INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 04
Determinants (Part- 1)

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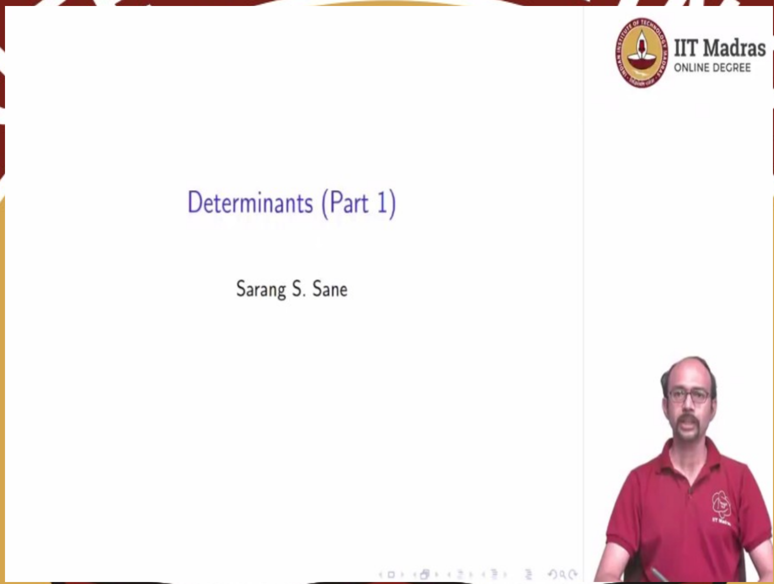

Hello, and welcome to the Maths 2 component of the BSc program on Data Science. So, far, in this in this component in Maths 2 we have looked at vectors, we have looked at matrices, and we have looked at Systems of Linear Equations, and its solutions. Today we are going to study meaning in this video, we are going to study something called the Determinant of a Square Matrix. We have two videos on this idea.

So, in this video, we will look at the determinant of small matrices. So, matrices of size 1 by 1, 2 by 2 and 3 by 3. And we will familiarize ourselves with some with how to compute them and some of the properties. And in the next video, we will see the more general version for any square matrix.

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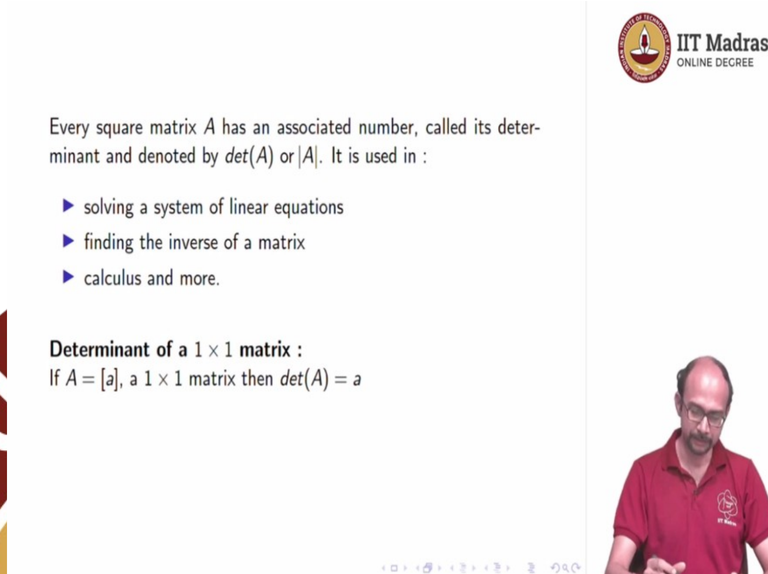

So, every square matrix A, we associate a number called the determinant of the square matrix. So, I will put in a caveat right at the start for this video and for the next, whenever I talk about matrix, I mean a square matrix if I do not say it explicitly. So, how do we write this number called the Determinant, we usually use the notation that. So, det(A) or the notation, |A|, So, it is one of these two. So, why do we want to study this? So, this is a very important question that one should ask whenever we study any concept.

So, determinants are going to be used in solving a system of linear equation, in finding the inverse of a matrix. And later on, when we do calculus, you will see determinants coming in again. And then there are many other uses. It is a very, very powerful concept. And unfortunately, we cannot, we would not be doing all those here. But it is a very, very useful concept beyond these 3 uses.

So, let us begin with the 1 by 1 matrix. So, when you have a 1 by 1 matrix, So, A is, it is a number with 2 brackets on the side. So, then the determinant is just that number. It is just A, So, if A is [a], then determinant of A is the number a. So, nothing very great happening So, far.

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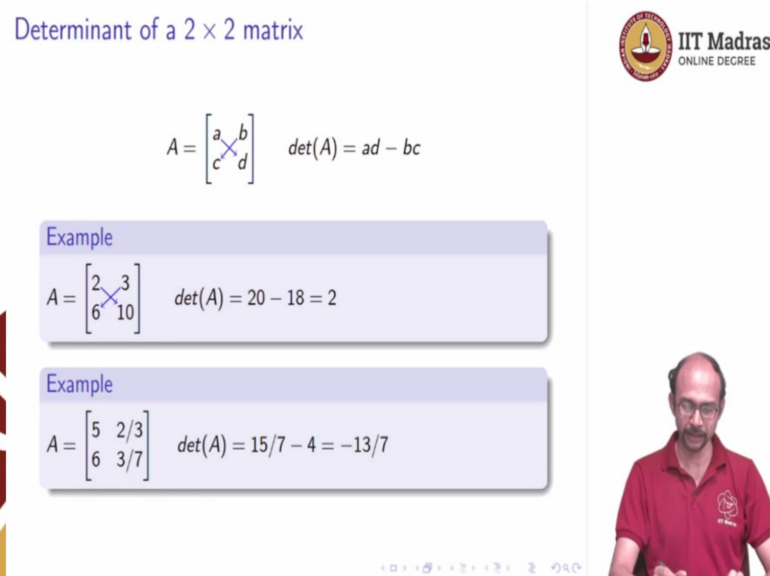

So, let us look at the determinant of a 2 by 2 matrix. So, here is your 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. I have written the entries, a, b in the first row and c, d in the second row. And how do we compute the determinant. So, we compute the determinant by looking at the diagonal. So, on the diagonal, we have a and d, So, we multiply those. And then we look at the non-diagonal entries. So, we have b and c.

So, we multiply those and then we subtract out bc from ad. So, determinant of A is $ad - bc$. The arrows here are very crucial. So, keep those in mind. So, let us do an example. This is a just for a start. So, here is your matrix A, which is $\begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$. It is a 2 by 2 matrix with those entries. So, what is determinant of A? Determinant of A is $2 \times 10 - 3 \times 6$. So, $20 - 18$, which is 2. Here is another example. So, $\begin{bmatrix} 5 & \frac{2}{3} \\ 6 & \frac{3}{7} \end{bmatrix}$.

So, if you multiply the diagonal entries, you get $\frac{15}{7}$, if you multiply those, the other 2 entries, you get $\frac{2}{3}\times 6$, which is 2×2, which is 4. So, then we get Determinant of A is $\frac{15}{7}$- 4, which is $\frac{-13}{7}$, So, $\frac{15-28}{7}$, which is $\frac{-13}{7}$. So, we know how to calculate the determinant of A 2 by 2 matrix.

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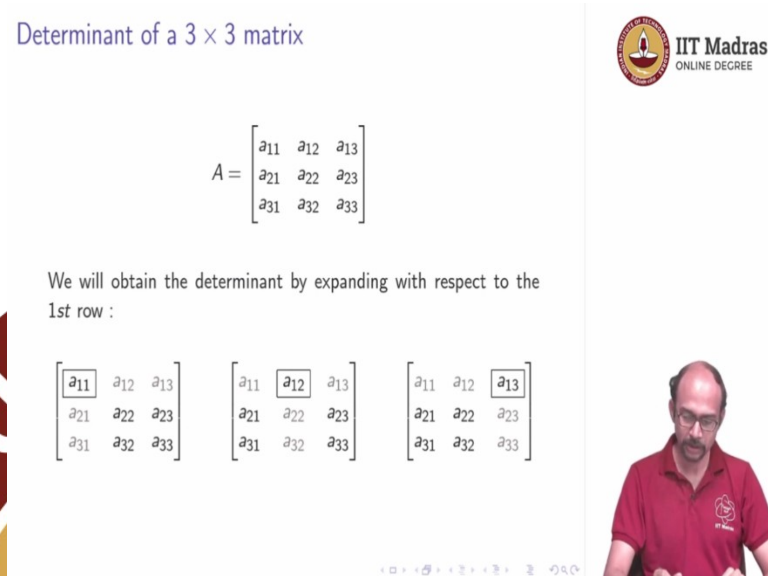

Let us move on to a 3 by 3 matrix. So, suppose we have a 3 by 3 matrix. Let us call it A and then now we write the entries in the way we learned in the matrix video, So, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. So, what is the determinant of A? So, we are going to obtain this in a very particular manner. So, we will say we will obtain the determinant by expanding with respect to the first row. So, what do we do in the first row, we write these 3 new expressions.

So, you can see a11 highlighted here and the entries in the first 1, the first column have been deleted. So, it is not as bold as the others. Similarly, for the next entry in the first row, a_{12}, we look at the entries are other than the ones in that row and column. So, we do not look at the other 2 entries in the first row. And we do not look at the other 2 entries in the first column in the second column. And similarly, we look at a_{13}, and we neglect the entries in the first row and the third column. So, keep this picture in mind as we go ahead. So, what do we do with these entries?

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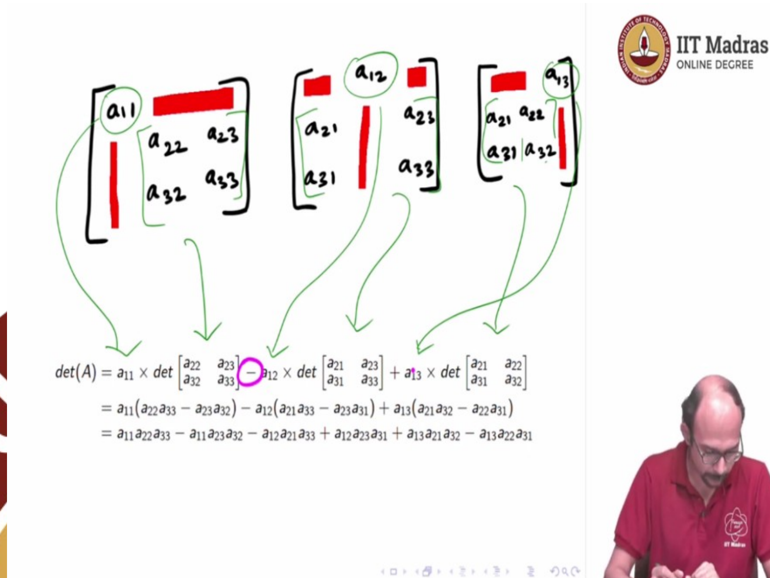

So, determinant of A is this rather strange looking expression, it is a_{11} × determinant of the 2 by 2 matrix that was left out in that first expression, - a_{12} × determinant of the 2 by 2 matrix that was left out in the second expression, + a_{13} × determinant of the 2 by 2 matrix which was left in the third expression.

So, let us just work this out, because now we know how to compute the determinant of 2 by 2 matrices. So, this is $a_{11} \times a_{22}$, $a_{33} - a_{23}$, $a_{32} - a_{12} \times a_{21}$, $a_{33} - a_{23}$, $a_{31} + a_{13}\, a_{21} a_{32}, -a_{22}$, a_{31}, and then we can expand the brackets. And we get some rather nasty expression with six terms. So, this is the determinant of A. Let us quickly recall from the previous page, what we had those three expressions were and maybe we can rewrite these terms.

So, this is what the expression for the first in the first expression look like. Similarly, the second expression looked like this a_{12}, a_{21}, a_{23}, a_{31}, a_{33}. And the third expression look like a_{13}, a_{21}, a_{22}, a_{31}, a_{32}. So, you can see what is happened. We have dropped the entries over here. And over here, similarly, here, we have dropped the entries over here and the rest of this row and we have dropped the entries over here, fine. So, what is the point?

So, the point is that the first expression over here, this a_{11} is what is coming here, and then this matrix is what is coming here. And you compute this determinant. Similarly, for the second expression, you take this a_{12} that is what coming here, and this matrix, So, we move it to the let us say it is a 2 by 2 matrix. And we compute its determinant and that is what is coming here. And then similarly, a_{13} is coming here.

And this 2 by 2 matrix is coming here, and you compute its determinant. So, remember, very important that over here, the first entry came with a + sign, the second entry came with a - sign. And then the third entry again, came with a + sign. So, it alternates. So, these signs alternate, this is a very important property than they alternate. And this, this is, what is maybe at the heart of this definition. So, I hope you understood how to compute the determinant, or how it is defined for a 3 by 3 matrix. So, let us do an example.

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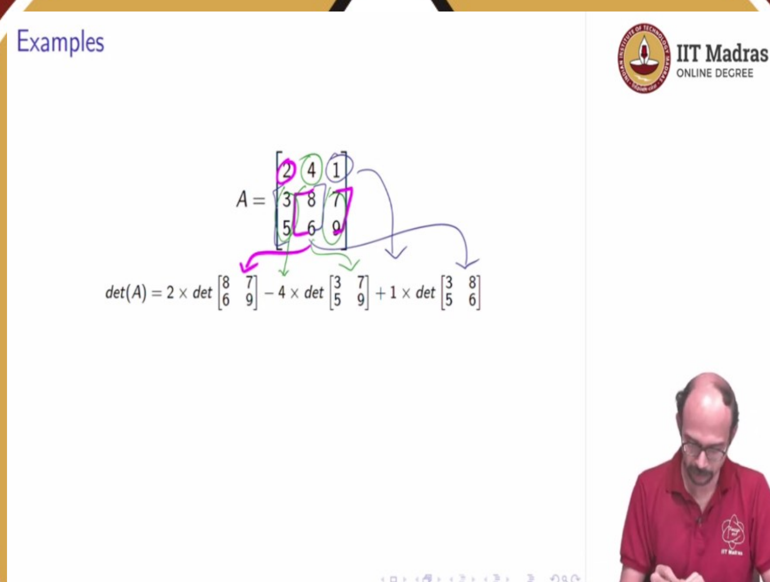

So, here is an example. So, A is this matrix here $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$. Let us see how to compute it? So, the 2 goes, you first look at the first and the 1 1 entry which is 2 So, that is over here. Then look at whatever is remaining, when you drop everything else in that corresponding row and column. So, that is what came here and compute this determinant multiply them.

Then you look at this 4 this 4 came here. Very importantly, with a negative sign, you look at $\begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix}$. And that matrix is what comes here, you take a determinant, multiply them, and then +

the third term, which is you take this 1 here, and then take the matrix left out, which is $\begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix}$, after you drop everything else in the corresponding row and column to the 1 and now we know how to compute determinant of 2 by 2 matrices.

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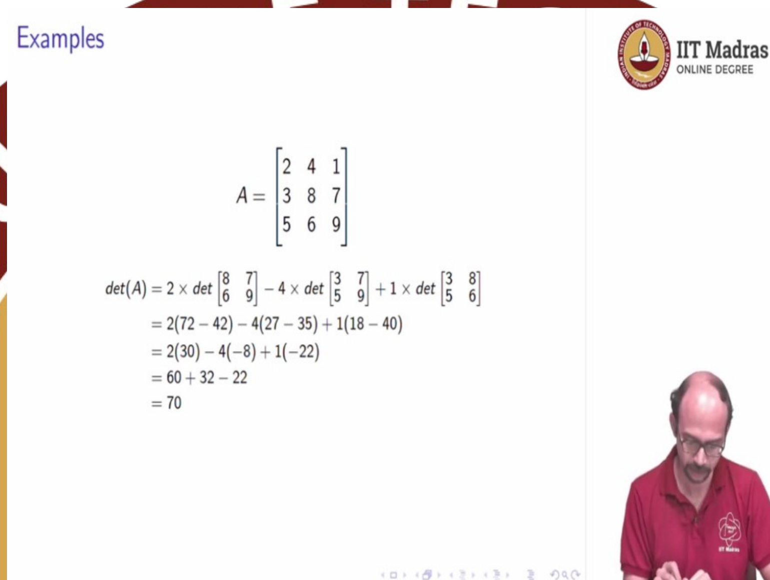

So, from there, we will be able to compute this determinant. So, let us come complete the calculation. So, this is 2 × determinant of this 2 by 2 matrix $\begin{bmatrix} 8 & 6 \\ 7 & 9 \end{bmatrix}$. So, 9 × 8 is 72. That is what we get in the first term here, - 7 × 6 is 42. That is what we get in the second term here. Similarly, here, we have - 4 × determinant of $\begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix}$. So, that determinant works out to be 3 × 9 - 7 × 5, which is 27, - 35.

And then + 1 × determinant of $\begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix}$, which works out to 3 × 6 – 8 × 5. So, that is 18 - 40. So, that is how you get this expression. And the rest of it is a standard computation, which if you complete you will get you should be getting 70. So, I hope this, this example, explained to you how to compute the determinant in case you got lost in the previous expressions. So, it is a very simple calculation. Maybe the important question is why we are indeed interested in this and very soon, in a couple of videos will answer that question.

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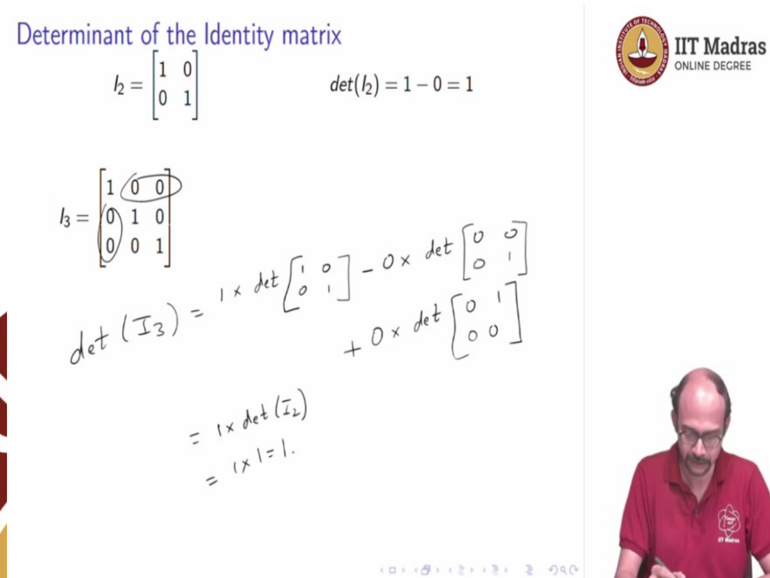

So, let us look at some special determinants. So, remember that we had some special matrix called the Identity Matrix. So, that was a matrix which had 1 in the diagonal entries and 0s everywhere else. So, the identity matrix of size, the 2 by 2 identity matrix, that is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. And what is its determinant its 1 × 1 - 0 × 0, which is 1. Now let us do the 3 by 3 determinant. So, maybe this one, I will work out the details over here.

So, for determinant of I_3, let us remember what we want how we got this expression. So, determinant of I_3 was, so, look at the 1 1 term, that is 1 × determinant of whatever you get after dropping everything else in that row and column. So, you drop these 0 0 and 0 0, when you write down this matrix, So, this is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so, this is the 2 by 2 identity matrix. And then you subtract out, you take the 1 2 term, and determinant of whatever you obtain after dropping that the first row and the second column.

So, which is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, you can see here, we do not really need to do this computation because I was 0 × something, but I am just writing down the expression. And then +, you look at the 1 3 term, which is again, 0 × determinant of whatever you get after dropping the first row and the third column, which is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This is the expression for the determinants. And as we noted, we do not really need to compute anything here, because this is just 1 × determinant of I_2.

The other 2 terms do not contribute anything because they are 0s and determinant of I2 we have computed upstairs it is 1. So, this is 1 × 1 is 1. So, I hope this, this showed you that the determinant calculation is really quite straightforward. And the determinant is computed in a sort of inductive way, meaning, if you want to compute the determinant of a 3 by 3 matrix, the computation reduces to determinants of a 2 by 2 matrix.

So, this is an important idea, which you should keep in mind as we go ahead, fine. So, let us look at some properties of the determinant. So, now, just to recall, we know what is the determinant of 1 by 1 matrix of a 2 by 2 matrix and a 3 by 3 matrix. So, let us go ahead.

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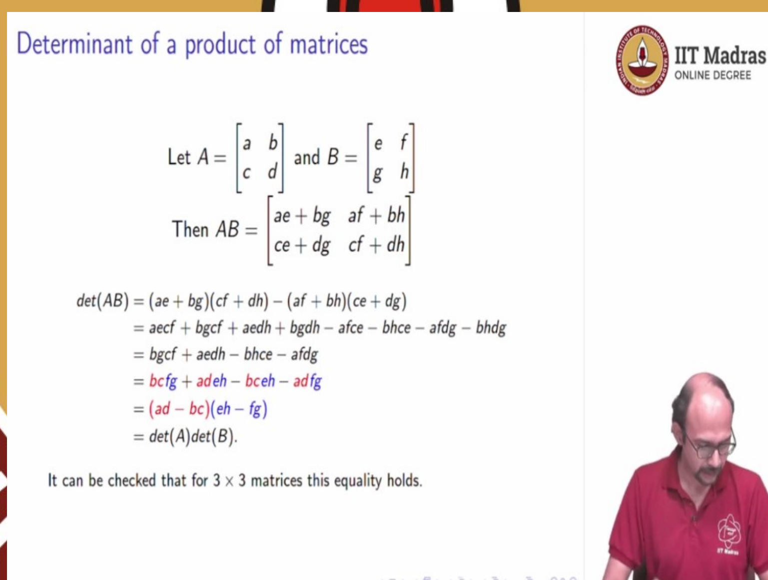

So, suppose we have a product of matrices. We know how to take the product of matrices. So, Let us say we have two 2 by 2 matrices, A and B. So, A is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B is $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$. So, let us compute A × B, A × B is $\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$. So, this is obtained by multiplication of matrices as we have seen in the previous video. Let us compute what is the determinant of A × B.

So, determinant of A × B by definition, is you take the diagonal entries. So, that is $(ae+bg)\times(cf+dh)$ -, you take the product of the non-diagonal entries. So, $(af+bh)\times(ce+dg)$. So, just to recall for you, this was how it worked. So, that is what we have done. So, now let us expand the brackets. So, if you expand the brackets, you get $ae\times cf, +bg\times cf+ae\times dh+bg\times dh-af\times ce-bh\times ce-af\times dg-bh\times dg$.

As you can see, this is something you should probably try to do on the side on your own. These are other long expressions. So, let us see what we get from there. So, what we get here is $bg\times cf$ +, So, there is some cancellation. So, $bg\times cf+ae\times dh$. So, what got cancelled, the *aecf* got cancelled with the *afce* and then the *bgdh* got cancelled with the *bhdg*. So, we have 4 terms remaining in this expression.

So, we get $bg\times cf+ae\times dh-bh\times ce-af\times dg$. So, let us now put these terms in a particular way. So, we will write this as $bc\times fg+ad\times eh-bc\times eh-ad\times fg$. So, what have we done? We have taken the entries coming from A first and the entries coming from B next, So, you can see some× a common here. So, let us take the common terms out. So, if you do that, you are going to get $ad-bc\times eh-fg$. So, the first term here is exactly determinant of A $ad-bc$.

And the second expression here is exactly the determinant of the B $eh-fg$. So, what does this show? This shows that if you take two 2 by 2 matrices, and take the product, then the determinant of the product is the product of the determinants, determinant of A B is determinant of A × determinant of B, you can check that this works for 3 by 3 matrices as well. So, that is going to be a slightly longer argument with more expressions involved, but it will work out exactly the same way. So, you will have to just play with more terms. So, I would not explicitly show this, but believe me on this, or rather, I would say work it out for yourself.

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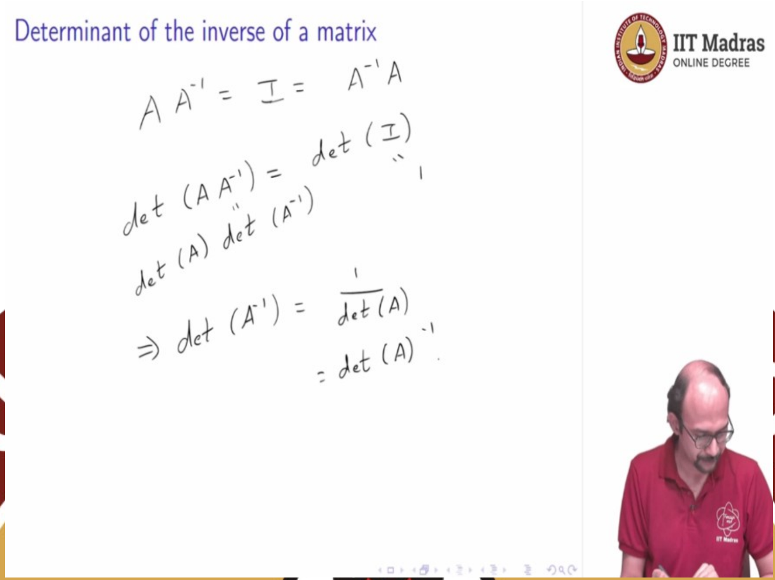

So, let us go on and ask ourselves. What is the determinant of the inverse of a matrix? Can I compute it from the determinant of a matrix, we define the inverse using this identity. So, the inverse satisfies A × A^{-1} is I, the identity matrix, and we can work out the A inverse × A is also I. So, now, we want to talk about the determinant of the inverse. So, let us take determinant on both sides. So, if we do that, we get determinant of A × A^{-1} is determinant of I. But we just saw that determinant of a product is the product of the determinants.

So, determinant of A × A inverse is the same as determinant of A determinant of A^{-1}. And on the other hand determinant of the identity matrix, we have computed to be 1. So, what does this mean? This means that determinant of A^{-1} is $\frac{1}{det\left(A\right)}$1 by determinant of A or I can write this in more, more fancy way as determinant of A^{-1}. So, determinant of A^{-1} is the reciprocal of the determinant of A, So, it is $\frac{1}{det\left(A\right)}$. So, we know explicitly how to compute the determinant of the inverse from the determinant of A.

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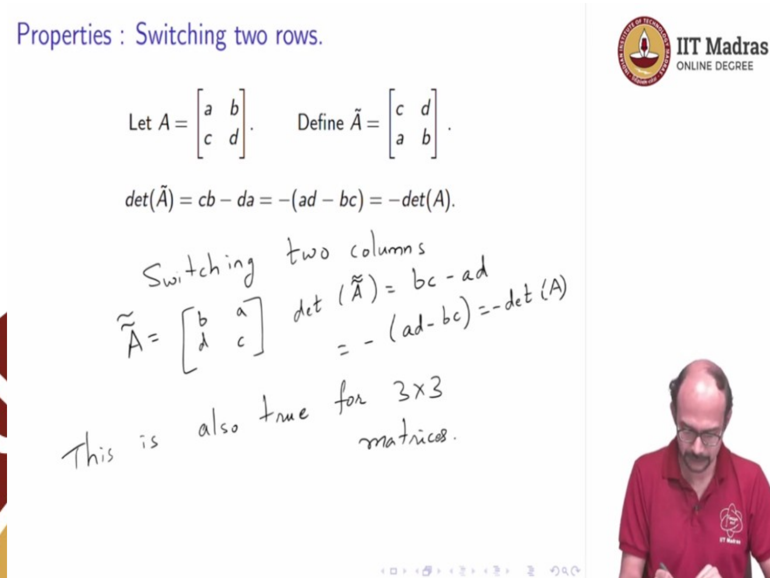

Let us look at some properties of the determinant. So, let us see what happens if we switch two rows. What does that mean? So, here is A, which is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let us define $\widetilde{A}$ by switching the first and the second row. So, what happened in $\widetilde{A}$, you swap the first and the second row. So, cd came first, and then ab was next. So, that is $\widetilde{A}$. So, we have switched two rows. Let us compute determinant of $\widetilde{A}$.

So, determinant of $\widetilde{A}$ is c × b, - d × a, which we can rewrite as - of ad - bc, which is exactly the determinant of A. So, this is - determinant of A. So, what happened when we switch two rows, we saw that the determinant of the new matrix was the same as the determinant of A except that it picked up an extra - sign. So, it is - of determinant of A, what happens when we switch two column.

So, let us do that same computation with a switching two columns. So, when we switch two columns, so, let us call it maybe $\widetilde{A}$. So, let us take the same matrix A and switch these two columns. So, we get bd ac, and what is it is determinant. So, its determinant is bc - ad. Yes. And I think you can identify this expression, this expression is exactly - of ad - bc, which is - of determinant of A.

So, the statement for rows also holds for columns. Namely, if we switch two columns, the determinant of the new matrix is the same as the determinant of A with a - sign. And the same thing can be checked for a 3 by 3 matrix. So, this is also true for 3 by 3 matrices. So, I leave that to you to check, it is a pretty easy check. Not, not hard at all.

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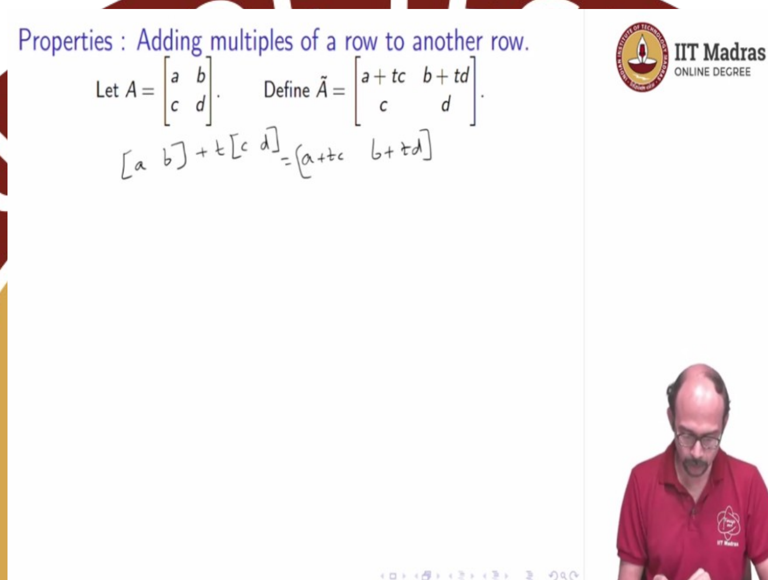

So, let us look at another property, if you add the multiple of a row to another row. So, what do we mean by that? First of all, so, here is A is abcd. And $\widetilde{A}$ is a + t × c, b + t × d, and then cd. How did we get $\widetilde{A}$? Well, the first row of A is ab. So, to this row, we added t × the row cd. And what we got was a + tc, b + td. So, here the t × the second row was added to the first row. So, remember here that when we say multiples here, they could be fractional multiples as well. It need not be integer multiples.

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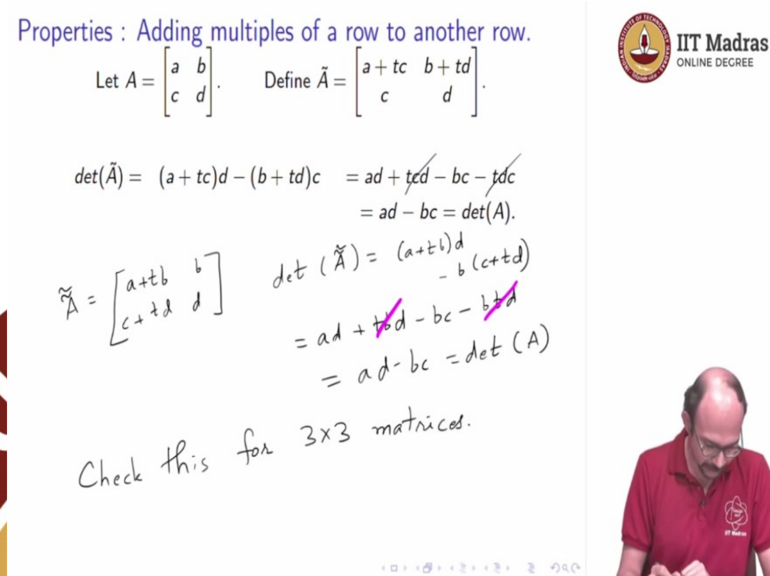

So, let us see what happens to the determinant. So, let us compute the determinant of $\tilde{A}$. So, determinant of $\tilde{A}$ is $a+t\times c\times d-b+t\times d\times c$. And let us expand the bracket. If you expand the bracket, you get $ad+t\times cd-bc,-t\times dc$. And you can see that the $t\times cd$ and the $t\times dc$ cancels out. And we are left with $ad-bc$, which is exactly the determinant of A.

So, what happened if you added a multiple of one row to another row, took that new matrix and computed its determinant, you got back the determinant of A. So, the determinant does has not change under this operation. Let us see what happens for columns. So, suppose I add t × the second column to the first column. So, a + t × b, a +, c + t × d. And then this is will bd. So, now what have we done? We have had it t × a second column to the first column.

This is our new matrix $\tilde{\tilde{A}}$A double tilde. Let us compute determinant of $\tilde{\tilde{A}}$. So, that is the product of the diagonal entries, so $a+tb\times d-¿$ the product of the non diagonal entries, which is b $\times c+t\times d$. So, that gives us $ad+t\times bd-bc+b\times td$. And you can see what is happening. t × bd and b × td, I want to cancel a, I have incorrectly put a sign this should be -.

So, now what we get here is $ad-bc$, which is exactly determinant of A. So, what happened? If you added a multiple of one column to another column, then the new matrix obtained has the same determinant as the original matrix. So, under this operation, the determinant remains unchanged. And you can take this for 3 by 3 matrices. The same results will work.

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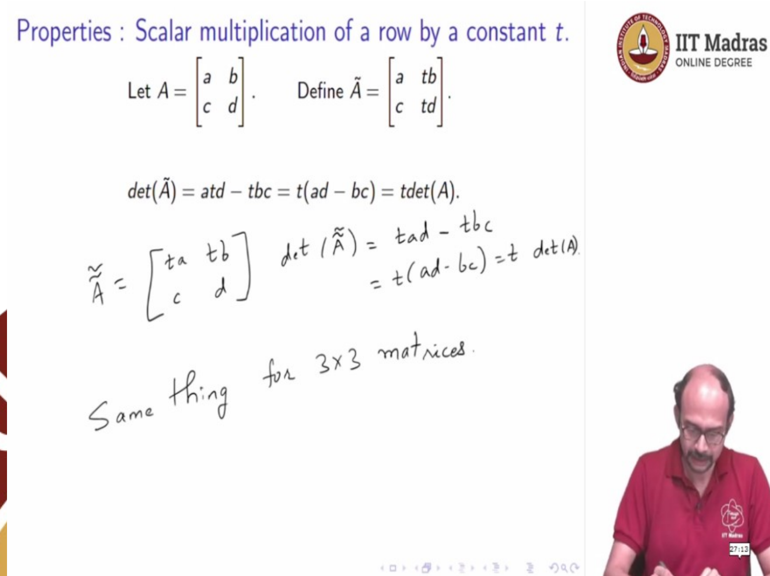

And finally, let us look at what happens when we multiply a row by a constant t. So, suppose A is abcd. And suppose we multiply. So, I think I made a mistake in the example. So, here we are multiplying a column and not a row. So, suppose we multiply this column by t. So, we get a tb c td. So, let us find out what the determinant of $\tilde{A}$ is? So, the determinant of $\tilde{A}$ is $a \times td - tb \times c$, which is if you take t common, So, $t \times ad - bc$.

So, this is exactly t × determinant of A. And now let us do what we originally intended to do, as in the title. So, suppose $\tilde{\tilde{A}} = \begin{bmatrix} ta & tb \\ c & d \end{bmatrix}$. So, in that case, determinant of $\tilde{\tilde{A}}$ is $t \times a \times d - t \times b \times c$, you can take the t common. So, you get $t \times ad - bc$, which is t × determinant of A. So, what have we shown, we have shown that if you multiply a particular row, meaning every entry have that row by some constant t.

Or you take some column and multiply every entry of that column by some constant t, then you compute the determinant of this new matrix, then the determinant of the new matrix is t × determinant of the original matrix. And again, you can check that the same thing happens. So, same results hold for 3 by 3 matrices. Fine, so, we have seen some properties of the determinant. So, let us quickly recall what we did today, we have defined something called the determinant for square matrices of size 1 by 1, 2 by 2 and 3 by 3. We saw that for the 3 by 3 case, they were defined by using what was the definition for the 2 by 2 case? So, some kind of inductive definition that is what we call it. And then we saw some properties, very important property was that, if you take the product of 2 matrices, then the determinant of that product is the product of the determinant. So, determinant of AB is determinant of A × determinant of B.

Then we saw the inverse of a matrix as determinant inverse of the, the reciprocal of the determinant of A and then we saw some other properties which the determinant satisfied. Namely, if you multiply a row by a constant or a column by a constant, that constant comes out of the determinant. If you add a multiple of one row to another or one column to another, the determinant is unaffected. And if you swap rows, then the determinant picks up a - sign.

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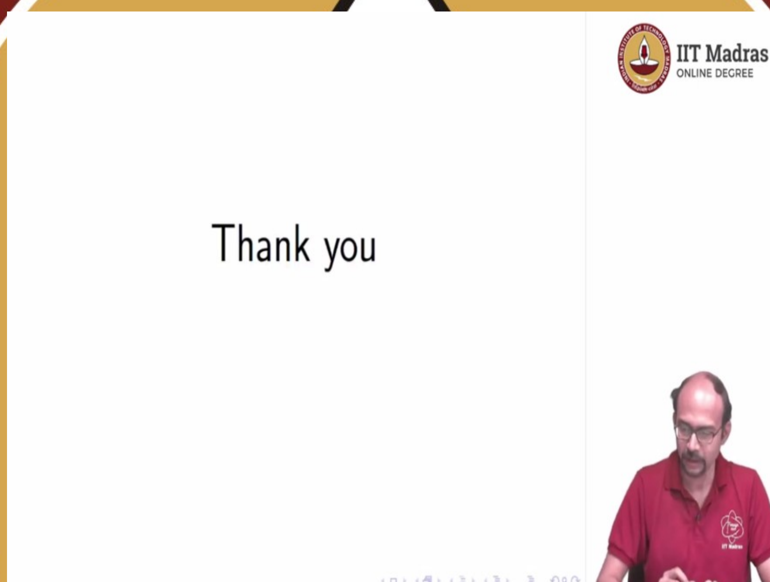

So, with that, I think we will end this video. Thank you.