IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
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Rank and dimension using Gaussian elimination

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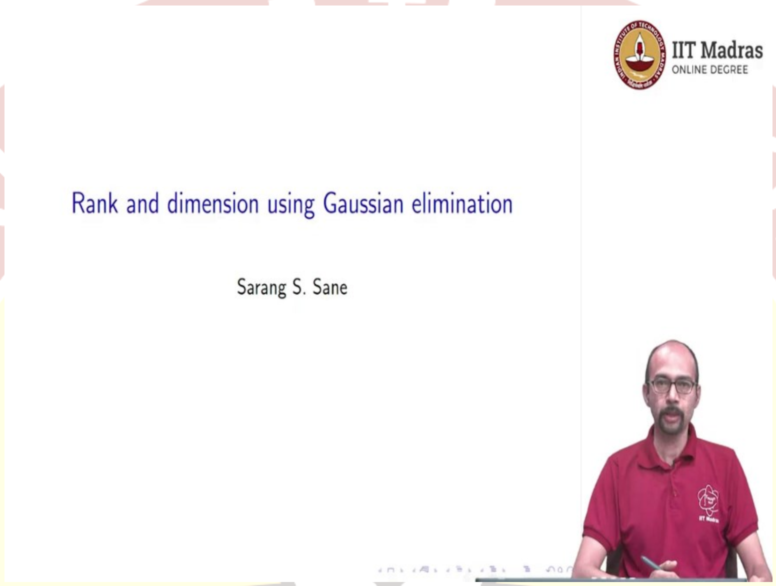

Hello, and welcome to the Maths 2 component of the online degree program in data science. In this short video, we are going to talk about how to compute rank and dimension using Gaussian elimination. So, in the previous videos, we have seen the notions of a basis and we have seen the notion of the rank of a matrix and the dimension of a vector space. So, we are going to see how to compute these using Gaussian elimination.

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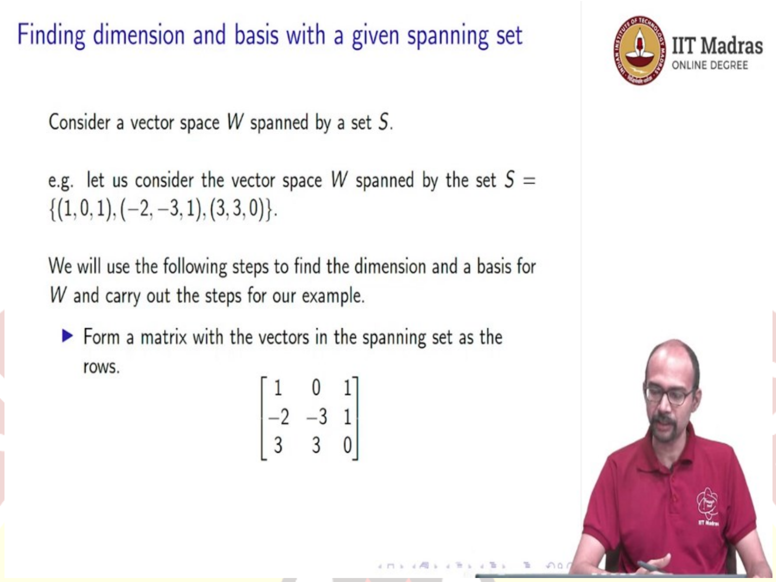

So, let us start finding dimension and basis with a given spanning set. So, the assumption here is that we know a spanning set for our vector space V and we are going to try using this spanning set, we are going to try and get a set of basis elements. So, we will try to make that spanning set linearly independent. Of course, this will involve maybe reducing the size, throwing away a few things or changing them all together.

So, we have seen some ideas of how to do this in the previous videos, where we saw that we could throw away things which were linear combinations of each other, but that was kind of an ad-hoc method and this is a formal rigorous method, which is not computationally intensive or uses any tricks, any observation skills. It is a very direct algebraic method to find the dimension and the basis and it uses the notion of Gaussian elimination.

So, let us recall that we have done Gaussian elimination in the previous weeks, where we used it to find the solutions to a system of linear equations. So, Gaussian elimination was that you have Ax equals B and then you look at the augmented matrix. And then we perform the row operations on A reduced it to reduced row echelon form and allowed the same operations on B. And then the idea is that we know how to read solutions of for a matrix which is in reduced row echelon form or row echelon form.

And then the point is that this process does not change the solutions. This was the idea. And we are going to use this idea again in order to find the basis. So, let us start. So, consider a vector space W, which is spanned by a set V, sorry, by a set S. For example, so I would have this running example, I will describe the method and each step that I do, we will run it on this particular example. So, what is the example?

You consider the vector space W spanned by the set S consisting of the vectors (1, 0, 1), (- 2, - 3, 1),(3, 3, 0). So, we are looking at W which spanned by these three vectors. So, we will use the following steps to find the dimension and a basis for W. And we, as I said, we will carry out the steps on this particular example. So, form a matrix with the vectors in the spanning set as the rows. So, you take these vectors and put them in the rows, so form a matrix.

So, this will be a three by three matrix in this case. So, in general, it will depend on how many vectors are given to you. So, if there are m vectors given and each of them is of length n, then this will be an m × n matrix. So, in this case, you get a three by three matrix, which is $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$.

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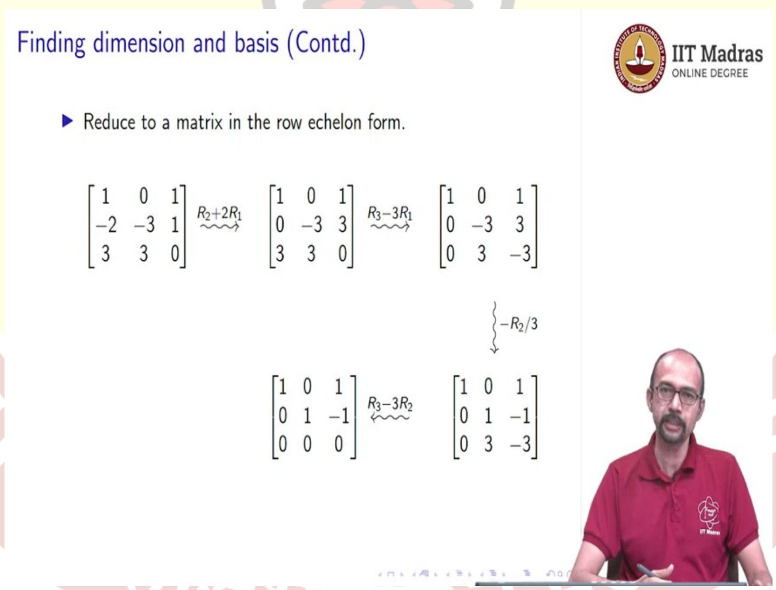

Reduce this matrix to a matrix in row echelon form. So, we know how to do this. This is row reduction. So, you have to use elementary row operations. So, let us do that. So, the first thing to do is get a 1 in the earliest possible position. So, here we have a 1 in the 1, 1 position. So, use that 1 to sweep out all the entries in the first column that is below 1. So, for this, we will do $\mathbb{R}^2 + 2\ \mathbb{R}^1$ since we have a - 2 over there.

So, that gives us a matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 3 & 3 & 0 \end{bmatrix}$. And then we will do $\mathbb{R}^3$- 3 × R1. So that will sweep out the 3 in the one 3th place rather the three 1th place and then we get the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$ and you can see this matrix is already in pretty good shape. So, we do not have to do a lot of further steps. So, we want now 1 in the 2, 2th position, if possible, and indeed it is possible by dividing by - 3, so the second row by - 3, so we get $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$.

And then we sweep out the remaining column, remaining entries in the second column below 1. So, that is $\mathbb{R}^3$ - 3 $\mathbb{R}^2$ and that gives us $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. And this is in reduced row echelon form. It is, so usually at this stage we each, row echelon form, but in this case we are lucky and that it is already in reduced row echelon form. And there is a 0 row in this at the bottom.

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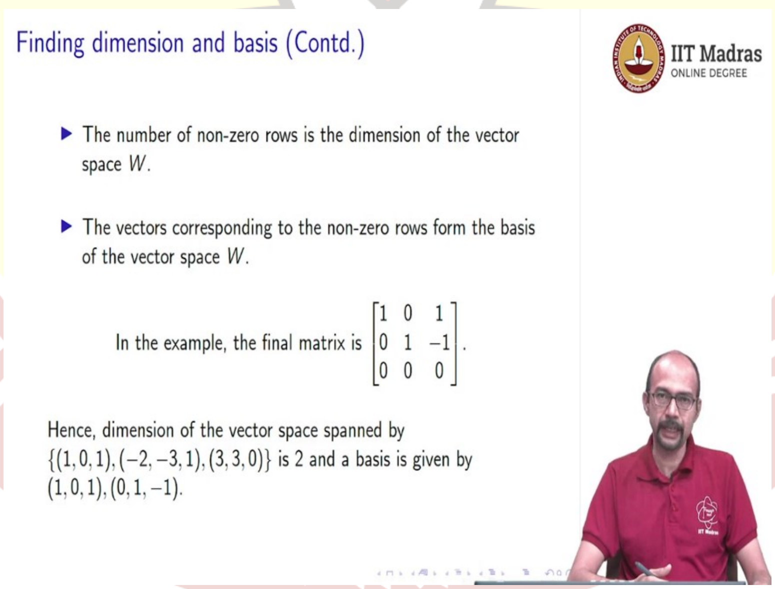

So, why did I talk about the 0 row, because the number of non-zero rows is exactly the dimension of the vector space W. So, this method of computation is already going to tell us what is the dimension without, we do not even have to know what the basis is, although we are going to get the basis very shortly. But we already know what is the dimension, namely the number of non-zero rows.

So, in this example, the number of non-zero rows is 2. So, that is the dimension of W. And the rows themselves, those vectors, the non-zero rows that is, vectors corresponding to those, those exactly are the ones which form the basis. So, in the example, the final matrix was $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

So, the dimension of the vector space spanned by the three vectors we started with (1, 0, 1), (- 2, - 3, 1),(3, 3, 0) that dimension is 2. And a basis is given by the vectors corresponding to these two rows 1, 0, 1 and 0, 1 - 1. So, I hope the method is clear. It is a very easy and fairly short method, as short as Gaussian elimination is.

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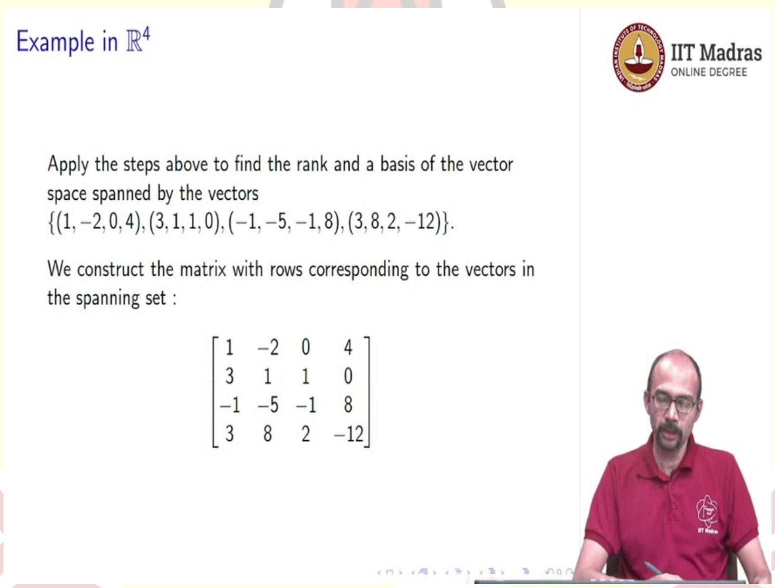

So, let us do an example in R4 to find the dimension and basis for the vector space spanned by the vectors $(1, -2, 4), (3, 1, 1), (0, -1, -5), (-1, 8, 3), (8, 2, -12)$. Now, I have written rank here, usually I use the word dimension. But as I commented in that previous video, they are often interchanged and this ambiguity is there depending on which source or text or person you interact with. So, let us construct the matrix with rows corresponding to these vectors. So, it is a four by four matrix now. That matrix is $\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & 5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix}$.

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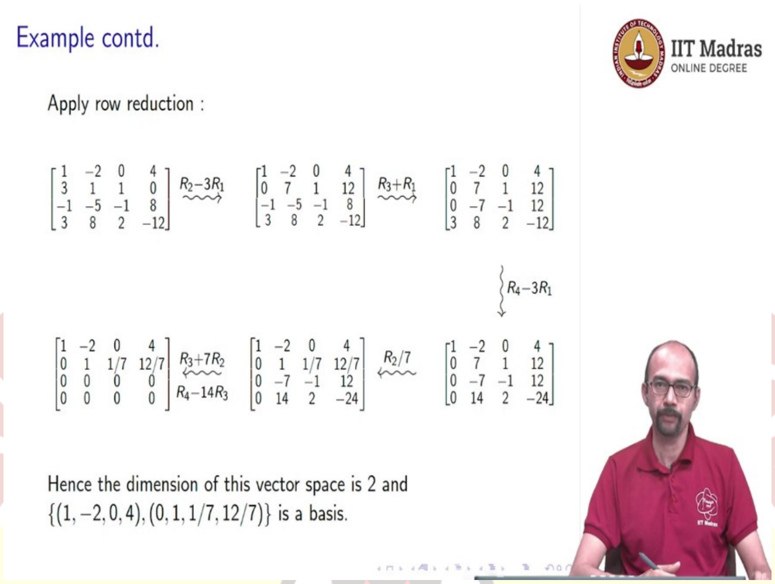

So, what do we have to do next, we have to row reduce this. Let us do that, apply reduction. So, here is your vector that, the matrix you start with. So, the, I see that. I have already given away the answer. But anyway, let us go through the row reduction. So, for row reduction, you need a 1 to start with. Well, there is a 1 in the 11 place. So, I sweep out everything in the column below that. We have to do $\mathbb{R}^2$ - $3\mathbb{R}^1$ and then we have $\mathbb{R}^3 + \mathbb{R}^1$ which knocks out the 3, 1 entry.

So, it sweeps out the third entry in the first column. And then we have R4 - 3R1. And now we have indeed swept out the first column. So, now we continue our process. So, now we want to look at this smaller matrix consisting of the three by three part, and we will try to row reduce that. So, we already have a 7, so I will divide $\mathbb{R}^2$ /7. So, if we do that, I get $\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & 12/7 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix}$.

And then we use that 1 to sweep out the second, the third and the fourth entries in the second column. So, that means we have to do $\mathbb{R}^3 + 7\mathbb{R}^2$ and $\mathbb{R}^4$- $14\mathbb{R}^3$. And it turns out that if you do that, we will actually, the entire third row and the fourth row become 0. So, you have to check that of course. So, this is in row echelon form. Note that this is not in reduced row echelon form because the entry - 2 of the 1 is not 0.

But that is okay. We do not need to put it into reduced row echelon form. Row echelon form is good enough for our purpose. So, if you put it into row echelon form, the number of non-zero rows tells you the dimension of this vector space. So, this dimension is 2 as we saw right at the start of the slide, and now we have a justification for that, because there are two non-zero rows.

And what is the basis, so the basis consists of the vectors corresponding to the non-zero rows. So, the basis is 1, - 2, 0, 4 and 0, 1, one-seventh, twelve-seventh. You could, of course, if you prefer having not fractional entries, you could multiply this second vector by 7 and instead of this basis you could write the same first vector and you could write the second vector as 7, 1, 0, 7, 1, 12. But, again, that is up to you. So, I hope the idea is clear.

So, the point one was trying to make here is that, we do not have to go all the way till the reduced row echelon form, although you could and that might be easier for some purposes. But if you just want a basis and the dimension then you can end at a matrix which is in, just in row echelon form and that will be sufficient.

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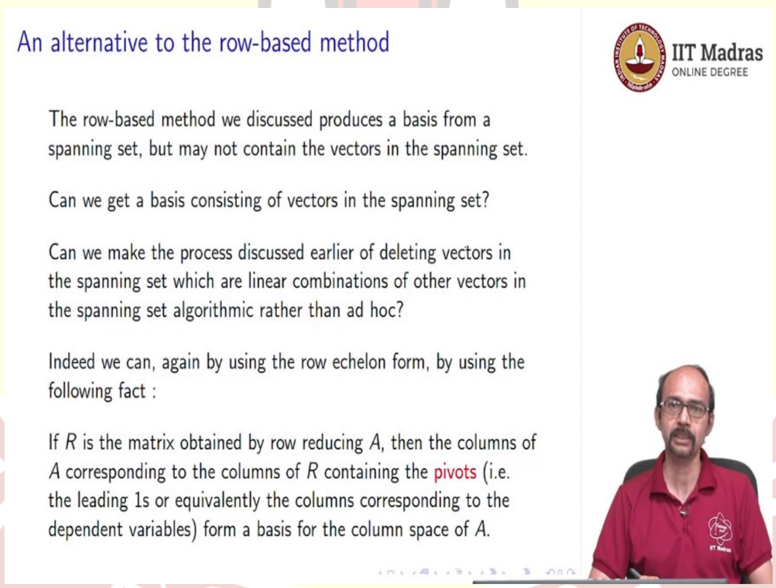

So, now that we have seen the row based method, let us take a step back and ask the following question. So, the row based method that we have discussed produces a basis from the spanning set, but note that it may not contain the vectors in the spanning set. So, if we started with some spending set and then we put it into the rows of a matrix and then we did row operations and then we got a matrix in row echelon form and then we took the rows, non-zero rows in that matrix.

So, note that the rows must have gotten changed in this. They may have. And so the new vectors that you got that are your basis vectors may not be from the original set of vectors that you started with. Can we get a basis consisting of vectors which actually lie in the set you start with in that spanning set. So, if you remember, we have done this before, but in an ad hoc way.

So, we discussed this, where you have a spanning set, and then you write one of the vectors as a linear combination of the others, then you throw it out if it can be written and then you continue this process, until you reach a point where you cannot do it. The only thing is that process was kind of an ad hoc process. So, we did not know which vector could be written as a linear combination of the other vectors. And so there was a little bit of a trial and error involved.

So, instead, we can make it a more algorithmic process. And to do it, we can use again the idea of the row echelon form and the following fact helps us in doing so. So, if R is the matrix obtained by row reducing A, where A is a matrix, then the columns have A corresponding to the columns of R containing the pivot elements.

So, the pivot elements are, remember, the places where you get 1, so those are the pivots, where there are 0s in that row before. So, you have a 1 and before that there are 0s. Those are your pivot elements. And those corresponding columns, so if you look at those columns from the matrix A then they form a basis for the column space of A. So, this fact is now going to help us to actually get vectors which are in the original spanning set which will form a basis.

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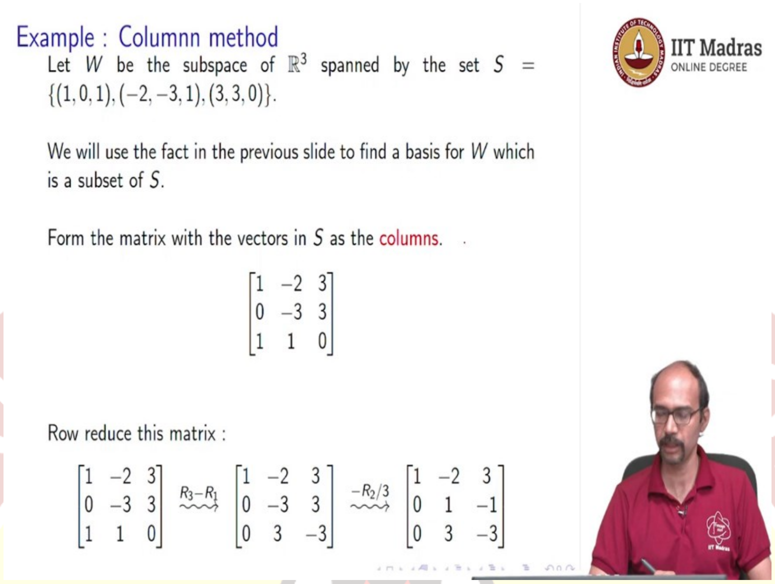

So, let us see this method. So, I will call it the column method. And we will do it on examples. And you will see how it works. So, let W be the subspace of $\mathbb{R}^3$ spanned by the set S consisting of these three vectors, $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$. So, we will use the fact in the previous slide to find a basis for W which is a subset of S, form the matrix with the vectors in S as the columns, not the rows.

So, in the previous method, we put them in the rows and then we row reduced. Here you want to put them as columns. So, here is your matrix, 1, - 2, 3, which are all the first entries of your vectors, 0, - 3, 3, and 1, 1, 0. So, notice that the first vector went into the first column, the second vector went into the second column, and the third vector went into the third column. Row reduce this matrix. So, you have $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & 0 \end{bmatrix}$.

So, if you do that, let us see how to go about this. So, for the first column you subtract 3 ×, sorry, you subtract the first row from the third row so as to make the 1 in that, in the 3, 1 position 0, so you get 1, - 2, 3, 0, - 3, 3, so no change there. And then 1, - 1 is 0, 1 -, - 2 is 1 + 2, which is 3, and then 0 - 3 is - 3, so you get 0, 3, - 3 as the third row. You, so your first column is set up in the way you want.

So, now you go to the second column and look at the 2, 2th position. And if you divide by - 3, you will get a 1 there. So, that is indeed what you do. So, then your second row becomes 0, 1, - 1, everything else remains the same.

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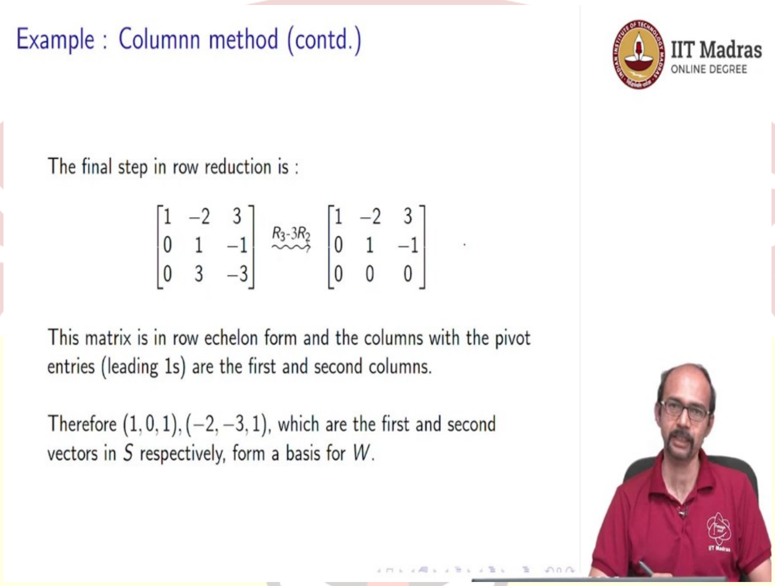

And the final step in the row reduction is, so this is the matrix that we had gotten to 1, - 2, 3, 0, 1, - 1, 0, 3, - 3, so you add - 3 × the second row to the third row or you subtract 3 × the second row from the third row, and you will get in the last row, you will get 0, 0, 0. So, now your matrix is in row echelon form, not in reduced row echelon form, but this is good enough for our purposes. And so your matrix is $\begin{bmatrix}1 & -2 & 3\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$.

And so we know now where the pivot elements are. The pivot elements are the one in the 1, 1th place and the one in the 2, 2th place. So, the columns containing the pivot elements are the first column and the second column. So, what that means is that the first vector and the second vector in your original spanning set form a basis. So, your first two vectors were 1, 0, 1 and - 2, - 3, - 1, sorry, - 2, - 3, 1. So, these are your first and second vectors and they form a basis for W. So, I hope the method is clear.

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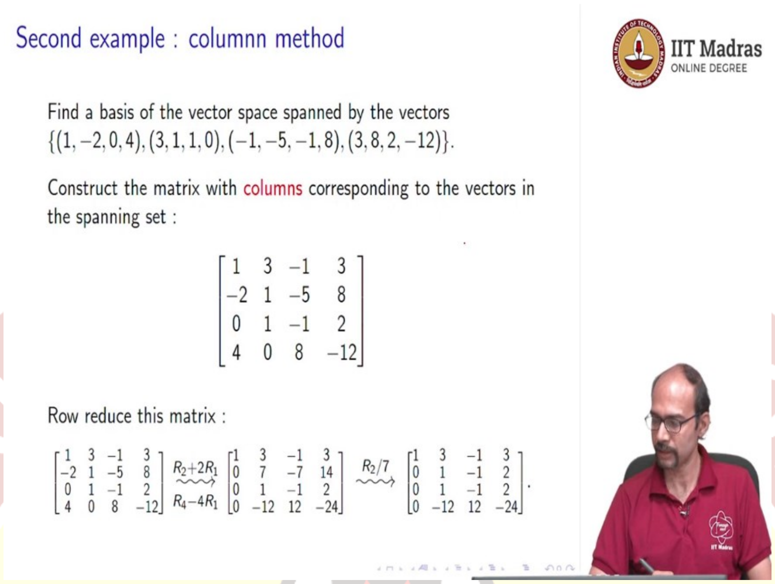

Let us do one more example. So, let us find a basis for the vector space spanned by the vectors, 1, - 2, 0, 4, 3, 1, 1, 0, - 1, - 5, - 1, 8, 3, 8, 2, - 12. So, we have seen this example before. So, to start with you have to put these into the columns, not the rows. So, you get 1, 3, - 1, 3, which are the first entries of each of these vectors, - 2, 1, - 5, 8, which are the entries of the, second entries of each of these vectors, and so on.

So, the first vector went into the first column, the second vector went into the second column, and so on. So, let us row reduce this matrix. So, you have a 1 conveniently in the 1, 1 position. So, you add 2 × the first row to the second row. So, if you do that, your second row changes to 0, then 1 + 2× × 3, which is 7, and then - 5 + 2 × - 1, which is - 7, and then 8 + 2 × 3, which is 14. Third row remains the same.

And the fourth row, you subtract 4 × the first row. So, that becomes 0, - 12, 8 + 7, which is 12 and then - 12 - 12, which is - 24. So, this is what you get. This is a nice looking matrix. Each of the second, third, fourth rows has a nice form. So you can divide by 7, the second row, so in the 2, 2th place you get 1. So, the second row becomes 0, 1, - 1, 2. And now you can see that the third and fourth rows are multiples of the second row.

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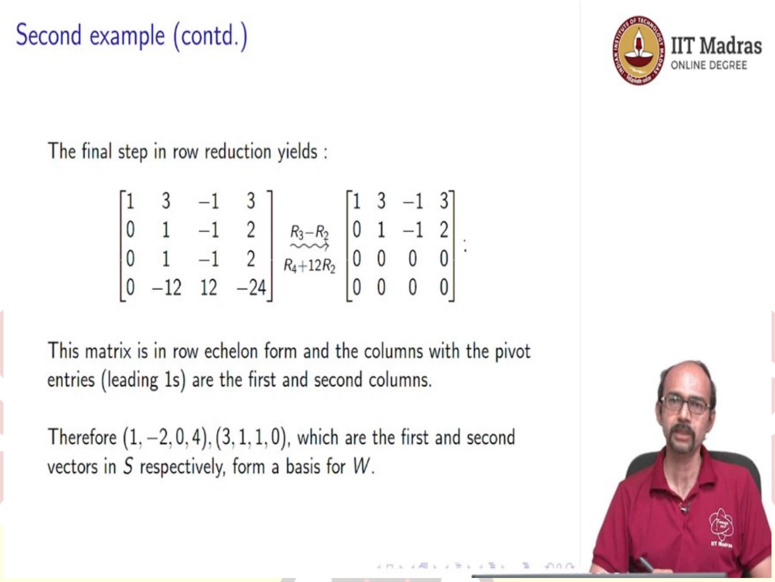

So, in the last step, you use multiple, suitable multiples to cancel those entries. So, you do R3 - R2 that give you 0 in the third row and then you do $\mathbb{R}^4 + 12 \times \mathbb{R}^2$ that gives you 0 in the fourth row. So, this matrix is now in row echelon form and the pivot elements that is the leading ones are in the 1, 1 and the 2, 2th positions.

So, In particular, they are in the first and the second columns, respectively. And so what we got is that the first vector and the second vector in your original spanning set which was 1, - 2, 0, 4 and 3, 1, 1, 0, they form a basis for W. So, this is the method, the column method of doing it.

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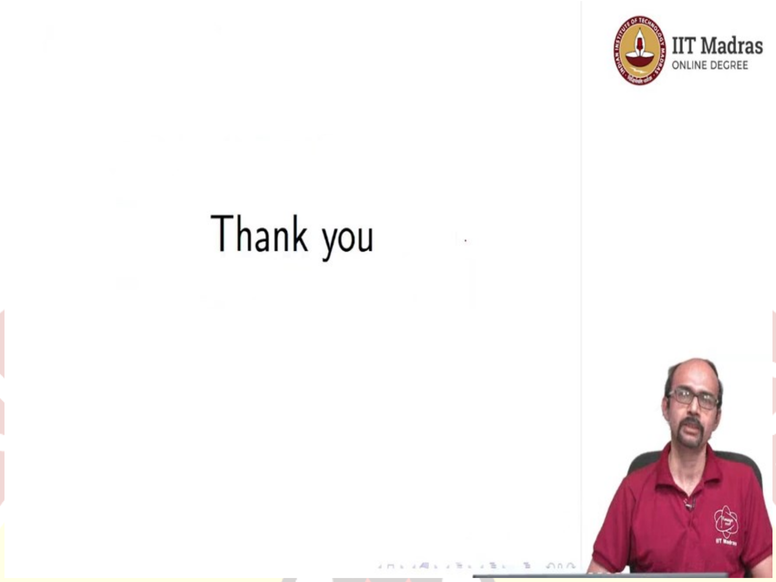

So, let us recall what we have done in this video. In this video, we have seen two methods to obtain two algorithmic methods, I will stress the fact that these are algorithmic methods, to obtain a basis from a spanning set. The first one was the, what I call the row method. So, in the row method, you will take the vectors that you have, put them into rows of a matrix, convert it to row echelon form or reduced row echelon form and then the non-zero rows that you get, the number of non-zero rows is the dimension of the space spanned by these vectors.

And the non-zero rows are actually a basis for this vector space. In the second method which I call the column method, also you obtain a basis and the dimension, but there is an additional advantage, namely you obtain vectors in the basis which are from your original spanning set. So, some× if you want vectors from your original spanning set which form a basis, you can apply the second method.

So, the second method is in some sense more general. And what was the second method, instead of writing the vectors as the rows of a matrix, you write them as the columns of a matrix, then you row reduce and you look at the pivot elements. So, again, first of all, the non-zero rows, number of non-zero rows which is the same as the number of pivot elements that is going to tell you the dimension of this, of the subspace spanned by this set of vectors.

And now if you look at the columns which the pivot elements belong to and look at the corresponding vectors in your basis, so for example, if the pivots are in the first and the fourth column or if they are in the third and the fifth column, then those corresponding vectors will give you a basis for your vector space. This is the, this is what we have seen in this video. Thank you.