Physical matrices

We have focused so far on vectors to represent position, deisplacement, etc.

That's not enough.

Attitude (rotation, orientantation) is equally important.

For attitude representation, we will introduce physical matrices and math matrices.

Particularly, if $\vec{r}$ is a physical vector, and $\vec{M}$ is a physical matrix representing a rotation of $\vec{r}$ to a new vector $\vec{n}$, then we'll see that:

$$\vec{n} = \vec{M}\,\vec{r}$$

To identify appropriatelly $\vec{M}$, let's build some background:

Physical covectors

Recall dot product:

(physical) $\vec{x} \cdot \vec{y} = |\vec{x}| \cdot |\vec{y}| \cdot \cos\theta_{\vec{x}|\vec{y}}$

(math) $\vec{x}|_A^T \cdot \vec{y}|_A = \vec{x} \cdot \vec{y}$

$$\underbrace{\quad}_{1\times 3}\ \underbrace{\quad}_{3\times 1}$$
$$\underbrace{\qquad\qquad}_{1\times 1}$$

Notation: $\vec{x}\cdot\vec{y} = \vec{x}'\,\vec{y}$

$\vec{x}'$ is call the physical covector.

Fact $\vec{x}'\big|_A = \vec{x}\big|_A^{T}$, transpose

given a frame A.

Having the notion of physical covector, let's define a physical matrix as follows:

$$\vec{M} \stackrel{\Delta}{=} \vec{\varphi}\,\vec{\xi}\,' \qquad \underbrace{\vec{n}}_{\text{vector}} = \underbrace{\vec{M}\,\vec{r}}_{\text{vector}}$$

given an $\vec{\varphi}$ and $\vec{\xi}$.

why this brings us closer to our end goal? Because $\vec{M}\,\vec{r}$ is a physical vector!

Let's validate this:

$$\vec{M}\,\vec{r} = \vec{\varphi}\,\underbrace{\vec{\xi}\,'\,\vec{r}}_{\text{scalar} \stackrel{\Delta}{=} s}$$

$$= s \cdot \vec{\varphi}$$

<u>Note</u>: $\vec{M}$ doesn't nessarilly repe–

sent rotation yet. We have to find appropriate $\vec{\varphi}, \vec{\xi}$ (if any).

<u>Note</u>: what is $\vec{M}\big|_A$?

$(\vec{M} = \vec{x}\,\vec{y}\,')$

$$\vec{M}\big|_A = \vec{x}\big|_A \, \vec{y}\,'\big|_A$$

$$= \underbrace{\underbrace{\vec{x}\big|_A}_{3\times1} \cdot \underbrace{\vec{y}\big|_A^T}_{1\times3}}_{3\times3}$$

$$\Rightarrow (\vec{M}\,\vec{r})\big|_A = \vec{M}\big|_A \cdot \vec{r}\big|_A \quad : 3\times1 \text{ vector}$$

<u>Note</u> (Physical matrix rules)

- Summation

$$\vec{M} = \vec{x}\,\vec{y}' \;,\; \vec{N} = \vec{z}\,\vec{w}'$$

$$\Rightarrow \vec{M} + \vec{N} = \vec{x}\,\vec{y}' + \vec{z}\,\vec{w}'$$

$$\Rightarrow (\vec{M} + \vec{N})\big|_A = \vec{M}\big|_A + \vec{N}\big|_A$$

- Multiplication

$$\vec{M}\,\vec{N} = \vec{x}\,\underbrace{\vec{y}' \cdot \vec{z}}_{\text{scalar}}\,\vec{w}' =$$

$$= (\vec{y}'\vec{z})\,\vec{x}\,\vec{w}'$$

$$(\vec{M}\,\vec{N})\big|_A = \underbrace{\vec{M}\big|_A}_{3\times 3} \cdot \underbrace{\vec{N}\big|_A}_{3\times 3} \;:\; 3\times 3 \text{ matrix}$$

· Dual physical matrix:

given $\vec{M} = \vec{x}\,\vec{y}'$, then its dual is $\vec{M}' \triangleq \vec{y}\,\vec{x}'$.

Its math version is the transpose:

$$\vec{M}'\big|_A = (\vec{y}\,\vec{x}')\big|_A$$

$$= \vec{y}\big|_A\,\vec{x}'\big|_A$$

$m = \begin{pmatrix} a & \beta \\ \gamma & c \end{pmatrix}$

$$= \vec{y}\big|_A\,\vec{x}\big|_A^T$$

$m^T \triangleq \begin{pmatrix} a & \gamma \\ \beta & c \end{pmatrix}$

$$= \left(\vec{x}\big|_A \cdot \vec{y}\big|_A^T\right)^T$$

$$= \vec{M}\big|_A^T$$

- How do we go from
  math $\longrightarrow$ physical?

(so far we've seen the
physical $\longrightarrow$ math)

Observe that, given frame

$$F_A = [\hat{i}_A \ \hat{j}_A \ \hat{k}_A]:$$

$$(\hat{i}_A \hat{i}'_A)\big|_A = \hat{i}_A\big|_A \hat{i}\big|_A^T$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1 \ 0 \ 0)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly for $\hat{i}_A \hat{j}_A'$, $\hat{i}_A \hat{k}_A'$

...

Then, given:

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$= M_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + M_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots$$

$$\Rightarrow M = M_{11} \hat{i}_A \hat{i}_A' + M_{12} \hat{i}_A \hat{j}_A' + \dots$$

Let's get closer to our goal in find $\vec{M}$ that captures rotation.

The most trivial rotation is the no-rotation, i.e., it must be

$$\vec{r} = \vec{M}\,\vec{r}.$$

That is, $\vec{M}$ should be an <u>identity</u> physical matrix, which we will denote by

$$\vec{I}.$$

Observe that in math we know:

$$\overleftrightarrow{I}\Big|_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \overleftrightarrow{I} = \hat{i}_A \hat{i}'_A + \hat{j}_A \hat{j}'_A + \hat{k}_A \hat{k}'_A$$

Fact (sanity check): $\overleftrightarrow{I}\,\vec{x} = \vec{x}$ for all $\vec{x}$.

Proof

$$\overleftrightarrow{I}\,\vec{x} = \left(\hat{i}_A \hat{i}'_A + \hat{j}_A \hat{j}'_A + \hat{k}_A \hat{k}'_A\right)\left(x_1 \hat{i}_A + x_2 \hat{j}_A + x_3 \hat{k}_A\right)$$

$$= x_1 \hat{i}_A + x_2 \hat{j}_A + x_3 \hat{k}_A$$

$$= \vec{x}. \qquad \blacksquare$$

Let's consider non-trivial rotations now.

Let F_A and F_B be frames.

We wish to rotate F_A so that it coincides with F_B.

Define:

$$\vec{R}_{B|A} \triangleq \hat{\imath}_B \hat{\imath}'_A + \hat{\jmath}_B \hat{\jmath}'_A + \hat{k}_B \hat{k}'_A$$

Then,

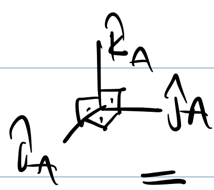
$$\boxed{F_B = \vec{R}_{B/A} F_A}$$

Proof We need to show:

$$[\hat{\imath}_B \;\; \hat{\jmath}_B \;\; \hat{k}_B] = \vec{R}_{B|A}\,[\hat{\imath}_A \;\; \hat{\jmath}_A \;\; \hat{k}_A]$$

We do this element-wise:

$$\vec{R}_{B|A}\,\hat{\imath}_A = (\hat{\imath}_B\,\hat{\imath}_A' + \hat{\jmath}_B\,\hat{\jmath}_A' + \hat{k}_B\,\hat{k}_A')\,\hat{\imath}_A$$



$$= \hat{\imath}_B$$

and so forth

POLL 1

Fact Let F_A, F_B, F_C be frames:

$$\vec{R}_{C|A} = \vec{R}_{C|B}\,\vec{R}_{B|A}$$

Fact

$$\vec{R}_{A|B} = \vec{R}'_{B|A}$$

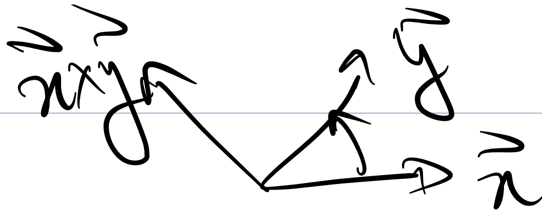
Fact $\vec{R}_{B|A}\vec{R}_{A|B} = \vec{R}_{A|B}\vec{R}_{B|A} = I$

Define $\vec{R}^{-1}_{B|A} \triangleq \vec{R}_{A|B} = \vec{R}'_{B|A}$

Fact $|\vec{R}_{B|A}\vec{x}| = |\vec{x}|$

[**Cross-product matrix**:

$$\vec{x} \times \vec{y} = |\vec{x}| \cdot |\vec{y}| \sin\theta_{\vec{x}|\vec{y}} \, \hat{\theta}_{\vec{y}|\vec{x}}$$



Then, $\vec{x}^{\times}$ is the super-cross matrix:

$$\underbrace{\vec{x}^{\times}}_{\text{a physical matrix}}\vec{y} \overset{\Delta}{=} \vec{x}\times\vec{y}$$

If $\vec{x}\big|_A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$(\cdot)^{\times}$

$$\Rightarrow \vec{x}^{\times}\big|_A = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

$$= \vec{x}\big|_A^{\times}$$

∎

Fact (Rodrigues's Formula)

Given F_A and F_B, there exist

1) Eigenaxis $\hat{n}$

2) Eigenvalue θ

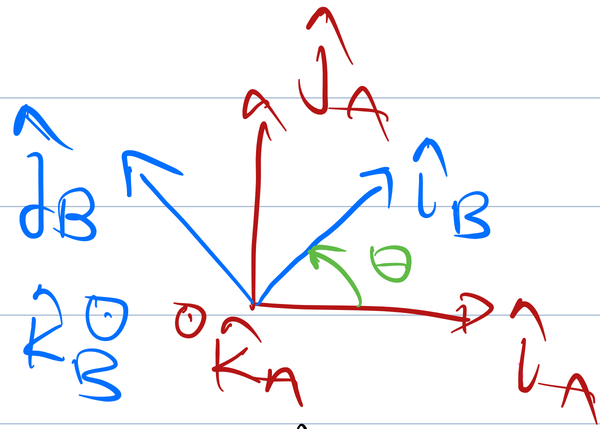
such that

$$\vec{R}_{B|A} = \vec{R}_{\hat{n}}(\theta)$$

where

$$\vec{R}_{\hat{n}}(\theta) = \cos\theta \cdot \vec{I} + (1-\cos\theta)\,\hat{n}\hat{n}' + \sin\theta \cdot \hat{n}^{\times}$$

Example:



$\Rightarrow$ $\hat{n}$ is the $\hat{k}_B$ $(= \hat{k}_A)$

POLL 2

Fact

$$\boxed{\vec{R}_{B|A}\big|_A = \vec{R}_{B|A}\big|_B}$$

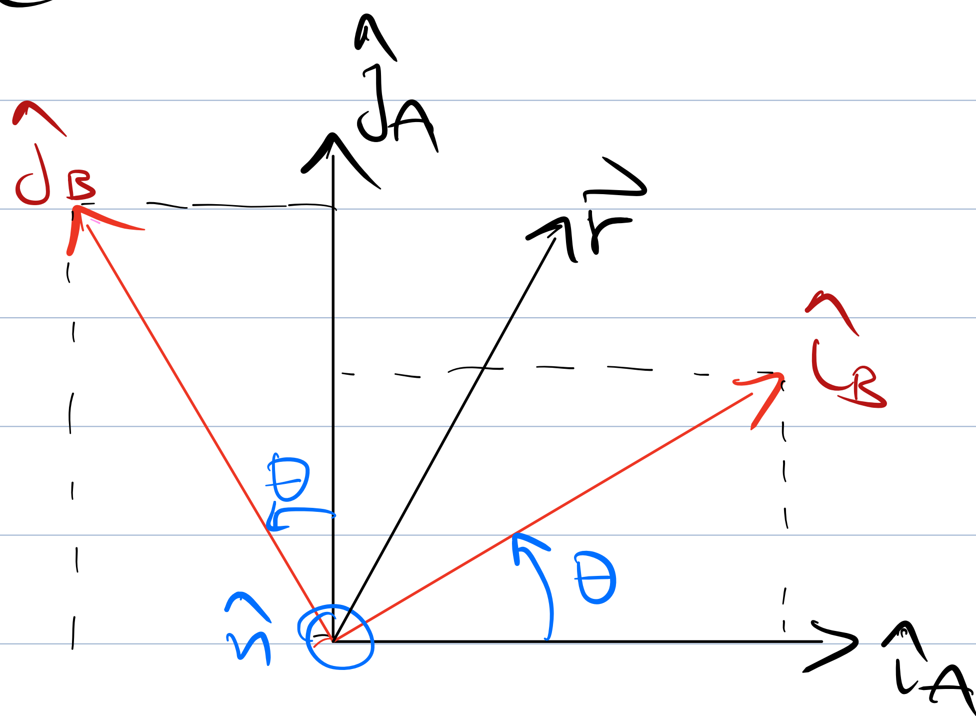
Now let's focus on math vectors:

Assume F_A and F_B.

Then, for what (math) matrix M it is:

$$\vec{x}\big|_A = M\,\vec{x}\big|_B \text{ ?}$$

Let's start from 2D case:



$\Rightarrow$ • $\hat{i}_B = \sin\theta \cdot \hat{j}_A + \cos\theta \, \hat{i}_A$

• $\hat{j}_B = -\sin\theta \cdot \hat{i}_A + \cos\theta \, \hat{j}_A$

Assume that I know $\vec{r}\big|_B = \begin{pmatrix} x_B \\ y_B \end{pmatrix}$,

$$\vec{r} = x_B \hat{i}_B + y_B \hat{j}_B$$

$$= x_B \left(\sin\theta \, \hat{j}_A + \cos\theta \, \hat{i}_A \right) + y_B \left(-\sin\theta \, \hat{i}_A + \cos\theta \, \hat{j}_A \right)$$

$$= \left(x_B \cos\theta - y_B \sin\theta \right) \hat{i}_A + \left(x_B \sin\theta + y_B \cos\theta \right) \hat{j}_A$$

$$\Rightarrow \vec{r}\big|_A = \begin{bmatrix} x_B \cos\theta - y_B \sin\theta \\ x_B \sin\theta + y_B \cos\theta \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{= \sigma_{A|B}} \underbrace{\begin{bmatrix} x_B \\ y_B \end{bmatrix}}_{= \vec{r}|_B}$$

$\Rightarrow$

$$\vec{r}|_A = \mathcal{O}_{A|B}\ \vec{r}|_B$$

$\hookrightarrow$ orientation matrix

(Notation in VNAV-W21:
$$R^A_B = \mathcal{O}_{A|B}$$
)

$$\Rightarrow \quad \vec{r}|_B = \mathcal{O}_{B|A}\ \vec{r}|_A$$

Fact

$$\boxed{\mathcal{O}_{A|B} = \vec{R}_{A|B}\Big|_A^T}$$

Proof.

$$\vec{R}_{A|B} = \hat{i}_A \hat{i}'_B + \hat{j}_A \hat{j}'_B$$

$$\Rightarrow \quad \vec{R}'_{A|B} = \hat{i}_B \hat{i}'_A + \hat{j}_B \hat{j}'_A$$

$$\Rightarrow \quad \vec{R}'_{A|B}\Big|_A = \vec{R}_{A|B}\Big|_A^T$$

$$= \hat{i}_B\Big|_A \hat{i}_A\Big|_A^T + \hat{j}_B\Big|_A \hat{j}_A\Big|_A^T$$

- $\hat{i}_B = \sin\theta \cdot \hat{j}_A + \cos\theta\, \hat{i}_A$
- $\hat{j}_B = -\sin\theta \cdot \hat{i}_A + \cos\theta\, \hat{j}_A$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \sigma_{A|B}. \qquad \blacksquare$$

Fact

$$\mathcal{O}_{B/A} \cdot \mathcal{O}_{B/A}^T = I$$

$$\mathcal{O}_{B/A}^T = \mathcal{O}_{B/A}^{-1} = \mathcal{O}_{A/B}$$

$$\det \mathcal{O}_{B/A} = \underline{1}$$

+1 because
right-handed
frame

Fact

$$\mathcal{O}_{C/A} = \mathcal{O}_{C/B}\, \mathcal{O}_{B/A}$$

Fact

$$\|x\| = \|O_{B|A}\, x\|$$

Fact

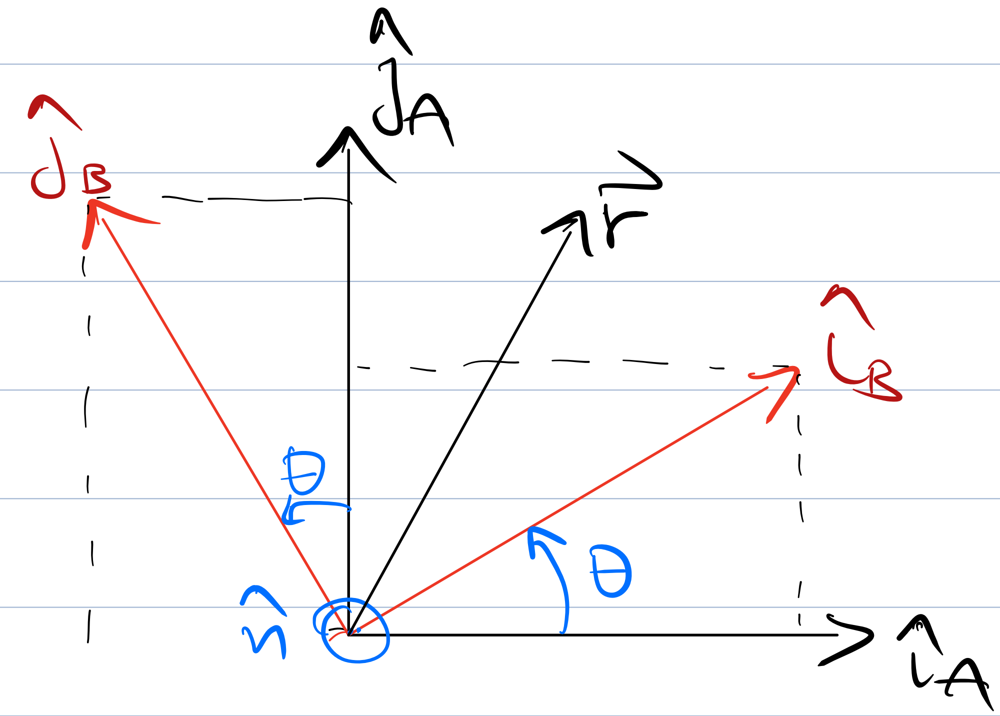
$$\begin{bmatrix} \hat{i}_B \\ \hat{j}_B \\ \hat{k}_B \end{bmatrix} = O_{B|A} \begin{bmatrix} \hat{i}_A \\ \hat{j}_A \\ \hat{k}_A \end{bmatrix}$$

Fact

$$O_{B|A} = \begin{bmatrix} \hat{i}_B' \hat{i}_A & \hat{i}_B' \hat{j}_A & \hat{i}_B' \hat{k}_A \\ \hat{j}_B' \hat{i}_A & \hat{j}_B' \hat{j}_A & \hat{j}_B' \hat{k}_A \\ \hat{k}_B' \hat{i}_A & \hat{k}_B' \hat{j}_A & \hat{k}_B' \hat{k}_A \end{bmatrix}$$

Why? Let's check 2D case

again (which results to $\mathcal{O}_{A|B} = \mathcal{O}_{B|A}^T$!):



$$\Rightarrow \cdot\ \hat{i}_B = \underbrace{\sin\theta}_{=\cos\left(\frac{\pi}{2}-\theta\right)} \hat{j}_A + \underbrace{\cos\theta}_{=\hat{i}_B' \hat{i}_A} \hat{i}_A$$

$$= \hat{j}_A' \hat{i}_B$$

$$\cdot\ \hat{j}_B = \underbrace{-\sin\theta}_{=\cos\left(\frac{\pi}{2}+\theta\right)} \hat{i}_A + \underbrace{\cos\theta}_{=\hat{j}_B' \hat{j}_A} \hat{j}_A$$

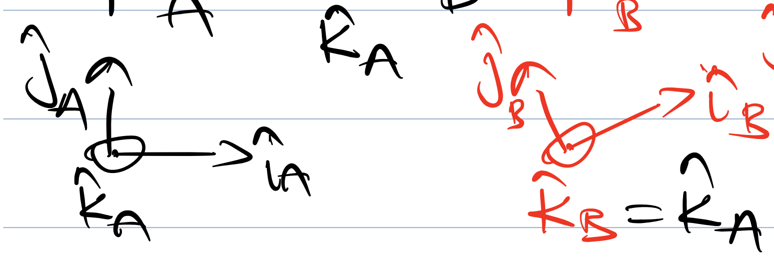
$$= \hat{j}_B' \hat{i}_A$$

Fact (Resolving matrices in different frames)

$$\vec{M}\big|_B = O_{B|A}\, \vec{M}\big|_A\, O_{A|B}$$

Euler orientation matrices and Euler angles

$$F_A \xrightarrow[\hat{k}_A]{\Psi} F_B \xrightarrow[\hat{j}_B]{H} F_C \xrightarrow[\hat{i}_C]{\Phi} F_D$$



Ψ, H, Φ are Euler angles

$$\Rightarrow \quad F_B = \underbrace{\vec{R}_{\hat{k}_A}(\Psi)}_{\triangleq \vec{R}_{B|A}} F_A$$

$$F_C = \underbrace{\vec{R}_{\hat{j}_B}(\theta)}_{\triangleq \vec{R}_{C|B}} F_B$$

$$F_D = \underbrace{\vec{R}_{\hat{i}_C}(\Phi)}_{\triangleq \vec{R}_{D|C}} F_C$$

$$\Rightarrow F_D = \vec{R}_{D|C} \cdot \vec{R}_{C|B} \vec{R}_{B|A} F_A$$