The solution of a system of linear equations with an invertible coefficient matrix

Sarang S. Sane

Square Matrix (Recall)

A square matrix is a matrix in which the number of rows is the same as the number of columns.

Square Matrix (Recall)

A square matrix is a matrix in which the number of rows is the same as the number of columns.

Example

$$\begin{bmatrix} 3 & 5 & -7 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}_{3\times 3}, \quad \begin{bmatrix} 2.5 & 1 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

The inverse of a Square Matrix (recall)

Let A be an $n \times n$ matrix. The inverse of A is another $n \times n$ matrix B such that $AB = BA = \mathrm{I}_{n\times n}$ and is denoted by A^{-1}.

Produced with a Trial Version of PDF Annotator - www.PDFAnno

The Inverse of a Square Matrix (recall)

Let A be an $n \times n$ matrix. The inverse of A is another $n \times n$ matrix B such that $\boxed{AB = BA = \mathrm{I}_{n\times n}}$ and is denoted by A^{-1}.

Example

$$\underbrace{\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

$A = B^{-1}$ $B = A^{-1}$.

Uniqueness of Inverse:

$AB = BA = I$, $AC = CA = I$.

$$C(AB) = (CA)B$$
$$C(AB) = CI_n = C, \quad (CA)B = I_nB = B$$

$\Rightarrow B = C$.

The determinant of an invertible matrix (recall)

The determinant of an invertible matrix (recall)

Recall that $det(A)det(A^{-1}) = 1$ and hence $det(A^{-1}) = \frac{1}{det(A)}$.

The determinant of an invertible matrix (recall)

Recall that $det(A)det(A^{-1}) = 1$ and hence $det(A^{-1}) = \frac{1}{det(A)}$.
Conclusion : inverse of A exists $\quad \Rightarrow \quad$ $det(A)$ has to be non-zero.

The determinant of an invertible matrix (recall)

Recall that $det(A)det(A^{-1}) = 1$ and hence $det(A^{-1}) = \frac{1}{det(A)}$.
Conclusion : inverse of A exists $\Rightarrow$ $det(A)$ has to be non-zero.

What about the converse i.e. does $det(A) \neq 0 \Rightarrow A$ is invertible?

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $det(A) = \boxed{ad - bc \neq 0}$

$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$ae + bg = 1$
$af + bh = 0$
$ce + dg = 0$
$cf + dh = 1$

$ade + bdg = d.$
$bce + bdg = 0.$
$(ad-bc)e = d.$
$\Rightarrow e = \frac{d}{ad-bc}$

The adjugate of a square matrix

Recall that the (i, j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column. Notation : M_{ij}.

The adjugate of a square matrix

Recall that the (i, j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column. Notation : M_{ij}.

The (i, j)-th cofactor is defined as : $C_{ij} := (-1)^{i+j} M_{ij}$.

The adjugate of a square matrix

Recall that the (i, j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column. Notation : M_{ij}.

The (i, j)-th cofactor is defined as : $C_{ij} := (-1)^{i+j} M_{ij}$.

The cofactor matrix C is the matrix whose (i, j)-th entry is C_{ij}.

The adjugate of a square matrix

Recall that the (i, j)-th minor is the determinant of the submatrix formed by deleting the i-th row and j-th column. Notation : M_{ij}.

The (i, j)-th cofactor is defined as : $C_{ij} := (-1)^{i+j} M_{ij}$.

The cofactor matrix C is the matrix whose (i, j)-th entry is C_{ij}.

Definition

The adjugate matrix of A is defined as : $adj(A) := C^T$.

A 3×3 example of adjugate and inverse

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$$

A 3×3 example of adjugate and inverse

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 1(2 \times 0 - 8 \times 6) - 2(0 \times 0 - 8 \times 5) + 3(0 \times 6 - 2 \times 5) \\ &= -48 + 80 - 30 = 2 \end{aligned}$$

A 3×3 example of adjugate and inverse

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 1(2 \times 0 - 8 \times 6) - 2(0 \times 0 - 8 \times 5) + 3(0 \times 6 - 2 \times 5) \\ &= -48 + 80 - 30 = 2 \end{aligned}$$

$$\begin{array}{lll} M_{11} = -48, & M_{12} = -40, & M_{13} = -10 \\ M_{21} = -18, & M_{22} = -15, & M_{23} = -4 \\ M_{31} = 10, & M_{32} = 8, & M_{33} = 2 \end{array}$$

A 3×3 example of adjugate and inverse

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 5 & 6 & 0 \end{bmatrix}$$

$$\begin{aligned} det(A) &= 1(2 \times 0 - 8 \times 6) - 2(0 \times 0 - 8 \times 5) + 3(0 \times 6 - 2 \times 5) \\ &= -48 + 80 - 30 = 2 \end{aligned}$$

$$\begin{aligned} M_{11} &= -48, & M_{12} &= -40, & M_{13} &= -10 \\ M_{21} &= -18, & M_{22} &= -15, & M_{23} &= -4 \\ M_{31} &= 10, & M_{32} &= 8, & M_{33} &= 2 \end{aligned}$$

The cofactor matrix $\quad C = \begin{bmatrix} -48 & 40 & -10 \\ 18 & -15 & 4 \\ 10 & -8 & 2 \end{bmatrix}$

A 3×3 example of adjugate and inverse (Contd.)

The adjugate matrix $\quad adj(A) = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}.$

A 3×3 example of adjugate and inverse (Contd.)

The adjugate matrix $\quad adj(A) = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}.$

Let us compute $A\boxed{\frac{1}{det(A)}adj(A)}$ [redacted] and $\boxed{\frac{1}{det(A)}adj(A)}A$.

A 3×3 example of adjugate and inverse (Contd.)

The adjugate matrix $\quad adj(A) = \begin{bmatrix} -48 & 18 & 10 \\ 40 & -15 & -8 \\ -10 & 4 & 2 \end{bmatrix}.$

Let us compute $A\frac{1}{det(A)}adj(A)$ [redacted] and $\frac{1}{det(A)}adj(A)A$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Hence $A^{-1} = \frac{1}{det(A)}adj(A)$.

Adjugate and Inverse

If A is an $n \times n$ matrix and $det(A) \neq 0$, then A^{-1} exists and equals

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

$$\sum_{j=1}^{n} a_{ij} C_{ij} = det(A)$$

$$\sum_{j=1}^{n} a_{ij} \boxed{\frac{1}{det(A)} C_{ij}} = 1.$$

$$\sum_{j=1}^{n} a_{ij} \left(adj(A)_{ji} \frac{1}{det(A)} \right) = 1.$$

$$\sum_{j=1}^{n} a_{ij} \left(\frac{1}{det(A)} adj(A) \right)_{ji} = 1.$$

$$\Rightarrow \left(A \frac{1}{det(A)} adj(A) \right)_{ii} = 1.$$

$i \neq k$

$$\frac{1}{det(A)} \boxed{\sum_{j=1}^{n} a_{ij} C_{kj}} = 0$$

The solution of a system of linear equations with an invertible coefficient matrix

Consider the system of linear equations $Ax = b$ where the coefficient matrix A is an invertible matrix.

The solution of a system of linear equations with an invertible coefficient matrix

Consider the system of linear equations $Ax = b$ where the coefficient matrix A is an invertible matrix.

Multiplying both sides by A^{-1} we obtain :

$$A x = b$$

$$A^{-1} A x = A^{-1} b$$

$$I_n x = A^{-1} b$$

$$x = A^{-1} b.$$

Example

$$\begin{aligned} 8x_1 + 8x_2 + 4x_3 &= 1960 \\ 12x_1 + 5x_2 + 7x_3 &= 2215 \\ 3x_1 + 2x_2 + 5x_3 &= 1135 \end{aligned}$$

Example

$$\begin{aligned} 8x_1 + 8x_2 + 4x_3 &= 1960 \\ 12x_1 + 5x_2 + 7x_3 &= 2215 \\ 3x_1 + 2x_2 + 5x_3 &= 1135 \end{aligned}$$

$$A = \begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$$

Example (Contd.)

$$det(A) = 8(25 - 14) - 8(60 - 21) + 4(24 - 15) = 88 - 312 + 36 = -188.$$

Example (Contd.)

$$det(A) = 8(25-14) - 8(60-21) + 4(24-15) = 88 - 312 + 36 = -188.$$

So A is invertible and we compute the inverse as follows :

Example (Contd.)

$$det(A) = 8(25-14) - 8(60-21) + 4(24-15) = 88 - 312 + 36 = -188.$$

So A is invertible and we compute the inverse as follows :

$$\begin{array}{lll} M_{11} = 11, & M_{12} = 39, & M_{13} = 9 \\ M_{21} = 32, & M_{22} = 28, & M_{23} = -8 \\ M_{31} = 36, & M_{32} = 8, & M_{33} = -56 \end{array}$$

$$\text{The cofactor matrix: } C = \begin{bmatrix} 11 & -39 & 9 \\ -32 & 28 & 8 \\ 36 & -8 & -56 \end{bmatrix}$$

Example (Contd.)

$$\text{The adjugate matrix } adj(A) = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}.$$

Example (Contd.)

The adjugate matrix $adj(A) = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$.

$$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$$

Example (Contd.)

$$\text{The adjugate matrix } adj(A) = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}.$$

$$A^{-1} = \tfrac{1}{det(A)} adj(A) = \tfrac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$$

$$\begin{aligned} x = A^{-1}b &= \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix} \\ &= -\frac{1}{188} \begin{bmatrix} -8460 \\ -23500 \\ -28200 \end{bmatrix} = \begin{bmatrix} 45 \\ 125 \\ 150 \end{bmatrix} \end{aligned}$$

Example (Contd.)

The adjugate matrix $adj(A) = \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$.

$$A^{-1} = \tfrac{1}{det(A)} adj(A) = \tfrac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix}$$

$$x = A^{-1}b = \frac{1}{-188} \begin{bmatrix} 11 & -32 & 36 \\ -39 & 28 & -8 \\ 9 & 8 & -56 \end{bmatrix} \begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$$

$$= -\frac{1}{188} \begin{bmatrix} -8460 \\ -23500 \\ -28200 \end{bmatrix} = \begin{bmatrix} 45 \\ 125 \\ 150 \end{bmatrix}$$

Hence the solution is $x_1 = 45, x_2 = 125, x_3 = 150$.

Homogeneous System of Linear Equations

A system of linear equations is homogeneous if all of the constant terms are 0 i.e. $b = 0$.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\
&\ldots \\
&\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}
$$

Solutions of a homogeneous system

The matrix form of a homogeneous system is $Ax = 0$.

Solutions of a homogeneous system

The matrix form of a homogeneous system is $Ax = 0$.

If A is an invertible matrix then multiplying both sides by A^{-1}, we obtain $x = A^{-1}0 = 0$.

Solutions of a homogeneous system

The matrix form of a homogeneous system is $Ax = 0$.

If A is an invertible matrix then multiplying both sides by A^{-1}, we obtain $x = A^{-1}0 = 0$.

A homogeneous system of linear equations with n equations in n unknowns :

- has a unique solution 0 if its coefficient matrix is invertible, i.e. its determinant is non-zero.

Solutions of a homogeneous system

The matrix form of a homogeneous system is $Ax = 0$.

If A is an invertible matrix then multiplying both sides by A^{-1}, we obtain $x = A^{-1}0 = 0$.

A homogeneous system of linear equations with n equations in n unknowns :

- has a unique solution 0 if its coefficient matrix is invertible, i.e. its determinant is non-zero.
- has an infinite number of solutions if its coefficient matrix is not invertible i.e. its determinant is 0.

Thank you