_{\mathrm{B/A}} + O(\theta_{\mathrm{B/A}}^3)]\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}, \tag{4.7.34}$$

$$|\vec{\omega}_{\rm B/A}| = \sqrt{\dot\theta^2_{\rm B/A} + [\theta^2_{\rm B/A} + O(\theta^4_{\rm B/A})]\,|\overset{\rm A\bullet}{\hat n}_{\rm B/A}|^2}, \tag{4.7.35}$$

where $\lim_{x\to 0} O(x)/x$ exists. We thus have the approximations

$$\vec{\omega}_{\rm B/A} \approx \dot\theta_{\rm B/A}\hat n_{\rm B/A} + \tfrac{1}{2}\theta^2_{\rm B/A}\hat n_{\rm B/A}\times \overset{\rm A\bullet}{\hat n}_{\rm B/A} + \theta_{\rm B/A}\overset{\rm A\bullet}{\hat n}_{\rm B/A}, \tag{4.7.36}$$

$$\vec{\omega}_{\rm B/A} \approx \dot\theta_{\rm B/A}\hat n_{\rm B/A} - \tfrac{1}{2}\theta^2_{\rm B/A}\hat n_{\rm B/A}\times \overset{\rm B\bullet}{\hat n}_{\rm B/A} + \theta_{\rm B/A}\overset{\rm B\bullet}{\hat n}_{\rm B/A}, \tag{4.7.37}$$

$$|\vec{\omega}_{\rm B/A}| \approx \sqrt{\dot\theta^2_{\rm B/A} + \theta^2_{\rm B/A}|\overset{\rm A\bullet}{\hat n}_{\rm B/A}|^2}. \tag{4.7.38}$$

Consequently,

$$\lim_{\theta_{\rm B/A}\to 0} \vec{\omega}_{\rm B/A} = \dot\theta_{\rm B/A}\hat n_{\rm B/A}, \tag{4.7.39}$$

$$\lim_{\theta_{\rm B/A}\to 0} |\vec{\omega}_{\rm B/A}| = |\dot\theta_{\rm B/A}|. \tag{4.7.40}$$

Therefore, for an infinitisimal rotation from $\rm F_A$ to $\rm F_B$, the angular velocity vector $\vec{\omega}_{\rm B/A}$ can be viewed as the eigenaxis of rotation scaled by the rate of rotation. In other words, the physical vector $\vec{\omega}_{\rm B/A}$ can be viewed as the instantaneous axis of rotation, where the rate of rotation is given by $|\vec{\omega}_{\rm B/A}|$ and the direction of rotation is given by the curled fingers of the right hand with the thumb pointing in the direction of $\vec{\omega}_{\rm B/A}$.

4.8 The Angular Acceleration Vector and Eigenaxis and Eigenangle†

Differentiating (4.7.18) and (4.7.19) yields

$$\begin{aligned}\vec{\alpha}_{\rm B/A} &= \ddot\theta_{\rm B/A}\hat n_{\rm B/A} + \dot\theta_{\rm B/A}[(1+\cos\theta_{\rm B/A})\overset{\rm A\bullet}{\hat n}_{\rm B/A} + (\sin\theta_{\rm B/A})\hat n_{\rm B/A}\times\overset{\rm A\bullet}{\hat n}_{\rm B/A}]\\ &\quad + (1-\cos\theta_{\rm B/A})\hat n_{\rm B/A}\times\overset{\rm A\bullet\bullet}{\hat n}_{\rm B/A} + (\sin\theta_{\rm B/A})\overset{\rm A\bullet\bullet}{\hat n}_{\rm B/A} \qquad (4.8.1)\\ &= \ddot\theta_{\rm B/A}\hat n_{\rm B/A} + \dot\theta_{\rm B/A}[(1+\cos\theta_{\rm B/A})\overset{\rm B\bullet}{\hat n}_{\rm B/A} - (\sin\theta_{\rm B/A})\hat n_{\rm B/A}\times\overset{\rm B\bullet}{\hat n}_{\rm B/A}]\\ &\quad - (1-\cos\theta_{\rm B/A})\hat n_{\rm B/A}\times\overset{\rm B\bullet\bullet}{\hat n}_{\rm B/A} + (\sin\theta_{\rm B/A})\overset{\rm B\bullet\bullet}{\hat n}_{\rm B/A}. \qquad (4.8.2)\end{aligned}$$

Fact 4.8.1. Let $\rm F_A$ and $\rm F_B$ be frames. Then, at each instant of time, the following statements hold:

i) If $\theta_{\rm B/A} = 0$, then $\vec{\alpha}_{\rm B/A} = \ddot\theta_{\rm B/A}\hat n_{\rm B/A} + 2\dot\theta_{\rm B/A}\overset{\rm A\bullet}{\hat n}_{\rm B/A}$.

ii) If $\vec{\alpha}_{\rm B/A} = 0$, then

$$\begin{aligned}\ddot\theta_{\rm B/A} &= \overset{\rm A\bullet\prime}{\hat n}_{\rm B/A}\vec{\omega}_{\rm B/A} \qquad (4.8.3)\\ &= (\sin\theta_{\rm B/A})\overset{\rm A\bullet\prime}{\hat n}_{\rm B/A}\overset{\rm A\bullet}{\hat n}_{\rm B/A} \qquad (4.8.4)\\ &= -(\sin\theta_{\rm B/A})\hat n'_{\rm B/A}\overset{\rm A\bullet\bullet}{\hat n}_{\rm B/A}, \qquad (4.8.5)\end{aligned}$$

$$|\vec{\omega}_{\rm B/A}|^2 = \dot\theta^2_{\rm B/A} + 2(\tan\tfrac{1}{2}\theta_{\rm B/A})\ddot\theta_{\rm B/A}. \tag{4.8.6}$$

iii) If $\theta_{\mathrm{B/A}} = 0$ and $\vec{\alpha}_{\mathrm{B/A}} = 0$, then $|\vec{\omega}_{\mathrm{B/A}}| = |\dot{\theta}_{\mathrm{B/A}}|$, $\ddot{\theta}_{\mathrm{B/A}} = 0$, $\dot{\theta}_{\mathrm{B/A}} \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$, and $\overset{\mathrm{A}\bullet\prime}{\hat{n}}_{\mathrm{B/A}}\, \vec{\omega}_{\mathrm{B/A}} = 0$.

iv) Assume that $\vec{\alpha}_{\mathrm{B/A}} = 0$ and $\sin\theta_{\mathrm{B/A}} \neq 0$. Then, $\ddot{\theta}_{\mathrm{B/A}} = 0$ if and only if $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$.

v) Assume that $\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$. Then, $\vec{\alpha}_{\mathrm{B/A}} = 0$ if and only if

$$\ddot{\theta}_{\mathrm{B/A}} = 0 \tag{4.8.7}$$

and

$$\dot{\theta}_{\mathrm{B/A}}[(1+\cos\theta_{\mathrm{B/A}}) \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}] = 0. \tag{4.8.8}$$

vi) Assume that $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$ and $\overset{\mathrm{A}\bullet\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$. Then, $\vec{\alpha}_{\mathrm{B/A}} = 0$ if and only if $\ddot{\theta}_{\mathrm{B/A}} = 0$.

Proof. All statements follow from (4.8.1). □

Fact 4.8.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, assume that there exists an instant of time t such that $\omega_{\mathrm{B/A}}(t) \neq 0$, assume that, for all t, $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = 0$, and assume that there exists an instant of time t_0 such that $\mathcal{R}_{\mathrm{B/A}}(t_0) = I_3$. Then, for all $t \geq t_0$ such that $\sin\theta_{\mathrm{B/A}}(t) \neq 0$, it follows that $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}(t) = 0$ and $\ddot{\theta}_{\mathrm{B/A}}(t) = 0$.

Proof. For convenience, let $t_0 = 0$. It follows from Fact 4.3.4 that both $\omega_{\mathrm{B/A|B}}$ and $\omega_{\mathrm{A/B|A}}$ are constant. Furthermore, since $\mathcal{O}_{\mathrm{A/B}}(0) = I_3$, it follows from (4.3.10) that $\omega_{\mathrm{B/A|B}} = \mathcal{O}_{\mathrm{A/B}}(0)\omega_{\mathrm{B/A|B}} = -\omega_{\mathrm{A/B|A}} \neq 0$.

Next, it follows from (4.3.17) that

$$\dot{\mathcal{R}}_{\mathrm{A/B}}(t) = -\omega^{\times}_{\mathrm{B/A|B}}\mathcal{R}_{\mathrm{A/B}}(t),$$
$$\dot{\mathcal{R}}_{\mathrm{B/A}}(t) = -\omega^{\times}_{\mathrm{A/B|A}}\mathcal{R}_{\mathrm{B/A}}(t).$$

Therefore,

$$\mathcal{R}_{\mathrm{A/B}}(t) = e^{-\omega^{\times}_{\mathrm{B/A|B}}t}\mathcal{R}_{\mathrm{A/B}}(0) = \mathcal{R}_{\mathrm{A/B}}(0)e^{\omega^{\times}_{\mathrm{A/B|A}}t},$$
$$\mathcal{R}_{\mathrm{B/A}}(t) = e^{-\omega^{\times}_{\mathrm{A/B|A}}t}\mathcal{R}_{\mathrm{B/A}}(0) = \mathcal{R}_{\mathrm{B/A}}(0)e^{\omega^{\times}_{\mathrm{B/A|B}}t}.$$

Hence, it follows from (2.11.45) that, for all $t \geq 0$,

$$\begin{aligned}
2\sin\theta_{\mathrm{B/A}}(t)n_{\mathrm{B/A}}(t) \times \omega_{\mathrm{B/A|B}} &= [\mathcal{R}_{\mathrm{B/A}}(t) - \mathcal{R}_{\mathrm{A/B}}(t)]\omega_{\mathrm{B/A|B}} \\
&= \mathcal{R}_{\mathrm{B/A}}(0)e^{\omega^{\times}_{\mathrm{B/A|B}}t}\omega_{\mathrm{B/A|B}} - \mathcal{R}_{\mathrm{A/B}}(0)e^{\omega^{\times}_{\mathrm{A/B|A}}t}\omega_{\mathrm{B/A|B}} \\
&= \mathcal{R}_{\mathrm{B/A}}(0)e^{\omega^{\times}_{\mathrm{B/A|B}}t}\omega_{\mathrm{B/A|B}} - \mathcal{R}_{\mathrm{A/B}}(0)e^{-\omega^{\times}_{\mathrm{B/A|B}}t}\omega_{\mathrm{B/A|B}} \\
&= \mathcal{R}_{\mathrm{B/A}}(0)\omega_{\mathrm{B/A|B}} - \mathcal{R}_{\mathrm{A/B}}(0)\omega_{\mathrm{B/A|B}} \\
&= [\mathcal{R}_{\mathrm{B/A}}(0) - \mathcal{R}_{\mathrm{A/B}}(0)]\omega_{\mathrm{B/A|B}} \\
&= 0.
\end{aligned}$$

Therefore, for all $t \geq 0$ such that $\sin\theta_{\mathrm{B/A}}(t) \neq 0$, it follows that $\hat{n}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = 0$, and thus $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = 0$. It thus follows from Problem 4.17.4 that, for all $t \geq 0$ such that $\sin\theta_{\mathrm{B/A}}(t) \neq 0$, it follows that $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} = 0$. Finally, for all $t \geq 0$ such that $\sin\theta_{\mathrm{B/A}}(t) \neq 0$, it follows from (4.8.3) that $\ddot{\theta}_{\mathrm{B/A}}(t) = 0$. □

4.9 The Angular Velocity Vector and Eigenaxis-Angle-Vector Derivative†

Recall from (2.16.5) that

$$\vec{\Theta}_{\mathrm{B/A}} \triangleq \theta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}}. \tag{4.9.1}$$

Fact 4.9.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}. \tag{4.9.2}$$

Proof. It follows from (2.11.43) and (4.7.13) that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} &= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}} \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}\\
&= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}}\vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}\\
&= \vec{R}_{\mathrm{A/B}}(\dot{\theta}_{\mathrm{B/A}}\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}})\\
&= \vec{R}_{\mathrm{A/B}}(\dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \theta_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}})\\
&= \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}.
\end{aligned}$$

□

Fact 4.9.2. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, and assume that the orientation of $\mathrm{F_B}$ relative to $\mathrm{F_A}$ is given by $\vec{R}_{\mathrm{B/A}} = \exp(\vec{\Theta}^{\times}_{\mathrm{B/A}})$. Then,

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet}{\vec{\Theta}}{}^{\times}_{\mathrm{B/A}} \exp\left((1-\tau)\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\mathrm{d}\tau, \tag{4.9.3}$$

$$\overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{A/B}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{A/B}}\right) \overset{\mathrm{A}\bullet}{\vec{\Theta}}{}^{\times}_{\mathrm{A/B}} \exp\left((1-\tau)\vec{\Theta}^{\times}_{\mathrm{A/B}}\right)\mathrm{d}\tau, \tag{4.9.4}$$

and thus

$$\vec{\omega}^{\times}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B/A}} \vec{R}_{\mathrm{A/B}} \tag{4.9.5}$$

$$= \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet}{\vec{\Theta}}{}^{\times}_{\mathrm{B/A}} \exp\left(\tau\vec{\Theta}^{\times\prime}_{\mathrm{B/A}}\right)\mathrm{d}\tau. \tag{4.9.6}$$

Consequently,

$$\vec{\omega}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\mathrm{d}\tau \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}. \tag{4.9.7}$$

Furthermore,

$$\vec{\omega}_{\mathrm{B/A}} = \frac{1}{\theta^2_{\mathrm{B/A}}}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right) \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{4.9.8}$$

$$= \frac{1}{\theta_{\mathrm{B/A}}^2}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} - (\vec{I} - \vec{R}_{\mathrm{A/B}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{4.9.9}$$

$$= \left(\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^3}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{4.9.10}$$

$$= \left(\vec{I} - \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^3}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}. \tag{4.9.11}$$

Equivalently,

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.9.12}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{A/B}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.9.13}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (1 - \cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \tag{4.9.14}$$

$$= \dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} - (1 - \cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}})\overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}. \tag{4.9.15}$$

In addition,

$$\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \left(\vec{I} - \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2 - \theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta_{\mathrm{B/A}}^2}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}, \tag{4.9.16}$$

$$\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \left(\vec{I} + \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2 - \theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta_{\mathrm{B/A}}^2}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}. \tag{4.9.17}$$

Finally, if $\vec{\Theta}_{\mathrm{B/A}}$ and $\overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}$ are parallel, then $\vec{\omega}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}$.

Proof. Fact 11.14.3 of [1] yields (4.9.3). It then follows that

$$\vec{\omega}^{\times}_{\mathrm{B/A}} = -\overset{\mathrm{B}\bullet}{\vec{R}}_{\mathrm{A/B}}\vec{R}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times\prime}_{\mathrm{B/A}}\right)\overset{\mathrm{B}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}}\exp\left(\tau\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\mathrm{d}\tau.$$

Therefore, it follows from Fact 2.9.8 that

$$\vec{\omega}_{\mathrm{B/A}} = \int_0^1 \exp\left(\tau\vec{\Theta}^{\times\prime}_{\mathrm{B/A}}\right)\mathrm{d}\tau\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}.$$

Using Fact 3.10.1, Fact 6.6.9, and Fact 11.13.14 of [1] yields (4.9.9). Using Fact 2.9.4, (4.9.10), and (4.9.11) yields (4.9.16) and (4.9.17). □

Note that

$$\lim_{\theta_{\mathrm{B/A}}\to 0}\frac{1-\cos\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^2} = \frac{1}{2}, \tag{4.9.18}$$

$$\lim_{\theta_{\mathrm{B/A}}\to 0}\frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^3} = \frac{1}{6}, \tag{4.9.19}$$

$$\lim_{\theta_{\mathrm{B/A}}\to 0}\frac{2 - \theta_{\mathrm{B/A}}\cot\frac{1}{2}\theta_{\mathrm{B/A}}}{2\theta_{\mathrm{B/A}}^2} = \frac{1}{12}. \tag{4.9.20}$$

However, note that, concerning (4.7.29) and (4.7.31),

$$\lim_{\theta_{\rm B/A}\to 0} \cot \tfrac{1}{2}\theta_{\rm B/A} = \infty. \tag{4.9.21}$$

The nonexistence of this limit reflects the fact that, if $\theta_{\rm B/A} = 0$, then (4.7.18) and (4.7.19) become $\vec{\omega}_{\rm B/A} = \dot{\theta}_{\rm B/A}\hat{n}_{\rm B/A}$, which is independent of $\overset{\rm A\bullet}{\hat{n}}_{\rm B/A}$ and $\overset{\rm B\bullet}{\hat{n}}_{\rm B/A}$.

The equivalence of (4.9.10) and (4.9.16) follows from the equality

$$\left(\vec{I} + \frac{1-\cos\theta_{\rm B/A}}{\theta_{\rm B/A}^2}\vec{\Theta}_{\rm B/A}^{\times} + \frac{\theta_{\rm B/A}-\sin\theta_{\rm B/A}}{\theta_{\rm B/A}^3}\vec{\Theta}_{\rm B/A}^{\times 2}\right)\left(\vec{I} - \frac{1}{2}\vec{\Theta}_{\rm B/A}^{\times} + \frac{2-\theta_{\rm B/A}\cot\frac{1}{2}\theta_{\rm B/A}}{2\theta_{\rm B/A}^2}\vec{\Theta}_{\rm B/A}^{\times 2}\right) = \vec{I}. \tag{4.9.22}$$

Likewise, the equivalence of (4.9.11) and (4.9.17) follows from the equality

$$\left(\vec{I} - \frac{1-\cos\theta_{\rm B/A}}{\theta_{\rm B/A}^2}\vec{\Theta}_{\rm B/A}^{\times} + \frac{\theta_{\rm B/A}-\sin\theta_{\rm B/A}}{\theta_{\rm B/A}^3}\vec{\Theta}_{\rm B/A}^{\times 2}\right)\left(\vec{I} + \frac{1}{2}\vec{\Theta}_{\rm B/A}^{\times} + \frac{2-\theta_{\rm B/A}\cot\frac{1}{2}\theta_{\rm B/A}}{2\theta_{\rm B/A}^2}\vec{\Theta}_{\rm B/A}^{\times 2}\right) = \vec{I}. \tag{4.9.23}$$

Both (4.9.22) and (4.9.23) are consequences of (2.9.4) with $\theta_{\rm B/A} = \|\vec{\Theta}_{\rm B/A}\|_2$.

4.10 The Angular Velocity Vector and Euler-Angle Derivatives

The angular velocity vector can be related to the derivatives of the Euler angles. For 3-2-1 (azimuth-elevation-bank) Euler angles Ψ, Θ, Φ (see (2.13.21)), we have

$$\vec{\omega}_{\rm D/A} = \vec{\omega}_{\rm D/C} + \vec{\omega}_{\rm C/B} + \vec{\omega}_{\rm B/A} \tag{4.10.1}$$

$$= \dot{\Phi}\hat{\imath}_{\rm C} + \dot{\Theta}\hat{\jmath}_{\rm B} + \dot{\Psi}\hat{k}_{\rm A}. \tag{4.10.2}$$

Since $\hat{\imath}_{\rm D} = \hat{\imath}_{\rm C}$, $\hat{\jmath}_{\rm C} = \hat{\jmath}_{\rm B}$, and $\hat{k}_{\rm B} = \hat{k}_{\rm A}$, resolving $\vec{\omega}_{\rm D/A}$ in $\mathrm{F_D}$ yields

$$\vec{\omega}_{\rm D/A} = \dot{\Phi}\hat{\imath}_{\rm D} + \dot{\Theta}\hat{\jmath}_{\rm C} + \dot{\Psi}\hat{k}_{\rm B} \tag{4.10.3}$$

$$= \dot{\Phi}\hat{\imath}_{\rm D} + \dot{\Theta}[(\cos\Phi)\hat{\jmath}_{\rm D} - (\sin\Phi)\hat{k}_{\rm D}] + \dot{\Psi}[(\cos\Theta)\hat{k}_{\rm C} - (\sin\Theta)\hat{\imath}_{\rm C}] \tag{4.10.4}$$

$$= \dot{\Phi}\hat{\imath}_{\rm D} + \dot{\Theta}(\cos\Phi)\hat{\jmath}_{\rm D} - \dot{\Theta}(\sin\Phi)\hat{k}_{\rm D} + \dot{\Psi}(\cos\Theta)[(\cos\Phi)\hat{k}_{\rm D} + (\sin\Phi)\hat{\jmath}_{\rm D}] - \dot{\Psi}(\sin\Theta)\hat{\imath}_{\rm D} \tag{4.10.5}$$

$$= [-\dot{\Psi}(\sin\Theta) + \dot{\Phi}]\hat{\imath}_{\rm D} + [\dot{\Psi}(\sin\Phi)\cos\Theta + \dot{\Theta}\cos\Phi]\hat{\jmath}_{\rm D} + [\dot{\Psi}(\cos\Phi)\cos\Theta - \dot{\Theta}(\sin\Phi)]\hat{k}_{\rm D}. \tag{4.10.6}$$

Hence

$$\omega_{\rm D/A|D} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}. \tag{4.10.7}$$

We rewrite (4.10.7) as

$$\omega_{\rm D/A|D} = S(\Phi,\Theta)\dot{\theta}, \tag{4.10.8}$$

where

$$S(\Phi,\Theta) \triangleq \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}, \quad \theta \triangleq \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}. \tag{4.10.9}$$

Note that $S(\Phi,\Theta)$ is independent of Ψ and $\det S(\Phi,\Theta) = \cos\Theta$. Assuming that $S(\Phi,\Theta)$ is nonsingular, solving (4.10.8) for $\dot\theta$ yields

$$\dot\theta = S(\Phi,\Theta)^{-1}\omega_{\mathrm{D/A|D}}, \tag{4.10.10}$$

where

$$S(\Phi,\Theta)^{-1} \triangleq \begin{bmatrix} 1 & (\sin\Phi)\tan\Theta & (\cos\Phi)\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & (\sin\Phi)\sec\Theta & (\cos\Phi)\sec\Theta \end{bmatrix}. \tag{4.10.11}$$

For 3-1-3 (precession, nutation, spin) Euler angles Φ, Θ, Ψ (see (2.13.34)), we have

$$\vec\omega_{\mathrm{D/A}} = \vec\omega_{\mathrm{D/C}} + \vec\omega_{\mathrm{C/B}} + \vec\omega_{\mathrm{B/A}} \tag{4.10.12}$$

$$= \dot\Psi\hat k_{\mathrm{C}} + \dot\Theta\hat\imath_{\mathrm{B}} + \dot\Phi\hat k_{\mathrm{A}}. \tag{4.10.13}$$

Since $\hat k_{\mathrm{A}} = \hat k_{\mathrm{B}}$, $\hat\imath_{\mathrm{B}} = \hat\imath_{\mathrm{C}}$, and $\hat k_{\mathrm{C}} = \hat k_{\mathrm{D}}$, resolving $\vec\omega_{\mathrm{D/A}}$ in $\mathrm{F_D}$ yields

$$\begin{aligned}
\vec\omega_{\mathrm{D/A}} &= \dot\Psi\hat k_{\mathrm{D}} + \dot\Theta\hat\imath_{\mathrm{C}} + \dot\Phi\hat k_{\mathrm{B}} \\
&= \dot\Psi\hat k_{\mathrm{D}} + \dot\Theta[(\cos\Psi)\hat\imath_{\mathrm{D}} - (\sin\Psi)\hat\jmath_{\mathrm{D}}] + \dot\Phi[(\cos\Theta)\hat k_{\mathrm{C}} + (\sin\Theta)\hat\jmath_{\mathrm{C}}] \\
&= \dot\Psi\hat k_{\mathrm{D}} + \dot\Theta(\cos\Psi)\hat\imath_{\mathrm{D}} - \dot\Theta(\sin\Psi)\hat\jmath_{\mathrm{D}} + \dot\Phi(\cos\Theta)\hat k_{\mathrm{D}} + \dot\Phi(\sin\Theta)[(\cos\Psi)\hat\jmath_{\mathrm{D}} + (\sin\Psi)\hat\imath_{\mathrm{D}}] \\
&= [\dot\Theta(\cos\Psi) + \dot\Phi(\sin\Psi)\sin\Theta]\hat\imath_{\mathrm{D}} + [\dot\Phi(\cos\Psi)\sin\Theta - \dot\Theta(\sin\Psi)]\hat\jmath_{\mathrm{D}} + [\dot\Psi + \dot\Phi(\cos\Theta)]\hat k_{\mathrm{D}}.
\end{aligned}$$

Hence,

$$\omega_{\mathrm{D/A|D}} = \begin{bmatrix} 0 & \cos\Psi & (\sin\Psi)\sin\Theta \\ 0 & -\sin\Psi & (\cos\Psi)\sin\Theta \\ 1 & 0 & \cos\Theta \end{bmatrix} \begin{bmatrix} \dot\Psi \\ \dot\Theta \\ \dot\Phi \end{bmatrix}. \tag{4.10.14}$$

We rewrite (4.10.14) as

$$\omega_{\mathrm{D/A|D}} = S(\Psi,\Theta)\dot\theta, \tag{4.10.15}$$

where

$$S(\Psi,\Theta) \triangleq \begin{bmatrix} 0 & \cos\Psi & (\sin\Psi)\sin\Theta \\ 0 & -\sin\Psi & (\cos\Psi)\sin\Theta \\ 1 & 0 & \cos\Theta \end{bmatrix}, \quad \theta \triangleq \begin{bmatrix} \Psi \\ \Theta \\ \Phi \end{bmatrix}. \tag{4.10.16}$$

Note that $S(\Psi,\Theta)$ is independent of Φ and $\det S(\Phi,\Theta) = \sin\Theta$. Assuming that $S(\Psi,\Theta)$ is nonsingular, solving (4.10.15) for $\dot\theta$ yields

$$\dot\theta = S(\Phi,\Theta)^{-1}\omega_{\mathrm{D/A|D}}, \tag{4.10.17}$$

where

$$S(\Psi,\Theta)^{-1} \triangleq \begin{bmatrix} -(\sin\Psi)\cot\Theta & -(\cos\Psi)\cot\Theta & 1 \\ \cos\Psi & -\sin\Psi & 0 \\ (\sin\Psi)\csc\Theta & (\cos\Psi)\csc\Theta & 0 \end{bmatrix}. \tag{4.10.18}$$

4.11 The Angular Velocity Vector and Euler-Vector Derivative

Recall from (2.17.15) that the eigenaxis angle vector is given by

$$\vec\varepsilon_{\mathrm{B/A}} = (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat n_{\mathrm{B/A}}. \tag{4.11.1}$$

Fact 4.11.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}}. \tag{4.11.2}$$

Proof. It follows from (2.11.43) and (4.7.13) that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} &= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\cos\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}}.
\end{aligned}$$

□

The following result relates the derivative of the Euler vector to the angular velocity vector.

Fact 4.11.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\eta_{\mathrm{B/A}}\dot{\eta}_{\mathrm{B/A}} + \vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} = 0, \tag{4.11.3}$$

$$\dot{\eta}_{\mathrm{B/A}} = -\tfrac{1}{2}\vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}, \tag{4.11.4}$$

$$\overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} = \tfrac{1}{2}(\eta_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}} + \vec{\varepsilon}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}}). \tag{4.11.5}$$

$$\vec{\omega}_{\mathrm{B/A}} = 2(\eta_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} - \dot{\eta}_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}} - \vec{\varepsilon}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}}). \tag{4.11.6}$$

Proof. To prove (4.11.3), differentiate (2.17.25) with respect to $\mathrm{F_B}$.

To prove (4.11.4), note that (4.7.18) implies that

$$\begin{aligned}
-\tfrac{1}{2}\vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}} &= -\tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}'_{\mathrm{B/A}}[\dot{\theta}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (1-\cos\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + (\sin\theta_{\mathrm{B/A}}) \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= -\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}\sin\tfrac{1}{2}\theta_{\mathrm{B/A}} \\
&= \dot{\eta}_{\mathrm{B/A}}.
\end{aligned}$$

Next, to prove (4.11.5), note that (4.7.31) and (4.7.22) imply that

$$\begin{aligned}
\overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} &= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}\eta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}\eta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})[\hat{n}^{\times}_{\mathrm{B/A}} + (\cot\tfrac{1}{2}\theta_{\mathrm{B/A}})(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}})]\vec{\omega}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}\eta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + [\tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} + \tfrac{1}{2}\eta_{\mathrm{B/A}}(\vec{I} - \hat{n}_{\mathrm{B/A}}\hat{n}'_{\mathrm{B/A}})]\vec{\omega}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}\eta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + [\tfrac{1}{2}(\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}} + \tfrac{1}{2}\eta_{\mathrm{B/A}}\vec{I}]\vec{\omega}_{\mathrm{B/A}} - \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}\eta_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}[\eta_{\mathrm{B/A}}\vec{I} + (\sin\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}^{\times}_{\mathrm{B/A}}]\vec{\omega}_{\mathrm{B/A}}
\end{aligned}$$

$$= \tfrac{1}{2}(\eta_{\mathrm{B/A}}\vec{I} + \vec{\varepsilon}^{\,\times}_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}}.$$

Finally, to prove (4.11.6), multiply both sides of (4.11.5) by $2\vec{\varepsilon}^{\,\times}_{\mathrm{B/A}}$ to obtain

$$\begin{aligned}
2\vec{\varepsilon}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} &= \eta_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} + \vec{\varepsilon}_{\mathrm{B/A}} \times (\vec{\varepsilon}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}})\\
&= \eta_{\mathrm{B/A}}(2\overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} - \eta_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}) + \vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}} - |\vec{\varepsilon}_{\mathrm{B/A}}|^2\vec{\omega}_{\mathrm{B/A}}\\
&= \eta_{\mathrm{B/A}}(2\overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} - \eta_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}) + \vec{\varepsilon}^{\,\prime}_{\mathrm{B/A}}\vec{\omega}_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}} - (1 - \eta^2_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}}\\
&= 2\eta_{\mathrm{B/A}}\overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}} - 2\dot{\eta}_{\mathrm{B/A}}\vec{\varepsilon}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{B/A}}.
\end{aligned}$$

□

Fact 4.11.3. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and define $\varepsilon_{\mathrm{B/A|B}} \triangleq \vec{\varepsilon}_{\mathrm{B/A}}\Big|_{\mathrm{B}}$. Then,

$$\eta_{\mathrm{B/A}}\dot{\eta}_{\mathrm{B/A}} + \varepsilon^{\mathsf{T}}_{\mathrm{B/A|B}}\dot{\varepsilon}_{\mathrm{B/A|B}} = 0, \tag{4.11.7}$$

$$\dot{\eta}_{\mathrm{B/A}} = -\tfrac{1}{2}\varepsilon^{\mathsf{T}}_{\mathrm{B/A|B}}\omega_{\mathrm{B/A|B}}, \tag{4.11.8}$$

$$\dot{\varepsilon}_{\mathrm{B/A|B}} = \tfrac{1}{2}(\eta_{\mathrm{B/A}}\omega_{\mathrm{B/A|B}} + \varepsilon_{\mathrm{B/A|B}} \times \omega_{\mathrm{B/A|B}}), \tag{4.11.9}$$

$$\omega_{\mathrm{B/A|B}} = 2(\eta_{\mathrm{B/A}}\dot{\varepsilon}_{\mathrm{B/A|B}} - \dot{\eta}_{\mathrm{B/A}}\varepsilon_{\mathrm{B/A|B}} - \varepsilon_{\mathrm{B/A|B}} \times \dot{\varepsilon}_{\mathrm{B/A|B}}). \tag{4.11.10}$$

In terms of the Euler parameter vector

$$q_{\mathrm{B/A}} = \begin{bmatrix} \eta_{\mathrm{B/A}} \\ \varepsilon_{\mathrm{B/A|B}} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{4.11.11}$$

defined by (2.17.20), it follows from (4.11.10) that $\omega_{\mathrm{B/A|B}}$ can be written in terms of $\dot{q}_{\mathrm{B/A}}$ as

$$\omega_{\mathrm{B/A|B}} = 2\begin{bmatrix} -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}\dot{q}_{\mathrm{B/A}}. \tag{4.11.12}$$

Conversely, it follows from (4.11.8) and (4.11.9) that $\dot{q}_{\mathrm{B/A}}$ can be written in terms of $\omega_{\mathrm{B/A|B}}$ as

$$\dot{q}_{\mathrm{B/A}} = \frac{1}{2}\begin{bmatrix} -\varepsilon^{\mathsf{T}}_{\mathrm{B/A|B}} \\ \eta_{\mathrm{B/A}}I_3 + \varepsilon^{\times}_{\mathrm{B/A|B}} \end{bmatrix}\omega_{\mathrm{B/A|B}} = \frac{1}{2}\begin{bmatrix} -b & -c & -d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{bmatrix}\omega_{\mathrm{B/A|B}}. \tag{4.11.13}$$

Equations (4.11.8) and (4.11.9) can be written as

$$\dot{q}_{\mathrm{B/A}} = Q(\omega_{\mathrm{B/A|B}})q_{\mathrm{B/A}}, \tag{4.11.14}$$

where

$$Q(\omega_{\mathrm{B/A|B}}) \triangleq \frac{1}{2}\begin{bmatrix} 0 & -\omega^{\mathsf{T}}_{\mathrm{B/A|B}} \\ \omega_{\mathrm{B/A|B}} & -\omega^{\times}_{\mathrm{B/A|B}} \end{bmatrix}. \tag{4.11.15}$$

Furthermore, if $\omega_{\mathrm{B/A|B}}$ is constant, then, for all $t \ge 0$,

$$q_{\mathrm{B/A}}(t) = e^{Q(\omega_{\mathrm{B/A|B}})t} q_{\mathrm{B/A}}(0), \tag{4.11.16}$$

where

$$e^{Q(\omega_{\mathrm{B/A|B}})t} = \cos(\tfrac{1}{2}\|\omega_{\mathrm{B/A|B}}\|t) I_4 + \frac{2\sin(\frac{1}{2}\|\omega_{\mathrm{B/A|B}}\|t)}{\|\omega_{\mathrm{B/A|B}}\|} Q(\omega_{\mathrm{B/A|B}}). \tag{4.11.17}$$

4.12 The Angular Velocity Vector and Gibbs-Vector Derivative†

Recall from (2.19.1) that

$$\vec{g}_{\mathrm{B/A}} \triangleq (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}}. \tag{4.12.1}$$

Therefore,

$$\vec{g}_{\mathrm{B/A}} = \frac{1}{\eta_{\mathrm{B/A}}} \vec{\varepsilon}_{\mathrm{B/A}}. \tag{4.12.2}$$

Note that $\vec{g}_{\mathrm{B/A}}$ is defined only if $\theta_{\mathrm{B/A}} \ne \pi$, that is, only if $\vec{\theta}_{\mathrm{B/A}} \in [0, \pi)$.

Fact 4.12.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{g}}_{\mathrm{B/A}} = \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}. \tag{4.12.3}$$

Proof. It follows from (2.11.43) and (4.7.13) that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{g}}_{\mathrm{B/A}} &= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2\tfrac{1}{2}\theta_{\mathrm{B/A}})\vec{R}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= \vec{R}_{\mathrm{A/B}}[\tfrac{1}{2}\dot{\theta}_{\mathrm{B/A}}(\sec^2\tfrac{1}{2}\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + (\tan\tfrac{1}{2}\theta_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}] \\
&= \vec{R}_{\mathrm{A/B}} \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}.
\end{aligned}$$

□

Fact 4.12.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and assume that $\theta_{\mathrm{B/A}} \ne \pi$. Then,

$$\overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}} = \tfrac{1}{2}\left(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}'_{\mathrm{B/A}} + \vec{g}^{\times}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}. \tag{4.12.4}$$

Furthermore,

$$\vec{\omega}_{\mathrm{B/A}} = \frac{2}{1 + |\vec{g}_{\mathrm{B/A}}|^2}(\vec{I} - \vec{g}^{\times}_{\mathrm{B/A}}) \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}. \tag{4.12.5}$$

Proof. It follows from (4.12.2), (4.11.4), and (4.11.5) that

$$\overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}} = -\frac{\dot{\eta}_{\mathrm{B/A}}}{\eta^2_{\mathrm{B/A}}}\vec{\varepsilon}_{\mathrm{B/A}} + \frac{1}{\eta_{\mathrm{B/A}}} \overset{\mathrm{B}\bullet}{\vec{\varepsilon}}_{\mathrm{B/A}}$$

$$
\begin{aligned}
&= -\frac{\dot{\eta}_{\mathrm{B/A}}}{\eta_{\mathrm{B/A}}^2}\vec{\varepsilon}_{\mathrm{B/A}} + \frac{1}{2\eta_{\mathrm{B/A}}}(\eta_{\mathrm{B/A}}\vec{I} + \vec{\varepsilon}_{\mathrm{B/A}}^{\times})\vec{\omega}_{\mathrm{B/A}} \\
&= -\frac{\dot{\eta}_{\mathrm{B/A}}}{\eta_{\mathrm{B/A}}}\vec{g}_{\mathrm{B/A}} + \tfrac{1}{2}(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times})\vec{\omega}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\frac{\vec{\varepsilon}_{\mathrm{B/A}}^{\,\prime}\vec{\omega}_{\mathrm{B/A}}}{\eta_{\mathrm{B/A}}}\vec{g}_{\mathrm{B/A}} + \tfrac{1}{2}(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times})\vec{\omega}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\vec{g}_{\mathrm{B/A}}^{\,\prime}\vec{\omega}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}} + \tfrac{1}{2}(\vec{I} + \vec{g}_{\mathrm{B/A}}^{\times})\vec{\omega}_{\mathrm{B/A}} \\
&= \tfrac{1}{2}\left(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\,\prime} + \vec{g}_{\mathrm{B/A}}^{\times}\right)\vec{\omega}_{\mathrm{B/A}}.
\end{aligned}
$$

Finally, it follows from (2.9.10) that

$$
\begin{aligned}
\vec{\omega}_{\mathrm{B/A}} &= 2\left(\vec{I} + \vec{g}_{\mathrm{B/A}}\vec{g}_{\mathrm{B/A}}^{\,\prime} + \vec{g}_{\mathrm{B/A}}^{\times}\right)^{-1} \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}} \\
&= \frac{2}{1 + |\vec{g}_{\mathrm{B/A}}|^2}(\vec{I} - \vec{g}_{\mathrm{B/A}}^{\times}) \overset{\mathrm{B}\bullet}{\vec{g}}_{\mathrm{B/A}}.
\end{aligned}
$$

□

4.13 The Instantaneous Velocity Center of Rotation†

Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p be a point fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$. Then, p is an *instantaneous velocity center of rotation* (IVCR) at time t if $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$ and $\vec{v}_{p/\mathrm{o_A}/\mathrm{A}}(t) = 0$. The motion of $\mathcal{B}$ can be viewed as instantaneously rotating around p. See Figure 4.13.1.

Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p be a point fixed in $\mathcal{B}$, let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$, and let q be a point fixed in $\mathcal{B}$. Then,

$$\vec{v}_{p/\mathrm{o_A}/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{o_A}/\mathrm{A}}, \tag{4.13.1}$$

and thus

$$\vec{\omega}_{\mathrm{B/A}}^{\,\prime}\vec{v}_{p/\mathrm{o_A}/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}}^{\,\prime}\vec{v}_{q/\mathrm{o_A}/\mathrm{A}}. \tag{4.13.2}$$

Fact 4.13.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$, and assume that p is an IVCR at time t. Then,

$$\vec{\omega}_{\mathrm{B/A}}(t) \times \vec{r}_{p/q}(t) + \vec{v}_{q/\mathrm{o_A}/\mathrm{A}}(t) = 0, \tag{4.13.3}$$

$$\vec{\omega}_{\mathrm{B/A}}(t)'\vec{v}_{p/\mathrm{o_A}/\mathrm{A}}(t) = \vec{\omega}_{\mathrm{B/A}}(t)'\vec{v}_{q/\mathrm{o_A}/\mathrm{A}}(t) = 0. \tag{4.13.4}$$

The following result follows from Fact 4.13.1.

Fact 4.13.2. Let $\mathcal{B}$ be a rigid body, and let p be a point fixed in $\mathcal{B}$. Then, the following statements are equivalent:

i) p is an IVCR at time t.

ii) $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$, and, for every point q fixed in $\mathcal{B}$,

$$\vec{\omega}_{\mathrm{B/A}}(t) \times \vec{r}_{p/q}(t) + \vec{v}_{q/\mathrm{o_A}/\mathrm{A}}(t) = 0. \tag{4.13.5}$$

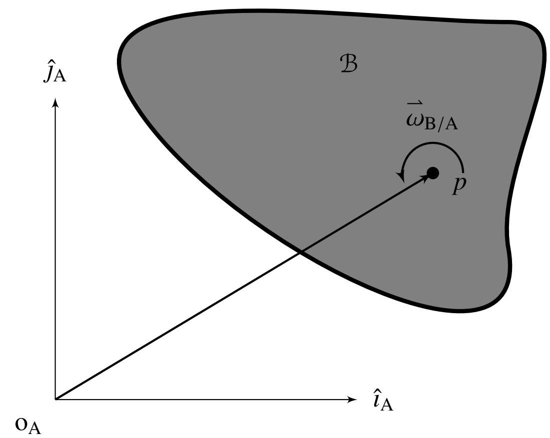


Figure 4.13.1: Instantaneous velocity center of rotation. At time t, $\vec{\omega}_{\rm B/A}(t) \neq 0$. At the same time t, the point p, which is fixed in $\mathcal{B}$, satisfies $\vec{v}_{p/{\rm o_A}/{\rm A}}(t) = 0$. Hence, $\mathcal{B}$ is instantaneously rotating around p.

iii) $\vec{\omega}_{\rm B/A}(t) \neq 0$, and there exists a point q fixed in $\mathcal{B}$ such that (4.13.5) is satisfied.

If $\vec{\omega}_{\rm B/A}(t)$ is nonzero, then Fact 4.13.2 implies that $\mathcal{B}$ has no IVCR if and only if there exists a point q fixed in $\mathcal{B}$ such that $\vec{\omega}_{\rm B/A}(t)'\vec{v}_{q/{\rm o_A}/{\rm A}}(t) \neq 0$. This situation occurs, for example, when $\vec{v}_{q/{\rm o_A}/{\rm A}}(t)$ is nonzero and parallel with $\vec{\omega}_{\rm B/A}(t)$.

Fact 4.13.3. Let $\mathcal{B}$ be a rigid body, and let p be a point fixed in $\mathcal{B}$. Then, p is an IVCR at time t if and only if $\vec{\omega}_{\rm B/A}(t) \neq 0$ and there exists a point q fixed in $\mathcal{B}$ such that the following conditions are satisfied:

i) $\vec{\omega}_{\rm B/A}(t)'\vec{v}_{q/{\rm o_A}/{\rm A}}(t) = 0$.

ii) $\vec{\omega}_{\rm B/A}(t) \times \left(\vec{r}_{p/q}(t) - \dfrac{1}{|\vec{\omega}_{\rm B/A}(t)|^2}\vec{\omega}_{\rm B/A}(t) \times \vec{v}_{q/{\rm o_A}/{\rm A}}(t)\right) = 0.$

If these conditions are satisfied, then

$$[|\vec{\omega}_{\rm B/A}(t)|^2\vec{I} - \vec{\omega}_{\rm B/A}(t)\vec{\omega}_{\rm B/A}'(t)]\,\vec{r}_{p/q}(t) = \vec{\omega}_{\rm B/A}(t) \times \vec{v}_{q/{\rm o_A}/{\rm A}}(t). \tag{4.13.6}$$

Proof. Assume that p is an IVCR at time t. Then, Fact 4.13.1 implies *i*). To prove *ii*), (4.13.5) and *i*) imply that

$$\vec{\omega}_{\rm B/A} \times \left(\vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{\rm B/A}|^2}\vec{\omega}_{\rm B/A} \times \vec{v}_{q/{\rm o_A}/{\rm A}}\right) = \vec{\omega}_{\rm B/A} \times \vec{r}_{p/q} + \vec{v}_{q/{\rm o_A}/{\rm A}} = 0.$$

Conversely, (4.13.1), *i*), and *ii*) yield

$$\vec{v}_{p/{\rm o_A}/{\rm A}} = \vec{\omega}_{\rm B/A} \times \vec{r}_{p/q} + \vec{v}_{q/{\rm o_A}/{\rm A}}$$

$$
\begin{aligned}
&= \vec{\omega}_{\mathrm{B/A}} \times \left(\frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2} \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{o_A}/\mathrm{A}} \right) + \vec{v}_{q/\mathrm{o_A}/\mathrm{A}} \\
&= -\vec{v}_{q/\mathrm{o_A}/\mathrm{A}} + \vec{v}_{q/\mathrm{o_A}/\mathrm{A}} \\
&= 0.
\end{aligned}
$$

To obtain (4.13.6), note that (4.13.3) implies

$$\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{v}_{q/\mathrm{o_A}/\mathrm{A}}) = 0,$$

and thus

$$(\vec{\omega}_{\mathrm{B/A}}' \vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2 \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{q/\mathrm{o_A}/\mathrm{A}} = 0. \tag{4.13.7}$$

Solving (4.13.7) for $\vec{r}_{p/q}$ yields (4.13.6). □

4.14 The Instantaneous Acceleration Center of Rotation†

Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p be a point fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$. Then, p is an *instantaneous acceleration center of rotation* (IACR) at time t if either $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$ or $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \neq 0$ is satisfied and $\vec{a}_{p/\mathrm{o_A}/\mathrm{A}}(t) = 0$. The motion of $\mathcal{B}$ can be viewed as instantaneously rotating or accelerating around p.

Fact 4.14.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$. Then, p is an IACR at time t if and only if either $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$ or $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \neq 0$ and

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \times \vec{r}_{p/q}(t) + \vec{\omega}_{\mathrm{B/A}}(t) \times [\vec{\omega}_{\mathrm{B/A}}(t) \times \vec{r}_{p/q}(t)] + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) = 0. \tag{4.14.1}$$

Fact 4.14.2. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$. Assume that $\vec{\omega}_{\mathrm{B/A}}(t) = 0$, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \neq 0$, and $\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) \neq 0$. Then, p is an IACR of $\mathcal{B}$ at time t if and only if the following conditions are satisfied:

i) $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)' \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) = 0.$

ii) $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \times \left(\vec{r}_{p/q}(t) - \dfrac{1}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) \right) = 0.$

If these conditions are satisfied, then

$$\left(|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)|^2 \vec{I} - \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}{}'(t) \right) \vec{r}_{p/q}(t) = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t). \tag{4.14.2}$$

Proof. Assume that p is an IACR of $\mathcal{B}$ at time t. Since $\vec{\omega}_{\mathrm{B/A}} = 0$, (4.14.1) implies

$$\begin{aligned}\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}}\, \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}} \left(-\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})\right)\\ &= -\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}} \left(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q}\right) = 0,\end{aligned}$$

which proves *i*). To prove *ii*), it follows from *i*) and (4.14.1) that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \frac{1}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\right) = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} = 0.$$

Conversely, *i*) implies

$$\begin{aligned}\vec{a}_{p/\mathrm{o_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{p/q} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{p/q} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\\ &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\frac{1}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\right) + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\\ &= \frac{\overset{\mathrm{B}\bullet}{\vec{\omega}}{}'_{\mathrm{B/A}}\, \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}}{|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} = 0.\end{aligned}$$

Hence, p is an IACR at time t.

To obtain (4.14.2), note that (4.14.1) implies

$$\begin{aligned}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{p/\mathrm{o_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times (\vec{a}_{p/q/\mathrm{A}} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}})\\ &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\right) = 0.\end{aligned}$$

Hence,

$$(\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}}\, \vec{r}_{p/q})\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - (\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}) \vec{r}_{p/q} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} = 0,$$

which yields (4.14.2). □

Fact 4.14.3. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$. Assume that $\vec{\omega}_{\mathrm{B/A}}(t) \neq 0$, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = 0$, and $\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) \neq 0$. Then, p is an IACR at time t if and only if the following conditions are satisfied:

i) $\vec{\omega}_{\mathrm{B/A}}(t)'\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) = 0$.

ii) $\vec{\omega}_{\mathrm{B/A}}(t) \times \left(\vec{r}_{p/q}(t) - \dfrac{1}{|\vec{\omega}_{\mathrm{B/A}}(t)|^2} \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t)\right) = 0.$

If these conditions are satisfied, then

$$[|\vec{\omega}_{\mathrm{B/A}}(t)|^2\vec{I} - \vec{\omega}_{\mathrm{B/A}}(t)\vec{\omega}_{\mathrm{B/A}}'(t)]\vec{r}_{p/q}(t) = \vec{a}_{q/\mathrm{o_A/A}}(t). \tag{4.14.3}$$

Proof. Assume that p is an IACR of $\mathcal{B}$ at time t. Since $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, (4.14.1) implies

$$\begin{aligned}\vec{\omega}_{\mathrm{B/A}}'\vec{a}_{q/\mathrm{o_A/A}} &= \vec{\omega}_{\mathrm{B/A}}'[-\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})]\\ &= -\vec{\omega}_{\mathrm{B/A}}'[\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})] = 0,\end{aligned}$$

which proves *i*). To prove *ii*), it follows from (4.14.1) that

$$\begin{aligned}&\vec{\omega}_{\mathrm{B/A}} \times \left(\vec{r}_{p/q} - \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2}\vec{a}_{q/\mathrm{o_A/A}}\right)\\ &\quad= \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2}[\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q})]\\ &\quad= \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} - \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q} = 0.\end{aligned}$$

Conversely, *i*) implies

$$\begin{aligned}\vec{a}_{p/\mathrm{o_A/A}} &= \vec{a}_{p/q/\mathrm{A}} + \vec{a}_{q/\mathrm{o_A/A}}\\ &= \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{o_A/A}}\\ &= \vec{\omega}_{\mathrm{B/A}} \times \left(\vec{\omega}_{\mathrm{B/A}} \times \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^2}\vec{a}_{q/\mathrm{o_A/A}}\right) + \vec{a}_{q/\mathrm{o_A/A}}\\ &= -\vec{a}_{q/\mathrm{o_A/A}} + \vec{a}_{q/\mathrm{o_A/A}} = 0.\end{aligned}$$

Hence, p is an IACR at time t.

To obtain (4.14.3), note that (4.14.1) implies

$$\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{o_A/A}} = 0.$$

Hence,

$$(\vec{\omega}_{\mathrm{B/A}}'\vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - (\vec{\omega}_{\mathrm{B/A}}'\vec{\omega}_{\mathrm{B/A}})\vec{r}_{p/q} + \vec{a}_{q/\mathrm{o_A/A}} = 0,$$

which yields (4.14.3). □

Fact 4.14.4. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let p and q be points fixed in $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$. Assume that $\vec{\omega}_{\mathrm{B/A}}(t)$ and $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)$ are nonzero and parallel. Then, p is an IACR at time t if and only if the following conditions are satisfied:

i) $\vec{\omega}_{\mathrm{B/A}}(t)'\vec{a}_{q/\mathrm{o_A/A}}(t) = 0$.

ii) $\vec{\omega}_{\mathrm{B/A}}(t)$ and $\vec{r}_{p/q}(t) - \dfrac{1}{|\vec{\omega}_{\mathrm{B/A}}(t)|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)|^2}\left(|\vec{\omega}_{\mathrm{B/A}}(t)|^2\vec{a}_{q/\mathrm{o_A/A}}(t) + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) \times \vec{a}_{q/\mathrm{o_A/A}}(t)\right)$

are parallel.

If these conditions are satisfied, then

$$\begin{aligned}&[(|\vec{\omega}_{\mathrm{B/A}}(t)|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)|^2)\vec{I} - \kappa(t)\vec{\omega}_{\mathrm{B/A}}(t)\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)' - |\vec{\omega}_{\mathrm{B/A}}(t)|^2\vec{\omega}_{\mathrm{B/A}}(t)\vec{\omega}_{\mathrm{B/A}}(t)']\vec{r}_{p/q}(t)\\ &\qquad = |\vec{\omega}_{\mathrm{B/A}}(t)|^2\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t) + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)\times\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}(t),\end{aligned} \tag{4.14.4}$$

where $\kappa(t) \triangleq \dfrac{\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t)'\vec{\omega}_{\mathrm{B/A}}(t)}{|\vec{\omega}_{\mathrm{B/A}}(t)|^2}$.

Proof. Assume that p is an IACR at time t. Then, it follows from (4.14.1) that $\vec{\omega}'_{\mathrm{B/A}}\vec{a}_{q/\mathrm{o_A}/\mathrm{A}} = 0$, which proves *i*). To prove *ii*), note that (4.14.1) implies

$$\begin{aligned}0 &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}}\times(\vec{\omega}_{\mathrm{B/A}}\times\vec{r}_{p/q}) + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\\ &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{r}_{p/q} + (\vec{\omega}'_{\mathrm{B/A}}\vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2\vec{r}_{p/q} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}.\end{aligned} \tag{4.14.5}$$

Therefore,

$$\begin{aligned}0 &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\left(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{r}_{p/q} + (\vec{\omega}'_{\mathrm{B/A}}\vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} - |\vec{\omega}_{\mathrm{B/A}}|^2\vec{r}_{p/q} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\right)\\ &= (\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}}\vec{r}_{p/q})\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2\vec{r}_{p/q} - |\vec{\omega}_{\mathrm{B/A}}|^2(\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{r}_{p/q}) + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}.\end{aligned} \tag{4.14.6}$$

Furthermore, (4.14.5) implies

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{r}_{p/q} = -(\vec{\omega}'_{\mathrm{B/A}}\vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}} + |\vec{\omega}_{\mathrm{B/A}}|^2\vec{r}_{p/q} - \vec{a}_{q/\mathrm{o_A}/\mathrm{A}}. \tag{4.14.7}$$

Substituting (4.14.7) into (4.14.6) and using $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \kappa\vec{\omega}_{\mathrm{B/A}}$ yields

$$\begin{aligned}0 &= (\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}}\vec{r}_{p/q})\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} - |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2\vec{r}_{p/q} + |\vec{\omega}_{\mathrm{B/A}}|^2(\vec{\omega}'_{\mathrm{B/A}}\vec{r}_{p/q})\vec{\omega}_{\mathrm{B/A}}\\ &\quad - |\vec{\omega}_{\mathrm{B/A}}|^4\vec{r}_{p/q} + |\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\\ &= [\kappa\,\overset{\mathrm{B}\bullet\prime}{\vec{\omega}}_{\mathrm{B/A}}\vec{r}_{p/q} + |\vec{\omega}_{\mathrm{B/A}}|^2(\vec{\omega}'_{\mathrm{B/A}}\vec{r}_{p/q})]\vec{\omega}_{\mathrm{B/A}} + |\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\\ &\quad + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{a}_{q/\mathrm{o_A}/\mathrm{A}} - (|\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2 + |\vec{\omega}_{\mathrm{B/A}}|^4)\vec{r}_{p/q},\end{aligned} \tag{4.14.8}$$

which implies *ii*).

Conversely, *ii*) implies that there exists $\alpha \in \mathbb{R}$ such that

$$\vec{r}_{p/q} = \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2}\left(|\vec{\omega}_{\mathrm{B/A}}|^2\vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\times\vec{a}_{q/\mathrm{o_A}/\mathrm{A}}\right) + \alpha\vec{\omega}_{\mathrm{B/A}}.$$

Using *i*) and *ii*), it follows that

$$
\begin{aligned}
\vec{a}_{p/\mathrm{o_A}/\mathrm{A}} &= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/q} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/q}) + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \\
&= \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left[\frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \right) + \alpha \vec{\omega}_{\mathrm{B/A}} \right] \\
&\quad + \vec{\omega}_{\mathrm{B/A}} \times \left(\vec{\omega}_{\mathrm{B/A}} \times \left[\frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \right) + \alpha \vec{\omega}_{\mathrm{B/A}} \right] \right) \\
&\quad + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \\
&= \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \right) \\
&\quad + \frac{1}{|\vec{\omega}_{\mathrm{B/A}}|^4 + |\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}|^2} \vec{\omega}_{\mathrm{B/A}} \times \left[\vec{\omega}_{\mathrm{B/A}} \times \left(|\vec{\omega}_{\mathrm{B/A}}|^2 \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \right) \right] \\
&\quad + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} \\
&= -\vec{a}_{q/\mathrm{o_A}/\mathrm{A}} + \vec{a}_{q/\mathrm{o_A}/\mathrm{A}} = 0.
\end{aligned}
$$

Hence, p is an IACR at time t.

Finally, (4.14.8) implies (4.14.4). □

4.15 Rolling With and Without Slipping

When a disk of radius r is moving in contact with a flat surface, the absence of slipping can be determined by comparing the velocity of its center to its angular velocity, that is, $v = r\omega$, where ω is the rate of rotation of the disk relative to the surface. The relation $v = r\omega$ equates the speed of the center of the disk to the rate of arc length along the path of motion. However, for more general bodies, it is difficult to determine the arc length. We therefore adopt a more general approach, which involves the relative velocity between two points that are in contact.

At a given instant of time, the points x_1 and x_2 are *colocated* if x_1 and x_2 are at the same location. The following result shows that, if two bodies are in contact, then the relative velocity of a pair of colocated body-fixed points is independent of the frame with respect to which the velocity is determined.

Fact 4.15.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let x_1 and x_2 be points that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact with x_1 and x_2 colocated. Then,

$$\vec{v}_{x_1/x_2/\mathrm{A}} = \vec{v}_{x_1/x_2/\mathrm{B}}. \tag{4.15.1}$$

Proof. Note that

$$\vec{v}_{x_1/x_2/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{x_1/x_2} = \overset{\mathrm{B}\bullet}{\vec{r}}_{x_1/x_2} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{x_1/x_2} = \vec{v}_{x_1/x_2/\mathrm{B}}. \qquad \square$$

Fact 4.15.1 shows that the choice of frame in the following definition is irrelevant.

Definition 4.15.2. Let $\mathrm{F_A}$ be a frame, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let x_1 and x_2 be points that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact with x_1 and x_2 colocated. Then, $\mathcal{B}_1$ and $\mathcal{B}_2$ are *rolling without slipping* if $\vec{v}_{x_1/x_2/\mathrm{A}} = 0$. Otherwise, $\mathcal{B}_1$ and $\mathcal{B}_2$ are *slipping*.

Fact 4.15.3. Let $\mathrm{F_A}$ be a frame, let w be a point, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let x_1 and x_2 be points that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are in contact with x_1 and x_2 colocated, and assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are rolling without slipping. Then,

$$\vec{v}_{x_1/w/\mathrm{A}} = \vec{v}_{x_2/w/\mathrm{A}}. \tag{4.15.2}$$

Proof. Note that

$$\vec{v}_{x_1/w/\mathrm{A}} = \vec{v}_{x_1/x_2/\mathrm{A}} + \vec{v}_{x_2/w/\mathrm{A}} = \vec{v}_{x_2/w/\mathrm{A}}. \qquad \square$$

Example 4.15.4. Consider a wheel that rolls without slipping in a straight line on a flat surface, let p denote a point fixed on the circumference of the disk, let $\mathrm{F_A}$ denote a body-fixed frame in the surface, and let x and q denote points that are fixed in the plane. When p is in contact with q, then $\vec{v}_{\mathrm{P}/x/\mathrm{A}} = \vec{v}_{\mathrm{Q}/x/\mathrm{A}} = 0$.

4.16 Examples

Example 4.16.1. Consider the 3-bar linkage shown from above in Figure 4.16.1 with links $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ of lengths ℓ_1, ℓ_2, and ℓ_3, and pin joints a, b, and c labeled as shown. The links are initially configured as shown, lying in a horizontal plane, that is, the plane spanned by $\hat{\imath}_{\rm A}$ and $\hat{\jmath}_{\rm A}$. The pins at joints a and c are vertical, that is, parallel with $\hat{k}_{\rm A}$, and the pin at joint b is horizontal, that is, lying in the plane spanned by $\hat{\imath}_{\rm A}$ and $\hat{\jmath}_{\rm A}$. The rotation angles at joints a, b, and c are ψ, θ, and ϕ, respectively, where a positive value of θ indicates that joint c moves in the negative $\hat{k}_{\rm A}$ direction. The rotation rates at joints a, b, and c are thus $\dot\psi$, $\dot\theta$, and $\dot\phi$, respectively. In terms of the frame $\rm F_A$, determine the velocity of the tip d of the linkage relative to a for arbitrary values of ψ, θ, and ϕ.

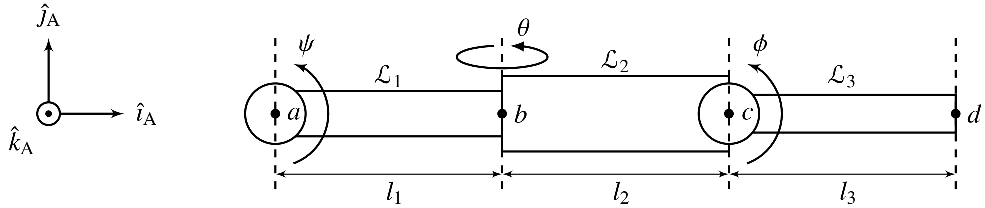


Figure 4.16.1: Example 4.16.1. Three-bar linkage. The orientation of the linkage as shown corresponds to $\psi = \theta = \phi = 0$.

Solution. Let $\rm F_B$ be a frame attached to link $\mathcal{L}_1$, $\rm F_C$ be a frame attached to link $\mathcal{L}_2$, and $\rm F_D$ be a frame attached to link $\mathcal{L}_3$. It thus follows that

$$\mathrm{F_A} \xrightarrow[3]{\psi} \mathrm{F_B} \xrightarrow[2]{\theta} \mathrm{F_C} \xrightarrow[3]{\phi} \mathrm{F_D},$$

and $\vec\omega_{\rm B/A} = \dot\psi \hat{k}_{\rm A}$, $\vec\omega_{\rm C/B} = \dot\theta \hat{\jmath}_{\rm B}$, and $\vec\omega_{\rm D/C} = \dot\phi \hat{k}_{\rm C}$. Note that the velocity of d relative to a in Frame $\rm F_A$ is given by

$$\begin{aligned}
\vec{v}_{d/a/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec{r}}_{d/a} \\
&= \overset{\mathrm{A}\bullet}{\vec{r}}_{d/c} + \overset{\mathrm{A}\bullet}{\vec{r}}_{c/b} + \overset{\mathrm{A}\bullet}{\vec{r}}_{b/a} \\
&= \ell_3 \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\rm D} + \ell_2 \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\rm C} + \ell_1 \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\rm B} \\
&= \ell_3 \vec\omega_{\rm D/A} \times \hat{\imath}_{\rm D} + \ell_2 \vec\omega_{\rm C/A} \times \hat{\imath}_{\rm C} + \ell_1 \vec\omega_{\rm B/A} \times \hat{\imath}_{\rm B} \\
&= \ell_3 (\vec\omega_{\rm D/C} + \vec\omega_{\rm C/B} + \vec\omega_{\rm B/A}) \times \hat{\imath}_{\rm D} + \ell_2 (\vec\omega_{\rm C/B} + \vec\omega_{\rm B/A}) \times \hat{\imath}_{\rm C} + \ell_1 \vec\omega_{\rm B/A} \times \hat{\imath}_{\rm B} \\
&= \ell_3 (\dot\phi \hat{k}_{\rm C} + \dot\theta \hat{\jmath}_{\rm B} + \dot\psi \hat{k}_{\rm A}) \times \hat{\imath}_{\rm D} + \ell_2 (\dot\theta \hat{\jmath}_{\rm B} + \dot\psi \hat{k}_{\rm A}) \times \hat{\imath}_{\rm C} + \ell_1 \dot\psi \hat{k}_{\rm A} \times \hat{\imath}_{\rm B} \qquad (4.16.1)
\end{aligned}$$

Resolving $\vec{v}_{d/a/\mathrm{A}}$ in $\rm F_A$ yields

$$\begin{aligned}
\vec{v}_{d/a/\mathrm{A}}\Big|_{\rm A} &= \ell_3 \left(+\dot\phi\, \hat{k}_{\rm C}\big|_{\rm A} + \dot\theta\, \hat{\jmath}_{\rm B}|_{\rm A} \dot\psi\, \hat{k}_{\rm A}\big|_{\rm A} \right) \times \hat{\imath}_{\rm D}|_{\rm A} + \ell_2 \left(\dot\theta\, \hat{\jmath}_{\rm B}|_{\rm A} + \dot\psi\, \hat{k}_{\rm A}\big|_{\rm A} \right) \times \hat{\imath}_{\rm C}|_{\rm A} + \ell_1 \dot\psi\, \hat{k}_{\rm A}\big|_{\rm A} \times \hat{\imath}_{\rm B}|_{\rm A} \\
&= \ell_3 \left(\dot\phi \mathcal{O}_3(\psi)^{\mathsf T} \mathcal{O}_2(\theta)^{\mathsf T}\, \hat{k}_{\rm C}\big|_{\rm C} + \dot\theta \mathcal{O}_3(\psi)^{\mathsf T}\, \hat{\jmath}_{\rm B}|_{\rm B} + \dot\psi e_3 \right) \times \mathcal{O}_3(\psi)^{\mathsf T} \mathcal{O}_2(\theta)^{\mathsf T} \mathcal{O}_3(\phi)^{\mathsf T}\, \hat{\imath}_{\rm D}|_{\rm D} \\
&\quad + \ell_2 \left(\dot\theta \mathcal{O}_3(\psi)^{\mathsf T}\, \hat{\jmath}_{\rm B}|_{\rm B} + \dot\psi e_3 \right) \times \mathcal{O}_3(\psi)^{\mathsf T} \mathcal{O}_2(\theta)^{\mathsf T}\, \hat{\imath}_{\rm C}|_{\rm C} + \ell_1 \dot\psi e_3 \times \mathcal{O}_3(\psi)^{\mathsf T}\, \hat{\imath}_{\rm B}|_{\rm B} \\
&= \ell_3 \left(\dot\phi \mathcal{O}_3(\psi)^{\mathsf T} \mathcal{O}_2(\theta)^{\mathsf T} e_3 + \dot\theta \mathcal{O}_3(\psi)^{\mathsf T} e_2 + \dot\psi e_3 \right) \times \mathcal{O}_3(\psi)^{\mathsf T} \mathcal{O}_2(\theta)^{\mathsf T} \mathcal{O}_3(\phi)^{\mathsf T} e_1 \\
&\quad + \ell_2 \left(\dot\theta \mathcal{O}_3(\psi)^{\mathsf T} e_2 + \dot\psi e_3 \right) \times \mathcal{O}_3(\psi)^{\mathsf T} \mathcal{O}_2(\theta)^{\mathsf T} e_1 + \ell_1 \dot\psi e_3 \times \mathcal{O}_3(\psi)^{\mathsf T} e_1. \qquad (4.16.2)
\end{aligned}$$

Note that, in the case where $\psi = \theta = \phi = 0$,

$$\begin{aligned}\vec{v}_{d/a/\mathrm{A}}\Big|_{\mathrm{A}} &= \ell_3\left(\dot{\phi}e_3 + \dot{\theta}e_2 + \dot{\psi}e_3\right)\times e_1 + \ell_2\left(\dot{\theta}e_2 + \dot{\psi}e_3\right)\times e_1 + \ell_1\dot{\psi}e_3\times e_1\\ &= \ell_3\left(\dot{\phi}e_2 - \dot{\theta}e_3 + \dot{\psi}e_2\right) + \ell_2\left(-\dot{\theta}e_3 + \dot{\psi}e_2\right) + \ell_1\dot{\psi}e_2\\ &= (\ell_3\dot{\phi} + (\ell_3 + \ell_2 + \ell_1)\dot{\psi})e_2 - (\ell_3 + \ell_2)\dot{\theta}e_3. \end{aligned} \tag{4.16.3}$$

◇

Example 4.16.2. The wheel shown in Figure 4.16.2 has radius R and rotates clockwise at a constant rate around its center point b, which is pinned to a fixed point in the ground. The point P denotes a pin located on the circumference of the wheel; this pin slides along the slot in the arm as shown. The arm is pinned to the ground at point a, which is fixed in the ground, and the distance from a to b is L. Using your intuition only, make a rough sketch of θ over the interval $[t_0, t_\mathrm{f}]$, where $\phi = 0$ deg at $t = 0$, and $\phi = 180$ deg at t_f. Mark the points on the plot at which *i*) θ achieves its maximum value, and *ii*) $\phi = 90$ deg. Next, derive expressions for $\dot{\theta}$ and $\ddot{\theta}$, and specialize these expressions to the case $\phi = 90$ deg. Finally, check the signs of $\dot{\theta}$ and $\ddot{\theta}$ and determine whether those signs are consistent with your sketch of θ versus t.

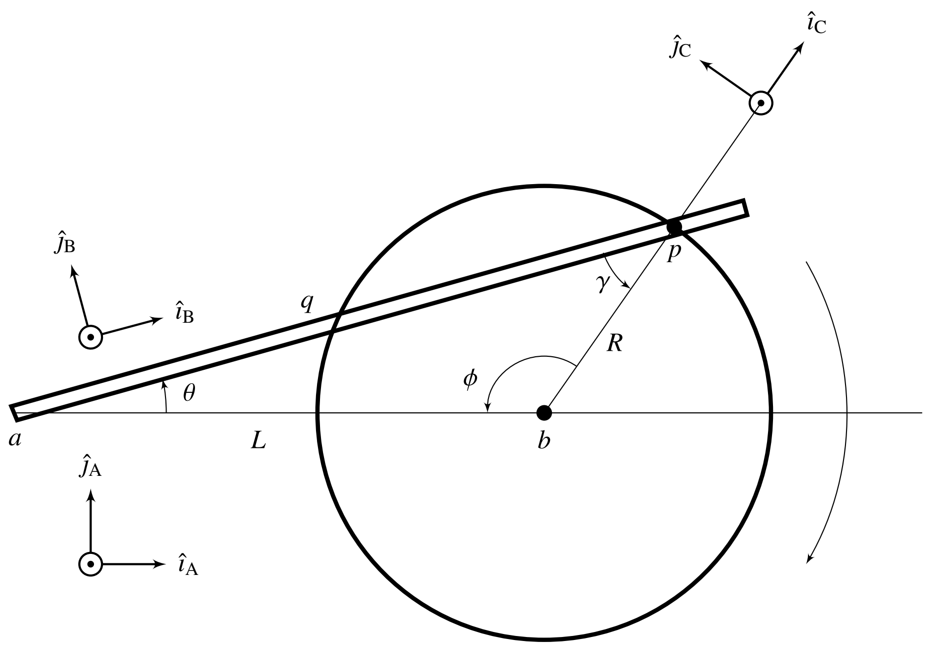


Figure 4.16.2: Example 4.16.2. Rotating wheel with slotted arm.

Solution. The frames are related by $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B} \xrightarrow[3]{\gamma} \mathrm{F_C}$, with angular velocities

$$\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{k}_\mathrm{A}, \qquad \vec{\omega}_{\mathrm{C/B}} = \dot{\gamma}\hat{k}_\mathrm{A}. \tag{4.16.4}$$

Since $\theta + \gamma + \phi = \pi$, it follows that

$$\vec{\omega}_{\mathrm{C/A}} = -\dot{\phi}\hat{k}_{\mathrm{A}}. \tag{4.16.5}$$

Next, let $q \triangleq |\vec{r}_{P/a}|$ so that $\vec{r}_{P/a} = q\hat{\imath}_{\mathrm{B}}$. Therefore,

$$\begin{aligned}\vec{v}_{P/a/\mathrm{A}} &= \dot{q}\hat{\imath}_{\mathrm{B}} + q\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}}\\ &= \dot{q}\hat{\imath}_{\mathrm{B}} + q\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}}\\ &= \dot{q}\hat{\imath}_{\mathrm{B}} + q\dot{\theta}\hat{\jmath}_{\mathrm{B}}.\end{aligned} \tag{4.16.6}$$

On the other hand, defining $v \triangleq R\dot{\phi}$, it follows that

$$\begin{aligned}\vec{v}_{P/a/\mathrm{A}} &= \vec{v}_{P/b/\mathrm{A}} + \vec{v}_{b/a/\mathrm{A}}\\ &= \overset{\mathrm{A}\bullet}{\vec{r}}_{p/b}\\ &= R\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{C}}\\ &= R\vec{\omega}_{\mathrm{C/A}} \times \hat{\imath}_{\mathrm{C}}\\ &= -v\hat{\jmath}_{\mathrm{C}}\\ &= v(\sin\gamma)\hat{\imath}_{\mathrm{B}} - v(\cos\gamma)\hat{\jmath}_{\mathrm{B}}.\end{aligned} \tag{4.16.7}$$

Equating the expressions for $\vec{v}_{P/a/\mathrm{A}}$ yields

$$\dot{\theta} = -\frac{v\cos\gamma}{q}. \tag{4.16.8}$$

The law of sines implies

$$\frac{\sin\theta}{R} = \frac{\sin\gamma}{L} = \frac{\sin\phi}{q}, \tag{4.16.9}$$

while the law of cosines implies

$$R^2 = q^2 + L^2 - 2qL\cos\theta \tag{4.16.10}$$

and

$$q^2 = R^2 + L^2 - 2RL\cos\phi, \tag{4.16.11}$$

and thus

$$\cos\theta = \frac{L - R\cos\phi}{q} = \frac{L - R\cos\phi}{\sqrt{R^2 + L^2 - 2RL\cos\phi}}. \tag{4.16.12}$$

It can be shown that θ is maximized when $\cos\phi = R/L$. Furthermore,

$$\dot{\theta} = \frac{(RL\cos\phi - R^2)\dot{\phi}}{R^2 + L^2 - 2RL\cos\phi}, \tag{4.16.13}$$

and thus

$$\ddot{\theta} = \frac{RL(R^2 - L^2)(\sin\phi)\dot{\phi}^2}{(R^2 + L^2 - 2RL\cos\phi)^2}. \tag{4.16.14}$$

Therefore, when $\phi = 90$ deg it follows that

$$\dot{\theta} = -\frac{R^2\dot{\phi}}{R^2 + L^2} \tag{4.16.15}$$

and

$$\ddot{\theta} = \frac{RL(R^2 - L^2)\dot{\phi}^2}{(R^2 + L^2)^2}. \tag{4.16.16}$$

Note that, when $\phi = 90$ deg, $\dot{\phi}$ is positive, and thus $\dot{\theta}$ is negative. Furthermore, since $R < L$, it follows that $\ddot{\theta}$ is negative. Figure 4.16.3 shows that the angle θ is maximized at a time t_1 at which the arm is tangent to the wheel, that is, when $\phi = 60$ deg. In addition, at time $t_2 > t_1$, $\phi(t_2) = \pi/2$ rad, where θ is decreasing, and the function $\theta(t)$ is convex. ◇

Example 4.16.3. Consider a small disk with radius r that rolls without slipping inside a large hoop of radius R as shown in Figure 4.16.4. Frame $\mathrm{F_A}$ is fixed to the hoop, frame $\mathrm{F_B}$ is fixed to the arm that connects the disk to the center of the hoop, and frame $\mathrm{F_C}$ is fixed to the disk. Point a is the center of the hoop, point b is the center of the disk, and point c is fixed in the disk. The distance from b to c is r_0. Point d is fixed to the edge of the disk and point e is fixed on the hoop. Define angles θ and ϕ as shown. Determine $\vec{a}_{c/a/\mathrm{A}}$ in terms of $r, R, r_0, \phi, \dot{\phi}, \ddot{\phi}$ resolved in $\mathrm{F_B}$ at the instant that d and e are colocated.

Solution. The frames are related by

$$\mathrm{F_C} \underset{3}{\overset{\phi}{\longrightarrow}} \mathrm{F_B} \underset{3}{\overset{\theta}{\longrightarrow}} \mathrm{F_A}, \tag{4.16.17}$$

with angular velocities

$$\vec{\omega}_{\mathrm{A/B}} = \dot{\theta}\hat{k}_{\mathrm{A}}, \qquad \vec{\omega}_{\mathrm{B/C}} = \dot{\phi}\hat{k}_{\mathrm{A}}. \tag{4.16.18}$$

so that

$$\vec{\omega}_{\mathrm{A/C}} = (\dot{\theta} + \dot{\phi})\hat{k}_{\mathrm{A}}. \tag{4.16.19}$$

Furthermore, the position vectors are

$$\vec{r}_{c/b} = r_0\hat{\imath}_{\mathrm{C}}, \qquad \vec{r}_{b/a} = (R - r)\hat{\imath}_{\mathrm{B}}, \tag{4.16.20}$$

so that

$$\vec{r}_{c/a} = r_0\hat{\imath}_{\mathrm{C}} + (R - r)\hat{\imath}_{\mathrm{B}}. \tag{4.16.21}$$

Hence,

$$\begin{aligned}
\vec{v}_{c/a/\mathrm{A}} &= r_0 \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{C}} + (R - r) \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\
&= r_0(\vec{\omega}_{\mathrm{C/A}} \times \hat{\imath}_{\mathrm{C}}) + (R - r)(\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}}) \\
&= r_0(-\dot{\phi} - \dot{\theta})\hat{k}_{\mathrm{C}} \times \hat{\imath}_{\mathrm{C}} + (R - r)(-\dot{\theta})\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{B}} \\
&= -r_0(\dot{\phi} + \dot{\theta})\hat{\jmath}_{\mathrm{C}} - (R - r)\dot{\theta}\hat{\jmath}_{\mathrm{B}} \\
&= -r_0(\dot{\phi} + \dot{\theta})[(\sin\phi)\hat{\imath}_{\mathrm{B}} + (\cos\phi)\hat{\jmath}_{\mathrm{B}}] - (R - r)\dot{\theta}\hat{\jmath}_{\mathrm{B}} \\
&= -r_0(\dot{\phi} + \dot{\theta})(\sin\phi)\hat{\imath}_{\mathrm{B}} - [r_0(\dot{\phi} + \dot{\theta})\cos\phi + (R - r)\dot{\theta}]\hat{\jmath}_{\mathrm{B}}.
\end{aligned} \tag{4.16.22}$$

Next, the no-slip condition implies

$$0 = \vec{v}_{d/e/\mathrm{A}} = \vec{v}_{d/a/\mathrm{A}} - \vec{v}_{e/a/\mathrm{A}}. \tag{4.16.23}$$

Since, in addition, $\vec{r}_{e/a}$ is fixed in $\mathrm{F_A}$, it follows that

$$\vec{v}_{d/a/\mathrm{A}} = \vec{v}_{e/a/\mathrm{A}} = 0. \tag{4.16.24}$$

Therefore, when d and e are colocated so that $\vec{r}_{d/b} = r\hat{\imath}_\mathrm{B}$, it follows that

$$\begin{aligned}
0 &= \vec{v}_{d/a/\mathrm{A}} \\
&= \vec{v}_{d/b/\mathrm{A}} + \vec{v}_{b/a/\mathrm{A}} \\
&= \overset{\mathrm{A}\bullet}{\vec{r}}_{d/b} + (R-r)\overset{\mathrm{A}\bullet}{\hat{\imath}}_\mathrm{B} \\
&= \overset{\mathrm{C}\bullet}{\vec{r}}_{d/b} + \vec{\omega}_{\mathrm{C/A}} \times \vec{r}_{d/b} + (R-r)\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_\mathrm{B} \\
&= \vec{0} + (-\dot{\phi} - \dot{\theta})\hat{k}_\mathrm{A} \times r\hat{\imath}_\mathrm{B} + (R-r)(-\dot{\theta})\hat{k}_\mathrm{A} \times \hat{\imath}_\mathrm{B} \\
&= -r(\dot{\phi} + \dot{\theta})\hat{\jmath}_\mathrm{B} - (R-r)\dot{\theta}\hat{\jmath}_\mathrm{B} \\
&= -(r\dot{\phi} + R\dot{\theta})\hat{\jmath}_\mathrm{B}.
\end{aligned} \tag{4.16.25}$$

Therefore,

$$\dot{\theta} = -\frac{r}{R}\dot{\phi}, \tag{4.16.26}$$

and thus

$$\vec{\omega}_{\mathrm{A/B}} = -\frac{r}{R}\dot{\phi}\hat{k}_\mathrm{A}, \qquad \vec{\omega}_{\mathrm{C/A}} = \rho\dot{\phi}\hat{k}_\mathrm{A}. \tag{4.16.27}$$

where

$$\rho \triangleq \frac{r}{R} - 1. \tag{4.16.28}$$

Consequently,

$$\vec{v}_{c/a/\mathrm{A}} = \alpha\hat{\imath}_\mathrm{B} + \beta\hat{\jmath}_\mathrm{B}, \tag{4.16.29}$$

where

$$\alpha \triangleq \rho r_0\dot{\phi}\sin\phi, \qquad \beta \triangleq \rho r_0\dot{\phi}\cos\phi - \rho r\dot{\phi}. \tag{4.16.30}$$

Finally,

$$\begin{aligned}
\vec{\alpha}_{c/a/\mathrm{A}} &= \dot{\alpha}\hat{\imath}_\mathrm{B} + \alpha\overset{\mathrm{A}\bullet}{\hat{\imath}}_\mathrm{B} + \dot{\beta}\hat{\jmath}_\mathrm{B} + \beta\overset{\mathrm{A}\bullet}{\hat{\jmath}}_\mathrm{B} \\
&= \dot{\alpha}\hat{\imath}_\mathrm{B} + \alpha\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_\mathrm{B} + \dot{\beta}\hat{\jmath}_\mathrm{B} + \beta\vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_\mathrm{B} \\
&= \dot{\alpha}\hat{\imath}_\mathrm{B} + \alpha\frac{r}{R}\dot{\phi}\hat{k}_\mathrm{A} \times \hat{\imath}_\mathrm{B} + \dot{\beta}\hat{\jmath}_\mathrm{B} + \beta\frac{r}{R}\dot{\phi}\hat{k}_\mathrm{A} \times \hat{\jmath}_\mathrm{B} \\
&= \left(\dot{\alpha} - \beta\frac{r}{R}\dot{\phi}\right)\hat{\imath}_\mathrm{B} + \left(\dot{\beta} + \alpha\frac{r}{R}\dot{\phi}\right)\hat{\jmath}_\mathrm{B} \\
&= \rho[r_0(\sin\phi)\ddot{\phi} + (r^2/R - \rho r_0\cos\phi)\dot{\phi}^2]\hat{\imath}_\mathrm{B} + \rho[(r_0\cos\phi - r)\ddot{\phi} + \rho r_0(\sin\phi)\dot{\phi}^2]\hat{\jmath}_\mathrm{B}.
\end{aligned} \tag{4.16.31}$$

◇

Example 4.16.4. Consider three accelerometers located at points p_1, p_2, and p_3 in $\mathcal{B}$ and whose measurement axes are parallel with, respectively, the axes $\hat{\imath}_{\mathrm{B}}$, $\hat{\jmath}_{\mathrm{B}}$, and $\hat{k}_{\mathrm{B}}$ of the body frame $\mathrm{F_B}$. Let p be a point fixed in $\mathcal{B}$. Determine the acceleration of p with respect to an unforced particle w relative to an inertial frame.

Solution. For $i = 1, 2, 3$ the acceleration measurement at p_i is given by

$$a_i = e_i^{\mathsf{T}} \vec{a}_{p_i/w/\mathrm{A}}\Big|_{\mathrm{B}}. \tag{4.16.32}$$

Next, note that, for $i = 1, 2, 3$,

$$\vec{a}_{p/w/\mathrm{A}}\Big|_{\mathrm{B}} = \vec{a}_{p/p_i/\mathrm{A}}\Big|_{\mathrm{B}} + \vec{a}_{p_i/w/\mathrm{A}}\Big|_{\mathrm{B}}, \tag{4.16.33}$$

and, furthermore,

$$\begin{aligned}\vec{a}_{p/p_i/\mathrm{A}} &= \vec{a}_{p/p_i/\mathrm{B}} + 2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{A}\bullet}{\vec{r}}_{p/p_i} + \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/p_i} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/p_i})\\ &= \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{p/p_i} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{p/p_i}).\end{aligned} \tag{4.16.34}$$

Substituting (4.16.34) in (4.16.33) yields

$$\vec{a}_{p/w/\mathrm{A}}\Big|_{\mathrm{B}} = \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} \times \vec{r}_{p/p_i}\Big|_{\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \vec{r}_{p/p_i}\Big|_{\mathrm{B}}\right) + \vec{a}_{p_i/w/\mathrm{A}}\Big|_{\mathrm{B}}, \tag{4.16.35}$$

and thus, for $i = 1, 2, 3$,

$$e_i^{\mathsf{T}} \vec{a}_{p/w/\mathrm{A}}\Big|_{\mathrm{B}} = e_i^{\mathsf{T}} \left[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} \times \vec{r}_{p/p_i}\Big|_{\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \vec{r}_{p/p_i}\Big|_{\mathrm{B}}\right)\right] + a_i. \tag{4.16.36}$$

Finally, stacking the components of $\vec{a}_{p/w/\mathrm{A}}\Big|_{\mathrm{B}}$ yields

$$\vec{a}_{p/w/\mathrm{A}}\Big|_{\mathrm{B}} = \begin{bmatrix} e_1^{\mathsf{T}} \left[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} \times \vec{r}_{p/p_1}\Big|_{\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \vec{r}_{p/p_1}\Big|_{\mathrm{B}}\right)\right] \\ e_2^{\mathsf{T}} \left[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} \times \vec{r}_{p/p_2}\Big|_{\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \vec{r}_{p/p_2}\Big|_{\mathrm{B}}\right)\right] \\ e_3^{\mathsf{T}} \left[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\Bigg|_{\mathrm{B}} \times \vec{r}_{p/p_3}\Big|_{\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \times \vec{r}_{p/p_3}\Big|_{\mathrm{B}}\right)\right] \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{4.16.37}$$

◇

4.17 Theoretical Problems

Problem 4.17.1. Let $\vec{x}$ be a nonzero physical vector, and let α be a real number. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\vec{x}|^{\alpha} = \alpha|\vec{x}|^{\alpha-2}\, \overset{\mathrm{A}\bullet\prime}{\vec{x}}\ \vec{x}. \tag{4.17.1}$$

Problem 4.17.2. Let $\vec{x}$ be a nonzero physical vector, and let $\mathrm{F_A}$ be a frame. Show that the following statements are equivalent:

i) $\overset{\widehat{\mathrm{A}\bullet}}{\vec{x}} = \pm\hat{x}.$

ii) $\overset{\mathrm{A}\bullet}{\hat{x}} = 0.$

(Hint: Use Problem 4.17.1 with $\alpha = -1$.)

Problem 4.17.3. Let $\mathrm{F_A}$ be a frame, let $\hat{y}$ be a unit vector, for all t, let $\hat{x}(t)$ be a unit vector, and assume that, for all t, the vectors $\hat{y}$ and $\hat{x}(t)$ lie in the same plane. Show that

$$\cos\theta_{\overset{\mathrm{A}\bullet}{\hat{x}}/\hat{y}} = \pm\sin\theta_{\hat{x}/\hat{y}}.$$

Now, assume that $\hat{y}$ is constant with respect to $\mathrm{F_A}$. Show that

$$\cos\theta_{\overset{\mathrm{A}\bullet}{\hat{x}}/\hat{y}} = -(\operatorname{sign}\dot{\theta}_{\hat{x}/\hat{y}})\sin\theta_{\hat{x}/\hat{y}}.$$

Problem 4.17.4. Let $\mathrm{F_A}$ be a frame, let $\hat{y}$ be a unit vector that is constant with respect to $\mathrm{F_A}$, and, for all t, let $\hat{x}(t)$ be a unit vector. Show that

$$\dot{\theta}_{\hat{x}/\hat{y}} = -\frac{\cos\theta_{\overset{\mathrm{A}\bullet}{\hat{x}}/\hat{y}}}{\sin\theta_{\hat{x}/\hat{y}}}|\overset{\mathrm{A}\bullet}{\hat{x}}|.$$

Now, assume that, for all t, the vectors $\hat{y}$ and $\hat{x}(t)$ lie in the same plane. Show that

$$|\dot{\theta}_{\hat{x}/\hat{y}}| = |\overset{\mathrm{A}\bullet}{\hat{x}}|.$$

(Hint: Use Problem 4.17.3.)

Problem 4.17.5. Apply Fact 4.1.4 to the following two cases:

i) The position of a point moving along a circle.

ii) The velocity of a point moving with constant speed along a circle.

Problem 4.17.6. Prove (4.5.1) by using (4.2.3) to represent each physical angular velocity matrix and by resolving each term in $\mathrm{F_C}$.

Problem 4.17.7. Let $\mathrm{F_A}$ be a frame, and, for all $t \in (t_1, t_2)$, let $\hat{x}(t)$ be a unit dimensionless vector and let $\vec{y}(t)$ be a vector such that $\hat{x}(t) \times \vec{y}(t) = 0$ and $\overset{\mathrm{A}\bullet}{\vec{y}}(t) = 0$. Furthermore, assume that there exists $t_0 \in (t_1, t_2)$ such that $\vec{y}(t_0) \neq 0$. Show that $\overset{\mathrm{A}\bullet}{\hat{x}}(t_0) = 0$. (Hint: Apply Problem 2.24.1.)

Problem 4.17.8. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, assume that, for all $t \in (t_1, t_2)$, $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}(t) = 0$ and $\vec{\omega}_{\mathrm{B/A}}(t) \times \hat{n}_{\mathrm{B/A}}(t) = 0$, and assume that there exists $t_0 \in (t_1, t_2)$ such that $\vec{\omega}_{\mathrm{B/A}}(t_0) \neq 0$. Show that $\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}}(t_0) = \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}}(t_0) = 0$. (Hint: Apply Problem 4.17.3.)

Problem 4.17.9. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that

$$\vec{\omega}_{\mathrm{B/A}} = \tfrac{1}{2}(\hat{\imath}_{\mathrm{B}} \times \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} + \hat{\jmath}_{\mathrm{B}} \times \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\mathrm{B}} + \hat{k}_{\mathrm{B}} \times \overset{\mathrm{A}\bullet}{\hat{k}}_{\mathrm{B}}), \tag{4.17.2}$$

$$\vec{\omega}_{\mathrm{B/A}} = (\overset{\mathrm{A}\bullet\prime}{\hat{\jmath}}_{\mathrm{B}}\;\hat{k}_{\mathrm{B}})\hat{\imath}_{\mathrm{B}} + (\overset{\mathrm{A}\bullet\prime}{\hat{k}}_{\mathrm{B}}\;\hat{\imath}_{\mathrm{B}})\hat{\jmath}_{\mathrm{B}} + (\overset{\mathrm{A}\bullet\prime}{\hat{\imath}}_{\mathrm{B}}\;\hat{\jmath}_{\mathrm{B}})\hat{k}_{\mathrm{B}}. \tag{4.17.3}$$

Problem 4.17.10. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ and $\vec{y}$ be position vectors that are constant with respect to $\mathrm{F_B}$. Show that

$$\overset{\mathrm{A}\bullet\prime}{\vec{x}}\;\vec{y}\vec{\omega}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet}{\vec{x}} \times \overset{\mathrm{A}\bullet}{\vec{y}}, \tag{4.17.4}$$

that is,

$$(\vec{\omega}_{\mathrm{B/A}} \times \vec{x})'\vec{y}\vec{\omega}_{\mathrm{B/A}} = (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}) \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{y}). \tag{4.17.5}$$

Problem 4.17.11. Consider 3-2-1 Euler angles Ψ, Θ, and Φ that transform $\mathrm{F_A}$ to $\mathrm{F_D}$.

i) Determine all values of the Euler angles such that *not* all angular velocities $\vec{\omega}_{\mathrm{D/A}}$ can be attained by Euler-angle derivatives $\dot{\Psi}$, $\dot{\Theta}$, and $\dot{\Phi}$. In particular, show that not all angular velocities $\vec{\omega}_{\mathrm{D/A}}$ can be attained by Euler-angle derivatives if and only if $\Theta = \pm\pi/2$.

ii) Show that, if $\Theta = \pm\pi/2$ and $\omega \neq 0$, then $\vec{\omega}_{\mathrm{D/A}} = \omega\hat{k}_{\mathrm{D}}$ is attainable by Euler-angle derivatives if and only if $\Phi = \pm\pi/2$.

(Remark: *ii*) illustrates *gimbal lock*.)

Problem 4.17.12. Let $\mathrm{F_A}$ be a frame, and let $\vec{x}$ be a physical vector. Show that

$$\left.\overset{\mathrm{A}\bullet\bullet}{\vec{x}}\right|_{\mathrm{A}} = \overbrace{\left.\vec{x}\right|_{\mathrm{A}}}^{\cdot\cdot}. \tag{4.17.6}$$

Problem 4.17.13. Show that the terms in the transport theorem (4.4.1) can be rearranged so that the equation has the same form but with $\mathrm{F_A}$ and $\mathrm{F_B}$ interchanged. Next, consider the same problem for the double transport theorem (4.6.1). (Hint: Move the last three terms in (4.6.1) to the left hand side and use the transport theorem.)

Problem 4.17.14. Consider the transport theorem (4.4.1) written in two forms, namely, for frames $\mathrm{F_A}$ and $\mathrm{F_B}$ and for frames $\mathrm{F_B}$ and $\mathrm{F_C}$. Combine these equations to obtain the transport theorem for frames $\mathrm{F_A}$ and $\mathrm{F_C}$. Next, consider the analogous problem for the double transport theorem (4.6.1).

Problem 4.17.15. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that

$$\overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}}, \quad \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}}, \tag{4.17.7}$$

$$\overset{\mathrm{A}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}, \quad \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{\omega}}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}. \tag{4.17.8}$$

Problem 4.17.16. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let $\vec{x}$ be a physical vector. Show that

$$\overset{\substack{\mathrm{A}\bullet\\ \mathrm{B}\bullet}}{\vec{x}} + \overset{\substack{\mathrm{B}\bullet\\ \mathrm{A}\bullet}}{\vec{x}} = \overset{\mathrm{A}\bullet\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet\bullet}{\vec{x}} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}), \tag{4.17.9}$$

$$\overset{\substack{\mathrm{A}\bullet\\ \mathrm{B}\bullet}}{\vec{x}} = \overset{\substack{\mathrm{B}\bullet\\ \mathrm{A}\bullet}}{\vec{x}} + \vec{x} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}\ . \tag{4.17.10}$$

If, in addition, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$, then

$$\overset{\substack{\mathrm{B}\bullet\\ \mathrm{A}\bullet}}{\vec{x}} = \overset{\substack{\mathrm{A}\bullet\\ \mathrm{B}\bullet}}{\vec{x}} = \tfrac{1}{2}[\overset{\mathrm{A}\bullet\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet\bullet}{\vec{x}} - \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x})]. \tag{4.17.11}$$

Finally, if $\vec{\omega}_{\mathrm{B/A}} = 0$, then

$$\overset{\substack{\mathrm{B}\bullet\\ \mathrm{A}\bullet}}{\vec{x}} = \overset{\substack{\mathrm{A}\bullet\\ \mathrm{B}\bullet}}{\vec{x}} = \tfrac{1}{2}(\overset{\mathrm{A}\bullet\bullet}{\vec{x}} + \overset{\mathrm{B}\bullet\bullet}{\vec{x}}). \tag{4.17.12}$$

Confirm these identities for the example $\vec{x} = \hat{\imath}_{\mathrm{B}}$ and $\vec{\omega}_{\mathrm{B/A}} = \hat{\imath}_{\mathrm{A}}$. (Hint: To prove the second equality, write $\vec{x} = x_1\hat{\imath}_{\mathrm{B}} + x_2\hat{\jmath}_{\mathrm{B}} + x_3\hat{k}_{\mathrm{B}}$.)

Problem 4.17.17. Derive the triple transport theorem

$$\begin{aligned}
\overset{\mathrm{A}\bullet\bullet\bullet}{\vec{x}} = {} & \overset{\mathrm{B}\bullet\bullet\bullet}{\vec{x}} + 3\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet\bullet}{\vec{x}} + 3\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}} + 3\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{x}}) \\
& + (\overset{\mathrm{B}\bullet\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}) \times \vec{x} + 2\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x}) \\
& + \vec{\omega}_{\mathrm{B/A}} \times (\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{x}) + \vec{\omega}_{\mathrm{B/A}} \times [\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{x})].
\end{aligned} \tag{4.17.13}$$

4.18 Applied Problems

Problem 4.18.1. For all parts below, assume that the given numbers are exact. In addition, assume that the Sun, Earth, and Moon all rotate and travel counterclockwise as viewed from "above" the solar system (that is, looking down on the North Pole of the Earth), and that all orbits are circular and lie in the same plane. Finally, assume that all stars (including the Sun) do not move relative to each other.

i) Assume that the length of the solar day on Earth is 24 hours, and assume that the Earth completes one orbit around the Sun in relation to the stars every 365.25 solar days (the sidereal year). Determine the length of the sidereal day, that is, the time it takes for the Earth to rotate around its axis once relative to a star frame, that is, a frame whose axes have fixed directions relative to the stars. State your solution in hours and minutes.

ii) In addition to the assumptions in *i*), assume that the Sun rotates around its axis relative to the star frame once every 27 solar days. Determine the length of time that it takes the Earth to rotate around its axis once relative to a Sun body-fixed frame.

iii) Assume that the Moon completes one orbit around the Earth in relation to the stars every 27.3 solar days (which is the Moon's sidereal period, also called the lunar month). For a given initial relative configuration, determine the time it takes for the Moon, Earth, and Sun to return to the same configuration (which is the Moon's synodic period).

(Hint: Create several frames, including a star frame and frames that are attached to the Sun, Earth, and Moon as well as the "invisible" arms connecting these bodies. Then, use sums of angular velocities to determine relationships between the periods.)

Problem 4.18.2. Let $\mathrm{F_A}$ be a frame that is fixed to a horizontal plane with origin $\mathrm{o_A}$. The axis $\hat{\imath}_\mathrm{A}$ points to the right, and the axis $\hat{\jmath}_\mathrm{A}$ points downward. A disk $\mathcal{B}$ of radius R rolls in a straight path to the right, which is in the direction of $\hat{\imath}_\mathrm{A}$. The center of $\mathcal{B}$ is the point $\mathrm{o_B}$, and $\mathrm{F_B}$ is fixed to the disk. The speed $v \ge 0$ of $\mathrm{o_B}$ along the path is not necessarily constant, and the angle ϕ of $\mathcal{B}$ relative to its starting angle has the rate $\dot{\phi} > 0$, which is not necessarily constant. As $\mathcal{B}$ rolls, it may also slip, and thus v is not necessarily equal to $R\dot{\phi}$. Let p denote a point fixed on the circumference of $\mathcal{B}$, and let q denote a point fixed in the path. Define $x \triangleq |\vec{r}_{q/\mathrm{o_A}}|$. Let c denote the instantaneous contact point between $\mathcal{B}$ and the path.

i) Determine the velocity of q relative to $\mathrm{o_A}$ with respect to $\mathrm{F_A}$ resolved in $\mathrm{F_A}$.

ii) Determine the velocity of q relative to $\mathrm{o_A}$ with respect to $\mathrm{F_B}$ resolved in $\mathrm{F_A}$.

iii) Determine the velocity of p relative to $\mathrm{o_A}$ with respect to $\mathrm{F_A}$ resolved in $\mathrm{F_A}$ when p is at the 9:00, 12:00, and 3:00 positions.

iv) Determine the velocity of p relative to $\mathrm{o_A}$ with respect to $\mathrm{F_A}$ resolved in $\mathrm{F_A}$ when p and q are colocated.

v) Determine the velocity of p relative to $\mathrm{o_A}$ with respect to $\mathrm{F_B}$ resolved in $\mathrm{F_A}$ when p and q are colocated.

vi) Determine the acceleration of p relative to $\mathrm{o_A}$ with respect to $\mathrm{F_A}$ resolved in $\mathrm{F_A}$ when p and q are colocated. If v is constant, what is the direction of this acceleration?

Specialize the solutions to *i*)–*v*) to the case where $\mathcal{B}$ rolls without slipping, that is, $v = R\dot{\phi}$. Now, assume that $\mathcal{B}$ rolls without slipping.

vii) Show that $\vec{v}_{c/\mathrm{o_A}/\mathrm{A}} = v\hat{\imath}_\mathrm{A} = \vec{v}_{c/\mathrm{o_B}/\mathrm{B}} = R\dot{\phi}\hat{\imath}_\mathrm{A}$.

viii) Determine the velocity of c relative to $\mathrm{o_A}$ with respect to $\mathrm{F_B}$ resolved in $\mathrm{F_A}$.

Problem 4.18.3. Consider the gimbal mechanism shown in Figure 4.18.2. Assume that the angle of the outer gimbal relative to the support is given by $\psi(t) = 0.6\sin(20\pi t)$, the angle of the inner gimbal relative to the outer gimbal is $\theta(t) = 0.3\sin(90\pi t)$, and the disk supported by the inner gimbal spins at 700 rpm with zero initial angle. Resolve the angular velocity and angular acceleration of the disk relative to the support in a frame attached to the disk, and compute these vectors at time $t = 0.01$ sec.

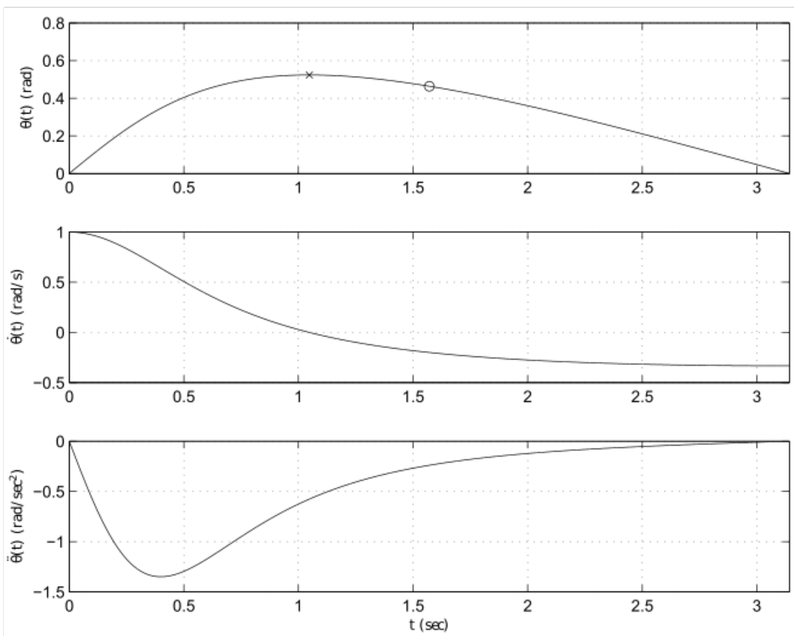


Figure 4.16.3: Example 4.16.2. Rotating wheel with slotted arm. The angle θ and angle rates $\dot\theta$ and $\ddot\theta$ are plotted for $0 \le t \le \pi$ sec assuming that $R = 2$ m, $L = 4$ m, and $\dot\phi = 1$ rad/sec. The points marked '×' and '∘' denote $\theta(t_1)$, where $\phi(t_1) = 60$ deg and thus the arm is tangent to the wheel, and $\theta(t_2)$, where $\phi(t_2) = 90$ deg, respectively.

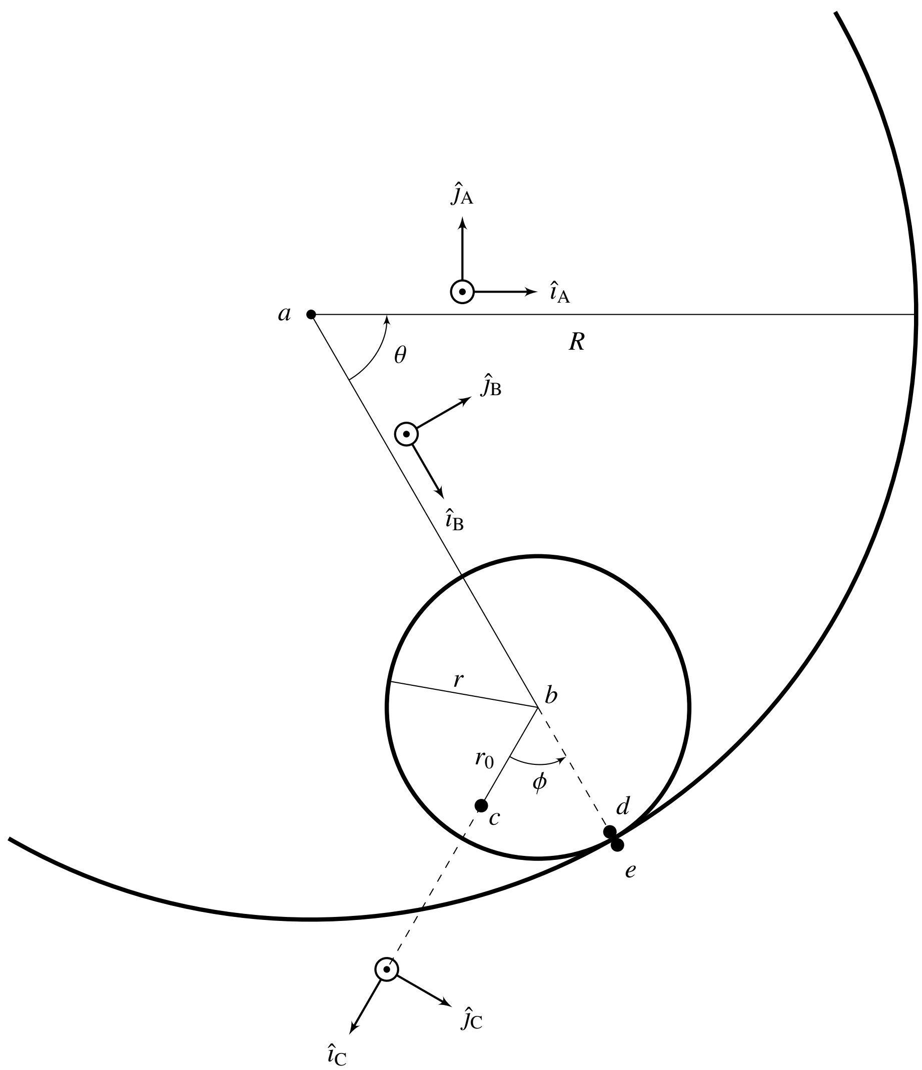
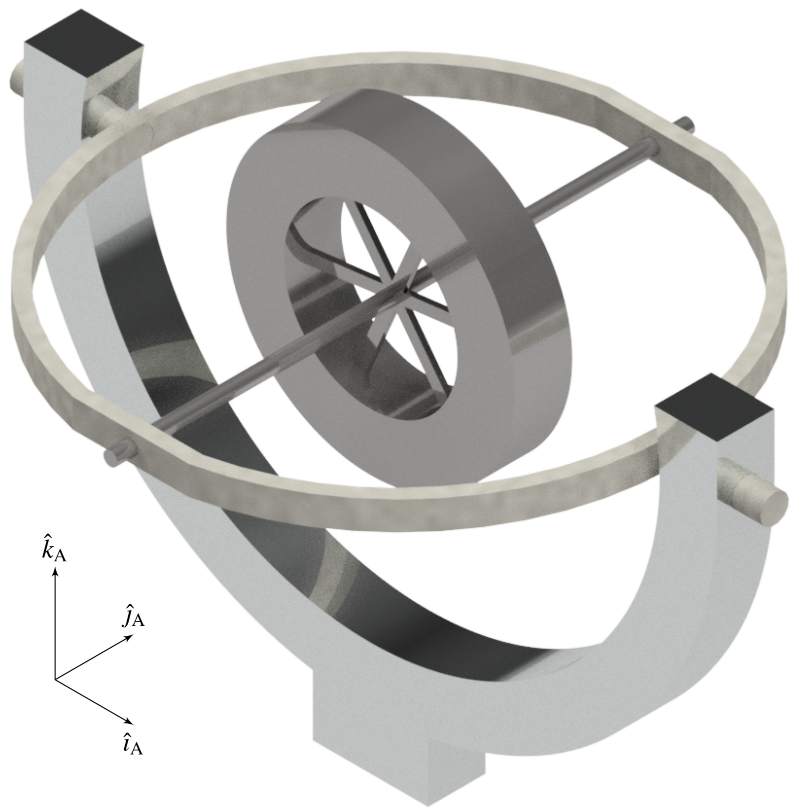
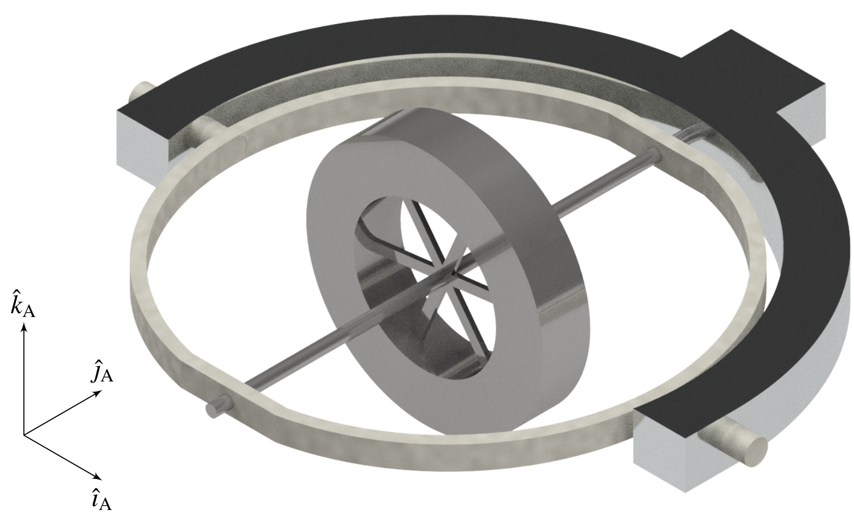


Figure 4.16.4: Example 4.16.3. Disk rolling inside a circle.



(a)



(b)

Figure 4.18.1: Gimbal mechanism for Problem 4.18.3 in a) unlocked position and b) locked position. Note that, in the locked position, the spin direction of the disk cannot be maintained if the outer gimbal rotates about the $\hat{k}_A$ axis.

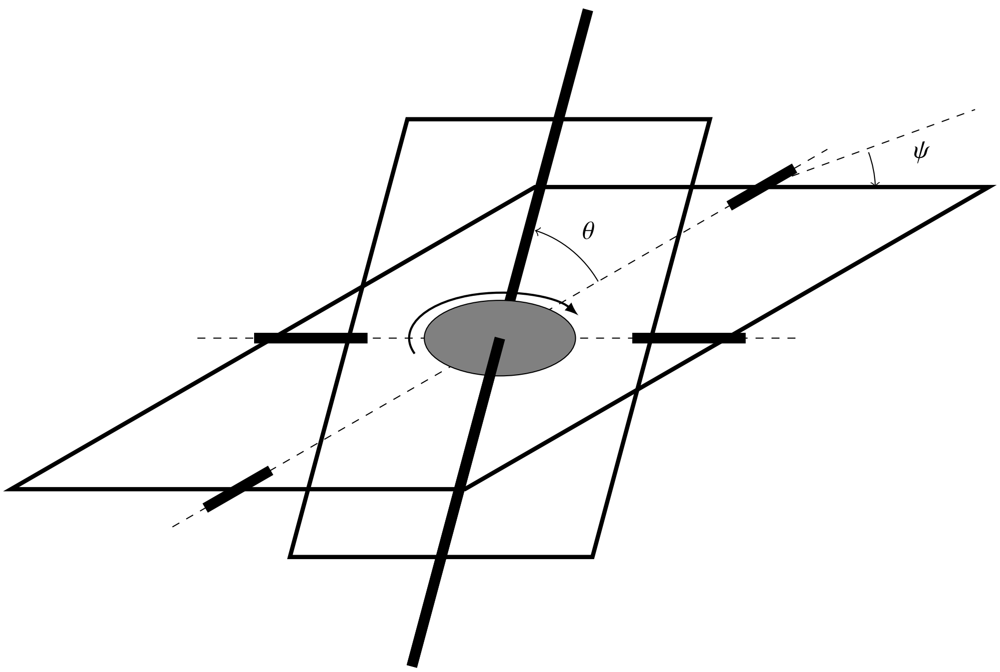


Figure 4.18.2: Gimbal mechanism for Problem 4.18.3.

Problem 4.18.4. Consider a horizontal platform connected to the horizontal ground by a vertical pin joint. The platform rotates at a constant rate relative to the ground. Let x and y be distinct points on the platform.

i) Show that the acceleration of y relative to x with respect to a frame fixed to the ground is independent of the location of the pin joint relative to x and y.

ii) Now, assume that the pin joint is mounted on a horizontal X-Y table that can move the pin joint along an arbitrary path, for example, a straight line or a curve. How does this additional motion affect the acceleration of y relative to x with respect to a frame fixed to the ground?

Problem 4.18.5. Points a and b are connected by a rigid bar with pin joints at both a and b as shown in Figure 4.18.3. The pin joint at a moves horizontally to the right with constant speed v. The pin joint at b is connected to a disk with center c that rolls clockwise without slipping. The length of the bar is $2R$, and the radius of the disk is R. At the time instant shown, the vector $\vec{r}_{c/b}$ is parallel with the horizontal surface. For the configuration shown, determine the velocity and acceleration of c relative to a point fixed in the horizontal surface and with respect to a frame that is also fixed to the horizontal surface. Resolve your solution in the frame that is fixed to the horizontal surface.

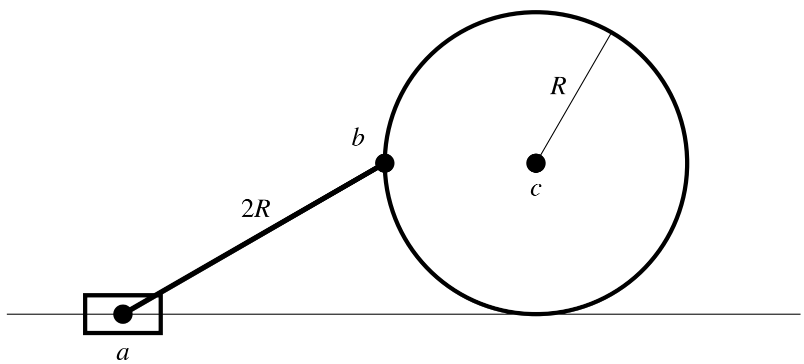


Figure 4.18.3: Bar and disk linkage for Problem 4.18.5.

Problem 4.18.6. The horizontal rod in Figure 4.18.4 moves to the right with constant speed v with respect to the ground. A pin at the end of the rod slides within the slot that passes through the center of the wheel. The radius of the wheel is R, and the distance from the center of the wheel to the rod is d. The wheel rolls clockwise in a straight line without slipping. Let θ denote the angle between the vertical direction and the direction of the slot. Determine $\dot{\theta}$ and $\ddot{\theta}$ as functions of d, R, v, and θ.

Problem 4.18.7. The wheel of radius r in Figure 4.18.5 is attached at its center b to an axle, whose other end is connected to a central hub at the point a. (Note that a is fixed relative to the ground.) The wheel rolls without slipping along a circular path on the ground and with radius R. The constant spin rate of the wheel is $\omega > 0$, and the constant spin rate of the hub is $\Omega > 0$. Both spin directions are indicated in the figure by the curved arrows. The point c is fixed on the edge of the wheel. At the instant at which c touches the ground, determine the velocity of c relative to a with respect to a frame F_A fixed to the ground and resolved in the ground frame. Next, derive an equation that relates r, R, ω, and Ω. Finally, at the instant at which c touches the ground, determine the acceleration of c relative to a with respect to the ground frame and resolved in a frame F_B fixed

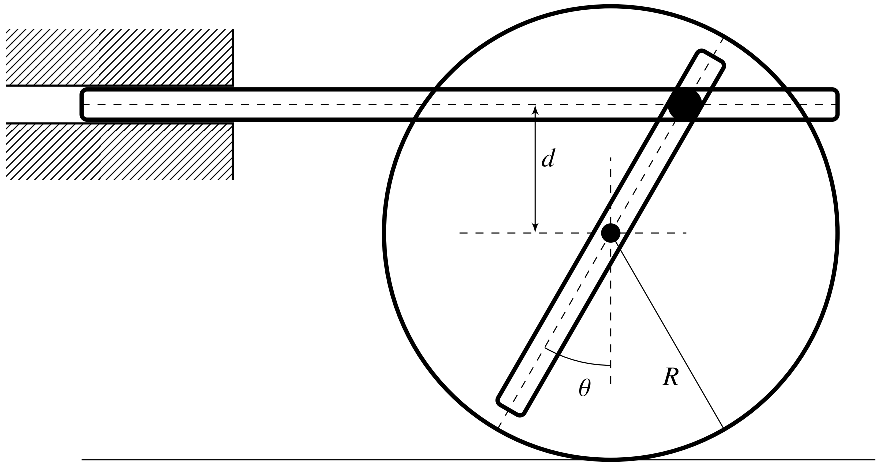


Figure 4.18.4: Wheel with slot and horizontal rod for Problem 4.18.6.

to the hub.

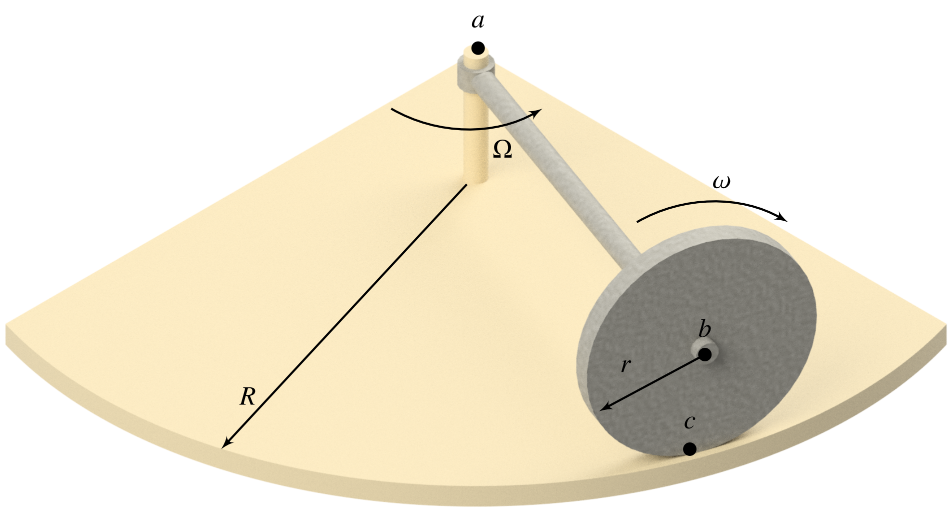


Figure 4.18.5: Wheel with hub for Problem 4.18.7.

4.19 Solutions to the Applied Problems

Solution to Problem 4.18.3.

$$\vec{\alpha}_{\mathrm{D/A}} = \begin{bmatrix} -3122 \\ -1158 \\ -7298 \end{bmatrix} \text{rad/sec}^2.$$

Solution to Problem 4.18.5.

$$\dot{\omega}_2 = \frac{4 + 3\sqrt{3}}{(1+\sqrt{3})^3} \frac{v^2}{R^2},$$

where ω_2 is the spin rate of the disk relative to the ground.

Solution to Problem 4.18.6.

$$\dot{\theta} = \frac{v \cos^2 \theta}{R \cos^2 \theta + d}.$$

Solution to Problem 4.18.7.

$$\omega r = \Omega R,$$

$$\vec{a}_{c/a/\mathrm{A}} = \Omega^2 R \hat{\jmath}_{\mathrm{B}} + \omega^2 r \hat{k}_{\mathrm{B}}.$$

Symbol	Definition
$\overset{\mathrm{A}\bullet}{\vec{x}}$	Derivative of $\vec{x}$ with respect to frame A
$\vec{v}_{y/x/\mathrm{A}}$	Velocity vector $\overset{\mathrm{A}\bullet}{\vec{r}}_{y/x}$
$\vec{a}_{y/x/\mathrm{A}}$	Acceleration vector $\overset{\mathrm{A}\bullet}{\vec{v}}_{y/x/\mathrm{A}} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/x}$
$\vec{\omega}_{\mathrm{B/A}}$	Angular velocity vector
$\vec{\Omega}_{\mathrm{B/A}}$	Angular velocity matrix
$\vec{\alpha}_{\mathrm{B/A}}$	Angular acceleration vector
$\vec{\alpha}_{\mathrm{B/A/C}}$	Angular acceleration vector

Table 4.19.1: Notation for Chapter 4.

Chapter Five

Geometry and Kinematics in Alternative Frames

If the orientation of a frame depends on the position of a point x, then the frame is a *position-dependent frame at x*. The cylindrical and spherical frames are position-dependent frames.

5.1 Cylindrical Frame

The cylindrical frame $\mathrm{F}_{\mathrm{cyl}}$ at the point x is obtained by rotating a given frame F_{A} around $\hat{k}_{\mathrm{A}}$ until $\hat{\imath}_{\mathrm{A}}$ is codirectional with the projection of $\vec{r}_{x/\mathrm{o_A}}$ onto the plane spanned by $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$. Consequently, the cylindrical frame at x is related to F_{A} by

$$\mathrm{F}_{\mathrm{cyl}} = \vec{R}_{\mathrm{cyl/A}}\mathrm{F}_{\mathrm{A}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\mathrm{F}_{\mathrm{A}}, \tag{5.1.1}$$

that is,

$$\mathrm{F}_{\mathrm{A}} \xrightarrow[3]{\theta} \mathrm{F}_{\mathrm{cyl}}, \tag{5.1.2}$$

where the *azimuthal angle* $\theta \in (-\pi, \pi]$ is the directed angle from $\hat{\imath}_{\mathrm{A}}$ to $\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{o_A}}$ around $\hat{k}_{\mathrm{A}}$, that is,

$$\theta \triangleq \theta_{\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{o_A}}/\hat{\imath}_{\mathrm{A}}/\hat{k}_{\mathrm{A}}}. \tag{5.1.3}$$

If $\vec{r}_{x/\mathrm{o_A}}$ is parallel with $\hat{k}_{\mathrm{A}}$, then θ is defined to be 0. The *radial*, *tangential*, and *axial* axes of the cylindrical frame $\mathrm{F}_{\mathrm{cyl}}$ at x are denoted by $\hat{e}_{\mathrm{r}}$, $\hat{e}_{\mathrm{t}}$, and $\hat{e}_{\mathrm{a}}$, respectively, and defined by

$$\hat{e}_{\mathrm{r}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\imath}_{\mathrm{A}}, \tag{5.1.4}$$

$$\hat{e}_{\mathrm{t}} = \vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\jmath}_{\mathrm{A}}, \tag{5.1.5}$$

$$\hat{e}_{\mathrm{a}} = \hat{k}_{\mathrm{A}}. \tag{5.1.6}$$

The *cylindrical frame* is thus given by

$$\mathrm{F}_{\mathrm{cyl}} = [\hat{e}_{\mathrm{r}} \ \ \hat{e}_{\mathrm{t}} \ \ \hat{e}_{\mathrm{a}}]. \tag{5.1.7}$$

It follows from (5.1.3) and (5.1.4) that

$$\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{o_A}} = |\vec{P}_{\hat{\imath}_{\mathrm{A}},\hat{\jmath}_{\mathrm{A}}}\vec{r}_{x/\mathrm{o_A}}|\hat{e}_{\mathrm{r}}. \tag{5.1.8}$$

See Figure 5.1.1.

The cylindrical frame can be viewed as a body-fixed frame. In particular, consider a shaft parallel with the axis $\hat{k}_{\mathrm{A}}$ of the frame F_{A} attached to the base of the shaft. The shaft has a collar. A

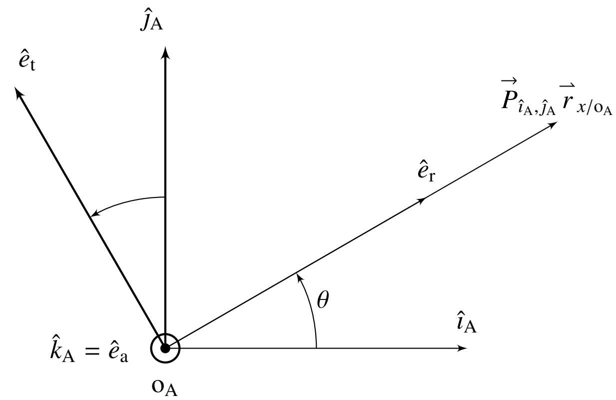


Figure 5.1.1: The directed angle θ that defines the cylindrical frame is the directed angle around $\hat{k}_A$ from $\hat{\imath}_A$ to the projection $\vec{P}_{\hat{\imath}_A,\hat{\jmath}_A}\vec{r}_{x/o_A}$ of $\vec{r}_{x/o_A}$ onto the plane spanned by $\hat{\imath}_A$ and $\hat{\jmath}_A$.

telescoping arm with a sleeve joint is attached to the collar at a right angle to the shaft. A rigid body $\mathcal{B}$ is rigidly attached to the tip of the telescoping arm. A body-fixed frame F_B is attached to $\mathcal{B}$ such that $\hat{k}_B$ is codirectional with $\hat{k}_A$ and $\hat{\imath}_B$ is parallel with the telescoping arm. The angle θ is defined to be $\theta \triangleq \theta_{\hat{\imath}_B/\hat{\imath}_A/\hat{k}_A}$. Therefore, if $\theta = 0$, then $F_A = F_B$. Consequently, the body-fixed frame F_B is the cylindrical frame F_{cyl}. See Figure 5.1.2.

Next, it follows from (5.1.6) that

$$\vec{P}_{\hat{\imath}_A,\hat{\jmath}_A} = \vec{P}_{\hat{e}_r,\hat{e}_t}, \tag{5.1.9}$$

that is,

$$\hat{\imath}_A\hat{\imath}_A + \hat{\jmath}_A\hat{\jmath}_A = \hat{e}_r\hat{e}_r + \hat{e}_t\hat{e}_t. \tag{5.1.10}$$

Using (5.1.8) and (5.1.9), it follows that

$$\vec{P}_{\hat{e}_r,\hat{e}_t}\vec{r}_{x/o_A} = \vec{P}_{\hat{\imath}_A,\hat{\jmath}_A}\vec{r}_{x/o_A} = |\vec{P}_{\hat{\imath}_A,\hat{\jmath}_A}\vec{r}_{x/o_A}|\hat{e}_r. \tag{5.1.11}$$

Next, since $\vec{R}_{cyl/A} = \vec{R}_{\hat{k}_A}(\theta)$, it follows from (5.1.4), (5.1.5), (5.1.6), and (2.10.13) that

$$\begin{bmatrix}\hat{e}_r\\ \hat{e}_t\\ \hat{e}_a\end{bmatrix} = \mathcal{O}_{cyl/A}\begin{bmatrix}\hat{\imath}_A\\ \hat{\jmath}_A\\ \hat{k}_A\end{bmatrix}, \tag{5.1.12}$$

where, using (2.13.12),

$$\mathcal{O}_{cyl/A} = \vec{R}_{\hat{k}_A}(\theta)\Big|_A^{\mathsf{T}} = \mathcal{O}_3(\theta) = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{5.1.13}$$

Consequently,

$$\hat{e}_r = (\cos\theta)\hat{\imath}_A + (\sin\theta)\hat{\jmath}_A, \tag{5.1.14}$$
$$\hat{e}_t = -(\sin\theta)\hat{\imath}_A + (\cos\theta)\hat{\jmath}_A, \tag{5.1.15}$$
$$\hat{e}_a = \hat{k}_A. \tag{5.1.16}$$

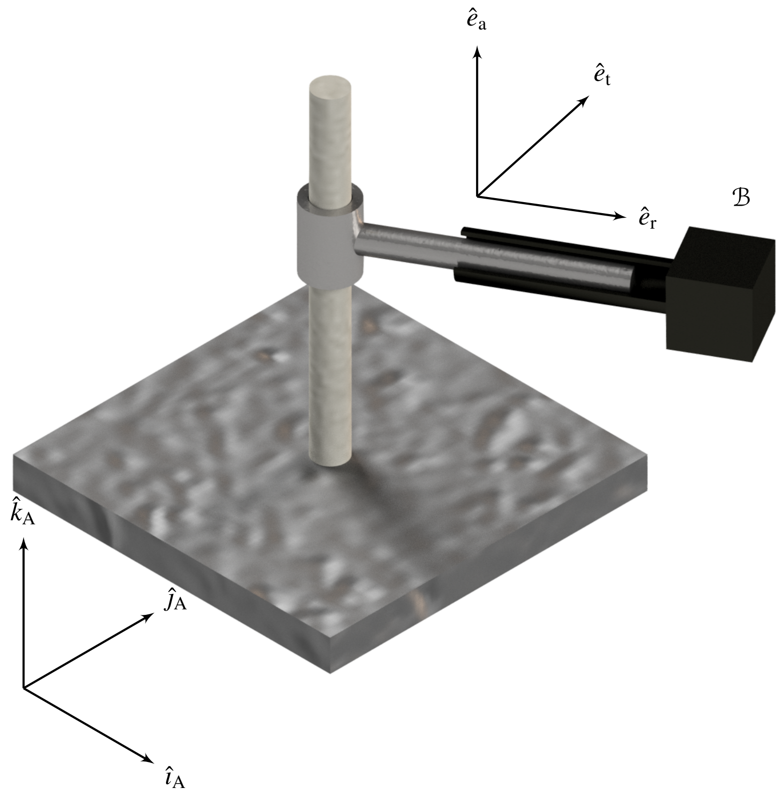


Figure 5.1.2: Cylindrical frame given as a body-fixed frame. The frame $\mathrm{F_A}$ is attached to the base of the shaft, and the cylindrical frame $\mathrm{F_{cyl}} = [\hat{e}_{\rm r} \ \ \hat{e}_{\rm t} \ \ \hat{e}_{\rm a}]$, which is obtained by rotating $\mathrm{F_A}$ about $\hat{k}_{\rm A}$, is attached to the body $\mathcal{B}$.

Next, write

$$r_{x/{\rm o_A}|{\rm A}} = \vec{r}_{x/{\rm o_A}}\Big|_{\rm A} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{5.1.17}$$

$$r_{x/{\rm o_A}|{\rm cyl}} = \vec{r}_{x/{\rm o_A}}\Big|_{\rm cyl} = \begin{bmatrix} r_{\rm r} \\ r_{\rm t} \\ r_{\rm a} \end{bmatrix}. \tag{5.1.18}$$

Then,

$$\begin{bmatrix} r_{\rm r} \\ r_{\rm t} \\ r_{\rm a} \end{bmatrix} = \mathcal{O}_{{\rm cyl/A}} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} (\cos\theta) r_1 + (\sin\theta) r_2 \\ -(\sin\theta) r_1 + (\cos\theta) r_2 \\ r_3 \end{bmatrix}. \tag{5.1.19}$$

Furthermore, with this notation (5.1.11) can be rewritten as

$$r_{\rm r}\hat{e}_{\rm r} + r_{\rm t}\hat{e}_{\rm t} = r_1\hat{\imath}_{\rm A} + r_2\hat{\jmath}_{\rm A} = |\vec{P}_{\hat{\imath}_{\rm A},\hat{\jmath}_{\rm A}}\vec{r}_{x/{\rm o_A}}|\hat{e}_{\rm r}, \tag{5.1.20}$$

which implies that

$$r_\mathrm{t} = 0, \tag{5.1.21}$$

$$r_\mathrm{r}\hat{e}_\mathrm{r} = r_1\hat{\imath}_\mathrm{A} + r_2\hat{\jmath}_\mathrm{A}, \tag{5.1.22}$$

$$\vec{r}_{x/\mathrm{o_A}} = r_1\hat{\imath}_\mathrm{A} + r_2\hat{\jmath}_\mathrm{A} + r_3\hat{k}_\mathrm{A} = r_\mathrm{r}\hat{e}_\mathrm{r} + r_\mathrm{a}\hat{e}_\mathrm{a}, \tag{5.1.23}$$

$$r_\mathrm{r} = |\vec{P}_{\hat{\imath}_\mathrm{A},\hat{\jmath}_\mathrm{A}}\vec{r}_{x/\mathrm{o_A}}| = \sqrt{r_1^2 + r_2^2}. \tag{5.1.24}$$

Next, it follows from the second equation in (5.1.19) that

$$(\cos\theta)r_2 = (\sin\theta)r_1, \tag{5.1.25}$$

If $r_1 = r_2 = 0$, then $\vec{r}_{x/\mathrm{o_A}}$ is parallel with $\hat{k}_\mathrm{A}$, and thus, by definition, $\theta = 0$. On the other hand, if $r_1 = 0$ and $r_2 \neq 0$, then $\cos\theta = 0$. In this case, $\theta = -\pi/2$ if and only if $r_2 < 0$, and $\theta = \pi/2$ if and only if $r_2 > 0$. In the case $r_1 \neq 0$, we have

$$\tan\theta = \frac{r_2}{r_1}. \tag{5.1.26}$$

Since θ is the angle of the complex number $r_1 + r_2\jmath$, it follows that

$$\theta = \mathrm{atan2}(r_2, r_1), \tag{5.1.27}$$

which determines θ for all values of r_1 and r_2. See (2.3.8). Finally, dotting (5.1.22) with $\hat{\imath}_\mathrm{A}$ yields

$$r_1 = (\cos\theta)r_\mathrm{r}, \tag{5.1.28}$$

while using (5.1.25) yields

$$r_2 = (\sin\theta)r_\mathrm{r}. \tag{5.1.29}$$

Note that (5.1.28) and (5.1.29) have the form of the planar polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, respectively.

The *radial*, *azimuthal*, and *vertical cylindrical coordinates* $(r_\mathrm{r}, \theta, r_\mathrm{a})$ at x are thus given by

$$r_\mathrm{r} = \sqrt{r_1^2 + r_2^2}, \tag{5.1.30}$$

$$\theta = \mathrm{atan2}(r_2, r_1), \tag{5.1.31}$$

$$r_\mathrm{a} = r_3. \tag{5.1.32}$$

5.2 Kinematics in the Cylindrical Frame

It follows from (5.1.2) that

$$\vec{\omega}_{\mathrm{cyl/A}} = \dot{\theta}\hat{e}_\mathrm{a}, \tag{5.2.1}$$

and thus

$$\omega_{\mathrm{cyl/A|cyl}} = \vec{\omega}_{\mathrm{cyl/A}}\Big|_{\mathrm{cyl}} = \dot{\theta}\ \hat{e}_\mathrm{a}\big|_{\mathrm{cyl}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}. \tag{5.2.2}$$

Alternatively, using (4.3.20) it follows that

$$\omega^{\times}_{\mathrm{cyl/A|cyl}} = -\dot{\mathcal{O}}_{\mathrm{cyl/A}}\mathcal{O}_{\mathrm{A/cyl}}$$

$$= -\begin{bmatrix} -\dot\theta\sin\theta & \dot\theta\cos\theta & 0 \\ -\dot\theta\cos\theta & -\dot\theta\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dot\theta & 0 \\ \dot\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.2.3}$$

Next, it follows from (4.3.21) that

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{r}} \\ \overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{t}} \\ \overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{a}} \end{bmatrix} = -\omega^{\times}_{\mathrm{cyl/A|cyl}}\begin{bmatrix} \hat e_{\mathrm{r}} \\ \hat e_{\mathrm{t}} \\ \hat e_{\mathrm{a}} \end{bmatrix}. \tag{5.2.4}$$

Therefore,

$$\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{r}} = \dot\theta\hat e_{\mathrm{t}} = \vec\omega_{\mathrm{cyl/A}} \times \hat e_{\mathrm{r}}, \tag{5.2.5}$$

$$\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{t}} = -\dot\theta\hat e_{\mathrm{r}} = \vec\omega_{\mathrm{cyl/A}} \times \hat e_{\mathrm{t}}, \tag{5.2.6}$$

$$\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{a}} = 0 = \vec\omega_{\mathrm{cyl/A}} \times \hat e_{\mathrm{a}}. \tag{5.2.7}$$

Now, let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$, and let x be a point. Then, $\vec r_{x/\mathrm{o_A}}$ can be expressed in the cylindrical frame as

$$\vec r_{x/\mathrm{o_A}} = r_{\mathrm{r}}\hat e_{\mathrm{r}} + r_{\mathrm{a}}\hat e_{\mathrm{a}}. \tag{5.2.8}$$

Therefore,

$$\begin{aligned}
\vec v_{x/\mathrm{o_A}/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec r}_{x/\mathrm{o_A}} \\
&= \dot r_{\mathrm{r}}\hat e_{\mathrm{r}} + r_{\mathrm{r}}\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{r}} + \dot r_{\mathrm{a}}\hat e_{\mathrm{a}} + r_{\mathrm{a}}\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{a}} \\
&= \dot r_{\mathrm{r}}\hat e_{\mathrm{r}} + \dot\theta r_{\mathrm{r}}\hat e_{\mathrm{t}} + \dot r_{\mathrm{a}}\hat e_{\mathrm{a}} \\
&= \dot r_{\mathrm{r}}\hat e_{\mathrm{r}} + \dot r_{\mathrm{a}}\hat e_{\mathrm{a}} + \dot\theta r_{\mathrm{r}}\hat e_{\mathrm{t}} \\
&= \overset{\mathrm{cyl}\bullet}{\vec r}_{x/\mathrm{o_A}} + \vec\omega_{\mathrm{cyl/A}} \times \vec r_{x/\mathrm{o_A}},
\end{aligned} \tag{5.2.9}$$

which is the transport theorem for the cylindrical frame. Furthermore,

$$\begin{aligned}
\vec a_{x/\mathrm{o_A}/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec v}_{x/\mathrm{o_A}/\mathrm{A}} \\
&= \ddot r_{\mathrm{r}}\hat e_{\mathrm{r}} + \dot r_{\mathrm{r}}\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{r}} + \ddot\theta r_{\mathrm{r}}\hat e_{\mathrm{t}} + \dot\theta\dot r_{\mathrm{r}}\hat e_{\mathrm{t}} + \dot\theta r_{\mathrm{r}}\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{t}} + \ddot r_{\mathrm{a}}\hat e_{\mathrm{a}} + \dot r_{\mathrm{a}}\overset{\mathrm{A}\bullet}{\hat e}_{\mathrm{a}} \\
&= \ddot r_{\mathrm{r}}\hat e_{\mathrm{r}} + \dot\theta\dot r_{\mathrm{r}}\hat e_{\mathrm{t}} + \ddot\theta r_{\mathrm{r}}\hat e_{\mathrm{t}} + \dot\theta\dot r_{\mathrm{r}}\hat e_{\mathrm{t}} - \dot\theta^2 r_{\mathrm{r}}\hat e_{\mathrm{r}} + \ddot r_{\mathrm{a}}\hat e_{\mathrm{a}} \\
&= (\ddot r_{\mathrm{r}} - \dot\theta^2 r_{\mathrm{r}})\hat e_{\mathrm{r}} + (2\dot\theta\dot r_{\mathrm{r}} + \ddot\theta r_{\mathrm{r}})\hat e_{\mathrm{t}} + \ddot r_{\mathrm{a}}\hat e_{\mathrm{a}} \\
&= \ddot r_{\mathrm{r}}\hat e_{\mathrm{r}} + \ddot r_{\mathrm{a}}\hat e_{\mathrm{a}} + \underbrace{2\dot\theta\dot r_{\mathrm{r}}\hat e_{\mathrm{t}}}_{\substack{\text{Coriolis}\\ \text{acceleration}}} + \underbrace{\ddot\theta r_{\mathrm{r}}\hat e_{\mathrm{t}}}_{\substack{\mathrm{A}^2\\ \text{acceleration}}} + \underbrace{-\dot\theta^2 r_{\mathrm{r}}\hat e_{\mathrm{r}}}_{\substack{\text{centripetal}\\ \text{acceleration}}} \\
&= \overset{\mathrm{cyl}\bullet\bullet}{\vec r}_{x/\mathrm{o_A}} + 2\vec\omega_{\mathrm{cyl/A}} \times \overset{\mathrm{cyl}\bullet}{\vec r}_{x/\mathrm{o_A}} + \overset{\mathrm{cyl}\bullet}{\vec\omega}_{\mathrm{cyl/A}} \times \vec r_{x/\mathrm{o_A}} + \vec\omega_{\mathrm{cyl/A}} \times (\vec\omega_{\mathrm{cyl/A}} \times \vec r_{x/\mathrm{o_A}}),
\end{aligned} \tag{5.2.10}$$

which is the double transport theorem for the cylindrical frame.

5.3 Spherical Frame

The spherical frame $\mathrm{F}_{\mathrm{sph}}$ at the point x is obtained by rotating the cylindrical frame around its tangential axis until the radial axis is codirectional with the position of x relative to the origin of F_{A}. Consequently, the spherical frame is related to the cylindrical frame $\mathrm{F}_{\mathrm{cyl}}$ by

$$\mathrm{F}_{\mathrm{sph}} = \vec{R}_{\mathrm{sph/cyl}}\mathrm{F}_{\mathrm{cyl}} = \vec{R}_{\mathrm{sph/A}}\mathrm{F}_{\mathrm{A}}, \tag{5.3.1}$$

that is,

$$\mathrm{F}_{\mathrm{A}} \xrightarrow[3]{\theta} \mathrm{F}_{\mathrm{cyl}} \xrightarrow[2]{\phi} \mathrm{F}_{\mathrm{sph}}, \tag{5.3.2}$$

where

$$\vec{R}_{\mathrm{sph/cyl}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi), \tag{5.3.3}$$

$\hat{e}_{\mathrm{t}}$ is the tangential axis (5.1.5) of the cylindrical frame, $\theta \in (-\pi, \pi]$ the azimuthal angle (5.1.3), and the *elevation angle* $\phi \in [-\pi/2, \pi/2]$ is the directed angle from the radial axis $\hat{e}_{\mathrm{r}}$ (defined by (5.1.4)) of the cylindrical frame to $\vec{r}_{x/\mathrm{o_A}}$ around $\hat{e}_{\mathrm{t}}$, that is,

$$\phi \triangleq \theta_{\vec{r}_{x/\mathrm{o_A}}/\hat{e}_{\mathrm{r}}/\hat{e}_{\mathrm{t}}}. \tag{5.3.4}$$

The elevation angle ϕ can be viewed as the latitude λ of a point on the Earth, where, by convention, $\lambda \triangleq -\phi$. Hence, $\lambda > 0$ corresponds to north latitude, and $\lambda < 0$ corresponds to south latitude. Note that $\vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)$ denotes a right-hand-rule clockwise rotation around $\hat{e}_{\mathrm{t}}$ for positive values of ϕ. Hence, if ϕ is positive, then the component $\vec{r}_{x/\mathrm{o_A}} \cdot \hat{k}_{\mathrm{A}}$ of $\vec{r}_{x/\mathrm{o_A}}$ in the direction of $\hat{k}_{\mathrm{A}}$ is negative. The *spherical frame* is thus given by

$$\mathrm{F}_{\mathrm{sph}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\mathrm{F}_{\mathrm{A}}. \tag{5.3.5}$$

Furthermore,

$$\vec{r}_{x/\mathrm{o_A}} = |\vec{r}_{x/\mathrm{o_A}}|\vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\imath}_{\mathrm{A}}. \tag{5.3.6}$$

The spherical frame can be viewed as a body-fixed frame. In particular, consider the vertical rotating shaft in Figure 5.3.3 whose longitudinal axis is parallel with $\hat{k}_{\mathrm{A}}$. A telescoping arm is connected to the shaft by means of a pin. The telescoping arm has a sleeve, and a rigid body $\mathcal{B}$ is rigidly attached to the end of the arm opposite to the pin. Now, consider a body-fixed frame F_{B} attached to $\mathcal{B}$ such that $\hat{\imath}_{\mathrm{B}}$ is parallel with the arm. As the shaft rotates, the arm rotates at the pin joint, and the arm extends and retracts, the frame F_{B} coincides with the spherical frame.

The axes of the spherical frame $\mathrm{F}_{\mathrm{sph}}$ at x are *up*, *east*, and *north*, denoted by $\hat{e}_{\mathrm{u}}$, $\hat{e}_{\mathrm{e}}$, and $\hat{e}_{\mathrm{n}}$, respectively, where

$$\hat{e}_{\mathrm{u}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\imath}_{\mathrm{A}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\hat{e}_{\mathrm{r}}, \tag{5.3.7}$$

$$\hat{e}_{\mathrm{e}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{\jmath}_{\mathrm{A}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\hat{e}_{\mathrm{t}} = \hat{e}_{\mathrm{t}}, \tag{5.3.8}$$

$$\hat{e}_{\mathrm{n}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\vec{R}_{\hat{k}_{\mathrm{A}}}(\theta)\hat{k}_{\mathrm{A}} = \vec{R}_{\hat{e}_{\mathrm{t}}}(\phi)\hat{e}_{\mathrm{a}}. \tag{5.3.9}$$

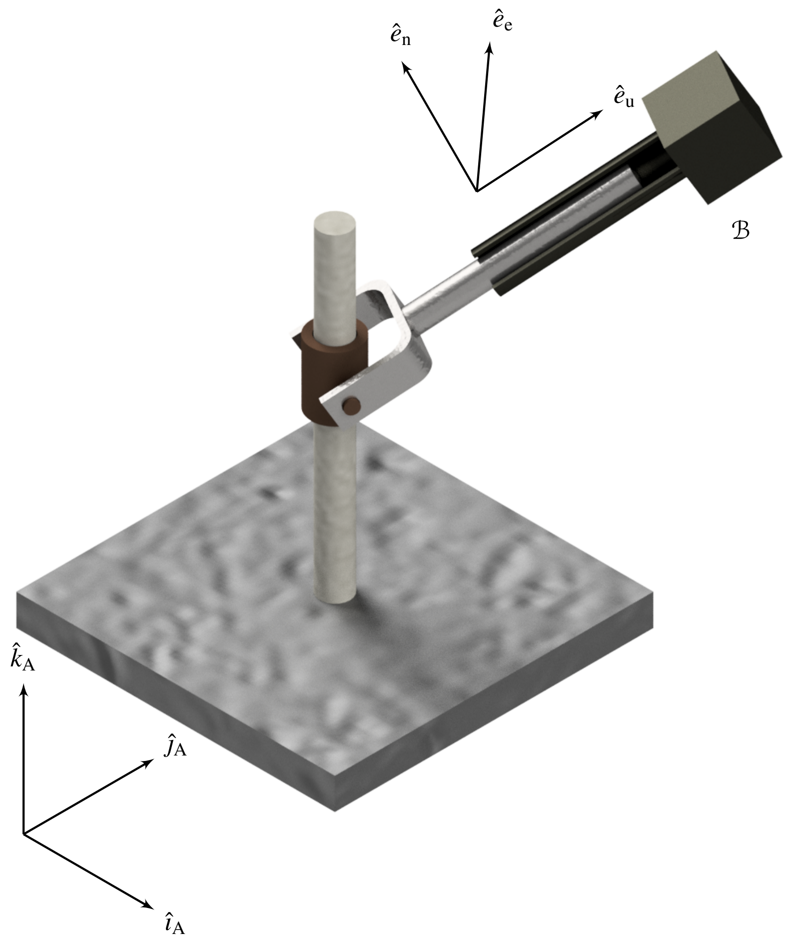


Figure 5.3.3: Spherical frame given as a body-fixed frame. The frame $\mathrm{F_A}$ is attached to the base of the shaft, and the spherical frame $\mathrm{F_{sph}} = [\hat{e}_{\rm u}\ \ \hat{e}_{\rm e}\ \ \hat{e}_{\rm n}]$, which is obtained by rotating $\mathrm{F_{cyl}}$ about $\hat{e}_{\rm t}$, is attached to the body $\mathcal{B}$.

Hence,

$$\mathrm{F_{sph}} = [\hat{e}_{\rm u}\ \ \hat{e}_{\rm e}\ \ \hat{e}_{\rm n}]. \tag{5.3.10}$$

Furthermore, it follows from (5.3.6) and (5.3.7) that

$$\vec{r}_{x/\mathrm{o_A}} = |\vec{r}_{x/\mathrm{o_A}}|\hat{e}_{\rm u}. \tag{5.3.11}$$

Next, it follows from (5.3.5) that

$$\begin{bmatrix} \hat{e}_{\rm u} \\ \hat{e}_{\rm e} \\ \hat{e}_{\rm n} \end{bmatrix} = \mathcal{O}_{\mathrm{sph/A}} \begin{bmatrix} \hat{\imath}_{\rm A} \\ \hat{\jmath}_{\rm A} \\ \hat{k}_{\rm A} \end{bmatrix}, \tag{5.3.12}$$

where

$$\mathcal{O}_{\mathrm{sph/A}} = \left(\vec{R}_{\hat{e}_{\rm t}}(\phi)\vec{R}_{\hat{k}_{\rm A}}(\theta)\right)\Big|_{\rm A}^{\mathsf{T}}$$

$$= \vec{R}_{\hat{k}_{\rm A}}(\theta)\Big|_{\rm A}^{\sf T} \vec{R}_{\hat{e}_{\rm t}}(\phi)\Big|_{\rm A}^{\sf T}$$

$$= \mathcal{O}_{\rm cyl/A}\mathcal{O}_{\rm A/cyl}\, \vec{R}_{\hat{e}_{\rm t}}(\phi)\Big|_{\rm cyl}^{\sf T} \mathcal{O}_{\rm cyl/A}$$

$$= \mathcal{O}_2(\phi)\mathcal{O}_3(\theta)$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\phi)\cos\theta & (\cos\phi)\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ (\sin\phi)\cos\theta & (\sin\phi)\sin\theta & \cos\phi \end{bmatrix}. \tag{5.3.13}$$

Consequently,

$$\hat{e}_{\rm u} = (\cos\phi)\hat{e}_{\rm r} - (\sin\phi)\hat{e}_{\rm a}, \tag{5.3.14}$$

$$\hat{e}_{\rm e} = \hat{e}_{\rm t}, \tag{5.3.15}$$

$$\hat{e}_{\rm n} = (\sin\phi)\hat{e}_{\rm r} + (\cos\phi)\hat{e}_{\rm a} \tag{5.3.16}$$

and

$$\hat{e}_{\rm u} = (\cos\phi)(\cos\theta)\hat{\imath}_{\rm A} + (\cos\phi)(\sin\theta)\hat{\jmath}_{\rm A} - (\sin\phi)\hat{k}_{\rm A}, \tag{5.3.17}$$

$$\hat{e}_{\rm e} = -(\sin\theta)\hat{\imath}_{\rm A} + (\cos\theta)\hat{\jmath}_{\rm A}, \tag{5.3.18}$$

$$\hat{e}_{\rm n} = (\sin\phi)(\cos\theta)\hat{\imath}_{\rm A} + (\sin\phi)(\sin\theta)\hat{\jmath}_{\rm A} + (\cos\phi)\hat{k}_{\rm A}. \tag{5.3.19}$$

Likewise,

$$\mathcal{O}_{\rm A/sph} = \begin{bmatrix} (\cos\phi)\cos\theta & -\sin\theta & (\sin\phi)\cos\theta \\ (\cos\phi)\sin\theta & \cos\theta & (\sin\phi)\sin\theta \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \tag{5.3.20}$$

and thus

$$\hat{\imath}_{\rm A} = (\cos\phi)(\cos\theta)\hat{e}_{\rm u} - (\sin\theta)\hat{e}_{\rm e} + (\sin\phi)(\cos\theta)\hat{e}_{\rm n}, \tag{5.3.21}$$

$$\hat{\jmath}_{\rm A} = (\cos\phi)(\sin\theta)\hat{e}_{\rm u} + (\cos\theta)\hat{e}_{\rm e} + (\sin\phi)(\sin\theta)\hat{e}_{\rm n}, \tag{5.3.22}$$

$$\hat{k}_{\rm A} = -(\sin\phi)\hat{e}_{\rm u} + (\cos\phi)\hat{e}_{\rm n}. \tag{5.3.23}$$

Next, writing

$$r_{x/{\rm o_A}|{\rm A}} = \vec{r}_{x/{\rm o_A}}\Big|_{\rm A} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \tag{5.3.24}$$

$$r_{x/{\rm o_A}|{\rm sph}}\ \vec{r}_{x/{\rm o_A}}\Big|_{\rm sph} = \begin{bmatrix} r_{\rm u} \\ r_{\rm e} \\ r_{\rm n} \end{bmatrix}, \tag{5.3.25}$$

it follows from (5.3.13) that

$$\begin{bmatrix} r_{\rm u} \\ r_{\rm e} \\ r_{\rm n} \end{bmatrix} = \mathcal{O}_{\rm sph/A} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} (\cos\phi)(\cos\theta)r_1 + (\cos\phi)(\sin\theta)r_2 - (\sin\phi)r_3 \\ -(\sin\theta)r_1 + (\cos\theta)r_2 \\ (\sin\phi)(\cos\theta)r_1 + (\sin\phi)(\sin\theta)r_2 + (\cos\phi)r_3 \end{bmatrix}. \tag{5.3.26}$$

Next, it follows from (5.3.11) that

$$r_{\rm u}\hat{e}_{\rm u} + r_{\rm e}\hat{e}_{\rm e} + r_{\rm n}\hat{e}_{\rm n} = |\vec{r}_{x/{\rm o_A}}|\hat{e}_{\rm u}, \tag{5.3.27}$$

and thus

$$r_{\rm e} = 0, \tag{5.3.28}$$

$$r_{\rm n} = 0, \tag{5.3.29}$$

$$\vec{r}_{x/{\rm o_A}} = r_1\hat{\imath}_{\rm A} + r_2\hat{\jmath}_{\rm A} + r_3\hat{k}_{\rm A} = r_{\rm u}\hat{e}_{\rm u}, \tag{5.3.30}$$

$$r_{\rm u} = |\vec{r}_{x/{\rm o_A}}| = \sqrt{r_1^2 + r_2^2 + r_3^2}. \tag{5.3.31}$$

Now, (5.3.19) and (5.3.30) imply that

$$r_1 = (\hat{e}_{\rm u} \cdot \hat{\imath}_{\rm A})r_{\rm u} = (\cos\phi)(\cos\theta)r_{\rm u}, \tag{5.3.32}$$

$$r_2 = (\hat{e}_{\rm u} \cdot \hat{\jmath}_{\rm A})r_{\rm u} = (\cos\phi)(\sin\theta)r_{\rm u}, \tag{5.3.33}$$

$$r_3 = (\hat{e}_{\rm u} \cdot \hat{k}_{\rm A})r_{\rm u} = -(\sin\phi)r_{\rm u}, \tag{5.3.34}$$

where the minus sign in (5.3.34) indicates that points for which ϕ is positive are located in the southern hemisphere. Note that $r_{\rm u} = 0$ if and only if $r_1 = r_2 = r_3 = 0$.

Since $r_{\rm e} = 0$, it follows from the second equation in (5.3.26) that

$$(\sin\theta)r_1 - (\cos\theta)r_2 = 0, \tag{5.3.35}$$

and thus, as in the case of the cylindrical frame,

$$\theta = \operatorname{atan2}(r_2, r_1). \tag{5.3.36}$$

Next, since $r_{\rm n} = 0$ it follows from the third equation in (5.3.26) that

$$(\sin\phi)(\cos\theta)r_1 + (\sin\phi)(\sin\theta)r_2 + (\cos\phi)r_3 = 0. \tag{5.3.37}$$

Recall that $\phi \in [-\pi/2, \pi/2]$. Now, assume that $r_1 = r_2 = 0$. Then, assuming that x is not located at $\rm o_A$, it follows from (5.3.37) that $\cos\phi = 0$, and thus either $\phi = \pi/2$ or $\phi = -\pi/2$. Conversely, assume that either $\phi = \pi/2$ or $\phi = -\pi/2$. Then, (5.3.37) implies that

$$(\cos\theta)r_1 + (\sin\theta)r_2 = 0. \tag{5.3.38}$$

It now follows from (5.3.35) and (5.3.38) that $r_1 = r_2 = 0$. Consequently, either $\phi = \pi/2$ or $\phi = -\pi/2$ if and only if $r_1 = r_2 = 0$.

Now, assume that r_1 and r_2 are not both zero. Then, $\cos\phi \neq 0$, and thus we can write

$$\tan\phi = -\frac{r_3}{(\cos\theta)r_1 + (\sin\theta)r_2}. \tag{5.3.39}$$

Hence, using (5.3.35) it follows that

$$\tan\phi = -\frac{r_3}{\sqrt{r_1^2 + r_2^2}} \in (-\pi/2, \pi/2), \tag{5.3.40}$$

and thus

$$\phi = -\mathrm{atan}\,\frac{r_3}{\sqrt{r_1^2 + r_2^2}} \tag{5.3.41}$$

Note that $r_3 > 0$ if and only if $\phi < 0$. Finally, including the case $r_1 = r_2 = 0$, we have

$$\phi = \begin{cases} -\pi/2, & r_1 = r_2 = 0, r_3 > 0, \\ \pi/2, & r_1 = r_2 = 0, r_3 < 0, \\ -\mathrm{atan}\,\frac{r_3}{\sqrt{r_1^2+r_2^2}}, & r_1^2 + r_2^2 > 0, \end{cases} \tag{5.3.42}$$

and thus, equivalently,

$$\phi = -\,\mathrm{atan2}\Big(r_3,\ \sqrt{r_1^2 + r_2^2}\Big). \tag{5.3.43}$$

The *radial*, *azimuthal*, and *polar spherical coordinates* $(r_\mathrm{u}, \theta, \phi)$ at x are thus given by

$$r_\mathrm{u} = \sqrt{r_1^2 + r_2^2 + r_3^2}, \tag{5.3.44}$$

$$\theta = \mathrm{atan2}(r_2, r_1), \tag{5.3.45}$$

$$\phi = -\,\mathrm{atan2}\Big(r_3,\ \sqrt{r_1^2 + r_2^2}\Big). \tag{5.3.46}$$

If, in addition, $r_\mathrm{u} > 0$, then

$$\phi = -\,\mathrm{asin}\,\frac{r_3}{r_\mathrm{u}}. \tag{5.3.47}$$

The elevation angle can be replaced by the *colatitude* $\phi_\mathrm{c} \in [0, \pi]$, which is the directed angle from the axial axis $\hat{e}_\mathrm{a}$ (defined by (5.1.16)) of the cylindrical frame to $\vec{r}_{x/\mathrm{o_A}}$ around $\hat{e}_\mathrm{t}$, that is,

$$\lambda_\mathrm{c} \triangleq \phi + \tfrac{\pi}{2}. \tag{5.3.48}$$

Hence,

$$\lambda_\mathrm{c} = \tfrac{\pi}{2} - \lambda. \tag{5.3.49}$$

Finally, an alternative spherical frame at x is $\mathrm{F_{NED}}$, whose axes are *north*, *east*, and *down*, denoted by $\hat{e}_\mathrm{n}$, $\hat{e}_\mathrm{e}$, and $\hat{e}_\mathrm{d}$, respectively, where

$$\mathrm{F_{sph}} \xrightarrow[2]{-\pi/2} \mathrm{F_{NED}}. \tag{5.3.50}$$

Therefore,

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_{cyl}} \xrightarrow[2]{\phi-\pi/2} \mathrm{F_{NED}}. \tag{5.3.51}$$

5.4 Kinematics in the Spherical Frame

It follows from (5.3.2) that

$$\vec{\omega}_{\mathrm{sph/A}} = \dot{\theta}\hat{e}_{\mathrm{a}} + \dot{\phi}\hat{e}_{\mathrm{e}}, \tag{5.4.1}$$

and thus

$$\begin{aligned}
\omega_{\mathrm{sph/A|sph}} &= \vec{\omega}_{\mathrm{sph/A}}\Big|_{\mathrm{sph}} \\
&= \dot{\theta}\ \hat{e}_{\mathrm{a}}|_{\mathrm{sph}} + \dot{\phi}\ \hat{e}_{\mathrm{e}}|_{\mathrm{sph}} \\
&= \dot{\theta}\mathcal{O}_{\mathrm{sph/cyl}}\ \hat{e}_{\mathrm{a}}|_{\mathrm{cyl}} + \dot{\phi}\ \hat{e}_{\mathrm{e}}|_{\mathrm{sph}} \\
&= \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\dot{\theta}\sin\phi \\ \dot{\phi} \\ \dot{\theta}\cos\phi \end{bmatrix}.
\end{aligned} \tag{5.4.2}$$

Alternatively, using (4.3.20) it follows that

$$\begin{aligned}
\omega^{\times}_{\mathrm{sph/A|sph}} &= \mathcal{O}_{\mathrm{sph/A}}\dot{\mathcal{O}}_{\mathrm{A/sph}} \\
&= \begin{bmatrix} (\cos\phi)\cos\theta & (\cos\phi)\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ (\sin\phi)\cos\theta & (\sin\phi)\sin\theta & \cos\phi \end{bmatrix} \\
&\quad \times \begin{bmatrix} -\dot{\phi}(\sin\phi)\cos\theta - \dot{\theta}(\cos\phi)\sin\theta & -\dot{\theta}\cos\theta & \dot{\phi}(\cos\phi)\cos\theta - \dot{\theta}(\sin\phi)\sin\theta \\ -\dot{\phi}(\sin\phi)\sin\theta + \dot{\theta}(\cos\phi)\cos\theta & -\dot{\theta}\sin\theta & \dot{\phi}(\cos\phi)\sin\theta + \dot{\theta}(\sin\phi)\cos\theta \\ -\dot{\phi}\cos\phi & 0 & -\dot{\phi}\sin\phi \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\dot{\theta}\cos\phi & \dot{\phi} \\ \dot{\theta}\cos\phi & 0 & \dot{\theta}\sin\phi \\ -\dot{\phi} & -\dot{\theta}\sin\phi & 0 \end{bmatrix}.
\end{aligned} \tag{5.4.3}$$

Next, it follows from (4.3.21) that

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{u}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{e}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{n}} \end{bmatrix} = -\omega^{\times}_{\mathrm{sph/A|sph}} \begin{bmatrix} \hat{e}_{\mathrm{u}} \\ \hat{e}_{\mathrm{e}} \\ \hat{e}_{\mathrm{n}} \end{bmatrix}. \tag{5.4.4}$$

Therefore,

$$\overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{u}} = \dot{\theta}(\cos\phi)\hat{e}_{\mathrm{e}} - \dot{\phi}\hat{e}_{\mathrm{n}} = \vec{\omega}_{\mathrm{sph/A}} \times \hat{e}_{\mathrm{u}}, \tag{5.4.5}$$

$$\overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{e}} = -\dot{\theta}(\cos\phi)\hat{e}_{\mathrm{u}} - \dot{\theta}(\sin\phi)\hat{e}_{\mathrm{n}} = \vec{\omega}_{\mathrm{sph/A}} \times \hat{e}_{\mathrm{e}}, \tag{5.4.6}$$

$$\overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{n}} = \dot{\phi}\hat{e}_{\mathrm{u}} + \dot{\theta}(\sin\phi)\hat{e}_{\mathrm{e}} = \vec{\omega}_{\mathrm{sph/A}} \times \hat{e}_{\mathrm{n}}. \tag{5.4.7}$$

Now, let $\mathrm{F_A}$ be a frame with origin $\mathrm{o_A}$, and let x be a point. Then, $\vec{r}_{x/\mathrm{o_A}}$ can be expressed in the spherical frame as

$$\vec{r}_{x/o_A} = r_u \hat{e}_u. \tag{5.4.8}$$

Therefore,

$$\begin{aligned}
\vec{v}_{x/o_A/A} &= \overset{A\bullet}{\vec{r}}_{x/o_A} \\
&= \dot{r}_u \hat{e}_u + r_u \overset{A\bullet}{\hat{e}}_u \\
&= \dot{r}_u \hat{e}_u + r_u \dot{\theta}(\cos\phi)\hat{e}_e - r_u \dot{\phi}\hat{e}_n \\
&= \overset{sph\bullet}{\vec{r}}_{x/o_A} + \vec{\omega}_{sph/A} \times \vec{r}_{x/o_A},
\end{aligned} \tag{5.4.9}$$

which is the transport theorem for the spherical frame. Furthermore,

$$\begin{aligned}
\vec{a}_{x/o_A/A} &= \overset{A\bullet}{\vec{v}}_{x/o_A/A} \\
&= \ddot{r}_u \hat{e}_u + \dot{r}_u \overset{A\bullet}{\hat{e}}_u + [\dot{r}_u \dot{\theta}(\cos\phi) + r_u \ddot{\theta}(\cos\phi) - r_u \dot{\theta}\dot{\phi}\sin\phi]\hat{e}_e + r_u \dot{\theta}(\cos\phi) \overset{A\bullet}{\hat{e}}_e \\
&\quad - (\dot{r}_u \dot{\phi} + r_u \ddot{\phi})\hat{e}_n - r_u \dot{\phi} \overset{A\bullet}{\hat{e}}_n \\
&= \ddot{r}_u \hat{e}_u + \dot{r}_u \dot{\theta}(\cos\phi)\hat{e}_e - \dot{r}_u \dot{\phi}\hat{e}_n + [\dot{r}_u \dot{\theta}(\cos\phi) + r_u \ddot{\theta}(\cos\phi) - r_u \dot{\theta}\dot{\phi}\sin\phi]\hat{e}_e \\
&\quad - r_u \dot{\theta}(\cos\phi)[\dot{\theta}(\cos\phi)\hat{e}_u + \dot{\theta}(\sin\phi)\hat{e}_n] - (\dot{r}_u \dot{\phi} + r_u \ddot{\phi})\hat{e}_n - r_u \dot{\phi}[\dot{\phi}\hat{e}_u + \dot{\theta}(\sin\phi)\hat{e}_e] \\
&= [\ddot{r}_u - r_u \dot{\theta}^2(\cos^2\phi) - r_u \dot{\phi}^2]\hat{e}_u + [2\dot{r}_u \dot{\theta}(\cos\phi) + r_u \ddot{\theta}(\cos\phi) - 2r_u \dot{\phi}\dot{\theta}\sin\phi]\hat{e}_e \\
&\quad + [r_u \dot{\theta}^2(\cos\phi)(\sin\phi) - 2\dot{r}_u \dot{\phi} - r_u \ddot{\phi}]\hat{e}_n \\
&= \ddot{r}_u \hat{e}_u + \underbrace{2\dot{r}_u[\dot{\theta}(\cos\phi)\hat{e}_e - \dot{\phi}\hat{e}_n]}_{\text{Coriolis acceleration}} + \underbrace{r_u([\ddot{\theta}(\cos\phi) - \dot{\phi}\dot{\theta}\sin\phi]\hat{e}_e - \ddot{\phi}\hat{e}_n)}_{\text{A}^2\text{ acceleration}} \\
&\quad + \underbrace{-r_u[(\dot{\phi}^2 + \dot{\theta}^2\cos^2\phi)\hat{e}_u + \dot{\phi}\dot{\theta}(\sin\phi)\hat{e}_e + \dot{\theta}^2(\cos\phi)(\sin\phi)\hat{e}_n]}_{\text{centripetal acceleration}} \\
&= \overset{sph\bullet\bullet}{\vec{r}}_{x/o_A} + 2\vec{\omega}_{sph/A} \times \overset{sph\bullet}{\vec{r}}_{x/o_A} + \overset{sph\bullet}{\vec{\omega}}_{sph/A} \times \vec{r}_{x/o_A} + \vec{\omega}_{sph/A} \times (\vec{\omega}_{sph/A} \times \vec{r}_{x/o_A}),
\end{aligned} \tag{5.4.10}$$

which is the double transport theorem for the spherical frame. Note that the Coriolis acceleration is zero if r_u is constant.

Note that the Coriolis acceleration is given by

$$\vec{a}_{Cor} = 2\vec{\omega}_{sph/A} \times \overset{sph\bullet}{\vec{r}}_{x/o_A} = 2\dot{r}_u[\dot{\theta}(\cos\phi)\hat{e}_e - \dot{\phi}\hat{e}_n]. \tag{5.4.11}$$

5.5 Frenet-Serret Frame†

Let x be a point whose position relative to a point w is parameterized by the real number α. The position of x relative to w can thus be written as $\vec{r}_{x/w}(\alpha)$. The Frenet-Serret frame is a frame that depends on the location of x and thus on α. The set of all locations of x over all possible values of α constitutes the curve $\mathcal{C}$. The parameter α may represent time, in which case the curve can be viewed as growing in time. Alternatively, α may represent arc length, in which case the location of x can be found by specifying the distance along the curve from w. Finally, the parameter α can itself be parameterized by time by writing $\alpha = \alpha(t)$.

This notation is incomplete, however, since it does not indicate which parameterization is chosen when α is set to a numerical value. For example, $\vec{r}_{x/w}(3)$ is ambiguous. To remove this ambiguity when confusion can arise, we may write $\vec{r}_{\alpha,x/w}(\beta)$, where the additional subscript α identifies the parameter that parameterizes the curve and β denotes a value of α. When an additional subscript is not used, the choice of parameterization is assumed to be inferred by the argument α in $\vec{r}_{x/w}(\alpha)$.

Resolving $\vec{r}_{x/w}(\alpha)$ in $\mathrm{F_A}$, we have

$$\vec{r}_{x/w}(\alpha) = r_1(\alpha)\hat{\imath}_{\mathrm{A}} + r_2(\alpha)\hat{\jmath}_{\mathrm{A}} + r_3(\alpha)\hat{k}_{\mathrm{A}}, \tag{5.5.1}$$

where

$$r_1(\alpha) \triangleq \vec{r}_{x/w}(\alpha) \cdot \hat{\imath}_{\mathrm{A}}, \tag{5.5.2}$$

$$r_2(\alpha) \triangleq \vec{r}_{x/w}(\alpha) \cdot \hat{\jmath}_{\mathrm{A}}, \tag{5.5.3}$$

$$r_3(\alpha) \triangleq \vec{r}_{x/w}(\alpha) \cdot \hat{k}_{\mathrm{A}}. \tag{5.5.4}$$

The *path derivative* $\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)$ is defined by

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \triangleq r_1'(\alpha)\hat{\imath}_{\mathrm{A}} + r_2'(\alpha)\hat{\jmath}_{\mathrm{A}} + r_3'(\alpha)\hat{k}_{\mathrm{A}}. \tag{5.5.5}$$

Hence,

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\Big|_{\mathrm{A}} = \begin{bmatrix} r_1'(\alpha) \\ r_2'(\alpha) \\ r_3'(\alpha) \end{bmatrix}. \tag{5.5.6}$$

Note that, if α denotes time t, then $\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)$ is the usual frame derivative $\overset{\mathrm{A}\bullet}{\vec{r}}_{x/w}$. Consequently, the path derivative is a generalization of the frame derivative.

Next, let $s(\alpha)$ denote the length of the path from $\vec{r}_{x/w}(0)$ to $\vec{r}_{x/w}(\alpha)$. Then,

$$s(\alpha) = \int_0^\alpha |\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\sigma)|\,\mathrm{d}\sigma, \tag{5.5.7}$$

which states that path length is the integral of the parametric speed along the path. Therefore,

$$s'(\alpha) = |\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)| = \sqrt{r_1'^2(\alpha) + r_2'^2(\alpha) + r_3'^2(\alpha)} \tag{5.5.8}$$

and, assuming that $s'(\alpha)$ is nonzero,

$$\begin{aligned} s''(\alpha) &= \frac{r_1'(\alpha)r_1''(\alpha) + r_2'(\alpha)r_2''(\alpha) + r_3'(\alpha)r_3''(\alpha)}{s'(\alpha)} \\ &= \frac{\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \cdot \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)}{s'(\alpha)}. \end{aligned} \tag{5.5.9}$$

It follows from (5.5.8) that $s'(\alpha)$ is nonzero if and only if $\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)$ is nonzero.

Next, assume that α is parameterized by s, that is, $\alpha = \alpha(s)$. Then, $\vec{r}_{s,x/w}(s) = \vec{r}_{\alpha,x/w}(\alpha(s))$, where the additional subscripts denote different parameterizations. Then, it follows from the chain rule that the *path-length derivative* $\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(\alpha)$ is given by

$$
\begin{aligned}
\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s) &= \overset{\mathrm{A}s\bullet}{\vec{r}}_{s,x/w}(s) \\
&= \overset{\mathrm{A}s\bullet}{\vec{r}}_{\alpha,x/w}(\alpha(s)) \\
&= \alpha'(s)\,\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha).
\end{aligned} \tag{5.5.10}
$$

Since $\alpha'(s)s'(\alpha) = 1$, we have

$$
\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(\alpha) = \frac{1}{s'(\alpha)}\,\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha). \tag{5.5.11}
$$

As a special case, assume that $\alpha(s) = s$. Then, the length of the path from $\vec{r}_{x/w}(0)$ to $\vec{r}_{x/w}(\alpha)$ is s, and thus $s(\alpha) = \alpha$. Therefore, $s'(\alpha) = 1$ and $s''(\alpha) = 0$, and thus it follows from (5.5.5) with $\alpha = s$ and (5.5.8) that

$$
|\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s)| = 1. \tag{5.5.12}
$$

Hence, $\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s)$ is a unit vector. Therefore,

$$
\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s) \cdot \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s) = 0. \tag{5.5.13}
$$

The axes of the Frenet-Serret frame F_{FS} at x are *tangential*, *normal*, and *binormal*, denoted by $(\hat{e}_{\mathrm{t}}, \hat{e}_{\mathrm{n}}, \hat{e}_{\mathrm{b}})$, respectively. The tangential axis $\hat{e}_{\mathrm{t}}$ is the unit tangent vector (see (5.5.12)) to the curve at the location of the point, that is,

$$
\hat{e}_{\mathrm{t}} \triangleq \overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s). \tag{5.5.14}
$$

Using (5.5.10) and (5.5.11) it follows that

$$
\hat{e}_{\mathrm{t}} = \alpha'(s)\,\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) = \frac{1}{s'(\alpha)}\,\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha). \tag{5.5.15}
$$

Furthermore,

$$
\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} = \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s), \tag{5.5.16}
$$

$$
\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = \frac{-s''(\alpha)}{s'^2(\alpha)}\,\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) + \frac{1}{s'(\alpha)}\,\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha). \tag{5.5.17}
$$

The normal axis $\hat{e}_{\mathrm{n}}$ is defined to be the unit vector in the direction of $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}$, that is,

$$\hat{e}_{\mathrm{n}} \triangleq \frac{\rho(\alpha)}{s'(\alpha)} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}, \tag{5.5.18}$$

where

$$\rho(\alpha) \triangleq \frac{|s'(\alpha)|}{|\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}|} > 0 \tag{5.5.19}$$

is the *radius of curvature*. However, $\hat{e}_{\mathrm{n}}$ is not defined when $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = 0$, that is, when $\hat{e}_{\mathrm{t}}$ is not changing, for example, if the curve is a straight line or at an inflection point. Furthermore, using (5.5.11) yields

$$\hat{e}_{\mathrm{n}} = \frac{\rho(\alpha)}{s'^3(\alpha)} \left(s'(\alpha) \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - s''(\alpha) \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \right), \tag{5.5.20}$$

which, using (5.5.9), can be written as

$$\hat{e}_{\mathrm{n}} = \frac{\rho(\alpha)}{s'^4(\alpha)} \left[s'^2(\alpha) \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - \left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \cdot \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) \right) \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \right]. \tag{5.5.21}$$

Note that it follows from (5.5.10) that

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) = s'(\alpha)\hat{e}_{\mathrm{t}} \tag{5.5.22}$$

and thus using (5.5.18) we have

$$\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) = s''(\alpha)\hat{e}_{\mathrm{t}} + \frac{s'^2(\alpha)}{\rho(\alpha)}\hat{e}_{\mathrm{n}}. \tag{5.5.23}$$

In the special case $\alpha = s$, (5.5.20) becomes

$$\hat{e}_{\mathrm{n}} = \rho(s) \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}}, \tag{5.5.24}$$

where

$$\rho(s) \triangleq \frac{1}{|\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}}|} > 0. \tag{5.5.25}$$

Hence, it follows from (5.5.24) and (5.5.13) that

$$\hat{e}_{\mathrm{n}} = \rho(s) \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s). \tag{5.5.26}$$

The vectors $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{n}}$ are orthogonal since $\hat{e}_{\mathrm{t}} \cdot \hat{e}_{\mathrm{t}} = 1$, and thus $\hat{e}_{\mathrm{t}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = 0$. For a circular path, $\hat{e}_{\mathrm{n}}$ points toward the center of the circle. The plane spanned by $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{n}}$ is called the *osculating plane*.

To complete the Frenet-Serret frame, the binormal axis is defined by

$$\hat{e}_{\mathrm{b}} \triangleq \hat{e}_{\mathrm{t}} \times \hat{e}_{\mathrm{n}}. \tag{5.5.27}$$

Hence, using (5.5.14), (5.5.26), (5.5.10), and (5.5.21), it follows that

$$\begin{aligned}\hat{e}_{\mathrm{b}} &= \rho(s)\left(\overset{\mathrm{A}s\bullet}{\vec{r}}_{x/w}(s) \times \overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s)\right)\\ &= \frac{\rho(\alpha)}{s'^3(\alpha)}\left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right).\end{aligned} \tag{5.5.28}$$

The plane spanned by $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{b}}$ is called the *rectifying plane*, while the plane spanned by $\hat{e}_{\mathrm{n}}$ and $\hat{e}_{\mathrm{b}}$ is called the *normal plane*. The *Frenet-Serret frame* is thus given by

$$\mathrm{F}_{\mathrm{FS}} = [\hat{e}_{\mathrm{t}}\ \hat{e}_{\mathrm{n}}\ \hat{e}_{\mathrm{b}}] = \vec{R}_{\mathrm{FS/A}}\mathrm{F}_{\mathrm{A}}, \tag{5.5.29}$$

so that

$$\begin{bmatrix}\hat{e}_{\mathrm{t}}\\ \hat{e}_{\mathrm{n}}\\ \hat{e}_{\mathrm{b}}\end{bmatrix} = \mathcal{O}_{\mathrm{FS/A}}\begin{bmatrix}\hat{\imath}_{\mathrm{A}}\\ \hat{\jmath}_{\mathrm{A}}\\ \hat{k}_{\mathrm{A}}\end{bmatrix}. \tag{5.5.30}$$

Next, we find the path-length derivatives of the unit vectors $(\hat{e}_{\mathrm{t}}, \hat{e}_{\mathrm{n}}, \hat{e}_{\mathrm{b}})$. It follows from (5.5.18) that

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} = \kappa(s)\hat{e}_{\mathrm{n}}, \tag{5.5.31}$$

and thus

$$\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = \kappa(\alpha)s'(\alpha)\hat{e}_{\mathrm{n}}, \tag{5.5.32}$$

where $\kappa(\alpha) \triangleq 1/\rho(\alpha)$ is the *curvature*. It follows from (5.5.28) that

$$\kappa(\alpha) = \frac{|\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)|}{|\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)|^3}. \tag{5.5.33}$$

Thus, if $\alpha = s$, then

$$\kappa(s) = |\overset{\mathrm{A}s\bullet\bullet}{\vec{r}}_{x/w}(s)|. \tag{5.5.34}$$

If $\kappa(\alpha) = 0$, then $\rho(\alpha) = \infty$. Taking the magnitude of (5.5.32) and using (5.5.11) yields

$$\begin{aligned}\kappa(\alpha) &= \frac{|\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}}|}{s'(\alpha)}\\ &= \frac{1}{s'^3(\alpha)}\left|s'(\alpha)\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - s''(\alpha)\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\right|\\ &= \frac{1}{s'^3(\alpha)}\sqrt{s'^2(\alpha)|\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)|^2 - (\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha)\cdot\overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha))^2}.\end{aligned} \tag{5.5.35}$$

Note that, since $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} = \kappa(\alpha)s'(\alpha)\hat{e}_{\mathrm{n}}$ and $\hat{e}_{\mathrm{b}}$ are perpendicular, it follows that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}s}(\hat{e}_{\mathrm{t}} \cdot \hat{e}_{\mathrm{b}}) \\ &= \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{t}} \cdot \hat{e}_{\mathrm{b}} + \hat{e}_{\mathrm{t}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} \\ &= \hat{e}_{\mathrm{t}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}}. \end{aligned} \tag{5.5.36}$$

In addition, since $\hat{e}_{\mathrm{b}} \cdot \hat{e}_{\mathrm{b}} = 1$, it follows that $\hat{e}_{\mathrm{b}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} = 0$. Consequently, $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}}$ is orthogonal to both $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{b}}$. Since $\hat{e}_{\mathrm{n}}$ is also orthogonal to both $\hat{e}_{\mathrm{t}}$ and $\hat{e}_{\mathrm{b}}$, it follows that $\overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}}$ and $\hat{e}_{\mathrm{n}}$ are parallel. Consequently, $\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} = \frac{1}{s'(\alpha)} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}}$ and $\hat{e}_{\mathrm{n}}$ are parallel. We thus define the *torsion* $\tau(\alpha)$ such that

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} = -\tau(\alpha)\hat{e}_{\mathrm{n}}. \tag{5.5.37}$$

Therefore,

$$\begin{aligned} \tau(\alpha) &= -\hat{e}_{\mathrm{n}} \cdot \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} \\ &= -\frac{1}{s'(\alpha)}\hat{e}_{\mathrm{n}} \cdot \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\mathrm{b}} \\ &= -\frac{\rho(\alpha)}{s'^{10}(\alpha)}\left(s'(\alpha) \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) - s''(\alpha) \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}\right) \\ &\quad \cdot \left[[s'^3(\alpha)\rho'(\alpha) - 3s'^2(\alpha)s''(\alpha)\rho(\alpha)]\left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right)\right. \\ &\quad \left. + s'^3(\alpha)\rho(\alpha)\left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right)\right] \\ &= -\frac{\rho^2(\alpha)}{s'^6(\alpha)} \overset{\mathrm{A}\alpha\bullet\bullet}{\vec{r}}_{x/w}(\alpha) \cdot \left(\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \times \overset{\mathrm{A}\alpha\bullet\bullet\bullet}{\vec{r}}_{x/w}(\alpha)\right). \end{aligned} \tag{5.5.38}$$

Next, since $\hat{e}_{\mathrm{n}} \cdot \hat{e}_{\mathrm{n}} = 1$, it follows that $\hat{e}_{\mathrm{n}} \cdot \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} = 0$. Therefore, there exist real numbers γ and δ such that

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} = \gamma\hat{e}_{\mathrm{t}} + \delta\hat{e}_{\mathrm{b}}. \tag{5.5.39}$$

Using (5.5.37) and taking the path-length derivative of (5.5.27) yields

$$\begin{aligned} -\tau(\alpha)\hat{e}_{\mathrm{n}} &= \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{b}} \\ &= \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{t}} \times \hat{e}_{\mathrm{n}} + \hat{e}_{\mathrm{t}} \times \overset{\mathrm{A}s\bullet}{\hat{e}}_{\mathrm{n}} \\ &= \kappa(\alpha)\hat{e}_{\mathrm{n}} \times \hat{e}_{\mathrm{n}} + \hat{e}_{\mathrm{t}} \times (\gamma\hat{e}_{\mathrm{t}} + \delta\hat{e}_{\mathrm{b}}) \\ &= -\delta\hat{e}_{\mathrm{n}}. \end{aligned} \tag{5.5.40}$$

Therefore, $\delta = \tau(\alpha)$. Next, since $\hat{e}_{\rm n} \cdot \hat{e}_{\rm t} = 0$, it follows that

$$\begin{aligned} \gamma = \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm n} \cdot \hat{e}_{\rm t} &= -\hat{e}_{\rm n} \cdot \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm t} \\ &= -\hat{e}_{\rm n} \cdot [\kappa(\alpha)\hat{e}_{\rm n}] \\ &= -\kappa(\alpha). \end{aligned} \tag{5.5.41}$$

Therefore,

$$\overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm n} = -\kappa(\alpha)\hat{e}_{\rm t} + \tau(\alpha)\hat{e}_{\rm b}. \tag{5.5.42}$$

In summary, the *Frenet-Serret relations* (5.5.31), (5.5.42), and (5.5.37) are given by the vectrix equation

$$\begin{bmatrix} \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}, \tag{5.5.43}$$

which can be written as

$$\begin{bmatrix} \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = -\begin{bmatrix} \tau(s) \\ 0 \\ \kappa(s) \end{bmatrix}^{\times} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}. \tag{5.5.44}$$

In terms of an arbitrary parameterization involving the parameter α, the Frenet-Serret relations are given by

$$\begin{bmatrix} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = \begin{bmatrix} 0 & s'(\alpha)\kappa(\alpha) & 0 \\ -s'(\alpha)\kappa(\alpha) & 0 & s'(\alpha)\tau(\alpha) \\ 0 & -s'(\alpha)\tau(\alpha) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}, \tag{5.5.45}$$

which can be written as

$$\begin{bmatrix} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = -\begin{bmatrix} s'(\alpha)\tau(\alpha) \\ 0 \\ s'(\alpha)\kappa(\alpha) \end{bmatrix}^{\times} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}. \tag{5.5.46}$$

In the special case where α denotes time, we have

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = -\begin{bmatrix} s'(t)\tau(t) \\ 0 \\ s'(t)\kappa(t) \end{bmatrix}^{\times} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}. \tag{5.5.47}$$

To illustrate the meaning of the Frenet-Serret relations, consider the case in which the curve lies in a plane. Then, $\hat{e}_{\rm t}$ and $\hat{e}_{\rm n}$ both lie in the plane, while $\hat{e}_{\rm b}$ is perpendicular to the plane, and thus $\overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b} = 0$. In this case, the curvature $\kappa(s)$ is the rate with respect to s of the rotation of $\hat{e}_{\rm t}$ around $\hat{e}_{\rm b}$.

If, however, the curve is not confined to a plane, then the torsion $\tau(s)$ is the rate with respect to s of the rotation of $\hat{e}_{\rm b}$ around $\hat{e}_{\rm n}$.

The Frenet-Serret relations can be viewed as analogous to the vectrix form of Poisson's equation given by (4.3.21), that is,

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{\imath}}_{\rm B} \\ \overset{\mathrm{A}\bullet}{\hat{\jmath}}_{\rm B} \\ \overset{\mathrm{A}\bullet}{\hat{k}}_{\rm B} \end{bmatrix} = -\omega_{\mathrm{B/A|B}}^{\times} \begin{bmatrix} \hat{\imath}_{\rm B} \\ \hat{\jmath}_{\rm B} \\ \hat{k}_{\rm B} \end{bmatrix}. \tag{5.5.48}$$

Recall that Poisson's equation applies to a rigid body and is parameterized by time. It is thus convenient to define the *angular velocity* of $\mathrm{F_{FS}}$ relative to $\mathrm{F_A}$ as

$$\vec{\omega}_{s,\mathrm{FS/A}}(s) \triangleq \tau(s)\hat{e}_{\rm t} + \kappa(s)\hat{e}_{\rm b}, \tag{5.5.49}$$

which lies in the rectifying plane. Defining

$$\omega_{s,\mathrm{FS/A|FS}} \triangleq \vec{\omega}_{s,\mathrm{FS/A}}\Big|_{\mathrm{FS}} = \begin{bmatrix} \tau(s) \\ 0 \\ \kappa(s) \end{bmatrix}, \tag{5.5.50}$$

we can rewrite (5.5.44) as

$$\begin{bmatrix} \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}s\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = -\omega_{s,\mathrm{FS/A|FS}}^{\times} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}. \tag{5.5.51}$$

For an arbitrary parameter α, we define

$$\vec{\omega}_{\alpha,\mathrm{FS/A}}(\alpha) \triangleq \tau(\alpha)\hat{e}_{\rm t} + \kappa(\alpha)\hat{e}_{\rm b}, \tag{5.5.52}$$

$$\omega_{\alpha,\mathrm{FS/A|FS}} \triangleq \vec{\omega}_{\alpha,\mathrm{FS/A}}\Big|_{\mathrm{FS}} = \begin{bmatrix} s'(\alpha)\tau(\alpha) \\ 0 \\ s'(\alpha)\kappa(\alpha) \end{bmatrix}. \tag{5.5.53}$$

We thus have

$$\begin{bmatrix} \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm t} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm n} \\ \overset{\mathrm{A}\alpha\bullet}{\hat{e}}_{\rm b} \end{bmatrix} = -\omega_{\alpha,\mathrm{FS/A|FS}}^{\times} \begin{bmatrix} \hat{e}_{\rm t} \\ \hat{e}_{\rm n} \\ \hat{e}_{\rm b} \end{bmatrix}. \tag{5.5.54}$$

Finally, in terms of time, we define

$$\vec{\omega}_{t,\mathrm{FS/A}}(t) \triangleq s'(t)\tau(t)\hat{e}_{\rm t} + s'(t)\kappa(t)\hat{e}_{\rm b}, \tag{5.5.55}$$

$$\omega_{t,\mathrm{FS/A|FS}} \triangleq \vec{\omega}_{t,\mathrm{FS/A}}\Big|_{\mathrm{FS}} = \begin{bmatrix} s'(t)\tau(t) \\ 0 \\ s'(t)\kappa(t) \end{bmatrix}. \tag{5.5.56}$$

We thus have

$$\begin{bmatrix} \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{t}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{n}} \\ \overset{\mathrm{A}\bullet}{\hat{e}}_{\mathrm{b}} \end{bmatrix} = -\omega^{\times}_{t,\mathrm{FS/A|FS}} \begin{bmatrix} \hat{e}_{\mathrm{t}} \\ \hat{e}_{\mathrm{n}} \\ \hat{e}_{\mathrm{b}} \end{bmatrix}. \tag{5.5.57}$$

Comparing (5.5.57) with (4.3.21), it can be seen that $\vec{\omega}_{t,\mathrm{FS/A}}$ is the angular velocity of $\mathrm{F_{FS}}$ relative to $\mathrm{F_A}$.

To obtain a matrix form of the Frenet-Serret relations, define

$$x_{\mathrm{t}} \triangleq \hat{e}_{\mathrm{t}}|_{\mathrm{A}}, \tag{5.5.58}$$

$$x_{\mathrm{n}} \triangleq \hat{e}_{\mathrm{n}}|_{\mathrm{A}}, \tag{5.5.59}$$

$$x_{\mathrm{b}} \triangleq \hat{e}_{\mathrm{b}}|_{\mathrm{A}}, \tag{5.5.60}$$

$$X \triangleq \begin{bmatrix} x_{\mathrm{t}} & x_{\mathrm{n}} & x_{\mathrm{b}} \end{bmatrix}. \tag{5.5.61}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} X(\alpha) = X(\alpha)\omega^{\times}_{\alpha,\mathrm{FS/A|FS}}, \tag{5.5.62}$$

which is analogous to the matrix form of Poisson's equation given by (4.3.17), that is,

$$\dot{\mathcal{R}}_{\mathrm{B/A}} = \mathcal{R}_{\mathrm{B/A}}\omega^{\times}_{\mathrm{B/A|B}}. \tag{5.5.63}$$

Let the Frenet-Serret frame be defined by the position vector $\vec{r}_{x/w}(\alpha)$, where α is a parameter, and let $\vec{x}(\alpha)$ be a physical vector that depends on α. Then, the transport theorem in terms of α has the form

$$\overset{\mathrm{FS}\alpha\bullet}{\vec{x}}(\alpha) = \overset{\mathrm{A}\alpha\bullet}{\vec{x}}(\alpha) + \vec{\omega}_{\alpha,\mathrm{FS/A}} \times \vec{x}(\alpha). \tag{5.5.64}$$

Finally, let α be a function of time t, and note that

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{x/w}(\alpha(t)) = \dot{\alpha}(t)\, \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}(\alpha) \tag{5.5.65}$$

Therefore,

$$\vec{v}_{x/w/\mathrm{A}}(\alpha(t)) = s'(\alpha(t))\dot{\alpha}(t)\hat{e}_{\mathrm{t}}, \tag{5.5.66}$$

$$\vec{a}_{x/w/\mathrm{A}}(\alpha(t)) = [s'(\alpha(t))\ddot{\alpha}(t) + s''(\alpha(t))\dot{\alpha}^2(t)]\hat{e}_{\mathrm{t}} + \frac{s'^2(\alpha(t))\dot{\alpha}^2(t)}{\rho(\alpha(t))}\hat{e}_{\mathrm{n}}. \tag{5.5.67}$$

Equivalently, we can write

$$\vec{v}_{x/w/\mathrm{A}}(\alpha(t)) = v(t)\hat{e}_{\mathrm{t}}, \tag{5.5.68}$$

$$\vec{a}_{x/w/\mathrm{A}}(\alpha(t)) = \dot{v}(t)\hat{e}_{\mathrm{t}} + \frac{v^2(t)}{\rho(\alpha(t))}\hat{e}_{\mathrm{n}}, \tag{5.5.69}$$

where $v(t) \triangleq s'(\alpha(t))\dot{\alpha}(t)$ is the speed.

The following result considers the rotational path of a rigid body whose attitude is given by a parameterization of (2.16.12). This result is analogous to Fact 4.9.2, where now the angular velocity is parameterized by α instead of time.

Fact 5.5.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, and assume that the physical rotation matrix that transforms $\mathrm{F_A}$ to $\mathrm{F_B}$ is given by $\vec{R}_{\mathrm{B/A}}(\alpha) = \exp(\vec{\Theta}^{\times}_{\mathrm{B/A}}(\alpha))$, where $\theta_{\mathrm{B/A}}(\alpha) \triangleq |\vec{\Theta}_{\mathrm{B/A}}(\alpha)| \in [0, \pi]$. Furthermore, define $\hat{n}_{\mathrm{B/A}}(\alpha) \triangleq \hat{\Theta}_{\mathrm{B/A}}(\alpha)$. Then,

$$\vec{\omega}_{\mathrm{B/A}}(\alpha) = \frac{1}{\theta^2_{\mathrm{B/A}}}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\overset{\mathrm{A}\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}} \tag{5.5.70}$$

$$= \left(\vec{I} + \frac{1-\cos\theta_{\mathrm{B/A}}}{\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta^3_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\overset{\mathrm{A}\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}}. \tag{5.5.71}$$

Furthermore,

$$\overset{\mathrm{A}\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \left(\vec{I} - \frac{1}{2}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \frac{2 - \theta_{\mathrm{B/A}}\cot\frac{\theta_{\mathrm{B/A}}}{2}}{2\theta^2_{\mathrm{B/A}}}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\right)\vec{\omega}_{\mathrm{B/A}}. \tag{5.5.72}$$

5.6 Theoretical Problems

Problem 5.6.1. Let x and w be points, and let $\vec{r}_{x/w}(\alpha)$ depend on a parameter α, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Show that, if

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathcal{O}_{\mathrm{B/A}} = 0,$$

then

$$\overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w} = \overset{\mathrm{A}\alpha\bullet}{\vec{r}}_{x/w}.$$

Problem 5.6.2. Consider a helix with radius r and distance between turns h, wrapped around a cylinder whose longitudinal axis is parallel with the $\hat{k}_{\mathrm{A}}$ axis. Show that this curve is parameterized in terms of arc length s by

$$\vec{r}_{x/w}(s) = \left(r\cos\frac{s}{\sqrt{r^2+h^2}}\right)\hat{\imath}_{\mathrm{A}} + \left(r\sin\frac{s}{\sqrt{r^2+h^2}}\right)\hat{\jmath}_{\mathrm{A}} + \left(\frac{hs}{\sqrt{r^2+h^2}}\right)\hat{k}_{\mathrm{A}}.$$

Furthermore, determine $\hat{e}_{\mathrm{t}}$, $\hat{e}_{\mathrm{n}}$, and $\hat{e}_{\mathrm{b}}$, and show that

$$\kappa(s) = \frac{r}{r^2+h^2}, \quad \tau(s) = \frac{h}{r^2+h^2}.$$

Problem 5.6.3. Consider the logarithmic spiral curve parameterized by

$$\vec{r}_{x/w}(\alpha) = (e^{\alpha}\cos\alpha)\hat{\imath}_{\mathrm{A}} + (e^{\alpha}\sin\alpha)\hat{\jmath}_{\mathrm{A}}.$$

Show that

$$\kappa(s) = \frac{1}{s}.$$

Problem 5.6.4. Consider the catenary curve parameterized by

$$\vec{r}_{x/w}(\alpha) = \alpha\hat{\imath}_{\mathrm{A}} + \frac{a}{2}(e^{\alpha/a} + e^{-\alpha/a})\hat{\jmath}_{\mathrm{A}},$$

where a is a positive constant. Show that

$$\kappa(s) = \frac{a}{s^2 + a}.$$

Problem 5.6.5. Show that if $\tau \neq 0$, then $\kappa \neq 0$.

Problem 5.6.6. Let $\mathrm{F_A}$ be a frame, let w be a point, let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let x and y be points that are fixed in $\mathcal{B}$, and let $\mathrm{F}_{\mathrm{FS}_x}$ and $\mathrm{F}_{\mathrm{FS}_y}$ be the Frenet-Serret frames associated with the curves generated by x and y, respectively. Find a relationship between $\vec{R}_{\mathrm{FS}_y/\mathrm{A}}$ and $\vec{R}_{\mathrm{FS}_x/\mathrm{A}}$.

5.7 Applied Problems

Problem 5.7.1. Consider a bowling alley located at λ degrees north latitude with bowling lanes that are l feet long. A ball is rolled down the center of the lane with speed v. Determine the Coriolis acceleration due to the rotation of the Earth and the resulting lateral position of the ball at the end of the lane. The bowling alley may be oriented in an arbitrary horizontal direction.

Symbol	Definition
$\hat{e}_\mathrm{r}, \hat{e}_\mathrm{t}, \hat{e}_\mathrm{a}$	Cylindrical (radial, tangential, axial) frame
$\hat{e}_\mathrm{u}, \hat{e}_\mathrm{e}, \hat{e}_\mathrm{n}$	Spherical (up, east, north) frame
$\hat{e}_\mathrm{t}, \hat{e}_\mathrm{n}, \hat{e}_\mathrm{b}$	Frenet-Serret (tangential, normal, binormal) frame

Table 5.7.1: Notation for Chapter 5.

Chapter Six

Mass Moments, Forces, Moments, Torques, and Statics

A *body* is a finite collection of particles connected by massless links as well as rotary and translational springs, dashpots and inerters. A single particle is a body, as is a pair of particles that are either freely moving or rigidly connected by a massless link. A *sub-body* is a subset of a body. A *rigid body* is a body containing at least two particles whose shape does not change. A *multi-rigid body* is a finite collection of interconnected rigid sub-bodies. A multi-rigid body consisting of a pair of freely moving rigid sub-bodies is a body but not a rigid body. If time is not relevant to the properties of interest, then there is no distinction between a body and a rigid body; in effect, the properties of interest can be viewed as instantaneous.

6.1 The Zeroth and First Moments of Mass

Definition 6.1.1. Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively. Then, the *zeroth moment of mass* of $\mathcal{B}$ is the *total mass* $m_\mathcal{B}$ of the body, that is,

$$m_\mathcal{B} \triangleq \sum_{i=1}^{l} m_i. \tag{6.1.1}$$

Now, let w be a point. Then, the *center of mass* c of $\mathcal{B}$ is the point c defined by the position vector

$$\vec{r}_{\mathrm{c}/w} \triangleq \frac{1}{m_\mathcal{B}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}, \tag{6.1.2}$$

and the *first moment of mass* of $\mathcal{B}$ is the physical vector

$$m_\mathcal{B} \vec{r}_{\mathrm{c}/w} = \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}. \tag{6.1.3}$$

The following result shows that the location of the center of mass is independent of the choice of the point w.

Fact 6.1.2. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_\mathcal{B}$ denote the mass of $\mathcal{B}$, let w and w' be points, and define the points c and c′ by

$$\vec{r}_{\mathrm{c}/w} \triangleq \frac{1}{m_\mathcal{B}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}, \tag{6.1.4}$$

$$\vec{r}_{c'/w'} \triangleq \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w'}. \tag{6.1.5}$$

Then, c and c′ are colocated.

Proof. Note that

$$\begin{aligned}
\vec{r}_{c'/c} &= \vec{r}_{c'/w'} + \vec{r}_{w'/w} + \vec{r}_{w/c} \\
&= \vec{r}_{c'/w'} - \vec{r}_{c/w} + \vec{r}_{w'/w} \\
&= \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w'} - \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w} + \vec{r}_{w'/w} \\
&= \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i (\vec{r}_{y_i/w'} - \vec{r}_{y_i/w}) + \vec{r}_{w'/w} \\
&= \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i (\vec{r}_{y_i/w'} + \vec{r}_{w/y_i}) + \vec{r}_{w'/w} \\
&= \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{r}_{w/w'} + \vec{r}_{w'/w} \\
&= \vec{r}_{w/w'} + \vec{r}_{w'/w} = \vec{r}_{w/w} = \vec{0}.
\end{aligned}$$

□

Fact 6.1.3. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$. Then, c satisfies

$$\sum_{i=1}^{l} m_i \vec{r}_{y_i/c} = 0. \tag{6.1.6}$$

Proof. Set $w = c$ in (6.1.2). □

6.2 The Second Moment of Mass

The *second moment of mass* of a body is given by the physical inertia matrix $\vec{J}_{\mathcal{B}/z}$, which characterizes the mass distribution of a body $\mathcal{B}$ relative to a reference point z.

Definition 6.2.1. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let z be a point. Then, the *physical inertia matrix* $\vec{J}_{\mathcal{B}/z}$ of $\mathcal{B}$ relative to z is defined by

$$\vec{J}_{\mathcal{B}/z} \triangleq \sum_{i=1}^{l} m_i \vec{r}^{\,\times\prime}_{y_i/z} \vec{r}^{\,\times}_{y_i/z}. \tag{6.2.1}$$

Fact 6.2.2. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let z be a point. Then,

$$\vec{J}_{\mathcal{B}/z} = -\sum_{i=1}^{l} m_i \vec{r}^{\times 2}_{y_i/z} \tag{6.2.2}$$

$$= \sum_{i=1}^{l} m_i \left(|\vec{r}_{y_i/z}|^2 \vec{I} - \vec{r}_{y_i/z} \vec{r}'_{y_i/z} \right). \tag{6.2.3}$$

Now, let $\mathrm{F_B}$ be a frame. Then,

$$\begin{aligned}\vec{J}_{\mathcal{B}/z} = {} & J_{xx/z|\mathrm{B}} \hat{\imath}_\mathrm{B} \hat{\imath}'_\mathrm{B} + J_{yy/z|\mathrm{B}} \hat{\jmath}_\mathrm{B} \hat{\jmath}'_\mathrm{B} + J_{zz/z|\mathrm{B}} \hat{k}_\mathrm{B} \hat{k}'_\mathrm{B} - J_{xy/z|\mathrm{B}} (\hat{\imath}_\mathrm{B} \hat{\jmath}'_\mathrm{B} + \hat{\jmath}_\mathrm{B} \hat{\imath}'_\mathrm{B}) \\ & - J_{xz/z|\mathrm{B}} (\hat{\imath}_\mathrm{B} \hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B} \hat{\imath}'_\mathrm{B}) - J_{yz/z|\mathrm{B}} (\hat{\jmath}_\mathrm{B} \hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B} \hat{\jmath}'_\mathrm{B}).\end{aligned} \tag{6.2.4}$$

Hence,

$$J_{\mathcal{B}/z|\mathrm{B}} = \begin{bmatrix} J_{xx/z|\mathrm{B}} & -J_{xy/z|\mathrm{B}} & -J_{xz/z|\mathrm{B}} \\ -J_{xy/z|\mathrm{B}} & J_{yy/z|\mathrm{B}} & -J_{yz/z|\mathrm{B}} \\ -J_{xz/z|\mathrm{B}} & -J_{yz/z|\mathrm{B}} & J_{zz/z|\mathrm{B}} \end{bmatrix}, \tag{6.2.5}$$

where

$$J_{xx/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i (\overline{y}_i^2 + \overline{z}_i^2), \qquad J_{xy/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i \overline{x}_i \overline{y}_i, \tag{6.2.6}$$

$$J_{yy/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i (\overline{x}_i^2 + \overline{z}_i^2), \qquad J_{xz/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i \overline{x}_i \overline{z}_i, \tag{6.2.7}$$

$$\underbrace{J_{zz/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i (\overline{x}_i^2 + \overline{y}_i^2),}_{\text{moments of inertia}} \qquad \underbrace{J_{yz/z|\mathrm{B}} \triangleq \sum_{i=1}^{l} m_i \overline{y}_i \overline{z}_i,}_{\text{products of inertia}} \tag{6.2.8}$$

and where

$$r_{y_i/z|\mathrm{B}} = \begin{bmatrix} \overline{x}_i \\ \overline{y}_i \\ \overline{z}_i \end{bmatrix}. \tag{6.2.9}$$

Proof. For $i = 1, \ldots, l$, $\vec{r}^{\times\prime}_{y_i/z} = \vec{r}^{\times}_{y_i/z}$, which implies that (6.2.1) can be written as (6.2.2). Furthermore, (2.9.15) implies that (6.2.1) can be written as (6.2.3). Next, resolving $\vec{J}_{\mathcal{B}/z}$ in $\mathrm{F_B}$ yields

$$\begin{aligned} J_{\mathcal{B}/z|\mathrm{B}} &= \sum_{i=1}^{l} m_i \left(|\vec{r}_{y_i/z}|^2 I_3 - r_{y_i/z|\mathrm{B}} r^{\mathsf{T}}_{y_i/z|\mathrm{B}} \right) \\ &= \sum_{i=1}^{l} m_i \left(\begin{bmatrix} \overline{x}_i^2 + \overline{y}_i^2 + \overline{z}_i^2 & 0 & 0 \\ 0 & \overline{x}_i^2 + \overline{y}_i^2 + \overline{z}_i^2 & 0 \\ 0 & 0 & \overline{x}_i^2 + \overline{y}_i^2 + \overline{z}_i^2 \end{bmatrix} - \begin{bmatrix} \overline{x}_i^2 & \overline{x}_i \overline{y}_i & \overline{x}_i \overline{z}_i \\ \overline{y}_i \overline{x}_i & \overline{y}_i^2 & \overline{y}_i \overline{z}_i \\ \overline{z}_i \overline{x}_i & \overline{z}_i \overline{y}_i & \overline{z}_i^2 \end{bmatrix} \right), \end{aligned}$$

where

$$\vec{r}_{y_i/z} = \overline{x}_i \hat{\imath}_\mathrm{B} + \overline{y}_i \hat{\jmath}_\mathrm{B} + \overline{z}_i \hat{k}_\mathrm{B}.$$

Hence, $J_{\mathcal{B}/z|\mathrm{B}}$ is given by (6.2.5)–(6.2.8). □

The diagonal entries $J_{xx/z|\mathrm{B}}, J_{yy/z|\mathrm{B}}, J_{zz/z|\mathrm{B}}$ of $J_{\mathcal{B}/z|\mathrm{B}}$ are the *moments of inertia of* $\mathcal{B}$ *relative to* z *determined by* $\mathrm{F_B}$, whereas the off-diagonal entries $J_{xy/z|\mathrm{B}}, J_{xz/z|\mathrm{B}}, J_{yz/z|\mathrm{B}}$ of $J_{\mathcal{B}/z|\mathrm{B}}$ are the *products of inertia of* $\mathcal{B}$ *relative to* z *determined by* $\mathrm{F_A}$.

The following result relates the physical inertia matrix resolved in different frames.

Fact 6.2.3. Let $\mathcal{B}$ be a body, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let z be a point. Then,

$$J_{\mathcal{B}/z|\mathrm{A}} = \mathcal{O}_{\mathrm{A/B}} J_{\mathcal{B}/z|\mathrm{B}} \mathcal{O}_{\mathrm{B/A}}. \tag{6.2.10}$$

Proof. The result follows from Fact 2.10.11. □

The following result relates the physical inertia matrix of a body to the physical inertia matrix of the rotated body.

Fact 6.2.4. Let $\mathcal{B}$ be a body, let z be point in $\mathcal{B}$, let $\vec{R}$ be a physical rotation matrix, let $\mathcal{B}'$ be the body $\mathcal{B}$ rotated by $\vec{R}$, and let z' denote the point in $\mathcal{B}'$ corresponding to z in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}'/z'} = \vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}'. \tag{6.2.11}$$

Proof. Note that

$$\begin{aligned}
\vec{J}_{\mathcal{B}'/z'} &= -\sum_{i=1}^{l} m_i \vec{r}_{y_i'/z'}^{\,\times 2} \\
&= -\sum_{i=1}^{l} m_i \left(\vec{R}\vec{r}_{y_i/z}\right)^{\times 2} \\
&= -\sum_{i=1}^{l} m_i \left(\vec{R}\vec{r}_{y_i/z}^{\,\times}\vec{R}'\right)^{2} \\
&= -\sum_{i=1}^{l} m_i \vec{R}\vec{r}_{y_i/z}^{\,\times 2}\vec{R}' \\
&= \vec{R}\left(-\sum_{i=1}^{l} m_i \vec{r}_{y_i/z}^{\,\times 2}\right)\vec{R}' \\
&= \vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}'.
\end{aligned}$$

□

Fact 6.2.5. Let $\mathcal{B}$ be a body, let z be a point in $\mathcal{B}$, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, let $\mathcal{B}'$ be the body $\mathcal{B}$ rotated by $\vec{R}_{\mathrm{A/B}}$, and let z' be the point in $\mathcal{B}'$ corresponding to z in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}'/z'} = \vec{R}_{\mathrm{A/B}}\vec{J}_{\mathcal{B}/z}\vec{R}_{\mathrm{B/A}}. \tag{6.2.12}$$

Consequently,

$$J_{\mathcal{B}'/z'|\mathrm{A}} = J_{\mathcal{B}/z|\mathrm{B}}. \tag{6.2.13}$$

Proof. The equality (6.2.12) follows from (6.2.11). Furthermore,

$$J_{\mathcal{B}'/z'|\mathrm{A}} = \left(\vec{R}_{\mathrm{A/B}} \vec{J}_{\mathcal{B}/z} \vec{R}_{\mathrm{B/A}}\right)\Big|_{\mathrm{A}} = \mathcal{R}_{\mathrm{A/B}} J_{\mathcal{B}/z|\mathrm{A}} \mathcal{R}_{\mathrm{B/A}} = \mathcal{O}_{\mathrm{B/A}} J_{\mathcal{B}/z|\mathrm{A}} \mathcal{O}_{\mathrm{A/B}} = J_{\mathcal{B}/z|\mathrm{B}}. \qquad \square$$

If $J_{\mathcal{B}/z|\mathrm{B}}$ is diagonal, that is,

$$J_{\mathcal{B}/z|\mathrm{B}} = \begin{bmatrix} J_{xx/z|\mathrm{B}} & 0 & 0 \\ 0 & J_{yy/z|\mathrm{B}} & 0 \\ 0 & 0 & J_{zz/z|\mathrm{B}} \end{bmatrix}, \tag{6.2.14}$$

then the axes of $\mathrm{F_B}$ are *principal axes* of $\mathcal{B}$ *relative to* z, $\mathrm{F_B}$ is a *principal-axis frame of* $\mathcal{B}$ *relative to* z, and $J_{xx/z|\mathrm{B}}$, $J_{yy/z|\mathrm{B}}$, $J_{zz/z|\mathrm{B}}$ are the *principal moments of inertia of* $\mathcal{B}$ *relative to* z. A principal axis of inertia is *degenerate* if the corresponding principal moment of inertia is zero.

The following result shows that every body $\mathcal{B}$ has a principal-axis frame relative to every point z.

Fact 6.2.6. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame. Then, the following statements hold:

i) There exists a rotation matrix S such that $SJ_{\mathcal{B}/z|\mathrm{B}}S^{\mathsf{T}}$ is diagonal.

ii) There exists a frame $\mathrm{F_A}$ such that $J_{\mathcal{B}/z|\mathrm{A}}$ is diagonal.

iii) There exists a physical rotation matrix $\vec{R}$ such that $(\vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}')\big|_{\mathrm{B}}$ is diagonal.

iv) The following statements are equivalent:

a) $\mathrm{F_B}$ is a principal-axis frame of $\mathcal{B}$ relative to z.

b) The axes of $\mathrm{F_B}$ are physical eigenvectors of $\vec{J}_{\mathcal{B}/z}$.

c) The vectors e_1, e_2, e_3 are eigenvectors of $J_{\mathcal{B}/z|\mathrm{B}}$.

v) The principal moments of inertia of $\mathcal{B}$ relative to z are the eigenvalues of $J_{\mathcal{B}/z|\mathrm{B}}$.

Proof. To prove *i*), note that it follows from the Schur decomposition given by Corollary 5.4.5 given in [1, p. 320] that there exists an orthogonal matrix $S \in \mathbb{R}^{3\times3}$ such that the positive-semidefinite matrix $D \triangleq SJ_{\mathcal{B}/z|\mathrm{B}}S^{\mathsf{T}}$ is diagonal. In the case where $\det S = 1$, it follows that S is a rotation matrix. In the case where $\det S = -1$, S can be replaced by $-S$, which is a rotation matrix that satisfies $D = (-S)J_{\mathcal{B}/z|\mathrm{B}}(-S)^{\mathsf{T}}$.

To prove *ii*), let S be given by *i*). Then, it follows from Problem 2.24.11 that there exists a frame $\mathrm{F_A}$ such that $\mathcal{O}_{\mathrm{A/B}} = S$. Then,

$$J_{\mathcal{B}/z|\mathrm{A}} = \mathcal{O}_{\mathrm{A/B}} J_{\mathcal{B}/z|\mathrm{B}} \mathcal{O}_{\mathrm{B/A}} = SJS^{\mathsf{T}}$$

is diagonal.

To prove *iii*), let the frame $\mathrm{F_A}$ be given by statement *ii*), and define $\vec{R} \triangleq \vec{R}_{\mathrm{B/A}}$. Then,

$$\begin{aligned}(\vec{R}\vec{J}_{\mathcal{B}/z}\vec{R}')\big|_{\mathrm{B}} &= (\vec{R}_{\mathrm{B/A}}\vec{J}_{\mathcal{B}/z}\vec{R}'_{\mathrm{B/A}})\big|_{\mathrm{B}} \\ &= \mathcal{O}_{\mathrm{A/B}} J_{\mathcal{B}/z|\mathrm{B}} \mathcal{O}_{\mathrm{B/A}} \\ &= J_{\mathcal{B}/z|\mathrm{A}}\end{aligned}$$

is diagonal. □

The following two results summarize properties of the inertia matrix.

Fact 6.2.7. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame.

Then, the following statements hold:

i) $J_{\mathcal{B}/z|\mathrm{B}}$ is positive semidefinite.

ii) $J_{\mathcal{B}/z|\mathrm{B}} = 0$ if and only if $\mathcal{B}$ consists of a single particle colocated with z.

iii) $\operatorname{rank} J_{\mathcal{B}/z|\mathrm{B}} \neq 1$.

iv) $\operatorname{rank} J_{\mathcal{B}/z|\mathrm{B}} = 2$ if and only if z and all of the particles of $\mathcal{B}$ are colinear.

v) $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite if and only if $\mathcal{B}$ contains at least two particles y_i and y_j such that $\vec{r}_{y_i/z}$ and $\vec{r}_{y_j/z}$ are linearly independent.

vi) If $\mathcal{B}$ contains three particles that are not colinear, then $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite.

vii) If z and $\mathcal{B}$ are not coplanar, then $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite.

Proof. To prove *i*), let $\vec{x}$ be a physical vector. Then, it follows from (7.9.2) that

$$\begin{aligned}\vec{x}\Big|_{\mathrm{B}}^{\mathsf{T}} J_{\mathcal{B}/z|\mathrm{B}}\, \vec{x}\Big|_{\mathrm{B}} &= \vec{x}\,'\vec{J}_{\mathcal{B}/z}\vec{x} = \vec{x}\,'\sum_{i=1}^{l} m_i \vec{r}_{y_i/z}^{\,\times\prime}\vec{r}_{y_i/z}^{\,\times}\vec{x}\\ &= \sum_{i=1}^{l} m_i \vec{x}\,'\vec{r}_{y_i/z}^{\,\times\prime}\vec{r}_{y_i/z}^{\,\times}\vec{x} = \sum_{i=1}^{l} m_i(\vec{r}_{y_i/z}^{\,\times}\vec{x})'\vec{r}_{y_i/z}^{\,\times}\vec{x}\\ &= \sum_{i=1}^{l} m_i|\vec{r}_{y_i/z}\times\vec{x}|^2 \ge 0.\end{aligned}$$

Statements *ii*)–*iv*) follow from Fact 8.93 and Fact 8.94 given in [1, p. 495].

To prove *v*), let $\vec{x}$ be a nonzero physical vector. Then, since $\vec{r}_{y_i/z}$ and $\vec{r}_{y_j/z}$ are linearly independent, it follows $\vec{r}_{y_i/z}\times\vec{x}$ and $\vec{r}_{y_y/z}\times\vec{x}$ are not both zero. Consequently,

$$\vec{x}\Big|_{\mathrm{B}}^{\mathsf{T}} J_{\mathcal{B}/z|\mathrm{B}}\, \vec{x}\Big|_{\mathrm{B}} = \sum_{i=1}^{l} m_i|\vec{r}_{y_i/z}\times\vec{x}|^2 > 0,$$

which implies that $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite.

To prove *vi*), let y_i, y_j, and y_k be particles of $\mathcal{B}$ that are not colinear.

To prove *vii*), note that, since z and $\mathcal{B}$ are not coplanar, it follows that $\mathcal{B}$ must have at least three particles that are not colinear, and thus the result follows from *vi*). □

For the following result, let J_1, J_2, J_3 be the moments of inertia of $\mathcal{B}$ relative to z determined by $\mathrm{F_B}$, that is, the diagonal entries of $J_{\mathcal{B}/z|\mathrm{B}}$. If, in addition, $\mathrm{F_B}$ is a principal-axis frame, then let $\lambda_1, \lambda_2, \lambda_3$ be the principal moments of inertia of $\mathcal{B}$ relative to z, that is, the diagonal entries of $J_{\mathcal{B}/z|\mathrm{B}}$. This result shows that the moments of inertia of $\mathcal{B}$ may represent the sides of a triangle.

Fact 6.2.8. Let $\mathcal{B}$ be a body, let $\mathrm{F_B}$ be a frame, and let z be a point. Then, the following statements hold:

i) The moments of inertia $J_1 \geq J_2 \geq J_3 \geq 0$ of $\mathcal{B}$ relative to z determined by $\mathrm{F_B}$ satisfy

$$J_1 \leq J_2 + J_3. \tag{6.2.15}$$

ii) If $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite, then the moments of inertia $J_1 \geq J_2 \geq J_3 > 0$ of $\mathcal{B}$ determined by $\mathrm{F_B}$ satisfy

$$1 \leq \min\left\{\frac{J_2}{J_3}, \frac{J_1}{J_2}\right\} \leq \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.16}$$

iii) If z and $\mathcal{B}$ do not lie in a single plane, then the moments of inertia $J_1 \geq J_2 \geq J_3 > 0$ of $\mathcal{B}$ determined by $\mathrm{F_B}$ satisfy

$$J_1 < J_2 + J_3 \tag{6.2.17}$$

and

$$1 \leq \min\left\{\frac{J_2}{J_3}, \frac{J_1}{J_2}\right\} < \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.18}$$

iv) The moments of inertia $J_1 \geq J_2 \geq J_3 \geq 0$ of $\mathcal{B}$ relative to z determined by $\mathrm{F_B}$ and the principal moments of inertia $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ of $\mathcal{B}$ satisfy

$$J_1 + J_2 + J_3 = \lambda_1 + \lambda_2 + \lambda_3 \tag{6.2.19}$$

and

$$\lambda_1 \leq \lambda_2 + \lambda_3. \tag{6.2.20}$$

Hence,

$$(\mathrm{tr}\, J_{\mathcal{B}/z|\mathrm{B}})I_3 \leq 2J_{\mathcal{B}/z|\mathrm{B}}. \tag{6.2.21}$$

v) If $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite, then the principal moments of inertia $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$ of $\mathcal{B}$ determined by $\mathrm{F_B}$ satisfy

$$1 \leq \min\left\{\frac{\lambda_2}{\lambda_3}, \frac{\lambda_1}{\lambda_2}\right\} \leq \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.22}$$

vi) If z and $\mathcal{B}$ do not lie in a single plane, then the principal moments of inertia $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$ of $\mathcal{B}$ satisfy

$$\lambda_1 < \lambda_2 + \lambda_3 \tag{6.2.23}$$

and

$$1 \leq \min\left\{\frac{\lambda_2}{\lambda_3}, \frac{\lambda_1}{\lambda_2}\right\} < \tfrac{1}{2}(1+\sqrt{5}) \approx 1.618. \tag{6.2.24}$$

Proof. To prove *i*), assume for convenience that $J_1 = J_{xx}$, $J_2 = J_{yy}$, and $J_3 = J_{zz}$. Then,

$$\begin{aligned} J_1 &= \sum_{i=1}^{l} m_i(\overline{y}_i^2 + \overline{z}_i^2) \\ &\leq \sum_{i=1}^{l} m_i(\overline{y}_i^2 + 2\overline{x}_i^2 + \overline{z}_i^2) \end{aligned}$$

$$= \sum_{i=1}^{l} m_i(\overline{x}_i^2 + \overline{z}_i^2) + \sum_{i=1}^{l} m_i(\overline{x}_i^2 + \overline{y}_i^2)$$
$$= J_2 + J_3.$$

To prove *iii*), note that, since z and $\mathcal{B}$ do not lie in a single plane, there exist three particles y_i, y_j, y_k such that $\vec{r}_{y_i/z}, \vec{r}_{y_j/z}, \vec{r}_{y_k/z}$ are linearly independent. Therefore, it follows from Fact 6.2.7 that $J_{\mathcal{B}/z|\mathrm{B}}$ is positive definite. Furthermore, $\overline{x}_m$ is nonzero for some particle y_m,, and thus the inequality in the proof of *i*) is strict, which proves (6.2.17). The right-hand inequality in (6.2.18) is a property of triangles given in [8, p. 145]. *ii*) is a limiting case of *iii*).

Finally, *iv*)–*vi*) follow from *i*)–*iii*) by choosing $\mathrm{F_B}$ to be a principal-axis frame. □

Figure 6.2.1 shows the triangular region of feasible principal moments of inertia of a rigid body. There are five cases that are highlighted for principal moments of inertia $\lambda_1 \ge \lambda_2 \ge \lambda_3 > 0$, where $\lambda_1, \lambda_2, \lambda_3$ satisfy the triangle inequality $\lambda_1 < \lambda_2 + \lambda_3$. Let m be the mass of the rigid body. The point $\lambda_1 = \lambda_2 = \lambda_3$ corresponds to a sphere of radius $R = \sqrt{\frac{5\lambda_1}{2m}}$, a cube whose sides have length $L = \sqrt{\frac{6\lambda_1}{m}}$, and a cylinder of length L and radius R, where $L/R = \sqrt{3}$ and $R = \sqrt{\frac{2\lambda_1}{m}}$. The point $\lambda_1 = \lambda_2 = 2\lambda_3$ corresponds to a cylinder of length L and radius R, where $L/R = 3$ and $R = \sqrt{\frac{2\lambda_1}{m}}$. The point $\lambda_1 = \frac{6}{5}\lambda_2 = 2\lambda_3$, located at the centroid of the triangular region, corresponds to a solid rectangular body with sides $L_1 = \sqrt{\frac{8\lambda_1}{m}} > L_2 = \sqrt{\frac{4\lambda_1}{m}} > L_3 = \sqrt{\frac{2\lambda_1}{m}}$.

The remaining cases in Figure 6.2.1 are limiting cases. The point $\lambda_1 = 2\lambda_2 = 2\lambda_3$ corresponds to a thin disk of radius $R = \sqrt{\frac{2\lambda_1}{m}}$. The point $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$ corresponds to a thin cylinder of radius $R = 0$ and length $L = \sqrt{\frac{12\lambda_1}{m}}$. Finally, points on the line segment $\lambda_1 = \lambda_2 + \lambda_3$, where $\lambda_2 > \lambda_3$ correspond to a thin rectangular plate with sides of length $L_1 = \sqrt{\frac{12\lambda_2}{m}} > L_2 = \sqrt{\frac{12\lambda_3}{m}}$.

The following result shows that the inertia of a body is the sum of the inertias of the components of the body.

Fact 6.2.9. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let $\mathcal{B}_3$ be the union of $\mathcal{B}_1$ and $\mathcal{B}_2$, and let z be a point. Then,

$$\vec{J}_{\mathcal{B}_3/z} = \vec{J}_{\mathcal{B}_1/z} + \vec{J}_{\mathcal{B}_2/z}. \tag{6.2.25}$$

The following result is an immediate consequence of Fact 6.2.9.

Fact 6.2.10. Let $\mathcal{B}_2$ be a body, let $\mathcal{B}_1$ be a body contained in $\mathcal{B}_2$, let $\mathcal{B}_3$ be the body $\mathcal{B}_2$ with the body $\mathcal{B}_1$ removed, and let z be a point. Then,

$$\vec{J}_{\mathcal{B}_3/z} = \vec{J}_{\mathcal{B}_2/z} - \vec{J}_{\mathcal{B}_1/z}. \tag{6.2.26}$$

For symmetric matrices $A, B \in \mathbb{R}^{3\times3}$, the notation "$A \le B$" means that $B - A$ is positive semidefinite. The following result is a consequence of Fact 6.2.9.

Fact 6.2.11. Let $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be bodies, assume that $\mathcal{B}$ is contained in $\tilde{\mathcal{B}}$, and let z be a point.

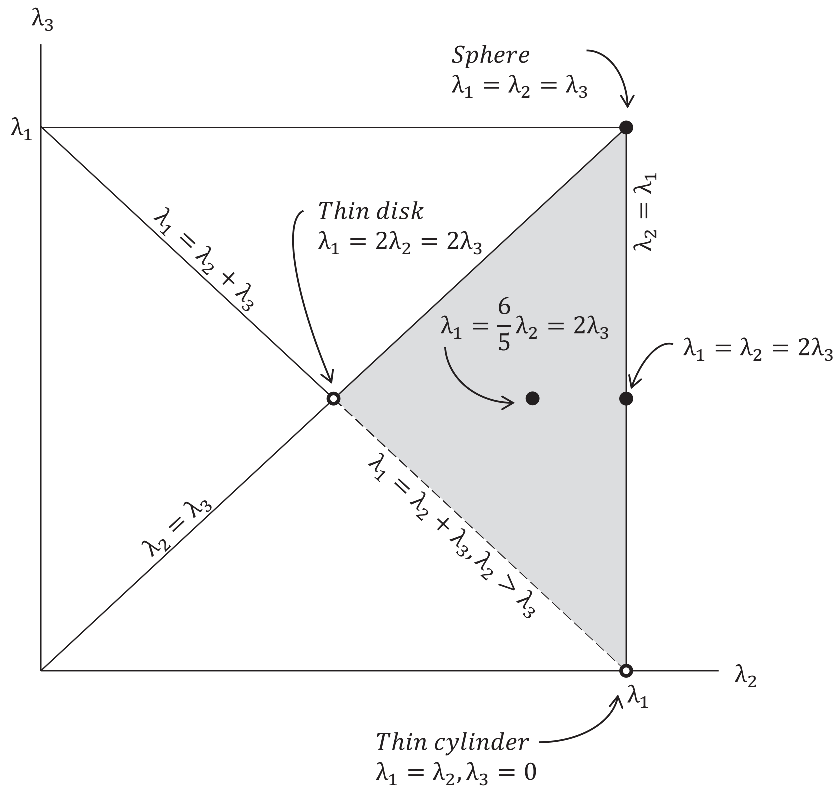


Figure 6.2.1: Feasible region of the principal moments of inertia $\lambda_1, \lambda_2, \lambda_3$ of a rigid body satisfying $0 < \lambda_3 \le \lambda_2 \le \lambda_1$, where $\lambda_1 < \lambda_2 + \lambda_3$. The shaded region shows all feasible values of λ_2 and λ_3 in terms of the largest principal moment of inertia λ_1. The open dots and dashed line segment indicate nonphysical, limiting cases.

Then,

$$J_{\mathcal{B}/z|\mathrm{B}} \le J_{\tilde{\mathcal{B}}/z|\mathrm{B}}. \tag{6.2.27}$$

The following result shows how the physical inertia matrix can be shifted from the center of mass to an arbitrary point. This result shows that every shift away from the center of mass increases the inertia of the body in the sense that $\vec{J}_{\mathcal{B}/\mathrm{c}} \le \vec{J}_{\mathcal{B}/z}$, that is, $\vec{J}_{\mathcal{B}/z} - \vec{J}_{\mathcal{B}/\mathrm{c}}$ is positive semidefinite. The physical inertia matrix relative to z turns out to be equivalent to the physical inertia matrix of a modified body $\mathcal{B}'$ consisting of $\mathcal{B}$ and a particle of mass $m_{\mathcal{B}}$ located at z relative to the center of mass of $\mathcal{B}$.

Fact 6.2.12. Let $\mathcal{B}$ be a body, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let z be a point, and let c be the center of mass of $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}/z} = \vec{J}_{\mathcal{B}/\mathrm{c}} - m_{\mathcal{B}} \overset{\rightharpoonup}{r}_{z/\mathrm{c}}^{\times 2} \tag{6.2.28}$$

$$= \vec{J}_{\mathcal{B}/\mathrm{c}} + m_{\mathcal{B}} \overset{\rightharpoonup}{r}_{z/\mathrm{c}}^{\times\prime} \overset{\rightharpoonup}{r}_{z/\mathrm{c}}^{\times} \tag{6.2.29}$$

$$= \overrightarrow{J}_{\mathcal{B}/\mathrm{c}} + m_{\mathcal{B}}(|\overrightarrow{r}_{z/\mathrm{c}}|^2 \overrightarrow{I} - \overrightarrow{r}_{z/\mathrm{c}}\overrightarrow{r}_{z/\mathrm{c}}'). \tag{6.2.30}$$

Proof. Note that

$$\begin{aligned}
\overrightarrow{J}_{\mathcal{B}/z} &= -\sum_{i=1}^{l} m_i \overrightarrow{r}_{y_i/z}^{\times 2} = -\sum_{i=1}^{l} m_i \left(\overrightarrow{r}_{y_i/\mathrm{c}}^{\times} + \overrightarrow{r}_{\mathrm{c}/z}^{\times}\right)^2 \\
&= -\sum_{i=1}^{l} m_i \left[\overrightarrow{r}_{y_i/\mathrm{c}}^{\times 2} + \overrightarrow{r}_{y_i/\mathrm{c}}^{\times}\overrightarrow{r}_{\mathrm{c}/z}^{\times} + \overrightarrow{r}_{\mathrm{c}/z}^{\times}\overrightarrow{r}_{y_i/\mathrm{c}}^{\times} + \overrightarrow{r}_{\mathrm{c}/z}^{\times 2}\right] \\
&= -\sum_{i=1}^{l} m_i \overrightarrow{r}_{y_i/\mathrm{c}}^{\times 2} - \left(\sum_{i=1}^{l} m_i \overrightarrow{r}_{y_i/\mathrm{c}}\right)^{\times} \overrightarrow{r}_{\mathrm{c}/z}^{\times} - \overrightarrow{r}_{\mathrm{c}/z}^{\times}\left(\sum_{i=1}^{l} m_i \overrightarrow{r}_{y_i/\mathrm{c}}\right)^{\times} - m_{\mathcal{B}}\overrightarrow{r}_{\mathrm{c}/z}^{\times 2} \\
&= \overrightarrow{J}_{\mathcal{B}/\mathrm{c}} - m_{\mathcal{B}}\overrightarrow{r}_{z/\mathrm{c}}^{\times 2} = \overrightarrow{J}_{\mathcal{B}/\mathrm{c}} + m_{\mathcal{B}}\overrightarrow{r}_{z/\mathrm{c}}^{\times\prime}\overrightarrow{r}_{z/\mathrm{c}}^{\times} = \overrightarrow{J}_{\mathcal{B}/\mathrm{c}} + m_{\mathcal{B}}(|\overrightarrow{r}_{z/\mathrm{c}}|^2\overrightarrow{I} - \overrightarrow{r}_{z/\mathrm{c}}\overrightarrow{r}_{z/\mathrm{c}}'). \qquad \square
\end{aligned}$$

The next result follows from Fact 6.2.12.

Fact 6.2.13. Let $\mathcal{B}$ be a body, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, and let z_1 and z_2 be points. Then,

$$\overrightarrow{J}_{\mathcal{B}/z_2} = \overrightarrow{J}_{\mathcal{B}/z_1} + m_{\mathcal{B}}[\overrightarrow{r}_{z_2/z_1}^{\times}\overrightarrow{r}_{\mathrm{c}/z_1}^{\times} + \overrightarrow{r}_{\mathrm{c}/z_1}^{\times}\overrightarrow{r}_{z_2/z_1}^{\times} - \overrightarrow{r}_{z_2/z_1}^{\times 2}]. \tag{6.2.31}$$

Fact 6.2.12 yields the *parallel axis theorem* given by the following result.

Fact 6.2.14. Let $\mathcal{B}$ be a body, let $\mathrm{F_B}$ be a body-fixed frame, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let z be a point fixed in $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, and assume that $\hat{\imath}_{\mathrm{B}}$ is perpendicular to $\overrightarrow{r}_{z/\mathrm{c}}$. Then,

$$J_{xx/z|\mathrm{B}} = J_{xx/\mathrm{c}|\mathrm{B}} + m_{\mathcal{B}}|\overrightarrow{r}_{z/\mathrm{c}}|^2. \tag{6.2.32}$$

Proof. Multiplying (6.2.31) on the left by $\hat{\imath}_{\mathrm{A}}'$ and on the right by $\hat{\imath}_{\mathrm{A}}$ yields

$$J_{xx/z|\mathrm{B}} = \hat{\imath}_{\mathrm{A}}'\overrightarrow{J}_{\mathcal{B}/z}\hat{\imath}_{\mathrm{A}} = \hat{\imath}_{\mathrm{A}}'[\overrightarrow{J}_{\mathcal{B}/\mathrm{c}} + m_{\mathcal{B}}(|\overrightarrow{r}_{z/\mathrm{c}}|^2\overrightarrow{I} - \overrightarrow{r}_{z/\mathrm{c}}\overrightarrow{r}_{z/\mathrm{c}}')]\hat{\imath}_{\mathrm{A}} = J_{xx/\mathrm{c}|\mathrm{B}} + m_{\mathcal{B}}|\overrightarrow{r}_{z/\mathrm{c}}|^2. \qquad \square$$

The radius of gyration is the distance from a point in a body to a fictitious particle that yields an equivalent moment of inertia.

Definition 6.2.15. Let $\mathcal{B}$ be a body, let $\mathrm{F_B}$ be a body-fixed frame, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let z and p be points fixed in $\mathcal{B}$, assume that $\hat{\imath}_{\mathrm{B}}$ is perpendicular to $\overrightarrow{r}_{z/p}$, define $r \triangleq |\overrightarrow{r}_{p/z}|$, and assume that $m_{\mathcal{B}}r^2 = J_{xx/z|\mathrm{B}}$. Then, r is the *radius of gyration of* $\mathcal{B}$ *relative to* z *around* $\hat{\imath}_{\mathrm{B}}$.

Fact 6.2.16. Let $\mathcal{B}$ be a body, let $\mathrm{F_B}$ be a body-fixed frame, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, let z be a point fixed in $\mathcal{B}$, and assume that $\hat{\imath}_{\mathrm{B}}$ is perpendicular to $\overrightarrow{r}_{z/\mathrm{c}}$. Then, the radius of gyration r of $\mathcal{B}$ relative to z around $\hat{\imath}_{\mathrm{B}}$ is given by

$$r = \sqrt{\frac{J_{xx/\mathrm{c}|\mathrm{B}}}{m_{\mathcal{B}}} + |\overrightarrow{r}_{\mathrm{c}/z}|^2}. \tag{6.2.33}$$

Now, assume that $\mathcal{B}$ consists of particles $y_1, \ldots, y_l$ with masses $m_1, \ldots, m_l$ whose distances to the line parallel with $\hat{\imath}_{\mathrm{B}}$ and passing through z are given by $r_1, \ldots, r_l$ respectively. Then,

$$r = \sqrt{\frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i r_i^2}. \tag{6.2.34}$$

A *plane of symmetry* of a body $\mathcal{B}$ is a plane that divides $\mathcal{B}$ into two "mirror-image" parts, which are identical in both geometry and mass properties. Note that the center of mass of $\mathcal{B}$ lies on every plane of symmetry of $\mathcal{B}$.

Fact 6.2.17. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame. Then, the following statements hold:

i) If z is an element of a plane of symmetry of $\mathcal{B}$ that is parallel with the $\hat{\imath}_{\mathrm{B}}$-$\hat{\jmath}_{\mathrm{B}}$ plane, then $J_{xz/z|\mathrm{B}} = J_{yz/z|\mathrm{B}} = 0$.

ii) If z is an element of a plane of symmetry of $\mathcal{B}$ that is parallel with the $\hat{\imath}_{\mathrm{B}}$-$\hat{k}_{\mathrm{B}}$ plane, then $J_{xy/z|\mathrm{B}} = J_{yz/z|\mathrm{B}} = 0$.

iii) If z is an element of a plane of symmetry of $\mathcal{B}$ that is parallel with the $\hat{\jmath}_{\mathrm{B}}$-$\hat{k}_{\mathrm{B}}$ plane, then $J_{xy/z|\mathrm{B}} = J_{xz/z|\mathrm{B}} = 0$.

iv) If $\mathcal{B}$ has orthogonal planes of symmetry P_1 and P_2 that are spanned by pairs of frame axes of $\mathrm{F_B}$, then $\mathrm{F_B}$ is a principal-axis frame of $\mathcal{B}$ relative to every point $z \in P_1 \cup P_2$.

Proof. To prove *i*), note that $\mathcal{B}$ contains an even number of particles $y_1, \ldots, y_{2r}$ whose masses are $m_1, \ldots, m_r, m_1, \ldots, m_r$ and whose locations relative to z and resolved in $\mathrm{F_B}$ have components $(\overline{x}_1, \overline{y}_1, \overline{z}_1), \ldots, (\overline{x}_r, \overline{y}_r, \overline{z}_r), (\overline{x}_1, \overline{y}_1, -\overline{z}_1), \ldots, (\overline{x}_r, \overline{y}_r, -\overline{z}_r)$, respectively, where $\overline{z}_1, \ldots, \overline{z}_r$ are nonzero. It thus follows from (6.2.8) that

$$J_{xz/z|\mathrm{B}} = \sum_{i=1}^{2r} m_i \overline{x}_i \overline{z}_i = \sum_{i=1}^{r} m_i \overline{x}_i \overline{z}_i + \sum_{i=r+1}^{2r} m_i \overline{x}_i \overline{z}_i = \sum_{i=1}^{r} m_i \overline{x}_i \overline{z}_i + \sum_{i=1}^{r} m_i \overline{x}_i (-\overline{z}_i) = 0.$$

Likewise, $J_{yz} = 0$. *iv*) follows from Fact 6.2.12. □

A body that has multiple planes of symmetry need not have two orthogonal planes of symmetry. For example, a bar whose cross section is an equilateral triangle whose size varies along the length of the bar has exactly three planes of symmetry, but no pair of these planes of symmetry is orthogonal.

6.3 The Physical Inertia Matrix for Continuum Bodies

For continuum bodies, we replace the finite sums in (6.2.6)–(6.2.8) with integrals.

Fact 6.3.1. Let $\mathcal{B}$ be a continuum body and let z be a point. Then,

$$\vec{J}_{\mathcal{B}/z} = -\int_{\mathcal{B}} \vec{r}_{\mathrm{d}m/z}^{\,\times 2}\, \mathrm{d}m \tag{6.3.1}$$

$$= \int_{\mathcal{B}} \vec{r}_{\mathrm{d}m/z}^{\,\times\prime}\, \vec{r}_{\mathrm{d}m/z}^{\,\times}\, \mathrm{d}m \tag{6.3.2}$$

$$= \int_{\mathcal{B}} |\vec{r}_{\mathrm{d}m/z}|^2 \vec{I} - \vec{r}_{\mathrm{d}m/z} \vec{r}_{\mathrm{d}m/z}^{\,\prime}\, \mathrm{d}m. \tag{6.3.3}$$

Now, let $\mathrm{F_B}$ be a frame. Then,

$$\vec{J}_{\mathcal{B}/z} = J_{xx/z|\mathrm{B}}\hat{\imath}_\mathrm{B}\hat{\imath}'_\mathrm{B} + J_{yy/z|\mathrm{B}}\hat{\jmath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + J_{zz/z|\mathrm{B}}\hat{k}_\mathrm{B}\hat{k}'_\mathrm{B} - J_{xy/z|\mathrm{B}}(\hat{\imath}_\mathrm{B}\hat{\jmath}'_\mathrm{B} + \hat{\jmath}_\mathrm{B}\hat{\imath}'_\mathrm{B})$$
$$- J_{xz/z|\mathrm{B}}(\hat{\imath}_\mathrm{B}\hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{\imath}'_\mathrm{B}) - J_{yz/z|\mathrm{B}}(\hat{\jmath}_\mathrm{B}\hat{k}'_\mathrm{B} + \hat{k}_\mathrm{B}\hat{\jmath}'_\mathrm{B}), \tag{6.3.4}$$

that is,

$$J_{\mathcal{B}/z|\mathrm{B}} = \begin{bmatrix} J_{xx/z|\mathrm{B}} & -J_{xy/z|\mathrm{B}} & -J_{xz/z|\mathrm{B}} \\ -J_{yx/z|\mathrm{B}} & J_{yy/z|\mathrm{B}} & -J_{yz/z|\mathrm{B}} \\ -J_{zx/z|\mathrm{B}} & -J_{zy/z|\mathrm{B}} & J_{zz/z|\mathrm{B}} \end{bmatrix}, \tag{6.3.5}$$

where

$$J_{xx/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} (y^2 + z^2)\,\mathrm{d}m, \qquad J_{xy/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} xy\,\mathrm{d}m, \tag{6.3.6}$$

$$J_{yy/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} (x^2 + z^2)\,\mathrm{d}m, \qquad J_{xz/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} xz\,\mathrm{d}m, \tag{6.3.7}$$

$$\underbrace{J_{zz/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} (x^2 + y^2)\,\mathrm{d}m,}_{\text{moments of inertia}} \qquad \underbrace{J_{yz/z|\mathrm{B}} \triangleq \int_{\mathcal{B}} yz\,\mathrm{d}m\,.}_{\text{products of inertia}} \tag{6.3.8}$$

If the density ρ of the material is constant, then $\mathrm{d}m = \rho\mathrm{d}V$, and each integral can be written as a volume integral. For example,

$$J_{xx/z|\mathrm{B}} = \rho\int_{\mathcal{B}} (y^2 + z^2)\,\mathrm{d}V. \tag{6.3.9}$$

For a flat plate, this integral becomes an integral over an area, and ρ is the area density, that is, mass per area. For a thin body, this integral becomes an integral over a length, and ρ is the linear density, that is, mass per length.

For a continuum body, the center of mass is the unique point fixed in the body and satisfies

$$\int_{\mathcal{B}} \vec{r}_{\mathrm{d}m/\mathrm{c}}\,\mathrm{d}m = 0. \tag{6.3.10}$$

Example 6.3.2. Let $\mathcal{B}$ be a homogeneous sphere of mass m and radius r, and let $\mathrm{F_B}$ be a frame. Then, the inertia matrix of the sphere relative to its center of mass c determined by $\mathrm{F_B}$ is given by

$$J_{\mathcal{B}/\mathrm{c}|\mathrm{B}} = \begin{bmatrix} J_{xx/\mathrm{c}|\mathrm{B}} & 0 & 0 \\ 0 & J_{yy/\mathrm{c}|\mathrm{B}} & 0 \\ 0 & 0 & J_{zz/\mathrm{c}|\mathrm{B}} \end{bmatrix}, \tag{6.3.11}$$

where $J_{xx/\mathrm{c}|\mathrm{B}} = J_{yy/\mathrm{c}|\mathrm{B}} = J_{zz/\mathrm{c}|\mathrm{B}} = \frac{2}{5}mr^2$ are the moments of inertia of $\mathcal{B}$ relative to the center of mass c determined by $\mathrm{F_B}$. Therefore,

$$J_{\mathcal{B}/\mathrm{c}|\mathrm{B}} = \frac{2}{5}mr^2 I_3. \tag{6.3.12}$$

◇

Example 6.3.3. Let $\mathcal{B}$ be a homogeneous rectangular solid, and let $\mathrm{F_B}$ be a frame whose axes

$\hat{\imath}_{\rm B}$, $\hat{\jmath}_{\rm B}$, and $\hat{k}_{\rm B}$ are parallel with the sides of length a, b, and c, respectively. Then,

$$J_{\mathcal{B}/{\rm c}|{\rm B}} = \begin{bmatrix} J_{xx/{\rm c}|{\rm B}} & 0 & 0 \\ 0 & J_{yy/{\rm c}|{\rm B}} & 0 \\ 0 & 0 & J_{zz/{\rm c}|{\rm B}} \end{bmatrix}, \tag{6.3.13}$$

where $J_{xx/{\rm c}|{\rm B}} = \frac{1}{12}m(b^2 + c^2)$, $J_{yy/{\rm c}|{\rm B}} = \frac{1}{12}m(a^2 + c^2)$, and $J_{zz/{\rm c}|{\rm B}} = \frac{1}{12}m(a^2 + b^2)$. If $a > b > c$, then $J_{zz/{\rm c}|{\rm B}} > J_{yy/{\rm c}|{\rm B}} > J_{xx/{\rm c}|{\rm B}}$, where $J_{zz/{\rm c}|{\rm B}}$, $J_{yy/{\rm c}|{\rm B}}$, and $J_{xx/{\rm c}|{\rm B}}$ are the major, intermediate, and minor principal moments of inertia, respectively, of $\mathcal{B}$ relative to the center of mass c determined by $\mathrm{F_B}$. If $b = c \approx 0$, then the rectangular solid approximates a thin bar, in which case $J_{xx/{\rm c}|{\rm B}} \approx 0$ and $J_{yy/{\rm c}|{\rm B}} = J_{zz/{\rm c}|{\rm B}} \approx \frac{1}{12}ma^2$. Alternatively, if $c \approx 0$, then the rectangular solid approximates a rectangular plate with sides a and b, in which case $J_{xx/{\rm c}|{\rm B}} \approx \frac{1}{12}mb^2$, $J_{yy/{\rm c}|{\rm B}} \approx \frac{1}{12}ma^2$, and $J_{zz/{\rm c}|{\rm B}} = \frac{1}{12}m(a^2 + b^2)$. ◇

Example 6.3.4. Let $\mathcal{B}$ be a homogeneous cylinder of length l and radius r, or, equivalently, a homogeneous disk of thickness l and radius r. Let $\mathrm{F_B}$ be a frame such that $\hat{\imath}_{\rm B}$ is parallel with the longitudinal axis of the cylinder. Then,

$$J_{\mathcal{B}/{\rm c}|{\rm B}} = \begin{bmatrix} J_{xx/{\rm c}|{\rm B}} & 0 & 0 \\ 0 & J_{yy/{\rm c}|{\rm B}} & 0 \\ 0 & 0 & J_{zz/{\rm c}|{\rm B}} \end{bmatrix}, \tag{6.3.14}$$

where $J_{xx/{\rm c}|{\rm B}} = \frac{1}{2}mr^2$ and $J_{yy/{\rm c}|{\rm B}} = J_{zz/{\rm c}|{\rm B}} = \frac{1}{12}m(3r^2 + l^2)$ are the principal moments of inertia of $\mathcal{B}$ relative to the center of mass c determined by $\mathrm{F_B}$. The cylinder approximates a thin bar if $r \approx 0$, in which case $J_{xx/{\rm c}|{\rm B}} \approx 0$ and $J_{yy/{\rm c}|{\rm B}} = J_{zz/{\rm c}|{\rm B}} \approx \frac{1}{12}ml^2$. The cylinder approximates a circular plate if $l \approx 0$, in which case $J_{xx/{\rm c}|{\rm B}} = \frac{1}{2}mr^2$ and $J_{yy/{\rm c}|{\rm B}} = J_{zz/{\rm c}|{\rm B}} \approx \frac{1}{4}mr^2$. Finally, if $l = \sqrt{3}r$, then

$$J_{\mathcal{B}/{\rm c}|{\rm B}} = \frac{1}{5}mr^2 I_3. \tag{6.3.15}$$

◇

6.4 Forces

Forces between interacting bodies arise from three types of mechanisms. *Contact* between two bodies gives rise to *reaction forces*. *Mechanical coupling* between two bodies connected by springs, dashpots, and inerters gives rise to *mechanical forces*. *Field coupling* between two bodies due to central gravity, electrostatics, electrodynamics, and magnetism gives rise to *field forces*.

An *internal force* on a body $\mathcal{B}$ is a force that originates from inside $\mathcal{B}$. For example, the reaction force on one sub-body due to contact with another sub-body is an internal force. An *external force* on $\mathcal{B}$ originates from outside of $\mathcal{B}$. For example, if $\mathcal{B}$ is in contact with another body, then the reaction force on $\mathcal{B}$ is an external force. Whether a force due to interacting bodies is internal or external depends on the extent of the body being considered.

Forces between two bodies are *reciprocal* if they are equal in magnitude, opposite in direction, and, for mechanical and field forces, parallel with the line that passes through the points or particles of interconnection. As shown in this section, forces due to contact and mechanical coupling are reciprocal, as are field-contact forces due to central gravity and electrostatics. Since magnetic forces involve field lines, the field forces due to magnetism are not reciprocal (see Section 6.9). Likewise, electrodynamic field forces are not reciprocal [4, pp. 349–351].

Definition 6.4.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies in contact. Then, the *reaction force on* $\mathcal{B}_1$ *due to*

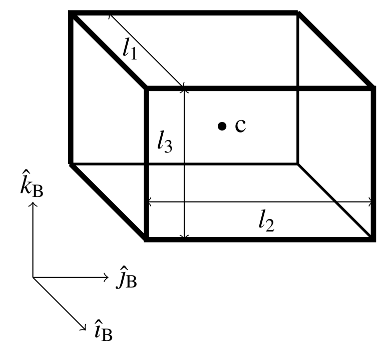
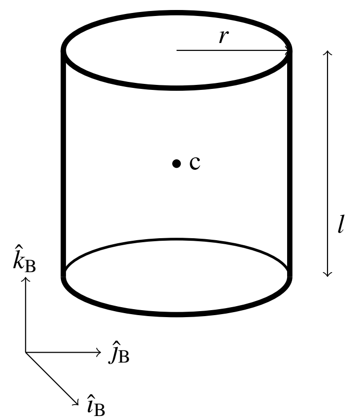
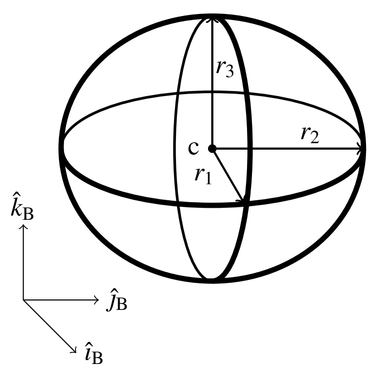
Body $\mathcal{B}$	Physical inertia matrix
	$\vec{J}_{\mathcal{B}/{\rm c}}\Big\vert_{\rm B} = \frac{m}{12}\begin{bmatrix} l_2^2 + l_3^2 & 0 & 0 \\ 0 & l_1^2 + l_3^2 & 0 \\ 0 & 0 & l_1^2 + l_2^2 \end{bmatrix}$
	$\vec{J}_{\mathcal{B}/{\rm c}}\Big\vert_{\rm B} = \frac{m}{12}\begin{bmatrix} 3r^2 + l^2 & 0 & 0 \\ 0 & 3r^2 + l^2 & 0 \\ 0 & 0 & 6r^2 \end{bmatrix}$
	$\vec{J}_{\mathcal{B}/{\rm c}}\Big\vert_{\rm B} = \frac{m}{5}\begin{bmatrix} r_2^2 + r_3^2 & 0 & 0 \\ 0 & r_1^2 + r_3^2 & 0 \\ 0 & 0 & r_1^2 + r_2^2 \end{bmatrix}$

Table 6.3.1: Physical inertia matrices of solid bodies relative to the center of mass resolved in principal-axis frames.

$\mathcal{B}_2$ is denoted by $\overrightarrow{f}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2}$. If, in addition, x is a point on $\mathcal{B}_1$, then the *reaction force on* $\mathcal{B}_1$ *due to contact at* x is denoted by $\overrightarrow{f}_{\mathrm{r}/\mathcal{B}_1/x}$.

The following is *Newton's third law*, which states that reaction forces are reciprocal. Since it is a law, no proof is given. Note that this law holds at each instant of time and thus is valid whether or not the bodies have nonzero relative velocity.

Fact 6.4.2. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies in contact at points x_1 and x_2 in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then, $\overrightarrow{f}_{\mathrm{r}/x_1/x_2} = -\overrightarrow{f}_{\mathrm{r}/x_2/x_1}$. Furthermore, $\overrightarrow{f}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2} = -\overrightarrow{f}_{\mathrm{r}/\mathcal{B}_2/\mathcal{B}_1}$.

Note that Newton's third law does not specify either the magnitude or direction of the reaction forces at a contact point. For example, for a sphere resting on a surface, the reaction force on the sphere due to the surface at the contact point is directed along the radial direction of the sphere. In other cases, the reaction force may be parallel with the tangent plane. The magnitude and direction of the reaction forces at a contact point must be determined on a case-by-case basis using free-body analysis.

Next, for mechanical forces, consider a spring connecting points y and w, where the stiffness of the spring is $k > 0$ and the relaxed length is $d \ge 0$. Let $\hat{z}$ be the unit vector that is equal to $\hat{r}_{yw}$ when the spring is relaxed. Then, the force $\overrightarrow{f}_{\mathrm{s}/y/w}$ on y due to the spring is given by

$$\overrightarrow{f}_{\mathrm{s}/y/w} = -k(\overrightarrow{r}_{y/w} - d\hat{z}). \tag{6.4.1}$$

Since $\overrightarrow{r}_{y/w}$ and $\hat{z}$ remain parallel when the spring is compressed or extended, (6.4.1) implies that $\overrightarrow{f}_{\mathrm{s}/y/w}$ is parallel with the line passing through y and w. Furthermore, since $\hat{r}_{w/y} = -\hat{r}_{y/w}$, it follows from (6.4.1) that the forces on y and w are equal and opposite, that is,

$$\overrightarrow{f}_{\mathrm{s}/w/y} = -\overrightarrow{f}_{\mathrm{s}/y/w}. \tag{6.4.2}$$

Hence, the mechanical forces due to a spring are reciprocal. Finally, if $d = 0$, then (6.4.1) becomes

$$\overrightarrow{f}_{\mathrm{s}/y/w} = -k\overrightarrow{r}_{y/w}. \tag{6.4.3}$$

Note that (6.4.1), (6.4.2), and (6.4.3) hold at each instant of time and thus are valid whether or not y and w have nonzero relative velocity.

Next, consider a dashpot connecting points y and w, where the damping of the dashpot is $c > 0$. Then, the force $\overrightarrow{f}_{y/w}$ on y due to the dashpot is given by

$$\overrightarrow{f}_{\mathrm{d}/y/w} = -c\frac{\mathrm{d}}{\mathrm{d}t}|\overrightarrow{r}_{y/w}|\hat{r}_{y/w}, \tag{6.4.4}$$

which shows that $\overrightarrow{f}_{\mathrm{d}/y/w}$ is codirectional with the line passing through y and w. Furthermore, $\overrightarrow{f}_{\mathrm{d}/y/w}$ pushes y in the direction $\hat{r}_{y/w}$ when the dashpot is compressing, and pushes y in the direction $-\hat{r}_{y/w}$ when the dashpot is extending. It follows from (6.4.4) the forces on y and w are equal and opposite, that is,

$$\overrightarrow{f}_{\mathrm{d}/w/y} = -\overrightarrow{f}_{\mathrm{d}/y/w}. \tag{6.4.5}$$

Hence, the mechanical forces due to a dashpot are reciprocal. Note that (6.4.4) and (6.4.5) hold at each instant of time and thus are valid whether or not y and w have nonzero relative acceleration.

Finally, consider an inerter connecting points y and w, where the inertance of the inerter is $b > 0$. Then, the force $\vec{f}_{y/w}$ on y due to the inerter is given by

$$\vec{f}_{\mathrm{i}/y/w} = -b\frac{\mathrm{d}^2}{\mathrm{d}t^2}|\vec{r}_{y/w}|\hat{r}_{y/w}, \tag{6.4.6}$$

which shows that $\vec{f}_{\mathrm{i}/y/w}$ is codirectional with the line passing through y and w. Furthermore, $\vec{f}_{\mathrm{i}/y/w}$ pushes y in the direction $\hat{r}_{y/w}$ when the inerter is compressing, and pushes y in the direction $-\hat{r}_{y/w}$ when the inerter is extending. It follows from (6.4.6) that the forces on y and w are equal and opposite, that is,

$$\vec{f}_{\mathrm{i}/w/y} = -\vec{f}_{\mathrm{i}/y/w}. \tag{6.4.7}$$

Hence, the mechanical forces due to an inerter are reciprocal. Note that (6.4.6) and (6.4.7) hold at each instant of time and thus are valid whether or not y and w have nonzero relative jerk (third derivative of relative displacement).

The following result is *Newton's law of universal gravitation*.

Fact 6.4.3. Let x be a particle whose mass is M, and let y be a particle whose mass is m. Then, the force due to central gravity on y due to x is given by

$$\vec{f}_{\mathrm{g}/y/x} = \frac{GMm}{|\vec{r}_{y/x}|^2}\hat{r}_{x/y}, \tag{6.4.8}$$

where the universal gravitational constant G is given by

$$G = 6.67428\ \mathrm{N\text{-}m^2/kg^2}. \tag{6.4.9}$$

Note that the force $\vec{f}_{\mathrm{g}/y/x}$ on y is attractive in the direction of x. A similar expression applies to charged particles, where the forces on the particles are attractive when the charges of the particles have opposite signs and repulsive when the charges of the particles have the same sign. Hence, the field forces due to central gravity and electrostatics are reciprocal. Since electrostatics is analogous to central gravity, we henceforth focus exclusively on central gravity.

The above discussion shows that reaction forces, mechanical forces due to springs, dashpots, and inerters, and central gravity are reciprocal. Henceforth, mechanical coupling refers to springs, dashpots, and inerters. Hence, for bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ that interact due to contact, mechanical coupling, or central gravity, let $\vec{f}_{\mathcal{B}_1/\mathcal{B}_2}$ denote the force on $\mathcal{B}_1$ due to $\mathcal{B}_2$. The following result extends Newton's third law to bodies that interact due to reciprocal forces.

Fact 6.4.4. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies that interact due to contact, mechanical coupling, or central gravity. Then, $\vec{f}_{\mathcal{B}_1/\mathcal{B}_2} = -\vec{f}_{\mathcal{B}_2/\mathcal{B}_1}$.

Note that Fact 6.4.4 does not specify the direction of $\vec{f}_{\mathcal{B}_1/\mathcal{B}_2}$. Problem 6.10.5 shows that, for central gravity, the net force vectors need not be parallel with the lines connecting either the centers of mass or the centers of gravity of the bodies.

Notation	Description	Defined in
$\vec{f}_y$	Force on y	
$\vec{f}_{\mathrm{r}/x_1/x_2}$	Reaction force on x_1 due to x_2	Definition 6.4.1
$\vec{f}_{\mathrm{s}/x_1/x_2}$	Force on x_1 due to a spring connecting x_1 and x_2	(6.4.1)
$\vec{f}_{\mathrm{d}/x_1/x_2}$	Force on x_1 due to a dashpot connecting x_1 and x_2	(6.4.4)
$\vec{f}_{\mathrm{i}/x_1/x_2}$	Force on x_1 due to an inerter connecting x_1 and x_2	(6.4.6)
$\vec{f}_{\mathrm{cg}/x_1/x_2}$	Force on x_1 due to central gravity due to x_2	(6.4.8)
$\vec{f}_{\mathrm{g}/x_1}$	Force on x_1 due to uniform gravity	(6.5.18)
$\vec{f}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2}$	Reaction force on $\mathcal{B}_1$ due to $\mathcal{B}_2$	Definition 6.4.1
$\vec{f}_{\mathrm{r}/\mathcal{B}_1/x}$	Reaction force on $\mathcal{B}_1$ due to contact at x	Definition 6.4.1
$\vec{f}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}$	Force on $\mathcal{B}_1$ due to a spring connecting $\mathcal{B}_1$ and $\mathcal{B}_2$	(6.4.1)
$\vec{f}_{\mathrm{d}/\mathcal{B}_1/\mathcal{B}_2}$	Force on $\mathcal{B}_1$ due to a dashpot connecting $\mathcal{B}_1$ and $\mathcal{B}_2$	(6.4.4)
$\vec{f}_{\mathrm{b}/\mathcal{B}_1/\mathcal{B}_2}$	Force on $\mathcal{B}_1$ due to an inerter connecting $\mathcal{B}_1$ and $\mathcal{B}_2$	(6.4.6)
$\vec{f}_{\mathrm{cg}/\mathcal{B}_1/\mathcal{B}_2}$	Force on $\mathcal{B}_1$ due to central gravity due to $\mathcal{B}_2$	(6.4.8)
$\vec{f}_{\mathrm{g}/\mathcal{B}_1}$	Force on $\mathcal{B}_1$ due to uniform gravity	(6.5.18)
$\vec{f}_{\mathrm{m}/\mathcal{B}_1/\mathcal{B}_2}$	Force on $\mathcal{B}_1$ due to $\mathcal{B}_2$ due to magnetism	(6.9.11)
$\vec{f}_{\mathcal{B}_1/\mathcal{B}_2}$	Net force on $\mathcal{B}_1$ due to $\mathcal{B}_2$	
$\vec{f}_{\mathcal{B}_1}$	Net force on $\mathcal{B}_1$	(6.5.2)

Table 6.4.1: Notation for forces.

6.5 Moments and Torques

Let y be a point in a body, let w be a point, and let $\vec{f}_y$ be the force on y. Then, the *moment* $\vec{M}_{y/w}$ *on y relative to w due to* $\vec{f}_y$ is defined by

$$\vec{M}_{y/w} \triangleq \vec{r}_{y/w} \times \vec{f}_y. \tag{6.5.1}$$

As illustrated in Figure 6.5.1, $\vec{M}_{y/w}$ induces a rotation of y around w in the direction given by the direction of the curled fingers of the right hand, where the right-hand thumb is pointing in the direction of $\vec{M}_{y/w}$.

Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, and, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force on y_i. Then, the *net force on $\mathcal{B}$ due to* $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ is defined by

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}. \tag{6.5.2}$$

The forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are *balanced* if $\vec{f}_{\mathcal{B}} = 0$.

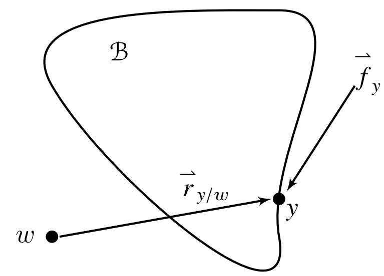


Figure 6.5.1: Representation of the moment $\vec{M}_{y/w} = \vec{r}_{y/w} \times \vec{f}_y$ on the point y in $\mathcal{B}$ relative to the point w due to the force $\vec{f}_y$ on y.

Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force on y_i, and let w be a point. Then, the *moment* $\vec{M}_{\mathcal{B}/w}$ *on* $\mathcal{B}$ *relative to* w *due to* $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ is defined by

$$\vec{M}_{\mathcal{B}/w} \triangleq \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{6.5.3}$$

Furthermore, let $y'_1, \ldots, y'_{l'}$ be points in $\mathcal{B}$, and, for all $i = 1, \ldots, l'$, let $\vec{f}_{y'_i}$ be the force on y'_i. Then, $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ and $\vec{f}_{y'_1}, \ldots, \vec{f}_{y'_{l'}}$ are *equivalent relative to* w if they have the same net force and

$$\sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l'} \vec{M}_{y'_i/w}. \tag{6.5.4}$$

As suggested by (6.5.4), different sets of forces with the same net force may produce the same moment relative to a specified point. It is thus convenient to consider a moment on a body relative to a point w without specifying the individual forces as long as the net force is specified. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, and let w be a point. Then, $\vec{M}$ is a *moment on* $\mathcal{B}$ *relative to* w *with net force* $\vec{f}_{\mathcal{B}}$ if there exist forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ on $y_1, \ldots, y_l$, respectively, with net force $\vec{f}_{\mathcal{B}}$ and such that $\vec{M} = \vec{M}_{\mathcal{B}/w}$. The forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ *realize* $\vec{f}_{\mathcal{B}}$ and $\vec{M}_{\mathcal{B}/w}$.

The following result shows that the moments $\vec{M}_{\mathcal{B}/w}$ and $\vec{M}_{\mathcal{B}/z}$ differ by a moment due to the net force on $\mathcal{B}$.

Fact 6.5.1. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force on y_i, and let w and z be points. Then,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}}, \tag{6.5.5}$$

where $\vec{f}_{\mathcal{B}}$ is defined by (6.5.2). If, in addition, $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/z}. \tag{6.5.6}$$

Proof. Note that

$$\begin{aligned}\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) = \sum_{i=1}^{l}(\vec{r}_{y_i/z} + \vec{r}_{z/w}) \times \vec{f}_{y_i} \\ &= \sum_{i=1}^{l}(\vec{r}_{y_i/z} \times \vec{f}_{y_i}) + \sum_{i=1}^{l}(\vec{r}_{z/w} \times \vec{f}_{y_i}) \\ &= \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \sum_{i=1}^{l} \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}}. \qquad \square\end{aligned}$$

Equation (6.5.6) shows that, if the forces on $\mathcal{B}$ are balanced, then the moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ relative to w is independent of the point w. The next two results focus on this case.

Fact 6.5.2. Let $\mathcal{B}$ be a body, let w be a point, let $\vec{f}_x$ and $\vec{f}_y$ be forces on points x and y in $\mathcal{B}$, respectively, assume that $\vec{f}_y = -\vec{f}_x$, and assume that $\vec{f}_x$ and $\vec{f}_y$ are the only forces on $\mathcal{B}$. Then,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{y/x} = \vec{M}_{x/y}. \tag{6.5.7}$$

Proof. Since the forces are balanced, Fact 6.5.1 implies that $\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/x}$. Therefore,

$$\vec{M}_{\mathcal{B}/w} = \vec{M}_{\mathcal{B}/x} = \vec{r}_{x/x} \times \vec{f}_x + \vec{r}_{y/x} \times \vec{f}_y = \vec{M}_{y/x}.$$

Interchanging x and y yields (6.5.7). $\square$

Fact 6.5.3. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the force on y_i, let w be a point, and assume that $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced. Then, for all $j = 1, \ldots, l$,

$$\vec{M}_{\mathcal{B}/w} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \vec{M}_{y_i/y_j}. \tag{6.5.8}$$

Proof. For convenience, set $j = l$. Then,

$$\begin{aligned}\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) + \vec{r}_{y_l/w} \times \vec{f}_{y_l} \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) + \vec{r}_{y_l/w} \times \left(-\sum_{i=1}^{l-1} \vec{f}_{y_i}\right) \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/w} \times \vec{f}_{y_i}) + \sum_{i=1}^{l-1}(\vec{r}_{w/y_l} \times \vec{f}_{y_i}) \\ &= \sum_{i=1}^{l-1}(\vec{r}_{y_i/y_l} \times \vec{f}_{y_i}) = \sum_{\substack{i=1 \\ i \neq l}}^{n} \vec{M}_{y_i/y_l}. \qquad \square\end{aligned}$$

Assume that $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced. Then, Fact 6.5.3 shows that the moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ is independent of w. In this case, we write $\vec{T}_{\mathcal{B}}$ instead of $\vec{M}_{\mathcal{B}/w}$, and we call $\vec{T}_{\mathcal{B}}$ the *torque on* $\mathcal{B}$ *due to* $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$. A torque is thus a moment due to balanced forces. Note that, unlike a moment, a torque does not require that a reference point be specified. Two sets of balanced forces are *equivalent* if the torques due to both sets are equal.

As in the case of moments, a torque on a body can be specified without specifying individual forces. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, and let w be a point. Then, $\vec{T}$ is a *torque on* $\mathcal{B}$ if there exist balanced forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ on $y_1, \ldots, y_l$, respectively, such that $\vec{T}$ is the torque on $\mathcal{B}$ due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$. The forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ *realize* $\vec{T}$.

For a body subject to a collection of forces, the following result shows that, in some cases, an additional force can be applied to the body, resulting in balanced forces and a torque that is equal to the original moment.

Fact 6.5.4. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ be forces on $y_1, \ldots, y_l$, respectively, let w be a point, let $\vec{M}_{\mathcal{B}/w}$ be the moment on $\mathcal{B}$ relative to w due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$, and define $\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}$. Furthermore, assume there exists a point z in $\mathcal{B}$ such that $\vec{r}_{z/w}$ is parallel with $\vec{f}_{\mathcal{B}}$, and let $\vec{T}_{\mathcal{B}}$ denote the torque on $\mathcal{B}$ due to the balanced forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ on $y_1, \ldots, y_l, z$, respectively. Then, $\vec{T}_{\mathcal{B}} = \vec{M}_{\mathcal{B}/w}$.

Proof. Since $\vec{r}_{z/w}$ is parallel with $\vec{f}_{\mathcal{B}}$ and $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ are balanced, it follows that the torque $\vec{M}_{\mathcal{B}}$ on $\mathcal{B}$ due to $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, -\vec{f}_{\mathcal{B}}$ on $y_1, \ldots, y_l, z$, respectively, is given by

$$\vec{T}_{\mathcal{B}} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} + \vec{r}_{z/w} \times (-\vec{f}_{\mathcal{B}}) = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/w}. \qquad \square$$

The following result shows that every collection of forces can be replaced by a single force at an arbitrary point and a torque.

Fact 6.5.5. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ be forces on $y_1, \ldots, y_l$, respectively, let z be a point in $\mathcal{B}$, let $n \le l$, define the force

$$\vec{f} \triangleq \sum_{i=1}^{n} \vec{f}_{y_i}, \tag{6.5.9}$$

let $\vec{T}$ be a torque that satisfies

$$\vec{T} = \sum_{i=1}^{n} \vec{r}_{y_i/z} \times \vec{f}_{y_i}, \tag{6.5.10}$$

and let w be a point. Furthermore, replace $\vec{f}_{y_1}, \ldots, \vec{f}_{y_n}$ by $\vec{f}$ on z, and apply $\vec{T}$ to $\mathcal{B}$. Then, the net force on $\mathcal{B}$ and the moment on $\mathcal{B}$ relative to w are unchanged.

Proof. With $\vec{f}_{y_1}, \ldots, \vec{f}_{y_n}$ replaced by $\vec{f}$, the total force on the body $\mathcal{B}$ is given by

$$\vec{f}_{\mathcal{B}} = \vec{f} + \sum_{i=n+1}^{l} \vec{f}_{y_i} = \sum_{i=1}^{l} \vec{f}_{y_i}.$$

Furthermore, with $\vec{f}_{y_1}, \ldots, \vec{f}_{y_n}$ replaced by $\vec{f}$ on z and $\vec{T}$ applied to $\mathcal{B}$, it follows that the total moment on $\mathcal{B}$ relative to w is given by

$$\begin{aligned}
\vec{M}_{\mathcal{B}/w} &= \vec{T} + \vec{r}_{z/w} \times \vec{f} + \sum_{i=n+1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} \\
&= \vec{T} + \sum_{i=1}^{n} \vec{r}_{z/w} \times \vec{f}_{y_i} + \sum_{i=n+1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} \\
&= \overset{\rightharpoonup}{\sum}_{i=1}^{n} \vec{r}_{y_i/z} \times \vec{f}_{y_i} + \sum_{i=1}^{n} (\vec{r}_{z/y_i} + \vec{r}_{y_i/w}) \times \vec{f}_{y_i} + \sum_{i=n+1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} \\
&= \sum_{i=1}^{n} \vec{r}_{y_i/w} \times \vec{f}_{y_i} + \sum_{i=n+1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}.
\end{aligned}$$

□

Note that the right-hand side of (6.6.2) is the moment of $\vec{f}_{y_1}, \ldots, \vec{f}_{y_n}$ relative to z, and the torque $\vec{T}$ is defined to be equal in magnitude and direction to this moment. This result shows that some or all of the forces applied to a body can be replaced by a single force and a torque applied to the body.

As in the case of forces, moments arise from three types of mechanisms. Contact between two bodies gives rise to *reaction moments* and *reaction torques*. Mechanical coupling between two bodies connected by rotary springs, dashpots, or inerters gives rise to *mechanical torques*. Field coupling between two bodies due to central gravity gives rise to *field moments* and *field torques*.

An *internal moment* on $\mathcal{B}$ is a moment that originates from inside $\mathcal{B}$. An *external moment* on $\mathcal{B}$ is a moment that originates from outside of $\mathcal{B}$. Whether a moment is internal or external depends on the extent of the body being considered. Likewise, an *internal torque* on $\mathcal{B}$ is a torque that originates from inside $\mathcal{B}$. An *external torque* on $\mathcal{B}$ is a torque that originates from outside of $\mathcal{B}$. Whether a torque is internal or external depends on the extent of the body being considered.

Moments between two bodies relative to a common reference point are *reciprocal* if they are equal in magnitude and opposite in direction. Torques between two bodies are *reciprocal* if they are equal in magnitude and opposite in direction.

Definition 6.5.6. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies in contact, and let w be a point. Then, $\vec{M}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2/w}$ denotes the moment on $\mathcal{B}_1$ relative to w due to all reaction forces on $\mathcal{B}_1$ due to contact with $\mathcal{B}_2$. If, in addition, $\vec{f}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2} = 0$, then $\vec{T}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2}$ denotes the torque on $\mathcal{B}_1$ due to all reaction forces on $\mathcal{B}_1$ due to contact with $\mathcal{B}_2$.

Definition 6.5.7. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies that interact due to contact, mechanical coupling, or

Notation	Description	Defined in
$\vec{M}_{\mathrm{y}/w}$	Moment due to a force applied to y relative to w	(6.5.1)
$\vec{M}_{\mathcal{B}/w}$	Moment on $\mathcal{B}$ relative to w	(6.5.3)
$\vec{M}_{\mathcal{B}_1/\mathcal{B}_2/w}$	Moment on $\mathcal{B}_1$ due to $\mathcal{B}_2$ relative to w	(6.5.7)
$\vec{M}_{\mathrm{r}/\mathrm{y}/w}$	Moment due to a reaction force applied to y relative to w	
$\vec{M}_{\mathrm{r}/\mathcal{B}_1/w}$	Moment on $\mathcal{B}_1$ due to all reaction forces on $\mathcal{B}_1$ relative to w	
$\vec{M}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2/w}$	Moment on $\mathcal{B}_1$ due to all reaction forces on $\mathcal{B}_1$ due to contact with $\mathcal{B}_2$ relative to w	(6.5.6)
$\vec{M}_{\mathrm{g}/\mathcal{B}_1/w}$	Moment on $\mathcal{B}_1$ due to uniform gravity relative to w	(6.5.19)
$\vec{M}_{\mathrm{cg}/\mathcal{B}_1/\mathcal{B}_2/w}$	Moment on $\mathcal{B}_1$ due to central gravity due to $\mathcal{B}_2$ relative to w	
$\vec{M}_{\mathrm{cg}/\mathcal{B}_1/w}$	Moment on $\mathcal{B}_1$ due to central gravity relative to w	
$\vec{T}_{\mathcal{B}_1/\mathcal{B}_2}$	Torque on $\mathcal{B}_1$ due to $\mathcal{B}_2$	Definition 6.5.7
$\vec{T}_{\mathrm{r}/\mathcal{B}_1/\mathcal{B}_2}$	Torque on $\mathcal{B}_1$ due to all reaction forces on $\mathcal{B}_1$ due to contact with $\mathcal{B}_2$	Definition 6.5.6
$\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}$	Torque on $\mathcal{B}_1$ due to a rotary spring connecting $\mathcal{B}_1$ and $\mathcal{B}_2$	(6.5.11)
$\vec{T}_{\mathrm{d}/\mathcal{B}_1/\mathcal{B}_2}$	Torque on $\mathcal{B}_1$ due to a rotary dashpot connecting $\mathcal{B}_1$ and $\mathcal{B}_2$	(6.5.14)
$\vec{T}_{\mathrm{i}/\mathcal{B}_1/\mathcal{B}_2}$	Torque on $\mathcal{B}_1$ due to a rotary inerter connecting $\mathcal{B}_1$ and $\mathcal{B}_2$	(6.5.16)

Table 6.5.1: Notation for moments and torques.

central gravity, and let w be a point. Then, the *moment* $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2/w}$ *on* $\mathcal{B}_1$ *due to* $\mathcal{B}_2$ *relative to* w is the moment on $\mathcal{B}_1$ relative to w due to all reaction forces, mechanical forces, and central gravity on $\mathcal{B}_1$ due to $\mathcal{B}_2$. If, in addition, $\vec{f}_{\mathcal{B}_1/\mathcal{B}_2} = 0$, then the *torque* $\vec{T}_{\mathcal{B}_1/\mathcal{B}_2}$ *on* $\mathcal{B}_1$ *due to* $\mathcal{B}_2$ is the torque on $\mathcal{B}_1$ due to all reaction forces, mechanical forces, and central gravity on $\mathcal{B}_1$ due to $\mathcal{B}_2$.

Next, consider rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ connected by a pin joint at a point that is fixed in both bodies. Let $\hat{x}_1$ and $\hat{x}_2$ be unit dimensionless vectors that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that are orthogonal to the axis of the pin joint. A rotary spring applies torques to $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel with the axis of the pin joint, where $\kappa > 0$ is the rotary stiffness of the rotary spring and $\theta_0 \in (0, \pi)$ is the angle between $\hat{x}_1$ and $\hat{x}_2$ when the rotary spring is relaxed. Let $\hat{z}$ denote $\hat{\theta}_{\hat{x}_1/\hat{x}_2}$ when the rotary spring is relaxed. Then, the torque $\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}$ on $\mathcal{B}_1$ due to the rotary spring is given by

$$\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2} = -\kappa(\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}\hat{\theta}_{\hat{x}_1/\hat{x}_2} - \theta_0\hat{z}). \tag{6.5.11}$$

Furthermore, since $\hat{\theta}_{\hat{x}_1/\hat{x}_2} = -\hat{\theta}_{\hat{x}_2/\hat{x}_1}$, it follows from (6.5.11) that the torques on $\mathcal{B}_1$ and $\mathcal{B}_2$ are equal and opposite, that is,

$$\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2} = -\vec{T}_{\mathrm{s}/\mathcal{B}_2/\mathcal{B}_1}, \tag{6.5.12}$$

which implies that the mechanical torques due to a rotary spring are reciprocal. Finally, if $\theta_0 = 0$, then (6.5.11) becomes

$$\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2} = -\kappa\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}\hat{\theta}_{\hat{x}_1/\hat{x}_2}. \tag{6.5.13}$$

Note that (6.5.11), (6.5.12), and (6.5.13) hold at each instant of time and thus are valid whether or not the bodies have nonzero relative angular velocity.

Next, in place of a rotary spring, consider a rotary dashpot with rotary damping $\gamma > 0$. Then, the mechanical torque $\vec{T}_{\mathrm{d}/\mathcal{B}_1/\mathcal{B}_2}$ on $\mathcal{B}_1$ due to the rotary dashpot is given by

$$\vec{T}_{\mathrm{d}/\mathcal{B}_1/\mathcal{B}_2} = -\gamma\frac{\mathrm{d}}{\mathrm{d}t}|\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}|\hat{\theta}_{\hat{x}_1/\hat{x}_2}. \tag{6.5.14}$$

Furthermore, it follows from (6.5.14) that

$$\vec{T}_{\mathrm{d}/\mathcal{B}_1/\mathcal{B}_2} = -\vec{T}_{\mathrm{d}/\mathcal{B}_2/\mathcal{B}_1}, \tag{6.5.15}$$

which implies that the mechanical torques on $\mathcal{B}_1$ and $\mathcal{B}_2$ are reciprocal, Note that (6.5.14) and (6.5.15) hold at each instant of time and thus are valid whether or not the bodies have nonzero relative angular acceleration.

Finally, in place of the rotary dashpot, consider an rotary inerter with rotary inertance $\beta > 0$. Then, the mechanical torque $\vec{T}_{\mathrm{i}/\mathcal{B}_1/\mathcal{B}_2}$ on $\mathcal{B}_1$ due to the rotary inerter is given by

$$\vec{T}_{\mathrm{i}/\mathcal{B}_1/\mathcal{B}_2} = -\beta\frac{\mathrm{d}^2}{\mathrm{d}t^2}|\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}|\hat{\theta}_{\hat{x}_1/\hat{x}_2}. \tag{6.5.16}$$

Furthermore, it follows from (6.5.16) that

$$\vec{T}_{\mathrm{i}/\mathcal{B}_1/\mathcal{B}_2} = -\vec{T}_{\mathrm{i}/\mathcal{B}_2/\mathcal{B}_1}, \tag{6.5.17}$$

which implies that the mechanical torques on $\mathcal{B}_1$ and $\mathcal{B}_2$ are reciprocal, Note that (6.5.16) and (6.5.17) hold at each instant of time and thus are valid whether or not the bodies have nonzero relative angular jerk.

The following result extends Newton's third law to reciprocal moments and torques.

Fact 6.5.8. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies that interact due to contact, mechanical coupling, or central gravity, and let w be a point. Then, $\vec{M}_{\mathcal{B}_1/\mathcal{B}_2/w} = -\vec{M}_{\mathcal{B}_2/\mathcal{B}_1/w}$. If, in addition, $\vec{f}_{\mathcal{B}_1/\mathcal{B}_2} = 0$, then $\vec{T}_{\mathcal{B}_1/\mathcal{B}_2} = -\vec{T}_{\mathcal{B}_2/\mathcal{B}_1}$.

Proof. Let l denote the number of pairs of points at which a reciprocal force is on $\mathcal{B}_1$ due to $\mathcal{B}_2$. For all $j = 1,\dots,l$, let $z_{1,j}$ denote the point in $\mathcal{B}_1$ at which a reciprocal force is on $\mathcal{B}_1$ due to $\mathcal{B}_2$, and let $z_{2,j}$ denote the corresponding point in $\mathcal{B}_2$. Furthermore, for all $j = 1,\dots,l$, let $\vec{f}_{z_{1,j}}$ denote the net force on $\mathcal{B}_1$ at $z_{1,j}$, and let $\vec{f}_{z_{2,j}}$ denote the net force on $\mathcal{B}_2$ at $z_{2,j}$. Therefore, for all $j = 1,\dots,l$, $\vec{f}_{z_{2,j}} = -\vec{f}_{z_{1,j}}$. Furthermore, for all $j = 1,\dots,l$, either $\vec{r}_{z_{1,j}/z_{2,j}} = 0$ or $\vec{r}_{z_{1,j}/z_{2,j}}$ and $\vec{f}_{z_{1,j}}$ are parallel. It thus follows that

$$\vec{M}_{\mathcal{B}_1/\mathcal{B}_2/w} + \vec{M}_{\mathcal{B}_2/\mathcal{B}_1/w} = \sum_{j=1}^{l} r_{z_{1,j}/w} \times \vec{f}_{z_{1,j}} + \sum_{j=1}^{l} r_{z_{2,j}/w} \times \vec{f}_{z_{2,j}}$$

$$
\begin{aligned}
&= \sum_{j=1}^{l} (\vec{r}_{z_{1,j}/z_{2,j}} + \vec{r}_{z_{2,j}/w}) \times \vec{f}_{z_{1,j}} + \sum_{j=1}^{l} r_{z_{2,j}/w} \times \vec{f}_{z_{2,j}} \\
&= \sum_{j=1}^{l} \vec{r}_{z_{2,j}/w} \times \vec{f}_{z_{1,j}} + \sum_{j=1}^{l} r_{z_{2,j}/w} \times \vec{f}_{z_{2,j}} \\
&= \sum_{j=1}^{l} \vec{r}_{z_{2,j}/w} \times \vec{f}_{z_{1,j}} - \sum_{j=1}^{l} \vec{r}_{z_{2,j}/w} \times \vec{f}_{z_{1,j}} \\
&= \vec{0}. \qquad \square
\end{aligned}
$$

An example of a reaction torque due to contact is a frictionless pin joint connecting two rigid bodies, where the bodies are subjected to external forces that produce reaction torques that are orthogonal to the pin axis.

We now consider the force and moment on a body due to uniform gravity.

Fact 6.5.9. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let $m_\mathcal{B}$ denote the mass of $\mathcal{B}$. Then, the gravitational force on $\mathcal{B}$ due to uniform gravity is given by

$$\vec{f}_{\mathrm{g}/\mathcal{B}} = m_\mathcal{B}\vec{g}. \tag{6.5.18}$$

Fact 6.5.10. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_\mathcal{B}$ denote the mass of $\mathcal{B}$, and let w be a point. Then, the moment $\vec{M}_{\mathrm{g}/\mathcal{B}/w}$ on $\mathcal{B}$ relative to w due to gravity is given by

$$\vec{M}_{\mathrm{g}/\mathcal{B}/w} = \vec{r}_{\mathrm{c}/w} \times m_\mathcal{B}\vec{g}. \tag{6.5.19}$$

In particular,

$$\vec{M}_{\mathrm{g}/\mathcal{B}/\mathrm{c}} = 0. \tag{6.5.20}$$

Proof. Note that

$$
\begin{aligned}
\vec{M}_{\mathrm{g}/\mathcal{B}/w} &= \sum_{i=1}^{l} (\vec{r}_{y_i/w} \times m_i\vec{g}) = \left(\sum_{i=1}^{l} m_i \vec{r}_{y_i/w}\right) \times \vec{g} \\
&= \left(\frac{1}{m_\mathcal{B}} \sum_{i=1}^{l} m_i \vec{r}_{y_i/w}\right) \times m_\mathcal{B}\vec{g} = \vec{r}_{\mathrm{c}/w} \times m_\mathcal{B}\vec{g}. \qquad \square
\end{aligned}
$$

Note that, although the location of the center of mass c is independent of the choice of w, the physical vector $\vec{r}_{\mathrm{c}/w}$ depends on w, and thus the moment $\vec{M}_{\mathrm{g}/\mathcal{B}/w}$ depends on w.

As an application of Newton's third law, we consider a body $\mathcal{B}$ composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and which are connected by massless links. As shown in Figure 6.5.2, external forces may be applied to the particles or to points along the massless links.

Fact 6.5.11. Let $\mathcal{B}$ be a body composed of particles connected by massless links and subject to external forces. Then, the net force and torque on each link are zero, the net internal force on the body is zero, and, for each link that is not subject to an external force, the internal forces on the

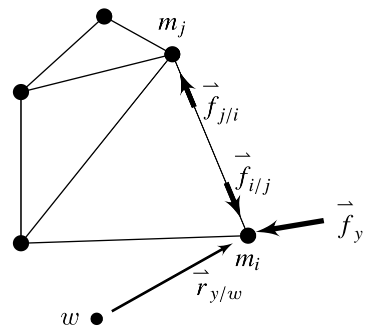


Figure 6.5.2: Body composed of particles and massless links.

ends of each link are equal in magnitude, opposite in direction, and parallel with the massless link.

Proof. Since each link is massless, it follows from free-body analysis that a nonzero net force or torque on a link yields infinite translational or rotary acceleration. Hence, the net force and torque on each link are zero. Furthermore, since Newton's third law applies to each connection between particles and links, it follows that the net internal force on the body is zero. Finally, for each link that is not subjected to an external force, the only forces on the link are reaction forces due to the particles at the ends of the link. To avoid infinite translational or rotary acceleration, it follows that these forces are equal in magnitude, opposite in direction, and parallel with the massless link $\square$

6.6 Statics

Finally, a rigid body $\mathcal{B}$ is in *static equilibrium* if its center of mass has zero inertial acceleration and its angular velocity relative to an inertial frame is constant with respect to an inertial frame. The concept of an inertial frame is defined in Chapter 7. For a static rigid body $\mathcal{B}$, the *laws of statics* are as follows:

i) The net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ is zero.

ii) The torque $\vec{T}_{\mathcal{B}}$ on $\mathcal{B}$ due to all of the forces on $\mathcal{B}$ is zero.

Note that, since the forces on $\mathcal{B}$ are balanced, it follows that the resulting moment $\vec{M}_{\mathcal{B}/w}$ on $\mathcal{B}$ relative to w is a torque and thus is independent of w.

Fact 6.6.1. Let $\mathcal{B}$ be a rigid body with points $y_1, \ldots, y_l$, let $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ be forces on $y_1, \ldots, y_l$, respectively, and assume that $\mathcal{B}$ is in static equilibrium. Let z be a point in $\mathcal{B}$, let $n \le l$, define the force

$$\vec{f} \triangleq \sum_{i=1}^{n} \vec{f}_{y_i} \tag{6.6.1}$$

and let $\vec{T}$ be a torque that satisfies

$$\vec{T} = \sum_{i=1}^{n} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{6.6.2}$$

Furthermore, replace $\vec{f}_{y_1}, \ldots, \vec{f}_{y_n}$ by $\vec{f}$ on z, and apply $\vec{T}$ to $\mathcal{B}$. Then $\mathcal{B}$ is in static equilibrium.

6.7 Reaction Forces, Moments, and Torques in Joints

A *joint* is a physical connection through which a body can apply force to another body. In a joint, reaction forces due to contact between the bodies constrain relative motion between them. In general, a reaction force is applied on the bodies along the direction in which the translation is constrained and a reaction torque is applied on the bodies about the axis around which rotation is constrained.

Consider two bodies $\mathcal{B}_\mathrm{A}$ and $\mathcal{B}_\mathrm{B}$ connected by a joint. Let $\mathrm{F_A}$ be fixed to $\mathcal{B}_\mathrm{A}$ and $\mathrm{F_B}$ be fixed to $\mathcal{B}_\mathrm{B}$. Based on the type of motion constrained, the joint can be of the following types.

i) A *weld joint* connecting two bodies constrains all translational and rotary relative motion between the two bodies. A weld joint thus does not allow any relative motion between the two bodies. Consequently, $\mathcal{B}_\mathrm{A}$ can apply a nonzero reaction force to the body $\mathcal{B}_\mathrm{B}$ along $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$ and a nonzero reaction torque about $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$.

ii) A *1D prismatic joint* connecting two bodies allows translational relative motion along one of the axes and constrains all rotary relative motion between the bodies. Letting $\hat{\imath}_\mathrm{A}$ be the axis of the 1D prismatic joint, $\mathcal{B}_\mathrm{A}$ can apply a nonzero reaction force to $\mathcal{B}_\mathrm{B}$ along $\hat{\jmath}_\mathrm{A}$ and $\hat{k}_\mathrm{A}$ and a nonzero reaction torque about $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$.

iii) A *2D prismatic joint* connecting two bodies constrains translational relative motion along one of the axes and constrains all rotary relative motion between the bodies. Letting $\hat{k}_\mathrm{A}$ be the axis along which translation relative motion is constrained, $\mathcal{B}_\mathrm{A}$ can apply a nonzero reaction force to $\mathcal{B}_\mathrm{B}$ along $\hat{\imath}_\mathrm{A}$ and $\hat{\jmath}_\mathrm{A}$, and a nonzero reaction torque about $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$.

iv) A *3D prismatic joint* connecting two bodies allows all translational relative motion along but constrains all rotary relative motion between the bodies. Consequently, $\mathcal{B}_\mathrm{A}$ cannot apply any reaction force to $\mathcal{B}_\mathrm{B}$, but can apply a nonzero reaction torque about $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$.

v) A *pin joint* connecting two bodies constrains all translational relative motion and rotary relative motion about axes that are perpendicular to the axis of the pin joint. A pin joint thus allows rotary relative motion between the two bodies about an axis that is fixed in both bodies. Letting $\hat{k}_\mathrm{A}$ be the axis of the pin joint, $\mathcal{B}_\mathrm{A}$ can apply a nonzero reaction force to $\mathcal{B}_\mathrm{B}$ along $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$ and a nonzero reaction torque about $\hat{\imath}_\mathrm{A}$ and $\hat{\jmath}_\mathrm{A}$.

vi) A *universal joint* connecting two bodies constrains all translational relative motion and rotary relative motion about one of the axes fixed in the bodies. A universal joint thus allows rotary relative motion between the two bodies about two independent axes. Figure 6.7.3 shows a universal joint between $\mathcal{B}_\mathrm{A}$ and $\mathcal{B}_\mathrm{B}$. Note that $\mathcal{B}_\mathrm{B}$ cannot rotate relative to $\mathcal{B}_\mathrm{A}$ about $\hat{k}_\mathrm{B}$. Consequently, a nonzero reaction force can be on applied on $\mathcal{B}_\mathrm{B}$ along $\hat{\imath}_\mathrm{A}$, $\hat{\jmath}_\mathrm{A}$, and $\hat{k}_\mathrm{A}$ and a nonzero reaction torque can be on applied $\mathcal{B}_\mathrm{B}$ about $\hat{k}_\mathrm{B}$. Note that $\vec{M}_{\mathrm{r}/\mathcal{B}_\mathrm{A}/\mathcal{B}_\mathrm{B}} \neq -\vec{M}_{\mathrm{r}/\mathcal{B}_\mathrm{B}/\mathcal{B}_\mathrm{A}}$ due to the fact that the reaction torque on $\mathcal{B}_\mathrm{A}$ is applied by the connector $\mathcal{C}$ and not by the body $\mathcal{B}_\mathrm{B}$. In fact, the reaction torque on $\mathcal{B}_\mathrm{A}$ is applied by $\mathcal{C}$ about $\hat{k}_\mathrm{A}$ and reaction torque on $\mathcal{B}_\mathrm{B}$ is applied by $\mathcal{C}$ about $\hat{k}_\mathrm{B}$.

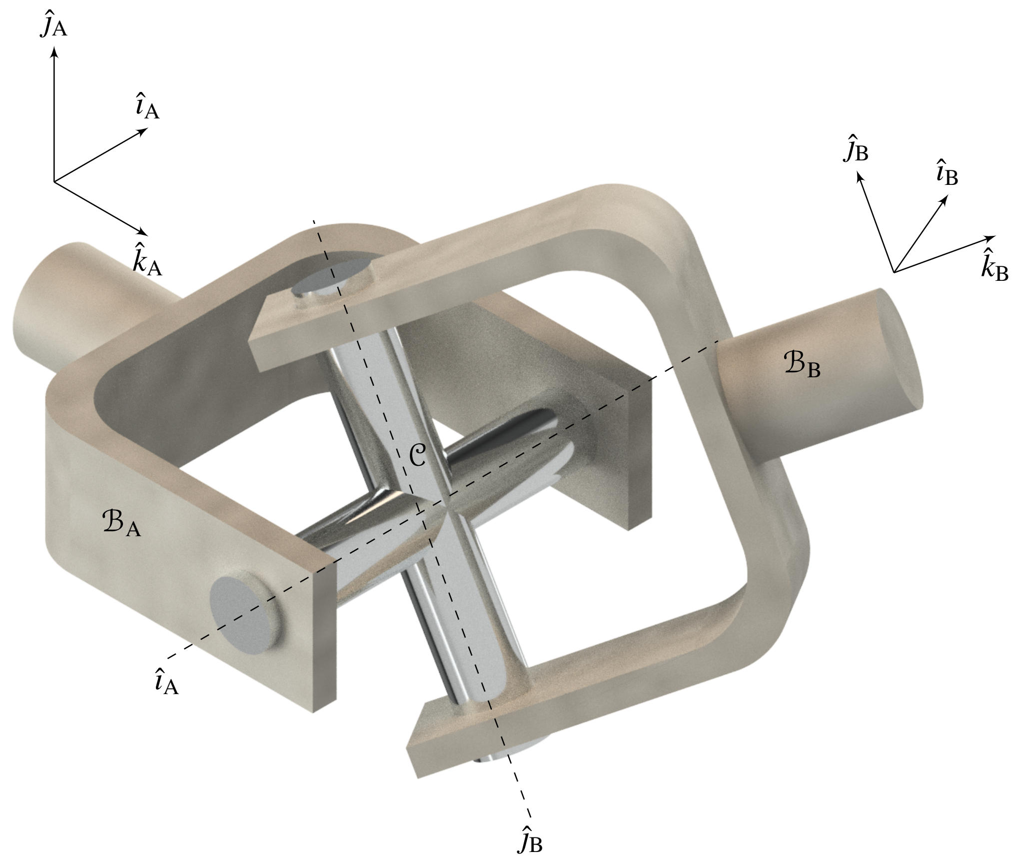


Figure 6.7.3: A universal joint.

vii) A *ball joint* connecting two bodies constrains all translational relative motion and allows all rotary relative motion. Consequently, $\mathcal{B}_{\rm A}$ can apply a nonzero reaction force to $\mathcal{B}_{\rm B}$ along $\hat{\imath}_{\rm A}$, $\hat{\jmath}_{\rm A}$, and $\hat{k}_{\rm A}$, but no reaction torque about $\hat{\imath}_{\rm A}$, $\hat{\jmath}_{\rm A}$, and $\hat{k}_{\rm A}$.

Finally, combination joints with more degrees of freedom, such as a slotted pin that allows rotation as well as translation about an axis, can be constructed by combining two or more joints described above. Table 6.7.1 shows the reaction forces and reaction torques that can be applied by various types of joints.

The next two examples use Fact 6.6.1 to model the reaction forces in a joint under static equilibrium. The first example replaces the joint with reaction forces at two points, and the second example replaces the joint with a reaction force and a reaction torque.

Example 6.7.1. Consider a link $\mathcal{L}$ shown in Figure 6.7.4. The link is in contact with the pinned supports at w_1 and w_2. The distance from z to y is l, and the distance from w_1 to w_2 is l_w. The external force $\vec{f}_y = f\hat{\imath}_{\rm A}$. Assuming that the link $\mathcal{L}$ is in static equilibrium, find the reaction forces applied to $\mathcal{L}$ by the supports at w_1 and w_2.

Joint	Translation constrained along	Reaction force on $\mathcal{B}_\mathrm{B}$	Rotation constrained about	Reaction torque on $\mathcal{B}_\mathrm{B}$
Weld	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$f_1\hat{\imath}_\mathrm{B} + f_2\hat{\jmath}_\mathrm{B} + f_3\hat{k}_\mathrm{B}$	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$T_1\hat{\imath}_\mathrm{B} + T_2\hat{\jmath}_\mathrm{B} + T_3\hat{k}_\mathrm{B}$
1D Prismatic	$\hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$f_2\hat{\jmath}_\mathrm{B} + f_3\hat{k}_\mathrm{B}$	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$T_1\hat{\imath}_\mathrm{B} + T_2\hat{\jmath}_\mathrm{B} + T_3\hat{k}_\mathrm{B}$
2D Prismatic	$\hat{k}_\mathrm{B}$	$f_3\hat{k}_\mathrm{B}$	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$T_1\hat{\imath}_\mathrm{B} + T_2\hat{\jmath}_\mathrm{B} + T_3\hat{k}_\mathrm{B}$
3D Prismatic	-	0	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$T_1\hat{\imath}_\mathrm{B} + T_2\hat{\jmath}_\mathrm{B} + T_3\hat{k}_\mathrm{B}$
Pin	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$f_1\hat{\imath}_\mathrm{B} + f_2\hat{\jmath}_\mathrm{B} + f_3\hat{k}_\mathrm{B}$	$\hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$T_2\hat{\jmath}_\mathrm{B} + T_3\hat{k}_\mathrm{B}$
Universal	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$f_1\hat{\imath}_\mathrm{B} + f_2\hat{\jmath}_\mathrm{B} + f_3\hat{k}_\mathrm{B}$	$\hat{k}_\mathrm{B}$	$T_3\hat{k}_\mathrm{B}$
Ball	$\hat{\imath}_\mathrm{B}, \hat{\jmath}_\mathrm{B}, \hat{k}_\mathrm{B}$	$f_1\hat{\imath}_\mathrm{B} + f_2\hat{\jmath}_\mathrm{B} + f_3\hat{k}_\mathrm{B}$	-	0

Table 6.7.1: Reaction forces and torques in various types of joints connecting bodies $\mathcal{B}_\mathrm{A}$ and $\mathcal{B}_\mathrm{B}$.

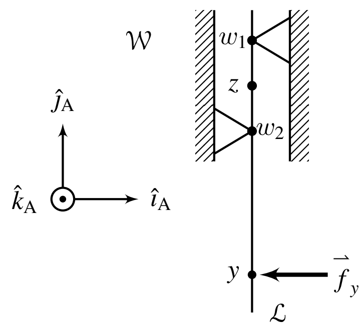


Figure 6.7.4: A link $\mathcal{L}$ supported at two points.

It follows from the free-body diagram of $\mathcal{L}$ in Figure 6.7.5 that the total force on $\mathcal{L}$ is given by

$$\vec{f}_{\mathcal{L}} = \vec{f}_y + \vec{f}_{\mathrm{r}/\mathcal{L}/w_1} + \vec{f}_{\mathrm{r}/\mathcal{L}/w_2}, \tag{6.7.1}$$

and the total moment relative to w_1 is given by

$$\begin{aligned}\vec{M}_{\mathcal{L}/w_1} &= \vec{r}_{y/w_1} \times \vec{f}_y + \vec{r}_{w_1/w_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}/w_1} + \vec{r}_{w_2/w_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}/w_2} \\ &= \vec{r}_{y/w_1} \times \vec{f}_y + \vec{r}_{w_2/w_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}/w_2}.\end{aligned} \tag{6.7.2}$$

Since the link is in static equilibrium, it follows from the laws of statics that

$$\vec{f}_{\mathrm{r}/\mathcal{L}/w_1} + \vec{f}_{\mathrm{r}/\mathcal{L}/w_2} = -\vec{f}_y. \tag{6.7.3}$$

and

$$\vec{r}_{w_2/w_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}/w_2} = -\vec{r}_{y/w_1} \times \vec{f}_y. \tag{6.7.4}$$

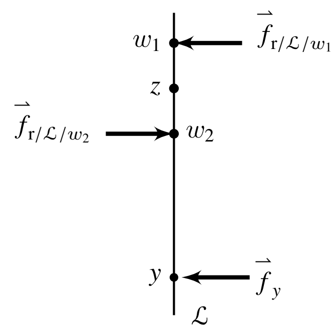


Figure 6.7.5: Free-body diagram of the link $\mathcal{L}$. The supports are replaced by the reaction forces at w_1 and w_2.

Letting $\vec{f}_y = f_y\hat{\imath}_{\mathrm{A}}$, $\vec{f}_{\mathrm{r}/\mathcal{L}/w_1} = f_{r1}\hat{\imath}_{\mathrm{A}}$, and $\vec{f}_{\mathrm{r}/\mathcal{L}/w_2} = f_{r2}\hat{\imath}_{\mathrm{A}}$, it follows from (6.7.3) and (6.7.4) that

$$f_{r1} + f_{r2} = -f_y, \tag{6.7.5}$$

$$2l_w f_{r2} = (l + l_w)f_y. \tag{6.7.6}$$

Solving (6.7.5) and (6.7.6) yields

$$f_{r1} = -\frac{f_y}{2} + \frac{f_y l}{2l_w}, \tag{6.7.7}$$

$$f_{r2} = -\frac{f_y}{2} - \frac{f_y l}{2l_w}. \tag{6.7.8}$$

Note that reaction forces $\vec{f}_{\mathrm{r}/\mathcal{L}/w_1}$ and $\vec{f}_{\mathrm{r}/\mathcal{L}/w_2}$ can be replaced by a force $\vec{f}$ and a torque $\vec{T}$ using Fact 6.6.1. It follows from fact 6.6.1 that

$$\vec{f} = \vec{f}_{\mathrm{r}/\mathcal{L}/w_1} + \vec{f}_{\mathrm{r}/\mathcal{L}/w_2} = -f_y\hat{\imath}_{\mathrm{A}} = -\vec{f}_y, \tag{6.7.9}$$

and

$$\begin{aligned}
\vec{T} &= \vec{r}_{w_1/z} \times \vec{f}_{\mathrm{r}/\mathcal{L}/w_1} + \vec{r}_{w_2/z} \times \vec{f}_{\mathrm{r}/\mathcal{L}/w_2} \\
&= l_w\hat{\jmath}_{\mathrm{A}} \times \left(-\frac{f_y}{2} + \frac{f_y l}{2l_w}\right)\hat{\imath}_{\mathrm{A}} - l_w\hat{\jmath}_{\mathrm{A}} \times \left(-\frac{f_y}{2} - \frac{f_y l}{2l_w}\right)\hat{\imath}_{\mathrm{A}} \\
&= -f_y l\hat{k}_{\mathrm{A}} \\
&= l\hat{\jmath}_{\mathrm{A}} \times f_y\hat{\imath}_{\mathrm{A}} \\
&= \vec{r}_{z/y} \times \vec{f}_y \\
&= -\vec{M}_{\mathrm{y}/z}
\end{aligned} \tag{6.7.10}$$

The force $\vec{f}$ and the torque $\vec{T}$ can thus be used to model the force interactions at a joint.

Example 6.7.2. Consider a link $\mathcal{L}$ connected to the ceiling $\mathcal{C}$ with a weld joint at w as shown in Figure 6.7.6. The external force $\vec{f}_y = f\hat{\imath}_{\mathrm{A}}$. Assuming that the link $\mathcal{L}$ is in static equilibrium, find the reaction force and the reaction torque applied to the link $\mathcal{L}$.

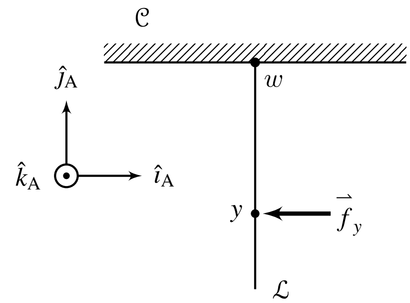


Figure 6.7.6: An abstract joint between the link $\mathcal{L}$ and the ceiling $\mathcal{C}$ that applies a reaction force and a reaction torque.

It follows from the free-body diagram of $\mathcal{L}$ in Figure 6.7.7 that the total force on $\mathcal{L}$ is given by

$$\overset{\rightharpoonup}{f}_{\mathcal{L}} = \overset{\rightharpoonup}{f}_y + \overset{\rightharpoonup}{f}_{{\rm r}/\mathcal{L}/\mathcal{C}}, \tag{6.7.11}$$

and letting z be a point, the total moment relative to z is given by

$$\begin{aligned}\overset{\rightharpoonup}{M}_{\mathcal{L}/z} &= \overset{\rightharpoonup}{r}_{y/z} \times \overset{\rightharpoonup}{f}_y + \overset{\rightharpoonup}{r}_{w/z} \times \overset{\rightharpoonup}{f}_{{\rm r}/\mathcal{L}/\mathcal{C}} + \overset{\rightharpoonup}{M}_{{\rm r}/\mathcal{L}/\mathcal{C}/z}\\ &= \overset{\rightharpoonup}{r}_{y/z} \times \overset{\rightharpoonup}{f}_y - \overset{\rightharpoonup}{r}_{w/z} \times \overset{\rightharpoonup}{f}_y + \overset{\rightharpoonup}{M}_{{\rm r}/\mathcal{L}/\mathcal{C}/z}\\ &= \overset{\rightharpoonup}{r}_{y/w} \times \overset{\rightharpoonup}{f}_y + \overset{\rightharpoonup}{M}_{{\rm r}/\mathcal{L}/\mathcal{C}/z}.\end{aligned} \tag{6.7.12}$$

Since the link is in static equilibrium, it follows from the laws of statics that $\overset{\rightharpoonup}{f}_{\mathcal{L}} = 0$, and thus

$$\overset{\rightharpoonup}{f}_{{\rm r}/\mathcal{L}/\mathcal{C}} = -\overset{\rightharpoonup}{f}_y, \tag{6.7.13}$$

and

$$\overset{\rightharpoonup}{M}_{{\rm r}/\mathcal{L}/\mathcal{C}/z} = -\overset{\rightharpoonup}{r}_{y/w} \times \overset{\rightharpoonup}{f}_y. \tag{6.7.14}$$

Since the reaction moment relative to z given by the expression above does not depend on z, it is a reaction torque.

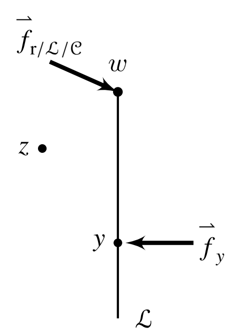


Figure 6.7.7: Free-body diagram of the link $\mathcal{L}$. The effect of ceiling is replaced by the reaction force $\overset{\rightharpoonup}{f}_{{\rm r}/\mathcal{L}/\mathcal{C}}$ applied at w.

6.8 Center of Gravity and Central Gravity†

In this section we do not assume that gravity is uniform, but rather consider central gravity. In this case, we define the center of gravity, which may be different from the center of mass. We also consider conditions under which the center of gravity coincides with the center of mass.

Let x and y be particles whose masses are M and m, respectively. Then, the force $\vec{f}_{\mathrm{g}/y/x}$ is the *central gravitational force* on y due to x, and the *weight* of y relative to x is given by

$$\mathrm{w}_{y/x} \triangleq |\vec{f}_{\mathrm{g}/y/x}| = \frac{GMm}{|\vec{r}_{y/x}|^2}. \tag{6.8.1}$$

Now, let $\mathcal{B}_1$ be a body consisting of particles $x_1, \ldots, x_m$ whose masses are $M_1, \ldots, M_j$, respectively. Then, the central gravitational force on y due to $\mathcal{B}_1$ is defined by

$$\vec{f}_{\mathrm{g}/y/\mathcal{B}_1} \triangleq \sum_{j=1}^{m} \vec{f}_{\mathrm{g}/y/x_j} = \sum_{j=1}^{m} \frac{GM_j m}{|\vec{r}_{y/x_j}|^2}\hat{r}_{y/x_j}, \tag{6.8.2}$$

and the *weight* of y relative to $\mathcal{B}_1$ is defined by

$$\mathrm{w}_{y/\mathcal{B}_1} \triangleq \sum_{j=1}^{m} \mathrm{w}_{y/x_j} = \sum_{j=1}^{m} |\vec{f}_{\mathrm{g}/y/x_j}| = \sum_{j=1}^{m} \frac{GM_j m}{|\vec{r}_{y/x_j}|^2}. \tag{6.8.3}$$

Finally, let $\mathcal{B}_2$ be a body consisting of particles $y_1, \ldots, y_l$. Then, the central gravitational force on $\mathcal{B}_2$ due to $\mathcal{B}_1$ is defined by

$$\vec{f}_{\mathrm{g}/\mathcal{B}_2/\mathcal{B}_1} \triangleq \sum_{i=1}^{l} \vec{f}_{\mathrm{g}/y_i/\mathcal{B}_1} = \sum_{i=1}^{l}\sum_{j=1}^{m} \vec{f}_{\mathrm{g}/y_i/x_j} = \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{GM_j m_i}{|\vec{r}_{y_i/x_j}|^2}\hat{r}_{y_i/x_j}, \tag{6.8.4}$$

and the *weight* of $\mathcal{B}_2$ relative to $\mathcal{B}_1$ is defined by

$$\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1} \triangleq \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} = \sum_{i=1}^{l}\sum_{j=1}^{m} \mathrm{w}_{y_i/x_j} = \sum_{i=1}^{l}\sum_{j=1}^{m} |\vec{f}_{\mathrm{g}/y_i/x_j}| = \sum_{i=1}^{l}\sum_{j=1}^{m} \frac{GM_j m_i}{|\vec{r}_{y_i/x_j}|^2}. \tag{6.8.5}$$

Fact 6.8.1. Let $\mathcal{B}_1$ be a body, let $\mathcal{B}_2$ be a body consisting of particles $y_1, \ldots, y_l$, let w and w' be points, and define the points g and g′ by

$$\vec{r}_{\mathrm{g}/w} \triangleq \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w}, \tag{6.8.6}$$

$$\vec{r}_{\mathrm{g}'/w'} \triangleq \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w'}. \tag{6.8.7}$$

Then, g and g′ are colocated.

Proof. Note that

$$\begin{aligned}\vec{r}_{\mathrm{g}'/\mathrm{g}} &= \vec{r}_{\mathrm{g}'/w'} + \vec{r}_{w'/w} + \vec{r}_{w/\mathrm{g}} = \vec{r}_{\mathrm{g}'/w'} - \vec{r}_{\mathrm{g}/w} + \vec{r}_{w'/w} \\ &= \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w'} - \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w} + \vec{r}_{w'/w}\end{aligned}$$

$$= \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1}(\vec{r}_{y_i/w'} - \vec{r}_{y_i/w}) + \vec{r}_{w'/w}$$

$$= \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1}(\vec{r}_{y_i/w'} + \vec{r}_{w/y_i}) + \vec{r}_{w'/w}$$

$$= \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{w/w'} + \vec{r}_{w'/w}$$

$$= \left(\frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \right) \vec{r}_{w/w'} + \vec{r}_{w'/w} = \vec{r}_{w/w'} + \vec{r}_{w'/w} = 0. \qquad \square$$

Fact 6.8.1 shows that the point g is uniquely defined irrespective of the reference point w.

The following definition is analogous to Definition 6.1.2.

Definition 6.8.2. Let $\mathcal{B}_1$ be a body, let $\mathcal{B}_2$ be a body composed of particles $y_1, \ldots, y_l$, and let w be a point. Then, the *center of gravity* $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}$ of $\mathcal{B}_2$ relative to $\mathcal{B}_1$ is defined by

$$\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} \triangleq \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w}. \tag{6.8.8}$$

The following result is analogous to Fact 6.1.3.

Fact 6.8.3. Let $\mathcal{B}_1$ be a body, let $\mathcal{B}_2$ be a body composed of particles $y_1, \ldots, y_l$, and let w be a point. Then, $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}$ satisfies

$$\sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} = 0. \tag{6.8.9}$$

Proof. It follows from (6.8.8) that

$$\begin{aligned}
\frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} &= \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1}(\vec{r}_{y_i/w} + \vec{r}_{w/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}}) \\
&= \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{y_i/w} + \frac{1}{\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1}} \sum_{i=1}^{l} \mathrm{w}_{y_i/\mathcal{B}_1} \vec{r}_{w/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} \\
&= \vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} + \vec{r}_{w/\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}} = \vec{r}_{w/w} = 0. \qquad \square
\end{aligned}$$

The following result gives conditions under which g is independent of G and M and under which $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}$ is colocated with the center of mass of $\mathcal{B}_2$.

Fact 6.8.4. Let $\mathcal{B}_1$ be a body with particles $x_1, \ldots, x_m$ whose masses are all equal to M, let $\mathcal{B}_2$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let w be a point, and let c_2 denote the center of mass of $\mathcal{B}_2$. Then,

$$\mathrm{w}_{\mathcal{B}_2/\mathcal{B}_1} = GM \sum_{i=1}^{l} \sum_{j=1}^{m} \frac{m_i}{|\vec{r}_{y_i/x_j}|^2}, \tag{6.8.10}$$

$$\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} = \frac{1}{\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{|\vec{r}_{y_i/x_j}|^2}}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{|\vec{r}_{y_i/x_j}|^2}\vec{r}_{y_i/w}. \tag{6.8.11}$$

If, in addition, $|\vec{r}_{y_i/x_j}|$ is independent of i and j, then $\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1} = \mathrm{c}_2$.

Proof. To prove the second statement, define $\alpha \triangleq |\vec{r}_{y_i/x_j}|^2$. Then,

$$\begin{aligned}
\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/w} &= \frac{1}{\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{\alpha}}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{m_i}{\alpha}\vec{r}_{y_i/w}\\
&= \frac{1}{\sum_{i=1}^{l}\frac{mm_i}{\alpha}}\sum_{i=1}^{l}\frac{mm_i}{\alpha}\vec{r}_{y_i/w} = \frac{1}{\sum_{i=1}^{l}m_i}\sum_{i=1}^{l}m_i\vec{r}_{y_i/w}\\
&= \frac{1}{m_{\mathcal{B}_2}}\sum_{i=1}^{l}m_i\vec{r}_{y_i/w} = \vec{r}_{\mathrm{c}_2/w}.
\end{aligned}$$

□

The following result shows that, if the distance between two bodies is large, then the center of gravity of each body is approximately colocated with its center of mass.

Fact 6.8.5. Let $\mathcal{B}_1$ be a body with particles $x_1,\ldots,x_m$ whose masses are $M_1,\ldots,M_m$, respectively, let $\mathcal{B}_2$ be a body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let c_1 and c_2 denote the centers of mass of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and define $\gamma \triangleq |\vec{r}_{\mathrm{c}_2/\mathrm{c}_1}|$. Then,

$$\lim_{\gamma\to\infty}\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/\mathrm{c}_2} = 0. \tag{6.8.12}$$

Proof. For convenience, define $\alpha_{ij} \triangleq |\vec{r}_{y_i/x_j}|^2$ and note that, for all $i_1, i_2 = 1,\ldots,l$ and all $j_1, j_2 = 1,\ldots,m$, $\frac{\alpha_{i_1j_1}}{\alpha_{i_2j_2}} \to 1$ as $\gamma\to\infty$. Then,

$$\begin{aligned}
\lim_{\gamma\to\infty}\vec{r}_{\mathrm{g}_{\mathcal{B}_2/\mathcal{B}_1}/\mathrm{c}_2} &= \lim_{\gamma\to\infty}\frac{1}{\sum_{i'=1}^{l}\sum_{j'=1}^{m}\frac{M_{j'}m_{i'}}{\alpha_{i'j'}}}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{\alpha_{ij}}\vec{r}_{y_i/\mathrm{c}_2}\\
&= \lim_{\gamma\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{\alpha_{ij}\sum_{i'=1}^{l}\sum_{j'=1}^{m}\frac{M_{j'}m_{i'}}{\alpha_{i'j'}}}\vec{r}_{y_i/\mathrm{c}_2}\\
&= \lim_{\gamma\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{\sum_{i'=1}^{l}\sum_{j'=1}^{m}\frac{\alpha_{ij}M_{j'}m_{i'}}{\alpha_{i'j'}}}\vec{r}_{y_i/\mathrm{c}_2}\\
&= \sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{\sum_{i'=1}^{l}\sum_{j'=1}^{m}M_{j'}m_{i'}}\vec{r}_{y_i/\mathrm{c}_2}\\
&= \sum_{i=1}^{l}\sum_{j=1}^{m}\frac{M_jm_i}{m_{\mathcal{B}_1}m_{\mathcal{B}_2}}\vec{r}_{y_i/\mathrm{c}_2}\\
&= \sum_{j=1}^{m}\frac{M_j}{m_{\mathcal{B}_1}}\sum_{i=1}^{l}\frac{m_i}{m_{\mathcal{B}_2}}\vec{r}_{y_i/\mathrm{c}_2}
\end{aligned}$$

$$= \sum_{i=1}^{l} \frac{m_i}{m_{\mathcal{B}_2}} \vec{r}_{y_i/\mathrm{c}_2} = \vec{r}_{\mathrm{c}_2/\mathrm{c}_2} = 0. \qquad \square$$

6.9 Magnetic Forces and Torques†

Let $\vec{m}_y$ and $\vec{m}_z$ be magnetic dipole moments located at the distinct points y and z, respectively. Let $\vec{r}_{z/y}$ denote the position of z relative to y. At the point z, the magnetic field $\vec{B}_y$ due to the magnetic dipole $\vec{m}_y$ is given by [6, p. 186]

$$\vec{B}_y(\vec{r}_{z/y}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}_{z/y}\cdot\vec{m}_y}{|\vec{r}_{z/y}|^5}\vec{r}_{z/y} - \frac{1}{|\vec{r}_{z/y}|^3}\vec{m}_y\right), \tag{6.9.1}$$

where μ_0 is the permeability of free space. For convenience, we adopt Gaussian units such that $\mu_0 = 4\pi$.

According to Maxwell's equations, the field $\vec{B}_y$ due to the dipole $\vec{m}_y$ is divergence free. To show this for (6.9.1), note that the Jacobian of $\vec{B}_y$ at $\vec{r}_{z/y}$ is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{B}_y(\vec{r})\bigg|_{\vec{r}=\vec{r}_{y/z}} &= \frac{\mathrm{d}}{\mathrm{d}\vec{r}}\left(\frac{3}{|\vec{r}|^5}\vec{m}_y'\vec{r}\,\vec{r} - \frac{1}{|\vec{r}|^3}\vec{m}_y\right)\bigg|_{\vec{r}=\vec{r}_{y/z}}\\
&= 3\vec{m}_y'\vec{r}_{y/z}\vec{r}_{y/z}\left(\vec{\partial}_{\vec{r}}\frac{1}{|\vec{r}|^5}\right)\bigg|_{\vec{r}=\vec{r}_{y/z}} + \frac{3}{|\vec{r}_{y/z}|^5}\vec{r}_{y/z}\,(\vec{\partial}_{\vec{r}}\vec{m}_y'\vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}}\\
&\quad + \frac{3}{|\vec{r}_{y/z}|^5}\vec{m}_y'\vec{r}_{y/z}\left(\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{r}\right)\bigg|_{\vec{r}=\vec{r}_{y/z}} - \vec{m}_y\left(\vec{\partial}_{\vec{r}}\frac{1}{|\vec{r}|^3}\right)\bigg|_{\vec{r}=\vec{r}_{y/z}}\\
&= 3\vec{m}_y'\vec{r}_{z/y}\vec{r}_{z/y}\left(\frac{-5}{|\vec{r}_{z/y}|^7}\vec{r}_{z/y}'\right) + \frac{3}{|\vec{r}_{z/y}|^5}\vec{r}_{z/y}\vec{m}_y'\\
&\quad + \frac{3}{|\vec{r}_{z/y}|^5}\vec{m}_y'\vec{r}_{z/y}\vec{I} - \vec{m}_y\left(\frac{-3}{|\vec{r}_{z/y}|^5}\vec{r}_{z/y}'\right)\\
&= \frac{3}{|\vec{r}_{z/y}|^5}\left[\vec{m}_y\vec{r}_{z/y}' + \vec{r}_{z/y}\vec{m}_y' + \vec{m}_y'\vec{r}_{z/y}\left(\vec{I} - \frac{5}{|\vec{r}_{z/y}|^2}\vec{r}_{z/y}\vec{r}_{z/y}'\right)\right].
\end{aligned} \tag{6.9.2}$$

Consequently, the divergence of $\vec{B}_y$ is given by

$$\vec{\nabla}_{\vec{r}}\cdot\vec{B}_y(\vec{r}) = \operatorname{tr}\vec{\partial}_{\vec{r}}\vec{B}_y(\vec{r}) = \operatorname{tr}\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{B}_y(\vec{r}) = 0. \tag{6.9.3}$$

Since $\vec{B}_y$ is divergence free, it can be written as the curl of a vector potential, namely,

$$\vec{A}_y(\vec{r}) = \frac{1}{|\vec{r}|^3}\vec{m}_y\times\vec{r}. \tag{6.9.4}$$

To confirm this, note first that

$$\vec{\nabla}_{\vec{r}} \cdot \left(\frac{1}{|\vec{r}|^3}\vec{r}\right) = 0. \tag{6.9.5}$$

It thus follows that

$$\begin{aligned}
\vec{\nabla}_{\vec{r}} \times \vec{A}_y(\vec{r}) &= \vec{\nabla}_{\vec{r}} \times \left(\vec{m}_y \times \frac{1}{|\vec{r}|^3}\vec{r}\right)\\
&= \vec{\nabla}_{\vec{r}} \cdot \left(\frac{1}{|\vec{r}|^3}\vec{r}\right)\vec{m}_y - (\vec{m}_y{}'\vec{\nabla}_{\vec{r}})\frac{1}{|\vec{r}|^3}\vec{r}\\
&= -\frac{\mathrm{d}}{\mathrm{d}\vec{r}}\left(\frac{1}{|\vec{r}|^3}\vec{r}\right)\vec{m}_y\\
&= \left(\frac{3}{|\vec{r}|^5}\vec{r}\,\vec{r}\,' - \frac{1}{|\vec{r}|^3}\vec{I}\right)\vec{m}_y\\
&= \frac{3\vec{r}\,'\vec{m}_y}{|\vec{r}|^5}\vec{r} - \frac{1}{|\vec{r}|^3}\vec{m}_y\\
&= \vec{B}_y(\vec{r}).
\end{aligned} \tag{6.9.6}$$

Note that (6.9.1) can be rewritten as

$$\vec{B}_y(\vec{r}_{z/y}) = \vec{J}(\vec{r}_{z/y})\vec{m}_y, \tag{6.9.7}$$

where the second-order tensor $\vec{J}(\vec{r}_{z/y})$ is defined by

$$\vec{J}(\vec{r}_{z/y}) \triangleq \frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}_{z/y}\hat{r}'_{z/y} - \vec{I}). \tag{6.9.8}$$

Note that $\operatorname{tr}\vec{J}(\vec{r}_{z/y}) = 0$, which reflects the divergence-free condition. Note that $\vec{J}(\vec{r}_{y/z}) = \vec{J}(\vec{r}_{z/y}) = \vec{J}'(\vec{r}_{y/z})$.

The force $\vec{f}_{\mathrm{m}/z/y}$ on the magnetic dipole $\vec{m}_z$ due to the magnetic dipole $\vec{m}_y$ is given by [6, p. 189]

$$\vec{f}_{\mathrm{m}/z/y} = \left.\vec{\nabla}_{\vec{r}}(\vec{m}_z{}'\vec{B}_y(\vec{r}))\right|_{\vec{r}=\vec{r}_{y/z}}. \tag{6.9.9}$$

Likewise, the torque $\vec{T}_{\mathrm{m}/z/y}$ on the magnetic dipole $\vec{m}_z$ due to the magnetic dipole $\vec{m}_y$ is given by [6, p. 190]

$$\vec{T}_{\mathrm{m}/z/y} = \vec{m}_z \times \vec{B}_y(\vec{r}_{z/y}). \tag{6.9.10}$$

6.9.1 Newton's Third Law for Magnetic Forces

The following result provides an expression for $\vec{f}_{\mathrm{m}/z/y}$ and states Newton's third law due to a pair of magnetic dipoles.

Fact 6.9.1. $\vec{f}_{\mathrm{m}/z/y}$ is given by

$$\vec{f}_{\mathrm{m}/z/y} = \frac{3}{|\vec{r}_{z/y}|^4}(\vec{m}_y'\hat{r}_{z/y}\vec{m}_z + \vec{m}_z'\hat{r}_{z/y}\vec{m}_y + \vec{m}_z'\vec{m}_y\hat{r}_{z/y} - 5\vec{m}_y'\hat{r}_{z/y}\vec{m}_z'\hat{r}_{z/y}\hat{r}_{z/y}). \tag{6.9.11}$$

Consequently, $\vec{f}_{\mathrm{m}/z/y}$ and $\vec{f}_{\mathrm{m}/y/z}$ satisfy

$$\vec{f}_{\mathrm{m}/z/y} = -\vec{f}_{\mathrm{m}/y/z}. \tag{6.9.12}$$

Proof. Using (2.22.11) it follows that, for all vectors $\vec{w}$ and $\vec{v}$,

$$\begin{aligned}
\vec{\partial}_{\vec{r}}(\vec{w}'\vec{J}(\vec{r})\vec{v})\Big|_{\vec{r}=\vec{r}_{y/z}} &= \vec{\partial}_{\vec{r}}\left(\frac{3}{|\vec{r}|^5}\vec{w}'\vec{r}\vec{v}'\vec{r} - \frac{1}{|\vec{r}|^3}\vec{w}'\vec{v}\right)\Bigg|_{\vec{r}=\vec{r}_{y/z}}\\
&= -\frac{15}{|\vec{r}_{z/y}|^7}\vec{w}'\vec{r}_{z/y}\vec{v}'\vec{r}_{z/y}\vec{r}_{z/y}' + \frac{3}{|\vec{r}_{z/y}|^5}\vec{v}'\vec{r}_{z/y}\vec{w}'\\
&\quad + \frac{3}{|\vec{r}_{z/y}|^5}\vec{w}'\vec{r}_{z/y}\vec{v}' + \frac{3}{|\vec{r}_{z/y}|^5}\vec{w}'\vec{v}\vec{r}_{z/y}'\\
&= \frac{3}{|\vec{r}_{z/y}|^4}(-5\vec{w}'\hat{r}_{z/y}\vec{v}'\hat{r}_{z/y}\hat{r}_{z/y}' + \vec{v}'\hat{r}_{z/y}\vec{w}' + \vec{w}'\hat{r}_{z/y}\vec{v}' + \vec{w}'\vec{v}\hat{r}_{z/y}').
\end{aligned} \tag{6.9.13}$$

Finally, setting $\vec{v} = \vec{m}_y$ and $\vec{w} = \vec{m}_z$ yields

$$\begin{aligned}
\vec{f}_{\mathrm{m}/z/y} &= \vec{\nabla}_{\vec{r}}(\vec{m}_z'\vec{J}(\vec{r})\vec{m}_y))\Big|_{\vec{r}=\vec{r}_{y/z}}\\
&= \frac{3}{|\vec{r}_{z/y}|^4}(\vec{m}_y'\hat{r}_{z/y}\vec{m}_z + \vec{m}_z'\hat{r}_{z/y}\vec{m}_y + \vec{m}_z'\vec{m}_y\hat{r}_{z/y} - 5\vec{m}_y'\hat{r}_{z/y}\vec{m}_z'\hat{r}_{z/y}\hat{r}_{z/y})\\
&= -\vec{f}_{\mathrm{m}/y/z}.
\end{aligned}$$

□

It can be seen from (6.9.11) that the direction of the magnetic force $\vec{f}_{\mathrm{m}/z/y}$ on the dipole $\vec{m}_z$ due to $\vec{m}_y$ is not necessarily codirectional with $\hat{r}_{z/y}$. This means that Newton's third law for magnetic forces does not share the alignment property that holds for mechanical forces, central gravity, and electrostatics. As shown below, this misalignment accounts for an additional contribution to the torque on each magnetic dipole.

Next, note that the directional derivative of the magnetic field vector $\vec{B}_y$ in the direction of the magnetic dipole $\vec{m}_z$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\vec{B}_y(\vec{r}_{y/z} + \alpha\vec{m}_z)\Big|_{\alpha=0} = \frac{\mathrm{d}}{\mathrm{d}\vec{r}}\vec{B}_y(\vec{r})\Big|_{\vec{r}=\vec{r}_{y/z}}\vec{m}_z = \vec{f}_{z/y}. \tag{6.9.14}$$

This shows that the misalignment between $\vec{f}_{\mathrm{m}/z/y}$ and $\hat{r}_{z/y}$ arises from the fact that the force $\vec{f}_{\mathrm{m}/z/y}$ on $\vec{m}_z$ due to $\vec{m}_y$ is parallel with the directional derivative of the magnetic field vector $\vec{B}_y$ in the direction of the magnetic dipole $\vec{m}_z$.

Example 6.9.2. Consider the dipoles $\vec{m}_y = m_y\hat{\imath}_{\rm A}$ and $\vec{m}_z = m_z\hat{\jmath}_{\rm A}$, where $\vec{r}_{z/y} = r\hat{\imath}_{\rm A}$. Then,

$$\vec{J}(\vec{r}_{z/y}) = \frac{1}{r^3}(3\hat{\imath}_{\rm A}\hat{\imath}'_{\rm A} - \vec{I}) = \frac{1}{r^3}(2\hat{\imath}_{\rm A}\hat{\imath}'_{\rm A} - \hat{\jmath}_{\rm A}\hat{\jmath}'_{\rm A} - \hat{k}_{\rm A}\hat{k}'_{\rm A}), \tag{6.9.15}$$

and thus

$$\vec{B}_y(\vec{r}_{z/y}) = \frac{2m_y}{r^3}\hat{\imath}_{\rm A}. \tag{6.9.16}$$

It follows from (6.9.11) that

$$\vec{f}_{{\rm m}/z/y} = \frac{3m_y m_z}{r^4}\hat{\jmath}_{\rm A}. \tag{6.9.17}$$

Alternatively, to obtain (6.9.17) directly without using (6.9.11), note that it follows from (6.9.7) and (6.9.9) that

$$\begin{aligned}
\vec{f}'_{{\rm m}/z/y} &= \vec{\partial}_{\vec{r}}(\vec{m}'_z\vec{J}(\vec{r})\vec{m}_y)\Big|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= m_z m_y\ \vec{\partial}_{\vec{r}}\left(\frac{1}{|\vec{r}|^3}\hat{\jmath}'_{\rm A}(3\hat{r}\hat{r}' - \vec{I})\hat{\imath}_{\rm A}\right)\Bigg|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= 3m_z m_y\ \vec{\partial}_{\vec{r}}\left(\frac{1}{|\vec{r}|^5}\hat{\jmath}'_{\rm A}\vec{r}\hat{\imath}'_{\rm A}\vec{r}\right)\Bigg|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= 3m_z m_y\left(\frac{-5}{|\vec{r}|^7}\hat{\jmath}'_{\rm A}\vec{r}\hat{\imath}'_{\rm A}\vec{r}\vec{r}' + \frac{1}{|\vec{r}|^5}\hat{\imath}'_{\rm A}\vec{r}\hat{\jmath}'_{\rm A} + \frac{1}{|\vec{r}|^5}\hat{\jmath}'_{\rm A}\vec{r}\hat{\imath}'_{\rm A}\right)\Bigg|_{\vec{r}=r\hat{\imath}_{\rm A}}\\
&= \frac{3m_z m_y}{r^4}\hat{\jmath}'_{\rm A}.
\end{aligned}$$

◇

The following result, which is an immediate consequence of Fact 6.9.1, shows that the net force due to an arbitrary collection of magnetic dipoles is zero.

Fact 6.9.3. Assume that a body consists of magnetic dipoles $\vec{m}_1, \ldots, \vec{m}_n$. Then, the net magnetic force on the body is zero.

6.9.2 Newton's Third Law for Magnetic Torques

The following result provides an expression for $\vec{T}_{z/y}$ and states Newton's third law for torques due to a pair of magnetic dipoles.

Fact 6.9.4. $\vec{T}_{{\rm m}/z/y}$ is given by

$$\vec{T}_{{\rm m}/z/y} = \frac{1}{|\vec{r}_{z/y}|^3}[3\hat{r}'_{z/y}\vec{m}_y(\vec{m}_z \times \hat{r}_{z/y}) - \vec{m}_z \times \vec{m}_y]. \tag{6.9.18}$$

Furthermore,

$$\vec{T}_{{\rm m}/z/y} + \vec{T}_{{\rm m}/y/z} + \vec{r}_{z/y} \times \vec{f}_{{\rm m}/z/y} = 0, \tag{6.9.19}$$

where

$$\vec{r}_{z/y} \times \vec{f}_{\mathrm{m}/z/y} = \frac{3}{|\vec{r}_{z/y}|^5}[\vec{r}\,'_{z/y}\vec{m}_y(\vec{r}_{z/y} \times \vec{m}_z) + \vec{r}\,'_{y/z}\vec{m}_z(\vec{r}_{y/z} \times \vec{m}_y)]. \tag{6.9.20}$$

Proof. It follows from (6.9.7), (6.9.8), and (6.9.10) that

$$\begin{aligned}\vec{T}_{\mathrm{m}/z/y} &= \vec{m}_z \times \vec{B}_y(\vec{r}_{z/y})\\ &= \vec{m}_z \times \vec{J}(\vec{r}_{z/y})\vec{m}_y\\ &= \vec{m}_z \times \left(\frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}_{z/y}\hat{r}'_{z/y} - \vec{I})\right)\vec{m}_y\\ &= \frac{1}{|\vec{r}_{z/y}|^3}[3\hat{r}'_{z/y}\vec{m}_y(\vec{m}_z \times \hat{r}_{z/y}) - \vec{m}_z \times \vec{m}_y]. \end{aligned} \tag{6.9.21}$$

It thus follows from (6.9.11) and (6.9.21) that

$$\begin{aligned}\vec{r}_{z/y} \times \vec{f}_{\mathrm{m}/z/y} &= \frac{3}{|\vec{r}_{z/y}|^4}[\vec{m}'_y\hat{r}_{z/y}(\vec{r}_{z/y} \times \vec{m}_z) + \vec{m}'_z\hat{r}_{z/y}(\vec{r}_{z/y} \times \vec{m}_y)]\\ &= -\frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}'_{z/y}\vec{m}_y(\vec{m}_z \times \hat{r}_{z/y}) - \vec{m}_z \times \vec{m}_y)\\ &\quad - \frac{1}{|\vec{r}_{z/y}|^3}(3\hat{r}'_{y/z}\vec{m}_z(\vec{m}_y \times \hat{r}_{y/z}) - \vec{m}_y \times \vec{m}_z)\\ &= -(\vec{T}_{\mathrm{m}/z/y} + \vec{T}_{\mathrm{m}/y/z}).\end{aligned}$$

□

Equation (6.9.19) involves the torque $\vec{T}_{\mathrm{m}/z/y}$, which is on $\vec{m}_z$ due to the magnetic field generated by $\vec{m}_y$, and the torque $\vec{T}_{\mathrm{m}/y/z}$, which is on $\vec{m}_y$ due to the magnetic field generated by $\vec{m}_z$. Furthermore, note that, since $\vec{f}_{\mathrm{m}/y/z} = -\vec{f}_{\mathrm{m}/z/y}$ and, as noted above, $\vec{f}_{\mathrm{m}/z/y}$ and $\vec{f}_{\mathrm{m}/y/z}$ are not codirectional with $\vec{r}_{z/y}$, these forces create an additional torque $\vec{T}_{\mathrm{m}}$; this torque can be computed relative to an arbitrary point. Choosing this point to be y, it follows that

$$\vec{T}_{\mathrm{m}} = \vec{r}_{z/y} \times \vec{f}_{\mathrm{m}/z/y} + \vec{r}_{y/y} \times \vec{f}_{\mathrm{m}/y/z} = \vec{r}_{z/y} \times \vec{f}_{\mathrm{m}/z/y}. \tag{6.9.22}$$

Consequently, $\vec{r}_{z/y} \times \vec{f}_{\mathrm{m}/z/y}$ in (6.9.19) is the torque due to $\vec{f}_{\mathrm{m}/z/y}$ and $\vec{f}_{\mathrm{m}/y/z}$.

The following result provides a symmetric version of Fact 6.9.4, where the torque $\vec{T}_{\mathrm{m}}$ is evaluated relative to an arbitrary point x.

Fact 6.9.5. Let x be a point. Then,

$$\vec{T}_{\mathrm{m}/z/y} + \vec{r}_{z/x} \times \vec{f}_{\mathrm{m}/z/y} = -(\vec{T}_{\mathrm{m}/y/z} + \vec{r}_{y/x} \times \vec{f}_{\mathrm{m}/y/z}). \tag{6.9.23}$$

Proof. Using $\vec{r}_{z/y} = \vec{r}_{z/x} + \vec{r}_{x/y}$, (6.9.19) implies

$$\vec{T}_{\mathrm{m}/z/y} + \vec{T}_{\mathrm{m}/y/z} = -\vec{r}_{z/y} \times \vec{f}_{\mathrm{m}/z/y}$$

$$= -(\vec{r}_{z/x} + \vec{r}_{x/y}) \times \vec{f}_{\mathrm{m}/z/y}$$
$$= -\vec{r}_{z/x} \times \vec{f}_{\mathrm{m}/z/y} - \vec{r}_{x/y} \times \vec{f}_{\mathrm{m}/z/y}$$
$$= -\vec{r}_{z/x} \times \vec{f}_{\mathrm{m}/z/y} - \vec{r}_{y/x} \times \vec{f}_{\mathrm{m}/y/z}. \qquad \square$$

Setting $x = y$ in (6.9.23) recovers (6.9.19).

The following result, which is an immediate consequence of Fact 6.9.4, shows that the torque due to an arbitrary collection of magnetic dipoles is zero.

Fact 6.9.6. Assume that a body consists of magnetic dipoles $\vec{m}_1, \ldots, \vec{m}_n$. Then, the magnetic torque on the body is zero.

Example 6.9.7. (Example 6.9.2 continued.) Using (6.9.18) it follows that

$$\vec{T}_{\mathrm{m}/z/y} = \frac{1}{r^3}[3\imath_{\mathrm{A}}' m_y \hat{\imath}_{\mathrm{A}}(m_z \hat{\jmath}_{\mathrm{A}} \times \hat{\imath}_{\mathrm{A}}) - m_z \hat{\jmath}_{\mathrm{A}} \times m_y \hat{\imath}_{\mathrm{A}}] = -\frac{2m_y m_z}{r^3}\hat{k}_{\mathrm{A}}, \tag{6.9.24}$$

$$\vec{T}_{\mathrm{m}/y/z} = \frac{1}{r^3}[-3\imath_{\mathrm{A}}' m_z \hat{\jmath}_{\mathrm{A}}(m_y \hat{\imath}_{\mathrm{A}} \times -\hat{\imath}_{\mathrm{A}}) - m_y \hat{\imath}_{\mathrm{A}} \times m_z \hat{\jmath}_{\mathrm{A}}] = -\frac{m_y m_z}{r^3}\hat{k}_{\mathrm{A}}, \tag{6.9.25}$$

$$\vec{T}_{\mathrm{m}} = r\hat{\imath}_{\mathrm{A}} \times \frac{3m_y m_z}{r^4}\hat{\jmath}_{\mathrm{A}} = \frac{3m_y m_z}{r^3}\hat{k}_{\mathrm{A}}. \tag{6.9.26}$$

Summing $\vec{T}_{\mathrm{m}/z/y}$, $\vec{T}_{\mathrm{m}/y/z}$, and $\vec{T}_{\mathrm{m}}$ verifies (6.9.19). $\Diamond$

Analogous results hold for electric dipoles.

6.10 Theoretical Problems

Problem 6.10.1. Let $\mathcal{B}$ be a rigid body consisting of particles $y_1, \ldots, y_l$ with masses $m_1, \ldots, m_l$, respectively, and let $\mathrm{F_B}$ be a body-fixed frame. Show that, for all $i = 1, \ldots, l$, $\vec{v}_{y_i/\mathrm{c/B}} = 0$.

Problem 6.10.2. Show that the moment on the rigid body $\mathcal{B}$ relative to the point w due to the force $\vec{f}$ on the point x in $\mathcal{B}$ does not change if the force $\vec{f}$ is applied instead to another point y in $\mathcal{B}$ located along the line that is parallel with $\vec{f}$ and that passes through x.

Problem 6.10.3. Consider a triangle $\mathcal{T}$ with vertices a, b, c, and define the following bodies:

i) $\mathcal{B}_1$ consists of three particles with the same mass located at a, b, c.

ii) $\mathcal{B}_2$ consists of three thin homogeneous links with the same linear mass density connecting a, b, c.

iii) $\mathcal{B}_3$ is a thin homogeneous triangular-shaped plate with vertices a, b, c.

Show that all three bodies have the same center of mass, which is located at the centroid of $\mathcal{T}$.

Problem 6.10.4. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, and let $\mathcal{B}_3$ denote the union of $\mathcal{B}_1$ and $\mathcal{B}_2$, that is, $\mathcal{B}_3$ is the body whose particles include all of the particles of $\mathcal{B}_1$ and $\mathcal{B}_2$. Let w be a point. Show that the center of mass of $\mathcal{B}_3$ relative to w lies on the line segment connecting the centers of mass of $\mathcal{B}_1$ and $\mathcal{B}_2$ relative to w. In particular, show that the location of the center of mass of $\mathcal{B}_3$ relative to w coincides with the center of mass of two "field" particles, namely, a particle y_1 located at the center of mass of $\mathcal{B}_1$ and whose mass is m_1, and a particle y_2 located at the center of mass of $\mathcal{B}_2$ and whose mass is m_2.

Problem 6.10.5. Consider a particle y whose mass is m and a body $\mathcal{B}$ that consists of two particles y_1 and y_2 with masses m_1 and m_2, respectively, connected by a massless link of length ℓ. The distance from y to y_1 is ℓ_1, and the distance from y to y_2 is ℓ_2, where $\ell_1^2 + \ell_2^2 = \ell^2$. Aside from the constraint forces on y_1 and y_2 due to the link, all forces on y, y_1, and y_2 are due to central gravity. Show that Newton's third law holds in the sense that the force $\vec{f}_y$ on y due to $\mathcal{B}$ is equal and opposite in direction to the net force on $\mathcal{B}$ due to y, but that the force $\vec{f}_y$ on y is not necessarily codirectional with either the center of mass of $\mathcal{B}$ or the center of gravity of $\mathcal{B}$. In particular, show that the following statements are equivalent:

i) $\ell_1 = \ell_2$.

ii) The center of mass of $\mathcal{B}$ coincides with the center of gravity of $\mathcal{B}$.

iii) The force $\vec{f}_y$ on y is codirectional with the center of mass of $\mathcal{B}$.

iv) The force $\vec{f}_y$ on y is codirectional with the center of gravity of $\mathcal{B}$.

Problem 6.10.6. Let $\mathcal{B}$ be a body subject to three forces that do not lie in a single plane, let z and w be distinct points, and assume that the moments $\vec{M}_{\mathcal{B}/w}$ and $\vec{M}_{\mathcal{B}/z}$ due to the forces are equal and nonzero. Does it follow that the forces are balanced? (Note that the converse is true, that is, if the forces are balanced, then $\vec{M}_{\mathcal{B}/w}$ and $\vec{M}_{\mathcal{B}/z}$ are independent of w and z and thus are equal.)

Problem 6.10.7. Let $\mathcal{B}$ be a body, let z be a point, and let $\mathrm{F_B}$ be a frame. Show that $J_{xx} = \hat{\imath}'_{\mathrm{B}} \vec{J}_{\mathcal{B}/z} \hat{\imath}_{\mathrm{B}}$, and provide similar expressions for the remaining components of $\vec{J}_{\mathcal{B}/z}\Big|_{\mathrm{B}}$. Furthermore, let $\hat{n}$ be a unit vector and show that $\hat{n}' \vec{J}_{\mathcal{B}/z} \hat{n}$ is a moment of inertia of $\mathcal{B}$. Finally, show that $\hat{n}' \vec{J}_{\mathcal{B}/z} \hat{n}$ is a principal moment of inertia of $\mathcal{B}$ if and only if $\hat{n}$ is an eigenvector of $\vec{J}_{\mathcal{B}/z}$.

Problem 6.10.8. Let $\mathcal{B}$ be a homogeneous cube, and let c be its center of mass. Show that every frame is a principal-axis frame relative to c.

Problem 6.10.9. Let $\mathcal{B}$ be a homogeneous rectangular solid whose mass is m, and let $\mathrm{F_B}$ be a frame whose axes $\hat{\imath}_{\mathrm{B}}$, $\hat{\jmath}_{\mathrm{B}}$, and $\hat{k}_{\mathrm{B}}$ are parallel with the sides of length a, b, and c, respectively, where $a > b > c$. Determine $J_{\mathcal{B}/z|\mathrm{B}}$ in the following cases:

i) z is the center of a face of $\mathcal{B}$ whose sides have lengths a and b.

ii) z is the center of an edge of $\mathcal{B}$ of length a.

iii) z is a vertex of $\mathcal{B}$.

Specialize *iii*) to the case where the rectangular solid approximates a thin bar, that is, $b = c \approx 0$.

Problem 6.10.10. Consider a rectangular plate whose sides have lengths $a > b > 0$, where $a = \sqrt{\frac{1+\sqrt{5}}{2}}b$. Show that $J_2 = \sqrt{J_1 J_3}$, and that the right-hand inequality in (6.2.16) holds as an equality.

Problem 6.10.11. Let $\mathcal{B}$ be an annulus, that is, a flat circular ring, with mass m, outer radius R, and inner radius r. Show that the principal moments of inertia of $\mathcal{B}$ relative to it center of mass are given by $m(R^2 + r^2)/4$, $m(R^2 + r^2)/4$, and $m(R^2 + r^2)/2$. Furthermore, show that, if the annulus is thin, that is, $R \approx r$, then the principal moments of inertia of $\mathcal{B}$ are given approximately by $\frac{1}{2}mR^2$, $\frac{1}{2}mR^2$, and mR^2. (Hint: Express the moments of inertia of the annulus in terms of the area density of the material.)

Problem 6.10.12. Let $\mathcal{B}$ be a spherical shell with mass m, outer radius R, and inner radius r. Show that the principal moments of inertia of $\mathcal{B}$ relative to it center of mass are given by $\frac{2m(R^5-r^5)}{5(R^3-r^3)}$. Furthermore, show that if the annulus is thin, that is, $R \approx r$, then the principal moments of inertia of $\mathcal{B}$ are given approximately by $2mR^2/3$.

Problem 6.10.13. Determine the physical inertia matrix of a triangular plate relative to its center of mass. Separately consider the cases where the triangle is a right triangle and the triangle is isosceles.

Problem 6.10.14. Let $\mathcal{B}$ be a body with particles $y_1, \dots, y_l$ whose masses are $m_1, \dots, m_l$, respectively, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, and let c be the center of mass of $\mathcal{B}$. Furthermore, let z be a point, and let $\mathcal{B}'$ be the body consisting of $\mathcal{B}$ and a particle y_{l+1} of mass $m_{\mathcal{B}}$ located at z. Finally, let c′ be the center of mass of $\mathcal{B}'$. Show that

$$\vec{J}_{\mathcal{B}/z} = \vec{J}_{\mathcal{B}'/\mathrm{c}} = 2\vec{J}_{\mathcal{B}'/\mathrm{c}'} - \vec{J}_{\mathcal{B}/\mathrm{c}}.$$

Problem 6.10.15. Let $\mathcal{B}$ be a body, let w be a point, and let $\vec{M}_1$ and $\vec{M}_2$ be moments on $\mathcal{B}$ relative to w with net forces $\vec{f}_{\mathcal{B},1}$ and $\vec{f}_{\mathcal{B},2}$, respectively. Show that $\vec{M}_1 + \vec{M}_2$ is a moment on $\mathcal{B}$ relative to w with net force $\vec{f}_{\mathcal{B},1} + \vec{f}_{\mathcal{B},2}$. Furthermore, show that, if $\vec{T}_1$ and $\vec{T}_2$ are torques on $\mathcal{B}$, then $\vec{T}_1 + \vec{T}_2$ is also a torque on $\mathcal{B}$.

6.11 Applied Problems

Problem 6.11.1. Consider the bar shown in Figure 6.11.1, which has a pin joint at a and whose opposite end c is attached to a translational spring connected to b. The stiffness of the translational spring is k, and the relaxed length of the translational spring is zero. In addition, a rotary spring, whose relaxed angle is zero and whose rotary stiffness κ, is attached to the bar around the pin joint. Determine the moment on the bar relative to a due to the force on the bar due to the translational spring. Furthermore, determine the longitudinal force on the bar due to the translational spring. Finally, determine κ.

Problem 6.11.2. The planar triangle shown in Figure 6.11.2 is attached to a vertical wall by the pin joint at P, and has point masses m_1 and m_2 at the remaining vertices y_1 and y_2, respectively. There is no mass at the pin joint P, and all links are massless. The angles $\theta_1, \theta_2, \theta_3$ and opposite sides ℓ_1, ℓ_2, ℓ_3 are defined in the figure. The side of length ℓ_3 is horizontal, and the direction of gravity is

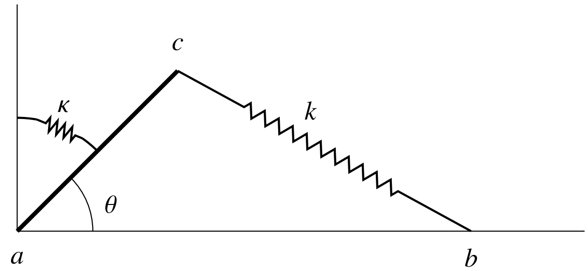


Figure 6.11.1: Bar with translational and rotary springs for Problem 6.11.1.

vertical as shown. A third mass M is connected to y_1 by a rope that passes around a small wheel at the point z, which is at the same height as y_1. Determine the location of the center of mass of the triangle. Furthermore, assuming that the triangle and mass M are in equilibrium, determine M. Finally, assume that the wheel at z and the mass M are moved horizontally to the right side of the triangle. Show that the triangle is not in equilibrium when the side of length ℓ_3 is horizontal.

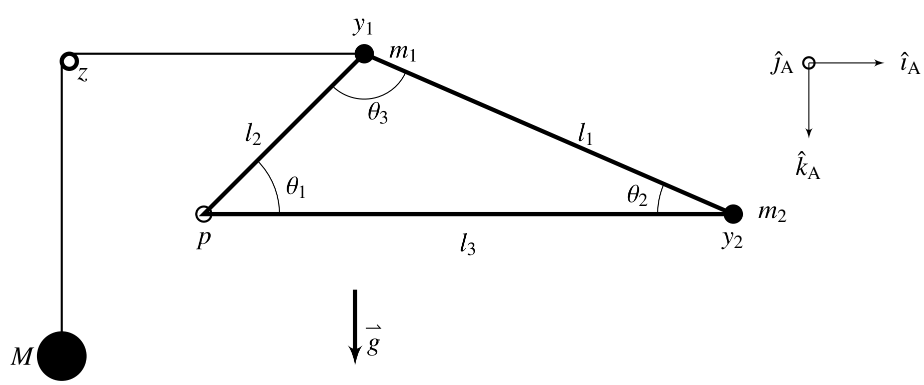


Figure 6.11.2: Triangular structure for Problem 6.11.2.

Problem 6.11.3. Consider three identical metal spheres that fit exactly inside a metal ring on a horizontal surface so that each sphere contacts the other two spheres as well as the inner surface of the ring. A fourth identical metal sphere is placed on top of the three spheres in a pyramid configuration. Determine the force that each of the three spheres applies to the ring due to the weight of the fourth sphere.

Symbol	Definition
c	Center of mass of $\mathcal{B}$
$\overset{\rightharpoonup}{f}$	Force vector
$\overset{\rightharpoonup}{M}_{x/y}$	Moment on x relative to y
$\overset{\rightharpoonup}{M}_{\mathcal{B}/y}$	Moment on $\mathcal{B}$ relative to y

Table 6.11.1: Notation for Chapter 6.

Chapter Seven

Theory of Newton-Euler Dynamics

Forces and moments can be applied to particles and bodies, resulting in changes in translational momentum and angular momentum. The basis for these changes is Newton's laws.

7.1 Kinematics of Straight-Line, Constant-Speed Motion

7.2 Newton's First Law for Particles

An *unforced particle* is a particle that has no forces on it. We use the concept of an unforced particle to define the concept of an inertial frame.

Definition 7.2.1. The frame $\mathrm{F_A}$ is an *inertial frame* if, for all unforced particles y and w,

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0. \tag{7.2.1}$$

Equation (7.2.1) can be written as

$$\vec{a}_{y/w/\mathrm{A}} = 0. \tag{7.2.2}$$

The following result shows that the acceleration of a point or particle relative to an unforced particle is independent of the choice of the unforced particle.

Fact 7.2.2. Let $\mathrm{F_A}$ be an inertial frame, let w and w' be unforced particles, and let y be a point or a particle. Then,

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w'}. \tag{7.2.3}$$

Proof. Note that

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w'} + \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{w'/w} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w'}. \qquad \square$$

The following terminology is convenient for referring to either an unforced particle or a point whose motion is the same as an unforced particle.

Definition 7.2.3. Let $\mathrm{F_A}$ be an inertial frame, let w be an unforced particle, and let y be a point or a particle. Then, the *inertial acceleration* of y is given by $\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w}$. Furthermore, y has *zero inertial* *acceleration* if

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0. \tag{7.2.4}$$

If w is an unforced particle and y is a point or particle, then $\vec{r}_{y/w}$ is the *inertial position of* y *relative to* w and $\overset{\mathrm{A}\bullet}{\vec{r}}_{y/w}$ is the *inertial velocity of* y *relative to* w. Note, however, that the terminology "inertial position of y" and "inertial velocity of y" is not meaningful when w is not specified.

The following result is *Newton's first law*. This statement is an axiom that concerns the existence of an inertial frame.

Fact 7.2.4. There exists an inertial frame.

Newton's first law cannot be proved mathematically; in fact, it is an approximation to observed motion. Assuming that Newton's first law is valid, it is shown in the following section that the stars provide an approximate inertial frame.

The following result shows that, for each pair of unforced particles y and w, the velocity of y relative to w with respect to an inertial frame $\mathrm{F_A}$ is constant. This means that the motion of y relative to w with respect to an inertial frame $\mathrm{F_A}$ is along a straight line whose direction is fixed with respect to $\mathrm{F_A}$ and with constant speed.

Fact 7.2.5. Let y and w be points, and let $\mathrm{F_A}$ be a frame. Then, (7.2.1) is satisfied if and only if there exist physical vectors $\vec{\alpha}$ and $\vec{\beta}$ such that $\overset{\mathrm{A}\bullet}{\vec{\alpha}} = 0$, $\overset{\mathrm{A}\bullet}{\vec{\beta}} = 0$, and

$$\vec{r}_{y/w} = t\vec{\alpha} + \vec{\beta}. \tag{7.2.5}$$

Proof. To prove sufficiency, note that it follows from (7.2.5) that the velocity of y relative to w with respect to $\mathrm{F_A}$ is given by

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{y/w} = \vec{\alpha},$$

while the acceleration of y relative to w with respect to $\mathrm{F_A}$ is given by

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0,$$

which verifies (7.2.1).

Conversely, resolving (7.2.1) in $\mathrm{F_A}$ yields

$$\overset{\,\,\cdot\cdot}{\overbrace{\vec{r}_{y/w}\Big|_{\mathrm{A}}}} = \left.\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w}\right|_{\mathrm{A}} = 0,$$

which implies that there exist $\alpha \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^3$ such that

$$\vec{r}_{y/w}\Big|_{\mathrm{A}} = t\alpha + \beta.$$

Therefore, (7.2.5) is satisfied with $\vec{\alpha} \triangleq \mathrm{F_A}\alpha$ and $\vec{\beta} \triangleq \mathrm{F_A}\beta$. $\square$

Note that $\vec{\alpha}$ and $\vec{\beta}$ are the physical velocity and physical position vectors given, respectively, by

$$\vec{\alpha} = \overset{\mathrm{A}\bullet}{\vec{r}}_{y/w}\,(t) \tag{7.2.6}$$

and

$$\vec{\beta} = \vec{r}_{y/w}(0). \tag{7.2.7}$$

Therefore, $\vec{\alpha}$ represents the velocity of y relative to w with respect to $\mathrm{F_A}$, while $\vec{\beta}$ represents the initial position of of y relative to w. Since both vectors are constant with respect to $\mathrm{F_A}$, the motion of y relative to w has the form of a straight line whose direction is constant with respect to $\mathrm{F_A}$.

The following result shows that all pairs of inertial frames have zero relative angular velocity.

Fact 7.2.6. Let $\mathrm{F_B}$ be an inertial frame, and let $\mathrm{F_A}$ be a frame. Then, $\mathrm{F_A}$ is an inertial frame if and only if $\vec{\omega}_{\mathrm{B/A}} = 0$.

Proof. Assume that $\vec{\omega}_{\mathrm{B/A}} = 0$, and let w and y be unforced particles. Since $\mathrm{F_B}$ is an inertial frame, it follows that $\overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w} = 0$. It then follows from (4.6.1) that $\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0$. Consequently, $\mathrm{F_A}$ is an inertial frame.

Conversely, let y and w be distinct unforced particles. Since $\mathrm{F_A}$ and $\mathrm{F_B}$ are inertial frames, it follows that $\overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w} = \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = 0$. It then follows from (4.6.1) that

$$2\vec{\omega}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{r}}_{y/w} + \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{y/w} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y/w}) = 0. \tag{7.2.8}$$

Now, choose distinct particles y and w such that $\vec{r}_{y/w} = \vec{\beta}$, where $\overset{\mathrm{B}\bullet}{\vec{\beta}} = 0$. Then, (7.2.8) implies that

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{\beta} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{\beta}) = 0. \tag{7.2.9}$$

Next, choosing y and w such that, at time t, $\vec{\omega}_{\mathrm{B/A}}$ and $\vec{\beta}$ are parallel, it follows from (7.2.9) that, at time t,

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{\omega}_{\mathrm{B/A}} = 0. \tag{7.2.10}$$

Alternatively, choosing y and w such that, at time t, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}$ and $\vec{\beta}$ are parallel, it follows from (7.2.9) and (7.2.10) that, at time t,

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0. \tag{7.2.11}$$

Hence, $\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = 0$. Therefore, for all choices of y and w, it follows from (7.2.9) that, at time t,

$$\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{\beta}) = 0, \tag{7.2.12}$$

and thus

$$(\vec{\beta}'\vec{\omega}_{\mathrm{B/A}})\vec{\omega}_{\mathrm{B/A}} = |\vec{\omega}_{\mathrm{B/A}}|^2\vec{\beta}. \tag{7.2.13}$$

Finally, choosing distinct y and w such that, at time t, $\vec{\beta}$ and $\vec{\omega}_{\mathrm{B/A}}$ are mutually orthogonal, it follows from (7.2.13) that $\vec{\omega}_{\mathrm{B/A}} = 0$. □

Fact 7.2.7. Let F_A and F_B be inertial frames, and let $\vec{x}(t)$ be a physical vector. Then,

$$\overset{\mathrm{A}\bullet}{\vec{x}}(t) = \overset{\mathrm{B}\bullet}{\vec{x}}(t), \tag{7.2.14}$$

$$\overset{\mathrm{A}\bullet\bullet}{\vec{x}}(t) = \overset{\mathrm{B}\bullet\bullet}{\vec{x}}(t). \tag{7.2.15}$$

7.3 Why the Stars Approximate an Inertial Frame

Assuming that Newton's first law is valid, we now show that the distant stars approximate an inertial frame. The distant stars, although visible to us, are at such a great distance that the angle between every pair does not change of short time periods. Therefore, the motion of these stars can be viewed as the motion of unforced particles. Consequently, Newton's first law implies that the relative motion of an unforced particle is along a straight line relative to the stars. This motion gives us the impression that the stars define an inertial frame. Notice that this argument assumes that Newton's first law is valid; in other words, the presence of visible stars per se does not imply that Newton's first law is true. It does show, however, that a frame can be defined that approximately satisfies Definition 7.2.1.

Let F_A be an inertial frame. Let w be a point with zero inertial acceleration, and let y_1, y_2, y_3 be distant stars that form three mutually orthogonal directions as viewed from w starting at time $t = 0$. Assuming that the stars are unforced particles, there exist velocity vectors $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$ and position vectors $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$ that are constant with respect to F_A and satisfy, for all $i = 1, 2, 3$,

$$\vec{r}_{y_i/w} = t\vec{\alpha}_i + \vec{\beta}_i. \tag{7.3.1}$$

Since $\vec{\beta}_1 \cdot \vec{\beta}_2 = \vec{\beta}_2 \cdot \vec{\beta}_3 = \vec{\beta}_3 \cdot \vec{\beta}_1 = 0$, the angle $\theta_{i,j}$ formed by the ith and jth stars satisfies

$$\cos\theta_{i,j} = \frac{\vec{\alpha}_i \cdot \vec{\alpha}_j t^2 + (\vec{\alpha}_i \cdot \vec{\beta}_j + \vec{\alpha}_j \cdot \vec{\beta}_i)t}{\|t\vec{\alpha}_i + \vec{\beta}_i\|\,\|t\vec{\alpha}_j + \vec{\beta}_j\|}. \tag{7.3.2}$$

Because the distances are large, $\|t\vec{\alpha}_i + \vec{\beta}_i\|$ and $\|t\vec{\alpha}_j + \vec{\beta}_j\|$ are large compared to the numerator of (7.3.2) over short time intervals. Hence, $\theta_{i,j}$ remains approximately $\pi/2$ rad over short time intervals.

Next, let F_S be the frame $[\hat{r}_{y_1/w}\ \hat{r}_{y_2/w}\ \hat{r}_{y_3/w}]$, which, over short time intervals, is given approximately by $[\hat{\beta}_1\ \hat{\beta}_2\ \hat{\beta}_3]$. Using (4.2.9), it follows that the physical angular velocity matrix of F_A relative to F_S is given by

$$\begin{aligned}\vec{\Omega}_{\mathrm{S/A}} &= -\vec{\Omega}_{\mathrm{A/S}} \qquad (7.3.3)\\ &= -[\hat{\imath}_{\mathrm{S}}\ \overset{\mathrm{A}\bullet\prime}{\hat{\imath}}_{\mathrm{S}} + \hat{\jmath}_{\mathrm{S}}\ \overset{\mathrm{A}\bullet\prime}{\hat{\jmath}}_{\mathrm{S}} + \hat{k}_{\mathrm{S}}\ \overset{\mathrm{A}\bullet\prime}{\hat{k}}_{\mathrm{S}}]\end{aligned}$$

$$
\begin{aligned}
&= -[\hat{r}_{y_1/w} \overset{\mathrm{A}\bullet\prime}{\hat{r}}_{y_1/w} + \hat{r}_{y_2/w} \overset{\mathrm{A}\bullet\prime}{\hat{r}}_{y_2/w} + \hat{r}_{y_3/w} \overset{\mathrm{A}\bullet\prime}{\hat{r}}_{y_3/w}] \\
&\approx -[\hat{\beta}_1 \overset{\mathrm{A}\bullet\prime}{\hat{\beta}}_1 + \hat{\beta}_2 \overset{\mathrm{A}\bullet\prime}{\hat{\beta}}_2 + \hat{\beta}_3 \overset{\mathrm{A}\bullet\prime}{\hat{\beta}}_3] \\
&= 0,
\end{aligned}
\tag{7.3.4}
$$

where the approximation holds over short time intervals and the last equality follows from the fact that $\vec{\beta}_i$ and thus $\hat{\beta}_i$ are constant with respect to $\mathrm{F_A}$. Finally, since $\mathrm{F_A}$ is an inertial frame, it follows from Fact 7.2.6 that $\mathrm{F_S}$ is approximately an inertial frame.

The observation that the stars determine an approximate inertial frame provides a practical framework for Newton's first law—whether or not it is true. These observations are consistent with the everyday experience that unforced motion evolves in straight lines with respect to the stars but not with respect to frames that are rotating relative to the stars.

7.4 Newton's Second Law for Particles

The following result is *Newton's second law*. This statement is an axiom that concerns the effect of forces on particles.

Fact 7.4.1. Let $\mathrm{F_A}$ be an inertial frame, let y be a particle with mass m, let $\vec{f}_y$ be the force acting on y, and let w be an unforced particle. Then,

$$m \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = \vec{f}_y. \tag{7.4.1}$$

Fact 7.4.2. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be inertial frames, let y be a particle, let $\vec{f}_y$ be a force acting on y, and let w be an unforced particle. Then,

$$\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w}. \tag{7.4.2}$$

Proof. Since $\mathrm{F_A}$ and $\mathrm{F_B}$ are inertial frames, the result follows from Fact 7.2.7. Alternatively, since $\mathrm{F_A}$ and $\mathrm{F_B}$ are inertial frames, it follows from Fact 8.1.5 that $m \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = \vec{f}_y$ and $m \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w} = \vec{f}_y$. Hence, $\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} = (1/m)\vec{f}_y = \overset{\mathrm{B}\bullet\bullet}{\vec{r}}_{y/w}$. □

The following result considers the acceleration of one particle relative to another particle in the case where forces are acting on both particles.

Fact 7.4.3. Let $\mathrm{F_A}$ be an inertial frame, let y_1 and y_2 be particles whose masses are m_1 and m_2, respectively, and let $\vec{f}_{y_1}$ and $\vec{f}_{y_2}$ be the forces acting on y_1 and y_2, respectively. Then,

$$m_1 \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} = \vec{f}_{y_1} - \frac{m_1}{m_2}\vec{f}_{y_2}. \tag{7.4.3}$$

Note that, if either y_1 or y_2 is unforced, then this result specializes to Fact 8.1.5. However, although (7.4.3) superficially has the form of (7.4.1), it is not a statement of Newton's second law since y_2 is forced. The distinction between (7.4.3) and (7.4.1) is due to the fact that the force on the right hand side of (7.4.1) is the force on y, whereas the force on the right hand side of (7.4.1) is not the force on y_1. Finally, note that, in the case where $m_2\vec{f}_{y_1} = m_1\vec{f}_{y_2}$, (7.4.3) becomes

$$m_1 \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} = 0. \tag{7.4.4}$$

Although (7.4.4) superficially has the form of (7.2.1), it is not a statement of Newton's second law since both y_1 and y_2 are forced.

It is often the case that the motion of a body is constrained due to its connection with a wall, ceiling, floor, or the ground. To address these problems, it is convenient to view these bodies as if they are unaffected by reaction forces. A *massive particle* is a particle with infinite mass and thus is unaffected by all forces, including uniform gravity. The motion of a massive particle is thus identical to the motion of an unforced particle. A body that contains at least one massive particle is a *massive body*. Consequently, a massive body that contains at least three massive particles that are not colinear is unaffected by all forces. An *inertially nonrotating massive body* (INMB), is a massive body that is not rotating relative an inertial frame. Therefore, every body-fixed frame for an INMB is an inertial frame. Walls, ceilings, floors, and the ground are assumed to be INMB's, and thus every point in these bodies has zero inertial acceleration.

It is useful to rewrite (7.4.3) as

$$\frac{m_1 m_2}{m_1 + m_2} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} = \frac{m_2}{m_1 + m_2}\vec{f}_{y_1} - \frac{m_1}{m_1 + m_2}\vec{f}_{y_2}, \tag{7.4.5}$$

where $\frac{m_1 m_2}{m_1+m_2}$ is the *reduced mass*. Now, assume that the forces $\vec{f}_{y_1}$ and $\vec{f}_{y_2}$ have approximately the same magnitude. Then, it can be seen from (7.4.5) that, if m_1 is much larger than m_2, then

$$m_2 \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_2/y_1} \approx \vec{f}_{y_2}, \tag{7.4.6}$$

whereas, if m_2 is much larger than m_1, then

$$m_1 \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \approx \vec{f}_{y_1}. \tag{7.4.7}$$

Consequently, the particle with significantly larger mass approximately plays the role of an unforced particle, and thus a massive particle plays the role of an unforced particle.

Let $\mathrm{F_A}$ be a frame, let y be a particle with mass m, and let w be a point. Then, the *translational momentum* $\vec{p}_{y/w/\mathrm{A}}$ *of* y *relative to* w *with respect to* $\mathrm{F_A}$ is defined by

$$\vec{p}_{y/w/\mathrm{A}} \triangleq m\vec{v}_{y/w/\mathrm{A}} = m \overset{\mathrm{A}\bullet}{\vec{r}}_{y/w}. \tag{7.4.8}$$

We can thus restate Newton's second law as follows.

Fact 7.4.4. Let $\mathrm{F_A}$ be an inertial frame, let y be a particle with mass m, let $\vec{f}_y$ be the force acting on y, and let w be an unforced particle. Then,

$$\overset{\mathrm{A}\bullet}{\vec{p}}_{y/w} = \vec{f}_y. \tag{7.4.9}$$

7.5 Newton's Second Law for Bodies

We now apply Newton's second law to a body $\mathcal{B}$. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$ as defined by (6.1.2), let $\mathrm{F_A}$ be a frame, and let w be a point. Then, the velocity and acceleration of the center of mass of $\mathcal{B}$ relative to w and with respect to $\mathrm{F_A}$ are given by

$$\vec{v}_{\mathrm{c}/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{\mathrm{c}/w} = \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{v}_{y_i/w/\mathrm{A}}, \tag{7.5.1}$$

$$\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{v}}_{\mathrm{c}/w} = \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{l} m_i \vec{a}_{y_i/w/\mathrm{A}}. \tag{7.5.2}$$

Hence,

$$m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}} = \sum_{i=1}^{l} m_i \vec{v}_{y_i/w/\mathrm{A}}, \tag{7.5.3}$$

$$m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \sum_{i=1}^{l} m_i \vec{a}_{y_i/w/\mathrm{A}}. \tag{7.5.4}$$

Recall that external forces on a particle in a body include all forces that are not due to interactions with other particles in the body. Note that, if a particle is not located at the point y_i, then the mass associated with y_i is defined to be zero.

The following result is Newton's second law for a body that is not necessarily rigid.

Fact 7.5.1. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let c be the center of mass of $\mathcal{B}$. Furthermore, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, and let w be a point with zero inertial acceleration. Then,

$$m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}, \tag{7.5.5}$$

where the mass $m_{\mathcal{B}}$ of $\mathcal{B}$ is defined by

$$m_{\mathcal{B}} \triangleq \sum_{i=1}^{l} m_i \tag{7.5.6}$$

and the net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ is defined by

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}. \tag{7.5.7}$$

Now, let z be a point. Then,

$$m_{\mathcal{B}} \vec{a}_{\mathrm{c}/z/\mathrm{A}} + m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}. \tag{7.5.8}$$

Finally, if the external forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then

$$\vec{a}_{\mathrm{c}/w/\mathrm{A}} = 0, \tag{7.5.9}$$

and thus

$$\vec{a}_{c/z/A} + \vec{a}_{z/w/A} = 0. \tag{7.5.10}$$

Proof. Let $\vec{f}_{ij}$ be the internal force on y_i due to y_j. Since each link is massless, it follows that $\sum_{j=1,\ldots,l,j\neq i} \vec{f}_{ij} = 0$. Since w is an unforced particle, we thus have

$$m_{\mathcal{B}}\vec{a}_{c/w/A} = \sum_{i=1}^{l} m_i \vec{a}_{y_i/w/A} = \sum_{i=1}^{l} m_i \overset{A\bullet\bullet}{\vec{r}}_{y_i/w} = \sum_{i=1}^{l}\left(\vec{f}_{y_i} + \sum_{\substack{j=1,\ldots,l\\ j\neq i}} \vec{f}_{ij}\right) = \sum_{i=1}^{l}\vec{f}_{y_i} = \vec{f}_{\mathcal{B}},$$

which proves (7.5.5). Furthermore,

$$\vec{f}_{\mathcal{B}} = \sum_{i=1}^{l} m_i(\vec{a}_{y_i/z/A} + \vec{a}_{z/w/A}),$$

which implies (7.5.8). □

Example 7.5.2. Consider a horizontal straight rigid wire $\mathcal{W}$ that rotates around a pin joint at the point w. As shown in Figure 7.5.1, a bead y slides without friction along the wire. The frame F_A is an inertial frame fixed to the ground, and the frame F_B is attached to the wire. Determine the equation of motion for the position of the bead along the wire, the acceleration of the bead relative to w with respect to F_A, and the reaction force on the bead due to the wire.

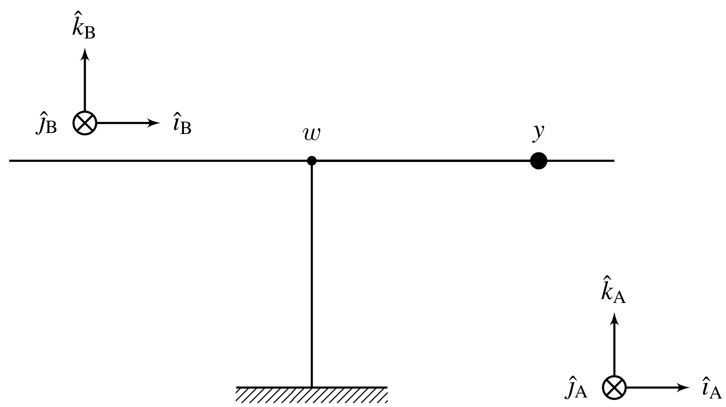


Figure 7.5.1: Bead on a rotating wire for Example 7.5.2.

Solution. Define $\omega \triangleq \dot\theta_{\hat\imath_{\rm B}/\hat\imath_{\rm A}/\hat k_{\rm B}}$. Since $\vec r_{y/w} = x\hat\imath_{\rm B}$, (4.6.4) implies that

$$\vec a_{y/w/{\rm A}} = \vec a_{y/w/{\rm B}} + \vec a_{\rm Cor} + \vec a_{\rm aa} + \vec a_{\rm cp}, \tag{7.5.11}$$

where

$$\vec a_{y/w/{\rm B}} = \ddot x\hat\imath_{\rm B}, \tag{7.5.12}$$
$$\vec a_{\rm Cor} = 2\vec\omega_{\rm B/A} \times \vec v_{y/w/{\rm B}} = 2\omega\dot x\hat\jmath_{\rm B}, \tag{7.5.13}$$
$$\vec a_{\rm aa} = \vec\alpha_{\rm B/A} \times \vec r_{y/w} = \dot\omega x\hat\jmath_{\rm B}, \tag{7.5.14}$$
$$\vec a_{\rm cp} = \vec\omega_{\rm B/A} \times (\vec\omega_{\rm B/A} \times \vec r_{y/w}) = -\omega^2 x\hat\imath_{\rm B}. \tag{7.5.15}$$

Hence,

$$\vec a_{y/w/{\rm A}} = (\ddot x - \omega^2 x)\hat\imath_{\rm B} + (2\omega\dot x + \dot\omega x)\hat\jmath_{\rm B}. \tag{7.5.16}$$

It follows from Newton's second law given by Fact 8.1.5 that the inertial acceleration $\vec a_{y/w/{\rm A}}$ is parallel with the force on the bead, which, since the wire is frictionless, is in the direction $\hat\jmath_{\rm B}$. Therefore,

$$\ddot x = \omega^2 x, \tag{7.5.17}$$

and thus

$$\vec a_{y/w/{\rm A}} = (2\omega\dot x + \dot\omega x)\hat\jmath_{\rm B}. \tag{7.5.18}$$

Hence, the reaction force on y due to $\mathcal{W}$ is given by

$$\vec f_{{\rm r}/y/\mathcal{W}} = m(2\omega\dot x + \dot\omega x)\hat\jmath_{\rm B}. \tag{7.5.19}$$

◇

In free-body analysis, it is often necessary to apply Fact 7.5.1 to a massless rigid body $\mathcal{B}$, such as a linkage. Since $m_\mathcal{B} = 0$, it follows from (7.5.9) that $\vec f_\mathcal{B} = 0$, where $\vec f_\mathcal{B}$ is the sum of all reaction and external forces on $\mathcal{B}$.

Let $\mathcal{B}$ be a body with mass $m_\mathcal{B}$, and let c be the center of mass of $\mathcal{B}$. Then, the *translational momentum* $\vec p_{\mathcal{B}/w/{\rm A}}$ *of* $\mathcal{B}$ *relative to* w *with respect to* $\rm F_A$ is defined by

$$\vec p_{\mathcal{B}/w/{\rm A}} \triangleq m_\mathcal{B}\vec v_{{\rm c}/w/{\rm A}} = m_\mathcal{B} \overset{{\rm A}\bullet}{\vec r}_{{\rm c}/w}. \tag{7.5.20}$$

The following result restates Fact 7.5.1 in terms of the translational momentum of $\mathcal{B}$.

Fact 7.5.3. Let $\rm F_A$ be an inertial frame, let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, and let c be the center of mass of $\mathcal{B}$. Furthermore, for all $i = 1, \ldots, l$, let $\vec f_{y_i}$ be the external force on y_i, and let w be a point with zero inertial acceleration. Then,

$$\overset{{\rm A}\bullet}{\vec p}_{\mathcal{B}/w/{\rm A}} = \vec f_\mathcal{B}, \tag{7.5.21}$$

where the mass $m_\mathcal{B}$ and the net force $\vec f_\mathcal{B}$ are defined by (7.5.6) and (7.5.7), respectively. Finally, if $\vec f_{y_1}, \ldots, \vec f_{y_l}$ are balanced, then

$$\overset{{\rm A}\bullet}{\vec p}_{\mathcal{B}/w/{\rm A}} = 0. \tag{7.5.22}$$

Fact 7.5.3 shows that the acceleration of a body due to all external forces on the body can be viewed as the acceleration of an equivalent particle located at the center of mass of the body, where the mass of the equivalent particle is equal to the mass of the body and where the applied force is given by the net force, that is, the sum of all of the forces on all of the particles in the body. Note that the internal forces are reciprocal and thus do not contribute to the net force since the body. If the net force is zero, then the equivalent particle has constant inertial velocity, that is, constant velocity relative to all unforced particles and with respect to all inertial frames. The external forces can also cause the body to rotate, as discussed in the following sections.

Assume that $\vec{f}_{\mathcal{B}}\big|_{\mathrm{A}}$ has a component that is identically zero. Then, it follows from (7.5.21) that the corresponding component of $\overset{\mathrm{A}\bullet}{\vec{p}}_{\mathcal{B}/w/\mathrm{A}}\big|_{\mathrm{A}}$ is identically zero. In this case, *conservation of momentum* holds along one of the axes of $\mathrm{F_A}$, and the corresponding component of the translational momentum is a constant of the motion relative to $\mathrm{F_A}$. If $\vec{f}_{\mathcal{B}} = 0$, then translational momentum is conserved along all three axes of $\mathrm{F_A}$, and all three components of the momentum are constants of the motion relative to $\mathrm{F_A}$.

The following result provides an alternative version of Fact 7.5.1 involving the first moment of inertia.

Fact 7.5.4. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body with mass $m_{\mathcal{B}}$, let c be the center of mass of $\mathcal{B}$, let $\vec{f}_{\mathcal{B}}$ be the net force on $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let w be a point with zero inertial acceleration, and let z be a point. Then,

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{\mathrm{c}/z/\mathrm{A}} + m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}. \tag{7.5.23}$$

If, in addition, $\mathcal{B}$ is a rigid body and z is fixed in $\mathcal{B}$, then

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}) = \vec{f}_{\mathcal{B}}. \tag{7.5.24}$$

Proof. Rewriting (7.5.5) as

$$m_{\mathcal{B}}\overset{\mathrm{A}\bullet}{\vec{v}}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}$$

and applying the transport theorem yields

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{\mathrm{c}/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_{\mathcal{B}},$$

which implies (7.5.23). Now, consider the case where $\mathcal{B}$ is a rigid body and z is fixed in $\mathcal{B}$. Then, using $\overset{\mathrm{B}\bullet}{\vec{r}}_{z/\mathrm{c}} = 0$ and $\overset{\mathrm{B}\bullet}{\vec{v}}_{\mathrm{c}/z/\mathrm{A}} = \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}/z}$, (7.5.23) implies (7.5.24). □

Example 7.5.5. A particle y with mass m is placed on the surface of the Earth at north latitude λ. The acceleration due to uniform gravity is $\vec{g}$, the radius of the Earth is r_{E}, the center w of the Earth has zero inertial acceleration, and the spin rate of the Earth about its north-south axis, whose direction is fixed relative to an inertial frame, is ω_{E}. Determine the apparent weight of the particle.

Solution. Let $\mathrm{F_A}$ denote an inertial frame whose axis $\hat{k}_{\mathrm{A}}$ is codirectional with the spin axis of the Earth, let $\vec{\omega}_{\mathrm{E/A}} = \omega_{\mathrm{E}}\hat{k}_{\mathrm{A}} = \omega_{\mathrm{E}}\hat{k}_{\mathrm{E}}$ denote the angular velocity of the Earth frame $\mathrm{F_E}$ relative to $\mathrm{F_A}$, where $\omega_{\rm E} > 0$, and let $\mathrm{F_{sph}}$ denote the spherical frame at y. Hence,

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_E} \xrightarrow[2]{-\lambda} \mathrm{F_{sph}}, \tag{7.5.25}$$

where $\omega_{\rm E} = \dot\theta$ and $\mathrm{F_{sph}} = [\hat e_{\rm u}\ \ \hat e_{\rm e}\ \ \hat e_{\rm n}]$. Since λ is constant, it follows that $\vec\omega_{\rm sph/E} = 0$, and thus $\vec\omega_{\rm sph/A} = \vec\omega_{\rm E/A} = \omega_{\rm E}\hat k_{\rm A} = \omega_{\rm E}\hat k_{\rm E}$. Since $\vec r_{y/w} = r_{\rm E}\hat e_{\rm u}$, it follows that

$$\begin{aligned}\vec v_{y/w/\mathrm{A}} &= \vec v_{y/w/\mathrm{E}} + \vec\omega_{\rm E/A} \times \vec r_{y/w}\\ &= r_{\rm E}\vec\omega_{\rm E/A} \times \hat e_{\rm u}\\ &= r_{\rm E}\omega_{\rm E}\hat k_{\rm E} \times [(\cos\lambda)\hat\imath_{\rm E} + (\sin\lambda)\hat k_{\rm E}]\\ &= r_{\rm E}\omega_{\rm E}(\cos\lambda)\hat\jmath_{\rm E},\end{aligned} \tag{7.5.26}$$

and thus

$$\begin{aligned}\vec a_{y/w/\mathrm{A}} &= r_{\rm E}\omega_{\rm E}(\cos\lambda)\vec\omega_{\rm E/A} \times \hat\jmath_{\rm E}\\ &= -r_{\rm E}\omega_{\rm E}^2(\cos\lambda)\hat\imath_{\rm E}\\ &= -r_{\rm E}\omega_{\rm E}^2(\cos\lambda)[(\cos\lambda)\hat e_{\rm u} - (\sin\lambda)\hat e_{\rm n}].\end{aligned} \tag{7.5.27}$$

Next, letting $\vec f_{\mathrm{r}/y}$ denote the reaction force on y due to its direct contact with the ground, it follows that

$$m\vec a_{y/w/\mathrm{A}} = m\vec g + \vec f_{\mathrm{r}/y}. \tag{7.5.28}$$

Therefore, with $\vec f_{\mathrm{r}/y} = f_{\rm u}\hat e_{\rm u} + f_{\rm n}\hat e_{\rm n} + f_{\rm e}\hat e_{\rm e}$ and noting that $\vec g = -g\hat e_{\rm u}$, it follows that

$$-mr_{\rm E}\omega_{\rm E}^2(\cos\lambda)[(\cos\lambda)\hat e_{\rm u} - (\sin\lambda)\hat e_{\rm n}] = -mg\hat e_{\rm u} + f_{\rm u}\hat e_{\rm u} + f_{\rm n}\hat e_{\rm n} + f_{\rm e}\hat e_{\rm e}, \tag{7.5.29}$$

and thus

$$f_{\rm u} = m(g - r_{\rm E}\omega_{\rm E}^2\cos^2\lambda), \tag{7.5.30}$$

$$f_{\rm n} = -mr_{\rm E}\omega_{\rm E}^2(\cos\lambda)\sin\lambda, \tag{7.5.31}$$

$$f_{\rm e} = 0. \tag{7.5.32}$$

The apparent weight of y is thus the upward reaction force $m(g - r_{\rm E}\omega_{\rm E}^2\cos^2\lambda)$. ◇

7.6 Newton's Second Law in Noninertial Frames

Newton's second law relates inertial acceleration to forces. In practice, however, the "observed" acceleration may be noninertial, that is, it may be defined with respect to a noninertial frame and relative to a point that does not have zero inertial acceleration. An expression for noninertial acceleration can be obtained by combining Newton's second law given by Fact 7.5.1 with the double transport theorem given by Fact 4.6.1.

Fact 7.6.1. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, define $m_{\mathcal{B}} \triangleq \sum_{i=1}^{l} m_i$, and let c be the center of mass of $\mathcal{B}$. Furthermore, for all $i = 1, \ldots, l$, let $\vec f_{y_i}$ be the external force on y_i, let z be a point, let $\mathrm{F_A}$ be an inertial frame, let w be a point with zero inertial acceleration, and let $\mathrm{F_B}$ be a frame. Then,

$$m_{\mathcal{B}}\vec{a}_{\mathrm{c}/z/\mathrm{B}} = \vec{f}_{\mathcal{B}/\mathrm{c}/z/\mathrm{B}}, \tag{7.6.1}$$

where

$$\vec{f}_{\mathrm{c}/z/\mathrm{B}} \triangleq \vec{f}_{\mathcal{B}} - \vec{f}_{\mathrm{rel}} - \vec{f}_{\mathrm{Cor}} - \vec{f}_{\mathrm{aa}} - \vec{f}_{\mathrm{cp}}, \tag{7.6.2}$$

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}, \tag{7.6.3}$$

$$\vec{f}_{\mathrm{rel}} \triangleq m_{\mathcal{B}}\vec{a}_{z/w/\mathrm{B}}, \tag{7.6.4}$$

$$\vec{f}_{\mathrm{Cor}} \triangleq m_{\mathcal{B}}2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{\mathrm{c}/w/\mathrm{B}}, \tag{7.6.5}$$

$$\vec{f}_{\mathrm{aa}} \triangleq m_{\mathcal{B}}\vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}/w}, \tag{7.6.6}$$

$$\vec{f}_{\mathrm{cp}} \triangleq m_{\mathcal{B}}\vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}/w}). \tag{7.6.7}$$

Furthermore, the following statements hold:

i) If $\mathrm{F_B}$ is an inertial frame, then $\vec{f}_{\mathrm{Cor}} = \vec{f}_{\mathrm{aa}} = \vec{f}_{\mathrm{cp}} = 0$, and thus $\vec{f}_{\mathcal{B}} = \vec{f}_{\mathrm{c}/z/\mathrm{B}} + \vec{f}_{\mathrm{rel}}$.

ii) If $\mathrm{F_B}$ is an inertial frame and z has zero inertial acceleration, then $\vec{f}_{\mathrm{rel}} = \vec{f}_{\mathrm{Cor}} = \vec{f}_{\mathrm{aa}} = \vec{f}_{\mathrm{cp}} = 0$, and thus $\vec{f}_{\mathrm{c}/z/\mathrm{B}} = \vec{f}_{\mathcal{B}}$.

iii) $\vec{f}_{\mathrm{c}/z/\mathrm{B}} = \vec{f}_{\mathcal{B}}$ if and only if $\vec{a}_{\mathrm{c}/z/\mathrm{B}} = \vec{a}_{\mathrm{c}/w/\mathrm{A}}$.

Note that the acceleration $\vec{a}_{\mathrm{c}/z/\mathrm{B}}$ in (7.6.1) is defined with respect to a frame $\mathrm{F_B}$ that is not necessarily inertial and relative to a point z that does not necessarily have zero inertial acceleration. Furthermore, (7.6.1) involves the frame-dependent, force-like term $\vec{f}_{\mathrm{c}/z/\mathrm{B}}$, which is a combination of the net force (7.6.3) and four force-like terms, namely, the relative force (7.6.4), the Coriolis force (7.6.5), the angular-acceleration force (7.6.6), and the centripetal force (7.6.7). The net force $\vec{f}_{\mathcal{B}}$ is thus the sum of five terms, that is,

$$\vec{f}_{\mathcal{B}} = \vec{f}_{\mathrm{c}/z/\mathrm{B}} + \vec{f}_{\mathrm{rel}} + \vec{f}_{\mathrm{Cor}} + \vec{f}_{\mathrm{aa}} + \vec{f}_{\mathrm{cp}}. \tag{7.6.8}$$

Unlike the net force $\vec{f}_{\mathcal{B}}$, which is independent of the choice of $\mathrm{F_B}$ and z, the force-like term $\vec{f}_{\mathrm{c}/z/\mathrm{B}}$ may be different for different choices of $\mathrm{F_B}$ and z. Real forces, however, which may arise from direct contact, springs, dashpots, inerters, uniform gravity, and magnetism, are independent of the choice of $\mathrm{F_B}$ and z.

If either $\mathrm{F_B}$ is not an inertial frame or z does not have zero inertial acceleration, then $\vec{f}_{\mathcal{B}/\mathrm{c}/z/\mathrm{B}}$ is not necessarily equal to the net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$. Furthermore, the difference

$$\vec{f}_{\mathcal{B}} - \vec{f}_{\mathrm{c}/z/\mathrm{B}} = \vec{f}_{\mathrm{rel}} + \vec{f}_{\mathrm{Cor}} + \vec{f}_{\mathrm{aa}} + \vec{f}_{\mathrm{cp}} \tag{7.6.9}$$

is an artifact of the choice of $\mathrm{F_B}$ and z. In the case where $\vec{f}_{\mathcal{B}/\mathrm{c}/z/\mathrm{B}}$ is not equal to the net force $\vec{f}_{\mathcal{B}}$ acting on $\mathcal{B}$, it follows that $\vec{f}_{\mathcal{B}/\mathrm{c}/z/\mathrm{B}}$ is not a real force, which motivates the following definition.

Definition 7.6.2. Let $\mathcal{B}$ be a body with mass $m_{\mathcal{B}}$ and center of mass c, let $\mathrm{F_B}$ be a frame, let z be a point, and define $\vec{f}_{\mathrm{c}/z/\mathrm{B}}$ by (7.6.2)–(7.6.7). Then $\vec{f}_{\mathrm{c}/z/\mathrm{B}}$ is a *fictitious force* if $\vec{f}_{\mathrm{c}/z/\mathrm{B}} \neq \vec{f}_{\mathcal{B}}$.

It follows from *iii*) of Fact 7.6.1 that $\vec{f}_{\mathrm{c}/z/\mathrm{B}}$ is a fictitious force if and only if $\vec{a}_{\mathrm{c}/z/\mathrm{B}} \neq \vec{a}_{\mathrm{c}/w/\mathrm{A}}$.

Defining the *centrifugal force* $\vec{f}_{\mathrm{cf}} \triangleq -\vec{f}_{\mathrm{cp}}$, (7.6.2) can be written as

$$\vec{f}_{\mathrm{fo}} \triangleq \vec{f}_{\mathrm{Cor}} + \vec{f}_{\mathrm{aa}} - \vec{f}_{\mathrm{cf}}. \tag{7.6.10}$$

If the centripetal force is a real reaction force, then the equal-and-opposite centrifugal force is also a real reaction force. However, if the centripetal force is a fictitious force, then so is the centrifugal force. Similarly, if the relative force, Coriolis force, or angular-acceleration force is a real reaction force, then the corresponding equal-and-opposite force is also a real reaction force; these reaction forces are denoted by $\vec{f}_{\mathrm{rel,opp}}$, $\vec{f}_{\mathrm{Cor,opp}}$, $\vec{f}_{\mathrm{aa,opp}}$, respectively Likewise, if the relative force, Coriolis force, or angular-acceleration force is fictitious, then the corresponding opposite force is also fictitious.

Example 7.6.3. As shown in Example 7.5.2,

$$\vec{a}_{y/w/\mathrm{A}} = \vec{a}_{y/w/\mathrm{B}} + \vec{a}_{\mathrm{Cor}} + \vec{a}_{\mathrm{aa}} + \vec{a}_{\mathrm{cp}}, \tag{7.6.11}$$

where

$$\vec{a}_{y/w/\mathrm{B}} = \ddot{x}\hat{\imath}_{\mathrm{B}}, \tag{7.6.12}$$

$$\vec{a}_{\mathrm{Cor}} = 2\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{y/w/\mathrm{B}} = 2\omega\dot{x}\hat{\jmath}_{\mathrm{B}}, \tag{7.6.13}$$

$$\vec{a}_{\mathrm{aa}} = \vec{\alpha}_{\mathrm{B/A}} \times \vec{r}_{y/w} = \dot{\omega}x\hat{\jmath}_{\mathrm{B}}, \tag{7.6.14}$$

$$\vec{a}_{\mathrm{cp}} = \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y/w}) = -\omega^2 x\hat{\imath}_{\mathrm{B}}. \tag{7.6.15}$$

Furthermore, since the reaction force on the bead due to the wire is in the direction of $\hat{\jmath}_{\mathrm{B}}$, it follows from (7.5.17) and (7.5.19) that

$$\ddot{x} = \omega^2 x \tag{7.6.16}$$

and

$$m\vec{a}_{y/w/\mathrm{A}} = \vec{f}_{\mathrm{r}/y/\mathcal{W}} = m(2\omega\dot{x} + \dot{\omega}x)\hat{\jmath}_{\mathrm{B}}. \tag{7.6.17}$$

Next, it follows from (7.6.12)–(7.6.15) that

$$\vec{f}_{\mathrm{rel}} = m\ddot{x}\hat{\imath}_{\mathrm{B}}, \tag{7.6.18}$$

$$\vec{f}_{\mathrm{Cor}} = 2m\omega\dot{x}\hat{\jmath}_{\mathrm{B}}, \tag{7.6.19}$$

$$\vec{f}_{\mathrm{aa}} = m\dot{\omega}x\hat{\jmath}_{\mathrm{B}}, \tag{7.6.20}$$

$$\vec{f}_{\mathrm{cp}} = -m\omega^2 x\hat{\imath}_{\mathrm{B}}, \tag{7.6.21}$$

which shows that the Coriolis force and angular-acceleration force are codirectional with the reaction force on y due to the wire and thus are real forces. In fact,

$$\vec{f}_{\mathrm{r}/y/\mathcal{W}} = \vec{f}_{\mathrm{Cor}} + \vec{f}_{\mathrm{aa}}. \tag{7.6.22}$$

Hence, the equal-and-opposite reaction force on the wire due to the bead is given by

$$\vec{f}_{\mathrm{r}/\mathcal{W}/y} = \vec{f}_{\mathrm{Cor,opp}} + \vec{f}_{\mathrm{aa,opp}}. \tag{7.6.23}$$

However, the relative force and centripetal force are orthogonal to $\vec{f}_{\mathrm{r}/y/\mathcal{W}}$, and thus are purely fictitious forces; hence, their sum is zero. Finally, note that

$$m\vec{a}_{y/w/\mathrm{B}} = \vec{f}_{\mathrm{cf}}, \tag{7.6.24}$$

which has the form of Newton's second law except that $\mathrm{F_B}$ is noninertial and $\vec{f}_{\mathrm{cf}}$ is fictitious.

Next, consider the case where the bead reaches a stopper $\mathcal{S}$ at $x = \ell$. Let $\vec{f}_{\mathrm{r}/y/\mathcal{S}} = f_{\mathrm{r}/y/\mathcal{S}}\hat{\imath}_{\mathrm{B}}$ denote the reaction force on y due to direct contact with $\mathcal{S}$. It thus follows from (7.5.16) that

$$\begin{aligned} m(\ddot{x} - \omega^2 x)\hat{\imath}_{\mathrm{B}} + m(2\omega\dot{x} + \dot{\omega}x)\hat{\jmath}_{\mathrm{B}}. &= \vec{f}_{\mathrm{r}/y/\mathcal{W}} + \vec{f}_{\mathrm{r}/y/\mathcal{S}} \\ &= f_{\mathrm{r}/y/\mathcal{W}}\hat{\jmath}_{\mathrm{B}} + f_{\mathrm{r}/y/\mathcal{S}}\hat{\imath}_{\mathrm{B}}. \end{aligned} \tag{7.6.25}$$

When the bead reaches the stopper, $x = \ell$, $\dot{x} = 0$, and $\ddot{x} = 0$, and thus

$$\vec{f}_{\mathrm{r}/y/\mathcal{W}} = m\dot{\omega}x\hat{\jmath}_{\mathrm{B}} = \vec{f}_{\mathrm{aa}}, \tag{7.6.26}$$

$$\vec{f}_{\mathrm{r}/y/\mathcal{S}} = -m\omega^2 x\hat{\imath}_{\mathrm{B}} = \vec{f}_{\mathrm{cp}}. \tag{7.6.27}$$

Therefore, the transverse reaction force $f_{\mathrm{r}/y/\mathcal{W}}$ on the bead due to the wire is the angular-acceleration reaction force, and the longitudinal reaction force $f_{\mathrm{r}/y/\mathcal{S}}$ on the bead due to the stopper is the centripetal reaction force. Hence, the transverse reaction force $f_{\mathrm{r}/\mathcal{W}/y}$ on the wire due to the bead is the equal-and-opposite angular-acceleration reaction force $\vec{f}_{\mathrm{Cor,opp}}$, and the longitudinal reaction force $f_{\mathrm{r}/\mathcal{S}/y}$ on the stopper due to the bead is the centrifugal reaction force $\vec{f}_{\mathrm{cf}}$. ◇

7.7 Newton's Second Law of Rotation

Newton's second law of rotation describes the relationship between the change in angular momentum of a body and the external moment on the body. Five versions of this relationship are presented in this section.

Let y be a particle with mass m, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *angular momentum of* y *relative to* w *with respect to* $\mathrm{F_A}$ is defined by

$$\vec{H}_{y/w/\mathrm{A}} \triangleq \vec{r}_{y/w} \times m\vec{v}_{y/w/\mathrm{A}}, \tag{7.7.1}$$

where

$$\vec{v}_{y/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}_{y/w}}. \tag{7.7.2}$$

Now, let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *angular momentum of* $\mathcal{B}$ *relative to* w *with respect to* $\mathrm{F_A}$ is defined by

$$\vec{H}_{\mathcal{B}/w/\mathrm{A}} \triangleq \sum_{i=1}^{l} \vec{H}_{y_i/w/\mathrm{A}}, \tag{7.7.3}$$

where

$$\vec{H}_{y_i/w/\mathrm{A}} = \vec{r}_{y_i/w} \times m_i \vec{v}_{y_i/w/\mathrm{A}}. \tag{7.7.4}$$

The first version of Newton's second law of rotation is given by the following result, which relates the inertial derivative of the angular momentum of a body, which is not necessarily rigid, to the moment on the body relative to an unforced particle. For this result, a force can be applied to a point along a massless link connecting two particles. The mass associated with such a point is defined to be zero.

Fact 7.7.1. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.7.5}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.7.6}$$

Proof. For all $i = 1, \ldots, l$, let m_i be the mass of y_i, and, for all $i, j = 1, \ldots, l$, let $\vec{f}_{ij}$ be the internal force on y_i due to y_j. Since w is an unforced particle, it follows from Fact 8.1.5 that, for all $i = 1, \ldots, l$,

$$m_i \vec{a}_{y_i/w/\mathrm{A}} = \vec{f}_{y_i} + \sum_{j=1}^{l} \vec{f}_{ij}. \tag{7.7.7}$$

Using (7.7.7) and Fact 4.1.7, it follows that the derivative of the angular momentum with respect to $\mathrm{F_A}$ is given by

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{H}}_{y_i/w/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w} \times m_i \vec{v}_{y_i/w/\mathrm{A}} + \vec{r}_{y_i/w} \times m_i \overset{\mathrm{A}\bullet}{\vec{v}}_{y_i/w/\mathrm{A}} \\
&= \vec{v}_{y_i/w/\mathrm{A}} \times m_i \vec{v}_{y_i/w/\mathrm{A}} + \vec{r}_{y_i/w} \times m_i \vec{a}_{y_i/w/\mathrm{A}} \\
&= \vec{r}_{y_i/w} \times m_i \vec{a}_{y_i/w/\mathrm{A}} \\
&= \vec{r}_{y_i/w} \times \left(\vec{f}_{y_i} + \sum_{j=1}^{l} \vec{f}_{ij} \right).
\end{aligned}$$

Summing over the particles $y_1, \ldots, y_l$ yields

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} &= \sum_{i=1}^{l} \overset{\mathrm{A}\bullet}{\vec{H}}_{y_i/w/\mathrm{A}} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \left(\vec{f}_{y_i} + \sum_{j=1}^{l} \vec{f}_{ij} \right) \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} + \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \sum_{j=1}^{l} \vec{f}_{ij} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/w}.
\end{aligned}$$

Note that Fact 6.5.11 implies that the internal forces are balanced and the torque $\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \sum_{j=1}^{l} \vec{f}_{ij}$ on $\mathcal{B}$ due to all of the internal forces is zero. Furthermore, this term is zero due to the fact that, for all distinct i and j, $\vec{f}_{ij} = -\vec{f}_{ji}$ and $\vec{f}_{ij}$ and $\vec{r}_{y_i/y_j}$ are parallel. □

Note that the change in angular momentum given by Fact 7.7.1 is not affected by internal forces.

Fact 7.7.1 relates the inertial change of the angular momentum of a body to the moment on the body. This result is a direct consequence of Newton's second law, and thus the angular momentum of the body and the moment on the body are both defined relative to the unforced particle w, while the angular momentum and its derivative are defined with respect to an inertial frame. The moment on the body is due to all of the external forces on points in the body. Equation (7.7.5) is applicable to a body that rotates around a pin joint w relative to which all moments are determined and whose motion coincides with the motion of an unforced particle.

Since the moment $\vec{M}_{\mathcal{B}/w}$ does not depend on the inertial frame $\mathrm{F_A}$, it follows that the change in angular momentum $\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}$ is independent of the choice of the inertial frame.

Fact 7.7.2. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be inertial frames. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}\,. \tag{7.7.8}$$

If the external forces $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, then the moment $\vec{M}_{\mathcal{B}/w}$ is independent of the point w. Consequently, the change in angular momentum $\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}$ is independent of the choice of the point w with zero inertial acceleration.

Fact 7.7.3. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, assume that $\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}$ are balanced, let w and w' be be points with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w'/\mathrm{A}}\,. \tag{7.7.9}$$

The following result, which is the second version of Newton's second law of rotation, considers the change in angular momentum of a body relative to an arbitrary point z.

Fact 7.7.4. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let m_B be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, let w and z be points, and let $\mathrm{F_A}$ be a frame. Then,

$$\vec{H}_{\mathcal{B}/w/\mathrm{A}} = \vec{H}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{z/w} \times m_\mathcal{B} \vec{v}_{\mathrm{c}/w/\mathrm{A}}. \tag{7.7.10}$$

Now, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, assume that w is an unforced particle, and assume that $\mathrm{F_A}$ is an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.7.11}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.7.12}$$

Proof. For all $i = 1, \ldots, l$, let m_i be the mass of y_i. To derive (7.7.10), note that

$$\begin{aligned}
\vec{H}_{\mathcal{B}/w/\mathrm{A}} &= \sum_{i=1}^{l} \vec{H}_{y_i/w/\mathrm{A}} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times m_i \vec{v}_{y_i/w/\mathrm{A}} = \sum_{i=1}^{l} (\vec{r}_{y_i/z} + \vec{r}_{z/w}) \times m_i (\vec{v}_{y_i/z/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}}) \\
&= \sum_{i=1}^{l} \vec{H}_{y_i/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{z/w} \times m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}} \\
&= \vec{H}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{z/w} \times m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}}.
\end{aligned}$$

Now, assume that $\mathrm{F_A}$ is an inertial frame and w is an unforced particle. Then,

$$\begin{aligned}
\vec{M}_{\mathcal{B}/w} &= \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i} = \sum_{i=1}^{l} (\vec{r}_{y_i/z} + \vec{r}_{z/w}) \times \vec{f}_{y_i} \\
&= \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i} \right) + \vec{r}_{z/w} \times \sum_{i=1}^{l} \vec{f}_{y_i} = \vec{M}_{\mathcal{B}/z} + \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}},
\end{aligned}$$

where the net force on the body is

$$\vec{f}_{\mathcal{B}} \triangleq \sum_{i=1}^{l} \vec{f}_{y_i}.$$

Using (7.7.5), differentiating (7.7.10), and using (7.5.5) we thus have

$$\begin{aligned}
\vec{M}_{\mathcal{B}/z} &= \vec{M}_{\mathcal{B}/w} - \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}} = \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} - \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}} \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{v}_{\mathrm{c}/z/\mathrm{A}} \times m_{\mathcal{B}} \vec{v}_{z/w/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} \\
&\qquad + \vec{v}_{z/w/\mathrm{A}} \times m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}} + \vec{r}_{z/w} \times m_{\mathcal{B}} \vec{a}_{\mathrm{c}/w/\mathrm{A}} - \vec{r}_{z/w} \times \vec{f}_{\mathcal{B}} \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}} (\vec{v}_{\mathrm{c}/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}} \times \vec{v}_{\mathrm{c}/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}} (\vec{v}_{\mathrm{c}/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}} - \vec{v}_{\mathrm{c}/w/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}} (\vec{v}_{\mathrm{c}/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}} + \vec{v}_{w/\mathrm{c}/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}})
\end{aligned}$$

$$
\begin{aligned}
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \vec{v}_{w/\mathrm{c}/\mathrm{A}}) \times \vec{v}_{z/w/\mathrm{A}} \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} + m_{\mathcal{B}}(\vec{v}_{w/z/\mathrm{A}} \times \vec{v}_{z/w/\mathrm{A}}) \\
&= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}}. \qquad \square
\end{aligned}
$$

If the point z in Fact 7.7.4 has zero inertial acceleration w', then $\vec{a}_{z/w/\mathrm{A}} = \vec{a}_{w'/w/\mathrm{A}} = 0$, and thus (7.7.11) specializes to (7.7.5). In particular, by setting $z = w$, Fact 7.7.4 specializes to Fact 7.7.1.

Note that the forces at z do not contribute to the moment $\vec{M}_{\mathcal{B}/z}$. Therefore, if z is a point at which reaction forces occur (such as a pin joint), then $\vec{M}_{\mathcal{B}/z}$ can be determined without needing to know the reaction forces at z.

Choosing z in Fact 7.7.4 to be the center of mass yields the following result, which is the third version of Newton's second law of rotation.

Fact 7.7.5. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let m_{B} be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, let w be a point, and let $\mathrm{F_A}$ be a frame. Then,

$$\vec{H}_{\mathcal{B}/w/\mathrm{A}} = \vec{H}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}} \vec{v}_{\mathrm{c}/w/\mathrm{A}}. \tag{7.7.13}$$

Now, assume that $\mathrm{F_A}$ is an inertial frame, and, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = \vec{M}_{\mathcal{B}/\mathrm{c}}, \tag{7.7.14}$$

where the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i}. \tag{7.7.15}$$

Note that (7.7.14) has the same form as (7.7.5) except that the unforced particle w in (7.7.5) is replaced by the center of mass c. This may seem surprising since there is no assumption in Fact 7.7.5 that c has zero inertial acceleration. To understand why (7.7.14) is valid, notice that (7.7.5) concerns only the rotational dynamics of the rigid body $\mathcal{B}$. These rotational dynamics are determined by the moment $\vec{M}_{\mathcal{B}/w}$ but not the details of the forces that give rise to $\vec{M}_{\mathcal{B}/w}$. In view of this fact, assume that an additional force $-\vec{f}_{\mathcal{B}}$ is applied to an arbitrary point in $\mathcal{B}$. Then, the net force on $\mathcal{B}$ is balanced, and the resulting torque $\vec{T}_{\mathcal{B}}$ is equal to the original moment $\vec{M}_{\mathcal{B}/w}$. Since $\vec{T}_{\mathcal{B}} = \vec{M}_{\mathcal{B}/w}$, it follows that the additional force $-\vec{f}_{\mathcal{B}}$ does not change the rotational dynamics given by (7.7.5). In addition, since, with the additional force $-\vec{f}_{\mathcal{B}}$, the net force is balanced, it follows from Fact 7.5.1 that the center of mass of $\mathcal{B}$ is unforced. Therefore, c can be chosen as the point w in (7.7.5). Consequently, from the point of view of the rotational dynamics, the center of mass plays the role of a point with zero inertial acceleration. It should be stressed, however, that the additional force $-\vec{f}_{\mathcal{B}}$ is not actually applied to $\mathcal{B}$, and thus c may have nonzero inertial acceleration.

The following result expresses the moment on a body relative to an arbitrary point in terms of the moment on the body relative to the center of mass of the body.

Fact 7.7.6. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let w be a point with zero inertial acceleration, let z be a point, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\vec{M}_{\mathcal{B}/z} = \vec{M}_{\mathcal{B}/\mathrm{c}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{a}_{\mathrm{c}/w/\mathrm{A}}. \tag{7.7.16}$$

Proof. Using Fact 7.5.1 it follows that

$$\begin{aligned}\vec{M}_{\mathcal{B}/z} &= \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i} = \sum_{i=1}^{l} (\vec{r}_{y_i/\mathrm{c}} + \vec{r}_{\mathrm{c}/z}) \times \vec{f}_{y_i} \\ &= \left(\sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i} \right) + \vec{r}_{\mathrm{c}/z} \times \sum_{i=1}^{l} \vec{f}_{y_i} \\ &= \vec{M}_{\mathcal{B}/\mathrm{c}} + \vec{r}_{\mathrm{c}/z} \times \vec{f}_\mathcal{B} = \vec{M}_{\mathcal{B}/\mathrm{c}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{a}_{\mathrm{c}/w/\mathrm{A}}. \end{aligned}$$ □

Applying Fact 7.7.6 to Fact 7.7.5 yields the following result, which is the fourth version of Newton's second law of rotation.

Fact 7.7.7. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let w be a point with zero inertial acceleration, let z be a point, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_\mathcal{B} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.7.17}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.7.18}$$

Setting $z = w$ in Fact (7.7.7) yields the following result, which is the fifth version of Newton's second law of rotation.

Fact 7.7.8. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/w} \times m_\mathcal{B} \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.7.19}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.7.20}$$

Let $\mathcal{B}$ be a body, and let w be either an unforced particle or the center of mass of $\mathcal{B}$. Then, Fact

7.7.1 and Fact 7.7.5 imply that

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.7.21}$$

Now, assume that $\vec{M}_{\mathcal{B}/w}\big|_{\mathrm{A}}$ has a component that is identically zero. Then, (7.7.21) implies that the corresponding component of $\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}}\Big|_{\mathrm{A}}$ is identically zero. In this case, *conservation of angular momentum* holds along one of the axes of $\mathrm{F_A}$, and the corresponding component of the angular momentum is a constant of the motion relative to $\mathrm{F_A}$. If $\vec{M}_{\mathcal{B}/w} = 0$, then angular momentum is conserved along all three axes of $\mathrm{F_A}$, and all three components of the momentum are constants of the motion relative to $\mathrm{F_A}$.

7.8 Effect of Gravity on Translational Momentum and Angular Momentum

If the bodies in Facts 7.5.1, 7.7.1, 7.7.4, 7.7.5, 7.7.7, and 7.7.8 are subject to uniform gravity, then the external force $\vec{f}_{y_i}$ on y_i includes the force $m_i\vec{g}$ due to uniform gravity, where $\vec{g}$ is the acceleration due to uniform gravity. When the body is subject to uniform gravity, we now consider the effect of uniform gravity separately from the nongravitational external forces. Throughout this chapter we assume that uniform gravity is uniform over the body.

The following result restates Fact 7.5.1 with uniform gravity separated from the nongravitational external forces.

Fact 7.8.1. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to uniform gravity, for all $i = 1, \ldots, l$, let $\vec{f}_{\mathrm{ng}/y_i}$ be the nongravitational external force on the particle m_i, let $\mathrm{F_A}$ be an inertial frame, let c be the center of mass of $\mathcal{B}$, and let w be a point with zero inertial acceleration. Then,

$$m_\mathcal{B}\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{f}_\mathcal{B}, \tag{7.8.1}$$

where the net force $\vec{f}_\mathcal{B}$ on $\mathcal{B}$ is given by

$$\vec{f}_\mathcal{B} \overset{\triangle}{=} \left(\sum_{i=1}^{l} \vec{f}_{\mathrm{ng}/y_i}\right) + m_\mathcal{B}\vec{g}. \tag{7.8.2}$$

Note that (7.8.1) and (7.8.2) imply that, if the only external force present is uniform gravity, then $\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{g}$.

The following result restates Fact 7.7.1 with uniform gravity separated from the nongravitational external forces. This result is based on Fact 6.5.10.

Fact 7.8.2. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to uniform gravity, for all $i = 1, \ldots, l$, let $\vec{f}_{\mathrm{ng}/y_i}$ be the nongravitational external force on y_i, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.8.3}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{\mathrm{ng}/y_i} \right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}} \vec{g}. \tag{7.8.4}$$

The following result follows from Fact 7.7.4 with uniform gravity separated from the nongravitational external forces.

Fact 7.8.3. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to uniform gravity, for all $i = 1, \ldots, l$, let $\vec{f}_{\mathrm{ng}/y_i}$ be the nongravitational external force on y_i, let w be a point with zero inertial acceleration, let z be a point, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.8.5}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{\mathrm{ng}/y_i} \right) + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}} \vec{g}. \tag{7.8.6}$$

The following result follows from Fact 7.7.5 with uniform gravity separated from the nongravitational external forces. In this case, uniform gravity has no effect on the change in angular momentum.

Fact 7.8.4. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, assume that $\mathcal{B}$ is subject to uniform gravity, for all $i = 1, \ldots, l$, let $\vec{f}_{\mathrm{ng}/y_i}$ be the nongravitational external force on y_i, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = \vec{M}_{\mathcal{B}/\mathrm{c}}, \tag{7.8.7}$$

where the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{\mathrm{ng}/y_i}. \tag{7.8.8}$$

Proof. For all $i = 1, \ldots, l$, let m_i be the mass of y_i. Note that

$$\begin{aligned}\vec{M}_{\mathcal{B}/\mathrm{c}} &= \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} (\vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{\mathrm{ng}/y_i} + \vec{r}_{y_i/\mathrm{c}} \times m_i \vec{g}) \\ &= \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{\mathrm{ng}/y_i} + \left(\sum_{i=1}^{l} m_i \vec{r}_{y_i/\mathrm{c}} \right) \times \vec{g} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{\mathrm{ng}/y_i}. \qquad \square\end{aligned}$$

The following result restates Fact 7.7.7 with uniform gravity separated from the nongravitational external forces.

Fact 7.8.5. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for all $i = 1, \ldots, l$, let $\vec{f}_{\mathrm{ng}/y_i}$ be the nongravitational external force on y_i, let w be a point with zero inertial acceleration, let z be a point, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.8.9}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{\mathrm{ng}/y_i} \right) + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{g}. \tag{7.8.10}$$

The following result restates Fact 7.7.8 with uniform gravity separated from the nongravitational external forces.

Fact 7.8.6. Let $\mathcal{B}$ be a body with points $y_1, \ldots, y_l$, let $\mathcal{B}$ be the mass of $\mathcal{B}$, let c be the center of mass of $\mathcal{B}$, for all $i = 1, \ldots, l$, let $\vec{f}_{\mathrm{ng}/y_i}$ be the nongravitational external force on y_i, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. Then,

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\, \vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.8.11}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{\mathrm{ng}/y_i} \right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\, \vec{g}. \tag{7.8.12}$$

7.9 Euler's Equation for the Rotational Dynamics of a Rigid Body

Specialization of Newton's second law of rotation to the case of a rigid body yields *Euler's equation*. This section presents five versions of Euler's equation. To develop these equations, the following result expresses the angular momentum of a body in terms of the physical inertia matrix defined by Definition 6.2.1.

Fact 7.9.1. Let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and let z be a point. Then,

$$\vec{H}_{\mathcal{B}/z/\mathrm{A}} = \vec{\vec{J}}_{\mathcal{B}/z}\, \vec{\omega}_{\mathrm{B/A}} + \vec{H}_{\mathcal{B}/z/\mathrm{B}}, \tag{7.9.1}$$

where

$$\vec{\vec{J}}_{\mathcal{B}/z} = \sum_{i=1}^{l} m_i\, \vec{r}^{\,\times\prime}_{y_i/z}\, \vec{r}^{\,\times}_{y_i/z}. \tag{7.9.2}$$

Proof. Using (7.7.4) and the transport theorem, it follows that

$$\begin{aligned}
\vec{H}_{y_i/z/\mathrm{A}} &= \vec{r}_{y_i/z} \times m_i \vec{v}_{y_i/z/\mathrm{A}} \\
&= \vec{r}_{y_i/z} \times m_i \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/z} \\
&= \vec{r}_{y_i/z} \times m_i \left(\overset{\mathrm{B}\bullet}{\vec{r}}_{y_i/z} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z} \right)
\end{aligned}$$

$$= \vec{r}_{y_i/z} \times m_i \left(\vec{v}_{y_i/z/\mathrm{B}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right).$$

Summing over the particles in the body and using (2.9.9) yields

$$\begin{aligned}
\vec{H}_{\mathcal{B}/z/\mathrm{A}} &= \sum_{i=1}^{l} \vec{r}_{y_i/z} \times m_i \left(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right) + \sum_{i=1}^{l} \vec{r}_{y_i/z} \times m_i \vec{v}_{y_i/z/\mathrm{B}} \\
&= \sum_{i=1}^{l} \vec{r}_{y_i/z} \times m_i \left(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{y_i/z}\right) + \vec{H}_{\mathcal{B}/z/\mathrm{B}} \\
&= -\sum_{i=1}^{l} m_i \vec{r}_{y_i/z} \times \left(\vec{r}_{y_i/z} \times \vec{\omega}_{\mathrm{B/A}}\right) + \vec{H}_{\mathcal{B}/z/\mathrm{B}} \\
&= \sum_{i=1}^{l} m_i \vec{r}^{\,\times\prime}_{y_i/z} \vec{r}^{\,\times}_{y_i/z} \vec{\omega}_{\mathrm{B/A}} + \vec{H}_{\mathcal{B}/z/\mathrm{B}} \\
&= \vec{J}_{\mathcal{B}/z} \vec{\omega}_{\mathrm{B/A}} + \vec{H}_{\mathcal{B}/z/\mathrm{B}}.
\end{aligned}$$

□

The following result specializes Fact 7.9.1 to the case where $\mathcal{B}$ is a rigid body and the point z is fixed in $\mathcal{B}$.

Fact 7.9.2. Let $\mathcal{B}$ be a rigid body with particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $\mathrm{F_A}$ be a frame, let $\mathrm{F_B}$ be a body-fixed frame, and let z be a point that is fixed in $\mathcal{B}$. Then,

$$\vec{H}_{\mathcal{B}/z/\mathrm{B}} = 0, \tag{7.9.3}$$

and thus

$$\vec{H}_{\mathcal{B}/z/\mathrm{A}} = \vec{J}_{\mathcal{B}/z} \vec{\omega}_{\mathrm{B/A}}. \tag{7.9.4}$$

Furthermore,

$$\overset{\mathrm{B}\bullet}{\vec{J}}_{\mathcal{B}/z} = 0, \tag{7.9.5}$$

and thus

$$\overset{\mathrm{B}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} = \vec{J}_{\mathcal{B}/z} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}}. \tag{7.9.6}$$

Proof. Since $\mathcal{B}$ is a rigid body and the point z is fixed in $\mathcal{B}$, it follows that, for all $i = 1, \ldots, l$, $\overset{\mathrm{B}\bullet}{\vec{r}}_{y_i/z} = 0$. Hence, (7.9.3) is satisfied, and thus (7.9.1) implies (7.9.4). Furthermore, differentiating (7.9.2) yields (7.9.5). Finally, (7.9.4), (7.9.5), and Fact 4.1.6 yield (7.9.6). □

The following result, which is the first version of Euler's equation, follows from Fact 7.7.1 and Fact 7.8.2 in the case where $\mathcal{B}$ is a rigid body.

Fact 7.9.3. Let $\mathcal{B}$ be a rigid body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, and let w be a point that is fixed in $\mathcal{B}$ and has zero inertial acceleration. Then,

$$\vec{J}_{\mathcal{B}/w}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.9.7}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l}\vec{M}_{y_i/w} = \sum_{i=1}^{l}\vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.9.8}$$

If, in addition, $\mathcal{B}$ is subject to uniform gravity, then $\vec{M}_{\mathcal{B}/w}$ is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l}\vec{M}_{y_i/w} = \left(\sum_{i=1}^{l}\vec{r}_{y_i/w} \times \vec{f}_{\mathrm{ng}/y_i}\right) + \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g}, \tag{7.9.9}$$

where, for all $i = 1, \ldots, l$, $\vec{f}_{\mathrm{ng}/y_i}$ is the nongravitational external force on y_i.

Proof. Using (7.7.5) as well as (7.9.4) and (7.9.6) with $z = w$, it follows that

$$\begin{aligned}
\vec{M}_{\mathcal{B}/w} &= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/w/\mathrm{A}} \\
&= \overset{\mathrm{A}\bullet}{\overbrace{\vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}}}} \\
&= \overset{\mathrm{B}\bullet}{\overbrace{\vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}}}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}} \\
&= \vec{J}_{\mathcal{B}/w}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}}.
\end{aligned}$$

□

The following result, which is the second version of Euler's equation, follows from Fact 7.7.4 and Fact 7.8.3.

Fact 7.9.4. Let $\mathcal{B}$ be a rigid body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, let w be a point with zero inertial acceleration, let c be the center of mass of $\mathcal{B}$, and let z be a point that is fixed in $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.9.10}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l}\vec{M}_{y_i/z} = \sum_{i=1}^{l}\vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.9.11}$$

If, in addition, $\mathcal{B}$ is subject to uniform gravity, then $\vec{M}_{\mathcal{B}/z}$ is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l}\vec{M}_{y_i/z} = \sum_{i=1}^{l}\vec{r}_{y_i/z} \times \vec{f}_{\mathrm{ng}/y_i} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\vec{g}, \tag{7.9.12}$$

where, for all $i = 1, \ldots, l$, $\vec{f}_{\mathrm{ng}/y_i}$ is the nongravitational external force on y_i

Proof. Using (7.7.11), (7.9.4), and (7.9.6), it follows that

$$\begin{aligned}
\vec{M}_{\mathcal{B}/z} &= \overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{B}/z/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} \\
&= \overbrace{\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}}}^{\mathrm{A}\bullet} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} \\
&= \overbrace{\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}}}^{\mathrm{B}\bullet} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}} \\
&= \vec{J}_{\mathcal{B}/z}\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \vec{r}_{\mathrm{c}/z} \times m_{\mathcal{B}}\, \vec{a}_{z/w/\mathrm{A}}.
\end{aligned}$$

□

The following result, which is the third version of Euler's equation, follows from Fact 7.7.5 and Fact 7.8.4.

Fact 7.9.5. Let $\mathcal{B}$ be a rigid body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, and let c be the center of mass of $\mathcal{B}$. Then,

$$\vec{J}_{\mathcal{B}/\mathrm{c}}\, \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B}/\mathrm{A}} = \vec{M}_{\mathcal{B}/\mathrm{c}}, \tag{7.9.13}$$

where the moment on $\mathcal{B}$ relative to c is given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{y_i}. \tag{7.9.14}$$

If, in addition, $\mathcal{B}$ is subject to uniform gravity, then (7.9.13) is satisfied with $\vec{M}_{\mathcal{B}/\mathrm{c}}$ given by

$$\vec{M}_{\mathcal{B}/\mathrm{c}} = \sum_{i=1}^{l} \vec{M}_{y_i/\mathrm{c}} = \sum_{i=1}^{l} \vec{r}_{y_i/\mathrm{c}} \times \vec{f}_{\mathrm{ng}/y_i}, \tag{7.9.15}$$

where, for all $i = 1, \ldots, l$, $\vec{f}_{\mathrm{ng}/y_i}$ is the nongravitational external force on y_i.

In free-body analysis, it is often necessary to apply Fact 7.9.5 to a massless rigid body $\mathcal{B}$, such as a massless link, or to a degenerate rigid body, such as a massless link with a particle attached to one of its endpoints. Since $\vec{J}_{\mathcal{B}/\mathrm{c}} = 0$, it follows from (7.9.13) that $\vec{M}_{\mathcal{B}/\mathrm{c}} = 0$, where $\vec{M}_{\mathcal{B}/\mathrm{c}}$ is the sum of the moments relative to c due to all reaction and external forces on $\mathcal{B}$ as well as all reaction and external torques on $\mathcal{B}$.

The following result, which is the fourth version of Euler's equation, follows from Fact 7.7.7 and Fact 7.8.5.

Fact 7.9.6. Let $\mathcal{B}$ be a rigid body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, let c be the center of mass of $\mathcal{B}$, let w be a point with zero inertial acceleration, and let z be a point.

Then,

$$\vec{J}_{\mathcal{B}/c}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/c}\vec{\omega}_{\mathrm{B/A}} + \vec{r}_{c/z} \times m_{\mathcal{B}}\vec{a}_{c/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{7.9.16}$$

where the moment on $\mathcal{B}$ relative to z is given by

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{y_i}. \tag{7.9.17}$$

If, in addition, $\mathcal{B}$ is subject to uniform gravity, then

$$\vec{M}_{\mathcal{B}/z} = \sum_{i=1}^{l} \vec{M}_{y_i/z} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/z} \times \vec{f}_{\mathrm{ng}/y_i}\right) + \vec{r}_{c/z} \times m_{\mathcal{B}}\vec{g}, \tag{7.9.18}$$

where, for all $i = 1, \ldots, l$, $\vec{f}_{\mathrm{ng}/y_i}$ is the nongravitational external force on y_i.

The following result, which is the fifth version of Euler's equation, follows from Fact 7.7.8 and Fact 7.8.6.

Fact 7.9.7. Let $\mathcal{B}$ be a rigid body with points $y_1, \ldots, y_l$, for all $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ be the external force on y_i, let $m_{\mathcal{B}}$ be the mass of $\mathcal{B}$, let $\mathrm{F_B}$ be a body-fixed frame, let $\mathrm{F_A}$ be an inertial frame, let c be the center of mass of $\mathcal{B}$, and let w be a point with zero inertial acceleration. Then,

$$\vec{J}_{\mathcal{B}/c}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/c}\vec{\omega}_{\mathrm{B/A}} + \vec{r}_{c/w} \times m_{\mathcal{B}}\vec{a}_{c/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/w}, \tag{7.9.19}$$

where the moment on $\mathcal{B}$ relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{y_i}. \tag{7.9.20}$$

If, in addition, $\mathcal{B}$ is subject to uniform gravity, then

$$\vec{M}_{\mathcal{B}/w} = \sum_{i=1}^{l} \vec{M}_{y_i/w} = \left(\sum_{i=1}^{l} \vec{r}_{y_i/w} \times \vec{f}_{\mathrm{ng}/y_i}\right) + \vec{r}_{c/w} \times m_{\mathcal{B}}\vec{g}, \tag{7.9.21}$$

where, for all $i = 1, \ldots, l$, $\vec{f}_{\mathrm{ng}/y_i}$ is the nongravitational external force on y_i.

7.10 Euler's Equation and the Eigenaxis Angle Vector†

Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames, and recall from (4.9.11) that

$$\vec{\omega}_{\mathrm{B/A}} = \vec{S}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}, \tag{7.10.1}$$

where

$$\vec{S} \triangleq \alpha\vec{\Theta}_{\mathrm{B/A}}^{\times 2} + \beta\vec{\Theta}_{\mathrm{B/A}}^{\times} + \vec{I}, \tag{7.10.2}$$

$$\alpha \triangleq \frac{\theta_{\mathrm{B/A}} - \sin\theta_{\mathrm{B/A}}}{\theta_{\mathrm{B/A}}^3}, \quad \beta \triangleq \frac{\cos\theta_{\mathrm{B/A}} - 1}{\theta_{\mathrm{B/A}}^2}. \tag{7.10.3}$$

Consequently,

$$\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \vec{S}\,\overset{\mathrm{B}\bullet\bullet}{\vec{\Theta}}_{\mathrm{B/A}} + \overset{\mathrm{B}\bullet\mathrm{B}\bullet}{\vec{S}\vec{\Theta}}_{\mathrm{B/A}}, \tag{7.10.4}$$

where

$$\overset{\mathrm{B}\bullet}{\vec{S}} = \dot{\alpha}\vec{\Theta}^{\times 2}_{\mathrm{B/A}} + \alpha\Big(\overset{\mathrm{B}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}}\,\vec{\Theta}^{\times}_{\mathrm{B/A}} + \vec{\Theta}^{\times}_{\mathrm{B/A}}\,\overset{\mathrm{B}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}}\Big) + \dot{\beta}\vec{\Theta}^{\times}_{\mathrm{B/A}} + \beta\,\overset{\mathrm{B}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}}\,. \tag{7.10.5}$$

Now, substituting (7.10.1) and (7.10.4) into (7.9.5) yields Euler's equation in the form

$$\vec{J}_{\mathcal{B}/\mathrm{c}}\Big(\vec{S}\,\overset{\mathrm{B}\bullet\bullet}{\vec{\Theta}}_{\mathrm{B/A}} + \overset{\mathrm{B}\bullet\mathrm{B}\bullet}{\vec{S}\vec{\Theta}}_{\mathrm{B/A}}\Big) + \Big(\vec{S}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\Big)\times\Big(\vec{J}_{\mathcal{B}/\mathrm{c}}\vec{S}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\Big) = \vec{M}_{\mathcal{B}/\mathrm{c}}. \tag{7.10.6}$$

Next, note that

$$\vec{S}\vec{\Theta}_{\mathrm{B/A}} = \vec{\Theta}_{\mathrm{B/A}}, \tag{7.10.7}$$

and thus,

$$\overset{\mathrm{B}\bullet}{\vec{S}}\,\vec{\Theta}_{\mathrm{B/A}} + \vec{S}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\,. \tag{7.10.8}$$

Furthermore, it follows from (7.10.5) that

$$\overset{\mathrm{B}\bullet}{\vec{S}}\,\vec{\Theta}_{\mathrm{B/A}} = \alpha\vec{\Theta}_{\mathrm{B/A}}\times\big(\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\times\vec{\Theta}_{\mathrm{B/A}}\big) + \beta\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\times\vec{\Theta}_{\mathrm{B/A}}. \tag{7.10.9}$$

Hence,

$$\vec{S}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} = \overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} - \alpha\vec{\Theta}_{\mathrm{B/A}}\times\big(\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\times\vec{\Theta}_{\mathrm{B/A}}\big) - \beta\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\times\vec{\Theta}_{\mathrm{B/A}}. \tag{7.10.10}$$

In addition,

$$\overset{\mathrm{B}\bullet\mathrm{B}\bullet}{\vec{S}\vec{\Theta}}_{\mathrm{B/A}} = \dot{\alpha}\vec{\Theta}^{\times 2}_{\mathrm{B/A}}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} + \alpha\,\overset{\mathrm{B}\bullet\times}{\vec{\Theta}}_{\mathrm{B/A}}\,\vec{\Theta}^{\times}_{\mathrm{B/A}}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}} + \dot{\beta}\vec{\Theta}^{\times}_{\mathrm{B/A}}\,\overset{\mathrm{B}\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\,. \tag{7.10.11}$$

Using (7.10.2), (7.10.10), and (7.10.11), it follows that (7.10.6) can be written without $\vec{S}$ appearing explicitly.

7.11 Free-Body Analysis

Consider a body comprised of multiple rigid sub-bodies. The dynamics of the body can be analyzed by considering the dynamics of each sub-body separately, that is, as a collection of free (unconstrained) rigid bodies subject to external forces and moments along with internal forces, moments, and torques. Unlike external forces, moments, and torques, the reaction forces and moments must be determined through simultaneous analysis of the statics or dynamics of each rigid body.

The reaction forces and moments can be represented as either physical vectors or they may be resolved in a convenient frame. Whenever convenient, it is advisable to solve for the reaction forces and moments as physical vectors.

If the reaction forces, moments,and torques are conservative, then Lagrangian dynamics can be used to derive the equations of motion without determining the reaction forces, moments, and torques. The conservative reaction forces, moments, and torques can be found subsequently by applying Newton-Euler dynamics. If, however, the reaction forces, moments, and torques dissipate energy, then Lagrangian dynamics cannot be used, and Newton-Euler methods must be used exclusively.

For a degenerate rigid body, such as a massless link with a single particle attached to one of its ends, all components of the net force and torque (arising from either external or internal forces, moments, and torques) that can produce infinite translational or angular acceleration must be zero. Hence, in the special case of a massless rigid link, the net forces, moments, and torques must be zero.

To convey the basic idea of free-body analysis, consider a body comprised of rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$, and let $\vec{f}_1$ and $\vec{f}_2$ denote the total external forces on $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Furthermore, let $\vec{f}_{\mathrm{R12}}$ denote the reaction force on $\mathcal{B}_1$ due to its interaction with $\mathcal{B}_2$, and let $\vec{f}_{\mathrm{R21}}$ denote the reaction force on $\mathcal{B}_2$ due to its interaction with $\mathcal{B}_1$. Letting c_1 and c_2 denote the centers of mass of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, w be an unforced particle, and $\mathrm{F_A}$ be an inertial frame, it follows from Fact 7.5.1 that

$$m_1\vec{a}_{\mathrm{c}_1/w/\mathrm{A}} = \vec{f}_1 + \vec{f}_{\mathrm{R12}}, \tag{7.11.1}$$

$$m_2\vec{a}_{\mathrm{c}_2/w/\mathrm{A}} = \vec{f}_2 + \vec{f}_{\mathrm{R21}}. \tag{7.11.2}$$

By considering (7.11.1) and (7.11.2) separately, the resulting equations can be combined to provide the components of the reaction forces. Substituting the reaction forces back into (7.11.1) and (7.11.2) yields the equations of motion for each rigid body.

Note that summing (7.11.1) and (7.11.2) and using the fact that $\vec{f}_{\mathrm{R21}} = -\vec{f}_{\mathrm{R12}}$ yields

$$m_1\vec{a}_{\mathrm{c}_1/w/\mathrm{A}} + m_2\vec{a}_{\mathrm{c}_2/w/\mathrm{A}} = \vec{f}_1 + \vec{f}_2. \tag{7.11.3}$$

Letting c_3 denote the center of mass of the body, it follows from (6.1.2) that

$$\vec{a}_{\mathrm{c}_3/w/\mathrm{A}} = \frac{1}{m_1+m_2}(m_1\vec{a}_{\mathrm{c}_1/w/\mathrm{A}} + m_2\vec{a}_{\mathrm{c}_2/w/\mathrm{A}}). \tag{7.11.4}$$

Now, combining (7.11.3) with (7.11.4) yields

$$(m_1+m_2)\vec{a}_{\mathrm{c}_3/w/\mathrm{A}} = \vec{f}_1 + \vec{f}_2, \tag{7.11.5}$$

which is Newton's second law for a body with two rigid sub-bodies.

7.12 Theoretical Problems

Problem 7.12.1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be rigid bodies whose masses are m_1 and m_2, respectively, let $\vec{f}$ be a force on $\mathcal{B}_1$, and assume that $\mathcal{B}_2$ is in direct contact with $\mathcal{B}_1$ in such a way that both bodies move along a straight line. Determine the magnitude of the reaction force between $\mathcal{B}_1$ and $\mathcal{B}_2$. Finally, extend this result to the case of $n \geq 3$ bodies.

Problem 7.12.2. Let $\mathrm{F_A}$ be an inertial frame, let y_1 and y_2 be particles whose masses are m_1 and m_2, respectively, and assume that the force $\vec{f}_1$ is applied to y_1 and the force $\vec{f}_2$ is applied to y_2. Show that

$$\frac{m_1 m_2}{m_1+m_2}\vec{a}_{y_1/y_2/\mathrm{A}} = \frac{m_2}{m_1+m_2}\vec{f}_1 - \frac{m_1}{m_1+m_2}\vec{f}_2. \tag{7.12.1}$$

In particular, show that, if $\vec{f}_1 = \vec{f}$ and $\vec{f}_2 = -\vec{f}$, then

$$\frac{m_1 m_2}{m_1+m_2}\vec{a}_{y_1/y_2/\mathrm{A}} = \vec{f}. \tag{7.12.2}$$

Now, specialize the result to the cases where the force is due to a spring, dashpot, or inerter connecting y_1 and y_2. In each case, provide a differential equation whose solution is the distance between the particles. In addition, in each case, use the solution of the differential equation to describe the qualitative behavior of the distance.

Problem 7.12.3. Let $\mathrm{F_A}$ be an inertial frame, let y_1, y_2, and y_3 be particles whose masses are m_1, m_2, and m_3, respectively, and assume that the forces $\vec{f}_{12}$ and $\vec{f}_{13}$ are applied to y_1, the forces $-\vec{f}_{12}$ and $\vec{f}_{23}$ are applied to y_2, and the forces $-\vec{f}_{13}$ and $-\vec{f}_{23}$ are applied to y_3. Show that

$$\frac{m_1 m_3}{m_1+m_3}\vec{a}_{y_1/y_3/\mathrm{A}} = \vec{f}_{13} + \frac{1}{m_1+m_3}(m_1\vec{f}_{23} + m_3\vec{f}_{12}), \tag{7.12.3}$$

$$\frac{m_2 m_3}{m_2+m_3}\vec{a}_{y_2/y_3/\mathrm{A}} = \vec{f}_{23} + \frac{1}{m_2+m_3}(m_2\vec{f}_{13} - m_3\vec{f}_{12}). \tag{7.12.4}$$

Now, specialize the result to the cases in which the forces are due to springs, dashpots, or inerters connecting y_1, y_2, and y_3. In each case, provide a differential equation whose solution is the distance between the particles. Finally, describe the qualitative behavior of the solution in each case.

Problem 7.12.4. Let $\mathrm{F_A}$ be an inertial frame, let y and z be unforced particles, let x be a particle, and let $\vec{f}$ be a force on x. Show that

$$\vec{a}_{x/z/\mathrm{A}} = \vec{a}_{x/y/\mathrm{A}}. \tag{7.12.5}$$

Finally, explain why the equation

$$\vec{v}_{x/z/\mathrm{A}} = \vec{v}_{x/y/\mathrm{A}}. \tag{7.12.6}$$

is not true in general.

Problem 7.12.5. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be inertial frames, and let w and y be unforced particles. Show that

$$\overset{\mathrm{B}\bullet}{\overset{\mathrm{A}\bullet}{\vec{r}}}_{y/w} = \overset{\mathrm{A}\bullet}{\overset{\mathrm{B}\bullet}{\vec{r}}}_{y/w} = 0. \tag{7.12.7}$$

Problem 7.12.6. Consider a rigid body spinning around a principal axis relative to its center of mass without any applied moments. Use Euler's equation to show that the body spins indefinitely around the principal axis.

Chapter Eight
Applications of Newton-Euler Dynamics

8.1 Accelerometer

Consider a mechanical accelerometer constructed from a box with mass M, a spring with stiffness k, a dashpot with damping c, and a proofmass with mass m. An external force $\vec{f}$ is applied to the box. As shown in Figure 8.1.1, the spring and dashpot are vertical, and thus uniform gravity affects both the box and the proofmass. A measurement of q is assumed to be available, and the objective is to determine the inertial acceleration a of z relative to w, that is,

$$\vec{a}_{z/w/\mathrm{A}} = a\hat{k}_{\mathrm{A}}, \tag{8.1.1}$$

where $\mathrm{F_A}$ is an inertial frame. We write $\vec{f} = f\hat{k}_{\mathrm{A}}$ and $\vec{g} = g\hat{k}_{\mathrm{A}}$, where $g = 9.8$ m/s^2.

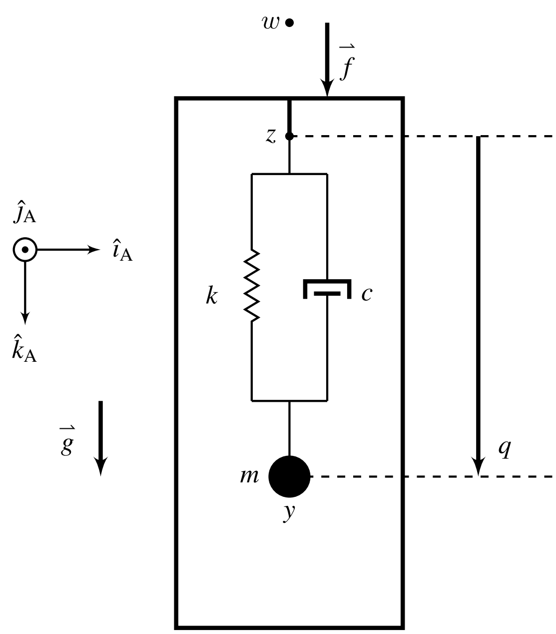


Figure 8.1.1: Accelerometer setup.

Let w be a reference point with zero inertial acceleration, let the point y be the location of the proofmass, let q denote the displacement of the proofmass, where $q = 0$ corresponds to the relaxed spring, and let z be a reference point fixed to the box at the location $q = 0$. For convenience, the center of mass of the box is assumed to be located at z, and thus the center of mass c of the accelerometer relative to z is located at

$$\vec{r}_{\mathrm{c}/z} = \frac{m}{M+m}\vec{r}_{y/z} = \frac{m}{M+m} q\hat{k}_{\mathrm{A}}. \tag{8.1.2}$$

Using

$$\vec{r}_{\mathrm{c}/w} = \vec{r}_{\mathrm{c}/z} + \vec{r}_{z/w}, \tag{8.1.3}$$

differentiating (8.1.3) twice with respect to $\mathrm{F_A}$ yields

$$\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{a}_{\mathrm{c}/z/\mathrm{A}} + \vec{a}_{z/w/\mathrm{A}}. \tag{8.1.4}$$

Now, it follows from Newton's second law that

$$(M+m)\vec{a}_{\mathrm{c}/w/\mathrm{A}} = \vec{f} + (M+m)\vec{g}, \tag{8.1.5}$$

and thus (8.1.4) and (8.1.5) imply that

$$(M+m)\vec{a}_{\mathrm{c}/z/\mathrm{A}} + (M+m)\vec{a}_{z/w/\mathrm{A}} = \vec{f} + (m+M)\vec{g}. \tag{8.1.6}$$

Next, note that (8.1.2) implies that

$$\vec{a}_{\mathrm{c}/z/\mathrm{A}} = \frac{m}{M+m}\ddot{q}\hat{k}_{\mathrm{A}}. \tag{8.1.7}$$

Hence, resolving (8.1.6) in $\mathrm{F_A}$ and using (8.1.2) yields

$$m\ddot{q} + (M+m)a = f + (m+M)g. \tag{8.1.8}$$

The acceleration to be measured is thus given by

$$a = \frac{1}{M+m}f + g - \frac{m}{M+m}\ddot{q}. \tag{8.1.9}$$

In practice, f is usually unknown, and thus (8.1.9) is not useful.

Now, letting $\vec{f}_{\mathrm{proofmass}} \triangleq -(c\dot{q}+kq)\hat{k}_{\mathrm{A}}$ denote the force on the proofmass due to the spring and dashpot, Newton's second law implies that

$$m\vec{a}_{y/w/\mathrm{A}} = \vec{f}_{\mathrm{proofmass}} + m\vec{g}. \tag{8.1.10}$$

Therefore,

$$m\vec{a}_{y/z/\mathrm{A}} + m\vec{a}_{z/w/\mathrm{A}} = \vec{f}_{\mathrm{proofmass}} + m\vec{g}, \tag{8.1.11}$$

which implies that

$$m\ddot{q} + ma = -(c\dot{q}+kq) + mg, \tag{8.1.12}$$

that is,

$$m\ddot{q} + c\dot{q} + kq = m(g-a). \tag{8.1.13}$$

In the case where the box has zero inertial acceleration, that is, $a \equiv 0$, it follows that

$$m\ddot{q} + c\dot{q} + kq = mg, \tag{8.1.14}$$

and thus $q(t) \to mg/k$ as $t \to \infty$, whereas, in the case where the box is freefalling, that is, $a \equiv g$, it follows that

$$m\ddot{q} + c\dot{q} + kq = 0, \tag{8.1.15}$$

and thus $q(t) \to 0$ as $t \to \infty$.

Next, it follows from (8.1.13) that the inertial acceleration is given by

$$a = g - \ddot{q} - \frac{c}{m}\dot{q} - \frac{k}{m}q. \tag{8.1.16}$$

Assuming that m, c, k, and g are known and that q, $\dot{q}$, and $\ddot{q}$ are measured, (8.1.16) provides a measurement

$$a_{\rm meas} = g - \ddot{q} - \frac{c}{m}\dot{q} - \frac{k}{m}q \tag{8.1.17}$$

of the inertial acceleration. When the box has zero inertial acceleration, and thus $q(t) \to mg/k$ as $t \to \infty$, it follows that $a_{\rm meas}(t) \to 0$ as $t \to \infty$, and, when the box is freefalling, and thus $q(t) \to 0$ as $t \to \infty$, it follows that $a_{\rm meas}(t) \to g$ as $t \to \infty$. In practice, however, it may be difficult to obtain estimates of $\dot{q}$ and $\ddot{q}$, and thus the measurement $a_{\rm meas}$ is given in practice by

$$a_{\rm meas} = g - \frac{k}{m}q. \tag{8.1.18}$$

Hence, if $\dot{q} \approx 0$ and $\ddot{q} \approx 0$, then $a_{\rm meas} \approx a$. The accuracy of these approximations depends on the settling time of the proofmass when a is constant over an interval of time.

Another impediment to using (8.1.16) to estimate the inertial acceleration of the box arises from the fact that the box may be rotated by the angle θ relative to the vertical direction. In this case, (8.1.18) becomes

$$a = (\cos\theta)g - \ddot{q} - \frac{c}{m}\dot{q} - \frac{k}{m}q. \tag{8.1.19}$$

Note that (8.1.22) involves θ, which must be known in order to correctly determine a. Now, suppose that the angle θ is unknown, and the accelerometer is implemented without accounting for the effect of uniform gravity. In this case, the resulting measurement after the transient response is given by

$$a_{\rm meas} = -\frac{k}{m}q, \tag{8.1.20}$$

and thus

$$a_{\rm meas} \approx a - (\cos\theta)g. \tag{8.1.21}$$

It follows from (8.1.21) that, when the box has zero inertial acceleration, $a_{\rm meas}(t) \to -g$ as $t \to \infty$, and, when the box is freefalling, $a_{\rm meas}(t) \to 0$ as $t \to \infty$. Both of these measurements are obviously erroneous. More generally, it can be seen that errors in θ entail erroneous measurements of the inertial acceleration. This shows that it is necessary to know the direction θ of uniform gravity relative to the direction of the accelerometer in order to obtain the “corrected” measurement

$$a_{\rm meas} = (\cos\theta)g - \frac{k}{m}q. \tag{8.1.22}$$

8.2 Dynamics of Interconnected Particles

Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, for all $i = 1, \ldots, l$, let $\vec{f}_i$ be the external force on y_i, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. Furthermore, assume that, for all distinct $i, j \in \{1, \ldots, l\}$, the particles y_i and y_j are connected by a dashpot with damping $c_{ij} \geq 0$ and a spring with stiffness $k_{ij} \geq 0$. The forces generated by springs and dashpots are described in Chapter 6. Note that, for all $i, j = 1, \ldots, l$,

$c_{ij} = c_{ji}$, $c_{ii} = 0$, $k_{ij} = k_{ji}$, and $k_{ii} = 0$. Then, for all $i \in \{1, \ldots, l\}$, it follows that

$$m_i \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w} + \sum_{j=1}^{l} c_{ij} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/y_j} + \sum_{j=1}^{l} k_{ij} \vec{r}_{y_i/y_j} = \vec{f}_i, \tag{8.2.1}$$

and thus

$$M \begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + C \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + K \begin{bmatrix} \vec{r}_{y_1/w} \\ \vdots \\ \vec{r}_{y_l/w} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_l \end{bmatrix}, \tag{8.2.2}$$

where

$$M \triangleq \operatorname{diag}(m_1, \ldots, m_l), \tag{8.2.3}$$

$$C \triangleq \begin{bmatrix} \sum_{j=1}^{l} c_{1j} & -c_{12} & -c_{13} & \cdots & -c_{1l} \\ -c_{12} & \sum_{j=1}^{l} c_{2j} & -c_{23} & \cdots & -c_{2l} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -c_{1l} & -c_{2l} & -c_{3l} & \cdots & \sum_{j=1}^{l} c_{lj} \end{bmatrix}, \tag{8.2.4}$$

$$K \triangleq \begin{bmatrix} \sum_{j=1}^{l} k_{1j} & -k_{12} & -k_{13} & \cdots & -k_{1l} \\ -k_{12} & \sum_{j=1}^{l} k_{2j} & -k_{23} & \cdots & -k_{2l} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -k_{1l} & -k_{2l} & -k_{3l} & \cdots & \sum_{j=1}^{l} k_{lj} \end{bmatrix}. \tag{8.2.5}$$

As a special case, assume that the motion of $y_1, \ldots, y_l$ is confined to a single line in the direction $\hat{n}$, which is fixed with respect to the inertial frame $\mathrm{F_A}$, and that the external forces $\vec{f}_1, \ldots, \vec{f}_l$ are parallel with $\hat{n}$. Then, for all $i = 1, \ldots, l$, it follows that

$$\vec{r}_{y_i/w} \triangleq q_i \hat{n}, \tag{8.2.6}$$

$$\vec{f}_i \triangleq f_i \hat{n}. \tag{8.2.7}$$

Then, (8.2.2) can be written as

$$M\ddot{q} + C\dot{q} + Kq = f, \tag{8.2.8}$$

where

$$q \triangleq \begin{bmatrix} q_1 \\ \vdots \\ q_l \end{bmatrix}, \quad f \triangleq \begin{bmatrix} f_1 \\ \vdots \\ f_l \end{bmatrix}. \tag{8.2.9}$$

Next, defining

$$\Gamma \triangleq \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \\ m_1 & m_2 & m_3 & \cdots & m_n \end{bmatrix}, \tag{8.2.10}$$

(8.2.2) can be rewritten as

$$\tilde{M}\Gamma \begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + \tilde{C}\Gamma \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/w} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_l/w} \end{bmatrix} + \tilde{K}\Gamma \begin{bmatrix} \vec{r}_{y_1/w} \\ \vdots \\ \vec{r}_{y_l/w} \end{bmatrix} = \Gamma^{-\mathrm{T}} \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_l \end{bmatrix}, \tag{8.2.11}$$

where

$$\tilde{M} \triangleq \Gamma^{-\mathrm{T}} M \Gamma^{-1}, \quad \tilde{C} \triangleq \Gamma^{-\mathrm{T}} C \Gamma^{-1}, \quad \tilde{K} \triangleq \Gamma^{-\mathrm{T}} K \Gamma^{-1}. \tag{8.2.12}$$

Note that

$$\Gamma \begin{bmatrix} \vec{r}_{y_1/w} \\ \vdots \\ \vec{r}_{y_l/w} \end{bmatrix} = \begin{bmatrix} \vec{r}_{y_1/y_2} \\ \vdots \\ \vec{r}_{y_{l-1}/y_l} \\ \sum_{i=1}^{l} m_i \vec{r}_{y_i/w} \end{bmatrix}. \tag{8.2.13}$$

Furthermore,

$$\det \Gamma = m_{\mathcal{B}} \triangleq \sum_{i=1}^{l} m_i, \tag{8.2.14}$$

$$\Gamma^{-\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \\ \frac{1}{m_{\mathcal{B}}} & \frac{1}{m_{\mathcal{B}}} & \frac{1}{m_{\mathcal{B}}} & \cdots & \frac{1}{m_{\mathcal{B}}} \end{bmatrix} - \frac{1}{m_{\mathcal{B}}} \begin{bmatrix} m_1 & m_1 & \cdots & m_1 \\ m_1 + m_2 & m_1 + m_2 & \cdots & m_1 + m_2 \\ \vdots & \vdots & \vdots & \vdots \\ m_{\mathcal{B}} - m_l & m_{\mathcal{B}} - m_l & \cdots & m_{\mathcal{B}} - m_l \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{8.2.15}$$

and thus

$$\Gamma^{-\mathrm{T}} \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_l \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_1 + \vec{f}_2 \\ \vdots \\ \vec{f}_{\mathcal{B}} - \vec{f}_l \\ \frac{1}{m_{\mathcal{B}}} \vec{f}_{\mathcal{B}} \end{bmatrix} - \begin{bmatrix} m_1 \\ m_1 + m_2 \\ \vdots \\ m_{\mathcal{B}} - m_l \\ 0 \end{bmatrix} \frac{1}{m_{\mathcal{B}}} \vec{f}_{\mathcal{B}}. \tag{8.2.16}$$

Therefore, (8.2.11) can be written as

$$\tilde{M}\begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_{l-1}/y_l} \\ \sum_{i=1}^{l} m_i \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w} \end{bmatrix} + \tilde{C}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_{l-1}/y_l} \\ \sum_{i=1}^{l} m_i \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w} \end{bmatrix} + \tilde{K}\begin{bmatrix} \vec{r}_{y_1/y_2} \\ \vdots \\ \vec{r}_{y_{l-1}/y_l} \\ \sum_{i=1}^{l} m_i \vec{r}_{y_i/w} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 - \frac{m_1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \\ \vec{f}_1 + \vec{f}_2 - \frac{m_1+m_2}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \\ \vdots \\ -\vec{f}_l + \frac{m_l}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \\ \frac{1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \end{bmatrix}. \tag{8.2.17}$$

In particular, if $l = 2$, then

$$\frac{1}{m_1+m_2}\begin{bmatrix} m_1 m_2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \\ \sum_{i=1}^{2} m_i \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w} \end{bmatrix} + \frac{1}{(m_1+m_2)^2}\begin{bmatrix} -2m_1m_2c_{12} & (m_2-m_1)c_{12} \\ (m_2-m_1)c_{12} & 2c_{12} \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2} \\ \sum_{i=1}^{2} m_i \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w} \end{bmatrix}$$
$$+ \frac{1}{(m_1+m_2)^2}\begin{bmatrix} -2m_1m_2k_{12} & (m_2-m_1)k_{12} \\ (m_2-m_1)k_{12} & 2k_{12} \end{bmatrix}\begin{bmatrix} \vec{r}_{y_1/y_2} \\ \sum_{i=1}^{2} m_i \vec{r}_{y_i/w} \end{bmatrix} = \begin{bmatrix} \frac{m_2}{m_1+m_2}\vec{f}_1 - \frac{m_1}{m_1+m_2}\vec{f}_2 \\ \frac{1}{m_1+m_2}(\vec{f}_1 + \vec{f}_2) \end{bmatrix}. \tag{8.2.18}$$

Furthermore, if $l = 3$, then

$$\frac{1}{m_{\mathcal{B}}}\begin{bmatrix} m_1(m_2+m_3) & m_1m_3 & 0 \\ m_1m_3 & m_3(m_1+m_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_2/y_3} \\ \sum_{i=1}^{3} m_i \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_i/w} \end{bmatrix} + \frac{1}{m_{\mathcal{B}}^2}\begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \tilde{c}_{13} \\ \tilde{c}_{12} & \tilde{c}_{22} & \tilde{c}_{23} \\ \tilde{c}_{13} & \tilde{c}_{23} & \tilde{c}_{33} \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2} \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_2/y_3} \\ \sum_{i=1}^{3} m_i \overset{\mathrm{A}\bullet}{\vec{r}}_{y_i/w} \end{bmatrix}$$
$$+ \frac{1}{m_{\mathcal{B}}^2}\begin{bmatrix} \tilde{k}_{11} & \tilde{k}_{12} & \tilde{k}_{13} \\ \tilde{k}_{12} & \tilde{k}_{22} & \tilde{k}_{23} \\ \tilde{k}_{13} & \tilde{k}_{23} & \tilde{k}_{33} \end{bmatrix}\begin{bmatrix} \vec{r}_{y_1/y_2/\mathrm{A}} \\ \vec{r}_{y_2/y_3} \\ \sum_{i=1}^{3} m_i \vec{r}_{y_i/w} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 - \frac{m_1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \\ -\vec{f}_3 + \frac{m_3}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \\ \frac{1}{m_{\mathcal{B}}}\vec{f}_{\mathcal{B}} \end{bmatrix}. \tag{8.2.19}$$

where

$$\begin{aligned}
\tilde{c}_{11} &\triangleq -2m_1(m_2+m_3)(c_{12}+c_{13}) + 2m_1^2c_{23},\\
\tilde{c}_{12} &\triangleq c_{23}m_{\mathcal{B}}^2 - m_3m_{\mathcal{B}}(c_{12}+2c_{13}+3c_{23}) - m_2m_{\mathcal{B}}(c_{13}+c_{23}) + 2m_3(m_2+m_3)(c_{12}+c_{13}+c_{23}),\\
\tilde{c}_{13} &\triangleq -m_1(c_{12}+c_{13}+m_2(c_{12}+c_{13})+2c_{23}) + m_3(c_{12}+c_{13}),\\
\tilde{c}_{22} &\triangleq 2m_3(m_2-m_1)(c_{13}+c_{23}) + 2m_3^2c_{12},\\
\tilde{c}_{23} &\triangleq -m_1(c_{13}+c_{23}) + m_3(2c_{12}+c_{13}+c_{23}) - m_2(c_{13}+c_{23}),\\
\tilde{c}_{33} &\triangleq 2(c_{12}+c_{13}+c_{23}),\\
\tilde{k}_{11} &\triangleq -2m_1(m_2+m_3)(k_{12}+k_{13}) + 2m_1^2k_{23},\\
\tilde{k}_{12} &\triangleq k_{23}m_{\mathcal{B}}^2 - m_3m_{\mathcal{B}}(k_{12}+2k_{13}+3k_{23}) - m_2m_{\mathcal{B}}(k_{13}+k_{23}) + 2m_3(m_2+m_3)(k_{12}+k_{13}+k_{23}),\\
\tilde{k}_{13} &\triangleq -m_1(k_{12}+k_{13}+m_2(k_{12}+k_{13})+2k_{23}) + m_3(k_{12}+k_{13}),\\
\tilde{k}_{22} &\triangleq 2m_3(m_2-m_1)(k_{13}+k_{23}) + 2m_3^2k_{12},\\
\tilde{k}_{23} &\triangleq -m_1(k_{13}+k_{23}) + m_3(2k_{12}+k_{13}+k_{23}) - m_2(k_{13}+k_{23}),\\
\tilde{k}_{33} &\triangleq 2(k_{12}+k_{13}+k_{23}).
\end{aligned}$$

Again, assume, as a special case, that the motion of $y_1, \ldots, y_l$ is confined to a single line in the direction $\hat{n}$, which is fixed with respect to the inertial frame $\mathrm{F_A}$, and that the external forces $\vec{f}_1, \ldots, \vec{f}_l$ are parallel with $\hat{n}$. Then, for all $i = 1, \ldots, l-1$, it follows that

$$\vec{r}_{y_i/y_{i+1}} \triangleq (q_i - q_{i+1})\hat{n}. \tag{8.2.20}$$

Now, defining

$$\tilde{q} \triangleq \Gamma q = \begin{bmatrix} q_1 - q_2 \\ \vdots \\ q_{l-1} - q_l \\ \sum_{i=1}^{l} m_i q_i \end{bmatrix} \tag{8.2.21}$$

and, with $f_{\mathcal{B}} \triangleq \sum_{i=1}^{l} f_i$,

$$\tilde{f} \triangleq \begin{bmatrix} f_1 - \frac{m_1}{m_{\mathcal{B}}} f_{\mathcal{B}} \\ f_1 + f_2 - \frac{m_1+m_2}{m_{\mathcal{B}}} f_{\mathcal{B}} \\ \vdots \\ -f_l + \frac{m_l}{m_{\mathcal{B}}} f_{\mathcal{B}} \\ \frac{1}{m_{\mathcal{B}}} f_{\mathcal{B}} \end{bmatrix}, \tag{8.2.22}$$

it follows from (8.2.17) that

$$\tilde{M}\ddot{\tilde{q}} + \tilde{C}\dot{\tilde{q}} + \tilde{K}\tilde{q} = \tilde{f}. \tag{8.2.23}$$

Finally, assume that $\vec{f}_{\mathcal{B}} = 0$, and thus the center of mass c of $\mathcal{B}$ is unforced. Then, choosing $w = \mathrm{c}$, it follows that $\sum_{i=1}^{l} m_i \vec{r}_{y_i/w} = 0$. Hence, (8.2.17) becomes

$$\tilde{M} \begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_{l-1}/y_l} \\ 0 \end{bmatrix} + \tilde{C} \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_{l-1}/y_l} \\ 0 \end{bmatrix} + \tilde{K} \begin{bmatrix} \vec{r}_{y_1/y_2} \\ \vdots \\ \vec{r}_{y_{l-1}/y_l} \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_1 + \vec{f}_2 \\ \vdots \\ -\vec{f}_l \\ 0 \end{bmatrix}. \tag{8.2.24}$$

Now, truncating the last row and last column of $\tilde{M}$, $\tilde{C}$, and $\tilde{K}$, yields

$$\tilde{M}_{[l,l]} \begin{bmatrix} \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y_{l-1}/y_l} \end{bmatrix} + \tilde{C}_{[l,l]} \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{r}}_{y_1/y_2} \\ \vdots \\ \overset{\mathrm{A}\bullet}{\vec{r}}_{y_{l-1}/y_l} \end{bmatrix} + \tilde{K}_{[l,l]} \begin{bmatrix} \vec{r}_{y_1/y_2} \\ \vdots \\ \vec{r}_{y_{l-1}/y_l} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_1 + \vec{f}_2 \\ \vdots \\ -\vec{f}_l \end{bmatrix}, \tag{8.2.25}$$

where $\tilde{M}_{[l,l]}$, $\tilde{C}_{[l,l]}$, and $\tilde{K}_{[l,l]}$ denote $\tilde{M}$, $\tilde{C}$, and $\tilde{K}$, respectively, after deleting the last row and last column of each matrix.

8.3 Rate Gyro

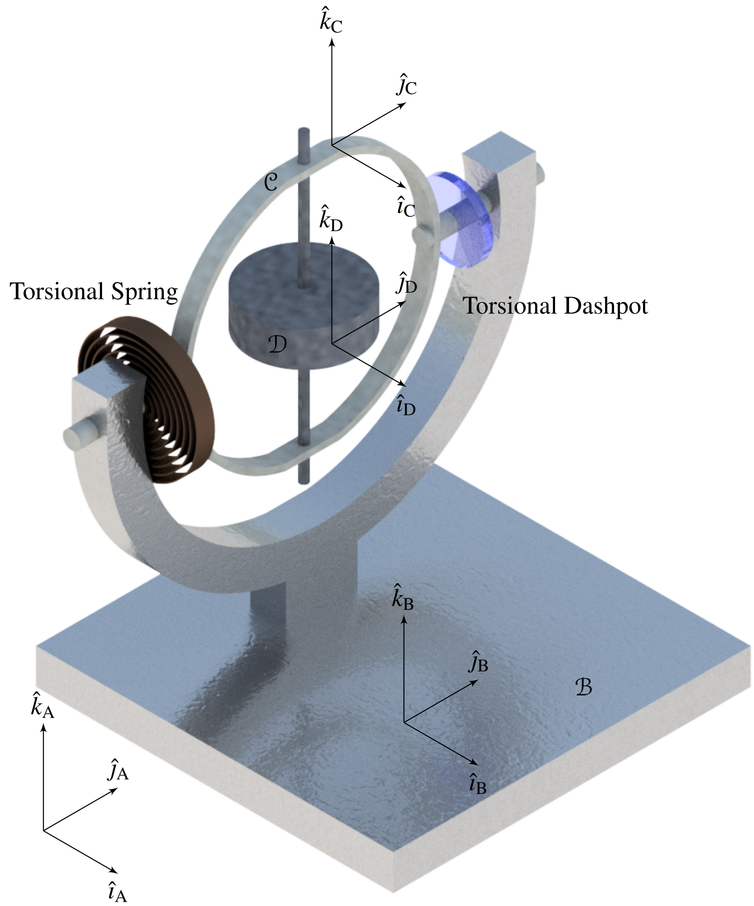


Figure 8.3.2: Rate-gyro setup. F_A is an inertial frame, F_B is attached to the rigid body $\mathcal{B}$, F_C is attached to the gimbal $\mathcal{C}$, and F_D is attached to the rotor $\mathcal{D}$.

Consider the single-axis rate gyro shown in Figure 8.3.2, which measures the component of the angular velocity of F_B relative to F_A along $\hat{\imath}_B$. The frame F_B is attached to the rigid body $\mathcal{B}$, and the frame F_C is attached to the gimbal $\mathcal{C}$. The rotor $\mathcal{D}$ is mounted on $\mathcal{C}$, and the frame F_D is attached to $\mathcal{D}$ such that $\hat{k}_D$ is aligned with $\hat{k}_C$. The axis $\hat{\jmath}_C$, which is aligned with $\hat{\jmath}_B$, is the measurement axis of the rate gyro. These frames are related by the Euler-angle rotations

$$F_A \underset{1}{\xrightarrow{\Psi}} F_B \underset{2}{\xrightarrow{\Theta}} F_C \underset{3}{\xrightarrow{\Phi}} F_D, \tag{8.3.1}$$

where the angles Ψ, Θ, and Φ correspond to rotations around $\hat{\imath}_A$, $\hat{\jmath}_B$, and $\hat{k}_C$, respectively. Note that Θ is the gimbal angle.

Henceforth, it is assumed that the angular velocity of $\mathcal{B}$ relative to F_A is aligned with $\hat{\imath}_B$, and thus $\mathcal{B}$ rotates relative to F_A about $\hat{\imath}_B$. It thus follows that

$$\vec{\omega}_{B/A} = \dot{\Psi}\hat{\imath}_B = \omega_{in}\hat{\imath}_B, \tag{8.3.2}$$

$$\vec{\omega}_{\mathrm{C/B}} = \dot{\Theta}\hat{\jmath}_{\mathrm{C}}, \tag{8.3.3}$$

$$\vec{\omega}_{\mathrm{D/C}} = \dot{\Phi}\hat{k}_{\mathrm{D}} = \omega\hat{k}_{\mathrm{D}}. \tag{8.3.4}$$

Therefore,

$$\begin{aligned}\vec{\omega}_{\mathrm{D/A}} &= \vec{\omega}_{\mathrm{D/C}} + \vec{\omega}_{\mathrm{C/B}} + \vec{\omega}_{\mathrm{B/A}}\\ &= \omega\hat{k}_{\mathrm{D}} + \dot{\Theta}\hat{\jmath}_{\mathrm{C}} + \omega_{\mathrm{in}}\hat{\imath}_{\mathrm{B}}\\ &= \omega\hat{k}_{\mathrm{D}} + \dot{\Theta}[(\sin\Phi)\hat{\imath}_{\mathrm{D}} + (\cos\Phi)\hat{\jmath}_{\mathrm{D}}] + \omega_{\mathrm{in}}[(\cos\Theta)\hat{\imath}_{\mathrm{C}} + (\sin\Theta)\hat{k}_{\mathrm{C}}]\\ &= \omega\hat{k}_{\mathrm{D}} + \dot{\Theta}[(\sin\Phi)\hat{\imath}_{\mathrm{D}} + (\cos\Phi)\hat{\jmath}_{\mathrm{D}}] + \omega_{\mathrm{in}}[\cos\Theta[(\cos\Phi)\hat{\imath}_{\mathrm{D}} - (\sin\Phi)\hat{\jmath}_{\mathrm{D}}] + (\sin\Theta)\hat{k}_{\mathrm{D}}]\\ &= [\dot{\Theta}\sin\Phi + \omega_{\mathrm{in}}(\cos\Theta)\cos\Phi]\hat{\imath}_{\mathrm{D}} + [\dot{\Theta}\cos\Phi - \omega_{\mathrm{in}}(\cos\Theta)\sin\Phi]\hat{\jmath}_{\mathrm{D}} + [\omega + \omega_{\mathrm{in}}\sin\Theta]\hat{k}_{\mathrm{D}}.\end{aligned} \tag{8.3.5}$$

Resolving (8.3.5) in $\mathrm{F_D}$ yields

$$\vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}} = \begin{bmatrix}(\cos\Theta)\cos\Phi & \sin\Phi & 0\\ -(\cos\Theta)\sin\Phi & \cos\Phi & 0\\ \sin\Theta & 0 & 1\end{bmatrix}\begin{bmatrix}\omega_{\mathrm{in}}\\ \dot{\Theta}\\ \omega\end{bmatrix}. \tag{8.3.6}$$

Next, let the moment of inertia of $\mathcal{D}$ relative to its center of mass $\mathrm{c}_{\mathcal{D}}$ be resolved in $\mathrm{F_D}$ as

$$\vec{J}_{\mathcal{D}/\mathrm{c}_{\mathcal{D}}}\Big|_{\mathrm{D}} = \begin{bmatrix}J_{\mathcal{D},11} & 0 & 0\\ 0 & J_{\mathcal{D},22} & 0\\ 0 & 0 & J_{\mathcal{D},33}\end{bmatrix}. \tag{8.3.7}$$

Then, Euler's equation has the form

$$\vec{J}_{\mathcal{D}/\mathrm{c}_{\mathcal{D}}} \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}} + \vec{\omega}_{\mathrm{D/A}} \times \vec{J}_{\mathcal{D}/\mathrm{c}_{\mathcal{D}}}\vec{\omega}_{\mathrm{D/A}} = \vec{T}_{\mathcal{D}}, \tag{8.3.8}$$

where

$$\begin{aligned}\vec{J}_{\mathcal{D}/\mathrm{c}_{\mathcal{D}}} \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}} = {} & J_{\mathcal{D},11}[\ddot{\Theta}\sin\Phi + \dot{\Theta}\omega\cos\Phi + \dot{\omega}_{\mathrm{in}}(\cos\Theta)\cos\Phi - \omega_{\mathrm{in}}\dot{\Theta}(\sin\Theta)\cos\Phi\\ & - \omega_{\mathrm{in}}\omega(\cos\Theta)\sin\Phi]\hat{\imath}_{\mathrm{D}} + J_{\mathcal{D},22}[\ddot{\Theta}\cos\Phi - \dot{\Theta}\omega\sin\Phi - \dot{\omega}_{\mathrm{in}}(\cos\Theta)\sin\Phi\\ & + \omega_{\mathrm{in}}\dot{\Theta}(\sin\Theta)\sin\Phi - \omega_{\mathrm{in}}\omega(\cos\Theta)\cos\Phi]\hat{\jmath}_{\mathrm{D}} + J_{\mathcal{D},33}[\dot{\omega} + \dot{\omega}_{\mathrm{in}}\sin\Theta + \omega_{\mathrm{in}}\dot{\Theta}\cos\Theta]\hat{k}_{\mathrm{D}},\end{aligned} \tag{8.3.9}$$

$$\begin{aligned}\vec{\omega}_{\mathrm{D/A}} \times \vec{J}_{\mathcal{D}/\mathrm{c}_{\mathcal{D}}}\vec{\omega}_{\mathrm{D/A}} = {} & [(\omega + \omega_{\mathrm{in}}\sin\Theta)(\dot{\Theta}\cos\Phi - \omega_{\mathrm{in}}(\cos\Theta)\sin\Phi)(J_{\mathcal{D},33} - J_{\mathcal{D},22})]\hat{\imath}_{\mathrm{D}}\\ & + [(\omega + \omega_{\mathrm{in}}\sin\Theta)(\dot{\Theta}\sin\Phi + \omega_{\mathrm{in}}(\cos\Theta)\cos\Phi)(J_{\mathcal{D},11} - J_{\mathcal{D},33})]\hat{\jmath}_{\mathrm{D}}\\ & + [(\dot{\Theta}\cos\Phi - \omega_{\mathrm{in}}(\cos\Theta)\sin\Phi)(\dot{\Theta}\sin\Phi + \omega_{\mathrm{in}}(\cos\Theta)\cos\Phi)(J_{\mathcal{D},22} - J_{\mathcal{D},11})]\hat{k}_{\mathrm{D}}.\end{aligned} \tag{8.3.10}$$

A motor mounted on $\mathcal{C}$ is used to keep the rotor spin rate ω constant. Letting $\vec{T}_{\mathrm{r}/\mathcal{D}/\mathcal{C}} = T_{\mathcal{D},1}\hat{\imath}_{\mathrm{D}} + T_{\mathcal{D},2}\hat{\jmath}_{\mathrm{D}}$ denote the reaction torque on $\mathcal{D}$ due to $\mathcal{C}$, and letting $\vec{T}_{\mathrm{m}} = T_{\mathrm{m}}\hat{k}_{\mathrm{D}}$ denote the torque applied to $\mathcal{D}$ by the motor, it follows that the total torque on $\mathcal{D}$ is given by

$$\begin{aligned}\vec{T}_{\mathcal{D}} &= \vec{T}_{\mathrm{r}/\mathcal{D}/\mathcal{C}} + \vec{T}_{\mathrm{m}}\\ &= T_{\mathcal{D},1}\hat{\imath}_{\mathrm{D}} + T_{\mathcal{D},2}\hat{\jmath}_{\mathrm{D}} + T_{\mathrm{m}}\hat{k}_{\mathrm{D}}.\end{aligned} \tag{8.3.11}$$

Using $J_{\mathcal{D},22} = J_{\mathcal{D},11}$ and substituting (8.3.9), (8.3.10), and (8.3.11) into (8.3.8) yields

$$\dot{\omega} + \dot{\omega}_{\rm in}\sin\Theta + \omega_{\rm in}(\cos\Theta)\dot{\Theta} = T_{\rm m}, \tag{8.3.12}$$

$$\begin{aligned} &J_{\mathcal{D},11}[\ddot{\Theta}\sin\Phi + \dot{\Theta}\omega\cos\Phi + \dot{\omega}_{\rm in}(\cos\Theta)\cos\Phi - \omega_{\rm in}\dot{\Theta}(\sin\Theta)\cos\Phi - \omega_{\rm in}\omega(\cos\Theta)\sin\Phi] \\ &\quad +(\omega + \omega_{\rm in}\sin\Theta)(\dot{\Theta}\cos\Phi - \omega_{\rm in}(\cos\Theta)\sin\Phi)(J_{\mathcal{D},33} - J_{\mathcal{D},22}) = T_{\mathcal{D},1}, \end{aligned} \tag{8.3.13}$$

$$\begin{aligned} &J_{\mathcal{D},22}[\ddot{\Theta}\cos\Phi - \dot{\Theta}\omega\sin\Phi - \dot{\omega}_{\rm in}(\cos\Theta)\sin\Phi + \omega_{\rm in}\dot{\Theta}(\sin\Theta)\sin\Phi - \omega_{\rm in}\omega(\cos\Theta)\cos\Phi] \\ &\quad +(\omega + \omega_{\rm in}\sin\Theta)(\dot{\Theta}\sin\Phi + \omega_{\rm in}(\cos\Theta)\cos\Phi)(J_{\mathcal{D},11} - J_{\mathcal{D},33}) = T_{\mathcal{D},2}. \end{aligned} \tag{8.3.14}$$

The rotor spin rate ω is kept constant by setting $T_{\rm m} = \dot{\omega}_{\rm in}\sin\Theta + \omega_{\rm in}(\cos\Theta)\dot{\Theta}$.

Next, it follows from (8.3.4) that

$$\begin{aligned} \vec{\omega}_{\rm C/A} &= \vec{\omega}_{\rm C/B} + \vec{\omega}_{\rm B/A} \\ &= \dot{\Theta}\hat{\jmath}_{\rm C} + \omega_{\rm in}\hat{\imath}_{\rm B} \\ &= \omega_{\rm in}(\cos\Theta)\hat{\imath}_{\rm C} + \dot{\Theta}\hat{\jmath}_{\rm C} + \omega_{\rm in}(\sin\Theta)\hat{k}_{\rm C}. \end{aligned} \tag{8.3.15}$$

Furthermore, let the moment of inertia of $\mathcal{C}$ relative to its center of mass $c_{\mathcal{C}}$ be resolved in $F_{\rm C}$ as

$$\vec{J}_{\mathcal{C}/c_{\mathcal{C}}}\Big|_{\rm C} = \begin{bmatrix} J_{\mathcal{C},11} & 0 & 0 \\ 0 & J_{\mathcal{C},22} & 0 \\ 0 & 0 & J_{\mathcal{C},33} \end{bmatrix}. \tag{8.3.16}$$

Then, Euler's equation has the form

$$\vec{J}_{\mathcal{C}/c_{\mathcal{C}}} \overset{\rm C\bullet}{\vec{\omega}}_{\rm C/A} + \vec{\omega}_{\rm C/A} \times \vec{J}_{\mathcal{C}/c_{\mathcal{C}}}\vec{\omega}_{\rm C/A} = \vec{T}_{\mathcal{C}}, \tag{8.3.17}$$

where

$$\vec{J}_{\mathcal{C}/c_{\mathcal{C}}} \overset{\rm C\bullet}{\vec{\omega}}_{\rm C/A} = J_{\mathcal{C},11}[\dot{\omega}_{\rm in}\cos\Theta - \omega_{\rm in}\dot{\Theta}\sin\Theta]\hat{\imath}_{\rm C} + J_{\mathcal{C},22}\ddot{\Theta}\hat{\jmath}_{\rm C} + J_{\mathcal{C},33}[\dot{\omega}_{\rm in}\sin\Theta + \omega_{\rm in}\dot{\Theta}\cos\Theta]\hat{k}_{\rm C}, \tag{8.3.18}$$

$$\begin{aligned} \vec{\omega}_{\rm C/A} \times \vec{J}_{\mathcal{C}/c_{\mathcal{C}}}\vec{\omega}_{\rm C/A} &= [(J_{\mathcal{C},33} - J_{\mathcal{C},22})\dot{\Theta}\omega_{\rm in}\sin\Theta]\hat{\imath}_{\rm C} + [(J_{\mathcal{C},11} - J_{\mathcal{C},33})\omega_{\rm in}^2\cos\Theta\sin\Theta]\hat{\jmath}_{\rm C} \\ &\quad + [(J_{\mathcal{C},22} - J_{\mathcal{C},11})\omega_{\rm in}\dot{\Theta}\cos\Theta]\hat{k}_{\rm C}, \end{aligned} \tag{8.3.19}$$

and, letting $\vec{T}_{{\rm r}/\mathcal{C}/\mathcal{B}} = T_{\mathcal{C},1}\hat{\imath}_{\rm C} + T_{\mathcal{C},3}\hat{k}_{\rm C}$ denote the reaction torque on $\mathcal{C}$ due to $\mathcal{B}$, the total torque on $\mathcal{C}$ is given by

$$\begin{aligned} \vec{T}_{\mathcal{C}} &= \vec{T}_{{\rm r}/\mathcal{C}/\mathcal{D}} + \vec{T}_{{\rm r}/\mathcal{C}/\mathcal{B}} - (k\Theta + c\dot{\Theta})\hat{\jmath}_{\rm C} - T_{\rm m}\hat{k}_{\rm D} \\ &= -T_{\mathcal{D},1}\hat{\imath}_{\rm D} - T_{\mathcal{D},2}\hat{\jmath}_{\rm D} + T_{\mathcal{C},1}\hat{\imath}_{\rm C} + T_{\mathcal{C},3}\hat{k}_{\rm C} - (k\Theta + c\dot{\Theta})\hat{\jmath}_{\rm C} - T_{\rm m}\hat{k}_{\rm C} \\ &= [-T_{\mathcal{D},1}\cos\Phi + T_{\mathcal{D},2}\sin\Phi + T_{\mathcal{C},1}]\hat{\imath}_{\rm C} + [-T_{\mathcal{D},1}\sin\Phi - T_{\mathcal{D},2}\cos\Phi - k\Theta - c\dot{\Theta}]\hat{\jmath}_{\rm C} \\ &\quad + (T_{\mathcal{C},3} - T_{\rm m})\hat{k}_{\rm C}, \end{aligned} \tag{8.3.20}$$

where $k > 0$ is the stiffness of the torsional spring and $c > 0$ is the damping coefficient of the torsional dashpot. Substituting (8.3.18), (8.3.19), and (8.3.20) into Euler's equation (8.3.17) and equating the coefficients of $\hat{\jmath}_{\rm C}$ yields

$$J_{\mathcal{C},22}\ddot{\Theta} + (J_{\mathcal{C},11} - J_{\mathcal{C},33})\omega_{\rm in}^2(\cos\Theta)\sin\Theta = -k\Theta - c\dot{\Theta} - T_{\mathcal{D},1}\sin\Phi - T_{\mathcal{D},2}\cos\Phi. \tag{8.3.21}$$

Finally, substituting $T_{\mathcal{D},1}$ given by (8.3.13) and $T_{\mathcal{D},2}$ given by (8.3.14) into (8.3.21) yields

$$\bar{J}_{22}\ddot{\Theta} + c\dot{\Theta} + k\Theta = (\bar{J}_{31}\omega_{\rm in}\sin\Theta + J_{\mathcal{D},33}\omega)\omega_{\rm in}\cos\Theta, \tag{8.3.22}$$

where

$$\bar{J}_{22} \triangleq J_{\mathcal{C},22} + J_{\mathcal{D},22}, \quad \bar{J}_{31} \triangleq J_{\mathcal{C},33} - J_{\mathcal{C},11} + J_{\mathcal{D},33} - J_{\mathcal{D},11}. \tag{8.3.23}$$

Next, defining $x \triangleq [\Theta \ \ \dot{\Theta}]^{\mathsf{T}}$ and $u \triangleq \omega_{\rm in}$, (8.3.22) can be written as

$$\dot{x} = f(x,u), \tag{8.3.24}$$

where

$$f(x,u) \triangleq \begin{bmatrix} x_2 \\ \dfrac{1}{\bar{J}_{22}}\left(\bar{J}_{31}(\sin x_1)(\cos x_1)u^2 + J_{\mathcal{D},33}\omega(\cos x_1)u - cx_2 - kx_1\right) \end{bmatrix}. \tag{8.3.25}$$

8.3.1 Local Analysis

Letting $(\bar{x},\bar{u}) = ([\bar{\Theta} \ \ 0]^{\mathsf{T}}, \bar{\omega}_{\rm in})$ denote an equilibrium of (8.3.24), the linearized dynamics in a neighborhood of $(\bar{x},\bar{u})$ are given by

$$\dot{\xi} = A\xi + Bv, \tag{8.3.26}$$

where

$$A \triangleq \begin{bmatrix} 0 & 1 \\ \dfrac{1}{\bar{J}_{22}}\left(\bar{J}_{31}\bar{u}^2\cos 2\bar{x}_1 - J_{\mathcal{D},33}\omega\bar{u}\sin\bar{x}_1 - k\right) & \dfrac{-c}{\bar{J}_{22}} \end{bmatrix}, \tag{8.3.27}$$

$$B \triangleq \begin{bmatrix} 0 \\ \dfrac{1}{\bar{J}_{22}}\left(\bar{J}_{31}(\sin\bar{x}_1)(\cos\bar{x}_1)2\bar{u} + J_{\mathcal{D},33}\omega(\cos\bar{x}_1)\right) \end{bmatrix}. \tag{8.3.28}$$

In particular, for $\bar{\Theta} = 0$ and $\bar{\omega}_{\rm in} = 0$, it follows that

$$A \triangleq \begin{bmatrix} 0 & 1 \\ \dfrac{-k}{\bar{J}_{22}} & \dfrac{-c}{\bar{J}_{22}} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ \dfrac{J_{\mathcal{D},33}\omega}{\bar{J}_{22}} \end{bmatrix}. \tag{8.3.29}$$

Defining $C \triangleq [1 \ \ 0]$, the transfer function of the linearized dynamics is given by

$$G(s) = C(sI - A)^{-1}B = \frac{J_{\mathcal{D},33}\omega}{\bar{J}_{22}s^2 + cs + k}. \tag{8.3.30}$$

Hence, for small $\omega_{\rm in}$, the Laplace transform of the gimbal angle perturbation θ is given approximately by

$$\hat{\theta}(s) = \frac{J_{\mathcal{D},33}\omega}{\bar{J}_{22}s^2 + cs + k}\hat{\omega}_{\rm in}(s), \tag{8.3.31}$$

and thus the gimbal angle perturbation satisfies

$$\theta(t) = -\frac{\bar{J}_{22}}{k}\ddot{\theta}(t) - \frac{c}{k}\dot{\theta}(t) + \frac{J_{\mathcal{D},33}\omega}{k}\omega_{\rm in}(t). \tag{8.3.32}$$

Since (8.3.32) represents a damped oscillator, it follows that, when $\dot{\theta}(t) \approx 0$ and $\ddot{\theta}(t) \approx 0$, the rate-gyro measurement is given by

$$\omega_{\rm in}(t) = \frac{k}{J_{\mathcal{D},33}\omega}\theta(t). \tag{8.3.33}$$

8.3.2 Global Analysis

Letting $(\bar{x}, \bar{u}) = ([\bar{\Theta}\ \ 0]^{\mathsf{T}}, \bar{\omega}_{\rm in})$ denote an equilibrium of (8.3.24), the set of input angular velocities $\bar{\omega}_{\rm in}$ corresponding to each $\bar{\Theta}$ is defined by

$$\mathcal{E}(\bar{\Theta}) \stackrel{\triangle}{=} \{\bar{\omega_{\rm in}}: \bar{J}_{31}(\sin\bar{\Theta})(\cos\bar{\Theta})\bar{\omega_{\rm in}}^2 + J_{\mathcal{D},33}\omega(\cos\bar{\Theta})\bar{\omega_{\rm in}} - k\bar{\Theta} = 0\}. \tag{8.3.34}$$

In particular, $\mathcal{E}(\bar{\Theta}) = \{\omega_{\rm in,1}, \bar{\omega}_{\rm in,2}\}$, where the values of angular velocity $\omega_{\rm in,1}$ and $\bar{\omega}_{\rm in,2}$ that satisfy (8.3.34) are given by

$$\bar{\omega}_{\rm in,1} = \frac{-J_{\mathcal{D},33}\omega\cos\bar{\Theta} + \sqrt{(J_{\mathcal{D},33}\omega\cos\bar{\Theta})^2 + 4\bar{J}_{31}(\sin\bar{\Theta})(\cos\bar{\Theta})\bar{\Theta}k}}{2\bar{J}_{31}(\sin\bar{\Theta})\cos\bar{\Theta}}, \tag{8.3.35}$$

$$\bar{\omega}_{\rm in,2} = \frac{-J_{\mathcal{D},33}\omega\cos\bar{\Theta} - \sqrt{(J_{\mathcal{D},33}\omega\cos\bar{\Theta})^2 + 4\bar{J}_{31}(\sin\bar{\Theta})(\cos\bar{\Theta})\bar{\Theta}k}}{2\bar{J}_{31}(\sin\bar{\Theta})\cos\bar{\Theta}}. \tag{8.3.36}$$

Hence, for each value of $\bar{\Theta}$, the corresponding equilibria of (8.3.24) are given by $([\bar{\Theta}\ \ 0]^{\mathsf{T}}, \bar{\omega}_{\rm in,1})$ and $([\bar{\Theta}\ \ 0]^{\mathsf{T}}, \bar{\omega}_{\rm in,2})$. In particular, for $\bar{\Theta} = 0$, (8.3.35) and (8.3.36) yield $\bar{\omega}_{\rm in,1} = 0$ and $\bar{\omega}_{\rm in,2} = \infty$. Hence, only $([\bar{\Theta}\ \ 0]^{\mathsf{T}}, \bar{\omega}_{\rm in,1})$ is physically meaningful.

Finally, using the fact that, for $x \approx 0$, $\sqrt{1+x} \approx 1 + \frac{x}{2}$, where x represents $\dfrac{4\bar{J}_{31}(\sin\bar{\Theta})(\cos\bar{\Theta})\bar{\Theta}k}{(J_{\mathcal{D},33}\omega\cos\bar{\Theta})^2}$, it follows that the equilibrium angular velocities are approximately given by

$$\bar{\omega}_{\rm in,1} \approx \frac{\bar{\Theta}k}{J_{\mathcal{D},33}\omega\cos\bar{\Theta}}, \tag{8.3.37}$$

$$\bar{\omega}_{\rm in,2} \approx \frac{-J_{\mathcal{D},33}\omega\cos\bar{\Theta} - \bar{J}_{31}(\sin\bar{\Theta})(\cos\bar{\Theta})\bar{\Theta}k}{\bar{J}_{31}(\sin\bar{\Theta})\cos\bar{\Theta}}. \tag{8.3.38}$$

where, for small $\bar{\Theta}$, (8.3.38) is not physically meaningful.

8.4 Examples Involving Interconnected Particles

Example 8.4.1. As shown in in Figure 8.4.1, the rigid body $\mathcal{B}$ consists of particles y_1 and y_2 whose masses are m_1 and m_2, respectively, connected by a massless rigid link $\mathcal{L}$ of length ℓ. The body-fixed frame $\mathrm{F_B}$ is chosen such that $\vec{r}_{y_2/y_1} = \ell\hat{\imath}_{\rm B}$. An external force $\vec{f}_{\rm ext}$ is applied to the link at the point z, whose distance from y_1 is $\ell_0 > 0$ so that $\vec{r}_{z/y_1} = \ell_0\hat{\imath}_{\rm B}$. $\mathcal{B}$ translates and rotates such that it remains in the $\hat{\imath}_{\rm A}$-$\hat{\jmath}_{\rm A}$ plane of the inertial frame $\mathrm{F_A}$. Determine the equation of motion for $\mathcal{B}$ as well as the reaction forces on y_1 and y_2. Separately consider the cases where the force $\vec{f}_{\rm ext}$ is applied to an internal point or an endpoint of $\mathcal{L}$.

Solution. Let w be a point with zero inertial acceleration, and define the rotation angle θ such that $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$, so that $\vec{\omega}_{\rm B/A} = \dot{\theta}\hat{k}_{\rm A}$. The location of the center of mass c of $\mathcal{B}$ is given by $\vec{r}_{{\rm c}/y_1} = \frac{m_2}{m_{\mathcal{B}}}\ell\hat{\imath}_{\rm B}$, where $m_{\mathcal{B}} = m_1 + m_2$. Note that $\vec{r}_{y_2/{\rm c}} = \frac{m_1}{m_{\mathcal{B}}}\ell\hat{\imath}_{\rm B}$.

Using (7.5.5) given by

$$m_{\mathcal{B}}\vec{a}_{{\rm c}/w/{\rm A}} = \vec{f}_{\rm ext} \tag{8.4.1}$$

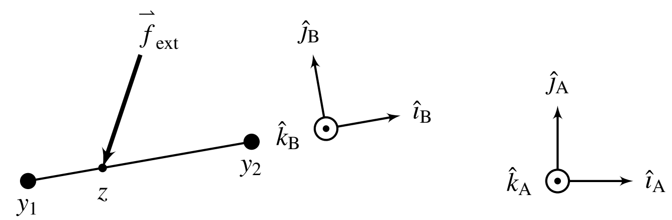


Figure 8.4.1: Example 8.4.1. Rigid body consisting of two particles connected by a massless link.

and defining the notation

$$\vec{r}_{{\rm c}/w/{\rm A}}\Big|_{\rm A} = \begin{bmatrix} x_{\rm c} \\ y_{\rm c} \\ z_{\rm c} \end{bmatrix}, \quad \vec{f}_{\rm ext}\Big|_{\rm B} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}, \tag{8.4.2}$$

it follows that

$$m_{\mathcal{B}} \begin{bmatrix} \ddot{x}_{\rm c} \\ \ddot{y}_{\rm c} \\ \ddot{z}_{\rm c} \end{bmatrix} = \mathcal{O}_{\rm A/B} \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}, \tag{8.4.3}$$

where

$$\mathcal{O}_{\rm A/B} = \mathcal{O}_3(-\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.4.4}$$

Let $\vec{f}_{{\rm r}/y_1/\mathcal{L}}$ and $\vec{f}_{{\rm r}/y_2/\mathcal{L}}$ denote the reaction forces on y_1 and y_2, respectively, so that $-\vec{f}_{{\rm r}/y_1/\mathcal{L}}$ and $-\vec{f}_{{\rm r}/y_2/\mathcal{L}}$ are the reaction forces on the endpoints x_1 and x_2, respectively, of $\mathcal{L}$. Using free-body analysis to determine the reaction forces on the particles and the endpoints of $\mathcal{L}$, $\mathcal{B}$ can be decomposed in three ways, as shown in Figure 8.4.2. Considering the decomposition in Figure 8.4.2(a), it follows that

$$m_1 \vec{a}_{y_1/w/{\rm A}} = \vec{f}_{{\rm r}/y_1/\mathcal{L}}, \tag{8.4.5}$$

and thus

$$m_1 \vec{a}_{y_1/{\rm c}/{\rm A}} = -m_1 \vec{a}_{{\rm c}/w/{\rm A}} + \vec{f}_{{\rm r}/y_1/\mathcal{L}} = -\frac{m_1}{m_{\mathcal{B}}} \vec{f}_{\rm ext} + \vec{f}_{{\rm r}/y_1/\mathcal{L}}. \tag{8.4.6}$$

Therefore,

$$\begin{aligned} \vec{f}_{{\rm r}/y_1/\mathcal{L}} &= \frac{m_1}{m_{\mathcal{B}}} \vec{f}_{\rm ext} + m_1 \vec{a}_{y_1/{\rm c}/{\rm A}} \\ &= \frac{m_1}{m_{\mathcal{B}}} \vec{f}_{\rm ext} + m_1[\alpha_{\rm B/A} \times \vec{r}_{y_1/{\rm c}} + \omega_{\rm B/A} \times (\omega_{\rm B/A} \times \vec{r}_{y_1/{\rm c}})] \\ &= \frac{m_1}{m_{\mathcal{B}}} \vec{f}_{\rm ext} + \frac{m_1 m_2 \ell \dot{\theta}^2}{m_{\mathcal{B}}} \hat{\imath}_{\rm B} - \frac{m_1 m_2 \ell \ddot{\theta}}{m_{\mathcal{B}}} \hat{\jmath}_{\rm B}. \end{aligned} \tag{8.4.7}$$

Writing $\vec{f}_{\mathrm{r}/y_1/\mathcal{L}} = f_{11}\hat{\imath}_\mathrm{B} + f_{12}\hat{\jmath}_\mathrm{B}$, it follows that

$$f_{11} = \frac{m_1}{m_\mathcal{B}} f_1 + \frac{m_1 m_2 \ell\dot{\theta}^2}{m_\mathcal{B}}, \tag{8.4.8}$$

$$f_{12} = \frac{m_1}{m_\mathcal{B}} f_2 - \frac{m_1 m_2 \ell\ddot{\theta}}{m_\mathcal{B}}. \tag{8.4.9}$$

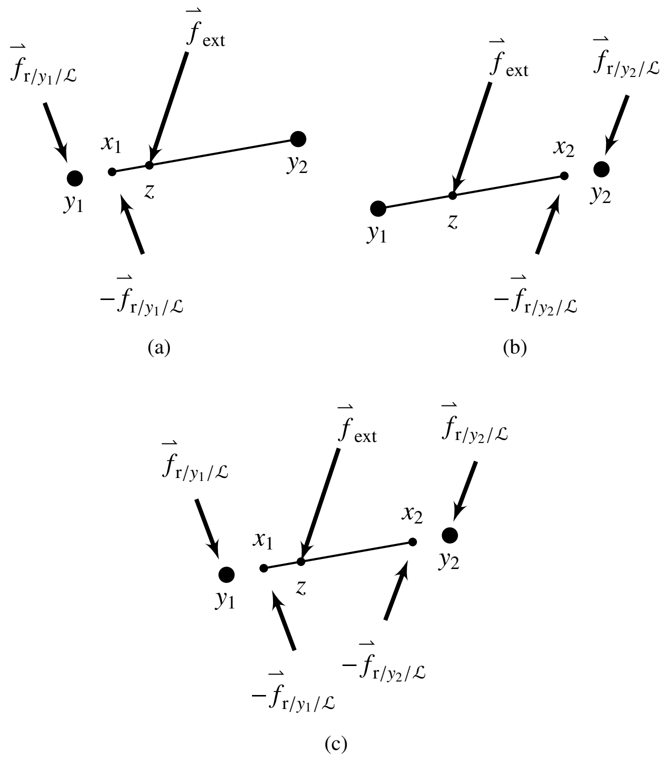


Figure 8.4.2: Example 8.4.1. Free-body analysis of a rigid body consisting of two particles connected by a massless link. Note that the external force $\vec{f}_\mathrm{ext}$ is applied to an interior point on the link.

Next, note that

$$\begin{aligned}\vec{J}_{\mathcal{B}/\mathrm{c}} &= \left[m_1\left(\frac{m_2}{m_\mathcal{B}}\ell\right)^2 + m_2\left(\frac{m_1}{m_\mathcal{B}}\ell\right)^2\right](\hat{\jmath}_\mathrm{B}\hat{\jmath}_\mathrm{B}' + \hat{k}_\mathrm{B}\hat{k}_\mathrm{B}')\\ &= \left(\frac{m_1 m_2 \ell^2}{m_\mathcal{B}}\right)(\hat{\jmath}_\mathrm{B}\hat{\jmath}_\mathrm{B}' + \hat{k}_\mathrm{B}\hat{k}_\mathrm{B}').\end{aligned} \tag{8.4.10}$$

Furthermore,

$$\begin{aligned}\vec{M}_{\mathcal{B}/\mathrm{c}} &= \vec{r}_{z/\mathrm{c}} \times \vec{f}_\mathrm{ext}\\ &= (\vec{r}_{z/y_1} + \vec{r}_{y_1/\mathrm{c}}) \times \vec{f}_\mathrm{ext}\\ &= \left[\ell_0\hat{\imath}_\mathrm{B} + \left(\frac{-m_2}{m_\mathcal{B}}\right)\ell\hat{\imath}_\mathrm{B}\right] \times \vec{f}_\mathrm{ext}\end{aligned}$$

$$= \left(\ell_0 - \frac{m_2}{m_{\mathcal{B}}}\ell\right) f_2 \hat{k}_{\mathrm{B}}. \tag{8.4.11}$$

Using (8.4.10) and (8.4.11), it follows from (7.9.13) that

$$\frac{m_1 m_2 \ell^2}{m_{\mathcal{B}}}\ddot{\theta} = \left(\ell_0 - \frac{m_2}{m_{\mathcal{B}}}\ell\right) f_2. \tag{8.4.12}$$

Now, using (8.4.12), it follows from (8.4.9) that

$$f_{12} = \frac{m_1}{m_{\mathcal{B}}} f_2 - \frac{m_{\mathcal{B}}\ell_0 - m_2\ell}{m_{\mathcal{B}}\ell} f_2 = \frac{\ell - \ell_0}{\ell} f_2. \tag{8.4.13}$$

Next, considering the decomposition in Figure 8.4.2(b), it follows that

$$m_2 \vec{a}_{y_2/w/\mathrm{A}} = \vec{f}_{\mathrm{r}/y_2/\mathcal{L}}, \tag{8.4.14}$$

and thus

$$m_2 \vec{a}_{y_2/\mathrm{c}/\mathrm{A}} = -m_2 \vec{a}_{\mathrm{c}/w/\mathrm{A}} + \vec{f}_{\mathrm{r}/y_2/\mathcal{L}} = -\frac{m_2}{m_{\mathcal{B}}} \vec{f}_{\mathrm{ext}} + \vec{f}_{\mathrm{r}/y_2/\mathcal{L}}. \tag{8.4.15}$$

Therefore,

$$\begin{aligned}
\vec{f}_{\mathrm{r}/y_2/\mathcal{L}} &= \frac{m_2}{m_{\mathcal{B}}} \vec{f}_{\mathrm{ext}} + m_2[\alpha_{\mathrm{B/A}} \times \vec{r}_{y_2/\mathrm{c}} + \omega_{\mathrm{B/A}} \times (\omega_{\mathrm{B/A}} \times \vec{r}_{y_2/\mathrm{c}})] \\
&= \frac{m_2}{m_{\mathcal{B}}} \vec{f}_{\mathrm{ext}} - \frac{m_1 m_2 \ell \dot{\theta}^2}{m_{\mathcal{B}}} \hat{\imath}_{\mathrm{B}} + \frac{m_1 m_2 \ell \ddot{\theta}}{m_{\mathcal{B}}} \hat{\jmath}_{\mathrm{B}}. \qquad (8.4.16)
\end{aligned}$$

Writing $\vec{f}_{\mathrm{r}/y_2/\mathcal{L}} = f_{21}\hat{\imath}_{\mathrm{B}} + f_{22}\hat{\jmath}_{\mathrm{B}}$, it follows that

$$f_{21} = \frac{m_2}{m_{\mathcal{B}}} f_1 - \frac{m_1 m_2 \ell \dot{\theta}^2}{m_{\mathcal{B}}}, \tag{8.4.17}$$

$$f_{22} = \frac{m_2}{m_{\mathcal{B}}} f_2 + \frac{m_1 m_2 \ell \ddot{\theta}}{m_{\mathcal{B}}}. \tag{8.4.18}$$

Defining $\ell_0 \triangleq |\vec{r}_{z/y_1}|$ and using (8.4.12), it follows from (8.4.18) that

$$f_{22} = \frac{m_2}{m_{\mathcal{B}}} f_2 + \frac{m_{\mathcal{B}}\ell_0 - m_2\ell}{m_{\mathcal{B}}\ell} f_2 = \frac{\ell_0}{\ell} f_2. \tag{8.4.19}$$

Finally, considering the decomposition in Figure 8.4.2(c), it follows that the net force and torque on the massless link must both be zero. Therefore,

$$\vec{f}_{\mathrm{ext}} = \vec{f}_{\mathrm{r}/y_1/\mathcal{L}} + \vec{f}_{\mathrm{r}/y_2/\mathcal{L}}, \tag{8.4.20}$$

which implies that

$$f_1 = f_{11} + f_{21}, \tag{8.4.21}$$

$$f_2 = f_{12} + f_{22}. \tag{8.4.22}$$

Furthermore,

$$\vec{M}_{\mathcal{L}/y_1} = \vec{r}_{z/y_2} \times \vec{f}_{\mathrm{ext}} + \vec{r}_{y_2/y_1} \times (-\vec{f}_{\mathrm{r}/y_2/\mathcal{L}}) = (\ell_0 f_2 - \ell f_{22})\hat{k}_{\mathrm{B}} = 0, \tag{8.4.23}$$

and thus it follows from (8.4.22) that

$$f_{12} = \frac{\ell - \ell_0}{\ell} f_2, \quad f_{22} = \frac{\ell_0}{\ell} f_2, \tag{8.4.24}$$

which agrees with (8.4.9) and (8.4.18).

Next, we consider the case where the external force $\vec{f}_{\text{ext}}$ is applied to the particle y_1. In this case, $\ell_0 = 0$, and the equations of motion are given by (8.4.3) and by (8.4.12) with $\ell_0 = 0$, that is,

$$m_1 \ell \ddot{\theta} = -f_2. \tag{8.4.25}$$

To determine the reaction forces, the external force $\vec{f}_{\text{ext}}$ can be applied to either the endpoint x_1 of $\mathcal{L}$ or to y_1. As a first approach, the free-body analysis and resulting reaction forces are identical to the free-body analysis with $\ell_0 > 0$ except that now $\ell_0 = 0$. Hence, f_{11} and f_{21} are given by (8.4.8) and (8.4.17), respectively, and it follows from (8.4.13) and (8.4.18) that f_{12} and f_{22} are given by

$$f_{12} = f_2, \quad f_{22} = 0. \tag{8.4.26}$$

Therefore, the net force $\vec{f}_{x_1}$ at x_1 is given by

$$\begin{aligned}\vec{f}_{x_1} &= \vec{f}_{\text{ext}} - \vec{f}_{\text{r}/y_1/\mathcal{L}} \\ &= f_1 \hat{\imath}_{\text{B}} + f_2 \hat{\jmath}_{\text{B}} - (f_{11}\hat{\imath}_{\text{B}} + f_{12}\hat{\jmath}_{\text{B}}) \\ &= \frac{m_2}{m_{\mathcal{B}}}(f_1 - m_1 \ell \dot{\theta}^2)\hat{\imath}_{\text{B}}. \end{aligned} \tag{8.4.27}$$

As an alternative approach, free-body analysis is based on Figure 8.4.3. It follows from Figure 8.4.3(a) that the moment $\vec{M}_{(\mathcal{L}\cup y_2)/y_2}$ relative to y_2 applied to the body $\mathcal{L} \cup y_2$ must be zero. Hence,

$$\vec{M}_{(\mathcal{L}\cup y_2)/y_2} = \vec{r}_{x_1/y_2} \times (-\vec{f}_{\text{r}/y_1/\mathcal{L}}) = -\ell \hat{\imath}_{\text{B}} \times (-f_{11}\hat{\imath}_{\text{B}} - f_{12}\hat{\jmath}_{\text{B}}) = \ell f_{12} \hat{k}_{\text{B}} = 0, \tag{8.4.28}$$

and thus $f_{12} = 0$. Next, it follows from Figure 8.4.3(b) that f_{21} is given by (8.4.17) and that $f_{22} = 0$. Furthermore, it follows from Figure 8.4.3(c) that $\vec{f}_{\text{r}/y_1/\mathcal{L}} + \vec{f}_{\text{r}/y_2/\mathcal{L}} = 0$, and thus $f_{11} + f_{21} = 0$ and $f_{12} + f_{22} = 0$. Hence,

$$f_{11} = -f_{21} = \frac{m_2}{m_{\mathcal{B}}}(m_1 \ell \dot{\theta}^2 - f_1), \tag{8.4.29}$$

and thus the net force $\vec{f}_{x_1}$ on x_1 is given by

$$\vec{f}_{x_1} = -f_{11}\hat{\imath}_{\text{B}} = \frac{m_2}{m_{\mathcal{B}}}(f_1 - m_1 \ell \dot{\theta}^2)\hat{\imath}_{\text{B}}, \tag{8.4.30}$$

which agrees with (8.4.27). ◇

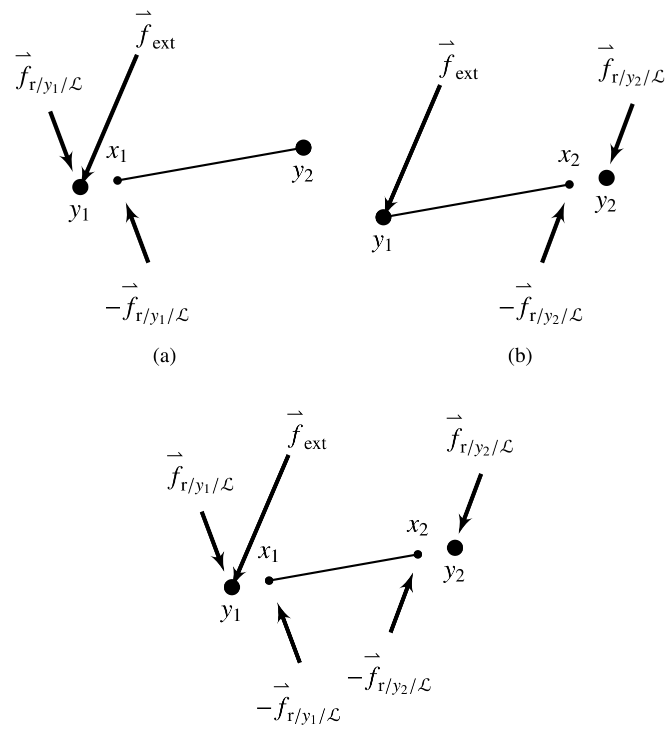


Figure 8.4.3: Example 8.4.1. Free-body analysis of a rigid body consisting of two particles connected by a massless link. Note that the external force $\vec{f}_{\rm ext}$ is applied to the particle y_1 at the end of the link.

8.5 Examples Involving a Pendulum

Example 8.5.1. As shown in Figure 8.5.4, a planar simple pendulum consists of a particle y with mass m and a massless rigid link $\mathcal{L}$ of length ℓ connected by a frictionless pin joint to the ceiling at the point w. Uniform uniform gravity acts on the pendulum, and an external force $\vec{f}_{\rm ext}$ is applied to $\mathcal{L}$ at the point z. Determine the reaction force on y due to $\mathcal{L}$ as well as the reaction force on the ceiling at w due to $\mathcal{L}$, and derive the equations of motion of the pendulum in terms of θ.

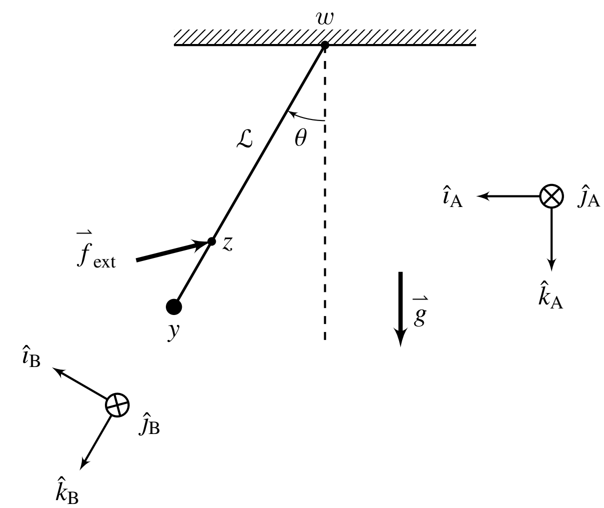


Figure 8.5.4: Planar simple pendulum for Example 8.5.1.

Solution. Let $\mathrm{F_A}$ be an inertial frame, and let $\mathrm{F_B}$ be a body-fixed frame. These frames are related by $\mathrm{F_A} \xrightarrow[2]{\theta} \mathrm{F_B}$, and thus

$$\begin{bmatrix} \hat\imath_{\rm B} \\ \hat\jmath_{\rm B} \\ \hat k_{\rm B} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \hat\imath_{\rm A} \\ \hat\jmath_{\rm A} \\ \hat k_{\rm A} \end{bmatrix}. \tag{8.5.1}$$

Hence, $\vec\omega_{\rm B/A} = \dot\theta\hat\jmath_{\rm A} = \dot\theta\hat\jmath_{\rm B}$.

First, consider the massless link $\mathcal{L}$ shown in Figure 8.5.5. Since $\mathcal{L}$ is massless, the net force on $\mathcal{L}$ is zero. Letting $\vec f_{{\rm r}/\mathcal{L}/w}$ denote the reaction force on $\mathcal{L}$ at w, it follows that

$$\vec f_{{\rm r}/\mathcal{L}/y} = -\vec f_{{\rm r}/\mathcal{L}/w} - \vec f_{\rm ext}. \tag{8.5.2}$$

Next, writing $\vec f_{{\rm r}/\mathcal{L}/w} = f_{w1}\hat\imath_{\rm B} + f_{w2}\hat\jmath_{\rm B} + f_{w3}\hat k_{\rm B}$ and $\vec f_{\rm ext} = f_1\hat\imath_{\rm A} + f_2\hat\jmath_{\rm A} + f_3\hat k_{\rm A}$, and letting ℓ_z denote the distance from w to z, the torque $\vec T_{\mathcal{L}}$ on $\mathcal{L}$ due to $\vec f_{{\rm r}/\mathcal{L}/w}$, $\vec f_{{\rm r}/\mathcal{L}/y}$, and $\vec f_{\rm ext}$ is given by

$$\begin{aligned} \vec T_{\mathcal{L}} &= \vec r_{y/w} \times \vec f_{{\rm r}/\mathcal{L}/y} + \vec r_{z/w} \times \vec f_{\rm ext} \\ &= \vec r_{y/w} \times (-\vec f_{{\rm r}/\mathcal{L}/w} - \vec f_{\rm ext}) + \vec r_{z/w} \times \vec f_{\rm ext} \\ &= -\vec r_{y/w} \times \vec f_{{\rm r}/\mathcal{L}/w} - \vec r_{y/z} \times \vec f_{\rm ext} \\ &= -\ell\hat k_{\rm B} \times (f_{w1}\hat\imath_{\rm B} + f_{w2}\hat\jmath_{\rm B} + f_{w3}\hat k_{\rm B}) - (\ell - \ell_z)\hat k_{\rm B} \times (f_1\hat\imath_{\rm A} + f_2\hat\jmath_{\rm A} + f_3\hat k_{\rm A}) \end{aligned}$$

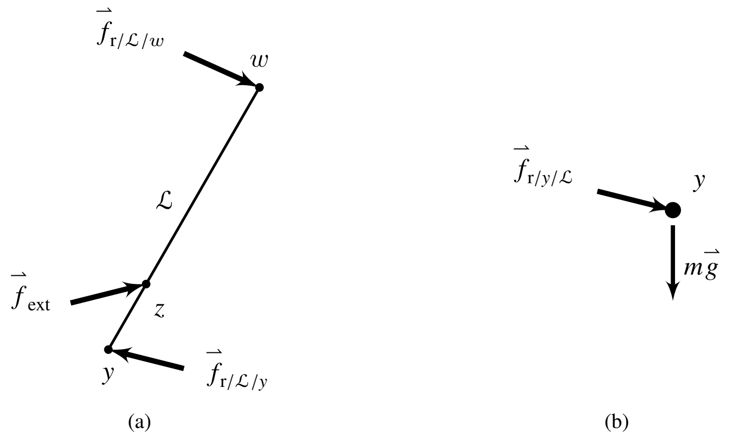


Figure 8.5.5: Example 8.5.1. Free-body diagram of the massless link $\mathcal{L}$. The reaction force on $\mathcal{L}$ due to the pin joint at w is $\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/w}$, and the the reaction force on $\mathcal{L}$ due to the particle y is $\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/y}$.

$$
\begin{aligned}
&= -\ell f_{w1}\hat{\jmath}_{\mathrm{B}} + \ell f_{w2}\hat{\imath}_{\mathrm{B}} \\
&\quad - (\ell - \ell_z)\hat{k}_{\mathrm{B}} \times [(f_1\cos\theta - f_3\sin\theta)\hat{\imath}_{\mathrm{B}} + f_2\hat{\jmath}_{\mathrm{B}} + (f_1\sin\theta + f_3\cos\theta)\hat{k}_{\mathrm{B}}] \\
&= -\ell f_{w1}\hat{\jmath}_{\mathrm{B}} + \ell f_{w2}\hat{\imath}_{\mathrm{B}} - (\ell - \ell_z)[(f_1\cos\theta - f_3\sin\theta)\hat{\jmath}_{\mathrm{B}} - f_2\hat{\imath}_{\mathrm{B}}] \\
&= [\ell f_{w2} + (\ell - \ell_z)f_2]\imath_{\mathrm{B}} + [-\ell f_{w1} - (\ell - \ell_z)(f_1\cos\theta - f_3\sin\theta)]\hat{\jmath}_{\mathrm{B}}.
\end{aligned} \tag{8.5.3}
$$

Since $\mathcal{L}$ is massless, it follows that $\overset{\rightharpoonup}{T}_{\mathcal{L}} = 0$, and thus (8.5.3) implies that

$$
f_{w1} = -\frac{\ell - \ell_z}{\ell}(f_1\cos\theta - f_3\sin\theta), \tag{8.5.4}
$$

$$
f_{w2} = -\frac{(\ell - \ell_z)}{\ell}f_2. \tag{8.5.5}
$$

Note that f_{w3} is not yet determined.

Next, consider the particle y. The reaction force on y due to $\mathcal{L}$ is given by $\overset{\rightharpoonup}{f}_{\mathrm{r}/y/\mathcal{L}} = -\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/y}$, and thus (8.5.2) implies that $\overset{\rightharpoonup}{f}_{\mathrm{r}/y/\mathcal{L}} = \overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/w} + \overset{\rightharpoonup}{f}_{\mathrm{ext}}$. The net force $\overset{\rightharpoonup}{f}_y$ on y is thus given by

$$
\begin{aligned}
\overset{\rightharpoonup}{f}_y &= m\overset{\rightharpoonup}{g} + \overset{\rightharpoonup}{f}_{\mathrm{r}/y/\mathcal{L}} \\
&= m\overset{\rightharpoonup}{g} + \overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/w} + \overset{\rightharpoonup}{f}_{\mathrm{ext}} \\
&= mg\hat{k}_{\mathrm{A}} + f_{w1}\hat{\imath}_{\mathrm{B}} + f_{w2}\hat{\jmath}_{\mathrm{B}} + f_{w3}\hat{k}_{\mathrm{B}} + f_1\hat{\imath}_{\mathrm{A}} + f_2\hat{\jmath}_{\mathrm{A}} + f_3\hat{k}_{\mathrm{A}} \\
&= (f_{w1} + f_1\cos\theta - f_3\sin\theta - mg\sin\theta)\hat{\imath}_{\mathrm{B}} + (f_{w2} + f_2)\hat{\jmath}_{\mathrm{B}} \\
&\quad + (f_{w3} + f_1\sin\theta + f_3\cos\theta + mg\cos\theta)\hat{k}_{\mathrm{B}}.
\end{aligned} \tag{8.5.6}
$$

Furthermore, since $\overset{\rightharpoonup}{r}_{y/w} = \ell\hat{k}_{\mathrm{B}}$, it follows that

$$
\overset{\rightharpoonup}{v}_{y/w/\mathrm{A}} = \ell \overset{\mathrm{A}\bullet}{\hat{k}_{\mathrm{B}}} = \ell\overset{\rightharpoonup}{\omega}_{\mathrm{B/A}} \times \hat{k}_{\mathrm{B}} = \ell\dot{\theta}\hat{\jmath}_{\mathrm{B}} \times \hat{k}_{\mathrm{B}} = \ell\dot{\theta}\hat{\imath}_{\mathrm{B}}, \tag{8.5.7}
$$

and thus

$$
\overset{\rightharpoonup}{a}_{y/w/\mathrm{A}} = \ell\ddot{\theta}\hat{\imath}_{\mathrm{B}} + \ell\dot{\theta}\overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}}
$$

$$
\begin{aligned}
&= \ell\ddot{\theta}\hat{\imath}_{\rm B} + \ell\dot{\theta}^2\hat{\jmath}_{\rm B}\times\hat{\imath}_{\rm B}\\
&= \ell\ddot{\theta}\hat{\imath}_{\rm B} - \ell\dot{\theta}^2\hat{k}_{\rm B}. \qquad (8.5.8)
\end{aligned}
$$

Next, it follows from Newton's second law applied to y that $m\overset{\rightharpoonup}{a}_{y/w/{\rm A}} = \overset{\rightharpoonup}{f}_y$, and thus

$$
\begin{aligned}
m\ell\ddot{\theta}\hat{\imath}_{\rm B} - m\ell\dot{\theta}^2\hat{k}_{\rm B} &= (f_{w1} + f_1\cos\theta - f_3\sin\theta - mg\sin\theta)\hat{\imath}_{\rm B} + (f_{w2} + f_2)\hat{\jmath}_{\rm B}\\
&\quad + (f_{w3} + f_1\sin\theta + f_3\cos\theta + mg\cos\theta)\hat{k}_{\rm B}, \qquad (8.5.9)
\end{aligned}
$$

and thus

$$m\ell\ddot{\theta} = f_{w1} + f_1\cos\theta - f_3\sin\theta - mg\sin\theta, \qquad (8.5.10)$$

$$0 = f_{w2} + f_2, \qquad (8.5.11)$$

$$-m\ell\dot{\theta}^2 = f_{w3} + f_1\sin\theta + f_3\cos\theta + mg\cos\theta. \qquad (8.5.12)$$

Using (8.5.4) and (8.5.10), it follows that

$$\ell\ddot{\theta} + g\sin\theta = \frac{\ell_z}{m\ell}(f_1\cos\theta - f_3\sin\theta). \qquad (8.5.13)$$

Furthermore,

$$f_{w2} = -f_2, \qquad (8.5.14)$$

$$f_{w3} = -m(\ell\dot{\theta}^2 + g\cos\theta) - f_1\sin\theta - f_3\cos\theta. \qquad (8.5.15)$$

Comparing (8.5.11) and (8.5.14) and assuming that $\ell_z \neq \ell$, it follows that $f_{w2} = f_2 = 0$. $\diamond$

Example 8.5.2. As shown in Figure 8.5.6, a planar physical pendulum consists of a body $\mathcal{B}$ connected to the ceiling by means of a frictionless pin joint at the point w. The center of mass of $\mathcal{B}$ is the point c, and the distance from w to c is ℓ. $\mathrm{F_A}$ is an inertial frame, and $\mathrm{F_B}$ is a body-fixed frame. The mass of $\mathcal{B}$ is m, and the component of the moment of inertia of $\mathcal{B}$ relative to c along $\hat{\jmath}_{\mathrm{B}}$ is J_0. The angle between the line passing through w and c and the vertical direction is θ. Uniform uniform gravity acts on $\mathcal{B}$, and an external force $\vec{f}_{\mathrm{ext}}$ is applied to the point z, which is fixed in $\mathcal{B}$ and located on the line passing through w and c at the distance ℓ_z from w.

i) Use Newton-Euler dynamics to derive the equations of motion for $\mathcal{B}$ in terms of θ. Also, determine the reaction force on $\mathcal{B}$ at w. Do this in two different ways, namely, by applying Newton's second law of rotation relative to w and relative to c.

ii) Specialize the equations of motion and the reaction force to the case where $\mathcal{B}$ is a uniform thin bar of length ℓ_0 and mass m.

iii) Specialize the equations of motion and reaction force on $\mathcal{B}$ at w to the case of a simple pendulum, that is, where $\mathcal{B}$ consists of a massless rigid link of length ℓ with a particle of mass m on its end.

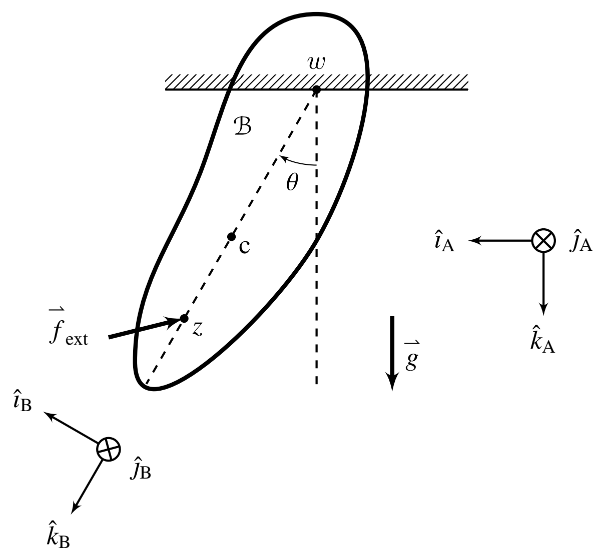


Figure 8.5.6: Physical pendulum for Example 8.5.2

Solution. *i*) Let $\mathrm{F_A}$ be an inertial frame, and let $\mathrm{F_B}$ be a body-fixed frame. The frames $\mathrm{F_A}$ and $\mathrm{F_B}$ are related by $\mathrm{F_A} \xrightarrow[2]{\theta} \mathrm{F_B}$, and thus

$$\begin{bmatrix}\hat{\imath}_{\mathrm{B}}\\ \hat{\jmath}_{\mathrm{B}}\\ \hat{k}_{\mathrm{B}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\imath}_{\mathrm{A}}\\ \hat{\jmath}_{\mathrm{A}}\\ \hat{k}_{\mathrm{A}}\end{bmatrix}, \quad \begin{bmatrix}\hat{\imath}_{\mathrm{A}}\\ \hat{\jmath}_{\mathrm{A}}\\ \hat{k}_{\mathrm{A}}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{\imath}_{\mathrm{B}}\\ \hat{\jmath}_{\mathrm{B}}\\ \hat{k}_{\mathrm{B}}\end{bmatrix}. \tag{8.5.16}$$

Hence, $\vec{\omega}_{\mathrm{B/A}} = \dot{\theta}\hat{\jmath}_{\mathrm{A}} = \dot{\theta}\hat{\jmath}_{\mathrm{B}}$.

First, consider the free-body diagram of the physical pendulum shown in Figure 8.5.7. Letting $\vec{f}_{\mathrm{r}/\mathcal{B}/w} = f_{w1}\hat{\imath}_{\mathrm{B}} + f_{w2}\hat{\jmath}_{\mathrm{B}} + f_{w3}\hat{k}_{\mathrm{B}}$ denote the reaction force on $\mathcal{B}$ due to the ceiling at w and writing

$\vec{f}_{\text{ext}} = f_1\hat{\imath}_{\text{A}} + f_2\hat{\jmath}_{\text{A}} + f_3\hat{k}_{\text{A}}$, the net force is given by

$$\begin{aligned}\vec{f}_{\mathcal{B}} &= \vec{f}_{\text{ext}} + \vec{f}_{\text{r}/\mathcal{B}/w} + m\vec{g}\\ &= (f_1\cos\theta - f_3\sin\theta + f_{w1} - mg\sin\theta)\hat{\imath}_{\text{B}} + (f_2 + f_{w2})\hat{\jmath}_{\text{B}}\\ &\quad + (f_1\sin\theta + f_3\cos\theta + f_{w3} + mg\cos\theta)\hat{k}_{\text{B}}.\end{aligned} \tag{8.5.17}$$

The moment $\vec{M}_{\mathcal{L}/w}$ on $\mathcal{B}$ relative to w is thus given by

$$\begin{aligned}\vec{M}_{\mathcal{B}/w} &= \vec{r}_{\text{c}/w} \times m\vec{g} + \vec{r}_{z/w} \times \vec{f}_{\text{ext}}\\ &= \ell\hat{k}_{\text{B}} \times mg\hat{k}_{\text{A}} + \ell_z\hat{k}_{\text{B}} \times (f_1\hat{\imath}_{\text{A}} + f_2\hat{\jmath}_{\text{A}} + f_3\hat{k}_{\text{A}})\\ &= -\ell_z f_2\hat{\imath}_{\text{B}} + (-mg\ell\sin\theta + \ell_z f_1\cos\theta - \ell_z f_3\sin\theta)\hat{\jmath}_{\text{B}}.\end{aligned} \tag{8.5.18}$$

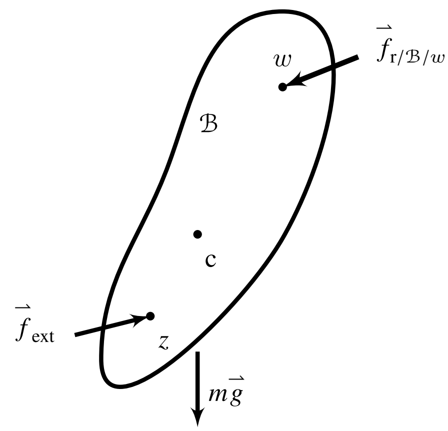


Figure 8.5.7: Physical pendulum for Example 8.5.2

Since $\vec{r}_{\text{c}/w} = \ell\hat{k}_{\text{B}}$, it follows that

$$\vec{v}_{\text{c}/w/\text{A}} = \overset{\text{A}\bullet}{\vec{r}}_{\text{c}/w} = \ell\,\overset{\text{A}\bullet}{\hat{k}}_{\text{B}} = \ell\vec{\omega}_{\text{B/A}} \times \hat{k}_{\text{B}} = \ell\dot{\theta}\hat{\jmath}_{\text{B}} \times \hat{k}_{\text{B}} = \ell\dot{\theta}\hat{\imath}_{\text{B}}, \tag{8.5.19}$$

and thus

$$\vec{a}_{\text{c}/w/\text{A}} = \ell\ddot{\theta}\hat{\imath}_{\text{B}} + \ell\dot{\theta}\,\overset{\text{A}\bullet}{\hat{\imath}}_{\text{B}} = \ell\ddot{\theta}\hat{\imath}_{\text{B}} + \ell\dot{\theta}^2\hat{\jmath}_{\text{B}} \times \hat{\imath}_{\text{B}} = \ell\ddot{\theta}\hat{\imath}_{\text{B}} - \ell\dot{\theta}^2\hat{k}_{\text{B}}. \tag{8.5.20}$$

It thus follows from Newton's second law $m\vec{a}_{\text{c}/w/\text{A}} = \vec{f}_{\mathcal{B}}$, (8.5.20), and (8.5.17) that

$$\begin{aligned}m(\ell\ddot{\theta}\hat{\imath}_{\text{B}} - \ell\dot{\theta}^2\hat{k}_{\text{B}}) &= (f_1\cos\theta - f_3\sin\theta + f_{w1} - mg\sin\theta)\hat{\imath}_{\text{B}} + (f_2 + f_{w2})\hat{\jmath}_{\text{B}}\\ &\quad + (f_1\sin\theta + f_3\cos\theta + f_{w3} + mg\cos\theta)\hat{k}_{\text{B}}.\end{aligned} \tag{8.5.21}$$

and thus

$$m\ell\ddot{\theta} = f_1\cos\theta - f_3\sin\theta + f_{w1} - mg\sin\theta, \tag{8.5.22}$$

$$-m\ell\dot{\theta}^2 = f_1\sin\theta + f_3\cos\theta + f_{w3} + mg\cos\theta, \tag{8.5.23}$$

which implies

$$f_{w1} = m\ell\ddot{\theta} + mg\sin\theta - f_1\cos\theta + f_3\sin\theta, \tag{8.5.24}$$

$$f_{w2} = -f_2, \tag{8.5.25}$$

$$f_{w3} = -m\ell\dot{\theta}^2 - mg\cos\theta - f_1\sin\theta - f_3\cos\theta. \tag{8.5.26}$$

Next, since $\mathcal{B}$ lies in the $\hat{\imath}_{\rm A}$-$\hat{k}_{\rm A}$ plane, it follows that $\vec{J}_{\mathcal{B}/w} = (J_0 + m\ell^2)\hat{\jmath}_{\rm A}\hat{\jmath}_{\rm A}'$. Hence, since $\vec{\omega}_{\rm B/A} = \dot{\theta}\hat{\jmath}_{\rm A}$, it follows that $\vec{\omega}_{\rm B/A} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\rm B/A} = 0$. Therefore, Newton's second law of rotation relative to w implies that

$$\vec{J}_{\mathcal{B}/w} \overset{\rm B\bullet}{\vec{\omega}}_{\rm B/A} = \vec{M}_{\mathcal{B}/w}. \tag{8.5.27}$$

Using $\overset{\rm B\bullet}{\vec{\omega}}_{\rm B/A} = \ddot{\theta}\hat{\jmath}_{\rm B}$ and (8.5.18), it follows that

$$(J_0 + m\ell^2)\ddot{\theta} = -mg\ell\sin\theta + \ell_z f_1\cos\theta - \ell_z f_3\sin\theta, \tag{8.5.28}$$

and thus

$$\ddot{\theta} + \frac{\ell mg}{J_0 + m\ell^2}\sin\theta = \frac{\ell_z}{J_0 + m\ell^2}(f_1\cos\theta - f_3\sin\theta). \tag{8.5.29}$$

Using (8.5.24) and (8.5.29) yields

$$f_{w1} = \frac{J_0}{J_0 + m\ell^2}mg\sin\theta + \frac{m\ell(\ell_z - \ell) - J_0}{J_0 + m\ell^2}(f_1\cos\theta - f_3\sin\theta). \tag{8.5.30}$$

ii) Now assume that $\mathcal{B}$ is a thin bar of length ℓ_0. Hence $\ell_0 = 2\ell$, and thus $J_0 = \frac{1}{12}m\ell_0^2 = \frac{1}{3}m\ell^2$ and $J_0 + m\ell^2 = \frac{1}{3}m\ell_0^2 = \frac{4}{3}m\ell^2$. Therefore, θ satisfies

$$\ddot{\theta} + \frac{3g}{2\ell_0}\sin\theta = \frac{3\ell_z}{4m\ell^2}(f_1\cos\theta - f_3\sin\theta). \tag{8.5.31}$$

In addition, the components of the reaction force are given by

$$f_{w1} = \frac{1}{4}mg\sin\theta + \frac{3\ell_z - 4\ell}{4\ell}(f_1\cos\theta - f_3\sin\theta), \tag{8.5.32}$$

$$f_{w3} = -m\ell\dot{\theta}^2 - mg\cos\theta - f_1\sin\theta - f_3\cos\theta. \tag{8.5.33}$$

iii) Now assume that $\mathcal{B}$ consists of a massless rigid link of length ℓ with a particle of mass m on its end. In this case, $J_0 = 0$. Therefore, $\ddot{\theta}$ satisfies

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = \frac{\ell_z}{m\ell^2}(f_1\cos\theta - f_3\sin\theta). \tag{8.5.34}$$

In addition, the components of the reaction force are given by

$$f_{w1} = \frac{(\ell_z - \ell)}{\ell}(f_1\cos\theta - f_3\sin\theta), \tag{8.5.35}$$

$$f_{w3} = -m\ell\dot{\theta}^2 - mg\cos\theta - f_1\sin\theta - f_3\cos\theta. \tag{8.5.36}$$

Note that these expressions agree with Example 8.5.1. ◇

Example 8.5.3. As shown in Figure 8.5.8, a spherical physical pendulum consists of a body $\mathcal{B}$ connected to an inertially nonrotating massive rigid body by means of a frictionless ball joint at the point w. The center of mass of $\mathcal{B}$ is the point c, and the distance from w to c is ℓ. The mass of $\mathcal{B}$ is $m_{\mathcal{B}}$, and the physical inertia matrix of $\mathcal{B}$ relative to c is $\vec{J}_{\mathcal{B}/\mathrm{c}}$. A force $\vec{f}_{\mathrm{ext}}$ is applied to the point z on $\mathcal{B}$. $\mathrm{F_A}$ is an inertial frame, and $\mathrm{F_B}$ is a body-fixed frame. Determine the equations of motion of $\mathcal{B}$ and the reaction force on $\mathcal{B}$ at w.

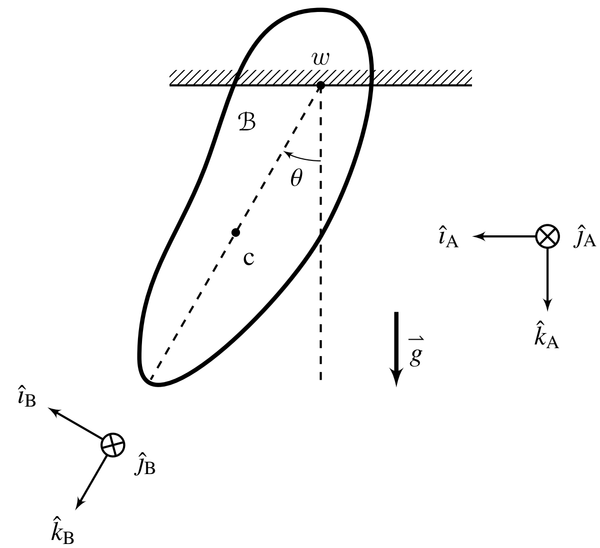


Figure 8.5.8: Spherical physical pendulum for Example 8.5.3

Solution. Euler's equation relative to w implies that

$$\vec{J}_{\mathcal{B}/w}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/w}, \tag{8.5.37}$$

where

$$\vec{M}_{\mathcal{B}/w} = \vec{r}_{\mathrm{c}/w} \times m_{\mathcal{B}}\vec{g} + \vec{r}_{z/w} \times \vec{f}_{\mathrm{ext}}. \tag{8.5.38}$$

Resolving (8.5.37) in $\mathrm{F_B}$ yields

$$J_{\mathcal{B}/w|\mathrm{B}}\dot{\omega}_{\mathrm{B/A|B}} + \omega_{\mathrm{B/A|B}} \times J_{\mathcal{B}/w|\mathrm{B}}\omega_{\mathrm{B/A|B}} = m_{\mathcal{B}}g r_{\mathrm{c}/w|\mathrm{B}} \times \mathcal{O}_{\mathrm{B/A}}e_3 + r_{z/w|\mathrm{B}} \times \mathcal{O}_{\mathrm{B/A}}f_{\mathrm{ext|A}}, \tag{8.5.39}$$

where $\mathcal{O}_{\mathrm{B/A}}$ is given by Poisson's equation (4.3.19) in the form

$$\dot{\mathcal{O}}_{\mathrm{B/A}} = -\omega^{\times}_{\mathrm{B/A|B}}\mathcal{O}_{\mathrm{B/A}}. \tag{8.5.40}$$

To recast (8.5.39) and (8.5.40) in terms of 3-2-1 Euler angles, let $\mathrm{F_{B'}}$ and $\mathrm{F_{B''}}$ be intermediate frames such that

$$\mathrm{F_A} \xrightarrow[3]{\Psi} \mathrm{F_{B'}} \xrightarrow[2]{\Theta} \mathrm{F_{B''}} \xrightarrow[1]{\Phi} \mathrm{F_B}. \tag{8.5.41}$$

Then, it follows from (4.10.7) that

$$\omega_{\mathrm{B/A|B}} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix}, \tag{8.5.42}$$

and thus

$$J_{\mathcal{B}/w|\mathrm{B}}\begin{bmatrix}1 & 0 & -\sin\Theta\\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta\\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta\end{bmatrix}\begin{bmatrix}\ddot\Phi\\ \ddot\Theta\\ \ddot\Psi\end{bmatrix}$$
$$+J_{\mathcal{B}/w|\mathrm{B}}\begin{bmatrix}0 & 0 & -(\cos\Theta)\dot\Theta\\ 0 & -(\sin\Phi)\dot\Phi & (\cos\Theta)(\cos\Phi)\dot\Phi-(\sin\Phi)(\sin\Theta)\dot\Theta\\ 0 & -(\cos\Phi)\dot\Phi & -(\cos\Theta)(\sin\Phi)\dot\Phi-(\cos\Phi)(\sin\Theta)\dot\Theta\end{bmatrix}\begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix}$$
$$+\,\omega_{\mathrm{B/A|B}}\times J_{\mathcal{B}/w|\mathrm{B}}\omega_{\mathrm{B/A|B}}=m_{\mathcal{B}}g r_{\mathrm{c}/w|\mathrm{B}}\times\mathcal{O}_{\mathrm{B/A}}e_3+r_{z/w|\mathrm{B}}\times\mathcal{O}_{\mathrm{B/A}}f_{\mathrm{ext|A}},\tag{8.5.43}$$

where

$$\mathcal{O}_{\mathrm{B/A}}=\mathcal{O}_{\mathrm{B/B''}}\mathcal{O}_{\mathrm{B''/B'}}\mathcal{O}_{\mathrm{B'/A}}=\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi).\tag{8.5.44}$$

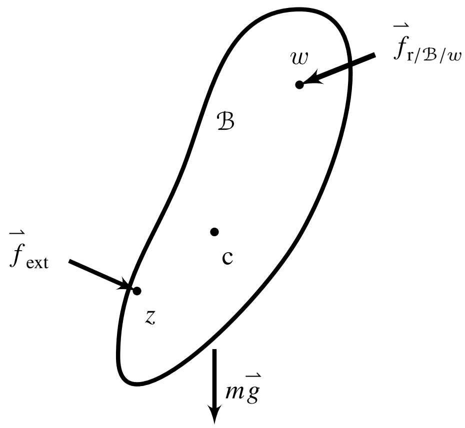


Figure 8.5.9: Free body diagram of the spherical pendulum for Example 8.5.3

Using $\vec r_{\mathrm{c}/w}=\ell\hat k_{\mathrm{B}}$, it follows that

$$\overset{\mathrm{A}\bullet}{\vec r}_{\mathrm{c}/w}=\ell\overset{\mathrm{A}\bullet}{\hat k}_{\mathrm{B}}=\ell\vec\omega_{\mathrm{B/A}}\times\hat k_{\mathrm{B}},\tag{8.5.45}$$

and thus

$$\begin{aligned}\overset{\mathrm{A}\bullet\bullet}{\vec r}_{\mathrm{c}/w}&=\ell\overset{\mathrm{A}\bullet}{\vec\omega}_{\mathrm{B/A}}\times\hat k_{\mathrm{B}}+\ell\vec\omega_{\mathrm{B/A}}\times\overset{\mathrm{A}\bullet}{\hat k}_{\mathrm{B}}\\ &=\ell\overset{\mathrm{B}\bullet}{\vec\omega}_{\mathrm{B/A}}\times\hat k_{\mathrm{B}}+\ell\vec\omega_{\mathrm{B/A}}\times\vec\omega_{\mathrm{B/A}}\times\hat k_{\mathrm{B}}.\end{aligned}\tag{8.5.46}$$

The net force $\vec f_{\mathcal{B}}$, as shown in Figure 8.5.9, is given by

$$\vec f_{\mathcal{B}}=\vec f_{\mathrm{r}/\mathcal{B}/w}+m_{\mathcal{B}}\vec g+\vec f_{\mathrm{ext}}.\tag{8.5.47}$$

Finally, it follows from Newton's second law $m\vec a_{\mathrm{c}/w/\mathrm{A}}=\vec f_{\mathcal{B}}$, (8.5.46), and (8.5.47) that

$$\vec f_{\mathrm{r}/\mathcal{B}/w}=m_{\mathcal{B}}\vec g-m_{\mathcal{B}}\ell\overset{\mathrm{B}\bullet}{\vec\omega}_{\mathrm{B/A}}\times\hat k_{\mathrm{B}}-m_{\mathcal{B}}\ell\vec\omega_{\mathrm{B/A}}\times\vec\omega_{\mathrm{B/A}}\times\hat k_{\mathrm{B}}+\vec f_{\mathrm{ext}}.\tag{8.5.48}$$

Resolving (8.5.48) in $\mathrm{F_B}$ yields

$$\left.\vec{f}_{\mathrm{r}/\mathcal{B}/w}\right|_{\mathrm{B}} = \mathcal{O}_{\mathrm{B/A}}(f_{\mathrm{ext|A}} + m_{\mathcal{B}} g_{\mathrm{A}}) - m_{\mathcal{B}} \ell(\dot{\omega}^{\times}_{\mathrm{B/A|B}} + \omega^{\times 2}_{\mathrm{B/A|B}})e_3. \tag{8.5.49}$$

To specialize (8.5.43) to the case of a simple pendulum of length ℓ supported by a pin joint whose axis of rotation is codirectional with $\hat{\jmath}_{\mathrm{A}}$, let $\Phi = \Psi = 0$ and note that $\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_2(\Theta)$. Therefore, writing $\vec{f}_{\mathrm{ext}} = f_1\hat{\imath}_{\mathrm{A}} + f_3\hat{k}_{\mathrm{A}}$, it follows that

$$\begin{aligned}
&\begin{bmatrix} m_{\mathcal{B}}\ell^2 & 0 & 0 \\ 0 & m_{\mathcal{B}}\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ 0 & 0 & \cos\Theta \end{bmatrix}\begin{bmatrix} 0 \\ \ddot{\Theta} \\ 0 \end{bmatrix} \\
&\qquad + \begin{bmatrix} m_{\mathcal{B}}\ell^2 & 0 & 0 \\ 0 & m_{\mathcal{B}}\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -(\cos\Theta)\dot{\Theta} \\ 0 & 0 & 0 \\ 0 & 0 & -(\sin\Theta)\dot{\Theta} \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \times \begin{bmatrix} m_{\mathcal{B}}\ell^2 & 0 & 0 \\ 0 & m_{\mathcal{B}}\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\Theta} \\ 0 \end{bmatrix} \\
&= m_{\mathcal{B}} g\ell e_3 \times \mathcal{O}_2(\Theta)e_3 + \ell e_2 \times \mathcal{O}_2(\Theta)(f_1e_1 + f_3e_3) \\
&= m_{\mathcal{B}} g\ell e_3 \times \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix} e_3 + \ell e_2 \times \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{bmatrix}(f_1e_1 + f_3e_3) \\
&= m_{\mathcal{B}} g\ell e_3 \times \begin{bmatrix} -\sin\Theta \\ 0 \\ \cos\Theta \end{bmatrix} + \ell e_2 \times \begin{bmatrix} f_1\cos\Theta - f_3\sin\Theta \\ 0 \\ f_1\sin\Theta + f_3\cos\Theta \end{bmatrix} \\
&= m_{\mathcal{B}} g\ell \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\sin\Theta \\ 0 \\ \cos\Theta \end{bmatrix} + \ell \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} f_1\cos\Theta - f_3\sin\Theta \\ 0 \\ f_1\sin\Theta + f_3\cos\Theta \end{bmatrix}.
\end{aligned} \tag{8.5.50}$$

Extracting the second equation of (8.5.50) yields

$$m_{\mathcal{B}}\ell^2\ddot{\Theta} = -m_{\mathcal{B}} g\ell \sin\Theta + \ell(f_1\sin\Theta + f_3\cos\Theta), \tag{8.5.51}$$

and thus

$$\ddot{\Theta} + \frac{g}{\ell}\sin\Theta = \frac{1}{\ell m_{\mathcal{B}}}(f_1\sin\Theta + f_3\cos\Theta). \tag{8.5.52}$$

◇

Example 8.5.4. As shown in Figure 8.5.10, a planar simple pendulum $\mathcal{B}$ consists of a particle y with mass m attached to the end of a massless link $\mathcal{L}$ of length ℓ. The pivot point x of the pendulum moves in a vertical plane with horizontal acceleration $a_{\rm h}$ and vertical acceleration $a_{\rm v}$, both of which are defined relative to an unforced particle w and with respect to an inertial frame $\rm F_A$. Determine the equation of motion for the pendulum in terms of the angle θ from the vertical direction as well as $a_{\rm h}$ and $a_{\rm v}$. Finally, determine the reaction forces on $\mathcal{L}$ at x and due to y.

Solution. Defining the body-fixed frame $\rm F_B$ as shown in Figure 8.5.10 and the rotation angle θ by $\mathrm{F_A} \xrightarrow[1]{\theta} \mathrm{F_B}$, it follows that $\vec{\omega}_{\rm B/A} = \dot{\theta}\hat{\imath}_{\rm A}$. Note that $\vec{r}_{y/x} = \ell\hat{k}_{\rm B}$ and that the acceleration of x relative to w with respect to $\rm F_A$ is given by $\vec{a}_{x/{\rm w/A}} = a_{\rm h}\hat{\jmath}_{\rm A} + a_{\rm v}\hat{k}_{\rm A}$. Furthermore,

$$\begin{aligned}
\vec{a}_{y/x/\rm A} &= \overset{\rm A\bullet}{\vec{\omega}}_{\rm B/A} \times \vec{r}_{y/x} + \vec{\omega}_{\rm B/A} \times (\vec{\omega}_{\rm B/A} \times \vec{r}_{y/x}) \\
&= -\ell\ddot{\theta}\hat{\jmath}_{\rm B} - \ell\dot{\theta}^2\hat{k}_{\rm B} \\
&= \ell[-(\cos\theta)\ddot{\theta} + (\sin\theta)\dot{\theta}^2]\hat{\jmath}_{\rm A} + \ell[-(\sin\theta)\ddot{\theta} - (\cos\theta)\dot{\theta}^2]\hat{k}_{\rm A}.
\end{aligned} \tag{8.5.53}$$

Alternatively,

$$\vec{r}_{y/x} = \ell[-(\sin\theta)\hat{\jmath}_{\rm A} + (\cos\theta)\hat{k}_{\rm A})], \tag{8.5.54}$$

and thus

$$\vec{v}_{y/x/\rm A} = \ell[-(\cos\theta)\dot{\theta}\hat{\jmath}_{\rm A} - (\sin\theta)\dot{\theta}\hat{k}_{\rm A}] \tag{8.5.55}$$

and

$$\vec{a}_{y/x/\rm A} = \ell[(\sin\theta)\dot{\theta}^2 - (\cos\theta)\ddot{\theta}]\hat{\jmath}_{\rm A} - \ell[(\cos\theta)\dot{\theta}^2 + (\sin\theta)\ddot{\theta}]\hat{k}_{\rm A}. \tag{8.5.56}$$

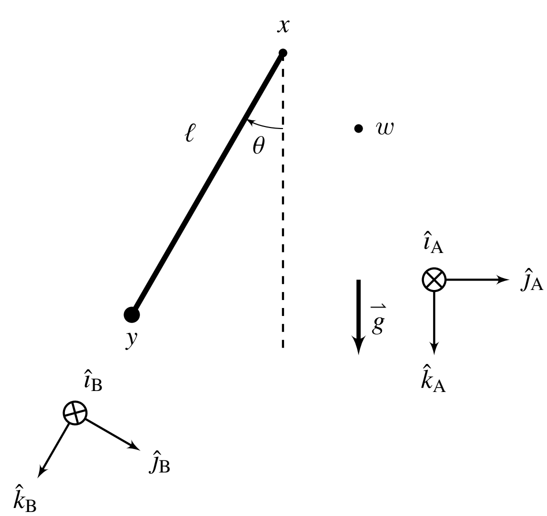


Figure 8.5.10: Example 8.5.4. Pendulum with pivot joint at x, which is subject to a prescribed acceleration.

Letting $\vec{f}_{{\rm r}/\mathcal{L}/x}$ denote the reaction force on $\mathcal{L}$ at x, it follows that the net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ is given by

$$\vec{f}_{\mathcal{B}} = m\vec{g} + \vec{f}_{{\rm r}/\mathcal{L}/x}. \tag{8.5.57}$$

Since y is the center of mass of $\mathcal{B}$, it follows from Fact 7.5.1 that

$$m\overset{\rightharpoonup}{a}_{y/w/\mathrm{A}} = \overset{\rightharpoonup}{f}_{\mathcal{B}} = m\overset{\rightharpoonup}{g} + \overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x}, \tag{8.5.58}$$

and thus, since

$$\overset{\rightharpoonup}{a}_{y/x/\mathrm{A}} = \overset{\rightharpoonup}{a}_{y/w/\mathrm{A}} - \overset{\rightharpoonup}{a}_{x/w/\mathrm{A}}, \tag{8.5.59}$$

it follows that

$$m\overset{\rightharpoonup}{a}_{y/x/\mathrm{A}} = m\overset{\rightharpoonup}{g} + \overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x} - \overset{\rightharpoonup}{f}_{\mathrm{car}}, \tag{8.5.60}$$

where

$$\overset{\rightharpoonup}{f}_{\mathrm{car}} \triangleq m\overset{\rightharpoonup}{a}_{x/w/\mathrm{A}} = m(a_{\mathrm{h}}\hat{\jmath}_{\mathrm{A}} + a_{\mathrm{v}}\hat{k}_{\mathrm{A}}) \tag{8.5.61}$$

is the carrying force. Since $\mathcal{L}$ is massless, it follows that the force on $\mathcal{L}$ at x is parallel with $\mathcal{L}$, that is,

$$\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x} = f_0\hat{k}_{\mathrm{B}} = f_0(-\sin\theta)\hat{\jmath}_{\mathrm{A}} + f_0(\cos\theta)\hat{k}_{\mathrm{A}}. \tag{8.5.62}$$

It thus follows from (8.5.53), (8.5.60), (8.5.61), and (8.5.62) that

$$-m\ell(\cos\theta)\ddot{\theta} + m\ell(\sin\theta)\dot{\theta}^2 = -f_0\sin\theta - ma_{\mathrm{h}}, \tag{8.5.63}$$

$$-m\ell(\sin\theta)\ddot{\theta} - m\ell(\cos\theta)\dot{\theta}^2 = mg + f_0\cos\theta - ma_{\mathrm{v}}. \tag{8.5.64}$$

Next, multiplying (8.5.63) by $-\sin\theta$ and (8.5.64) by $\cos\theta$, adding the resulting equations, and solving for f_0 yields

$$f_0 = -m\ell\dot{\theta}^2 - mg\cos\theta - ma_{\mathrm{h}}\sin\theta + ma_{\mathrm{v}}\cos\theta, \tag{8.5.65}$$

which yields the reaction force $\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x} = f_0\hat{k}_{\mathrm{B}}$ on $\mathcal{L}$ at x. Since $\mathcal{L}$ is massless, the reaction force $\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/y}$ on $\mathcal{L}$ due to y is $-\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x} = -f_0\hat{k}_{\mathrm{B}}$. Alternatively, (8.5.60) implies that

$$\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x} = m\overset{\rightharpoonup}{a}_{y/x/\mathrm{A}} - m\overset{\rightharpoonup}{g} + \overset{\rightharpoonup}{f}_{\mathrm{car}}, \tag{8.5.66}$$

and thus

$$\begin{aligned}
f_0 &= \hat{k}'_{\mathrm{B}}\overset{\rightharpoonup}{f}_{\mathrm{r}/\mathcal{L}/x}\\
&= m[-(\sin\theta)\hat{\jmath}'_{\mathrm{A}} + (\cos\theta)\hat{k}'_{\mathrm{A}}](\ell[(\sin\theta)\dot{\theta}^2 - (\cos\theta)\ddot{\theta}]\hat{\jmath}_{\mathrm{A}} - \ell[(\cos\theta)\dot{\theta}^2 + (\sin\theta)\ddot{\theta}]\hat{k}_{\mathrm{A}})\\
&\quad - m[-(\sin\theta)\hat{\jmath}'_{\mathrm{A}} + (\cos\theta)\hat{k}'_{\mathrm{A}}]g\hat{k}_{\mathrm{A}} + [-(\sin\theta)\hat{\jmath}'_{\mathrm{A}} + (\cos\theta)\hat{k}'_{\mathrm{A}}]m(a_{\mathrm{h}}\hat{\jmath}_{\mathrm{A}} + a_{\mathrm{v}}\hat{k}_{\mathrm{A}})\\
&= -m\ell(\sin\theta)[(\sin\theta)\dot{\theta}^2 - (\cos\theta)\ddot{\theta}] - m\ell(\cos\theta)[(\cos\theta)\dot{\theta}^2 + (\sin\theta)\ddot{\theta}]\\
&\quad - mg\cos\theta - ma_{\mathrm{h}}\sin\theta + ma_{\mathrm{v}}\cos\theta\\
&= -m\ell\dot{\theta}^2 - mg\cos\theta - ma_{\mathrm{h}}\sin\theta + ma_{\mathrm{v}}\cos\theta.
\end{aligned} \tag{8.5.67}$$

Finally, substituting (8.5.65) into either (8.5.63) or (8.5.64) yields

$$\ell\ddot{\theta} + (g - a_{\mathrm{v}})\sin\theta - a_{\mathrm{h}}\cos\theta = 0. \tag{8.5.68}$$

In the case where $a_{\mathrm{v}} = 0$, (8.5.68) specializes to

$$\ell\ddot{\theta} + g\sin\theta - a_{\mathrm{h}}\cos\theta = 0, \tag{8.5.69}$$

which, under the steady condition $\ddot{\theta} = 0$, yields

$$a_{\mathrm{h}} = g \tan\theta. \tag{8.5.70}$$

Note that setting $a_{\mathrm{h}} = a_{\mathrm{v}} = 0$ in (8.5.68) yields the standard pendulum dynamics $\ell\ddot{\theta} + g\sin\theta = 0$. ◇

In Example 8.5.4, the pivot of the pendulum translates in a vertical plane, within which the pendulum swings. This vertical plane is fixed with respect to an inertial frame. In the following example, the pivot is located at a fixed point above the Earth, which rotates relative to an inertial frame. The pendulum is connected to a universal joint and thus can swing in all directions but it cannot rotate around its longitudinal axis. This is the Foucault pendulum, whose motion demonstrates the rotation of the Earth.

Example 8.5.5. As shown in Figure 8.5.11, a Foucault pendulum consists of a particle y with mass m attached to the end of a massless rigid link $\mathcal{L}$ of length ℓ. The other end of the link is connected to a universal joint at the point z, which is located above the Earth at north latitude λ. Assume that the axis of rotation of the Earth is fixed with respect to an inertial frame, the rate $\omega_{\rm E}$ of rotation of the Earth relative to an inertial frame is constant, and the center w of the Earth has zero inertial acceleration. Determine the reaction force on y due to $\mathcal{L}$ and the equations of motion of the pendulum.

Figure 8.5.11: Foucault pendulum with universal joint for Example 8.5.5.

Solution. Let $\rm F_A$ denote an inertial frame whose axis $\hat{k}_{\rm A}$ is codirectional with the spin axis of the Earth, let $\vec{\omega}_{\rm E/A} = \omega_{\rm E}\hat{k}_{\rm A} = \omega_{\rm E}\hat{k}_{\rm E}$ denote the angular velocity of the Earth frame $\rm F_E$ relative to $\rm F_A$, where $\omega_{\rm E} > 0$, and let $\rm F_{sph}$ denote the spherical frame at z. Hence,

$$\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_E} \xrightarrow[2]{-\lambda} \mathrm{F_{sph}} \xrightarrow[3]{\Psi} \mathrm{F_{B'}} \xrightarrow[2]{\Theta} \mathrm{F_B}, \tag{8.5.71}$$

where $\omega_{\rm E} = \dot{\theta}$, $\rm F_{sph} = [\hat{e}_u\ \ \hat{e}_e\ \ \hat{e}_n]$, and

$$\begin{bmatrix}\hat{\imath}_{\rm E}\\ \hat{\jmath}_{\rm E}\\ \hat{k}_{\rm E}\end{bmatrix} = \begin{bmatrix}\cos\lambda & 0 & -\sin\lambda\\ 0 & 1 & 0\\ \sin\lambda & 0 & \cos\lambda\end{bmatrix}\begin{bmatrix}\hat{e}_{\rm u}\\ \hat{e}_{\rm e}\\ \hat{e}_{\rm n}\end{bmatrix}, \quad \begin{bmatrix}\hat{\imath}_{\rm B'}\\ \hat{\jmath}_{\rm B'}\\ \hat{k}_{\rm B'}\end{bmatrix} = \begin{bmatrix}\cos\Psi & \sin\Psi & 0\\ -\sin\Psi & \cos\Psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{e}_{\rm u}\\ \hat{e}_{\rm e}\\ \hat{e}_{\rm n}\end{bmatrix}, \tag{8.5.72}$$

$$\begin{bmatrix}\hat{\imath}_{\rm B}\\ \hat{\jmath}_{\rm B}\\ \hat{k}_{\rm B}\end{bmatrix} = \begin{bmatrix}\cos\Theta & 0 & -\sin\Theta\\ 0 & 1 & 0\\ \sin\Theta & 0 & \cos\Theta\end{bmatrix}\begin{bmatrix}\hat{\imath}_{\rm B'}\\ \hat{\jmath}_{\rm B'}\\ \hat{k}_{\rm B'}\end{bmatrix}. \tag{8.5.73}$$

Since λ is constant, it follows that $\vec{\omega}_{\rm sph/E} = 0$, and thus $\vec{\omega}_{\rm sph/A} = \vec{\omega}_{\rm E/A} = \omega_{\rm E}\hat{k}_{\rm A} = \omega_{\rm E}\hat{k}_{\rm E}$. Note that $\omega_{\rm E} = \dot{\theta}$, $\vec{r}_{y/z} = -\ell\hat{\imath}_{\rm B}$, and $\vec{g} = -g\hat{e}_{\rm u}$.

Newton's second law implies that

$$m\vec{a}_{y/w/{\rm A}} = m\vec{g} + \vec{f}_{{\rm r}/y/\mathcal{L}}, \tag{8.5.74}$$

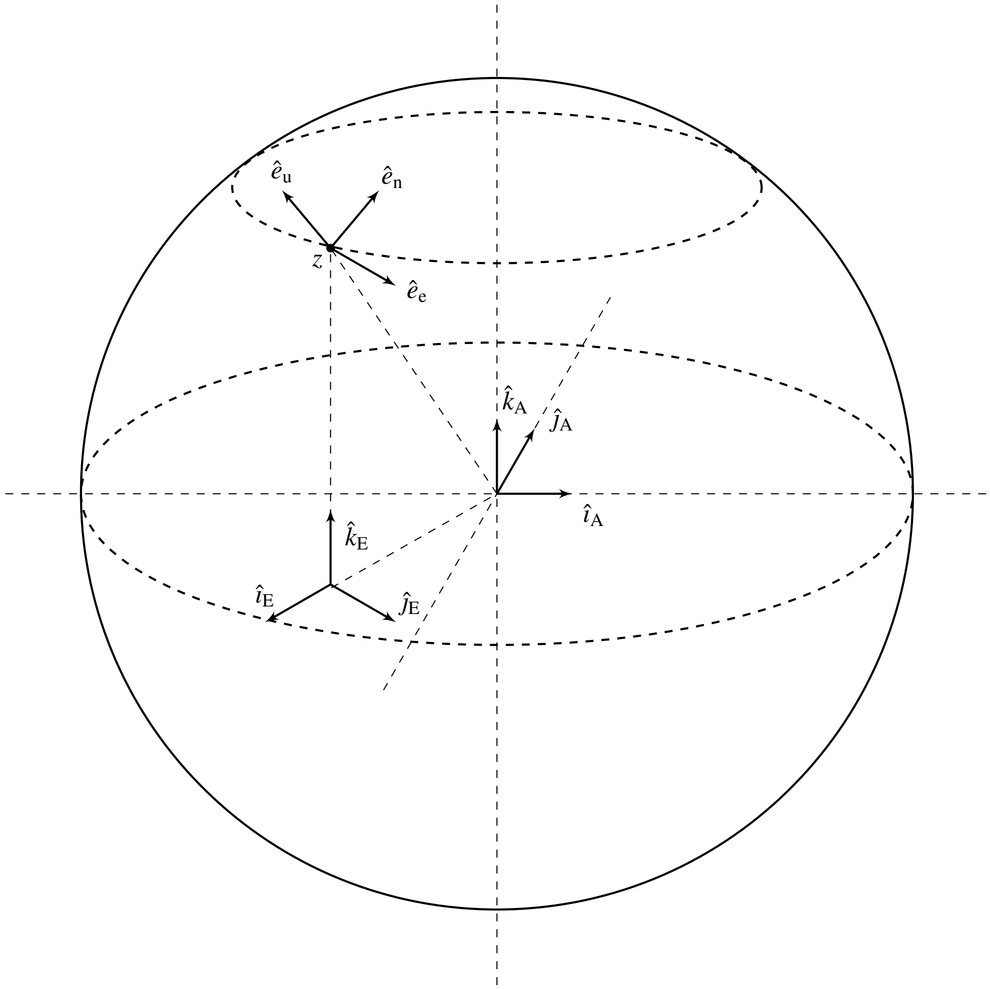


Figure 8.5.12: Frames in Example 8.5.5.

where $\vec{f}_{{\rm r}/y/\mathcal{L}}$ is the reaction force on y due to $\mathcal{L}$. Hence,

$$m\vec{a}_{y/z/{\rm A}} = -m\vec{a}_{z/w/{\rm A}} + m\vec{g} + \vec{f}_{{\rm r}/y/\mathcal{L}}. \tag{8.5.75}$$

Using (7.5.27), it follows that

$$\vec{a}_{z/w/{\rm A}} = -r_{\rm E}\omega_{\rm E}^2(\cos\lambda)^2\hat{e}_{\rm u} + r_{\rm E}\omega_{\rm E}^2(\sin\lambda)(\cos\lambda)\hat{e}_{\rm n}, \tag{8.5.76}$$

and thus (8.5.75) implies that

$$m\vec{a}_{y/z/{\rm A}} = mr_{\rm E}\omega_{\rm E}^2(\cos\lambda)^2\hat{e}_{\rm u} - mr_{\rm E}\omega_{\rm E}^2(\sin\lambda)(\cos\lambda)\hat{e}_{\rm n} + m\vec{g} + \vec{f}_{{\rm r}/y/\mathcal{L}}. \tag{8.5.77}$$

Next, using double transport, $\vec{\omega}_{{\rm sph}/{\rm A}} = \vec{\omega}_{{\rm E}/{\rm A}}$, and $\overset{{\rm E}\bullet}{\vec{\omega}}_{{\rm E}/{\rm A}} = 0$, it follows from (8.5.77) that

$$m\vec{a}_{y/z/{\rm sph}} + 2m\vec{\omega}_{{\rm E}/{\rm A}} \times \vec{v}_{y/z/{\rm sph}} + m\vec{\omega}_{{\rm E}/{\rm A}} \times (\vec{\omega}_{{\rm E}/{\rm A}} \times \vec{r}_{y/z})$$

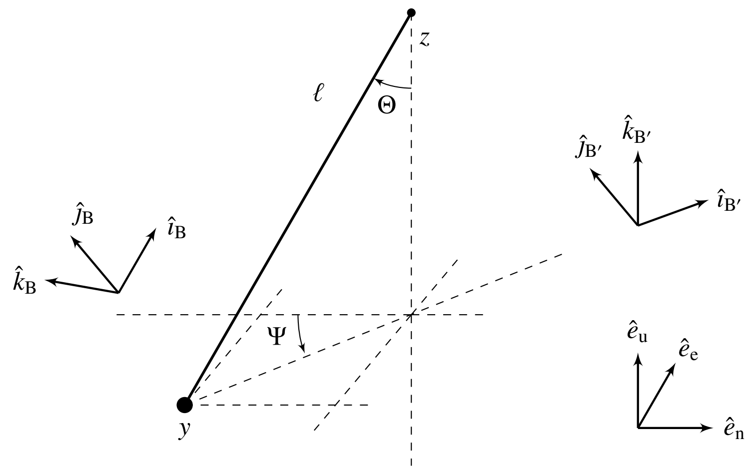


Figure 8.5.13: Simple pendulum for Example 8.5.5.

$$= mr_{\mathrm{E}}\omega_{\mathrm{E}}^2(\cos\lambda)^2\hat{e}_{\mathrm{u}} - mr_{\mathrm{E}}\omega_{\mathrm{E}}^2(\sin\lambda)(\cos\lambda)\hat{e}_{\mathrm{n}} + m\vec{g} + \vec{f}_{\mathrm{r}/y/\mathcal{L}}. \tag{8.5.78}$$

Furthermore, note that

$$\begin{aligned}\hat{\imath}_{\mathrm{B}} &= (\cos\Theta)\hat{\imath}_{\mathrm{B}'} - (\sin\Theta)\hat{k}_{\mathrm{B}'}\\ &= (\cos\Theta)(\cos\Psi)\hat{e}_{\mathrm{u}} + (\cos\Theta)(\sin\Psi)\hat{e}_{\mathrm{e}} - (\sin\Theta)\hat{e}_{\mathrm{n}}\\ &= c_{\mathrm{u}}\hat{e}_{\mathrm{u}} + c_{\mathrm{e}}\hat{e}_{\mathrm{e}} + c_{\mathrm{n}}\hat{e}_{\mathrm{n}},\end{aligned} \tag{8.5.79}$$

where

$$c_{\mathrm{u}} \triangleq (\cos\Theta)\cos\Psi, \tag{8.5.80}$$

$$c_{\mathrm{e}} \triangleq (\cos\Theta)\sin\Psi, \tag{8.5.81}$$

$$c_{\mathrm{n}} \triangleq -\sin\Theta. \tag{8.5.82}$$

Therefore,

$$\vec{r}_{y/z} = -\ell c_{\mathrm{u}}\hat{e}_{\mathrm{u}} - \ell c_{\mathrm{e}}\hat{e}_{\mathrm{e}} - \ell c_{\mathrm{n}}\hat{e}_{\mathrm{n}}, \tag{8.5.83}$$

$$\vec{v}_{y/z/\mathrm{sph}} = -\ell\dot{c}_{\mathrm{u}}\hat{e}_{\mathrm{u}} - \ell\dot{c}_{\mathrm{e}}\hat{e}_{\mathrm{e}} - \ell\dot{c}_{\mathrm{n}}\hat{e}_{\mathrm{n}}, \tag{8.5.84}$$

$$\vec{a}_{y/z/\mathrm{sph}} = -\ell\ddot{c}_{\mathrm{u}}\hat{e}_{\mathrm{u}} - \ell\ddot{c}_{\mathrm{e}}\hat{e}_{\mathrm{e}} - \ell\ddot{c}_{\mathrm{n}}\hat{e}_{\mathrm{n}}. \tag{8.5.85}$$

In addition,

$$\vec{\omega}_{\mathrm{E/A}} = \omega_{\mathrm{E}}(\sin\lambda)\hat{e}_{\mathrm{u}} + \omega_{\mathrm{E}}(\cos\lambda)\hat{e}_{\mathrm{n}}, \tag{8.5.86}$$

and thus

$$\begin{aligned}\vec{\omega}_{\mathrm{E/A}}\times\vec{v}_{y/z/\mathrm{sph}} &= -\ell\omega_{\mathrm{E}}[(\sin\lambda)\hat{e}_{\mathrm{u}} + (\cos\lambda)\hat{e}_{\mathrm{n}}]\times(\dot{c}_{\mathrm{u}}\hat{e}_{\mathrm{u}} + \dot{c}_{\mathrm{e}}\hat{e}_{\mathrm{e}} + \dot{c}_{\mathrm{n}}\hat{e}_{\mathrm{n}})\\ &= -\ell\omega_{\mathrm{E}}[-\dot{c}_{\mathrm{e}}(\cos\lambda)\hat{e}_{\mathrm{u}} + (-\dot{c}_{\mathrm{n}}\sin\lambda + \dot{c}_{\mathrm{u}}\cos\lambda)\hat{e}_{\mathrm{e}} + \dot{c}_{\mathrm{e}}(\sin\lambda)\hat{e}_{\mathrm{n}}]\\ &= \ell\omega_{\mathrm{E}}[\dot{c}_{\mathrm{e}}(\cos\lambda)\hat{e}_{\mathrm{u}} + (\dot{c}_{\mathrm{n}}\sin\lambda - \dot{c}_{\mathrm{u}}\cos\lambda)\hat{e}_{\mathrm{e}} - \dot{c}_{\mathrm{e}}(\sin\lambda)\hat{e}_{\mathrm{n}}].\end{aligned} \tag{8.5.87}$$

It thus follows from (8.5.78) that

$$\begin{aligned}&m\ell(\ddot{c}_{\mathrm{u}}\hat{e}_{\mathrm{u}} + \ddot{c}_{\mathrm{e}}\hat{e}_{\mathrm{e}} + \ddot{c}_{\mathrm{n}}\hat{e}_{\mathrm{n}}) - 2m\ell\omega_{\mathrm{E}}[\dot{c}_{\mathrm{e}}(\cos\lambda)\hat{e}_{\mathrm{u}} + (\dot{c}_{\mathrm{n}}\sin\lambda - \dot{c}_{\mathrm{u}}\cos\lambda)\hat{e}_{\mathrm{e}} - \dot{c}_{\mathrm{e}}(\sin\lambda)\hat{e}_{\mathrm{n}}]\\ &\quad + m\ell\omega_{\mathrm{E}}^2(c_{\mathrm{u}}\sin\lambda + c_{\mathrm{n}}\cos\lambda)[(\sin\lambda)\hat{e}_{\mathrm{u}} + (\cos\lambda)\hat{e}_{\mathrm{n}}] - m\ell\omega_{\mathrm{E}}^2(c_{\mathrm{u}}\hat{e}_{\mathrm{u}} + c_{\mathrm{e}}\hat{e}_{\mathrm{e}} + c_{\mathrm{n}}\hat{e}_{\mathrm{n}})\end{aligned}$$

$$= -r_E\omega_E^2(\cos\lambda)^2\hat{e}_u + mr_E\omega_E^2(\sin\lambda)(\cos\lambda)\hat{e}_n + mg\hat{e}_u - \vec{f}_{r/y/\mathcal{L}}. \tag{8.5.88}$$

Next, writing $\vec{f}_{r/y/\mathcal{L}} = f_r\hat{\imath}_B$, solving (8.5.88) for $f_r\hat{\imath}_B$, and multiplying by $\hat{\imath}_B'$ yields

$$\begin{aligned} f_r &= -m\ell(c_u\ddot{c}_u + c_e\ddot{c}_e + c_n\ddot{c}_n) + 2m\ell\omega_E[c_u\dot{c}_e(\cos\lambda) + c_e(\dot{c}_n\sin\lambda - \dot{c}_u\cos\lambda) - c_n\dot{c}_e(\sin\lambda)] \\ &\quad - m\ell\omega_E^2(c_u\sin\lambda + c_n\cos\lambda)^2 + m\ell\omega_E^2 - c_umr_E\omega_E^2(\cos\lambda)^2 + c_nmr_E\omega_E^2(\sin\lambda)(\cos\lambda) + c_umg \\ &= -m\ell(c_u\ddot{c}_u + c_e\ddot{c}_e + c_n\ddot{c}_n) + 2m\ell\omega_E[(c_u\dot{c}_e - c_e\dot{c}_u)\cos\lambda + (c_e\dot{c}_n - c_n\dot{c}_e)\sin\lambda] \\ &\quad - m\ell\omega_E^2(c_u\sin\lambda + c_n\cos\lambda)^2 + m\ell\omega_E^2 - c_umr_E\omega_E^2(\cos\lambda)^2 + c_nmr_E\omega_E^2(\sin\lambda)(\cos\lambda) + c_umg. \end{aligned} \tag{8.5.89}$$

Extracting the scalar components from (8.5.88) yields

$$m\ell\ddot{c}_u - 2m\ell\omega_E\dot{c}_e\cos\lambda + m\ell\omega_E^2(\cos\lambda)(c_n\sin\lambda - c_u\cos\lambda) - c_um\ell\omega_E^2 = -r_E\omega_E^2(\cos\lambda)^2 + mg - f_rc_u, \tag{8.5.90}$$

$$m\ell\ddot{c}_e + 2m\ell\omega_E(\dot{c}_u\cos\lambda - \dot{c}_n\sin\lambda) - m\ell\omega_E^2c_e = -f_rc_e, \tag{8.5.91}$$

$$m\ell\ddot{c}_n + 2m\ell\omega_E\dot{c}_e\sin\lambda + m\ell\omega_E^2(\sin\lambda)(c_u\cos\lambda - c_n\sin\lambda) = mr_E\omega_E^2(\sin\lambda)(\cos\lambda) - f_rc_n. \tag{8.5.92}$$

Next, eliminating all terms arising from centripetal acceleration, that is, all terms involving ω_E^2, yields

$$m\ell\ddot{c}_u - 2m\ell\omega_E\dot{c}_e\cos\lambda = mg - f_rc_u, \tag{8.5.93}$$

$$m\ell\ddot{c}_e + 2m\ell\omega_E(\dot{c}_u\cos\lambda - \dot{c}_n\sin\lambda) = -f_rc_e, \tag{8.5.94}$$

$$m\ell\ddot{c}_n + 2m\ell\omega_E\dot{c}_e\sin\lambda = -f_rc_n, \tag{8.5.95}$$

where

$$f_r = -m\ell(c_u\ddot{c}_u + c_e\ddot{c}_e + c_n\ddot{c}_n) + 2m\ell\omega_E[(c_u\dot{c}_e - c_e\dot{c}_u)\cos\lambda + (c_e\dot{c}_n - c_n\dot{c}_e)\sin\lambda] + c_umg. \tag{8.5.96}$$

Now define

$$r_u \triangleq -\ell c_u, \quad r_e \triangleq -\ell c_e, \quad r_n \triangleq -\ell c_n, \tag{8.5.97}$$

so that

$$\vec{r}_{y/z} = r_u\hat{e}_u + r_e\hat{e}_e + r_n\hat{e}_n. \tag{8.5.98}$$

Then, (8.5.93)–(8.5.96) can be written as

$$\ddot{r}_u - 2\omega_E\dot{r}_e\cos\lambda + \frac{f_r}{m\ell}r_u = g, \tag{8.5.99}$$

$$\ddot{r}_e + 2\omega_E(\dot{r}_u\cos\lambda - \dot{r}_n\sin\lambda) + \frac{f_r}{m\ell}r_e = 0, \tag{8.5.100}$$

$$\ddot{r}_n + 2\omega_E\dot{r}_e\sin\lambda + \frac{f_r}{m\ell}r_n = 0, \tag{8.5.101}$$

$$f_r = -\frac{m}{\ell}(r_u\ddot{r}_u + r_e\ddot{r}_e + r_n\ddot{r}_n) + \frac{2m}{\ell}\omega_E[(r_u\dot{r}_e - r_e\dot{r}_u)\cos\lambda + (r_e\dot{r}_n - r_n\dot{r}_e)\sin\lambda] - \frac{mg}{\ell}r_u. \tag{8.5.102}$$

Next, we assume that Θ and Ψ are small angles so that $r_u = -\ell$. Since the Coriolis term in

(8.5.99) is small, it follows from (8.5.99) that

$$f_\mathrm{r} = -mg, \tag{8.5.103}$$

and thus (8.5.100) and (8.5.101) become

$$\ddot{r}_\mathrm{e} - 2\omega_\mathrm{E}(\sin\lambda)\dot{r}_\mathrm{n} + \omega^2 r_\mathrm{e} = 0, \tag{8.5.104}$$

$$\ddot{r}_\mathrm{n} + 2\omega_\mathrm{E}(\sin\lambda)\dot{r}_\mathrm{e} + \omega^2 r_\mathrm{n} = 0. \tag{8.5.105}$$

where $\omega \triangleq \sqrt{\frac{g}{\ell}}$. Using $z \triangleq r_\mathrm{e} + \jmath r_\mathrm{n}$ to represent the horizontal position as a complex variable, it follows that

$$\ddot{z} + 2\jmath\omega_\mathrm{E}(\sin\lambda)\dot{z} + \omega^2 z = 0. \tag{8.5.106}$$

An approximate solution is given by

$$z(t) = e^{\jmath\omega_\mathrm{E}(\sin\lambda)t}(c_1 e^{\jmath\omega t} + c_2 e^{-\jmath\omega t}), \tag{8.5.107}$$

which shows that the vertical plane of oscillation of the pendulum rotates at the rate of $\omega_\mathrm{E}\sin\lambda$ rad/sec. At the north pole of the Earth, where $\lambda = 90$ deg, the plane of oscillation of the pendulum rotates once per day. ◇

Example 8.5.6. As shown in Figure 8.5.14, a planar simple pendulum $\mathcal{B}$ consists of a particle y with mass m attached to the end of a massless rigid link $\mathcal{L}$ of length ℓ. The pivot point x of the pendulum is attached to the endpoint of a thin horizontal arm $\mathcal{A}$ of length r and mass $m_{\mathcal{A}}$ whose other endpoint is attached to a massless rotating vertical shaft $\mathcal{S}$ of length L. The pivot is mounted on the arm such that the pendulum is constrained to lie in a plane that is orthogonal to the arm. The shaft is attached to the ground at w by means of a vertical frictionless pin joint. A vertical external torque $\vec{T}_{\rm ext}$ is applied to the shaft. Determine the equations of motion for the pendulum in terms of the angle θ of $\mathcal{B}$ from the vertical direction and the angular rate ω of the arm around the shaft. In addition, determine the reaction force on $\mathcal{L}$ at x, the reaction force on $\mathcal{L}$ due to y, the reaction torque on $\mathcal{L}$ at x, and the reaction force and reaction torque on the ground at w.

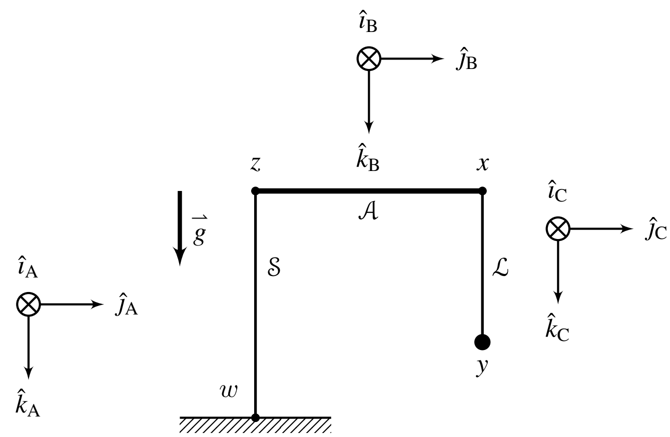


Figure 8.5.14: Example 8.5.6. Pendulum with pivot mounted on the endpoint of a rotating horizontal arm. The pendulum is constrained to swing in a plane that is orthogonal to the arm and thus tangential to the circle traced by x. As shown, the pendulum swings into and out of the page.

Solution. As shown in Figure 8.5.14, define the frame $\rm F_C$ attached to $\mathcal{B} = \mathcal{L} \cup y$, the frame $\rm F_B$ attached to the arm, and the inertial frame $\rm F_A$. Furthermore, defining the rotation angles ϕ and θ by $\mathrm{F_A} \xrightarrow[3]{\phi} \mathrm{F_B} \xrightarrow[2]{\theta} \mathrm{F_C}$, it follows that $\vec{\omega}_{\rm C/B} = \dot\theta \hat{\jmath}_{\rm C} = \dot\theta \hat{\jmath}_{\rm B}$, $\vec{\omega}_{\rm B/A} = \omega \hat{k}_{\rm B} = \omega \hat{k}_{\rm A}$, where $\omega = \dot\phi$, and $\vec{\omega}_{\rm C/A} = \dot\theta \hat{\jmath}_{\rm B} + \omega \hat{k}_{\rm B}$. Note that

$$\mathcal{O}_{\rm C/B} = \mathcal{O}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \tag{8.5.108}$$

and thus

$$\begin{bmatrix} \hat{\imath}_{\rm C} \\ \hat{\jmath}_{\rm C} \\ \hat{k}_{\rm C} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\rm B} \\ \hat{\jmath}_{\rm B} \\ \hat{k}_{\rm B} \end{bmatrix}. \tag{8.5.109}$$

Finally, note that w has zero inertial acceleration.

First, consider the massless link $\mathcal{L}$ shown in Figure 8.5.15(a). Note that, since $\mathcal{L}$ is massless, the net force and torque on $\mathcal{L}$ are both zero. Letting $\vec{f}_{{\rm r}/\mathcal{L}/x} = f_1 \hat{\imath}_{\rm C} + f_2 \hat{\jmath}_{\rm C} + f_3 \hat{k}_{\rm C}$ denote the reaction force on $\mathcal{L}$ at x, it follows that the reaction force on $\mathcal{L}$ at y is $\vec{f}_{{\rm r}/\mathcal{L}/y} = -\vec{f}_{{\rm r}/\mathcal{L}/x}$, and the reaction force on

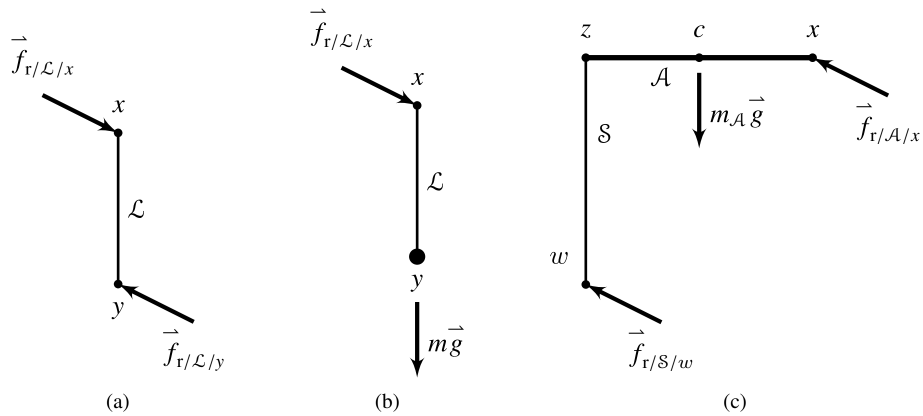


Figure 8.5.15: Example 8.5.7. Free-body diagram of (a) the massless link $\mathcal{L}$, showing the reaction force $\vec{f}_{\mathrm{r}/\mathcal{L}/x}$ on $\mathcal{L}$ due to the pin joint at the point x as well as the reaction force $\vec{f}_{\mathrm{r}/\mathcal{L}/y}$ on $\mathcal{L}$ due to particle at y, (b) the pendulum $\mathcal{B}$, and (c) the arm $\mathcal{A}$ and the shaft $\mathcal{S}$, showing the reaction force $\vec{f}_{\mathrm{r}/\mathcal{A}/x}$ on $\mathcal{A}$ due to the pin joint at x and the reaction force $\vec{f}_{\mathrm{r}/\mathcal{S}/w}$ on $\mathcal{S}$ due to the ground.

y due to $\mathcal{L}$ is $\vec{f}_{\mathrm{r}/y/\mathcal{L}} = -\vec{f}_{\mathrm{r}/\mathcal{L}/y} = \vec{f}_{\mathrm{r}/\mathcal{L}/x}$. Furthermore, let $\vec{T}_{\mathrm{r}/\mathcal{L}/x} = T_1\hat{\imath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}}$ denote the reaction torque on $\mathcal{L}$ due to the pin joint at x, where $\vec{T}_{\mathrm{r}/\mathcal{L}/x}$ has a zero component in the direction $\hat{\jmath}_{\mathrm{C}}$, which is the axis of the pin joint. The torque $\vec{T}_{\mathcal{L}}$ on $\mathcal{L}$ is thus given by

$$\begin{aligned}
\vec{T}_{\mathcal{L}} &= \vec{T}_{\mathrm{r}/\mathcal{L}/x} + \vec{r}_{y/x} \times \vec{f}_{\mathrm{r}/\mathcal{L}/y} \\
&= \vec{T}_{\mathrm{r}/\mathcal{L}/x} + \vec{r}_{y/x} \times (-\vec{f}_{\mathrm{r}/\mathcal{L}/x}) \\
&= T_1\hat{\imath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}} - \ell\hat{k}_{\mathrm{C}} \times (f_1\hat{\imath}_{\mathrm{C}} + f_2\hat{\jmath}_{\mathrm{C}} + f_3\hat{k}_{\mathrm{C}}) \\
&= T_1\hat{\imath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}} - \ell f_1\hat{\jmath}_{\mathrm{C}} + \ell f_2\hat{\imath}_{\mathrm{C}} \\
&= (T_1 + \ell f_2)\hat{\imath}_{\mathrm{C}} - \ell f_1\hat{\jmath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}}.
\end{aligned} \tag{8.5.110}$$

Since $\vec{T}_{\mathcal{L}} = 0$, it follows that $T_1 = -\ell f_2$, $f_1 = 0$, and $T_3 = 0$.

Next, consider the pendulum $\mathcal{B}$ shown in Figure 8.5.15(b). The net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ is given by

$$\begin{aligned}
\vec{f}_{\mathcal{B}} &= \vec{f}_{\mathrm{r}/\mathcal{L}/x} + m\vec{g} \\
&= f_2\hat{\jmath}_{\mathrm{C}} + f_3\hat{k}_{\mathrm{C}} + mg\hat{k}_{\mathrm{B}} \\
&= f_2\hat{\jmath}_{\mathrm{B}} + f_3[(\sin\theta)\hat{\imath}_{\mathrm{B}} + (\cos\theta)\hat{k}_{\mathrm{B}}] + mg\hat{k}_{\mathrm{B}} \\
&= f_3(\sin\theta)\hat{\imath}_{\mathrm{B}} + f_2\hat{\jmath}_{\mathrm{B}} + (f_3\cos\theta + mg)\hat{k}_{\mathrm{B}},
\end{aligned} \tag{8.5.111}$$

and the reaction torque on $\mathcal{L}$ is given by

$$\vec{T}_{\mathrm{r}/\mathcal{L}/x} = T_1\hat{\imath}_{\mathrm{C}} = -\ell f_2\hat{\imath}_{\mathrm{C}}. \tag{8.5.112}$$

Using

$$\begin{aligned}\vec{r}_{y/w} &= \vec{r}_{y/x} + \vec{r}_{x/z} + \vec{r}_{z/w} \\ &= \ell\hat{k}_{\mathrm{C}} + r\hat{\jmath}_{\mathrm{B}} - L\hat{k}_{\mathrm{A}},\end{aligned} \tag{8.5.113}$$

it follows that

$$\begin{aligned}\vec{v}_{y/w/\mathrm{A}} &= \ell\vec{\omega}_{\mathrm{C/A}} \times \hat{k}_{\mathrm{C}} + r\vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_{\mathrm{B}} \\ &= \ell(\dot{\theta}\hat{\jmath}_{\mathrm{B}} + \omega\hat{k}_{\mathrm{B}}) \times [(\sin\theta)\hat{\imath}_{\mathrm{B}} + (\cos\theta)\hat{k}_{\mathrm{B}}] + r\omega\hat{k}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}} \\ &= \alpha\hat{\imath}_{\mathrm{B}} + \beta\hat{\jmath}_{\mathrm{B}} + \gamma\hat{k}_{\mathrm{B}} \\ &= \alpha\hat{\imath}_{\mathrm{B}} + \beta\hat{\jmath}_{\mathrm{B}} + \gamma\hat{k}_{\mathrm{A}},\end{aligned} \tag{8.5.114}$$

where

$$\alpha \triangleq \ell(\cos\theta)\dot{\theta} - r\omega, \quad \beta \triangleq \ell\omega\sin\theta, \quad \gamma \triangleq -\ell(\sin\theta)\dot{\theta}. \tag{8.5.115}$$

Furthermore,

$$\vec{a}_{y/w/\mathrm{A}} = (\dot{\alpha} - \omega\beta)\hat{\imath}_{\mathrm{B}} + (\omega\alpha + \dot{\beta})\hat{\jmath}_{\mathrm{B}} + \dot{\gamma}\hat{k}_{\mathrm{B}}, \tag{8.5.116}$$

where

$$\dot{\alpha} = \ell(\cos\theta)\ddot{\theta} - \ell(\sin\theta)\dot{\theta}^2 - r\dot{\omega}, \tag{8.5.117}$$

$$\dot{\beta} = \ell\dot{\omega}\sin\theta + \ell\omega(\cos\theta)\dot{\theta}, \tag{8.5.118}$$

$$\dot{\gamma} = -\ell(\sin\theta)\ddot{\theta} - \ell(\cos\theta)\dot{\theta}^2. \tag{8.5.119}$$

Using Fact 7.5.1, it follows from $m\vec{a}_{y/w/\mathrm{A}} = \vec{f}_{\mathcal{B}}$, (8.5.116), and (8.5.111) that

$$m(\dot{\alpha} - \omega\beta) = f_3\sin\theta, \tag{8.5.120}$$

$$m(\omega\alpha + \dot{\beta}) = f_2, \tag{8.5.121}$$

$$m\dot{\gamma} = f_3\cos\theta + mg. \tag{8.5.122}$$

It thus follows from (8.5.121) that

$$f_2 = m(2\ell\omega\dot{\theta}\cos\theta + \ell\dot{\omega}\sin\theta - r\omega^2), \tag{8.5.123}$$

and thus

$$T_1 = -m\ell(2\ell\omega\dot{\theta}\cos\theta + \ell\dot{\omega}\sin\theta - r\omega^2). \tag{8.5.124}$$

Furthermore, multiplying (8.5.120) by $\sin\theta$ and (8.5.122) by $\cos\theta$, adding the resulting equations, and solving for f_3 yields

$$\begin{aligned}f_3 &= m[(\dot{\alpha} - \omega\beta)\sin\theta + (\dot{\gamma} - g)\cos\theta] \\ &= -m(\ell\dot{\theta}^2 + \ell\omega^2\sin^2\theta + r\dot{\omega}\sin\theta + g\cos\theta).\end{aligned} \tag{8.5.125}$$

Now, substituting (8.5.125) into (8.5.122) yields

$$-m[\ell(\sin\theta)\ddot{\theta} + \ell(\cos\theta)\dot{\theta}^2] = -m(\ell\dot{\theta}^2 + \ell\omega^2\sin^2\theta + r\dot{\omega}\sin\theta + g\cos\theta)\cos\theta + mg, \tag{8.5.126}$$

which can be simplified as

$$\ell\ddot{\theta} + (g - \ell\omega^2\cos\theta)\sin\theta - r\dot{\omega}\cos\theta = 0. \tag{8.5.127}$$

In the case where ω is constant, (8.5.127) specializes to

$$\ell\ddot{\theta} + (g - \ell\omega^2\cos\theta)\sin\theta = 0. \tag{8.5.128}$$

If, in addition, $\ddot{\theta} = 0$, then (8.5.128) implies that either $\sin\theta = 0$, that is, $\theta \in \{0, \pi\}$, or, in the case where $g \le \ell\omega^2$,

$$\cos\theta = \frac{g}{\ell\omega^2}. \tag{8.5.129}$$

Alternatively, when $\omega = 0$, (8.5.128) yields the standard pendulum equation $\ell\ddot{\theta} + g\sin\theta = 0$. Finally, when θ is constant, it follows from (8.5.127)

$$(g - \ell\omega^2\cos\theta)\sin\theta - r\dot{\omega}\cos\theta = 0. \tag{8.5.130}$$

If, in addition, $\theta \approx 0$, then

$$\theta \approx \frac{r\dot{\omega}}{g - \ell\omega^2}. \tag{8.5.131}$$

Next, consider the arm and shaft shown in Figure 8.5.15(c). The net force $\vec{f}_{\mathcal{S}\cup\mathcal{A}}$ on $\mathcal{S}\cup\mathcal{A}$ is given by

$$\begin{aligned}
\vec{f}_{\mathcal{S}\cup\mathcal{A}} &= \vec{f}_{\mathrm{r}/\mathcal{A}/x} + \vec{f}_{\mathrm{r}/\mathcal{S}/w} + m_{\mathcal{A}}\vec{g}\\
&= -\vec{f}_{\mathrm{r}/\mathcal{L}/x} + \vec{f}_{\mathrm{r}/\mathcal{S}/w} + m_{\mathcal{A}}\vec{g}\\
&= -(f_2\hat{\jmath}_{\mathrm{C}} + f_3\hat{k}_{\mathrm{C}}) + \vec{f}_{\mathrm{r}/\mathcal{S}/w} + m_{\mathcal{A}}\vec{g}\\
&= -f_2\hat{\jmath}_{\mathrm{B}} - f_3[(\sin\theta)\hat{\imath}_{\mathrm{B}} + (\cos\theta)\hat{k}_{\mathrm{B}})] + \vec{f}_{\mathrm{r}/\mathcal{S}/w} + m_{\mathcal{A}}g\hat{k}_{\mathrm{B}}.
\end{aligned} \tag{8.5.132}$$

Furthermore, let $\vec{T}_{\mathrm{r}/\mathcal{A}/x}$ denote the reaction torque on the arm by the link at x, and let $\vec{T}_{\mathrm{r}/\mathcal{S}/w}$ denote the reaction torque on the shaft due to the ground at w. Then, the moment $\vec{M}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}$ on the shaft and arm relative to c, where c is the center of mass of $\mathcal{S}\cup\mathcal{A}$, is given by

******************torque and moment notation needs to be checked here

$$\begin{aligned}
\vec{M}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}} &= \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} + \vec{r}_{w/\mathrm{c}} \times \vec{f}_{\mathrm{r}/\mathcal{S}/w} + \vec{M}_{\mathrm{r}/\mathcal{A}/x} + \vec{r}_{x/\mathrm{c}} \times \vec{f}_{\mathrm{r}/\mathcal{A}/x}\\
&= \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} + \vec{r}_{w/\mathrm{c}} \times \vec{f}_{\mathrm{r}/\mathcal{S}/w} - \vec{T}_{\mathrm{r}/\mathcal{L}/x} + \vec{r}_{\mathrm{c}/x} \times \vec{f}_{\mathrm{r}/\mathcal{L}/x}\\
&= \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} + \vec{r}_{w/\mathrm{c}} \times \vec{f}_{\mathrm{r}/\mathcal{S}/w} + \ell f_2\hat{\imath}_{\mathrm{C}} + (-\tfrac{1}{2}r\hat{\jmath}_{\mathrm{B}}) \times [f_3(\sin\theta)\hat{\imath}_{\mathrm{B}} + f_2\hat{\jmath}_{\mathrm{B}} + (f_3\cos\theta)\hat{k}_{\mathrm{B}}]\\
&= \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} + \vec{r}_{w/\mathrm{c}} \times \vec{f}_{\mathrm{r}/\mathcal{S}/w} + \ell f_2[(\cos\theta)\hat{\imath}_{\mathrm{B}} - (\sin\theta)\hat{k}_{\mathrm{B}}] - \tfrac{1}{2}rf_3(\cos\theta)\hat{\imath}_{\mathrm{B}}\\
&\quad + \tfrac{1}{2}rf_3(\sin\theta)\hat{k}_{\mathrm{B}}\\
&= \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} + \vec{r}_{w/\mathrm{c}} \times \vec{f}_{\mathrm{r}/\mathcal{S}/w} + (\ell f_2 - \tfrac{1}{2}rf_3)(\cos\theta)\hat{\imath}_{\mathrm{B}} + (-\ell f_2 + \tfrac{1}{2}rf_3)(\sin\theta)\hat{k}_{\mathrm{B}}.
\end{aligned} \tag{8.5.133}$$

Next, using $\vec{r}_{\mathrm{c}/w} = \frac{1}{2}r\hat{\jmath}_{\mathrm{B}} - L\hat{k}_{\mathrm{B}}$, it follows that

$$\vec{v}_{\mathrm{c}/w/\mathrm{A}} = \tfrac{1}{2}r\overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} = \tfrac{1}{2}r\vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_{\mathrm{B}} = -\tfrac{1}{2}r\omega\hat{\imath}_{\mathrm{B}}. \tag{8.5.134}$$

Furthermore,

$$\begin{aligned}
\vec{a}_{\mathrm{c}/w/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec{v}}_{\mathrm{c}/w/\mathrm{A}} \\
&= -\tfrac{1}{2}r\dot{\omega}\hat{\imath}_{\mathrm{B}} - \tfrac{1}{2}r\omega\,\overset{\mathrm{A}\bullet}{\hat{\imath}}_{\mathrm{B}} \\
&= -\tfrac{1}{2}r\dot{\omega}\hat{\imath}_{\mathrm{B}} - \tfrac{1}{2}r\omega\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}} \\
&= -\tfrac{1}{2}r\dot{\omega}\hat{\imath}_{\mathrm{B}} - \tfrac{1}{2}r\omega^2\hat{\jmath}_{\mathrm{B}}. \qquad (8.5.135)
\end{aligned}$$

Next, applying Newton's second law to the arm and shaft yields

$$-\tfrac{1}{2}rm_{\mathcal{A}}\dot{\omega}\hat{\imath}_{\mathrm{B}} - \tfrac{1}{2}rm_{\mathcal{A}}\omega^2\hat{\jmath}_{\mathrm{B}} = -f_2\hat{\jmath}_{\mathrm{B}} - f_3[(\sin\theta)\hat{\imath}_{\mathrm{B}} + (\cos\theta)\hat{k}_{\mathrm{B}})] + \vec{f}_{\mathrm{r}/\mathcal{S}/w} + m_{\mathcal{A}}g\hat{k}_{\mathrm{B}}. \qquad (8.5.136)$$

The reaction force on the shaft at w is thus given by

$$\begin{aligned}
\vec{f}_{\mathrm{r}/\mathcal{S}/w} &= -\tfrac{1}{2}rm_{\mathcal{A}}\dot{\omega}\hat{\imath}_{\mathrm{B}} - \tfrac{1}{2}rm_{\mathcal{A}}\omega^2\hat{\jmath}_{\mathrm{B}} + f_2\hat{\jmath}_{\mathrm{B}} + f_3[(\sin\theta)\hat{\imath}_{\mathrm{B}} + (\cos\theta)\hat{k}_{\mathrm{B}})] - m_{\mathcal{A}}g\hat{k}_{\mathrm{B}} \\
&= (-\tfrac{1}{2}rm_{\mathcal{A}}\dot{\omega} + f_3\sin\theta)\hat{\imath}_{\mathrm{B}} + (f_2 - \tfrac{1}{2}rm_{\mathcal{A}}\omega^2)\hat{\jmath}_{\mathrm{B}} + (f_3\cos\theta - m_{\mathcal{A}}g)\hat{k}_{\mathrm{B}}. \qquad (8.5.137)
\end{aligned}$$

Thus, using (8.5.133) and (8.5.137), it follows that

$$\begin{aligned}
\vec{M}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}} = \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} &+ (\ell f_2 - \tfrac{1}{2}rf_3)(\cos\theta)\hat{\imath}_{\mathrm{B}} + \varepsilon_1\hat{\imath}_{\mathrm{B}} + \varepsilon_2\hat{\jmath}_{\mathrm{B}} \\
&+ (-\tfrac{1}{4}r^2m_{\mathcal{A}}\dot{\omega} - \ell f_2\sin\theta + rf_3\sin\theta)\hat{k}_{\mathrm{B}}. \qquad (8.5.138)
\end{aligned}$$

where ε_1 and ε_1 are inconsequential for the subsequent analysis. It thus follows from (7.9.13), (8.5.112), and (8.5.138) that

$$\begin{aligned}
\vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}\,\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}} = \vec{M}_{\mathrm{ext}} + \vec{M}_{\mathrm{r}/\mathcal{S}/w} &+ (\ell f_2 - \tfrac{1}{2}rf_3)(\cos\theta)\hat{\imath}_{\mathrm{B}} + \varepsilon_1\hat{\imath}_{\mathrm{B}} + \varepsilon_2\hat{\jmath}_{\mathrm{B}} \\
&+ (-\tfrac{1}{4}r^2m_{\mathcal{A}}\dot{\omega} - \ell f_2\sin\theta + rf_3\sin\theta)\hat{k}_{\mathrm{B}}. \qquad (8.5.139)
\end{aligned}$$

Since $\vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}} = J\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + J\hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}$ and $\omega_{\mathrm{B/A}} = \omega\hat{k}_{\mathrm{B}}$, where $J = \frac{1}{12}m_{\mathcal{A}}r^2$, it follows that

$$\vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}\,\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}} = \tfrac{1}{12}m_{\mathcal{A}}r^2\dot{\omega}\hat{k}_{\mathrm{B}}. \qquad (8.5.140)$$

Furthermore, since the pin joint at w is parallel with $\hat{k}_{\mathrm{B}}$, it follows that the component of $\vec{M}_{\mathrm{r}/\mathcal{S}/w}$ along $\hat{k}_{\mathrm{B}}$ is zero. It thus follows from (8.5.139) that

$$\tfrac{1}{12}m_{\mathcal{A}}r^2\dot{\omega} + \tfrac{1}{4}m_{\mathcal{A}}r^2\dot{\omega} + \ell f_2\sin\theta - rf_3\sin\theta = \tau, \qquad (8.5.141)$$

and thus

$$\begin{aligned}
\tfrac{1}{3}m_{\mathcal{A}}r^2\dot{\omega} &+ m\ell(2\ell\omega\dot{\theta}\cos\theta + \ell\dot{\omega}\sin\theta - r\omega^2)\sin\theta \\
&+ mr(\ell\dot{\theta}^2 + \ell\omega^2\sin^2\theta + r\dot{\omega}\sin\theta + g\cos\theta)\sin\theta = \tau, \qquad (8.5.142)
\end{aligned}$$

which can be rewritten as

$$[\tfrac{1}{3}m_{\mathcal{A}}r^2 + m(\ell^2 + r^2)\sin^2\theta]\dot{\omega} + m[2\ell^2(\cos\theta)\dot{\theta}\omega + r\ell\dot{\theta}^2 + r(\cos\theta)(g - \ell\omega^2\cos\theta)]\sin\theta = \tau. \qquad (8.5.143)$$

This equation along with (8.5.127) describes the dynamics of the rotating pendulum. This example is revisited using Lagrangian dynamics in Example 11.17.12. ◇

Example 8.5.7. As shown in Figure 8.5.16, reconsider Example 8.5.6 except that now the pivot of the planar simple pendulum is mounted on the arm such that the pendulum swings radially, that is, at each time instant, the link and the particle lie in a vertical plane that contains the arm and the shaft.

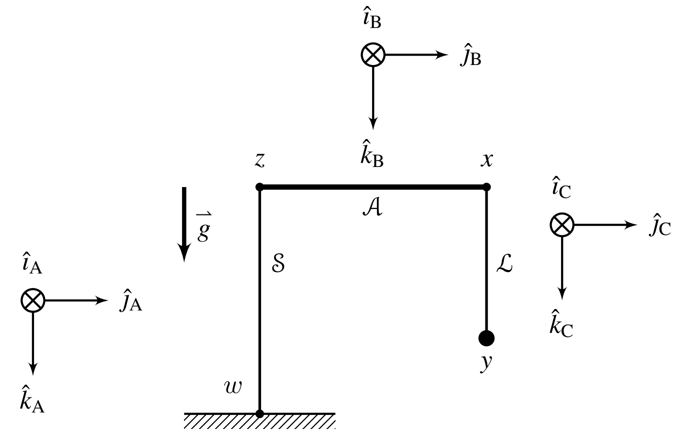


Figure 8.5.16: Example 8.5.7. Pendulum with pivot mounted on the endpoint of a rotating horizontal arm. The pivot is mounted on the arm such that the pendulum swings radially, that is, at each time instant, the link and the particle lie in a vertical plane that contains the arm and the shaft.

Solution. As shown in Figure 8.5.16, the frame $\mathrm{F_C}$ is attached to $\mathcal{B} = \mathcal{L} \cup y$, the frame $\mathrm{F_B}$ is attached to the arm, and $\mathrm{F_A}$ is an inertial frame. Furthermore, defining the rotation angles ϕ and θ by $\mathrm{F_A} \xrightarrow[3]{\phi} \mathrm{F_B} \xrightarrow[1]{\theta} \mathrm{F_C}$, it follows that $\vec{\omega}_{\rm C/B} = \dot{\theta}\hat{\imath}_{\rm C} = \dot{\theta}\hat{\imath}_{\rm B}$ and $\vec{\omega}_{\rm B/A} = \omega\hat{k}_{\rm B} = \omega\hat{k}_{\rm A}$, where $\omega = \dot{\phi}$ and $\vec{\omega}_{\rm C/A} = \dot{\theta}\hat{\imath}_{\rm B} + \omega\hat{k}_{\rm B}$. Note that

$$\mathcal{O}_{\rm C/B} = \mathcal{O}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{8.5.144}$$

and thus

$$\begin{bmatrix} \hat{\imath}_{\rm C} \\ \hat{\jmath}_{\rm C} \\ \hat{k}_{\rm C} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\imath}_{\rm B} \\ \hat{\jmath}_{\rm B} \\ \hat{k}_{\rm B} \end{bmatrix}. \tag{8.5.145}$$

Finally, note that w has zero inertial acceleration.

First, consider the massless link $\mathcal{L}$ shown in Figure 8.5.17(a). Note that, since $\mathcal{L}$ is massless, the net force on $\mathcal{L}$ is zero. Letting $\vec{f}_{{\rm r}/\mathcal{L}/x} = f_1\hat{\imath}_{\rm C} + f_2\hat{\jmath}_{\rm C} + f_3\hat{k}_{\rm C}$ denote the reaction force on $\mathcal{L}$ at x, it follows that the reaction force on $\mathcal{L}$ due to y is $\vec{f}_{{\rm r}/\mathcal{L}/y} = -\vec{f}_{{\rm r}/\mathcal{L}/x}$, and the reaction force on y due to $\mathcal{L}$ is $\vec{f}_{{\rm r}/y/\mathcal{L}} = -\vec{f}_{{\rm r}/\mathcal{L}/y} = \vec{f}_{{\rm r}/\mathcal{L}/x}$.

Next, let $\vec{T}_{{\rm r}/\mathcal{L}/x} = T_2\hat{\jmath}_{\rm C} + T_3\hat{k}_{\rm C}$ denote the reaction torque on $\mathcal{L}$ due to the pin joint at x, where $\vec{T}_{{\rm r}/\mathcal{L}/x}$ has a zero component in the direction $\hat{\imath}_{\rm C}$, which is the axis of the pin joint. The torque $\vec{T}_{\mathcal{L}}$ on

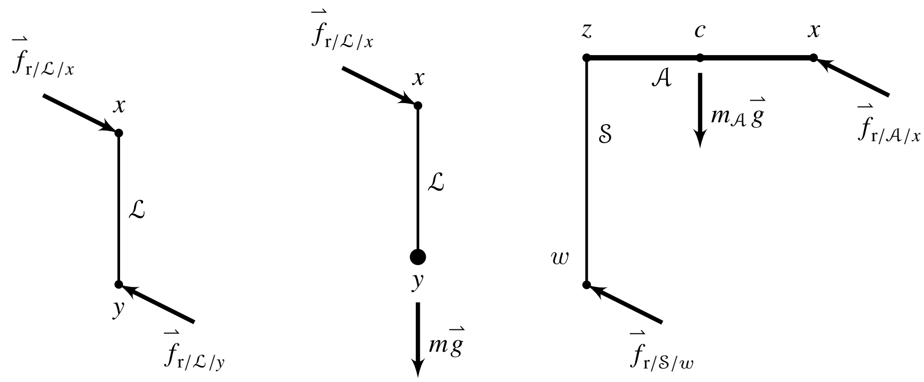


Figure 8.5.17: Example 8.5.7. Free-body diagram of (a) the massless link $\mathcal{L}$, where the pin joint at x is replaced by the reaction force $\overrightarrow{f}_{\mathrm{r}/\mathcal{L}/x}$ on $\mathcal{L}$ due to the pin joint at the point x and the particle at y is replaced by the reaction force $\overrightarrow{f}_{\mathrm{r}/\mathcal{L}/y}$ on $\mathcal{L}$ due to particle at y, (b) the pendulum $\mathcal{B}$, and (c) the arm $\mathcal{A}$ and the shaft $\mathcal{S}$, where the pin joint at x is replaced by the reaction force $\overrightarrow{f}_{\mathrm{r}/\mathcal{A}/x}$ on $\mathcal{A}$ due to the pin joint at x and the ground is replaced by the reaction force $\overrightarrow{f}_{\mathrm{r}/\mathcal{S}/w}$ on $\mathcal{S}$ due to the ground.

$\mathcal{L}$ is thus given by

$$\begin{aligned}
\overrightarrow{T}_{\mathcal{L}} &= \overrightarrow{T}_{\mathrm{r}/\mathcal{L}/x} + \overrightarrow{r}_{y/x} \times \overrightarrow{f}_{\mathrm{r}/\mathcal{L}/y} \\
&= \overrightarrow{T}_{\mathrm{r}/\mathcal{L}/x} + \overrightarrow{r}_{y/x} \times (-\overrightarrow{f}_{\mathrm{r}/\mathcal{L}/x}) \\
&= T_2\hat{\jmath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}} - \ell\hat{k}_{\mathrm{C}} \times (f_1\hat{\imath}_{\mathrm{C}} + f_2\hat{\jmath}_{\mathrm{C}} + f_3\hat{k}_{\mathrm{C}}) \\
&= T_2\hat{\jmath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}} - \ell f_1\hat{\jmath}_{\mathrm{C}} + \ell f_2\hat{\imath}_{\mathrm{C}} \\
&= \ell f_2\hat{\imath}_{\mathrm{C}} + (T_2 - \ell f_1)\hat{\jmath}_{\mathrm{C}} + T_3\hat{k}_{\mathrm{C}}.
\end{aligned} \tag{8.5.146}$$

However, since $\mathcal{L}$ is massless, it follows that $\overrightarrow{T}_{\mathcal{L}} = 0$, and thus (8.5.146) implies that $T_2 = \ell f_1$, $f_2 = 0$, and $T_3 = 0$.

Next, consider the pendulum $\mathcal{B}$ shown in Figure 8.5.17(b). The net force $\overrightarrow{f}_{\mathcal{B}}$ on $\mathcal{B}$ is given by

$$\begin{aligned}
\overrightarrow{f}_{\mathcal{B}} &= \overrightarrow{f}_{\mathrm{r}/\mathcal{L}/x} + m\overrightarrow{g} \\
&= f_1\hat{\imath}_{\mathrm{C}} + f_3\hat{k}_{\mathrm{C}} + mg\hat{k}_{\mathrm{B}} \\
&= f_1\hat{\imath}_{\mathrm{B}} + f_3(-\sin\theta\hat{\jmath}_{\mathrm{B}} + \cos\theta\hat{k}_{\mathrm{B}}) + mg\hat{k}_{\mathrm{B}} \\
&= f_1\hat{\imath}_{\mathrm{B}} - f_3\sin\theta\hat{\jmath}_{\mathrm{B}} + (f_3\cos\theta + mg)\hat{k}_{\mathrm{B}},
\end{aligned} \tag{8.5.147}$$

and the reaction torque on $\mathcal{L}$ is given by

$$\overrightarrow{T}_{\mathrm{r}/\mathcal{L}/x} = T_2\hat{\jmath}_{\mathrm{C}} = \ell f_1\hat{\jmath}_{\mathrm{C}}. \tag{8.5.148}$$

Using

$$\overrightarrow{r}_{y/w} = \overrightarrow{r}_{y/x} + \overrightarrow{r}_{x/z} + \overrightarrow{r}_{z/w}$$

$$= \ell\hat{k}_{\rm C} + r\hat{\jmath}_{\rm B} - L\hat{k}_{\rm A}, \tag{8.5.149}$$

it follows that

$$\begin{aligned}\vec{v}_{y/w/{\rm A}} &= \ell\vec{\omega}_{\rm C/A} \times \hat{k}_{\rm C} + r\vec{\omega}_{\rm B/A} \times \hat{\jmath}_{\rm B} \\ &= \ell(\dot{\theta}\hat{\imath}_{\rm B} + \omega\hat{k}_{\rm B}) \times [-\sin\theta\hat{\jmath}_{\rm B} + \cos\theta\hat{k}_{\rm B}] + r\omega\hat{k}_{\rm B} \times \hat{\jmath}_{\rm B} \\ &= \alpha\hat{\imath}_{\rm B} + \beta\hat{\jmath}_{\rm B} + \gamma\hat{k}_{\rm B} \\ &= \alpha\hat{\imath}_{\rm B} + \beta\hat{\jmath}_{\rm B} + \gamma\hat{k}_{\rm A},\end{aligned} \tag{8.5.150}$$

where

$$\alpha \triangleq \ell\omega\sin\theta - r\omega, \quad \beta \triangleq -\ell\dot{\theta}\cos\theta, \quad \gamma \triangleq -\ell\dot{\theta}\sin\theta. \tag{8.5.151}$$

Furthermore,

$$\begin{aligned}\vec{a}_{y/w/{\rm A}} &= \overset{\rm A\bullet}{\vec{v}}_{y/w/{\rm A}} \\ &= \dot{\alpha}\hat{\imath}_{\rm B} + \alpha\omega\hat{k}_{\rm B} \times \hat{\imath}_{\rm B} + \dot{\beta}\hat{\jmath}_{\rm B} + \beta\omega\hat{k}_{\rm B} \times \hat{\jmath}_{\rm B} + \dot{\gamma}\hat{k}_{\rm B} \\ &= (\dot{\alpha} - \omega\beta)\hat{\imath}_{\rm B} + (\dot{\beta} + \omega\alpha)\hat{\jmath}_{\rm B} + \dot{\gamma}\hat{k}_{\rm B},\end{aligned} \tag{8.5.152}$$

where

$$\dot{\alpha} = \ell\dot{\omega}\sin\theta + \ell\omega\dot{\theta}\cos\theta - r\dot{\omega}, \tag{8.5.153}$$

$$\dot{\beta} = -\ell\ddot{\theta}\cos\theta + \ell\dot{\theta}^2\sin\theta, \tag{8.5.154}$$

$$\dot{\gamma} = -\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta. \tag{8.5.155}$$

Using Fact 7.5.1, it follows from $m\vec{a}_{y/w/{\rm A}} = \vec{f}_{\mathcal{B}}$, (8.5.147), and (8.5.152) that

$$m(\dot{\alpha} - \omega\beta) = f_1, \tag{8.5.156}$$

$$m(\dot{\beta} + \omega\alpha) = -f_3\sin\theta, \tag{8.5.157}$$

$$m\dot{\gamma} = f_3\cos\theta + mg. \tag{8.5.158}$$

Furthermore, multiplying (8.5.121) by $\cos\theta$, multiplying (8.5.122) by $\sin\theta$, and adding the resulting equations yields

$$\ell\ddot{\theta} + g\sin\theta + \omega^2\cos\theta(r - \ell\sin\theta) = 0, \tag{8.5.159}$$

and multiplying (8.5.121) by $-\sin\theta$, multiplying (8.5.122) by $\cos\theta$, adding the resulting equations, and solving for f_3 yields

$$f_3 = -m(\ell\dot{\theta}^2 + \ell\omega^2\sin^2\theta - r\omega^2\sin\theta + g\cos\theta). \tag{8.5.160}$$

Furthermore, it follows from (8.5.156) and (8.5.146) that

$$f_1 = m(\ell\dot{\omega}\sin\theta + 2\ell\omega\dot{\theta}\cos\theta - r\dot{\omega}), \tag{8.5.161}$$

$$T_2 = m\ell(\ell\dot{\omega}\sin\theta + 2\ell\omega\dot{\theta}\cos\theta - r\dot{\omega}). \tag{8.5.162}$$

Next, consider the arm and shaft shown in Figure 8.5.17(c). The net force $\vec{f}_{\mathcal{S}\cup\mathcal{A}}$ on $\mathcal{S} \cup \mathcal{A}$ is given by

$$\begin{aligned}\vec{f}_{\mathcal{S}\cup\mathcal{A}} &= \vec{f}_{{\rm r}/\mathcal{A}/x} + \vec{f}_{{\rm r}/\mathcal{S}/w} + m_{\mathcal{A}}\vec{g} \\ &= -\vec{f}_{{\rm r}/\mathcal{L}/x} + \vec{f}_{{\rm r}/\mathcal{S}/w} + m_{\mathcal{A}}\vec{g}\end{aligned}$$

$$= -(f_1\hat{\imath}_C + f_3\hat{k}_C) + \vec{f}_{r/\mathcal{S}/w} + m_{\mathcal{A}}\vec{g}$$
$$= -f_1\hat{\imath}_B - f_3(-\sin\theta\hat{\jmath}_B + \cos\theta\hat{k}_B) + \vec{f}_{r/\mathcal{S}/w} + m_{\mathcal{A}}g\hat{k}_B. \tag{8.5.163}$$

Furthermore, let $\vec{M}_{r/\mathcal{A}/x}$ denote the reaction torque on the arm due to the link at x, and let $\vec{M}_{r/\mathcal{S}/w}$ denote the reaction torque on the shaft due to the ground at w. Then, the moment $\vec{M}_{\mathcal{S}\cup\mathcal{A}/c}$ on the shaft and arm relative to c, where c is the center of mass of $\mathcal{S}\cup\mathcal{A}$, is given by

$$\begin{aligned}\vec{M}_{\mathcal{S}\cup\mathcal{A}/c} &= \vec{M}_{ext} + \vec{M}_{r/\mathcal{S}/w} + \vec{r}_{w/c}\times\vec{f}_{r/\mathcal{S}/w} + \vec{M}_{r/\mathcal{A}/x} + \vec{r}_{x/c}\times\vec{f}_{r/\mathcal{A}/x}\\ &= \vec{M}_{ext} + \vec{M}_{r/\mathcal{S}/w} + \vec{r}_{w/c}\times\vec{f}_{r/\mathcal{S}/w} - \vec{T}_{r/\mathcal{L}/x} + \vec{r}_{c/x}\times\vec{f}_{r/\mathcal{L}/x}\\ &= \vec{M}_{ext} + \vec{M}_{r/\mathcal{S}/w} + \vec{r}_{w/c}\times\vec{f}_{r/\mathcal{S}/w} - \ell f_1\hat{\jmath}_C - \tfrac{1}{2}r\hat{\jmath}_B\times(f_1\hat{\imath}_B - f_3\sin\theta\hat{\jmath}_B + f_3\cos\theta\hat{k}_B)\\ &= \vec{M}_{ext} + \vec{M}_{r/\mathcal{S}/w} + \vec{r}_{w/c}\times\vec{f}_{r/\mathcal{S}/w} - \tfrac{1}{2}rf_3\cos\theta\hat{\imath}_B - \ell f_1\cos\theta\hat{\jmath}_B + (-\ell f_1\sin\theta + \tfrac{1}{2}rf_1)\hat{k}_B.\end{aligned} \tag{8.5.164}$$

Next, using $\vec{r}_{c/w} = \frac{1}{2}r\hat{\jmath}_B - L\hat{k}_B$, it follows that

$$\vec{v}_{c/w/A} = \tfrac{1}{2}r\overset{A\bullet}{\hat{\jmath}_B} = \tfrac{1}{2}r\vec{\omega}_{B/A}\times\hat{\jmath}_B = -\tfrac{1}{2}r\omega\hat{\imath}_B. \tag{8.5.165}$$

Furthermore,

$$\begin{aligned}\vec{a}_{c/w/A} &= \overset{A\bullet}{\vec{v}}_{c/w/A}\\ &= -\tfrac{1}{2}r\dot{\omega}\hat{\imath}_B - \tfrac{1}{2}r\omega\overset{A\bullet}{\hat{\imath}_B}\\ &= -\tfrac{1}{2}r\dot{\omega}\hat{\imath}_B - \tfrac{1}{2}r\omega\vec{\omega}_{B/A}\times\hat{\imath}_B\\ &= -\tfrac{1}{2}r\dot{\omega}\hat{\imath}_B - \tfrac{1}{2}r\omega^2\hat{\jmath}_B.\end{aligned} \tag{8.5.166}$$

Next, applying Newton's second law to the arm and shaft yields

$$-\tfrac{1}{2}rm_{\mathcal{A}}\dot{\omega}\hat{\imath}_B - \tfrac{1}{2}rm_{\mathcal{A}}\omega^2\hat{\jmath}_B = -f_1\hat{\imath}_B - f_3(-\sin\theta\hat{\jmath}_B + \cos\theta\hat{k}_B)) + \vec{f}_{r/\mathcal{S}/w} + m_{\mathcal{A}}g\hat{k}_B. \tag{8.5.167}$$

The reaction force on the shaft at w is thus given by

$$\begin{aligned}\vec{f}_{r/\mathcal{S}/w} &= -\tfrac{1}{2}rm_{\mathcal{A}}\dot{\omega}\hat{\imath}_B - \tfrac{1}{2}rm_{\mathcal{A}}\omega^2\hat{\jmath}_B + f_1\hat{\imath}_B + f_3(-\sin\theta\hat{\jmath}_B + \cos\theta\hat{k}_B)] - m_{\mathcal{A}}g\hat{k}_B\\ &= (-\tfrac{1}{2}rm_{\mathcal{A}}\dot{\omega} + f_1)\hat{\imath}_B + (-\tfrac{1}{2}rm_{\mathcal{A}}\omega^2 - f_3\sin\theta)\hat{\jmath}_B + (f_3\cos\theta - m_{\mathcal{A}}g)\hat{k}_B.\end{aligned} \tag{8.5.168}$$

Thus, using (8.5.168) and (8.5.164), it follows that

$$\vec{M}_{\mathcal{S}\cup\mathcal{A}/c} = \vec{M}_{ext} + \vec{M}_{r/\mathcal{S}/w} + \varepsilon_1\hat{\imath}_B + \varepsilon_2\hat{\jmath}_B + (-\tfrac{1}{4}r^2m_{\mathcal{A}}\dot{\omega} + rf_1 - \ell f_1\sin\theta)\hat{k}_B, \tag{8.5.169}$$

where ε_1 and ε_2 are inconsequential for the subsequent analysis. It thus follows from (7.9.13) and (8.5.169) that

$$\begin{aligned}\vec{J}_{\mathcal{S}\cup\mathcal{A}/c}\overset{B\bullet}{\vec{\omega}}_{B/A} + \vec{\omega}_{B/A}\times\vec{J}_{\mathcal{S}\cup\mathcal{A}/c}\vec{\omega}_{B/A} &= \vec{M}_{ext} + \vec{M}_{r/\mathcal{S}/w} + \varepsilon_1\hat{\imath}_B + \varepsilon_2\hat{\jmath}_B\\ &\quad + (-\tfrac{1}{4}r^2m_{\mathcal{A}}\dot{\omega} + rf_1 - \ell f_1\sin\theta)\hat{k}_B\end{aligned} \tag{8.5.170}$$

Since $\vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}} = J\hat{\imath}_{\mathrm{B}}\hat{\imath}_{\mathrm{B}}' + J\hat{k}_{\mathrm{B}}\hat{k}_{\mathrm{B}}'$ and $\omega_{\mathrm{B/A}} = \omega\hat{k}_{\mathrm{B}}$, where $J = \frac{1}{12}m_{\mathcal{A}}r^2$, it follows that

$$\vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{S}\cup\mathcal{A}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}} = \tfrac{1}{12}m_{\mathcal{A}}r^2\dot{\omega}\hat{k}_{\mathrm{B}}. \tag{8.5.171}$$

Furthermore, since the pin joint at w is parallel with $\hat{k}_{\mathrm{B}}$, it follows that the component of $\vec{M}_{\mathrm{r}/\mathcal{S}/w}$ along $\hat{k}_{\mathrm{B}}$ is zero. It thus follows from (8.5.170) that

$$[\tfrac{1}{3}r^2m_{\mathcal{A}} + m(\ell\sin\theta - r)^2]\dot{\omega} + 2m\ell\omega\dot{\theta}\cos\theta(\ell\sin\theta - r) = \tau. \tag{8.5.172}$$

This equation along with (8.5.159) describes the dynamics of the rotating pendulum. This example is revisited using Lagrangian dynamics in Example 11.17.12. ◇

Example 8.5.8. Consider a multi-rigid body $\mathcal{B}$ consisting of two rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ whose masses are m_1 and m_2, respectively, connected by massless rigid links $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, to a joint. The centers of mass c_1 and c_2 of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, may be offset from the ends of $\mathcal{L}_1$ and $\mathcal{L}_2$. An external force $\vec{f}$ is applied to the point z on $\mathcal{B}_1$. Determine the equations of motion and the reaction forces and torques on $\mathcal{L}_1$ and $\mathcal{L}_2$ at the pin joint as well as at the endpoints of $\mathcal{L}_1$ and $\mathcal{L}_2$. Consider the cases where the joint is *i*) a pin joint, and *ii*) a ball joint.

Solution. To be written.

Example 8.5.9. A cart with mass m_1 is connected to the point w in a wall by a spring with stiffness k. The relaxed length of the spring is zero, and the distance from the wall to the center of mass c_1 of the cart is q. A massless rotating arm with length ℓ is connected to the cart at c_1, and a particle with mass m_2 is mounted on the end of the arm, as shown in Figure 8.5.18. The rotating arm is connected to a torsional spring with torsional stiffness κ Derive the equations of motion.

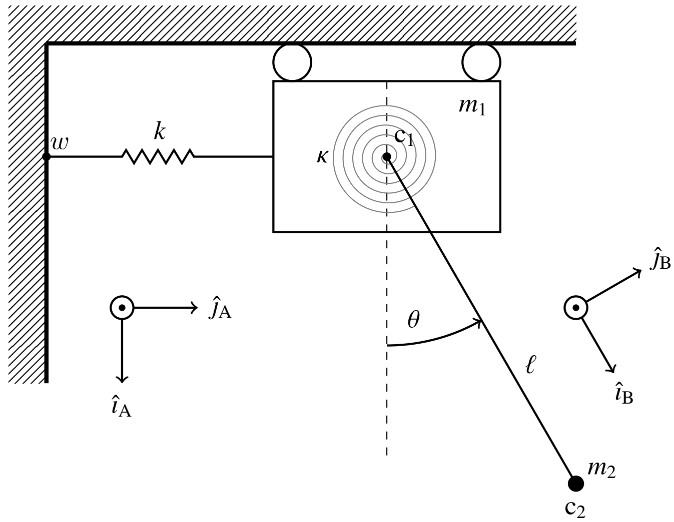


Figure 8.5.18: Cart with rotating arm. This mechanism has one translational degree of freedom and one rotational degree of freedom.

Solution. Free-body diagrams of the cart $\mathcal{C}$ and arm $\mathcal{A}$ are shown in Figure 8.5.19. It follows from Fact 6.4.2 and Fact 6.5.8 that

$$\vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{A}} = -\vec{f}_{\mathrm{r}/\mathcal{A}/\mathcal{C}}, \tag{8.5.173}$$

$$\vec{T}_{\mathrm{s}/\mathcal{A}/\mathcal{C}} = -\vec{T}_{\mathrm{s}/\mathcal{C}/\mathcal{A}}. \tag{8.5.174}$$

Furthermore, the spring force and the torque are given by

$$\vec{f}_{\mathrm{s}/\mathcal{C}/w} = -k\vec{r}_{\mathrm{c}_1/w}, \tag{8.5.175}$$

$$\vec{T}_{\mathrm{s}/\mathcal{C}/\mathcal{A}} = \kappa\theta\hat{k}_{\mathrm{A}}. \tag{8.5.176}$$

Note that the rotary spring torque is given by (6.5.11). The total force $\vec{f}_{\mathcal{C}}$ on the cart is

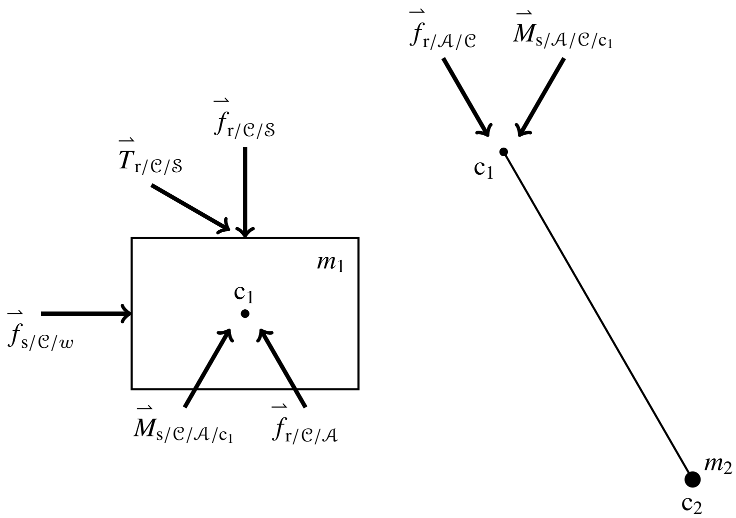
$$\vec{f}_{\mathcal{C}} = \vec{f}_{\mathrm{s}/\mathcal{C}/w} + \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}} + \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{A}}, \tag{8.5.177}$$

and the total torque on $\mathcal{C}$ is

$$\vec{T}_{\mathcal{C}/\mathrm{c}_1} = \vec{T}_{\mathrm{s}/\mathcal{C}/\mathcal{A}} + \vec{T}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}. \tag{8.5.178}$$

Now, applying Newton's second law to the cart yields

$$m_1\vec{a}_{\mathrm{c}_1/w/\mathrm{A}} = \vec{f}_{\mathrm{s}/\mathcal{C}/w} + \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}} + \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{A}}. \tag{8.5.179}$$



(a) Free-body diagram of the cart $\mathcal{C}$ (b) Free-body diagram of the arm $\mathcal{A}$

Figure 8.5.19: Free-body diagram for the cart with rotating arm.

Furthermore, since the cart does not rotate relative to $\mathrm{F_A}$, Euler's equation implies

$$0 = \vec{T}_{\mathrm{s}/\mathcal{C}/\mathcal{A}} + \vec{T}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}. \tag{8.5.180}$$

Next, the total force $\vec{f}_{\mathcal{A}}$ on $\mathcal{A}$ is

$$\vec{f}_{\mathcal{A}} = \vec{f}_{\mathrm{r}/\mathcal{A}/\mathcal{C}}, \tag{8.5.181}$$

and the total moment on $\mathcal{A}$ relative to c_2 is

$$\vec{M}_{\mathcal{A}/\mathrm{c}_2} = \vec{r}_{\mathrm{c}_1/\mathrm{c}_2} \times \vec{f}_{\mathrm{r}/\mathcal{A}/\mathcal{C}} + \vec{T}_{\mathrm{s}/\mathcal{A}/\mathcal{C}} \tag{8.5.182}$$

Applying Newton's second law to $\mathcal{A}$ yields

$$m_2(\vec{a}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + \vec{a}_{\mathrm{c}_1/w/\mathrm{A}}) = \vec{f}_{\mathrm{r}/\mathcal{A}/\mathcal{C}}, \tag{8.5.183}$$

and, since $\vec{J}_{\mathcal{A}/c_2} = 0$, Euler's equation implies

$$0 = \vec{r}_{\mathrm{c}_1/\mathrm{c}_2} \times \vec{f}_{\mathrm{r}/\mathcal{A}/\mathcal{C}} + \vec{T}_{\mathrm{s}/\mathcal{A}/\mathcal{C}}. \tag{8.5.184}$$

Next, substituting (8.5.183), (8.5.174), and (8.5.176) into (8.5.184) yields

$$0 = \vec{r}_{\mathrm{c}_1/\mathrm{c}_2} \times m_2[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1}) + \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{\mathrm{c}_1/w}] - \kappa\theta\hat{k}_{\mathrm{A}}, \tag{8.5.185}$$

which can be written as

$$0 = -\ell\hat{\imath}_{\mathrm{B}} \times m_2(\ddot{\theta}\hat{k}_{\mathrm{B}} \times \ell\hat{\imath}_{\mathrm{B}} + \ddot{q}\hat{\jmath}_{\mathrm{A}}) - \kappa\theta\hat{k}_{\mathrm{A}}. \tag{8.5.186}$$

Therefore,

$$0 = -\ell^2 m_2 \ddot{\theta}\hat{k}_{\mathrm{B}} - m_2 \ell \ddot{q} \cos\theta \hat{k}_{\mathrm{B}} - \kappa\theta\hat{k}_{\mathrm{A}}, \tag{8.5.187}$$

which implies

$$m_2 \ell \cos\theta \ddot{q} + m_2 \ell^2 \ddot{\theta} + \kappa\theta = 0. \tag{8.5.188}$$

Next, adding (8.5.183) and (8.5.179) yields

$$(m_1 + m_2) \overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{\mathrm{c}_1/w} + m_2[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1})] = -k\vec{r}_{\mathrm{c}_1/w} + \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}, \tag{8.5.189}$$

which can be written as

$$[(m_1 + m_2)\ddot{q} + kq + m_2\ell\ddot{\theta}\cos\theta - m_2\ell\dot{\theta}^2 \sin\theta]\hat{\jmath}_{\mathrm{A}} + (-m_2\ell\ddot{\theta}\sin\theta - m_2\ell\dot{\theta}^2\cos\theta)\hat{\imath}_{\mathrm{A}} = f_r\hat{\imath}_{\mathrm{A}}. \tag{8.5.190}$$

Resolving (8.5.190) yields

$$(m_1 + m_2)\ddot{q} + m_2\ell\cos\theta\ddot{\theta} + kq - m_2\ell\dot{\theta}^2\sin\theta = 0 \tag{8.5.191}$$

and

$$f_r = -m_2\ell\ddot{\theta}\sin\theta - m_2\ell\dot{\theta}^2\cos\theta. \tag{8.5.192}$$

Finally, combining (8.5.188) and (8.5.191) yields

$$\begin{bmatrix} m_1 + m_2 & m_2\ell\cos\theta \\ m_2\ell\cos\theta & m_2\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} kq - m_2\ell\dot{\theta}^2\sin\theta \\ \kappa\theta \end{bmatrix} = 0. \tag{8.5.193}$$

8.6 Examples Involving Sliding and Rolling

Example 8.6.1. For the bead on the wire considered in Example 7.5.2 and Example 7.6.3, the wire $\mathcal{W}$ rotates at the rate $\omega = \dot\theta$ around a frictionless pin joint at w, which has zero inertial acceleration. As shown in Figure 7.5.1, a bead y with mass m slides without friction along the wire. The distance from w to the bead is x, where $x(0) > 0$ and $\dot x(0) = 0$. The moment of inertia of the wire around the axis of rotation is J.

i) Assume that an external moment is applied to the wire such that $\omega > 0$ is constant. Derive the equations of motion for the bead and determine the reaction force on the bead due the wire.

ii) Assume that the wire has nonzero initial angular rate $\dot\theta(0) > 0$ and that no external moments are applied to the wire, and thus, as the bead moves, the rotation rate $\dot\theta$ is not constant. Determine the equations of motion of the body $\mathcal{W} \cup y$ consisting of the bead and the wire, and use conservation of angular momentum with respect to an inertial frame to determine a constant of the motion.

iii) Use the equations from *ii*) to show that the bead speeds up and the wire slows down. In particular, show that, as $t \to \infty$, $\dot\theta(t) \to 0$, $x(t) \to \infty$, and $\dot x(t) \to v_\infty$. In addition, determine an expression for the terminal velocity v_∞. Finally, show that $\theta(t)$ converges and find its limiting value. (Note: $\dot\theta \to 0$ does not imply that θ converges.)

iv) Returning to *i*) where ω is constant, consider the case where, on both sides of w, the wire is tilted upward by the angle $\phi > 0$. Determine the dynamics of the bead in this case.

Solution. *i*) Assume that $\mathrm{F_A}$ is an inertial frame, and let $\mathrm{F_B}$ be a body-fixed frame. These frames are related by $\mathrm{F_A} \xrightarrow[3]{\theta} \mathrm{F_B}$, and thus

$$\begin{bmatrix} \hat\imath_\mathrm{B} \\ \hat\jmath_\mathrm{B} \\ \hat k_\mathrm{B} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat\imath_\mathrm{A} \\ \hat\jmath_\mathrm{A} \\ \hat k_\mathrm{A} \end{bmatrix} \tag{8.6.1}$$

and $\vec\omega_{\mathrm{B/A}} = \omega \hat k_\mathrm{A} = \omega \hat k_\mathrm{B}$, where $\omega \triangleq \dot\theta$. The position of y relative to w is given by $\vec r_{y/w} = x\hat\imath_\mathrm{B}$. Therefore,

$$\vec v_{y/w/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec r}_{y/w} = \overset{\mathrm{B}\bullet}{\vec r}_{y/w} + \vec\omega_{\mathrm{B/A}} \times \vec r_{y/w} = \dot x\hat\imath_\mathrm{B} + \omega\hat k_\mathrm{B} \times x\hat\imath_\mathrm{B} = \dot x\hat\imath_\mathrm{B} + \omega x\hat\jmath_\mathrm{B}. \tag{8.6.2}$$

Furthermore,

$$\begin{aligned} \vec a_{y/w/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec v}_{y/w/\mathrm{A}} = \overset{\mathrm{B}\bullet}{\vec v}_{y/w/\mathrm{A}} + \vec\omega_{\mathrm{B/A}} \times \vec v_{y/w/\mathrm{A}} \\ &= \ddot x\hat\imath_\mathrm{B} + \omega\dot x\hat\jmath_\mathrm{B} + \omega\hat k_\mathrm{B} \times (\dot x\hat\imath_\mathrm{B} + \omega x\hat\jmath_\mathrm{B} \\ &= \ddot x\hat\imath_\mathrm{B} + \omega\dot x\hat\jmath_\mathrm{B} + \omega\dot x\hat\jmath_\mathrm{B} - \omega^2 x\hat\imath_\mathrm{B} \\ &= (\ddot x - \omega^2 x)\hat\imath_\mathrm{B} + 2\omega\dot x\hat\jmath_\mathrm{B}, \end{aligned} \tag{8.6.3}$$

where $\omega^2 x\hat\imath_\mathrm{B}$ is the centripetal acceleration and $2\omega\dot x\hat\jmath_\mathrm{B}$ is the Coriolis acceleration.

Next, since the bead slides without friction along the wire, it follows that the reaction force $\vec f_{\mathrm{r}/y/\mathcal{W}}$ on the bead due to the wire is in the direction $\hat\jmath_\mathrm{B}$. Hence,

$$\vec f_{\mathrm{r}/y/\mathcal{W}} = f_{\mathrm{r}/y/\mathcal{W}}\hat\jmath_\mathrm{B}, \tag{8.6.4}$$

where $\omega = \dot\theta > 0$ and thus $f_{\mathrm{r}/y/\mathcal{W}} > 0$. Note that, although $\hat\imath_\mathrm{B}$ is the direction of the acceleration of the bead with respect to the body frame, the inertial acceleration of the bead is codirectional with

$\hat{\jmath}_{\rm B}$. Now, it follows from Newton's second law $m\vec{a}_{y/w/{\rm A}} = \vec{f}_{{\rm r}/y/\mathcal{W}}$ that

$$m(\ddot{x} - \omega^2 x)\hat{\imath}_{\rm B} + 2m\omega\dot{x}\hat{\jmath}_{\rm B} = f_{{\rm r}/y/\mathcal{W}}\hat{\jmath}_{\rm B}. \tag{8.6.5}$$

Therefore,

$$f_{{\rm r}/y/\mathcal{W}} = 2m\omega\dot{x}, \tag{8.6.6}$$

$$\ddot{x} = \omega^2 x. \tag{8.6.7}$$

The solution of (8.6.7) is given by

$$x(t) = \tfrac{1}{2}(e^{\omega t} + e^{-\omega t})x(0), \tag{8.6.8}$$

and thus

$$\dot{x}(t) = \tfrac{1}{2}\omega(e^{\omega t} - e^{-\omega t})x(0). \tag{8.6.9}$$

Note that, since $x(0) > 0$, it follows that, for all $t \geq 0$, $x(t) > 0$, and thus $f_{{\rm r}/y/\mathcal{W}} > 0$, which is consistent with the assumption that $\omega > 0$. Furthermore, note that $x(t) \to \infty$ as $t \to \infty$.

ii) We no longer assume that $\omega = \dot{\theta}$ is constant. Following the same steps as in the case of constant ω and using (7.5.18) yields

$$\ddot{x} = \dot{\theta}^2 x, \tag{8.6.10}$$

$$\vec{a}_{y/w/{\rm A}} = (2\omega\dot{x} + \dot{\omega}x)\hat{\jmath}_{\rm B}, \tag{8.6.11}$$

$$f_{{\rm r}/y/\mathcal{W}} = 2m\dot{\theta}\dot{x}. \tag{8.6.12}$$

To determine $\omega(t)$, note that the moment of inertia of the wire relative to w is given by

$$J_{\mathcal{W}/w|{\rm B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}. \tag{8.6.13}$$

It thus follows from Euler's equation relative to w resolved in $\rm F_B$ that

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ J\ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ J\dot{\theta} \end{bmatrix} &= [\vec{r}_{y/w} \times (-\vec{f}_{{\rm r}/y/\mathcal{W}})]\Big|_{\rm B} \\ &= \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -f_{{\rm r}/y/\mathcal{W}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -xf_{{\rm r}/\mathcal{W}} \end{bmatrix}. \end{aligned} \tag{8.6.14}$$

Therefore,

$$J\ddot{\theta} = -xm(\ddot{\theta}x + 2\dot{\theta}\dot{x}), \tag{8.6.15}$$

and thus

$$(J\ddot{\theta} + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} = 0. \tag{8.6.16}$$

Next, since the external moment applied to the bead and wire is zero, angular momentum is conserved. To determine this constant of the motion, note that

$$\begin{aligned} \vec{H}_{y/w/{\rm A}} &= \vec{r}_{y/w} \times m\vec{v}_{y/w/{\rm A}} = x\hat{\imath}_{\rm B} \times m \overset{{\rm A}\bullet}{\overbrace{x\hat{\imath}_{\rm B}}} \\ &= mx\hat{\imath}_{\rm B} \times (\dot{x}\hat{\imath}_{\rm B} + x \overset{{\rm A}\bullet}{\hat{\imath}}_{\rm B}) = mx\hat{\imath}_{\rm B} \times x(\vec{\omega}_{{\rm B/A}} \times \hat{\imath}_{\rm B}) \end{aligned}$$

$$= mx^2\hat{\imath}_{\mathrm{B}} \times \dot{\theta}(\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{B}}) = m\dot{\theta}x^2\hat{\imath}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}} = m\dot{\theta}x^2\hat{k}_{\mathrm{A}} \tag{8.6.17}$$

and

$$\vec{H}_{\mathcal{W}/w/\mathrm{A}} = \vec{J}_{\mathcal{W}/w}\vec{\omega}_{\mathrm{A/B}} = J\dot{\theta}\hat{k}_{\mathrm{A}}. \tag{8.6.18}$$

Therefore,

$$\vec{H}_{\mathcal{W}\cup\mathrm{y}/w/\mathrm{A}} = (J + mx^2)\dot{\theta}\hat{k}_{\mathrm{A}}. \tag{8.6.19}$$

It thus follows from Newton's second law of rotation relative to w that

$$\overset{\mathrm{A}\bullet}{\vec{H}}_{\mathcal{W}\cup\mathrm{y}/w/\mathrm{A}} = \vec{M}_{\mathcal{W}\cup\mathrm{y}/w} = 0. \tag{8.6.20}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}[(J + mx^2)\dot{\theta}] = 0. \tag{8.6.21}$$

Thus, $(J + mx^2)\dot{\theta}$ is a constant of the motion, that is, there exists c_0 such that, for all $t \geq 0$,

$$(J + mx^2)\dot{\theta} = c_0. \tag{8.6.22}$$

Since, by assumption, $\dot{\theta}(0) > 0$, it follows that $c_0 > 0$.

iii) It follows from *ii*) that, for all $t \geq 0$,

$$\dot{x}(t) = \int_0^t \ddot{x}(s)\,\mathrm{d}s = \int_0^t \dot{\theta}^2(s)x(s)\,\mathrm{d}s. \tag{8.6.23}$$

By assumption, $x(0) > 0$. Assume that there exists $\varepsilon > 0$ such that, for all $t \in [0, \varepsilon)$, $x(t) > 0$ and such that $x(\varepsilon) = 0$. It thus follows from (8.6.23) that, for all $t \in [0, \varepsilon)$, $\dot{x}(t) > 0$. Therefore,

$$x(\varepsilon) = x(0) + \int_0^\varepsilon \dot{x}(s)\,\mathrm{d}s > 0, \tag{8.6.24}$$

which is a contradiction. Hence, $x(t) > 0$ for all $t \geq 0$. Consequently, (8.6.23) implies that, for all $t \geq 0$, $\dot{x}(t) > 0$. Furthermore, it follows from (8.6.10) that, for all $t \geq 0$, $\ddot{x}(t) > 0$. Hence, $\dot{x}(t)$ is positive and increasing on $[0, \infty)$, and it follows from

$$x(t) = x(0) + \int_0^t \dot{x}(s)\,\mathrm{d}s \tag{8.6.25}$$

that $x(t) \to \infty$ as $t \to \infty$. Hence,

$$\lim_{t\to\infty} \dot{\theta}(t) = \lim_{t\to\infty} \frac{c_0}{J + mx^2(t)} = 0. \tag{8.6.26}$$

The fact that $\lim_{t\to\infty} \dot{\theta}(t) = 0$ is not sufficient to conclude that the angle of the wire converges. To analyze the convergence of θ, we note that the total energy $E(t)$ of the system is conserved, where

$$E(t) = \tfrac{1}{2}[J + mx^2(t)]\dot{\theta}^2(t) + \tfrac{1}{2}m\dot{x}^2(t). \tag{8.6.27}$$

In fact, note that

$$\begin{aligned}
\dot{E}(t) &= mx\dot{x}\dot{\theta}^2 + (J + mx^2)\dot{\theta}\ddot{\theta} + m\dot{x}\ddot{x} \\
&= 2mx\dot{x}\dot{\theta}^2 + (J + mx^2)\dot{\theta}\ddot{\theta} \\
&= 2mx\dot{x}\dot{\theta}^2 + (J + mx^2)\dot{\theta}\frac{-2mx\dot{x}\dot{\theta}}{J + mx^2} = 0.
\end{aligned} \tag{8.6.28}$$

Therefore, for all $t \ge 0$,

$$\tfrac{1}{2}c_0\dot\theta(t) + \tfrac{1}{2}m\dot x^2(t) = E(0), \tag{8.6.29}$$

and thus, for all $t \ge 0$,

$$\dot x(t) = \sqrt{\frac{2E(0) - c_0\dot\theta(t)}{m}}. \tag{8.6.30}$$

Consequently, the terminal velocity v_∞ of the bead is given by

$$v_\infty = \lim_{t\to\infty} \dot x(t) = \sqrt{\frac{2E(0)}{m}}. \tag{8.6.31}$$

Finally, to determine whether $\lim_{t\to\infty}\dot\theta$ is finite, note that

$$\theta(t) = \int_0^t \frac{c_0}{J + mx^2(s)}\,{\rm d}s, \tag{8.6.32}$$

which implies that $\theta(t)$ is increasing on $[0,\infty)$. Now, let ε satisfy $0 < \varepsilon v_\infty$. Since $v(t) \triangleq \dot x(t)$ is increasing and $v(t) \to \infty$ as $t \to \infty$, it follows that there exists $T > 0$ such that, for all $t > T$, $v_\infty - \varepsilon < v(t) < v_\infty$. Therefore, since, for all $s \ge T$, $x(s) \ge x(T) + (v_\infty - \varepsilon)(s - T)$, it follows that, for all $t \ge T$,

$$\begin{aligned}
\theta(t) &= \theta(T) + \int_T^\infty \frac{c_0}{J + mx^2(s)}\,{\rm d}s \\
&\le \theta(T) + \int_T^\infty \frac{c_0}{J + m[x(T) + (v_\infty - \varepsilon)\sigma]^2}\,{\rm d}s \\
&= \theta(T) + \frac{c_0}{m(v_\infty - \varepsilon)^2}\int_0^\infty \frac{1}{\sigma^2 + \frac{2x(T)}{v_\infty - \varepsilon}\sigma + \frac{J + mx^2(T)}{m(v_\infty-\varepsilon)^2}}\,{\rm d}\sigma \\
&= \theta(T) + \frac{c_0}{\sqrt{mJ}(v_\infty - \varepsilon)}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{mx^2(T)}{J}}\right).
\end{aligned} \tag{8.6.33}$$

Since $\theta(t)$ is increasing on $[0,\infty)$, $\lim_{t\to\infty}\dot\theta(t)$ exists, and thus the wire comes to rest.

iv) Assuming that ω is constant and the wire is tilted upward on both sides of w by the angle $\phi > 0$, it follows from Newton's second law $m\vec{a}_{y/w/{\rm A}} = \vec{f}_{{\rm r}/y/\mathcal{W}} + m\vec{g}$, where $\vec{g} = -g\hat{k}_{\rm B}$, that

$$m(\ddot x - \omega^2 x)\hat\imath_{\rm B} + 2m\omega\dot x\hat\jmath_{\rm B} = f_{{\rm r}/y/\mathcal{W}2}\hat\jmath_{\rm B} + f_{{\rm r}/y/\mathcal{W}3}\hat k_{\rm B} - mg\sin\phi\hat\imath_{\rm B} - mg\cos\phi k_{\rm B}, \tag{8.6.34}$$

where $f_{{\rm r}/y/\mathcal{W}2}$ and $f_{{\rm r}/y/\mathcal{W}3}$ are the components of $\vec{f}_{{\rm r}/y/\mathcal{W}}$ along $\hat\jmath_{\rm B}$ and $\hat k_{\rm B}$, respectively. Therefore,

$$f_{{\rm r}/y/\mathcal{W}2} = 2m\omega\dot x, \tag{8.6.35}$$

$$f_{{\rm r}/y/\mathcal{W}3} = mg\cos\phi, \tag{8.6.36}$$

$$\ddot x = \omega^2 x - g\sin\phi. \tag{8.6.37}$$

The solution of (8.6.37) is given by

$$x(t) = \tfrac{1}{2}(e^{\omega t} + e^{-\omega t})\left[x(0) - \frac{g\sin\phi}{\omega^2}\right] + \frac{g\sin\phi}{\omega^2}. \tag{8.6.38}$$

Note that, if $x(0) = g(\sin\phi)/\omega^2$, then the bead remains at $x(0)$; otherwise, the bead either slides toward w or diverges exponentially with $x(t) \to \infty$ as $t \to \infty$. ◇

Example 8.6.2. In Figure 8.6.20, the cart $\mathcal{C}$ moves in a straight line along the ground with prescribed horizontal acceleration a relative to the point w fixed in the ground, where $a > 0$ corresponds to increasing speed to the right. The particle y with mass m slides along the slanted surface of the cart, whose angle relative to the ground is θ. The distance from the point x fixed in the cart to y is d. Determine the normal force f_N on y due to contact with the cart, and determine $\ddot{d}$.

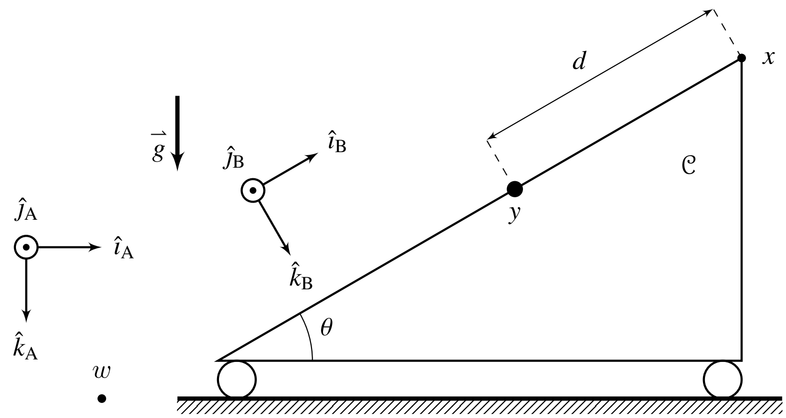


Figure 8.6.20: Example 8.6.2. Particle y sliding on the slanted surface of a cart, which has prescribed horizontal acceleration.

Solution. As shown in Figure 8.6.20, let F_A be an inertial frame, and let F_B be a frame fixed to the cart such that $F_A \xrightarrow[2]{\theta} F_B$. Note that w has zero inertial acceleration. Using

$$\begin{aligned}\vec{r}_{y/w} &= \vec{r}_{y/x} + \vec{r}_{x/w}\\ &= -d\hat{\imath}_B + \vec{r}_{x/w}\end{aligned} \tag{8.6.39}$$

and $\vec{\omega}_{B/A} = 0$, it follows that

$$\begin{aligned}\vec{a}_{y/w/A} &= \vec{a}_{y/x/A} + \vec{a}_{x/w/A}\\ &= \vec{a}_{y/x/B} + a\hat{\imath}_A\\ &= (-\ddot{d} + a\cos\theta)\hat{\imath}_B + a\sin\theta\hat{k}_B.\end{aligned} \tag{8.6.40}$$

Letting $\vec{f}_{r/y/\mathcal{C}} = f_N\hat{k}_B$, Figure 8.6.21 shows that the net force $\vec{f}_y$ on y is given by

$$\vec{f}_y = m\vec{g} + \vec{f}_{r/y/\mathcal{C}}$$

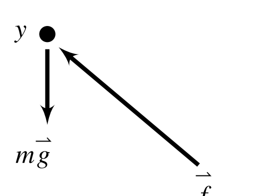


Figure 8.6.21: Example 8.6.2. Free-body diagram of the particle y.

$$= mg\hat{k}_{\mathrm{A}} + f_{\mathrm{N}}\hat{k}_{\mathrm{B}}$$
$$= (-mg\sin\theta)\hat{\imath}_{\mathrm{B}} + (f_{\mathrm{N}} + mg\cos\theta)\hat{k}_{\mathrm{B}}. \tag{8.6.41}$$

Using (8.6.40) and (8.6.41), Fact 7.5.1 implies that

$$-mg\sin\theta = m(-\ddot{d} + a\cos\theta), \tag{8.6.42}$$
$$f_{\mathrm{N}} + mg\cos\theta = ma\sin\theta. \tag{8.6.43}$$

Rearranging (8.6.42) into (8.6.43) yields

$$\ddot{d} = a\cos\theta + g\sin\theta, \tag{8.6.44}$$
$$f_{\mathrm{N}} = m(a\sin\theta - g\cos\theta). \tag{8.6.45}$$

◇

Example 8.6.3. In Figure 8.6.22, the cart moves in a straight line along the ground with prescribed horizontal acceleration a relative to the point w fixed in the ground, where $a > 0$ corresponds to increasing speed to the right. The ball $\mathcal{B}$ with mass m and radius r rolls without slipping along the slanted surface of the cart, whose angle relative to the ground is θ. The distance from the point x, which is fixed in the cart, to the center y of $\mathcal{B}$ is d. Determine the normal and tangential forces $f_{\rm N}$ and $f_{\rm T}$ on $\mathcal{B}$ due to contact with the cart, and determine $\ddot{d}$.

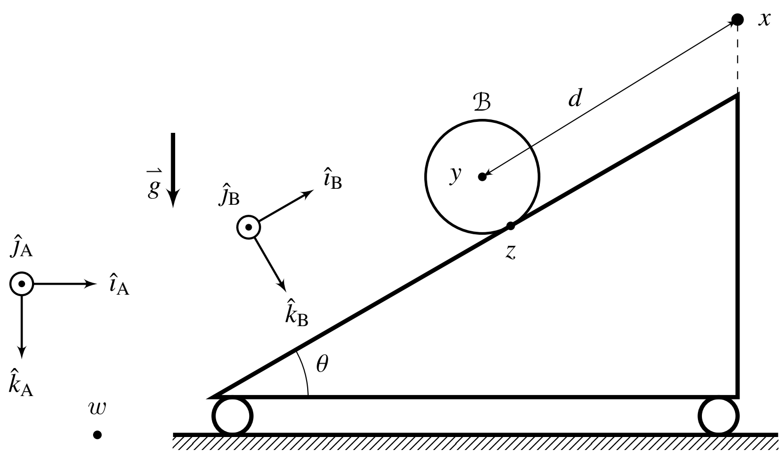


Figure 8.6.22: Example 8.6.3. Ball rolling without slipping on the slanted surface of a cart, which has prescribed horizontal acceleration.

Solution. As shown in Figure 8.6.22, let $\rm F_A$ be an inertial frame, and let $\rm F_B$ be a frame fixed to the cart such that $\mathrm{F_A} \xrightarrow[2]{\theta} \mathrm{F_B}$. Furthermore, let $\rm F_C$ be attached to $\mathcal{B}$ such that $\mathrm{F_A} \xrightarrow[2]{\phi} \mathrm{F_C}$, and thus $\vec{\omega}_{\rm C/A} = \dot{\phi}\hat{\jmath}_{\rm C}$. Finally, note that w has zero inertial acceleration. Using

$$\begin{aligned}\vec{r}_{y/w} &= \vec{r}_{y/x} + \vec{r}_{x/w}\\ &= -d\hat{\imath}_{\rm B} + \vec{r}_{x/w}\end{aligned} \tag{8.6.46}$$

and $\vec{\omega}_{\rm B/A} = 0$, it follows that

$$\begin{aligned}\vec{a}_{y/w/\rm A} &= \vec{a}_{y/x/\rm A} + \vec{a}_{x/w/\rm A}\\ &= \vec{a}_{y/x/\rm B} + a\hat{\imath}_{\rm A}\\ &= (-\ddot{d} + a\cos\theta)\hat{\imath}_{\rm B} + a\sin\theta\hat{k}_{\rm B}.\end{aligned} \tag{8.6.47}$$

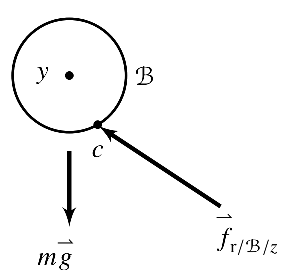


Figure 8.6.23: Example 8.6.3. Free-body diagram of the ball $\mathcal{B}$.

Letting $\vec{f}_{\mathrm{r}/\mathcal{B}/z} = f_{\mathrm{T}}\hat{\imath}_{\mathrm{B}} + f_{\mathrm{N}}\hat{k}_{\mathrm{B}}$, Figure 8.6.23 shows that the net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ is given by

$$\begin{aligned}\vec{f}_{\mathcal{B}} &= m\vec{g} + \vec{f}_{\mathrm{r}/\mathcal{B}/z}\\ &= mg\hat{k}_{\mathrm{A}} + f_{\mathrm{T}}\hat{\imath}_{\mathrm{B}} + f_{\mathrm{N}}\hat{k}_{\mathrm{B}}\\ &= (f_{\mathrm{T}} - mg\sin\theta)\hat{\imath}_{\mathrm{B}} + (f_{\mathrm{N}} + mg\cos\theta)\hat{k}_{\mathrm{B}}.\end{aligned} \tag{8.6.48}$$

Using (8.6.47) and (8.6.48), Fact 7.5.1 implies that

$$f_{\mathrm{T}} - mg\sin\theta = m(-\ddot{d} + a\cos\theta), \tag{8.6.49}$$

$$f_{\mathrm{N}} + mg\cos\theta = ma\sin\theta. \tag{8.6.50}$$

Next, the moment $\vec{M}_{\mathcal{B}}$ on $\mathcal{B}$ is given by

**************torque??

$$\begin{aligned}\vec{M}_{\mathcal{B}} &= \vec{r}_{z/y} \times \vec{f}_{\mathrm{r}/\mathcal{B}/z}\\ &= r\hat{k}_{\mathrm{B}} \times (f_{\mathrm{T}}\hat{\imath}_{\mathrm{B}} + f_{\mathrm{N}}\hat{k}_{\mathrm{B}})\\ &= rf_{\mathrm{T}}\hat{\jmath}_{\mathrm{B}}.\end{aligned} \tag{8.6.51}$$

It thus follows from (7.9.13) that

$$J\ddot{\phi} = rf_{\mathrm{T}}. \tag{8.6.52}$$

Since the ball rolls without slipping, it follows that $r\ddot{\phi} = \ddot{d}$, and thus

$$\ddot{d} = \frac{r^2 f_{\mathrm{T}}}{J} = \frac{5f_{\mathrm{T}}}{2m}. \tag{8.6.53}$$

where $J = \frac{2}{5}mr^2$. Finally, substituting (8.6.49) into (8.6.53) yields

$$\ddot{d} = \frac{mr^2}{J + mr^2}(a\cos\theta + g\sin\theta) = \frac{5}{7}(a\cos\theta + g\sin\theta). \tag{8.6.54}$$

Furthermore,

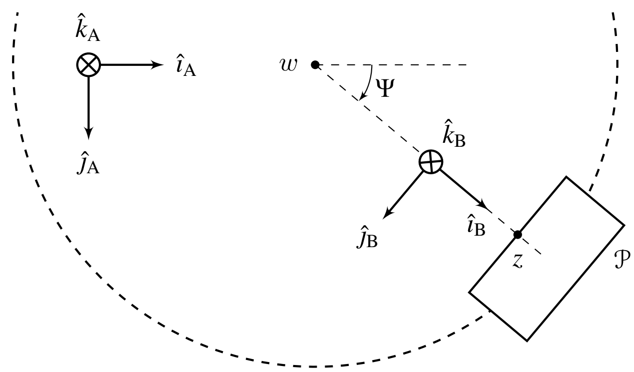
$$f_{\mathrm{T}} = \frac{J}{J + mr^2}m(a\cos\theta + g\sin\theta) = \frac{2}{7}m(a\cos\theta + g\sin\theta), \tag{8.6.55}$$

$$f_{\mathrm{N}} = m(a\sin\theta - g\cos\theta). \tag{8.6.56}$$

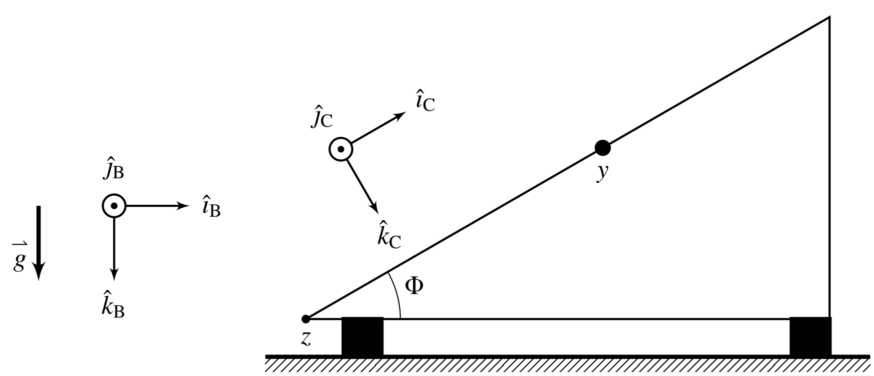
Note that, if $\theta = 0$, then $\vec{f}_{\mathrm{r}/\mathcal{B}/z} = \frac{2}{7}ma\hat{\imath}_{\mathrm{B}} - mg\hat{k}_{\mathrm{B}}$. ◇

Example 8.6.4. A flat platform $\mathcal{P}$ moves at constant speed v in a horizontal circle whose radius is r. When viewed from above, the point z on $\mathcal{P}$ moves clockwise along the circle. The platform is horizontal along the tangent of the circle but is tilted toward the center of the circle with tilt angle $\Phi > 0$. The platform is frictionless, and a particle y with mass m slides on $\mathcal{P}$. The forces on the particle include uniform uniform gravity as well as the reaction force due to contact with the platform. Determine the acceleration of y relative to the point z and with respect to a frame attached to the platform when y is at z.

Solution.



(a) Top view



(b) Tangential view

Figure 8.6.24: Example 8.6.4. Platform moving along a horizontal circle.

As shown in Figure 8.6.24, let F_A be an inertial frame such that $\vec{g} = g\hat{k}_A$, let Ψ denote the angle from $\hat{\imath}_A$ to $\vec{r}_{z/w}$ around $\hat{k}_A$, and let F_B and F_C denote frames attached to the platform defined by

$$F_A \xrightarrow[3]{\Psi} F_B \xrightarrow[2]{\Phi} F_C, \tag{8.6.57}$$

and thus $\vec{\omega}_{B/A} = \omega\hat{k}_A$, where $\omega \triangleq \dot{\Psi}$. Using

$$\begin{aligned} \vec{r}_{y/w} &= \vec{r}_{y/z} + \vec{r}_{z/w} \\ &= d_1\hat{\imath}_C + d_2\hat{\jmath}_C + r\hat{\imath}_B, \end{aligned} \tag{8.6.58}$$

it follows that

$$
\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{r}}_{y/w} &= \dot{d}_1\hat{\imath}_{\mathrm{C}} + d_1 \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{C}}} + \dot{d}_2\hat{\jmath}_{\mathrm{C}} + d_2 \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{C}}} + r \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} \\
&= \dot{d}_1\hat{\imath}_{\mathrm{C}} + d_1\vec{\omega}_{\mathrm{C/A}} \times \hat{\imath}_{\mathrm{C}} + \dot{d}_2\hat{\jmath}_{\mathrm{C}} + d_2\vec{\omega}_{\mathrm{C/A}} \times \hat{\jmath}_{\mathrm{C}} + r\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}} \\
&= \dot{d}_1\hat{\imath}_{\mathrm{C}} + d_1\omega\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{C}} + \dot{d}_2\hat{\jmath}_{\mathrm{C}} + d_2\omega\hat{k}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{C}} + r\omega\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{B}} \\
&= \dot{d}_1\hat{\imath}_{\mathrm{C}} + d_1\omega\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{C}} + \dot{d}_2\hat{\jmath}_{\mathrm{C}} + d_2\omega\hat{k}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}} + r\omega\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{B}} \\
&= \dot{d}_1\hat{\imath}_{\mathrm{C}} + d_1\omega\hat{k}_{\mathrm{B}} \times [(\cos\Phi)\hat{\imath}_{\mathrm{B}} - (\sin\Phi)\hat{k}_{\mathrm{B}}] + \dot{d}_2\hat{\jmath}_{\mathrm{C}} - d_2\omega\hat{\imath}_{\mathrm{B}} + r\omega\hat{\jmath}_{\mathrm{B}} \\
&= \dot{d}_1\hat{\imath}_{\mathrm{C}} + \dot{d}_2\hat{\jmath}_{\mathrm{C}} - d_2\omega\hat{\imath}_{\mathrm{B}} + (d_1\cos\Phi + r)\omega\hat{\jmath}_{\mathrm{B}}
\end{aligned}
\tag{8.6.59}
$$

and

$$
\begin{aligned}
\overset{\mathrm{A}\bullet\bullet}{\vec{r}}_{y/w} &= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + \ddot{d}_2\hat{\jmath}_{\mathrm{C}} + \dot{d}_1 \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{C}}} + \dot{d}_2 \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{C}}} - \dot{d}_2\omega\hat{\imath}_{\mathrm{B}} - d_2\omega \overset{\mathrm{A}\bullet}{\hat{\imath}_{\mathrm{B}}} \\
&\quad + \dot{d}_1(\cos\Phi)\omega\hat{\jmath}_{\mathrm{B}} + (d_1\cos\Phi + r)\omega \overset{\mathrm{A}\bullet}{\hat{\jmath}_{\mathrm{B}}} \\
&= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + \ddot{d}_2\hat{\jmath}_{\mathrm{C}} + \dot{d}_1\vec{\omega}_{\mathrm{C/A}} \times \hat{\imath}_{\mathrm{C}} + \dot{d}_2\vec{\omega}_{\mathrm{C/A}} \times \hat{\jmath}_{\mathrm{C}} - \dot{d}_2\omega\hat{\imath}_{\mathrm{B}} - d_2\omega\vec{\omega}_{\mathrm{B/A}} \times \hat{\imath}_{\mathrm{B}} \\
&\quad + \dot{d}_1(\cos\Phi)\omega\hat{\jmath}_{\mathrm{B}} + (d_1\cos\Phi + r)\omega\vec{\omega}_{\mathrm{B/A}} \times \hat{\jmath}_{\mathrm{B}} \\
&= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + \ddot{d}_2\hat{\jmath}_{\mathrm{C}} + \dot{d}_1\omega\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{C}} + \dot{d}_2\omega\hat{k}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{C}} - \dot{d}_2\omega\hat{\imath}_{\mathrm{B}} - d_2\omega\omega\hat{k}_{\mathrm{B}} \times \hat{\imath}_{\mathrm{B}} \\
&\quad + \dot{d}_1(\cos\Phi)\omega\hat{\jmath}_{\mathrm{B}} + (d_1\cos\Phi + r)\omega\omega\hat{k}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}} \\
&= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + \ddot{d}_2\hat{\jmath}_{\mathrm{C}} + \dot{d}_1\omega\hat{k}_{\mathrm{B}} \times [(\cos\Phi)\hat{\imath}_{\mathrm{B}} - (\sin\Phi)\hat{k}_{\mathrm{B}}] - 2\dot{d}_2\omega\hat{\imath}_{\mathrm{B}} - d_2\omega^2\hat{\jmath}_{\mathrm{B}} \\
&\quad + \dot{d}_1(\cos\Phi)\omega\hat{\jmath}_{\mathrm{B}} - (d_1\cos\Phi + r)\omega^2\hat{\imath}_{\mathrm{B}} \\
&= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + \ddot{d}_2\hat{\jmath}_{\mathrm{C}} + \dot{d}_1\omega(\cos\Phi)\hat{\jmath}_{\mathrm{B}} - 2\dot{d}_2\omega\hat{\imath}_{\mathrm{B}} - d_2\omega^2\hat{\jmath}_{\mathrm{B}} + \dot{d}_1(\cos\Phi)\omega\hat{\jmath}_{\mathrm{B}} - (d_1\cos\Phi + r)\omega^2\hat{\imath}_{\mathrm{B}} \\
&= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + [\ddot{d}_2 + \dot{d}_1\omega(\cos\Phi) - d_2\omega^2 + \dot{d}_1(\cos\Phi)\omega]\hat{\jmath}_{\mathrm{C}} + [-(d_1\cos\Phi + r)\omega^2 - 2\dot{d}_2\omega]\hat{\imath}_{\mathrm{B}} \\
&= \ddot{d}_1\hat{\imath}_{\mathrm{C}} + [\ddot{d}_2 + \dot{d}_1\omega(\cos\Phi) - d_2\omega^2 + \dot{d}_1(\cos\Phi)\omega]\hat{\jmath}_{\mathrm{C}} + \\
&\quad [-(d_1\cos\Phi + r)\omega^2 - 2\dot{d}_2\omega][(\cos\Phi)\hat{\imath}_{\mathrm{C}} + (\sin\Phi)\hat{k}_{\mathrm{C}}] \\
&= [\ddot{d}_1 + [-(d_1\cos\Phi + r)\omega^2 - 2\dot{d}_2\omega](\cos\Phi)]\hat{\imath}_{\mathrm{C}} + [\ddot{d}_2 + 2\dot{d}_1\omega(\cos\Phi) - d_2\omega^2]\hat{\jmath}_{\mathrm{C}} + \\
&\quad [-(d_1\cos\Phi + r)\omega^2 - 2\dot{d}_2\omega](\sin\Phi)\hat{k}_{\mathrm{C}}
\end{aligned}
\tag{8.6.60}
$$

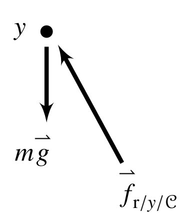


Figure 8.6.25: Example 8.6.4. Free-body diagram of the particle y.

Next, letting $\vec{f}_{\mathrm{r}/y/\mathcal{C}} = f_{\mathrm{N}}\hat{k}_{\mathrm{C}}$, the net force $\vec{f}_y$ on the particle y, as shown in Figure 8.6.25, is given by

$$
\begin{aligned}
\vec{f}_y &= m\vec{g} + \vec{f}_{\mathrm{r}/y/\mathcal{C}} \\
&= mg\hat{k}_{\mathrm{B}} + f_{\mathrm{N}}\hat{k}_{\mathrm{C}}
\end{aligned}
$$

$$= mg[-(\sin\Phi)\hat{\imath}_{\rm C} + (\cos\Phi)\hat{k}_{\rm C}] + f_{\rm N}\hat{k}_{\rm C}$$
$$= -mg(\sin\Phi)\hat{\imath}_{\rm C} + (f_{\rm N} + mg\cos\Phi)\hat{k}_{\rm C}. \tag{8.6.61}$$

Using (8.6.60) and (8.6.61), Fact 7.5.1 implies that

$$\ddot{d}_1 = [(d_1\cos\Phi + r)\omega^2 + 2\dot{d}_2\omega]\cos\Phi - g\sin\Phi, \tag{8.6.62}$$
$$\ddot{d}_2 = -2\dot{d}_1\omega(\cos\Phi) + d_2\omega^2, \tag{8.6.63}$$
$$f_{\rm N} = -m(d_1\cos\Phi + r)\omega^2 - 2m\dot{d}_2\omega - mg\cos\Phi. \tag{8.6.64}$$

Defining

$$x \triangleq \begin{bmatrix} d_1 \\ \dot{d}_1 \\ d_2 \\ \dot{d}_2 \end{bmatrix}, \tag{8.6.65}$$

it follows from (8.6.62) and (8.6.63) that

$$\dot{x} = Ax + Bu, \tag{8.6.66}$$

where

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2\cos^2\Phi & 0 & 0 & 2\omega\cos\Phi \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega\cos\Phi & \omega^2 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{8.6.67}$$

and $u \triangleq r\omega^2\cos\Phi - g\sin\Phi$. The eigenvalues of A are given by

$$\lambda_1(A) = -\frac{\omega}{\sqrt{2}}\sqrt{1 - 3\cos^2\Phi - \sqrt{9\cos^4\Phi - 10\cos^2\Phi + 1}} \tag{8.6.68}$$
$$\lambda_2(A) = \frac{\omega}{\sqrt{2}}\sqrt{1 - 3\cos^2\Phi - \sqrt{9\cos^4\Phi - 10\cos^2\Phi + 1}} \tag{8.6.69}$$
$$\lambda_3(A) = -\frac{\omega}{\sqrt{2}}\sqrt{1 - 3\cos^2\Phi + \sqrt{9\cos^4\Phi - 10\cos^2\Phi + 1}} \tag{8.6.70}$$
$$\lambda_4(A) = \frac{\omega}{\sqrt{2}}\sqrt{1 - 3\cos^2\Phi + \sqrt{9\cos^4\Phi - 10\cos^2\Phi + 1}} \tag{8.6.71}$$

Figure 8.6.26 shows the eigenvalues of A for $\omega = 1$ and various values of Φ. Note that for $\Phi \in [0, \pi/2]$, there are two positive eigenvalues and two negative eigenvalues. Furthermore, for $\Phi > \Phi_{tbd}$, all eigenvalues are real. Therefore

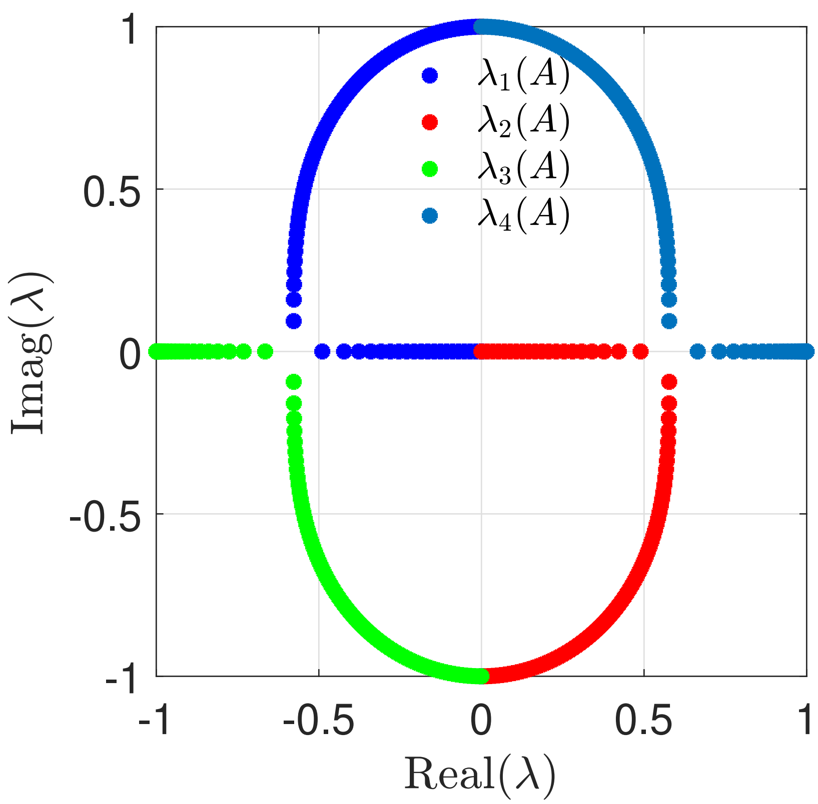


Figure 8.6.26: Eigenvalues of A for $\omega = 1$ and various values of Φ.

Define

$$\Phi_0 \triangleq \arctan \frac{r\omega^2}{g}. \tag{8.6.72}$$

First, let $\Phi = \Phi_0$ and $d_1(0) = 0$ and $\dot{d}_2(0) = 0$. Then,

$$\ddot{d}_1(0) = r\omega^2 \cos\Phi_0 - g\sin\Phi_0 = 0. \tag{8.6.73}$$

Furthermore, letting $d_2(0) = 0$ and $\dot{d}_1(0) = 0$, it follows that

$$\ddot{d}_2(0) = 0. \tag{8.6.74}$$

Next, let $\Phi = \Phi_0 + \phi$. For $\dot{d}_1(0) = \dot{d}_2(0) = d_1(0) = d_2(0) = 0$, $r = 1$, and $\omega = 1$, Figure 8.6.27 shows the trajectory of the particle y for various values of ϕ.

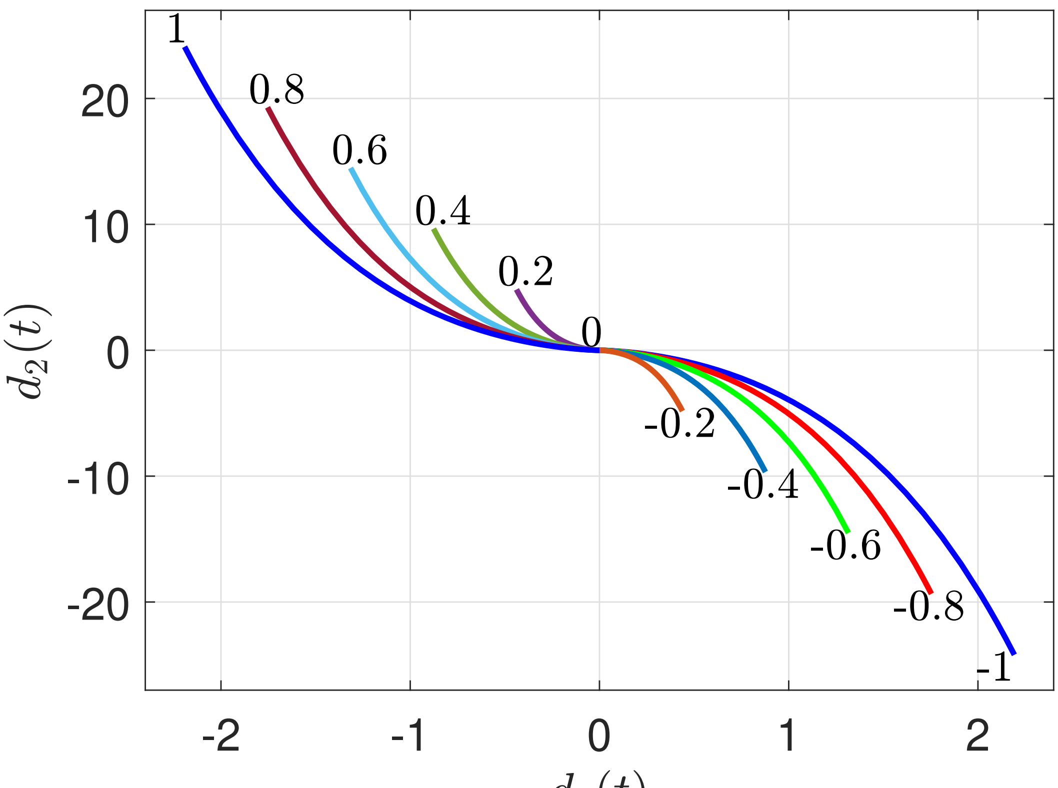


Figure 8.6.27: Trajectory of the particle y for various values of ϕ, where $r = 1, \omega = 1$, and $\Phi = \Phi_0 + \phi$.

$$\begin{aligned} \ddot{d}_1 &= [(d_1\cos\Phi_0 + r)\omega^2 + 2\dot{d}_2\omega]\cos\Phi_0 - g\sin\Phi_0 \\ &= \cos\Phi_0\left(d_1\cos\Phi_0\omega^2 + 2\dot{d}_2\omega\right), \\ &= \frac{g}{\sqrt{r\omega^2 - g}}\left(d_1\frac{g}{\sqrt{r\omega^2 - g}}\omega^2 + 2\dot{d}_2\omega\right), \end{aligned} \tag{8.6.75}$$

$$\ddot{d}_2 = -2\dot{d}_1 \frac{g}{\sqrt{r\omega^2 - g}} + d_2\omega^2, \tag{8.6.76}$$

$$\begin{aligned}
\begin{bmatrix} \dot{d}_1 \\ \ddot{d}_1 \\ \dot{d}_2 \\ \ddot{d}_2 \end{bmatrix} &= \begin{bmatrix} \dot{d}_1 \\ [(d_1 \cos\Phi + r)\omega^2 + 2\dot{d}_2\omega]\cos\Phi - g\sin\Phi \\ \dot{d}_2 \\ -2\dot{d}_1\omega(\cos\Phi) + d_2\omega^2, \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2\cos^2\Phi & 0 & 0 & 2\omega\cos\Phi \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega\cos\Phi & \omega^2 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ \dot{d}_1 \\ d_2 \\ \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ r\omega^2\cos\Phi - g\sin\Phi \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{8.6.77}$$

$$\begin{bmatrix} \dot{d}_1 \\ \ddot{d}_1 \\ \dot{d}_2 \\ \ddot{d}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2\cos^2\Phi_0 & 0 & 0 & 2\omega\cos\Phi_0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega\cos\Phi_0 & \omega^2 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ \dot{d}_1 \\ d_2 \\ \dot{d}_2 \end{bmatrix} \tag{8.6.78}$$

Next, let $\dot{d}_1(0) = \dot{d}_2(0) = 0$. Then,

$$\ddot{d}_1(0) = (d_1 \cos\Phi + r)\omega^2 \cos\Phi - g\sin\Phi, \tag{8.6.79}$$

$$\ddot{d}_2(0) = d_2\omega^2. \tag{8.6.80}$$

Furthermore, let Φ be given by the solution of

$$(d_1(0)\cos\Phi + r)\omega^2\cos\Phi - g\sin\Phi = 0, \tag{8.6.81}$$

and let $d_2(0) = 0$. Then, $\ddot{d}_1(0) = \ddot{d}_2(0) = 0$.

In order to obtain Φ that satisfies (8.6.81), note that

$$(d_1(0)\cos\Phi + r)\omega^2 = g\tan\Phi. \tag{8.6.82}$$

Squaring both sides and rearranging,

$$d_1(0)^2\omega^4\cos^4\Phi + 2rd_1(0)\omega^4\cos^3\Phi + (r^2\omega^4 - g)\cos^2\Phi - g = 0. \tag{8.6.83}$$

Assume that $d_2 \equiv 0$. Then,

$$\ddot{d}_1 = (d_1 \cos\Theta + r)\dot{\Psi}^2 \cos\Theta - mg\sin\Theta. \tag{8.6.84}$$

Setting $\ddot{d}_1 = 0$,

$$(d_1 \cos\Theta + r)\dot{\Psi}^2 = mg\tan\Theta = mg\sqrt{1 + \sec^2\Theta}. \tag{8.6.85}$$

Defining $x \triangleq \cos\Theta$,

$$(d_1 x + r)\dot{\Psi}^2 x = mg\sqrt{1 + x^2}. \tag{8.6.86}$$

$$(d_1\dot{\Psi}^2 x^2 + r\dot{\Psi}^2 x) = Ax^2 + Bx = C\sqrt{1 + x^2}. \tag{8.6.87}$$

$$A^2x^4 + (B^2 - C)x^2 + 2ABx^3 - C = 0. \tag{8.6.88}$$

Example 8.6.5. Consider a ball y rolling along a beam $\mathcal{B}$ subject to uniform gravity. The beam rotates around a pin joint at its center w due to the weight of the ball as well as an external torque τ. The point w has zero inertial acceleration. Determine the equations of motion for the ball and beam as well as the reaction force on the ball due to the beam.

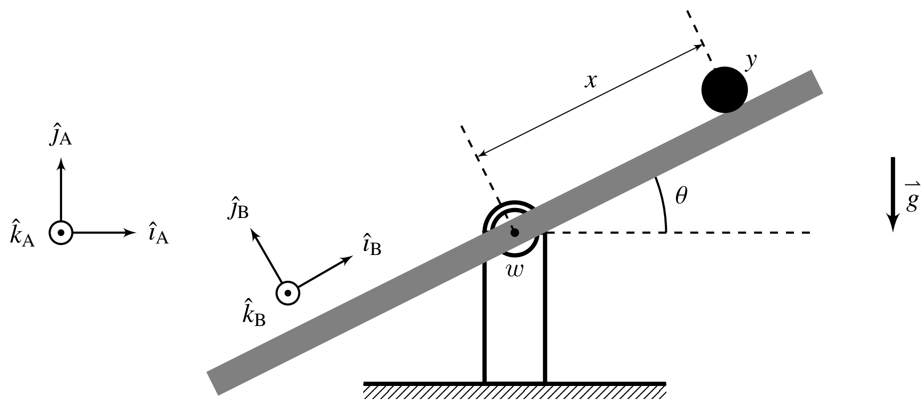


Figure 8.6.28: Ball and beam for Example 8.6.5.

Solution. Let F_A be fixed to the ground such that $\hat{\imath}_A$ is horizontal and $\hat{\jmath}_A$ points upward, and let F_B be fixed to $\mathcal{B}$ such that $\hat{\imath}_B$ and $\hat{\imath}_A$ are parallel when the beam is horizontal. Then, F_A and F_B are related by $F_A \xrightarrow[3]{\theta} F_B$, and thus

$$\begin{bmatrix}\hat{\imath}_B\\ \hat{\jmath}_B\\ \hat{k}_B\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\imath}_A\\ \hat{\jmath}_A\\ \hat{k}_A\end{bmatrix} \tag{8.6.89}$$

and $\vec{\omega}_{B/A} = \dot{\theta}\hat{k}_A$. Furthermore, since $\vec{r}_{y/w} = x\hat{\imath}_B$, it follows that

$$\vec{v}_{y/w/A} = \overset{A\bullet}{\vec{r}}_{y/w} = \dot{x}\hat{\imath}_B + x\overset{A\bullet}{\hat{\imath}}_B = \dot{x}\hat{\imath}_B + x\vec{\omega}_{B/A}\times\hat{\imath}_B = \dot{x}\hat{\imath}_B + x\dot{\theta}\hat{\jmath}_B, \tag{8.6.90}$$

$$\begin{aligned}\vec{a}_{y/w/A} &= \overset{A\bullet\bullet}{\vec{r}}_{y/w}\\ &= \overset{B\bullet\bullet}{\vec{r}}_{y/w} + 2\vec{\omega}_{B/A}\times\overset{B\bullet}{\vec{r}}_{y/w} + \overset{B\bullet}{\vec{\omega}}_{B/A}\times\vec{r}_{y/w} + \vec{\omega}_{B/A}\times\vec{\omega}_{B/A}\times\vec{r}_{y/w}\\ &= (\ddot{x}-\dot{\theta}^2x)\hat{\imath}_B + (2\dot{\theta}\dot{x}+\ddot{\theta}x)\hat{\jmath}_B.\end{aligned} \tag{8.6.91}$$

The reaction force on y due to $\mathcal{B}$ is given by $\vec{f}_{r/y/\mathcal{B}} = f_r\hat{\jmath}_B$, and thus the net force on y is given by

$$\vec{f}_y = \vec{f}_{r/y/\mathcal{B}} + m\vec{g} = f_r\hat{\jmath}_B - mg\hat{\jmath}_A = -mg\sin\theta\hat{\imath}_B + (f_r - mg\cos\theta)\hat{\jmath}_B. \tag{8.6.92}$$

It thus follows from Newton's second law $m\vec{a}_{y/w/A} = \vec{f}_y$ that

$$m(\ddot{x}-\dot{\theta}^2x)\hat{\imath}_B + m(2\dot{\theta}\dot{x}+\ddot{\theta}x)\hat{\jmath}_B = -mg\sin\theta\hat{\imath}_B + (f_r - mg\cos\theta)\hat{\jmath}_B. \tag{8.6.93}$$

Hence,

$$\ddot{x} = \dot{\theta}^2 x - g \sin\theta, \tag{8.6.94}$$

$$f_{\mathrm{r}} = mg\cos\theta + 2m\dot{\theta}\dot{x} + m\ddot{\theta}x. \tag{8.6.95}$$

Next, writing $\vec{J}_{\mathcal{B}/w} = J_{xx}\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + J_{yy}\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + J_{zz}\hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}$, it follows that

$$\vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/w}\vec{\omega}_{\mathrm{B/A}} = 0, \tag{8.6.96}$$

$$\vec{J}_{\mathcal{B}/w} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = J_{zz}\ddot{\theta}\hat{k}_{\mathrm{B}}. \tag{8.6.97}$$

The moment on the beam relative to w is given by

$$\vec{M}_{\mathcal{B}/w} = \vec{r}_{y/w} \times (-\vec{f}_{\mathrm{r}/y/\mathcal{B}}) + \vec{M}_{\mathrm{ext}} = (-xf_{\mathrm{r}} + \tau)\hat{k}_{\mathrm{B}}. \tag{8.6.98}$$

Newton's second law of rotation relative to w implies

$$\vec{J}_{\mathcal{B}/w} \overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} = \vec{M}_{\mathcal{B}/w}. \tag{8.6.99}$$

Therefore,

$$J_{zz}\ddot{\theta}\hat{k}_{\mathrm{B}} = (-xf_{\mathrm{r}} + \tau)\hat{k}_{\mathrm{B}}, \tag{8.6.100}$$

and thus

$$(J_{zz} + mx^2)\ddot{\theta} = -x(mg\cos\theta + 2m\dot{\theta}\dot{x}) + \tau. \tag{8.6.101}$$

In summary, (8.6.94) and (8.6.101) imply that the equations of motion for the ball and beam are given by

$$\ddot{x} - \dot{\theta}^2 x + g\sin\theta = 0, \tag{8.6.102}$$

$$(J_{zz} + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} + mgx\cos\theta = \tau. \tag{8.6.103}$$

◇

Example 8.6.6. Consider the planar linkage shown in Figure 8.6.29 consisting of two uniform bars $\mathcal{L}_1$ and $\mathcal{L}_2$ with lengths ℓ_1 and ℓ_2, masses m_1 and m_2, and centers of mass c_1 and c_2, respectively. One end of $\mathcal{L}_1$ is attached to the ground $\mathcal{G}$ at the point z_1 by means of a frictionless pin joint. Likewise, $\mathcal{L}_2$ is connected to $\mathcal{L}_1$ by means of a frictionless pin joint at the point z_2. Since the mechanism lies in the horizontal plane, gravity has no effect. For $i = 1, 2$, the frame $\mathrm{F}_{\mathrm{B}_i}$ is fixed to $\mathcal{L}_i$, and F_A is an inertial frame. For $i = 1, 2$, let $\vec{J}_i$ denote $\vec{J}_{\mathcal{L}_i/\mathrm{c}_i}$, and let $\vec{\omega}_i$ denote $\vec{\omega}_{\mathrm{B}_i/\mathrm{A}}$.

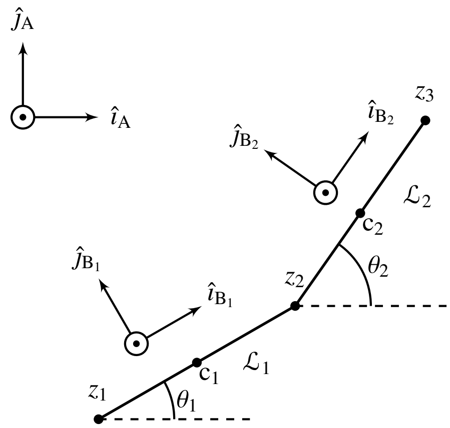


Figure 8.6.29: Two-bar linkage. Ankit: Please put arrows on the angles to show that they are directed angles. Thanks.

Solution. Euler's equation given by Fact 7.9.5 implies that

$$\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = \vec{r}_{z_1/\mathrm{c}_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} + \vec{r}_{z_2/\mathrm{c}_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_2}, \tag{8.6.104}$$

$$\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = \vec{r}_{z_2/\mathrm{c}_2} \times \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2}. \tag{8.6.105}$$

Furthermore, Newton's second law given by Fact 7.5.1 implies that

$$m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} + \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_2}, \tag{8.6.106}$$

$$m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2}. \tag{8.6.107}$$

Using $\vec{f}_{\mathrm{r}/\mathcal{L}_1/z_2} = -\vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2}$ and writing (8.6.106) and (8.6.107) as

$$\begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{I} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} \\ \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2} \end{bmatrix}, \tag{8.6.108}$$

it follows that

$$\begin{aligned} \begin{bmatrix} \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} \\ \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2} \end{bmatrix} &= \begin{bmatrix} \vec{I} & -\vec{I} \\ 0 & \vec{I} \end{bmatrix}^{-1} \begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \end{bmatrix} \\ &= \begin{bmatrix} \vec{I} & \vec{I} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} + m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \end{bmatrix}. \tag{8.6.109}$$

Substituting (8.6.109) into (8.6.104) and (8.6.105) yields

$$\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = \vec{r}_{z_1/\mathrm{c}_1} \times (m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} + m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}}) - \vec{r}_{z_2/\mathrm{c}_1} \times (m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}}), \tag{8.6.110}$$

$$\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = \vec{r}_{z_2/\mathrm{c}_2} \times (m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}}), \tag{8.6.111}$$

which can be rewritten as

$$\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = -m_1 \vec{r}_{\mathrm{c}_1/z_1} \times \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} + m_2 \vec{r}_{z_1/z_2} \times \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}}, \tag{8.6.112}$$

$$\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = -m_2 \vec{r}_{\mathrm{c}_2/z_2} \times \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}}. \tag{8.6.113}$$

Next, the double transport theorem implies

$$\begin{aligned} \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} &= \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{\mathrm{c}_1/z_1} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1}) \\ &= \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{\mathrm{c}_1/z_1} - \vec{\omega}_1' \vec{\omega}_1 \vec{r}_{\mathrm{c}_1/z_1}, \end{aligned} \tag{8.6.114}$$

and

$$\begin{aligned} \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} &= \vec{a}_{\mathrm{c}_2/z_2/\mathrm{A}} + \vec{a}_{z_2/z_1/\mathrm{A}} \\ &= \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \times \vec{r}_{\mathrm{c}_2/z_2} + \vec{\omega}_2 \times (\vec{\omega}_2 \times \vec{r}_{\mathrm{c}_2/z_2}) + \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{z_2/z_1} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{z_2/z_1}) \\ &= \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \times \vec{r}_{\mathrm{c}_2/z_2} - \vec{\omega}_2' \vec{\omega}_2 \vec{r}_{\mathrm{c}_2/z_2} + \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{z_2/z_1} - \vec{\omega}_1' \vec{\omega}_1 \vec{r}_{z_2/z_1}. \end{aligned} \tag{8.6.115}$$

Now, substituting

$$\begin{aligned} \vec{r}_{\mathrm{c}_1/z_1} \times \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} &= \vec{r}_{\mathrm{c}_1/z_1} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{\mathrm{c}_1/z_1} - \vec{\omega}_1' \vec{\omega}_1 \vec{r}_{\mathrm{c}_1/z_1}) \\ &= \vec{r}'_{\mathrm{c}_1/z_1} \vec{r}_{\mathrm{c}_1/z_1} \overset{\mathrm{A}\bullet}{\vec{\omega}}_1, \end{aligned} \tag{8.6.116}$$

$$\begin{aligned} \vec{r}_{z_2/z_1} \times \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} &= \vec{r}_{z_2/z_1} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \times \vec{r}_{\mathrm{c}_2/z_2} - \vec{\omega}_2' \vec{\omega}_2 \vec{r}_{\mathrm{c}_2/z_2} + \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{z_2/z_1} - \vec{\omega}_1' \vec{\omega}_1 \vec{r}_{z_2/z_1}) \\ &= \vec{r}_{z_2/z_1} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \times \vec{r}_{\mathrm{c}_2/z_2}) - \vec{\omega}_2' \vec{\omega}_2 \vec{r}_{z_2/z_1} \times \vec{r}_{\mathrm{c}_2/z_2} + \vec{r}_{z_2/z_1} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{z_2/z_1}) \\ &= \vec{r}'_{z_2/z_1} \vec{r}_{\mathrm{c}_2/z_2} \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 - \vec{\omega}_2' \vec{\omega}_2 \vec{r}_{z_2/z_1} \times \vec{r}_{\mathrm{c}_2/z_2} + \vec{r}'_{z_2/z_1} \vec{r}_{z_2/z_1} \overset{\mathrm{A}\bullet}{\vec{\omega}}_1, \end{aligned} \tag{8.6.117}$$

and

$$\begin{aligned} r_{\mathrm{c}_2/z_2} \times \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} &= r_{\mathrm{c}_2/z_2} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \times \vec{r}_{\mathrm{c}_2/z_2} - \vec{\omega}_2' \vec{\omega}_2 \vec{r}_{\mathrm{c}_2/z_2} + \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{z_2/z_1} - \vec{\omega}_1' \vec{\omega}_1 \vec{r}_{z_2/z_1}) \\ &= r_{\mathrm{c}_2/z_2} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \times \vec{r}_{\mathrm{c}_2/z_2}) + r_{\mathrm{c}_2/z_2} \times (\overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \times \vec{r}_{z_2/z_1}) - \vec{\omega}_1' \vec{\omega}_1 r_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1} \\ &= r'_{\mathrm{c}_2/z_2} \vec{r}_{\mathrm{c}_2/z_2} \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 + r'_{\mathrm{c}_2/z_2} \vec{r}_{z_2/z_1} \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 - \vec{\omega}_1' \vec{\omega}_1 r_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1} \end{aligned} \tag{8.6.118}$$

into (8.6.112) and (8.6.113) yields

$$\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = -m_1 \vec{r}\,'_{\mathrm{c}_1/z_1}\vec{r}_{\mathrm{c}_1/z_1}\overset{\mathrm{A}\bullet}{\vec{\omega}}_1 - m_2 \vec{r}\,'_{z_2/z_1}\vec{r}_{\mathrm{c}_2/z_2}\overset{\mathrm{A}\bullet}{\vec{\omega}}_2 + m_2\vec{\omega}\,'_2\vec{\omega}_2\vec{r}_{z_2/z_1}\times\vec{r}_{\mathrm{c}_2/z_2} - m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\overset{\mathrm{A}\bullet}{\vec{\omega}}_1, \tag{8.6.119}$$

$$\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = -m_2 \vec{r}\,'_{\mathrm{c}_2/z_2}\vec{r}_{\mathrm{c}_2/z_2}\overset{\mathrm{A}\bullet}{\vec{\omega}}_2 - m_2 \vec{r}\,'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}\overset{\mathrm{A}\bullet}{\vec{\omega}}_1 + m_2\vec{\omega}\,'_1\vec{\omega}_1\vec{r}_{\mathrm{c}_2/z_2}\times\vec{r}_{z_2/z_1}, \tag{8.6.120}$$

which can be written as

$$\begin{bmatrix} \vec{J}_1 + m_1\vec{r}\,'_{\mathrm{c}_1/z_1}\vec{r}_{\mathrm{c}_1/z_1}\vec{I} + m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I} & m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{\mathrm{c}_2/z_2}\vec{I} \\ m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{\mathrm{c}_2/z_2}\vec{I} & \vec{J}_2 + m_2\vec{r}\,'_{\mathrm{c}_2/z_2}\vec{r}_{\mathrm{c}_2/z_2}\vec{I} \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \\ \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \end{bmatrix} + \begin{bmatrix} m_2(\vec{r}_{\mathrm{c}_2/z_2}\times\vec{r}_{z_2/z_1})\vec{\omega}\,'_2\vec{\omega}_2 \\ -m_2(\vec{r}_{\mathrm{c}_2/z_2}\times\vec{r}_{z_2/z_1})\vec{\omega}\,'_1\vec{\omega}_1 \end{bmatrix} = 0. \tag{8.6.121}$$

Next, note that, for $i = 1, 2$, $\vec{\omega}_i = \dot{\theta}_1\hat{k}_{\mathrm{A}}$ and $\vec{J}_i = J_{i1}\hat{\imath}_{\mathrm{B}_i}\hat{\imath}'_{\mathrm{B}_i} + J_{i2}\hat{\jmath}_{\mathrm{B}_i}\hat{\jmath}'_{\mathrm{B}_i} + J_{i3}\hat{k}_{\mathrm{B}_i}\hat{k}'_{\mathrm{B}_i}$, and thus

$$\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = J_{13}\ddot{\theta}_1\hat{k}_{\mathrm{A}}, \quad \vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = J_{23}\ddot{\theta}_2\hat{k}_{\mathrm{A}}. \tag{8.6.122}$$

Furthermore,

$$\vec{r}_{z_2/z_1} = \ell_1\hat{\imath}_{\mathrm{B}_1}, \quad \vec{r}_{\mathrm{c}_2/z_2} = \tfrac{1}{2}\ell_2\hat{\imath}_{\mathrm{B}_2}, \tag{8.6.123}$$

and thus

$$\begin{aligned} \vec{r}_{\mathrm{c}_2/z_2}\times\vec{r}_{z_2/z_1} &= \tfrac{1}{2}\ell_2\hat{\imath}_{\mathrm{B}_2}\times\ell_1\hat{\imath}_{\mathrm{B}_1} \\ &= \tfrac{1}{2}\ell_1\ell_2(-\sin\theta_2\hat{\imath}_{\mathrm{A}} + \cos\theta_2\hat{\jmath}_{\mathrm{A}})\times(-\sin\theta_1\hat{\imath}_{\mathrm{A}} + \cos\theta_1\hat{\jmath}_{\mathrm{A}}) \\ &= \tfrac{1}{2}\ell_1\ell_2(-\sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1)\hat{k}_{\mathrm{A}} \\ &= \tfrac{1}{2}\ell_1\ell_2\sin(\theta_1-\theta_2)\hat{k}_{\mathrm{A}}. \end{aligned} \tag{8.6.124}$$

Therefore,

$$\begin{bmatrix} J_{13} + \frac{1}{4}m_1\ell_1^2 + m_2\ell_1^2 & \frac{1}{2}m_2\ell_1\ell_2\cos(\theta_2-\theta_1) \\ \frac{1}{2}m_2\ell_1\ell_2\cos(\theta_2-\theta_1) & J_{23} + \frac{1}{4}m_2\ell_2^2 \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} \frac{1}{2}m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 \\ -\frac{1}{2}m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 \end{bmatrix} = 0, \tag{8.6.125}$$

and thus

$$\begin{bmatrix} \frac{1}{3}m_1\ell_1^2 + m_2\ell_1^2 & \frac{1}{2}m_2\ell_1\ell_2\cos(\theta_2-\theta_1) \\ \frac{1}{2}m_2\ell_1\ell_2\cos(\theta_2-\theta_1) & \frac{1}{3}m_2\ell_2^2 \end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} \frac{1}{2}m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 \\ -\frac{1}{2}m_2\ell_1\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 \end{bmatrix} = 0. \tag{8.6.126}$$

Example 8.6.7. Consider the spatial linkage shown in Figure 8.6.30 consisting of three uniform bars $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ with lengths ℓ_1, ℓ_2, and ℓ_3, masses m_1, m_2, and m_3, and centers of mass c_1, c_2, and c_3, respectively. One end of $\mathcal{L}_1$ is attached to the ground $\mathcal{G}$ at the point z_1 by means of a frictionless ball joint. Likewise, $\mathcal{L}_2$ is connected to $\mathcal{L}_1$ by means of a frictionless ball joint at the point z_2, and $\mathcal{L}_3$ is connected to $\mathcal{L}_2$ by means of a frictionless ball joint at the point z_3. Since the linkage is unaffected by gravity. For $i = 1,2,3$, the frame $\mathrm{F}_{\mathrm{B}_i}$ is fixed to $\mathcal{L}_i$, and F_A is an inertial frame. For $i = 1,2,3$, let $\vec{J}_i$ denote $\vec{J}_{\mathcal{L}_i/\mathrm{c}_i}$, and let $\vec{\omega}_i$ denote $\vec{\omega}_{\mathrm{B}_i/\mathrm{A}}$.

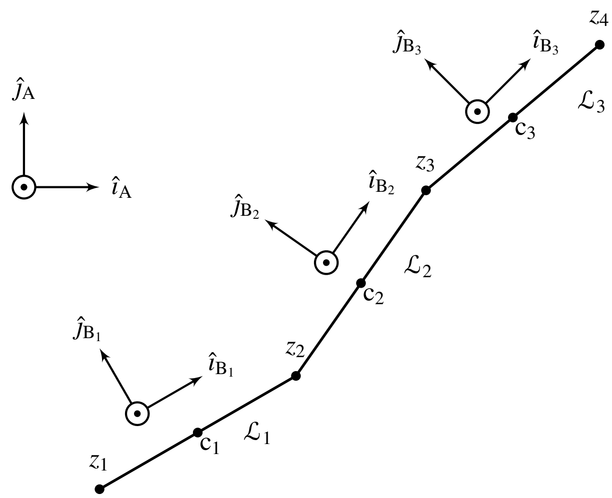


Figure 8.6.30: Three-bar linkage. The point z_1 has zero inertial acceleration, and the links are connected by frictionless ball joints at z_1, z_2, and z_3.

Solution. It follows from Euler's equation Fact 7.9.5 that

$$\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = \vec{r}_{\mathrm{c}_1/z_1} \times \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} + \vec{r}_{\mathrm{c}_1/z_2} \times \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_2}, \tag{8.6.127}$$

$$\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = \vec{r}_{\mathrm{c}_2/z_2} \times \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2} + \vec{r}_{\mathrm{c}_2/z_3} \times \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_3}, \tag{8.6.128}$$

$$\vec{J}_3 \overset{\mathrm{A}\bullet}{\vec{\omega}}_3 = \vec{r}_{\mathrm{c}_3/z_3} \times \vec{f}_{\mathrm{r}/\mathcal{L}_3/z_3}. \tag{8.6.129}$$

Furthermore, Newton's second law Fact 7.5.1 implies that

$$m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} + \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_2}, \tag{8.6.130}$$

$$m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2} + \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_3}, \tag{8.6.131}$$

$$m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{L}_3/z_3}. \tag{8.6.132}$$

Using $\vec{f}_{\mathrm{r}/\mathcal{L}_1/z_2} = -\vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2}$ and $\vec{f}_{\mathrm{r}/\mathcal{L}_2/z_3} = -\vec{f}_{\mathrm{r}/\mathcal{L}_3/z_3}$ and writing (8.6.130)–(8.6.132) as

$$\begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \\ m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{I} & 0 \\ 0 & \vec{I} & -\vec{I} \\ 0 & 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} \\ \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2} \\ \vec{f}_{\mathrm{r}/\mathcal{L}_3/z_3} \end{bmatrix}, \tag{8.6.133}$$

it follows that

$$\begin{aligned}
\begin{bmatrix} \vec{f}_{\mathrm{r}/\mathcal{L}_1/z_1} \\ \vec{f}_{\mathrm{r}/\mathcal{L}_2/z_2} \\ \vec{f}_{\mathrm{r}/\mathcal{L}_3/z_3} \end{bmatrix} &= \begin{bmatrix} \vec{I} & -\vec{I} & 0 \\ 0 & \vec{I} & -\vec{I} \\ 0 & 0 & \vec{I} \end{bmatrix}^{-1} \begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \\ m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} \end{bmatrix} \\
&= \begin{bmatrix} \vec{I} & \vec{I} & \vec{I} \\ 0 & \vec{I} & \vec{I} \\ 0 & 0 & \vec{I} \end{bmatrix} \begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} \\ m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} \end{bmatrix} \\
&= \begin{bmatrix} m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} + m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} + m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} \\ m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} + m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} \\ m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}} \end{bmatrix}.
\end{aligned} \tag{8.6.134}$$

Substituting (8.6.134) in (8.6.127)–(8.6.129) yields

$$\begin{aligned}
\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = {}& \vec{r}_{\mathrm{c}_1/z_1} \times (m_1 \vec{a}_{\mathrm{c}_1/z_1/\mathrm{A}} + m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} + m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}}) \\
& - \vec{r}_{\mathrm{c}_1/z_2} \times (m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} + m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}}),
\end{aligned} \tag{8.6.135}$$

$$\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = \vec{r}_{\mathrm{c}_2/z_2} \times (m_2 \vec{a}_{\mathrm{c}_2/z_1/\mathrm{A}} + m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}}) - \vec{r}_{\mathrm{c}_2/z_3} \times m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}}, \tag{8.6.136}$$

$$\vec{J}_3 \overset{\mathrm{A}\bullet}{\vec{\omega}}_3 = \vec{r}_{\mathrm{c}_3/z_3} \times m_3 \vec{a}_{\mathrm{c}_3/z_1/\mathrm{A}}. \tag{8.6.137}$$

Finally, substituting

$$\vec{a}_{\mathrm{c}_i/z_i/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{\omega}}_i \times \vec{r}_{\mathrm{c}_i/z_i} + \vec{\omega}_i \times (\vec{\omega}_i \times \vec{r}_{\mathrm{c}_i/z_i}) \tag{8.6.138}$$

$$\vec{a}_{z_{i+1}/z_i/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{\omega}}_i \times \vec{r}_{z_{i+1}/z_i} + \vec{\omega}_i \times (\vec{\omega}_i \times \vec{r}_{z_{i+1}/z_i}) \tag{8.6.139}$$

in (8.6.135), (8.6.136), and (8.6.137) yields

$$\begin{aligned}
\vec{J}_1 \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 = {}& \Big(m_1(\vec{r}\,'_{\mathrm{c}_1/z_1} \vec{r}_{\mathrm{c}_1/z_1} \vec{I} - \vec{r}_{\mathrm{c}_1/z_1} \vec{r}\,'_{\mathrm{c}_1/z_1}) + m_2(\vec{r}\,'_{z_2/z_1} \vec{r}_{z_2/z_1} \vec{I} - \vec{r}_{z_2/z_1} \vec{r}\,'_{z_2/z_1}) + m_3(\vec{r}\,'_{z_2/z_1} \vec{r}_{z_2/z_1} \vec{I} - \vec{r}_{z_2/z_1} \vec{r}\,'_{z_2/z_1}) \Big) \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \\
& + \Big(m_2(\vec{r}\,'_{z_2/z_1} \vec{r}_{\mathrm{c}_2/z_2} \vec{I} - \vec{r}_{\mathrm{c}_2/z_2} \vec{r}\,'_{z_2/z_1}) + m_3(\vec{r}\,'_{z_2/z_1} \vec{r}_{z_3/z_2} \vec{I} - \vec{r}_{z_3/z_2} \vec{r}\,'_{z_2/z_1}) \Big) \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \\
& + m_3 \left(\vec{r}\,'_{z_2/z_1} \vec{r}_{\mathrm{c}_3/z_3} \vec{I} - \vec{r}_{\mathrm{c}_3/z_3} \vec{r}\,'_{z_2/z_1} \right) \overset{\mathrm{A}\bullet}{\vec{\omega}}_3 \\
& + m_2 \vec{r}_{z_2/z_1} \times (\vec{\omega}_2 \times (\vec{\omega}_2 \times \vec{r}_{\mathrm{c}_2/z_2})) + m_3 \vec{r}_{z_2/z_1} \times (\vec{\omega}_3 \times (\vec{\omega}_3 \times \vec{r}_{\mathrm{c}_3/z_3})) + m_3 \vec{r}_{z_2/z_1} \times (\vec{\omega}_2 \times (\vec{\omega}_2 \times \vec{r}_{z_3/z_2})),
\end{aligned} \tag{8.6.140}$$

$$\begin{aligned}
\vec{J}_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 = {}& m_3(\vec{r}\,'_{z_3/z_2} \vec{r}_{\mathrm{c}_3/z_3} \vec{I} - \vec{r}_{\mathrm{c}_3/z_3} \vec{r}\,'_{z_3/z_2}) \overset{\mathrm{A}\bullet}{\vec{\omega}}_3 \\
& + \Big(m_2(\vec{r}\,'_{\mathrm{c}_2/z_2} \vec{r}_{\mathrm{c}_2/z_2} \vec{I} - \vec{r}_{\mathrm{c}_2/z_2} \vec{r}\,'_{\mathrm{c}_2/z_2}) + m_3(\vec{r}\,'_{z_3/z_2} \vec{r}_{z_3/z_2} \vec{I} - \vec{r}_{z_3/z_2} \vec{r}\,'_{z_3/z_2}) \Big) \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \\
& + \Big(m_2(\vec{r}\,'_{\mathrm{c}_2/z_2} \vec{r}_{z_2/z_1} \vec{I} - \vec{r}_{z_2/z_1} \vec{r}\,'_{\mathrm{c}_2/z_2}) + m_3(\vec{r}\,'_{z_3/z_2} \vec{r}_{z_2/z_1} \vec{I} - \vec{r}_{z_2/z_1} \vec{r}\,'_{z_3/z_2}) \Big) \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 +
\end{aligned}$$

$$m_3\vec{r}_{z_3/z_2}\times(\vec{\omega}_3\times(\vec{\omega}_3\times\vec{r}_{c_3/z_3}))+(m_2\vec{r}_{c_2/z_2}+m_3\vec{r}_{z_3/z_2})\times(\vec{\omega}_1\times(\vec{\omega}_1\times\vec{r}_{z_2/z_1})), \quad (8.6.141)$$

$$\begin{aligned}\vec{J}_3\overset{\mathrm{A}\bullet}{\vec{\omega}}_3 &= m_3\left(\vec{r}'_{c_3/z_3}\vec{r}_{c_3/z_3}\vec{I}-\vec{r}_{c_3/z_3}\vec{r}'_{c_3/z_3}\right)\overset{\mathrm{A}\bullet}{\vec{\omega}}_3+m_3\left(\vec{r}'_{c_3/z_3}\vec{r}_{z_3/z_2}\vec{I}-\vec{r}_{z_3/z_2}\vec{r}'_{c_3/z_3}\right)\overset{\mathrm{A}\bullet}{\vec{\omega}}_2\\ &\quad+m_3\left(\vec{r}'_{c_3/z_3}\vec{r}_{z_2/z_1}\vec{I}-\vec{r}_{z_2/z_1}\vec{r}'_{c_3/z_3}\right)\overset{\mathrm{A}\bullet}{\vec{\omega}}_1\\ &\quad+m_3\vec{r}_{c_3/z_3}\times(\vec{\omega}_2\times(\vec{\omega}_2\times\vec{r}_{z_3/z_2}))+m_3\vec{r}_{c_3/z_3}\times(\vec{\omega}_1\times(\vec{\omega}_1\times\vec{r}_{z_2/z_1})). \end{aligned} \quad (8.6.142)$$

which can be written as

$$A\begin{bmatrix}\overset{\mathrm{A}\bullet}{\vec{\omega}}_1\\ \overset{\mathrm{A}\bullet}{\vec{\omega}}_2\\ \overset{\mathrm{A}\bullet}{\vec{\omega}}_3\end{bmatrix}-b=0, \quad (8.6.143)$$

where

$$\begin{aligned}A_{11} &\triangleq \vec{J}_1-m_1(\vec{r}'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{I}-\vec{r}_{c_1/z_1}\vec{r}'_{c_1/z_1})-m_2(\vec{r}'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I}-\vec{r}_{z_2/z_1}\vec{r}'_{z_2/z_1})\\ &\quad-m_3(\vec{r}'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I}-\vec{r}_{z_2/z_1}\vec{r}'_{z_2/z_1})\end{aligned} \quad (8.6.144)$$

$$A_{12}\triangleq -m_2(\vec{r}'_{z_2/z_1}\vec{r}_{c_2/z_2}\vec{I}-\vec{r}_{c_2/z_2}\vec{r}'_{z_2/z_1})-m_3(\vec{r}'_{z_2/z_1}\vec{r}_{z_3/z_2}\vec{I}-\vec{r}_{z_3/z_2}\vec{r}'_{z_2/z_1}), \quad (8.6.145)$$

$$A_{13}\triangleq -m_3(\vec{r}'_{z_2/z_1}\vec{r}_{c_3/z_3}\vec{I}-\vec{r}_{c_3/z_3}\vec{r}'_{z_2/z_1}), \quad (8.6.146)$$

$$A_{21}\triangleq -m_2(\vec{r}'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{I}-\vec{r}_{z_2/z_1}\vec{r}'_{c_2/z_2})-m_3(\vec{r}'_{z_3/z_2}\vec{r}_{z_2/z_1}\vec{I}-\vec{r}_{z_2/z_1}\vec{r}'_{z_3/z_2}), \quad (8.6.147)$$

$$A_{22}\triangleq \vec{J}_2-m_2(\vec{r}'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{I}-\vec{r}_{c_2/z_2}\vec{r}'_{c_2/z_2})-m_3(\vec{r}'_{z_3/z_2}\vec{r}_{z_3/z_2}\vec{I}-\vec{r}_{z_3/z_2}\vec{r}'_{z_3/z_2}), \quad (8.6.148)$$

$$A_{23}\triangleq -m_3(\vec{r}'_{z_3/z_2}\vec{r}_{c_3/z_3}\vec{I}-\vec{r}_{c_3/z_3}\vec{r}'_{z_3/z_2}), \quad (8.6.149)$$

$$A_{31}\triangleq -m_3(\vec{r}'_{c_3/z_3}\vec{r}_{z_2/z_1}\vec{I}-\vec{r}_{z_2/z_1}\vec{r}'_{c_3/z_3}), \quad (8.6.150)$$

$$A_{32}\triangleq -m_3(\vec{r}'_{c_3/z_3}\vec{r}_{z_3/z_2}\vec{I}-\vec{r}_{z_3/z_2}\vec{r}'_{c_3/z_3}), \quad (8.6.151)$$

$$A_{33}\triangleq \vec{J}_3-m_3(\vec{r}'_{c_3/z_3}\vec{r}_{c_3/z_3}\vec{I}-\vec{r}_{c_3/z_3}\vec{r}'_{c_3/z_3}). \quad (8.6.152)$$

and

$$\begin{aligned}b_1 &\triangleq m_2\vec{r}_{z_2/z_1}\times(\vec{\omega}_2\times(\vec{\omega}_2\times\vec{r}_{c_2/z_2}))+m_3\vec{r}_{z_2/z_1}\times(\vec{\omega}_2\times(\vec{\omega}_2\times\vec{r}_{z_3/z_2}))\\ &\quad+m_3\vec{r}_{z_2/z_1}\times(\vec{\omega}_3\times(\vec{\omega}_3\times\vec{r}_{c_3/z_3})),\end{aligned} \quad (8.6.153)$$

$$b_2\triangleq m_3\vec{r}_{z_3/z_2}\times(\vec{\omega}_3\times(\vec{\omega}_3\times\vec{r}_{c_3/z_3}))+(m_2\vec{r}_{c_2/z_2}+m_3\vec{r}_{z_3/z_2})\times(\vec{\omega}_1\times(\vec{\omega}_1\times\vec{r}_{z_2/z_1})), \quad (8.6.154)$$

$$b_3\triangleq m_3\vec{r}_{c_3/z_3}\times(\vec{\omega}_2\times(\vec{\omega}_2\times\vec{r}_{z_3/z_2}))+m_3\vec{r}_{c_3/z_3}\times(\vec{\omega}_1\times(\vec{\omega}_1\times\vec{r}_{z_2/z_1})). \quad (8.6.155)$$

8.7 Collisions†

Consider a body $\mathcal{B}$ consisting of particles y_1 and y_2 whose masses are m_1 and m_2, respectively, let c be the center of mass of $\mathcal{B}$, let w be a point with zero inertial acceleration, and let $\mathrm{F_A}$ be an inertial frame. It thus follows from (7.5.3) and (7.5.4) that

$$(m_1+m_2)\vec{v}_{c/w/\mathrm{A}}=m_1\vec{v}_{y_1/w/\mathrm{A}}+m_2\vec{v}_{y_2/w/\mathrm{A}}, \quad (8.7.1)$$

$$(m_1 + m_2)\vec{a}_{\mathrm{c}/w/\mathrm{A}} = m_1\vec{a}_{y_1/w/\mathrm{A}} + m_2\vec{a}_{y_2/w/\mathrm{A}}. \tag{8.7.2}$$

Assuming that the external force on $\mathcal{B}$ is zero, it follows that $\vec{a}_{\mathrm{c}/w/\mathrm{A}} = 0$. It thus follows from (8.7.1) and (8.7.2) that

$$\overbrace{m_1\vec{v}_{y_1/w/\mathrm{A}} + m_2\vec{v}_{y_2/w/\mathrm{A}}}^{\mathrm{A}\bullet} = 0, \tag{8.7.3}$$

which shows that the translational momentum of $\mathcal{B}$ is conserved with respect to $\mathrm{F_A}$.

Next, writing (8.7.3) as

$$\overbrace{m_1\ \vec{v}_{y_1/w/\mathrm{A}}\Big|_{\mathrm{A}} + m_2\ \vec{v}_{y_2/w/\mathrm{A}}\Big|_{\mathrm{A}}}^{\cdot} = 0 \tag{8.7.4}$$

and letting

$$\vec{v}_{y_1/w/\mathrm{A}}(t)\Big|_{\mathrm{A}} = \begin{bmatrix} u_1(t) \\ v_1(t) \\ w_1(t) \end{bmatrix}, \qquad \vec{v}_{y_2/w/\mathrm{A}}(t)\Big|_{\mathrm{A}} = \begin{bmatrix} u_2(t) \\ v_2(t) \\ w_2(t) \end{bmatrix}, \tag{8.7.5}$$

it follows that, for all times t_1 and t_2,

$$m_1\begin{bmatrix} u_1(t_1) \\ v_1(t_1) \\ w_1(t_1) \end{bmatrix} + m_2\begin{bmatrix} u_2(t_1) \\ v_2(t_1) \\ w_2(t_1) \end{bmatrix} = m_1\begin{bmatrix} u_1(t_2) \\ v_1(t_2) \\ w_1(t_2) \end{bmatrix} + m_2\begin{bmatrix} u_2(t_2) \\ v_2(t_2) \\ w_2(t_2) \end{bmatrix}. \tag{8.7.6}$$

Now, we view y_1 and y_2 as small spheres, and assume that y_1 and y_2 collide and thus are in direct contact during a short time interval centered at time t_c, where $t_1 < t_\mathrm{c} < t_2$. When y_1 and y_2 are in direct contact, it follows from Newton's third law that equal and opposite reaction forces $\vec{f}_{\mathrm{r}/r/y_1}$ and $f_{\mathrm{r}/y_2} = -f_{\mathrm{r}/y_1}$ are applied to the particles y_1 and y_2, respectively, at the point of direct contact. Therefore, during direct contact, it follows that

$$m_1\vec{a}_{y_1/w/\mathrm{A}} = \vec{f}_{\mathrm{r}/r/y_1}, \tag{8.7.7}$$

$$m_2\vec{a}_{y_2/w/\mathrm{A}} = -\vec{f}_{\mathrm{r}/r/y_1}. \tag{8.7.8}$$

For convenience, we assume that these reaction forces are parallel with $\hat{\imath}_\mathrm{B}$, and thus the force on the particles in the directions of $\hat{\jmath}_\mathrm{A}$ and $\hat{k}_\mathrm{A}$ are zero. Therefore, the components of $\vec{v}_{y_1/w/\mathrm{A}}$ and $\vec{v}_{y_2/w/\mathrm{A}}$ in the directions $\hat{\jmath}_\mathrm{A}$ and $\hat{k}_\mathrm{A}$ are constant, that is,

$$v_1(t_1) = v_1(t_2), \qquad v_2(t_1) = v_2(t_2), \tag{8.7.9}$$

$$w_1(t_1) = w_1(t_2), \qquad w_2(t_1) = w_2(t_2). \tag{8.7.10}$$

In the direction $\hat{\imath}_\mathrm{A}$ it follows from (8.7.6) that

$$m_1u_1(t_1) + m_2u_2(t_1) = m_1u_1(t_2) + m_2u_2(t_2). \tag{8.7.11}$$

Next, define

$$u_\mathrm{c} \triangleq u_1(t_\mathrm{c}) = u_2(t_\mathrm{c}), \tag{8.7.12}$$

which is the common speed of y_1 and y_2 in the direction $\hat{\imath}_\mathrm{A}$ at t_c. Therefore, by conservation of translational momentum, it follows from (8.7.11) that

$$m_1u_1(t_1) + m_2u_2(t_1) = m_1u_\mathrm{c} + m_2u_\mathrm{c} = m_1u_1(t_2) + m_2u_2(t_2). \tag{8.7.13}$$

Hence,

$$m_1u_1(t_2) - m_1u_c = -[m_2u_2(t_2) - m_2u_c] \tag{8.7.14}$$

and

$$m_1u_c - m_1u_1(t_1) = -[m_2u_c - m_2u_2(t_1)]. \tag{8.7.15}$$

Dividing the left- and right-hand sides of (8.7.14) and (8.7.15) yields

$$e \triangleq \frac{u_1(t_2) - u_c}{u_c - u_1(t_1)} = \frac{u_2(t_2) - u_c}{u_c - u_2(t_1)}, \tag{8.7.16}$$

where e is the *coefficient of restitution*.

Next, it follows from (8.7.16) that

$$u_c = \frac{u_1(t_1)u_2(t_2) - u_1(t_2)u_2(t_1)}{u_1(t_1) + u_2(t_2) - u_1(t_2) - u_2(t_1)}. \tag{8.7.17}$$

Substituting this expression into (8.7.16) yields

$$e = \frac{u_1(t_2) - u_2(t_2)}{u_2(t_1) - u_1(t_1)}. \tag{8.7.18}$$

Hence, e is the ratio of the relative velocities of y_1 and y_2 in the direction $\hat{\imath}_A$ before and after the collision. In particular, $e = 0$ corresponds to the case in which the particles are stuck together after the collision, whereas $e = 1$ captures the case in which the relative speed reverses its sign.

For the following result, let $T_{\mathcal{B}/w/A}(t)$ be the kinetic energy of $\mathcal{B}$ relative to w with respect to F_A, which is given by

$$T_{\mathcal{B}/w/A}(t) = \tfrac{1}{2}m_1[u_1^2(t) + v_1^2(t) + w_1^2(t)] + \tfrac{1}{2}m_2[u_2^2(t) + v_2^2(t) + w_2^2(t)]). \tag{8.7.19}$$

Fact 8.7.1. Let y_1 and y_2 be small spheres whose masses are m_1 and m_2, respectively, let w be a point with zero inertial acceleration, and let F_A be an inertial frame. Assume that a collision between y_1 and y_2 occurs at time t_c, assume that the reaction forces are parallel with $\hat{\imath}_A$, let $u_1(t)$ and $u_2(t)$, respectively, be the speeds of y_1 and y_2 in the direction $\hat{\imath}_A$, that is, $u_1(t) = \hat{\imath}_A' \vec{v}_{y_1/w/A}(t)$ and $u_2(t) = \hat{\imath}_A' \vec{v}_{y_2/w/A}(t)$, and let $t_1 < t_c < t_2$. Then,

$$u_1(t_2) = \frac{m_1 - em_2}{m_1 + m_2}u_1(t_1) + \frac{(1+e)m_2}{m_1 + m_2}u_2(t_1), \tag{8.7.20}$$

$$u_2(t_2) = \frac{(1+e)m_1}{m_1 + m_2}u_1(t_1) + \frac{m_2 - em_1}{m_1 + m_2}u_2(t_1). \tag{8.7.21}$$

Furthermore,

$$T_{\mathcal{B}/w/A}(t_2) = T_{\mathcal{B}/w/A}(t_1) - (1 - e^2)\frac{m_1m_2}{m_1 + m_2}[u_1(t_1) - u_2(t_1)]^2. \tag{8.7.22}$$

Finally,

$$0 \le e \le 1. \tag{8.7.23}$$

Proof. Using (8.7.11) and (8.7.18) yields (8.7.20) and (8.7.21), which in turn imply (8.7.22). To prove (8.7.23), we consider the case where y_1 and y_2 move along a line that is parallel with $\hat{\imath}_A$. Consider the case where $\hat{\imath}_A' \vec{r}_{y_1/y_2}(t_1) > 0$, $u_2(t_1) > 0$, and $u_1(t_1) < u_2(t_1)$. With these relative locations and speeds, a collision occurs at some time $t_c > t_1$. At time t_c, $\hat{\imath}_A' \vec{r}_{y_1/y_2}(t_c) = 0$ and

$u_1(t_c) = u_2(t_c)$. Furthermore, the reaction force $\vec{f}_{r/r/y_1}$ on y_1 is given by $\vec{f}_{r/r/y_1} = f_r\hat{\imath}_A$, where $f_r > 0$, whereas the reaction force $\vec{f}_{r/y_2} = -\vec{f}_{r/r/y_1}$ on y_2 is given by $-\vec{f}_{r/r/y_1} = -f_r\hat{\imath}_A$, where $-f_r < 0$. Consequently, at time t_2 after the collision, it follows that $\hat{\imath}'_A \vec{r}_{y_1/y_2}(t_2) > 0$ and $u_2(t_2) \le u_1(t_2)$, where equality holds if and only if the particles stick together. Since $u_1(t_2) - u_2(t_2) \ge 0$ and $u_2(t_1) - u_1(t_1) > 0$, it follows that $e \ge 0$. An analogous argument applies in the case where $\hat{\imath}'_A \vec{r}_{y_2/y_1}(t_1) > 0$, $u_2(t_1) > 0$, and $u_2(t_1) < u_1(t_1)$. In both cases, $e \ge 0$. Using (8.7.22), conservation of energy implies that $e \le 1$.

The case where y_1 and y_2 are not necessarily moving along the same line is left to the reader. □

Note that the second term on the right-hand side of (8.7.22) represents the amount of energy dissipated in the collision. Since the occurrence of a collision implies that $u_1(t_1) \ne u_2(t_1)$, it follows that energy is conserved if and only if $e = 1$. Furthermore, the amount of dissipated energy is larger for smaller values of e.

8.8 Center of Percussion and Percussive Center of Rotation†

It follows from (7.5.5) that a force on a rigid body produces an acceleration of the center of mass of the body. Furthermore, it follows from (7.9.13) that, if the line of force does not pass through the center of mass of the body, then the moment on the body relative to the center of mass is nonzero and produces an angular acceleration of the body. It is reasonable to expect that the combined translational acceleration of the center of mass and angular acceleration of the body induces a change in the velocity of every point on the body. If, however, an impulsive force is applied to a point P in the body, then it turns out that there exists a point R in the body that does not initially accelerate. The point P is the *center of percussion*, and the point R is the *percussive center of rotation.* This property is made precise by the following result. The notation "0^+" denotes the limit toward zero through positive numbers, which can be viewed as the response immediately after an impulse at time $t = 0$.

Fact 8.8.1. Let $\mathcal{B}$ be a rigid body with body-fixed frame $\mathrm{F_B}$, let $\mathrm{F_A}$ be an inertial frame, assume that $\vec{\omega}_{B/A}(0) = 0$, let w be a point with zero inertial acceleration, let P be a point that is fixed in $\mathcal{B}$, let $\vec{f}(t) = f_0\delta(t)\hat{n}_P$ denote an impulsive force on $\mathcal{B}$ at P, and assume that $\hat{n}_P$ and $\vec{r}_{P/c}$ are not parallel and $\hat{n}_P$ is a principal axis of $\vec{J}_{\mathcal{B}/c}$. Then, there exists a point R fixed in $\mathcal{B}$ such that $\vec{v}_{R/w/A}(0^+) = 0$. In particular, the location of one such point R is given by

$$\vec{r}_{R/c} = \frac{1}{m|\vec{J}^{-1}_{\mathcal{B}/c}(\vec{r}_{P/c} \times \hat{n}_P)|^2}[\vec{J}^{-1}_{\mathcal{B}/c}(\vec{r}_{P/c} \times \hat{n}_P)] \times \hat{n}_P. \tag{8.8.1}$$

Proof. It follows from (7.5.5) that

$$m\vec{v}_{c/w/A}(0^+) = f_0\hat{n}_P \tag{8.8.2}$$

and, since $\vec{\omega}_{B/A}(0) = 0$, it follows from (7.9.13) that

$$\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A}(0^+) = f_0\,\vec{r}_{P/c} \times \hat{n}_P. \tag{8.8.3}$$

Using (8.8.2) and (8.8.3) yields

$$\vec{v}_{R/w/A}(0^+) = \vec{v}_{R/c/A}(0^+) + \vec{v}_{c/w/A}(0^+)$$

$$
\begin{aligned}
&= \vec{\omega}_{\mathrm{B/A}}(0^+) \times \vec{r}_{\mathrm{R/c}} + \vec{v}_{\mathrm{c}/w/\mathrm{A}}(0^+) \\
&= [f_0 \vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})] \times \vec{r}_{\mathrm{R/c}} + \vec{v}_{\mathrm{c}/w/\mathrm{A}}(0^+) \\
&= f_0(\vec{a} \times \vec{r}_{\mathrm{R/c}} + \frac{1}{m}\hat{n}_{\mathrm{P}}),
\end{aligned} \tag{8.8.4}
$$

where $\vec{a} \triangleq \vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}})$. Since, by assumption, $\hat{n}_{\mathrm{P}}$ and $\vec{r}_{\mathrm{P/c}}$ are not parallel, it follows that $\vec{a}$ is not zero. In addition, since, by assumption, $\hat{n}_{\mathrm{P}}$ is a principal axis of $\vec{J}_{\mathcal{B}/\mathrm{c}}$, it follows that there exists $\gamma > 0$ such that $\vec{J}_{\mathcal{B}/\mathrm{c}}\hat{n}_{\mathrm{P}} = \gamma\hat{n}_{\mathrm{P}}$, and thus $\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}\hat{n}_{\mathrm{P}} = (1/\gamma)\hat{n}_{\mathrm{P}}$. Therefore,

$$
\hat{n}'_{\mathrm{P}}\vec{a} = \hat{n}'_{\mathrm{P}}\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}) = (1/\gamma)\hat{n}'_{\mathrm{P}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}) = 0. \tag{8.8.5}
$$

Since $\vec{a}$ and $\hat{n}_{\mathrm{P}}$ are mutually orthogonal, it follows that $\vec{a} \times \hat{n}_{\mathrm{P}}$ is not zero. Hence, let R be the point that is fixed in $\mathcal{B}$ such that

$$
\vec{r}_{\mathrm{R/c}} = \alpha\vec{a} \times \hat{n}_{\mathrm{P}}, \tag{8.8.6}
$$

where $\alpha \triangleq 1/(m|\vec{a}|^2)$. It thus follows from (8.8.5) that

$$
\begin{aligned}
\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^+) &= f_0(\vec{a} \times \vec{r}_{\mathrm{R/c}} + \frac{1}{m}\hat{n}_{\mathrm{P}}) \\
&= f_0[\vec{a} \times (\alpha\vec{a} \times \hat{n}_{\mathrm{P}}) + \frac{1}{m}\hat{n}_{\mathrm{P}}] \\
&= f_0(-\alpha|\vec{a}|^2\hat{n}_{\mathrm{P}} + \frac{1}{m}\hat{n}_{\mathrm{P}}) = 0,
\end{aligned}
$$

which shows that R is the percussive center of rotation of $\mathcal{B}$. □

Example 8.8.2. Consider a free rigid body $\mathcal{B}$ consisting of particles y_1 and y_2 whose masses are m_1 and m_2, respectively, connected by a massless rigid link of length ℓ. The body frame is chosen such that $\vec{r}_{y_2/y_1} = \ell\hat{\imath}_{\mathrm{B}}$. The location of the center of mass is $\vec{r}_{\mathrm{c}/y_1} = \ell_{\mathrm{c}}\hat{\imath}_{\mathrm{B}}$, where $\ell_{\mathrm{c}} \triangleq \frac{m_2\ell}{m_1+m_2}$, and the physical inertia matrix is given by

$$
\vec{J}_{\mathcal{B}/\mathrm{c}} = \varepsilon\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + \beta(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}), \tag{8.8.7}
$$

where ε is a small positive number that allows inversion of $\vec{J}_{\mathcal{B}/\mathrm{c}}$ and

$$
\beta \triangleq m_1\ell_{\mathrm{c}}^2 + m_2(\ell - \ell_{\mathrm{c}})^2 = \frac{m_1m_2\ell^2}{m_1 + m_2}. \tag{8.8.8}
$$

Hence,

$$
\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}} = \frac{1}{\varepsilon}\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + \frac{1}{\beta}(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}). \tag{8.8.9}
$$

The impulsive force on $\mathcal{B}$ at the point P is given by $\vec{f}(t) = f_0\delta(t)\hat{n}_{\mathrm{P}}$, where $\hat{n}_{\mathrm{P}} = \hat{\jmath}_{\mathrm{B}}$ and $\vec{r}_{\mathrm{P}/y_1} = \ell_{\mathrm{P}}\hat{\imath}_{\mathrm{B}}$. It thus follows that

$$
\begin{aligned}
\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}} \times \hat{n}_{\mathrm{P}}) &= \left[\frac{1}{\varepsilon}\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}} + \frac{1}{\beta}(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}} + \hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}})\right][(\ell_{\mathrm{P}} - \ell_{\mathrm{c}})\hat{\imath}_{\mathrm{B}} \times \hat{\jmath}_{\mathrm{B}}] \\
&= \frac{\ell_{\mathrm{P}} - \ell_{\mathrm{c}}}{\beta}\hat{k}_{\mathrm{B}}.
\end{aligned} \tag{8.8.10}
$$

Therefore, it follows from (8.8.1) that

$$\begin{aligned}\vec{r}_{\mathrm{R/c}} &= \frac{1}{(m_1+m_2)\left|\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_{\mathrm{P}})\right|^2}[\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_{\mathrm{P}})]\times\hat{n}_{\mathrm{P}}\\ &= -\frac{\beta^2}{(m_1+m_2)(\ell_{\mathrm{P}}-\ell_{\mathrm{c}})^2}\frac{\ell_{\mathrm{P}}-\ell_{\mathrm{c}}}{\beta}\hat{\imath}_{\mathrm{B}}\\ &= \frac{m_1m_2\ell^2}{(m_1+m_2)^2(\ell_{\mathrm{c}}-\ell_{\mathrm{P}})}\hat{\imath}_{\mathrm{B}}. \end{aligned}\tag{8.8.11}$$

To confirm this result, note that

$$\begin{aligned}\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^+) &= f_0([\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_{\mathrm{P}})]\times\vec{r}_{\mathrm{R/c}}+\frac{1}{m_1+m_2}\hat{n}_{\mathrm{P}})\\ &= f_0\left(\frac{(m_1+m_2)(\ell_{\mathrm{P}}-\ell_{\mathrm{c}})}{m_1m_2\ell^2}\hat{k}_{\mathrm{B}}\times\frac{m_1m_2\ell^2}{(m_1+m_2)^2(\ell_{\mathrm{c}}-\ell_{\mathrm{P}})}\hat{\imath}_{\mathrm{B}}+\frac{1}{m_1+m_2}\hat{\jmath}_{\mathrm{B}}\right)\\ &= 0.\end{aligned}\tag{8.8.12}$$

As a special case, letting $\mathrm{P}=y_2$ yields $\vec{r}_{\mathrm{R/c}}=-\ell_{\mathrm{c}}\hat{\imath}_{\mathrm{B}}$, and thus the percussive center of rotation is y_1. ◇

Example 8.8.3. Consider a rigid body $\mathcal{B}$ consisting of a uniform thin bar of length ℓ and mass m. Determine the percussive center of rotation.

Solution. Let y_1 and y_2 denote the endpoints of the bar, and let the body frame be chosen such that $\vec{r}_{y_2/y_1}=\ell\hat{\imath}_{\mathrm{B}}$. The location of the center of mass is $\vec{r}_{\mathrm{c}/y_1}=\frac{\ell}{2}\hat{\imath}_{\mathrm{B}}$, and the physical inertia matrix is given by

$$\vec{J}_{\mathcal{B}/\mathrm{c}}=\varepsilon\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}}+\beta(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}}+\hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}),\tag{8.8.13}$$

where ε is a small positive number that allows inversion of $\vec{J}_{\mathcal{B}/\mathrm{c}}$ and

$$\beta\triangleq\frac{1}{12}m\ell^2.\tag{8.8.14}$$

Hence,

$$\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}=\frac{1}{\varepsilon}\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}}+\frac{1}{\beta}(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}}+\hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}}).\tag{8.8.15}$$

The impulsive force on $\mathcal{B}$ at the point P is given by $\vec{f}(t)=f_0\delta(t)\hat{n}_{\mathrm{P}}$, where $\hat{n}_{\mathrm{P}}=\hat{\jmath}_{\mathrm{B}}$ and $\vec{r}_{\mathrm{P}/y_1}=\ell_{\mathrm{P}}\hat{\imath}_{\mathrm{B}}$. It thus follows that

$$\begin{aligned}\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_{\mathrm{P}}) &= \left[\frac{1}{\varepsilon}\hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}}+\frac{1}{\beta}(\hat{\jmath}_{\mathrm{B}}\hat{\jmath}'_{\mathrm{B}}+\hat{k}_{\mathrm{B}}\hat{k}'_{\mathrm{B}})\right][(\ell_{\mathrm{P}}-\ell/2)\hat{\imath}_{\mathrm{B}}\times\hat{\jmath}_{\mathrm{B}}]\\ &= \frac{12(\ell_{\mathrm{P}}-\ell/2)}{m\ell^2}\hat{k}_{\mathrm{B}}.\end{aligned}\tag{8.8.16}$$

Therefore, it follows from (8.8.1) that

$$\vec{r}_{\mathrm{R/c}}=\frac{1}{m|\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_{\mathrm{P}})|^2}[\vec{J}^{-1}_{\mathcal{B}/\mathrm{c}}(\vec{r}_{\mathrm{P/c}}\times\hat{n}_{\mathrm{P}})]\times\hat{n}_{\mathrm{P}}$$

$$
\begin{aligned}
&= -\frac{m^2\ell^4}{144m(\ell_{\mathrm{P}} - \ell/2)^2}\frac{12(\ell_{\mathrm{P}} - \ell/2)}{m\ell^2}\hat{\imath}_{\mathrm{B}} \\
&= \frac{\ell^2}{12(\ell/2 - \ell_{\mathrm{P}})}\hat{\imath}_{\mathrm{B}}. \qquad (8.8.17)
\end{aligned}
$$

To confirm this result, note that

$$
\begin{aligned}
\vec{v}_{\mathrm{R}/w/\mathrm{A}}(0^+) &= f_0([\vec{J}_{\mathcal{B}/\mathrm{c}}^{-1}(\vec{r}_{\mathrm{P}/\mathrm{c}} \times \hat{n}_{\mathrm{P}})] \times \vec{r}_{\mathrm{R}/\mathrm{c}} + \frac{1}{m_1 + m_2}\hat{n}_{\mathrm{P}}) \\
&= f_0\left(\frac{12(\ell_{\mathrm{P}} - \ell_{\mathrm{c}})}{m\ell^2}\hat{k}_{\mathrm{B}} \times \frac{\ell^2}{12(\ell_{\mathrm{c}} - \ell_{\mathrm{P}})}\hat{\imath}_{\mathrm{B}} + \frac{1}{m}\hat{\jmath}_{\mathrm{B}}\right) \\
&= 0. \qquad (8.8.18)
\end{aligned}
$$

As a special case, letting $\ell_{\mathrm{P}} = \frac{2}{3}\ell$ yields $\vec{r}_{\mathrm{R}/\mathrm{c}} = -\frac{1}{2}\ell\hat{\imath}_{\mathrm{B}}$, and thus the percussive center of rotation is y_1. ◇

8.9 Applied Problems

Problem 8.9.1. In Figure 8.6.22, the cart moves in a straight line along the ground with a prescribed acceleration relative to a point in the ground and with respect to an inertial frame. The angle between the surface of the cart and the ground is θ. A ball with mass m and radius r rolls on the top of the cart. Determine the reaction force between the ball and the cart as well as the acceleration of the ball relative to the cart with respect to an inertial frame.

Problem 8.9.2. In Figure 8.9.1, the ball y under the influence of uniform gravity rolls without slipping down a slanted surface of a moving cart. The angle between the slanted surface and the ground is θ, the distance between y and the point x is d_1, and the distance between the vertical edge of the cart and a point fixed in the ground is d_2. The mass of the ball is m, the radius of the ball is r, and the mass of the cart is M. The cart is mounted on frictionless, massless wheels, which allow the cart to translate on the ground. Derive the equations of motion for the ball and the cart in terms of d_1 and d_2, and determine the normal and tangential components of the reaction force between the ball and the slanted surface as well as the vertical reaction force between the slanted surface and the ground.

Problem 8.9.3. The rotating platform in Figure 8.9.2 is connected to the ground by a pin joint at the point w. The inertial frame $\mathrm{F_A}$ is fixed to the ground. The body $\mathcal{B}$ consists of a massless bar of length $2R$ with center at y and with identical small spheres of mass m mounted on each end. The distance from w to y is L. A motor spins the bar relative to the platform at the rate $\omega_1 > 0$, and the platform spins at the rate $\omega_2 > 0$ relative to the ground. Neither ω_1 nor ω_2 is necessarily constant, and the spin directions are shown in the figure.

i) Determine the net force $\vec{f}_{\mathcal{B}}$ on $\mathcal{B}$ resolved in a platform-fixed frame.

ii) Determine the angular momentum $\vec{H}_{\mathcal{B}/y/\mathrm{A}}$ of $\mathcal{B}$ relative to y with respect to $\mathrm{F_A}$ resolved in a platform-fixed frame.

iii) Determine the moment $\vec{M}_{\mathcal{B}/y}$ on $\mathcal{B}$ relative to y resolved in a platform-fixed frame.

Problem 8.9.4. The uniform disk and uniform shaft shown in Figure 8.9.3 are welded together

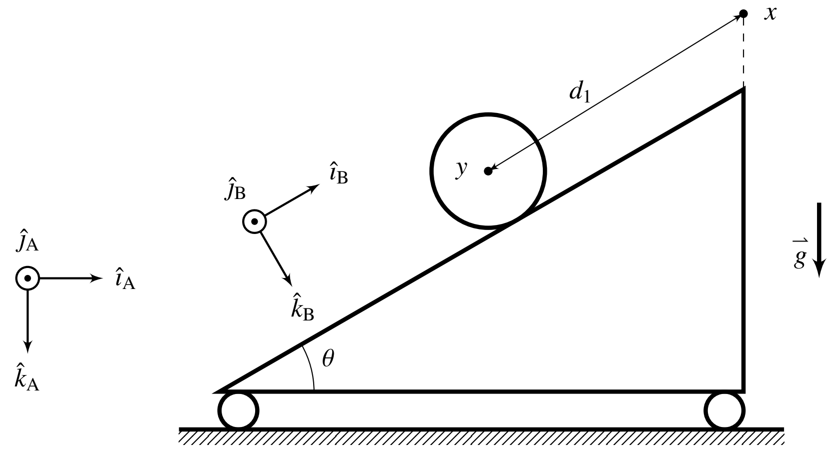


Figure 8.9.1: Ball rolling down the slanted surface of a moving cart for Problem 8.9.2.

in such a way that the angle between the shaft and the line perpendicular to the disk and passing through its center is θ. The distance from the left end of the shaft to the center of the disk is d. The mass, length, and radius of the shaft are m_1, l_1, and r_1, respectively, while the mass, thickness, and radius of the disk are m_2, l_2, and r_2, respectively. Resolve the physical inertia matrix of the assembly relative to its center of mass in a frame F_A whose axis $\hat{\imath}_A$ is codirectional with the longitudinal axis of the shaft and whose axis $\hat{\jmath}_A$ is such that the line perpendicular to the disk and passing through its center lies in the $\hat{\imath}_A$-$\hat{\jmath}_A$ plane. (Remark: Ignore the extra mass due to the intersection of the disk and shaft.)

Problem 8.9.5. The vertical rotating shaft in Figure 8.9.4 rotates at the constant rate $\Omega \geq 0$ in the direction shown. The cable, which is horizontal, supports a uniform bar of length l and mass m attached to the vertical shaft by a horizontal pin joint. The angle between the bar and the vertical direction is θ. Assume that the bearings that support the shaft are mounted on an inertially nonrotating massive body. Gravity acts in the direction shown. Determine the tension in the cable, the reaction force on the bar at point a, and the reaction torque on the vertical shaft at point a. (Hint: Consider the case $\Omega = 0$ first, and note that every point on the shaft has zero inertial acceleration.)

Problem 8.9.6. The massless rotating shaft in Figure 8.9.5 is welded to a cylinder with mass m so that the center of mass of the cylinder lies on the shaft and the angle between the shaft and centerline of the cylinder is θ. The ends of the shaft are prevented from moving transversely by bearings, which are mounted on an inertially nonrotating massive body. No uniform gravity is present, and a torque $\vec{T}$ is applied to the shaft. The length of the shaft is l, the radius and length of the cylinder are r and h, respectively, and the rotation rate of the assembly relative to the bearings is $\Omega > 0$, which is not necessarily constant. Determine the reaction forces on the shaft due to the bearings and resolve these vectors in a frame that is attached to the shaft and one of whose axes is parallel with the shaft.

Problem 8.9.7. The vertical rotating shaft in Figure 8.9.6 is welded to a horizontal arm, which is connected by a frictionless pin joint to a slanted bar, where the angle between the slanted bar and the horizontal direction is θ. The tip of the slanted bar slides over the ground as the shaft rotates in the direction shown at the constant angular rate $\Omega > 0$. Assume that the ground is an inertially

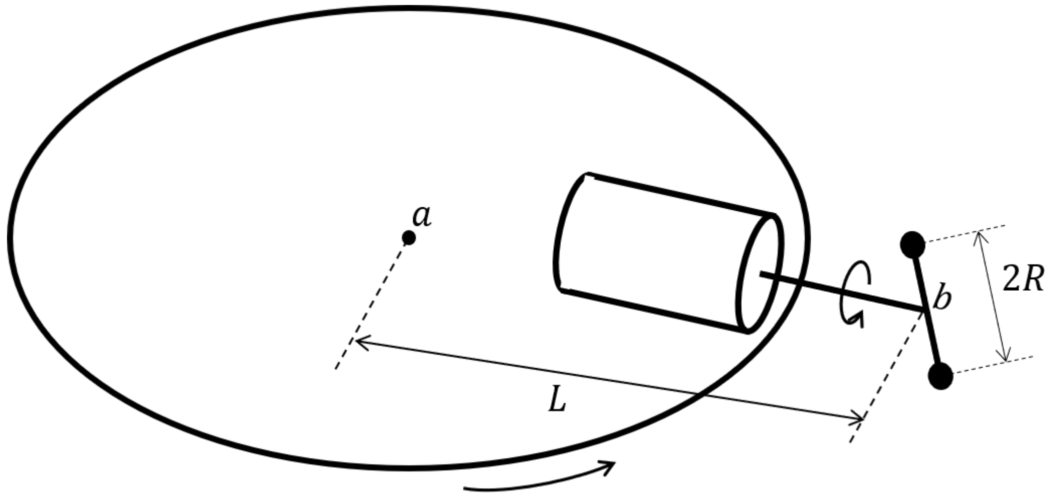


Figure 8.9.2: Rotating platform with a rotating bar for Problem 8.9.3

nonrotating massive body, and that uniform gravity acts in the direction shown. The length of the horizontal arm is l_1, and the mass and length of the slanted bar are m and l_2, respectively. The coefficient of friction between the tip of the slanted bar and the ground is $\mu > 0$. Determine the reaction force acting on the slanted bar due to the horizontal arm, the reaction torque acting on the slanted bar due to the pin joint, the upward reaction force of the ground acting on the slanted bar, and the horizontal friction force acting on the slanted bar. Resolve these quantities in a frame fixed to the slanted bar, and express all of their components in terms of $\theta, \Omega, l_1, m, l_2, g, \mu$. (Hint: The friction force opposes the motion of the slanted bar; its magnitude is obtained by multiplying μ by the magnitude of the upward reaction force of the ground on the slanted bar.)

Problem 8.9.8. The wheel in Figure 8.9.7 is attached to a thin bar by means of a frictionless pin joint at the center of the wheel. The wheel rolls without slipping on the ground, which is an inertially nonrotating massive body. Gravity acts in the direction shown. At the time instant $t = 0$ shown, the disk rotates in the direction shown at the angular rate $\dot{\phi}(0) > 0$, and the angle θ between the arm and the horizontal direction has the value $\theta(0)$ and the rate $\dot{\theta}(0) > 0$. The radius and mass of the wheel are r and m_1, respectively, while the length and mass of the arm are l and m_2, respectively. At time $t = 0$, determine $\ddot{\phi}(0)$ and $\ddot{\theta}(0)$, as well as the horizontal and vertical components of the reaction force at the pin joint.

Problem 8.9.9. The simple pendulum on a cart shown in Figure 8.9.8 consists of a cart with mass M that slides without friction along a horizontal surface of an inertially nonrotating massive rigid body. The force $\vec{f}$ is applied to the cart. At the point x, the cart is connected by a massless rigid link of length ℓ to a particle y of mass m. Determine the reaction force on the pendulum and derive the equations of motion for the cart and pendulum in terms of the directed angle θ and the distance d from x to the point w fixed in the inertially nonrotating massive rigid body. Show that the equations of motion for the simple pendulum are recovered in the limit as $M \to \infty$.

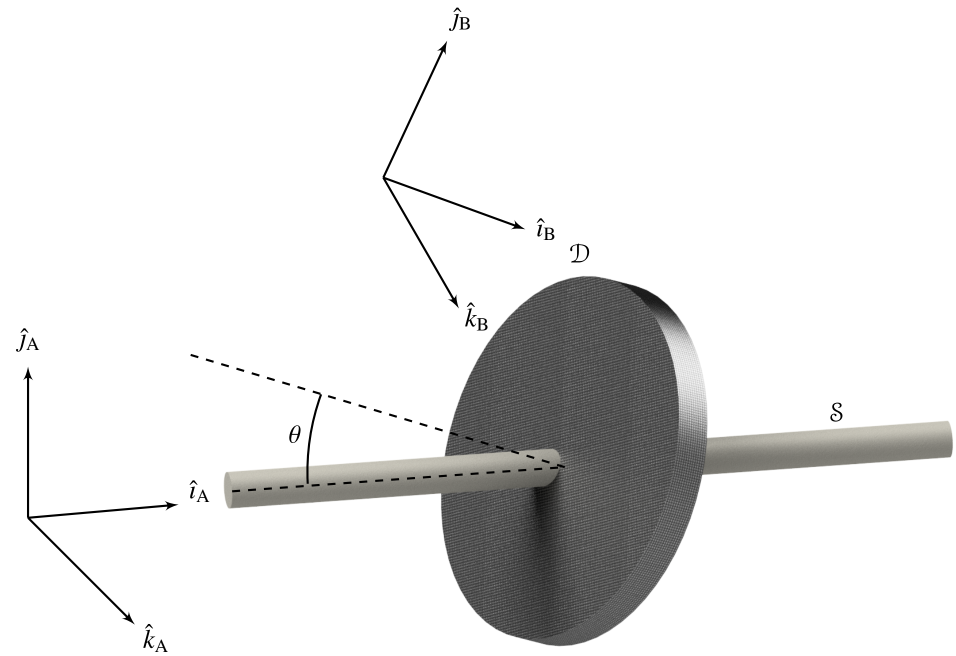


Figure 8.9.3: Disk welded to a shaft for Problem 8.9.4. The frame F_A is attached to the shaft $\mathcal{S}$ and frame F_B is attached to disk $\mathcal{D}$.

8.10 Solutions to the Applied Problems

Solution to Problem 8.9.2.

$$\ddot{d}_1 = \frac{5(m+M)g\sin\theta}{7M + (2+5\sin^2\theta)m}, \quad \ddot{d}_2 = g\tan\theta - \frac{7}{5\cos\theta}\ddot{x},$$

$$f_\mathsf{T} = \frac{-2(m+M)mg\sin\theta}{7M+(2+5\sin^2\theta)m}, \quad f_\mathsf{N} = \frac{mg}{\cos\theta}\left(1 - \frac{7(m+M)\sin^2\theta}{7M+(2+5\sin^2\theta)m}\right),$$

where d_1 is the distance from x to the center of the ball, and d_2 is the distance from the vertical edge of the cart to a point on the ground.

Solution to Problem 8.9.3.

$$\vec{f}_{\mathcal{B}} = -2mL\dot{\omega}_2\hat{\imath}_B - 2mL\omega_2^2\hat{\jmath}_B, \quad \vec{H}_{\mathcal{B}/y/A} = 2mR^2[-\omega_2(\sin\theta)(\cos\theta)\hat{\imath}_B + \omega_1\hat{\jmath}_B + \omega_2(\sin^2\theta)\hat{k}_B],$$

$$\vec{M}_{\mathcal{B}/b} = mR^2[-(\dot{\omega}_2\sin 2\theta + 4\omega_1\omega_2\cos^2\theta)\hat{\imath}_B + (2\dot{\omega}_1 - \omega_2^2\sin 2\theta)\hat{\jmath}_B + (2\dot{\omega}_2\sin^2\theta + 2\omega_1\omega_2\sin 2\theta)\hat{k}_B],$$

where $\hat{\jmath}_B$ is parallel with the motor axis and $\hat{k}_B$ is pointing up. F_C is defined such that $\hat{k}_C$ is parallel with the masses, and $\hat{\jmath}_C$ is parallel with the motor axis.

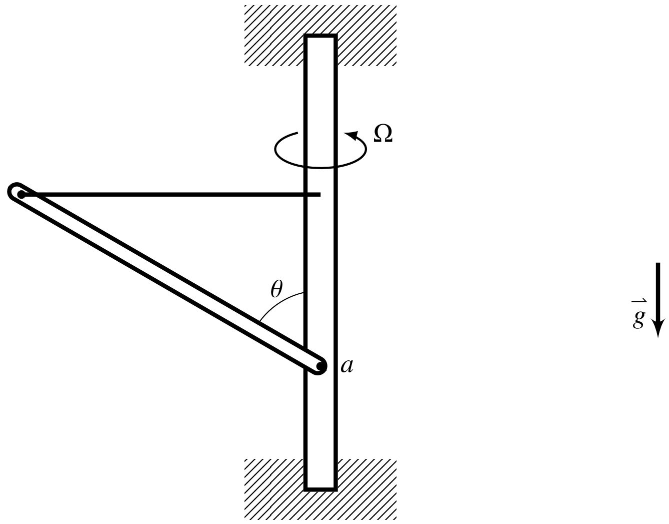


Figure 8.9.4: Rotating shaft with an attached bar and cable for Problem 8.9.5

Solution to Problem 8.9.4.

$$J_{\mathcal{B}_3/c_3|\mathrm{A}} = \begin{bmatrix} \alpha_1 + \delta_1 & \delta_2 & 0 \\ \delta_3 & \beta_1 + m_1 l_4^2 + \delta_4 + m_2 l_5^2 & 0 \\ 0 & 0 & \beta_1 + m_1 l_4^2 + \beta_2 + m_2 l_5^2 \end{bmatrix},$$

$$m_3 = m_1 + m_2, \quad l_3 = \frac{l_1}{2} + \frac{m_2}{m_3}\left(d - \frac{l_1}{2}\right), \quad l_4 = \frac{m_2}{m_3}\left(\frac{l_1}{2} - d\right), \quad l_5 = d - \frac{l_1}{2} + \frac{m_2}{m_3}\left(\frac{l_1}{2} - d\right),$$

$$\alpha_1 = \frac{1}{2} m_1 r_1^2, \quad \alpha_2 = \frac{1}{2} m_2 r_2^2, \quad \beta_1 = \frac{1}{12} m_1 (3r_1^2 + l_1^2), \quad \beta_2 = \frac{1}{12} m_2 (3r_2^2 + l_2^2),$$

$$\delta_1 = \alpha_2 \cos^2\theta + \beta_2 \sin^2\theta, \quad \delta_2 = (-\alpha_2 + \beta_2)(\cos\theta)\sin\theta,$$

$$\delta_3 = (-\alpha_2 + \beta_2)(\cos\theta)\sin\theta, \quad \delta_4 = \alpha_2 \sin^2\theta + \beta_2 \cos^2\theta,$$

where $\mathcal{B}_3 = \mathcal{B}_1 \cup \mathcal{B}_2$ and c_3 is the center of mass of $\mathcal{B}_3$.

Solution to Problem 8.9.5.

$$T = \frac{1}{2} mg \tan\theta + \frac{1}{3} ml(\sin\theta)\Omega^2, \quad f_{ay} = -\frac{1}{2} mg\tan\theta + \frac{1}{6} ml(\sin\theta)\Omega^2, \quad f_{az} = mg.$$

Solution to Problem 8.9.6.

$$\dot{\Omega} = \frac{12\tau}{m[6r^2 + (h^2 - 3r^2)\sin^2\theta]}, \quad f_{ax} = \frac{m\Omega^2 \sin 2\theta}{24L}(h^2 - 3r^2), \quad f_{ay} = \frac{-m\dot{\Omega}\sin 2\theta}{24L}(h^2 - 3r^2).$$

Solution to Problem 8.9.7.

$$f_{ay} = -\frac{1}{2} mg\sin\theta - \frac{1}{6} m\Omega^2 [2l_2 + 6l_1\cos\theta + l_2\cos^2\theta + 3l_1(\tan\theta)\sin\theta],$$

$$f_{az} = -\frac{1}{6} m\Omega^2 (\sin\theta)(3l_1 + l_2\cos\theta) + \frac{1}{2} mg\cos\theta, \quad f_b = \frac{1}{6} m[3g - \Omega^2(2l_2\sin\theta + 3l_1\tan\theta)],$$

$$f_{ax} = -\mu f_b, \quad f_{\mathrm{f}} = \mu f_b, \quad M_{az} = \mu l_2 f_b.$$

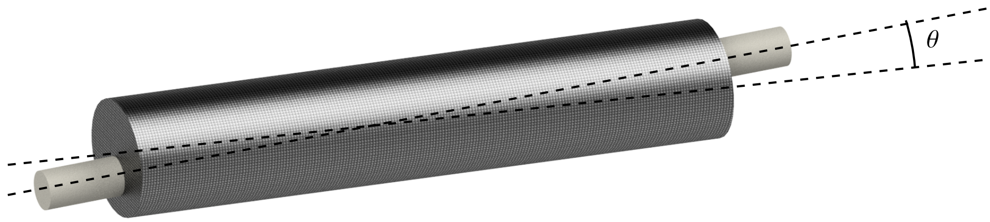


Figure 8.9.5: Rotating shaft with a rigidly attached cylinder for Problem 8.9.6

f_{ax}, f_{ay}, f_{az} are the components of $\vec{f}_a$ resolved in F_C, where F_C is defined such that $\hat{\jmath}_C$ is parallel with the slanted bar pointing toward the ground and such that $\hat{\imath}_C$ points out of the page toward the reader. f_f is the friction force on the bar due to the ground, and f_b is the force on the ground due to the bar. M_{az} is a component of $\vec{M}_{\mathcal{B}/c}$ resolved in F_C, where $\mathcal{B}$ is the slanted bar, and c is its center of mass. The remaining components of $\vec{M}_{\mathcal{B}/c}$ resolved in F_C are zero.

Solution to Problem 8.9.8.

$$(3m_1 + 2m_2)r\ddot{\phi} + m_2\ell(\sin\theta)\ddot{\theta} + m_2\ell(\cos\theta)\dot{\theta}^2 = 0,$$
$$2\ell\ddot{\theta} + 3g\cos\theta + 3r(\sin\theta)\ddot{\phi} = 0,$$
$$f_h = \tfrac{3}{2}m_1 r\ddot{\phi}, \quad f_v = \tfrac{1}{2}m_2\ell[(\cos\theta)\ddot{\theta} - (\sin\theta)\dot{\theta}^2 + 2g/\ell],$$

where f_h and f_v are, respectively, the horizontal and vertical components of the reaction force on the bar.

Solution to Problem 8.9.9.

$$(M + m\sin^2\theta)\ddot{d} + m\ell(\sin\theta)\dot{\theta}^2 + mg(\sin\theta)\cos\theta = f,$$
$$(M + m\sin^2\theta)\ell\ddot{\theta} + m\ell(\sin\theta)(\cos\theta)\dot{\theta}^2 + (M + m\sin\theta)g = (\cos\theta)f,$$
$$f_R = -m\ell\dot{\theta}^2 - m(\sin\theta)\ddot{d} - mg\cos\theta, \quad f_N = f_R\cos\theta - Mg.$$

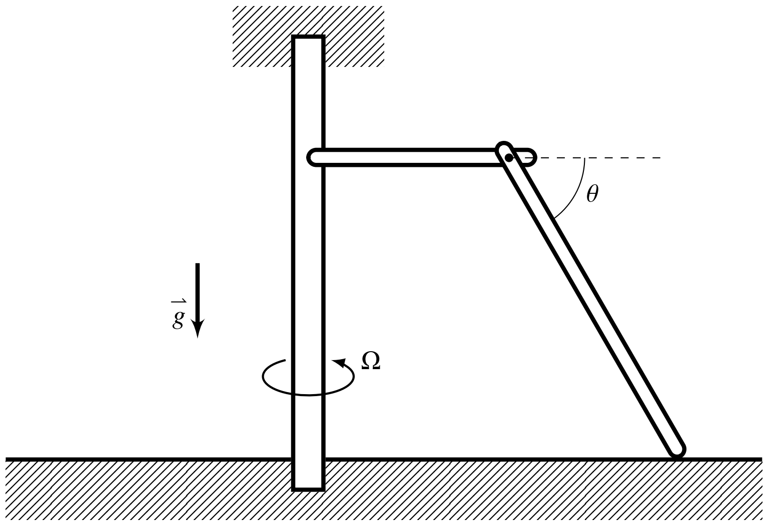


Figure 8.9.6: Rotating shaft with a slanted bar contacting the ground for Problem 8.9.7

Symbol	Definition
dm	Mass element of $\mathcal{B}$
$\vec{p}_{y/w/\mathrm{A}}$	Momentum of the particle y relative to w with respect to $\mathrm{F_A}$
$\vec{p}_{\mathcal{B}/x/\mathrm{A}}$	Momentum of the body $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$
$\vec{J}_{\mathcal{B}/w}$	Physical inertia matrix of body $\mathcal{B}$ relative to w
$\vec{H}_{\mathcal{B}/w/\mathrm{A}}$	Angular momentum of the body $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$

Table 8.10.1: Notation for Chapter 7.

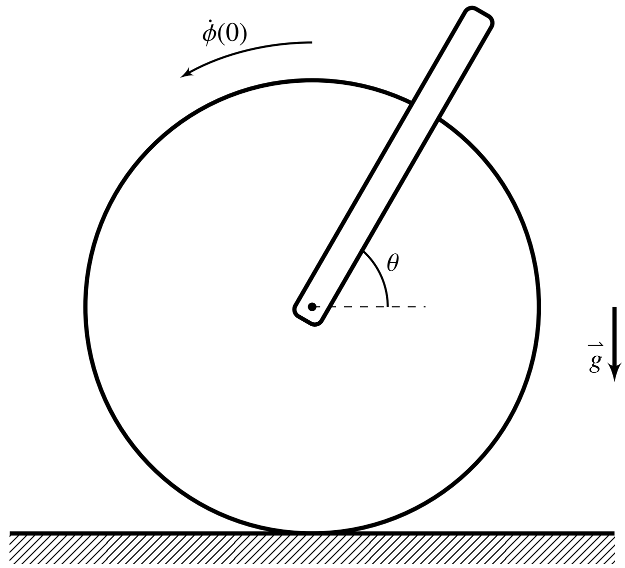


Figure 8.9.7: Rotating wheel with a pinned bar for Problem 8.9.8

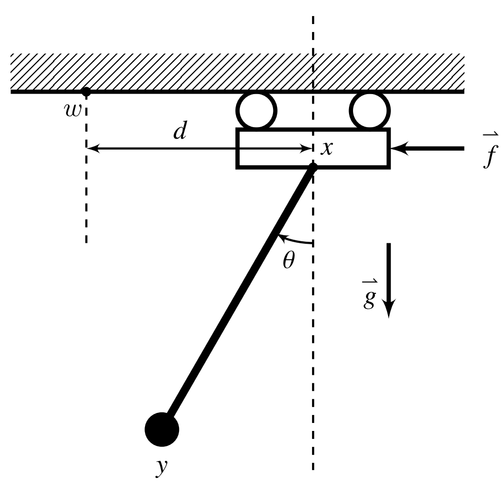


Figure 8.9.8: Simple pendulum attached to a cart for Problem 8.9.9.

Chapter Nine
Twists, Screws, and Wrenches

9.1 Chasles's Theorem

In Section 2.12, it was shown that rotating a rigid body around a point is equivalent to rotating the body around a line parallel to the eigenaxis of the physical rotation matrix. In this section we consider a transformation of a rigid body consisting of rotating the body around a line followed by a translation.

Fact 9.1.1. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$ be identical rigid bodies, let x be a point, let $\mathrm{F_B}$ and $\mathrm{F_{B'}}$ be frames that are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively, and let x be a point. Then, there exists a unique physical position vector $\vec{r}$ such that, for all $i = 1, \ldots, l$,

$$\vec{r}_{y'_i/x} = \vec{R}_{\mathrm{B'/B}} \vec{r}_{y_i/x} + \vec{r}. \tag{9.1.1}$$

Proof. Since $\mathcal{B}$ and $\mathcal{B}'$ are identical, it follows that, for all $i, j \in \{1, \ldots, l\}$,

$$\vec{r}_{y'_i/y'_j} = \vec{R}_{\mathrm{B'/B}} \vec{r}_{y_i/y_j}.$$

Therefore, for all $i, j \in \{1, \ldots, l\}$,

$$\vec{r}_{y'_i/x} + \vec{r}_{x/y'_j} = \vec{R}_{\mathrm{B'/B}} \vec{r}_{y_i/x} + \vec{R}_{\mathrm{B'/B}} \vec{r}_{x/y_j},$$

and thus

$$\vec{r}_{y'_i/x} = \vec{R}_{\mathrm{B'/B}} \vec{r}_{y_i/x} + \vec{r}_j, \tag{9.1.2}$$

where, for all $j = 1, \ldots, l$, $\vec{r}_j \triangleq \vec{R}_{\mathrm{B/A}} \vec{r}_{x/y_j} - \vec{r}_{x/y'_j}$. It follows from (9.1.2) that $\vec{r}_j$ is independent of j. Hence, writing $\vec{r}$ for $\vec{r}_j$, (9.1.2) implies that, for all $i = 1, \ldots, l$, (9.1.1) is satisfied. Finally, (9.1.1) implies that $\vec{r} = \vec{r}_{y'_1/x} - \vec{R}_{\mathrm{B'/B}} \vec{r}_{y_1/x}$, and thus $\vec{r}$ is uniquely determined. □

The next result, which is *Chasles's theorem*, is a stronger version of Fact 9.1.1. This result states that an arbitrary rotation and displacement of a rigid body can be expressed in terms of a rotation around a line that is parallel to the eigenaxis and a displacement along the line.

Fact 9.1.2. Let $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y'_1, \ldots, y'_l\}$ be identical rigid bodies. Then, there exist a point z, an eigenaxis $\hat{n}$, an eigenangle θ, and a real number α such that, for all $i = 1, \ldots, l$,

$$\vec{r}_{y'_i/z} = \vec{R}_{\hat{n}}(\theta) \vec{r}_{y_i/z} + \alpha\hat{n}. \tag{9.1.3}$$

Proof. Let $\mathrm{F_B}$ and $\mathrm{F_{B'}}$ be frames that are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively. Let $\hat{n}$ and θ be an eigenaxis and eigenangle such that $\vec{R}_{\mathrm{B'/B}} = \vec{R}_{\hat{n}}(\theta)$. We first consider the case where $\theta \in (0, \pi)$.

Let $\mathcal{P}$ be the plane that is orthogonal to $\hat{n}$ and that contains y_1. Furthermore, consider the isosceles triangle $\mathcal{T}$ contained in $\mathcal{P}$ with vertices y_1, w_1, and z, where w_1 is the projection of y_1' onto $\mathcal{P}$, the angle between the sides $[y_1, z]$ and $[w_1, z]$ is θ, and the angles opposite the sides $[y_1, z]$ and $[w_1, z]$ are $\pi - \theta/2$. Analogously, let $\mathcal{P}'$ be the plane that is orthogonal to $\hat{n}$ and that contains y_1'. Furthermore, consider the isosceles triangle $\mathcal{T}'$ contained in $\mathcal{P}'$ with vertices y_1', w_1', and z', where w_1' is the projection of y_1 onto $\mathcal{P}'$, the angle between the sides $[y_1', z']$ and $[w_1', z']$ is θ, and the angles opposite the sides $[y_1', z']$ and $[w_1', z']$ are $\pi - \theta/2$. It can thus be seen that there exists a real number α such that $\vec{r}_{z'/z} = \alpha\hat{n}$. Since $\vec{r}_{y_1'/z'} = \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_1/z}$, it follows that

figure needed

$$\begin{aligned}\vec{r}_{y_1'/z} &= \vec{r}_{y_1'/z'} + \vec{r}_{z'/z}\\ &= \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_1/z} + \alpha\hat{n}.\end{aligned} \tag{9.1.4}$$

Next, Fact 9.1.1 implies that there exists a unique physical position vector such that, for all $i = 1, \ldots, l$,

$$\vec{r}_{y_i'/z} = \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_i/z} + \vec{r}. \tag{9.1.5}$$

Hence,

$$\vec{r} = \vec{r}_{y_1'/z} - \vec{R}_{\hat{n}}(\theta)\vec{r}_{y_1/z} = \alpha\hat{n}. \tag{9.1.6}$$

Therefore, since, for all $i = 1, \ldots, l$, (9.1.5) is satisfied, (9.1.6) implies that, for all $i = 1, \ldots, l$, (9.1.3) is satisfied

The case where $\theta \in (-\pi, 0)$ is proved by replacing θ with $|\theta|$. □

Fact 9.1.2 states that rotating the body $\mathcal{B}$ around the line parallel to the eigenaxis $\hat{n}$ and passing through the point z and then translating the rotated body along the direction of $\hat{n}$ yields the body $\mathcal{B}'$. However, since $\vec{R}_{\hat{n}}(\theta)\hat{n} = \hat{n}$, it follows that (9.1.3) can be written equivalently as

$$\vec{r}_{y_i'/z} = \vec{R}_{\hat{n}}(\theta)(\vec{r}_{y_i/z} + \alpha\hat{n}). \tag{9.1.7}$$

This equation states that translating the body $\mathcal{B}$ along the direction of $\hat{n}$ and then rotating the translated by around the line parallel to the eigenaxis $\hat{n}$ and passing through the point z yields the body $\mathcal{B}'$. These interpretations are both valid. However, it is especially useful to view the rotation and translation as occurring simultaneously in the from of a *screw motion,* where the body rotates with a constant angular rate as it translates with a constant translational speed.

Note that (9.1.3) can be written in the matrix-vector form

$$\begin{bmatrix}\vec{r}_{y_i'/z}\\ 1\end{bmatrix} = \begin{bmatrix}\vec{R}_{\hat{n}}(\theta) & \alpha\hat{n}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\vec{r}_{y_i/z}\\ 1\end{bmatrix}. \tag{9.1.8}$$

We thus define the *physical screw matrix*

$$\vec{S}_{\hat{n}}(\theta, \alpha) \triangleq \begin{bmatrix}\vec{R}_{\hat{n}}(\theta) & \alpha\hat{n}\\ 0 & 1\end{bmatrix}, \tag{9.1.9}$$

which can be resolved in $\mathrm{F_A}$ as the 4×4 matrix

$$\vec{S}_{\hat{n}}(\theta, \alpha)\Big|_{\mathrm{A}} \triangleq \begin{bmatrix}\vec{R}_{\hat{n}}(\theta)\Big|_{\mathrm{A}} & \alpha\ \hat{n}|_{\mathrm{A}}\\ 0_{1\times 3} & 1\end{bmatrix}. \tag{9.1.10}$$

9.2 Kinematics Based on Chasle's Theorem

In Chasle's theorem, the identical bodies $\mathcal{B} = \{y_1, \ldots, y_l\}$ and $\mathcal{B}' = \{y_1', \ldots, y_l'\}$ can be viewed as a single body that undergoes translation and rotation, with $\mathcal{B}$ denoting the body before translation and rotation and $\mathcal{B}'$ denoting the body after translation and rotation. To use Chasle's theorem for kinematics, we consider only $\mathcal{B}$, and we model its motion in terms of physical screw matrices. To do this, let $\mathrm{F_A}$ be a frame, and let $\mathrm{F_B}$ be a body-fixed frame. Initially, $\mathrm{F_A}$ and $\mathrm{F_B}$ are identical, and the physical rotation matrix is given by $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}})$, where the eigenaxis $\hat{n}_{\mathrm{B/A}}$ and the eigenangle $\theta_{\mathrm{B/A}} \in (-\pi, \pi]$ may be time-dependent. Then, Chasle's theorem implies that there exists a point z such that, for each point y on $\mathcal{B}$ and all $t \geq 0$,

$$\vec{r}_{y/z}(t) = \vec{R}_{\hat{n}_{\mathrm{B/A}}(t)}(\theta_{\mathrm{B/A}}(t))\vec{r}_{y/z}(0) + \alpha(t)\hat{n}_{\mathrm{B/A}}(t). \tag{9.2.1}$$

Note that (9.2.1) relates the initial and final locations of $y_1, \ldots, y_l$, but does not determine how the motion evolves over time.

Next, note that (9.2.1) can be written as

$$\begin{bmatrix} \vec{r}_{y/z}(t) \\ 1 \end{bmatrix} = \vec{S}(\vec{R}_{\mathrm{B/A}}(t), \vec{p}(t)) \begin{bmatrix} \vec{r}_{y/z}(0) \\ 1 \end{bmatrix}, \tag{9.2.2}$$

where, omitting the time argument,

$$\vec{S}(\vec{R}_{\mathrm{B/A}}, \vec{p}) \triangleq \begin{bmatrix} \vec{R}_{\mathrm{B/A}} & \vec{p} \\ 0 & 1 \end{bmatrix} \tag{9.2.3}$$

and

$$\vec{p} \triangleq \alpha\hat{n}. \tag{9.2.4}$$

Note that

$$\vec{S}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}, \alpha) = \begin{bmatrix} \vec{R}_{\mathrm{B/A}} & \alpha\hat{n}_{\mathrm{B/A}} \\ 0 & 1 \end{bmatrix}, \tag{9.2.5}$$

and thus

$$\vec{S}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}, \alpha)^{-1} = \begin{bmatrix} \vec{R}_{\mathrm{A/B}} & -\alpha\vec{R}_{\mathrm{A/B}}\hat{n}_{\mathrm{B/A}} \\ 0 & 1 \end{bmatrix} \tag{9.2.6}$$

Furthermore, note that

$$\overset{\mathrm{A}\bullet}{\vec{S}}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}, \alpha) = \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{R}}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}) & \overbrace{\alpha\hat{n}_{\mathrm{B/A}}}^{\mathrm{A}\bullet} \\ 0 & 0 \end{bmatrix}. \tag{9.2.7}$$

Therefore, it follows from (4.2.4) and (4.2.19) that

$$\begin{aligned}
\overset{\mathrm{A}\bullet}{\vec{S}}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}, \alpha)\vec{S}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}}, \alpha)^{-1} &= \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{R}}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}})\vec{R}_{\hat{n}_{\mathrm{B/A}}}(-\theta_{\mathrm{B/A}}) & \dot{\alpha}\hat{n}_{\mathrm{B/A}} - \alpha\overset{\mathrm{A}\bullet}{\vec{R}}_{\hat{n}_{\mathrm{B/A}}}(\theta_{\mathrm{B/A}})\vec{R}_{\hat{n}_{\mathrm{B/A}}}(-\theta_{\mathrm{B/A}})\hat{n}_{\mathrm{B/A}} + \alpha\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\vec{\Omega}_{\mathrm{A/B}} & \dot{\alpha}\hat{n}_{\mathrm{B/A}} + \alpha\vec{\Omega}_{\mathrm{A/B}}\hat{n}_{\mathrm{B/A}} + \alpha\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \vec{\Omega}_{\mathrm{B/A}} & \dot{\alpha}\hat{n}_{\mathrm{B/A}} - \alpha\vec{\Omega}_{\mathrm{B/A}}\hat{n}_{\mathrm{B/A}} + \alpha\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} \vec{\omega}^{\times}_{\mathrm{B/A}} & \dot{\alpha}\hat{n}_{\mathrm{B/A}} + \alpha(\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} - \vec{\omega}_{\mathrm{B/A}} \times \hat{n}_{\mathrm{B/A}}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \vec{\omega}^{\times}_{\mathrm{B/A}} & \dot{\alpha}\hat{n}_{\mathrm{B/A}} + \alpha(\overset{\mathrm{A}\bullet}{\hat{n}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{A/B}} \times \hat{n}_{\mathrm{B/A}}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \vec{\omega}^{\times}_{\mathrm{B/A}} & \dot{\alpha}\hat{n}_{\mathrm{B/A}} + \alpha\, \overset{\mathrm{B}\bullet}{\hat{n}}_{\mathrm{B/A}} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \vec{\omega}^{\times}_{\mathrm{B/A}} & \overbrace{\alpha\hat{n}_{\mathrm{B/A}}}^{\mathrm{B}\bullet} \\ 0 & 0 \end{bmatrix}
\end{aligned}
\tag{9.2.8}
$$

Now, let z' denote the location of z as $\mathcal{B}$ moves.

In the special case where the motion from the initial orientation and location of $\mathcal{B}$ to the final orientation and location of $\mathcal{B}$ occurs such that $\hat{n}_{\mathrm{B/A}}$ is constant with respect to both $\mathrm{F_A}$ and $\mathrm{F_B}$, that is, $\mathcal{B}$ rotates around the line through z parallel to $\hat{n}_{\mathrm{B/A}}$.

Letting $\mathrm{F_B}$ and $\mathrm{F_{B'}}$ be frames that are fixed identically in $\mathcal{B}$ and $\mathcal{B}'$, respectively, and differentiating (9.2.1) yields

$$
\vec{v}_{y_i'/z/\mathrm{B}} = \overset{\mathrm{B}\bullet}{\vec{R}_{\hat{n}}}(\theta_{\mathrm{B/A}})\vec{r}_{y_i/z} + \vec{\xi}, \tag{9.2.9}
$$

where

$$
\vec{\xi} \triangleq \dot{\alpha}\hat{n} + \alpha \overset{\mathrm{B}\bullet}{\hat{n}}. \tag{9.2.10}
$$

Solving (9.2.1) for $\vec{r}_{y_i/z}$ and substituting into (9.2.9) yields

$$
\vec{v}_{y_i'/z/\mathrm{B}} = \overset{\mathrm{B}\bullet}{\vec{R}_{\hat{n}}}(\theta_{\mathrm{B/A}})\vec{R}'_{\hat{n}}(\theta_{\mathrm{B/A}})(\vec{r}_{y_i'/z} - \alpha\hat{n}) + \vec{\xi}. \tag{9.2.11}
$$

Noting from (4.2.22) that

$$
\overset{\mathrm{A}\bullet}{\vec{R}_{\hat{n}}}(\theta_{\mathrm{B/A}})\vec{R}'_{\hat{n}}(\theta_{\mathrm{B/A}}) = \overset{\mathrm{A}\bullet}{\vec{R}}_{\mathrm{B'/B}}\,\vec{R}_{\mathrm{B/B'}} = \vec{\Omega}_{\mathrm{B'/B}}, \tag{9.2.12}
$$

(9.2.11) can be written as

$$
\vec{v}_{y_i'/z/\mathrm{B}} = \vec{\Omega}_{\mathrm{B'/B}}(\vec{r}_{y_i'/z} - \alpha\hat{n}) + \vec{\xi}. \tag{9.2.13}
$$

Using $\vec{\Omega}_{\mathrm{B'/B}} = \vec{\omega}^{\times}_{\mathrm{B'/B}}$ and rearranging (9.2.13) yields

$$
\vec{v}_{y_i'/z/\mathrm{B}} = \vec{\omega}_{\mathrm{B'/B}} \times \vec{r}_{y_i'/z} + \alpha\hat{n} \times \vec{\omega}_{\mathrm{B'/B}} + \vec{\xi}. \tag{9.2.14}
$$

9.3 6D Dynamics of a Rigid Body

In this section we express the translational and rotational dynamics of a rigid body in terms of 6D vectors. In the next section, this formulation is applied to the chain of rigid bodies considered in Section 9.4, Section 9.5, and Section 9.6.

Let $\mathcal{B}$ be a rigid body, let c denote the center of mass of $\mathcal{B}$, let z be a point that is fixed in $\mathcal{B}$, let w be a point with zero inertial acceleration, let $\mathrm{F_B}$ be a body-fixed frame, and let $\mathrm{F_A}$ be an inertial frame. In addition, let $\vec{f}_{\mathcal{B}}$ denote the net force on $\mathcal{B}$, and let $\vec{M}_{\mathcal{B}/z}$ denote the moment on $\mathcal{B}$ relative to z. It follows from (7.5.24) that the translational dynamics of $\mathcal{B}$ are given by

$$m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}) = \vec{f}_{\mathcal{B}}. \tag{9.3.1}$$

and from (7.9.10) that the rotational dynamics of $\mathcal{B}$ are given by

$$\vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \vec{a}_{z/w/\mathrm{A}} = \vec{M}_{\mathcal{B}/z}, \tag{9.3.2}$$

which can be rewritten as

$$\vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{z/w/\mathrm{A}}) = \vec{M}_{\mathcal{B}/z}, \tag{9.3.3}$$

To construct a single equation for both the translational and rotational dynamics, note that (9.3.1) and (9.3.3) can be written as the single equation

$$\begin{bmatrix} m_{\mathcal{B}}\overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + \vec{\alpha}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} + \vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{v}_{z/w/\mathrm{A}} + \vec{\omega}_{\mathrm{B/A}} \times (\vec{\omega}_{\mathrm{B/A}} \times m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}) \\ \vec{J}_{\mathcal{B}/z}\overset{\mathrm{B}\bullet}{\vec{\omega}}_{\mathrm{B/A}} + \vec{\omega}_{\mathrm{B/A}} \times \vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B/A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \overset{\mathrm{B}\bullet}{\vec{v}}_{z/w/\mathrm{A}} + m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{z/w/\mathrm{A}}) \end{bmatrix} = \begin{bmatrix} \vec{f}_{\mathcal{B}} \\ \vec{M}_{\mathcal{B}/z} \end{bmatrix}. \tag{9.3.4}$$

Using Jacobi's identity given by Problem 2.24.6 in the form

$$m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{v}_{z/w/\mathrm{A}}) = \vec{\omega}_{\mathrm{B/A}} \times (m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \vec{v}_{z/w/\mathrm{A}}) - \vec{v}_{z/w/\mathrm{A}} \times (m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z} \times \vec{\omega}_{\mathrm{B/A}}), \tag{9.3.5}$$

(9.3.4) can be rewritten as

$$\begin{bmatrix} m_{\mathcal{B}}\vec{I} & -m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\,\times} \\ m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\,\times} & \vec{J}_{\mathcal{B}/z} \end{bmatrix} \overset{\mathrm{B}\bullet}{\overbrace{\begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}}} + \begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}^{\times} \begin{bmatrix} m_{\mathcal{B}}\vec{I} & -m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\,\times} \\ m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\,\times} & \vec{J}_{\mathcal{B}/z} \end{bmatrix} \begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix} = \begin{bmatrix} \vec{f}_{\mathcal{B}} \\ \vec{M}_{\mathcal{B}/z} \end{bmatrix}, \tag{9.3.6}$$

where

$$\begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}^{\times} \triangleq \begin{bmatrix} \vec{\omega}_{\mathrm{B/A}}^{\,\times} & 0 \\ \vec{v}_{z/w/\mathrm{A}}^{\,\times} & \vec{\omega}_{\mathrm{B/A}}^{\,\times} \end{bmatrix}. \tag{9.3.7}$$

By defining

$$\vec{\mathcal{J}}_{\mathcal{B}/z} \triangleq \begin{bmatrix} m_{\mathcal{B}}\vec{I} & -m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\,\times} \\ m_{\mathcal{B}}\vec{r}_{\mathrm{c}/z}^{\,\times} & \vec{J}_{\mathcal{B}/z} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\mathcal{B}} \triangleq \begin{bmatrix} \vec{v}_{z/w/\mathrm{A}} \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}, \quad \vec{\mathcal{F}}_{\mathcal{B}/z} \triangleq \begin{bmatrix} \vec{f}_{\mathcal{B}} \\ \vec{M}_{\mathcal{B}/z} \end{bmatrix}, \tag{9.3.8}$$

(9.3.6) can be written as

$$\vec{\mathcal{J}}_{\mathcal{B}/z}\vec{\mathcal{A}}_{\mathcal{B}} + \vec{\mathcal{V}}_{\mathcal{B}}^{\,\times}\vec{\mathcal{J}}_{\mathcal{B}/z}\vec{\mathcal{V}}_{\mathcal{B}} = \vec{\mathcal{F}}_{\mathcal{B}/z}. \tag{9.3.9}$$

where

$$\vec{\mathcal{A}}_{\mathcal{B}} \triangleq \overset{\mathrm{B}\bullet}{\vec{\mathcal{V}}}_{\mathcal{B}}\ . \tag{9.3.10}$$

Resolving (9.3.9) in $\mathrm{F_B}$ yields

$$\mathcal{J}_{\mathcal{B}/z|\mathrm{B}}\mathcal{A}_{\mathcal{B}|\mathrm{B}} + \mathcal{V}^{\times}_{\mathcal{B}|\mathrm{B}}\mathcal{J}_{\mathcal{B}/z|\mathrm{B}}\mathcal{V}_{\mathcal{B}|\mathrm{B}} = \mathcal{F}_{\mathcal{B}/z|\mathrm{B}}, \tag{9.3.11}$$

where

$$\mathcal{J}_{\mathcal{B}/z|\mathrm{B}} \triangleq \begin{bmatrix} m_{\mathcal{B}} I & -m_{\mathcal{B}} r^{\times}_{\mathrm{c}/z|\mathrm{B}} \\ m_{\mathcal{B}} r^{\times}_{\mathrm{c}/z|\mathrm{B}} & J_{\mathcal{B}/z|\mathrm{B}} \end{bmatrix}, \quad \mathcal{V}_{\mathcal{B}|\mathrm{B}} \triangleq \begin{bmatrix} v_{z/w/\mathrm{A}|\mathrm{B}} \\ \omega_{\mathrm{B}/\mathrm{A}|\mathrm{B}} \end{bmatrix}, \quad \mathcal{A}_{\mathcal{B}|\mathrm{B}} \triangleq \dot{\mathcal{V}}_{\mathcal{B}|\mathrm{B}} = \begin{bmatrix} \dot{v}_{z/w/\mathrm{A}|\mathrm{B}} \\ \alpha_{\mathrm{B}/\mathrm{A}|\mathrm{B}} \end{bmatrix}, \tag{9.3.12}$$

$$\mathcal{V}^{\times}_{\mathcal{B}|\mathrm{B}} \triangleq \begin{bmatrix} \omega^{\times}_{\mathrm{B}/\mathrm{A}|\mathrm{B}} & 0 \\ v^{\times}_{z/w/\mathrm{A}|\mathrm{B}} & \omega^{\times}_{\mathrm{B}/\mathrm{A}|\mathrm{B}} \end{bmatrix}, \quad \mathcal{F}_{\mathcal{B}/z|\mathrm{B}} \triangleq \begin{bmatrix} f_{\mathcal{B}|\mathrm{B}} \\ M_{\mathcal{B}/z|\mathrm{B}} \end{bmatrix}. \tag{9.3.13}$$

9.4 Geometry of a Chain of Rigid Bodies

Consider rigid bodies $\mathcal{B}_{\rm B}$, $\mathcal{B}_{\rm C}$, and $\mathcal{B}_{\rm D}$ connected to the base rigid body $\mathcal{B}_{\rm A}$ in the form of a chain with three links as shown in Figure 9.4.1. In particular, $\mathcal{B}_{\rm B}$ is connected to $\mathcal{B}_{\rm A}$ at the point $z_{\rm A}$, $\mathcal{B}_{\rm C}$ is connected to $\mathcal{B}_{\rm B}$ at the point $z_{\rm B}$, and $\mathcal{B}_{\rm D}$ is connected to $\mathcal{B}_{\rm C}$ at the point $z_{\rm C}$. The point $z_{\rm D}$ is the *end effector*. The point $z_{\rm A}$ is fixed in $\mathcal{B}_{\rm B}$ and $\mathcal{B}_{\rm A}$; the point $z_{\rm B}$ is fixed in $\mathcal{B}_{\rm C}$ and $\mathcal{B}_{\rm B}$; the point $z_{\rm C}$ is fixed in $\mathcal{B}_{\rm D}$ and $\mathcal{B}_{\rm C}$; and the point $z_{\rm D}$ is fixed in $\mathcal{B}_{\rm D}$. Each attachment point is assumed to represent a rotary joint, such as a pin, universal joint, or ball joint. Note that

$$\vec{r}_{z_{\rm D}/z_{\rm A}} = \vec{r}_{z_{\rm D}/z_{\rm C}} + \vec{r}_{z_{\rm C}/z_{\rm B}} + \vec{r}_{z_{\rm B}/z_{\rm A}}. \tag{9.4.1}$$

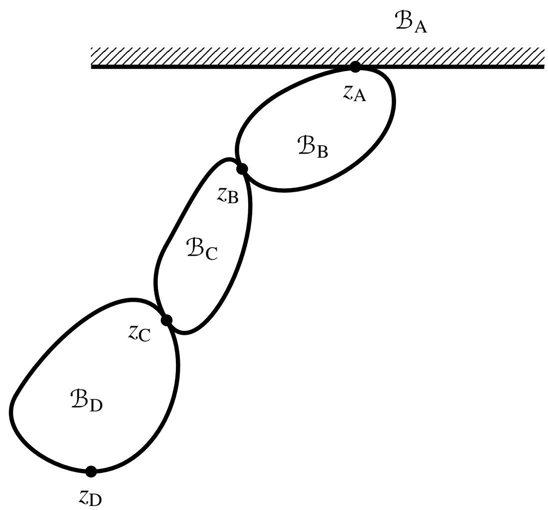


Figure 9.4.1: Chain of rigid bodies.

We assume that $\vec{r}_{z_{\rm D}/z_{\rm C}}$ is known in $\rm F_D$, $\vec{r}_{z_{\rm C}/z_{\rm B}}$ is known in $\rm F_C$, and $\vec{r}_{z_{\rm B}/z_{\rm A}}$ is known in $\rm F_B$. Hence, $r_{z_{\rm D}/z_{\rm C}|{\rm D}}$, $r_{z_{\rm C}/z_{\rm B}|{\rm C}}$, and $r_{z_{\rm B}/z_{\rm A}|{\rm B}}$ are known. Furthermore, we assume that $\mathcal{O}_{\rm D/C}$, $\mathcal{O}_{\rm C/B}$, and $\mathcal{O}_{\rm B/A}$ are known. In order to express $r_{z_{\rm D}/z_{\rm A}|{\rm A}}$ in terms of the known quantities, note that

$$\begin{aligned} r_{z_{\rm D}/z_{\rm A}|{\rm A}} &= r_{z_{\rm D}/z_{\rm C}|{\rm A}} + r_{z_{\rm C}/z_{\rm B}|{\rm A}} + r_{z_{\rm B}/z_{\rm A}|{\rm A}} \\ &= \mathcal{O}_{\rm A/D} r_{z_{\rm D}/z_{\rm C}|{\rm D}} + \mathcal{O}_{\rm A/C} r_{z_{\rm C}/z_{\rm B}|{\rm C}} + \mathcal{O}_{\rm A/B} r_{z_{\rm B}/z_{\rm A}|{\rm B}} \\ &= \mathcal{O}_{\rm A/C}\mathcal{O}_{\rm C/D} r_{z_{\rm D}/z_{\rm C}|{\rm D}} + \mathcal{O}_{\rm A/B}\mathcal{O}_{\rm B/C} r_{z_{\rm C}/z_{\rm B}|{\rm C}} + \mathcal{O}_{\rm A/B} r_{z_{\rm B}/z_{\rm A}|{\rm B}}. \end{aligned} \tag{9.4.2}$$

Note that (9.4.2) has the form of a recursive algorithm that proceeds from the top link of the chain to the bottom link of the chain, where $r_{z_{\rm B}/z_{\rm A}|{\rm B}}$ and $\mathcal{O}_{\rm A/B}$ are used first, $r_{z_{\rm C}/z_{\rm B}|{\rm C}}$ and $\mathcal{O}_{\rm B/C}$ are used second, where $\mathcal{O}_{\rm B/C}$ is multiplied by $\mathcal{O}_{\rm A/B}$ to obtain $\mathcal{O}_{\rm A/C}$, and $r_{z_{\rm D}/z_{\rm C}|{\rm D}}$ and $\mathcal{O}_{\rm C/D}$ are used last, where $\mathcal{O}_{\rm C/D}$ is multiplied by $\mathcal{O}_{\rm A/C}$ computed in the second step to obtain $\mathcal{O}_{\rm A/D}$. Alternatively, we can write

$$\begin{aligned} r_{z_{\rm D}/z_{\rm A}|{\rm A}} &= r_{z_{\rm D}/z_{\rm C}|{\rm A}} + r_{z_{\rm C}/z_{\rm B}|{\rm A}} + r_{z_{\rm B}/z_{\rm A}|{\rm A}} \\ &= \mathcal{O}_{\rm A/D} r_{z_{\rm D}/z_{\rm C}|{\rm D}} + \mathcal{O}_{\rm A/C} r_{z_{\rm C}/z_{\rm B}|{\rm C}} + \mathcal{O}_{\rm A/B} r_{z_{\rm B}/z_{\rm A}|{\rm B}} \\ &= \mathcal{O}_{\rm A/B}\mathcal{O}_{\rm B/C}\mathcal{O}_{\rm C/D} r_{z_{\rm D}/z_{\rm C}|{\rm D}} + \mathcal{O}_{\rm A/B}\mathcal{O}_{\rm B/C} r_{z_{\rm C}/z_{\rm B}|{\rm C}} + \mathcal{O}_{\rm A/B} r_{z_{\rm B}/z_{\rm A}|{\rm B}} \\ &= \mathcal{O}_{\rm A/B}[\mathcal{O}_{\rm B/C}\mathcal{O}_{\rm C/D} r_{z_{\rm D}/z_{\rm C}|{\rm D}} + \mathcal{O}_{\rm B/C} r_{z_{\rm C}/z_{\rm B}|{\rm C}} + r_{z_{\rm B}/z_{\rm A}|{\rm B}}] \\ &= \mathcal{O}_{\rm A/B}[\mathcal{O}_{\rm B/C}(\mathcal{O}_{\rm C/D} r_{z_{\rm D}/z_{\rm D}|{\rm D}} + r_{z_{\rm C}/z_{\rm B}|{\rm C}}) + r_{z_{\rm B}/z_{\rm A}|{\rm B}}]. \end{aligned} \tag{9.4.3}$$

Note that (9.4.3) has the form of a recursive algorithm that proceeds from the bottom link of the chain to the top link of the chain, where $r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{D}}$ and $\mathcal{O}_{\mathrm{C/D}}$ are used first, $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{C}}$ and $\mathcal{O}_{\mathrm{B/C}}$ are used second, and $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{B}}$ and $\mathcal{O}_{\mathrm{A/B}}$ are used last. For a chain consisting of $n \geq 4$ rigid bodies, recursive equations of the form (9.4.2) and (9.4.3) can be derived.

9.5 6D Velocity Kinematics of a Chain of Rigid Bodies

Consider the chain of rigid bodies shown in Figure 9.4.1. We assume that $\omega_{\mathrm{B/A|B}}(t)$, $\omega_{\mathrm{C/A|C}}(t)$, and $\omega_{\mathrm{D/A|D}}(t)$ are known for all time t and that the initial orientations $\mathcal{O}_{\mathrm{A/B}}(0)$, $\mathcal{O}_{\mathrm{B/C}}(0)$, and $\mathcal{O}_{\mathrm{C/D}}(0)$ and the initial displacements $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}}(0)$, $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}}(0)$, and $r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}}(0)$ are known. The goal is to determine $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}}(t)$, $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}}(t)$, and $r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}}(t)$ as functions of time.

First, to determine the position of z_B relative to z_A in $\mathrm{F_A}$, note that

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{z_\mathrm{B}/z_\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{z_\mathrm{B}/z_\mathrm{A}}. \tag{9.5.1}$$

Resolving (9.5.1) in $\mathrm{F_A}$ yields

$$\dot{r}_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}} = \omega_{\mathrm{B/A|A}} \times r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}} \tag{9.5.2}$$

$$= (\mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{B/A|B}}) \times r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}}. \tag{9.5.3}$$

Since $\omega_{\mathrm{B/A|B}}(t)$ is known for all t and $\mathcal{O}_{\mathrm{A/B}}(0)$ is known, integrating Poisson's equation (4.3.20) in the form

$$\dot{\mathcal{O}}_{\mathrm{A/B}} = \mathcal{O}_{\mathrm{A/B}}\omega_{\mathrm{B/A|B}}^{\times} \tag{9.5.4}$$

yields $\mathcal{O}_{\mathrm{A/B}}(t)$. Since $\mathcal{O}_{\mathrm{A/B}}(t)\omega_{\mathrm{B/A|B}}(t)$ is known for all t and $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}}(0)$ is known, integrating (9.5.3) yields $r_{z_\mathrm{B}/z_\mathrm{A}|\mathrm{A}}(t)$.

Next, to determine the position of z_C relative to z_B in $\mathrm{F_A}$, note that

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{z_\mathrm{C}/z_\mathrm{B}} = \vec{\omega}_{\mathrm{C/A}} \times \vec{r}_{z_\mathrm{C}/z_\mathrm{B}}. \tag{9.5.5}$$

Resolving (9.5.5) in $\mathrm{F_A}$ yields

$$\dot{r}_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}} = \omega_{\mathrm{C/A|A}} \times r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}} \tag{9.5.6}$$

$$= (\mathcal{O}_{\mathrm{A/C}}\omega_{\mathrm{C/A|C}}) \times r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}}. \tag{9.5.7}$$

Since $\omega_{\mathrm{C/A|C}}(t)$ is known for all t and $\mathcal{O}_{\mathrm{A/C}}(0)$ is known, integrating Poisson's equation (4.3.20) in the form

$$\dot{\mathcal{O}}_{\mathrm{A/C}} = \mathcal{O}_{\mathrm{A/C}}\omega_{\mathrm{C/A|C}}^{\times} \tag{9.5.8}$$

yields $\mathcal{O}_{\mathrm{A/C}}(t)$. Since $\mathcal{O}_{\mathrm{A/C}}(t)\omega_{\mathrm{C/A|C}}(t)$ is known for all t and $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}}(0)$ is known, integrating (9.5.7) yields $r_{z_\mathrm{C}/z_\mathrm{B}|\mathrm{A}}(t)$.

Next, to determine the position of z_D relative to z_C in $\mathrm{F_A}$, note that

$$\overset{\mathrm{A}\bullet}{\vec{r}}_{z_\mathrm{D}/z_\mathrm{C}} = \vec{\omega}_{\mathrm{D/A}} \times \vec{r}_{z_\mathrm{D}/z_\mathrm{C}}. \tag{9.5.9}$$

Resolving (9.5.9) in $\mathrm{F_A}$ yields

$$\dot{r}_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}} = \omega_{\mathrm{D/A|A}} \times r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}} \tag{9.5.10}$$

$$= (\mathcal{O}_{\mathrm{A/D}}\omega_{\mathrm{D/A|D}}) \times r_{z_\mathrm{D}/z_\mathrm{C}|\mathrm{A}}. \tag{9.5.11}$$

Since $\omega_{\mathrm{D/A|D}}(t)$ is known for all t and $\mathcal{O}_{\mathrm{A/D}}(0)$ is known, integrating Poisson's equation (4.3.20) in the form

$$\dot{\mathcal{O}}_{\mathrm{A/D}} = \mathcal{O}_{\mathrm{A/D}}\omega^{\times}_{\mathrm{D/A|D}} \tag{9.5.12}$$

yields $\mathcal{O}_{\mathrm{A/D}}(t)$. Since $\mathcal{O}_{\mathrm{A/D}}(t)\omega_{\mathrm{D/A|D}}(t)$ is known for all t and $r_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{A}}(0)$ is known, integrating (9.5.11) yields $r_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{A}}(t)$. Finally, since $r_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{A}}$, $r_{z_{\mathrm{C}}/z_{\mathrm{B}}|\mathrm{A}}$, and $r_{z_{\mathrm{B}}/z_{\mathrm{A}}|\mathrm{A}}$ are known, it follows that $r_{z_{\mathrm{D}}/z_{\mathrm{A}}|\mathrm{A}} = r_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{A}} + r_{z_{\mathrm{C}}/z_{\mathrm{B}}|\mathrm{A}} + r_{z_{\mathrm{B}}/z_{\mathrm{A}}|\mathrm{A}}$ is known.

As an alternative approach, note that

$$\vec{r}_{z_{\mathrm{C}}/z_{\mathrm{A}}} = \vec{r}_{z_{\mathrm{C}}/z_{\mathrm{B}}} + \vec{r}_{z_{\mathrm{B}}/z_{\mathrm{A}}}, \tag{9.5.13}$$

$$\vec{r}_{z_{\mathrm{D}}/z_{\mathrm{A}}} = \vec{r}_{z_{\mathrm{D}}/z_{\mathrm{C}}} + \vec{r}_{z_{\mathrm{C}}/z_{\mathrm{A}}}. \tag{9.5.14}$$

Since

$$\overset{\mathrm{B}\bullet}{\vec{r}}_{z_{\mathrm{B}}/z_{\mathrm{A}}} = \overset{\mathrm{C}\bullet}{\vec{r}}_{z_{\mathrm{C}}/z_{\mathrm{B}}} = \overset{\mathrm{D}\bullet}{\vec{r}}_{z_{\mathrm{D}}/z_{\mathrm{C}}} = 0, \tag{9.5.15}$$

it follows that

$$\vec{v}_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A}} = \vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{z_{\mathrm{B}}/z_{\mathrm{A}}}, \tag{9.5.16}$$

$$\vec{v}_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A}} = \vec{\omega}_{\mathrm{C/A}} \times \vec{r}_{z_{\mathrm{C}}/z_{\mathrm{B}}} + \vec{v}_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A}}, \tag{9.5.17}$$

$$\vec{v}_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A}} = \vec{\omega}_{\mathrm{D/A}} \times \vec{r}_{z_{\mathrm{D}}/z_{\mathrm{C}}} + \vec{v}_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A}}. \tag{9.5.18}$$

Combining (9.5.16)–(9.5.18) with angular velocities yields

$$\begin{bmatrix} \vec{v}_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{C/A}} \end{bmatrix} = \begin{bmatrix} 0 & -\vec{r}^{\,\times}_{z_{\mathrm{B}}/z_{\mathrm{A}}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{C/B}} \end{bmatrix}, \tag{9.5.19}$$

$$\begin{bmatrix} \vec{v}_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{D/A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\,\times}_{z_{\mathrm{C}}/z_{\mathrm{B}}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{v}_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{C/A}} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{D/C}} \end{bmatrix}, \tag{9.5.20}$$

$$\begin{bmatrix} \vec{v}_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{D/A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\,\times}_{z_{\mathrm{D}}/z_{\mathrm{C}}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{v}_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{D/A}} \end{bmatrix}. \tag{9.5.21}$$

Resolving (9.5.16)–(9.5.18) yields

$$v_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A|B}} = \omega_{\mathrm{B/A|B}} \times r_{z_{\mathrm{B}}/z_{\mathrm{A}}|\mathrm{B}}, \tag{9.5.22}$$

$$v_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A|C}} = \omega_{\mathrm{C/A|C}} \times r_{z_{\mathrm{C}}/z_{\mathrm{B}}|\mathrm{C}} + v_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A|C}}, \tag{9.5.23}$$

$$v_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A|D}} = \omega_{\mathrm{D/A|D}} \times r_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{D}} + v_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A|D}}, \tag{9.5.24}$$

and thus resolving (9.5.19)–(9.5.21) yields

$$\begin{bmatrix} v_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A|B}} \\ \omega_{\mathrm{C/A|B}} \end{bmatrix} = \begin{bmatrix} 0 & -r^{\times}_{z_{\mathrm{B}}/z_{\mathrm{A}}|\mathrm{B}} \\ 0 & I_3 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_{\mathrm{B/A|B}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{O}_{\mathrm{B/C}}\omega_{\mathrm{C/B|C}} \end{bmatrix}, \tag{9.5.25}$$

$$\begin{bmatrix} v_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A|C}} \\ \omega_{\mathrm{D/A|C}} \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{\mathrm{C/B}} & -r^{\times}_{z_{\mathrm{C}}/z_{\mathrm{B}}|\mathrm{C}}\mathcal{O}_{\mathrm{C/B}} \\ 0 & \mathcal{O}_{\mathrm{C/B}} \end{bmatrix} \begin{bmatrix} v_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A|B}} \\ \omega_{\mathrm{C/A|B}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{O}_{\mathrm{C/D}}\omega_{\mathrm{D/C|D}} \end{bmatrix}, \tag{9.5.26}$$

$$\begin{bmatrix} v_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A|D}} \\ \omega_{\mathrm{D/A|D}} \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{\mathrm{D/C}} & -r^{\times}_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{D}}\mathcal{O}_{\mathrm{D/C}} \\ 0 & \mathcal{O}_{\mathrm{D/C}} \end{bmatrix} \begin{bmatrix} v_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A|C}} \\ \omega_{\mathrm{D/A|C}} \end{bmatrix}. \tag{9.5.27}$$

Now, assume that, for all time t, the angular velocities $\omega_{\mathrm{B/A|B}}(t)$, $\omega_{\mathrm{C/B|C}}(t)$, and $\omega_{\mathrm{D/C|D}}(t)$, the orientation matrices $\mathcal{O}_{\mathrm{C/B}}(t)$ and $\mathcal{O}_{\mathrm{D/C}}(t)$, and the position vectors $r_{z_{\mathrm{B}}/z_{\mathrm{A}}|\mathrm{B}}(t)$, $r_{z_{\mathrm{C}}/z_{\mathrm{B}}|\mathrm{C}}(t)$, and $r_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{D}}(t)$ are known. Then, (9.5.25)–(9.5.27) yield $v_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A|B}}(t)$, $v_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A|C}}(t)$, $v_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A|D}}(t)$, $\omega_{\mathrm{C/A|B}}(t)$, $\omega_{\mathrm{D/A|C}}(t)$, and $\omega_{\mathrm{D/A|D}}(t)$ as functions of time. Note that $\mathcal{O}_{\mathrm{B/A}}(0)$ is not needed for these computations.

For a chain consisting of $n \ge 4$ rigid bodies, n equations of the form (9.5.25)–(9.5.27) can be derived, where one equation is of the form (9.5.25) for the top link of the chain, $n-2$ equations are of the form (9.5.26) for the intermediate links of the chain, and one equation is of the form (9.5.27) for the bottom link of the chain.

Next, defining the column vectrices

$$\vec{\mathcal{V}}_{\mathrm{A}} \triangleq \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{B/A}} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\mathrm{B}} \triangleq \begin{bmatrix} \vec{v}_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{C/A}} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\mathrm{C}} \triangleq \begin{bmatrix} \vec{v}_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{D/A}} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\mathrm{D}} \triangleq \begin{bmatrix} \vec{v}_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{D/A}} \end{bmatrix} \tag{9.5.28}$$

and the physical matrices

$$\vec{\mathcal{T}}_{\mathrm{B/A}} \triangleq \begin{bmatrix} 0 & -\vec{r}^{\,\times}_{z_{\mathrm{B}}/z_{\mathrm{A}}} \\ 0 & \vec{I} \end{bmatrix}, \quad \vec{\mathcal{T}}_{\mathrm{C/B}} \triangleq \begin{bmatrix} \vec{I} & -\vec{r}^{\,\times}_{z_{\mathrm{C}}/z_{\mathrm{B}}} \\ 0 & \vec{I} \end{bmatrix}, \quad \vec{\mathcal{T}}_{\mathrm{D/C}} \triangleq \begin{bmatrix} \vec{I} & -\vec{r}^{\,\times}_{z_{\mathrm{D}}/z_{\mathrm{C}}} \\ 0 & \vec{I} \end{bmatrix}, \tag{9.5.29}$$

(9.5.19)–(9.5.21) can be written as

$$\vec{\mathcal{V}}_{\mathrm{B}} = \vec{\mathcal{T}}_{\mathrm{B/A}}\vec{\mathcal{V}}_{\mathrm{A}} + \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{C/B}} \end{bmatrix}, \tag{9.5.30}$$

$$\vec{\mathcal{V}}_{\mathrm{C}} = \vec{\mathcal{T}}_{\mathrm{C/B}}\vec{\mathcal{V}}_{\mathrm{B}} + \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{D/C}} \end{bmatrix}, \tag{9.5.31}$$

$$\vec{\mathcal{V}}_{\mathrm{D}} = \vec{\mathcal{T}}_{\mathrm{D/C}}\vec{\mathcal{V}}_{\mathrm{C}}. \tag{9.5.32}$$

Furthermore, defining the 6×1 vectors

$$\mathcal{V}_{\mathrm{A}} \triangleq \begin{bmatrix} 0 \\ \omega_{\mathrm{B/A|B}} \end{bmatrix}, \quad \mathcal{V}_{\mathrm{B}} \triangleq \begin{bmatrix} v_{z_{\mathrm{B}}/z_{\mathrm{A}}/\mathrm{A|B}} \\ \omega_{\mathrm{C/A|B}} \end{bmatrix}, \quad \mathcal{V}_{\mathrm{C}} \triangleq \begin{bmatrix} v_{z_{\mathrm{C}}/z_{\mathrm{A}}/\mathrm{A|C}} \\ \omega_{\mathrm{D/A|C}} \end{bmatrix}, \quad \mathcal{V}_{\mathrm{D}} \triangleq \begin{bmatrix} v_{z_{\mathrm{D}}/z_{\mathrm{A}}/\mathrm{A|D}} \\ \omega_{\mathrm{D/A|D}} \end{bmatrix} \tag{9.5.33}$$

and the 6×6 matrices

$$\mathcal{T}_{\mathrm{B/A}} \triangleq \begin{bmatrix} 0 & -r^{\times}_{z_{\mathrm{B}}/z_{\mathrm{A}}|\mathrm{B}} \\ 0 & I_3 \end{bmatrix}, \quad \mathcal{T}_{\mathrm{C/B}} \triangleq \begin{bmatrix} \mathcal{O}_{\mathrm{C/B}} & -r^{\times}_{z_{\mathrm{C}}/z_{\mathrm{B}}|\mathrm{C}}\mathcal{O}_{\mathrm{C/B}} \\ 0 & \mathcal{O}_{\mathrm{C/B}} \end{bmatrix}, \tag{9.5.34}$$

$$\mathcal{T}_{\mathrm{D/C}} \triangleq \begin{bmatrix} \mathcal{O}_{\mathrm{D/C}} & -r^{\times}_{z_{\mathrm{D}}/z_{\mathrm{C}}|\mathrm{D}}\mathcal{O}_{\mathrm{D/C}} \\ 0 & \mathcal{O}_{\mathrm{D/C}} \end{bmatrix}, \tag{9.5.35}$$

(9.5.25)–(9.5.27) can be written as

$$\mathcal{V}_{\mathrm{B}} = \mathcal{T}_{\mathrm{B/A}}\mathcal{V}_{\mathrm{A}} + \begin{bmatrix} 0 \\ \omega_{\mathrm{C/B|B}} \end{bmatrix}, \tag{9.5.36}$$

$$\mathcal{V}_{\mathrm{C}} = \mathcal{T}_{\mathrm{C/B}}\mathcal{V}_{\mathrm{B}} + \begin{bmatrix} 0 \\ \omega_{\mathrm{D/C|C}} \end{bmatrix}, \tag{9.5.37}$$

$$\mathcal{V}_{\mathrm{D}} = \mathcal{T}_{\mathrm{D/C}}\mathcal{V}_{\mathrm{C}}. \tag{9.5.38}$$

9.6 6D Acceleration Kinematics of a Chain of Rigid Bodies

Differentiating (9.5.19), (9.5.20), and (9.5.21) with respect to $\mathrm{F_B}$, $\mathrm{F_C}$, and $\mathrm{F_D}$, respectively, yields

$$\begin{bmatrix} \vec{a}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}/\mathrm{B}} \\ \vec{\alpha}_{\mathrm{C}/\mathrm{A}} \end{bmatrix} = \begin{bmatrix} 0 & -\vec{r}^{\times}_{z_\mathrm{B}/z_\mathrm{A}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{B}/\mathrm{A}} \end{bmatrix} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{C}/\mathrm{B}} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{\omega}_{\mathrm{C}/\mathrm{B}} \end{bmatrix}, \tag{9.6.1}$$

$$\begin{bmatrix} \vec{a}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}/\mathrm{C}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\times}_{z_\mathrm{C}/z_\mathrm{B}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{a}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}/\mathrm{B}} \\ \vec{\alpha}_{\mathrm{C}/\mathrm{A}} \end{bmatrix} + \begin{bmatrix} \vec{\omega}_{\mathrm{B}/\mathrm{C}} \times \vec{v}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{C}} + \vec{\omega}_{\mathrm{C}/\mathrm{A}} \times \vec{\omega}_{\mathrm{D}/\mathrm{C}} \end{bmatrix}, \tag{9.6.2}$$

$$\begin{bmatrix} \vec{a}_{z_\mathrm{D}/z_\mathrm{A}/\mathrm{A}/\mathrm{D}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \vec{I} & -\vec{r}^{\times}_{z_\mathrm{D}/z_\mathrm{C}} \\ 0 & \vec{I} \end{bmatrix} \begin{bmatrix} \vec{a}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}/\mathrm{C}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{A}} \end{bmatrix} + \begin{bmatrix} \vec{\omega}_{\mathrm{C}/\mathrm{D}} \times \vec{v}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}} \\ 0 \end{bmatrix}. \tag{9.6.3}$$

Defining

$$\vec{\mathcal{A}}_\mathrm{A} \triangleq \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{B}/\mathrm{A}} \end{bmatrix}, \quad \vec{\mathcal{A}}_\mathrm{B} \triangleq \begin{bmatrix} \vec{a}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}/\mathrm{B}} \\ \vec{\alpha}_{\mathrm{C}/\mathrm{A}} \end{bmatrix}, \quad \vec{\mathcal{A}}_\mathrm{C} \triangleq \begin{bmatrix} \vec{a}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}/\mathrm{C}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{A}} \end{bmatrix}, \quad \vec{\mathcal{A}}_\mathrm{D} \triangleq \begin{bmatrix} \vec{a}_{z_\mathrm{D}/z_\mathrm{A}/\mathrm{A}/\mathrm{D}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{A}} \end{bmatrix}, \tag{9.6.4}$$

and using (9.5.29), (9.6.1)–(9.6.3) can be written as

$$\vec{\mathcal{A}}_\mathrm{B} = \vec{\mathcal{T}}_{\mathrm{B}/\mathrm{A}} \vec{\mathcal{A}}_\mathrm{A} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{C}/\mathrm{B}} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{\omega}_{\mathrm{C}/\mathrm{B}} \end{bmatrix}, \tag{9.6.5}$$

$$\vec{\mathcal{A}}_\mathrm{C} = \vec{\mathcal{T}}_{\mathrm{C}/\mathrm{B}} \vec{\mathcal{A}}_\mathrm{B} + \begin{bmatrix} \vec{\omega}_{\mathrm{B}/\mathrm{C}} \times \vec{v}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}} \\ \vec{\alpha}_{\mathrm{D}/\mathrm{C}} + \vec{\omega}_{\mathrm{C}/\mathrm{A}} \times \vec{\omega}_{\mathrm{D}/\mathrm{C}} \end{bmatrix}, \tag{9.6.6}$$

$$\vec{\mathcal{A}}_\mathrm{D} = \vec{\mathcal{T}}_{\mathrm{D}/\mathrm{C}} \vec{\mathcal{A}}_\mathrm{C} + \begin{bmatrix} \vec{\omega}_{\mathrm{C}/\mathrm{D}} \times \vec{v}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}} \\ 0 \end{bmatrix}. \tag{9.6.7}$$

Next, by defining the acceleration vectrices that depend on angular accelerations

$$\vec{\mathcal{A}}_{\mathrm{B},\alpha} \triangleq \vec{\mathcal{T}}_{\mathrm{B}/\mathrm{A}} \vec{\mathcal{A}}_\mathrm{A} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{C}/\mathrm{B}} \end{bmatrix}, \tag{9.6.8}$$

$$\vec{\mathcal{A}}_{\mathrm{C},\alpha} \triangleq \vec{\mathcal{T}}_{\mathrm{C}/\mathrm{B}} \vec{\mathcal{A}}_{\mathrm{B},\alpha} + \begin{bmatrix} 0 \\ \vec{\alpha}_{\mathrm{D}/\mathrm{C}} \end{bmatrix}, \tag{9.6.9}$$

$$\vec{\mathcal{A}}_{\mathrm{D},\alpha} \triangleq \vec{\mathcal{T}}_{\mathrm{D}/\mathrm{C}} \vec{\mathcal{A}}_{\mathrm{C},\alpha} \tag{9.6.10}$$

and the acceleration vectrices that depend on angular velocities

$$\vec{\mathcal{A}}_{\mathrm{B},\omega} \triangleq \begin{bmatrix} 0 \\ \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{\omega}_{\mathrm{C}/\mathrm{B}} \end{bmatrix}, \tag{9.6.11}$$

$$\vec{\mathcal{A}}_{\mathrm{C},\omega} \triangleq \vec{\mathcal{T}}_{\mathrm{C}/\mathrm{B}} \vec{\mathcal{A}}_{\mathrm{B},\omega} + \begin{bmatrix} \vec{\omega}_{\mathrm{B}/\mathrm{C}} \times \vec{v}_{z_\mathrm{B}/z_\mathrm{A}/\mathrm{A}} \\ \vec{\omega}_{\mathrm{C}/\mathrm{A}} \times \vec{\omega}_{\mathrm{D}/\mathrm{C}} \end{bmatrix}, \tag{9.6.12}$$

$$\vec{\mathcal{A}}_{\mathrm{D},\omega} \triangleq \vec{\mathcal{T}}_{\mathrm{D}/\mathrm{C}} \vec{\mathcal{A}}_{\mathrm{C},\omega} + \begin{bmatrix} \vec{\omega}_{\mathrm{C}/\mathrm{D}} \times \vec{v}_{z_\mathrm{C}/z_\mathrm{A}/\mathrm{A}} \\ 0 \end{bmatrix}, \tag{9.6.13}$$

it follows that

$$\vec{\mathcal{A}}_\mathrm{B} = \vec{\mathcal{A}}_{\mathrm{B},\alpha} + \vec{\mathcal{A}}_{\mathrm{B},\omega}, \tag{9.6.14}$$

$$\vec{\mathcal{A}}_\mathrm{C} = \vec{\mathcal{A}}_{\mathrm{C},\alpha} + \vec{\mathcal{A}}_{\mathrm{C},\omega}, \tag{9.6.15}$$

$$\vec{\mathcal{A}}_{\rm D} = \vec{\mathcal{A}}_{\rm D,\alpha} + \vec{\mathcal{A}}_{\rm D,\omega}. \tag{9.6.16}$$

9.7 6D Dynamics of a Chain of Rigid Bodies

For the chain of rigid bodies shown in Figure 9.4.1, assume that $z_{\rm A}$ has zero inertial acceleration, let $c_{\rm B}$, $c_{\rm C}$, and $c_{\rm D}$ denote the center of mass of $\mathcal{B}_{\rm B}$, $\mathcal{B}_{\rm C}$, and $\mathcal{B}_{\rm D}$, respectively. and define

$$\vec{\mathcal{J}}_{{\rm B}/z_{\rm B}} \triangleq \begin{bmatrix} m_{\rm B}\vec{I} & -m_{\rm B}\vec{r}^{\times}_{c_{\rm B}/z_{\rm B}} \\ m_{\rm B}\vec{r}^{\times}_{c_{\rm B}/z_{\rm B}} & \vec{J}_{{\rm B}/z_{\rm B}} \end{bmatrix}, \quad \vec{\mathcal{V}}_{\rm B} \triangleq \begin{bmatrix} \vec{v}_{z_{\rm B}/w/{\rm A}} \\ \vec{\omega}_{\rm B/A} \end{bmatrix}, \quad \vec{\mathcal{F}}_{{\rm B}/z_{\rm B}} \triangleq \begin{bmatrix} \vec{f}_{\rm B} \\ \vec{M}_{{\rm B}/z_{\rm B}} \end{bmatrix}, \tag{9.7.1}$$

and likewise for $\mathcal{B}_{\rm C}$ and $\mathcal{B}_{\rm D}$. Note that the moment $\vec{M}_{{\rm B}/z_{\rm B}}$ on $\mathcal{B}_{\rm B}$ relative to $z_{\rm B}$ includes external moments on $\mathcal{B}_{\rm B}$ relative to $z_{\rm B}$, reaction moments on $\mathcal{B}_{\rm B}$ due to forces on $\mathcal{B}_{\rm B}$ at the joints connecting $\mathcal{B}$ to $\mathcal{A}$ and $\mathcal{C}$, and reaction torques on $\mathcal{B}_{\rm B}$ due to the joints connecting $\mathcal{B}$ to $\mathcal{A}$ and $\mathcal{C}$. It thus follows from (9.3.9) that

$$\vec{\mathcal{J}}_{{\rm B}/z_{\rm B}}\vec{\mathcal{A}}_{\rm B} + \vec{\mathcal{V}}^{\times}_{\rm B}\vec{\mathcal{J}}_{{\rm B}/z_{\rm B}}\vec{\mathcal{V}}_{\rm B} = \vec{\mathcal{F}}_{{\rm B}/z_{\rm B}}, \tag{9.7.2}$$

$$\vec{\mathcal{J}}_{{\rm C}/z_{\rm C}}\vec{\mathcal{A}}_{\rm C} + \vec{\mathcal{V}}^{\times}_{\rm C}\vec{\mathcal{J}}_{{\rm C}/z_{\rm C}}\vec{\mathcal{V}}_{\rm C} = \vec{\mathcal{F}}_{{\rm C}/z_{\rm C}}, \tag{9.7.3}$$

$$\vec{\mathcal{J}}_{{\rm D}/z_{\rm D}}\vec{\mathcal{A}}_{\rm D} + \vec{\mathcal{V}}^{\times}_{\rm D}\vec{\mathcal{J}}_{{\rm D}/z_{\rm D}}\vec{\mathcal{V}}_{\rm D} = \vec{\mathcal{F}}_{{\rm D}/z_{\rm D}}. \tag{9.7.4}$$

$$\tag{9.7.5}$$

where

$$\vec{\mathcal{A}}_{\rm B} \triangleq \overset{{\rm B}\bullet}{\vec{\mathcal{V}}}_{\rm B}, \quad \vec{\mathcal{A}}_{\rm C} \triangleq \overset{{\rm C}\bullet}{\vec{\mathcal{V}}}_{\rm C}, \quad \vec{\mathcal{A}}_{\rm D} \triangleq \overset{{\rm D}\bullet}{\vec{\mathcal{V}}}_{\rm D}. \tag{9.7.6}$$

The forces and moments can be written as

$$\vec{\mathcal{F}}_{{\rm B}/z_{\rm B}} = \vec{\mathcal{F}}_{{\rm ext,B}/z_{\rm B}} + \vec{\mathcal{F}}_{\rm int,B}, \tag{9.7.7}$$

$$\vec{\mathcal{F}}_{{\rm C}/z_{\rm C}} = \vec{\mathcal{F}}_{{\rm ext,C}/z_{\rm C}} + \vec{\mathcal{F}}_{\rm int,C}, \tag{9.7.8}$$

$$\vec{\mathcal{F}}_{{\rm D}/z_{\rm D}} = \vec{\mathcal{F}}_{{\rm ext,D}/z_{\rm D}} + \vec{\mathcal{F}}_{\rm int,D}, \tag{9.7.9}$$

where "ext" denotes externally applied forces and moments, and "int" denotes reaction forces and torques applied at the joints. Hence,

$$\vec{\mathcal{F}}_{\rm int,B} = \vec{\mathcal{F}}_{{\rm int,B}/z_{\rm A}} + \vec{\mathcal{F}}_{{\rm int,B}/z_{\rm B}} = \vec{\mathcal{F}}_{{\rm int,B}/z_{\rm A}} - \vec{\mathcal{F}}_{{\rm int,C}/z_{\rm B}}, \tag{9.7.10}$$

$$\vec{\mathcal{F}}_{\rm int,C} = -\vec{\mathcal{F}}_{{\rm int,B}/z_{\rm B}} + \vec{\mathcal{F}}_{{\rm int,C}/z_{\rm C}} = \vec{\mathcal{F}}_{{\rm int,C}/z_{\rm B}} - \vec{\mathcal{F}}_{{\rm int,D}/z_{\rm C}}, \tag{9.7.11}$$

$$\vec{\mathcal{F}}_{\rm int,D} = -\vec{\mathcal{F}}_{{\rm int,C}/z_{\rm C}} = \vec{\mathcal{F}}_{{\rm int,D}/z_{\rm C}}. \tag{9.7.12}$$

Combining (9.7.2)–(9.7.4), (9.7.7)–(9.7.9), and (9.7.10)–(9.7.12) yields

$$\vec{\mathcal{J}}_{{\rm B}/z_{\rm B}}\vec{\mathcal{A}}_{\rm B} + \vec{\mathcal{V}}^{\times}_{\rm B}\vec{\mathcal{J}}_{{\rm B}/z_{\rm B}}\vec{\mathcal{V}}_{\rm B} = \vec{\mathcal{F}}_{{\rm ext,B}/z_{\rm B}} + \vec{\mathcal{F}}_{{\rm int,B}/z_{\rm A}} - \vec{\mathcal{F}}_{{\rm int,C}/z_{\rm B}}, \tag{9.7.13}$$

$$\vec{\mathcal{J}}_{{\rm C}/z_{\rm C}}\vec{\mathcal{A}}_{\rm C} + \vec{\mathcal{V}}^{\times}_{\rm C}\vec{\mathcal{J}}_{{\rm C}/z_{\rm C}}\vec{\mathcal{V}}_{\rm C} = \vec{\mathcal{F}}_{{\rm ext,C}/z_{\rm C}} + \vec{\mathcal{F}}_{{\rm int,C}/z_{\rm B}} - \vec{\mathcal{F}}_{{\rm int,D}/z_{\rm C}}, \tag{9.7.14}$$

$$\vec{\mathcal{J}}_{{\rm D}/z_{\rm D}}\vec{\mathcal{A}}_{\rm D} + \vec{\mathcal{V}}^{\times}_{\rm D}\vec{\mathcal{J}}_{{\rm D}/z_{\rm D}}\vec{\mathcal{V}}_{\rm D} = \vec{\mathcal{F}}_{{\rm ext,D}/z_{\rm D}} + \vec{\mathcal{F}}_{{\rm int,D}/z_{\rm C}}. \tag{9.7.15}$$

Next, writing $\vec{\mathcal{A}}_{\mathrm{B}}$, $\vec{\mathcal{A}}_{\mathrm{B}}$, and $\vec{\mathcal{A}}_{\mathrm{B}}$ as

$$\vec{\mathcal{A}}_{\mathrm{B}} = \vec{\mathcal{A}}_{\mathrm{B},\alpha} + \vec{\mathcal{A}}_{\mathrm{B},\omega}, \quad \vec{\mathcal{A}}_{\mathrm{C}} = \vec{\mathcal{A}}_{\mathrm{C},\alpha} + \vec{\mathcal{A}}_{\mathrm{C},\omega}, \quad \vec{\mathcal{A}}_{\mathrm{D}} = \vec{\mathcal{A}}_{\mathrm{D},\alpha} + \vec{\mathcal{A}}_{\mathrm{D},\omega}. \tag{9.7.16}$$

where $\vec{\mathcal{A}}_{\mathrm{B},\alpha}$, $\vec{\mathcal{A}}_{\mathrm{C},\alpha}$, $\vec{\mathcal{A}}_{\mathrm{D},\alpha}$, $\vec{\mathcal{A}}_{\mathrm{B},\omega}$, $\vec{\mathcal{A}}_{\mathrm{C},\omega}$, $\vec{\mathcal{A}}_{\mathrm{D},\omega}$ are defined by (9.6.8)–(9.6.13), it follows that (9.7.13)–(9.7.15) can be written as

$$\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\alpha} = -\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\omega} - \vec{\mathcal{V}}_{\mathrm{B}}^{\times}\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{V}}_{\mathrm{B}} + \vec{\mathcal{F}}_{\mathrm{ext,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} - \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}}, \tag{9.7.17}$$

$$\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\alpha} = -\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\omega} - \vec{\mathcal{V}}_{\mathrm{C}}^{\times}\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{V}}_{\mathrm{C}} + \vec{\mathcal{F}}_{\mathrm{ext,C}/z_{\mathrm{C}}} + \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}} - \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}, \tag{9.7.18}$$

$$\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\alpha} = -\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\omega} - \vec{\mathcal{V}}_{\mathrm{D}}^{\times}\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{V}}_{\mathrm{D}} + \vec{\mathcal{F}}_{\mathrm{ext,D}/z_{\mathrm{D}}} + \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}. \tag{9.7.19}$$

Finally, we reorder and rewrite (9.7.17)–(9.7.19) as

$$\vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}} = \vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\alpha} + \vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{A}}_{\mathrm{D},\omega} + \vec{\mathcal{V}}_{\mathrm{D}}^{\times}\vec{\mathcal{J}}_{\mathrm{D}/z_{\mathrm{D}}}\vec{\mathcal{V}}_{\mathrm{D}} - \vec{\mathcal{F}}_{\mathrm{ext,D}/z_{\mathrm{D}}}, \tag{9.7.20}$$

$$\vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}} = \vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\alpha} + \vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{A}}_{\mathrm{C},\omega} + \vec{\mathcal{V}}_{\mathrm{C}}^{\times}\vec{\mathcal{J}}_{\mathrm{C}/z_{\mathrm{C}}}\vec{\mathcal{V}}_{\mathrm{C}} - \vec{\mathcal{F}}_{\mathrm{ext,C}/z_{\mathrm{C}}} + \vec{\mathcal{F}}_{\mathrm{int,D}/z_{\mathrm{C}}}, \tag{9.7.21}$$

$$\vec{\mathcal{F}}_{\mathrm{int,B}/z_{\mathrm{A}}} = \vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\alpha} + \vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{A}}_{\mathrm{B},\omega} + \vec{\mathcal{V}}_{\mathrm{B}}^{\times}\vec{\mathcal{J}}_{\mathrm{B}/z_{\mathrm{B}}}\vec{\mathcal{V}}_{\mathrm{B}} - \vec{\mathcal{F}}_{\mathrm{ext,B}/z_{\mathrm{B}}} + \vec{\mathcal{F}}_{\mathrm{int,C}/z_{\mathrm{B}}}. \tag{9.7.22}$$

Note that, assuming that the velocities and accelerations are known, (9.7.20)–(9.7.22) can be solved recursively for the reaction torques and moments. The same technique can be applied to a chain consisting of $n \geq 4$ rigid bodies, where the recursion proceeds from the bottom link of the chain to the top link of the chain.

9.8 Theoretical Problems

Problem 9.8.1. Let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames with origins $\mathrm{o_A}$ and $\mathrm{o_B}$, respectively, and let x be a point. Show that

$$\begin{bmatrix} \vec{r}_{x/\mathrm{o_B}}\big|_{\mathrm{B}} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{\mathrm{B/A}} & \vec{r}_{\mathrm{o_A}/\mathrm{o_B}}\big|_{\mathrm{B}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{r}_{x/\mathrm{o_A}}\big|_{\mathrm{A}} \\ 1 \end{bmatrix}.$$

Furthermore, show that

$$\begin{bmatrix} \mathcal{O}_{\mathrm{B/A}} & \vec{r}_{\mathrm{o_A}/\mathrm{o_B}}\big|_{\mathrm{B}} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{O}_{\mathrm{A/B}} & -\mathcal{O}_{\mathrm{A/B}}\,\vec{r}_{\mathrm{o_A}/\mathrm{o_B}}\big|_{\mathrm{B}} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{\mathrm{A/B}} & \vec{r}_{\mathrm{o_B}/\mathrm{o_A}}\big|_{\mathrm{A}} \\ 0 & 1 \end{bmatrix}.$$

Chapter Ten
Kinetic and Potential Energy

10.1 Kinetic Energy of Particles and Bodies

Definition 10.1.1. Let y be a particle with mass m, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *kinetic energy of y relative to w with respect to* $\mathrm{F_A}$ is defined by

$$T_{y/w/\mathrm{A}} \triangleq \tfrac{1}{2}m|\vec{v}_{y/w/\mathrm{A}}|^2. \tag{10.1.1}$$

Definition 10.1.2. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let w be a point, and let $\mathrm{F_A}$ be a frame. Then, the *kinetic energy* of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is defined by

$$T_{\mathcal{B}/w/\mathrm{A}} \triangleq \frac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/w/\mathrm{A}}|^2. \tag{10.1.2}$$

The follow result shows that the kinetic energy of a body is the sum of the kinetic energies of its component bodies.

Fact 10.1.3. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bodies, let $\mathcal{B}$ be the body consisting of $\mathcal{B}_1$ and $\mathcal{B}_2$, let w be a point, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = T_{\mathcal{B}_1/w/\mathrm{A}} + T_{\mathcal{B}_2/w/\mathrm{A}}. \tag{10.1.3}$$

The following result relates the kinetic energies of a body relative to two different points.

Fact 10.1.4. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w and z be points, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = T_{\mathcal{B}/z/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2. \tag{10.1.4}$$

Proof. Note that

$$\begin{aligned}
T_{\mathcal{B}/w/\mathrm{A}} &= \frac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/w/\mathrm{A}}\vec{v}_{y_i/w/\mathrm{A}}\\
&= \frac{1}{2}\sum_{i=1}^{l} m_i(\vec{v}_{y_i/z/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}})'(\vec{v}_{y_i/z/\mathrm{A}} + \vec{v}_{z/w/\mathrm{A}})\\
&= \frac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{A}} + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{z/w/\mathrm{A}} + \frac{1}{2}m_{\mathcal{B}}\sum_{i=1}^{l}\vec{v}'_{z/w/\mathrm{A}}\vec{v}_{z/w/\mathrm{A}}
\end{aligned}$$

$$= T_{\mathcal{B}/z/\mathrm{A}} + m_{\mathcal{B}}\vec{v}'_{z/w/\mathrm{A}}\vec{v}_{\mathrm{c}/z/\mathrm{A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/\mathrm{A}}|^2. \qquad \square$$

Choosing z to be the center of mass yields the following result, which shows that the kinetic energy of a body relative to an arbitrary point can be viewed as the kinetic energy of the body relative to its center of mass plus the kinetic energy of the mass of the body concentrated at the center of mass.

Fact 10.1.5. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w be a point, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2. \tag{10.1.5}$$

The following result expresses the kinetic energy $T_{\mathcal{B}/\mathrm{c}/\mathrm{A}}$ of $\mathcal{B}$ relative to its center of mass in terms of the relative velocities of its particles.

Fact 10.1.6. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, and let $\mathrm{F_A}$ be a frame. Then,

$$T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = \tfrac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{l} \frac{m_i m_j}{m_{\mathcal{B}}}|\vec{v}_{y_j/y_i/\mathrm{A}}|^2. \tag{10.1.6}$$

Now assume that $\mathcal{B}$ is rigid and $\mathrm{F_A}$ is body fixed. Then,

$$T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = 0. \tag{10.1.7}$$

Finally, let w be a point. Then,

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2. \tag{10.1.8}$$

Proof. Note that, for all $i = 1, \ldots, l$,

$$\begin{aligned}\vec{v}_{\mathrm{c}/y_i/\mathrm{A}} &= \frac{1}{m_{\mathcal{B}}}\sum_{j=1}^{l} m_j \vec{v}_{y_j/y_i/\mathrm{A}}\\ &= \frac{1}{m_{\mathcal{B}}}\sum_{j=1}^{l} m_j(\vec{v}_{y_j/\mathrm{c}/\mathrm{A}} - \vec{v}_{y_i/\mathrm{c}/\mathrm{A}}).\end{aligned}$$

Next, we note the identity

$$\sum_{i=1}^{l} m_i \left|\sum_{j=1}^{l} m_j(\vec{v}_{y_j/\mathrm{c}/\mathrm{A}} - \vec{v}_{y_i/\mathrm{c}/\mathrm{A}})\right|^2 = \frac{m_{\mathcal{B}}}{2}\sum_{i,j=1}^{l} m_i m_j|\vec{v}_{y_j/\mathrm{c}/\mathrm{A}} - \vec{v}_{y_i/\mathrm{c}/\mathrm{A}}|^2.$$

Therefore,

$$\begin{aligned}T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} &= \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/\mathrm{c}/\mathrm{A}}|^2\\ &= \tfrac{1}{2}\sum_{i=1}^{l} \frac{m_i}{m_{\mathcal{B}}^2}\left|\sum_{j=1}^{l} m_j(\vec{v}_{y_j/\mathrm{c}/\mathrm{A}} - \vec{v}_{y_i/\mathrm{c}/\mathrm{A}})\right|^2\end{aligned}$$

$$= \tfrac{1}{4}\sum_{i,j=1}^{l}\frac{m_i m_j}{m_{\mathcal{B}}}|\vec{v}_{y_j/\mathrm{c}/\mathrm{A}} - \vec{v}_{y_i/\mathrm{c}/\mathrm{A}}|^2$$

$$= \tfrac{1}{4}\sum_{i,j=1}^{l}\frac{m_i m_j}{m_{\mathcal{B}}}|\vec{v}_{y_j/y_i/\mathrm{A}}|^2$$

$$= \tfrac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{l}\frac{m_i m_j}{m_{\mathcal{B}}}|\vec{v}_{y_j/y_i/\mathrm{A}}|^2.$$

Now assume that $\mathcal{B}$ is rigid and $\mathrm{F_A}$ is body fixed. Then, for all $i, j \in \{1,\ldots,l\}$, it follows that $\vec{v}_{y_j/y_i/\mathrm{A}} = 0$, which implies (10.1.7). Finally, (10.1.8) follows from (10.1.7) and (10.1.5). □

The following result relates the kinetic energies of a body relative to the same point but with respect to two different frames.

Fact 10.1.7. Let $\mathcal{B}$ be a body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let z be a point, and let $\mathrm{F_A}$ and $\mathrm{F_B}$ be frames. Then,

$$T_{\mathcal{B}/z/\mathrm{A}} + T_{\mathcal{B}/z/\mathrm{B}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{B}}. \tag{10.1.9}$$

If $\vec{\omega}_{\mathrm{B}/\mathrm{A}} = 0$, then

$$T_{\mathcal{B}/z/\mathrm{A}} = T_{\mathcal{B}/z/\mathrm{B}}. \tag{10.1.10}$$

Alternatively, if $\mathcal{B}$ is rigid, $\mathrm{F_B}$ is body fixed, and z is fixed in $\mathcal{B}$, then

$$T_{\mathcal{B}/z/\mathrm{A}} = \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}}. \tag{10.1.11}$$

Proof. Note that

$$\begin{aligned}
T_{\mathcal{B}/z/\mathrm{A}} + T_{\mathcal{B}/z/\mathrm{B}} &= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{A}} + \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{B}}\vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i(\vec{v}_{y_i/z/\mathrm{A}} - \vec{v}_{y_i/z/\mathrm{B}})'(\vec{v}_{y_i/z/\mathrm{A}} - \vec{v}_{y_i/z/\mathrm{B}}) + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z})'(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{y_i/z}) + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{r}^{\times\prime}_{y_i/z}\vec{r}^{\times}_{y_i/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{B}}\\
&= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{\mathrm{B}/\mathrm{A}} + \sum_{i=1}^{l} m_i\vec{v}'_{y_i/z/\mathrm{A}}\vec{v}_{y_i/z/\mathrm{B}}.
\end{aligned}$$

□

Choosing z to be c in Fact 10.1.7 and using Fact 10.1.6 yields the following result.

Fact 10.1.8. Let $\mathrm{F_A}$ be a frame, let $\mathcal{B}$ be a rigid body composed of particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let $\mathrm{F_B}$ be a body-fixed frame, and let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$.

Then,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} = \tfrac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{l}\frac{m_i m_j}{m_{\mathcal{B}}}|\vec{v}_{y_j/y_i/A}|^2. \tag{10.1.12}$$

The following result expresses the kinetic energy of a body in terms of its physical inertia matrix and center of mass.

Fact 10.1.9. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w and z be points, and let F_A and F_B be frames. Then,

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{B/A} + m_{\mathcal{B}}\vec{v}_{z/w/A}^{\,\prime}(\vec{\omega}_{B/A}\times\vec{r}_{c/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/A}|^2 + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/z/B}|^2. \tag{10.1.13}$$

If, in addition, $\mathcal{B}$ is rigid, F_B is body fixed, and z is fixed in $\mathcal{B}$, then

$$\begin{aligned} T_{\mathcal{B}/w/A} &= \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{B/A} + m_{\mathcal{B}}\vec{v}_{z/w/A}^{\,\prime}(\vec{\omega}_{B/A}\times\vec{r}_{c/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/A}|^2 \\ &= \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{H}_{\mathcal{B}/z/A} + m_{\mathcal{B}}\vec{v}_{z/w/A}^{\,\prime}(\vec{\omega}_{B/A}\times\vec{r}_{c/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/A}|^2. \end{aligned} \tag{10.1.14}$$

Proof. It follows from (10.1.4) that

$$\begin{aligned} T_{\mathcal{B}/w/A} &= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}_{y_i/z/A}^{\,\prime}\vec{v}_{y_i/z/A} + m_{\mathcal{B}}\vec{v}_{z/w/A}^{\,\prime}\vec{v}_{c/z/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/A}|^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{l} m_i\vec{v}_{y_i/z/B}^{\,\prime}\vec{v}_{y_i/z/B} + \tfrac{1}{2}\sum_{i=1}^{l} m_i(\vec{\omega}_{B/A}\times\vec{r}_{y_i/z})'(\vec{\omega}_{B/A}\times\vec{r}_{y_i/z}) \\ &\quad + \vec{v}_{z/w/A}^{\,\prime}\sum_{i=1}^{l} m_i(\vec{\omega}_{B/A}\times\vec{r}_{y_i/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/A}|^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/z/B}|^2 + \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{B/A} + m_{\mathcal{B}}\vec{v}_{z/w/A}^{\,\prime}(\vec{\omega}_{B/A}\times\vec{r}_{c/z}) + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{z/w/A}|^2. \quad \square \end{aligned}$$

Choosing z to be colocated with w in Fact 10.1.9 yields the following result.

Fact 10.1.10. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let z be a point, and let F_A and F_B be frames. Then,

$$T_{\mathcal{B}/z/A} = \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{B/A} + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/z/B}|^2. \tag{10.1.15}$$

If, in addition, $\mathcal{B}$ is rigid, F_B is body fixed, and z is fixed in $\mathcal{B}$, then

$$\begin{aligned} T_{\mathcal{B}/z/A} &= \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{J}_{\mathcal{B}/z}\vec{\omega}_{B/A} \\ &= \tfrac{1}{2}\vec{\omega}_{B/A}^{\,\prime}\vec{H}_{\mathcal{B}/z/A}. \end{aligned} \tag{10.1.16}$$

In particular,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{H}_{\mathcal{B}/c/A}. \tag{10.1.17}$$

Alternatively, choosing z to be the center of mass in Fact 10.1.9 yields the following result.

Fact 10.1.11. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, let w be a point, and let F_A and F_B be frames. Then,

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2 + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/c/B}|^2. \tag{10.1.18}$$

Now assume that $\mathcal{B}$ is rigid and F_B is body fixed. Then,

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2 = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{H}_{\mathcal{B}/c/A} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2. \tag{10.1.19}$$

If, in addition, F_A is body fixed, then

$$T_{\mathcal{B}/w/A} = \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{c/w/A}|^2. \tag{10.1.20}$$

Note that (10.1.20) is identical to (10.1.8).

Finally, choosing w to be the center of mass in Fact 10.1.11 yields the following result.

Fact 10.1.12. Let $\mathcal{B}$ be a body composed of particles $y_1, \ldots, y_l$ whose masses are $m_1, \ldots, m_l$, respectively, let $m_{\mathcal{B}}$ denote the mass of $\mathcal{B}$, and let F_A and F_B be frames. Then,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} + \tfrac{1}{2}\sum_{i=1}^{l} m_i|\vec{v}_{y_i/c/B}|^2. \tag{10.1.21}$$

Now assume that $\mathcal{B}$ is rigid and F_B is body fixed. Then,

$$T_{\mathcal{B}/c/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{J}_{\mathcal{B}/c}\vec{\omega}_{B/A} = \tfrac{1}{2}\vec{\omega}_{B/A}'\vec{H}_{\mathcal{B}/c/A}. \tag{10.1.22}$$

If, in addition, F_A is body fixed, then

$$T_{\mathcal{B}/c/A} = 0. \tag{10.1.23}$$

Note that (10.1.23) is identical to (10.1.7).

It follows from Fact 10.1.12 that, if $\mathcal{B}$ is rigid and F_A is a body-fixed frame, then $T_{\mathcal{B}/c/A} = 0$, which is the second statement of Fact 10.1.6.

10.2 Work Done by Forces and Moments on a Body

Energy is a relative concept. Potential energy depends on position relative to a specified reference point, while kinetic energy depends on velocity relative to a reference point and with respect to a specified frame. The *work done* on a body by a force or moment is the energy transferred to or removed from the body due to the force or moment.

The following result is a law of physics, and thus is not proved. Since this result concerns the effect of a force on a particle, and since this effect is governed by Newton's second law, the reference point is taken to be an unforced particle.

Fact 10.2.1. Let y be a particle and let w be an unforced particle. Then, the work done on y relative to w by the force $\vec{f}_y$ applied to y as y moves along the path $\mathcal{C}_y$ is given by

$$W_{y/w}(\vec{f}_y, \mathcal{C}_y) = \int_{\mathcal{C}_y} \vec{f}_y \cdot \mathrm{d}\vec{r}_{y/w}. \tag{10.2.1}$$

Now, let $\mathcal{B}$ be a body with particles $y_1, \ldots, y_l$. The work done on $\mathcal{B}$ relative to the point w by the force $\vec{f}_{y_i}$ applied to y_i as y_i moves along the path $\mathcal{C}_{y_i}$ for all $i = 1, \ldots, l$ is given by

$$W_{\mathcal{B}/w}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_{y_1}, \ldots, \mathcal{C}_{y_l}) = \sum_{i=1}^{l} W_{y_i/w}(\vec{f}_{y_i}, \mathcal{C}_{y_i}). \tag{10.2.2}$$

Notice that the mass of y plays no role. Therefore, we can consider a point y in place of a particle with the understanding that $W_{y/w}(\vec{f}_y, \mathcal{C}_y)$ denotes the energy transferred to y if y were a particle.

Let $\mathrm{F_A}$ be a frame. Then, we can rewrite (10.2.1) as

$$\begin{aligned} W_{y/w}(\vec{f}_y, \mathcal{C}_y) &= \int_0^{s_\mathrm{f}} \vec{f}_y(s) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s \\ &= \int_0^{s_\mathrm{f}} \vec{f}_y(s) \cdot \hat{e}_\mathrm{t}(s)\,\mathrm{d}s, \end{aligned} \tag{10.2.3}$$

where the path $\mathcal{C}$ is parameterized by the path length s in the interval $[0, s_\mathrm{f}]$ and, by (5.5.14), $\hat{e}_\mathrm{t}(s) = \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)$, which is the unit tangent vector to $\mathcal{C}_y$ at s.

The following result is the analogue of Fact 10.2.1 for the case of forces applied to a rigid body.

Fact 10.2.2. Let $\mathcal{B}$ be a rigid body with particles $y_1, \ldots, y_l$, for $i = 1, \ldots, l$, let $\vec{f}_{y_i}$ denote the force applied to y_i, and let $\mathcal{C}_{y_i}$ denote the path of y_i. Furthermore, let y be a point fixed in $\mathcal{B}$, let $\mathcal{C}_y$ denote the path of y, let $\vec{f}_\mathcal{B}$ denote the total force on $\mathcal{B}$, let w be an unforced particle, and let $\mathrm{F_A}$ be a frame. In addition, let the rotational path $\mathcal{C}_\mathcal{B}$ of $\mathcal{B}$ be given by $\vec{R}_{\mathrm{B/A}}(\alpha) = e^{\vec{\Theta}^{\times}_{\mathrm{B/A}}(\alpha)}$, where $\alpha \in [0, \alpha_\mathrm{f}]$. Then, the work done on $\mathcal{B}$ relative to the point w by the forces $\vec{f}_1, \ldots, \vec{f}_l$ is given by

$$W_{\mathcal{B}/w}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_{y_1}, \ldots, \mathcal{C}_{y_l}) = W_{y/w}(\vec{f}_\mathcal{B}, \mathcal{C}_y) + W_{\mathcal{B}/\mathrm{A}}(\vec{M}_\mathcal{B}, \mathcal{C}_\mathcal{B}), \tag{10.2.4}$$

where

$$W_{\mathcal{B}/\mathrm{A}}(\vec{M}_\mathcal{B}, \mathcal{C}_\mathcal{B}) \triangleq \int_{\mathcal{C}_\mathcal{B}} \vec{M}_\mathcal{B} \cdot G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}}, \tag{10.2.5}$$

$$\vec{M}_\mathcal{B} \triangleq \sum_{i=1}^{l} \vec{r}_{y_i/y} \times \vec{f}_{y_i}, \tag{10.2.6}$$

and

$$G(\vec{\Theta}_{\mathrm{B/A}}) \triangleq \frac{1}{|\vec{\Theta}_{\mathrm{B/A}}|^2}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right). \tag{10.2.7}$$

Proof. Let s denote the path length variable for the point y, for $i = 1, \ldots, l$, let s_i denote the path length variable for the point y_i. Therefore, for $i = 1, \ldots, l$, using the fact that $\vec{r}_{y_i/y}(\alpha) = \vec{R}_{\mathrm{B/A}}(\alpha)\vec{r}_{y_i/y}(0)$ as well as (4.9.8), we have

$$\begin{aligned}
\mathrm{d}\vec{r}_{y_i/y} &= \stackrel{\alpha\bullet}{\vec{r}}_{y_i/y}(\alpha)\mathrm{d}\alpha \\
&= \stackrel{\alpha\bullet}{\vec{R}}_{\mathrm{B/A}}(\alpha)\mathrm{d}\alpha\,\vec{r}_{y_i/y}(0) \\
&= \vec{\omega}^{\times}_{\mathrm{B/A}}(\alpha)\vec{R}_{\mathrm{B/A}}(\alpha)\mathrm{d}\alpha\,\vec{r}_{y_i/y}(0) \\
&= \vec{\omega}^{\times}_{\mathrm{B/A}}(\alpha)\vec{r}_{y_i/y}(\alpha)\mathrm{d}\alpha \\
&= \vec{\omega}_{\mathrm{B/A}}(\alpha) \times \vec{r}_{y_i/y}(\alpha)\mathrm{d}\alpha \\
&= -\vec{r}_{y_i/y}(\alpha) \times \vec{\omega}_{\mathrm{B/A}}(\alpha)\mathrm{d}\alpha \\
&= -\vec{r}_{y_i/y}(\alpha) \times \frac{1}{|\vec{\Theta}_{\mathrm{B/A}}|^2}\left(\vec{\Theta}_{\mathrm{B/A}}\vec{\Theta}'_{\mathrm{B/A}} + (\vec{I} - \vec{R}_{\mathrm{B/A}})\vec{\Theta}^{\times}_{\mathrm{B/A}}\right)\stackrel{\alpha\bullet}{\vec{\Theta}}_{\mathrm{B/A}}\,\mathrm{d}\alpha \\
&= -\vec{r}_{y_i/y} \times G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&W_{\mathcal{B}/w}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_{y_1}, \ldots, \mathcal{C}_{y_l}) \\
&\quad = \sum_{i=1}^{l} W_{y_i/w}(\vec{f}_{y_i}, \mathcal{C}_{y_i}) \\
&\quad = \sum_{i=1}^{l} \int_{\mathcal{C}_{y_i}} \vec{f}_{y_i} \cdot \mathrm{d}\vec{r}_{y_i/w} \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \mathrm{d}\vec{r}_{y_i/w}(s) \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \stackrel{\mathrm{A}s\bullet}{\vec{r}}_{y_i/w}(s)\,\mathrm{d}s \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot [\stackrel{\mathrm{A}s\bullet}{\vec{r}}_{y_i/y}(s) + \stackrel{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)]\,\mathrm{d}s \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \stackrel{\mathrm{A}s\bullet}{\vec{r}}_{y_i/y}(s)\,\mathrm{d}s + \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \stackrel{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s \\
&\quad = \sum_{i=1}^{l} \int_0^{s_\mathrm{f}} \vec{f}_{y_i}(s) \cdot \stackrel{\mathrm{A}s\bullet}{\vec{r}}_{s_i, y_i/y}(s_i(s))\,\mathrm{d}s + \int_0^{s_\mathrm{f}} \sum_{i=1}^{l} \vec{f}_{y_i}(s) \cdot \stackrel{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{l} \int_0^{s_{\mathrm{f}}} \vec{f}_{y_i}(s) \cdot \overset{\mathrm{A}s_i\bullet}{\vec{r}}_{y_i/y}(s_i) \frac{\mathrm{d}s_i}{\mathrm{d}s}\mathrm{d}s + \int_0^{s_{\mathrm{f}}} \vec{f}_{\mathcal{B}}(s)\cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{y/w}(s)\,\mathrm{d}s \\
&= \sum_{i=1}^{l} \int_0^{s_{i,\mathrm{f}}} \vec{f}_{y_i}(s_i) \cdot \overset{\mathrm{A}s_i\bullet}{\vec{r}}_{y_i/y}(s_i)\,\mathrm{d}s_i + \int_0^{s_{\mathrm{f}}} \vec{f}_{\mathcal{B}}(s)\cdot \mathrm{d}\vec{r}_{y/w}(s) \\
&= \sum_{i=1}^{l} \int_{\mathcal{C}_{y_i}} \vec{f}_{y_i} \cdot \mathrm{d}\vec{r}_{y_i/y} + \int_0^{s_{\mathrm{f}}} \vec{f}_{\mathcal{B}}(s)\cdot \mathrm{d}\vec{r}_{y/w}(s) \\
&= -\int_{\mathcal{C}_{\mathcal{B}}} \sum_{i=1}^{l} \vec{f}_{y_i} \cdot (\vec{r}_{y_i/y} \times G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}}) + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= -\int_{\mathcal{C}_{\mathcal{B}}} \sum_{i=1}^{l} (\vec{f}_{y_i} \times \vec{r}_{y_i/y}) \cdot G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}} + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= \int_{\mathcal{C}_{\mathcal{B}}} \sum_{i=1}^{l} (\vec{r}_{y_i/y} \times \vec{f}_{y_i}) \cdot G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}} + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= \int_{\mathcal{C}_{\mathcal{B}}} \vec{M}_{\mathcal{B}} \cdot G(\vec{\Theta}_{\mathrm{B/A}})\,\mathrm{d}\vec{\Theta}_{\mathrm{B/A}} + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y) \\
&= W_{\mathcal{B}/\mathrm{A}}(\vec{M}_{\mathcal{B}}, \mathcal{C}_{\mathcal{B}}) + W_{y/w}(\vec{f}_{\mathcal{B}}, \mathcal{C}_y). \qquad \square
\end{aligned}
$$

10.3 Potential Energy of Particles and Bodies

The following result is fundamental.

Fact 10.3.1. Let y be a particle, let $\vec{f}_y$ be a force acting on y that depends only on the position of y, and let w be a point. Then, the following statements are equivalent:

i) For every path $\mathcal{C}$, $W_{y/w}(\vec{f}_y, \mathcal{C})$ depends on the initial and final endpoints z_0 and z_1, respectively, of $\mathcal{C}$ but is otherwise independent of $\mathcal{C}$.

ii) There exists a function $U_{y/w}$ that maps physical position vectors into real numbers such that

$$\vec{f}_y = -\vec{\partial} U_{y/w}(\vec{r}_{y/w}). \tag{10.3.1}$$

In this case,

$$W_{y/w}(\vec{f}_y, \mathcal{C}) = U_{y/w}(\vec{r}_{z_0/w}) - U_{y/w}(\vec{r}_{z_1/w}), \tag{10.3.2}$$

that is,

$$U_{y/w}(\vec{r}_{z_1/w}) - U_{y/w}(\vec{r}_{z_0/w}) = -\int_{\mathcal{C}} \vec{f}_y \cdot \mathrm{d}\vec{r}_{y/w}. \tag{10.3.3}$$

Proof. To prove *ii*) implies *i*), let $z(s)$ denote the path of y along $\mathcal{C}$. Then, note that

$$
\begin{aligned}
W_{y/w}(\vec{f}_y, \mathcal{C}) &= \int_{\mathcal{C}} \vec{f}_y \cdot \mathrm{d}\vec{r}_{y/w} \\
&= -\int_{\mathcal{C}} \vec{\partial} U_{y/w}(\vec{r}_{z/w}) \cdot \mathrm{d}\vec{r}_{y/w}
\end{aligned}
$$

$$= -\int_{s_0}^{s_1} \overset{\rightharpoonup}{\partial} U_{y/w}(\vec{r}_{z(s)/w}) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{z(s)/w}\ \mathrm{d}s$$
$$= -\int_{s_0}^{s_1} \partial U_{y/w|\mathrm{A}}\left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}}\right) \cdot \overset{\mathrm{A}s\bullet}{\vec{r}}_{z(s)/w}\Big|_{\mathrm{A}}\ \mathrm{d}s$$
$$= -\int_{s_0}^{s_1} \partial U_{y/w|\mathrm{A}}\left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}s}\left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}}\right) \mathrm{d}s$$
$$= -\int_{s_0}^{s_1} \frac{\mathrm{d}}{\mathrm{d}s} U_{y/w|\mathrm{A}}\left(\vec{r}_{z(s)/w}\Big|_{\mathrm{A}}\right) \mathrm{d}s$$
$$= U_{y/w|\mathrm{A}}\left(\vec{r}_{z(s_0)/w}\Big|_{\mathrm{A}}\right) - U_{y/w|\mathrm{A}}\left(\vec{r}_{z(s_1)/w}\Big|_{\mathrm{A}}\right)$$
$$= U_{y/w}(\vec{r}_{z(s_0)/w}) - U_{y/w}(\vec{r}_{z(s_1)/w})$$
$$= U_{y/w}(\vec{r}_{z_0/w}) - U_{y/w}(\vec{r}_{z_1/w}).$$

To prove *i*) implies *ii*), suppose that y is located at z_0, and let $\mathcal{C}$ denote a path from z_0 to z_1. Since $W_{y/w}(\vec{f}_y, \mathcal{C})$ is independent of $\mathcal{C}$, we define

$$U_{y/w}(\vec{r}_{z_1/w}) \triangleq -W_{y/w}(\vec{f}_y, \mathcal{C}).$$

Next, let $\varepsilon > 0$, and let $z(s)$ for $s \in [0, \varepsilon]$ denote the path $\mathcal{C}_\varepsilon$ given by $\vec{r}_{z(s)/w} = s\hat{r}_{z_1/w}$. Hence $z(\varepsilon)$ denotes the point such that $\vec{r}_{z(\varepsilon)/z_0} = \varepsilon\hat{r}_{z_1/w}$. Note that $|\vec{r}_{z(\varepsilon)/w}| = \varepsilon$. We thus have

$$U_{y/w}(\vec{r}_{z(\varepsilon)/w}) = -\int_{\mathcal{C}_\varepsilon} \vec{f}_y \cdot \mathrm{d}\vec{r}_{z(s)/w} = -\int_0^\varepsilon \vec{f}_y(s)\,\mathrm{d}s \cdot \hat{r}_{z_1/w}.$$

□ Needs work

The force $\vec{f}_y$ acting on the particle y moving along the path $\mathcal{C}$ is a *potential force* if, for every point w, condition *i*) or, equivalently, condition *ii*), is satisfied. Furthermore, $U_{y/w}(\vec{r}_{z/w})$ is the *potential energy* of y located at z relative to w associated with the force $\vec{f}_y$. Note that $U_{y/w}(\vec{0}) = 0$.

Now consider a body $\mathcal{B}$ each of whose particles $y_1, \ldots, y_l$ is acted on by a potential force f_{y_i} associated with the potential energy $U_{y_i}(\vec{r}_{z_i/w})$, where z_{i0} and z_{i1} are the initial and final locations of y_i along $\mathcal{C}_i$, and w is a point. Then,

$$W_{\mathcal{B}}(\vec{f}_{y_1}, \ldots, \vec{f}_{y_l}, \mathcal{C}_1, \ldots, \mathcal{C}_l, w) = \sum_{i=1}^{l} [U_{y_i/w}(\vec{r}_{z_{i0}/w}) - U_{y_i/w}(\vec{r}_{z_{i1}/w})]. \tag{10.3.4}$$

The moment $\vec{M}_{\mathcal{B}/w}$ acting on $\mathcal{B}$ relative to w is a *potential moment* if it arises from forces that are potential.

We first consider the gravitational potential energy in a uniform gravitational field. Let $\vec{g}$ denote the acceleration due to gravity.

Fact 10.3.2. Let y be a particle located at the point z, let m be the mass of y, let w be a point, let $\mathrm{F_A}$ be a frame, and let the acceleration due to gravity be given by $\vec{g}$. Then, the force acting on y

due to gravity is given by $\vec{f}_y = -m\vec{g}$, and the potential energy of y relative to w is given by

$$U_{y/w}(\vec{r}_{z/w}) = -m\vec{r}^{\,\prime}_{z/w}\vec{g}. \tag{10.3.5}$$

Proof. Need proof. □

Let $\mathrm{F_A}$ be as in Fact 10.3.2, let $\vec{g} = -g\hat{k}_\mathrm{A}$, and let $\vec{r}\Big|_\mathrm{A} = [r_1\ \ r_2\ \ r_3]^\mathsf{T}$. Then,

$$U_{y/w|\mathrm{A}}(r) = mgr_3. \tag{10.3.6}$$

If, in addition, $\mathrm{F_B}$ is a frame, then

$$U_{y/w|\mathrm{B}}(r) = mge_3^\mathsf{T}\mathcal{O}_{\mathrm{B/A}}r. \tag{10.3.7}$$

where e_3 is the third column of I_3.

The gravitational potential energy of a body $\mathcal{B}$ due to a uniform gravitational field is defined to be the sum of the gravitational potential energy of each particle in $\mathcal{B}$.

Definition 10.3.3. Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ located at points $z_1, \ldots, z_l$, respectively. Then, the gravitational potential energy of $\mathcal{B}$ relative to w is defined by

$$U_{\mathcal{B}/w} \triangleq \sum_{i=1}^{l} U_{y_i w}(\vec{r}_{z_i/w}). \tag{10.3.8}$$

Fact 10.3.4. Let $\mathcal{B}$ be a body with total mass $m_\mathcal{B}$, let w be a point, let c denote the center of mass of $\mathcal{B}$, and let $\vec{g}$ be the acceleration due to gravity. Then, the gravitational potential energy of $\mathcal{B}$ relative to w is given by

$$U_{\mathcal{B}/w} = -m_\mathcal{B}\vec{r}^{\,\prime}_{\mathrm{c}/w}\vec{g}. \tag{10.3.9}$$

Proof. Note that

$$\begin{aligned} U_{\mathcal{B}/w} &= -\sum_{i=1}^{l}(m_i\vec{r}^{\,\prime}_{y_i/w}\vec{g}) \\ &= -\left(\sum_{i=1}^{l} m_i\vec{r}_{y_i/w}\right)^{\prime}\vec{g} \\ &= -m_\mathcal{B}\vec{r}^{\,\prime}_{\mathrm{c}/w}\vec{g}. \end{aligned}$$

□

Next we consider the potential energy due to a central gravitational field on the Earth with origin $\mathrm{o_E}$ at the center of the Earth. Let $\mu_\mathrm{E} = GM_\mathrm{E}$ denote the gravitational constant for the Earth, and

Fact 10.3.5. Let y be a particle with mass m, let $\mathrm{o_E}$ be the center of the Earth, let $\mathrm{F_E} = \mathrm{F_{sph}}$, and let the potential energy of y relative to $\mathrm{o_E}$ due to gravity be given by

$$U_{y/\mathrm{o_E}}(\vec{r}_{y/\mathrm{o_E}}) = -\frac{\mu_\mathrm{E} m}{\vec{r}^{\,\prime}_{y/\mathrm{o_E}}\hat{e}_\mathrm{u}}. \tag{10.3.10}$$

Then, the force $\vec{f}_{\rm grav}$ due to gravity is given by

$$\vec{f}_{\rm grav} = -mg\hat{e}_{\rm u},$$

where $g \triangleq \mu_{\rm E}/|\vec{r}_{y/{\rm o_E}}|^2$.

Proof. Recall that $\mathrm{F}_{\rm sph} = [\hat{e}_{\rm u}\ \ \hat{e}_{\rm e}\ \ \hat{e}_{\rm n}]$. Writing

$$U_{y/{\rm o_E}|{\rm E}}(\vec{r}_{y/{\rm o_E}}) = -\frac{\mu_{\rm E} m}{r_1},$$

where $\vec{r}_{y/{\rm o_E}}\Big|_{\rm E} = [r_1\ \ r_2\ \ r_3]^{\sf T}$, it follows that

$$\partial U_{y/{\rm o_E}|{\rm E}}(\vec{r}_{y/{\rm o_E}}) = \begin{bmatrix} \frac{\mu_{\rm E} m}{r_1^2} & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \vec{f}_{\rm grav}\Big|_{\rm E}^{\sf T} &= -\vec{\partial} U_{y/{\rm o_E}}(\vec{r}_{y/{\rm o_E}})\Big|_{\rm E} \\ &= -\partial U_{y/{\rm o_E}|{\rm E}}(|\vec{r}_{y/{\rm o_E}}|e_1) \\ &= -\frac{\mu_{\rm E} m}{|\vec{r}_{y/{\rm o_E}}|^2} e_1^{\sf T} \\ &= -mg\ \hat{e}_{\rm u}|_{\rm sph}^{\sf T}. \end{aligned}$$ □

Next, we consider the potential energy of a spring. Recall from (2.22.7) that the physical gradient $\vec{\partial} f(\vec{r})$ of f at $\vec{r}$ is defined by

$$\vec{\partial} f(\vec{r}) \triangleq \partial_{r_{|\rm A}} f_{\rm A}(r_{|\rm A}) \mathrm{F}_{\rm A}'^{\sf T}. \tag{10.3.11}$$

Fact 10.3.6. Assume that the point y is connected to the point w by a spring with stiffness k and whose relaxed length is d. Then, the force acting on y due to the spring is given by

$$\vec{f}_{{\rm s}/y/w} = -k(|\vec{r}_{y/w}| - d)\hat{r}_{y/w} = -\vec{\partial} U_{{\rm s}/y/w}(\vec{r}_{y/w}), \tag{10.3.12}$$

where the potential energy $U_{{\rm s}/y/w}$ of the spring connecting y and w is given by

$$U_{{\rm s}/y/w}(\vec{r}_{y/w}) = \tfrac{1}{2}k(|\vec{r}_{y/w}| - d)^2. \tag{10.3.13}$$

Proof. Let $\vec{r}_{z/w}\Big|_{\rm A} = \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix}$. Then,

$$U_{y/w|{\rm A}}\left(\vec{r}_{z/w}\Big|_{\rm A}\right) = \tfrac{1}{2}k(\overline{x}^2 + \overline{y}^2 + \overline{z}^2 - 2d\sqrt{\overline{x}^2 + \overline{y}^2 + \overline{z}^2} + d^2). \tag{10.3.14}$$

Therefore,

$$\partial_{\overline{x}} U_{y/w|{\rm A}}\left(\vec{r}_{z/w}\Big|_{\rm A}\right) = \frac{k}{|\vec{r}_{y/w}|}(|\vec{r}_{y/w}| - d)\overline{x},$$

and thus

$$\partial^{\mathsf{T}} U_{y/w|\mathrm{A}}\left(\vec{r}_{z/w}\Big|_{\mathrm{A}}\right) = \frac{k}{|\vec{r}_{y/w}|}(|\vec{r}_{y/w}| - d)\begin{bmatrix}\overline{x}\\ \overline{y}\\ \overline{z}\end{bmatrix}$$
$$= -k(|\vec{r}_{y/w}| - d)\ \hat{r}_{y/w}\Big|_{\mathrm{A}}$$
$$= -\vec{f}_{\mathrm{s}/y/w}\Big|_{\mathrm{A}},$$

which confirms (10.3.12). □

Finally, we consider the potential energy of a rotary spring.

Fact 10.3.7. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be rigid bodies connected by a pin joint at a point that is fixed in both bodies. Let $\hat{x}_1$ and $\hat{x}_2$ be unit dimensionless vectors that are fixed in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and that are orthogonal to the axis of the pin joint. Consider a rotary spring that applies torques to $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel with the axis of the pin joint, where $\kappa > 0$ is the rotary stiffness of the rotary spring and $\theta_0 \in (0, \pi)$ is the angle between $\hat{x}_1$ and $\hat{x}_2$ when the rotary spring is relaxed. Let $\hat{z}$ denote $\hat{\theta}_{\hat{x}_1/\hat{x}_2}$ when the rotary spring is relaxed. Then, the torque $\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}$ on $\mathcal{B}_1$ due to the rotary spring is given by

$$\vec{T}_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2} = -\kappa(\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}\hat{\theta}_{\hat{x}_1/\hat{x}_2} - \theta_0\hat{z}) = -\vec{\partial}\, U_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}(\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}\hat{\theta}_{\hat{x}_1/\hat{x}_2}). \tag{10.3.15}$$

where the potential energy $U_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}$ of the rotary spring connecting $\mathcal{B}_1$ and $\mathcal{B}_2$ is given by

$$U_{\mathrm{s}/\mathcal{B}_1/\mathcal{B}_2}(\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}\hat{\theta}_{\hat{x}_1/\hat{x}_2}) = \tfrac{1}{2}\kappa(|\theta_{\hat{x}_1/\hat{x}_2/\hat{z}}| - \theta_0)^2. \tag{10.3.16}$$

10.4 Conservation of Energy

The *total energy* of a body $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is defined by

$$E_{\mathcal{B}/w/\mathrm{A}} \triangleq T_{\mathcal{B}/w/\mathrm{A}} + U_{\mathcal{B}/w}. \tag{10.4.1}$$

Note that $E_{\mathcal{B}/w/\mathrm{A}}$ includes the kinetic energy of $\mathcal{B}$ as well as the potential energy associated with all potential forces acting on the particles in $\mathcal{B}$, such as gravity (which may be central or uniform) and springs. The body $\mathcal{B}$ is assumed to have finite mass. If $\mathcal{B}$ interacts with a massive body, then all reaction forces due to the massive body are viewed as internal forces, but the kinetic and potential energy of the massive body are not included. If $\mathcal{B}$ consists of particles $y_1, \ldots, y_l$, then

$$E_{\mathcal{B}/w/\mathrm{A}} = \sum_{i=1}^{l}[T_{y_i/w/\mathrm{A}} + U_{y_i/w}(\vec{r}_{y_i/w})]. \tag{10.4.2}$$

where, for $i = 1, \ldots, l$, $T_{y_i/w/\mathrm{A}} = \frac{1}{2}m_i|\vec{v}_{y_i/w/\mathrm{A}}|^2$.

The particles in a body can be subjected to both internal forces (that is, reaction forces) and external forces. Internal forces, such as sliding with friction, can lead to a decrease in energy, while external forces can lead to either a decrease or increase in energy. If the internal or external force $\vec{f}$ does not increase or decrease the total energy of the body, then $\vec{f}$ is a *conservative force*. The following result shows that all potential forces are conservative forces.

Fact 10.4.1. Let $\mathcal{B}$ be a body, let $\mathrm{F_A}$ be an inertial frame, let w be an unforced particle, and assume that all internal and external forces applied to $\mathcal{B}$ are potential forces. Then, the total energy of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is conserved.

Proof. For $i = 1, \ldots, l$, let $\overset{\rightharpoonup}{f}_{y_i}$ denote the total force applied to y_i. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathcal{B}/w/\mathrm{A}} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{l}[\tfrac{1}{2}m_i|\overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}}|^2 + U_{y_i/w}(\overset{\rightharpoonup}{r}_{y_i/w})] \\
&= \sum_{i=1}^{l}\left[\tfrac{1}{2}m_i\frac{\mathrm{d}}{\mathrm{d}t}(\overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}} \cdot \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}}) + \partial U_{y_i/w}(\overset{\rightharpoonup}{r}_{y_i/w}) \cdot \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}}\right] \\
&= \sum_{i=1}^{l}(m_i\overset{\rightharpoonup}{a}_{y_i/w/\mathrm{A}} \cdot \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}} - \overset{\rightharpoonup}{f}_{y_i} \cdot \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}}) \\
&= \sum_{i=1}^{l}[(m_i\overset{\rightharpoonup}{a}_{y_i/w/\mathrm{A}} - \overset{\rightharpoonup}{f}_{y_i}) \cdot \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}}] \\
&= 0.
\end{aligned}$$

□

10.5 Theoretical Problems

Problem 10.5.1. Let $\mathcal{B}$ be a body consisting of particles $y_1, \ldots, y_l$ with masses $m_1, \ldots, m_l$, respectively, let c denote the center of mass of $\mathcal{B}$, and let w be an unforced particle. Assume that, for all distinct $i, j \in \{1, \ldots, l\}$, the particles y_i and y_j are connected by a dashpot with viscosity c_{ij}. (Note that $c_{ij} = c_{ji}$ and $c_{ii} = 0$.) Assume, in addition, that no external forces are applied to $\mathcal{B}$.

i) Show that, for all $i \in \{1, \ldots, l\}$, $\lim_{t\to\infty} \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}}$ exists and, for all distinct $i, j \in \{1, \ldots, l\}$, $\lim_{t\to\infty} \overset{\rightharpoonup}{v}_{y_i/w/\mathrm{A}} = \lim_{t\to\infty} \overset{\rightharpoonup}{v}_{y_j/w/\mathrm{A}}$.

ii) Show that, for all distinct $i, j \in \{1, \ldots, l\}$, $\lim_{t\to\infty} \overset{\rightharpoonup}{v}_{y_i/y_j/\mathrm{A}} = 0$.

iii) Show that $\lim_{t\to\infty} T_{\mathcal{B}/\mathrm{c}/\mathrm{A}} = 0$.

iv) Show that $\lim_{t\to\infty} T_{\mathcal{B}/w/\mathrm{A}} = m_{\mathcal{B}}|\overset{\rightharpoonup}{v}_{\mathrm{c}/w/\mathrm{A}}|^2$.

Chapter Eleven
Lagrangian Dynamics

11.1 Lagrangian Dynamics versus Newton-Euler Dynamics

The equations of motion for a body that consists of multiple rigid bodies can be derived by applying Newton-Euler dynamics individually to each rigid body. Rigid bodies can interact with each other through rolling, sliding (with or without friction), and pivoting (with or without friction). This interaction can occur through joints, which may be revolute (rotatable) or prismatic (extendable).

To apply Newton-Euler dynamics to each rigid body, we must determine the forces and moments acting on each rigid body. Since each rigid body can interact with all other rigid bodies (including massive bodies) through reaction forces and moments as determined by Newton's third law, the total force and moment on each rigid body must include the forces and moments due to the interaction with the remaining bodies as well as the forces and moments due to gravity.

A rigid body may also be subject to a constraint, such as a connection to another, possibly massive, rigid body by means of a revolute or prismatic joint. The constraint can be viewed as equivalent to forces and moments that prevent the interconnected rigid bodies from moving in ways that violate the constraint. The effect of constraints on the rigid bodies in a body is thus to introduce contact reaction forces and moments between the rigid bodies; these forces are known as *constraint forces and moments*. The massive body is unaffected by constraint forces and moments.

Unfortunately, the task of determining reaction forces and moments due to the interactions between rigid bodies can be difficult. In 1788, Lagrange discovered that the equations of motion for a body consisting of multiple rigid bodies can be obtained by performing operations on the kinetic and potential energies. Lagrangian dynamics circumvents the need to determine the contact forces and moments between particles and rigid bodies. If the contact forces and moments are of interest, then these can be determined by using Newton-Euler methods in conjunction with the equations derived from Lagrangian dynamics.

11.2 Generalized Coordinates

We consider a discrete or continuum body $\mathcal{B}$ that consists of particles that can translate as well as rigid bodies that can translate and rotate. The *configuration* of $\mathcal{B}$ refers to the spatial arrangement of the particles and rigid bodies in $\mathcal{B}$. The configuration of $\mathcal{B}$ can be modeled by using *generalized coordinates* q_i, each of which is either a position or an angle. More specifically, q_i is either a scalar position along a dimensionless unit vector or a directed angle around a dimensionless unit vector. The maximum number of generalized coordinates needed to determine the configuration of a body is the number of *degrees of freedom* of the body. Each particle that is not constrained is described by three coordinates, and thus has three degrees of freedom. Likewise, each rigid body that is not constrained can be modeled by six coordinates and thus has six degrees of freedom. Cylindrical and spherical coordinates are generalized coordinates. If $\mathcal{B}$ consists of l_1 particles and l_2 rigid bodies, then $\mathcal{B}$ can have up to $3l_1 + 6l_2$ degrees of freedom. If the particles and rigid bodies in $\mathcal{B}$ are constrained, for example, through revolute or prismatic joints, then the number of degrees of freedom is less than this number.

Ignoring constraints that may be present, the position of the particle y_i in the body $\mathcal{B}$ relative to the point w can be modeled by the three coordinates

$$\vec{r}_{y_i/w}\Big|_{\mathrm{A}} = \begin{bmatrix} \overline{x}_i \\ \overline{y}_i \\ \overline{z}_i \end{bmatrix}. \tag{11.2.1}$$

However, when constraints are considered, it is often convenient to model the configuration of the body by using r generalized coordinates

$$q \triangleq \begin{bmatrix} q_1 \\ \vdots \\ q_r \end{bmatrix} \in \mathbb{R}^r. \tag{11.2.2}$$

The coordinates $\overline{x}_i, \overline{y}_i, \overline{z}_i$ for y_i can thus be written as

$$\overline{x}_i = \overline{x}_i(q, t), \tag{11.2.3}$$

$$\overline{y}_i = \overline{y}_i(q, t), \tag{11.2.4}$$

$$\overline{z}_i = \overline{z}_i(q, t), \tag{11.2.5}$$

For example, the position of the tip of a pendulum connected by a massless arm to an inertially nonrotating massive rigid body by means of a revolute joint is constrained to lie on a circle centered at the joint, while a rigid body can be thought of as being composed of innumerable particles constrained to move in such a manner that the distance between each pair of particles is constant. It is, therefore, useful to consider a body of l particles with p independent physical constraints modeled by

$$\phi_j(\overline{x}_1, \overline{y}_1, \overline{z}_1, \ldots, \overline{x}_l, \overline{y}_l, \overline{z}_l) = 0, \ \ j = 1, \ldots, p. \tag{11.2.6}$$

This body possesses $r = 3l - p$ degrees of freedom since only r independent quantities need be known to determine the configuration. The configuration space of a pendulum is the circle on which the tip of the pendulum moves, while the configuration space of a rigid body—regardless of the number of particles comprising the body—is the set of all positions of its center of mass together with the set of all possible orientations.

Constraints such as (11.2.6), which constrain the positions but not the derivatives of position, are *holonomic*. When the constraints on a body are holonomic, it is often possible to find independent, that is, unconstrained, generalized coordinates that describe the configuration of the body. The positions of all of the particles, when expressed in terms of these independent coordinates, automatically satisfy the physical constraints of the body. For instance, the generalized coordinate for the pendulum is its angular position. Likewise, the coordinates of the center of mass of a rigid body together with its three Euler angles provide six generalized coordinates for the rigid body regardless of the number of particles comprising the body. For the remainder of this chapter we assume that all bodies are holonomic with configurations that are described by independent, unconstrained generalized coordinates.

11.3 Generalized Velocities and Kinetic Energy

The rate of change of the configuration of the body $\mathcal{B}$ induces the generalized coordinates to change with rates called *generalized velocities*, which are denoted by $\dot{q} = [\dot{q}_1 \ \cdots \ \dot{q}_r]^{\mathrm{T}}$. If $q_1, \ldots, q_r$ are independent, then so are $\dot{q}_1, \ldots, \dot{q}_r$. By applying the chain rule to (11.2.3)–(11.2.5), we obtain the components of the velocity of the ith particle relative to the point w and with respect to $\mathrm{F_A}$ as

$$\vec{v}_{y_i/w/\mathrm{A}}\Big|_{\mathrm{A}} = \begin{bmatrix} \overline{u}_i \\ \overline{v}_i \\ \overline{w}_i \end{bmatrix}, \tag{11.3.1}$$

where

$$\overline{u}_i = \overline{u}_i(q, \dot{q}, t) = \sum_{j=1}^{r} \partial_{q_j} \overline{x}_i(q, t) \dot{q}_j + \partial_t \overline{x}_i(q, t), \tag{11.3.2}$$

$$\overline{v}_i = \overline{v}_i(q, \dot{q}, t) = \sum_{j=1}^{r} \partial_{q_j} \overline{y}_i(q, t) \dot{q}_j + \partial_t \overline{y}_i(q, t), \tag{11.3.3}$$

$$\overline{w}_i = \overline{w}_i(q, \dot{q}, t) = \sum_{j=1}^{r} \partial_{q_j} \overline{z}_i(q, t) \dot{q}_j + \partial_t \overline{z}_i(q, t). \tag{11.3.4}$$

Defining the gradient $\partial_q \overline{x}_i(q, t) \in \mathbb{R}^{1 \times r}$ of $x_i(q, t)$ by

$$\partial_q \overline{x}_i(q, t) \triangleq \begin{bmatrix} \partial_{q_1} \overline{x}_i(q, t) & \cdots & \partial_{q_r} \overline{x}_i(q, t) \end{bmatrix}, \tag{11.3.5}$$

which is a row vector, and

$$\alpha_i(q, t) \triangleq \partial_t \overline{x}_i(q, t), \quad \beta_i(q, t) \triangleq \partial_t \overline{y}_i(q, t), \quad \gamma_i(q, t) \triangleq \partial_t \overline{z}_i(q, t), \tag{11.3.6}$$

(11.3.2)–(11.3.4) can be rewritten as

$$\overline{u}_i = \partial_q \overline{x}_i(q, t) \dot{q} + \alpha_i(q, t), \tag{11.3.7}$$

$$\overline{v}_i = \partial_q \overline{y}_i(q, t) \dot{q} + \beta_i(q, t), \tag{11.3.8}$$

$$\overline{w}_i = \partial_q \overline{z}_i(q, t) \dot{q} + \gamma_i(q, t). \tag{11.3.9}$$

Note that, if the dynamics of the body are time invariant, then $\alpha_i(q, t)$, $\beta_i(q, t)$, and $\gamma_i(q, t)$ are zero.

Next, letting m_i denote the mass of the ith particle of the body $\mathcal{B}$, we write the kinetic energy of $\mathcal{B}$ relative to the point w with respect to $\mathrm{F_A}$ in terms of its generalized coordinates and generalized velocities as

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}, t) &= \tfrac{1}{2} \sum_{i=1}^{l} m_i |\vec{v}_{y_i/w/\mathrm{A}}|^2 \\ &= \tfrac{1}{2} \sum_{i=1}^{l} m_i [\overline{u}_i^2(q, \dot{q}, t) + \overline{v}_i^2(q, \dot{q}, t) + \overline{w}_i^2(q, \dot{q}, t)] \\ &= \tfrac{1}{2} \dot{q}^{\mathsf{T}} \sum_{i=1}^{l} m_i [\partial_q^{\mathsf{T}} \overline{x}_i(q, t) \partial_q \overline{x}_i(q, t) + \partial_q^{\mathsf{T}} \overline{y}_i(q, t) \partial_q \overline{y}_i(q, t) + \partial_q^{\mathsf{T}} \overline{z}_i(q, t) \partial_q \overline{z}_i(q, t)] \dot{q} \\ &\quad + \sum_{i=1}^{l} m_i [\alpha_i(q, t) \partial_q \overline{x}_i(q, t) + \beta_i(q, t) \partial_q \overline{y}_i(q, t) + \gamma_i(q, t) \partial_q \overline{z}_i(q, t)] \dot{q} \end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{l} m_i[\alpha_i^2(q,t)+\beta_i^2(q,t)+\gamma_i^2(q,t)]. \tag{11.3.10}$$

Hence

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) = \tfrac{1}{2}\dot q^{\mathsf{T}} M(q,t)\dot q + F(q,t)\dot q + G(q,t), \tag{11.3.11}$$

where the *mass matrix* $M(q,t)\in\mathbb{R}^{r\times r}$ is defined by

$$M(q,t) \triangleq \sum_{i=1}^{l} m_i[\partial_q^{\mathsf{T}}\overline{x}_i(q,t)\partial_q\overline{x}_i(q,t) + \partial_q^{\mathsf{T}}\overline{y}_i(q,t)\partial_q\overline{y}_i(q,t) + \partial_q^{\mathsf{T}}\overline{z}_i(q,t)\partial_q\overline{z}_i(q,t)] \tag{11.3.12}$$

and $F(q,t)\in\mathbb{R}^{1\times l}$ and $G(q,t)\in\mathbb{R}$ are defined by

$$F(q,t) \triangleq \sum_{i=1}^{l} m_i[\alpha_i(q,t)\partial_q\overline{x}_i(q,t) + \beta_i(q,t)\partial_q\overline{y}_i(q,t) + \gamma_i(q,t)\partial_q\overline{z}_i(q,t)], \tag{11.3.13}$$

$$G(q,t) \triangleq \frac{1}{2}\sum_{i=1}^{l} m_i[\alpha_i^2(q,t)+\beta_i^2(q,t)+\gamma_i^2(q,t)]. \tag{11.3.14}$$

Note that the (j,k) entry of $M(q,t)$ is given by

$$M_{jk}(q,t) = \sum_{i=1}^{l} m_i[\partial_{q_j}\overline{x}_i(q,t)\partial_{q_k}\overline{x}_i(q,t) + \partial_{q_j}\overline{y}_i(q,t)\partial_{q_k}\overline{y}_i(q,t) + \partial_{q_j}\overline{z}_i(q,t)\partial_{q_k}\overline{z}_i(q,t)]. \tag{11.3.15}$$

Finally, if the dynamics of the body are time invariant, then (11.3.11) becomes

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) = \tfrac{1}{2}\dot q^{\mathsf{T}} M(q,t)\dot q. \tag{11.3.16}$$

Next, assume that $\mathcal{B}$ is a rigid body with mass m, and let $\mathrm{F_B}$ be a body-fixed frame. Then, (10.1.19) implies that

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) = \tfrac{1}{2}m\vec{v}'_{\mathrm{c}/w/\mathrm{A}}\vec{v}_{\mathrm{c}/w/\mathrm{A}} + \tfrac{1}{2}\vec{\omega}'_{\mathrm{B/A}}\vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{B/A}}. \tag{11.3.17}$$

In order to express $\vec{\omega}_{\mathrm{B/A}}$ as the derivative of components of q, define $q\in\mathbb{R}^6$ by

$$q = \begin{bmatrix} r_{\mathrm{c}} \\ \Phi \\ \Theta \\ \Psi \end{bmatrix}, \tag{11.3.18}$$

where

$$r_{\mathrm{c}} \triangleq \vec{r}_{\mathrm{c}/w}\Big|_{\mathrm{A}} \tag{11.3.19}$$

and Ψ,Θ,Φ are (3,2,1) Euler angles that define the orientation of $\mathrm{F_B}$ relative to $\mathrm{F_A}$. It thus follows from (2.13.32) that $\mathcal{O}_{\mathrm{B/A}} = \mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi)$. Consequently, $T_{\mathcal{B}/w/\mathrm{A}}$ can be written as

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) &= \tfrac{1}{2}m\, \vec{v}_{\mathrm{c}/w/\mathrm{A}}\Big|_{\mathrm{A}}^{\mathsf{T}}\, \vec{v}_{\mathrm{c}/w/\mathrm{A}}\Big|_{\mathrm{A}} + \tfrac{1}{2}\, \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}}^{\mathsf{T}}\, \vec{J}_{\mathcal{B}/\mathrm{c}}\Big|_{\mathrm{B}}\, \vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{B}} \\ &= \tfrac{1}{2}m\|v_{\mathrm{c}}\|^2 + \tfrac{1}{2}\begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix}^{\mathsf{T}} S^{\mathsf{T}}(\Phi,\Theta)\; \vec{J}_{\mathcal{B}/\mathrm{c}}\Big|_{\mathrm{B}} S(\Phi,\Theta)\begin{bmatrix}\dot\Phi\\ \dot\Theta\\ \dot\Psi\end{bmatrix}, \end{aligned} \tag{11.3.20}$$

where

$$v_{\mathrm{c}} \triangleq \dot{r}_{\mathrm{c}} = \left.\overset{\mathrm{A}\bullet}{\vec{r}}_{\mathrm{c}/w}\right|_{\mathrm{A}} = \left.\vec{v}_{\mathrm{c}/w/\mathrm{A}}\right|_{\mathrm{A}}, \tag{11.3.21}$$

and, using (4.10.7),

$$\left.\vec{\omega}_{\mathrm{B}/\mathrm{A}}\right|_{\mathrm{B}} = S(\Phi,\Theta)\begin{bmatrix}\dot{\Phi}\\ \dot{\Theta}\\ \dot{\Psi}\end{bmatrix}, \tag{11.3.22}$$

where

$$S(\Phi,\Theta) \triangleq \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\cos\Theta)\sin\Phi \\ 0 & -\sin\Phi & (\cos\Theta)\cos\Phi \end{bmatrix}. \tag{11.3.23}$$

Hence,

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) = \tfrac{1}{2}\dot{q}^{\mathsf{T}}M(q)\dot{q}, \tag{11.3.24}$$

where $M(q) \in \mathbb{R}^{6\times 6}$ is defined by

$$M(q) \triangleq \begin{bmatrix} mI_3 & 0 \\ 0 & S^{\mathsf{T}}(\Phi,\Theta)\left.\vec{J}_{\mathcal{B}/\mathrm{c}}\right|_{\mathrm{B}} S(\Phi,\Theta) \end{bmatrix}. \tag{11.3.25}$$

As shown by Problem 4.18.2, however, (11.3.22) is not able to represent all possible angular velocities due to gimbal lock. To address this issue, (4.9.11) can be used to express $\vec{\omega}_{\mathrm{B}/\mathrm{A}}$ in terms of the eigenaxis angle vector and its derivative. In this case, define $q \in \mathbb{R}^6$ by

$$q = \begin{bmatrix} r_{\mathrm{c}} \\ \Theta_{\mathrm{B}/\mathrm{A}} \end{bmatrix}, \tag{11.3.26}$$

where

$$r_{\mathrm{c}} \triangleq \left.\vec{r}_{\mathrm{c}/w}\right|_{\mathrm{A}} \tag{11.3.27}$$

and, using (2.16.8),

$$\Theta_{\mathrm{B}/\mathrm{A}} \triangleq \left.\vec{\Theta}_{\mathrm{B}/\mathrm{A}}\right|_{\mathrm{B}} = \left.\vec{\Theta}_{\mathrm{B}/\mathrm{A}}\right|_{\mathrm{A}}. \tag{11.3.28}$$

Consequently, $T_{\mathcal{B}/w/\mathrm{A}}$ can be written as

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) &= \tfrac{1}{2}m \left.\vec{v}_{\mathrm{c}/w/\mathrm{A}}\right|_{\mathrm{A}}^{\mathsf{T}} \left.\vec{v}_{\mathrm{c}/w/\mathrm{A}}\right|_{\mathrm{A}} + \tfrac{1}{2} \left.\vec{\omega}_{\mathrm{B}/\mathrm{A}}\right|_{\mathrm{B}}^{\mathsf{T}} \left.\vec{J}_{\mathcal{B}/\mathrm{c}}\right|_{\mathrm{B}} \left.\vec{\omega}_{\mathrm{B}/\mathrm{A}}\right|_{\mathrm{B}} \\ &= \tfrac{1}{2}m\|v_{\mathrm{c}}\|^2 + \tfrac{1}{2}\dot{\Theta}_{\mathrm{B}/\mathrm{A}}^{\mathsf{T}}K^{\mathsf{T}}(\Theta_{\mathrm{B}/\mathrm{A}}) \left.\vec{J}_{\mathcal{B}/\mathrm{c}}\right|_{\mathrm{B}} K(\Theta_{\mathrm{B}/\mathrm{A}})\dot{\Theta}_{\mathrm{B}/\mathrm{A}}, \end{aligned} \tag{11.3.29}$$

where

$$v_{\mathrm{c}} \triangleq \dot{r}_{\mathrm{c}} = \left.\vec{v}_{\mathrm{c}/w/\mathrm{A}}\right|_{\mathrm{A}} \tag{11.3.30}$$

and, using (4.9.11),

$$\left.\vec{\omega}_{\mathrm{B}/\mathrm{A}}\right|_{\mathrm{B}} = G(\Theta_{\mathrm{B}/\mathrm{A}})\dot{\Theta}_{\mathrm{B}/\mathrm{A}}, \tag{11.3.31}$$

where

$$K(\Theta_{\rm B/A}) \triangleq I_3 - \frac{1-\cos\theta_{\rm B/A}}{\theta_{\rm B/A}^2}\Theta_{\rm B/A}^{\times} + \frac{\theta_{\rm B/A}-\sin\theta_{\rm B/A}}{\theta_{\rm B/A}^3}\Theta_{\rm B/A}^{\times 2} \tag{11.3.32}$$

and $\theta_{\rm B/A} \triangleq \|\Theta_{\rm B/A}\|$. Hence,

$$T_{\mathcal{B}/w/{\rm A}}(q,\dot q) = \tfrac{1}{2}\dot q^{\mathsf T} M(q)\dot q, \tag{11.3.33}$$

where $M(q) \in \mathbb{R}^{6\times 6}$ is defined by

$$M(q) \triangleq \begin{bmatrix} mI_3 & 0 \\ 0 & G^{\mathsf T}(\Theta_{\rm B/A})\ \vec{J}_{\mathcal{B}/c}\Big|_{\rm B} G(\Theta_{\rm B/A}) \end{bmatrix}. \tag{11.3.34}$$

Alternatively, the angular velocity vector can be expressed in terms of Euler angles and their derivatives as well as Euler parameters and their derivatives.

The following result shows that the kinetic energy is a positive-definite quadratic form in the generalized velocities.

Fact 11.3.1. For all generalized coordinates q, the mass matrix $M(q)$ is positive definite.

Proof. Note that $M(q)$ is symmetric and positive semidefinite. To show that $M(q)$ is positive definite, suppose that $\dot q$ is nonzero. Then, there exists an inertia point y_i in $\mathcal{B}$ such that $\vec{v}_{y_i/w/{\rm A}}$ is nonzero. Consequently, $T_{\mathcal{B}/w/{\rm A}}$ is positive. Hence $M(q)$ is positive definite. □

Note that Fact 11.3.1 does not apply to (11.3.25) since $S(\Phi,\Theta)$ is singular when gimbal lock occurs.

11.4 Generalized Forces and Moments for Bodies with Forces

Each particle in a body $\mathcal{B}$ moves under the influence of *external forces*, which are external to the body in origin, as well as *internal forces*, which result from interactions between the particles and rigid bodies in $\mathcal{B}$. Internal forces can include reaction forces due to contact between particles and rigid bodies, as well as forces due to interconnections comprised of springs and dashpots. We assume in this section that all moments on $\mathcal{B}$ are expressed in terms of forces.

Assume that the body $\mathcal{B}$ has inertia points $y_1,\ldots,y_l$, let $q_1,\ldots,q_r$ be independent generalized coordinates for $\mathcal{B}$, let w be an unforced particle in (11.2.1) and (11.3.1), and let $\mathrm{F_A}$ be an inertial frame, as necessitated by the Lagrangian dynamics later in this chapter. Furthermore, for $i = 1,\ldots,l$, let $\vec{f}_i$ be the total internal and external force acting on y_i excluding conservative contact forces, and let

$$\vec{f}_i(t)\Big|_{\rm A} = \begin{bmatrix} f_{x_i}(t) \\ f_{y_i}(t) \\ f_{z_i}(t) \end{bmatrix}. \tag{11.4.1}$$

Then, for $j = 1,\ldots,r$, the *generalized force or moment* Q_j due to $\vec{f}_1,\ldots,\vec{f}_l$ is defined by

$$Q_j(q,\dot q,t) \triangleq \sum_{i=1}^{l}[f_{\overline{x}_i}(t)\partial_{q_j}\overline{x}_i(q) + f_{\overline{y}_i}(t)\partial_{q_j}\overline{y}_i(q) + f_{\overline{z}_i}(t)\partial_{q_j}\overline{z}_i(q)], \tag{11.4.2}$$

which we can be rewritten as

$$Q_j(q,\dot q,t) = \sum_{i=1}^{l} \vec{f}_i\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{q_j}\left(\vec{r}_{y_i/w}\Big|_{\mathrm{A}}\right). \tag{11.4.3}$$

Equivalently, (11.4.2) can be written in terms of velocities as

$$Q_j(q,\dot q,t) = \sum_{i=1}^{l} \vec{f}_i\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{\dot q_j}\left(\vec{v}_{y_i/w/\mathrm{A}}\Big|_{\mathrm{A}}\right). \tag{11.4.4}$$

Note that, if q_j is a position, then Q_j is a force, whereas, if q_j is an angle, then Q_j is a moment. Furthermore, note that the ith term in the summation in (11.4.3) is a measure of the effect of the force $\vec{f}_i$ on the sensitivity of the position of the inertia point y_i to changes in the generalized coordinate q_j.

11.5 Generalized Forces and Moments for Bodies with Moments

If the body $\mathcal{B}$ consists of at least one rigid body, then moments may be specified rather than forces. In this case, each moment can be replaced by a pair of balanced forces, and (11.4.3) can be used to determine the resulting generalized forces and moments.

Alternatively, (11.4.3) can be replaced by an expression involving moments. In particular, assume that $\mathcal{B}$ is a rigid body subject to exactly two nonzero forces that are not due to conservative contact. For convenience, assume that these forces are $\vec{f}$ and $-\vec{f}$ applied to inertia points y_1 and y_2, respectively. Let $\mathrm{F_A}$ be an inertial frame, and let $\mathrm{F_A}$ be a body-fixed frame attached to $\mathcal{B}$. Then, the generalized force Q_j corresponding to the generalized coordinate q_j is given by

$$\begin{aligned}
Q_j(q,\dot q,t) &= \vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{\dot q_j}\left(\vec{v}_{y_1/w/\mathrm{A}}\Big|_{\mathrm{A}}\right) - \vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{\dot q_j}\left(\vec{v}_{y_2/w/\mathrm{A}}\Big|_{\mathrm{A}}\right)\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \left[\partial_{\dot q_j}\left(\vec{v}_{y_1/w/\mathrm{A}}\Big|_{\mathrm{A}}\right) - \partial_{\dot q_j}\left(\vec{v}_{y_2/w/\mathrm{A}}\Big|_{\mathrm{A}}\right)\right]\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \left[\partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} \times \vec{r}_{y_1/w}\Big|_{\mathrm{A}}\right) - \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}} \times \vec{r}_{y_2/w}\Big|_{\mathrm{A}}\right)\right]\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \left[\partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right) \times \left(\vec{r}_{y_1/w}\Big|_{\mathrm{A}} - \vec{r}_{y_2/w}\Big|_{\mathrm{A}}\right)\right]\\
&= \vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \left[\partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right) \times \vec{r}_{y_1/y_2}\Big|_{\mathrm{A}}\right]\\
&= -\vec{f}\Big|_{\mathrm{A}}^{\mathsf{T}} \left[\vec{r}_{y_1/y_2}\Big|_{\mathrm{A}} \times \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right)\right]\\
&= -\left(\vec{f}\Big|_{\mathrm{A}} \times \vec{r}_{y_1/y_2}\Big|_{\mathrm{A}}\right)^{\mathsf{T}} \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right)\\
&= \left(\vec{r}_{y_1/y_2}\Big|_{\mathrm{A}} \times \vec{f}\Big|_{\mathrm{A}}\right)^{\mathsf{T}} \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right)\\
&= \vec{M}_k\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{\dot q_j}\left(\vec{\omega}_{\mathrm{B/A}}\Big|_{\mathrm{A}}\right).
\end{aligned} \tag{11.5.1}$$

More generally, assume that the body $\mathcal{B}$ consists of m rigid bodies $\mathcal{B}_1,\ldots,\mathcal{B}_m$, and, for $k = 1,\ldots,m$, let $\mathrm{F}_{\mathrm{B}_k}$ be a body-fixed frame for $\mathcal{B}_k$ and let $\vec{M}_k$ denote the total internal and external torque applied to $\mathcal{B}_k$ excluding conservative contact torques. Then, for $j = 1, \ldots, r$, the generalized force or moment Q_j due to $\vec{M}_1, \ldots, \vec{M}_m$ is given in analogy with (11.4.4) by

$$Q_j(q, \dot{q}, t) = \sum_{k=1}^{m} \vec{M}_k\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{\dot{q}_j}\left(\vec{\omega}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}}\right). \tag{11.5.2}$$

As in the case of forces, if q_j is a position, then Q_j is a force, whereas, if q_j is an angle, then Q_j is a moment.

In order to apply (11.5.2), each angular velocity vector $\vec{\omega}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}}$ must be expressed in terms of the derivatives of the generalized coordinates. To demonstrate this approach in terms of the Cartesian coordinates of the centers of mass and eigenaxis angle vectors of the rigid bodies, we define $q \in \mathbb{R}^{9m}$ by

$$q = \begin{bmatrix} r_{\mathrm{c},1} \\ \vdots \\ r_{\mathrm{c},m} \\ \Theta_{\mathrm{B}_1/\mathrm{A}} \\ \vdots \\ \Theta_{\mathrm{B}_m/\mathrm{A}} \\ \dot{\Theta}_{\mathrm{B}_1/\mathrm{A}} \\ \vdots \\ \dot{\Theta}_{\mathrm{B}_m/\mathrm{A}} \end{bmatrix}, \tag{11.5.3}$$

where, using (2.16.8),

$$\Theta_{\mathrm{B}_k/\mathrm{A}} \triangleq \vec{\Theta}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}} = \vec{\Theta}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{B}}. \tag{11.5.4}$$

Next, it follows from (4.9.10) that

$$\vec{\omega}_{\mathrm{B}_k/\mathrm{A}}\Big|_{\mathrm{A}} = H(\Theta_{\mathrm{B}_k/\mathrm{A}})\dot{\Theta}_{\mathrm{B}_m/\mathrm{A}}, \tag{11.5.5}$$

where

$$H(\Theta_{\mathrm{B}_k/\mathrm{A}}) \triangleq I_3 + \frac{1 - \cos\theta_{\mathrm{B}_k/\mathrm{A}}}{\theta_{\mathrm{B}_k/\mathrm{A}}^2}\Theta_{\mathrm{B}_k/\mathrm{A}}^{\times} + \frac{\theta_{\mathrm{B}_k/\mathrm{A}} - \sin\theta_{\mathrm{B}_k/\mathrm{A}}}{\theta_{\mathrm{B}_k/\mathrm{A}}^3}\Theta_{\mathrm{B}_k/\mathrm{A}}^{\times 2} \tag{11.5.6}$$

and $\theta_{\mathrm{B}_k/\mathrm{A}} \triangleq \|\Theta_{\mathrm{B}_k/\mathrm{A}}\|$. Now, using the fact that $\partial_{\dot{q}_j}\dot{\Theta}_{\mathrm{B}_k/\mathrm{A}} = \partial_{q_j}\Theta_{\mathrm{B}_k/\mathrm{A}}$, it follows from (11.5.2) and (11.5.5) that

$$Q_j(q, \dot{q}, t) = \sum_{k=1}^{m} \vec{M}_k\Big|_{\mathrm{A}}^{\mathsf{T}} H(\Theta_{\mathrm{B}_k/\mathrm{A}})\partial_{q_j}(\Theta_{\mathrm{B}_k/\mathrm{A}}). \tag{11.5.7}$$

If both forces and moments are present, then (11.4.3) and (11.5.7) can be combined to obtain

$$Q_j(q, \dot{q}, t) = \sum_{i=1}^{l} \vec{f}_i\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{q_j}\left(\vec{r}_{y_i/w}\Big|_{\mathrm{A}}\right) + \sum_{k=1}^{m} \vec{M}_k\Big|_{\mathrm{A}}^{\mathsf{T}} H(\Theta_{\mathrm{B}_k/\mathrm{A}})\partial_{q_j}(\Theta_{\mathrm{B}_k/\mathrm{A}}). \tag{11.5.8}$$

11.6 Lagrange's Equations: Kinetic Energy Form

The following result provides equations of motion for a body in terms of generalized coordinates. Generalized forces and moments that are due to conservative internal contact forces do not need to be considered, and thus these forces and moments are excluded.

Fact 11.6.1. Let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^\mathsf{T}$, let w be an unforced particle, and let Q denote all generalized forces and moments except those that arise from conservative contact, and let $\mathrm{F_A}$ be an inertial frame. Then, $q(t)$ satisfies

$$\mathrm{d}_t\partial_{\dot{q}}^\mathsf{T} T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t),t) - \partial_q^\mathsf{T} T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t),t) = Q(q(t),\dot{q}(t),t). \tag{11.6.1}$$

Proof. Prove this result. □

Equation (11.6.1) can be rewritten as

$$\mathrm{d}_t[M(q(t),t)\dot{q}(t) + F(q(t),t)] - \partial_q^\mathsf{T}[\dot{q}^\mathsf{T}(t)M(q(t),t)\dot{q}(t) + F(q(t),t)\dot{q}(t) + G(q(t),t)] = Q(q(t),\dot{q}(t),t). \tag{11.6.2}$$

In terms of components $j = 1,\ldots,r$, (11.6.1) can be rewritten as

$$\mathrm{d}_t\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t),t) - \partial_{q_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t),t) = Q_j(q(t),\dot{q}(t),t). \tag{11.6.3}$$

Furthermore,

$$\partial_q^\mathsf{T} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = \begin{bmatrix} \frac{1}{2}\dot{q}^\mathsf{T}\partial_{q_1}M(q,t)\dot{q} \\ \vdots \\ \frac{1}{2}\dot{q}^\mathsf{T}\partial_{q_r}M(q,t)\dot{q} \end{bmatrix} + \begin{bmatrix} \partial_{q_1}F(q,t)\dot{q} \\ \vdots \\ \partial_{q_r}F(q,t)\dot{q} \end{bmatrix} + \partial_q^\mathsf{T} G(q,t). \tag{11.6.4}$$

Equations (11.6.1) are *Lagrange's equations*. These equations consist of r second-order ordinary differential equations. Each solution, therefore, requires $2r$ initial values of the variables $q_1,\ldots,q_r,\dot{q}_1,\ldots,\dot{q}_r$. Thus the motion of a body depends on its initial configuration as well as the initial velocities of all of its constituent particles. The generalized coordinates and the corresponding generalized velocities at each instant together uniquely determine the subsequent motion of the body.

For the following identity, we suppress the arguments of $T_{\mathcal{B}/w/\mathrm{A}}$.

Fact 11.6.2. Let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^\mathsf{T}$, let w be a point, and let $\mathrm{F_A}$ be a frame. For all $j = 1,\ldots,r$, the kinetic energy $T_{\mathcal{B}/w/\mathrm{A}}$ satisfies

$$\mathrm{d}_t\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}} = \partial_{\dot{q}_j}^2 T_{\mathcal{B}/w/\mathrm{A}}\ddot{q}_j + \mathrm{d}_t[\partial_{\dot{q}_j}^2 T_{\mathcal{B}/w/\mathrm{A}}]\dot{q}_j + \mathrm{d}_t[\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}})\big|_{\dot{q}_j=0}]. \tag{11.6.5}$$

Proof. For each $j \in \{1,\ldots,r\}$, we can write

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) &= \tfrac{1}{2}\dot{q}^\mathsf{T}M(q,t)\dot{q} + F(q,t)\dot{q} + G(q,t) \\ &= \tfrac{1}{2}M_j(q_j,\overline{q}_j,\dot{\overline{q}}_j,t)\dot{q}_j{}^2 + L_j(q_j,\overline{q}_j,\dot{\overline{q}}_j,t)\dot{q}_j + K_j(\overline{q}_j,\dot{\overline{q}}_j,t), \end{aligned}$$

where $\overline{q}_j$ denotes q with q_j omitted, and M_j, L_j, and K_j are appropriate functions of the indicated arguments. We thus have

$$\begin{aligned} \mathrm{d}_t\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) &= \tfrac{1}{2}\mathrm{d}_t\partial_{\dot{q}_j}[M_j(q_j,\overline{q}_j,\dot{\overline{q}}_j,t)\dot{q}_j^2] + \mathrm{d}_t\partial_{\dot{q}_j}[L_j(q_j,\overline{q}_j,\dot{\overline{q}}_j,t)\dot{q}_j] + \mathrm{d}_t\partial_{\dot{q}_j}[K_j(\overline{q}_j,\dot{\overline{q}}_j,t)] \\ &= \mathrm{d}_t[M_j(q_j,\overline{q}_j,\dot{\overline{q}}_j,t)\dot{q}_j] + \mathrm{d}_t L_j(q_j,\overline{q}_j,\dot{\overline{q}}_j,t) \end{aligned}$$

$$= M_j(q_j, \overline{q}_j, \dot{\overline{q}}_j, t)\ddot{q}_j + \mathrm{d}_t[M_j(q_j, \overline{q}_j, \dot{\overline{q}}_j, t)]\dot{q}_j + \mathrm{d}_t L_j(q_j, \overline{q}_j, \dot{\overline{q}}_j, t)$$

$$= \partial^2_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q_j, \overline{q}_j, \dot{\overline{q}}_j, t)\ddot{q}_j + \mathrm{d}_t[\partial^2_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q_j, \overline{q}_j, \dot{\overline{q}}_j, t)]\dot{q}_j$$

$$+ \mathrm{d}_t[\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q_j, \overline{q}_j, \dot{\overline{q}}_j, t))\big|_{\dot{q}_j=0}]. \qquad \square$$

Using Fact 11.6.2 we can rewrite Fact 11.6.1 as follows.

Fact 11.6.3. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^\mathsf{T}$, let w be an unforced particle, and let Q denote all generalized forces and moments except those that arise from conservative contact. Then, for all $j = 1, \ldots, r$, $q(t)$ satisfies

$$\partial^2_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}\ddot{q}_j + \mathrm{d}_t[\partial^2_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}]\dot{q}_j + \mathrm{d}_t[\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}})\big|_{\dot{q}_j=0}] - \partial_{q_j} T_{\mathcal{B}/w/\mathrm{A}} = Q_j. \tag{11.6.6}$$

The following observation provides a constant of the motion in special cases.

Fact 11.6.4. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^\mathsf{T}$, let w be an unforced particle, and let Q denote all generalized forces and moments except those that arise from conservative contact. Furthermore, let $j \in \{1, \ldots, r\}$ and assume that $Q_j = 0$ and $T_{\mathcal{B}/w/\mathrm{A}}$ does not depend on q_j. Then,

$$\mathrm{d}_t \partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t) = 0. \tag{11.6.7}$$

That is, $\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t)$ is a constant of the motion.

If q_j is a position, then the constant $\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t)$ is a constant of momentum, whereas, if q_j is an angle, then the constant $\partial_{\dot{q}_j} T_{\mathcal{B}/w/\mathrm{A}}(q(t), \dot{q}(t), t)$ is a constant of angular momentum

11.7 Derivation of Euler's Equation from Lagrange's Equations

For the case of a single rigid body, we now use Lagrange's equations to derive Euler's equation given by Fact 7.9.5. We do this for the case where the angular velocity is represented in terms of 3-2-1 Euler angles. To begin, we rewrite (4.10.6) as

$$\vec{\omega}_{\mathrm{D/A}} = f(\Theta, \dot{\Phi}, \dot{\Psi})\hat{\imath}_{\mathrm{D}} + g(\Phi, \Theta, \dot{\Theta}, \dot{\Psi})\hat{\jmath}_{\mathrm{D}} + h(\Phi, \Theta, \dot{\Theta}, \dot{\Psi})\hat{k}_{\mathrm{D}}, \tag{11.7.1}$$

where

$$f(\Theta, \dot{\Phi}, \dot{\Psi}) \triangleq -\dot{\Psi}(\sin\Theta) + \dot{\Phi}, \tag{11.7.2}$$

$$g(\Phi, \Theta, \dot{\Theta}, \dot{\Psi}) \triangleq \dot{\Psi}(\sin\Phi)(\cos\Theta) + \dot{\Theta}\cos\Phi, \tag{11.7.3}$$

$$h(\Phi, \Theta, \dot{\Theta}, \dot{\Psi}) \triangleq \dot{\Psi}(\cos\Phi)(\cos\Theta) - \dot{\Theta}(\sin\Phi). \tag{11.7.4}$$

Alternatively, we can rewrite $\vec{\omega}_{\mathrm{D/A}}$ as

$$\vec{\omega}_{\mathrm{D/A}} = \dot{\Phi}\vec{x} + \dot{\Theta}\vec{y}(\Phi) + \dot{\Psi}\vec{z}(\Phi, \Theta), \tag{11.7.5}$$

where

$$\vec{x} \triangleq \hat{\imath}_{\mathrm{D}}, \tag{11.7.6}$$

$$\vec{y}(\Phi) \triangleq (\cos\Phi)\hat{\jmath}_{\mathrm{D}} - (\sin\Phi)\hat{k}_{\mathrm{D}}, \tag{11.7.7}$$

$$\vec{z}(\Phi, \Theta) \triangleq -(\sin\Theta)\hat{\imath}_{\mathrm{D}} + (\sin\Phi)(\cos\Theta)\hat{\jmath}_{\mathrm{D}} + (\sin\Phi)(\cos\Theta)\hat{k}_{\mathrm{D}}. \tag{11.7.8}$$

By writing (11.7.5) as

$$\vec{\omega}_{\mathrm{D/A}} = \begin{bmatrix} \vec{x} & \vec{y}(\Phi) & \vec{z}(\Phi,\Theta) \end{bmatrix} \dot{\theta}, \tag{11.7.9}$$

where

$$\theta \triangleq \begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix}, \tag{11.7.10}$$

it follows that

$$\vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}} = S(\Phi,\Theta)\dot{\theta}, \tag{11.7.11}$$

where

$$\begin{aligned} S(\Phi,\Theta) &\triangleq \begin{bmatrix} \vec{x}\Big|_{\mathrm{D}} & \vec{y}(\Phi)\Big|_{\mathrm{D}} & \vec{z}(\Phi,\Theta)\Big|_{\mathrm{D}} \end{bmatrix} \\ &= \begin{bmatrix} S_1 & S_2(\Phi) & S_3(\Phi,\Theta) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & (\sin\Phi)\cos\Theta \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \end{bmatrix}. \end{aligned} \tag{11.7.12}$$

Next, we have

$$\begin{aligned} \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}} &= \dot{f}(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \dot{g}(\Phi,\Theta,\dot{\Theta},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \dot{h}(\Phi,\Theta,\dot{\Theta},\dot{\Psi})\hat{k}_{\mathrm{D}} \\ &= \nabla f\dot{\theta}\hat{\imath}_{\mathrm{D}} + \nabla g\dot{\theta}\hat{\jmath}_{\mathrm{D}} + \nabla h\dot{\theta}k_{\mathrm{D}} \\ &= \dot{\Phi}[\partial_\Phi f(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \partial_\Phi g(\Theta,\dot{\Phi},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \partial_\Phi h(\Theta,\dot{\Phi},\dot{\Psi})\hat{k}_{\mathrm{D}}] \\ &\quad + \dot{\Theta}[\partial_\Theta f(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \partial_\Theta g(\Theta,\dot{\Phi},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \partial_\Theta h(\Theta,\dot{\Phi},\dot{\Psi})\hat{k}_{\mathrm{D}}] \\ &\quad + \dot{\Psi}[\partial_\Psi f(\Theta,\dot{\Phi},\dot{\Psi})\hat{\imath}_{\mathrm{D}} + \partial_\Psi g(\Theta,\dot{\Phi},\dot{\Psi})\hat{\jmath}_{\mathrm{D}} + \partial_\Psi h(\Theta,\dot{\Phi},\dot{\Psi})\hat{k}_{\mathrm{D}}] \\ &= \dot{\Phi}\partial_\Phi\vec{\omega}_{\mathrm{D/A}} + \dot{\Theta}\partial_\Theta\vec{\omega}_{\mathrm{D/A}} + \dot{\Psi}\partial_\Psi\vec{\omega}_{\mathrm{D/A}}, \end{aligned} \tag{11.7.13}$$

where

$$\partial_\Phi\vec{\omega}_{\mathrm{D/A}} = [\dot{\Psi}(\cos\Phi)(\cos\Theta) - \dot{\Theta}\sin\Phi]\hat{\jmath}_{\mathrm{D}} + [-\dot{\Psi}(\sin\Phi)\cos\Theta - \dot{\Theta}(\cos\Phi)]\hat{k}_{\mathrm{D}}, \tag{11.7.14}$$

$$\partial_\Theta\vec{\omega}_{\mathrm{D/A}} = -\dot{\Psi}(\cos\Theta)\hat{\imath}_{\mathrm{D}} - \dot{\Psi}(\sin\Phi)(\sin\Theta)\hat{\jmath}_{\mathrm{D}} - \dot{\Psi}(\cos\Phi)(\sin\Theta)\hat{k}_{\mathrm{D}}, \tag{11.7.15}$$

$$\partial_\Psi\vec{\omega}_{\mathrm{D/A}} = 0. \tag{11.7.16}$$

Furthermore, (11.7.5) implies that

$$\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{D/A}} = \dot{\Phi}\overset{\mathrm{A}\bullet}{\vec{x}} + \dot{\Theta}\overset{\mathrm{A}\bullet}{\vec{y}}(\Phi) + \dot{\Psi}\overset{\mathrm{A}\bullet}{\vec{z}}(\Phi,\Theta). \tag{11.7.17}$$

Since $\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{D/A}} = \overset{\mathrm{D}\bullet}{\vec{\omega}}_{\mathrm{D/A}}$, it follows from (11.7.13) and (11.7.17) that

$$\partial_\Phi\vec{\omega}_{\mathrm{D/A}} = \overset{\mathrm{A}\bullet}{\vec{x}} = \overset{\mathrm{D}\bullet}{\vec{x}} + \vec{\omega}_{\mathrm{D/A}} \times \vec{x}, \tag{11.7.18}$$

$$\partial_\Theta\vec{\omega}_{\mathrm{D/A}} = \overset{\mathrm{A}\bullet}{\vec{y}}(\Phi) = \overset{\mathrm{D}\bullet}{\vec{y}}(\Phi) + \vec{\omega}_{\mathrm{D/A}} \times \vec{y}(\Phi), \tag{11.7.19}$$

$$\partial_\Psi \vec{\omega}_{\mathrm{D/A}} = \overset{\mathrm{A}\bullet}{\vec{z}}\,(\Phi,\Theta) = \overset{\mathrm{D}\bullet}{\vec{z}}\,(\Phi,\Theta) + \vec{\omega}_{\mathrm{D/A}} \times \vec{z}(\Phi,\Theta). \tag{11.7.20}$$

Resolving (11.7.18)–(11.7.20) in $\mathrm{F_D}$ yields

$$\partial_\Phi S(\Phi,\Theta)\dot{\theta} = \dot{S}_1 + [S(\Phi,\Theta)\dot{\theta}] \times S_1, \tag{11.7.21}$$

$$\partial_\Theta S(\Phi,\Theta)\dot{\theta} = \dot{S}_2(\Phi) + [S(\Phi,\Theta)\dot{\theta}] \times S_2(\Phi), \tag{11.7.22}$$

$$\partial_\Psi S(\Phi,\Theta)\dot{\theta} = \dot{S}_3(\Phi,\Theta) + [S(\Phi,\Theta)\dot{\theta}] \times S_3(\Phi,\Theta), \tag{11.7.23}$$

where

$$\partial_\Phi S(\Phi,\Theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\Phi & (\cos\Phi)\cos\Theta \\ 0 & -\cos\Phi & -(\sin\Phi)\cos\Theta \end{bmatrix}, \tag{11.7.24}$$

$$\partial_\Theta S(\Phi,\Theta) = \begin{bmatrix} 0 & 0 & -\cos\Phi \\ 0 & 0 & -(\sin\Phi)\sin\Theta \\ 0 & 0 & -(\cos\Phi)\sin\Theta \end{bmatrix}, \tag{11.7.25}$$

$$\partial_\Psi S(\Phi,\Theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11.7.26}$$

Note that (11.7.21)–(11.7.23) can be written as

$$\begin{bmatrix} \partial_\Phi S(\Phi,\Theta)\dot{\theta} & \partial_\Theta S(\Phi,\Theta)\dot{\theta} & \partial_\Psi S(\Phi,\Theta)\dot{\theta} \end{bmatrix} = \dot{S}(\Phi,\Theta) + [S(\Phi,\Theta)\dot{\theta}]^{\times} S(\Phi,\Theta). \tag{11.7.27}$$

Now, let $\mathcal{B}$ be a rigid body, let $\vec{M}$ denote the moment applied to $\mathcal{B}$, let $\mathrm{F_A}$ be an inertial frame, let w be an unforced particle, and let $\mathrm{F_D}$ be a body-fixed frame. Furthermore, let Ψ, Θ, and Φ denote 3-2-1 Euler angles so that, by (2.13.28), $\vec{R}_{\mathrm{B/A}} = \vec{R}_{\hat{\imath}_{\mathrm{C}}}(\Phi)\vec{R}_{\hat{\jmath}_{\mathrm{B}}}(\Theta)\vec{R}_{\hat{k}_{\mathrm{A}}}(\Psi)$, where $\mathrm{F_B} \triangleq \vec{R}_{\hat{k}_{\mathrm{A}}}(\Psi)\mathrm{F_A}$, $\mathrm{F_C} \triangleq \vec{R}_{\hat{\jmath}_{\mathrm{B}}}(\Theta)\mathrm{F_B}$ and $\mathrm{F_D} \triangleq \vec{R}_{\hat{\imath}_{\mathrm{C}}}(\Phi)\mathrm{F_C}$. Next, define the generalized coordinates $q_1 \triangleq \Phi$, $q_2 \triangleq \Theta$, and $q_3 \triangleq \Psi$ so that $q = \theta$. It thus follows from (11.5.2) that the generalized moment $Q_1(q,\dot{q})$ is given by

$$\begin{aligned}
Q_1(q,\dot{q}) &= \vec{M}\Big|_{\mathrm{A}}^{\mathsf{T}} \partial_{\dot{\Phi}}\left(\vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{A}}\right) \\
&= \left(\mathcal{O}_{\mathrm{A/D}}\, \vec{M}\Big|_{\mathrm{D}}\right)^{\mathsf{T}} \partial_{\dot{\Phi}}\left(\mathcal{O}_{\mathrm{A/D}}\, \vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}}\right) \\
&= \vec{M}\Big|_{\mathrm{D}}^{\mathsf{T}} \partial_{\dot{\Phi}}\left(\vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}}\right) \\
&= M^{\mathsf{T}} \partial_{\dot{\Phi}}(\omega_{\mathrm{D/A}}) \\
&= M^{\mathsf{T}} \partial_{\dot{\Phi}}[S(\Phi,\Theta)\dot{\theta}] \\
&= M^{\mathsf{T}} \partial_{\dot{\Phi}}[\dot{\Phi}S_1 + \dot{\Theta}S_2(\Phi) + \dot{\Psi}S_3(\Phi,\Theta)] \\
&= M^{\mathsf{T}} S_1 \\
&= S_1^{\mathsf{T}} M.
\end{aligned} \tag{11.7.28}$$

where $M \triangleq \vec{M}\Big|_{\mathrm{D}}$ and $\omega_{\mathrm{D/A}} \triangleq \vec{\omega}_{\mathrm{D/A}}\Big|_{\mathrm{D}}$. Therefore,

$$Q(q,\dot q) = \begin{bmatrix} Q_1(q,\dot q)\\ Q_2(q,\dot q)\\ Q_3(q,\dot q)\end{bmatrix} = \begin{bmatrix} S_1^{\mathsf T}M\\ S_2^{\mathsf T}(\Phi)M\\ S_3^{\mathsf T}(\Phi,\Theta)M\end{bmatrix} = \begin{bmatrix} S_1^{\mathsf T}\\ S_2^{\mathsf T}(\Phi)\\ S_3^{\mathsf T}(\Phi,\Theta)\end{bmatrix} M = S^{\mathsf T}(\Phi,\Theta)M. \tag{11.7.29}$$

Next, define $J \triangleq \vec{J}_{\mathcal{B}/\mathrm{c}}\Big|_{\mathrm{D}}$. Then, it follows from (10.1.19) that the kinetic energy of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ is given by

$$\begin{aligned} T_{\mathcal{B}/w/\mathrm{A}} &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{D/A}}\vec{J}_{\mathcal{B}/\mathrm{c}}\vec{\omega}_{\mathrm{D/A}} + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2\\ &= \tfrac{1}{2}\dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta + \tfrac{1}{2}m_{\mathcal{B}}|\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2. \end{aligned} \tag{11.7.30}$$

Therefore, using (11.7.11) it follows that

$$\begin{aligned} \mathrm{d}_t\partial_{\dot q}^{\mathsf T}T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) &= \mathrm{d}_t\partial_{\dot q}^{\mathsf T}[\tfrac{1}{2}\dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta]\\ &= \mathrm{d}_t[S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta]\\ &= \mathrm{d}_t[S^{\mathsf T}(\Phi,\Theta)J\omega_{\mathrm{D/A}}]\\ &= S^{\mathsf T}(\Phi,\Theta)J\dot\omega_{\mathrm{D/A}} + \dot S^{\mathsf T}(\Phi,\Theta)J\omega_{\mathrm{D/A}}. \end{aligned} \tag{11.7.31}$$

Furthermore, using (11.7.27) we obtain

$$\begin{aligned} \partial_q^{\mathsf T}T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) &= \partial_q^{\mathsf T}[\tfrac{1}{2}\dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta]\\ &= \begin{bmatrix} \partial_\Phi[\tfrac{1}{2}\dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta]\\ \partial_\Theta[\tfrac{1}{2}\dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta]\\ \partial_\Psi[\tfrac{1}{2}\dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)JS(\Phi,\Theta)\dot\theta]\end{bmatrix}\\ &= \begin{bmatrix} \dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)J\partial_\Phi S(\Phi,\Theta)\dot\theta\\ \dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)J\partial_\Theta S(\Phi,\Theta)\dot\theta\\ \dot\theta^{\mathsf T}S^{\mathsf T}(\Phi,\Theta)J\partial_\Psi S(\Phi,\Theta)\dot\theta\end{bmatrix}\\ &= \left(\omega_{\mathrm{D/A}}^{\mathsf T}J\begin{bmatrix}\partial_\Phi S(\Phi,\Theta)\dot\theta & \partial_\Theta S(\Phi,\Theta)\dot\theta & \partial_\Psi S(\Phi,\Theta)\dot\theta\end{bmatrix}\right)^{\mathsf T}\\ &= \left(\omega_{\mathrm{D/A}}^{\mathsf T}J[\dot S(\Phi,\Theta) + \omega_{\mathrm{D/A}}^{\times}S(\Phi,\Theta)]\right)^{\mathsf T}\\ &= \dot S^{\mathsf T}(\Phi,\Theta)J\omega_{\mathrm{D/A}} - S^{\mathsf T}(\Phi,\Theta)(\omega_{\mathrm{D/A}}\times J\omega_{\mathrm{D/A}}). \end{aligned} \tag{11.7.32}$$

Consequently, using (11.7.29), (11.7.31), and (11.7.32), it follows from Fact 11.6.1 that

$$S^{\mathsf T}(\Phi,\Theta)J\dot\omega_{\mathrm{D/A}} + S^{\mathsf T}(\Phi,\Theta)(\omega_{\mathrm{D/A}}\times J\omega_{\mathrm{D/A}}) = S^{\mathsf T}(\Phi,\Theta)M. \tag{11.7.33}$$

Therefore, assuming that $\det S(\Phi,\Theta) = \cos\Theta \neq 0$ so that $S(\Phi,\Theta)$ is nonsingular, it follows that

$$J\dot\omega_{\mathrm{D/A}} + \omega_{\mathrm{D/A}}\times J\omega_{\mathrm{D/A}} = M, \tag{11.7.34}$$

which is Euler's equation given by Fact 7.9.5.

Alternatively, we can parameterize the angular velocity in terms of the Euler parameters

$$q_{\mathrm{D/A}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \eta_{\mathrm{D/A}} \\ \varepsilon_{\mathrm{D/A}} \end{bmatrix} \triangleq \begin{bmatrix} \cos \frac{1}{2}\theta_{\mathrm{D/A}} \\ (\sin \frac{1}{2}\theta_{\mathrm{D/A}}) n_{\mathrm{D/A}} \end{bmatrix}. \tag{11.7.35}$$

It follows from (4.11.12) that

$$\begin{aligned} \omega_{\mathrm{D/A}} &= 2(\eta_{\mathrm{D/A}}\dot{\varepsilon}_{\mathrm{D/A}} - \dot{\eta}_{\mathrm{D/A}}\varepsilon_{\mathrm{D/A}} - \varepsilon_{\mathrm{D/A}} \times \dot{\varepsilon}_{\mathrm{D/A}}) \\ &= \mathcal{Q}(q_{\mathrm{D/A}})\dot{q}_{\mathrm{D/A}}, \end{aligned} \tag{11.7.36}$$

where

$$\mathcal{Q}(q_{\mathrm{D/A}}) \triangleq 2 \begin{bmatrix} -q_2 & q_1 & q_4 & -q_3 \\ -q_3 & -q_4 & q_1 & q_2 \\ -q_4 & q_3 & -q_2 & q_1 \end{bmatrix}. \tag{11.7.37}$$

Since $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, we can write

$$q_1 = \sqrt{1 - q_2^2 - q_3^2 - q_4^2} \tag{11.7.38}$$

and define the vector of generalized coordinates

$$q \triangleq \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}. \tag{11.7.39}$$

Therefore,

$$q_{\mathrm{D/A}} \triangleq \begin{bmatrix} \sqrt{1 - q_2^2 - q_3^2 - q_4^2} \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{11.7.40}$$

and, assuming that $q_1 \neq 0$,

$$\dot{q}_{\mathrm{D/A}} = \begin{bmatrix} \dfrac{-(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}. \tag{11.7.41}$$

Furthermore,

$$\omega_{\mathrm{D/A}} = 2 \begin{bmatrix} \dfrac{q_2(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} + q_1\dot{q}_2 + q_4\dot{q}_3 - q_3\dot{q}_4 \\ \dfrac{q_3(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} - q_4\dot{q}_2 + q_1\dot{q}_3 + q_2\dot{q}_4 \\ \dfrac{q_4(q_2\dot{q}_2 + q_3\dot{q}_3 + q_4\dot{q}_4)}{q_1} + q_3\dot{q}_2 - q_2\dot{q}_3 + q_1\dot{q}_4 \end{bmatrix}$$

$$= 2\begin{bmatrix} \left(q_1 + \frac{q_2^2}{q_1}\right)\dot{q}_2 + \left(q_4 + \frac{q_2 q_3}{q_1}\right)\dot{q}_3 + \left(-q_3 + \frac{q_2 q_4}{q_1}\right)\dot{q}_4 \\ \left(-q_4 + \frac{q_2 q_3}{q_1}\right)\dot{q}_2 + \left(q_1 + \frac{q_3^2}{q_1}\right)\dot{q}_3 + \left(q_2 + \frac{q_3 q_4}{q_1}\right)\dot{q}_4 \\ \left(q_3 + \frac{q_2 q_4}{q_1}\right)\dot{q}_2 + \left(-q_2 + \frac{q_3 q_4}{q_1}\right)\dot{q}_3 + \left(q_1 + \frac{q_4^2}{q_1}\right)\dot{q}_4 \end{bmatrix}$$

$$= S(q)\dot{q}, \tag{11.7.42}$$

where

$$S(q) \triangleq 2\begin{bmatrix} q_1 + \frac{q_2^2}{q_1} & q_4 + \frac{q_2 q_3}{q_1} & -q_3 + \frac{q_2 q_4}{q_1} \\ -q_4 + \frac{q_2 q_3}{q_1} & q_1 + \frac{q_3^2}{q_1} & q_2 + \frac{q_3 q_4}{q_1} \\ q_3 + \frac{q_2 q_4}{q_1} & -q_2 + \frac{q_3 q_4}{q_1} & q_1 + \frac{q_4^2}{q_1} \end{bmatrix}. \tag{11.7.43}$$

For convenience, we write $S(q) = [S_1(q) \;\; S_2(q) \;\; S_3(q)]$.

Next, it follows from (11.5.2) that, for $i = 2, 3, 4$, the generalized moment $Q_i(q, \dot{q})$ is given by

$$\begin{aligned} Q_i(q, \dot{q}) &= M^{\mathsf{T}} \partial_{\dot{q}_i}(\omega_{\mathrm{D/A}}) \\ &= M^{\mathsf{T}} \partial_{\dot{q}_i}[S(q)\dot{q}] \\ &= M^{\mathsf{T}} \partial_{\dot{q}_i}[\dot{q}_2 S_1(q) + \dot{q}_3 S_2(q) + \dot{q}_4 S_3(q)] \\ &= M^{\mathsf{T}} S_i(q) \\ &= S_i^{\mathsf{T}}(q) M. \end{aligned} \tag{11.7.44}$$

Therefore,

$$Q(q, \dot{q}) = S^{\mathsf{T}}(q) M. \tag{11.7.45}$$

Next, since

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2}\dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J S(q)\dot{q} + \tfrac{1}{2} m_{\mathcal{B}} |\vec{v}_{\mathrm{c}/w/\mathrm{A}}|^2, \tag{11.7.46}$$

it follows that

$$\begin{aligned} \mathrm{d}_t \partial_{\dot{q}}^{\mathsf{T}} T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}) &= \mathrm{d}_t \partial_{\dot{q}}^{\mathsf{T}}[\tfrac{1}{2}\dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J S(q)\dot{q}] \\ &= \mathrm{d}_t[S^{\mathsf{T}}(q) J \omega_{\mathrm{D/A}}] \\ &= S^{\mathsf{T}}(q) J \dot{\omega}_{\mathrm{D/A}} + \dot{S}^{\mathsf{T}}(q) J \omega_{\mathrm{D/A}}. \end{aligned} \tag{11.7.47}$$

Next, a lengthy calculation shows that

$$\begin{bmatrix} \partial_{q_2}[S(q)\dot{q}] & \partial_{q_3}[S(q)\dot{q}] & \partial_{q_4}[S(q)\dot{q}] \end{bmatrix} = \dot{S}(q) + [S(q)\dot{q}]^{\times} S(q). \tag{11.7.48}$$

Therefore,

$$\partial_q^{\mathsf{T}} T_{\mathcal{B}/w/\mathrm{A}}(q, \dot{q}) = \partial_q^{\mathsf{T}}[\tfrac{1}{2}\dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J S(q)\dot{q}]$$

$$
\begin{aligned}
&= \begin{bmatrix} \partial_{q_2}[\frac{1}{2}\dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J S(q)\dot{q}] \\ \partial_{q_3}[\frac{1}{2}\dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J S(q)\dot{q}] \\ \partial_{q_4}[\frac{1}{2}\dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J S(q)\dot{q}] \end{bmatrix} \\
&= \begin{bmatrix} \dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J \partial_{q_2}[S(q)\dot{q}] \\ \dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J \partial_{q_3}[S(q)\dot{q}] \\ \dot{q}^{\mathsf{T}} S^{\mathsf{T}}(q) J \partial_{q_4}[S(q)\dot{q}] \end{bmatrix} \\
&= \begin{bmatrix} \omega_{\mathrm{D/A}}^{\mathsf{T}} J \partial_{q_2}[S(q)\dot{q}] \\ \omega_{\mathrm{D/A}}^{\mathsf{T}} J \partial_{q_3}[S(q)\dot{q}] \\ \omega_{\mathrm{D/A}}^{\mathsf{T}} J \partial_{q_4}[S(q)\dot{q}] \end{bmatrix} \\
&= \left(\omega_{\mathrm{D/A}}^{\mathsf{T}} J \begin{bmatrix} \partial_{q_2}[S(q)\dot{q}] & \partial_{q_3}[S(q)\dot{q}] & \partial_{q_4}[S(q)\dot{q}] \end{bmatrix}\right)^{\mathsf{T}} \\
&= \left(\omega_{\mathrm{D/A}}^{\mathsf{T}} J [\dot{S}(q) + \omega_{\mathrm{D/A}}^{\times} S(q)]\right)^{\mathsf{T}} \\
&= \dot{S}^{\mathsf{T}}(q) J \omega_{\mathrm{D/A}} - S^{\mathsf{T}}(q)(\omega_{\mathrm{D/A}} \times J\omega_{\mathrm{D/A}}).
\end{aligned} \tag{11.7.49}
$$

Consequently, using (11.7.45), (11.7.47), and (11.7.49), it follows from Fact 11.6.1 that

$$
S^{\mathsf{T}}(q) J \dot{\omega}_{\mathrm{D/A}} + S^{\mathsf{T}}(q)(\omega_{\mathrm{D/A}} \times J\omega_{\mathrm{D/A}}) = S^{\mathsf{T}}(q) M. \tag{11.7.50}
$$

Therefore, since by assumption $\det S(q) = 8/q_1 \neq 0$ and thus $S(q)$ is nonsingular, it follows that

$$
J\dot{\omega}_{\mathrm{D/A}} + \omega_{\mathrm{D/A}} \times J\omega_{\mathrm{D/A}} = M, \tag{11.7.51}
$$

which is Euler's equation given by Fact 7.9.5.

11.8 Lagrange's Equations: Potential Energy Form

Potential energy gives rise to generalized forces and moments, and it is convenient to distinguish generalized forces and moments arising from potential energy from the remaining generalized forces and moments. We thus write

$$
Q(q, \dot{q}, t) = Q_{\mathrm{p}}(q) + Q_{\mathrm{np}}(q, \dot{q}, t), \tag{11.8.1}
$$

where Q_{p} denotes the generalized forces and moments arising from potential energy and Q_{np} denotes the remaining generalized forces and moments. To determine the generalized forces and moments arising from potential energy, we write $U_{\mathcal{B}}(q)$ to denote the potential energy of $\mathcal{B}$ in terms of generalized coordinates. The following result is analogous to Fact 10.3.1.

Fact 11.8.1. Let $\mathcal{B}$ be a body, let $U_{\mathcal{B}}(q)$ denote the potential energy of $\mathcal{B}$ in terms of the generalized coordinates q. Then, the generalized force Q associated with $U_{\mathcal{B}}(q)$ is given by

$$
Q_{\mathrm{p}}(q) = -\partial_q^{\mathsf{T}} U_{\mathcal{B}}(q). \tag{11.8.2}
$$

Proof. Need proof. □

A generalized force or moment associated with the potential energy is called a *generalized potential force or moment*. The potential energy and associated generalized potential force for a collection of springs is given by the following result.

Fact 11.8.2. Consider a collection of s springs such that, for each $i = 1, \ldots, s$, the ith spring has stiffness $k_i > 0$, relaxed length $d_i \geq 0$, and connects inertia points y_i and w_i, respectively. Furthermore, assume that the force of the ith spring is given by

$$\vec{f}_{y_i/w_i} = -k_i(|\vec{r}_{y_i/w_i}| - d_i)\hat{r}_{y_i/w_i}. \tag{11.8.3}$$

Furthermore, let $q \in \mathbb{R}^r$ denote generalized coordinates, and assume that

$$U(q) \triangleq \tfrac{1}{2}\sum_{i=1}^{s} k_i(|\vec{r}_{y_i/w_i}| - d_i)^2 \tag{11.8.4}$$

can be written as

$$U(q) = \tfrac{1}{2}(q-d)^{\mathsf{T}}K(q-d), \tag{11.8.5}$$

where $K \in \mathbb{R}^{r\times r}$ is a symmetric matrix and $d \triangleq [d_1 \ \cdots \ d_s]^{\mathsf{T}}$. Then, K is positive semidefinite, and the corresponding generalized force is given by

$$Q_{\mathrm{p}}(q) = -K(q-d). \tag{11.8.6}$$

The following result considers rotary springs.

Fact 11.8.3. Consider a collection of s rotary springs such that, for each $i = 1, \ldots, s$, let $\mathcal{B}_i$ and $\hat{\mathcal{B}}_i$ be rigid bodies that are connected by a pin joint at a point fixed in both bodies. Let $\hat{z}$ be a unit dimensionless vector that is parallel with the pin joint, and assume that the ith rotary spring has rotary stiffness $\kappa_i > 0$, and applies torques that are parallel with $\hat{z}$. Let $\hat{x}_{i1}$ and $\hat{x}_{i2}$ be unit dimensionless vectors that are fixed in $\mathcal{B}_i$ and $\hat{\mathcal{B}}_i$, respectively, and that are orthogonal to $\hat{z}$. Assume that the rotary spring is relaxed when $\hat{x}_{i1}$ and $\hat{x}_{i2}$ are parallel and that the torque of the ith rotary spring is given by

$$\vec{M}_{\mathcal{B}_i/\hat{\mathcal{B}}_i} = -\kappa_i\theta_{\hat{x}_{i1}/\hat{x}_{i2}/\hat{z}_i}\hat{z}_i. \tag{11.8.7}$$

Furthermore, let $q \in \mathbb{R}^r$ denote generalized coordinates, and assume that

$$U(q) \triangleq \tfrac{1}{2}\sum_{i=1}^{s} \kappa_i\theta^2_{\hat{x}_1/\hat{x}_2/\hat{z}} \tag{11.8.8}$$

can be written as

$$U(q) = \tfrac{1}{2}q^{\mathsf{T}}\mathcal{K}q, \tag{11.8.9}$$

where $\mathcal{K} \in \mathbb{R}^{r\times r}$ is a symmetric matrix. Then, $\mathcal{K}$ is positive semidefinite, and the corresponding generalized force is given by

$$Q_{\mathrm{p}}(q) = -\mathcal{K}q. \tag{11.8.10}$$

The following result reconsiders Fact 10.4.1 in terms of generalized forces and moments. This result uses Lagrange's equations to obtain a constant of the motion, namely, the total energy.

Fact 11.8.4. Let $\mathcal{B}$ be a body, let w be an unforced particle, let $\mathrm{F_A}$ be an inertial frame, and assume that all generalized forces and moments are potential forces and moments. Then, the total energy of $\mathcal{B}$ is conserved.

Proof. First we consider the simpler case where $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of q. In this case, it follows from (11.6.6) that

$$M\ddot{q}(t) = Q_{\mathrm{p}}(q(t)).$$

Omitting the argument t for convenience and using (11.8.2) yields

$$\begin{aligned}
\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) &= \mathrm{d}_t T_{\mathcal{B}/w/\mathrm{A}}(\dot{q}) + \mathrm{d}_t U_{\mathcal{B}}(q)\\
&= \dot{q}^{\mathsf{T}} M\ddot{q} + \partial_q U_{\mathcal{B}}(q)\dot{q}\\
&= \dot{q}^{\mathsf{T}} Q_{\mathrm{p}}(q) - Q_{\mathrm{p}}^{\mathsf{T}}(q)\dot{q}\\
&= 0.
\end{aligned}$$

Next, we consider the case where $T_{\mathcal{B}/w/\mathrm{A}}$ may depend on q. In this case, it follows from (11.6.1) that

$$\begin{aligned}
\dot{q}^{\mathsf{T}} Q(q) &= \tfrac{1}{2}\dot{q}^{\mathsf{T}}\mathrm{d}_t\partial_{\dot{q}}^{\mathsf{T}}[\dot{q}^{\mathsf{T}} M(q)\dot{q}] - \tfrac{1}{2}\dot{q}^{\mathsf{T}}\partial_q^{\mathsf{T}}[\dot{q}^{\mathsf{T}} M(q)\dot{q}]\\
&= \dot{q}^{\mathsf{T}}\mathrm{d}_t[M(q)\dot{q}] - \tfrac{1}{2}\dot{q}^{\mathsf{T}}\partial_q^{\mathsf{T}}\left[\sum_{j,k=1}^{r} M_{j,k}(q)\dot{q}_j\dot{q}_k\right]\\
&= \dot{q}^{\mathsf{T}}\mathrm{d}_t\begin{bmatrix}\sum_{j=1}^r M_{1,j}(q)\dot{q}_j\\ \vdots\\ \sum_{j=1}^r M_{r,j}(q)\dot{q}_j\end{bmatrix} - \tfrac{1}{2}\dot{q}^{\mathsf{T}}\begin{bmatrix}\sum_{j,k}^r \partial_{q_1} M_{j,k}(q)\dot{q}_j\dot{q}_k\\ \vdots\\ \sum_{j,k}^r \partial_{q_r} M_{j,k}(q)\dot{q}_j\dot{q}_k\end{bmatrix}\\
&= \dot{q}^{\mathsf{T}}\begin{bmatrix}\sum_{j=1}^r\sum_{k=1}^r \partial_{q_k} M_{1,j}(q)\dot{q}_k\dot{q}_j + M_{1,j}(q)\ddot{q}_j\\ \vdots\\ \sum_{j=1}^r\sum_{k=1}^r \partial_{q_k} M_{r,j}(q)\dot{q}_k\dot{q}_j + M_{r,j}(q)\ddot{q}_j\end{bmatrix} - \tfrac{1}{2}\dot{q}^{\mathsf{T}}\begin{bmatrix}\sum_{j,k}^r \partial_{q_1} M_{j,k}(q)\dot{q}_j\dot{q}_k\\ \vdots\\ \sum_{j,k}^r \partial_{q_r} M_{j,k}(q)\dot{q}_j\dot{q}_k\end{bmatrix}\\
&= \sum_{i,j,k=1}^{r}\dot{q}_i\partial_{q_k} M_{i,j}(q)\dot{q}_k\dot{q}_j + \sum_{i,j=1}^{r} M_{i,j}(q)\dot{q}_i\ddot{q}_j - \tfrac{1}{2}\sum_{i,j,k=1}^{r}\dot{q}_i\partial_{q_k} M_{i,j}(q)\dot{q}_k\dot{q}_j\\
&= \tfrac{1}{2}\sum_{i,j,k=1}^{r}\partial_{q_k} M_{i,j}(q)\dot{q}_i\dot{q}_j\dot{q}_k + \sum_{i,j=1}^{r} M_{i,j}(q)\dot{q}_i\ddot{q}_j.
\end{aligned}\tag{11.8.11}$$

Therefore, using (11.8.11) it follows that

$$\begin{aligned}
\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) &= \tfrac{1}{2}\mathrm{d}_t[\dot{q}^{\mathsf{T}} M(q)\dot{q}] + \mathrm{d}_t U_{\mathcal{B}}(q)\\
&= \tfrac{1}{2}\mathrm{d}_t\sum_{i,j=1}^{r} M_{i,j}\dot{q}_i\dot{q}_j + \partial_q U_{\mathcal{B}}(q)\dot{q}\\
&= \tfrac{1}{2}\sum_{i,j=1}^{r}\left[\sum_{k=1}^{r}\partial_{q_k} M_{i,j}\dot{q}_k\dot{q}_i\dot{q}_j + M_{i,j}\ddot{q}_i\dot{q}_j + M_{i,j}\dot{q}_i\ddot{q}_j\right] - Q_{\mathrm{p}}(q)^{\mathsf{T}}\dot{q}\\
&= \tfrac{1}{2}\sum_{i,j,k=1}^{r}\partial_{q_k} M_{i,j}\dot{q}_i\dot{q}_j\dot{q}_k + \sum_{i,j=1}^{r} M_{i,j}\dot{q}_i\ddot{q}_j - \dot{q}^{\mathsf{T}} Q_{\mathrm{p}}(q)\\
&= 0.
\end{aligned}$$

□

The *Lagrangian* of a system with potential energy $U_{\mathcal{B}}(q)$ and kinetic energy $T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q})$ is the function

$$L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) \triangleq T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q}) - U_{\mathcal{B}}(q). \tag{11.8.12}$$

Lagrange's equations for a system with potential force are given by the following result. This result is a specialization of Fact 11.6.1.

Fact 11.8.5. Let $\mathrm{F_A}$ be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^\mathsf{T}$, let w be an unforced particle, define $L_{\mathcal{B}/w/\mathrm{A}}$ by (11.8.12), where $U_{\mathcal{B}}$ includes all internal and external potential forces and moments, and let Q_{np} denote all generalized forces and moments except those that arise from $U_{\mathcal{B}}$ or conservative contact. Then, $q(t)$ satisfies

$$\mathrm{d}_t\partial_{\dot{q}}^\mathsf{T} T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t),t) - \partial_q^\mathsf{T} L_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot{q}(t),t) = Q_{\mathrm{np}}(q(t),\dot{q}(t),t). \tag{11.8.13}$$

Proof. Show algebra to go from (11.6.1) to (11.8.13). □

If $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of q, then (11.8.13) can be written as

$$M\ddot{q}(t) + \partial_q^\mathsf{T} U_{\mathcal{B}}(q(t)) = Q_{\mathrm{np}}(q(t),\dot{q}(t),t). \tag{11.8.14}$$

11.9 Lagrange's Equations: Rayleigh Dissipation Function Form

In some cases, it is convenient to decompose the nonpotential force Q_{np} in terms of a function R called a *Rayleigh dissipation function*. We thus write

$$Q_{\mathrm{np}}(q,\dot{q},t) = Q_{\mathrm{R}}(q,\dot{q},t) + Q_{\mathrm{npnR}}(q,\dot{q},t), \tag{11.9.1}$$

where Q_{R} arises from a Rayleigh dissipation function and Q_{npnR} are generalized forces and moments that do not arise from a Rayleigh dissipation function. The force or moment Q_{R} arising from a Rayleigh dissipation function $R = R(q,\dot{q})$ has the form

$$Q_{\mathrm{R}}(q,\dot{q}) = -\partial_{\dot{q}}^\mathsf{T} R(q,\dot{q}). \tag{11.9.2}$$

It is usually the case that $\partial_{\dot{q}}R(q,0) = 0$ for all q, which implies that no energy dissipation occurs when the generalized velocities are zero.

The Rayleigh dissipation function for a collection of dashpots is given by the following result.

Fact 11.9.1. Consider a collection of s dashpots such that, for each $i = 1,\ldots,s$, the ith dashpot has viscosity $c_i > 0$ and connects inertia points y_i and w_i. Furthermore, assume that the force applied by the ith dashpot is given by

$$\vec{f}_{y_i/w_i} = -c_i(\mathrm{d}_t|\vec{r}_{y_i/w_i}|)\hat{r}_{y_i/w_i}. \tag{11.9.3}$$

Furthermore, let $q \in \mathbb{R}^r$ denote generalized coordinates, and let

$$R(q,\dot{q}) \triangleq \tfrac{1}{2}\sum_{i=1}^{s} c_i(\mathrm{d}_t|\vec{r}_{y_i/w_i}|)^2 \tag{11.9.4}$$

be written as

$$R(q,\dot{q}) = \tfrac{1}{2}\dot{q}^\mathsf{T} C(q)\dot{q}, \tag{11.9.5}$$

where, for all $q \in \mathbb{R}^r$, $C(q) \in \mathbb{F}^{r\times r}$ is a symmetric matrix. Then, for all $q \in \mathbb{R}^r$, $C(q)$ is positive semidefinite, and the corresponding generalized force is given by

$$Q_{\mathrm{R}}(q,\dot q) = -C(q)\dot q. \tag{11.9.6}$$

The following result presents Lagrange's equations for a body with Lagrangian $L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q)$, Rayleigh dissipation function $R(q,\dot q)$, and generalized forces and moments $Q_{\mathrm{npnR}}(q,\dot q)$ that do not arise from the Rayleigh dissipation function, a potential, or conservative contact.

Fact 11.9.2. Let F_{A} be an inertial frame, let $\mathcal{B}$ be a body described by generalized coordinates $q = [q_1 \ \cdots \ q_r]^{\mathsf{T}}$, let w be an unforced particle, define $L_{\mathcal{B}/w/\mathrm{A}}$ by (11.8.12), where $U_{\mathcal{B}}$ includes all potential generalized forces and moments, let R be a Rayleigh dissipation function, and let Q_{npnR} denote all generalized forces and moments except those that arise from R, $U_{\mathcal{B}}$, or conservative contact. Then, q satisfies

$$\mathrm{d}_t\partial_{\dot q}^{\mathsf{T}}T_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot q(t)) - \partial_q^{\mathsf{T}}L_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot q(t)) + \partial_{\dot q}^{\mathsf{T}}R(q(t),\dot q(t)) = Q_{\mathrm{npnR}}(q(t),\dot q(t),t). \tag{11.9.7}$$

The following result shows that the system dissipates energy if $q(t)$ and $\dot q(t)$ are such that $[\partial_{\dot q}R(q(t),\dot q(t))]\dot q(t) > 0$.

Fact 11.9.3. Let $\mathcal{B}$ be a body, let w be an unforced particle, let F_{A} be an inertial frame, and assume that all generalized forces or moments that are neither conservative reaction forces or moments nor potential forces or moments are given by the Rayleigh dissipation function R. Then

$$\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}}(q(t),\dot q(t)) = -[\partial_{\dot q}R(q(t),\dot q(t))]\dot q(t). \tag{11.9.8}$$

Proof. First we consider the simpler case where $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of q. In this case, it follows from (11.9.7) that

$$M\ddot q(t) = Q_{\mathrm{p}}(q(t)) - \partial_{\dot q}^{\mathsf{T}}R(q,\dot q).$$

Omitting the argument t for convenience, it follows from (11.8.14) that

$$\begin{aligned}\mathrm{d}_t E_{\mathcal{B}/w/\mathrm{A}} &= \dot q^{\mathsf{T}}M\ddot q + \partial_q U_{\mathcal{B}}(q)\dot q\\ &= \dot q^{\mathsf{T}}[-\partial_q U_{\mathcal{B}}(q) + Q_{\mathrm{R}}(q,\dot q)] + \partial_q U_{\mathcal{B}}(q)\dot q\\ &= Q_{\mathrm{R}}^{\mathsf{T}}(q,\dot q)\dot q\\ &= -[\partial_{\dot q}R(q,\dot q)]\dot q.\end{aligned}$$

Finally, for the case where $T_{\mathcal{B}/w/\mathrm{A}}$ depends on q, see the proof of Fact 11.8.4. □

11.10 Examples

Example 11.10.1. Derive the equations of motion for the damped SDOF oscillator.

Solution. A convenient generalized coordinate for such an oscillator is the extension q of the spring, while the corresponding generalized velocity is the instantaneous rate of extension $\dot q$ of the spring. The kinetic energy of the oscillator is $T_{\mathcal{B}/w/\mathrm{A}}(\dot q) = \frac{1}{2}m\dot q^2$, while the potential energy is $U_{\mathcal{B}}(q) = \frac{1}{2}kq^2$. The Rayleigh dissipation function for a dashpot with viscosity c is $R(q,\dot q) = \frac{1}{2}c\dot q^2$. The remaining nonconservative generalized force is the external applied force f. Lagrange's equations (11.9.7) then yield

$$m\ddot q + c\dot q + kq = f. \tag{11.10.1}$$

Example 11.10.2. Derive the equations of motion for the simple pendulum.

Solution. Let $q_1 = \theta$ denote the angle of the pendulum as shown in Figure xxxxx. Then, the kinetic and potential energy are given by

$$T(\dot{q}) = \tfrac{1}{2}\vec{v}'_{y/w/\mathrm{A}}\vec{v}_{y/w/\mathrm{A}} \tag{11.10.2}$$

and

$$U(q) = \vec{r}'_{y/w}\hat{k}_{\mathrm{A}}. \tag{11.10.3}$$

Note that

$$\begin{aligned}\vec{v}_{y/w/\mathrm{A}} &= \vec{\omega} \times r_{y/w}\\ &= \dot{\theta}\hat{\imath} \times \ell\hat{\imath}\end{aligned} \tag{11.10.4}$$

Example 11.10.3. Derive the equations of motion for the bead on the wire considered in Example 8.6.1.

Solution. First we consider the case where $\omega = \dot{\theta}$ is constant, and thus $q_1 = x$ is the only generalized coordinate. The kinetic energy of the body relative to w with respect to $\mathrm{F_A}$ is given by

$$\begin{aligned}T_{\mathcal{B}/w/\mathrm{A}} &= T_{\mathrm{wire}/w/\mathrm{A}} + T_{y/w/\mathrm{A}}\\ &= \tfrac{1}{2}\vec{\omega}'_{\mathrm{B/A}}\vec{J}_{\mathrm{wire}/w}\vec{\omega}_{\mathrm{B/A}} + \tfrac{1}{2}m|v_{y/w/\mathrm{A}}|^2\\ &= \tfrac{1}{2}\begin{bmatrix}0\\0\\\omega\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}0&0&0\\0&I&0\\0&0&I\end{bmatrix}\begin{bmatrix}0\\0\\\omega\end{bmatrix} + \tfrac{1}{2}m(\dot{x}^2 + x^2\omega^2)\\ &= \tfrac{1}{2}(I + mx^2)\omega^2 + \tfrac{1}{2}m\dot{x}^2.\end{aligned}$$

It thus follows from Lagrange's equations (11.6.1) that

$$\mathrm{d}_t\partial_{\dot{x}}T_{\mathcal{B}/w/\mathrm{A}} - \partial_x T_{\mathcal{B}/w/\mathrm{A}} = 0$$

that

$$\mathrm{d}_t(m\dot{x}) - m\omega^2 x = 0$$

that is,

$$\ddot{x} = \omega^2 x.$$

To determine the reaction force between the bead and the wire, Newton's second law implies that

$$\begin{aligned}\vec{f}_{\mathrm{R}} &= m\vec{a}_{y/w/\mathrm{A}}\\ &= (\ddot{x} - \omega^2 x)\hat{\imath}_{\mathrm{B}} + 2\omega\dot{x}\hat{\jmath}_{\mathrm{B}}\\ &= 2\omega\dot{x}\hat{\jmath}_{\mathrm{B}}.\end{aligned}$$

Therefore, the force applied to the bead is transverse to the wire. Although the bead moves away from w, the component of the force along the wire is zero. Now, assume that a stopper is placed on the wire preventing the bead from moving further along the wire. Then, the reaction force on the stopper due to the bead is the *centrifugal reaction force*, whereas the reaction force on the bead due to the stopper is the *centripetal reaction force*.

Next, we consider the case where $\dot\theta$ is not necessarily constant, and thus the generalized coordinates are $q_1 = x$ and $q_2 = \theta$. The kinetic energy thus has the form

$$T_{\mathcal{B}/w/\mathrm{A}} = \tfrac{1}{2}\begin{bmatrix}\dot x\\ \dot\theta\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}m & 0\\ 0 & I+mx^2\end{bmatrix}\begin{bmatrix}\dot x\\ \dot\theta\end{bmatrix}.$$

For q_1 it follows from Lagrange's equations that

$$\ddot x = \dot\theta^2 x.$$

Since $T_{\mathcal{B}/w/\mathrm{A}}$ is independent of $q_2 = \theta$, it follows from Lagrange's equations that

$$\mathrm{d}_t\partial_{\dot\theta}T_{\mathcal{B}/w/\mathrm{A}} = 0,$$

and thus

$$\mathrm{d}_t[(I+mx^2)\dot\theta] = 0,$$

which shows that the component of momentum $(I+mx^2)\dot\theta$ is a constant of the motion. Consequently,

$$(I+mx^2)\ddot\theta + 2mx\dot x\dot\theta = 0.$$

To determine the reaction force between the bead and the wire, Newton's second law implies that

$$\begin{aligned}\vec{f}_{\mathrm{R}} &= m\vec{a}_{y/w/\mathrm{A}}\\ &= (\ddot x - \dot\theta^2 x)\hat\imath_{\mathrm{B}} + (\ddot\theta x + 2\dot\theta\dot x)\hat\jmath_{\mathrm{B}}\\ &= (\ddot\theta x + 2\dot\theta\dot x)\hat\jmath_{\mathrm{B}}.\end{aligned}$$

As in the case where $\dot\theta$ is constant, the reaction force on the bead due to the wire is transverse to the wire.

Example 11.10.4. Reconsider the cart with rotating arm described in Example 8.5.9 using Lagrangian dynamics.

Solution. In terms of the generalized coordinates $q_1 = q$ and $q_2 = \theta$, (11.2.3) and (11.2.4) have the form

$$x_1 = q_1, \tag{11.10.5}$$

$$y_1 = 0, \tag{11.10.6}$$

$$x_2 = q_1 + \ell\sin q_2, \tag{11.10.7}$$

$$y_2 = -\ell\cos q_2, \tag{11.10.8}$$

and the generalized velocities (11.3.2) and (11.3.3) become

$$\overline{u}_1 = \dot q_1, \tag{11.10.9}$$

$$\overline{v}_1 = 0, \tag{11.10.10}$$

$$\overline{u}_2 = \dot q_1 + \ell\dot q_2\cos q_2, \tag{11.10.11}$$

$$\overline{v}_2 = \ell\dot q_2\sin q_2. \tag{11.10.12}$$

Substituting (11.10.9)–(11.10.12) into (11.3.11), the kinetic energy of the cart with rotating arm is given by

$$T_{\mathcal{B}/w/\mathrm{A}}(q_1,q_2,\dot q_1,\dot q_2) = \tfrac{1}{2}(m_1+m_2)\dot q_1^2 + m_2\ell\dot q_1\dot q_2\cos q_2 + \tfrac{1}{2}m_2\ell^2\dot q_2^2. \tag{11.10.13}$$

Alternatively, note that

$$
\begin{aligned}
T &= \tfrac{1}{2}m_1\vec{v}'_{c_1/w/A}\vec{v}_{c_1/w/A} + \tfrac{1}{2}m_2\vec{v}'_{c_2/w/A}\vec{v}_{c_2/w/A} \\
&= \tfrac{1}{2}m_1\vec{v}'_{c_1/w/A}\vec{v}_{c_1/w/A} + \tfrac{1}{2}m_2(\vec{v}_{c_2/c_1/A} + \vec{v}_{c_1/w/A})'(\vec{v}_{c_2/c_1/A} + \vec{v}_{c_1/w/A}) \\
&= \tfrac{1}{2}(m_1+m_2)\vec{v}'_{c_1/w/A}\vec{v}_{c_1/w/A} + \tfrac{1}{2}m_2\vec{v}'_{c_2/c_1/A}\vec{v}_{c_2/c_1/A} + m_2\vec{v}'_{c_1/w/A}\vec{v}_{c_2/c_1/A} \\
&= \tfrac{1}{2}(m_1+m_2)\vec{v}'_{c_1/w/A}\vec{v}_{c_1/w/A} + \tfrac{1}{2}m_2(\vec{\omega}_{B/A}\times r_{c_2/c_1})'(\vec{\omega}_{B/A}\times\vec{r}_{c_2/c_1}) + m_2\vec{v}'_{c_1/w/A}(\vec{\omega}_{B/A}\times\vec{r}_{c_2/c_1}) \\
&= \tfrac{1}{2}(m_1+m_2)\dot{q}^2 + \tfrac{1}{2}m_2(\dot{\theta}\hat{k}_A\times\ell\hat{\imath}_B)'(\dot{\theta}\hat{k}_A\times\ell\hat{\imath}_B) + m_2\dot{q}\hat{\jmath}'_A(\dot{\theta}\hat{k}_A\times\ell\hat{\imath}_B) \\
&= \tfrac{1}{2}(m_1+m_2)\dot{q}^2 + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2(\hat{k}_A\times\hat{\imath}_B)'(\hat{k}_A\times\hat{\imath}_B) + m_2\ell\dot{q}\dot{\theta}\hat{\jmath}'_A(\hat{k}_A\times\hat{\imath}_B) \\
&= \tfrac{1}{2}(m_1+m_2)\dot{q}^2 + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2\hat{\jmath}'_B\hat{\jmath}_B + m_2\ell\dot{q}\dot{\theta}\hat{\jmath}'_A\hat{\jmath}_B \\
&= \tfrac{1}{2}(m_1+m_2)\dot{q}^2 + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2 + m_2\ell\dot{q}\dot{\theta}\hat{\jmath}'_A(\sin\theta\hat{\imath}_A + \cos\theta\hat{\jmath}_A) \\
&= \tfrac{1}{2}(m_1+m_2)\dot{q}^2 + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2 + m_2\ell\dot{q}\dot{\theta}\cos\theta. \qquad (11.10.14)
\end{aligned}
$$

The potential energy of the springs is

$$U_{\mathcal{B}}(q_1,q_2) = \tfrac{1}{2}kq_1^2 + \tfrac{1}{2}\kappa q_2^2, \qquad (11.10.15)$$

that is,

$$U_{\mathcal{B}}(q,\theta) = \tfrac{1}{2}kq^2 + \tfrac{1}{2}\kappa\theta^2. \qquad (11.10.16)$$

Next, noting that

$$
\begin{aligned}
\mathrm{d}_t\partial_{\dot{q}}T - \partial_q T &= \mathrm{d}_t[(m_1+m_2)\dot{q} + m_2\ell\dot{\theta}\cos\theta] \\
&= (m_1+m_2)\ddot{q} + m_2\ell\ddot{\theta}\cos\theta - m_2\ell\dot{\theta}^2\sin\theta \qquad (11.10.17)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}_t\partial_{\dot{\theta}}T - \partial_\theta T &= \mathrm{d}_t(m_2\ell^2\dot{\theta} + m_2\ell\dot{q}\cos\theta) - m_2\ell\dot{q}\dot{\theta}(-\sin\theta) \\
&= m_2\ell\mathrm{d}_t(\ell\dot{\theta} + \dot{q}\cos\theta) + m_2\ell\dot{q}\dot{\theta}\sin\theta \\
&= m_2\ell(\ell\ddot{\theta} + \ddot{q}\cos\theta - \dot{q}\dot{\theta}\sin\theta) + m_2\ell\dot{q}\dot{\theta}\sin\theta \\
&= m_2\ell^2\ddot{\theta} + m_2\ell\ddot{q}\cos\theta, \qquad (11.10.18)
\end{aligned}
$$

it follows from (11.8.12) and (11.8.13) that

$$(m_1+m_2)\ddot{q} + m_2\ell\ddot{\theta}\cos\theta - m_2\ell\dot{\theta}^2\sin\theta + kq = 0, \qquad (11.10.19)$$

$$m_2\ell\ddot{q}\cos\theta + m_2\ell^2\ddot{\theta} + \kappa\theta = 0, \qquad (11.10.20)$$

which can be written as

$$\begin{bmatrix} m_1+m_2 & m_2\ell\cos\theta \\ m_2\ell\cos\theta & m_2\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} kq - m_2\ell\dot{\theta}^2\sin\theta \\ \kappa\theta \end{bmatrix} = 0. \qquad (11.10.21)$$

Note that (11.10.21) is identical to (8.5.193).

11.11 Lagrangian Dynamics with Constraints

Use Lagrange multipliers to address constraints on generalized coordinates.

Consider a linkage consisting of 3 links, with the ends pinned to a base. This mechanical system cannot be modeled by independent generalized coordinates. The sum of the internal angles is 360 degrees.

11.12 Lagrangian Dynamics for Nonholonomic Systems

Consider systems with constraints on the generalized velocities that cannot be integrated to yield constraints on generalized coordinates.

11.13 Lagrangian Dynamics Using Physical Vectors

For the translation of each rigid body with center of mass y,

$$\overbrace{\vec{\partial}_{\vec{v}_{y/w/\mathrm{A}}} T}^{\mathrm{A}\bullet} - \vec{\partial}_{\vec{r}_{y/w}} T = 0 \tag{11.13.1}$$

For each rotating rigid body in SO(3)????

$$\overbrace{\vec{\partial}_{\vec{\omega}_i} T}^{\mathrm{A}\bullet} - \sum_{j=1}^{3} \vec{\partial}_{\vec{R}} T = 0 \tag{11.13.2}$$

Separate versions for S^1 and S^2

Example 11.13.1. Reconsider Example 8.6.6 with Lagrangian dynamics using physical vectors. Note that the kinetic energy of the two-bar planar linkage is given by

$$T = \tfrac{1}{2}\sum_{i=1}^{2} \vec{\omega}_i{}' \vec{J}_i \vec{\omega}_i + \tfrac{1}{2}\sum_{i=1}^{2} m_i \vec{v}_{\mathrm{c}_i/z_1/\mathrm{A}}{}' \vec{v}_{\mathrm{c}_i/z_1/\mathrm{A}}. \tag{11.13.3}$$

Since

$$\vec{v}_{\mathrm{c}_1/z_1/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{\mathrm{c}_1/z_1} = \vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1}, \tag{11.13.4}$$

$$\vec{v}_{\mathrm{c}_2/z_1/\mathrm{A}} = \overset{\mathrm{A}\bullet}{\vec{r}}_{\mathrm{c}_2/z_2} + \overset{\mathrm{A}\bullet}{\vec{r}}_{z_2/z_1} = \vec{\omega}_2 \times \vec{r}_{\mathrm{c}_2/z_2} + \vec{\omega}_1 \times \vec{r}_{z_2/z_1}, \tag{11.13.5}$$

it follows that

$$\begin{aligned} T = {} & \tfrac{1}{2}\vec{\omega}_1{}' \vec{J}_1 \vec{\omega}_1 + \tfrac{1}{2}\vec{\omega}_2{}' \vec{J}_2 \vec{\omega}_2 + \tfrac{1}{2} m_1 (\vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1})'(\vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1}) \\ & + \tfrac{1}{2} m_2 (\vec{\omega}_2 \times \vec{r}_{\mathrm{c}_2/z_2} + \vec{\omega}_1 \times \vec{r}_{z_2/z_1})'(\vec{\omega}_2 \times \vec{r}_{\mathrm{c}_2/z_2} + \vec{\omega}_1 \times \vec{r}_{z_2/z_1}). \end{aligned} \tag{11.13.6}$$

Note that

$$\begin{aligned} (\vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1})'(\vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1}) &= \vec{\omega}_1{}'[\vec{r}_{\mathrm{c}_1/z_1} \times (\vec{\omega}_1 \times \vec{r}_{\mathrm{c}_1/z_1})] \\ &= \vec{\omega}_1{}'[\vec{r}_{\mathrm{c}_1/z_1}{}' \vec{r}_{\mathrm{c}_1/z_1} \vec{\omega}_1] \end{aligned}$$

$$= \vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{\omega}\,'_1\vec{\omega}_1 \tag{11.13.7}$$

and

$$\begin{aligned}
(\vec{\omega}_2 \times \vec{r}_{c_2/z_2} + \vec{\omega}_1 \times \vec{r}_{z_2/z_1})'(\vec{\omega}_2 \times \vec{r}_{c_2/z_2} + \vec{\omega}_1 \times \vec{r}_{z_2/z_1}) \\
&= \vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}\,'_2\vec{\omega}_2 + \vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}\,'_1\vec{\omega}_1 + 2(\vec{\omega}_2 \times \vec{r}_{c_2/z_2})'(\vec{\omega}_1 \times \vec{r}_{z_2/z_1}) \\
&= \vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}\,'_2\vec{\omega}_2 + \vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}\,'_1\vec{\omega}_1 + 2\vec{\omega}\,'_2[\vec{r}_{c_2/z_2} \times (\vec{\omega}_1 \times \vec{r}_{z_2/z_1})] \\
&= \vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}\,'_2\vec{\omega}_2 + \vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}\,'_1\vec{\omega}_1 + 2\vec{\omega}\,'_2(\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}_1) \\
&= \vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}\,'_2\vec{\omega}_2 + \vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}\,'_1\vec{\omega}_1 + 2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}\,'_2\vec{\omega}_1. \qquad (11.13.8)
\end{aligned}$$

Therefore,

$$\begin{aligned}
T &= \tfrac{1}{2}\vec{\omega}\,'_1\vec{J}_1\vec{\omega}_1 + \tfrac{1}{2}\vec{\omega}\,'_2\vec{J}_2\vec{\omega}_2 + \tfrac{1}{2}m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{\omega}\,'_1\vec{\omega}_1 \\
&\quad + \tfrac{1}{2}m_2[\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}\,'_2\vec{\omega}_2 + \vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}\,'_1\vec{\omega}_1 + 2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}\,'_2\vec{\omega}_1], \qquad (11.13.9)
\end{aligned}$$

and thus

$$\begin{aligned}
\vec{\partial}_{\vec{\omega}_1}T &= \vec{J}_1\vec{\omega}_1 + m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{\omega}_1 + m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}_1 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}_2 \\
&= (\vec{J}_1 + m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{I} + m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I})\vec{\omega}_1 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}_2 \qquad (11.13.10)
\end{aligned}$$

and

$$\begin{aligned}
\vec{\partial}_{\vec{\omega}_2}T &= \vec{J}_2\vec{\omega}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}_1 \\
&= (\vec{J}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{I})\vec{\omega}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}_1. \qquad (11.13.11)
\end{aligned}$$

$$(\vec{I} - \hat{r}_{c_1/z_1}\hat{r}'_{c_1/z_1})[\overbrace{\vec{\partial}_{\vec{\omega}_1} T}^{A\bullet} - \vec{\partial}_{\vec{r}_{c_1/z_1}} T] = 0 \tag{11.13.12}$$

$$(\vec{I} - \hat{r}_{c_2/z_2}\hat{r}'_{c_2/z_2})[\overbrace{\vec{\partial}_{\vec{\omega}_2} T}^{A\bullet} - \vec{\partial}_{\vec{r}_{c_2/z_2}} T] = 0 \tag{11.13.13}$$

$$\begin{aligned} T = {} & \tfrac{1}{2}\vec{\omega}_1'\vec{J}_1\vec{\omega}_1 + \tfrac{1}{2}\vec{\omega}_2'\vec{J}_2\vec{\omega}_2 + \tfrac{1}{2}m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{\omega}_1'\vec{\omega}_1 \\ & + \tfrac{1}{2}m_2[\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{\omega}_2'\vec{\omega}_2 + \vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{\omega}_1'\vec{\omega}_1 + 2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\vec{\omega}_2'\vec{\omega}_1], \end{aligned} \tag{11.13.14}$$

$$\vec{\partial}_{\vec{r}_{c_1/z_1}} T = m_1\vec{\omega}_1'\vec{\omega}_1\vec{r}_{c_1/z_1} + m_2(???) \tag{11.13.15}$$

$$\vec{\partial}_{\vec{r}_{c_2/z_2}} T = m_2(\vec{\omega}_2'\vec{\omega}_2\vec{r}_{c_2/z_2} + \vec{\omega}_2'\vec{\omega}_1\vec{r}_{z_2/z_1}) \tag{11.13.16}$$

$$\begin{aligned} (\vec{I} - \hat{r}_{c_1/z_1}\hat{r}'_{c_1/z_1})[(\vec{J}_1 + m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1} + m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1})\overset{A\bullet}{\vec{\omega}_1} + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\overset{A\bullet}{\vec{\omega}_2} \\ + m_2\overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_2 - m_1\vec{\omega}_1'\vec{\omega}_1\vec{r}_{c_1/z_1} + m_2(???)] = 0 \end{aligned} \tag{11.13.17}$$

and

$$\begin{aligned} (\vec{I} - \hat{r}_{c_2/z_2}\hat{r}'_{c_2/z_2})[(\vec{J}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2})\overset{A\bullet}{\vec{\omega}_2} + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\overset{A\bullet}{\vec{\omega}_1} \\ + m_2\overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_1 - m_2(\vec{\omega}_2'\vec{\omega}_2\vec{r}_{c_2/z_2} + \vec{\omega}_2'\vec{\omega}_1\vec{r}_{z_2/z_1})] = 0. \end{aligned} \tag{11.13.18}$$

Next, note that

$$\begin{aligned} \overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_2 &= [\vec{v}\,'_{c_2/z_2/A}\vec{r}_{z_2/z_1} + \vec{r}\,'_{c_2/z_2}\vec{v}_{z_2/z_1/A}]\vec{\omega}_2 \\ &= [(\vec{\omega}_2 \times \vec{r}_{c_2/z_2})'\vec{r}_{z_2/z_1} + \vec{r}\,'_{c_2/z_2}(\vec{\omega}_1 \times \vec{r}_{z_2/z_1})]\vec{\omega}_2 \\ &= [\vec{\omega}_2'(\vec{r}_{c_2/z_2} \times \vec{r}_{z_2/z_1}) + \vec{\omega}_1'(\vec{r}_{z_2/z_1} \times \vec{r}_{c_2/z_2})]\vec{\omega}_2 \\ &= (\vec{\omega}_2 - \vec{\omega}_1)'(\vec{r}_{c_2/z_2} \times \vec{r}_{z_2/z_1})\vec{\omega}_2 \\ &= (\vec{\omega}_2 - \vec{\omega}_1)'\vec{\omega}_2(\vec{r}_{c_2/z_2} \times \vec{r}_{z_2/z_1}), \end{aligned} \tag{11.13.19}$$

where the last equality follows from the fact that $\vec{r}_{c_2/z_2} \times \vec{r}_{z_2/z_1}$, $\vec{\omega}_2 - \vec{\omega}_1$, and $\vec{\omega}_2$ are scalar multiples of $\hat{k}_A$.

Combining (11.13.30) and (11.13.31) thus yields

$$\overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_2 - m_1\vec{\omega}\,'_1\vec{\omega}_1\vec{r}_{c_1/z_1} + m_2(???) = (\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1})(\vec{\omega}_2-\vec{\omega}_1)'\vec{\omega}_2 + m_2(???)$$
$$= ?(\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1})\vec{\omega}\,'_2\vec{\omega}_2. \qquad (11.13.20)$$

Likewise,

$$\overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_1 - (\vec{\omega}\,'_2\vec{\omega}_2\vec{r}_{c_2/z_2} + \vec{\omega}\,'_2\vec{\omega}_1\vec{r}_{z_2/z_1}) = (\vec{\omega}_2-\vec{\omega}_1)'\vec{\omega}_1(\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1}) - (\vec{\omega}\,'_2\vec{\omega}_2\vec{r}_{c_2/z_2} + \vec{\omega}\,'_2\vec{\omega}_1\vec{r}_{z_2/z_1}) \qquad (11.13.21)$$

$$(\vec{I} - \hat{r}_{c_2/z_2}\hat{r}'_{c_2/z_2})[(\vec{J}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2})\overset{\mathrm{A}\bullet}{\vec{\omega}_2} + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}\overset{\mathrm{A}\bullet}{\vec{\omega}_1}$$
$$+ m_2(\vec{\omega}_2-\vec{\omega}_1)'\vec{\omega}_1(\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1}) - m_2(\vec{\omega}\,'_2\vec{\omega}_2\vec{r}_{c_2/z_2} + \vec{\omega}\,'_2\vec{\omega}_1\vec{r}_{z_2/z_1})] = 0. \qquad (11.13.22)$$

$$(\vec{I} - \hat{r}_{c_2/z_2}\hat{r}'_{c_2/z_2})\vec{\omega}\,'_2\vec{\omega}_1\vec{r}_{z_2/z_1} = \vec{\omega}\,'_2\vec{\omega}_1(\vec{r}_{z_2/z_1} - \hat{r}_{c_2/z_2}\hat{r}'_{c_2/z_2}\vec{r}_{z_2/z_1})$$
$$=$$
$$= ?\vec{\omega}\,'_2\vec{\omega}_1(\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1}) \qquad (11.13.23)$$

Therefore, (11.13.29) yields

$$\begin{bmatrix} \vec{J}_1 + m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{I} + m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I} & m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{c_2/z_2}\vec{I} \\ m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{c_2/z_2}\vec{I} & \vec{J}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{I} \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{\omega}_1} \\ \overset{\mathrm{A}\bullet}{\vec{\omega}_2} \end{bmatrix}$$
$$+ \begin{bmatrix} m_2(\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1})\vec{\omega}\,'_2\vec{\omega}_2 \\ -m_2(\vec{r}_{c_2/z_2}\times\vec{r}_{z_2/z_1})\vec{\omega}\,'_1\vec{\omega}_1 \end{bmatrix} = 0, \qquad (11.13.24)$$

which coincides with (8.6.121).

Combining (11.13.27) and (11.13.28) yields

$$\begin{bmatrix} \vec{J}_1 + m_1\vec{r}\,'_{c_1/z_1}\vec{r}_{c_1/z_1}\vec{I} + m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I} & m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{c_2/z_2}\vec{I} \\ m_2\vec{r}\,'_{z_2/z_1}\vec{r}_{c_2/z_2}\vec{I} & \vec{J}_2 + m_2\vec{r}\,'_{c_2/z_2}\vec{r}_{c_2/z_2}\vec{I} \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{\omega}_1} \\ \overset{\mathrm{A}\bullet}{\vec{\omega}_2} \end{bmatrix}$$
$$+ \begin{bmatrix} m_2[\overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_2 - \vec{\partial}_{\theta_1\hat{k}_{\mathrm{A}}}(\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}\,'_1\vec{\omega}_2] \\ m_2[\overbrace{\vec{r}\,'_{c_2/z_2}\vec{r}_{z_2/z_1}}^{\cdot}\vec{\omega}_1 - (\vec{\omega}\,'_2\vec{\omega}_2\vec{r}_{c_2/z_2} + \vec{\omega}\,'_2\vec{\omega}_1\vec{r}_{z_2/z_1})] \end{bmatrix} = 0. \qquad (11.13.25)$$

Therefore, for $i = 1, 2$, it follows from

$$\overset{\mathrm{A}\bullet}{\overbrace{\vec{\partial}_{\vec{\omega}_i} T}} - \vec{\partial}_{\theta_i \hat{k}_{\mathrm{A}}} T = 0 \tag{11.13.26}$$

that

$$\begin{aligned}(\vec{J}_1 + m_1 \vec{r}'_{\mathrm{c}_1/z_1}\vec{r}_{\mathrm{c}_1/z_1} + m_2 \vec{r}'_{z_2/z_1}\vec{r}_{z_2/z_1})\, \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 + m_2 \vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}\, \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \\ + m_2 \overset{\cdot}{\overbrace{\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}}}\,\vec{\omega}_2 - m_2 \vec{\partial}_{\theta_1\hat{k}_{\mathrm{A}}}(\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}'_2\vec{\omega}_1 = 0\end{aligned} \tag{11.13.27}$$

and

$$\begin{aligned}(\vec{J}_2 + m_2 \vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{\mathrm{c}_2/z_2})\, \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 + m_2 \vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}\, \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \\ + m_2 \overset{\cdot}{\overbrace{\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}}}\,\vec{\omega}_1 - m_2 \vec{\partial}_{\theta_2\hat{k}_{\mathrm{A}}}(\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}'_2\vec{\omega}_1 = 0.\end{aligned} \tag{11.13.28}$$

Combining (11.13.27) and (11.13.28) yields

$$\begin{aligned}&\begin{bmatrix} \vec{J}_1 + m_1 \vec{r}'_{\mathrm{c}_1/z_1}\vec{r}_{\mathrm{c}_1/z_1}\vec{I} + m_2 \vec{r}'_{z_2/z_1}\vec{r}_{z_2/z_1}\vec{I} & m_2 \vec{r}'_{z_2/z_1}\vec{r}_{\mathrm{c}_2/z_2}\vec{I} \\ m_2 \vec{r}'_{z_2/z_1}\vec{r}_{\mathrm{c}_2/z_2}\vec{I} & \vec{J}_2 + m_2 \vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{\mathrm{c}_2/z_2}\vec{I} \end{bmatrix}\begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \\ \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \end{bmatrix} \\ &+ \begin{bmatrix} m_2[\overset{\cdot}{\overbrace{\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}}}\,\vec{\omega}_2 - \vec{\partial}_{\theta_1\hat{k}_{\mathrm{A}}}(\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}'_1\vec{\omega}_2] \\ m_2[\overset{\cdot}{\overbrace{\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}}}\,\vec{\omega}_1 - \vec{\partial}_{\theta_2\hat{k}_{\mathrm{A}}}(\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}'_2\vec{\omega}_1] \end{bmatrix} = 0.\end{aligned} \tag{11.13.29}$$

Next, note that

$$\begin{aligned}\overset{\cdot}{\overbrace{\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}}}\,\vec{\omega}_2 &= [\vec{v}'_{\mathrm{c}_2/z_2/\mathrm{A}}\vec{r}_{z_2/z_1} + \vec{r}'_{\mathrm{c}_2/z_2}\vec{v}_{z_2/z_1/\mathrm{A}}]\vec{\omega}_2 \\ &= [(\vec{\omega}_2 \times \vec{r}_{\mathrm{c}_2/z_2})'\vec{r}_{z_2/z_1} + \vec{r}'_{\mathrm{c}_2/z_2}(\vec{\omega}_1 \times \vec{r}_{z_2/z_1})]\vec{\omega}_2 \\ &= [\vec{\omega}'_2(\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1}) + \vec{\omega}'_1(\vec{r}_{z_2/z_1} \times \vec{r}_{\mathrm{c}_2/z_2})]\vec{\omega}_2 \\ &= (\vec{\omega}_2 - \vec{\omega}_1)'(\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})\vec{\omega}_2 \\ &= (\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})(\vec{\omega}_2 - \vec{\omega}_1)'\vec{\omega}_2,\end{aligned} \tag{11.13.30}$$

where the last equality follows from the fact that $\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1}$, $\vec{\omega}_2 - \vec{\omega}_1$, and $\vec{\omega}_2$ are scalar multiples of $\hat{k}_{\mathrm{A}}$. Furthermore, Fact 2.22.2 implies that

$$\vec{\partial}_{\theta_1\hat{k}_{\mathrm{A}}}(\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}'_1\vec{\omega}_2 = -(\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})\vec{\omega}'_1\vec{\omega}_2. \tag{11.13.31}$$

Combining (11.13.30) and (11.13.31) thus yields

$$\overset{\cdot}{\overbrace{\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1}}}\,\vec{\omega}_2 - \vec{\partial}_{\theta_1\hat{k}_{\mathrm{A}}}(\vec{r}'_{\mathrm{c}_2/z_2}\vec{r}_{z_2/z_1})\vec{\omega}'_1\vec{\omega}_2 = (\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})(\vec{\omega}_2 - \vec{\omega}_1)'\vec{\omega}_2 - [-(\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1}\vec{\omega}'_1\vec{\omega}_2)]$$

$$= (\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})\vec{\omega}_2'\vec{\omega}_2. \tag{11.13.32}$$

Likewise,

$$\overset{\cdot}{\overbrace{\vec{r}_{\mathrm{c}_2/z_2}' \vec{r}_{z_2/z_1}}}\vec{\omega}_1 - \vec{\partial}_{\theta_2\hat{k}_{\mathrm{A}}}(\vec{r}_{\mathrm{c}_2/z_2}' \vec{r}_{z_2/z_1})\vec{\omega}_2'\vec{\omega}_1 = -m_2\vec{\omega}_1'\vec{\omega}_1\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1}. \tag{11.13.33}$$

Therefore, (11.13.29) yields

$$\begin{bmatrix} \vec{J}_1 + m_1\vec{r}_{\mathrm{c}_1/z_1}' \vec{r}_{\mathrm{c}_1/z_1}\vec{I} + m_2\vec{r}_{z_2/z_1}' \vec{r}_{z_2/z_1}\vec{I} & m_2\vec{r}_{z_2/z_1}' \vec{r}_{\mathrm{c}_2/z_2}\vec{I} \\ m_2\vec{r}_{z_2/z_1}' \vec{r}_{\mathrm{c}_2/z_2}\vec{I} & \vec{J}_2 + m_2\vec{r}_{\mathrm{c}_2/z_2}' \vec{r}_{\mathrm{c}_2/z_2}\vec{I} \end{bmatrix} \begin{bmatrix} \overset{\mathrm{A}\bullet}{\vec{\omega}}_1 \\ \overset{\mathrm{A}\bullet}{\vec{\omega}}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_2(\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})\vec{\omega}_2'\vec{\omega}_2 \\ -m_2(\vec{r}_{\mathrm{c}_2/z_2} \times \vec{r}_{z_2/z_1})\vec{\omega}_1'\vec{\omega}_1 \end{bmatrix} = 0, \tag{11.13.34}$$

which coincides with (8.6.121). ◇

Example 11.13.2. Reconsider the cart with rotating arm described in Example 11.10.4 using Lagrangian dynamics with physical vectors. Note that the kinetic energy of the cart and arm is given by

$$T = \tfrac{1}{2}m_1\vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \tfrac{1}{2}m_2\vec{v}'_{\mathrm{c}_2/w/\mathrm{A}}\vec{v}_{\mathrm{c}_2/w/\mathrm{A}}. \tag{11.13.35}$$

Noting that

$$\begin{aligned}\vec{v}_{\mathrm{c}_2/w/\mathrm{A}} &= \vec{v}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}}\\ &= \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}},\end{aligned} \tag{11.13.36}$$

it follows that

$$T = \tfrac{1}{2}m_1\vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \tfrac{1}{2}m_2(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}})'(\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}}). \tag{11.13.37}$$

The potential energy is given by

$$U = \tfrac{1}{2}k\vec{r}'_{\mathrm{c}_1/w}\vec{r}_{\mathrm{c}_1/w} + \tfrac{1}{2}\kappa\theta^2. \tag{11.13.38}$$

Now, using

$$\overbrace{\vec{\partial}_{\vec{v}_{\mathrm{c}_1/w/\mathrm{A}}}T}^{\mathrm{A}\bullet} - \vec{\partial}_{\vec{r}_{\mathrm{c}_1/w}}T + \vec{\partial}_{\vec{r}_{\mathrm{c}_1/w}}U = \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}, \tag{11.13.39}$$

it follows that

$$\overbrace{(m_1 + m_2)\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + m_2\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1}}^{\mathrm{A}\bullet} - 0 + k\vec{r}_{\mathrm{c}_1/w} = \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}, \tag{11.13.40}$$

which implies

$$(m_1 + m_2)\vec{a}_{\mathrm{c}_1/w/\mathrm{A}} + m_2 \overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + m_2\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{v}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + k\vec{r}_{\mathrm{c}_1/w} = \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}, \tag{11.13.41}$$

where $\vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}$ is the reaction force on the cart to due the supporting structure $\mathcal{S}$. Therefore,

$$(m_1 + m_2)\ddot{q}\hat{\jmath}_{\mathrm{A}} + m_2\ddot{\theta}\hat{k}_{\mathrm{A}} \times \ell\hat{\imath}_{\mathrm{B}} + m_2\dot{\theta}\hat{k}_{\mathrm{A}} \times (\dot{\theta}\hat{k}_{\mathrm{A}} \times \ell\hat{\imath}_{\mathrm{B}}) + kq\hat{\jmath}_{\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}, \tag{11.13.42}$$

which implies that

$$(m_1 + m_2)\ddot{q}\hat{\jmath}_{\mathrm{A}} + m_2\ell\ddot{\theta}\hat{\jmath}_{\mathrm{B}} - m_2\ell\dot{\theta}^2\hat{\imath}_{\mathrm{B}} + kq\hat{\jmath}_{\mathrm{A}} = \vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}}. \tag{11.13.43}$$

Therefore, writing $\vec{f}_{\mathrm{r}/\mathcal{C}/\mathcal{S}} = f_{\mathrm{r}}\hat{\imath}$, it follows that

$$(m_1 + m_2)\ddot{q}\hat{\jmath}_{\mathrm{A}} + m_2\ell\ddot{\theta}(-\sin\theta\hat{\imath}_{\mathrm{A}} + \cos\theta\hat{\jmath}_{\mathrm{A}}) - m_2\ell\dot{\theta}^2(\cos\theta\hat{\imath}_{\mathrm{A}} + \sin\theta\hat{\jmath}_{\mathrm{A}}) + kq\hat{\jmath}_{\mathrm{A}} = f_{\mathrm{r}}\hat{\imath}_{\mathrm{A}}. \tag{11.13.44}$$

Hence,

$$f_{\mathrm{r}} = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta. \tag{11.13.45}$$

and

$$(m_1 + m_2)\ddot{q} + m_2\ell\ddot{\theta}\cos\theta + kq - m_2\ell\dot{\theta}^2\sin\theta = 0. \tag{11.13.46}$$

$$T = \tfrac{1}{2}m_1\vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \tfrac{1}{2}m_2(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}})'(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}}). \quad (11.13.47)$$

$$T = \tfrac{1}{2}m_1\vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \tfrac{1}{2}m_2[(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1})'(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1}) + 2(\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1})'\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}}]. \quad (11.13.48)$$

$$T = \tfrac{1}{2}m_1\vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \tfrac{1}{2}m_2[\vec{\omega}_{\mathrm{B/A}} \times (\vec{r}_{\mathrm{c}_2/\mathrm{c}_1} \times (\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1}) + 2(\vec{\omega}'_{\mathrm{B/A}}(\vec{r}_{\mathrm{c}_2/\mathrm{c}_1} \times \vec{v}_{\mathrm{c}_1/w/\mathrm{A}}) + \vec{v}'_{\mathrm{c}_1/w/\mathrm{A}}\vec{v}_{\mathrm{c}_1/w/\mathrm{A}}]. \quad (11.13.49)$$

Prop 4.2

$$\overbrace{\vec{\partial}_{\vec{\omega}_{\mathrm{B/A}}}}^{\mathrm{A}\bullet} T - \hat{r}_{\mathrm{c}_2/\mathrm{c}_1} \times (\vec{\partial}_{\vec{r}_{\mathrm{c}_2/\mathrm{c}_1}} T - \vec{\partial}_{\vec{r}_{\mathrm{c}_1/w}} U) = 0, \quad (11.13.50)$$

it follows that

$$\overbrace{(m_1 + m_2)\vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + m_2\vec{\omega}_{\mathrm{B/A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1}}^{\mathrm{A}\bullet} - 0 + k\vec{r}_{\mathrm{c}_1/w} = 0, \quad (11.13.51)$$

which implies

Use Prop 4.2

Next, the equation of motion for the arm is given by

$$(\vec{I} - \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}\hat{r}'_{\mathrm{c}_2/\mathrm{c}_1})[\overbrace{\vec{\partial}_{\vec{v}_{\mathrm{c}_2/w/\mathrm{A}}} T}^{\mathrm{A}\bullet} + \vec{\partial}_{\vec{r}_{\mathrm{c}_2/\mathrm{c}_1}} U] = 0. \tag{11.13.52}$$

Note that

$$\vec{\partial}_{\vec{r}_{\mathrm{c}_2/\mathrm{c}_1}} \theta = \hat{k}_{\mathrm{A}} \times \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}. \tag{11.13.53}$$

Now, writing

$$T = \tfrac{1}{2} m_1 \vec{v}'_{\mathrm{c}_1/w/\mathrm{A}} \vec{v}_{\mathrm{c}_1/w/\mathrm{A}} + \tfrac{1}{2} m_2 (\vec{v}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}})'(\vec{v}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + \vec{v}_{\mathrm{c}_1/w/\mathrm{A}}), \tag{11.13.54}$$

it follows that

$$(\vec{I} - \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}\hat{r}'_{\mathrm{c}_2/\mathrm{c}_1})[\overbrace{m_2 \vec{v}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + m_2 \vec{v}_{\mathrm{c}_1/w/\mathrm{A}}}^{\mathrm{A}\bullet} + \frac{1}{\ell}\kappa\theta\hat{k}_{\mathrm{A}} \times \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}] = 0, \tag{11.13.55}$$

and thus

$$(\vec{I} - \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}\hat{r}'_{\mathrm{c}_2/\mathrm{c}_1})[m_2 \vec{a}_{\mathrm{c}_2/\mathrm{c}_1/\mathrm{A}} + m_2 \vec{a}_{\mathrm{c}_1/w/\mathrm{A}} + \frac{1}{\ell}\kappa\theta\hat{k}_{\mathrm{A}} \times \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}] = 0. \tag{11.13.56}$$

Hence,

$$(\vec{I} - \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}\hat{r}'_{\mathrm{c}_2/\mathrm{c}_1})[m_2[\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} + \vec{\omega}_{\mathrm{B}/\mathrm{A}} \times (\vec{\omega}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1})] + m_2 \vec{a}_{\mathrm{c}_1/w/\mathrm{A}} + \frac{1}{\ell}\kappa\theta\hat{k}_{\mathrm{A}} \times \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}] = 0, \tag{11.13.57}$$

and thus

$$(\vec{I} - \hat{r}_{\mathrm{c}_2/\mathrm{c}_1}\hat{r}'_{\mathrm{c}_2/\mathrm{c}_1})[m_2(\overset{\mathrm{A}\bullet}{\vec{\omega}}_{\mathrm{B}/\mathrm{A}} \times \vec{r}_{\mathrm{c}_2/\mathrm{c}_1} - \vec{\omega}'_{\mathrm{B}/\mathrm{A}}\vec{\omega}_{\mathrm{B}/\mathrm{A}}\vec{r}_{\mathrm{c}_2/\mathrm{c}_1}) + m_2 \vec{a}_{\mathrm{c}_1/w/\mathrm{A}} + \frac{1}{\ell}\kappa\theta\hat{k}_{\mathrm{A}} \times \hat{\imath}_{\mathrm{B}}] = 0. \tag{11.13.58}$$

Using $\hat{r}_{\mathrm{c}_2/\mathrm{c}_1} = \hat{\imath}_{\mathrm{B}}$ yields

$$(\vec{I} - \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}})[m_2(\ddot{\theta}\hat{k}_{\mathrm{A}} \times \ell\hat{\imath}_{\mathrm{B}} - \dot{\theta}^2\ell\hat{\imath}_{\mathrm{B}}) + m_2\ddot{q}\hat{\jmath}_{\mathrm{A}} + \frac{1}{\ell}\kappa\theta\hat{\jmath}_{\mathrm{B}}] = 0, \tag{11.13.59}$$

and thus

$$(\vec{I} - \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}})[m_2(\ell^2\ddot{\theta}\hat{\jmath}_{\mathrm{B}} - \dot{\theta}^2\ell^2\hat{\imath}_{\mathrm{B}}) + m_2\ell\ddot{q}\hat{\jmath}_{\mathrm{A}} + \kappa\theta\hat{\jmath}_{\mathrm{B}}] = 0. \tag{11.13.60}$$

Therefore,

$$(\vec{I} - \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}})[m_2\ell\ddot{q}\hat{\jmath}_{\mathrm{A}} + (m_2\ell^2\ddot{\theta} + \kappa\theta)\hat{\jmath}_{\mathrm{B}} - m_2\dot{\theta}^2\ell^2\hat{\imath}_{\mathrm{B}}] = 0, \tag{11.13.61}$$

and thus

$$(\vec{I} - \hat{\imath}_{\mathrm{B}}\hat{\imath}'_{\mathrm{B}})[m_2\ell\ddot{q}(\sin\theta\hat{\imath}_{\mathrm{B}} + \cos\theta\hat{\jmath}_{\mathrm{B}}) + (m_2\ell^2\ddot{\theta} + \kappa\theta)\hat{\jmath}_{\mathrm{B}} - m_2\dot{\theta}^2\ell^2\hat{\imath}_{\mathrm{B}}] = 0. \tag{11.13.62}$$

Hence,

$$m_2\ell\ddot{q}\cos\theta + m_2\ell^2\ddot{\theta} + \kappa\theta = 0, \tag{11.13.63}$$

which agrees with (11.10.20).

11.14 Hamiltonian Dynamics

Consider a particle y with mass m and generalized coordinates $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The total energy of y relative to an unforced particle w with respect to an inertial frame $\mathrm{F_A}$ is given by

$$\begin{aligned} \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) &= \tfrac{1}{2}m(\dot x^2+\dot y^2+\dot z^2)+V(q,t) \\ &= T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)+V(q,t) \\ &= 2T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)-L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t). \end{aligned} \tag{11.14.1}$$

In terms of the components of the momentum $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} m\dot x \\ m\dot y \\ m\dot z \end{bmatrix} = m\dot q$, it follows that

$$\begin{aligned} \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) &= \sum_{i=1}^{3} p_i\dot q_i - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) \\ &= p^{\mathsf T}\dot q - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t). \end{aligned} \tag{11.14.2}$$

Note that $p = \partial_{\dot q} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)$.

More generally, let $\mathcal{B}$ be a body with particles $y_1,\ldots,y_l$ whose masses are $m_1,\ldots,m_l$, respectively, let w be an unforced particle, and let $\mathrm{F_A}$ be a frame. As in the case of a single particle, define the *conjugate momentum* $p \in \mathbb{R}^l$ by

$$p \triangleq \partial_{\dot q}^{\mathsf T} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t), \tag{11.14.3}$$

and, in analogy with (11.14.2), define the *Hamiltonian* $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)$ of $\mathcal{B}$ relative to w with respect to $\mathrm{F_A}$ by

$$H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) \triangleq p^{\mathsf T}\dot q - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t). \tag{11.14.4}$$

Next, recall from (11.3.11) that $T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)$ is given by

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) = \tfrac{1}{2}\dot q^{\mathsf T}M(q,t)\dot q + F(q,t)\dot q + G(q,t), \tag{11.14.5}$$

where $M(q,t)$, $F(q,t)$, and $G(q,t)$ are defined by (11.3.12), (11.3.13), and (11.3.14), respectively. It thus follows from (11.14.3) and (11.14.5) that

$$p = M(q,t)\dot q + F^{\mathsf T}(q,t). \tag{11.14.6}$$

Now, substituting (11.14.6) into (11.14.4) and using (11.14.5) yields

$$\begin{aligned} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= [M(q,t)\dot q + F^{\mathsf T}(q,t)]^{\mathsf T}\dot q - L_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) \\ &= \dot q^{\mathsf T}M(q,t)\dot q + F(q,t)\dot q - T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) + V(q,t) \\ &= \tfrac{1}{2}\dot q^{\mathsf T}M(q,t)\dot q - G(q,t) + V(q,t) \\ &= \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t) - F(q,t)\dot q - 2G(q,t). \end{aligned} \tag{11.14.7}$$

Therefore, if $F(q,t) = 0$ and $G(q,t) = 0$, then $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) = \mathcal{E}_{\mathcal{B}/w/\mathrm{A}}(q,\dot q,t)$, as in the case of a single particle.

Next, to express $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)$ in terms of p rather than $\dot q$, note that it follows from (11.14.6) that

$$\dot q = M^{-1}(q,t)[p - F^{\mathsf T}(q,t)]. \tag{11.14.8}$$

Therefore, substituting (11.14.8) into (11.14.7) yields

$$\begin{aligned}H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= \tfrac{1}{2}[p - F^{\mathsf{T}}(q,t)]^{\mathsf{T}} M^{-1}(q,t)[p - F^{\mathsf{T}}(q,t)] - G(q,t) + V(q,t)\\ &= \tfrac{1}{2}p^{\mathsf{T}} M^{-1}(q,t)p - F(q,t)M^{-1}(q,t)p - \tfrac{1}{2}F(q,t)M^{-1}(q,t)F^{\mathsf{T}}(q,t) - G(q,t) + V(q,t).\end{aligned} \tag{11.14.9}$$

Next, note that

$$\mathrm{d}_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) = \partial_p H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)\dot{p} + \partial_q H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)\dot{q} + \partial_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t). \tag{11.14.10}$$

On the other hand, it follows from (11.14.4) and $p = \partial_{\dot{q}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)$ that

$$\begin{aligned}\mathrm{d}_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= \dot{p}^{\mathsf{T}}\dot{q} + p^{\mathsf{T}}\ddot{q} - [\partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} + \partial_{\dot{q}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\ddot{q} + \partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)]\\ &= \dot{p}^{\mathsf{T}}\dot{q} + p^{\mathsf{T}}\ddot{q} - [\partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} + p^{\mathsf{T}}\ddot{q} + \partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)]\\ &= \dot{p}^{\mathsf{T}}\dot{q} - \partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} - \partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t).\end{aligned} \tag{11.14.11}$$

Comparing (11.14.10) and (11.14.11), it follows that

$$\dot{q} = \partial_p^{\mathsf{T}} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t), \tag{11.14.12}$$

$$\partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = -\partial_q H_{\mathcal{B}/w/\mathrm{A}}(q,p,t), \tag{11.14.13}$$

and

$$\partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = -\partial_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t). \tag{11.14.14}$$

Finally, using (11.14.3), the fact that $\partial_{\dot{q}}^{\mathsf{T}} T_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = \partial_{\dot{q}}^{\mathsf{T}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)$, and (11.14.13), it follows from (11.8.13) that

$$\dot{p} = -\partial_q^{\mathsf{T}} H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) + Q_{\mathrm{np}}. \tag{11.14.15}$$

Equations (11.14.12) and (11.14.15) are *Hamilton's equations*.

If $\partial_t L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t) = 0$, then it follows from (11.8.13) that

$$\begin{aligned}\mathrm{d}_t H_{\mathcal{B}/w/\mathrm{A}}(q,p,t) &= \dot{p}^{\mathsf{T}}\dot{q} - \partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q}\\ &= \mathrm{d}_t\partial_{\dot{q}} L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q} - \partial_q L_{\mathcal{B}/w/\mathrm{A}}(q,\dot{q},t)\dot{q}\\ &= Q_{\mathrm{np}}^{\mathsf{T}}\dot{q}.\end{aligned} \tag{11.14.16}$$

Therefore, if $Q_{\mathrm{np}} = 0$, then $H_{\mathcal{B}/w/\mathrm{A}}(q,p,t)$ is constant.

11.15 GAK Dynamics

Include Gibbs-Appel-Kane dynamics.

11.16 Theoretical Problems

Problem 11.16.1. Let $q \in \mathbb{R}^r$ denote generalized coordinates for a body $\mathcal{B}$, and let $B \in \mathbb{R}^{r\times r}$ be a positive-semidefinite matrix. For a collection of inerters with generalized force

$$Q_{\mathrm{inert}}(\ddot{q}) = -B\ddot{q},$$

define the *inertance function*

$$I(\ddot{q}) = \tfrac{1}{2}\ddot{q}^{\mathsf{T}} B\ddot{q}.$$

Show that the kinetic energy of the body with the inerters is given by

$$T_{\mathcal{B}/w/\mathrm{A}}(q,\dot q) = \tfrac{1}{2}\dot q^{\mathsf{T}}[M(q)+B]\dot q,$$

where $\frac{1}{2}\dot q^{\mathsf{T}}M(q)\dot q$ is the kinetic energy of $\mathcal{B}$ without the inerters.

11.17 Applied Problems

Problem 11.17.1. Two particles, three springs, and one dashpot are interconnected as shown in Figure 11.17.1. The masses are constrained to move along straight, frictionless tracks that are mutually orthogonal, as shown. No gravity is present. Springs k_1 and k_2 are connected to the point labeled a. The tracks and the point a are embedded in an inertially nonrotating massive body. The particles are interconnected by a spring with stiffness k and a dashpot with viscosity c. The relaxed lengths of the springs with stiffnesses k_1, k_2, and k are, respectively, r_1, r_2, and r. Derive the equations of motion in terms of the distance q_1 from m_1 to a, the distance q_2 from m_2 to a, and the distance $l = \sqrt{q_1^2+q_2^2}$ from m_1 to m_2. Then, specialize these equations to the case where m_2 is fixed at a.

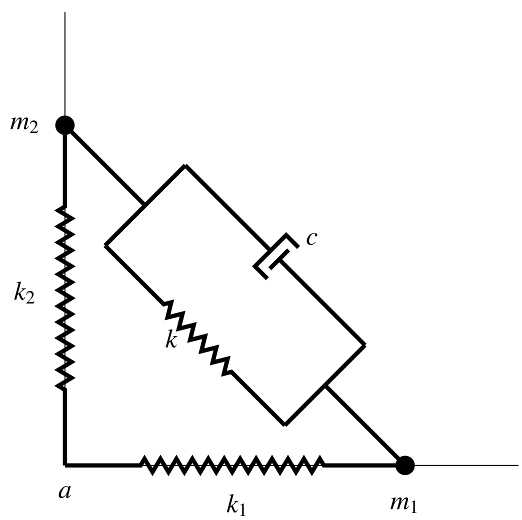


Figure 11.17.1: Two-particle body with springs and dashpot for Problem 11.17.1

Problem 11.17.2. In Figure 11.17.2, a rectangular rigid body whose mass is M and a thin bar whose mass is m and length is l move without friction (due to mounting on small wheels) over the surface of an inertially nonrotating massive body. The upper end of the thin bar is attached to a pin that moves without friction along a vertical track on the left edge of the rectangular rigid body. The pin at the end of the bar is attached to a spring with stiffness k and relaxed length r. The rectangular rigid body is connected to the right wall of the massive body by a dashpot whose viscosity is c. Finally, a force $\vec{f}$ is applied as shown to the lower left corner of the rectangular body. Derive the equations of motion.

Problem 11.17.3. The triangular cart in Figure 11.17.3 with mass m_1 is connected to a massive nonrotating body by means of a spring with stiffness k_1. The angle between the slanted surface of the cart and the horizontal direction is θ. The particle y, whose mass is m_2, slides without friction

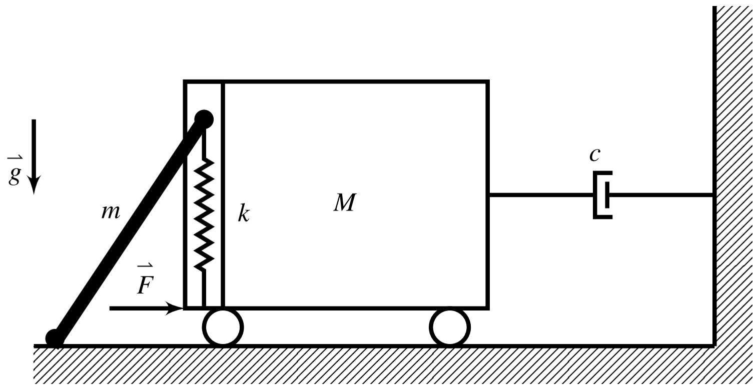


Figure 11.17.2: Rectangular rigid body and thin bar for Problem 11.17.2

along the slanted surface attached to a spring with stiffness k_2. The relaxed length of both springs is zero, and gravity acts in the direction shown. Derive the equations of motion.

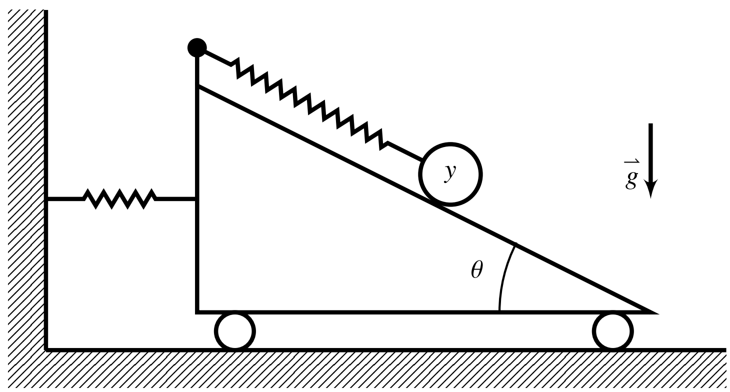


Figure 11.17.3: Triangular cart with particle and springs for Problem 11.17.3

Problem 11.17.4. The body in Figure 11.17.4 is a rigid bar that has two springs and two particles that slide without friction along the bar. The bar is attached by a frictionless pin joint to an inertially nonrotating massive body. Both particles y_1 and y_2 have mass m, and both springs have stiffness k. The relaxed length of each spring is l, and gravity acts in the direction shown. The angle θ between the bar and the direction shown is a prescribed function of time, and thus the inertia of the rigid bar is irrelevant. Derive the equations of motion.

Problem 11.17.5. The spherical pendulum in Figure 11.17.5 consists of a particle y attached by a rope to the top of a rigid bar. The length l of the rope is a prescribed function of time. The bar is attached to an inertially nonrotating massive body. Gravity acts in the vertical direction. Derive the equations of motion for this body in terms of the angle θ around the bar and the angle ϕ between the rope and the bar.

Problem 11.17.6. The two-link pendulum in Figure 11.17.6 is connected to a massive nonrotating body by means of a pin joint at the point a. The first link, whose ends are points a and b, has mass m_1 and length l_1. The second link, which is connected to the first link at point b by means of a

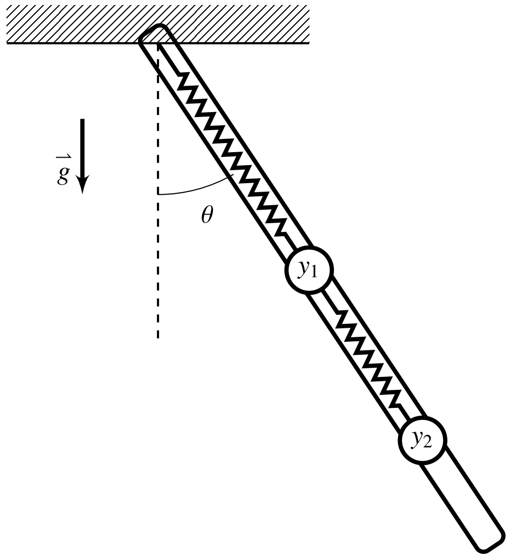


Figure 11.17.4: Rotating bar with two particles and two springs for Problem 11.17.4

pin joint, has mass m_2 and length l_2. The force $\vec{f}$ is applied to the tip of the second link at point c in a direction that is perpendicular to the second link. Derive the equations of motion for this body.

Problem 11.17.7. The rotating disk $\mathcal{D}$ in Figure 11.17.7 is subjected to a moment $\vec{\Gamma}$. The particle y slides without friction along a linear slot on the platform and is attached to a pair of identical springs, which are relaxed when y is at point b. The center of the disk is the point a, which is attached by a frictionless pin joint to a massive nonrotating body, and the moment of inertia of the disk around an axis that is perpendicular to the disk and relative to the point a is $J_{\mathcal{D}/a}$. The path of y is orthogonal to the line segment connecting a and b. The mass of the particle is m, the distance from a to b is h, and the stiffness of each spring is k. Derive the equations of motion.

Problem 11.17.8. Consider the two-bar linkage shown in Figure 11.17.8 consisting of two thin bars, one spring, and one particle. a, b, c, d, e are points. There are pin joints at a, b, c. The thin bar $\mathcal{B}_1$ between a and b has length l and mass m_1. The thin bar $\mathcal{B}_2$ between b and c has length l and mass m_2. The center of mass of $\mathcal{B}_1$ is at d, while the center of mass of $\mathcal{B}_2$ is at e. The spring connects points d and e. The relaxed length of the spring is $l/2$, and its stiffness is k. Because the bars $\mathcal{B}_1$ and $\mathcal{B}_2$ have equal length, there are four angles labeled θ. The particle y located at c is attached to the end of $\mathcal{B}_2$ and is mounted on a slider that allows it to move horizontally. The mass of y is m_3. All motion is frictionless. Uniform gravity acts in the direction shown, which is perpendicular to the line passing through a and c. The force $\vec{f}$ is applied to $\mathcal{B}_2$ at point e. The direction of $\vec{f}$ is the same as the direction of gravity. The moment $\vec{M}$ is a torque applied to $\mathcal{B}_1$ at the point a. Derive the equation of motion for this body in terms of θ.

Problem 11.17.9. Use Lagrangian dynamics to derive the equations of motion for the ball rolling down the inclined plane considered in Problem 8.9.2. Use these equations to determine the normal and tangential components of the reaction force between the ball and the inclined plane as well as the vertical reaction force between the inclined plane and the ground.

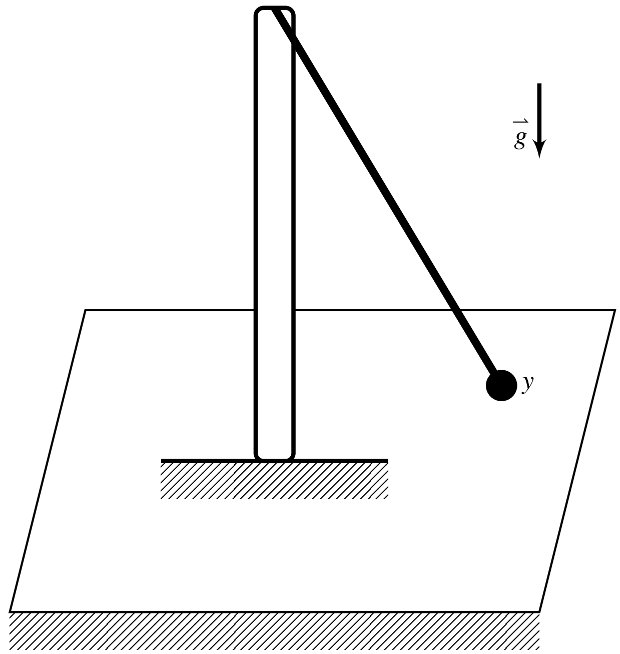


Figure 11.17.5: Spherical pendulum with variable length for Problem 11.17.5

Problem 11.17.10. Use Lagrangian dynamics to derive the equations of motion for the wheel with bar considered in Problem 8.9.8. Use these equations to determine the components of the reaction force at the pin joint.

Problem 11.17.11. Use Lagrangian dynamics to derive the equations of motion for the physical pendulum considered in Example 8.5.2. Use these equations to determine the reaction forces at the pin joint.

Problem 11.17.12. Use Lagrangian dynamics to derive the equations of motion for the ball and beam considered in Example 8.6.5. Use these equations to determine the reaction force on the ball due to direct contact with the beam.

Problem 11.17.13. Reconsider the rotating pendulum in Example 8.5.6.

i) Use Lagrangian dynamics to obtain a differential equation that describes the motion of the pendulum in terms of $\ddot{\theta}$, θ, ω, $\dot{\omega}$, r, ℓ, and g.

ii) Use Newton's second law to determine the components of the reaction force $\vec{f}_{\mathrm{r}/\mathcal{L}/y} = f_1\hat{\imath}_{\mathrm{C}} + f_2\hat{\jmath}_{\mathrm{C}} + f_3\hat{k}_{\mathrm{C}}$ on $\mathcal{L}$ due to the particle y.

iii) Determine the reaction force and reaction torque on the horizontal arm at z.

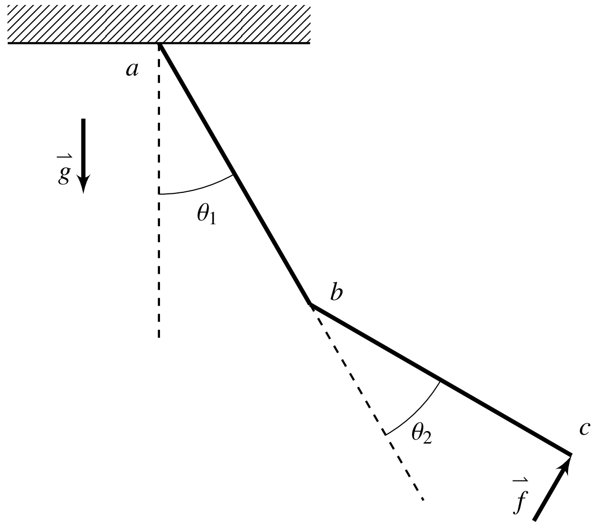


Figure 11.17.6: Two-link pendulum for Problem 11.17.6

11.18 Solutions to the Applied Problems

Solution to Problem 11.17.1.

$$m_1\ddot{q}_1 + \frac{c}{l^2}(q_1^2\dot{q}_1 + q_1q_2\dot{q}_2) + k_1(q_1 - r_1) + \frac{k}{l}(l - r)q_1 = 0,$$
$$m_2\ddot{q}_2 + \frac{c}{l^2}(q_2^2\dot{q}_2 + q_1q_2\dot{q}_1) + k_2(q_2 - r_2) + \frac{k}{l}(l - r)q_2 = 0.$$

Solution to Problem 11.17.2.

$$(m + M)\ddot{x} + c\dot{x} - \tfrac{1}{2}ml[(\cos\theta)\dot{\theta}^2 + (\sin\theta)\ddot{\theta}] = f,$$
$$\tfrac{1}{3}ml\ddot{\theta} - \tfrac{1}{2}m(\sin\theta)\ddot{x} + \tfrac{1}{2}mg\cos\theta + k(\cos\theta)(l\sin\theta - r) = 0,$$

where x is the distance from the lower left corner of the cart to the wall, and θ is the angle between the inclined link and the floor.

Solution to Problem 11.17.3.

$$(m_1 + m_2)\ddot{q}_1 + m_2\cos\theta\ddot{q}_2 + k_1q_1 = 0,$$
$$m_2[\ddot{q}_2 + (\cos\theta)\ddot{q}_1] + k_2q_2 = m_2g\sin\theta,$$

where q_1 is the distance between the vertical edge of the inclined plane and the wall, and q_2 is the distance between m_2 and the vertical edge of the inclined plane along the inclined plane.

Solution to Problem 11.17.4.

$$2\ddot{q}_1 + \ddot{q}_2 - (2q_1 + q_2)\dot{\theta}^2 + \frac{k}{m}(q_1 - l) = 2g\cos\theta,$$
$$\ddot{q}_1 + \ddot{q}_2 - (q_1 + q_2)\dot{\theta}^2 + \frac{k}{m}(q_2 - l) = g\cos\theta,$$

where q_1 is the distance between the pin joint and first mass, and q_2 is the distance between the first and second mass.

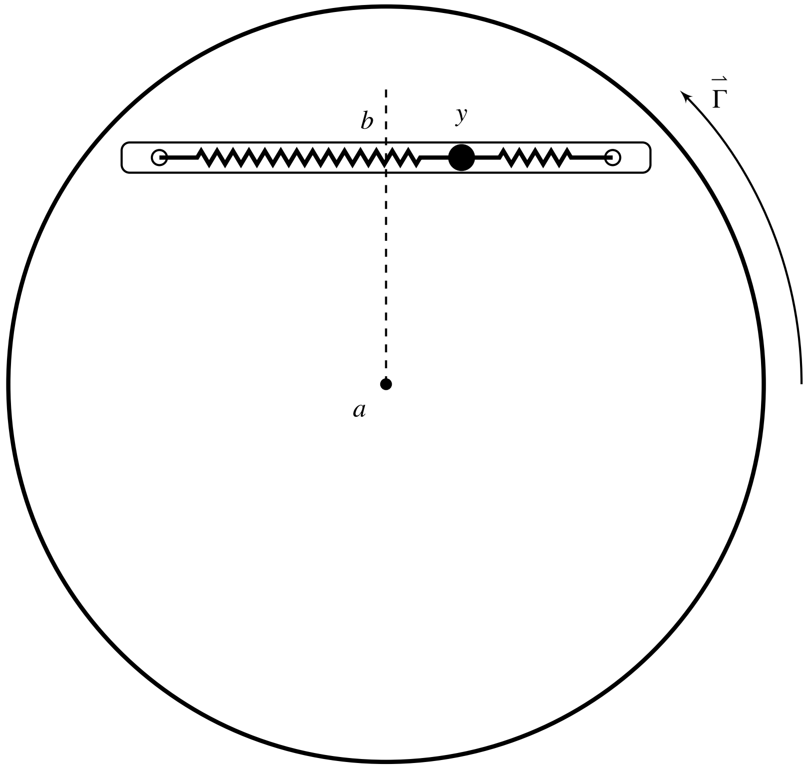


Figure 11.17.7: Rotating disk with translating particle for Problem 11.17.7

Solution to Problem 11.17.5.

$$l^2\ddot{\phi} + 2l\dot{l}\dot{\phi} - l^2\dot{\theta}^2(\sin\phi)\cos\phi + gl\sin\phi = 0,$$
$$l(\sin\phi)\ddot{\theta} + [2\dot{l}\sin\phi + 2l(\cos\phi)\dot{\phi}]\dot{\theta} = 0,$$

where θ is the angle around the bar, and ϕ is the angle between the rope and the bar.

Solution to Problem 11.17.6.

$$\beta\ddot{\theta}_1 + \alpha\ddot{\theta}_2 - \tfrac{1}{2}m_2l_1l_2(\sin\theta_2)\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2) + (\tfrac{1}{2}m_1 + m_2)gl_1\sin\theta_1 + \gamma = f(\cos\theta_2 l_1 + l_2),$$
$$\alpha\ddot{\theta}_1 + J_2\ddot{\theta}_2 + \tfrac{1}{2}m_2l_1l_2\sin\theta_2\dot{\theta}_1^2 + \gamma = fl_2,$$

where

$$\alpha = J_2 + \tfrac{1}{2}m_2l_1l_2\cos\theta_2, \quad \beta = J_1 + J_2 + m_2l_1^2 + m_2l_1l_2\cos\theta_2, \quad \gamma = \tfrac{1}{2}m_2gl_2\sin(\theta_1 + \theta_2).$$

Note that J_1 is the moment of inertia of bar 1 with respect to a around an axis coming out of the page, and J_2 is the moment of inertia of bar 2 with respect to b around an axis coming out of the page.

Solution to Problem 11.17.7.

$$(J_{\mathrm{D}} + mh^2 + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} - mh\ddot{x} = \tau, \quad \ddot{x} - h\ddot{\theta} - x\dot{\theta}^2 + \frac{2k}{m}x = 0,$$

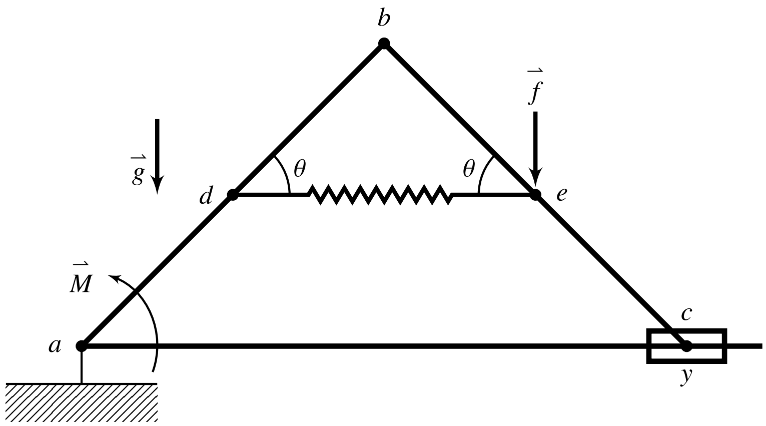


Figure 11.17.8: Two-bar linkage with spring for Problem 11.17.8

where x is the distance between y and b, θ is the angle between $\vec{r}_{b/a}$ and $\hat{\imath}_{\mathrm{A}}$, and F_{A} is defined such that $\hat{\imath}_{\mathrm{A}}$ and $\hat{\jmath}_{\mathrm{A}}$ are in the plane of the disk, and $\hat{k}_{\mathrm{A}}$ is perpendicular to the disk.

Solution to Problem 11.17.8.

$$\frac{1}{3}(m_1 + m_2)l^2\ddot{\theta} + (2m_2 + 4m_3)l^2 \sin\theta(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2) + kl^2 \sin\theta(\tfrac{1}{2} - \cos\theta)$$
$$+ \frac{1}{2}(m_1 + m_2)gl\cos\theta = M - \frac{1}{2}fl\cos\theta.$$

Note that y can move only horizontally.

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