INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

# IIT Madras

ONLINE DEGREE

**Mathematics for Data Science 2**
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**Lecture 07**
**Cramer's Rule**

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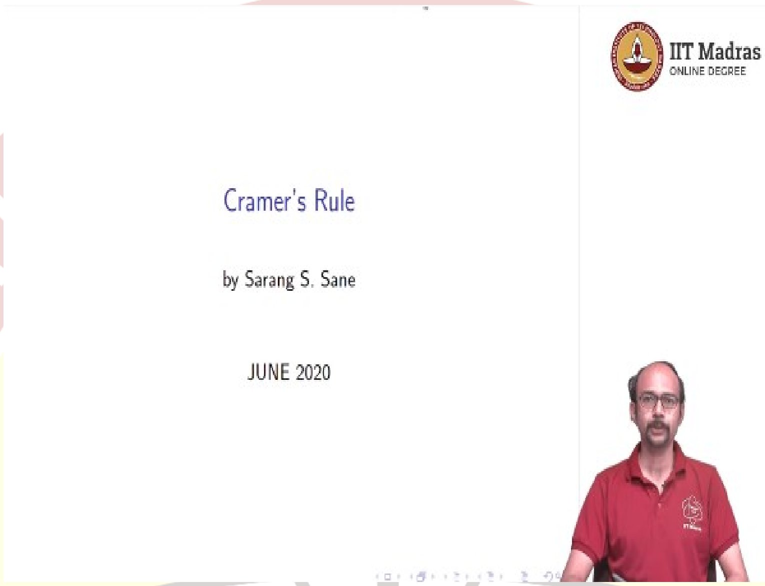


Hello and welcome to the Maths 2 component of the online BSC degree on Data Science. In today's video we are going to look at Cramer's Rule which employs the determinant in order to find the solutions of a system of linear equations, when the coefficient matrix is invertible. Let us go on and do an example.

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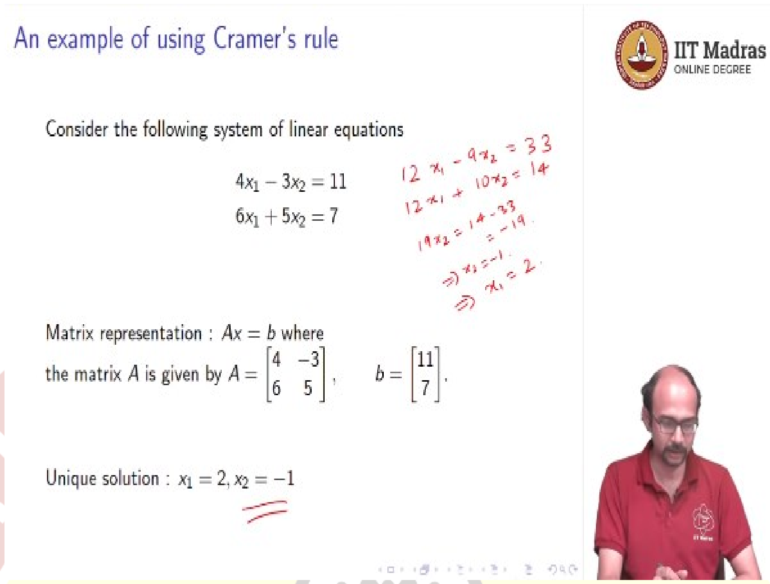


So, let us look at this linear system, $4x_1 - 3x_2 = 11$ and $6x_1 + 5x_2 = 7$. So, let us recall what was the matrix representation. So, that is Ax = b, so the matrix A is given by the coefficients, $\begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$ and b is the constants on the right, so it is a column vector or column matrix $\begin{bmatrix} 11 \\ 7 \end{bmatrix}$ and x is the unknowns $\begin{bmatrix} x1 \\ x2 \end{bmatrix}$. So, this is easy to compute the solution, it has a unique solution $x_1$ is 2 and $x_2$ is minus 1.

Let us quickly go through how, so we can multiply the first equation by 3 to get $12x_1 - 9x_2 = 33$ , we can multiply the second one by 2 to get $12x_1 + 10x_2 = 14$ and then we subtract let us say the first one from the second one, so in that case we get $19x_2 = -19$ , that tells us $x_2 = -1$ and then we can substitute in one of these expressions and get $x_1 = 2$. So, we can easily check that this is a unique solution. So, now let us do something slightly different and reinterpret these using determinants.

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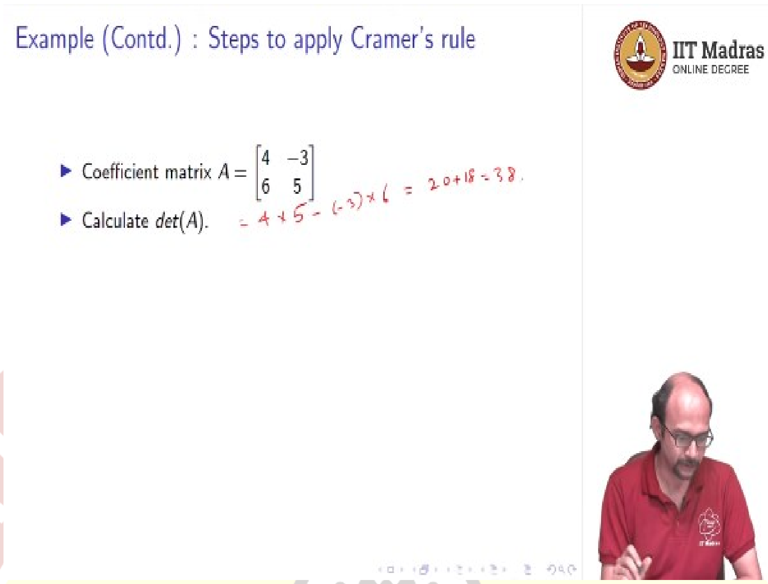


So, here is the same example and here is how we will apply Cramer's rule. So, we look at the coefficient matrix A. Let us compute the determinant of A, so if we do that that $4x_5 - (-3x_6) = 20 + 18 = 38$. So, the determinant of A is 38.

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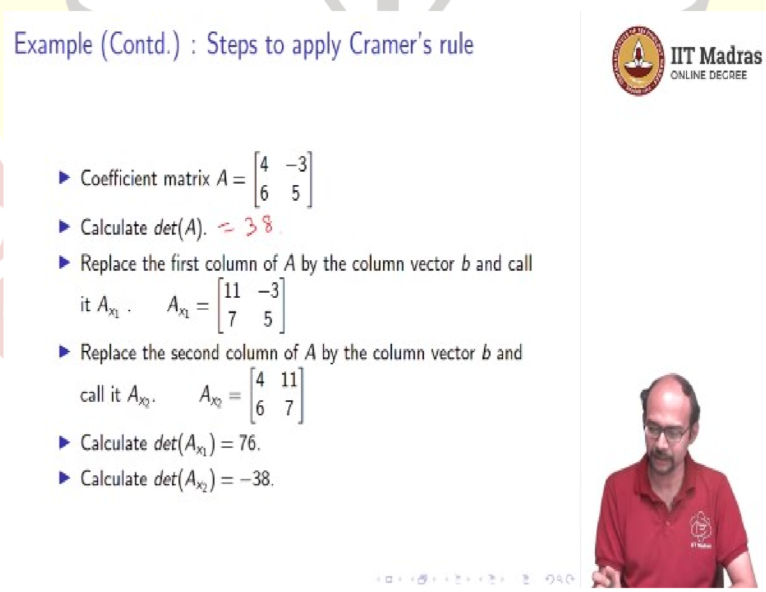


So, replace the first column of A by the column vector b and we will call that matrix $A_{x1}$ because the first column are the coefficients corresponding to $x_1$ in the two equations. So,

$A_{x1}$ is $\begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}$. Replace the second column of A by the column vector b and call $A_{x2}$, so we are doing the same thing for the second column.

So, now the first column is the same as the first column of A and the second column is replaced by 11, 7 so we get $\begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}$, let us calculate determinant of $A_{x1}$, so you can do this, $11x5 - (-3x7) = 76$ and then let us calculate the determinant of $A_{x2}$. So, if you do that that is $4x7 - 11x6 = -38$, so remember that the determinant of A was calculated to be 38. So, let us keep this in mind 38, 76 and -38. So, now what do we do?

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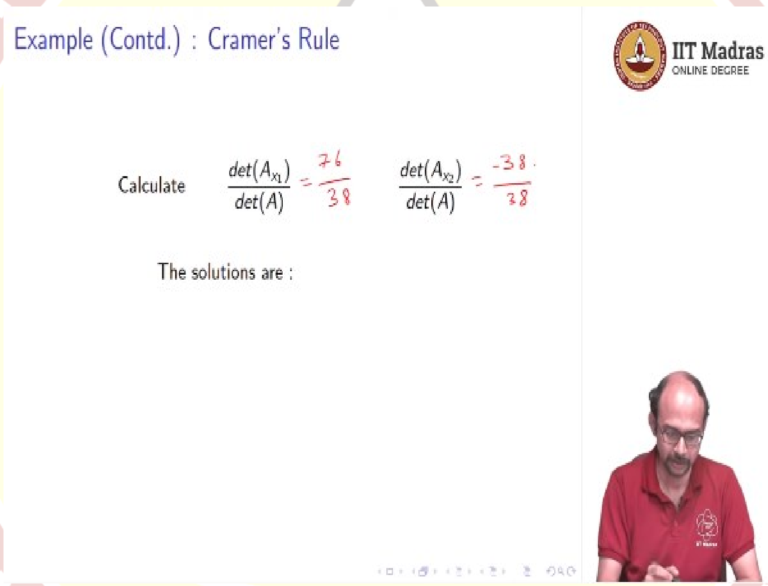


So Cramer's rule will say, let us calculate $\frac{det(A_{x_1})}{det(A)} = \frac{76}{38} = 2$ and $\frac{det(A_{x_2})}{det(A)} = \frac{-38}{38} = -1$ and these are exactly the solutions of $x_1$ and $x_2$ respectively.

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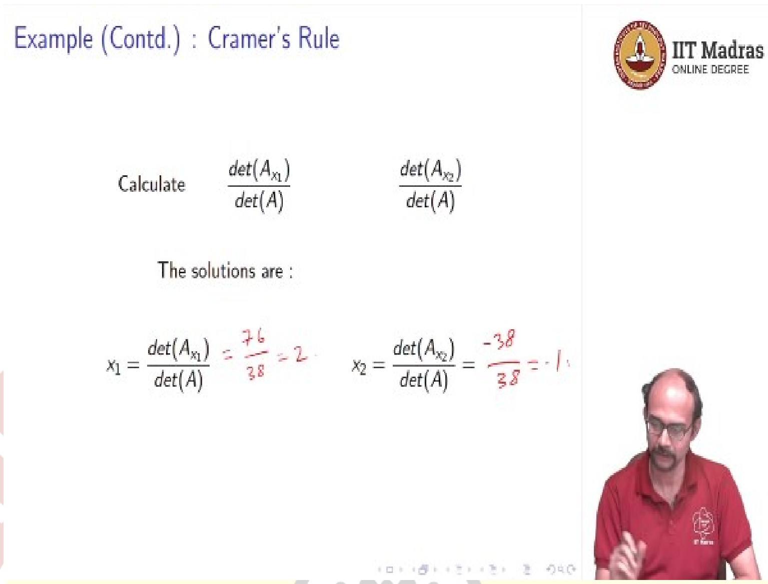


So, at least in this example we have this strange method using determinants and it gives us the actual solutions.

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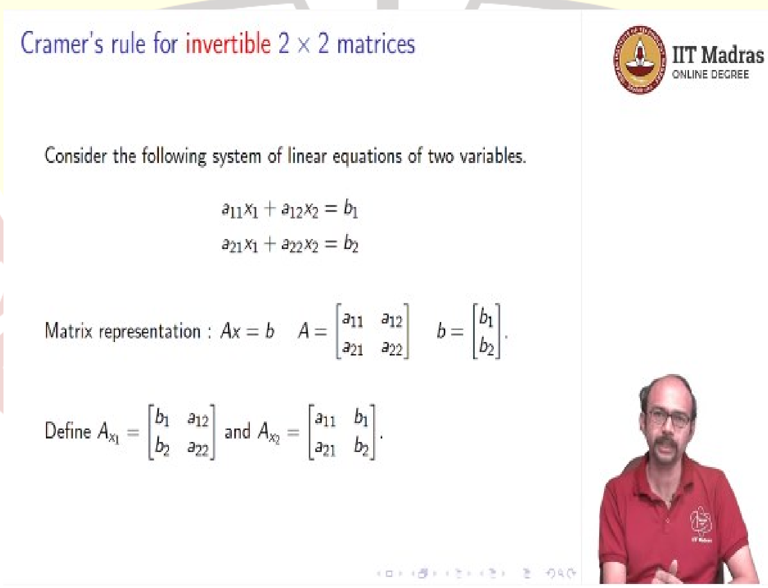


So, let us study what is Cramer's rule in general, so this is for when the coefficient matrix is an invertible matrix, so what do you mean by invertible? Invertible means that the inverse exists. So, consider the following system of linear equations of 2 variables $a_{11}x_1 +$

$a_{12}x_2 = b_1, a_{21}x_1 + a_{22}x_2$. So, the matrix representation here is $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, b = $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

And the assumption here is that the matrix A is invertible, meaning its inverse exists and because its inverse exists, remember that in the previous video with determinants we have seen that determinant of A inverse is $1/det(A)$ which in particular means that determinant of A is non-zero, so $1/det(A)$ exists. So, let us define these 2 matrices as in the previous example. Define $A_{x1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$ and $A_{x2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$.

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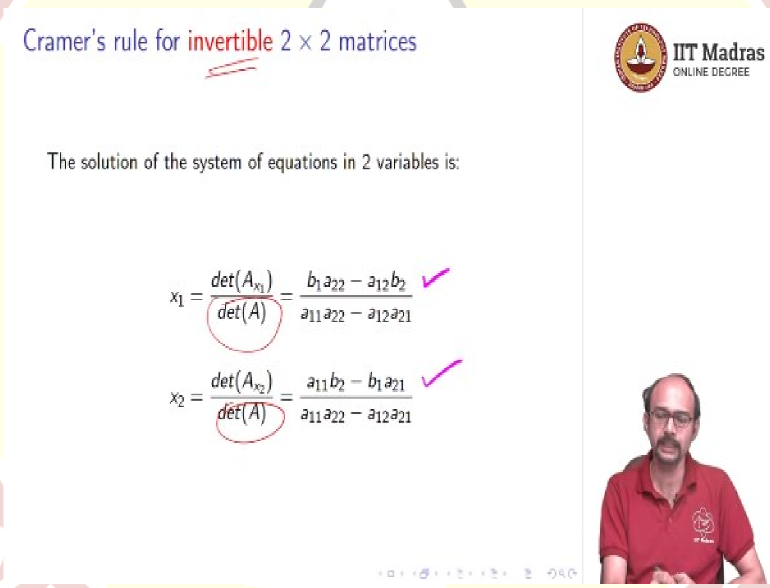


And then Cramer's rule says the solution of the system of equations and I am saying the which means it is a unique solution, in two variables, so $x_1 = \frac{det(A_{x_1})}{det(A)}$, note that this division makes sense because its invertible and $x_2 = \frac{det(A_{x_2})}{det(A)}$, again makes sense and we know explicitly what these expressions are, for $2 \times 2$ matrices the determinant is easy to compute.

So $x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11}a_{22} - a_{12}a_{21}}$, $x_2 = \frac{b_2 a_{11} - a_{21} b_1}{a_{11}a_{22} - a_{12}a_{21}}$. So, that is the algorithm you have to follow for Cramer's rule to find the solutions of a system of equations where the coefficient matrix A has non-zero determinant.

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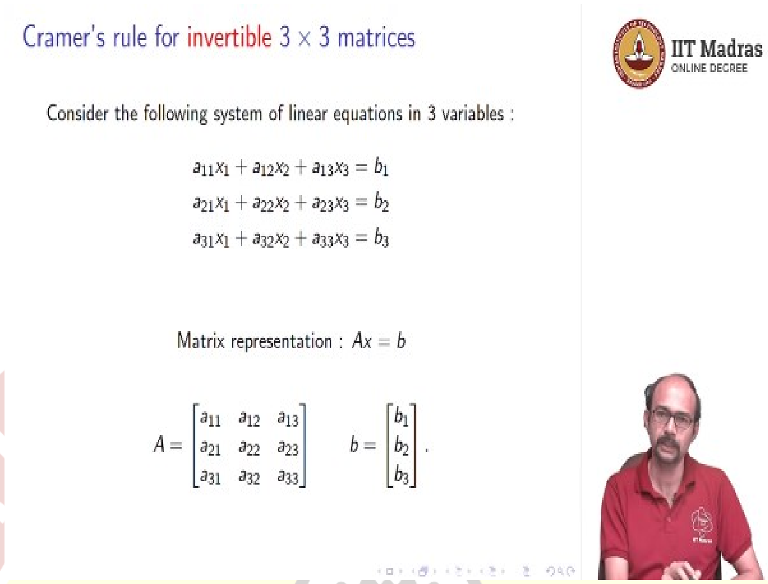


We can make this more general, so let us see Cramer's rule for $3 \times 3$ matrices. So, again here the coefficient matrix must be invertible, so the determinant of A must be non-zero because we divide by the determinant. So, here is the system of linear equations in three variables, so recall that its matrix representation is $Ax = b$, where A is the coefficient matrix and b is this column vector of constants $b_1$, $b_2$, $b_3$.

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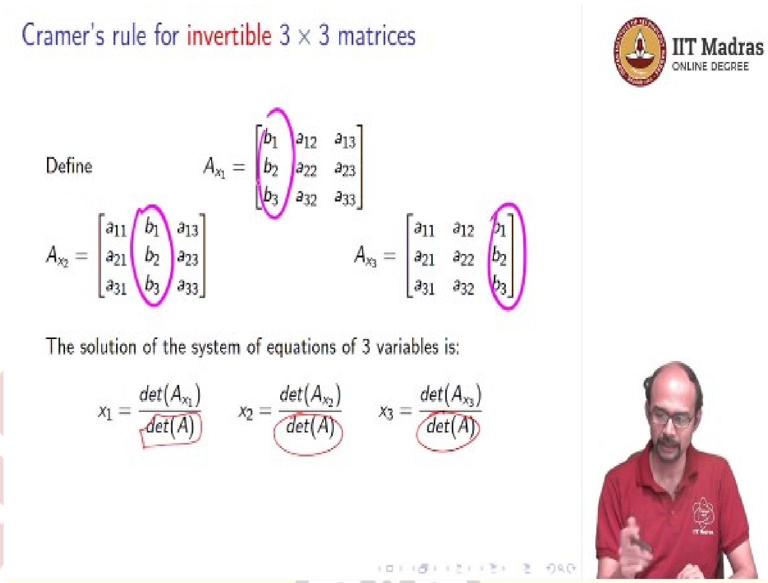


So, in that case what is Cramer's rule? So, we will have to define these new matrices $A_{x_1} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$, $A_{x_2} = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$, $A_{x_3} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$ And then how do we get the solution?

So, the solution meaning again its unique of the system of equations that we get $x_1 = \frac{det(A_{x_1})}{det(A)}$, $x_2 = \frac{det(A_{x_2})}{det(A)}$, $x_3 = \frac{det(A_{x_3})}{det(A)}$ . Again, it is very important that the determinant here must be non-zero, otherwise of course, we cannot divide and assuming that A has an inverse ensures this or explicitly we can say that determinant of A is non-zero.

So, how did we define these $A_{x_1}$, $A_{x_2}$ and $A_{x_3}$? So, in the first one the first column was replaced, in the second one the second column was replaced and in the third one the third column was replaced. So, these and the notations should tell you which columns are replaced because $x_1$ means the coefficients corresponding to $x_1$ column which is replaced. Similarly, for $x_2$ and $x_3$. So, this is Cramer's rule for 3 × 3 matrices or for a system of linear equations in three variables, maybe let us do a quick example.

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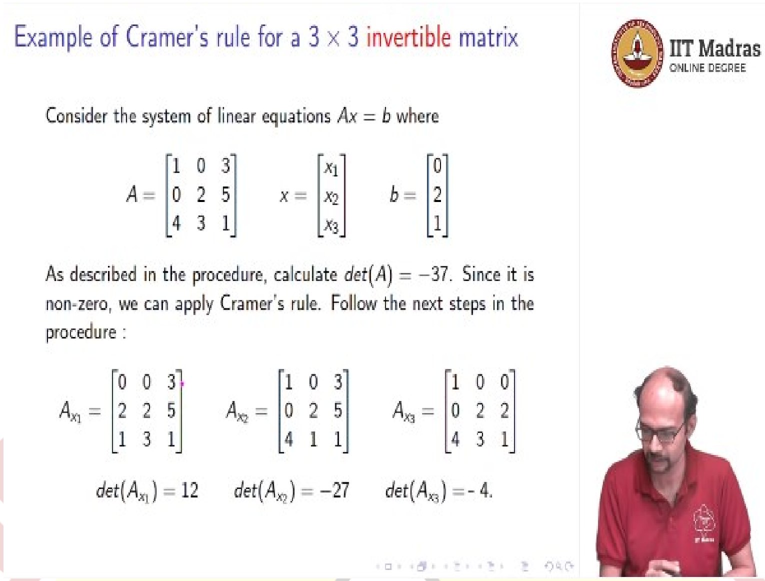


So, consider this system of equations where the coefficient matrix, so I am instead of writing down the system I am writing down the coefficient matrix and the constants. So, the coefficient matrix is $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 3 & 1 \end{bmatrix}$ and the constants are 0, 2 and 1 respectively for the three equations. So, to use Cramer's rule first let us look at the determinant.

So, the determinant of A, so we will have to compute this, so the determinant is -37, so I encourage you to compute this. So, since it is non-zero we can apply Cramer's rule. So, let us follow the next steps in the procedure for Cramer's rule. So, we have to compute these matrices $A_{x_1}$, $A_{x_2}$ and $A_{x_3}$. So, how do we compute these matrices? We replace the corresponding columns by the column b.

Then we compute their determinants, so the determinant of det($A_{x_1}$) = 12, you can check this. So, here we have first row has two 0s, so this is an easy computation similarly, for det($A_{x_2}$) = - 27 and det($A_{x_3}$) = -4.

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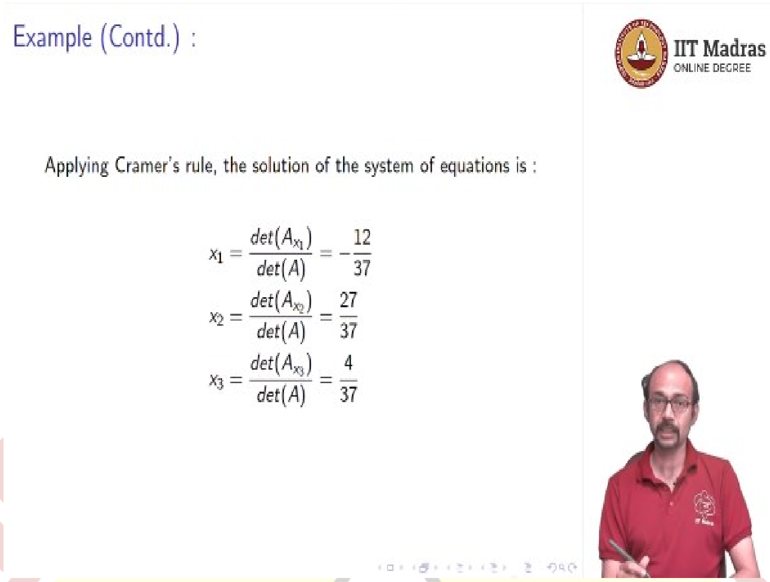


And then by Cramer's rule we get that the solution to this system of equations is $x_1 = -12/37$, $x_2 = 27/37$ and $x_3 = 4/37$. So, I will encourage you to substitute these $x_1$, $x_2$, $x_3$ and check that indeed this is the solution and you can, you know do a usual method of solving equations and check that indeed this is the solution.

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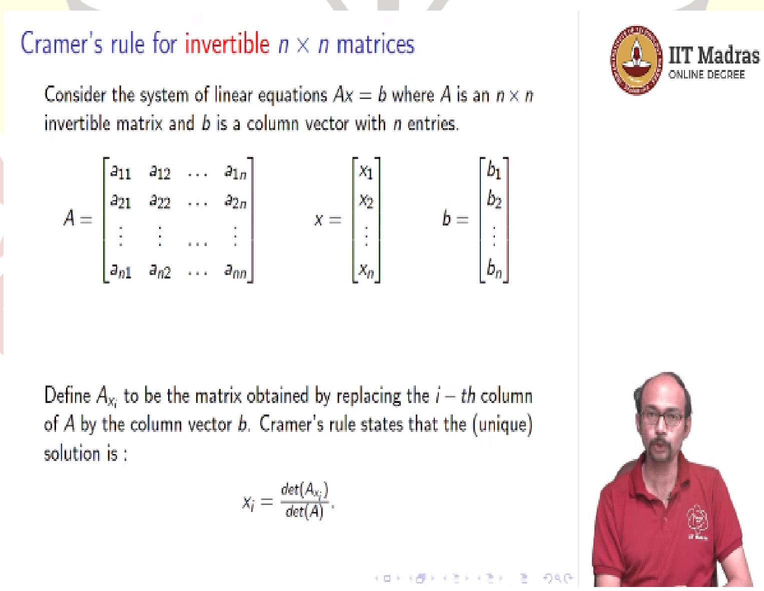


So, finally let us end with Cramer's rule for the $n \times n$ system, so now if you have n equations in a system with n unknowns. So, remember here that we should have, the coefficient matrix has to be square first of all, so we need n equations in n unknowns, this will not work otherwise and the second thing we need here is that the coefficient matrix is invertible or that the determinant is non-zero.

So, this is the general system of equations, A is the coefficient matrix and b is the column vector containing $b_1$, $b_2$, .., $b_n$, then you define $A_{xi}$ and how do we obtain this? We obtain this by for the $i^{th}$ column you replace it by the column vector b. So, if you do that then you compute its determinant and then the solution to the system of linear equations.

So, Cramer's rule states that $x_i = det(A_{xi})/det(A)$, so we compute for each i. So, $x_1 = \frac{det(A_{x_1})}{det(A)}$, $x_2 = \frac{det(A_{x_2})}{det(A)}$, $x_3 = \frac{det(A_{x_3})}{det(A)}$. So, in Cramer's rule remember that we need, this is very important n equations in n unknowns, that is when we can apply it and further we need that the determinant of the coefficient matrix A is non-zero and then we can explicitly work out what the solution is. Thank you.