Orthogonality and linear independence

Sarang S. Sane

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Recall also that the dot product and the length are special cases of an inner product $\langle \; . \; , \; . \; \rangle$ is and a norm on $\mathbb{R}^n$.

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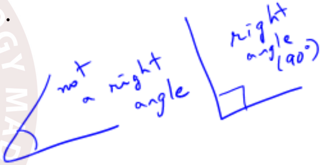

Then $u \cdot v = 0$.

e.g. $(1,2,3)$ and $(2,2,-2))$ are orthogonal.

$$(1,2,3) \cdot (2,2,-2) = 1 \times 2 + 2 \times 2 + 3 \times (-2)$$
$$= 2 + 4 - 6 = 0.$$

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where $u = (x_1, x_2)$ and $v = (y_1, y_2)$.

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Then the vectors $(1, 1)$ and $(1, 0)$ are orthogonal (w.r.t. this inner product).

$$\langle (1,1), (1,0) \rangle = 1 \times 1 - (1 \times 0 + 1 \times 1) + 2 \times 1 \times 0$$

$$= 1 - 0 - 1 + 0$$

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Explicitly, if $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$, then S is an orthogonal set of vectors if

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e.g. consider $\mathbb{R}^3$ with the usual inner product i.e. the dot product. Then the set $S = \{(4, 3, -2), (-3, 2, -3), (-5, 18, 17)\}$ is an orthogonal set of vectors.

$(4,3,-2) \cdot (-3,2,-3) = 4 \times (-3) + 3 \times 2 + (-2) \times (-3)$
$= -12 + 6 + 6 = 0.$

$(4,3,-2) \cdot (-5,18,17) = -20 + 54 - 34 = 0$

$(-3,2,-3) \cdot (-5,18,17) = 15 + 36 - 51 = 0.$

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Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set of non-zero vectors in the inner product space V.

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Suppose $\sum_{i=1}^{k} c_i v_i = 0$

Then $\left\langle \sum_{i=1}^{k} c_i v_i, v_1 \right\rangle = \langle 0, v_1 \rangle = 0.$

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$\therefore c_1 \underbrace{\langle v_1, v_1 \rangle}_{\neq 0} = 0 \Rightarrow c_1 = 0.$

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3. consider $\mathbb{R}^2$ with the inner product $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2x_2 y_2$. Then $\{(1, 1), (1, 0)\}$ is an orthogonal basis.