The direction of steepest ascent/descent

Sarang S. Sane

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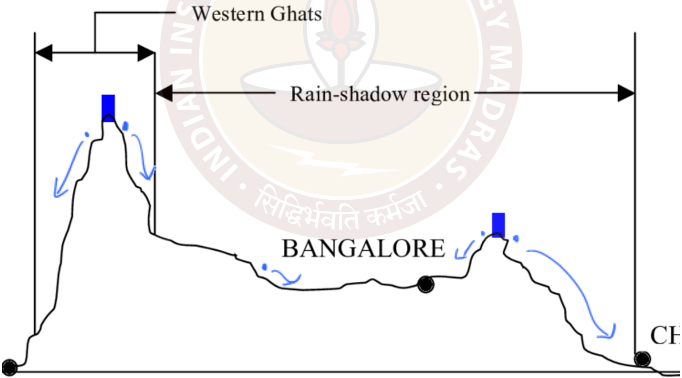

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- If the derivative is positive at a point, water flows to the right from that point.
- If the derivative is 0 at a point, water will remain stationary at that point.

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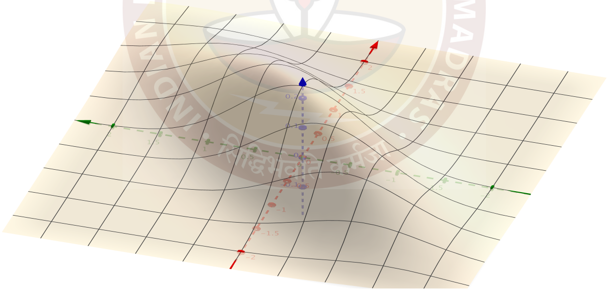

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This is same as finding the vector u in which the directional derivative h_u is largest in absolute value amongst those for which it has negative sign i.e. u such that

1. $h_u \leq h_v \quad$ for all $\quad v \in \mathbb{R}^2$ and
2. $h_u < 0$.

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$$f_u = \nabla f(\underset{\sim}{a}) \cdot u$$
$$= \|\nabla f(\underset{\sim}{a})\| \, \|u\| \cos(\theta)$$
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where θ is the angle between $\nabla f(\underset{\sim}{a})$ and u.

$\cos(\theta)$ is minimized when $\theta = \pi$

i.e. u is pointing in the direction opposite to $\nabla f(\underset{\sim}{a})$.

$\therefore$ The minimum value of f_u is attained when $u = -\nabla f(\underset{\sim}{a}) / \|\nabla f(\underset{\sim}{a})\|$ & is equal to $-\|\nabla f(\underset{\sim}{a})\|$.

Directions in which the directional derivative is maximized or remains unchanged

Assume the same hypothesis as the previous slide.

$$f_u = \|\nabla f(\underset{\sim}{a})\| \|u\| \cos(\theta) = \|\nabla f(\underset{\sim}{a})\| \cos(\theta)$$

It is maximized when $\theta = 0$, i.e. u is in the same direction as $\nabla f(\underset{\sim}{a})$ i.e. $u = \dfrac{\nabla f(\underset{\sim}{a})}{\|\nabla f(\underset{\sim}{a})\|}$.

It remains unchanged when $f_u = 0$ i.e. $\theta = \pi/2$ i.e. u is orthogonal/perpendicular to $\nabla f(\underset{\sim}{a})$.

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Property	In terms of directional derivatives	Direction
Steepest ascent	f_u is positive and maximum	$u = \nabla f / \|\|\nabla f\|\|$
Steepest descent	f_u is negative and minimum	$u = -\nabla f / \|\|\nabla f\|\|$
No change	$f_u = 0$	u is orthogonal to ∇f

Examples

1. $f(x, y) = sin(xy)$

   $\nabla f(x,y) = (y\cos(xy),\ x\cos(xy))$

   At $(\pi, 1)$ what is the direction of steepest descent on the graph of this function?

   $\nabla f(\pi, 1) = (1 \times \cos(\pi),\ \pi \times \cos(\pi)) = (-1, -\pi)$.

   $u = \dfrac{-\nabla f(\pi,1)}{\|\nabla f(\pi,1)\|} = \dfrac{(1, \pi)}{\sqrt{1+\pi^2}}$.

2. $f(x, y, z) = x^2 + y^2 + z^2$

   $\nabla f(x,y,z) = (2x, 2y, 2z)$.

   At $(1,1,1)$ what is the direction in which the fn. increases fastest?

   $\nabla f(1,1,1) = (2,2,2)$. In the direction $u = \dfrac{(2,2,2)}{\sqrt{2^2+2^2+2^2}} = \dfrac{1}{\sqrt{3}}(1,1,1)$.

   In which direction does the fn. remain constant?

   e.g. $(1,-1,0)\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}(1,0,-1)$.

A cautionary tale

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$\nabla f(0,0) = (0,0)$.

$f_u(0,0)$ DNE unless $u = e_1$ or e_2.

Thank you