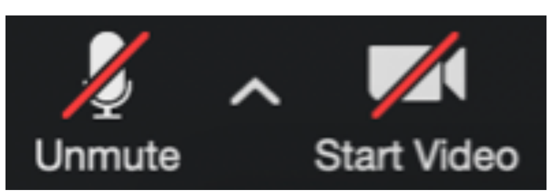
AEROSP 584 - Navigation and Guidance: From Perception to Control



Lectures start at 10:30am EST

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Lecture 14a

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To ask questions:

Raise Hand or Chat

Review

$$x_{k+1} = Ax_k + Bu_k + D_1 w_k$$
$$y_k = Cx_k + D_2 w_k$$

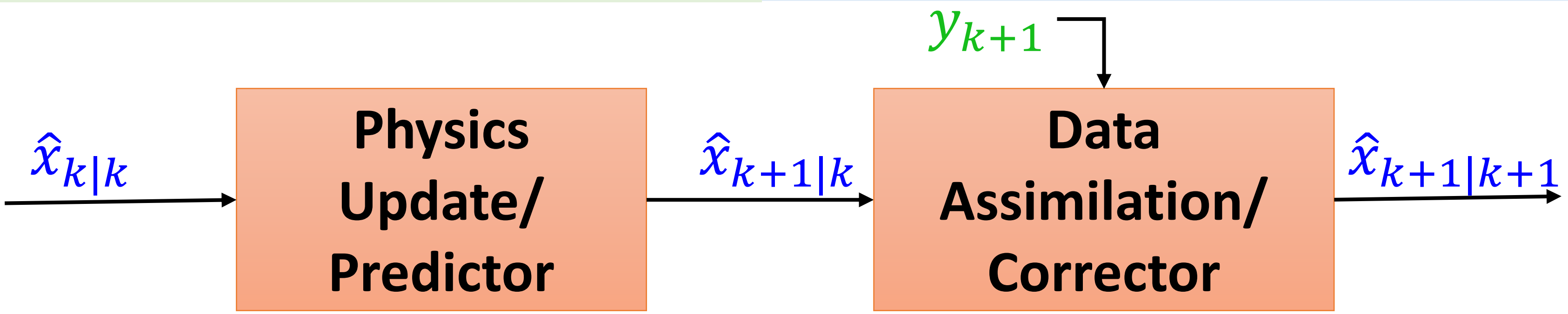
$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - C\hat{x}_{k+1|k}\right)$$

$$P_{k+1|k} = AP_{k|k}A^{\mathrm{T}} + Q$$
$$K_k = P_{k+1|k}C^{\mathrm{T}}\left(CP_{k+1|k}C^{\mathrm{T}} + R\right)^{-1}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_k C P_{k+1|k}$$

$$x_{k+1} = A_k x_k + B_k u_k + D_{1,k} w_k$$
$$y_k = C_k x_k + D_{2,k} w_k$$

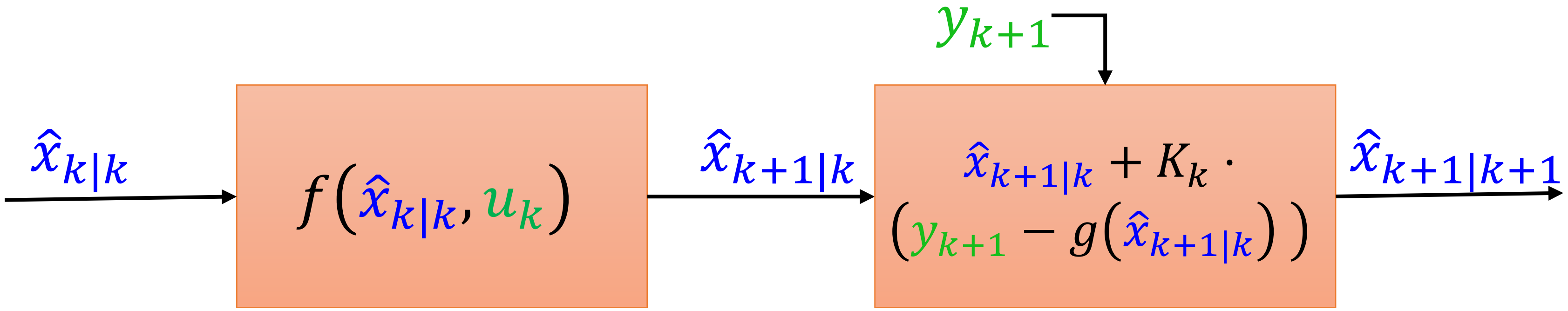
$$\hat{x}_{k+1|k} = A_k\hat{x}_{k|k} + B_k u_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - C_{k+1}\hat{x}_{k+1|k}\right)$$

$$P_{k+1|k} = A_k P_{k|k} A_k^{\mathrm{T}} + Q_k$$
$$K_k = P_{k+1|k}C_{k+1}^{\mathrm{T}}\left(C_{k+1}P_{k+1|k}C_{k+1}^{\mathrm{T}} + R_{k+1}\right)^{-1}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}$$



Extended Kalman Filter

$$x_{k+1} = f(x_k, u_k) + D_{1,k} w_k$$
$$y_{k+1} = g(x_{k+1}) + D_{2,k+1} w_{k+1}$$



$$P_{k+1|k} = A_k P_{k|k} A_k^{\mathrm{T}} + Q_k$$
$$K_k = P_{k+1|k} C_{k+1}^{\mathrm{T}} \left(C_{k+1} P_{k+1|k} C_{k+1}^{\mathrm{T}} + R_{k+1} \right)^{-1}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}$$

$$A_k = \left. \frac{\partial f}{\partial x} \right|_{(\hat{x}_{k|k}, u_k)}, C_{k+1} = \left. \frac{\partial g}{\partial x} \right|_{\hat{x}_{k+1|k}}$$

Static Estimation Cases

	Single Measurement	Multiple Measurements
Single Sensor		
Multiple Sensors	Case 1 (Sensor Fusion)	

Sensor Fusion

- Suppose we measured the velocity using different methods
  - Pitot tube
  - Tachometer
  - GPS

- How do we merge these measurements to get the best estimate of the velocity?

Optimal Static Estimation

- Consider two measurements y_1 and y_2 of a variable x

$$y_1 = x + w_1$$
$$y_2 = x + w_2$$

What is the best estimate of x?

$$J(x) = \alpha_1(y_1 - x)^2 + \alpha_2(y_2 - x)^2$$

- The minimizer $x^* = \min_{x \in \mathbb{R}} J(x)$ satisfies $\left.\frac{\partial J}{\partial x}\right|_{x=x^*} = 0$

$$\frac{\partial J}{\partial x} = -2\alpha_1(y_1 - x) - 2\alpha_2(y_2 - x)$$
$$(\alpha_1 y_1 + \alpha_2 y_2) - (\alpha_1 + \alpha_2)x^* = 0$$
$$x^* = \frac{\alpha_1 y_1 + \alpha_2 y_2}{\alpha_1 + \alpha_2}$$

Optimal Static Estimation - Weights

$$x^* = \frac{\alpha_1 y_1 + \alpha_2 y_2}{\alpha_1 + \alpha_2}$$

- What should the weights α_1, α_2 be?
  - w_1 is the noise in measurement y_1
  - If $|w_1|$ is large, we want to weigh it less
  - If $|w_1|$ is large, then variance σ_i^2 is large, so let $\alpha_i = \left(\sigma_i^2\right)^{-1}$

$$x^* = \frac{(\sigma_1^2)^{-1} y_1 + (\sigma_2^2)^{-1} y_2}{(\sigma_1^2)^{-1} + (\sigma_2^2)^{-1}} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} y_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} y_2$$

- For n measurements of x

$$x^* = \frac{\sum_{i=1}^{n} \left(\sigma_i^2\right)^{-1} y_i}{\sum_{i=1}^{n} \left(\sigma_i^2\right)^{-1}}$$

Static Estimation – Probabilistic Approach

- We can model the measurement as a random variable

$$y \sim \mathcal{N}(\bar{y}, \sigma^2)$$

- Suppose we are going at 60 mph and we record our speed every second

$$\bar{y} = \mathbb{E}[y] = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} y_i$$

$$\sigma^2 = \mathrm{Cov}[y] = \mathbb{E}[(y - \mathbb{E}[y])^2] = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

Static Estimation – Probabilistic Approach

- Suppose that we have two sensors

$$y_1 \sim \mathcal{N}(\bar{y}_1, \sigma_1^2)$$
$$y_2 \sim \mathcal{N}(\bar{y}_2, \sigma_2^2)$$

- We want to construct a random variable

$$x = \beta y_1 + (1 - \beta) y_2$$

with the smallest possible variance.

Static Estimation – Probabilistic Approach

$$\begin{aligned}\mathbb{E}[x] &= \beta\mathbb{E}[y_1] + (1-\beta)\mathbb{E}[y_2] \\ &= \beta\bar{y}_1 + (1-\beta)\bar{y}_2\end{aligned}$$

$$\mathrm{Cov}[x] = \mathbb{E}[(x-\mathbb{E}[x])^2] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= \mathbb{E}[(\beta y_1 + (1-\beta)y_2)^2] - (\beta\bar{y}_1 + (1-\beta)\bar{y}_2)^2$$

$$= \mathbb{E}[\beta^2 y_1^2 + (1-\beta)^2 y_2^2 + 2\beta(1-\beta)y_1 y_2] - (\beta\bar{y}_1 + (1-\beta)\bar{y}_2)^2$$

$$= \beta^2\mathbb{E}[y_1^2] + (1-\beta)^2\mathbb{E}[y_2^2] + 2\beta(1-\beta)\mathbb{E}[y_1 y_2]$$

$$-(\beta\bar{y}_1 + (1-\beta)\bar{y}_2)^2$$

Static Estimation – Probabilistic Approach

- $\sigma_1^2 = \mathrm{Cov}[y_1] = \mathbb{E}[(y_1 - \bar{y}_1)^2] = \mathbb{E}[y_1^2] - \bar{y}_1^2$
- $\mathbb{E}[y_1 y_2] = \mathbb{E}[y_1]\mathbb{E}[y_2] = \bar{y}_1\bar{y}_2$

$$
\begin{aligned}
\mathrm{Cov}[x] = {} & \beta^2(\sigma_1^2 + \bar{y}_1^2) + (1-\beta)^2(\sigma_2^2 + \bar{y}_2^2) + 2\beta(1-\beta)\bar{y}_1\bar{y}_2 \\
& -\beta^2\bar{y}_1^2 - (1-\beta)^2\bar{y}_2^2 - 2\beta(1-\beta)\bar{y}_1\bar{y}_2 \\
& = \beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2
\end{aligned}
$$

$$J(\beta) = \beta^2\sigma_1^2 + (1-\beta)^2\sigma_2^2$$

- Setting the derivative of $J(\beta)$ wrt β equal to zero yields

$$2\beta\sigma_1^2 - 2(1-\beta)\sigma_2^2 = 0$$

$$\beta^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Static Estimation – Probabilistic Approach

$$x = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} y_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} y_2$$

$$\mathbb{E}[x] = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \bar{y}_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \bar{y}_2$$

$$\mathrm{Cov}[x] = \frac{\sigma_2^2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2} = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1} < \min(\sigma_1^2, \sigma_2^2)$$

Static Estimation – Kalman Filter

$$x_{k+1} = A_k x_k + B_k u_k + D_{1,k} w_k$$
$$y_k = C_k x_k + D_{2,k} w_k$$

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k \left(y_{k+1} - C_{k+1} \hat{x}_{k+1|k} \right)$$

$$P_{k+1|k} = A_k P_{k|k} A_k^{\mathrm{T}} + Q_k$$
$$K_k = P_{k+1|k} C_{k+1}^{\mathrm{T}} \left(C_{k+1} P_{k+1|k} C_{k+1}^{\mathrm{T}} + R_{k+1} \right)^{-1}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}$$

$$x_{k+1} = x_k$$
$$y_k = x_k + D_{2,k} w_k$$

$$\hat{x}_{k+1|k} = \hat{x}_{k|k}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k \left(y_{k+1} - \hat{x}_{k+1|k} \right)$$

$$P_{k+1|k} = P_{k|k}$$
$$K_k = P_{k+1|k} \left(P_{k+1|k} + R_{k+1} \right)^{-1}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_k P_{k+1|k}$$

Static Estimation – Kalman Filter

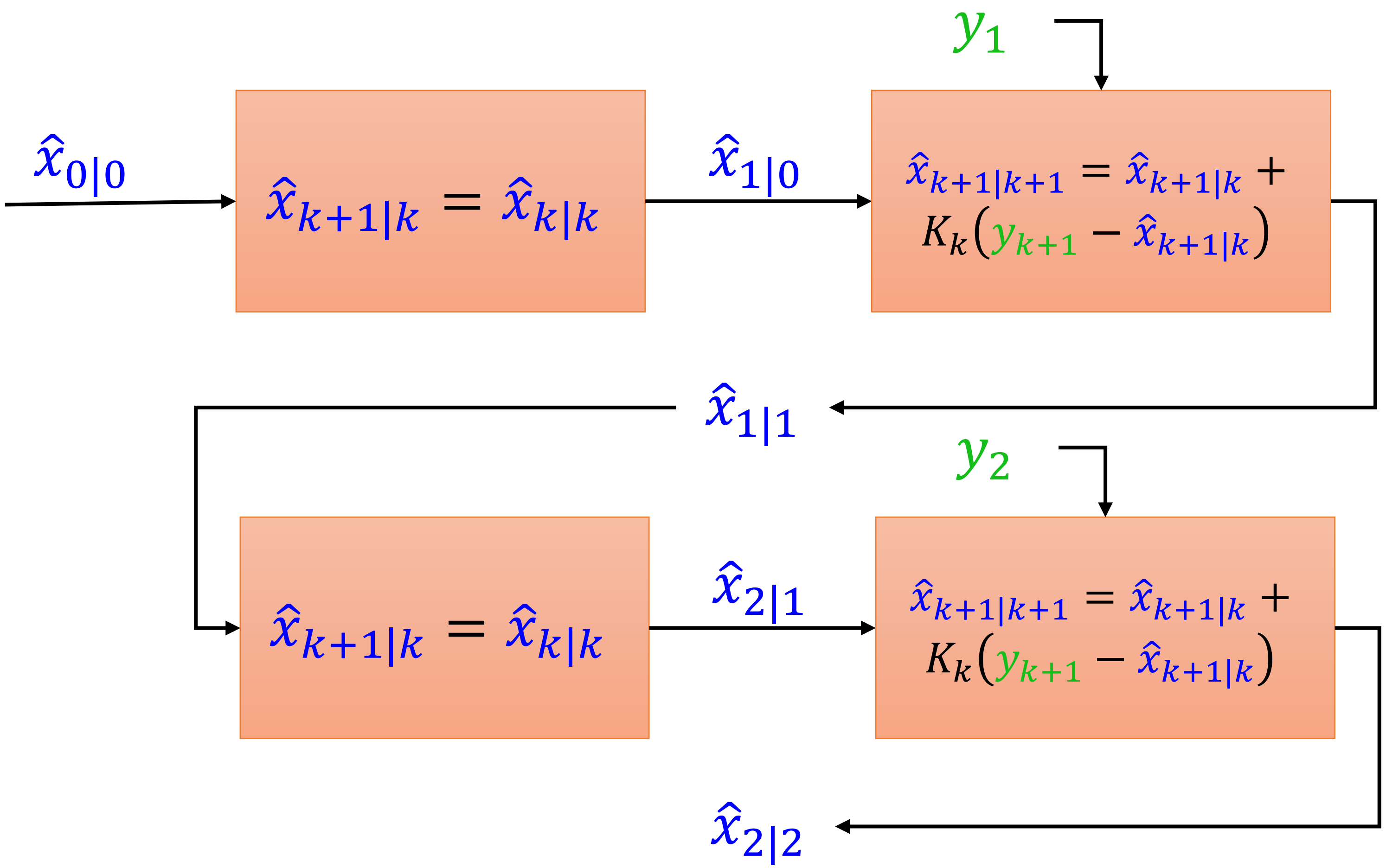
$$x_{k+1} = x_k$$
$$y_{k+1} = x_{k+1} + D_2 w_{k+1}$$

$$\hat{x}_{k+1|k} = \hat{x}_{k|k}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - \hat{x}_{k+1|k}\right)$$
$$= (1 - K_k)\hat{x}_{k+1|k} + K_k y_{k+1}$$

$$P_{k+1|k} = P_{k|k}$$

$$K_k = \frac{P_{k|k}}{P_{k|k} + R_{k+1}}$$

$$P_{k+1|k+1} = (1 - K_k)P_{k+1|k} = \frac{P_{k|k} R_{k+1}}{P_{k|k} + R_{k+1}}$$



Static Estimation – Kalman Filter

$$x_{k+1} = x_k$$
$$y_{k+1} = x_{k+1} + D_2 w_{k+1}$$
$$\hat{x}_{k+1|k} = \hat{x}_{k|k}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - \hat{x}_{k+1|k}\right)$$
$$= (1 - K_k)\hat{x}_{k+1|k} + K_k y_{k+1}$$
$$P_{k+1|k} = P_{k|k}$$
$$K_k = \frac{P_{k|k}}{P_{k|k} + R_{k+1}}$$
$$P_{k+1|k+1} = (1 - K_k)P_{k+1|k}$$
$$= \frac{P_{k|k}R_{k+1}}{P_{k|k} + R_{k+1}}$$

$$\hat{x}_{1|0} = \hat{x}_{0|0} = \bar{x}, \qquad P_{1|0} = P_{0|0} = \bar{P}$$
$$K_0 = \frac{\bar{P}}{\bar{P} + R_1}$$
$$\hat{x}_{1|1} = \frac{R_1}{\bar{P} + R_1}\bar{x} + \frac{\bar{P}}{\bar{P} + R_1}y_1$$
$$P_{1|1} = \frac{\bar{P}R_1}{\bar{P} + R_1}$$

$$\hat{x}_{2|1} = \hat{x}_{1|1} = \frac{\bar{P}y_1}{\bar{P} + R_1}, \qquad P_{2|1} = P_{1|1} = \frac{\bar{P}R_1}{\bar{P} + R_1}$$
$$K_1 = \frac{\dfrac{\bar{P}R_1}{\bar{P} + R_1}}{\dfrac{\bar{P}R_1}{\bar{P} + R_1} + R_2} = \frac{\bar{P}R_1}{\bar{P}(R_1 + R_2) + R_1R_2}$$

$$\hat{x}_{2|2}$$
$$= \frac{(\bar{P}+R_1)R_2}{\bar{P}(R_1+R_2)+R_1R_2}\left(\frac{R_1}{\bar{P}+R_1}\bar{x} + \frac{\bar{P}}{\bar{P}+R_1}y_1\right) + \frac{\bar{P}R_1}{\bar{P}(R_1+R_2)+R_1R_2}y_2$$
$$= \frac{R_1R_2}{\bar{P}(R_1+R_2)+R_1R_2}\bar{x} + \frac{\bar{P}R_2}{\bar{P}(R_1+R_2)+R_1R_2}y_1 + \frac{\bar{P}R_1}{\bar{P}(R_1+R_2)+R_1R_2}y_2$$

$$P_{2|2} = \frac{P_{1|1}R_2}{P_{1|1} + R_2} = \frac{\dfrac{\bar{P}R_1}{\bar{P} + R_1}R_2}{\dfrac{\bar{P}R_1}{\bar{P} + R_1} + R_2} = \frac{\bar{P}R_1R_2}{\bar{P}(R_1 + R_2) + R_1R_2}$$

Static Estimation – Kalman Filter

- If $\bar{P} \rightarrow \infty$

$$\hat{x}_{2|2} \rightarrow \frac{R_2}{R_1 + R_2} y_1 + \frac{R_1}{R_1 + R_2} y_2$$

$$P_{2|2} \rightarrow \frac{R_1 R_2}{R_1 + R_2}$$

Static Estimation Cases

	Single Measurement	Multiple Measurements
Single Sensor		Case 2
Multiple Sensors	Case 1 (Sensor Fusion)	

Static Estimation – Multiple Measurements

- Suppose we are going at 60 mph and we record our speed every second

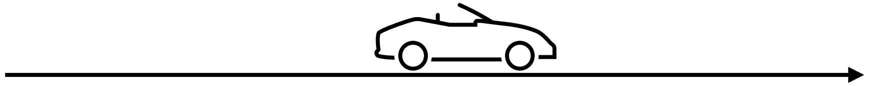
$$x_{k+1} = x_k$$
$$y_{k+1} = x_{k+1} + D_2 w_{k+1}$$

$$\hat{x}_{k+1|k} = \hat{x}_{k|k}$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_k\left(y_{k+1} - \hat{x}_{k+1|k}\right)$$

$$P_{k+1|k} = P_{k|k}$$
$$K_k = \frac{P_{k|k}}{P_{k|k} + R_{k+1}}$$
$$P_{k+1|k+1} = (1 - K_k)P_{k+1|k}$$

- $\bar{y} = \mathbb{E}[y] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} y_i$
- $\sigma^2 = \mathrm{Cov}[y] = \mathbb{E}[(y - \mathbb{E}[y])^2]$
  $= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (y_i - \bar{y})^2$

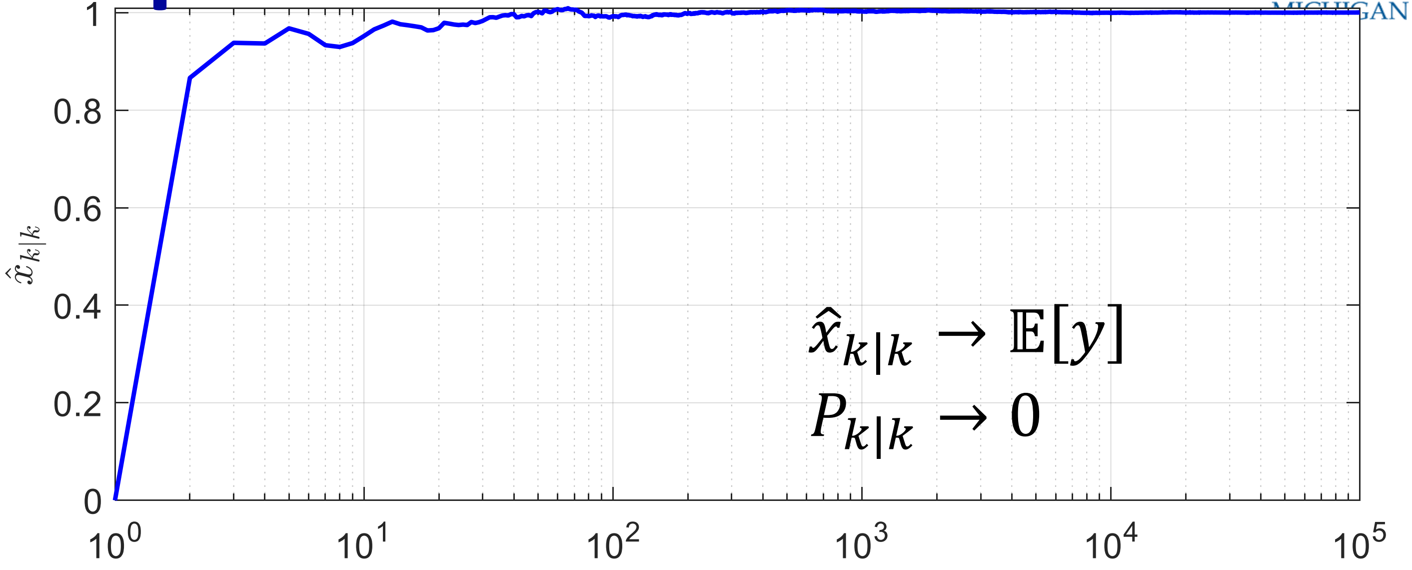
Static Estimation – Multiple Measurements



- Let $x_k = 1$ and $D_{2,k} = 0.1 \Rightarrow R_k = 0.01$

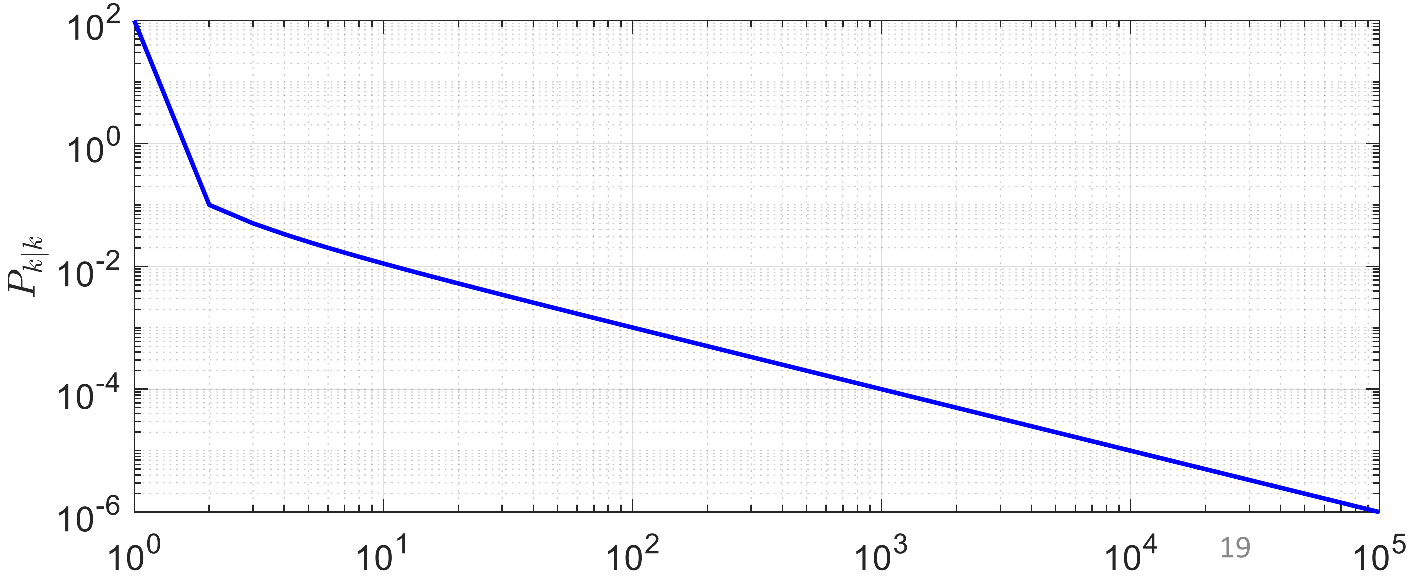
- $\bar{y} = \mathbb{E}[y] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} y_i$
- $\sigma^2 = \mathrm{Cov}[y] = \mathbb{E}[(y-\mathbb{E}[y])^2]$

  $= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (y_i-\bar{y})^2$
- $y \sim \mathcal{N}(1, 0.01)$



$\hat{x}_{k|k} \to \mathbb{E}[y]$

$P_{k|k} \to 0$



Static Estimation - Observations

$$K_k = \frac{P_{k|k}}{P_{k|k} + R_{k+1}}, \qquad P_{k+1|k+1} = \frac{P_{k|k} R_{k+1}}{P_{k|k} + R_{k+1}}$$

- $P_{k|k} > 0$ and $P_{k+1|k+1} < \min\left(P_{k|k}, R_{k+1}\right) \Rightarrow P_{k|k} \to 0$
- $K_k \to 0 \Rightarrow$ Filter ignores the measurements
- $P_{k|k} \to 0 \Rightarrow \frac{R_{k+1}}{P_{k|k}+R_{k+1}} \to 1 \Rightarrow$the covariance decreases slower

Static Estimation Cases

	Single Measurement	Multiple Measurements
Single Sensor		Case 2
Multiple Sensors	Case 1 (Sensor Fusion)	Case 3 (Problem 3, HW1)

Position Fixing Using Kalman Filter

- Let the location of beacon L_i be (x_i, y_i)
- Let the distance from beacon L_i be R_i

$$(x - x_i)^2 + (y - y_i)^2 = R_i^2$$

- $Y_i = R_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$

- $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \sqrt{(x - x_1)^2 + (y - y_1)^2} \\ \sqrt{(x - x_2)^2 + (y - y_2)^2} \\ \sqrt{(x - x_3)^2 + (y - y_3)^2} \end{bmatrix} = g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = g(X)$

Position Fixing Using Kalman Filter

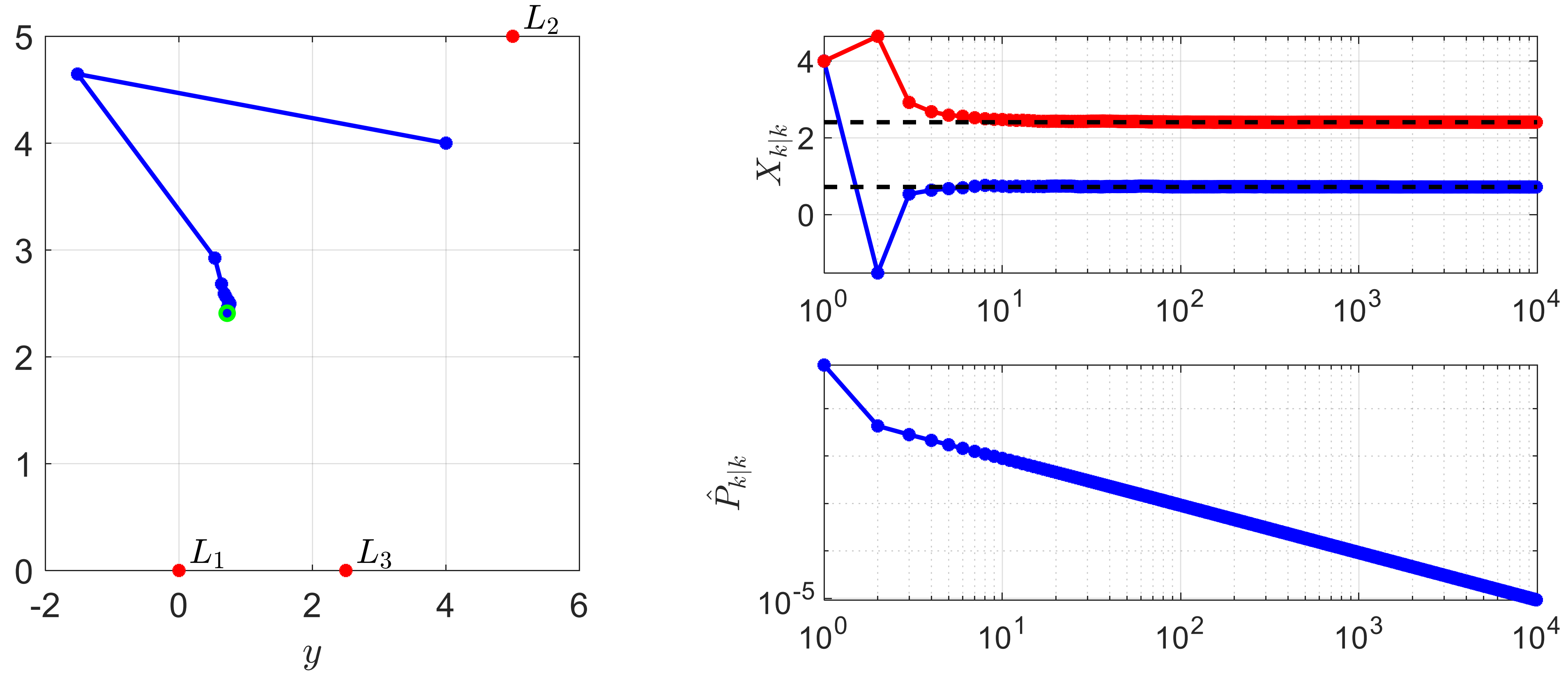
$$X_{k+1} = X_k$$
$$Y_{k+1} = g(X_{k+1}) + D_2 w_{k+1}$$

$$\hat{X}_{k+1|k} = \hat{X}_{k|k} \triangleq \hat{X}_k$$
$$\hat{X}_{k+1|k+1} = \hat{X}_{k+1|k} + K_k\left(Y_{k+1} - g(\hat{X}_{k+1|k})\right)$$

$$P_{k+1|k} = P_{k|k}$$
$$K_k = P_{k+1|k} C_{k+1}^{\mathrm{T}} \left(C_{k+1} P_{k+1|k} C_{k+1}^{\mathrm{T}} + R_{k+1}\right)^{-1}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_k C_{k+1} P_{k+1|k}$$

$$c_i = \left.\frac{\partial g_i}{\partial X}\right|_{\hat{X}_k} = \frac{1}{\sqrt{(\hat{x}_k - x_i)^2 + (\hat{y}_k - y_i)^2}} \begin{bmatrix} \hat{x}_k - x_i & \hat{y}_k - y_i \end{bmatrix}$$

Position Fixing Using Kalman Filter



Position Fixing Using Kalman Filter