Statistics for Data Science - 2

Week 6 Practice Assignment Solution

1. Let $X \sim$ Bernoulli(0.6). Let $(Y \mid X = 0) \sim$ Exp(1) and $(Y \mid X = 1) \sim$ Exp(3). Find the marginal of Y.

   a) $0.6e^{-y} + 0.4e^{-3y}$

   b) $0.4e^{-y} + 0.6e^{-3y}$

   c) $0.6e^{-y} + 1.2e^{-3y}$

   d) $0.4e^{-y} + 1.8e^{-3y}$

   Solution:
   Given that, $X \sim$ Bernoulli(0.6), therefore $p_X(1) = 0.6$ and $p_X(0) = 0.4$.
   The marginal density of Y is given by

$$\begin{aligned} f_Y(y) &= \sum_{x \in T_X} p_X(x) f_{Y|X=x}(y) \\ &= p_X(1) f_{Y|X=1}(y) + p_X(0) f_{Y|X=0}(y) \\ &= 0.6 \times 3e^{-3y} + 0.4e^{-y} \\ &= 1.8e^{-3y} + 0.4e^{-y} \end{aligned}$$

2. Let $X \sim$ Uniform$\{1, 2, 3\}$. Let $(Y \mid X = 1) \sim$ Exp(1), $(Y \mid X = 2) \sim$ Exp(2) and $(Y \mid X = 3) \sim$ Normal(0, 4). What is the marginal of Y?

   a) $e^{-y} + 2e^{-2y} + \dfrac{1}{2\sqrt{2\pi}} e^{-y^2/8}$

   b) $\dfrac{1}{3}[e^{-y} + 2e^{-2y} + \dfrac{1}{2\sqrt{2\pi}} e^{-y^2/8}]$

   c) $\dfrac{1}{3}[e^{-y} + e^{-2y} + \dfrac{1}{\sqrt{2\pi}} e^{-y^2/4}]$

   d) $e^{-y} + e^{-2y} + \dfrac{1}{2\sqrt{2\pi}} e^{-y^2/4}$

   Solution:
   Given that, $X \sim$ Uniform$\{1, 2, 3\}$, therefore $p_X(1) = p_X(2) = p_X(3) = \frac{1}{3}$.

The marginal density of Y is given by

$$\begin{aligned}
f_Y(y) &= \sum_{x \in T_X} p_X(x) f_{Y|X=x}(y) \\
&= p_X(1) f_{Y|X=1}(y) + p_X(2) f_{Y|X=2}(y) + p_X(3) f_{Y|X=3}(y) \\
&= \frac{1}{3} \times e^{-y} + \frac{1}{3} \times 2e^{-2y} + \frac{1}{3} \times \frac{e^{-y^2/8}}{2\sqrt{2\pi}} \\
&= \frac{1}{3}[e^{-y} + 2e^{-2y} + \frac{1}{2\sqrt{2\pi}} e^{-y^2/8}]
\end{aligned}$$

3. Let $X \sim \text{Uniform}\{1,2\}$. Let $(Y \mid X = 1) \sim \text{Exp}(2)$ and $(Y \mid X = 2) \sim \text{Exp}(4)$. Find the value of $f_{X|Y=3}(2)$.

   a) $\dfrac{2e^{-12}}{e^{-6} + 2e^{-12}}$

   b) $\dfrac{e^{-6}}{e^{-6} + 2e^{-12}}$

   c) $\dfrac{e^{-12}}{e^{-6} + e^{-12}}$

   d) $\dfrac{e^{-6}}{e^{-6} + e^{-12}}$

Solution:
Given that, $X \sim \text{Uniform}\{1,2\}$, therefore $p_X(1) = p_X(2) = \frac{1}{2}$.
The marginal density of Y is given by

$$\begin{aligned}
f_Y(y) &= \sum_{x \in T_X} p_X(x) f_{Y|X=x}(y) \\
&= p_X(1) f_{Y|X=1}(y) + p_X(2) f_{Y|X=2}(y) \\
&= \frac{1}{2} \times 2e^{-2y} + \frac{1}{2} \times 4e^{-4y} \\
&= e^{-2y} + 2e^{-4y}
\end{aligned}$$

And

$$\begin{aligned}
f_{X|Y=3}(2) &= \frac{p_X(2) f_{Y|X=2}(3)}{f_Y(3)} \\
&= \frac{\frac{1}{2} \times 4e^{-4\times3}}{e^{-2\times3} + 2e^{-4\times3}} \\
&= \frac{2e^{-12}}{e^{-6} + 2e^{-12}}
\end{aligned}$$

4. The joint density function of two continuous random variables X and Y is given as

$$f_{XY}(x,y) = \begin{cases} kxy & 0<x<4, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of k. Enter your answer correct to two decimals accuracy.
Solution:
We know that for joint PDF, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$
Since $f_{XY}(x,y)$ is nonzero in the region $0<x<4, 0<y<1$.

$$\Rightarrow \int_0^1\int_0^4 f_{XY}(x,y)dxdy = 1$$
$$\Rightarrow \int_0^1\int_0^4 kxy\ dxdy = 1$$
$$\Rightarrow \int_0^1 kx\frac{y^2}{2}\Big|_0^4 dx = 1$$
$$\Rightarrow \int_0^1 8kxdx = 1$$
$$\Rightarrow 8k\frac{x^2}{2}\Big|_0^1 = 1$$
$$\Rightarrow k = \frac{1}{4} = 0.25$$

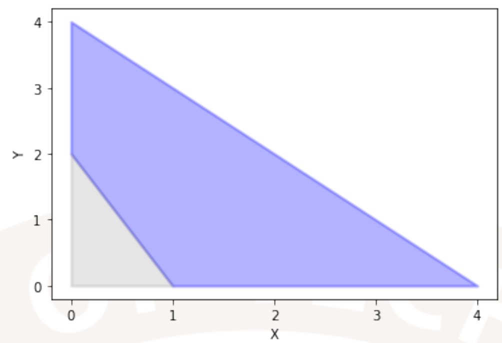
5. Let $(X,Y) \sim \text{Uniform}(D)$, where $D = \{(x,y) : x+y<4, x>0, y>0\}$. Find the value of $P(2X+Y>2)$.

   a) $\frac{1}{8}$
   b) $\frac{7}{8}$
   c) $\frac{3}{4}$
   d) $\frac{1}{4}$

Solution:

$(X,Y) \sim \text{Uniform}(D)$, therefore

$$f_{XY}(x,y) = \begin{cases} \frac{1}{8} & (x,y)\in D \\ 0 & \text{otherwise} \end{cases}$$



Area of the lower shaded region (A) will be $\frac{1}{2} \times 1 \times 2 = 1$

$$\begin{aligned} P(2X + Y > 2) &= 1 - P(2X + Y \leq 2) \\ &= 1 - \frac{|A|}{|D|} \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8} \end{aligned}$$

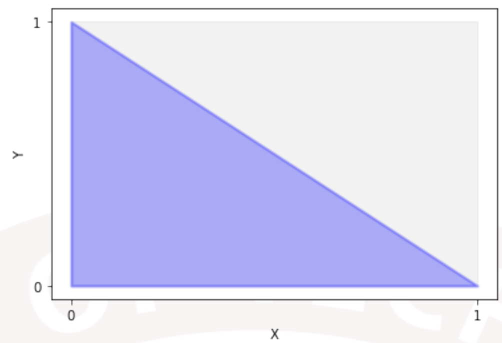
6. The joint density function of the random variables X and Y is given by

$$f_{XY}(x, y) = \begin{cases} x + y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(X + Y < 1)$.

a) $\frac{1}{3}$
b) $\frac{2}{3}$
c) $\frac{1}{6}$
d) $\frac{3}{4}$

Solution:



$$
\begin{aligned}
P(X+Y<1) &= \int_0^1 \int_0^{1-y} (x+y)dxdy \\
&= \int_0^1 \left(\frac{x^2}{2} + xy\right)\Big|_0^{1-y} dy \\
&= \int_0^1 \left(\frac{(1-y)^2}{2} + (1-y)y\right) dy \\
&= \left(-\frac{(1-y)^3}{6} + \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 \\
&= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(-\frac{1}{6}\right) \\
&= \frac{1}{3}
\end{aligned}
$$

7. The joint PDF of two continuous random variables X and Y is given by

$$
f_{XY}(x,y) = \begin{cases} \frac{2}{7}(5x+2y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

Find the marginal PDF of X.

a)

$$
f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

b)

$$
f_X(x) = \begin{cases} \frac{2}{7}(5x+1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

c)

$$
f_X(x) = \begin{cases} \frac{2}{7}(3x+2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

d)

$$f_X(x) = \begin{cases} \frac{2}{7}(5y+1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
For $0 \le x \le 1$

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{2}{7}(5x+2y)dy \\ &= \frac{2}{7}\left(5xy + \frac{2y^2}{2}\right)\Bigg|_0^1 \\ &= \frac{2}{7}(5x+1) \end{aligned}$$

8. Let X and Y be jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} k(2-y) & 0 < x < 4, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal PDF of Y.

a)

$$f_Y(y) = \begin{cases} \frac{3}{2}y(2-y) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

b)

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

c)

$$f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

d)

$$f_Y(y) = \begin{cases} \frac{2}{3}(2-y) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
We know that for joint PDF, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

Since $f_{XY}(x,y)$ is nonzero in the region $0 < x < 4, 0 < y < 1$.

$$\Rightarrow \int_0^1 \int_0^4 f_{XY}(x,y)dxdy = 1$$
$$\Rightarrow \int_0^1 \int_0^4 k(2-y)dxdy = 1$$
$$\Rightarrow \int_0^1 k(2-y)x\Big|_0^4 dy = 1$$
$$\Rightarrow \int_0^1 4k(2-y)dy = 1$$
$$\Rightarrow 4k\left(2y - \frac{y^2}{2}\right)\Bigg|_0^1 = 1$$
$$\Rightarrow 4k \times \frac{3}{2} = 1$$
$$\Rightarrow k = \frac{1}{6}$$

For $0 < y < 1$

$$f_Y(y) = \int_0^4 \frac{1}{6}(2-y)dx$$
$$= \frac{1}{6}(2-y)x\Big|_0^4$$
$$= \frac{2}{3}(2-y)$$

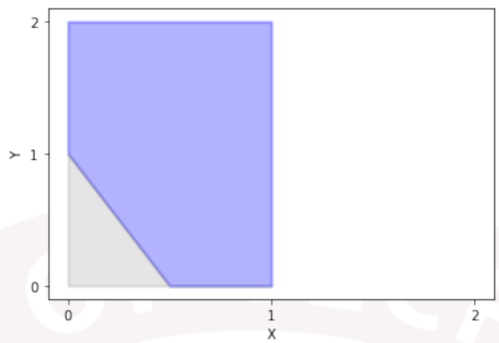
9. Let X and Y be two independent continuous random variables with PDFs $f_X(x)$ and $f_Y(y)$ given as

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(2X + Y > 1)$.

a) $\frac{1}{24}$
b) $\frac{11}{12}$
c) $\frac{1}{12}$
d) $\frac{23}{24}$



Solution:
Given that X and Y be two independent continuous random variables, therefore $f_{XY}(x, y) = f_X(x)f_Y(y)$.

$$f_{XY}(x, y) = \begin{cases} y/2 & 0 \leq x < 1, 0 \leq y < 2 \\ 0 & \text{otherwise} \end{cases}$$

We have to find the value of $P(2X + Y > 1)$.
And
$P(2X + Y > 1) = 1 - P(2X + Y \leq 1)$

$$\begin{aligned} P(2X + Y \leq 1) &= \int_0^1 \int_0^{\frac{1-y}{2}} \frac{y}{2}\ dxdy \\ &= \int_0^1 \frac{y}{2} x \Big|_0^{\frac{1-y}{2}} dy \\ &= \int_0^1 \frac{1}{4} y(1-y) dy \\ &= \frac{1}{4}\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 \\ &= \frac{1}{24} \end{aligned}$$

$\Rightarrow P(2X + Y > 1) = 1 - \frac{1}{24} = \frac{23}{24}$

10. The joint density function of two random variables X and Y is given by

$$f_{XY}(x, y) = \begin{cases} 8xy & 0 \leq x \leq 1, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

a) Yes

b) No

Solution:

$$
\begin{aligned}
f_X(x) &= \int_0^x 8xy\ dy \\
&= 8x\frac{y^2}{2}\Big|_0^x \\
&= 4x^3
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \int_0^1 8xy\ dx \\
&= 8y\frac{x^2}{2}\Big|_0^1 \\
&= 4y
\end{aligned}
$$

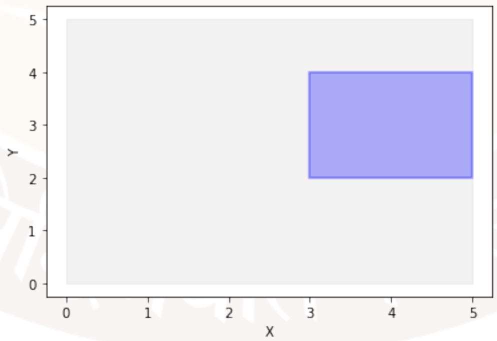
$f_X(x)f_Y(y) = 4x^3 \times 4y = 16x^3y \neq f_{XY}(x, y)$.
Hence X and Y are not independent.

11. Let $(X, Y) \sim \text{Uniform}(D)$, where $D = [3, 5] \times [2, 4]$. Are X and Y independent?

a) Yes

b) No

Solution:
$(X, Y) \sim \text{Uniform}(D)$, therefore



$$f_{XY}(x,y) = \begin{cases} \frac{1}{4} & 3 \le x \le 5, 2 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} f_X(x) &= \int_2^4 \frac{1}{4}\, dy \\ &= \frac{1}{4} y \Big|_2^4 \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_3^5 \frac{1}{4}\, dx \\ &= \frac{1}{4} x \Big|_3^5 \\ &= \frac{1}{2} \end{aligned}$$

$f_X(x)f_Y(y) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = f_{XY}(x,y)$.
Hence X and Y are independent.

12. The joint PDF of two random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution of $X \mid Y = 0.5$. ($f_{X|Y=0.5}(x)$)

a)

$$f_{X|Y=0.5}(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

b)

$$f_{X|Y=0.5}(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

c)

$$f_{X|Y=0.5}(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

d)

$$f_{X|Y=0.5}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
For $0 < y < 1$

$$\begin{aligned} f_Y(y) &= \int_0^1 4xy \ dx \\ &= 4y\frac{x^2}{2}\Big|_0^1 \\ &= 2y \end{aligned}$$

The distribution of $X \mid Y = 0.5, (0 < x < 1)$ is given by

$$\begin{aligned} f_{X|Y=0.5}(x) &= \frac{f_{XY}(x, 0.5)}{f_Y(0.5)} \\ &= \frac{4x \times 0.5}{2 \times 0.5} \\ &= 2x \end{aligned}$$

13. The joint PDF of two random variables X and Y is given by

$$f_{XY}(x, y) = \begin{cases} x^2 + \frac{xy}{3} & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(\frac{1}{4} < X < \frac{1}{2} \mid Y = 1)$.

a) $\frac{83}{96}$
b) $\frac{13}{96}$
c) $\frac{13}{48}$
d) $\frac{35}{48}$

Solution:
For $0 < y < 1$

$$\begin{aligned} f_Y(y) &= \int_0^1 \left(x^2 + \frac{xy}{3}\right) \ dx \\ &= \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)\Big|_0^1 \\ &= \frac{1}{3} + \frac{1}{6}y \end{aligned}$$

$$
\begin{aligned}
f_{X|Y=1}(x) &= \frac{f_{XY}(x,1)}{f_Y(1)} \\
&= \frac{x^2 + \frac{x\times 1}{3}}{\frac{1}{3} + \frac{1}{6}\times 1} \\
&= 2\left(x^2 + \frac{x}{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
P\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = 1\right) &= \int_{1/4}^{1/2} 2\left(x^2 + \frac{x}{3}\right) dx \\
&= 2\left(\frac{x^3}{3} + \frac{x^2}{6}\right)\Bigg|_{1/4}^{1/2} \\
&= 2\left[\left(\frac{1}{24} + \frac{1}{24}\right) - \left(\frac{1}{192} + \frac{1}{96}\right)\right] \\
&= 2\left(\frac{1}{12} - \frac{1}{64}\right) \\
&= \frac{13}{96}
\end{aligned}
$$