# Linear Integer Optimization problems

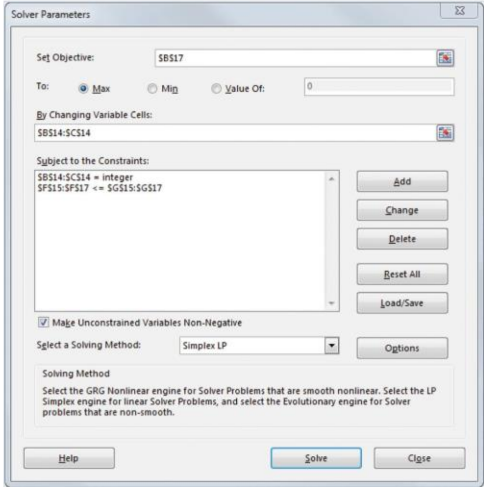
- **Integer linear programs**: Problems that are modeled as linear programs with the additional requirement that one or more variables must be integer.
- **All-integer linear program**: If all variables are required to be integer.
- **LP Relaxation** (linear programming relaxation) of the integer linear program: The linear program that results from dropping the integer requirements.
- **Mixed-integer linear program**: If some, but not necessarily all, variables are required to be integer.
- **Binary integer linear program**: The integer variables may take on only the values 0 or 1.

# Eastborne Realty

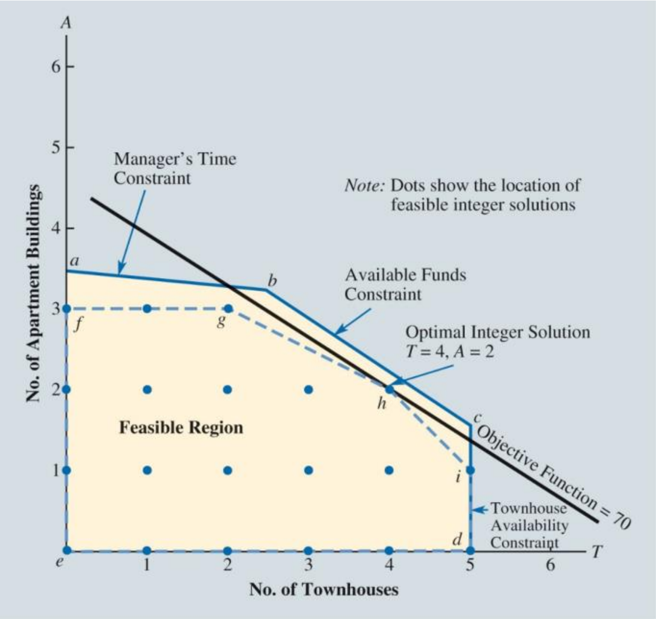
Decision variables
- T = number of townhouses
- A = number of apartment buildings

Objective
- Maximize 10**T** + 15**A**

Constraints
- $282T + 400A \leq 2000$ *Funds available*
- $4T + 40A \leq 140$ *Manager's time*
- $T \leq 5$ *Townhouses available*
- **T, A** - *Non-negative and integer*



# Eastborne Realty




- Eastborne cannot purchase fractional numbers of townhouses and apartment buildings—further analysis is necessary.
- In many cases, a non-integer solution can be rounded to obtain an acceptable integer solution.
- Optimal non-integer solution: **T = 2.479** *townhouses* and **A = 3.252** *apartment buildings.*
- Rounding - trial and error approach

# Applications with binary variables

- **Capital budgeting problem**: A binary integer programming problem that involves choosing which possible projects or activities provide the best investment return.
- In a capital budgeting problem, the objective function is to maximize the net present value of the capital budgeting projects.
- The Ice-Cold Refrigerator Company is considering investing in several projects that have varying capital requirements over the next four years.
- Faced with limited capital each year, management would like to select the most profitable projects that it can afford.

Decision variables

- P = plant expansion
- W = warehouse expansion
- M = machinery project
- R = product research project

Objective

Maximize 90**T** + 40**W** + 10**M** + 37**R**

Constraints

$$
\begin{array}{llll}
15P + 10W + 10M + 15R \le 40 & \text{(Year 1 capital available)} \\
20P + 15W \phantom{+10M} + 10R \le 50 & \text{(Year 2 capital available)} \\
20P + 20W \phantom{+10M+} 10R \le 40 & \text{(Year 3 capital available)} \\
15P + \ 5W + \ 4M + 10R \le 35 & \text{(Year 4 capital available)}
\end{array}
$$

$$P, W, M, R = 0, 1$$

# RMC – fixed cost

Decision variables
- F = fuel additive
- S = solvent base
- C = carpet cleaning fluid

Objective → Maximize 40**F** + 30**S** + 50**C**

Constraints

$$
\begin{aligned}
0.4F + 0.5S + 0.6C &\le 20 \quad \text{(Material 1)} \\
0.2S + 0.1C &\le \ \ 5 \quad \text{(Material 2)} \\
0.6F + 0.3S + 0.3C &\le 21 \quad \text{(Material 3)}
\end{aligned}
$$

$F, S, C, \ge 0$

SF, SS, SC are binary variables - 1, if produced, else 0

Objective → Maximize 40**F** + 30**S** + 50**C** - 200**SF** - 50**SS** - 400**SC**

Constraints

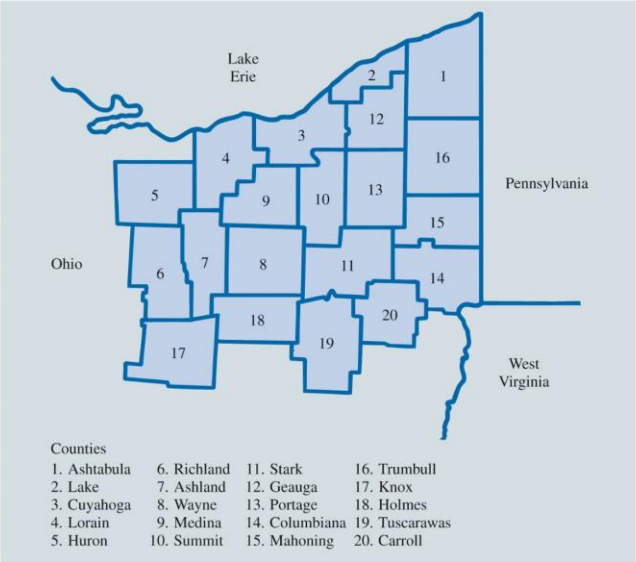
$$
\begin{aligned}
0.4F + 0.5S + 0.6C &\le 20 && \text{Material 1} \\
0.2S + 0.1C &\le \ \ 5 && \text{Material 2} \\
0.6F + 0.3S + 0.3C &\le 21 && \text{Material 3} \\
F &\le 50SF && \text{Maximum Fuel Additive} \\
S &\le 25SS && \text{Maximum Solvent Base} \\
C &\le 40SC && \text{Maximum Carpet Cleaning}
\end{aligned}
$$

$F, S, C, \ge 0;\ SF, SS, SC = 0 \text{ or } 1$

# Ohio trust company

- As an initial step in its planning, Ohio Trust would like to determine the minimum number of PPBs necessary to do business throughout the 20-county region.
- A binary integer programming model can be used to solve this location problem for Ohio Trust.
- $x_i$=1, if a PPB is established in county, i; else ‘0’

$$
\begin{array}{llll}
\text{Min} & x_1 + x_2 + \quad \cdots \quad + x_{20} & & \\
\text{s.t.} & & & \\
 & x_1 + x_2 \quad + x_{12} + x_{16} & \geq 1 & \text{Ashtabula} \\
 & x_1 + x_2 + x_3 + x_{12} & \geq 1 & \text{Lake} \\
 & \vdots & \vdots & \\
 & \quad x_{11} + x_{14} + x_{19} + x_{20} & \geq 1 & \text{Carroll} \\
 & x_i = 0, 1 \quad i = 1, 2, \ldots, 20 & &
\end{array}
$$




**Adjacent counties**

1. Ashtabula 2, 12, 16

# Salem foods

- Salem must design a pizza (choose the type of crust, cheese, sauce, and sausage flavor) that will have the highest utility for a sufficient number of people to ensure sufficient sales to justify making the product.

- Assuming the sample of eight consumers in the current study is representative of the marketplace for frozen sausage pizza:
  - Formulate and solve an integer programming model that can help Salem come up with such a design.
  - In marketing literature, the problem being solved is called the share of choice problem.

| Consumer | Thin Crust | Thick Crust | Mozzarella Cheese | Cheese Blend | Smooth Sauce | Chunky Sauce | Mild Sausage | Medium Sausage | Hot Sausage |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 11 | 2 | 6 | 7 | 3 | 17 | 26 | 27 | 8 |
| 2 | 11 | 7 | 15 | 17 | 16 | 26 | 14 | 1 | 10 |
| 3 | 7 | 5 | 8 | 14 | 16 | 7 | 29 | 16 | 19 |
| 4 | 13 | 20 | 20 | 17 | 17 | 14 | 25 | 29 | 10 |
| 5 | 2 | 8 | 6 | 11 | 30 | 20 | 15 | 5 | 12 |
| 6 | 12 | 17 | 11 | 9 | 2 | 30 | 22 | 12 | 20 |
| 7 | 9 | 19 | 12 | 16 | 16 | 25 | 30 | 23 | 19 |
| 8 | 5 | 9 | 4 | 14 | 23 | 16 | 16 | 30 | 3 |

# Salem foods

$l_{ij}$ = 1, if Salem chooses level *i* for attribute *j*; 0 otherwise.
$y_k$ = 1, if consumer k chooses the Salem brand; 0 otherwise.

$$\text{Max } y_1 + y_2 + \ldots + y_8$$

$$\begin{aligned}
&11l_{11} + 2l_{21} + 6l_{12} + 7l_{22} + 3l_{13} + 17l_{23} + 26l_{14} + 27l_{24} + 8l_{34} \geq 1 + 52y_1 \\
&11l_{11} + 7l_{21} + 15l_{12} + 17l_{22} + 16l_{13} + 26l_{23} + 14l_{14} + 1l_{24} + 10l_{34} \geq 1 + 58y_2 \\
&7l_{11} + 5l_{12} + 8l_{12} + 14l_{22} + 16l_{13} + 7l_{23} + 29l_{14} + 16l_{24} + 19l_{34} \geq 1 + 66y_3 \\
&13l_{11} + 20l_{21} + 20l_{12} + 17l_{22} + 17l_{13} + 14l_{23} + 25l_{14} + 29l_{24} + 10l_{34} \geq 1 + 83y_4 \\
&2l_{11} + 8l_{21} + 6l_{12} + 11l_{22} + 30l_{13} + 20l_{23} + 15l_{14} + 5l_{24} + 12l_{34} \geq 1 + 58y_5 \\
&12l_{11} + 17l_{21} + 11l_{12} + 9l_{22} + 2l_{13} + 30l_{23} + 22l_{14} + 12l_{24} + 20l_{34} \geq 1 + 70y_6 \\
&9l_{11} + 19l_{21} + 12l_{12} + 16l_{22} + 16l_{13} + 25l_{23} + 30l_{14} + 23l_{24} + 19l_{34} \geq 1 + 79y_7 \\
&5l_{11} + 9l_{21} + 4l_{12} + 14l_{22} + 23l_{13} + 16l_{23} + 16l_{14} + 30l_{24} + 3l_{34} \geq 1 + 59y_8
\end{aligned}$$

$$l_{11} = l_{22} = l_{23} = l_{14} = 1 \text{ and } y_2 = y_5 = y_6 = y_7 = 1.$$

# Modeling flexibility - binary variables

- Binary integer variables can be used to model:
    - Multiple-choice and mutually exclusive constraints.
    - Situations in which *k* projects out of a set of *n* projects must be selected.
    - Situations in which the acceptance of one project is conditional on the acceptance of another project.

Multiple-Choice and Mutually Exclusive Constraints:

- **Multiple-choice constraint**: A constraint requiring that the sum of two or more binary variables equals one.
- Any feasible solution makes a choice of which variable to set equal to one.
- **Mutually exclusive constraint**: A constraint requiring that the sum of two or more binary variables be less than or equal to one.
    - If one of the variables equals one, the others must equal zero.
    - All variables could equal zero.