The Hessian matrix and local extrema for $f(x, y)$

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A saddle point is a critical point $\underset{\sim}{a}$ such that $\nabla f(\underset{\sim}{a})$ exists and $\nabla f(\underset{\sim}{a}) = 0$ but $\underset{\sim}{a}$ is not a local extremum.

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4. If $det(Hf(\underset{\sim}{a})) = 0$ then the test is inconclusive.

Examples

$f(x,y) = x^2 + y^2$

Critical pt. : $(0,0)$.

$\nabla f = (2x, 2y)$.

$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

$\det(Hf(0,0)) = 4 > 0$.

$f_{xx}(0,0) = 2 > 0$.

$\therefore (0,0)$ is a local minimum.

$f(x,y) = -x^2 - y^2$

$\nabla f = (-2x, -2y)$. Critical pt. : $(0,0)$

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$\det(Hf(0,0)) = 4 > 0$

$f_{xx}(0,0) = -2 < 0$.

$\therefore (0,0)$ is a local maximum.

$f(x,y) = x^2 - y^2$

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$f(x,y) = x^4 + y^4$

$\nabla f = (4x^3, 4y^3)$. Critical pt. $(0,0)$.

$Hf(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$Hf = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$.

$\det(Hf(0,0)) = 0$.

$\therefore$ The test is inconclusive.

Examples (contd.)

$f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$

$\nabla f = (2x + 6y + 2,\ 6x + 8y - 4)$. Equating to 0, we get

the critical pt. $(2, -1)$.

$Hf = \begin{bmatrix} 2 & 6 \\ 6 & 8 \end{bmatrix} = Hf(2,-1)$.

$\det(Hf(2,-1)) = 16 - 36 = -20 < 0$.

$\therefore (2,-1)$ is a saddle point of $f(x,y)$.

$f(x, y) = xy - x^3 - y^2$

$\nabla f = (y - 3x^2,\ x - 2y)$. Equating to 0, we get:

$y = 3x^2,\ x = 2y \Rightarrow y = 3(2y)^2 = 12y^2 \Rightarrow y(1 - 12y) = 0$.

Critical pts: $(0,0)$, $(\frac{1}{6}, \frac{1}{12})$.

$Hf = \begin{bmatrix} -6x & 1 \\ 1 & -2 \end{bmatrix}$.

$Hf(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ $\det(Hf(0,0)) = -1 < 0$.

$\therefore (0,0)$ is a saddle pt.

$Hf(\frac{1}{6}, \frac{1}{12}) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$ $\det(Hf(\frac{1}{6}, \frac{1}{12})) = 2 - 1 = 1 > 0$.

$f_{xx}(\frac{1}{6}, \frac{1}{12}) = -1 < 0$.

$\therefore (\frac{1}{6}, \frac{1}{12})$ is a local max.

Examples (contd.)

$f(x, y) = sin(xy)$

$$\nabla f = (y\cos(xy),\ x\cos(xy)).$$

$$Hf = \begin{bmatrix} -y^2\sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{bmatrix}$$

Equating ∇f to 0, we get : ① $\cos(xy) = 0$ or ② $x = y = 0$.

$\Updownarrow$

$\sin(xy) = \pm 1$.

$$Hf(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det(Hf(0,0)) = -1 < 0.$$

$\Rightarrow (0,0)$ is a saddle pt. for f.

For pts. such that $\cos(xy) = 0$

$$Hf(x,y) = \begin{bmatrix} -y^2 & -xy \\ -xy & -x^2 \end{bmatrix}$$ if $\sin(xy) = 1$

or

$$Hf(x,y) = \begin{bmatrix} y^2 & xy \\ xy & x^2 \end{bmatrix}$$ if $\sin(xy) = -1$.

$\det(Hf(x,y)) = 0$.

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Thank you