Image and kernel of linear transformations

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$Im(f)$ is another name for the "range of the function f" which we have studied in Maths-1.

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Then $ker(f) = \{(0,y) | y \in \mathbb{R}\}$ i.e. the Y-axis.

Also $Im(f) = \{(x,0) | x \in \mathbb{R}\}$ i.e. the X-axis.

The kernel and injectivity of a linear transformation

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A linear transformation f is 1-1 if and only if $ker(f) = 0$.

$ker(f) = \{0\}$.

$ker(f)$ is the 0 subspace

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Writing this out for linear transformations, we see that : a linear transformation $f : V \to W$ is onto if and only if $Im(f) = W$

Kernels and null spaces

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Thus, $v = \sum_{j=1}^{n} c_j v_j \in ker(f)$

$$\iff c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ is in the null space of } A.$$

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Let $w = \sum_{i=1}^{m} d_i w_i \in W$. Then $w \in Im(f)$ precisely when there exist scalars $c_j; j = 1, 2, \ldots, n$ such that $\sum_{j=1}^{n} A_{ij} c_j = d_i$ for all i.

Images and column spaces (contd.)

Equivalently $w = \sum_{i=1}^{m} d_i w_i \in Im(f)$ if there exists a column vector $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ such that the column vector $d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} = Ac$.

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Hence, $w = \sum_{i=1}^{m} d_i w_i \in Im(f)$

$$\iff d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} \text{ is in the column space of } A.$$

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In particular, the vectors $\begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \end{bmatrix}, \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2n} \end{bmatrix}, \ldots, \begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kn} \end{bmatrix}$ form a basis for the null space of A precisely when $v'_1, v'_2, \ldots, v'_k \in ker(f)$, where $v'_i = \sum_{j=1}^{n} c_{ij} v_j$, form a basis for $ker(f)$.

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Note further that under this isomorphism, the columns of A, which form a spanning set of the column space of A, correspond to the images $f(v_i)$, which form a spanning set for $im(f)$.

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We can thus use row reduction to obtain these bases.