Week 4
Solve with us

Question 1

Let the random variables X and Y be independent with means equal to 5 and 10, and variances equal to 1 and 2, respectively.

Hint: If X and Y are independent, X^2 and Y^2 are also independent.

Find the value of Var(XY).

Solution

$$
\begin{aligned}
Var(XY) &= E\left[(XY)^2\right] - E[XY]^2 \\
&= E[X^2Y^2] - E[X]^2E[Y]^2 \\
&= E[X^2]E[Y^2] - E[X]^2E[Y]^2 \\
&= E[X^2]E[Y^2] - E[X]^2E[Y^2] + E[X]^2E[Y^2] - E[X]^2E[Y]^2 \\
&= E[Y^2]\left(E[X^2] - E[X]^2\right) + E[X]^2\left(E[Y^2] - E[Y]^2\right) \\
&= E[Y^2]Var(X) + E[X]^2Var(Y)
\end{aligned}
$$

Continue...

$E(X) = 5$, $Var(X) = 1$

$E(Y) = 10$, $Var(Y) = 2$

$E(X)^2 = 25$

$E(Y^2) = Var(Y) + E[Y]^2 = 2 + 100 = 102$

Therefore, $Var(XY) = (102*1) + (25*2) = 152$.

Question 2

The joint probability mass function of three discrete random variables X, Y and Z is given as

$p(0, 1, 2) = ¼$

$p(0, 1, 3) = ¼$

$p(1, 0, -2) = ¼$

$p(1, 1, -3) = ¼$

Calculate Var(XY+2Z).

Solution:

XY+2Z will take values in {-5, -4, 4, 6} each with probability ¼.

Therefore,

$$
\begin{aligned}
E(XY+2Z) &= \frac{1}{4}[4+6-4-5] \\
&= \frac{1}{4}
\end{aligned}
$$

And

$$
\begin{aligned}
E\left[(XY+2Z)^2\right] &= \frac{1}{4}\left[4^2+6^2+(-4)^2+(-5)^2\right] \\
&= \frac{1}{4}[16+36+16+25] \\
&= \frac{93}{4}
\end{aligned}
$$

Continue...

Now,

$$Var(XY+2Z) = E\left[(XY+2Z)^2\right] - E[XY+2Z]^2$$
$$= \frac{93}{4} - \left(\frac{1}{4}\right)^2$$
$$= 23.1875$$

Prelude 1 to Q.3

A share of a company costs Rs 100 today. Suppose today’s share price increases by 50% with probability 0.5 and decreases by 20% with probability 0.5.

If you decide to buy one share today, find the expected profit at the end of the day.

Solution:

The price of a share is Rs 100.

Price can either go up by 50% with probability 0.5 or can go down by 20% with probability 0.5.

i.e. If the share price increases by 50%, the price of the share will become Rs 150.

And if the share price decreases by 20%, the price of the share will become Rs 80.

Continue..

Let the random variable X represent the price of share at the end of the day.

Therefore, X will take values in {80, 150}.

Now, expected value of the share is

$$\begin{aligned} E(X) &= \frac{1}{2}(150 + 80) \\ &= 115 \end{aligned}$$

Therefore, expected profit at the end of the day is 115-100 = Rs 15.

Prelude 2 to Q.3

A share of a company costs Rs 100 today. Suppose today's share price increases by 50% with probability 0.5 and decreases by 20% with probability 0.5. Independent of today, suppose that tomorrow's share price increases by 30% with probability 0.6 and decreases by 20% with probability 0.4.

If you decide to buy a share today, find the expected profit at the end of 2 days.

Solution:

Let the random variable X represent the price of the share at the end of 2 days.

As we have seen in the previous question, the price of the share will either be Rs 150 or Rs 80 with probabilities 0.5 each at the end of one day.

It is given that the price of the share will either go up by 30% with probability 0.6 or go down by 20% with probability 0.4.

Continue..

If the price of the share at the end of one day is 150, the price at the end of two days can be either (150*30/100)+150 or 150-(150*20/100).

Therefore, the price of the share can either be 195 with probability 0.3 or 120 with probability 0.2.

If the price of the share at the end of one day is 80, the price at the end of two days can be either (80*30/100)+80 or 80-(80*20/100).

Therefore, the price of the share can either be 104 with probability 0.3 or 64 with probability 0.2.

Continue...

If the random variable X represent the price of the share at the end of two days, then X will take values in {195, 120, 104, 64}.

Since the change in the prices are independent,

$P(X=195) = 0.5*0.6 = 0.3$

$P(X=120) = 0.5*0.4 = 0.2$

$P(X=104) = 0.5*0.6 = 0.3$

$P(X=64) = 0.5*0.4 = 0.2$

Continue...

Now,

Expected value of the price of the share at the end of 2 days is

$$\begin{aligned} E(X) &= (195 \times 0.3) + (120 \times 0.2) + (104 \times 0.3) + (64 \times 0.2) \\ &= 126.5 \end{aligned}$$

Now, expected profit at the end of 2 days is 126.5 - 100 = Rs 26.5.

Question 3

A share of a company costs Rs 100 today. Suppose today's share price increases by 50% with probability 0.5 and decreases by 20% with probability 0.5. Independent of today, suppose that tomorrow's share price increases by 30% with probability 0.6 and decreases by 20% with probability 0.4.

If you decide to buy 3 shares today, find the expected profit at the end of 2 days.

Solution:

The price of one share costs Rs 100.

If you buy 3 shares, total cost price of the shares will be Rs 300.

Expected profit in one share at the end of 2 days is Rs 26.5.

Therefore, expected profit in 3 shares at the end of 2 days is

$$26.5 \times 3 = 79.5$$

Question 4.

Number of people (X) who make reservation in a restaurant a day is a random variable with mean 5 and variance 4.

Using Markov's inequality, find a bound on the probability that on a particular day, the number of reservations will exceed 20.

Choose the correct options from the following:

1. At least 5/20.
2. At most 5/21
3. At least 5/21
4. At most 5/20

Solution:

Using Markov's inequality,

$$P(X \geq c) \leqslant \frac{\mu}{c}$$

Random variable X represents the number of people who make reservation in a restaurant.

To find: P(X>20)

$$P(X > 20) = P(X \geq 21) \leqslant \frac{\mu}{21} = \frac{5}{21}$$

Therefore, the correct option is (b).

Question 5

Number of people (X) who make reservation in a restaurant a day is a random variable with mean equal to 5 and variance equal to 4.

Find a bound on the probability that on a particular day, the number of reservations made will lie in between 1 and 9 using Chebyshev’s inequality.

Choose the correct options from the following:

1. At most $\frac{1}{4}$.
2. At most $\frac{3}{4}$.
3. At least $\frac{3}{4}$.
4. At least $\frac{1}{2}$.

Continue...

Using Chebyshev's inequality, we know that

$$P(\,\mu - k\sigma \; < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

where

$$\mu = \; E[X], \sigma = Var[X]$$ and k is a constant.

It is given that E[X] = 5 and Var[X] = 4.

Continue...

To find: P(1 < X < 9)

$$P(1 < X < 9) = P(5 - 4 < X < 5 + 4)$$

>= 1- ¼ = 3/4

Here k = 2.

Therefore, the correct option is (c).

Prelude 1 to Q6

Number of cars (X) that visit Garage A each day is a random variable with mean 15 and variance 6 while the number of cars (Y) that visit Garage B each day is a random variable with mean 15 and variance 30. The arrival of cars on Garages A and B are independent.

Find E(X-Y) and Var(X-Y).

1. E(X-Y) = 0 and Var(X-Y) = 24
2. E(X-Y) = 0 and Var(X-Y) = 36
3. E(X-Y) = 30 and Var(X-Y) = 24
4. E(X-Y) = 30 and Var(X-Y) = 36

Solution:

Given, $E(X) = 15$, $E(Y) = 15$

Therefore, $E(X-Y) = E(X) - E(Y) = 0$.

Now $Var(X) = 6$, $Var(Y) = 30$.

Since X and Y are independent, $Var(X-Y) = Var(X) + Var(Y)$

$= 6 + 30 = 36$

Therefore, the correct option is (b).

Question 6

Number of cars (X) that visit Garage A each day is a random variable with mean 15 and variance 6 while the number of cars (Y) that visit Garage B each day is a random variable with mean 15 and variance 30. The arrival of cars on Garages A and B are independent.

Find an upper bound on the probability that the difference in the number of cars arriving in Garage A and Garage B on a particular day is greater than or equal to 8.

Enter your answer in decimal.

Solution:

Difference in the number of cars arriving in Garage A and Garage B will be denoted by $| X - Y |$.

where X represents the number of cars arriving in Garage A and Y represents the number of cars arriving in Garage B.

To find: $P(| X - Y | >= 8)$.

Continued...

Using Chebyshev's inequality, we know that

$$P(\,|X-\mu| \geq k\sigma) \leqslant \frac{1}{k^2}$$

where

$$\mu = \; E[X],\, \sigma = Var[X]$$

Here, random variable is X - Y.

E[X-Y] = 0, Var[X-Y] = 36 and SD[X-Y] = 6.

Continue...

Therefore,

$$P(|X-Y| \geq 8) = P\left(|X-Y| \geq \frac{8}{6} \times 6\right) \leqslant \frac{9}{16}$$

Here, k = 8/6.

Thank You