Non-linear Optimization

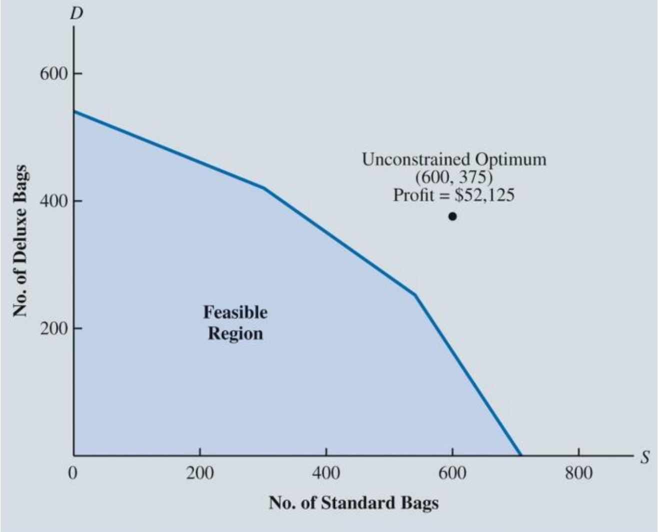
Any optimization problem in which at least one term in the **objective function or a constraint** is nonlinear.

Assuming an inverse relationship between demand and price,

$$S = 2{,}250 - 15P_S$$
$$D = 1{,}500 - 5P_D$$

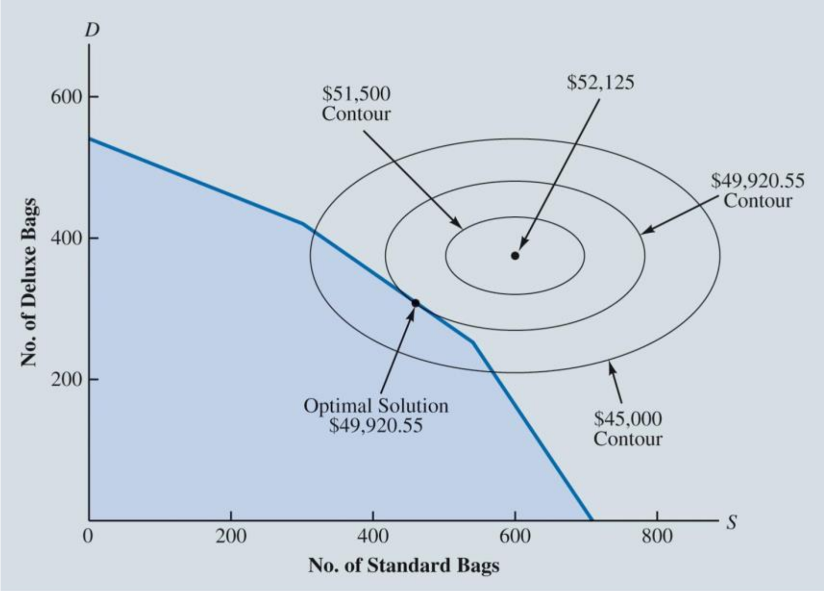
$$\text{Total profit contribution} = 80S - \left(\frac{1}{15}\right)S^2 + 150D - \left(\frac{1}{5}\right)D^2$$

Equation of an ellipse, $\left(\frac{x}{a}\right)^2 + y^2 = 1$



A Production Application: Par, Inc. : Unconstrained

Max	$80S - \frac{1}{15}S^2 + 150D - \frac{1}{5}D^2$	
s.t.		
	$\frac{7}{10}S + 1D \leq 630$	Cutting and dyeing
	$\frac{1}{2}S + \frac{5}{6}D \leq 600$	Sewing
	$1S + \frac{2}{3}D \leq 708$	Finishing
	$\frac{1}{10}S + \frac{1}{4}D \leq 135$	Inspection and packaging
	$S, D \geq 0$	



Sensitivity and Shadow prices

- **Reduced gradient & Lagrange multiplier**
    - The reduced cost/gradient for a variable is nonzero only when the variable's value is equal to its upper or lower bound at the optimal solution.
    - Non-critical constraints will have zero shadow prices as slack (*amount by which a resource is under-utilized*) exists already.

	A	B	C	D	E	F
4						
5						
6	Variable Cells					
7						
8		Cell	Name	Final Value	Reduced Gradient	
9		B14	Bags Produced Standard	459.7166	0	
10		C14	Bags Produced Deluxe	308.19838	0	
11						
12	Constraints					
13						
14		Cell	Name	Final Value	Lagrange Multiplier	
15		B19	Cutting and Dyeing Hours Used	630	26.720587	
16		B20	Sewing Hours Used	486.69028	0	
17		B21	Finishing Hours Used	665.18219	0	
18		B22	Inspection and Packaging Hours Used	123.02126	0	
19						

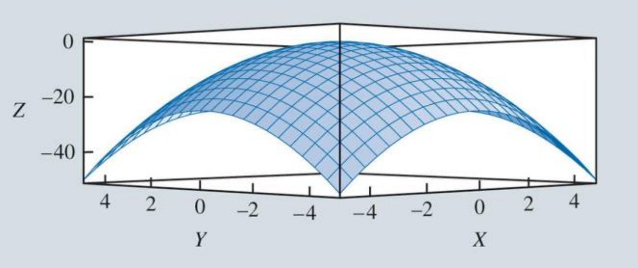
Local and Global optima

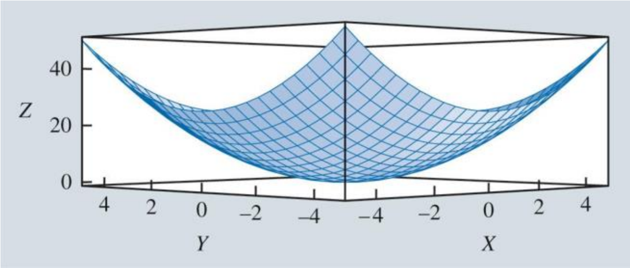
- **Local optimimum**: A feasible solution when there are no other feasible solutions with a better objective function value in the immediate neighborhood; may be either a local maximum or a local minimum.
  - **Local maximum**: A feasible solution when there are no other feasible solutions with a larger objective function value in the immediate neighborhood.
  - **Local minimum**: A feasible solution when there are no other feasible solutions with a smaller objective function value in the immediate neighborhood.

- **Global optimum**: A feasible solution when there are no other feasible points with a better objective function value in the entire feasible region; may be either a global maximum or a global minimum.
  - **Global maximum**: A feasible solution when there are no other feasible points with a larger objective function value in the entire feasible region; also a local maximum.
  - **Global minimum**: A feasible solution when there are no other feasible points with a smaller objective function value in the entire feasible region; also a local minimum.

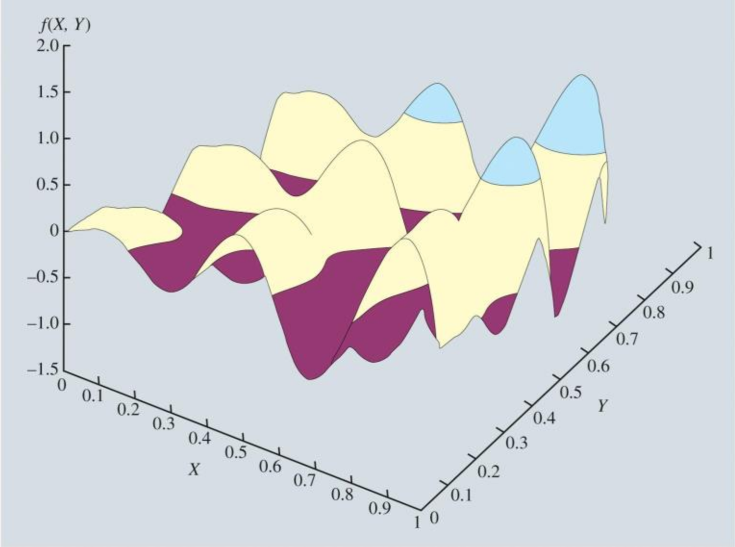
Local and Global optima

$$f(X,Y) = -X^2 - Y^2$$





$$f(X,Y) = X^2 + Y^2$$



$$\text{Max } f(X,Y) = X\sin(5\pi X) + Y\sin(5\pi Y)$$

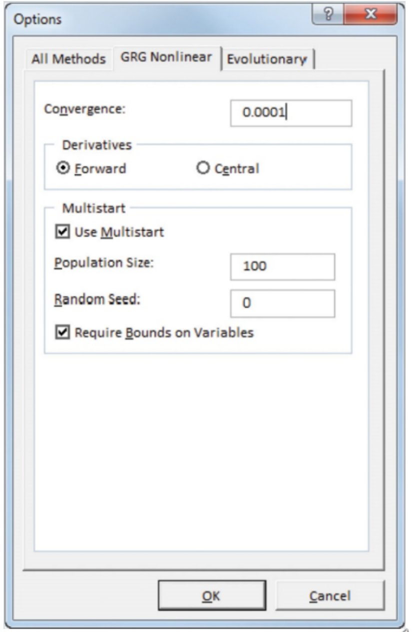
s.t.

$$0 \le X \le 1, \qquad 0 \le Y \le 1$$

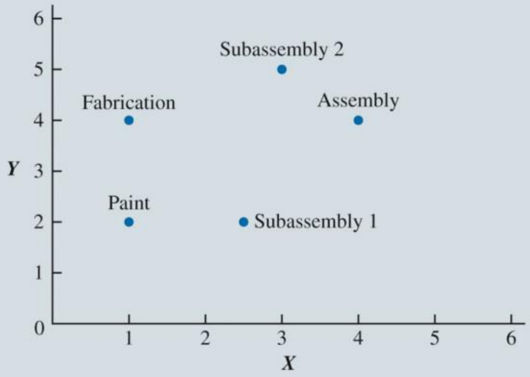
Overcoming Local optima

Overcoming Local Optima with Excel Solver (cont.):

- Clicking the **Use Multistart** option causes Solver to use multiple starting solutions and report the best solution found from all of the starting points.
- The **Population Size** is the number of starting points used.
- Solver selects starting points randomly using the **Random Seed** (an integer value) such that the points are within the bounds specified.
- Although providing simple lower and upper bounds is not required (unless the Bounds on Variables option is selected), the procedure is much more effective when bounds are provided.
- Recommend selecting the **Require Bounds** on Variables checkbox and providing bounds before you use the Multistart option.



A Location problem (LaRosa Machine Shop)



Station	Location X	Location Y
Fabrication	1	4
Paint	1	2
Subassembly 1	2.5	2
Subassembly 2	3	5
Assembly	4	4

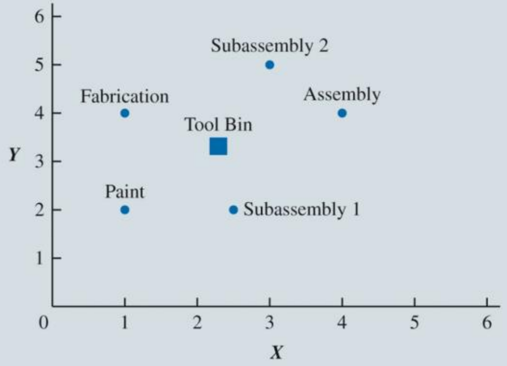
Decision variables:

- X = horizontal location of the tool bin.
- Y = vertical location of the tool bin.

$$\text{Min} \left(\sqrt{(X-1)^2 + (Y-4)^2} + \sqrt{(X-1)^2 + (Y-2)^2} + \sqrt{(X-2.5)^2 + (Y-2)^2} + \sqrt{(X-3)^2 + (Y-5)^2} + \sqrt{(X-4)^2 + (Y-4)^2} \right)$$

A Location problem

- Note that we do not require that the variables X or Y be non-negative.
- The optimal solution found by Excel Solver is $X = 2.230$, $Y = 3.349$.
- Location models are used for determining the optimal locations for:
  - Drilling holes in computer circuit boards.
  - Locating distribution centers and retail stores in supply chains.
- A variety of different location models can be created by using different objective functions or by adding additional constraints on distances traveled.



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Markowitz Portfolio Model

Mutual Fund	Annual Return (%) Year 1	Year 2	Year 3	Year 4	Year 5
Foreign Stock	10.06	13.12	13.47	45.42	−21.93
Intermediate-Term Bond	17.64	3.25	7.51	−1.33	7.36
Large-Cap Growth	32.41	18.71	33.28	41.46	−23.26
Large-Cap Value	32.36	20.61	12.93	7.06	−5.37
Small-Cap Growth	33.44	19.40	3.85	58.68	−9.02
Small-Cap Value	24.56	25.32	−6.70	5.43	17.31

FS = Proportion of portfolio invested in the foreign stock mutual fund.

IB = Proportion of portfolio invested in the intermediate-term bond fund.

LG = Proportion of portfolio invested in the large-cap growth fund.

LV = Proportion of portfolio invested in the large-cap value fund.

SG = Proportion of portfolio invested in the small-cap growth fund.

SV = Proportion of portfolio invested in the small-cap value fund.

Markowitz Portfolio Model

$$FS + IB + LG + LV + SG + SV = 1$$

(sum of all investment proportions)

$$\bar{R} = \sum_{s=1}^{n} p_s R_s$$

Expected return

$$\bar{R} = \sum_{s=1}^{5} \tfrac{1}{5} R_s = \tfrac{1}{5} \sum_{s=1}^{5} R_s$$

Assuming equal probability of each of the scenarios

$$Var = \sum_{s=1}^{5} \tfrac{1}{5} \left(R_s - \bar{R}\right)^2$$

$$\text{Min } \tfrac{1}{5} \sum_{s=1}^{5} \left(R_s - \bar{R}\right)^2$$

Minimizing risk (*variance*)

Markowitz Portfolio Model

s.t.

$$10.06FS + 17.64IB + 32.41LG + 32.36LV + 33.44SG + 24.56SV = R_1 \quad \textbf{(14.9)}$$

$$13.12FS + +3.25IB + 18.71LG + 20.61LV + 19.40SG + 25.32SV = R_2 \quad \textbf{(14.10)}$$

$$13.47FS + 7.51IB + 33.28LG + 12.93LV + 3.85SG - 6.70SV = R_3 \quad \textbf{(14.11)}$$

$$45.42FS - 1.33IB + 41.46LG + 7.06LV + 58.68SG + 5.43SV = R_4 \quad \textbf{(14.12)}$$

$$-21.93FS + 7.36IB - 23.26LG - 5.37LV - 9.02SG + 17.31SV = R_5 \quad \textbf{(14.13)}$$

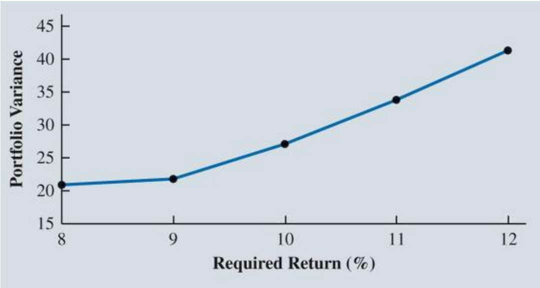
$$FS + IB + LG + LV + SG + SV = 1 \quad \textbf{(14.14)}$$

$$\tfrac{1}{5}\sum_{s=1}^{5} R_s = \bar{R} \quad \textbf{(14.15)}$$

$$\bar{R} \geq 10 \quad \textbf{(14.16)}$$

$$FS, IB, LG, LV, SG, SV \geq 0 \quad \textbf{(14.17)}$$

Efficient frontier



- The minimum value for the portfolio variance is 27.136.
- This solution implies that the clients will get an expected return of 10% and minimize their risk as measured by portfolio variance by investing approximately:
  - 16% of the portfolio in the foreign stock fund (*FS* = 0.158), 53% in the intermediate bond fund (*IB* = 0.525), 4% in the large-cap growth fund (*LG* = 0.042), 27% in the small-cap value fund (*SV* = 0.274).