Orthogonal transformations and rotations

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What are orthogonal transformations

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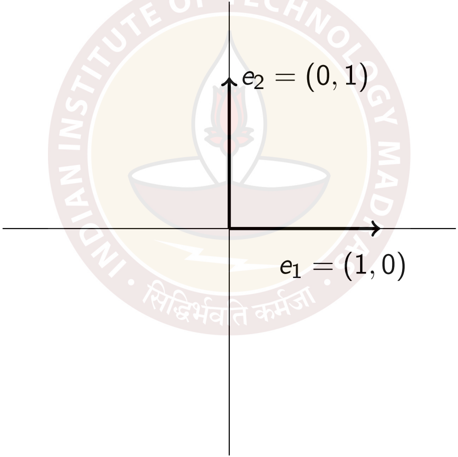
Fact : It is enough to demand that the linear transformation preserves lengths. In that case, angles automatically get preserved (think of triangle congruences).

Example

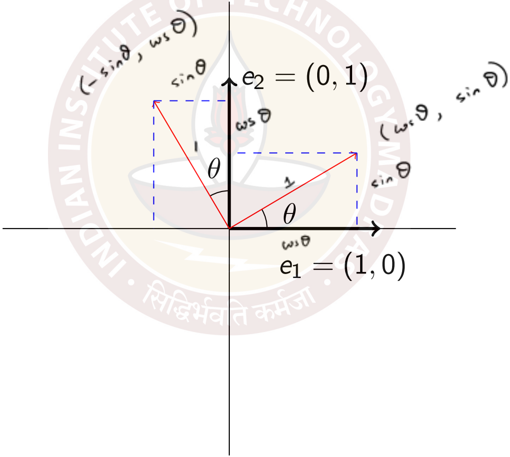
Finding the rotation matrix in $\mathbb{R}^2$

Consider the standard basis$\{(1,0),(0,1)\}$ of $\mathbb{R}^2$.



Finding the rotation matrix in $\mathbb{R}^2$

Consider the standard basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. Rotate the plane by an angle θ. The vectors obtained after rotation tell us the matrix corresponding to this linear transformation.



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Note that $R_\theta^T = R_{-\theta}$ and $\boxed{R_\theta^T R_\theta = R_\theta R_\theta^T = I.}$

Further note that since angles and lengths ~~and angles~~ are preserved and the standard basis is orthonormal, the rotated vectors are also orthonormal and therefore yield an orthonormal basis of $\mathbb{R}^2$.

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When considering the rotation about the Z-axis, e_3 remains unchanged and the XY-plane gets rotated exactly as in the previous case of $\mathbb{R}^2$. Therefore its matrix is

$$T_3(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotations in $\mathbb{R}^3$ (contd.)

Similarly, the matrix corresponding to rotation about the X-axis is

$$T_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix}$$

Rotations in $\mathbb{R}^3$ (contd.)

Similarly, the matrix corresponding to rotation about the X-axis is

$$T_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix}$$

and the matrix corresponding to rotation about the Y-axis is

$$T_2(\theta) = \begin{pmatrix} cos(\theta) & 0 & -sin(\theta) \\ 0 & 1 & 0 \\ sin(\theta) & 0 & cos(\theta) \end{pmatrix}.$$

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Notice : $T_i(\theta)^T = T_i(-\theta)$ and $T_i(\theta)^T T_i(\theta) = T_i(\theta) T_i(\theta)^T = I$.

Another example of an orthogonal transformation

Let us define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$T(x_1, x_2, x_3) = \frac{1}{3}(x_1 - 2x_2 + 2x_3, 2x_1 - x_2 - 2x_3, 2x_1 + 2x_2 + x_3).$$

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$$\begin{aligned} T(e_1) &= v_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \\ T(e_2) &= v_2 = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) \\ T(e_3) &= v_3 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \end{aligned}.$$

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$$T(e_3) = v_3 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \quad . \leftarrow$$

Thus, the matrix corresponding to T is $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

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$$\|Tx\| = \langle Tx, Tx\rangle = (Tx)\cdot(Tx) = (Ax)^T Ax = x^T A^T A x = x^T x = \|x\|.$$

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Thank you