The Gaussian Elimination Method

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- If A is in (reduced) row echelon form, we can find all the solutions as follows :

  1. Find the dependent variables (corr. to columns with leading entries) and independent variables (corr. to other columns).

  2. Assign a value to each independent variable.Calculate the values of each dependent variable using the unique equation in which it occurs.

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- Computing the inverse using Gaussian elimination.

The augmented matrix

Let $Ax = b$ be a system of linear equations where A is an $m \times n$ matrix and b is a $m \times 1$ column vector.

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We denote the augmented matrix by $[A|b]$ and put a vertical line between the first n columns and the last column b while writing it.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$$\Vert$$

$$A$$

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Example

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + x_4 &= 6 \\ x_1 + x_2 &= 2 \\ 7x_2 + x_3 + x_4 &= 8 \end{aligned}$$

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$$\text{where } A = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}.$$

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The augmented matrix is $[A|b] = \left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$.

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The solutions of $Ax = b$ are precisely the solutions of $Rx = c$.

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Since R is in reduced row echelon form, we can find ALL its solutions (as described earlier).

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$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \overset{R_1/3}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

$$\downarrow R_2 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1/3 & -1/3 & -1/3 & 0 \\ 0 & 7 & 1 & 1 & 8 \end{array}\right]$$

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$$\xrightarrow{R_3-7R_2} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array}\right]$$

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$$\overset{R_3-7R_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 8 & 8 & 8 \end{array}\right] \overset{R_3/8}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Example (contd.)

$$\left[\begin{array}{cccc|c} 1 & 2/3 & 1/3 & 1/3 & 2 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - \frac{1}{3}R_3]{R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 2/3 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\Big\downarrow\, R_1 - \tfrac{2}{3}R_2$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{R} \quad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{c}$$

$Rx = c$

$x_1, x_2, x_3 \rightarrow$ dependent

$x_4 \rightarrow$ independent

$x_4 = c$.

$x_1 = 1$

$x_2 = 1$

$x_3 + x_4 = 1 \Rightarrow x_3 = 1 - c$

Set of solns. of $Rx = c$ & hence $Ax = b$

is $\left\{ \begin{array}{l} x_1 = 1, x_2 = 1, \\ x_3 = 1-c, x_4 = c \end{array} \;\middle|\; c \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1-c \\ c \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$.

Another example

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\begin{aligned}
x_1 + x_2 + x_3 &= 2 \\
x_2 - 3x_3 &= 1 \\
2x_1 + x_2 + 5x_3 &= 0
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The matrix representation of this system of linear equations is:

$$
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}
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The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right]$

Another example (contd.)

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$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 2 & 1 & 5 & 0 \end{array}\right] \overset{R_3-2R_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & -1 & 3 & -4 \end{array}\right]$$

Another example (contd.)

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$R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$R \qquad c$

$Rx = c.$

This system does not have solutions.

$\therefore$ $Ax = b$ does not have solutions

Homogeneous system of linear equations

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- 0 is the unique solution.
- there are infinitely many solutions other than 0.

$Ax = 0$
$x_1 = w_1, x_2 = w_2, \dots, x_n = w_n$ is a soln.
then so is
$x_1 = tw_1, x_2 = tw_2, \dots, x_n = tw_n$.

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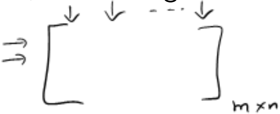
In a homogeneous system of equations, if there are more variables than equations, then it is guaranteed to have nontrivial solutions.



Computing the inverse

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Finding solutions of $Ax = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $Ay = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $Az = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$x' \quad y' \quad z'$

$\left[A \,\middle|\, \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] = [A \mid I] \xrightarrow[\text{row echelon form}]{\text{reduction to reduced}} [I \mid \underbrace{Q}_{A^{-1}}]$

$A \begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} Ax' & Ay' & Az' \end{bmatrix} \overset{3\times 3}{=} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I.$

$\left[\begin{matrix}1&1&1\\2&4&8\\3&9&27\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{matrix}1&1&1\\0&2&6\\0&6&24\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-2&1&0\\-3&0&1\end{matrix}\right] \xrightarrow{R_2/2} \left[\begin{matrix}1&1&1\\0&1&3\\0&6&24\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-1&1/2&0\\-3&0&1\end{matrix}\right] \xrightarrow{R_3-6R_2} \left[\begin{matrix}1&1&1\\0&1&3\\0&0&6\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-1&1/2&0\\3&-3&1\end{matrix}\right]$

$\xrightarrow{R_3/6} \left[\begin{matrix}1&1&1\\0&1&3\\0&0&1\end{matrix}\,\middle|\,\begin{matrix}1&0&0\\-1&1/2&0\\1/2&-1/2&1\end{matrix}\right] \longleftarrow$

Thank you