IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
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Orthogonality and Linear Independence

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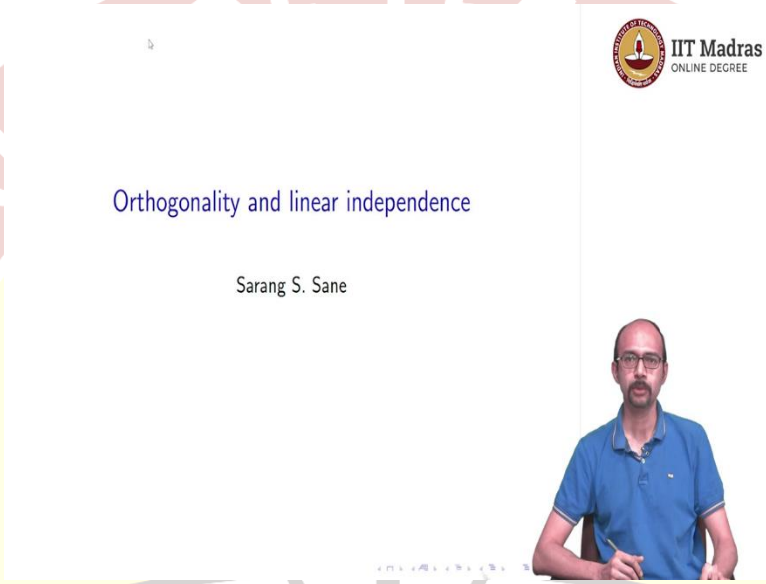

Hello, and welcome to the Maths 2 component of the online B.Sc. program in data science. In this video, we are going to talk about orthogonality and linear independence. So, we have seen before the notion of an inner product and in this video, we are going to continue using the ideas of inner products and norms.

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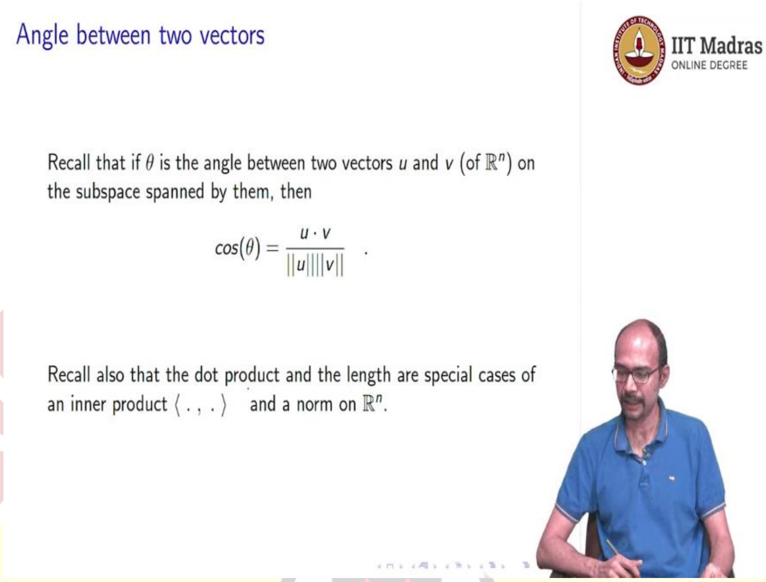

So, let us recall first that we have expressed the angle between two vectors in terms of the special inner product, namely the dot product in $\mathbb{R}^n$, and the special norm, namely the length in $\mathbb{R}^n$. So, how do we do that? We take the angle θ between two vectors, let us say, the vectors u and v. And of course, when we talk about angle, we first, we have to first look at the plane that they span and then we look at the angle on that plane.

And how do we compute it? To compute it, we can use the dot product or the norm and the norm, namely, $\cos(\theta) = \frac{u.v}{||u||||v||}$. So, the dot product is a special case of an inner product on $\mathbb{R}^n$ and the length is a special case of a norm on $\mathbb{R}^n$.

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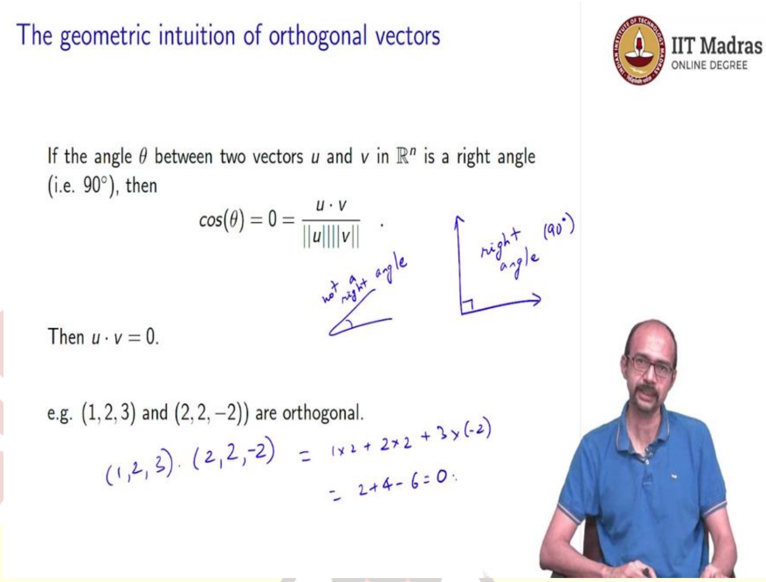

So, let us first ask what is the geometric intuition of what are called orthogonal vectors. We want to study orthogonality in this video. So, what is a pair of orthogonal vectors? So, the angle θ between two vectors u and v in $\mathbb{R}^n$, if it is a right angle, which means it is 90 degrees, then $\cos(\theta) = 0$.

And we can, we have just seen in the previous slide that we can compute it by looking at the dot product of these two vectors divided by the product of the norms. So, that means, 0 is $\frac{u.v}{\|u\|\|v\|}$. Of course, if we have a, if we have a ratio, which is 0, then the, that means the numerator is 0. So, this means $u.v$ is 0.

So, this is exactly what we mean by saying two vectors are orthogonal. So, two vectors are orthogonal when in $\mathbb{R}^n$ when they form a right angle, they are at right angles to each other. So, in $\mathbb{R}^2$, for example, this is what we mean by two vectors are orthogonal. So, this is a geometric intuition.

Let us, as an example, let us look at the vectors (1, 2, 3) and (2, 2, −2). They are orthogonal, because if you compute the dot product, then you will see this is $1 \times 2 + 2 \times 2 + 3 \times (-2)$, which is 0. So, that means cosine of θ is 0. And $\cos(\theta) = 0$ means you can back calculate to say that θ is 90 degrees.

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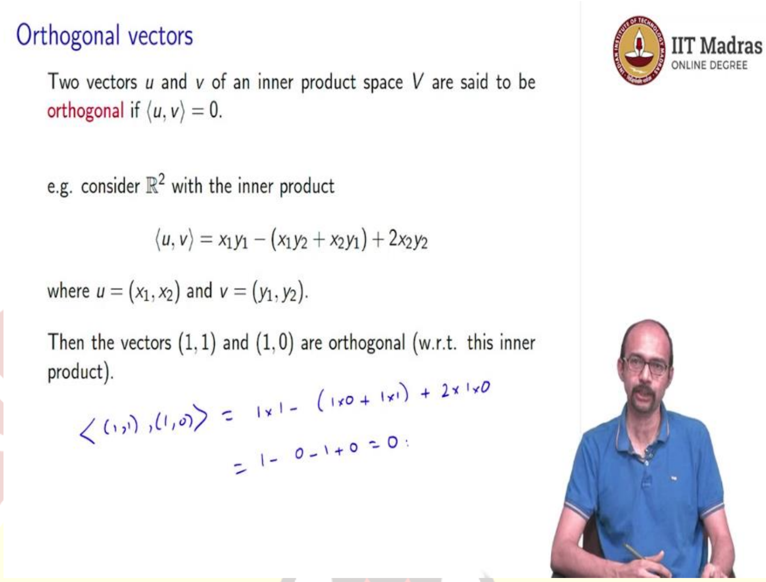

So, let us define the notion of an orthogonal vectors in the general vector space v. And the intuition we have just seen is that if the dot product is 0, then in $\mathbb{R}^n$ the vectors are said to be orthogonal. So, we can make a more general definition now. So, suppose you have an inner product space v and you have two vectors u and v then we say that they are orthogonal if you compute the inner product u, v and that is 0. So, the, let us do an example.

Suppose you take $\mathbb{R}^2$ with the inner product given by this complicated looking function, which we have seen in the previous video. So, you have the inner product of $\langle u, v \rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$. So, then the vectors (1, 1) and (1, 0) are orthogonal in this inner product.

Let us compute why that is the case. So, if you compute it for this inner product, what we get, so we get $1 \times 1 - (1 \times 0 + 1 \times 1) + 2 \times 1 \times 0 = 1 - 0 - 1 + 0 = 0$. So, the inner product is 0 and that is why these vectors are orthogonal by definition.

I want to point out that in this case we chose this particular inner product on $\mathbb{R}^2$. Had we chosen the usual inner product namely the dot product, then these vectors are not orthogonal. So, the notion of orthogonality depends on the chosen inner product, even for $\mathbb{R}^n$, it depends on which product to choose.

Now, if we say without alluding to any inner product that the vectors are orthogonal then by convention it means that we are talking about the usual inner product namely the dot product. But if we specify some other inner product and talk about orthogonality, it means with respect to that inner product.

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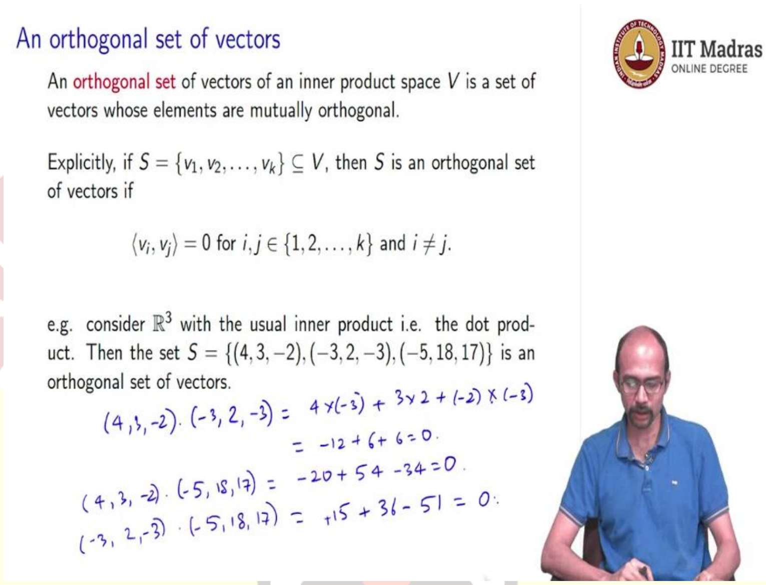

So, let us talk about an orthogonal set of vectors. So, an orthogonal set of vectors in an inner product space V is a set of vectors whose elements are mutually orthogonal, that means they are pair-wise orthogonal. So, if you take any two of them, then they are orthogonal. So, explicitly what that means is if you take the set $v_1, v_2, \ldots, v_k$ in V then S is an orthogonal set of vectors.

If you look at $\langle v_i, v_j \rangle$ that is 0, of course, you need that $i \neq j$. So, if you take v_1 and v_2 are orthogonal v_1 and v_3 are orthogonal, v_1 and v_4 are orthogonal, all the way up to v_1 and v_k, similarly v_2 and v_3, v_2 and v_4, all the way up to v_k and so on, that is what we mean by mutually orthogonal.

For example, let us consider $\mathbb{R}^3$ with the usual inner product, the dot product, then the set $S = \{(4,3,-2), (-3,2,-3), (-5,18,17)\}$ is an orthogonal set of vectors. Let us quickly see why that is the case. So, let us compute these dot products. So, if you dot this with $(4,3,-2).(-3,2,-3) = 4 \times (-3) + 3 \times 2 + (-2)(-3) = -12 + 6 + 6 = 0$.

So, certainly the first two are orthogonal. Let us look at the next, maybe the first and the third. So, if you do this, you get $(4,3,-2).(-5,18,17) = -20 + 54 - 34 = 0$. And finally, if you do the second and the third one, then we get $(-3, 2, -3).(-5, 18, 17) = +5 + 36 - 51 = 0$. So, these are indeed mutually orthogonal. So, this is an orthogonal set of vectors.

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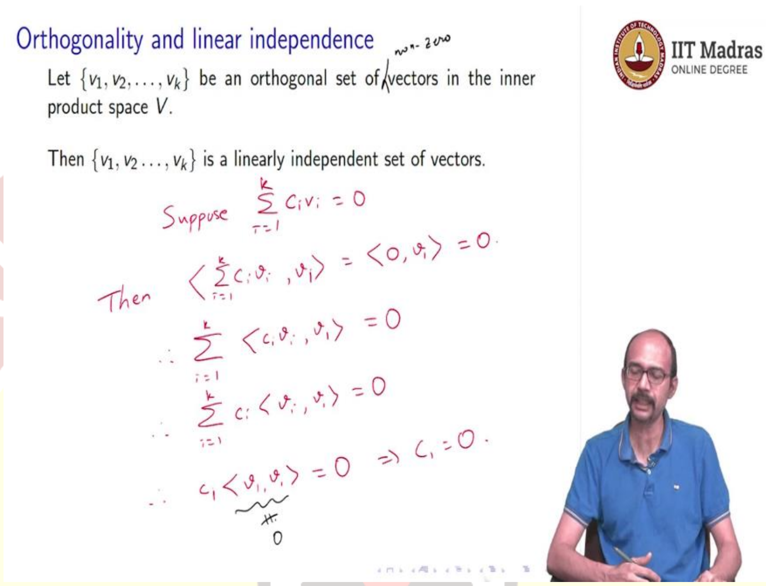

So, this is, now let us talk about orthogonality and linear independence. This is the title of our video and clearly this is the most important slide in this video. So, the reason we want to discuss orthogonality, is one of the reasons at least, is because we want to, because it has some relation with when vectors are linearly independent. So, let us see what we want to say.

So, suppose we have a orthogonal set of vectors in the inner product space V, then $v_1, v_2, \ldots, v_k$ is a linearly independent set of vectors. So, if you have an orthogonal set, automatically it is a linearly independent set. Now, let us recall, to check something is linearly independent was a fairly hard process. We had to write down $\sum_{i=1}^{k} c_i v_i = 0$ and then by some way we had to say that each of the coefficients c_i is 0.

And in $\mathbb{R}^n$ if we had to do this, that means we had to solve a system of linear equations and check that indeed each coefficient is ending up being 0. So, that was a fairly non-trivial process. Instead, here what we are saying is, if it so happens that these $v_1, v_2, \ldots, v_k$ are an orthogonal set of vectors and notice that to check something is orthogonal is quite easy. You have to just check that the inner products are 0, which is a fairly mechanical process.

Let us see a quick proof or at least idea of the proof of this statement why is this happening. So, suppose, maybe I will do this in red. So, suppose $\sum_{i=1}^{k} c_i v_i = 0$, well, then what I can do is I can take the inner product of $\langle \sum_{i=1}^{k} c_i v_i, v_1 \rangle$ or maybe let us say with v_1, let us do it for v_1, and you will see the same thing happens for any v_i.

So, if you do that, then this is the inner product of 0 with v_1 and the inner product of 0 with anything is 0. This is one of the things that we know about inner products. You can check this from the definition, because you can write $0, v_1, 0, v_1 + 0, v_1$ and then cancel $0, v_1$ on both sides. So, this is 0.

But on the other end we know what this is, the first term is. So, this is a sum. So, I can take out my summation. So, therefore, $\langle \sum_{i=1}^{k} c_i v_i, v_1 \rangle = \langle 0, v_1 \rangle = 0$. So, what are the non-zero terms here?

Remember, that these are an orthogonal set of vectors. So, unless $i = 1$, everything is 0, because v_i, v_1 is equal to inner product of v_1, v_i. So, inner product of v_i and v_1 is 0 for all $i \neq 1$. So, the only term that I am left with here is $c_1 \times v_1, v_1$ and this is 0. And so, if v_1 is not 0, then this inner product is non-zero, so and that will imply that c_1 is 0.

So, maybe here I should make a small correction to what I have written here. Namely, I should say, let v_1, v_2, ...,v_k be an orthogonal set of non-zero vectors. So, typically, when we say orthogonal set, we often mean non-zero vectors. But that is not, that is not the definition per set. So, we will say orthogonal set of non-zero vectors.

So, if you have an orthogonal set of non-zero vectors, then this part v_1, v_1 is not 0 and as a result c_1 is equal to 0. And you can do this argument for instead of v_1 you could have done it with v_2 or v_3 or v_j in general and you get that c_j is 0. So, that will tell you that each of these constants is 0, which is exactly the definition of linear independence. So, that is the reason why every orthogonal set of non-zero vectors is indeed a linearly independent set of vectors.

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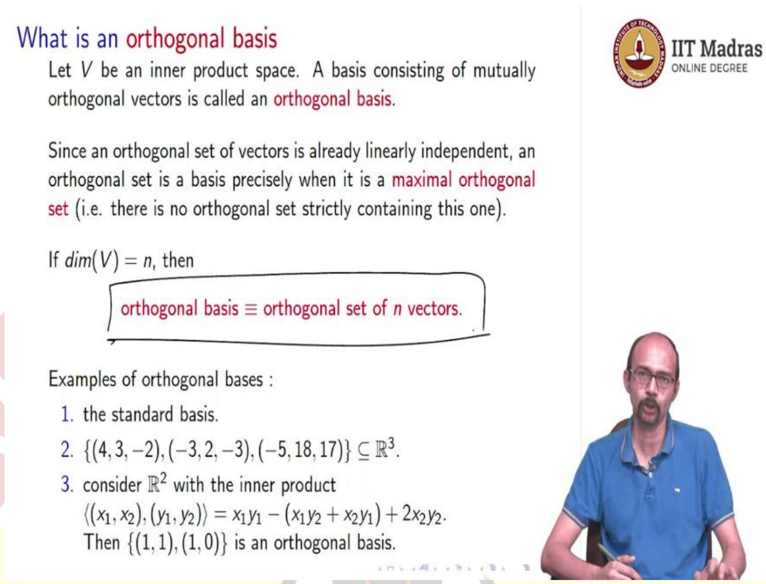

So, what is an orthogonal basis? So, let V be an inner product space. A basis consisting of mutually orthogonal vectors is called an orthogonal basis. So, this is the same as saying that an orthogonal basis is an orthogonal set which is also a basis or a basis which is also an orthogonal set.

So, since an orthogonal set we saw is linearly independent, an orthogonal set is a basis exactly when it is a maximal orthogonal set. Because how do we, when is a linearly independent set a basis? When it is the maximal linearly independent set. So, if it is already orthogonal, then that is the same as saying it is a maximal orthogonal set or maximal linearly independent set.

So, what does maximal orthogonal set mean? It means there is no orthogonal set which contains strictly contains this set that is being referred to. So, a set is called a maximal orthogonal set if there is no super set which is also an orthogonal set. So, specifically, what this means is if the dimension of V is n, then an orthogonal basis is exactly an orthogonal set of n vectors. So, this is what you should keep in mind.

So, let us look at some examples of orthogonal basis. The standard basis that we have seen is an orthogonal basis. Of course, I have not referred to an inner product here. So, as I mentioned before, the convention means that is that if I do not refer to any particular inner product, you have to take the dot product. So, the standard basis in the inner product space $\mathbb{R}^n$ with the dot product.

Let us look at this example which we have done before. So, we have seen already that this is a orthogonal set. What is the size of the set? It is a set of size 3. And what is the dimension of $\mathbb{R}^3$? It is 3. So, this is a orthogonal set with three vectors, where 3 is the dimension of the vector space $\mathbb{R}^3$. So, this is going to be an orthogonal basis. We are using this thing here, the statement here.

And finally, let us look at $\mathbb{R}^2$ with the inner product that we, the non-standard inner product that we defined earlier, then we saw that this is a orthogonal set of vectors, and it has size 2. So, there are two vectors and the dimension of $\mathbb{R}^2$ is 2. So, this must be an orthogonal basis. So, this is also an orthogonal. Again, we are using this fact that is written here.

So, let us recall what we saw in this video. So, we have seen the notion of an orthogonal, when two vectors are orthogonal, so a pair of orthogonal vectors or we can just say just orthogonal vectors, remember that this depends on the inner product chosen and the intuition is coming from the fact that in $\mathbb{R}^2$ or in $\mathbb{R}^n$, if, with respect to the usual dot product, if vectors are orthogonal, then that exactly means that they are at a right angle to each other, like this perpendicular to each other.

So, in general, we can talk about the notion of orthogonal vectors for any inner product space. And this is a very useful notion, because if you have an orthogonal set, which means all the vectors are mutually orthogonal, then that set is actually linearly independent. So, now, we can check linear independence using the notion of orthogonality, this is the main point.

And after that we defined the notion of an orthogonal basis, namely its orthogonal set which is also a basis and if you have your vector space as dimension V, essentially what this means is that you have a set of n or n vectors which formed an orthogonal set and we have seen the examples out here. Thank you.

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