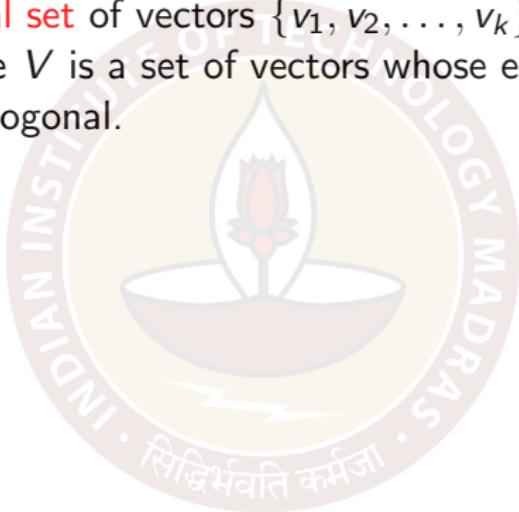


What is an orthonormal basis?

Sarang S. Sane

Recall :

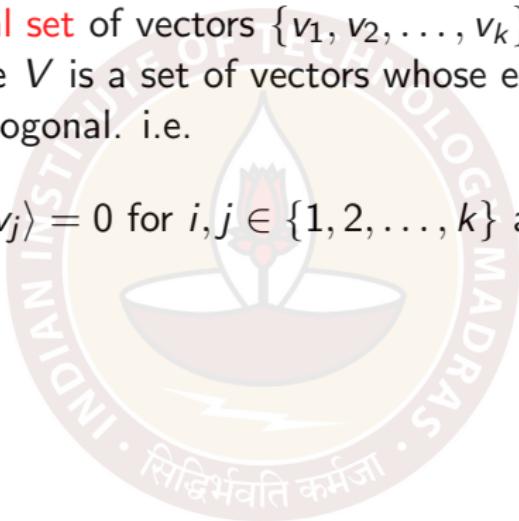
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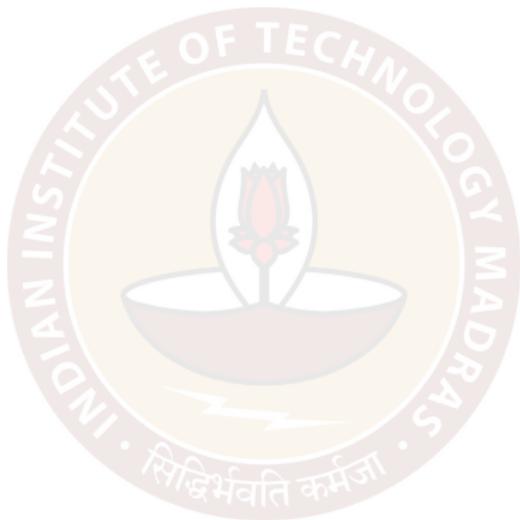
- ▶ An orthogonal set of vectors is linearly independent.
- ▶ A maximal orthogonal set is a basis and is called an **orthogonal basis**.

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Explicitly, if $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, then S is an orthonormal set of vectors if

$$\langle v_i, v_j \rangle = 0 \quad \text{for } i, j \in \{1, 2, \dots, k\} \text{ and } i \neq j$$

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Then the set

$$\left\{ \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0 \right), \left(\frac{2}{\sqrt{42}}, \frac{1}{\sqrt{42}}, \frac{1}{\sqrt{42}}, \frac{6}{\sqrt{42}} \right), \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{-1}{3} \right) \right\}$$

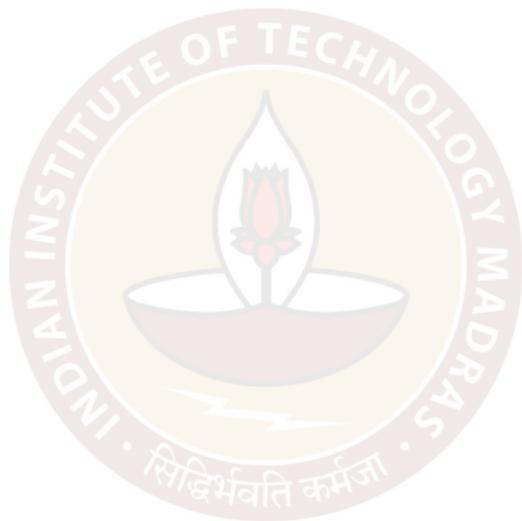
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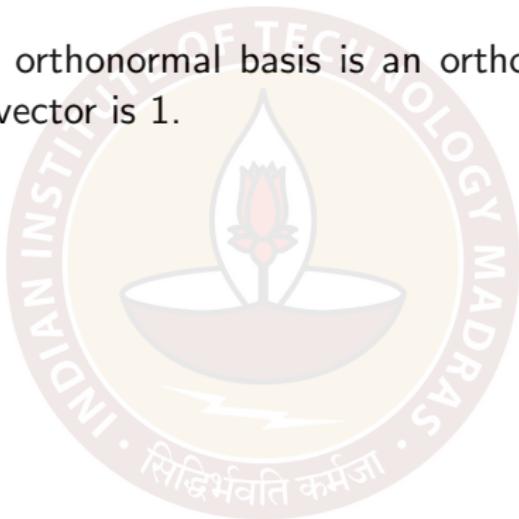
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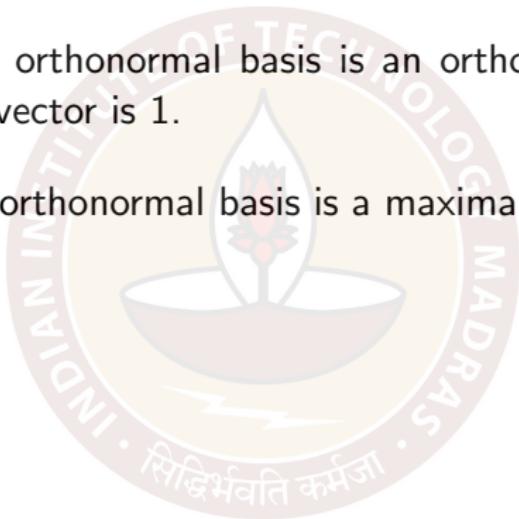


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Example : The standard basis w.r.t. the usual inner product forms an orthonormal basis.

$$\checkmark \langle e_i, e_j \rangle = (0, 0, \dots, 0, \underset{i^{\text{th}}}{1}, 0, \dots, 0) \cdot (0, 0, \dots, \underset{j^{\text{th}}}{1}, 0, \dots, 0) \quad i \neq j \\ = 0 \times 0 + \dots + 1 \times 0 + 0 \dots + 0 + 0 \times 1 + 0 + \dots + 0 = 0 \\ \checkmark \|e_i\| = \sqrt{\langle e_i, e_i \rangle} = \sqrt{0 \times 0 + \dots + 1 \times 1 + 0 + \dots + 0} = \sqrt{1} = 1.$$

Another example

Consider \mathbb{R}^3 with the usual inner product and the set

$\beta = \left\{ \frac{1}{3}(1, 2, 2), \frac{1}{3}(-2, -1, 2), \frac{1}{3}(2, -2, 1) \right\}$. Then β forms an orthonormal basis of \mathbb{R}^3 .

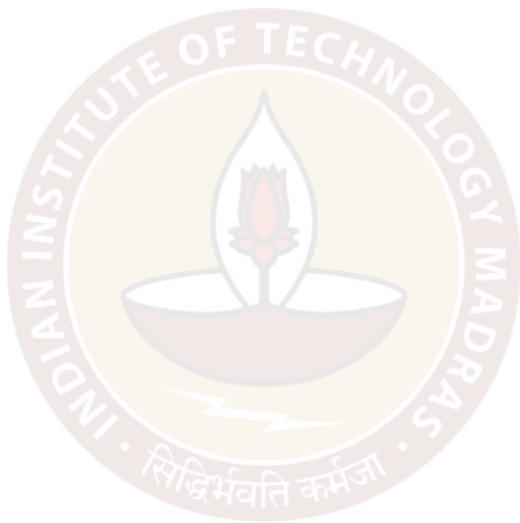
$$\left\{ \begin{array}{l} v_1, v_2, v_3 \\ \|v_1\|^2 = \langle v_1, v_1 \rangle = \left\langle \frac{1}{3}(1, 2, 2), \frac{1}{3}(1, 2, 2) \right\rangle = \frac{1}{9} (1 \times 1 + 2 \times 2 + 2 \times 2) = \frac{1}{9} (9) = 1. \\ \|v_2\|^2 = \langle v_2, v_2 \rangle = \left\langle \frac{1}{3}(-2, -1, 2), \frac{1}{3}(-2, -1, 2) \right\rangle = \frac{1}{9} \{ (-2) \times (-2) + (-1) \times (-1) + 2 \times 2 \} = \frac{1}{9} (9) = 1. \\ \|v_3\|^2 = \langle v_3, v_3 \rangle = \left\langle \frac{1}{3}(2, -2, 1), \frac{1}{3}(2, -2, 1) \right\rangle = \frac{1}{9} (1 \times (-2) + 2 \times (-1) + 2 \times 2) = \frac{1}{9} (0) = 0. \\ \langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = 0 \\ \langle v_1, v_3 \rangle = \langle v_1, v_2 \rangle = 0 \\ |\beta| = 3 \text{ & } \beta \text{ is lin. indept.} \end{array} \right.$$

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$\dim(\mathbb{R}^3) = 3$
 $\Rightarrow \beta$ is an o.n. basis.

Obtaining orthonormal sets from orthogonal sets

Let V be an inner product space.



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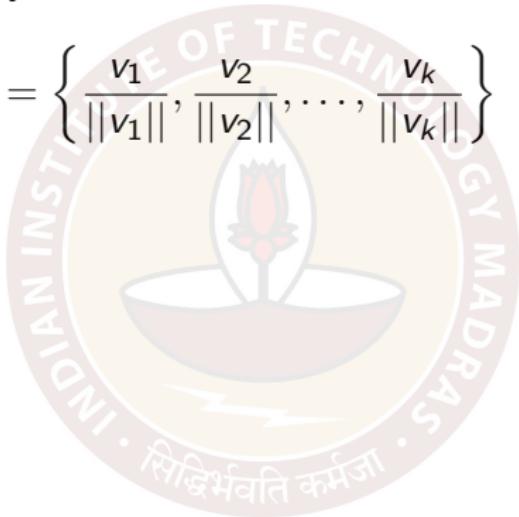
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$$\langle v_i, v_j \rangle = 0$$

$$\Rightarrow \left\langle \frac{v_i}{\|v_i\|}, \frac{v_j}{\|v_j\|} \right\rangle = 0$$

$$\left\| \frac{v_i}{\|v_i\|} \right\| = \frac{1}{\|v_i\|} \|v_i\| = 1.$$

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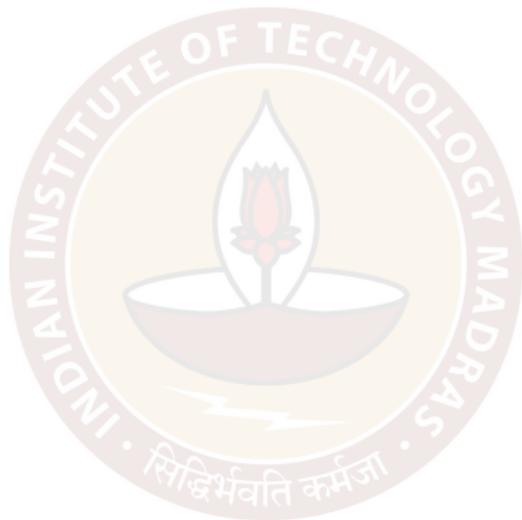
Then $\beta = \left\{ \frac{1}{\sqrt{10}}(1, 3), \frac{1}{\sqrt{10}}(-3, 1) \right\}$ is an orthonormal basis of \mathbb{R}^2 .

$$\begin{aligned} \checkmark \langle v_i, v_j \rangle &= 0 \\ \Rightarrow \left\langle \frac{v_i}{\|v_i\|}, \frac{v_j}{\|v_j\|} \right\rangle &= \frac{1}{\|v_i\| \|v_j\|} \langle v_i, v_j \rangle \\ &= 0 . \end{aligned}$$

$\left| \left| \frac{v_i}{\|v_i\|} \right| \right| = \frac{1}{\|v_i\|} \|v_i\| = 1 .$

Why are orthonormal bases important?

Suppose $\Gamma = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

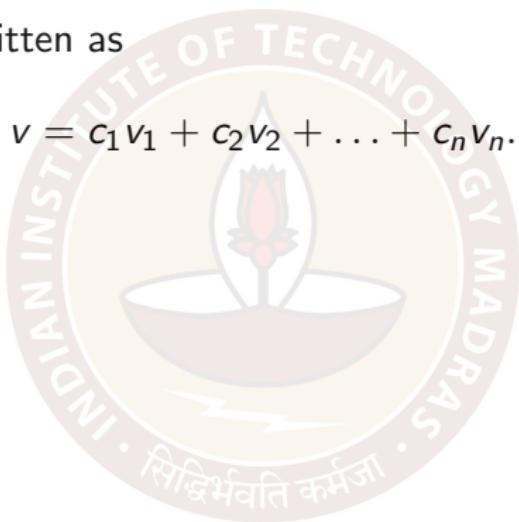


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Suppose $\Gamma = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Then v can be written as

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$



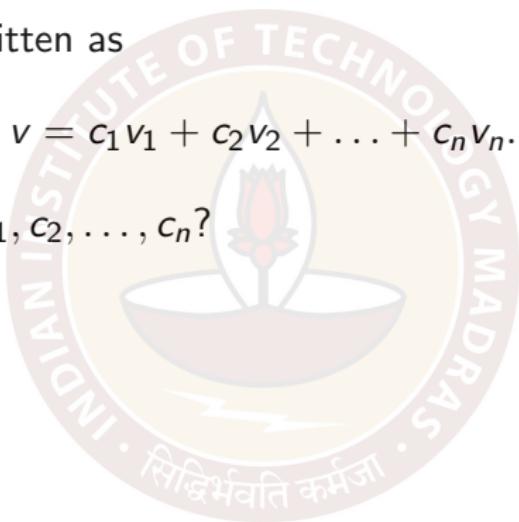
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But since Γ is orthonormal, we can use the inner product and compute $c_i = \langle v, v_i \rangle$.

$$\begin{aligned}\langle v, v_i \rangle &= \left\langle c_1 v_1 + c_2 v_2 + \dots + c_i v_i + \dots + c_n v_n, v_i \right\rangle \\ &= c_1 \cancel{\langle v_1, v_i \rangle} + c_2 \cancel{\langle v_2, v_i \rangle} + \dots + c_i \boxed{\cancel{\langle v_i, v_i \rangle}} + \dots \\ &\quad + c_n \cancel{\langle v_n, v_i \rangle} \\ &= c_i \langle v_i, v_i \rangle = c_i \|v_i\|^2 = c_i.\end{aligned}$$

Example

$\left\{ \frac{1}{\sqrt{10}}(1, 3), \frac{1}{\sqrt{10}}(-3, 1) \right\}$ is an orthonormal basis of \mathbb{R}^2 . Write $(2, 5)$ as a linear combination in terms of these basis vectors.

$$(2, 5) = c_1 \frac{1}{\sqrt{10}} (1, 3) + c_2 \frac{1}{\sqrt{10}} (-3, 1)$$

$$c_1 = \left\langle (2, 5), \frac{1}{\sqrt{10}} (1, 3) \right\rangle = \frac{1}{\sqrt{10}} (2 \times 1 + 5 \times 3) = \frac{1}{\sqrt{10}} 17.$$

$$c_2 = \left\langle (2, 5), \frac{1}{\sqrt{10}} (-3, 1) \right\rangle = \frac{1}{\sqrt{10}} (2 \times (-3) + 5 \times 1) = -\frac{1}{\sqrt{10}}.$$

$$(2, 5) = \frac{17}{\sqrt{10}} v_1 + -\frac{1}{\sqrt{10}} v_2 = \frac{17}{\sqrt{10}} \times \frac{1}{\sqrt{10}} (1, 3) - \frac{1}{\sqrt{10}} \times \frac{1}{\sqrt{10}} (-3, 1) \\ = \frac{17}{10} (1, 3) - \frac{1}{10} (-3, 1).$$

Thank you

