Week 10
Graded Assignment
Mathematics for Data Science - 2

1 Multiple Choice Questions (MCQ)

1. Match the functions of two variables in Column A with their graphs given in Column B.

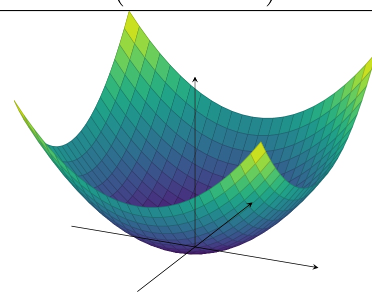
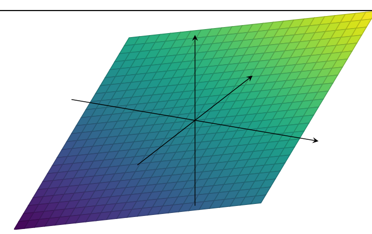
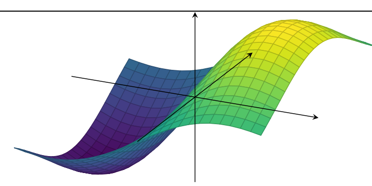
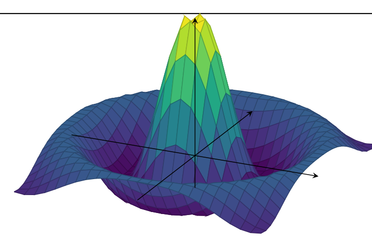
	Function of two variables (Column A)		Graph of the function (Column B)
i)	$f(x,y) = 2x + 3y$	1)	
ii)	$f(x,y) = x^2 + y^2$	2)	
iii)	$f(x,y) = \frac{sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$	3)	
iv)	$f(x,y) = xe^{(-x^2-y^2)}$	4)	

Table: M2W10G1

- ◯ Option 1: i → 2, ii → 3, iii → 4, iv → 1.
- ◯ **Option 2:** i → 2, ii → 1, iii → 4, iv → 3.
- ◯ Option 3: i → 3, ii → 2, iii → 4, iv → 1.
- ◯ Option 4: i → 2, ii → 4, iii → 3, iv → 1.

Solution:-

(i) $f(x,y) = 2x+3y$, which is a linear function of two variable. So, it represents a plane.

$i \to 2$

(ii) $f(x,y) = x^2+y^2$

So, $f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$

& $f(0,0) = 0$

$(ii) \to 1$

(iii) $f(x,y) = \dfrac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$

Let $\sqrt{x^2+y^2} = t$

So, $(x,y) \to 0 \Rightarrow \sqrt{x^2+y^2} \to 0$

$\Rightarrow t \to 0$

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

which is represented by figure 4

So, $(iii) \to 4$

(iv) $f(x,y) = x e^{-(x^2+y^2)}$

So, $f(0,0) = 0$, $f(x,y) < 0 \ \forall x \in (-\infty, 0)$

and $f(x,y) \geq 0 \ \forall x \in [0, \infty)$.

And $f(x,y)$ has exponetially growth.

(iv) $\Rightarrow$ 3.

2. Which of the following functions f and g from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous at the origin?

$$f(x,y) = \begin{cases} \frac{x^3}{3x^2y} & \text{if } x, y \neq 0, \\ 0 & \text{if } x = y = 0 \end{cases}$$

$$g(x,y) = \begin{cases} x^4 + x^3y + xy^3 + y^4 & \text{if } x, y \neq 0, \\ 0 & \text{if } x = y = 0 \end{cases}$$

- ◯ Option 1: Both f and g are continuous at the origin.
- ◯ Option 2: f is continuous at the origin but g is not.
- ◯ **Option 3:** g is continuous at the origin but f is not.
- ◯ Option 4: Neither f nor g is continuous at the origin.

Solution :- Given, $f(x,y) = \begin{cases} \frac{x^3}{3x^2y} & \text{if } x, y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$

Let $y = 2x$

We get, $f(x,y) = \frac{x^3}{6x^3}$ when $x \neq 0$

$= \frac{1}{6}$ when $x \neq 0$

So, if $(x,y) \to (0,0)$ along the path $y = 2x$, then

$f(x,y) \to \frac{1}{6}$

and if $(x,y) \to (0,0)$ along the path $y = x$, then

$f(x,y) \to \frac{1}{3}$

That means, as $(x,y) \to (0,0)$ along different-different path then function has different limit. That means limit of $f(x,y)$ does not exist. So function $f(x,y)$ is not continuous at $(0,0)$.

Given $g(x,y) = \begin{cases} x^4 + x^3y + xy^3 + y^4 & \text{if } x, y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$

Observe that $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$

So function $g(x,y)$ is continuous at the origin.

2 Multiple Select Questions (MSQ)

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} \frac{x+y}{\sqrt{x^2+y^2}} & \text{if } x, y \neq 0, \\ 0 & \text{if } x = y = 0 \end{cases}$$

Choose the correct options from the following.

- ◯ **Option 1:** $f(x,y) \to 1$ if $x \to 0$ and $y \to b$, for some positive real number b.
- ◯ Option 2: $f(x,y) \to 0$ if $(x,y) \to (0,0)$.
- ◯ **Option 3:** $f(x,y) \to 1$ if $(x,y) \to (0,0)$ along the positive X-axis.
- ◯ Option 4: $f(x,y) \to 0$ if $(x,y) \to (0,0)$ along the positive Y-axis.
- ◯ Option 5: $f(x,y) \to 0$ if $(x,y) \to (0,0)$ along the line $y = 2x$.
- ◯ **Option 6:** $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution:- Given, $f(x,y) = \begin{cases} \frac{x+y}{\sqrt{x^2+y^2}} & \text{if } x, y \neq 0 \\ 0 & \text{if } x = y = 0 \end{cases}$

Option 1:

$\lim_{(x,y)\to(0,b)} f(x,y) = \frac{b}{\sqrt{b^2}} = 1$, where b is a positive real number.

option 2:

let $y = x$

we get, $f(x,y) = \frac{2x}{\sqrt{x^2+x^2}} = \sqrt{2}$ when $x \neq 0$

Again, let $y = 2x$, we get $f(x,y) = \frac{3}{\sqrt{5}}$

So, if $(x,y) \to (0,0)$ along the path $y = x$, $f(x,y) \to \sqrt{2}$

and if $(x,y) \to (0,0)$ along the path $y = 2x$, $f(x,y) \to \frac{3}{\sqrt{5}}$

So, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Option 3: $(x,y) \to (0,0)$ along the positive X-axis

$\Rightarrow (x,0) \to (0,0)$ when $(x,y) \to (0,0)$, where $x>0$

So, $\displaystyle\lim_{(x,0)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2}} = 1$

Option 4 :- $(x,y) \to (0,0)$ along the positive Y-axis

$\Rightarrow (0,y) \to (0,0)$ when $(x,y) \to (0,0)$, where $y>0$

So, $\displaystyle\lim_{(0,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{y}{\sqrt{y^2}} = 1$

Option 5 :- As given path $y=2x$, $f(x,y) = \dfrac{3x}{\sqrt{5x^2}} = \dfrac{3}{\sqrt{5}}$

So. $(x,y) \to (0,0)$ along the path $y=2x$, $f(x,y) \to \frac{3}{\sqrt{5}}$

Option 6: Observe that from options 1, 3, 4, 5; existance of limit of the function $f(x,y)$ is path dependent. Hence $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

4. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$, such that $\frac{\partial f}{\partial x} = y\cos(xy)$. Which of the following are possible candidates for the function f?

- ○ **Option 1:** $f(x, y) = \sin(xy)$.
- ○ Option 2: $f(x, y) = xy\cos(xy)$.
- ○ Option 3: $f(x, y) = \sin(xy) + xy$.
- ○ **Option 4:** $f(x, y) = \sin(xy) + \cos(y)$.
- ○ Option 5: $f(x, y) = \sin(xy) + \phi(x)$, for some function $\phi(x)$.
- ○ **Option 6:** $f(x, y) = \sin(xy) + \phi(y)$, for some function $\phi(y)$.

Solution:- Option 1 :- If $f(x,y) = \sin(xy)$,

then $\frac{\partial f}{\partial x} = y\cos(xy)$ which is equal to the given partial derivative of $f(x,y)$ w.r.t x.

Option 2 :- If $f(x,y) = xy\cos(xy)$, then

$\frac{\partial f}{\partial x} = y\cos(xy) - xy^2\sin(xy)$ which is not equal to the given partial derivative of $f(x,y)$ w.r.t x.

Similarly, we can check for other options.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x, y) = \sqrt{x^2 + y^2}$$

Choose the set of correct options.

- ◯ Option 1: $f_x(0,0) = 0$, and $f_y(0,0) = 0$.
- ◯ **Option 2:** $f_x(0,0)$ does not exist.
- ◯ **Option 3:** $f_y(0,0)$ does not exist.
- ◯ **Option 4:** $f_x(1,1) = \frac{1}{\sqrt{2}}$.

Solution:- Given $f(x,y) = \sqrt{x^2+y^2}$

Now, $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h}$

Observe that, left limit, $\lim_{h\to 0^-} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0} \frac{h}{-h} = -1$

& right limit, $\lim_{h\to 0^+} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$

So, $f_x(0,0)$ does not exist.

Similarly, $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h}$

which is the same as above case.

Hence $f_y(0,0)$ does not exist.

Option 4:- $f_x(x,y) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

So, $f_x(1,1) = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$.

3 Numerical Answer Type (NAT):

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:

$$f(x, y) = (f_1(x, y), f_2(x, y))$$

where $f_1(x, y) = xy + y^2$ and $f_2(x, y) = x + xy + 1$. Define a matrix A as follows:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x}(1,1) & \frac{\partial f_1}{\partial y}(1,1) \\ \\ \frac{\partial f_2}{\partial x}(1,1) & \frac{\partial f_2}{\partial y}(1,1) \end{bmatrix}$$

What will be the determinant of A? [Answer: -5]

Solution:- Given $f_1(x,y) = xy + y^2$

So $\frac{\partial f_1}{\partial x} = y . \Rightarrow \frac{\partial f_1}{\partial x}(1,1) = 1$

& $\frac{\partial f_1}{\partial y} = x+2y \Rightarrow \frac{\partial f_1}{\partial y}(1,1) = 1+2 = 3$

Again, $f_2(x,y) = x + xy + 1$

So, $\frac{\partial f_2}{\partial x} = 1+y \Rightarrow \frac{\partial f_2}{\partial x}(1,1) = 1+1 = 2$

& $\frac{\partial f_2}{\partial y} = x \Rightarrow \frac{\partial f_2}{\partial y}(1,1) = 1$

So, matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

Now, $det(A) = 1 - 6 = -5$

7. Find out the directional derivative of the function $f(x, y) = x^2y^3$ at $(1, 2)$, in the direction of the vector $(0, 2)$. [Answer: 12]

Solution :- Given $f(x,y) = x^2y^3$

Directional derivative of $f(x,y)$ at $(1,2)$ in direction

of $(0,2)$ $= \nabla f(1,2) \cdot \frac{(0,2)}{2}$

$\because$ Unit vector along the direction $(0,2)$ is $\frac{(0,2)}{2} = (0,1)$

Now, $\nabla f = (f_x, f_y)$

$= (2xy^3, 3x^2y^2)$

So, $\nabla f(1,2) = (2\times1\times8, 3\times1\times4) = (16, 12)$

Now, $\nabla f(1,2) \cdot \frac{(0,2)}{2} = (16,12)\cdot(0,1) = 12$

4 Comprehension Type Question:

The price of a product ($f(x, y)$) depends on the price (x) of the raw materials and the price (y) of transportation of the product to the market. The dependency of these quantities is given by the function

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$f(x, y) = x^2 + xy + y^2$$

Answer Questions 8,9 and 10 using the information above.

8. Which of the following statements is true if the rate of change of the price of the product with respect to the price of the raw materials is the same as the rate of change of price of the product with respect to the price of transportation of the product to the market? (MCQ)

   - ◯ Option 1: The price of the raw materials is twice the price of the transportation of the product to the market.
   - ◯ Option 2: The price of the raw materials is half the price of the transportation of the product to the market.
   - ◯ **Option 3:** The price of the raw materials is the same as the price of the transportation of the product to the market.
   - ◯ Option 4: The price of the raw materials is 3 times the price of the transportation of the product to the market.

Solution :-

Given $f(x,y) = x^2 + xy + y^2$

The rate of change of the price of the product w.r.t the price of raw matrials $= \frac{\partial f}{\partial x}(x,y) = 2x + y$.

And the rate of change of the price of the product w.r.t the price of transportation of the product to the market $= \frac{\partial f}{\partial y}(x,y) = x + 2y$

Now, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow 2x + y = x + 2y \Rightarrow x = y$

Hence, the price of raw materials is the same as the price of the transportation of the product to the market.

9. Which of the following statements are true? (MSQ)

- ◯ Option 1: $f(x, y)$ is a linear function.
- ◯ **Option 2:** $f(cx, cy) = c^2 f(x, y)$ for any real number c.
- ◯ **Option 3:** $f(x, y)$ is continuous at $(0, 0)$.
- ◯ Option 4: If the price of transportation and raw material of the product approaches 0 and 5 respectively, then the price of the product approaches 30.
- ◯ **Option 5:** If the price of transportation and raw material of the product approaches 0 and 5 respectively, then the price of the product approaches 25.

Solution :- Given $f(x,y) = x^2 + xy + y^2$

Option 1 :- Observe that degree of $f(x,y) = 2$. So, $f(x,y)$ is not linear function.

Option 2 :- $f(cx, cy) = (cx)^2 + (cx)(cy) + (cy)^2 = c^2(x^2 + xy + y^2)$

$\Rightarrow f(cx, cy) = c^2 f(x,y)$

Option 3 :- $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2 + xy + y^2) = 0 = f(0,0)$

So, $f(x,y)$ is continuous.

Options 4, 5 :- $\lim_{(x,y)\to(0,5)} f(x,y) = \lim_{(x,y)\to(0,5)} x^2 + xy + y^2 = 5^2 = 25.$

10. What will be the rate of change of the price of the product along the direction of the vector $(1, m)$, when the price of raw material is a and the price of transportation of the product to the market is b? (MCQ)

- ◯ Option 1: $\frac{1}{\sqrt{1+m^2}}(1+2m)(a+b)$
- ◯ Option 2: $\frac{1}{\sqrt{1+m^2}}[(2+2m)a+(1+2m)b]$
- ◯ Option 3: $\frac{1}{\sqrt{1+m^2}}(a+mb)$
- ◯ **Option 4:** $\frac{1}{\sqrt{1+m^2}}[(2+m)a+(1+2m)b]$

Solution: Given $f(x,y) = x^2+xy+y^2$ & point (a,b)

The rate of change of the price of the product along the direction of the vector $(1,m)$ = directional derivative of $f(x,y)$ in the direction of $(1,m)$ at point (a,b) $= \nabla f(a,b) \cdot \frac{(1,m)}{\sqrt{1+m^2}}$

Now $\nabla f(x,y) = (f_x, f_y) = (2x+y, x+2y)$

$\nabla f(a,b) = (2a+b, a+2b)$

so, $\nabla f(a,b) \cdot \frac{(1,m)}{\sqrt{1+m^2}} = (2a+b, a+2b)\cdot(1, m)\frac{1}{\sqrt{1+m^2}}$

$$= \frac{1}{\sqrt{1+m^2}}(2a+b+am+2bm)$$

$$= \frac{1}{\sqrt{1+m^2}}\left[(2+m)a+(1+2m)b\right]$$