Maths 2 : Activity Questions - Selected solutions
Week-6

1 Lecture 6.1

1. **Question 3:** Consider the set $V = \{(z - x, -y) \mid x, y, z \in \mathbb{R}\} \subseteq \mathbb{R}^2$ with the usual addition and scalar multiplication as in $\mathbb{R}^2$ and associated statements given below.

   - **P:** V is closed under addition i.e ($v_1, v_2 \in V \implies v_1 + v_2 \in V$).
   - **Q:** V is closed under scalar multiplication i.e ($c \in \mathbb{R}, v \in V \implies cv \in V$).
   - **R:** V has a zero element with respect to addition. i.e., there exists some element 0 such that $v + 0 = v$, for all $v \in V$.
   - **S:** V is not a vector space.
   - **T:** $(a + b)v = av + bv$ where $a, b \in \mathbb{R}$ and $v \in V$.

   Choose the set of correct options.

   - ○ Option 1: Only P is true.
   - ○ Option 2: Only S is true.
   - ○ **Option 3:** Both P and Q are true.
   - ○ **Option 4:** Both R and T are true.
   - ○ **Option 5:** All statements are true except S.

Soln: Let $v_1 = (z_1 - x_1, -y_1)$ and $v_2 = (z_2 - x_2, -y_2)$ be in V.

$v_1 + v_2 = (z_1 - x_1 + z_2 - x_2, -y_1 - y_2)$

$= ((z_1 + z_2) - (x_1 + x_2), -(y_1 + y_2)) \in V$

Hence, P is true.

Let $v = (z - x, -y) \in V$. $cv = (c(z - x), -cy)$

$= (cz - cx, -cy) \in V$

Hence, Q is true.

$v + (0,0) = (z - x, -y) + (0,0) = (z - x, -y) = v$

Hence $\exists$ zero element in V. Hence R is true.

verify all the axioms of vector space.

$(a+b)(z - x, -y) = ((a+b)(z - x), (a+b)(-y)) = a(z-x,-y) + b(z-x,-y)$

Hence, S is true.

2. **Question 6:** Consider the following sets:

- $V_1 = \{A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a symmetric matrix, i.e., } A = A^T\}$
- $V_2 = \{A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a scalar matrix}\}$
- $V_3 = \{A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a diagonal matrix}\}$
- $V_4 = \{A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a upper triangular matrix}\}$
- $V_5 = \{A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a lower triangular matrix}\}$

Choose the set of correct options.

- ◯ Option 1: Only V_1 is a subspace of $M_{2\times 2}(\mathbb{R})$.
- ◯ Option 2: Only V_4 is a subspace of $M_{2\times 2}(\mathbb{R})$.
- ◯ **Option 3:** Both V_2 and V_3 are subspaces of $M_{2\times 2}(\mathbb{R})$
- ◯ **Option 4:** All are subspaces of $M_{2\times 2}(\mathbb{R})$

Soln: V_1 $\quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a symmetric matrix. So $0 \in V_1$

let $A, B \in V_1$, then $(A+B)^T = A^T + B^T = A + B$. Hence, $A+B \in V_1$

$(cA)^T = cA^T = cA$. Hence $cA \in V_1$

Hence, V_1 is a subspace of $M_{2\times 2}(\mathbb{R})$

V_2 $\quad 0$ is a scalar matrix. So $0 \in V_2$.

Let $A, B \in V_2$. let $A = \begin{pmatrix} x_1 & 0 \\ 0 & x_1 \end{pmatrix}$ and $B = \begin{pmatrix} y_1 & 0 \\ 0 & y_1 \end{pmatrix}$ where $x_1, y_1 \in \mathbb{R}$.

$A + B = \begin{pmatrix} x_1+y_1 & 0 \\ 0 & x_1+y_1 \end{pmatrix}$ where $x_1 + y_1 \in \mathbb{R}$.

$A+B$ is a scalar matrix.

$cA = \begin{pmatrix} cx_1 & 0 \\ 0 & cx_1 \end{pmatrix}$ is a scalar matrix. Hence, $A+B \in V_2$, $cA \in V_2$.
So, V_2 is a subspace.

V_3 $\quad 0$ is a diagonal matrix. So, $0 \in V_3$

let $A, B \in V_3$ let $A = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ and $B = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$ where. $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

$A+B = \begin{pmatrix} x_1+y_1 & 0 \\ 0 & x_2+y_2 \end{pmatrix}$, $cA = \begin{pmatrix} cx_1 & 0 \\ 0 & cx_2 \end{pmatrix}$. Hence, $A+B \in V_3$, $cA \in V_3$.
So, V_3 is a subspace.

V_4 $\quad 0$ is a upper triangular matrix. So, $0 \in V_4$

Let $A, B \in V_4$. Let $A = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}$ and $B = \begin{pmatrix} y_1 & y_2 \\ 0 & y_3 \end{pmatrix}$

$A+B = \begin{pmatrix} x_1+y_1 & x_2+y_2 \\ 0 & x_3+y_3 \end{pmatrix}$, $cA = \begin{pmatrix} cx_1 & cx_2 \\ 0 & cx_3 \end{pmatrix}$ Hence, $A+B \in V_4$, $cA \in V_4$
So, V_4 is a subspace.

Similarly V_5 is a subspace.

3. **Question 8:** Consider a set $V = \{(x, y) \mid x, y \in \mathbb{R}\} \subseteq \mathbb{R}^2$ with the usual addition as in $\mathbb{R}^2$ and scalar multiplication is defined as

$$c(x,y) = \begin{cases} (0,0) & c = 0 \\ (\frac{cx}{2}, \frac{y}{c}) & c \neq 0 \end{cases} \qquad (x,y) \in V,\ c \in \mathbb{R}$$

Consider the statements given below.

- **P:** V is closed under addition.
- **Q:** V has zero element with respect to addition. i.e., there exists some element 0 such that $v + 0 = v$, for all $v \in V$.
- **R:** $1.v = v$ where $1 \in \mathbb{R}$ and $v \in V$.
- **S:** $a(v_1 + v_2) = av_1 + av_2$ where $v_1, v_2 \in V$ and $a \in \mathbb{R}$.
- **T:** $(a + b)v = av + bv$ where $a, b \in \mathbb{R}$ and $v \in V$.

Choose the set of correct options.

- ○ Option 1: Only P is true.
- ○ Option 2: Only Q is true.
- ○ **Option 3:** Both P and Q are true.
- ○ **Option 4:** P,Q and S are true.
- ○ **Option 5:** Both R and T are not true.

For T:

Let $a = 2,\ b = 2 \qquad v = (1,1)$

$(a+b)v = 4(1,1) = (2, 1/4)$

$av + bv = 2(1,1) + 2(1,1)$

$= (1, 1/2) + (1, 1/2)$

$= (2, 1)$

Hence, $(a+b)v \neq av + bv$.

Soln. The addition on V is defined as usual one.

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$

So, P and Q are true.

Now, $1.(x,y) = (\frac{x}{2}, y)$ (as $1 \neq 0$)

$\neq (x,y)$ (in general, it is true only when $x = 0$ not for all $x \in \mathbb{R}$)

Hence, R is not true.

Let $v_1 = (x_1, y_1),\ v_2 = (x_2, y_2)$

$$a(v_1 + v_2) = a(x_1 + x_2, y_1 + y_2) = \begin{cases} (0,0) & \text{, if } a = 0 \\ \left(\frac{a(x_1+x_2)}{2}, \frac{y_1+y_2}{a}\right), & \text{if } a \neq 0 \end{cases}$$

$$av_1 + av_2 = a(x_1, y_1) + a(x_2, y_2) = \begin{cases} (0,0) & \text{, if } a = 0 \\ \left(\frac{ax_1}{2}, \frac{y_1}{a}\right) + \left(\frac{ax_2}{2}, \frac{y_2}{a}\right) & \text{, if } a \neq 0 \end{cases}$$

$$= \begin{cases} (0,0) & \text{, if } a = 0 \\ \left(\frac{a(x_1+x_2)}{2}, \frac{y_1+y_2}{a}\right) & \text{, if } a \neq 0. \end{cases}$$

Hence, $\underline{a(v_1 + v_2) = av_1 + av_2}$. $\forall v_1, v_2 \in V, a \in \mathbb{R}$.

2 Lecture 6.2

Let V be a plane plane parallel to the XY-plane. Any plane parallel to XY-plane is given by $z = c$. We define addition of two vectors $v_1 = (x_1, y_1, c)$ and $v_2 = (x_2, y_2, c)$ on V as follows: First project v_1 and v_2 on the XY-plane (we will get the vectors $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ by projection on the XY-plane) and then calculate the addition of the vectors we obtained by the projection on the XY-plane (we will obtain $(x_1 + x_2, y_1 + y_2, 0)$). Then project the obtained vector back to the plane V (we will obtain the vector $(x_1 + x_2, y_1 + y_2, c)$).
Let V be a plane which is parallel to the XY-plane, defined by $z = 2$. Let $v_1 = (1, 2, 2)$ and $v_2 = (0, 3, 2)$ be in V. Answer questions 5 and 6.

4. **Question 5:** Which of the following pairs of vectors will be the projections of v_1 and v_2 on the XY-plane?
   - ○ Option 1: $(1, 0, 2)$ and $(0, 0, 2)$
   - ○ **Option 2:** $(1, 2, 0)$ and $(0, 3, 0)$
   - ○ Option 3: $(0, 2, 2)$ and $(0, 3, 2)$
   - ○ Option 4: $(1, 0, 0)$ and $(0, 0, 0)$

5. **Question 6:** Which of the following vectors will be $v_1 + v_2$ as per the addition defined above?
   - ○ Option 1: $(1, 0, 4)$
   - ○ Option 2: $(1, 5, 0)$
   - ○ **Option 3:** $(1, 5, 2)$
   - ○ Option 4: $(1, 5, 4)$

Soln 4) $v_1 = (1,2,2)$ $\quad v_2 = (0,3,2)$

Projection of v_1 on XY-plane will be $(1,2,0)$

Projection of v_2 on XY-plane will be $(0,3,0)$

[Making z-coordinate 0].

5) Adding the vectors obtained by projection will give

$(1,2,0) + (0,3,0) = (1,5,0)$

Now projecting back to the plane $z = 2$ will give

$(1, 5, 2)$

Let V be a plane plane parallel to the XY-plane. Any plane parallel to the XY-plane is given by $z = c$. We define scalar multiplication of a vector $v = (x, y, c)$ on V as follows: First project v on the XY-plane first (we will get the vector $(x, y, 0)$ by projection on the XY-plane) and then calculate the scalar (α) multiple of the vector we obtained by the projection on the XY-plane (we will obtain $(\alpha x, \alpha y, 0)$). Then project the obtained vector back to the plane V (we will obtain the vector $(\alpha x, \alpha y, c)$).
Let V be a plane which is parallel to the XY-plane, defined by $z = 2$. Let $v = (1, 2, 2)$ be in V. Answer questions 7 and 8.

6. **Question 7:** Which of the following vectors will be the projection of v on XY-plane?
   - ○ Option 1: $(1, 0, 2)$
   - ○ Option 2: $(1, 0, 0)$
   - ○ **Option 3:** $(1, 2, 0)$
   - ○ Option 4: $(0, 2, 2)$
7. **Question 8:** Which of the following vectors will be $4v$ as per the scalar multiplication defined above?
   - ○ Option 1: $(4, 2, 8)$
   - ○ Option 2: $(4, 8, 8)$
   - ○ **Option 3:** $(4, 8, 2)$
   - ○ Option 4: $(4, 2, 2)$

<u>Soln:</u> 6) Projection of v on XY-plane will give $(1, 2, 0)$.

7) If we multiply the new vector by 4 we get $(4, 8, 0)$.
By projecting back to the plane $z = 2$, we get $(4, 8, 2)$.

3 Lecture 6.3

8. **Question 5:** Consider three vectors $v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (1,1,1)$ in the vector space $\mathbb{R}^3$, with usual addition and scalar multiplication. Which of the following sets is (are) true?

   - ○ Option 1: $\{v_1, v_2, v_3\}$ is a linearly dependent set.
   - ○ Option 2: $\{v_1 + v_2, v_1 + v_3, v_3\}$ is a linearly dependent set.
   - ○ **Option 3:** $\{v_2 - v_1, v_1, v_2\}$ is a linearly dependent set.
   - ○ **Option 4:** $\{v_1, v_2 - v_1, v_3 - v_2, v_3\}$ is a linearly dependent set.

Solⁿ: let $av_1 + bv_2 + cv_3 = 0$

$\Rightarrow a(1,0,0) + b(1,1,0) + c(1,1,1) = 0$

$\Rightarrow (a+b+c, b+c, c) = 0$

$\Rightarrow a+b+c = 0, \; b+c = 0, \; c = 0$

$\Rightarrow \underline{a = b = c = 0}$

Hence, $\{v_1, v_2, v_3\}$ is linearly independent set

Let $a(v_1 + v_2) + b(v_1 + v_3) + cv_3 = 0$

$\Rightarrow av_1 + av_2 + bv_1 + bv_3 + cv_3 = 0$

$\Rightarrow (a+b)v_1 + av_2 + (b+c)v_3 = 0$

As $\{v_1, v_2, v_3\}$ is lin. ind. set, $a+b = 0, \; a = 0, \; b+c = 0$

Hence, $a = b = c = 0$

So, $\{v_1 + v_2, v_1 + v_3, v_3\}$ is lin. ind. set.

$v_2 - v_1 = v_2 + (-1)v_1$. So the first vector of the set $\{v_2 - v_1, v_1, v_2\}$ is a linear combination of the other two. Hence the set is lin. dep.

$v_3 = (v_3 - v_2) + (v_2 - v_1) + v_1$. So the 4-th vector of the set $\{v_1, v_2 - v_1, v_3 - v_2, v_3\}$ is a linear combination of the other three. Hence the set is lin. dep.

$M_{3\times 3}(\mathbb{R})$ denotes the vector space consisting of real square matrices of order 3 with usual matrix addition and scalar multiplication. Consider the following matrices:

$$M_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

Use the above information to answer questions 7 and 8.

9. **Question 7:** Which of the following options are correct?

   - ○ Option 1: $M_1 - M_2 = 0$, where 0 denotes the zero matrix of order 3.
   - ○ Option 2: $2M_1 + M_2 = M_3$
   - ○ **Option 3:** $2M_1 - M_2 - M_3 = 0$, where 0 denotes the zero matrix of order 3.
   - ○ **Option 4:** $2M_2 - M_1 - M_4 = 0$, where 0 denotes the zero matrix of order 3.
   - ○ Option 5: The tuple (a, b, c) is unique for which $aM_1 + bM_3 + cM_4 = 0$ holds, where 0 denotes the zero matrix of order 3.
   - ○ **Option 6:** The pair (a, b) is unique for which $aM_1 + bM_2 = 0$ holds, where 0 denotes the zero matrix of order 3.

<u>Soln.</u>

$$M_1 - M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \neq 0.$$

$$2M_1 + M_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix} \neq M_3$$

$$2M_1 - M_2 - M_3 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = 0$$

$$2M_2 - M_1 - M_4 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = 0$$

$$aM_1 + bM_3 + cM_4 = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ a & a & a \end{bmatrix} + \begin{bmatrix} b & 0 & b \\ 0 & b & -b \\ b & 2b & b \end{bmatrix} + \begin{bmatrix} c & 0 & c \\ 0 & c & 2c \\ c & -c & c \end{bmatrix} = \begin{bmatrix} a+b+c & 0 & a+b+c \\ 0 & a+b+c & -b+2c \\ a+b+c & a+2b-c & a+b+c \end{bmatrix}$$

$aM_1 + bM_3 + cM_4 = 0 \Rightarrow a+b+c=0, \quad -b+2c=0, \quad a+2b-c=0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, there are infinitely many solutions for a, b, c.

$$aM_1 + bM_2 = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ a & a & a \end{bmatrix} + \begin{bmatrix} b & 0 & b \\ 0 & b & b \\ b & 0 & b \end{bmatrix} = \begin{bmatrix} a+b & 0 & a+b \\ 0 & a+b & b \\ a+b & a & a+b \end{bmatrix}$$

$$aM_1 + bM_2 = 0 \Rightarrow a+b = 0, \quad b = 0, \quad a = 0.$$

Hence, there is a unique solution for a, b.

i.e. $\underline{a = b = 0}$.

10. **Question 8:** Which of the following options is(are) true?

- ○ Option 1: $\{M_1, M_2\}$ is a linearly dependent set.
- ○ **Option 2:** $\{M_1, M_2, M_3\}$ is a linearly dependent set.
- ○ **Option 3:** $\{M_1, M_2, M_4\}$ is a linearly dependent set.
- ○ **Option 4:** $\{M_1, M_3, M_4\}$ is a linearly dependent set.

Sol: $aM_1 + bM_2 = 0 \Rightarrow a = b = 0$ (we have shown this while verifying the last option of previous question)

Hence, $\{M_1, M_2\}$ is lin. ind.

$$2M_1 - M_2 - M_3 = 0 \longrightarrow (1)$$

Here, we have a linear combination of M_1, M_2 and M_3 which gives 0 and the coefficients are not all zero.

So, $\{M_1, M_2, M_3\}$ is a lin. dep. set.

$$2M_2 - M_1 - M_4 = 0 \longrightarrow (2)$$

By similar argument as above, $\{M_1, M_2, M_4\}$ is a lin. dep. set.

From (1) we get, $M_2 = 2M_1 - M_3$

putting M_2 in (2) we get,

$$2(2M_1 - M_3) - M_1 - M_4 = 0 \Rightarrow 3M_1 - 2M_3 - M_4 = 0$$

By similar argument as above, $\{M_1, M_3, M_4\}$ is a lin. dep. set.

4 Lecture 6.4

11. **Question 7:** Consider a set of vectors $S = \{(1,1,-1), (-1,1,1), (0,\frac{1}{2},0), (0,1,-2), (1,0,-2)\}$. Choose the set of correct options.

   - ○ Option 1: The singleton set $\{(0,1,-2)\}$ is linearly dependent.
   - ○ **Option 2:** If $\alpha, \beta \in S$ and α, β are distinct then $\{\alpha, \beta\}$ is a linearly independent set of vectors.
   - ○ **Option 3:** The set {(-1, 1, 1), (0, $\frac{1}{2}$, 0),(0, 1, -2)} is a linearly independent set of vectors.
   - ○ Option 4: The set S is a linearly independent set of vectors.
   - ○ Option 5: The set $\{\alpha, \beta, \gamma\}$ is a linearly dependent set of vectors for any $\alpha, \beta, \gamma \in S$, where all the three are distinct vectors.
   - ○ Option 6: The set $\{\alpha, \beta, \gamma, \delta\}$ is a linearly independent set of vectors for any $\alpha, \beta, \gamma, \delta \in S$, where all the four are distinct vectors.

Soln: Any non-zero singleton set is lin. ind. So the first option is not correct.

If $\{\alpha, \beta\}$ is lin. dep. then β should be scalar multiple of α.

But here no two vectors are scalar multiple of other.

Hence, $\{\alpha, \beta\}$ is lin. ind. for any two distinct $\alpha, \beta \in S$.

Let $a(-1,1,1) + b(0,\frac{1}{2},0) + c(0,1,-2) = 0$

$\Rightarrow (-a, \ a + \frac{1}{2}b + c, \ a - 2c) = 0$

$\Rightarrow a = 0, \ b = 0, \ c = 0$.

Hence, $\{(-1,1,1), (0,\frac{1}{2},0), (0,1,-2)\}$ is lin. ind.

$$(1,1,-1) + (-1,1,1) - 4(0,\tfrac{1}{2},0) + 0(0,1,-2) + 0(1,0,-2) = 0$$

Hence, S is lin. dep.

$\alpha = (-1,1,1), \ \beta = (0,\frac{1}{2},0), \ \gamma = (0,1,-2)$. We have already show that $\{\alpha,\beta,\gamma\}$ is lin. ind.

Hence for this choice of α, β, γ, the set $\{\alpha,\beta,\gamma\}$ is not lin. dep.

$\alpha = (1,1,-1), \ \beta = (-1,1,1), \ \gamma = (0,\frac{1}{2},0), \ \delta = (0,1,-2)$

$\{\alpha, \beta, \gamma, \delta\}$ is lin. dep. for this choice of $\alpha, \beta, \gamma, \delta$.

12. **Question 8:** Let S be the solution set of a system of homogeneous linear equations with 3 variables and 3 equations, whose matrix representation is as follows:

$$Ax = 0$$

where A is the 3×3 coefficient matrix and x denotes the column vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Choose the set of correct options.

- ○ **Option 1:** If v_1 and v_2 are in S, then any linear combination of v_1 and v_2 will also be in S.
- ○ **Option 2:** The set S will be a subspace of $\mathbb{R}^3$, with respect to usual addition and scalar multiplication as in $\mathbb{R}^3$.
- ○ **Option 3:** The set $\{v_1, v_2, v_1 - v_2\}$ is a linearly dependent subset in S.
- ○ **Option 4:** The set $\{v_1, v_2\}$ is a linearly independent subset in S if v_1 is not a scalar multiple of v_2.

Soln: Let $v_1, v_2 \in S$. Then $Av_1 = 0 = Av_2$.

Let $\alpha v_1 + \beta v_2$ be an arbitrary linear combination of v_1 and v_2.

$$A(\alpha v_1 + \beta v_2) = A(\alpha v_1) + B(\beta v_2)$$
$$= \alpha Av_1 + \beta Bv_2$$
$$= \alpha \cdot 0 + \beta \cdot 0 = 0.$$

Hence, $\alpha v_1 + \beta v_2$ is also a solution of $Ax = 0$.

So, option 1 is true.

From the previous argument we already have,

i) $v_1, v_2 \in S \Rightarrow v_1 + v_2 \in S$

ii) $v \in S \Rightarrow \alpha v \in S$ for any $\alpha \in \mathbb{R}$.

iii) $0 \in S$ as. $A0 = 0$.

Hence, S is a subspace of $\mathbb{R}^3$. So, option 2 is true.

$v_1 - v_2$ is a linear combination of v_1 and v_2.

Hence $\{v_1, v_2, v_1 - v_2\}$ is lin. dep. set.

So option 3 is true.

If v_1 is scalar multiple of v_2, then $\{v_1, v_2\}$ is lin. dep.

So, $\{v_1, v_2\}$ is lin. ind. if v_1 is not a scalar multiple of v_2.

Hence, option 4 is true.

5 Lecture 6.5

13. **Question 5:** If the set $\{(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)\}$ is given to be linearly dependent, then the following system of linear equations

$$
\begin{aligned}
a_1x + b_1y + c_1z &= 1 \\
a_2x + b_2y + c_2z &= 2 \\
a_3x + b_2y + c_3z &= 3
\end{aligned}
$$

has

- ◯ Option 1: either a unique solution or no solution.
- ◯ **Option 2:** either no solution or infinitely many solutions.
- ◯ Option 3: either infinitely many solutions or a unique solution.
- ◯ Option 4: a unique solution.

Soln: coefficient matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

$\det(A) = \det(A^T) = 0$

as $\{(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)\}$ is lin. dep.

Hence, $\underline{\text{Rank}(A) < 3}$.

So the system of linear equations has either no solution or infinitely many solution.

14. **Question 6:** Let the set $\{(a,b),(c,d)\}$ be a linearly independent subset of $\mathbb{R}^2$. Choose the set of correct options.

- ○ **Option 1:** $\{(a,b,0),(c,d,0)\}$ must be a linearly independent subset of $\mathbb{R}^3$.
- ○ **Option 2:** $\{(a,b,0),(c,d,0),(0,0,1)\}$ must be a linearly independent subset of $\mathbb{R}^3$.
- ○ Option 3: $\{(a,b,0),(c,d,0),(1,0,0)\}$ must be a linearly independent subset of $\mathbb{R}^3$.
- ○ Option 4: $\{(a,b,0),(c,d,0),(0,1,0)\}$ must be a linearly independent subset of $\mathbb{R}^3$.

Soln:

$$\alpha_1(a,b,0)+\alpha_2(c,d,0)=0$$

$$\Rightarrow (\alpha_1 a+\alpha_2 c,\ \alpha_1 b+\alpha_2 d,\ 0)=0$$

$$\Rightarrow \alpha_1 a+\alpha_2 c=0,\quad \alpha_1 b+\alpha_2 d=0$$

$$\Rightarrow \alpha_1(a,b)+\alpha_2(c,d)=0$$

As $\{(a,b),(c,d)\}$ is lin. ind., $\alpha_1=\alpha_2=0$.

So, $\{(a,b,0),(c,d,0)\}$ is lin. ind. subset of $\mathbb{R}^3$.

$$\alpha_1(a,b,0)+\alpha_2(c,d,0)+\alpha_3(0,0,1)=0$$

$$\Rightarrow (\alpha_1 a+\alpha_2 c,\ \alpha_1 b+\alpha_2 d,\ \alpha_3)=0$$

$$\Rightarrow \alpha_1 a+\alpha_2 c=0,\quad \alpha_1 b+\alpha_2 d=0,\quad \alpha_3=0$$

$$\Rightarrow \alpha_1(a,b)+\alpha_2(c,d)=0,\quad \alpha_3=0$$

$\Rightarrow \alpha_1=\alpha_2=0$. (as $\{(a,b),(c,d)\}$ is lin. ind)

$\alpha_3=0$

Hence, $\{(a,b,0),(c,d,0),(0,0,1)\}$ is lin. ind.

Let $(a,b)=(2,0)$, $(c,d)=(0,1)$

So, $\{(a,b),(c,d)\}$ is lin. ind.

$\{(a,b,0),(c,d,0),(1,0,0)\}=\{(2,0,0),(0,1,0),(1,0,0)\}$ is not lin. ind.

So, option 3 is not true.

Similar argument suggests that option 4 is not true.

6 Lecture 6.6

15. **Question 2:** Consider the following set of matrices: $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. What is $Span(S)$?

    ◯ Option 1: The vector space consisting of only lower triangular square matrices of order 2.

    ◯ **Option 2:** The vector space consisting of only upper triangular square matrices of order 2.

    ◯ Option 3: The vector space consisting of all the square matrices of order 2.

    ◯ Option 4: The vector space consisting of only scalar matrices of order 2.

Soln: An element of Span (S) is a finite linear combination of elements of S.

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

which is an upper triangular matrix.

So, option 2 is correct.

16. **Question 7:** Let V be the subspace of $\mathbb{R}^3$ defined as follows:

$$V = \{(x, y, z) \mid x = y - z, \text{ and } x, y, z \in \mathbb{R}\}$$

Choose the set of correct options from the following.

- ○ **Option 1:** $\{(1, 1, 0), (1, 0, -1)\}$ is a linearly independent set of V.
- ○ Option 2: $\{(1, 1, 0), (1, 0, -1), (0, 1, 1)\}$ is a linearly independent set of V.
- ○ **Option 3:** $\{(0, 1, 1), (1, 0, -1)\}$ is a spanning set of V.
- ○ Option 4: $\{(1, 1, 0)\}$ is a spanning set of V.

Soln: $(1,1,0), (1,0,-1) \in V$.

$a(1,1,0) + b(1,0,-1) = 0$

$\Rightarrow (a+b, a, -b) = 0 \Rightarrow a = 0 = b$.

Hence, $\{(1,1,0), (1,0,-1)\}$ is lin. ind.

Observe that,

$(1,1,0) - (1,0,-1) = (0,1,1)$

So, $\{(1,1,0), (1,0,-1), (0,1,1)\}$ is lin. dep.

Let $S = \{(0,1,1), (1,0,-1)\}$

$$a(0,1,1) + b(1,0,-1) = (\underset{\substack{\uparrow \\ x}}{b}, \underset{\substack{\uparrow \\ y}}{a}, \underset{\substack{\uparrow \\ z}}{a-b})$$

Hence $\text{Span}(S) = V$.

$z = y - x$

$\Rightarrow \underline{x = y - z}$.

$(1,0,-1) \in V$

But it can not be written as a linear combination of $(1,1,0)$. Because if $(1,0,-1)$ can be written as a linear combination of $(1,1,0)$, then we have, $(1,0,-1) = \alpha(1,1,0)$

$\Rightarrow (1,0,-1) = (\alpha, \alpha, 0)$

From this, we get, $0 = 1$ which is absurd.

Hence, $(1,0,-1) \notin \text{span}\{(1,1,0)\}$

So, $\{(1,1,0)\}$ is not a spanning set of V.

17. **Question 8:** Let V and W be vector spaces which are defined as follows:

$V = \{(x, y) \mid y = mx, \text{ where } m \neq 0 \text{ and } x, y, m \in \mathbb{R}\}$ with usual addition and scalar multiplication as in $\mathbb{R}^2$.
$W = \{(x, y) \mid x = 0\}$ with usual addition and scalar multiplication as in $\mathbb{R}^2$.

Choose the correct set of options.

- ○ Option 1: The set $\{(1, m), (\frac{1}{m}, 1)\}$ is a linearly independent set in V.
- ○ **Option 2:** The set $\{(1, m), (\frac{1}{m}, 1)\}$ is a spanning set for V.
- ○ **Option 3:** The set $\{(1, m)\}$ is a linearly independent set in V.
- ○ **Option 4:** The set $\{(\frac{1}{m}, 1)\}$ is a linearly independent set in V.
- ○ Option 5: The set $\{(0, 1), (0, 2)\}$ is a linearly independent set in W.
- ○ **Option 6:** The set $\{(0, 1)\}$ is a linearly independent set in W.
- ○ **Option 7:** The set $\{(0, 5)\}$ is a spanning set for W.

Soln. 1) $\frac{1}{m}(1, m) = (\frac{1}{m}, 1)$, so $\{(1, m), (1/m, 1)\}$ is lin. dep.

2) $x(1, m) = (x, mx)$

Hence, $V = \text{Span}\{(1, m)\}$

Now, $\{(1, m)\} \subseteq \{(1, m), (1/m, 1)\}$

so, $V = \text{Span}\{(1, m), (1/m, 1)\}$.

3) Any nonzero singleton set is lin. ind.

4) Same argument as 3.

5) $2(0, 1) = (0, 2)$, so $\{(0, 1), (0, 2)\}$ is lin. dep.

6 and 7) Same argument as 3.

7 Lecture 6.7

18. **Question 6:** Which of the following sets form a basis of the vector space of 2×2 lower triangular real matrices with usual matrix addition and scalar multiplication? (More than one option may be correct)

let the given vector space be V.

- ○ **Option 1:** $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} = S$
- ○ **Option 2:** $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = T$
- ○ Option 3: $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} = U$
- ○ **Option 4:** $\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} = W$

<u>Soln:</u> $a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = b = c = 0$

Hence, S is lin ind.

$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \rightarrow$ lower triangular matrix.

S spans V.

So, S is a basis of V.

$a \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+c & 0 \\ a+b & b+c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a+c = 0,\ a+b = 0,\ b+c = 0$

$\Rightarrow a = b = c = 0$

Hence, T is lin ind.

Any lower triangular matrix is of the form

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

Let us try to write it as a linear

combinations of matrices in T.

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x+z & 0 \\ x+y & y+z \end{pmatrix}$$

$x + z = a$, $x + y = b$, $y + z = c$

$x - y = a - c$

$x + y = b$

$x = \dfrac{a+b-c}{2}$,

$$y = b - \frac{a+b-c}{2}$$

$$= \frac{2b - a - b + c}{2}$$

$$= \frac{b - a + c}{2}$$

$$z = a - \frac{a+b-c}{2} = \frac{2a - a - b + c}{2}$$

$$= \frac{a - b + c}{2}$$

Hence,

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \frac{a+b-c}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{b-a+c}{2}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$+ \frac{a-b+c}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, T spans V.

So, T is a basis of V.

U is lin ind. but U cannot span V.

as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in V$ but it cannot be written as a linear combination of $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

So, U is not a basis of V.

By giving similar argument as we have given for S, we can conclude that W is a basis for V

19. **Question 7:** If $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$, then which of the following are true?

- ○ **Option 1:** $\{v_1, v_2, v_1 + v_3\}$ forms a basis of $\mathbb{R}^3$.
- ○ **Option 2:** $\{v_1, v_1 + v_2, v_1 + v_3\}$ forms a basis of $\mathbb{R}^3$.
- ○ **Option 3:** $\{v_1, v_1 + v_2, v_1 - v_3\}$ forms a basis of $\mathbb{R}^3$.
- ○ **Option 4:** $\{v_1, v_1 - v_2, v_1 - v_3\}$ forms a basis of $\mathbb{R}^3$.

<u>Soln:</u> 1) $a v_1 + b v_2 + c(v_1 + v_3) = 0$

$\Rightarrow (a+c) v_1 + b v_2 + c v_3 = 0$

$\Rightarrow a+c = 0, \; b = 0, \; c = 0$ as $\{v_1, v_2, v_3\}$ is lin ind.

$\Rightarrow a = b = c = 0$.

So, $\{v_1, v_2, v_1 + v_3\}$ is lin. ind.

$v_3 = (v_1 + v_3) - v_1$

Hence, $v_3 \in \text{span}\{v_1, v_2, v_1 + v_3\}$

Hence, $\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_2, v_1 + v_3\}$

So, $\{v_1, v_2, v_1 + v_3\}$ is a basis for $\mathbb{R}^3$.

2) $a v_1 + b(v_1 + v_2) + c(v_1 + v_3) = 0$

$\Rightarrow (a+b+c) v_1 + b v_2 + c v_3 = 0$

$\Rightarrow a+b+c = 0, \; b = 0, \; c = 0$

$\Rightarrow a = b = c = 0$

Hence, $\{v_1, v_1 + v_2, v_1 + v_3\}$ is lin ind.

Now, $v_2 = (v_1 + v_2) - v_1$

$v_3 = (v_1 + v_3) - v_1$

So, $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_1 + v_2, v_1 + v_3\}$

So, $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Give similar argument for options 3 an 4.

20. **Question 8:** Which of the following options is(are) true?

- ◯ **Option 1:** Any minimal spanning set of a vector space V must be a basis of V.
- ◯ Option 2: Any maximal spanning set of a vector space V must be a basis of V.
- ◯ Option 3: Any minimal linear independent set of vector space V must be a basis of V.
- ◯ **Option 4:** Any maximal linear independent set of vector space V must be a basis of V.
- ◯ Option 5: The basis of a vector space is unique.
- ◯ **Option 6:** The number of elements in a basis of $\mathbb{R}^3$ is 3.
- ◯ Option 7: The number of elements in a basis of $M_{3\times3}(\mathbb{R})$ is 3.
- ◯ **Option 8:** There are infinite number of bases of $\mathbb{R}^3$.
- ◯ **Option 9:** Any subset of a minimal spanning set of V cannot be a spanning set.

Soln:

Basis: a) Minimal spanning set of V

b) Maximal lin. ind. set of V.

(a) and (b) are two equivalent definitions of basis.

(*) Basis of ~~vector~~ space is not unique.

$\{(1,0),(0,1)\}$, $\{(1,0),(1,1)\}$ both are basis of $\mathbb{R}^2$.

(*) $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$.

So the number of elements of a basis of $\mathbb{R}^3$ is 3.

(*) An arbitrary element of the vector space $M_{3\times3}(\mathbb{R})$ looks like,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$ is a basis of $M_{3\times 3}(\mathbb{R})$.

It has 9 elements.

(*) $\{(\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)\}$ forms a basis of $\mathbb{R}^3$ if $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$.

α, β, γ can take any real numbers. So, there are infinite number of bases of $\mathbb{R}^3$.

(*) Minimal spanning set of V is defined as the set such that there does not exist any proper subset of it which spans V.

Week 7 Activity Questions (Selected Solutions):

Lecture 7.1

Question 4:

$V_1 = \{ A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a symmetric matrix} \}$

A general form of 2×2 symmetric matrix is

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ where, $a, b, c \in \mathbb{R}$.

Basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\dim(V_1) = 3$.

$V_2 = \{ A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a scalar matrix} \}$

A general form of 2×2 scalar matrix is

$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ where, $a \in \mathbb{R}$

Basis: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ $\dim(V_2) = 1$

$V_3 = \{ A \mid A \in M_{2\times 2}(\mathbb{R}) \text{ and } A \text{ is a diagonal matrix} \}$

A general form of 2×2 diagonal matrix is

$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a, b \in \mathbb{R}$.

Basis : $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ $\dim(V_3) = 2$.

$V_4 = \{ A \mid A \in M_{2\times2}(\mathbb{R})$ and A is an upper triangular matrix $\}$

A general form of 2x2 upper triangular matrix is

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $a, b, c \in \mathbb{R}$.

Basis : $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ $\dim(V_4) = 3$

$V_5 = \{ A \mid A \in M_{2\times2}(\mathbb{R})$ and A is lower triangular matrix $\}$

A general form of 2x2 lower triangular matrix is.

$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $a, b, c \in \mathbb{R}$.

Basis : $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ $\dim V_5 = 3$.

Question 6:

$V = \{ A \mid$ sum of entries in each row is 0, and $A \in M_{3\times 2}(\mathbb{R}) \}$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \;\middle|\; \begin{array}{c} a+b=0,\ c+d=0,\ e+f=0 \\ \text{where, } a,b,c,d,e,f \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} a & -a \\ c & -c \\ e & -e \end{pmatrix} \;\middle|\; a, c, e \in \mathbb{R} \right\}$$

Basis: $\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$ $\dim(V) = 3$

Question 7:

$V = \{ (x, y, z, \omega) \mid x+y = z+\omega,\ x+\omega = y+z,$ and $x, y, z, \omega \in \mathbb{R} \}$

$x = z + \omega - y$

$x = y + z - \omega$

$\Rightarrow z + \omega - y = y + z - \omega$

$\Rightarrow \omega - y = y - \omega \Rightarrow \omega = y.$

Hence, $\quad x = z + \omega - y = z + \omega - \omega = z$

So, $V = \{ (x, y, z, \omega) \mid x + y = z + \omega,\ x + \omega = y + z,$
$\text{and } x, y, z, \omega \in \mathbb{R} \}$

$= \{ (x, y, x, y) \mid x, y \in \mathbb{R} \}$

Basis: $\{ (1, 0, 1, 0), (0, 1, 0, 1) \}$

$\dim(V) = 2$.

Lecture 7.2

Question 6

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

If $f=0$, then the last row is 0.

So, the number linerly independent rows is at most 2.

Hence, the rank of the matrix must be less than or equal to 2.

Determinant of the given matrix is adf

If a, b, c, d, e, f all are non-zero, then

$$adf \neq 0.$$

Hence rank of the matrix is 3.

Similarly if a, d, f are nonzero, then also

$$adf \neq 0$$

Hence rank of the matrix is 3.

Question 7:

$$\begin{pmatrix} 2 & -3 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{pmatrix} \xrightarrow{R_1/2} \begin{pmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 1 & -3 & a \end{pmatrix}$$

$$\Big\downarrow R_3 - R_1$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & -3/2 & a-2 \end{pmatrix} \xleftarrow{R_1 + 3/2\, R_2} \begin{pmatrix} 1 & -3/2 & 2 \\ 0 & 1 & -2 \\ 0 & -3/2 & a-2 \end{pmatrix}$$

$$\Big\downarrow R_3 + 3/2\, R_2$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & a-5 \end{pmatrix}$$

Rank of this matrix will be 2 if

$$a - 5 = 0$$

$$\Rightarrow a = 5$$

Question 8 :

$A = (a_{ij})$ $\qquad\qquad a_{ij} = \min\{i, j\}$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\left.\begin{matrix} R_3 - R_2 \end{matrix}\right\} R_1 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, $\operatorname{rank}(A) = 3$.

Lecture 7.3

Question 6:

$$x_1 + x_2 + x_4 = 0$$
$$x_2 + x_3 = 0$$
$$x_1 - x_3 + x_4 = 0$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 - x_3 + x_4 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$$

The first two columns have the pivot elements.

So, x_3 and x_4 are independent variable.

$$x_2 = -x_3$$

$$x_1 = x_3 - x_4$$

Let $x_3 = t_1$ and $x_4 = t_2$, then

$x_1 = t_1 - t_2$, $x_2 = -t_1$, $x_3 = t_1$, $x_4 = t_2$.

So, null space $(A) = \{(t_1 - t_2, -t_1, t_1, t_2) \mid t_1, t_2 \in \mathbb{R}\}$

Question 7:

$A = (a_{ij})$ $\qquad a_{ij} = \min\{i, j\}$

We have already calculated that,

$$\text{rank}(A) = 3$$

Rank-Nullity theorem states that,

$$\text{rank}(A) + \text{nullity}(A) = 3$$

$$\Rightarrow 3 + \text{nullity}(A) = 3$$

$$\Rightarrow \text{nullity}(A) = 0 .$$

—

Question 8:

General form of non-zero scalar matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{where } a \in \mathbb{R} \setminus \{0\}$$

$\text{rank}(A) = 3$

$\Rightarrow \text{nullity}(A) = 0$

General form of non-zero diagonal matrix

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$a, b, c \in \mathbb{R}$.

where atleast one of a, b, c is non-zero.

If all of them are non-zero, then

$\text{rank}(B) = 3 \Rightarrow \text{nullity}(B) = 0$

If one of them is zero, then

$\text{rank}(B) = 2 \Rightarrow \text{nullity}(B) = 1$

If two of them is zero, then

$\text{rank}(B) = 1 \Rightarrow \text{nullity}(B) = 2$

Now all of them cannot be zero.

So, nullity of B can be atmost 2.

AQ 7.4

Rank-nullity-dimension theorem

Let $T: V \to W$ (V and W are finite dimensional vector spaces) be a linear transformation. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

2. $Ax = 0$ always has $x = 0$ as a solution.

If A is invertible, then $Ax = 0 \Rightarrow A^{-1}(Ax) = A^{-1}0 = 0$
$\Rightarrow x = 0$ (unique soln).

If $\text{rank}(A) < m, n$ (A is an $m \times n$ matrix), then $[A|0]$ will have rank $= \text{rank}(A) < m, n$

$$[A|0] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right].$$

Thus, in this case there are infinitely many solutions.

If $x \neq 0$ is a soln. of $Ax = 0$, then αx is a soln. of $Ax = 0$ for all $\alpha \in \mathbb{R}$.

3. If A is invertible, it is 1-1 and onto.

$\Rightarrow \text{nullity}(A) = 0$ (hence $\text{rank}(A) = n$ (A-$n \times n$ matrix)).

If $\text{nullity}(A) \neq 0$, then A is not 1-1. Since A is a square matrix, A is not invertible.

4. <u>Option 1</u>: No. of linearly independent rows of A and $-A$ are same. ($\{a, b\}$ is linearly independent iff $\{-a, -b\}$ are linearly independent).

So $\text{rank}(A) = \text{rank}(-A) \Rightarrow \text{nullity}(A) = \text{nullity}(-A)$.

<u>Option 2</u>: A = Identity matrix, $B = -I$

$A + B = 0 \Rightarrow \text{nullity}(A+B) = n$.

But $\text{nullity}(A) = \text{nullity}(B) = 0$

<u>Option 3</u>: $\text{nullity}(0) = n$. $(\because \underset{\text{no lin ind rows}}{\text{rank}(0) = 0})$

5. $\text{rank}(A) \leq \min\{m, n\}$ if A is an $m \times n$ matrix.

Recall $\text{rank}(A)$ = number of linearly independent rows (columns) of A.

<u>7.7</u>

1. T is 1-1 $\Rightarrow$ $T(x) = T(y)$ implies $x = y$.

So $T(v) = 0 \; (= T(0)) \Rightarrow v = 0$.

<u>Option 2</u>: $T = I_{2\times 2}$ is an injective transformation that is also surjective (any invertible transformation will work)

<u>Option 3</u>: $T: \mathbb{R} \to \mathbb{R}$ is not injective. Then $T(v) = 0$ for some $v \neq 0$. But then $T(\alpha v) \overset{=\alpha T(v)}{=} 0 \;\forall\, \alpha \in \mathbb{R}$. Since $\mathbb{R}$ is one-dimensional, $\{v\}$ is a basis for $\mathbb{R}$.

$\therefore T(x) = 0 \;\forall\, x \in \mathbb{R}$.

<u>Option 4</u>: $\exists\, v \neq 0$ such that $T(v) = 0$. Thus $v \in \ker(T)$ ~~(Thus nullity (T) ≥ 1)~~ Thus T is not 1-1. So T cannot be an isomorphism.

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (x, 0)$.

$T(0,1) = (0,0)$. Thus $\exists\, v \neq 0$ s.t. $T(v) = 0$.

$\therefore$ T is not 1-1.

$\Rightarrow$ nullity $(T) \geq 1 \Rightarrow$ rank$(T) \leq 1$ ($\because$ rank(T) + nullity$(T) = 2$)

$\Rightarrow$ T is not onto (for T to be onto, rank$(T) = 2$)

6. $S : V_1 \longrightarrow V_2$, $T : V_2 \longrightarrow V_3$

$T \circ S(v) = T(S(v))$.

Let $V_1 = V_3 = \mathbb{R}$ and $V_2 = \mathbb{R}^2$.

$T(x, y) = x$ (not injective : $T(0,1) = 0$)

$S(x) = (x, 0)$ (not surjective : no pre-image for $(0,1)$)

$T \circ S(x) = x$ (Identity map on $\mathbb{R}$).

If $T \circ S$ is injective, then S must be injective.

Suppose $Sx = 0$

$\Rightarrow T(Sx) = 0$

$\Rightarrow (T \circ S)(x) = 0 \Rightarrow x = 0$ ($\because$ $T \circ S$ is injective)

$\therefore$ S is injective.

If $T \circ S$ is surjective, then T must be surjective.

Let $x \in$ Co-domain of $T = \mathbb{R}$.

Since $T \circ S : \mathbb{R} \to \mathbb{R}$ is surjective, $\exists\, v \in \mathbb{R}$ s.t. $T(S(v)) = x$.

$(T \circ S)(v) = x$ ie., $T(S(v)) = x$

$\therefore \exists\, Sv \in \underbrace{\mathbb{R}^2}_{\text{domain of } T}$ s.t. $T(Sv) = x$.

We have found a pre-image for x under T.

$\therefore$ T is surjective.

LECTURE 8.5

3. Affine subspace of $\mathbb{R}^3$ can be
$\{(0,0,0)\}$, $\{(x,y,z) : ax+by+cz = d\}$,
$\{(x,y,z) : \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = d\}$ or $\mathbb{R}^3$.

Note option 1:
$\{(x,y,z) : x^2+y^2+z^2=0\}$ is nothing but $\{(0,0,0)\}$.

6. From $x+z=0$, we get $x=-z$.
$\therefore$ $x-y+z=-1$ gives $y=1$.
$\therefore$ Solutions of the system are of the form $(x,1,-x)$.
or $\{(-t,1,t) : t \in \mathbb{R}\}$.

LECTURE 8.4

6. Suppose $\exists P$ s.t. $A = P^{-1} E P$

$\Rightarrow PAP^{-1} = E$

$\Rightarrow P(3I)P^{-1} = E$

$\Rightarrow 3(PIP^{-1}) = E$

$\Rightarrow 3I = E$

$\Rightarrow A = E$.

7. $A = P^{-1}BP \Rightarrow A^{-1} = (P^{-1}BP)^{-1} = P^{-1}B^{-1}P$

$\Rightarrow A^{-1}$ & B^{-1} are similar

$A = P^{-1}BP \Rightarrow A^2 = (P^{-1}BP)^2 = (P^{-1}BP)(P^{-1}BP)$

$= P^{-1}B^2P$

$\Rightarrow A^2$ & B^2 are similar

<u>Option 3</u>: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Check A & B are <u>not</u> similar but A^2 & B^2 are similar

④ $A = P^{-1}BP \Rightarrow A^T = (P^{-1}BP)^T = P^T B^T (P^{-1})^T$

$= P^T B^T (P^T)^{-1}$

$\Rightarrow A^T$ & B^T are similar

8. $T : M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$

$T(A) = PA$.

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$$

$$= a\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$$

$$= b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

$$= b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ matrix of T

$$= \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

LECTURE 8.3

6 rank (T) + nullity (T) = dim (domain)

Option 1: T is injective $\Rightarrow$ nullity $= 0$

$\Rightarrow$ rank (T) = dim (domain)

= dim (codomain) [given]

$\Rightarrow$ T is onto. and hence is an isomorphism.

Option 2: similar to Option 1

Option 3: dim $(\mathbb{R}^2) = 2$ (dim of domain)

dim (codomain) = 3.

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

rank (T) + nullity (T) = 2

$\Rightarrow$ rank $(T) < 2$

$\Rightarrow$ T cannot be onto.

Option 4: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that

$T(x, y) = (x, x, y)$

is an example of an injective transformati

7. $V = \{(x, y, z) : x = y - z\}$

$W = \{(x, y, z) : x = y\}$.

$T : V \to W$. so $T(x, y, z)$ should be of the form $(x, x, *)$ or $(y, y, *)$

$*$ can be anything.

Option 3 is thus ruled out.

Option 1 : $\ker(T) = \{(x, y, z) : y = 0\}$.

$T(1, 0, -1) = (0, 0, 0)$

$\Rightarrow T$ is not one-one.

Option 2 : $\ker(T) = \{(x, y, z) \in V : y = z = 0\}$

Since $(x, y, z) \in V$, $x = y - z$.

But for $(x, y, z) \in \ker(T)$, $y = z = 0$ and so $x = 0$. Thus T is 1-1.

$\dim(V) = 2$, $\dim W = 2$.

$\dim(\ker(T)) = 0$ ($\because$ T is 1-1)

$\Rightarrow \dim(\mathrm{Im}(T)) = 2 = \dim(W)$

$\therefore$ T is onto.

LECTURE 8.2

3. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T(x, y, z) = (2x + 3z, 4y + z)$

$\ker(T) = \{(x, y, z) \in \mathbb{R}^3 : T(x, y, z) = 0\}$

$= \{(x, y, z) : 2x + 3z = 0, 4y + z = 0\}$

$= \{(x, y, z) : 2x = -3z, 4y = -z\}$

$= \{(x, y, z) : 2x = -3z = 12y\}$

$= \{(x, y, z) : \frac{-2}{3}x = -4y = z\}$.

To get a basis for $\ker(T)$, check options that satisfy $\frac{-2}{3}x = -4y = z$.

4. $\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^3) = 3$

Since $\text{nullity}(T) = 1$, $\text{rank}(T) = 2$.

4. From (3), we know $T(x,y,z) = (y,z)$.

First, let us check if T is one-one.

ie., $T(x,y,z) = 0 \Rightarrow (x,y,z) = 0$.

$\cancel{T_1(x_1,y_1)}$

$$0 = T(x,y,z) = (y,z)$$

$$\Rightarrow y = 0,\ z = 0.$$

Since $T : W \longrightarrow \mathbb{R}^2$, we have $x = 2y + z$.

But $y = z = 0$. Thus $x = 0$.

$$\therefore T(x,y,z) = 0 \Rightarrow (x,y,z) = 0.$$

$\therefore$ T is one-one.

Now, let us check if T is onto.

ie., for $(x,y) \in \mathbb{R}^2$, $\exists\ (\alpha,\beta,\gamma) \in W$ s.t. $T(\alpha,\beta,\gamma) = (x,y)$.

Let $(x,y) \in \mathbb{R}^2$.

Then $(2x+y, x, y) \in W$ such that

$T(2x+y, x, y) = (x,y)$.

$\therefore$ T is onto.

Thus T is an isomorphism ($\because$ T is onto and one-one).

MATHEMATICS FOR DATA SCIENCE - 2

WEEK 8 - ACTIVITY QUESTIONS - SOLUTIONS

LECTURE 8.1

2. Let A be the matrix of $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with respect to the standard basis.

Then, $Tx = Ax, \quad \forall\, x \in \mathbb{R}^2$

ie., $T(x,y) = A(x,y) \;, \quad (x,y) \in \mathbb{R}^2$.

Here $A(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$.

Now, given $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

$$\Rightarrow T(1,0) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (2,1)$$

$$T(0,1) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (3,4)$$

$$\begin{aligned} \therefore T(x,y) &= T\big(x(1,0) + y(0,1)\big) \\ &= x\,T(1,0) + y\,T(0,1) \\ &= x(2,1) + y(3,4) \\ &= (2x+3y,\; x+4y) \end{aligned}$$

Maths 2 : Activity Questions
Week-9

Contents

1 AQ 9.1

1. **Q4:** If $a = (6, -1, 3), b = (4, c, 2)$ where $a, b \in \mathbb{R}^3$ and a and b are perpendicular to each other, then find the value of c? (Answer: 30)

<u>Solution</u>: Since a & b are perpendicular to each other

So, $\langle a, b \rangle = 0$

$\Rightarrow \langle (6,-1,3), (4,c,2) \rangle = 0$

$\Rightarrow 24 - c + 6 = 0$

$\Rightarrow c = 30.$

2. **Q5:** Consider two vectors $a = (1, 2), b = (2, 2)$ and θ is the angle between them. The sum of the two vectors is given by $c = a + b$. Choose the correct options.

   - ◯ **Option 1:** Length/Norm of c is 5.
   - ◯ Option 2: Length/Norm of c is 25.
   - ◯ Option 3: Length/Norm of c is 4.
   - ◯ **Option 4:** $\cos\theta = \frac{3}{\sqrt{10}}$.
   - ◯ Option 5: $\cos\theta = \frac{3}{\sqrt{5}}$.

Solution: $c = a + b = (1,2) + (2,2) = (3,4)$

Now, $\|c\| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Since θ is angle between a & b

So $\cos\theta = \dfrac{a \cdot b}{\|a\| \, \|b\|} = \dfrac{(1,2)\cdot(2,2)}{\sqrt{1+4}\,\sqrt{4+4}} = \dfrac{6}{\sqrt{5}\cdot 2\sqrt{2}}$

$\Rightarrow \cos\theta = \dfrac{3}{\sqrt{10}}$

2 AQ 9.2

3. **Q1:** Consider a function $f : V \times V \to \mathbb{R}$ where $V \subseteq \mathbb{R}^2$ defined by $f(v, w) = 2v_1w_1 + 5v_2w_2$, where $v = (v_1, v_2), w = (w_1, w_2)$. Choose the set of correct options.

   ◯ **Option 1:** f satisfies the symmetry condition of the inner product.

   ◯ **Option 2:** f satisfies the bilinearity condition of the inner product.

   ◯ **Option 3:** f satisfies the positivity condition of the inner product.

   ◯ **Option 4:** f is an inner product.

   ◯ Option 5: f is not an inner product.

Solution:- Given $f(v,w) = 2v_1w_1 + 5v_2w_2$, where $v = (v_1, v_2)$
$w = (w_1, w_2)$

Option 1: Symmetry Condition, i.e $f(v,w) = f(w,v)$, $v, w \in V$

Consider $f(w,v) = 2w_1v_1 + 5w_2v_2$
$= 2v_1w_1 + 5v_2w_2$ (Since; $v_1, v_2, w_1, w_2 \in \mathbb{R}$ & $\mathbb{R}$ has commutative property).
$= f(v,w)$

Bilinearity condition, i.e $f(u+v, w) = f(u,w) + f(v,w)$
& $f(cv, w) = cf(v,w)$
$\forall\ u, v, w \in V$, $c \in \mathbb{R}$

let $u = (u_1, u_2)$

Consider $f(u+v, w) = f((u_1+v_1, u_2+v_2), (w_1, w_2))$
$= 2(u_1+v_1)w_1 + 5(u_2+v_2)w_2$
$= 2u_1w_1 + 5u_1w_2 + 2v_1w_1 + 5v_2w_2$
$= f(u,w) + f(v,w)$

Similarly, we can prove $f(cv, w) = cf(v,w)$, $c \in \mathbb{R}$.

Positivity condition, i.e $f(v,v) \geq 0$, $v \in V$

observe $f(v,v) = 2v_1^2 + 5v_2^2 \geq 0$
& $f(v,v) = 0 \Rightarrow 2v_1^2 + 5v_2^2 = 0 \Rightarrow v_1 = 0, v_2 = 0$
$\Rightarrow v = 0$

So, $f(v,v) > 0$ & $f(v,v) = 0 \Leftrightarrow v = 0$

Hence $f : V \times V \to \mathbb{R}$ is an inner product space.

4. **Q5:** Consider a vector $a = (a_1, b_1, c_1)$ in $\mathbb{R}^3$. Which of these is (are) possible candidates for a norm?

- ○ **Option 1:** $\sqrt{a_1^2 + b_1^2 + c_1^2}$
- ○ Option 2: $a_1 - b_1$
- ○ Option 3: $a_1 + b_1 + c_1$
- ○ Option 4: $max(a_1, b_1, c_1)$
- ○ Option 5: $min(a_1, b_1, c_1)$

Solution :. $a \in \mathbb{R}^3$, where $a = (a_1, b_1, c_1)$

Option 1: $\|a\| = \sqrt{a_1^2 + b_1^2 + c_1^2}$

This is a length function & length function induces a norm (from video lecture)

Hence option 1 is true. (We can check all three condition of a norm & find that $\sqrt{a_1^2 + b_1^2 + c_1^2}$ induces norm.)

Option 2 let $a = (0, 0, -1)$

Observe that $a_1 - b_1 = 0$ but $a \neq 0$

this fails the condition $\|a\| = 0 \Leftrightarrow a = 0$

Option 3: let $a = (0, 0, -1)$

observe that $a_1 + b_1 + c_1 = -1 < 0$

this fails the condition $\|a\| \geq 0$

Option 4: Let $a = (0, 0, -1)$

$\max\{a_1, b_1, c_1\} = 0$ but $a \neq 0$

Same as option 2.

Option 5: let $a = (0, 0, -1)$

$\min\{a_1, b_1, c_1\} = \min\{0, 0, -1\} = -1$, but $a \neq 0$

this same as option 3.

3 AQ 9.3

5. **Q2:** Choose the set of correct statements.

   - ◯ Option 1: In an orthogonal set, the norms of all the vectors are equal.
   - ◯ **Option 2:** In an orthogonal set, the vectors are linearly independent.
   - ◯ Option 3: In an orthogonal set, the vectors are linearly dependent.
   - ◯ **Option 4:** If the columns of an $n \times n$ coefficient matrix A comprises the individual vectors of an orthogonal set in $\mathbb{R}^n$, then there must be a unique solution to the system $AX = b$, where X, b are $n \times 1$ vectors.
   - ◯ Option 5: If the columns of an $n \times n$ coefficient matrix A comprises the individual vectors of an orthogonal set in $\mathbb{R}^n$, then there are no solutions to the system $AX = b$, where X, b are $n \times 1$ vectors.
   - ◯ Option 6: The determinant of a square matrix formed by a set of orthogonal vectors in $\mathbb{R}^n$ is zero.
   - ◯ **Option 7:** A set of n vectors can never form an orthogonal basis in $\mathbb{R}^{n-1}$.

Solution :- Option 1: Consider two vectors $(1,1)$ & $(2,-2)$ from $\mathbb{R}^2$ w.r.t. dot product.

Observe $\langle (1,1),(2,-2)\rangle = 0$. but $(1,1)$ & $(2,-2)$ have different norm which are $\sqrt{2}$ & 2 resp.

Option 2 :- Let $\{a_1, a_2 \cdots a_k\}$ are orthogonal set, where $a_i \neq 0$ $\forall i$

i.e $\langle a_i, a_j\rangle = 0$ when $i \neq j$

To show linearly independent:

Let $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_k a_k = 0$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.

Consider $\langle \alpha_1 a_1 + \cdots + \alpha_k a_k, a_1\rangle = \langle 0, a_1\rangle = 0$

$\Rightarrow \alpha_1 \langle a_1, a_1\rangle + \cdots + \alpha_k \langle a_k, a_1\rangle = 0$

$\Rightarrow \alpha_1 \|a_1\|^2 + 0 + \cdots + 0 = 0$

$\Rightarrow \alpha_1 = 0$ as $a_1 \neq 0$

Similarly we can show $\alpha_i = 0$ $\forall i = 1, 2, \cdots k$

Hence $\{a_1, \cdots, a_k\}$ is linearly independent set.

Option 4: Consider a system of linear equation $An = b$ where the column of A are orthogonal.

Lets assume for $\mathbb{R}^2$ & $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

where $\langle (a_{11}, a_{21}), (a_{12}, a_{22}) \rangle = 0$

Now Consider $AA^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{12}a_{22} \\ a_{11}a_{21} + a_{12}a_{22} & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}^2 & 0 \\ 0 & a_{21}^2 + a_{22}^2 \end{bmatrix},$$ Actually diagonal matrix

Here A is not a zero matrix.

So $AA^T \neq 0 \Rightarrow A$ is invertible

So $An = b \Rightarrow x = A^{-1}b$. So $An = b$ has unique solution.

Similarly, we can prove for $n = 3, \cdots$ (finite)

Hence, option 4 is true.

Option 6: Refer to option 4. Observe that A is invertible.

Option 7: Observe that set of n vector are Linearly dependent in $\mathbb{R}^{n-1}$

6. **Q6:** Consider the system of linear equations:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 &= 5 \\ 3x_1 - 6x_2 - 5x_3 &= 9. \end{aligned}$$

Choose the correct option.

- ◯ **Option 1:** The system has a unique solution.
- ◯ Option 2: The system has no solution.
- ◯ Option 3: The system can have infinitely many solutions.
- ◯ Option 4: None of the above.

Solution: Matrix representation of the System of linear equation is $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$

Observe that columns of coefficient matrix are Orthogonal. And we have seen Question 2 that when a system has such property then system has a unique Solution.

So, above system has a unique solution.

4 AQ 9.4:

7. **Q6:** Choose the correct option(s).
   - ◯ **Option 1:** The vectors in an orthonormal set are linearly independent.
   - ◯ Option 2: A set of linearly dependent vectors can be orthonormal.
   - ◯ Option 3: A set of linearly independent vectors is always orthonormal.
   - ◯ Option 4: A set of linearly independent vectors is always orthogonal but not orthonormal.

Solution: Since orthonormal veetor are orthogonal veetor also.

Refer to AQ 9.3, Question 2, that orthogonal vectors are linearly independent.

Consider the inner product $\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$ where $u = (u_1, u_2), v = (v_1, v_2)$ are vectors in $\mathbb{R}^2$. Answer questions 7 and 8 based on this information.

8. **Q7:** Which of the following is an orthonormal basis with respect to the inner product defined above?

   - ◯ Option 1: $\{(2, 3), (-4, 4)\}$
   - ◯ **Option 2:** $\{\frac{1}{\sqrt{30}}(2, 3), \frac{1}{\sqrt{80}}(-4, 4)\}$
   - ◯ Option 3: $\{\frac{1}{\sqrt{13}}(2, 3), \frac{1}{\sqrt{32}}(-4, 4)\}$
   - ◯ Option 4: $\{(2, 3), (-3, 2)\}$
   - ◯ Option 5: $\{\frac{1}{\sqrt{13}}(2, 3), \frac{1}{\sqrt{13}}(-3, 2)\}$

<u>Solution</u>: <u>Option 1</u>: $\langle (2,3), (-4,4)\rangle = 3.2.(-4) + 2.3.4$

$= 0$

but $\|(2,3)\| = \sqrt{\langle (2,3),(2,3)\rangle} = \sqrt{3.2^2 + 2.3^2}$

$= \sqrt{30} \neq 1$

Similarly $\|(-4,4)\| = \sqrt{80} \neq 1$

So $\{(2,3), (-4,4)\}$ is orthogonal basis but not orthonormal basis.

<u>Option 2</u>: Refer to option 1.

$\{\frac{1}{\sqrt{30}}(2,3), \frac{1}{\sqrt{80}}(4,-4)\}$ is an orthonormal basis.

Option 3: $\langle (2,3), (-3,2) \rangle = 3 \cdot 2 \cdot (-3) + 2 \cdot 3 \cdot 2$

$\neq 0$

So $(2,3)$, $(-3,2)$ are not orthgonal.

Option 4: Refer to option 3.

9. **Q8:** Use the orthonormal basis $\{u, v\}$ obtained in question 7 with respect to the defined inner product. Express the vector $(4, 0)$ as a linear combination of the basis vectors u and v, as $(4, 0) = c_1 u + c_2 v$. Which of the following gives the coefficients of the linear combination?

   - ◯ **Option 1:** $c_1 = \frac{24}{\sqrt{30}}, c_2 = \frac{-48}{\sqrt{80}}$
   - ◯ Option 2: $c_1 = \frac{24}{\sqrt{13}}, c_2 = \frac{-48}{\sqrt{32}}$
   - ◯ Option 3: $c_1 = \frac{8}{\sqrt{13}}, c_2 = \frac{-16}{\sqrt{32}}$
   - ◯ Option 4: $c_1 = \frac{24}{\sqrt{30}}, c_2 = \frac{48}{\sqrt{80}}$

Solution :. Let $(4,0) = c_1 u + c_2 v$ for orthonormal basis

$$\left\{ \frac{1}{\sqrt{30}}(2,3), \frac{1}{\sqrt{80}}(-4,4) \right\}$$

Observe that

$$\langle (4,0), u \rangle = \langle c_1 u + c_2 v, u \rangle$$

$$= c_1$$

$$\Rightarrow c_1 = \langle (4,0), \tfrac{1}{\sqrt{30}}(2,3) \rangle = \frac{24}{\sqrt{30}}$$

Similarly

$$c_2 = \langle (4,0), v \rangle$$

$$= \langle (4,0), \tfrac{1}{\sqrt{80}}(-4,4) \rangle$$

$$= \frac{-48}{\sqrt{80}}$$

5 AQ 9.5

10. **Q3:** Consider an orthonormal basis $\{(1,0,0),(0,1,0)\}$ for a subspace W in $\mathbb{R}^3$. If $x = (1,2,3)$ is a vector in $\mathbb{R}^3$, then which of the following represents a vector in W whose distance from x is the least? Consider dot product as the standard inner product.

   - ◯ Option 1: $(2,4,0)$
   - ◯ Option 2: $(3,4,0)$
   - ◯ Option 3: $(4,5,0)$
   - ◯ **Option 4:** $(1,2,0)$

Solution: Observe that distance between projection vector v' of a vector v on a subspace U and v, is the smallest distance between the vector v & the subspace U.

Now, projection of v on a subspace U which having Orthonormal basis $\{u_1, u_2, \dots u_k\}$ is $= \sum_{i=1}^{k} \langle v, u_i \rangle u_i$.

So projection of $(1,2,3)$ on the subspace which having basis $\{(1,0,0),(0,1,0)\}$

is $= \langle (1,2,3),(1,0,0) \rangle (1,0,0) + \langle (1,2,3),(0,1,0) \rangle (0,1,0)$

$= 1(1,0,0) + 2(0,1,0)$

$= (1,2,0)$.

11. **Q6:** Consider an orthogonal basis $\{(1,2,1),(-2,0,2)\}$ of a subspace W, of the inner product space $\mathbb{R}^3$ with respect to the dot product. If $y=(1,2,3)\in\mathbb{R}^3$, then find $Proj_W(y)$.

   - ◯ **Option 1:** $(\frac{1}{3},\frac{8}{3},\frac{7}{3})$
   - ◯ Option 2: $(\frac{18}{\sqrt{6}}-\frac{32}{\sqrt{8}},\frac{36}{\sqrt{8}},\frac{18}{\sqrt{6}}+\frac{32}{\sqrt{8}})$
   - ◯ Option 3: $(\frac{1}{3},-\frac{8}{3},\frac{7}{3})$
   - ◯ Option 4: $(\frac{18}{\sqrt{6}}-\frac{32}{\sqrt{8}},\frac{36}{\sqrt{8}},-\frac{18}{\sqrt{6}}+\frac{32}{\sqrt{8}})$

Solution:

$$Proj_W(y) = \left\langle (1,2,3), \frac{(1,2,1)}{\sqrt{6}} \right\rangle \frac{(1,2,1)}{\sqrt{6}} + \left\langle (1,2,3), \frac{(-2,0,2)}{2\sqrt{2}} \right\rangle \frac{(-2,0,2)}{2\sqrt{2}}$$

$$= \frac{(1+4+3)(1,2,1)}{6} + \frac{1}{2}(-1+3)(-1,0,1)$$

$$= \frac{4}{3}(1,2,1) + (-1,0,1)$$

$$= \frac{1}{3}(4-3,\ 8,\ 4+3)$$

$$= \frac{1}{3}(1,8,7)$$

6 AQ 9.6

12. **Q3:** Let $v_1 = (1,0,1,1)$ and $v_2 = (0,1,1,1)$ be the vectors from the inner product space $\mathbb{R}^4$ with respect to the dot product. If $v_3 = v_2 + av_1$ where $a \in \mathbb{R}$ and v_1, v_3 are orthogonal, then

   ◯ **Option 1:** $a = -2/3$

   ◯ Option 2: $a = 2/3$

   ◯ Option 3: $a = 1/3$

   ◯ Option 4: $a = -1/3$

Solution :-

$$v_3 = v_2 + a v_1$$
$$= (0,1,1,1) + a(1,0,1,1)$$

Now v_1 & v_3 are orthogonal

so $\langle v_3, v_1 \rangle = 0$

$$\Rightarrow \langle (0,1,1,1) + a(1,0,1,1), (1,0,1,1) \rangle = 0$$

$$\Rightarrow \langle (0,1,1,1), (1,0,1,1) \rangle + a \langle (1,0,1,1), (1,0,1,1) \rangle = 0$$

$$\Rightarrow 2 + 3a = 0$$

$$\Rightarrow a = -2/3$$

13. **Q5:** Let $a = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ be a vector from the inner product space $\mathbb{R}^3$ with respect to dot product and $W = \{(x, y, z) \in \mathbb{R}^3 \mid \langle (x, y, z), (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}) \rangle = 0\}$ be a subspace of $\mathbb{R}^3$. Then which of the following is (are) a basis of W?

- ○ Option 1: $\{(1, 0, 2), (0, 1, 2)\}$
- ○ **Option 2:** $\{(1, 0, -2), (0, 1, -2)\}$
- ○ **Option 3:** $\{(-1, 0, 2), (0, -1, 2)\}$
- ○ Option 4: $\{(1, 0, 2), (0, 1, -2)\}$

Solution:

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid \langle (x, y, z), (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}) \rangle = 0 \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 \mid 2x + 2y + z = 0 \}$$

Now $2x + 2y + z = 0$

$x = -y - z/2$

let $y = 0, z = -2 \Rightarrow x = -1$

and $y = 1, z = -2 \Rightarrow x = 0$

So basis of $W = \{ (1, 0, -2), (0, 1, -2) \}$

or basis of $W = \{ -1(1, 0, -2), -1(0, 1, -2) \}$

Or basis of $W = \{ (-1, 0, 2), (0, -1, 2) \}$

7 AQ 9.7

14. **Q1:** Choose the correct options.

- ○ Option 1: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, where $\mathbb{R}^2$ is the inner product space with respect to the dot product. Then $Tu \cdot Tv = u \cdot v$.
- ○ **Option 2:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal linear transformation, where $\mathbb{R}^2$ is the inner product space with respect to the dot product. Then $Tu \cdot Tv = u \cdot v$.
- ○ Option 3: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, where $\mathbb{R}^2$ is the inner product space with respect to the inner product given by $< a, b >= 2a_1b_1 + 5a_2b_2$. Then $\langle Tu, Tv\rangle = \langle u, v\rangle$.
- ○ **Option 4:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal linear transformation, where $\mathbb{R}^2$ is the inner product space with respect to the inner product given by $< a, b >= 2a_1b_1 + 5a_2b_2$. Then $\langle Tu, Tv\rangle = \langle u, v\rangle$.

Solution:- Option 1: let $T : \mathbb{R}^2 \to \mathbb{R}^2 \ni T(x,y) = (2x+y, x-y)$

Again let $u = (1,0)$, $v = (0,1)$

$T(u) = T(1,0) = (2,1)$,

$T(v) = T(0,1) = (1,-1)$

$Tu \cdot Tv = (2,1)\cdot(1,-1) = 2-1 = 1 \neq 0 = (1,0)\cdot(0,1)$

Option 2:- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal transformation

then $\langle Tu, Tv\rangle = \langle u, v\rangle$, where $u, v \in \mathbb{R}^2$

But here inner product is dot product

So $\langle Tu, Tv\rangle = Tu \cdot Tv$

& $\langle u, v\rangle = u \cdot v$

So $Tu \cdot Tv = u \cdot v$

Obtion3 ! Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ and u, v are as in option 1.

then $\langle Tu, Tv \rangle = \langle (2,1), (1,-1) \rangle$

$= 2 \cdot 2 \cdot 1 + 5 \cdot 1 \cdot (-1)$

$= 4 - 5$

$\langle Tu, Tv \rangle = -1$

for $\langle u, v \rangle = \langle (1,0), (0,1) \rangle$

$= 2 \cdot 1 \cdot 0 + 5 \cdot 0 \cdot 1$

$\langle u, v \rangle = 0$

So, $\langle Tu, Tv \rangle \neq \langle u, v \rangle$

Option 4 :- Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ an be orthogonal transformation w.r.t inner product $\langle (a_1, a_2), (b_1, b_2) \rangle = 2a_1b_1 + 5a_2b_2$

From the defination of orthogonal transformation

$\Rightarrow \langle Tu, Tv \rangle = \langle u, v \rangle \quad \forall \; u, v \in \mathbb{R}^2$

w.r.t orthonormal basis,

15. **Q2:** If A is the matrix representation of an orthogonal transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, then choose the set of correct statements.

- ◯ **Option 1:** $AA^T = I$
- ◯ Option 2: $A = A^T$
- ◯ **Option 3**: AA^T is an upper triangular matrix
- ◯ **Option 4:** AA^T is a lower triangular matrix
- ◯ **Option 5:** $A^T = A^{-1}$
- ◯ Option 6: $A^T = -A^{-1}$
- ◯ Option 7: A^{-1} does not exist
- ◯ **Option 8:** A^{-1} is also an orthogonal matrix

Solution:- Given $T : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation and A be the matrix representation of T w.r.t orthonormal basis

then $AA^T = I$ (from video Lecture)

Observe that I (Identity matrix) is both upper & lower triangular matrix.

Since $\det(AA^T) = \det(I)$

$\Rightarrow \det(A) \cdot \det(A^T) = 1$

$\Rightarrow \det(A) \neq 0$

So A is invertible matrix.

Now $AA^T = I$

$\Rightarrow A^T = A^{-1}$ or $A = (A^T)^{-1}$

Option 8:

Consider $A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1}$ $\left(\because (A^T)^{-1} = (A^{-1})^T\right)$

$= (A^T A)^{-1}$

$= (I)^{-1}$

$= I$

So A^{-1} is also an orthogonal matrix.

How $(A^{-1})^T = (A^T)^{-1}$?

We know that $AA^{-1} = I$

$\Rightarrow (AA^{-1})^T = I^T$

$\Rightarrow (A^{-1})^T A^T = I$ $\left(\because (AB)^T = B^T A^T\right)$

$\Rightarrow (A^{-1})^T = (A^T)^{-1}$

Maths-II
Activity Questions
Week-10

1 Lecture 10.1

Functions of several variables:

1. **Question 8:** Which of the following are correct with respect to $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{16}}$, where x, y are real? (MSQ Ans: a,c)

   (a) Domain of f is $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{16} \leq 1\}$

   (b) Domain of f is $D = \{(x, y) | \frac{x^2}{9} + \frac{y^2}{16} \geq 1\}$

   (c) Range of f is $[0, 1]$

   (d) Range of f is $[0, 1)$

   (e) Range of f is $[1, \infty)$

Solution :- $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{16}}$

For domain $\left(1 - \frac{x^2}{9} - \frac{y^2}{16}\right) \geq 0$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{16} \leq 1$ So, domain $f(x,y) = \left\{(x,y) \middle| \frac{x^2}{9} + \frac{y^2}{16} \leq 1\right\}$

For range, observe that $0 \leq \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{16}} \leq 1$

So range of $f(x,y)$ is $[0, 1]$

2. **Question 11:** Consider $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Note: An ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a, b are the dimensions of the ellipse.

   (a) Ellipses of increasing size represent the curves $\{(x,y)|f(x,y) = c\}$, where c increases from 0 towards 1

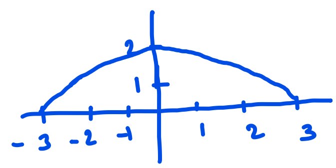
   (b) Ellipses of decreasing size represent the curves $\{(x,y)|f(x,y) = c\}$, where c increases from 0 towards 1

   (c) Ellipses of constant size represent the curves $\{(x,y)|f(x,y) = c\}$, where c increases from 0 towards 1

   (d) The curves $\{(x,y)|f(x,y) = c\}$, where $c \in [0,1)$, cannot be represented by an ellipse

   (e) $f(x,y) = 1$ is a point

Solution:- Observe that $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$ is half ellipse in positive side of y-axis with x-intercept is 3 & y-intercept is 2 i.e



Now, $f(x,y) = c \Rightarrow 1 - c = \frac{x^2}{9} + \frac{y^2}{4} \Rightarrow \frac{x^2}{9(1-c)} + \frac{y^2}{4(1-c)}$ which is an ellipse.

Also, Area of half ellipse $= \frac{\pi}{2} ab$

Observe that as $c \geq 0$ & $c \to 1^- \Rightarrow 9(1-c) \to 0$ & $4(1-c) \to 0$

i.e x-intercept & y-intercept is decreasing as $c \to 1^-$

that means Area of ellipse is also decreasing

So, ellipse of decreasing size represent the curve $\{(x,y) | f(x,y) = c\}$, where c increases from 0 toward 1.

And $f(x,y) = 1 \Rightarrow \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} = 1 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = 0, y = 0$

So $f(x,y) = 1$ represents a point which is $(0,0)$.

3. **Question 11:** Consider a point source $S = (1, 2, 3)$ radiating energy. The intensity I at a given point $P = (x, y, z)$ in space is inversely related to the square of the distance d between S and P : $I(x, y, z) = \frac{k}{d^2}$, where k is a real positive constant. Choose the correct option(s).

   (a) Intensity I is constant on $\{(x, y, z)|(x-1)^2 + (y-2)^2 + (z-3)^2 = 1\}$
   (b) Intensity I is constant on $\{(x, y, z)|(x-1)^2 + (y-2)^2 + (z-3)^2 = 2\}$
   (c) Intensity I is constant on $\{(x, y, z)|x^2 + y^2 + z^2 = 1\}$
   (d) Intensity I is constant on $\{(x, y, z)|x^2 + y^2 + z^2 = 2\}$

Solution: Distance of the point $P(x,y,z)$ from point source $S = (1,2,3)$

$d^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$

from option (a) $d^2 = 1$ so $I(x,y,z) = \frac{K}{d^2} = \frac{K}{1} = K$

so $I(x,y,z)$ is constant.

Similarly, for option (b) also.

For option (c), let $(a,b,c) \in \{(x,y,z) \mid x^2+y^2+z^2 = 1\} \Rightarrow a^2+b^2+c^2 = 1$

let h is the distence of the point (a,b,c) from S

$\Rightarrow h^2 = (a-1)^2 + (b-2)^2 + (c-3)^2$

$= a^2+b^2+c^2 + 14 - 2a - 4b - 6c$

$= 1 + 14 - 2a - 4b - 6c$

$= 15 - 2a - 4b - 6c$

So $I(a,b,c) = \frac{K}{15-2a-4b-6c}$ So, as (a,b,c) varies in the set $\{(x,y,z) \mid x^2+y^2+z^2=1\}$, $I(a,b,c)$ also varies.

that means $I(x,y,z)$ is not constant on the set

$\{x,y,z \mid x^2+y^2+z^2=1\}$

similarly, we can show for option (d).

.

2 Lecture 10.2

Partial Derivatives:

1. **Question 45** Suppose $f(x,y,z) = xy + yz + zx - 3xyz$ is a multivariable function defined on domain $D \subset \mathbb{R}^3$. Which of the following is(are) correct?
   (a) f is a scalar valued multivariable function.
   (b) The rate of change of the function f at (0,1,1) with respect to x is 1.
   (c) The rate of change of the function f at (2,2,2) with respect to x is same as the rate of change of the function f at (2,2,2) with respect to z.
   (d) The rate of change of the function f at (1,3,5) with respect to y is -9.

Solution:- $f(x,y,z) = xy + yz + zx - 3xyz$

Observe that $f(x,y,z)$ is a scalar valued multivariable function.

Now the rate of change of $f(x,y,z)$ at (a,b,c) w.r.t x = value of $\frac{\partial}{\partial x} f(x,y,z)$ at (a,b,c)

So $\frac{\partial f}{\partial x} = y + z - 3yz$

So $\frac{\partial}{\partial x} f(a,b,c) = b + c - 3bc$

Now $\frac{\partial}{\partial x}(0,1,1) = 1 + 1 - 3 = -1$

Similarly $\frac{\partial}{\partial y} f(x,y,z) = x + z - 3xz$

$\frac{\partial}{\partial y} f(2,2,2) = 2 + 2 - 12 = -8$

$\frac{\partial}{\partial y} f(1,3,5) = 1 + 5 - 15 = -9$

and $\frac{\partial}{\partial z} f(x,y,z) = y + x - 3xy$

So $\frac{\partial f}{\partial z}(2,2,2) = 2+2-12 = -8$

2. **Question 6:** Suppose $f : D \longrightarrow \mathbb{R}$ is a multivariable function defined on domain $D \subset \mathbb{R}^2$ and also satisfying the following conditions:

   (i) $\frac{\partial f}{\partial x} = 6x + 4xy + 3y$

   (ii) $\frac{\partial f}{\partial y} = 2x^2 + 3x - 3y$

   Which of the following functions can be f?

   (a) $f(x, y) = 3x^2 + 2x^2y - 3xy - \frac{3}{2}y^2$

   (b) $f(x, y) = 3x^2 + 2x^2y + 3xy - \frac{3}{2}y^2$

   (c) $f(x, y) = 3x^2 - 3xy - 2x^2y + 3y^2$

   (d) $f(x, y) = 3x^2 + 3y^2 + 3xy$

Solution:. Method 1 :-

Option (a): Given $f(x,y,z) = 3x^2 + 2x^2y - 3xy - 3y^2$

Now $\frac{\partial}{\partial x} f(x,y,z) = 6x + 4xy - 3y$ which is not equal to given $\frac{\partial}{\partial x} f(x,y,z)$

Similarly, we can check for each option.

Method: 2: Let $\frac{\partial f}{\partial x} = g(x,y)$, for some function $g(x,y)$

and $\frac{\partial f}{\partial y} = h(x,y)$, for some function $h(x,y)$

then $f(x,y) = \int g(x,y)\, dx + p(y)$ for some function $p(y)$, where $p(y)$ can be a constant function &

$f(x,y) = \int h(x,y)\, dy + q(x)$ for some function $q(x)$, where $q(x)$ can be a constant function.

Now, choose suitable $f(x,y)$ such that

$\frac{\partial}{\partial x} f(x,y) = g(x,y)$ & $\frac{\partial}{\partial y} f(x,y) = h(x,y)$ satisfied.

3. **Question 7:** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (P(x, y), Q(x, y))$, where $P(x, y) = 3x^2y, Q(x, y) = 5x + y^3$. Consider a matrix $A = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}$. Find the determinant of A.

(a) $18xy^3 - 15x^2$

(b) $18x^3y - 15x^2$

(c) $15x^3y - 18x^2$

(d) $15xy^3 - 18x^2$

Solution:- Given $P(x,y) = 3x^2y$, $Q(x,y) = 5x + y^3$

So $\frac{\partial P}{\partial x} = 6xy$, $\frac{\partial P}{\partial y} = 3x^2$

&

$\frac{\partial Q}{\partial x} = 5$, $\frac{\partial Q}{\partial y} = 3y^2$

So $A = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} = \begin{bmatrix} 6xy & 3x^2 \\ 5 & 3y^2 \end{bmatrix}$

Now $\det(A) = 18xy^3 - 15x^2$

3 Lecture 10.3

Directional Derivatives:

1. **Question 45:** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & x,y \neq 0 \\ 0 & x = y = 0 \end{cases}$$

Find the directional derivative of f at (0,0) in the direction of the vector $u = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

(a) $\frac{3}{2}$
(b) $\frac{3}{8}$
(c) 3
(d) $\sqrt{3}$

Solution :. Given $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Now, directional derivative of $f(x,y)$ at $(0,0)$ in the direction of $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = D_{\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)} f(0,0)$

So $D_{(\sqrt{3}/2,1/2)} f(0,0) = \lim_{h\to 0} \dfrac{f\left((0,0)+h\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)\right) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{f\left(\frac{h\sqrt{3}}{2}, \frac{h}{2}\right) - 0}{h} = \lim_{h\to 0} \frac{\frac{h^2 3}{4} \cdot \frac{h}{2}}{h\left(\frac{h^4 9}{16} + \frac{h^2}{4}\right)}$$

$$= \lim_{h\to 0} \frac{h^3 \cdot \frac{3}{8}}{h^3\left(\frac{h^2 9}{16} + \frac{1}{4}\right)}$$

$$= \lim_{h\to 0} \frac{3/8}{\frac{h^2 9}{16} + 1/4} = \frac{3/8}{1/4} = 3/2$$

2. **Question 7:** Suppose $f(x, y, z) = ax^2y + yz^2 - z^2x$, where $a \in \mathbb{R}$, is a scalar valued multivariable function defined on domain $D \subset \mathbb{R}^3$. For what value of a, does the directional derivative of the given function at the point (1,1,1) in the direction of the vector $u = (1, 2, -2)$ equal 3?
(Answer:) $1/2$

Solution: Given $f(x,y,z) = ax^2y + yz^2 - z^2x$

Now $$D_u f(1,1,1) = \lim_{h\to 0} \frac{f((1,1,1)+h(1,2,-2)) - f(1,1,1)}{h}$$

$$= \lim_{h\to 0} \frac{f((1+h),(1+2h),(1-2h)) - a}{h}$$

$$= \lim_{h\to 0} \frac{a(1+h)^2(1+2h) + (1+2h)(1-2h)^2 - (1-2h)^2(1+h) - a}{h}$$

$$= 4a+1$$

Now $D_u f(1,1,1) = 3$

$$\Rightarrow 4a+1 = 3 \Rightarrow a = 1/2$$

4 Lecture 10.4

Limits:

1. **Question 3:** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as:

$$f(x,y) = \begin{cases} \frac{4x^2y^2}{x^2+y^2} & x, y \neq 0 \\ 0 & x = y = 0 \end{cases}$$

What is the value of $\underset{(x,y)\to(0,0)}{lim}(f(x,y))$?

(a) 4

(b) 0

(c) $\frac{1}{4}$

(d) Limit does not exist at (0,0).

Solution: Given $f(x,y) = \begin{cases} \frac{4x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Observe that if $x, y \neq 0$ then $0 < x^2 \leq x^2+y^2$; $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\Rightarrow 0 < \frac{x^2}{x^2+y^2} \leq 1 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

Observe that $\frac{x^2}{x^2+y^2}$ is a bounded function & it's upper bound is 1.

So $$0 \leq \frac{4x^2y^2}{x^2+y^2} \leq 4y^2$$

Observe, $\lim\limits_{(x,y)\to(0,0)} 0 = 0$ & $\lim\limits_{(x,y)\to(0,0)} 4y^2 = 0$

Now, Sandwitch theorem, $\lim\limits_{(x,y)\to(0,0)} \frac{4x^2y^2}{x^2+y^2} = 0$

2. **Question 7:** Which of the following functions have limit value equal to 5 at the point (1,1,-1)?

   (a) $f(x,y,z) = \frac{x+3yz^2-zx}{xy+yz+xz^2}$

   (b) $f(x,y,z) = \frac{3(x+y-z)+7x^2y+yz}{xyz+zy+xz}$

   (c) $f(x,y,z) = x^3+y^3+z^3-2xy-2yz-4zx$

   (d) $f(x,y,z) = \frac{x^2e^{\frac{x}{2}}+y^2e^{\frac{y}{2}}+z^2e^{\frac{-z}{2}}}{e^{\frac{x+y+z}{2}}}$

Solution:

(a) $$\lim_{(x,y,z)\to(1,1,-1)} \frac{x+3yz^2-zx}{xy+yz+xz^2} = \frac{1+3+1}{1-1+1} = 5$$

(b) $$\lim_{(x,y,z)\to(1,1,-1)} \frac{3(x+y-z)+7x^2y+yz}{xyz+zy+xz} = \frac{9+7-1}{-1-1-1} = -15$$

(c) $$\lim_{(x,y,z)\to(1,1,-1)} x^3+y^3+z^3-2xy-2yz-4zx = 1+1-1-2+2+4$$
$$= 5$$

(d) $$\lim_{(x,y,z)\to(1,1,-1)} \frac{x^2e^{x/2}+y^2e^{y/2}+z^2e^{-z/2}}{e^{\frac{x+y+z}{2}}} = \frac{e^{1/2}+e^{1/2}+e^{1/2}}{e^{1/2}}$$
$$= \frac{3e^{1/2}}{e^{1/2}} = 3.$$

.

3. **Question 8:** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as:

$$f(x,y) = \begin{cases} \frac{x^k y}{x^{2n}+y^{2n}} & x, y \neq 0 \\ 0 & x = y = 0 \end{cases}$$

where $k, n \in \mathbb{N} \setminus \{0\}$. Which of the following statements is(are) true about f?

(a) If $k = 2n-1$, then the limit at (0,0) exists and is equal to 0.

(b) If $k < 2n-1$, then the limit at (0,0) does not exist.

(c) If $k > 2n$, then the limit at (0,0) always exists and is equal to 0.

(d) If $k > 2n$, then the limit at (0,0) does not exist.

Solution:- Given $f(x,y) = \begin{cases} \frac{x^k y}{x^{2n}+y^{2n}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Options (a,b), Given, $K < 2n-1$

let $K = 2n - m$, where $m \geq 1$, $m < 2n$

let $x, y \neq 0$
then, $f(x,y) = \dfrac{x^{2n-m}\, y}{x^{2n}+y^{2n}} = \dfrac{\frac{y}{x^m}\cdot x^{2n}}{x^{2n}+y^{2n}}$

Observe that $0 \leq (x^2)^n \leq (x^2)^n + (y^2)^n$, $\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow \quad 0 < \dfrac{x^{2n}}{x^{2n}+y^{2n}} \leq 1$, $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\Rightarrow \quad 0 < \dfrac{y\, x^{2n}}{x^{2n}+y^{2n}} \leq y$, $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $y > 0$

(if $y < 0$ then inequality will reverse but still)
Using sandwich theorem, $\lim_{(x,y)\to(0,0)} \dfrac{y\, x^{2n}}{x^{2n}+y^{2n}} = 0$

but $\lim_{(x,y)\to(0,0)} \dfrac{1}{x^m}$ does not exist $\forall\, m \geq 1$, $m < 2n$

So $\lim_{(x,y)\to(0,0)} \dfrac{\frac{y}{x^m}\, x^{2n}}{x^{2n}+y^{2n}} = \lim_{(x,y)\to(0,0)} \dfrac{y\, x^{2n}}{x^{2n}+y^{2n}} \cdot \lim_{(x,y)\to(0,0)} \dfrac{1}{x^m}$

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^{2n-m}y}{x^{2n}+y^{2n}}$ does not exist.

Similarly, for options (c,d), Given $K > 2n$

let $K = 2n + m$ where $m \geq 1$

let $(x,y) \neq (0,0)$

then $f(x,y) = \dfrac{x^{2n+m}y}{x^{2n}+y^{2n}} = \dfrac{x^{2n}\, x^m y}{x^{2n}+y^{2n}}$

again $\quad 0 \leq x^{2n} \leq x^{2n}+y^{2n}, \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{0,0\}$

$\Rightarrow \quad 0 \leq \dfrac{x^{2n}}{x^{2n}+y^{2n}} \leq 1 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{0,0\}$

$\Rightarrow \quad 0 < \dfrac{y\,x^{2n}}{x^{2n}+y^{2n}} \leq y \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, y > 0$

(If $y<0$ then inequality will reverse but still)

using Sandwitch theorem, $\lim\limits_{(x,y)\to(0,0)} \dfrac{y\,x^{2n}}{x^{2n}+y^{2n}} = 0$

Now $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^{2n+m}y}{x^{2n}+y^{2n}} = \lim\limits_{(x,y)\to(0,0)} x^m \cdot \lim\limits_{(x,y)\to(0,0)} \dfrac{y\,x^{2n}}{x^{2n}+y^{2n}}$

$= 0 \cdot 0$

$= 0$

5 Lecture 10.5

Continuity: (Level 1)

1. **Question 2:** Let $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Choose the correct options.

   (a) f is not defined at $(0,0)$

   (b) f is defined at $(0,0)$

   (c) f is continuous at $(0,0)$

   (d) f is not continuous at $(0,0)$

Solution :- Given $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$

Observe $f(x,y)$ is defined at $(0,0)$ & $f(0,0) = 1$

Now $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} =$

$$= \lim_{(x,y)\to(0,0)} \cos y \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$$

$$= \lim_{y\to 0} \cos y \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1 \cdot 1 = 1$$

$$= f(0,0)$$

Hence function $f(x,y)$ is continuous at $(0,0)$

2. **Question 5:** Let $f(x,y) = ln(x^2+y^2-1)$. Identify where f is continuous. Note: Equation of a circle with radius r and center (a,b) is given by $(x-a)^2+(y-b)^2 = r^2$

   (a) Everywhere in $\mathbb{R}^2$

   (b) Everywhere in $\mathbb{R}^2 \setminus \{(1,0),(0,1)\}$

   (c) Everywhere in $\mathbb{R}^2$ outside the circle with radius 1 and center $(0,0)$

   (d) None of the above

Solution :- Given $f(x,y) = \ln(x^2+y^2-1)$

Observe, domain of $f(x,y) = \{(x,y) \mid x^2+y^2>1\}$

let $(a,b) \in \text{Domain}(f)$

then $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \ln(x^2+y^2-1)$

$= \ln(a^2+b^2-1)$, where $(a,b) \in \text{Domain}(f)$

$= f(a,b)$

So $f(x,y)$ is continuous every where in $\mathbb{R}^2$ outside of the circle of radius 1 centre $(0,0)$

3. **Question 6:** Let $g(u,v,w) = \frac{uv}{w}$ and $f(x,y,z) = g(P(x,y,z), Q(x,y,z), R(x,y,z))$ where $P(x,y,z) = e^{x^2+y}$, $Q(x,y,z) = \sqrt{y^2+z^2+3}$, $R(x,y,z) = \sin(xyz)+5$. Choose the correct option(s).

   (a) f is a vector-valued function

   (b) f is a scalar-valued function

   (c) f is continuous everywhere in its domain

   (d) f is continuous everywhere in its domain. P is also continuous everywhere in its domain, while Q is not continuous everywhere in its domain

Solution:- $g(u,v,w) = \frac{uv}{w}$

Observe that $P(x,y,z) = e^{x^2+y}$

$Q(x,y,z) = \sqrt{y^2+z^2+3}$

$R(x,y,z) = \sin(xyz) + 5$ are continuous functions in its domain

Now $f(x,y,z) = g(P,Q,R)$

$$= \frac{PQ}{R} = \frac{e^{x^2+y}\cdot\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$

Observe that $\sin(xyz)+5 \neq 0, \ \forall (x,y,z) \in \mathbb{R}^3$.

Now, let $(a,b,c) \in \text{Domain}(f) = \mathbb{R}^3$

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = \frac{e^{a^2+b}\sqrt{b^2+c^2+3}}{\sin(abc)+5} = f(a,b,c)$$

Hence, $f(x,y,z)$ is a continuous in $\mathbb{R}^3$.

4. **Question 9:** Let $f(r) = \frac{1}{<r,r>-1}$, where $r = (x, y, z)$ and the inner product $<,>$ is the dot product. Choose the correct option.

   (a) f is continuous throughout $\mathbb{R}^3$
   (b) f is continuous throughout $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$
   (c) f is discontinuous on the unit sphere $\{r \in \mathbb{R}^3 : \|r\| = 1\}$
   (d) f is continuous throughout $\mathbb{R}^3 \setminus \{(1, 0, 0)\}$
   (e) None of the above

Solution:- $f(r) = \dfrac{1}{\langle r, r\rangle - 1} = \dfrac{1}{x^2+y^2+z^2-1}$

Observe that domain of $f(x,y,z) = \{(x,y,z) \mid x^2+y^2+z^2 \neq 1\}$

In other words $\text{Domain}(f) = \{r \in \mathbb{R}^3 \mid \|r\| \neq 1\}$

Now, let $(a,b,c) \in \text{Domain}(f)$

Now $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = \dfrac{1}{a^2+b^2+c^2-1} = f(a,b,c)$

So, $f(r)$ is continuous in its domain

Let $(a,b,c) \in \mathbb{R}^3 \Rightarrow a^2+b^2+c^2 = 1$

Now $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = \lim\limits_{(x,y,z)\to(a,b,c)} \dfrac{1}{x^2+y^2+z^2-1}$

let $x^2+y^2+z^2 = t \Rightarrow$ As $(x,y,z) \to (a,b,c)$, $t \to 1$

So $\lim\limits_{(x,y,z)\to(a,b,c)} \dfrac{1}{x^2+y^2+z^2-1} = \lim\limits_{t\to 1} \dfrac{1}{t-1}$ which does not exist

Hence $f(x,y,z)$ is discontinuous on unit sphere.

6 Lecture 10.6

Gradients

1. **Question 4:** Let $p = -\frac{k}{(\|r\|)^3} r$, where $r = (x, y, z) \in \mathbb{R}^3 \setminus (0, 0, 0), k \in \mathbb{R}$. Amongst the options below, choose those for which the function f satisfies $\nabla f = p$ for some $k \in \mathbb{R}$. Consider dot product as the inner product for finding the norm.

   (a) $f(x, y, z) = \frac{1}{\|r\|}$

   (b) $f(x, y, z) = \frac{1}{2\|r\|}$

   (c) $f(x, y, z) = \frac{1}{\|r\|+2}$

   (d) $f(x, y, z) = \|r\|$

Solution :- Given $p = -\frac{k r}{\|r\|^3}$

option (a) $f(x,y,z) = \frac{1}{\|r\|} = \frac{1}{\sqrt{(x^2+y^2+z^2)}}$

Now,

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}} = \frac{-x}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} = \frac{-x}{\|r\|^2 \cdot \|r\|} = -\frac{x}{\|r\|^3}.$$

Similarly, $\frac{\partial f}{\partial y} = \frac{-y}{\|r\|^3}$, $\frac{\partial f}{\partial z} = \frac{-z}{\|r\|^3}$

So $\nabla f = \left(\frac{-x}{\|r\|^3}, \frac{-y}{\|r\|^3}, \frac{-z}{\|r\|^3}\right) = \frac{-1}{\|r\|^3}(x,y,z) = \frac{-1}{\|r\|} r$

option (b): $f = \frac{1}{2\|r\|} \Rightarrow \nabla f = \frac{1}{2}\left(\frac{-1}{\|r\|^3} r\right) = \frac{-1}{2} \frac{r}{\|r\|^3}$

Similarly, we check for other options.

2. **Question 8:** Consider a function defined as $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Choose the correct option(s).

(a) $\frac{\partial f}{\partial x}(0,0) = 0, \frac{\partial f}{\partial y}(0,0) = 0$

(b) $\frac{\partial f}{\partial x} = 0$

(c) $\frac{\partial f}{\partial y} = 0$

(d) None of the above

Solution:-

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} \frac{\sin 1/h}{1/h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin 1/h}{h} = \lim_{h\to 0} h \sin 1/h = 0$$

(use sandwich theorem)

Observe that $\frac{\partial f}{\partial x} = 2x \sin\frac{1}{\sqrt{x^2+y^2}} + (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\frac{(-x)}{(x^2+y^2)^{3/2}}$

$$\neq 0, \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

similarly $\frac{\partial}{\partial y} f \neq 0, \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

3. **Question 9:** Consider the function definition given in Q8 above. Choose the correct option(s) with respect to the partial derivative of f. Note $sign(x) = \pm 1$ depending on the sign of x.

   (a) $\frac{\partial f}{\partial x}(x,0) = 2x\sin(\frac{1}{|x|}) - sign(x)\cos(\frac{1}{|x|})$ for $x \neq 0$

   (b) $\frac{\partial f}{\partial x}(x,0) = 2x\sin(\frac{1}{|x|}) - sign(x)\cos(\frac{1}{|x|})$

   (c) $\frac{\partial f}{\partial x}(x,0) = 2x\sin(\frac{1}{|x|}) - \cos(\frac{1}{|x|})$ for $x \neq 0$

   (d) $\frac{\partial f}{\partial y}(0,y) = 2y\sin(\frac{1}{|y|}) - sign(y)\cos(\frac{1}{|y|})$ for $y \neq 0$

   (e) $\frac{\partial f}{\partial y}(0,y) = 2y\sin(\frac{1}{|y|}) - sign(y)\cos(\frac{1}{|y|})$

   (f) $\frac{\partial f}{\partial y}(0,y) = 2y\sin(\frac{1}{|y|}) - \cos(\frac{1}{|y|})$ for $y \neq 0$

Solution: As we get in solution of Q8, when $x, y \neq 0$

$$\frac{\partial f}{\partial x}(x,y) = 2x\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$$

So $\frac{\partial f}{\partial x}(x,0) = 2x\sin\frac{1}{|x|} - \frac{x}{|x|}\cos\frac{1}{|x|}$

$\Rightarrow \frac{\partial f}{\partial x}(x,0) = 2x\sin\frac{1}{|x|} - sign(x)\cos\frac{1}{|x|}$ for $x \neq 0$

Similarly, when $x, y \neq 0$, $\frac{\partial f}{\partial y}(x,y) = 2y\sin\frac{1}{\sqrt{x^2+y^2}} - (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\frac{-y}{(x^2+y^2)^{3/2}}$

$$= 2y\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$$

So $\frac{\partial f}{\partial y}(0,y) = 2y\sin\frac{1}{|y|} - \frac{y}{|y|}\cos\frac{1}{|y|}$

$= 2y\sin\frac{1}{|y|} - sign(y)\cos\frac{1}{|y|}$, when $y \neq 0$

4. **Question 10:** Consider the function definition given in Q8 above. Choose the correct option.

   (a) ∇f is continuous everywhere

   (b) ∇f is discontinuous at the origin $(0, 0)$

   (c) $\frac{\partial f}{\partial x}$ is continuous at the origin but $\frac{\partial f}{\partial y}$ is discontinuous at the origin

   (d) $\frac{\partial f}{\partial y}$ is continuous at the origin but $\frac{\partial f}{\partial x}$ is discontinuous at the origin

   (e) Directional derivative of f in the direction of a vector $a \in \mathbb{R}^2$ at $(0, 0)$ can be obtained using the gradient of f

Solution: As we know, $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Now, if $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$ then ∇f is continuous at $(0,0)$

As we get, $\frac{\partial f}{\partial x} = 2x \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}$

Now, $\frac{\partial f}{\partial x}(0,0) = \lim\limits_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \frac{h^2 \sin \frac{1}{|h|} - 0}{h} = \lim\limits_{h \to 0} h \sin \frac{1}{|h|}$

Now, left limit, $\lim\limits_{h \to 0^-} h \sin \frac{1}{-h} = -1 \neq 1 = \lim\limits_{h \to 0^+} h \sin \frac{1}{h}$, right limit

Hence $\frac{\partial}{\partial x} f(0,0)$ does not exist.

$\Rightarrow \nabla f(x,y)$ is not continuous at the origin.