6 Forces in equilibrium and resultant forces

A child on a sledge is being pulled up a smooth slope of 20° by a rope which makes an angle of 40° with the slope. The mass of the child and sledge together is 20 kg and the tension in the rope is 170 N. Draw a diagram to show the forces acting on the child and sledge together. In what direction is the resultant of these forces?

When the child and sledge are modelled as a particle, all the forces can be assumed to be acting at a point. There is no friction force because the slope is smooth. Here is the force diagram.

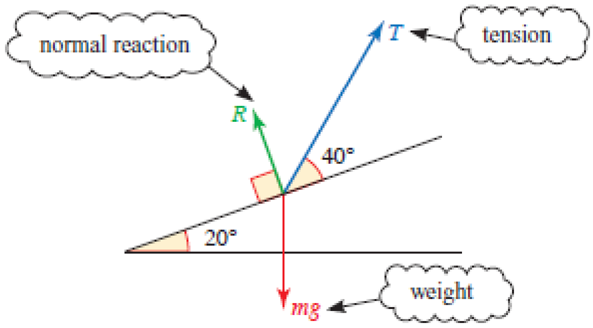


Figure 6.1

❓ The sledge is sliding along the slope. What direction is the resultant force acting on it?

Ans:

Parallel to the slope up the slope.

You can find the normal reaction and the resultant force on the sledge using two methods.

Method 1: Using components

This method involves resolving forces into components in two perpendicular directions as in Chapter 5. It is easiest to use the components of the forces parallel and perpendicular to the slope in the directions shown.

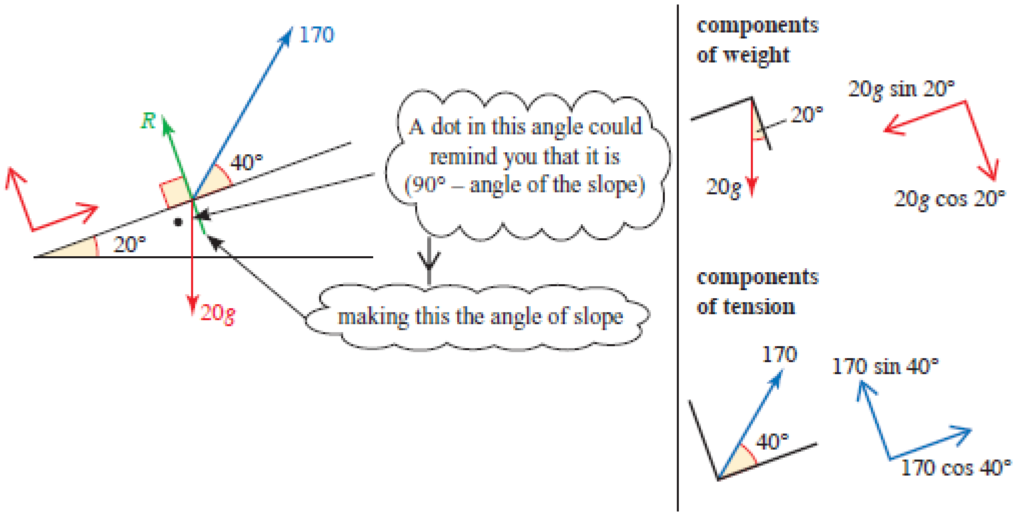


Figure 6.2

Resolve parallel to the slope (↗):

The force R is perpendicular to the slope so it has no component in this direction.

The resultant $\quad F = 170 \cos 40° - 20g \sin 20°$

$= 61.8$ (to 3 s.f.)

cos (90° – 20°) = sin 20°

Resolve perpendicular to the slope (↖):

$R + 170 \sin 40° - 20g \cos 20° = 0$

$R = 20g \cos 20° - 170 \sin 40°$

$= 78.7$ (to 3 s.f.)

There is no resultant in this direction because the motion is parallel to the slope.

The normal reaction is 78.7 N and the resultant is 61.8 N up the slope.

Alternatively, you could have worked in column vectors as follows.

$$\begin{array}{ll} \text{Parallel to slope} & \\ \text{Perpendicular to slope} & \end{array} \underbrace{\begin{pmatrix}0\\R\end{pmatrix}}_{\text{Normal reaction}}+\underbrace{\begin{pmatrix}170\cos 40^\circ\\170\sin 40^\circ\end{pmatrix}}_{\text{Tension}}+\underbrace{\begin{pmatrix}-20g\sin 20^\circ\\-20g\cos 20^\circ\end{pmatrix}}_{\text{Weight}}=\underbrace{\begin{pmatrix}F\\0\end{pmatrix}}_{\text{Resultant}}$$

Once you know the resultant force, you can work out the acceleration of the sledge using Newton's second law.

$$\begin{aligned} F &= ma \\ 61.8 &= 20a \end{aligned}$$

The acceleration is $3.1\ \mathrm{m\,s^{-2}}$ (correct to 1 d.p.).

Method 2: Scale drawing

An alternative is to draw a scale diagram with the three forces represented by three of the sides of a quadrilateral taken in order (with the arrows following each other) as shown in figure 6.3. The resultant is represented by the fourth side AD. This must be parallel to the slope.

? In what order would you draw the lines in the diagram?

From the diagram you can estimate the normal reaction to be about 80 N and the resultant 60 N. This is a reasonable estimate, but components are more precise.

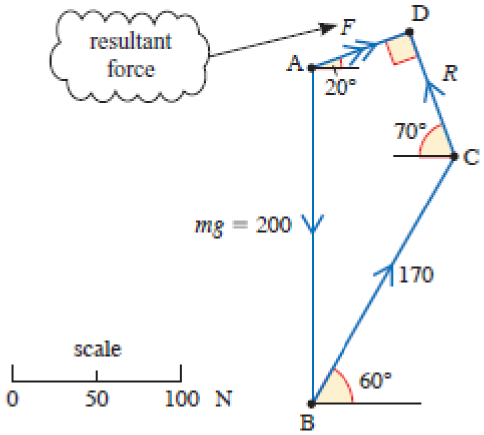


Figure 6.3

EXERCISE 6A

For questions 1 to 6, carry out the following steps. All forces are in newtons.

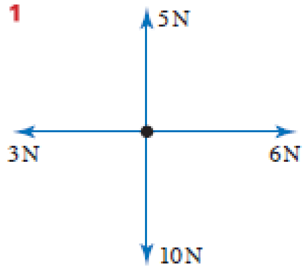
(i) Draw a scale diagram to show the polygon of the forces and the resultant.

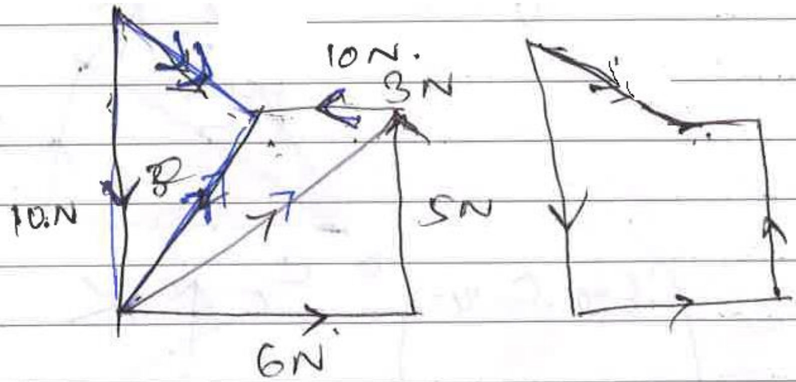
(ii) State whether you think the forces are in equilibrium and, if not, estimate the magnitude and direction of the resultant.

(iii) Write the forces in component form, using the directions indicated and so obtain the components of the resultant.
Hence find the magnitude and direction of the resultant as on page 95.

(iv) Compare your answers to parts **(ii)** and **(iii)**.

1





$$\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}0\\5\end{pmatrix}=\begin{pmatrix}6\\5\end{pmatrix}+\begin{pmatrix}-3\\0\end{pmatrix}$$

$$=\begin{pmatrix}3\\5\end{pmatrix}+\begin{pmatrix}0\\-10\end{pmatrix}$$

Direction

$$\theta = \tan\left(\frac{-5}{3}\right) = -59^{\circ}$$

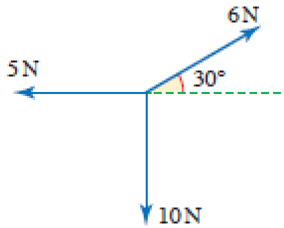
magnitude

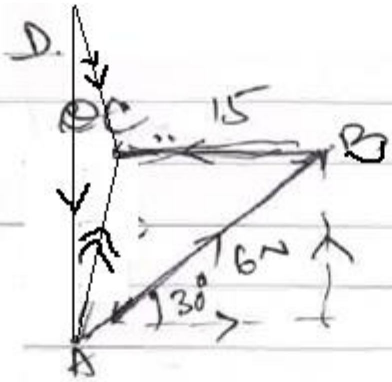
$$= \sqrt{3^2 + 5^2} = \sqrt{9+25}$$

$$= \sqrt{34} = 5.83N$$

6N

2





$$\vec{AB} \cdot \cancel{6N} = \begin{pmatrix} 6N \cos 30 \\ 6N \sin 30 \end{pmatrix}$$

$$= \begin{pmatrix} 6N \times \frac{\sqrt{3}}{2} \\ 3N \end{pmatrix}$$

$$AB + BC = \begin{pmatrix} 5.196 \\ 3 \end{pmatrix} + \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 0.196 \\ 3 \end{pmatrix}$$

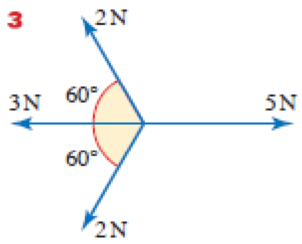
$$\overrightarrow{DC} = \begin{pmatrix} 0.196 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -10 \end{pmatrix}$$

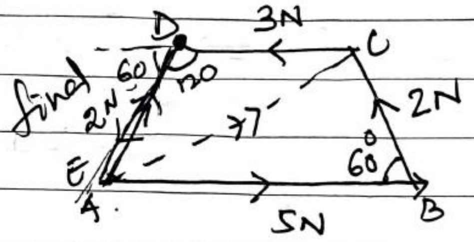
$$= \begin{pmatrix} 0.196 \\ -7 \end{pmatrix}$$

$$|DC| = \sqrt{0.196^2 + 7^2} = 7N$$

$$\theta = \tan^{-1}\left(\frac{-7}{0.196}\right) = -88.4^\circ$$

3





$$\vec{AC} = \vec{AB} + \vec{BC} = \begin{pmatrix}5\\0\end{pmatrix} + \begin{pmatrix}-2\cos 60\\2\sin 60\end{pmatrix}$$

$$= \begin{pmatrix}5\\0\end{pmatrix} + \begin{pmatrix}-1\\1.732\end{pmatrix}$$

$$= \begin{pmatrix}4\\1.732\end{pmatrix}.$$

$$\vec{AD} = \vec{AC} + \vec{CD} = \begin{pmatrix}4\\1.732\end{pmatrix} + \begin{pmatrix}-3\\0\end{pmatrix}$$

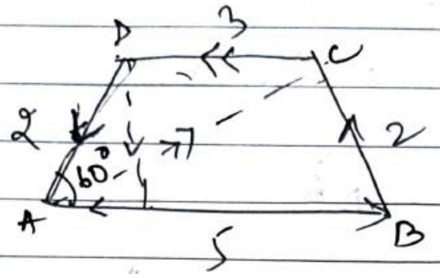
$$= \begin{pmatrix}1\\1.732\end{pmatrix}$$

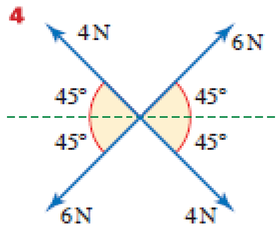
~~EoA~~ $\vec{AE} = \vec{AD} + \vec{DE}$

$$= \begin{pmatrix}1\\1.732\end{pmatrix} + \begin{pmatrix}-2\cos 120\\-2\sin 120\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

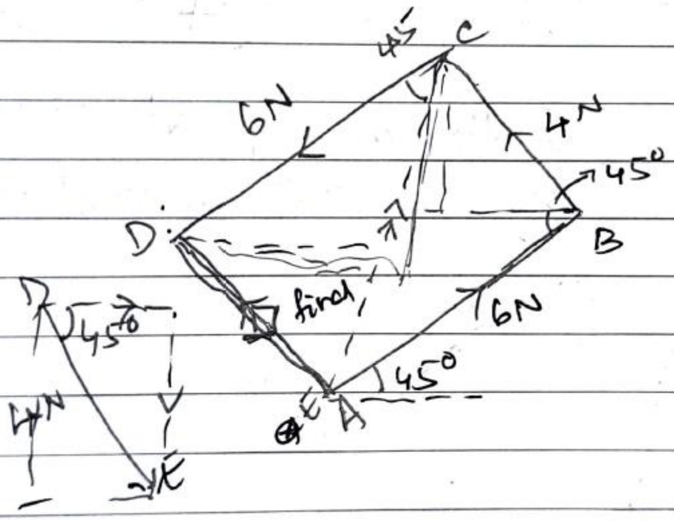
Equilibrium and trapezium

$$\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}-1\\1.732\end{pmatrix}+\begin{pmatrix}-3\\0\end{pmatrix}+\begin{pmatrix}1\\1.732\end{pmatrix}$$





4



$$\vec{AB}=\begin{pmatrix}6\cos 45\\6\sin 45\end{pmatrix},\quad \vec{BC}=\begin{pmatrix}-4\cos 45\\4\sin 45\end{pmatrix}.$$

$$\vec{AC}=\begin{pmatrix}6\cos 45-4\cos 45\\6\sin 45+4\sin 45\end{pmatrix}=\begin{pmatrix}1.414\\7.071\end{pmatrix}$$

$$\vec{CD}=\begin{pmatrix}-6\cos 45\\-6\sin 45\end{pmatrix}\Rightarrow \vec{AD}=\vec{AC}+\vec{CD}$$

PAGE No.
DATE / /

$$\overrightarrow{AD} = \begin{pmatrix} 1.414 \\ 7.071 \end{pmatrix} + \begin{pmatrix} -6\cos 45 \\ -6\sin 45 \end{pmatrix}$$

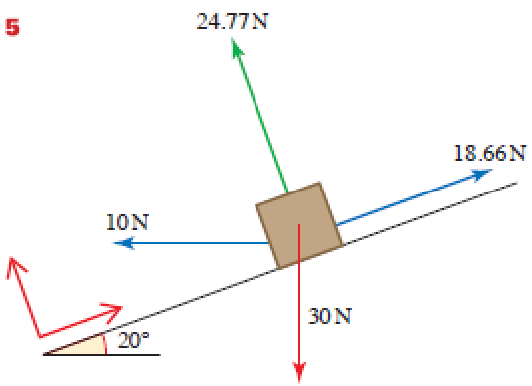
$$= \begin{pmatrix} -2.828 \\ 2.828 \end{pmatrix}.$$

$$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$$

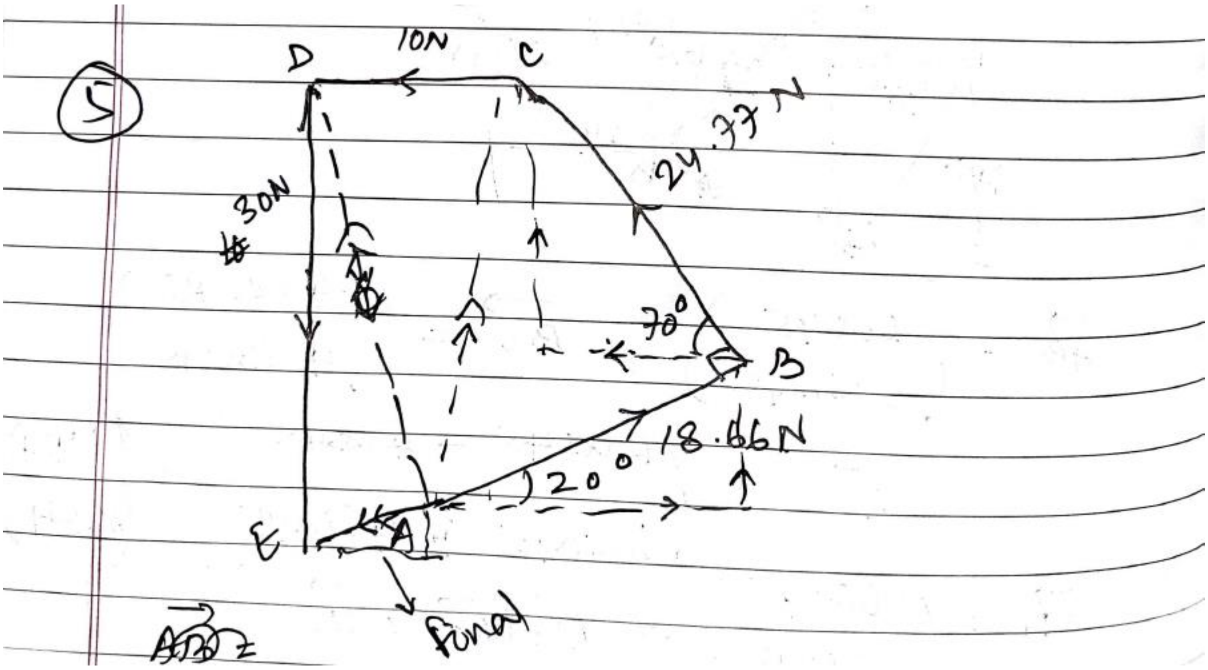
$$= \begin{pmatrix} -2.828 \\ 2.828 \end{pmatrix} + \begin{pmatrix} 4\cos 45 \\ -4\sin 45 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence equilibrium and a rectangle.

5



24.77 N
18.66 N
10 N
30 N
20°



5
D
10N
C
24.77 N
30N
70°
B
18.66N
20°
E
A
final
$\vec{AE} =$

$$\vec{AB} = \begin{pmatrix} 18.66 \cos 20^\circ \\ 18.66 \sin 20 \end{pmatrix} = \Big(\ .$$

$$\vec{BC} = \begin{pmatrix} -24.77 \cos 70^\circ \\ 24.77 \sin 70^\circ \end{pmatrix}.$$

$$\vec{AC} = \begin{pmatrix} 9.063 \\ 29.66 \end{pmatrix}. \quad \left(\because \ \vec{AB} + \vec{BC} \right)$$

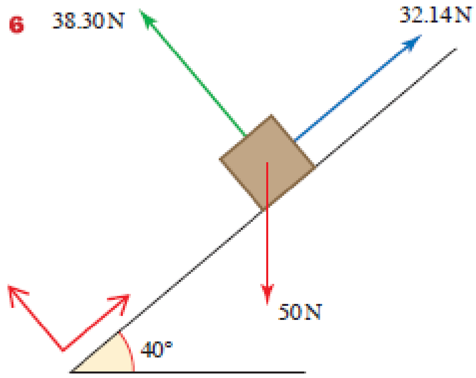
$$\vec{AD} = \vec{AC} + \vec{CD}$$

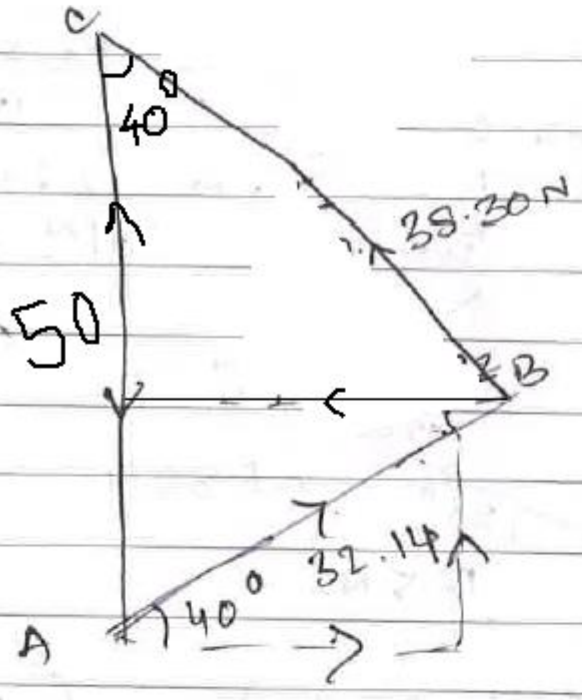
$$= \begin{pmatrix} 9.063 \\ 29.66 \end{pmatrix} + \begin{pmatrix} -10 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.937 \\ 29.660 \end{pmatrix}.$$

$$\vec{AE} = \vec{AD} + \vec{DE}$$

$$= \begin{pmatrix} -0.937 \\ 29.660 \end{pmatrix} + \begin{pmatrix} 0 \\ -30 \end{pmatrix} = \begin{pmatrix} -0.937 \\ -0.34 \end{pmatrix}.$$

6





$$AB = \begin{pmatrix} 32.14 \cos 40 \\ 32.14 \sin 40 \end{pmatrix} = \begin{pmatrix} 24.6 \\ 20.7 \end{pmatrix}$$

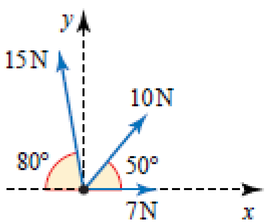
$$BC = \begin{pmatrix} -38.30 \sin 40 \\ 38.30 \cos 40 \end{pmatrix} = \begin{pmatrix} -24.6 \\ 29.3 \end{pmatrix}$$

$$CA = \begin{pmatrix} 0 \\ -50 \end{pmatrix}$$

$$\text{Resultant} = \begin{pmatrix} 24.6 \\ 20.7 \end{pmatrix} + \begin{pmatrix} -24.6 \\ 29.3 \end{pmatrix} + \begin{pmatrix} 0 \\ -50 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \text{equilibrium.}$$

7 Forces of magnitudes 7 N, 10 N and 15 N act on a particle in the directions shown in the diagram.



(i) Find the component of the resultant of the three forces

(a) in the x direction,

(b) in the y direction.

(ii) Hence find the direction of the resultant.

[Cambridge AS and A Level Mathematics 9709, Paper 4 Q3 June 2009]

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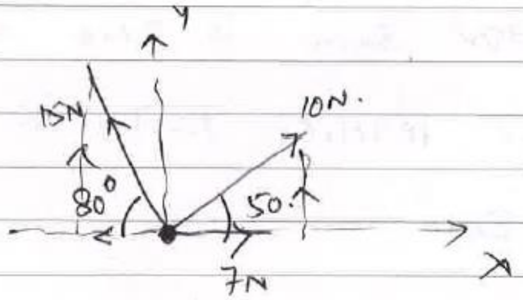

$$7N = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

$$10N = \begin{pmatrix} 10\cos 50 \\ 10\sin 50 \end{pmatrix} = \begin{pmatrix} 6.43 \\ 7.66 \end{pmatrix}$$

$$15N = \begin{pmatrix} -15\cos 80 \\ 15\sin 80 \end{pmatrix} = \begin{pmatrix} -2.6 \\ 14.77 \end{pmatrix}$$

In x-axis

$= 7 + 6.43 - 2.6$

$= 10.83.$

In y-axis

$= 0 + 7.66 + 14.77$

$= 22.43.$

Resultant

$$= \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \begin{pmatrix} 6.43 \\ 7.66 \end{pmatrix} + \begin{pmatrix} -2.6 \\ 14.77 \end{pmatrix}$$

$$= \begin{pmatrix} 10.83 \\ 22.43 \end{pmatrix}.$$

$$\theta = \tan^{-1}\left(\frac{22.43}{10.83}\right) = 64.22^\circ.$$

anti-clockwise because the angle is +ve.