Week-1
Mathematics for Data Science - 2
Some topics of Maths 1, Functions of one variable, Graphs and tangents, Limits for sequence, Limits for functions of one variable
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Consider the following statements.

   - **Statement 1:** If $\{a_n\}$ and $\{b_n\}$ both converge to some non-zero real number, then $\{a_n + b_n\}$ also converges to some non-zero real number.
   - **Statement 2:** If $\{a_n\}$ is an increasing sequence, i.e., $a_i \leq a_{i+1}$, for all $i \in \mathbb{N}$, then $\{(-1)^n a_n\}$ is a decreasing sequence.
   - **Statement 3:** If $\{a_n\}$ and $\{b_n\}$ both converge to the same real number, then $\{a_n - b_n\}$ must converge to 0.

   Choose the correction option from the following.

   ◯ Option 1: All the three statements are true.

   ◯ Option 2: Statements 1 and 2 are true, but Statement 3 is false.

   ◯ Option 3: Statements 1 and 3 are true, but Statement 2 is false.

   ◯ **Option 4:** Only Statement 3 is true.

   ◯ Option 5: None of the statements is true.

   Solution:

   - **Statement 1:** Suppose $\{a_n\}$ and $\{b_n\}$ both are constant sequences, such that $a_n = -1$ and $b_n = 1$, for all n. Both of them converges to some non-zero real number. As $\{a_n\}$ converges to -1 and $\{b_n\}$ converges to 1. But $a_n + b_n = 0$ for all n. Hence $\{a_n + b_n\}$ converges to 0. Hence the statement is false.
   - **Statement 2:** Suppose $a_n = n$. Hence the sequence $\{a_n\} = \{1, 2, 3, 4, \ldots\}$ is an increasing sequence. So the sequence $\{(-1)^n a_n\} = \{-1, 2, -3, 4, \ldots\}$, which is not a decreasing sequence. Hence the statement is false.
   - **Statement 3:** Suppose $\lim a_n = c = \lim b_n$ for some real number c. We know that, $\lim(a_n - b_n) = \lim a_n - \lim b_n = c - c = 0$. Hence the statement is true.

2. Match the given functions in Column A with their types in column B and their graphs in Column C, given in Table M2W1T1.

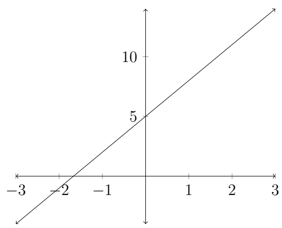
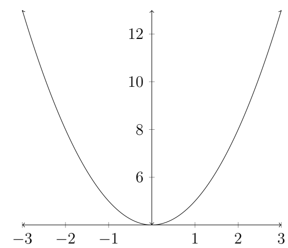
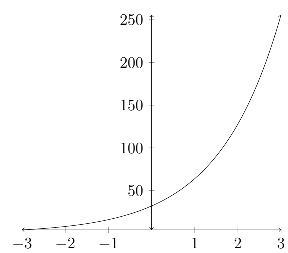
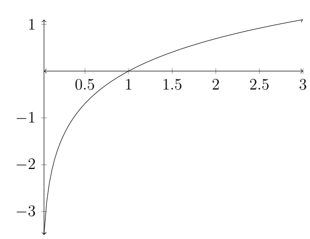
	Functions (Column A)		Types of functions (Column B)		Graphs (Column C)
i)	$f(x) = x^2 + 4$	a)	Logarithmic function	1)	
ii)	$f(x) = ln(x)$	b)	Exponential function	2)	
iii)	$f(x) = 2^{x+5}$	c)	Linear function	3)	
iv)	$f(x) = 3x + 5$	d)	Quadratic function	4)	

Table: M2W1G1

- ◯ Option 1: i) → d) → 2), ii) → a) → 4), iii) → c) → 3), iv) → b) → 1).
- ◯ **Option 2:** i) → d) → 2), ii) → a) → 4), iii) → b) → 3), iv) → c) → 1).

◯ Option 3: i) $\rightarrow$ d) $\rightarrow$ 2), ii) $\rightarrow$ a) $\rightarrow$ 3), iii) $\rightarrow$ b) $\rightarrow$ 4), iv) $\rightarrow$ c) $\rightarrow$ 1).

◯ Option 4: i) $\rightarrow$ d) $\rightarrow$ 3), ii) $\rightarrow$ a) $\rightarrow$ 4), iii) $\rightarrow$ b) $\rightarrow$ 2), iv) $\rightarrow$ c) $\rightarrow$ 1).

Solution:

- $f(x) = x^2 + 4$, is a quadratic function. The curve represented by the function f is a parabola. So, i) $\rightarrow$ d) $\rightarrow$ 2).
- $f(x) = ln(x)$ is a logarithmic function. So, ii) $\rightarrow$ a) $\rightarrow$ 4).
- $f(x) = 2^{x+5}$ is a exponential function. So, iii) $\rightarrow$ b) $\rightarrow$ 3).
- $f(x) = 3x + 5$ is a linear function. The curve represented by the function f is a straight line. So, iv) $\rightarrow$ c) $\rightarrow$ 1).

2 Multiple Select Questions (MSQ)

3. Limits of some standard functions are given below:

- $\lim_{x\to 0} \frac{sin(x)}{x} = 1$
- $\lim_{x\to 0} \frac{log(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

Using this given information choose the correct options.

○ **Option 1:** $\lim_{x\to 0} \frac{log(1+x)}{sin(x)} = 1.$

○ Option 2: $\lim_{x\to 0} \frac{log(1+x)}{sin(x)}$ is undefined.

○ Option 3: $\lim_{x\to 0} \frac{sin(5x)}{x} = 1.$

○ **Option 4:** $\lim_{x\to 0} \frac{sin(5x)}{x} = 5.$

○ Option 5: $\lim_{x\to 0} \frac{sin(5x)}{x} = \frac{1}{5}$

○ Option 6: $\lim_{x\to 0} \frac{e^{\frac{x}{2}} - 1}{sin2x} = 1.$

○ Option 7: $\lim_{x\to 0} \frac{e^{\frac{x}{2}} - 1}{sin2x} = 2.$

○ **Option 8:** $\lim_{x\to 0} \frac{e^{\frac{x}{2}} - 1}{sin2x} = \frac{1}{4}.$

○ Option 9: $\lim_{x\to 0} \frac{e^{\frac{x}{2}} - 1}{sin2x} = \frac{1}{2}.$

Solution:

- $\lim_{x\to 0} \frac{log(1+x)}{sin(x)} = \lim_{x\to 0} \frac{\frac{log(1+x)}{x}}{\frac{sin(x)}{x}} = \frac{\lim_{x\to 0} \frac{log(1+x)}{x}}{\lim_{x\to 0} \frac{sin(x)}{x}} = \frac{1}{1} = 1$
- $\lim_{x\to 0} \frac{sin(5x)}{x} = \lim_{5x\to 0} 5\frac{sin(5x)}{5x} = 5 \lim_{5x\to 0} \frac{sin(5x)}{5x} = 5$ (Since $x \to 0$, we have $5x \to 0$).
- $\lim_{x\to 0} \frac{e^{\frac{x}{2}} - 1}{sin2x} = \lim_{x\to 0} \frac{1}{4} \frac{\frac{e^{\frac{x}{2}}-1}{\frac{x}{2}}}{\frac{sin2x}{2x}} = \frac{1}{4} \frac{\lim_{\frac{x}{2}\to 0} \frac{e^{\frac{x}{2}}-1}{\frac{x}{2}}}{\lim_{2x\to 0} \frac{sin2x}{2x}} = \frac{1}{4}$ (Since $x \to 0$, we have $\frac{x}{2} \to 0$ and $2x \to 0$).

4. The graph of some function is drawn below in Figure M2W1G1. Choose the set of correct statements about it.

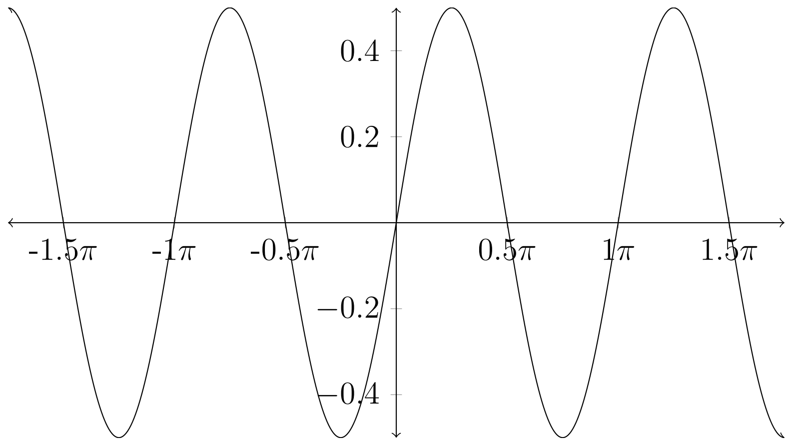
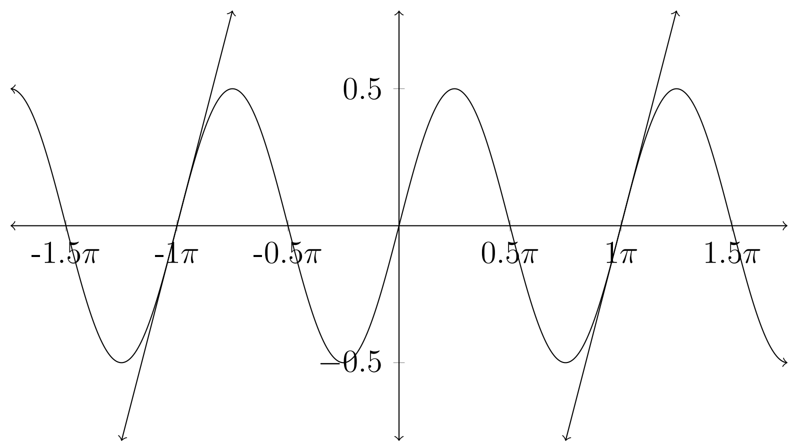


Figure: M2W1G1

- ◯ Option 1: Limit of the function as x tends to 0 is 1.
- ◯ **Option 2:** Limit of the function as x tends to 0 is 0.
- ◯ Option 3: Limit of the function as x tends to 0 is undefined.
- ◯ Option 4: There is a (unique) tangent at the point $x = \pi$, but not at $x = -\pi$.
- ◯ **Option 5:** There is a (unique) tangent at $x = \pi$, as well as at $x = -\pi$.
- ◯ Option 6: The given function is monotonically increasing in the interval $[-0.5\pi, 0.5\pi]$.
- ◯ Option 7: The given function is monotonically decreasing in the interval $[-0.5\pi, 0.5\pi]$.

Solution:

- As we are approaching from right of 0 towards 0, the value of the function is also approaching to 0. Similarly as we are approaching from the left of 0 towards 0, the value of the function is also approaching to 0. Hence limit of the function as x tends to 0 is 0.
- There is a (unique) tangent at $x = \pi$, as well as at $x = -\pi$ as shown in the figure below.



- The function is decreasing in the interval $[-0.5\pi, -0.25\pi]$ and increasing in the interval $[-0.25\pi, 0.25\pi]$. Again the function decreases in the interval $[0.25\pi, 0.5\pi]$.

5. Depending on the graphs given below, predict which have a (unique) tangent at the origin (i.e., $(0,0)$) ?

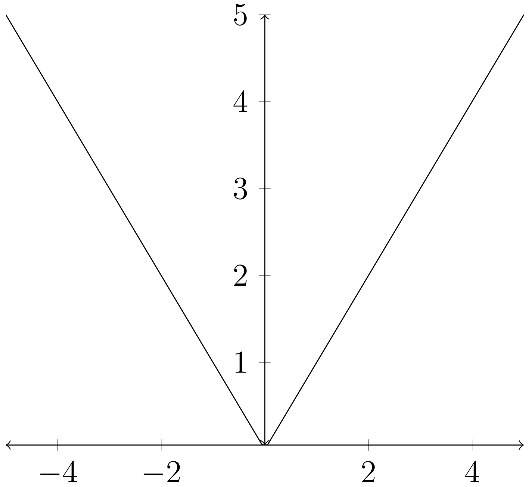


Figure: Curve 1

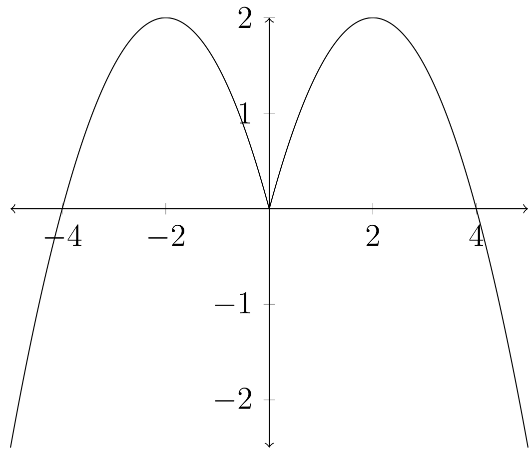


Figure: Curve 2

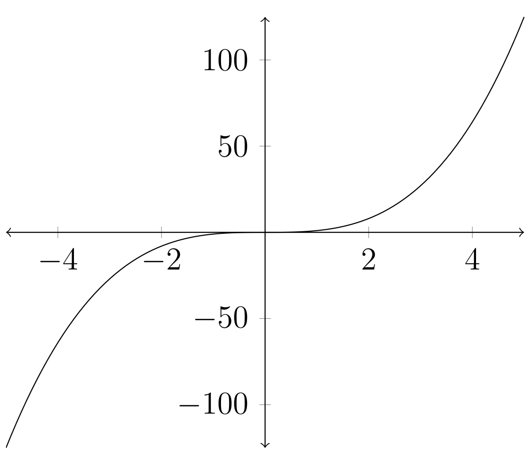


Figure: Curve 3

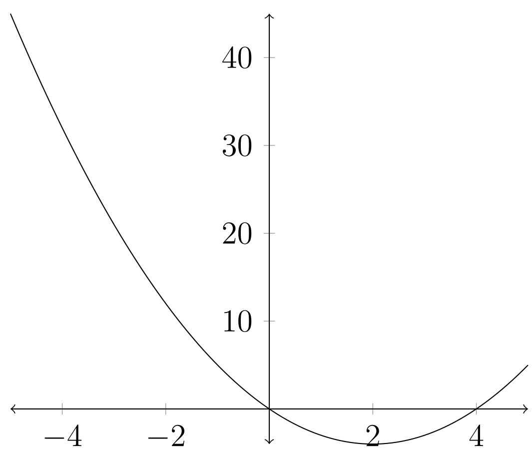


Figure: Curve 4

- ◯ Option 1: Curve 1
- ◯ Option 2: Curve 2
- ◯ **Option 3:** Curve 3
- ◯ **Option 4:** Curve 4

Solution:

- There are sudden changes in the slopes of Curve 1 and Curve 2, at the origin. So from the graph we can predict that the these two curves do not have a (unique) tangent at the origin $(0,0)$, where as Curve 3 and Curve 4 have.

3 Numerical Answer Type (NAT)

6. Find the limit of the sequence given by $a_n = \dfrac{2+4+6+\ldots+2n}{n^2}$, (where $n \in \mathbb{N} \setminus \{0\}$). (Answer: 1)

   Solution: $a_n = \dfrac{2(1+2+3+\ldots+n)}{n^2} = \dfrac{2\frac{n(n+1)}{2}}{n^2} = \dfrac{n(n+1)}{n^2} = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$
   As n increases, $\frac{1}{n} \to 0$. Hence $a_n \to 1$.

7. What will be the value of $\lim_{x \to 2+} \lfloor x \rfloor - \lim_{x \to 2-} \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x? (Answer: 1)

   Solution: $\lim_{x \to 2^+} \lfloor x \rfloor = 2$ and $\lim_{x \to 2^-} \lfloor x \rfloor = 1$. Hence $\lim_{x \to 2+} \lfloor x \rfloor - \lim_{x \to 2-} \lfloor x \rfloor = 1$

4 Comprehension Type Question:

Suppose a company runs three algorithms to predict its future growth. Suppose the error in the estimation depends on the available number (n) (where $n \in \mathbb{N} \setminus \{0\}$) of data as follows:

- Error in estimation by Algorithm 1: $a_n = \dfrac{n^2 + 5n}{3n^2 + 1}$.
- Error in estimation by Algorithm 2: $b_n = \dfrac{1}{2} + (-1)^n \dfrac{1}{n}$
- Error in estimation by Algorithm 3: $c_n = \dfrac{e^n + 4}{4e^n}$

Suppose the company has a large amount of data in their hand (we can assume n tends to ∞). Using the above set of information answer the questions 8, 9 and 10.

8. Which of the given algorithms should the company use to get the minimum error in the prediction of its growth? (MCQ)

   ◯ Option 1: Algorithm 1

   ◯ Option 2: Algorithm 2

   ◯ **Option 3:** Algorithm 3

   ◯ Option 4: Both Algorithm 1 and Algorithm 2 will give the same error and that will be the minimum.

   Solution: We can write $a_n = \dfrac{1 + \frac{5}{n}}{3 + \frac{1}{n^2}}$ and $c_n = \dfrac{1}{4} + \dfrac{1}{e^n}$.

   As $n \to \infty$, $a_n \to \frac{1}{3}$, $b_n \to \frac{1}{2}$, and $c_n \to \frac{1}{4}$, among which $\frac{1}{4}$ is the minimum.

   Hence, Algorithm 3 will give the minimum error in the prediction.

9. Which of the given algorithms gives the maximum error? (MCQ)

   ◯ Option 1: Algorithm 1

   ◯ **Option 2:** Algorithm 2

   ◯ Option 3: Algorithm 3

   ◯ Option 4: Both Algorithm 1 and Algorithm 2 will give the same error and that will be the maximum.

   Solution: From the solution of Question 8, it is clear that Algorithm 2 will give the maximum error in the prediction.

10. Suppose a new algorithm is designed to predict the growth of the company in future and the error in estimation by the new algorithm is given by $b_n - a_n$, where a_n and b_n are the same as defined earlier. Choose the set of correct options. (MSQ)

- ◯ **Option 1:** The error in estimation using the new algorithm is less than the error in estimation using Algorithm 1.
- ◯ Option 2: The error in estimation using the new algorithm is more than the error in estimation using Algorithm 2.
- ◯ **Option 3:** The error in estimation using the new algorithm is less than the error in estimation using Algorithm 3.
- ◯ Option 4: The error in estimation using the new algorithm cannot be compared with the error in estimation using Algorithm 3.

Solution: As $a_n \to \frac{1}{3}$ and $b_n \to \frac{1}{2}$, we have $b_n - a_n \to \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Hence, the error in estimation using the new algorithm is less than the error in estimation using Algorithm 1, Algorithm 2, or Algorithm 3.