Finding the tangent (hyper)plane

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If u is a unit vector in the direction of the line L, then the tangent (if it exists) will be the line with slope $f_u(\underset{\sim}{a})$ passing through the point $(\underset{\sim}{a}, f(\underset{\sim}{a}))$ and so its parametric equation is :

$x(t) = a + t u_1, \quad y(t) = b + t u_2, \quad z(t) = f(a,b) + t f_u(a,b)$

$u = (u_1, u_2)$

$\underset{\sim}{a} = (a, b).$

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Then all the tangent lines at the point (a, b) exist and we can rewrite the equation of a tangent line in the direction of the unit vector u as :

$$f_u(a,b) = \nabla f(a,b) \cdot u = \frac{\partial f}{\partial x}(a,b)\, u_1 + \frac{\partial f}{\partial y}(a,b)\, u_2 .$$

$$x(t) = a + u_1 t, \quad y(t) = b + u_2 t, \quad z(t) = f(a,b) + f_u(a,b)\, t = f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\, u_1 + \frac{\partial f}{\partial y}(a,b)\, u_2\right) t$$

$$z(t) - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x(t)-a) + \frac{\partial f}{\partial y}(a,b)(y(t)-b)$$

Tangent lines in terms of linear algebra for $f(x, y)$

$$\underbrace{(x(t), y(t), z(t))} = (a, b, f(a,b)) + t(u_1, u_2, f_u(a,b)).$$

Tangent line to f at (a,b) in the direction of u

$$= (a, b, f(a,b)) + W_u$$

W_u: Line passing through the vector $(u_1, u_2, f_u(a,b))$.

$$f_u(a,b) = u_1 \frac{\partial f}{\partial x}(a,b) + u_2 \frac{\partial f}{\partial y}(a,b)$$

The lines W_u all lie on the plane

$$P: \quad z = \frac{\partial f}{\partial x}(a,b)\, x + \frac{\partial f}{\partial y}(a,b)\, y.$$

Tangent plane of f at (a,b) $= (a, b, f(a,b)) + P.$

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Suppose ∇f exists and is continuous on some open ball around the point (a, b).

Then the equation of the tangent plane to f at (a, b) is given by :

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

$$\frac{\partial f}{\partial x}(a,b)x + \frac{\partial f}{\partial y}(a,b)y - z = \frac{\partial f}{\partial x}(a,b)a + \frac{\partial f}{\partial y}(a,b)b.$$

Examples

$f(x, y) = x + y$; tangent at $(1, 1)$

$\nabla f(1,1) = (1,1).$

$z = 2 + 1(x-1) + 1(y-1)$

$z = 2 + x - 1 + y - 1 = x+y.$

$z = x+y.$

$f(x, y) = xy$; tangent at $(1, 1)$

$\nabla f(1,1) = (1,1).$

$z = 1 + 1(x-1) + 1(y-1)$

$= x+y-1.$

$f(x, y) = sin(xy)$; tangent at $(1, 0)$

$\nabla f(1,0) = (0,1).$

$z = 0 + 0(x-1) + 1(y-0)$

$= y.$

Eqn. : $z = y.$

The tangent hyperplane

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Suppose ∇f exists and is continuous on some open ball around the point $\underset{\sim}{a}$.

$$(\underset{\sim}{x}(t), z(t)) = (\underset{\sim}{a}, f(\underset{\sim}{a})) + t(u, \underbrace{\boxed{f_u(\underset{\sim}{a})}})$$

$$\sum u_i \frac{\partial f}{\partial x_i}(\underset{\sim}{a})$$

$$z = \sum x_i \frac{\partial f}{\partial x_i}(\underset{\sim}{a})$$

Then the equation of the tangent hyperplane to f at (a, b) is given by :

$$z = f(\underset{\sim}{a}) + \sum \frac{\partial f}{\partial x_i}(\underset{\sim}{a})(x_i - a_i)$$

$$= f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a}) \cdot (\underset{\sim}{x} - \underset{\sim}{a}).$$

Examples

$f(x, y, z) = xy + yz + zx$; tangent at $(1, 1, 1)$

Tangent hyperplane eqn. is :

$\nabla f(x,y,z) = (x+y,\ y+z,\ z+x)$

$\nabla f(1,1,1) = (2,2,2)$.

$u = f(1,1,1) + \nabla f(1,1,1) \cdot (x-1, y-1, z-1)$

$= 3 + (2,2,2) \cdot (x-1, y-1, z-1)$

Eqn. is $u = 3 + 2(x-1) + 2(y-1) + 2(z-1)$.

$f(x, y, z) = x^2 + y^2 + z^2$; tangent at $(2, 3, -1)$

$\nabla f = (2x, 2y, 2z)$

$\nabla f(2,3,-1) = (4,6,-2)$.

$u = f(2,3,-1) + \nabla f(2,3,-1) \cdot (x-2, y-3, z+1)$

$= 14 + (4,6,-2) \cdot (x-2, y-3, z+1)$

Eqn. is : $u = 14 + 4(x-2) + 6(y-3) - 2(z+1)$.

Caution : tangent planes need not always exist.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$f(x, y) = |x| + |y|$$

Linear approximation

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Suppose ∇f exists and is continuous on some open ball around the point $\underset{\sim}{a}$.

Then the function $L_f(\underset{\sim}{x}) = f(\underset{\sim}{a}) + \nabla f(\underset{\sim}{a}) \cdot (\underset{\sim}{x} - \underset{\sim}{a})$ is the best linear approximation for the function f close to $\underset{\sim}{a}$.

Examples

Linear approximation to $f(x, y) = xy$ at $(1, 1)$

$\nabla f(x,y) = (y, x)$

$\nabla f(1,1) = (1, 1)$.

$L_f(x,y) = f(1,1) + \nabla f(1,1) \cdot (x-1, y-1)$

$= 1 + (1,1) \cdot (x-1, y-1)$

$= 1 + x - 1 + y - 1 = x + y - 1$. is the best linear approx. to f close to $(1,1)$.

Linear approximation to $f(x, y, z) = x^2 + y^2 + z^2$ at $(2, 3, -1)$

$\nabla f(2,3,-1) = (4, 6, -2)$.

$L_f(x,y,z) = 3 + (4, 6, -2) \cdot (x-2, y-3, z+1)$

$= 3 + 4(x-2) + 6(y-3) - 2(z+1)$

$= 4x + 6y - 2z - 29$. is the best linear approx. to f close to $(2,3,-1)$.

Thank you