Hypothesis testing

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Section 1

Introduction

What is hypothesis testing?

Motivating example: Is a coin authentic or counterfeit?

An authentic coin is known to have $P(H) = 0.5$ when tossed, while a counterfeit coin has $P(H) = 0.6$. Suppose you have a coin that could be authentic or counterfeit. You may toss the coin multiple times and observe the results. How will you test whether the coin is authentic or counterfeit?

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- Using samples, decide between a *null hypothesis* denoted H_0 and an *alternative hypothesis* denoted H_A

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  - Counterfeit coin example: H_0: $P(H) = 0.5$ and H_A: $P(H) = 0.6$
- One of the most important statistical analysis methods with a wide range of applications

Accepting or Rejecting the Null Hypothesis

Example: Is a coin authentic or counterfeit?

- Suppose we toss the coin 3 times
  - Possible outcomes are $HHH, HHT, \ldots, TTT$
  - For some outcomes, we will accept H_0 and the others, we will reject H_0
  - Let A be the set of all outcomes for which we accept H_0
- Every *acceptance* subset A corresponds to a *test*

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Acceptance set and test

$X_1, \ldots, X_n \sim$ iid X, H_0: null hypothesis, H_A: alternative

- Suppose $X \in \mathcal{X}$. Then, the samples $X_1, \ldots, X_n \in \mathcal{X}^n$
- Subset $A \subseteq \mathcal{X}^n \leftrightarrow$ a hypothesis test

If $X_1, \ldots, X_n \in A$, we accept H_0; otherwise, we reject H_0

Metrics for hypothesis testing

Example: Is a coin authentic or counterfeit?

H_0: $P(H) = 0.5$ and H_A: $P(H) = 0.6$

- Suppose we toss the coin 3 times: 8 outcomes
  - $2^8 = 256$ subsets $\leftrightarrow$ 256 tests
- How to define a *good* acceptance set or a test?

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  - $\beta = P(\text{Type II error}) = P(\text{Accept } H_0 | H_A \text{is true})$
  - Power $= 1 - \beta = P(\text{Reject } H_0 | H_A \text{is true})$

Counterfeit coin: Computing α, β

H_0: $P(H) = 0.5$ and H_A: $P(H) = 0.6$

Toss 3 times. $\mathcal{X}^3 = \{HHH, HHT, \ldots, TTT\}$

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- $A = \phi$

There is a tradeoff between α and β

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  - $\alpha = P(A^c | P(H) = 0.5) = 2/8 = 0.25$

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  - $\alpha = P(A^c | P(H) = 0.5) = 2/8 = 0.25$
  - $\beta = P(A | P(H) = 0.6) = 3\underbrace{(0.4)^2(0.6)}_{2T,\,1H} + \underbrace{3(0.4)(0.6)^2}_{2H,\,1T} = 0.72$

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- $A = \{TTT, TTH, THT, HTT\}$
  - $\alpha = 4/8 = 0.5$

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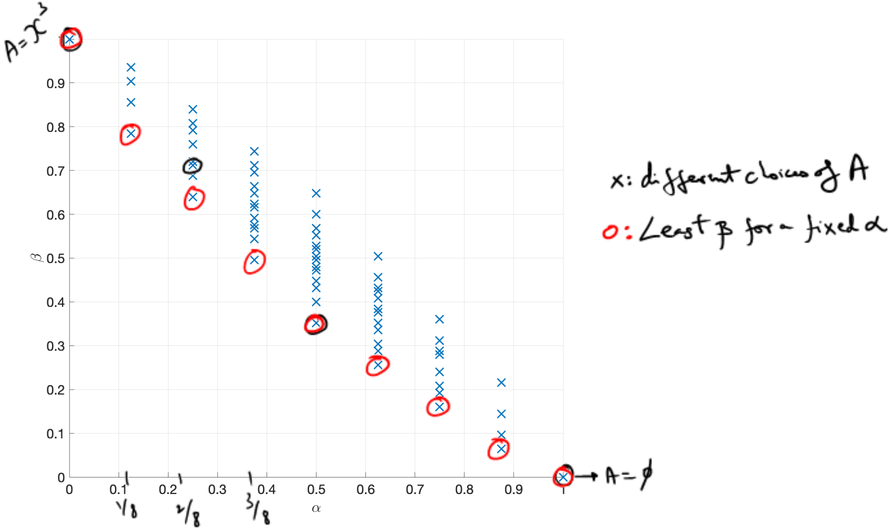
  - $\beta = P(A | P(H) = 0.6) = 3(0.4)^2(0.6) + 3(0.4)(0.6)^2 = 0.72$
- $A = \{TTT, TTH, THT, HTT\}$
  - $\alpha = 4/8 = 0.5$
  - $\beta = 0.4^3 + 3(0.4)^2(0.6) = 0.352$

There is a tradeoff between α and β

Counterfeit coin: α, β for all 256 tests



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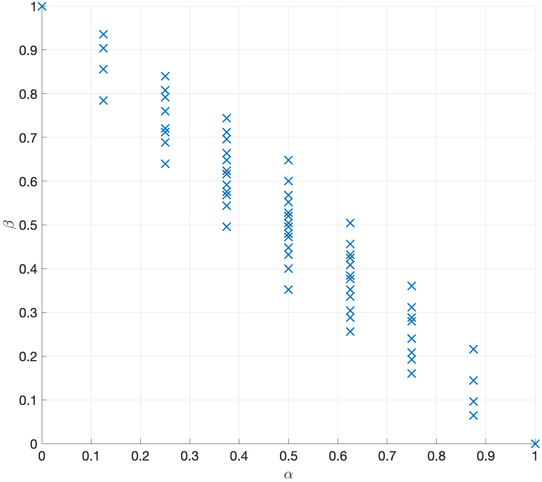


What if we toss 100 times? What about other distributions?

Neyman-Pearson paradigm of hypothesis testing

$$X_1, \dots, X_n \sim \text{iid } X$$

- H_0: null hypothesis on distribution of X, H_A: alternative hypothesis
- Test: defined by an acceptance set A
  - If samples fall in A, accept H_0; otherwise, reject H_0
- Two errors
  - Type I error: Reject H_0 when H_0 is true
  - Type II error: Accept H_0 when H_A is true
- Two metrics
  - *Significance level*, α
    - $\alpha = P(\text{Type I error}) = P(\text{Reject } H_0 | H_0)$
  - *Power* of a test, $1 - \beta$
    - $\beta = P(\text{Type II error}) = P(\text{Accept } H_0 | H_A)$

Section 2

Problems

Problem 1

Consider 100 tosses of a coin, which could be either authentic with probabiltiy of heads equal to 0.5, or counterfeit with probability of heads 0.6. Suppose T is the number of heads seen. Consider a test that rejects H_0 if $T > c$ for some constant c. What is the significance level of the test? What is the power of the test?

$A = \{\text{outcomes}: T \leq c\}$ $\quad T \sim \text{Binomial}(100, P(H))$

$$\alpha = P(\underbrace{\text{Reject } H_0}_{\text{outcome} \in A^c} \mid \underbrace{H_0}_{P(H)=0.5}) = P(A^c \mid P(H)=1/2) = \sum_{k=c+1}^{100} \binom{100}{k} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{100-k}$$

→ Fix an α and find c

$$1-\beta = P(\underbrace{\text{Reject } H_0}_{\text{outcome} \in A^c} \mid \underbrace{H_A}_{P(H)=0.6}) = P(A^c \mid P(H)=0.6) = \sum_{k=c+1}^{100} \binom{100}{k} 0.6^k \, 0.4^{100-k}$$

→ lower $c \Rightarrow$ higher power.

→ Fix α. Find lowest possible c.

Problem 2

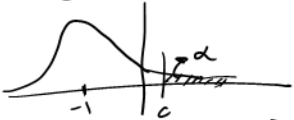
Consider one sample $X \sim \text{Normal}(\mu, 1)$. Let the null and alternative hypothesis be $H_0 : \mu = -1$ and $H_A : \mu = 1$. Consider a test that rejects H_0 if $X > c$ for some constant c. What is the significance level of the test? What is the power of the test?

$$A = \{X \le c\}$$

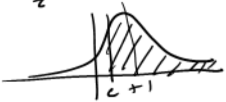
$$\alpha = P(A^c \mid \mu = -1) = P(N(-1,1) > c)$$

$$= P\Big(\underbrace{\frac{N(-1,1)+1}{1}}_{Z \sim N(0,1)} > \frac{c+1}{1}\Big) = P(Z > c+1)$$

$$= 1 - F_Z(c+1)$$



F_Z : CDF of $N(0,1)$



$$1-\beta = P(A^c \mid \mu = 1) = P(N(1,1) > c)$$

$$= P\Big(\frac{N(1,1)-1}{1} > \frac{c-1}{1}\Big) = P(Z > c-1)$$

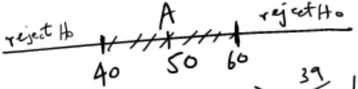
$$= 1 - F_Z(c-1)$$

$\underline{\underline{c = -1}}$ $\alpha = 1/2$, $\beta = 1 - F_Z(-2)$

$\underline{\underline{c = 0}}$ $\alpha = 1 - F_Z(1)$, $\beta = 1 - F_Z(-1)$

Problem 3

Consider one sample $X \sim$ Binomial$(100, p)$. Let the null and alternative hypothesis be $H_0 : p = 0.5$ and $H_A : p \neq 0.5$. Consider a test that rejects H_0 if $|X - 50| > 10$. What is the significance level of the test? What is the power of the test as a function of p? Use the normal approximation.



$A = \{40, 41, \ldots, 60\}$

$$\alpha = P(A^c \mid p = 0.5) = \sum_{k=0}^{39} \binom{100}{k} \left(\frac{1}{2}\right)^{100} + \sum_{k=61}^{100} \binom{100}{k} \left(\frac{1}{2}\right)^{100}$$

$$1 - \beta = P(A^c \mid p) = P(|X-50| > 10 \mid p) = P(X > 60 \mid p) + P(X < 40 \mid p)$$

$$= P\left(\frac{X - 100p}{\sqrt{100p(1-p)}} > \frac{60 - 100p}{\sqrt{100p(1-p)}}\right) + P\left(\frac{X - 100p}{\sqrt{100p(1-p)}} < \frac{40 - 100p}{\sqrt{100p(1-p)}}\right)$$

$$\simeq P\left(N(0,1) > \frac{6 - 10p}{\sqrt{p(1-p)}}\right) + P\left(N(0,1) < \frac{4 - 10p}{\sqrt{p(1-p)}}\right)$$

$$= 1 - F_Z\left(\frac{6 - 10p}{\sqrt{p(1-p)}}\right) + F_Z\left(\frac{4 - 10p}{\sqrt{p(1-p)}}\right)$$

mean $= 100p$, Var $= 100p(1-p)$

$X \approx$ Normal$(100p, 100p(1-p))$

$Z \sim N(0,1)$

Problem 4

Consider 100 samples $X_1, \ldots, X_{100} \sim$ iid Normal$(\mu, 1)$. Let the null and alternative hypothesis be $H_0 : \mu = -1$ and $H_A : \mu = 1$. Suppose $T = (X_1 + \cdots + X_{100})/100$. Consider a test that rejects H_0 if $T > c$ for some constant c. What is the significance level of the test? What is the power of the test?

$T \sim N\left(\mu, \frac{1}{100}\right) \qquad A = \{T \le c\}$

$$\alpha = P(T > c \mid \mu = -1) = P\left(\underbrace{\frac{N(-1, \frac{1}{100}) + 1}{1/10}}_{Z \sim N(0,1)} > \frac{c+1}{1/10}\right) = P(Z > 10(c+1))$$

$$= 1 - F_Z(10(c+1))$$

$$\overset{c=0}{=} \alpha = 1 - \underbrace{F_Z(10)}_{\approx 1}$$

$$1 - \beta = P(T > c \mid \mu = 1) = P(Z > 10(c-1))$$

$$= 1 - F_Z(10(c-1))$$

$$\overset{c=0}{=} 1 - \beta = 1 - \underbrace{F_Z(-10)}_{\approx 0}$$

Section 3

Types of hypothesis testing

Simple hypothesis

Definition (Simple hypothesis)

A hypothesis that completely specifies the distribution of the samples is called a simple hypothesis.

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Examples of simple hypothesis

- Coin toss
  - P(Heads) $= 0.5$, P(Heads) $= 0.9$ etc.
- Normal$(\mu, 3)$ samples
  - $\mu = 1$, $\mu = -1$ etc.

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- Normal$(\mu, 3)$ samples
  - $\mu = 1$, $\mu = -1$ etc.

Simple null vs simple alternative

- Very well understood, best approach is known
- Rarely occurs

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Examples

- Coin toss
  - Null: P(Heads) $= 0.5$ (coin is fair), simple
  - Alternative: P(Heads) $\neq 0.5$ (coin is unfair), composite
- Normal$(\mu, 3)$ samples
  - Null: $\mu = 0$ (some effect is not present), simple
  - Alternative: $\mu > 1$ (effect is present), composite

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- Normal$(\mu, 3)$ samples
  - Null: $\mu = 0$ (some effect is not present), simple
  - Alternative: $\mu > 1$ (effect is present), composite

Simple/composite null vs composite alternative

- Well studied, but multiple approaches are possible
- Most common

Standard tests: One sample

$$X_1, \ldots, X_n \sim iid X, E[X] = \mu, \mathrm{Var}(X) = \sigma^2$$

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- Testing for mean, null $H_0 : \mu = c$
  - Alternative
    - Right tail test, $H_A : \mu > c$
    - Left tail test, $H_A : \mu < c$
    - Two tail test, $H_A : \mu \neq c$
  - Two cases: known or unknown variance

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    - Right tail test, $H_A : \mu > c$
    - Left tail test, $H_A : \mu < c$
    - Two tail test, $H_A : \mu \neq c$
  - Two cases: known or unknown variance
- Testing for variance
  - Null $H_0 : \sigma = c$
  - Alternative $H_A : \sigma > c$

Standard tests: Two samples

$$X_1, \ldots, X_{n_1} \sim iid X, E[X] = \mu_1, \mathrm{Var}(X) = \sigma_1^2$$

$$Y_1, \ldots, Y_{n_2} \sim iid Y, E[X] = \mu_2, \mathrm{Var}(X) = \sigma_2^2$$

Standard tests: Two samples

$$X_1, \ldots, X_{n_1} \sim iid X, E[X] = \mu_1, \text{Var}(X) = \sigma_1^2$$

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- Testing to compare means
  - Null $H_0 : \mu_1 = \mu_2$
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- Testing to compare means
  - Null $H_0 : \mu_1 = \mu_2$
  - Alternative $H_A : \mu_1 \neq \mu_2$
- Testing to compare variances
  - Null $H_0 : \sigma_1 = \sigma_2$
  - Alternative $H_A : \sigma_1 \neq \sigma_2$

Goodness of fit testing

Samples: $X_1, \ldots, X_n$

Problem: Do the samples follow a certain distribution?

Goodness of fit testing

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Problem: Do the samples follow a certain distribution?

Examples

- Integer samples $X_i \in \{0, 1, 2, \ldots\}$. Is the distribution Poisson?

  H_0: $X \sim \text{Poisson}(\lambda)$

  H_A: X : not Poisson

Goodness of fit testing

Samples: $X_1, \ldots, X_n$

Problem: Do the samples follow a certain distribution?

Examples

- Integer samples $X_i \in \{0, 1, 2, \ldots\}$. Is the distribution Poisson?
- Continuous positive samples $X_i \in [0, \infty)$. Is the distribution Gamma?

Goodness of fit testing

Samples: $X_1, \ldots, X_n$

Problem: Do the samples follow a certain distribution?

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- Integer samples $X_i \in \{0, 1, 2, \ldots\}$. Is the distribution Poisson?
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- Continuous samples $X_i \in (-\infty, \infty)$. Is the distribution normal?

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Samples: $X_1, \ldots, X_n$

Problem: Do the samples follow a certain distribution?

Examples

- Integer samples $X_i \in \{0, 1, 2, \ldots\}$. Is the distribution Poisson?
- Continuous positive samples $X_i \in [0, \infty)$. Is the distribution Gamma?
- Continuous samples $X_i \in (-\infty, \infty)$. Is the distribution normal?
- Multinomial $X_i \in \{1, 2, \ldots, M\}$. Is the distribution $\{f_1(\theta), \ldots, f_M(\theta)\}$?

$P(X_i = 1) \quad P(X_i = M)$

Ex: $\frac{2}{3}(1-\theta) \quad \ldots \quad \frac{1}{3}\theta$

Section 4

Answering questions using data

Questions

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- Breaking down the question so that it becomes a hypothesis test is an important *design* step

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- A person claims magical powers in being able to predict something. How will you design a statistical hypothesis test?
- A company claims a new treatment method for a disease. How will you test for the effectiveness of the treatment?
- Data of accidental deaths in a country: Is there a seasonal or monthly pattern in this data?
- Data on hiring by an organization: Is there any gender or geographical bias in the hiring?

Example 1: Magical powers

Suppose a person claims magical powers to predict the throw of a die. Here is one possible way to post it as a hypothesis testing problem.

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- Throw the die n times and record the predictions. Let T be the number of correct predictions.
- Null $H_0 : T \sim \text{Binomial}(n, 1/6)$ No magical power; random guess.
- Alternative $H_A : T \sim$ any other distribution

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In the above test, we need to measure or quantify the *confidence* of our conclusion and justify its *statistical significance*. The number of trials n will be an important factor to decide.

Example 2: New medical treatment

A company claims a new drug is effective in reducing heart attacks in a certain segment of the population. Here is a common way in which drugs are tested.

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- n volunteers are chosen randomly for the trial from the population segment. About $n/2$ of them (Group I) are chosen randomly and given the drug, and the remaining (Group II) are given a placebo. Volunteers are not told what they got.
- Over a time period, the volunteers are observed for heart attacks. Suppose the fraction of volunteers who got a heart attack in Group I is f_1, and the same fraction in Group II is f_2
- Null $H_0 : f_1 \approx f_2$
- Alternative $H_A : \dfrac{f_2 - f_1}{f_2} = c$

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It is important to find c, and to quantify *confidence* and *statistical significance*. Once again, n will be an important factor.

Example 3: Pattern in accidental deaths

The number of accidental deaths in a country are tabulated every month over a year. Here is one way to test if there is a constant number of deaths per day, i.e. a constant rate.

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- Estimate the overall rate, $\theta =$ Total deaths / Total number of days
- Estimated monthly deaths: $\{31\theta, 28\theta, 31\theta, \ldots, 31\theta\}$
- Null H_0 : Estimated deaths fits the observed deaths
- Alternative H_A: Estimated and observed deaths do not fit for some months

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Assessing goodness of fit is an important ingredient here. We need to quantify the *confidence* in the fit.

Example 4: Gender bias in hiring

Consider the following cross-tabulation of hires made by a company.

	Female	Male	Total
Hired	6	12	18
Not hired	9	25	34
Total	15	37	52

Is there a gender bias in the hiring? Here is an approach.

Example 4: Gender bias in hiring

Consider the following cross-tabulation of hires made by a company.

	Female	Male	Total
Hired	6	12	18
Not hired	9	25	34
Total	15	37	52

$T = 12 - 6 = 6$

Is there a gender bias in the hiring? Here is an approach.

- Pick 18 out of 52 uniformly at random, $T = M - F \rightarrow$ Distribution
- Null H_0 : Distribution of T is as given above
- Alternative H_A : Any other distribution

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Is there a gender bias in the hiring? Here is an approach.

- Pick 18 out of 52 uniformly at random, $T = M - F$
- Null H_0 : Distribution of T is as given above
- Alternative H_A : Any other distribution

Is 6 a reasonable value for T? How to quantify this?

Observations

- In all examples, the question seems to be reasonably posed in a statistical hypothesis testing framework
- In most cases, the null and/or alternative are composite
- In all cases, the *confidence* of the testing is very important
- How do you quantify *confidence*?
  - We use ideas from confidence interval of estimation
  - A notion called P-value is used to quantify confidence

Section 5

Standard testing methods: z-test

General methodology of testing

$$X_1, \ldots, X_n \sim \text{iid } X$$

- *Test statistic*, denoted T
  - Some function of the samples
  - Example: Sample mean $\overline{X}$, Sample variance S^2 etc
- Acceptance and rejection regions are specified through T
  - Examples
    - Reject H_0 if $T > c$ (right)
    - Reject H_0 if $T < -c$ (left)
    - Reject H_0 if $|T| > c$ (two-sided)
- Significance level α depends on c and the distribution of $T|H_0$
  - Right-sided: $\alpha = P(T > c|H_0)$ (similar for others)
  - Fix α and find c

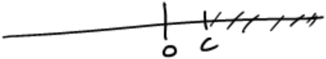
Testing for mean (normal samples, known variance)

$$X_1, \ldots, X_n \sim \text{iid } N(\mu, 4^2)$$

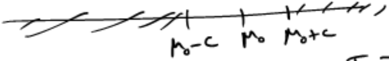
- Test statistic $T = \overline{X} \triangleq \frac{1}{n}(X_1 + \cdots + X_n)$
- Null $H_0 : \mu = 0$, Alternative $H_A : \mu > 0$
- Test: Reject H_0 if $T > c$

(Could be $\overset{H_0}{\mu=\mu_0}$ & $\overset{H_A}{\mu>\mu_0}$ (or) $\mu<\mu_0$ (or) $\mu\neq\mu_0$)

$H_0: \mu=0 \quad H_A: \mu>0$



$H_0: \mu=\mu_0$, $H_A: \mu\neq\mu_0$



Reject H_0: $T>\mu_0+c$ (or) $T<\mu_0-c$

Testing for mean (normal samples, known variance)

$$X_1, \ldots, X_n \sim \text{iid } N(\mu, 4^2)$$

- Test statistic $T = \overline{X} \triangleq \frac{1}{n}(X_1 + \cdots + X_n)$
- Null $H_0 : \mu = 0$, Alternative $H_A : \mu > 0$
- Test: Reject H_0 if $T > c$
- Different samplings, $n = 10$
  - [-6.9, 0.6, -0.6, -4.8, -1.9, -5.1, 7.5, 6.1, 0.5, 3.3], $T = -0.14$ Acc
  - [-1.8, -1.8, 4.1, 3.4, 1.9, 0.6, 1.7, -6.9, 0.3, -4.0], $T = -0.25$ Acc
  - [-5.8, 2.0, 2.5, 1.7, -2.8, 0.9, -0.4, 0.6, -8.5, -2.9], $T = -1.25$ Acc
  - [4.2, 14.2, 7.1, -5.1, -2.3, -3.9, -3.2, -0.9, -1.4, -6.4], $T = 0.23$ Acc
  - [1.0, 3.6, 5.9, -2.2, 2.3, 6.9, 1.7, 0.1, 6.3, 4.0], $T = 2.96$ Rej ?
  - [1.7, 3.9, -1.6, 3.8, 4.0, 1.9, -1.8, 10.3, 4.2, 4.6], $T = 3.10$ Rej
  - [9.4, 2.2, 13.8, 3.1, 6.3, 7.0, 5.8, 1.0, 7.6, 5.7], $T = 6.20$ Rej

1.9...?

Stronger

Testing for mean (normal samples, known variance)

$$X_1, \ldots, X_n \sim \text{iid } N(\mu, 4^2)$$

- Test statistic $T = \overline{X} \triangleq \frac{1}{n}(X_1 + \cdots + X_n)$
- Null $H_0 : \mu = 0$, Alternative $H_A : \mu > 0$
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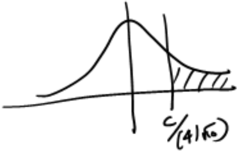
  - [9.4, 2.2, 13.8, 3.1, 6.3, 7.0, 5.8, 1.0, 7.6, 5.7], $T = 6.20$

Higher values of T give us more confidence in rejecting null

Testing for mean: Significance level

$$X_1, \ldots, X_{10} \sim \text{iid } N(\mu, 4^2)$$



- Significance level $\alpha = P(\overline{X} > c \mid \mu = 0)$
- Since $(\overline{X} \mid \mu = 0) \sim \text{Normal}(0, 4^2/10)$, we have

$Z \sim$ Standard normal

$\sim$ Normal$(0,1)$

$$\alpha = P\left(\frac{\overline{X}}{4/\sqrt{10}} > \frac{c}{4/\sqrt{10}}\right) = 1 - F_Z(\sqrt{10}c/4)$$

$\to \sqrt{n}$

CDF of Z

c	0	1.62	2.08	2.94	3.26	3.91
α	0.5	0.1	0.05	0.01	0.005	0.001

depends on $n=10$

very common

Testing for mean: Significance level

$$X_1, \ldots, X_{10} \sim \text{iid } N(\mu, 4^2)$$

- Significance level $\alpha = P(\overline{X} > c | \mu = 0)$
- Since $(\overline{X} | \mu = 0) \sim \text{Normal}(0, 4^2/10)$, we have

critical value

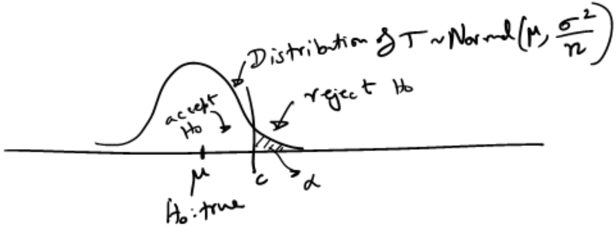
$$\alpha = P\left(\frac{\overline{X}}{4/\sqrt{10}} > \frac{c}{4/\sqrt{10}}\right) = 1 - F_Z(\sqrt{10}c/4)$$

c	0	1.62	2.08	2.94	3.26	3.91
α	0.5	0.1	0.05	0.01	0.005	0.001

z-**test at significance level** α: Reject H_0 if $T > c$, where c is as above.

Testing for mean: Results and P-value

c	0	1.62	2.08	2.94	3.26	3.91
α	0.5	0.1	0.05	0.01	0.005	0.001
$T = 0.23$	Rej	Acc	Acc	Acc	Acc	Acc
$T = 2.96$	Rej	Rej	Rej	Rej	Acc	Acc
$T = 6.20$	Rej	Rej	Rej	Rej	Rej	Rej



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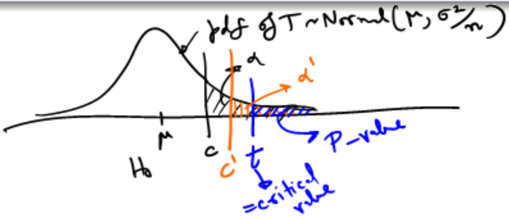
$T = 6.20$	Rej	Rej	Rej	Rej	Rej	Rej

Definition (P-value)

Suppose the test statistic $T = t$ in one sampling. The lowest significance level α at which the null will be rejected for $T = t$ is said to be the P-value of the sampling.



$t > c \Rightarrow$ reject H_0 at α

$t > c' \Rightarrow$ reject H_0 at $\alpha' < \alpha$

critical value $= t \Rightarrow$ lowest significance level at which we reject H_0

Testing for mean: Results and P-value

c	0	1.62	2.08	2.94	3.26	3.91
α	0.5	0.1	0.05	0.01	0.005	0.001
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$T = 6.20$	Rej	Rej	Rej	Rej	Rej	Rej

Definition (P-value)

Suppose the test statistic $T = t$ in one sampling. The lowest significance level α at which the null will be rejected for $T = t$ is said to be the P-value of the sampling.

- Finding P-value for $T = t$: Put $c = t$ in computation of α

T	-0.14	0.23	2.96	6.20
P-value	0.544	0.428	0.00964	4.755e-07

If P-value is low enough, we reject H_0

What to choose? Significance level or P-value

- Samples are given, and there is some hypothesis that needs to be tested
- Step 1: Decide on the null and alternative hypotheses H_0 and H_A
- Step 2: Decide on the test statistic T
- Step 3: "Philosophy" of testing
  - Choice 1: Pick a significance level first
    - ★ Probability of Type I error can be fixed in some applications. In those cases, significance level is easy to fix
    - ★ Historically, in many applications, 0.05 or 0.01 is accepted as a common significance level
    - ★ Find rejection region (find the *critical value* c and reject H_0 if $T > c$, for example)
  - Choice 2: Use P-value
    - ★ Report the P-value
    - ★ If P-value is *low enough*, choose to reject H_0; otherwise, accept H_A
    - ★ How low is low enough? Depends on applications and other information

Section 6

z-test problems

Problem 1

Suppose $X \sim \text{Normal}(\mu, 9)$. For $n = 16$ iid samples of X, the observed sample mean is 10.2. What conclusion would a z-test reach if the null hypothesis assumes $\mu = 9.5$ (against an alternative hypothesis $\mu > 9.5$) at a signifiance level of $\alpha = 0.05$? What if the null hypothesis assumes $\mu = 8.5$ (against an alternative hypothesis $\mu > 8.5$)?

$H_0: \mu = 9.5$, $H_A: \mu > 9.5$, $\bar{X} \sim N(\mu, 9/16)$, $\bar{X} = 10.2$

$\sigma^2 = 9/16$

Test: Reject H_0 if $\bar{X} > c$

$$\alpha = P(\bar{X} > c \mid \mu = 9.5) = 1 - F_Z\left(\frac{c - 9.5}{(3/4)}\right) = 0.05$$

$$\frac{\bar{X} - 9.5}{\underbrace{(3/4)}_{\sigma}} \sim Z$$

$$c = 9.5 + \frac{3}{4} F_Z^{-1}(0.95) = 10.73$$

z-test @ $\alpha = 0.05$: Accept H_0

$H_0: \mu = 8.5$, $H_A: \mu > 8.5$

$$c = 8.5 + \frac{3}{4} F_Z^{-1}(0.95) = 9.73$$

z-test @ $\alpha = 0.05$: Reject H_0.

Problem 2

Suppose an app is desired to make an accurate identification of faces in photographs more than 90% of the time in the long run. For a random sample of 500 such photos, the app makes the correct identification 462 times - a 92.4% success rate. What does a z-test say about a null hypothesis that the app is only 90% accurate (compared to an alternative hypothesis that the app is more than 90% accurate with a significance level of $\alpha = 0.05$)?

Samples: $X_1, \ldots, X_{500} \overset{iid}{\sim} \text{Bernoulli}(p)$ $\quad X_i = \begin{cases} 1 & \text{if app correctly identifies} \\ 0 & \text{else} \end{cases}$

$$\bar{X} = \frac{462}{500} = 0.924$$

$H_0: p = 0.9$, $H_A: p > 0.9$ $\quad$ Test: Reject H_0 if $\bar{X} > c$

$$\bar{X} \sim N\left(p, \frac{p(1-p)}{500}\right) \quad \frac{\bar{X} - p}{\sqrt{\frac{p(1-p)}{500}}} \sim Z$$

$$\alpha = P(\bar{X} > c \mid p = 0.9) = 1 - F_Z\left(\frac{c - 0.9}{\sqrt{0.09/500}}\right) = 0.05$$

$$c = 0.9 + \sqrt{\frac{0.09}{500}}\, F_Z^{-1}(0.95) = 0.922$$

Since $\bar{X} > c$, z-test @ $\alpha = 0.05$: Reject H_0

Problem 3

Suppose $X \sim \text{Normal}(\mu, 36)$. For $n = 25$ iid samples of X, the observed sample mean is 6.2. What conclusion would a z-test reach if the null hypothesis assumes $\mu = 4$ (against an alternative hypothesis $\mu \neq 4$) at a signifiance level of $\alpha = 0.05$? What if the null hypothesis assumes $\mu = 8$ (against an alternative hypothesis $\mu < 8$)?

$H_0: \mu = 4$, $H_A: \mu \neq 4$, Test: Reject H_0, if $|\bar{X} - 4| > c$

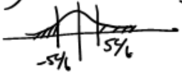
$\Updownarrow$

$\bar{X} > 4 + c$ (or) $\bar{X} < 4 - c$

$\bar{X} \sim N\left(\mu, \frac{36}{25}\right)$ $\quad \frac{\bar{X} - \mu}{(6/5)} \sim Z$

$\alpha = P(|\bar{X} - 4| > c \mid \mu = 4) = P\left(\left|\underbrace{\frac{\bar{X}-4}{6/5}}_{\sim |Z|}\right| > \frac{c}{(6/5)}\right) = P\left(|Z| > \frac{c}{6/5}\right) = 2 F_Z\left(\frac{-5c}{6}\right)$

$c = -\frac{6}{5} F_Z^{-1}\left(\frac{0.05}{2}\right) = 2.352$



Since $|6.2 - 4| = 2.2 < 2.352$, Accept H_0.

$H_0: \mu = 8$, $H_A: \mu < 8$, Test: Reject H_0 if $\bar{X} < c$

$\alpha = P(\bar{X} < c \mid \mu = 8) = F_Z\left(\frac{c - 8}{(6/5)}\right) = 0.05 \Rightarrow c = 8 + \frac{6}{5} F_Z^{-1}(0.05) = 6.02$

Since $\bar{X} = 6.2 > 6.02$, z-test @ $\alpha = 0.05$: Accept H_0

Section 7

More problems on z-test

Problem 1a (with binomial distribution)

Vaccine hesitancy (percentage of people who are unwilling to vaccinate) in a town has been reported to be 20%. To test whether the fraction is 20%, you call a randomly selected group of 10 people and find out that 3 of them are vaccine hesitant. What is the null hypothesis? Will you accept or reject null at a significance level of $\alpha = 0.05$ against an alternative that the fraction is above 20%? What is power against an alternative that the fraction is 30%?

Samples: $X_1, X_2, \ldots, X_{10} \overset{iid}{\sim} \text{Bernoulli}(p)$ $\quad X_i = \begin{cases} 1 & \text{if } i\text{-th person is hesitant} \\ 0 & \text{else} \end{cases}$

$T = X_1 + \cdots + X_{10}$

$H_0: p = 0.2$, $H_A: p > 0.2$

Test: Reject H_0 if $T > c$

$T \sim \text{Binomial}(10, p)$ $\quad \alpha = P(T > c \mid p = 0.2)$

$$0.05 = 1 - \sum_{k=0}^{c} \binom{10}{k} p^k (1-p)^{10-k}$$

$\Rightarrow c = 4$

c	α
0	0.893
1	0.624
2	0.322
3	0.121
4	0.033
5	0.0064
6	0.00086

Type II error:

$\beta = P(\text{Accept } H_0 \mid p = 0.3) = P(T \le c \mid p = 0.3) = \underline{0.85}$

Problem 1b (with normal approximation)

Vaccine hesitancy (percentage of people who are unwilling to vaccinate) in a town has been reported to be 20%. To test whether the fraction is 20%, you call a randomly selected group of 100 people and find out that 28 of them are vaccine hesitant. What is the null hypothesis? Will you accept or reject null at a significance level of $\alpha = 0.05$ against an alternative that the fraction is above 20%? What is power against an alternative that the fraction is 30%?

$$T = \text{Binomial}(100, p) \approx \text{Normal}(100p, 100p(1-p))$$

$$\alpha = P(T > c \mid p = 0.2) = 1 - F_Z\left(\frac{c - 100 \times 0.2}{\underbrace{\sqrt{100 \times 0.2 \times 0.8}}_{4}}\right) = 0.05$$

$$c = 20 + 4 F_Z^{-1}(0.95) = 26.58$$

Since $T = 28 > c = 26.58$, Z-test @ $\alpha = 0.05$: Reject H_0

$$\beta = P(T \le c \mid p = 0.3) = F_Z\left(\frac{26.58 - 100 \times 0.3}{\sqrt{100 \times 0.3 \times 0.7}}\right) = 0.23$$

Problem 2a

The current-carrying capacity of a resistor manufactured at your company is supposed to be 3.0 A (A stands for Amperes). Because of a recent change in the manufacturing process, you suspect that the current-carrying capacity might actually be lesser than 3.0 A. You decide to test by measuring current-carrying capacities of 10 resistors with a test measurement has a standard deviation of 0.05 A. If the sample mean of the test measurements $T_{10} < 2.95$, you will conclude that the manufacturing process is faulty.

1. What is the null hypothesis? What is the alternative hypothesis? What are the samples?
2. What is the significance level α of the test?
3. If the current-carrying capacity falls to 2.9 A, there could be serious safety issues. Against the alternative hypothesis of 2.9 A, what is the power $(1 - \beta)$ of the test?

Samples: $X_1, X_2, \ldots, X_{10} \sim N(\mu, 0.05^2)$ X_i = measured capacity of i-th test resistor

$H_0: \mu = 3$ A, $H_A: \mu < 3$ A.

Working

Test: Reject H_0 if $\bar{X} < 2.95$

$\bar{X} \sim N\left(\mu, \frac{0.05^2}{10}\right)$

$$\alpha = P(\bar{X} < 2.95 \mid \mu = 3) = F_Z\left(\frac{2.95-3}{0.05/\sqrt{10}}\right) = 0.00078\ldots$$

$$\beta = P(\bar{X} \geq 2.95 \mid \mu = 2.9) = 1 - F_Z\left(\frac{2.95-2.9}{0.05/\sqrt{10}}\right) = 0.00078\ldots$$

Problem 2b

In the same problem, suppose you can test n resistors. If the sample mean $T_n < c$, you conclude that the manufacturing process is faulty. You need to determine suitable values for n and c under the following conditions:

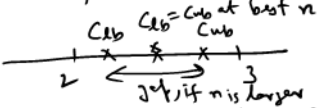
1. Significance level or probability of Type I error, $\alpha \leq 10^{-6}$
2. Probability of Type II error, $\beta \leq 10^{-12}$ (against an alternative of 2.9 A)

$\bar{X} \sim N\left(\mu, \frac{0.05^2}{n}\right)$ $\quad \alpha = P(\bar{X} < c \mid \mu = 3) = F_Z\left(\frac{c-3}{0.05/\sqrt{n}}\right) \leq 10^{-6}$

$$\frac{c-3}{0.05/\sqrt{n}} \leq F_Z^{-1}(10^{-6})$$

$$c \leq \overbrace{3 + \frac{0.05}{\sqrt{n}}(-4.753)}^{c_{ub}} \qquad \text{—— (1)}$$

$$\beta = P(\bar{X} \geq c \mid \mu = 2.9) = 1 - F_Z\left(\frac{c-2.9}{0.05/\sqrt{n}}\right) \leq 10^{-12} \Rightarrow F_Z\left(\frac{c-2.9}{0.05/\sqrt{n}}\right) \geq 1 - 10^{-12}$$



$$\frac{c-2.9}{0.05/\sqrt{n}} \geq 7.034$$

$$c \geq \underbrace{2.9 + \frac{0.05}{\sqrt{n}}(7.034)}_{c_{lb}} \qquad \text{—— (2)}$$

Best choice of n: $\quad 0.1 = \frac{0.05}{\sqrt{n}}(4.753 + 7.034) \Rightarrow n = 35$

Problem 3

The average CGPA of students in a college is reported to be 8.0 with a standard deviation of 1. You suspect that the average may be lower, possibly 7.5, and decide to sample students to find their CGPA. What sample size do you need for a test at a significance level of 0.05 and power of 0.95? How will the sample size change if you suspect the CGPA to be 7.0?

$X_1, \ldots, X_n \sim N(\mu, 1)$ $\quad H_0 : \mu = 8 \quad H_A : \mu < 8$

$\bar{X} \sim N(\mu, 1/n)$ $\quad$ Test: Reject H_0 if $\bar{X} < c$

$$\alpha = P(\bar{X} < c \mid \mu = 8) = F_Z\left(\frac{c-8}{1/\sqrt{n}}\right) = 0.05$$

$$c = 8 - \frac{1.645}{\sqrt{n}} \quad \text{——} ①$$

Case 1: $$\beta = P(\bar{X} \geq c \mid \mu = 7.5) = 1 - F_Z\left(\frac{c-7.5}{1/\sqrt{n}}\right) = 0.05$$

$$c = 7.5 + \frac{1.645}{\sqrt{n}} \quad \text{——} ②$$

$$① \,\&\, ② : \quad 0.5 = \frac{2 \times 1.645}{\sqrt{n}} \Rightarrow n \approx 44$$

Case 2: $$\beta = 1 - F_Z\left(\frac{c-7}{1/\sqrt{n}}\right) = 0.05 \Rightarrow c = 7 + \frac{1.645}{\sqrt{n}} \quad \text{——} ③$$

$$① \,\&\, ③ : \quad 1 = \frac{2 \times 1.645}{\sqrt{n}} \Rightarrow n \approx 11$$