IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
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The null space of a matrix finding nullity and a basis - Part 2

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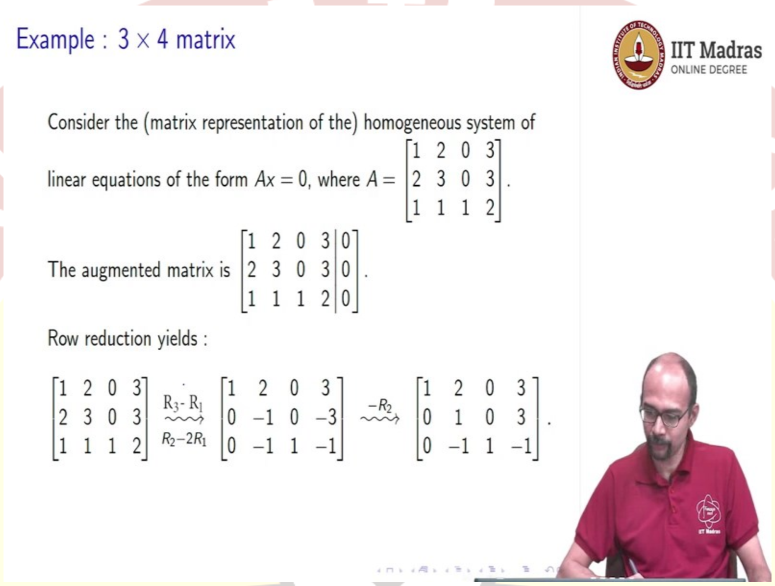

So, let us do a three by four example. So, here your matrix is $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ So, in this example what do we want to do? We want to get what is the nullity and a basis for the null space. That is what we want to do. So, the augmented matrix is this. So, let us see what row reduction yields.

So, row reduction is going to give me $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ I wrote that matrix. And as I commented in my previous slide, I am going to drop the augmented part.

So, if we do that, we have swept out our first column, everything below that one. Now, let us continue with the process. So, we get, we want a 1 in the $(2, 2)^{th}$ place if possible. And indeed, we can multiply the second row by -1 and obtain that. So, we get 0, 1, 0, 3 in the second row, everything else remains the same.

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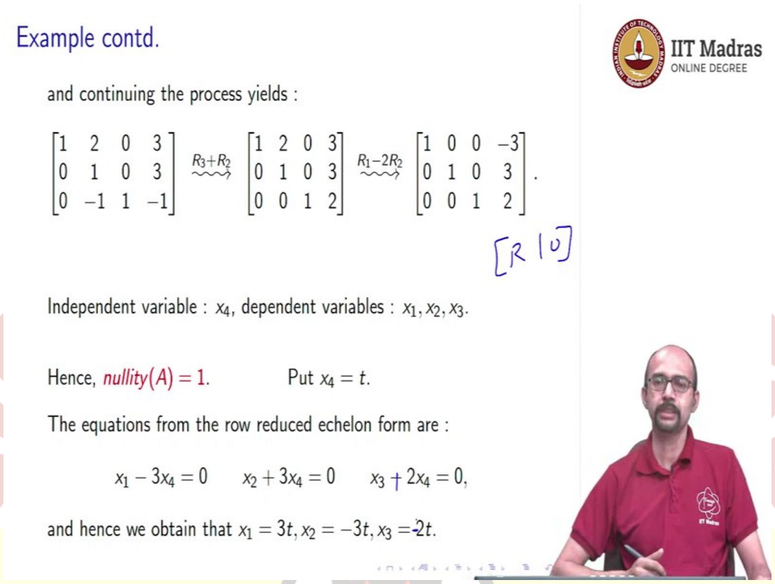

So, let us continue the process. This is the same matrix we had in the previous slide. So, now, let us sweep everything below the second, in the second column below the 1. So, if you do that, we get $\begin{bmatrix}1&2&0&3\\0&1&0&3\\0&0&1&2\end{bmatrix}$ And this is already been reduced, not reduced in the row echelon form. But we can go ahead one step and do reduced row echelon form. And if we do that, we get $\begin{bmatrix}1&0&0&-3\\0&1&0&3\\0&0&1&2\end{bmatrix}$

So, we have reached our matrix which is the reduced row echelon form. Now, let us ask what are the dependent and independent variables. The leading ones are in the first, second and third column that means x_1, x_2, x_3 are dependent variables. That means x_4 is an independent variable. So, the independent variable is x_4 dependent variables are x_1, x_2, x_3 So, the nullity of A is 1, because the nullity corresponds to the number of independent variables.

And what is the null space of A? So, for the null space, we put the independent variables to be the t_is. In this case, there is only one independent variable. So, instead of putting t_1 I have put, just called it t. So, put $x_4 = t$ So, the equations from the row reduced echelon form are

$x_1 - 3x_4 = 0$ So, here I am using the augmented matrix. So, I have this matrix $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ And I am using the augmented matrix.

So, if I call this R, I am using the augmented matrix [R |0]. And from there, I am getting these equations. So, I get $x_1 - 3x_4 = 0, x_2 + 3x_4 = 0$ and $x_3 + 2x_4 = 0.$ So, from here we get that x_1 must be 3t by putting x_4 as t, x_2 is -3t and x_3 is actually -2t. So, now we know what is the general form of a solution.

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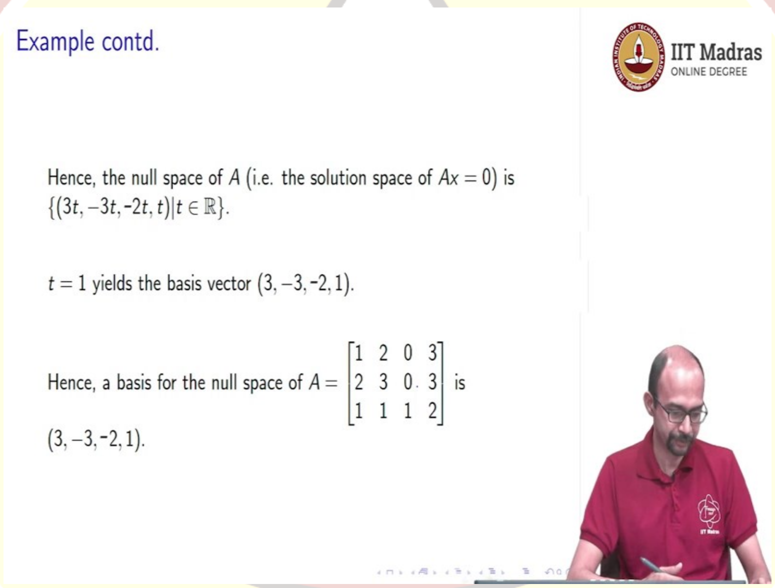

It is of the form, so $\{(3t, -3t, -2t, t) | t \in R\}$ So, some possible solutions are given by, let us say, if you put t = 5, you get (15,-15,-10,5) or if you put a t = 100, you get (300,-300,-200,100) or you could of course put t = 0 also. That is also a solution. That is the trivial solution, (0, 0, 0, 0).

So, actually, other than the last one the (0, 0, 0, 0) any of these we could have chosen as a basis vector. But just to follow a procedure, we put t = 1. So, t = 1 yields a basis vector (3, - 3, - 2, 1). So, a basis for the null space of this matrix is just a single basis vector (3, - 3, - 2, 1). So, you can check that this is actually inside your null space by multiplying this to A on the right of course as a column and seeing that the result is 0.

So, we have done these two examples of computing the nullity of a matrix and the null space of a matrix and its basis. So, now let us take a step back and ask what did we exactly do? So, we found a general solution for the system of linear equations Ax = 0. And then we put the t_i's to be 1 and everything, all other t_j's to be 0 and each such will give you a basis vector. This is the main point, the main idea.

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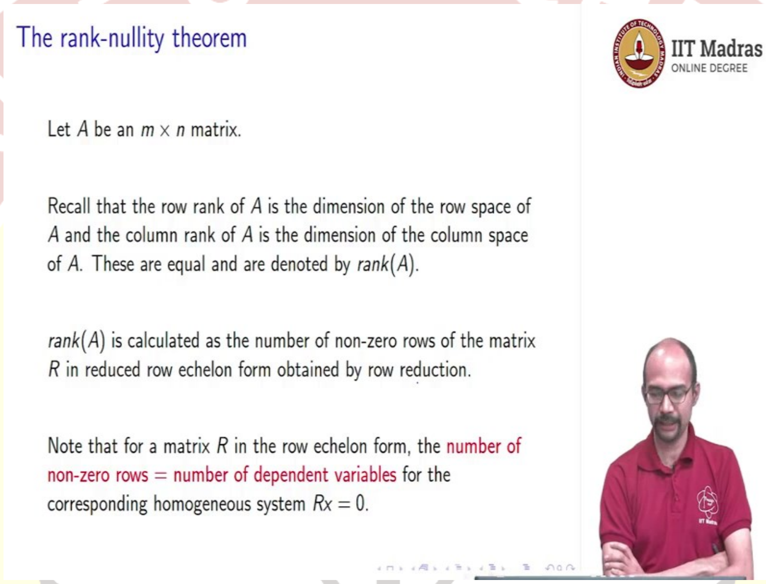

So, let us do some observations and these observations are going to yield a theorem, which is called the rank nullity theorem. So, what is the rank nullity theorem? So, let A be an m x n matrix. So, recall that the row rank of A is the dimension of the row space of A and the column rank of A is the dimension of the column space of A.

So, what do we mean by the row space of A? It is the vector space or the subspace of $\mathbb{R}^n$ if it is an m by n matrix of $\mathbb{R}^n$ which is spanned by the rows of A, which we treat as vectors. And what is the column space of A? It is a the subspace of $\mathbb{R}^m$ which is given as the span of the columns of A, meaning you treat each column as a vector in $\mathbb{R}^m$ and that span is going to give you the column space.

So, recall that these are equal. This is something I stated, but did not prove and they are denoted by rank of A. So, the rank of A is any one of these two, either the row rank or the column rank, because they are both the same. So, we used in a previous video, we used the fact that rank of A is the same as the row rank to compute this rank. How did we do that?

We said, you take this matrix A, you do row reduction, and once you do row reduction, you look at the non-zero rows of this matrix. And the number of non-zero rows is exactly the rank of A. So, recall this if you do not remember. So, now this is an important point. So, note that for a matrix R in the row echelon form, so suppose you have a matrix which is in row echelon form, the number of non-zero rows is the same as the number of dependent variables.

How do we get this? Each non-zero row has a leading one, because it is in row echelon form. And each leading one that corresponding column corresponds to a dependent variable. So, the non-zero rows correspond to the leading ones correspond to the columns with leading ones correspond to the dependent variables. That is how this correspondence is. So, the number of non-zero rows is the same as the number of dependent variables.

Of course, dependent variables were, because we there is no system of equations in this place. So, as I said, here you think of the, in your mind, you think of the system of linear, the homogeneous system of linear equations, $Rx = 0$. So, you have this matrix A, you row reduce it to R, think of the system of equations $Rx = 0$, think of how many dependent variables there are, that is exactly the same as the number of non-zero rows. That is what we are saying. And what is the number of non-zero rows?

That is the same as the rank of the matrix A. Why is that the case by the way, because remember that in the previous video we have seen that, if you have a spanning set for vector space, you put those in into your matrix as rows. But in this case, we have already done that. That matrix is by construction that we started with the rows are the vectors which span the row space. So, that is how we obtain the statement.

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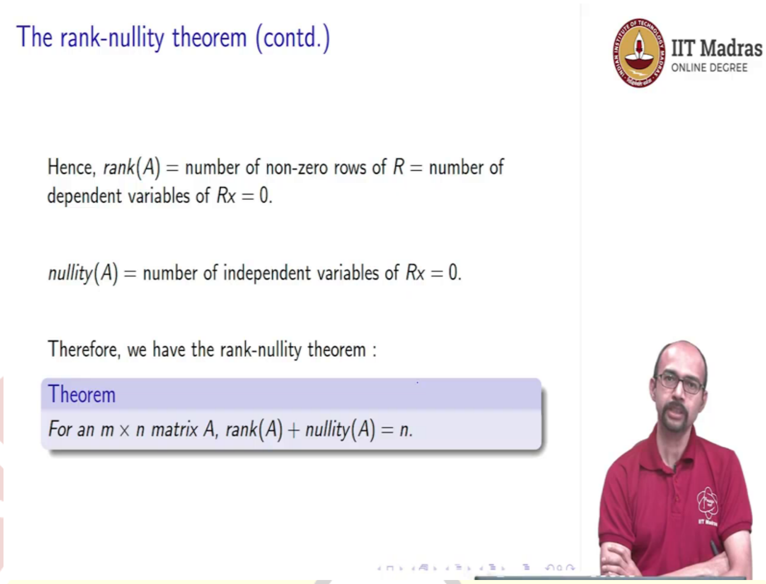

So, the moral of the story is, the rank of A is the number of non-zero rows of R, R was remember obtained as, from A by doing row reduction, which is the number of dependent variables of Rx=0. How is this related to nullity? Remember that the nullity of A is the number of independent variables of Rx=0. So, rank is the number of dependent variables, nullity is the number of independent variables.

But well, if you have a variable either it is a dependent variable or an independent variables, so the number of dependent variables plus the number of independent variables is the total number of variables which is exactly n, remember, it is an m by n matrix. So, this gives us the rank nullity theorem that the rank of A + the nullity of A = n.

Why, because the rank is a number of dependent variables and the nullity is a number of independent variables and we just saw that the sum is the total number of variables which is n. So, this is a heuristic proof of the, or I let me remove the word proof, this is a heuristic for this theorem. So, this is a useful and often quoted theorem in linear algebra.

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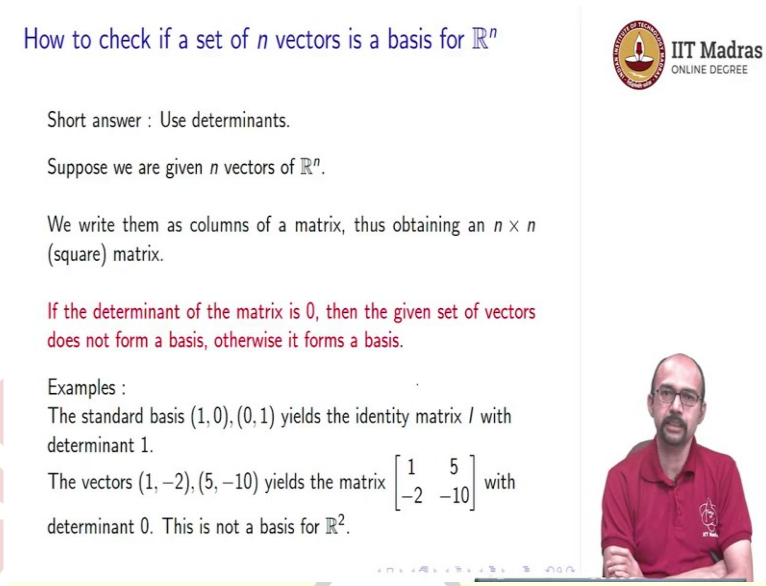

So, with that, let us do the final topic that we want to do in this video. How do we check if a set of n vectors is a basis for $\mathbb{R}^n$ So, remember that for $\mathbb{R}^n$ we have a standard basis. Namely, the e_is which is 1 in the i-th place, i-th coordinate and 0 is everywhere else. This is the standard basis for $\mathbb{R}^n$ And we know that the size of any basis is the same. So, the, that was what we had called dimension. So, we know that the dimension of Rn is n.

So, we know that if at all you have a basis it must consist of n vectors. So, now suppose we are given n vectors, how do I check whether or not this is a basis? So, here, we use the idea from a previous video about linear independence and dependence. Namely, that if you have a set of vectors and you want to check whether or not they are linearly independent, so if you have, for example, n vectors in $\mathbb{R}^n$ you look at the determinant and that is exactly what we are going to do now. So, that will tell you whether or not they are linearly independent.

But then the point is if you have, if they are independent, when you have n linearly independent vectors and they must form a basis, because if not, once there will be some vector which is not in the span of these vectors which is linearly independent so you add that, and you can keep doing this process as we did. And in the end you, after appending a bunch of vectors, you will get a basis. But then remember that the dimension will be strictly bigger than n.

So, the only way that this is possible is if your original set was itself a basis. So, what is the moral? The upshot is that if you have n vectors, it is enough to check that they are linearly independent. Other way of doing this is to say it is enough to check that there are a spanning set. In either of these two cases, there must be a basis.

So, how do I check that in this case? In this case, it so happens that since you have n and n, the corresponding matrix is a square matrix and I can look for its determinant. So, let us see how to check this. So, the short answer is use determinants. And I sketched out how. Suppose we are given n vectors of $\mathbb{R}^n$ we write them as columns of a matrix.

So, we will obtain an n by n square matrix. If the determinant of the matrix is 0, then the given set of vectors does not form a basis, otherwise it forms a basis. So, what it saying is that in a very large number of cases, almost always, which has a precise meaning, but we would not say what that is now, almost always, they do form a basis. So, what do you have to do to check? You have to compute its determinant.

So, let us do examples. The quick examples are in $\mathbb{R}^2$ you take the standard basis (1, 0) and (0, 1). We already know it is a basis. So, let us just check that the statement is correct. So, what is the corresponding matrix? It is the identity matrix. The determinant is 1. And indeed, this forms a basis. So, the determinant is non-zero.

And let us look at the vectors let us say (1, - 2) and (5, - 10). So, this yields a matrix 1, - 2 in the first column, 5, - 10 in the second column. And if you look at its determinant, it is 0. Of course, we know this, because they are linear multiples of each other. And as a result, this is not linearly independent and so it cannot form a basis.

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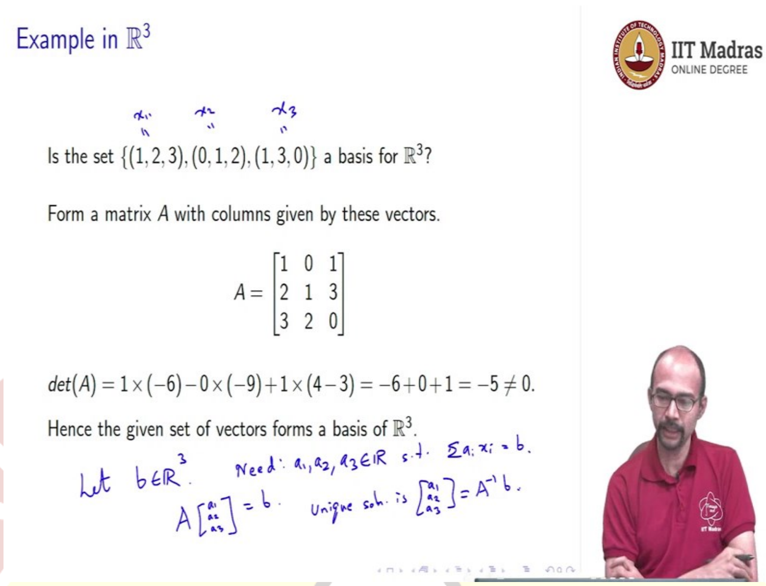

Let us do this for a slightly more larger example and maybe a more involved example. So, is the set {(1, 2, 3), (0, 1, 2), (1, 3, 0)} a basis for R^3 Let us form the matrix. So, form a matrix A with columns given by these vectors, so that is, the first column is (1, 2, 3) the second column is (0, 1, 2) the third column is (1, 3, 0). Let us find its determinant. Well, we know how to do this.

So, I am, I have just directly done this. So, it is 1 times minus 6, minus 0 times minus 9, which will anyway does not count, plus 1 times 4 minus 3 i.e., ($1\times-6-0\times-9+1\times(4-3)$). So, that is -6+0+1, which is - 5, and that is non-zero. So, this does give you a set of vectors which form a basis. Let me directly show you that this is indeed a basis by proving that this is a spanning set for $\mathbb{R}^3$ So, let B belong to $\mathbb{R}^3$

So, I want to get a bunch of scalars such that a_1 times, if I call these vectors x_1, x_2, x_3 so this is x_1 this is x_2 this is x_3 So, I want to get, so need $a_1, a_2, a_3 \in \mathbb{R}$ such that $\sum a_i x_i = b$ I want this.

So, in other words, I want to solve $A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = b$ Why is that, because the first column remember is 1, 2, 3, which corresponds to x_1 the second column is 0, 1, 2, which corresponds to x_2 and the third column is 1, 3, 0, which corresponds to x_3

So, if you write out this equation in terms of the columns, you will exactly have $a_1 x_1 + a_2 x_2 + a_3 x_3$ So, we want to solve this for a_1, a_2, a_3 But now we know A is invertible. So, we know a solution. In fact, there is a unique solution. So, unique solution is $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A^{-1} b$ And why do we, I mean, where am I getting A inverse from, well, because the determinant is non-zero. So, the inverse exists.

In fact, we know explicitly how to compute it. The point is that there is a solution. So, that means a_1, a_2, a_3 exists. So, scalars a_1, a_2, a_3 exists such that s $\sum a_i x_i = b$ for any b in $\mathbb{R}^3$ That means it is a spanning set for $\mathbb{R}^3$ Already, the fact that determinant is non-zero tells you it is linearly independent that means it is a basis. So, that is why it is enough to check for the determinant.

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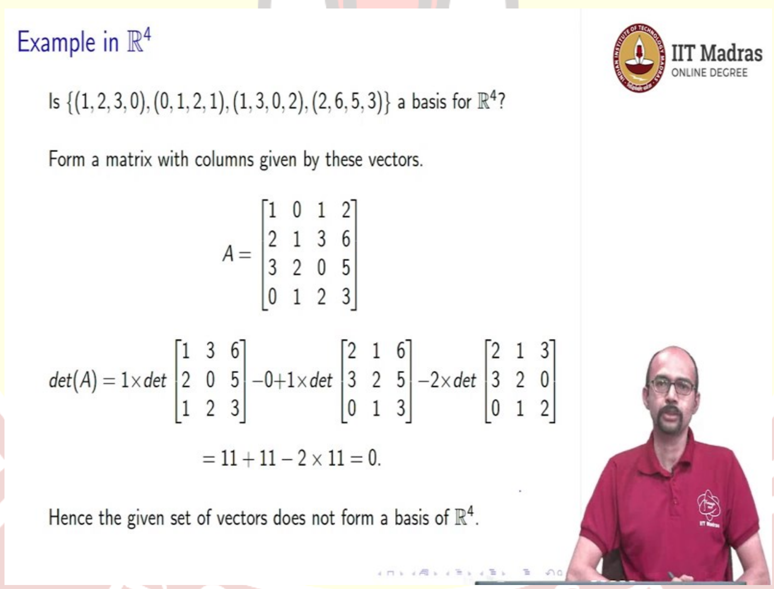

Let us do one last example in $\mathbb{R}^4$ So, let us have these four vectors {(1, 2, 3, 0), (0, 1, 2, 1), (1, 3, 0, 2),(2, 6, 5, 3)} is this a basis for $\mathbb{R}^4$ What do I do? I make the matrix with these columns. Put these vectors in the columns. So, in the first column, I have 1, 2, 3, 0. In the second column, I have 0, 1, 2, 1. In the third column, I have 1, 3, 0, 2. In the fourth column, I have 2, 6, 5, 3. Compute its determinant. We know how to compute determinants.

The, we will do this by, well, the easiest way would have been by doing Gaussian elimination or row reduction rather, but let me do it by hand. So, by hand, let us do it by expansion along the first row. So, I get 1 times the determinant of the (1, 1) minor, minus 0 times the (1, 2) minor, plus 1 times the (1, 3) minor, minus 2 times the (1, 4) minor. So, the 0 times part I have removed, because anyway I am going to get the, that will contribute nothing and the rest are written down here.

So, you can compute this. The first term is 11, second is 0, so I have removed it altogether, the third term is 11 and the fourth term is 2 times 11 and so this is 0. And as a result, the given set of vectors do not form a basis of $\mathbb{R}^4$

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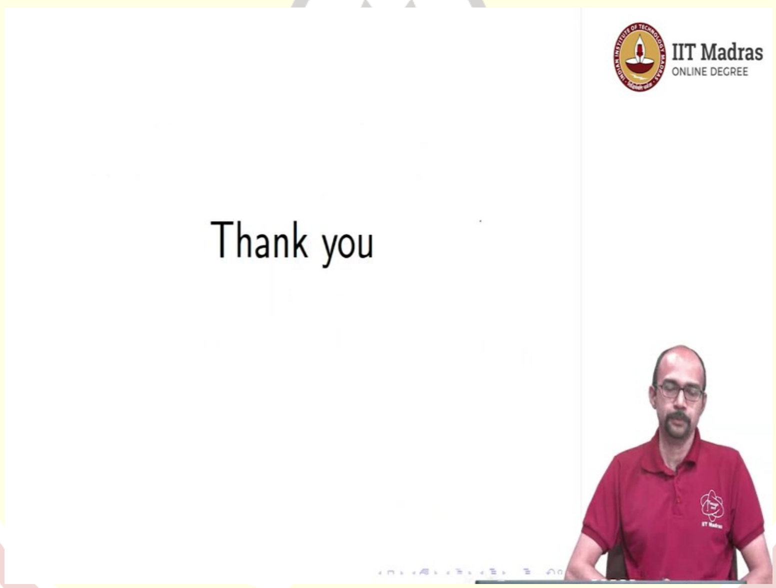

Thank you.