IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 02
Functions of one variable

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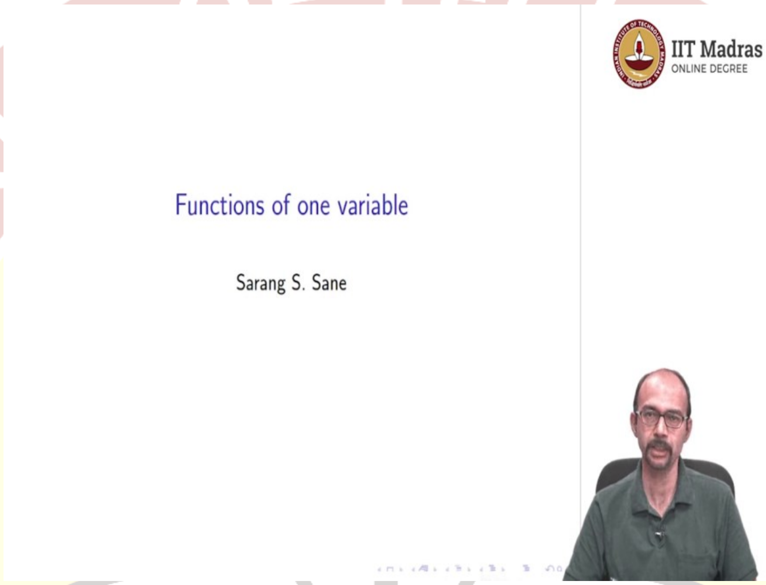

Hello and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to talk about Functions of One Variable. So, this was an idea that we introduced in the previous video, namely these are functions from some subset of R to R.

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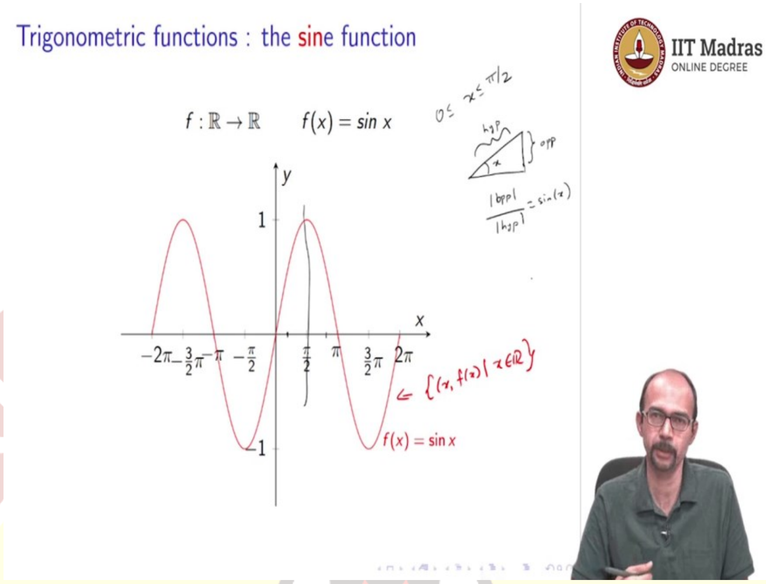

And here is an example of such thing. So, the common examples that we have seen, I mean, that we have heard of, but we may not have studied in Maths 1 are the trigonometric functions. So, I am going to do a large part of this video is going to be about that the trigonometric functions and then we will study some other generalities. So, the sine function, so it is a function from R → R, defined as $\sin x$ and how do you define it?

So, here is the graph. It is a very nice graph. It is periodic. So, such graphs are called periodic or sinusoidal and it varies between -1 and 1 and it has a period 2π. So, as you can see it, if you look at its trajectory between minus 2π and 0, we start like this and come here and then come here, and then it restarts. So, again go back and again 2π. So, it has period 2 π that is what we mean. So, if you shift it by 2 π. It does not really change.

So, this here is the graph of that function, f of x is $\sin x$, which means it is the set of values x, where $x \epsilon R$. Of course, the entire graph it keeps going. It does not end at minus 2 π , it keeps going over the entire real line and how do we define this?

So, to compute this, what you do is, you first define it between 0 and 2 π. So, I mean, I am giving you the heuristic definition. There is another definition which is a technical one, which I am not, I am going to avoid in this course, till later on where we may be forced to use it. So, between 0 and 2 π, what you do is or at least let us say between 0 and $\pi/2$. So, suppose x is between 0 and $\pi/2$.

So, this is x. Now, we know what is $\sin x$. So, $\sin x$ is the opposite side by the hypotenuse, the length of the opposite side divided by the hypotenuse. So, length of the opposite side divided by the length of the hypotenuse, this was exactly how we defined $\sin x$.

So, we know what is $\sin x$when x is between 0 and $\pi/2$ and now if you carefully look at the picture that should tell you how to define $\sin x$ for any number, because once I know what it is between 0 and $\pi/2$, for $\pi/2$ to π, I use the fact that this is symmetric. So, I from there, I will be able to say what is, if you give me let us say this value here, this point here, then I look at the corresponding value here and that value is what I get here. So, then I know what it is between 0 and π.

And again between π and $2\,\pi$, you reflect it like this and then you reflect it like this or if you want, you rotate it like this and that gives you, so you either reflect like this and like this or you rotate it like this. So, that tells you what is the sine function value of any particular point. So, this definition will be able to tell you what a sin for between 0 and $2\,\pi$ and then if you have something which is not between 0 and $2\,\pi$, you just add or subtract multiples of $2\,\pi$, so as to get it within 0 to $2\,\pi$. And then you define the, we know what the definition is over there and you use that to define this function.

So, I am giving you a heuristic definition of what is a sine function. There is a more formal mathematical definition, which I will avoid for now. But the main point is it is a very beautiful looking function and it is a very useful function as well.

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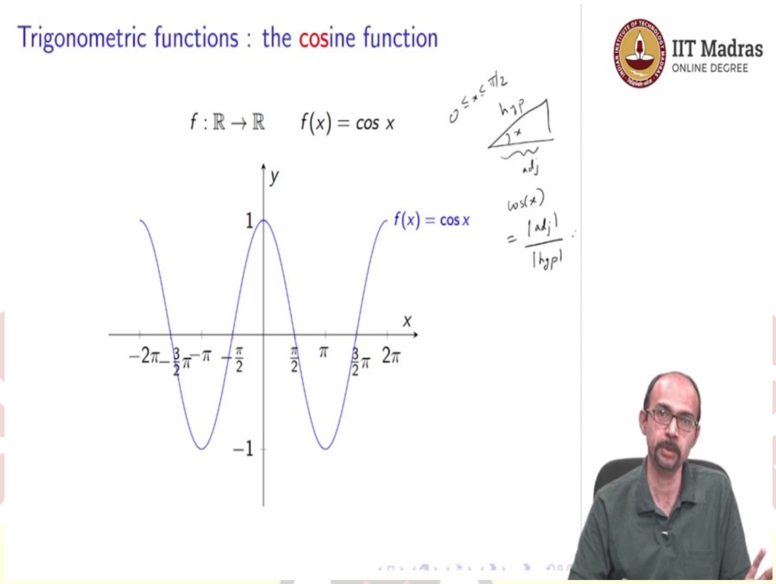

The next function that we may have already encountered before, not as a function, but as cosine of an angle is, so that is the cosine function and again the idea is the same, we define it between 0 and $\pi/2$ and for the rest of the picture, we extend it by some, from the picture we can see how to extend it and between 0 and $\pi/2$ what is $\cos x$.

So, you, if x is between 0 and $\pi/2$, then you draw x over here, this is the adjacent side, this is the hypotenuse and then $\cos x$ is the length of the adjacent side divided by the length of the hypotenuse. So, this defines it between 0 and $\pi/2$ and now from the picture you will be able to see how to extend it beyond that because of the periodic nature of the function. So, we will study some relations like this later on in this video.

So, the cosine function has similar properties to the sine. The cosine function is periodic of period 2π. It is an example of a sinusoidal function and it takes values between -1 and 1. So, that part is clear because the adjacent side cannot have larger length than the hypotenuse. So, this value can be at most 1.

But of course there is, we are also allowing minus because we kind of flip it below. So, the minus is, for example, to keep track of which quadrant we are talking about. These ideas we have seen in when we did, we changed our orthonormal basis, if you remember that video. So, I will suggest you go back and check that video.

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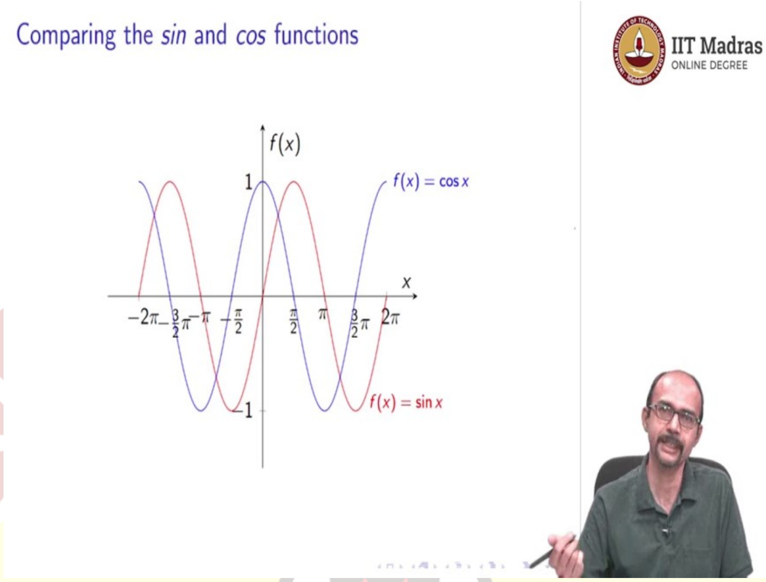

Here is how the sine and cosine function look. When we put them together, it is a very beautiful picture and you can see that the cosine function or the sine function, whichever you prefer is just the shift of the other one. So, if you shift it by $\pi/2$, then you get the other one. So, this is something you can keep in mind and we will, maybe we will use this, we would not use it directly, but we will mention this later on also in the properties.

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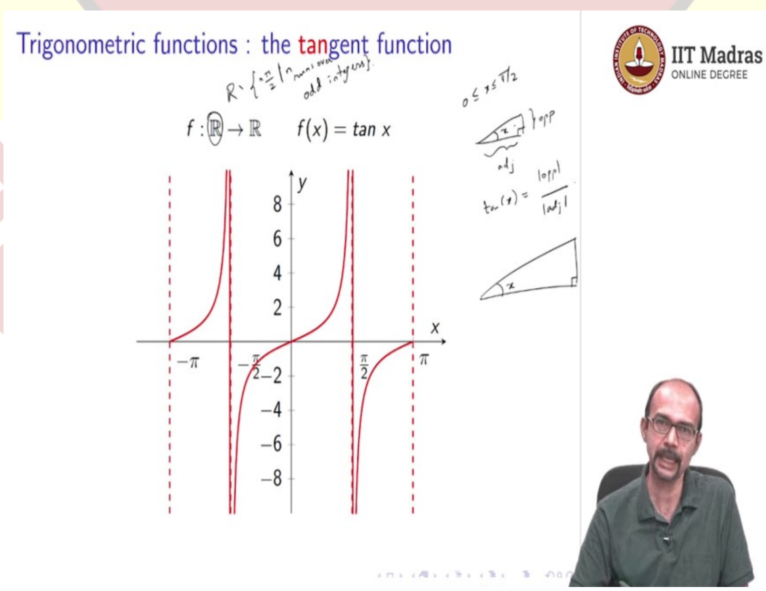

So, what are the other trigonometric functions that we may have heard of, the tangent function, often called the tan of an angle. So, f is from $R \to R$. It is f(x) = $\tan x$ and now here we need a little bit of a caveat, because this f is actually not defined on the entire R. So, here, let me first make this, let me first show this graph and then make the statement I want to. So, as you can see, this is again a function which is periodic and here from $-\pi/2$ to $\pi/2$, it rises very rapidly. So, it increases very, very, very fast until it hits a point then it is relatively slower. It passes through 0, 0 and then again it shows the opposite kind of behaviour.

So, again, it rises slightly slowly and then takes off and what happens on this red line, which is the two red lines which are on, so those are the - $\pi/2$ and $\pi/2$. So, at - $\pi/2$ and $\pi/2$, this function takes the values minus infinity and infinity. So, you can either think of it as taking those values or you can think of it as an undefined function. So, since we are talking about real numbers, we think of these as undefined which means that this function is not really defined on the entire real line, it is defined on $R\setminus n\pi/2$, where n runs over odd integers.

So, we do not ask what is f($-\pi/2$) or f($\pi/2$) or f(-5 $\pi/2$) or f(7 $\pi/2$). So, for such numbers tangent is not defined. For the other numbers, it is defined by these graphs and it is, again, this is periodic of period π. So, if you want something in here, you just shift it by π each time and then you can find the value and how do we define between minus $\pi/2$ and $\pi/2$. So, for this, again, we will define it only between 0 and $\pi/2$ and the symmetry of the function will allow you to see what is it for minus $\pi/2$ to 0.

So, now, this is if x is between 0 and $\pi/2$. This time you look at the adjacent side and the opposite side and you define tangent of x as the length of the opposite side by the length of the adjacent side. So, this function is called the tangent function. So, the main point is here is that we are, I mean, what we are exploiting here is that this ratio is constant. It depends only on the angle. It does not depend on how large the triangle is. I mean, I could, instead of this, I could draw another triangle like this, where this is x.

This is, of course, a right angle triangle. But the ratio will not change. This is something you can prove by similarity of triangles. So, this is how you get $\tan x$. I will also point out that the tangent function should have something to do with the tangent that we have seen in the previous video towards the end and indeed it does and that is something we will explore using calculus.

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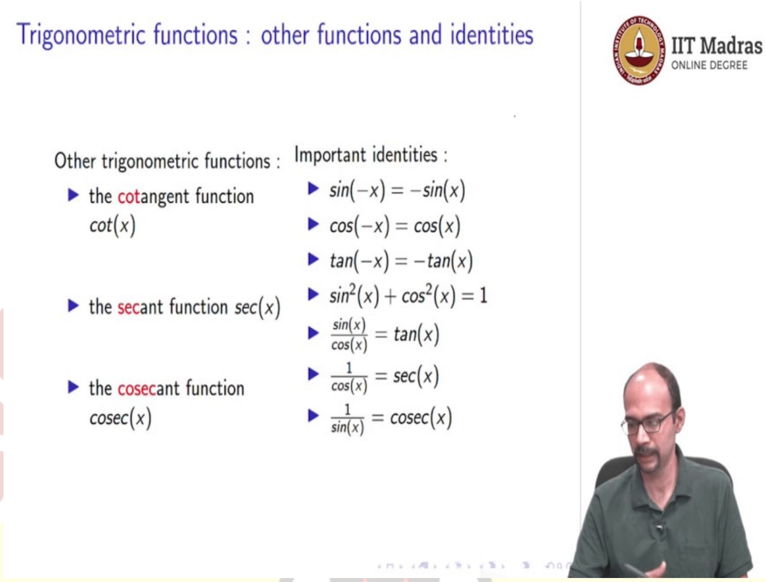

So, let us look at some other trigonometric functions and identities governing them. So, we also have the cotangent function $\cot x$, which is $\frac{1}{\tan x}$ and then the secant function, $\sec x$ and the cosecant function $\csc x$. Here are some identities on these functions shown in the image above. Very important identity, $\sin^2 x + \cos^2 x = 1$. This is a straightforward consequence of the Pythagoras theorem, so called Pythagoras theorem, which of course was known much before Pythagoras in many civilizations.

And then what is $\tan x$, that is $\frac{\sin x}{\cos x}$. Here, of course, there is a caveat. The denominator has to be non-zero. So, we do not talk about it when the denominator is 0 and when is the denominator 0 that is exactly those points where you have n $\pi/2$, where n is odd. That is exactly where cosine function takes 0 values. That is, so we have already ruled those out when we define the tangent. So, that we have to keep in mind when we write down this identity.

And then $\sec x = \frac{1}{\sin x}$. You can think of this as the definition. And $\csc x = \frac{1}{\sin x}$. Again, one can think of this as the definition. And of course, again, in these cases, we have to be careful about what is the domain, because not all of our will work. So, you have to throw out those points where cosecant or where cosine or sine functions are 0. So, these are some identities.

There are other identities. I am not writing down all of them. For example, $\sin\left(x+\frac{\pi}{2}\right)=\cos x$ or $\cos\left(x+\frac{\pi}{2}\right)=-\cos x$ and so on. So, I am not getting into that, but the picture should tell you what the identity is if you go back to the graph.

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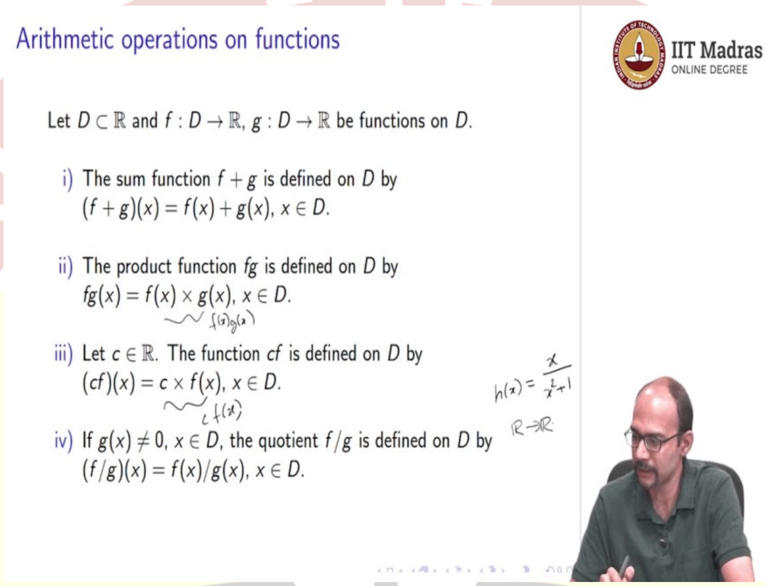

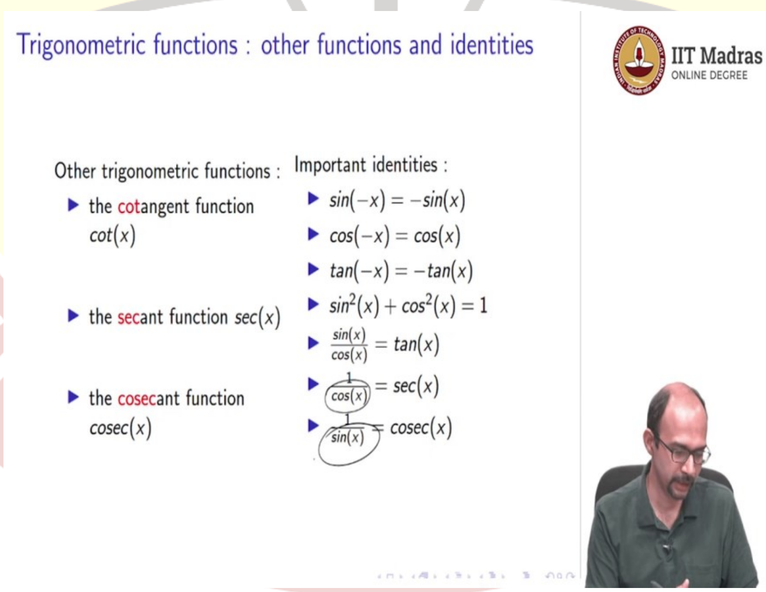

So, finally, so we have studied some nice functions, the trigonometric functions. Finally, let us study some general properties of functions of one variable. So, these are all, remember, functions of one variable. You have input x, which is a real number, may not be all real numbers, it may be in some smaller set. That is what we saw, for example, for the tangent function and the same thing happens by the way for secant and cosecant. They are not defined on all of R, because it will depend on where these become 0 and that we know from the graphs of cosine and sine.

So, these are all functions of one variable. So, here is now some general definitions. So, arithmetic operations on functions. So, we would like to add and subtract and divide and multiply functions, and indeed, we can do that without in problem. So, if D is a subset of R and we have two functions f and g, both of which have domain D. So, the sum function f+g is defined on D by (f+g)(x) = f(x) + g(x), where x ϵ D. So, for example, $(sin + \cos)x = \sin x + \cos x$.

Already, we have seen this idea of the sum when we constructed polynomials, because we know what monomials are and then polynomials are sums of monomials with coefficients with scalar multiplication. So, look at the product function. So, fg. So, this is where you take f(x) and you multiply it to g(x), so that gives you the product function fg. So, sometimes this is, maybe not even something, maybe often this is written as f(x)×g(x). So, be aware of that. This is often written like this.

Let c ϵ R, the function cf is defined on D by (cf)x =c×f(x). Finally, if g(x) is non-zero, we can divide by g. So, f/g is defined on x by looking at f(x)/g(x) and it makes sense because g(x) is not 0. So, the quotient is always defined when g(x) is not 0. So, sometimes what may happen is that f and g are both defined on the domain D, for example sine and cosine are defined on the entire real line. But there are values for which cosine is 0.

So, when you divide sine by cosine, you have to restrict your domain to those values where $\cos x$ is not 0. So, that is why the tangent function is defined on all values of R except those points which are odd multiples of $\pi/2$. That is an example. So, it is not defined on all of D. So, here also the same thing. That is an example of this fourth arithmetic operation. So, again, this is f(x) times g(x) and this is cf(x). So, I hope these are clear. So, in particular, the fourth one you can use this to create rational functions.

For example, you could take h(x)to be $\frac{x}{x^2+1}$, $x^2 + 1$ is never 0, because you have a square term plus 1. So, it is always bigger than equal to 1. So, this function makes sense. It is a function from R→ R and how did I get it, I got it by looking at f(x)= x, g(x)= $x^2 + 1$ nd then looking at f/ g.

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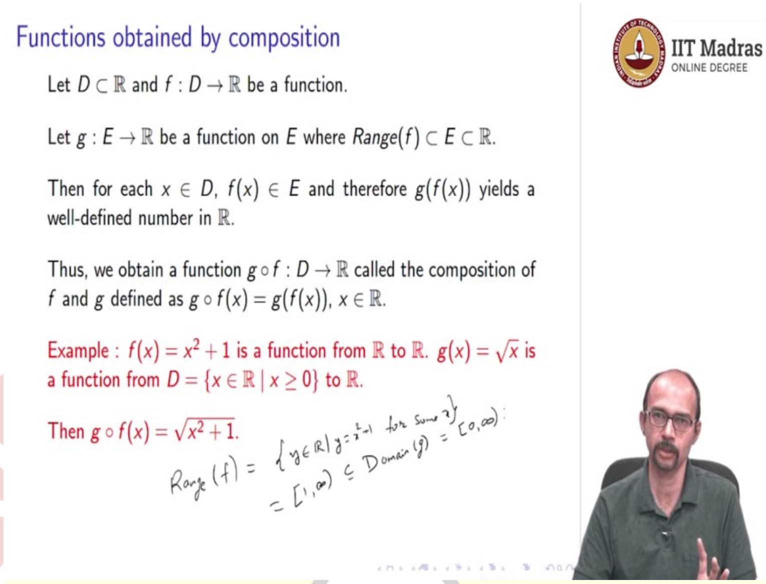

So, finally, let us look at functions obtained by composition. So, let D be a subset of R and f be a function on D. G is a function from E→ R, where the range of f ϵ E. This is very important that the range of f ϵ E. Then for each x ϵ D we have f(x) ϵ E and so we can talk about g(f(x)). So, this yields a well defined number in R and thus we obtain a function g composed f, D→ R is called the composition of f and g defined as g composed f(x) is g(f(x)).

So, here is an example. Let us take f(x) to be $x^2 + 1$. This is a function from R→R. So, let us take g(x)=$\sqrt{x}$. This is again a function from the positive side or non-negative side of the real line to R. So, again, note here that the domain of g is restricted. Then we can talk about g composed f and that is just the $\sqrt{x^2 + 1}$. So, why can we talk about g composed f? Well, let us look at what is the range of f.

So, range of f is all those values y ϵ R such that y is equal to $x^2 + 1$ for some x. But $x^2 + 1$, what values does it take. So, the smallest value it takes is 1 and then after that it takes all values.. So, it does not contain negative numbers and that is why taking square root makes sense. So, therefore, this belongs to the domain of g, which we know as (0, ∞). So, this, that is why we can do this idea of composition. So, we have seen in this video the trigonometric functions. I have not defined them, but I have told you how to think of them and how to compute using them.

We have seen some identities concerning trigonometric functions. So, these are all functions of one variable and then in the last two slides, we saw some general properties of, some general operations we can do on functions of one variable, namely we can add them, we can scalar multiply them, we can multiply them and if they have the same domain all this, and we can even divide one by the other provided the denominators are non-zero. And then if you are, if you have two functions with the range and the domain sort of match nicely, then you can compose them. That is what we did at the end. Thank you.