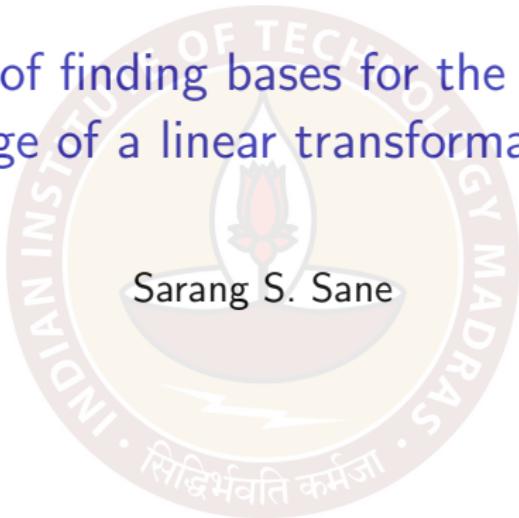


Examples of finding bases for the kernel and image of a linear transformation



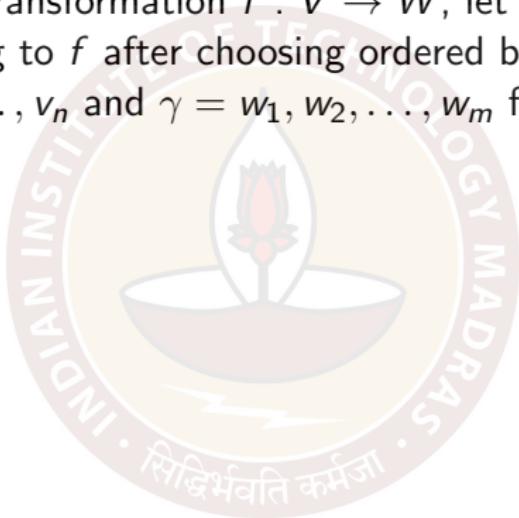
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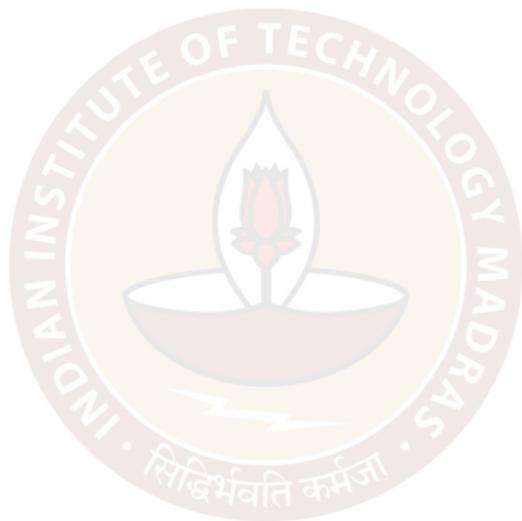
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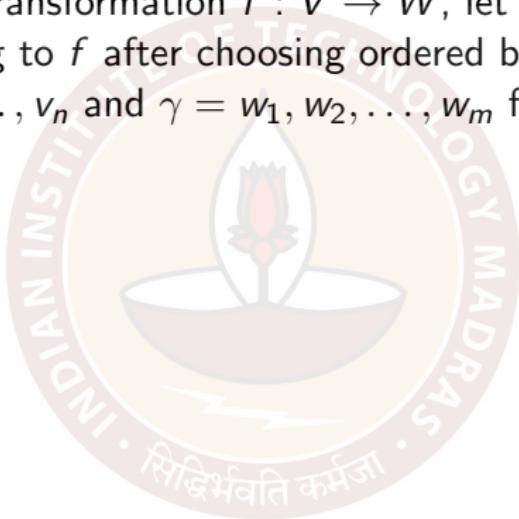
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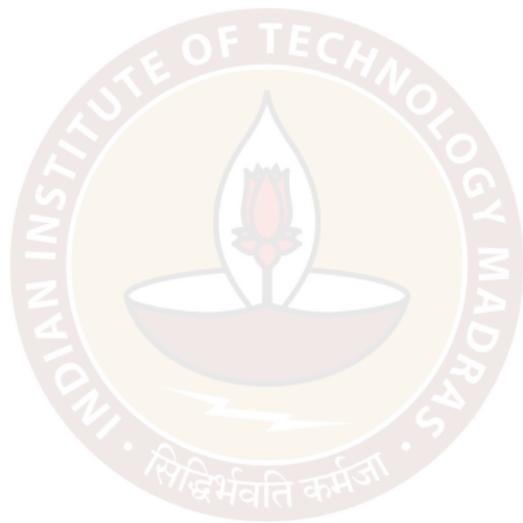
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Example

Consider $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4) = (2x_1 + 4x_2 + 6x_3 + 8x_4, x_1 + 3x_2 + 5x_4, x_1 + x_2 + 6x_3 + 3x_4)$



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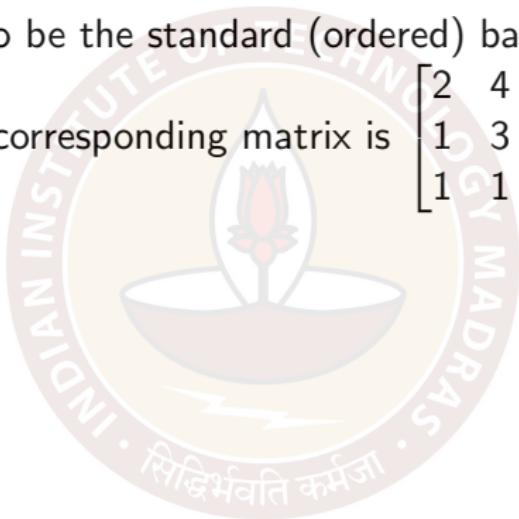


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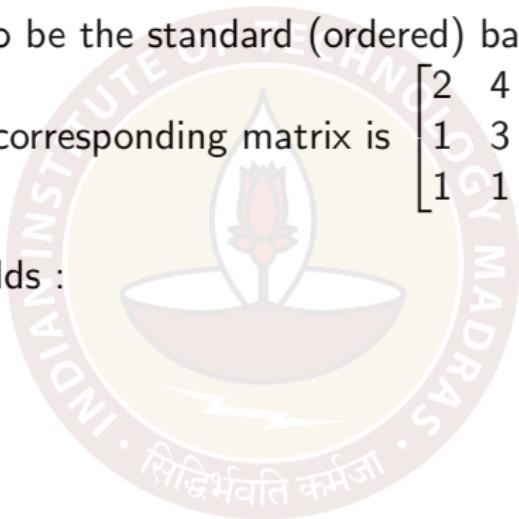
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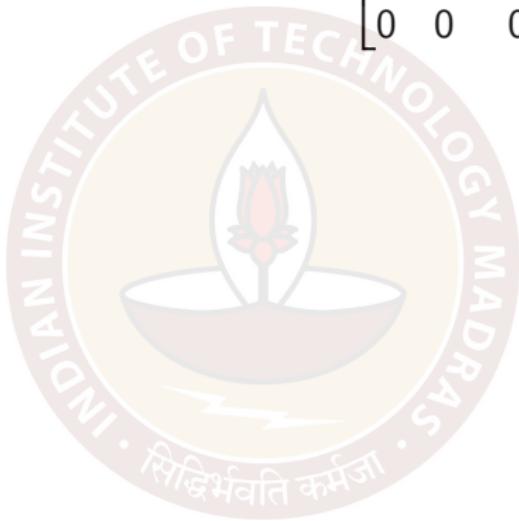
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Since we have chosen β to be the standard ordered basis, the basis for the $\ker(T)$ is also the same, i.e.

$$-9e_1 + 3e_2 + 1e_3 + 0e_4 = (-9, 3, 1, 0) \text{ and} \\ (-2e_1 - 1e_2 + 0e_3 + 1e_4) = (-2, -1, 0, 1).$$

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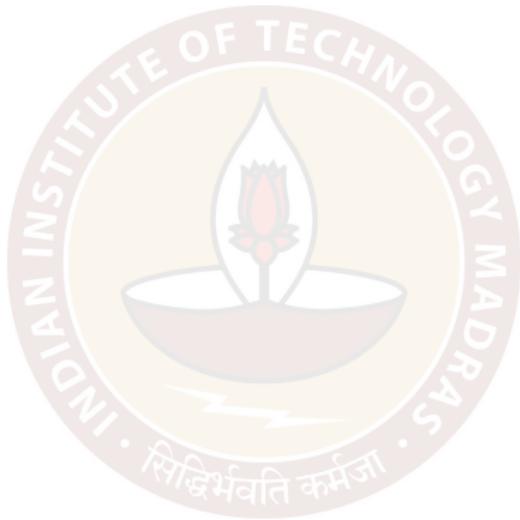
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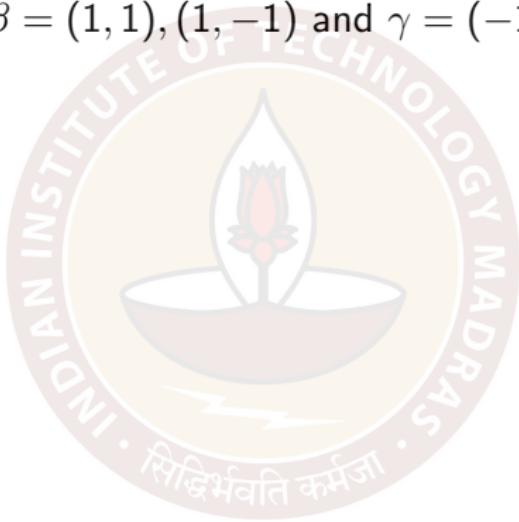
Another example

Let $V = \mathbb{R}^2$, $W = \{(x, y, z) | x + y + z = 0\}$.



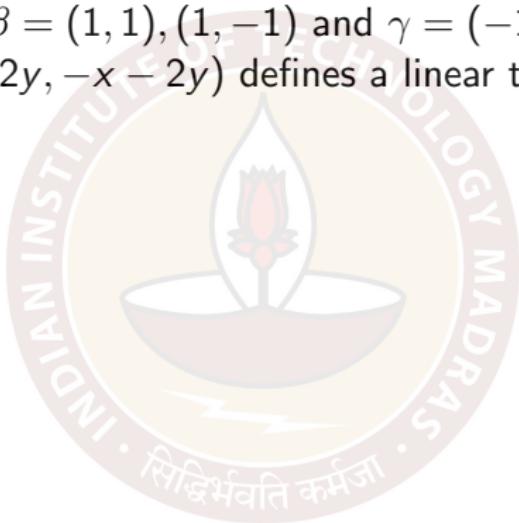
Another example

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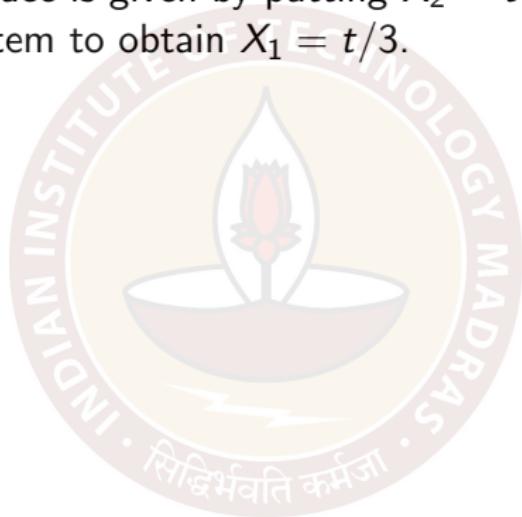
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Example(contd.)

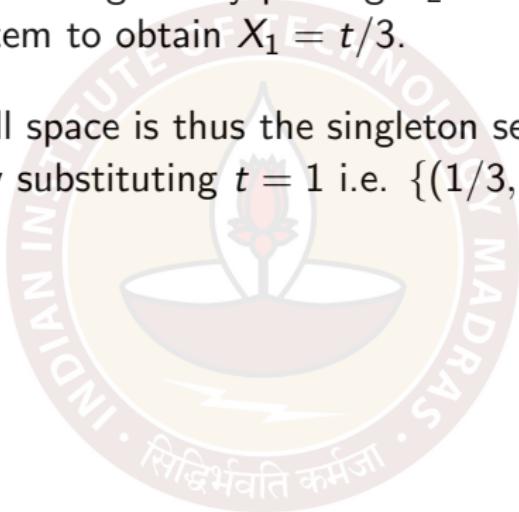
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A basis for the null space is thus the singleton set consisting of the vector obtained by substituting $t = 1$ i.e. $\{(1/3, 1)\}$.



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The rank-nullity theorem for linear transformations

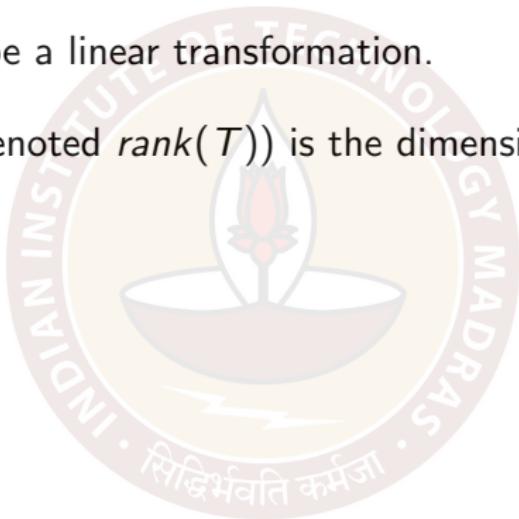
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The rank-nullity theorem for linear transformations

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The rank of T (denoted $\text{rank}(T)$) is the dimension of $\text{Im}(T)$.

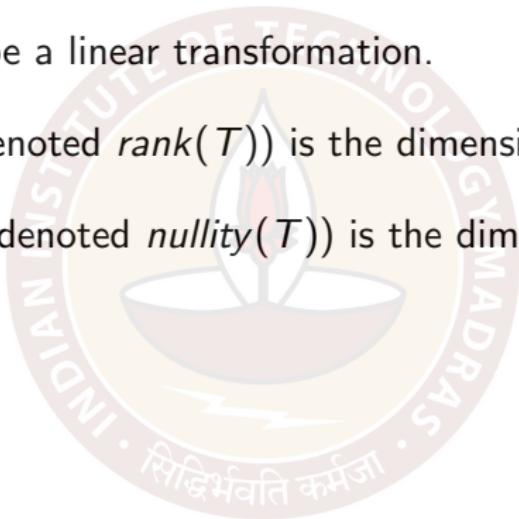


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Reinterpreting the rank-nullity theorem for matrices, we obtain :

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Thank you

