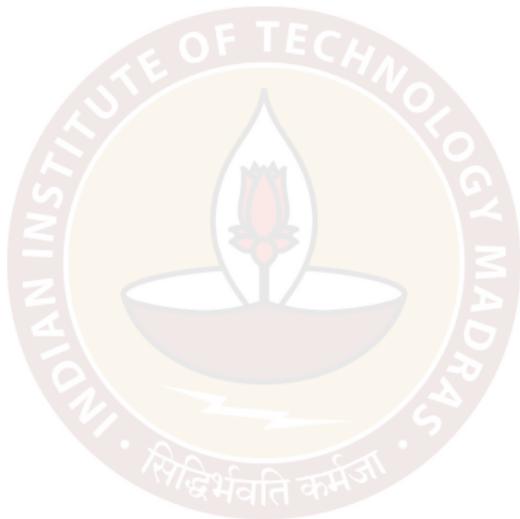


Finding the tangent (hyper)plane

Sarang S. Sane

Recall : Tangent lines for $f(x, y)$

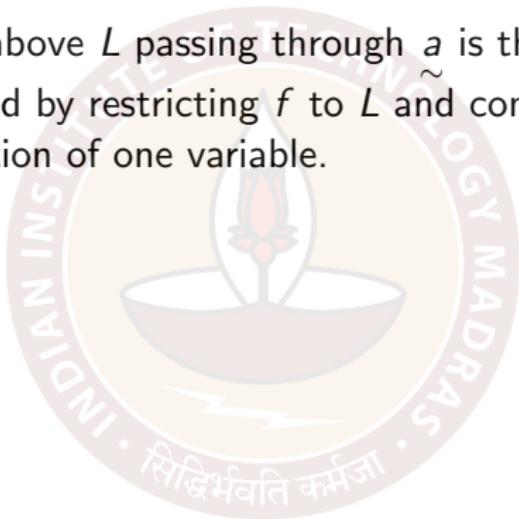
Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point a .



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Recall : Tangent lines for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point $\underline{\underline{a}}$.

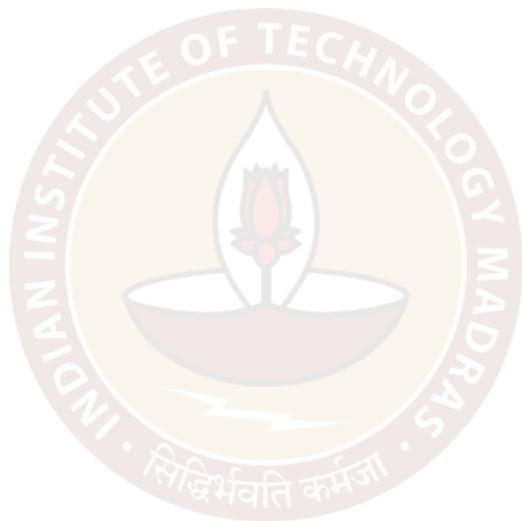
The tangent line above L passing through $\underline{\underline{a}}$ is the tangent to the function f obtained by restricting f to L and considering its tangent as a function of one variable.

If u is a unit vector in the direction of the line L , then the tangent (if it exists) will be the line with slope $f_u(\underline{\underline{a}})$ passing through the point $(\underline{\underline{a}}, f(\underline{\underline{a}}))$ and so its parametric equation is :

$$x(t) = \underline{\underline{a}} + t u_1, \quad y(t) = b + t u_2, \quad z(t) = f(\underline{\underline{a}}, b) + t f_u(\underline{\underline{a}}, b)$$

$u = (u_1, u_2)$
 $\underline{\underline{a}} = (a, b)$.

The collection of all tangents



The collection of all tangents

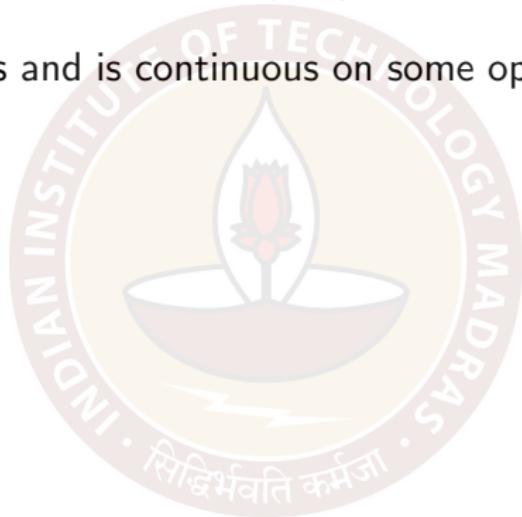
Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point (a, b) .



The collection of all tangents

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point (a, b) .

Suppose ∇f exists and is continuous on some open ball around the point (a, b) .



The collection of all tangents

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point (a, b) .

Suppose ∇f exists and is continuous on some open ball around the point (a, b) .

Then all the tangent lines at the point (a, b) exist and we can rewrite the equation of a tangent line in the direction of the unit vector u as :

$$\begin{aligned}f_u(a, b) &= \nabla f(a, b) \cdot u \\&= \frac{\partial f}{\partial x}(a, b) u_1 + \frac{\partial f}{\partial y}(a, b) u_2.\end{aligned}$$
$$x(t) = a + u_1 t, \quad y(t) = b + u_2 t, \quad z(t) = f(a, b) + f_u(a, b) t$$
$$z(t) - f(a, b) = \frac{\partial f}{\partial x}(a, b)(x(t) - a) + \frac{\partial f}{\partial y}(a, b)(y(t) - b)$$
$$= \frac{\partial f}{\partial x}(a, b) u_1 t + \frac{\partial f}{\partial y}(a, b) u_2 t$$

Tangent lines in terms of linear algebra for $f(x, y)$

$$\underbrace{(x(t), y(t), z(t))}_{\text{Tangent line to } f \text{ at } (a, b) \text{ in the direction of } u} = (a, b, f(a, b)) + t(u_1, u_2, f_u(a, b)).$$

$= (a, b, f(a, b)) + W_u$ Line passing through the vector $(u_1, u_2, f_u(a, b))$.

$$f_u(a, b) = u_1 \frac{\partial f}{\partial x}(a, b) + u_2 \frac{\partial f}{\partial y}(a, b)$$

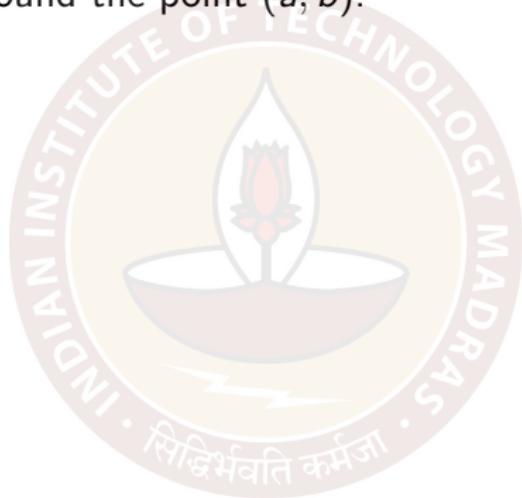
The lines W_u all lie on the plane

$$P: z = \frac{\partial f}{\partial x}(a, b)x + \frac{\partial f}{\partial y}(a, b)y.$$

Tangent plane of f at (a, b) = $(a, b, f(a, b)) + P$.

The equation of the tangent plane

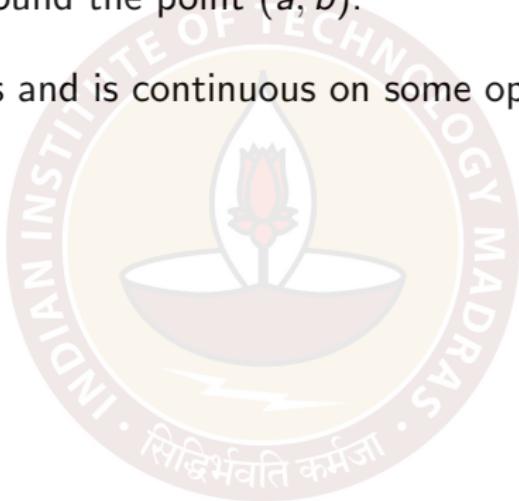
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Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point (a, b) .

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The equation of the tangent plane

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing some open ball around the point (a, b) .

Suppose ∇f exists and is continuous on some open ball around the point (a, b) .

Then the equation of the tangent plane to f at (a, b) is given by :

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$
$$\frac{\partial f}{\partial x}(a, b)x + \frac{\partial f}{\partial y}(a, b)y - z = \frac{\partial f}{\partial x}(a, b)a + \frac{\partial f}{\partial y}(a, b)b.$$

Examples

$$f(x, y) = x + y ; \text{ tangent at } (1, 1)$$

$$\nabla f(1, 1) = (1, 1).$$

$$z = 1 + 1(x-1) + 1(y-1)$$

$$z = 1 + x - 1 + y - 1 = x + y.$$

$$z = x + y.$$

$$f(x, y) = xy ; \text{ tangent at } (1, 1)$$

$$\nabla f(1, 1) = (1, 1).$$

$$z = 1 + 1(x-1) + 1(y-1)$$

$$= x + y - 1.$$

$$f(x, y) = \sin(xy) ; \text{ tangent at } (1, 0)$$

$$\nabla f(1, 0) = (0, 1).$$

$$z = 0 + 0(x-1) + 1(y-0)$$

$$= y.$$

$$\text{Eqn.} : z = y.$$

The tangent hyperplane

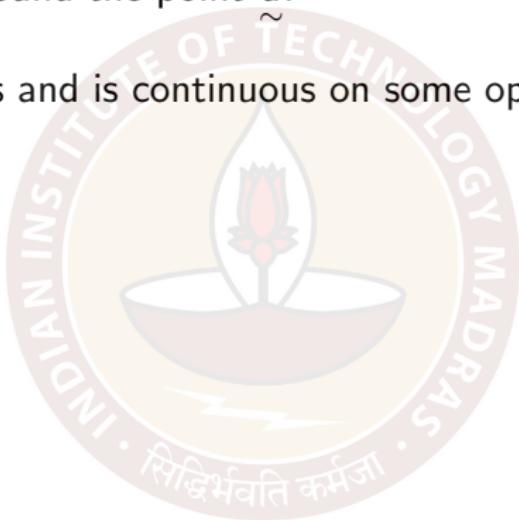
Let $\tilde{f}(\tilde{x})$ be a function defined on a domain D in \mathbb{R}^n containing some open ball around the point \tilde{a} .



The tangent hyperplane

Let $\tilde{f}(\tilde{x})$ be a function defined on a domain D in \mathbb{R}^n containing some open ball around the point \tilde{a} .

Suppose $\nabla \tilde{f}$ exists and is continuous on some open ball around the point \tilde{a} .



The tangent hyperplane

Let $\tilde{f}(\tilde{x})$ be a function defined on a domain D in \mathbb{R}^n containing some open ball around the point a .

Suppose ∇f exists and is continuous on some open ball around the point a .

$$(\tilde{x}(t), z(t)) = (a, f(a) + t(u, f_u(a)))$$
$$z = \sum x_i \frac{\partial f}{\partial x_i}(a)$$
$$\sum u_i \frac{\partial f}{\partial x_i}(a)$$

Then the equation of the tangent hyperplane to f at (a, b) is given by :

$$z = f(a) + \sum \frac{\partial f}{\partial x_i}(a) (x_i - a_i)$$
$$= f(a) + \nabla f(a) \cdot (\tilde{x} - a)$$

Examples

$$f(x, y, z) = xy + yz + zx ; \text{ tangent at } (1, 1, 1)$$

$$\nabla f(x, y, z) = (x+y, y+z, z+x)$$

$$\nabla f(1, 1, 1) = (2, 2, 2).$$

Tangent hyperplane eqn.
is:

$$u = f(1, 1, 1) + \nabla f(1, 1, 1) \cdot (x-1, y-1, z-1)$$

$$u = 3 + (2, 2, 2) \cdot (x-1, y-1, z-1)$$

Eqn. is

$$u = 3 + 2(x-1) + 2(y-1) + 2(z-1).$$

$$f(x, y, z) = x^2 + y^2 + z^2 ; \text{ tangent at } (2, 3, -1)$$

$$\nabla f = (2x, 2y, 2z)$$

$$u = f(2, 3, -1) + \nabla f(2, 3, -1) \cdot (x-2, y-3, z+1)$$

$$\nabla f(2, 3, -1) = (4, 6, -2).$$

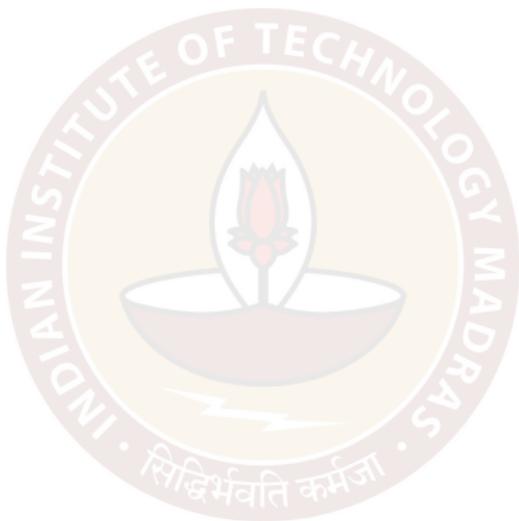
$$= 14 + (4, 6, -2) \cdot (x-2, y-3, z+1)$$

$$\text{Eqn. is : } u = 14 + 4(x-2) + 6(y-3) - 2(z+1).$$

Caution : tangent planes need not always exist.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$f(x, y) = |x| + |y|$$



Linear approximation

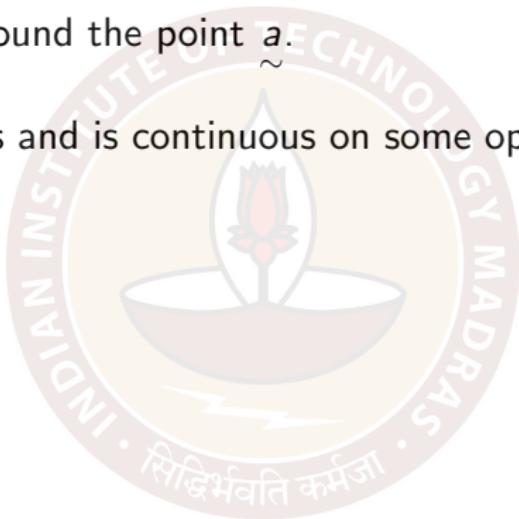
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Linear approximation

Let $\underset{\sim}{f(x)}$ be a function defined on a domain D in \mathbb{R}^n containing some open ball around the point $\underset{\sim}{a}$.

Suppose ∇f exists and is continuous on some open ball around the point $\underset{\sim}{a}$.



Linear approximation

Let $\tilde{f}(\tilde{x})$ be a function defined on a domain D in \mathbb{R}^n containing some open ball around the point \tilde{a} .

Suppose ∇f exists and is continuous on some open ball around the point \tilde{a} .

Then the function $L_f(\tilde{x}) = \tilde{f}(\tilde{a}) + \nabla f(\tilde{a}) \cdot (\tilde{x} - \tilde{a})$ is the best linear approximation for the function f close to \tilde{a} .

Examples

Linear approximation to $f(x, y) = xy$ at $(1, 1)$

$$\nabla f(x, y) = (y, x)$$
$$\nabla f(1, 1) = (1, 1).$$

$$L_f(x, y) = f(1, 1) + \nabla f(1, 1) \cdot (x-1, y-1)$$
$$= 1 + (1, 1) \cdot (x-1, y-1)$$
$$= 1 + x-1 + y-1 = x+y-1.$$

is the best linear
approx. to f close
to $(1, 1)$.

Linear approximation to $f(x, y, z) = x^2 + y^2 + z^2$ at $(2, 3, -1)$

$$\nabla f(2, 3, -1) = (4, 6, -2).$$

$$L_f(x, y, z) = 3 + (4, 6, -2) \cdot (x-2, y-3, z+1)$$
$$= 3 + 4(x-2) + 6(y-3) - 2(z+1)$$
$$= 4x + 6y - 2z - 29.$$

is the best linear
approx. to f close to $(2, 3, -1)$.

Thank you

