Statistics for Data Science - 2

Week 1 Graded Assignment

Basic Probability

1. The probability that an electrical machine will work more than 5 years but less than 8 years is 0.6 and the probability that it will work at least 8 years is 0.1. What is the probability that the machine will work for more than 5 years? [1 mark]

   Solution:
   Define events A and B as follows:
   $A =$ Event that electrical machine will work more than 5 years.
   $B =$ Event that electrical machine will work more than 8 years.
   From the given information,

$$P(A \setminus B) = 0.6$$
$$P(B) = 0.1$$

   Now,

$$\begin{aligned}
&A = (A \setminus B) \cup (A \cap B) \\
&\text{Note that } A \cap B = B \\
\Rightarrow &A = (A \setminus B) \cup B \\
\Rightarrow &P(A) = P((A \setminus B) \cup B) \\
\Rightarrow &P(A) = P(A \setminus B) + P(B) \qquad \text{(Since, } A \setminus B \text{ and } B \text{ are disjoints events.)} \\
\Rightarrow &P(A) = 0.6 + 0.1 = 0.7
\end{aligned}$$

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2. Five cards are drawn from a well-shuffled pack of playing cards with replacement. Find the probability that there will be at least two aces. [1 mark]

   (a) $\left(\dfrac{1}{13}\right)^5$

   (b) $\left(\dfrac{12}{13}\right)^5$

   (c) $1 - \left(\dfrac{12}{13}\right)^5 - 5\left(\dfrac{12^4}{13^5}\right)$

   (d) $1 - \left(\dfrac{1}{13}\right)^5 - 5\left(\dfrac{12}{13^5}\right)$

   Solution:
   Since, cards are drawn with replacement, probability of drawing ace in every draw will

be same and equal to $\dfrac{4}{52} = \dfrac{1}{13}$

$$
\begin{aligned}
P(\text{There will be at least two aces}) &= 1 - P(\text{There will be no ace}) - P(\text{There will be one ace}) \\
&= 1 - \left({}^5C_0 \left(\frac{1}{13}\right)^0 \left(\frac{12}{13}\right)^5\right) - \left({}^5C_1 \left(\frac{1}{13}\right)^1 \left(\frac{12}{13}\right)^4\right) \\
&= 1 - \left(\frac{12}{13}\right)^5 - 5\left(\frac{12^4}{13^5}\right)
\end{aligned}
$$

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3. Choose the correct statements for any two non empty events A and B. [2 mark]

   (a) $P(A \setminus B) = P(A) - P(B)$
   (b) $P(A \setminus B) = P(A) - P(A \cap B)$
   (c) $P(A \setminus B) = P(A \cup B) - P(B)$
   (d) If $B \subset A$, then $P(A \setminus B) = P(A) - P(B)$
   (e) If A and B are disjoint events, then $P(A \setminus B) = P(A)$

Solution:
We know that $A \cap B \subseteq A$,
Then by using subset property, we have

$$
\begin{aligned}
&P(A) = P(A \cap B) + P(A \setminus (A \cap B)) \\
\Rightarrow &P(A) = P(A \cap B) + P(A \setminus B) && (\text{Since,} A \setminus (A \cap B) = A \setminus B) \\
\Rightarrow &P(A \setminus B) = P(A) - P(A \cap B) && ....(1)
\end{aligned}
$$

Therefore, option (a) is not necessarily true while option (b) is correct.

From equation (1),

$$
\begin{aligned}
P(A \setminus B) &= P(A) - P(A \cap B) \\
&= P(A) - [P(A) + P(B) - P(A \cup B)] && (\text{By addition rule}) \\
&= P(A \cup B) - P(B)
\end{aligned}
$$

Therefore, option (c) is correct.

$B \subset A \Rightarrow A \cap B = B$. ...(2)
From equation (1) and (2), we have
If $B \subset A$, then $P(A \setminus B) = P(A) - P(B)$
Therefore, option (d) is correct.

If A and B are disjoint events, then $P(A \cap B) = 0$ .. (3)
From equation (1) and (3), we have
If A and B are disjoint events, then $P(A \setminus B) = P(A)$
Therefore, option (e) is correct.

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4. Let A, B, and C be three events of a random experiment such that $A \cup B \cup C = S$, where S is the sample space. The probability that at least one of the events A or B will occur is $\frac{1}{2}$. What is the value of $P(C \setminus (A \cup B))$? [2 mark]
Solution:
Given that
A, B, and C are the three events of a random experiment such that

$$A \cup B \cup C = S \qquad ...(1)$$

And

$$P(A \cup B) = \frac{1}{2} \qquad ...(2)$$

Now, we know that (Proved in the previous question): $P(A \setminus B) = P(A \cup B) - P(B)$ for any two events A and B. Using this, we have

$$\begin{aligned} P(C \setminus (A \cup B)) &= P(A \cup B \cup C) - P(A \cup B) \\ &= P(S) - P(A \cup B) \\ &= 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

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5. Two friends Ravi and Sonali are playing a game in which they are hitting a target in rounds. In each round, both hit the target independent of each other with a probability of 0.5. The first one who hits the target three times wins the game. What is the probability that in the fifth round Sonali wins the game? [2 marks]

   1. $6 \times (0.5)^5$
   2. $30 \times (0.5)^{10}$
   3. $96 \times (0.5)^5$
   4. $96 \times (0.5)^{10}$

   Solution:
   Define Events A and B as follows:
   $A =$ Ravi hits the target.
   $B =$ Sonali hits the target.
   Given that

$$P(A) = P(B) = 0.5 \qquad ...(1)$$

Sonali will win in the fifth round if Sonali hits her target third time in the fifth round and Ravi hits target 0 or 1 or 2 times out of five rounds.
Probability that Sonali will hit the target third time in her fifth round $= {}^4C_2(0.5)^2(0.5)^3$

$$= 6 \times (0.5)^5$$

Prbability that Ravi hits target 0 or 1 or 2 times out of five rounds $= ({}^5C_0 + {}^5C_1 + {}^5C_2)(0.5)^5$

$$= 16 \times (0.5)^5$$

Therefore, Probability that Sonali wins in the fifth round $= 6 \times (0.5)^5 \times 16 \times (0.5)^5$

$$= 96 \times (0.5)^{10}$$

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6. A family has three children each of which is equally likely to be a boy or a girl independently to each other. Let A be the event that at most one child is a boy. B be the event that the family has at least one girl and one boy. C be the event that all three children are of same-sex. Choose the correct options. [2 mark]

   (a) A and B are independent events.
   (b) A and C are independent events.
   (c) B and C are independent events.
   (d) B and C are disjoint events.

   Solution:
   Since, a family has three children each of which is equally likely to be a boy or a girl independently to each other, sample space of gender of all three children (in the order of elder to younger) will be

$$S = \{bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg\}$$

   Where b and g stand for boy and girl, respectively.
   Events A, B, and C are defined as:

$$A = \text{At most one child is a boy}$$
$$A = \{bgg, gbg, ggb, ggg\}$$

   Since each outcome in S is equally likely, we have

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } S}$$
$$P(A) = \frac{4}{8} = \frac{1}{2} \qquad ...(1)$$

$$B = \text{Family has at least one girl and one boy}$$
$$B = \{bbg, bgb, gbb, bgg, gbg, ggb\}$$

Since each outcome in S is equally likely, we have

$$P(B) = \frac{\text{Number of outcomes in } B}{\text{Number of outcomes in } S}$$
$$P(B) = \frac{6}{8} = \frac{3}{4} \qquad ...(2)$$

$$C = \text{All three children are of same-sex}$$
$$C = \{bbb, ggg\}$$

Since each outcome in S is equally likely, we have

$$P(C) = \frac{\text{Number of outcomes in } C}{\text{Number of outcomes in } S}$$
$$P(C) = \frac{2}{8} = \frac{1}{4} \qquad ...(3)$$

Now,

$$A \cap B = \text{Event that family has at least one boy and one girl and at most one child is a boy}$$
$$\Rightarrow A \cap B = \text{Event that family has one boy and two girls}$$
$$A \cap B = \{bgg, gbg, ggb\}$$

Since each outcome in S is equally likely, we have

$$P(A \cap B) = \frac{\text{Number of outcomes in } A \cap B}{\text{Number of outcomes in } S}$$
$$P(A \cap B) = \frac{3}{8} \qquad ...(4)$$

$$A \cap C = \text{Event that at most one child is a boy and all three children are of same sex.}$$
$$\Rightarrow A \cap C = \text{Event that all three children are girls}$$
$$A \cap C = \{ggg\}$$

Since each outcome in S is equally likely, we have

$$P(A \cap C) = \frac{\text{Number of outcomes in } A \cap C}{\text{Number of outcomes in } S}$$
$$P(A \cap C) = \frac{1}{8} \qquad ...(5)$$

$B \cap C =$ Family has at least one boy and one girl and all three children are of same sex.
$\Rightarrow B \cap C =$ Empty event

$P(B \cap C) = 0$ ...(6)
$\Rightarrow B$ and C are disjoint events.

Option (c) is wrong and (d) is right.

From equation (1) and (4), we have

$$P(A \cap B) = \frac{3}{8} = \frac{1}{2}.\frac{3}{4} = P(A).P(B)$$
$$\Rightarrow A \text{ and } B \text{ are independent events}$$

Option (a) is right.

From equation (1) and (5), we have

$$P(A \cap C) = \frac{1}{8} = \frac{1}{2}.\frac{1}{4} = P(A).P(C)$$
$$\Rightarrow A \text{ and } C \text{ are independent events}$$

Option (b) is right.

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7. In a town, 60% of the residents are eligible for voting in an election but only 80 % of the eligible residents voted in the election. A person is randomly selected from the town. What is the conditional probability that the person is eligible for the voting given that he or she did not vote? [2 mark]

    1. $\frac{2}{13}$
    2. $\frac{3}{13}$
    3. $\frac{4}{13}$
    4. $\frac{6}{13}$

    Define events A and B as follows:
    $A =$ randomly selected person is eligible for voting.

$B =$ randomly selected person has voted.
Given that

$$
\begin{aligned}
P(A) &= 0.6 \\
P(B|A) &= 0.8 \\
\Rightarrow P(B^C|A) &= 0.2 \\
\text{Note that} \\
P(B^C|A^C) &= 1
\end{aligned}
$$

To find: $P(A|B^C)$

$$
\begin{aligned}
p(A|B^C) &= \frac{P(B^C|A).P(A)}{P(B^C|A).P(A) + P(B^C|A^C).P(A^C)} \\
&= \frac{(0.2)(0.6)}{(0.2)(0.6) + (1)(0.4)} \\
&= \frac{0.12}{0.52} = \frac{3}{13}
\end{aligned}
$$

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8. Urn A contains 3 red and 2 blue marbles while urn B contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up head, a marble is chosen from urn A. If it turns up tail, a marble is chosen from urn B. Suppose Shreya who tosses the coin gets a red color marble. What is the conditional probability that the marble is drawn from the urn A? (Answer the question correctly up to two decimal points.) [2 marks]

Solution:
Define the events as follows:
$H =$ Coin turns up head.
$T =$ Coin turns up tail.
$R =$ Red marble is drawn.
$B =$ Blue marble is drawn.

From the given information, we have

$$
\begin{aligned}
P(R|H) &= \frac{3}{5} \\
P(B|H) &= \frac{2}{5} \\
P(R|T) &= \frac{2}{10} = \frac{1}{5} \\
P(B|T) &= \frac{8}{10} = \frac{4}{5}
\end{aligned}
$$

Marble is drawn from urn A if the coin turns up head.

$$
\begin{aligned}
P(H|R) &= \frac{P(R|H).P(H)}{P(R|H).P(H)+P(R|T).P(T)} \\
&= \frac{\frac{3}{5}.\frac{1}{2}}{\frac{3}{5}.\frac{1}{2}+\frac{1}{5}.\frac{1}{2}} \\
&= \frac{3}{4}
\end{aligned}
$$

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9. Three different tasks were assigned to three persons A, B, and C. Previous records show that A, B, and C will complete their tasks independent of each other with probabilities of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, respectively. If it is known that exactly two of them have completed their tasks, then what is the conditional probability that A has not completed his task? [3 marks]

   (a) $\frac{3}{4}$
   (b) $\frac{3}{11}$
   (c) $\frac{6}{11}$
   (d) $\frac{9}{11}$

   Define events A, B, and C as follows:
   $A =$ A has completed his task.
   $B =$ B has completed his task.
   $C =$ C has completed his task.
   Given that

$$
\begin{aligned}
P(A) &= \frac{1}{2} \\
P(B) &= \frac{2}{3} \\
P(C) &= \frac{3}{4}
\end{aligned}
$$

   Let D be the event that exactly two of them have completed their tasks, then

$$
\begin{aligned}
D &= (A \cap B \cap C^C) \cup (A \cap B^C \cap C) \cup (A^C \cap B \cap C) \\
P(D) &= P((A \cap B \cap C^C) \cup (A \cap B^C \cap C) \cup (A^C \cap B \cap C)) \\
&\quad \text{Since, } A \cap B \cap C^C,\ A \cap B^C \cap C, \text{ and } A^C \cap B \cap C \text{ are disjoint events, we have} \\
P(D) &= P(A \cap B \cap C^C) + P(A \cap B^C \cap C) + P(A^C \cap B \cap C) \\
&\quad \text{Since, } A,\ B, \text{ and } C \text{ are independent events, we have} \\
P(D) &= P(A)P(B)P(C^C) + P(A)P(B^C)P(C) + P(A^C)P(B)P(C)
\end{aligned}
$$

Now,

$$P(A^C|D) = \frac{P(A^C \cap D)}{P(D)}$$

$$= \frac{P(A^C \cap B \cap C)}{P(A)P(B)P(C^C) + P(A)P(B^C)P(C) + P(A^C)P(B)P(C)}$$

$$= \frac{P(A^C)P(B)P(C)}{P(A)P(B)P(C^C) + P(A)P(B^C)P(C) + P(A^C)P(B)P(C)}$$

$$= \frac{\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}}{\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}}$$

$$= \frac{6}{11}$$

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10. There are twenty boxes out of which exactly fifteen contains gifts and five are empty. Five boxes are removed randomly. Now, a person selects one box from the remaining boxes, then what is the probability that the person selects the empty box? [3 marks] (Hint: Consider all the cases of removing empty boxes and apply the law of total probability)

    (a) $\frac{1}{4}$
    (b) $\frac{2}{3}$
    (c) $\frac{3}{4}$
    (d) $\frac{1}{3}$

    Solution:
    Define the events A, B, C, D, E,and F as follows:
    A = Removed boxes contain no empty box.
    B = Removed boxes contain one empty box.
    C = Removed boxes contain two empty boxes.
    D = Removed boxes contain three empty boxes.
    E = Removed boxes contain four empty boxes.
    F = Removed boxes contain five empty boxes.
    Let X be the event that person selects the empty box.

$$
\begin{aligned}
P(X) &= P(A).P(X|A) + P(B).P(X|B) + P(C).P(X|C) + P(D).P(X|D) + P(E).P(X|E) \\
&\qquad + P(F).P(X|F) \\
&= \frac{{}^{15}C_5 {}^5C_0}{{}^{20}C_5}\frac{5}{15} + \frac{{}^{15}C_4 {}^5C_1}{{}^{20}C_5}\frac{4}{15} + \frac{{}^{15}C_3 {}^5C_2}{{}^{20}C_5}\frac{3}{15} + \frac{{}^{15}C_2 {}^5C_3}{{}^{20}C_5}\frac{2}{15} + \frac{{}^{15}C_1 {}^5C_4}{{}^{20}C_5}\frac{1}{15} + \frac{{}^{15}C_0 {}^5C_5}{{}^{20}C_5}\frac{0}{15} \\
&= \frac{1}{15504 \times 15}\left(15015 + 27300 + 13650 + 2100 + 75\right) \\
&= \frac{1}{4}
\end{aligned}
$$