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ONLINE DEGREE

Mathematics for Data Science- 2

Professor Sarang Sane
Department of Mathematics
Indian Institute of Madras

Tangents for scalar-valued multivariable functions

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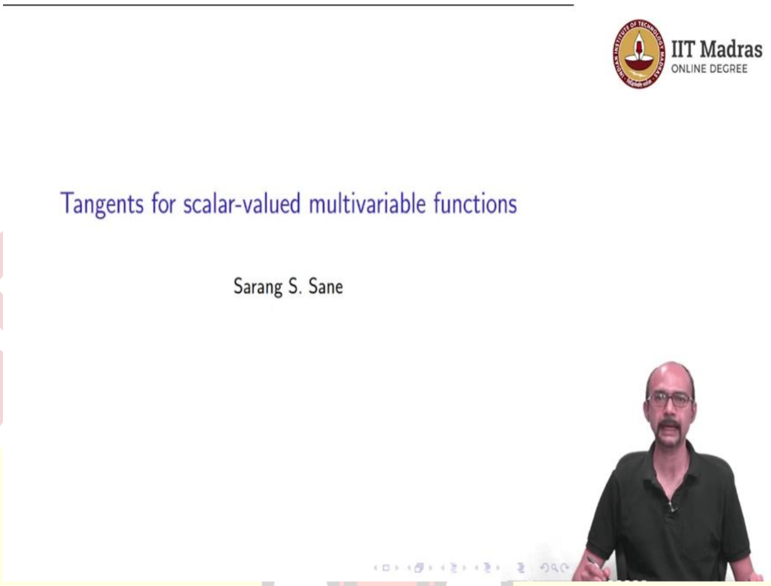

Hello, and welcome to the Maths 2 component of the Online BSc program on Data Science and Programming. In this video, we are going to talk about tangents for scalar-valued multivariable functions. So, we have seen the notion of tangents for functions of one variable. This is a topic that we studied extensively when we read one variable calculus, and we used it to study the ideas of the tangent line, and the idea of differentiability and the derivative. So, let us recall first what are tangent lines to curves.

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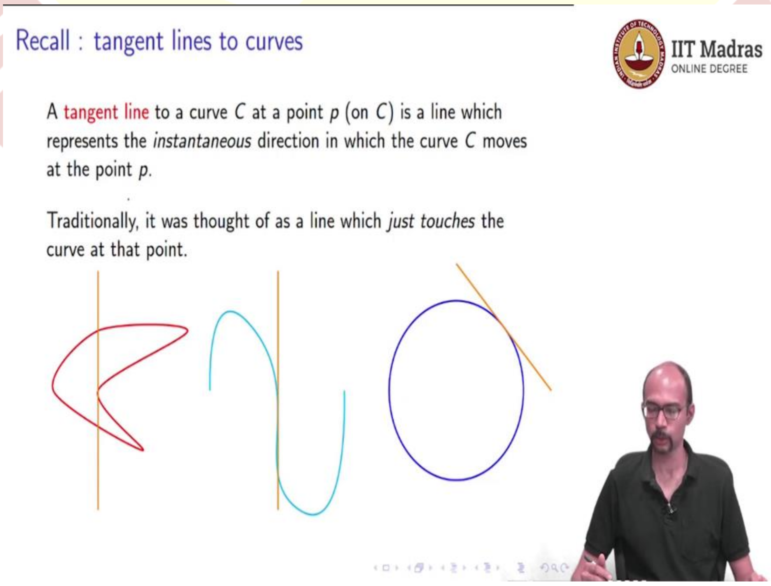

So, a tangent line to a curve C at a point P on the curve C is a line which represents the instantaneous direction in which the curve C moves at the point P. So, later on, we summarize this information in terms of the derivative. And traditionally, we have thought of, we think of tangents as lines, which just touch the curve at that point.

As we now know, the idea of just touches is a little vague, which is why you have to use more involved definitions like the derivative. And as examples, here are curves that we had actually seen in our study of one variable calculus, along with the tangents to those curves. So, this is just a recollection of what we had done in one variable calculus. So, from here, how did we move to functions?

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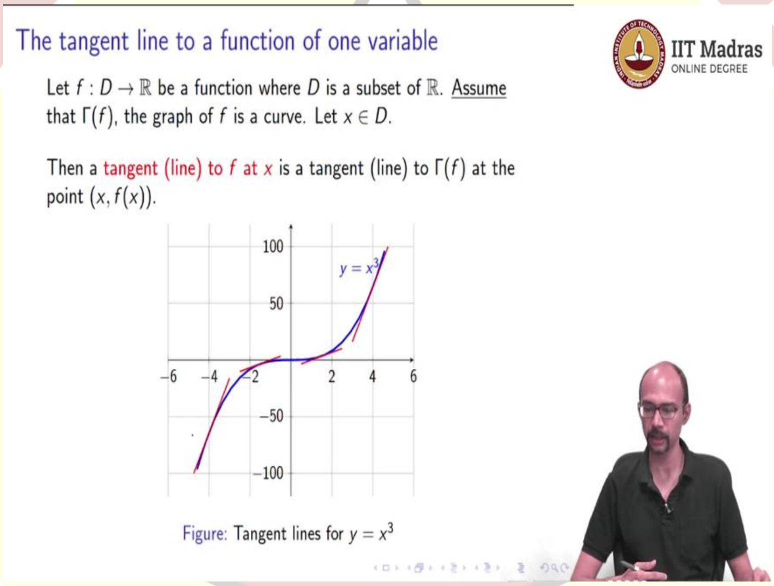

So, we looked at the graph of the function. So, suppose we have $f : D \to R$, which is a function, where D is, of course, now a subset of R. So, if we assume that the graph is a curve and we take a point in this domain D then the tangent line to f at that point, x is the tangent to the graph of that function f at the point $(x, f(x))$, this was our definition.

And just as an example, here is the function $f(x) = x^3$ and you can see these tangent lines. So, if your function is nice, by which we mean it is differentiable then you can compute the explicit equations for these tangent lines by looking at $y - f(a) = f'(a)\,(x - a)$, this is the equation of the tangent line at the point a.

And we saw that this notion of just touches, can have interesting implications. For example, for $y = x^3$, if you will draw the tangent at 0 then it is actually the $x -$ axis and that actually cuts the curve of the, that actually cuts the graph of the function $f(x) = x^3$. So just touches can have various implications for what it means.

It can be like any of these lines, the red lines which are drawn or it can actually be a line, which cuts the curve, and those are kind of special or interesting points, as we later saw in our study of maxima and minimum. So, this is just a collection of tangents for functions of one variable.

Now, we want to study tangents for scalar-valued multivariable functions. So, what do we mean by tangents? What do, what does the collection of tangents look like and so on? So, we already have some idea about this, when we studied the notion of partial derivatives and directional derivatives because what we did there was we restricted ourselves to a particular line, then we said, let us look at how the function looks on top of that line. And over there, we said, well, now we, over here we ask what is the rate of change of the function when we restricted to that line, which made sense because we had restricted to a function of one variable. We will use the same idea here.

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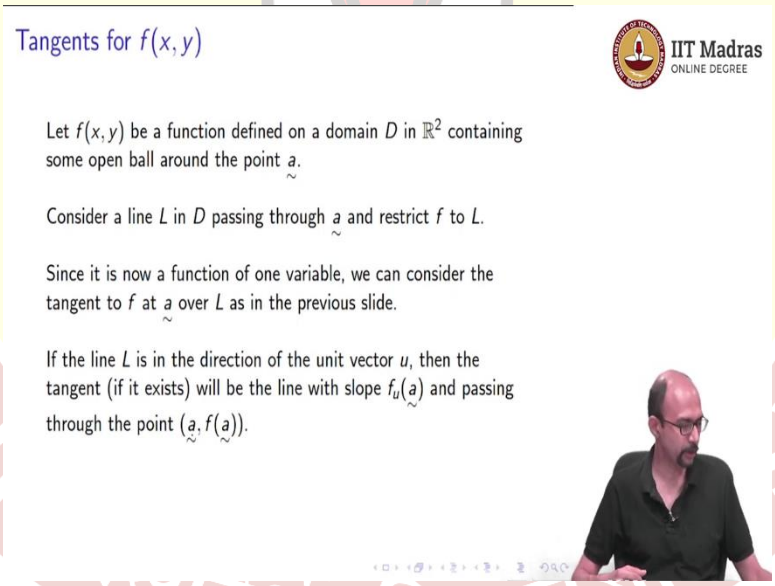

So, suppose you have for now let us start with two variables. So, suppose you have a function of two variables, which is defined on a domain D in $\mathbb{R}^2$ and contains some open ball around the point (a, b), which sometimes are represented by a tilde. So, consider a line L in D passing through a tilde and restrict f to L. This is exactly what we did in when we defined the notion of the directional derivative.

And since it is now a function of one variable, we can consider the tangent to f at a tiled over L, as in the previous slide. By previous slide, I mean, since it is a function of one variable we know exactly what we mean by tangent. It is the line which represents the instantaneous direction of the graph. And that is the line that we consider.

So, if the line L is in the direction of the unit vector u, then the tangent if it exists and we know examples from one variable calculus where it does not exist, will be the line with slope $f_u\ (a)$ tilde and passing through the point $(a\ tilde,\ f(a\ tilde))$. So, let us understand this statement better. So, we know from one variable calculus, as we just observed that the equation of the tangent is given by the derivative at that point, so once we know the derivative at that point in one variable calculus that is, then we can compute what is the equation of the tangent line.

So, now what we have done here is we have restricted ourselves to the line L. So, we have a one variable function and then using that one variable function, what is the derivative, that is exactly the directional derivative, where how do you compute the directional derivative, well you take this line L, and you look at unit vector, which is pointing in the same direction as that line L.

Of course, this line L may not pass to the origin. So, what you have to do is you have to translate this line L parallelly back to the origin, and then on that line L, you have to take a unit vector. Once you take this unit vector, you have to compute the directional derivative. And once we compute that directional derivative, you can use that to compute the equations of the point, so it will be a line.

So, if this is the line L. And earlier when we had the one variable situation, this was your $x-$ axis, this was your $y-$axis, and so you had this line below, and the derivative was something like this, so it was with respect to the $x-$ axis, when we took the derivative, it told you the slope, meaning the angle with respect to the $x-$ axis.

So now your line is some arbitrary line in our $\mathbb{R}^2$, so what you do is above this line you take a line with the slope prescribed by $f_u(atilde)$. And so, this is a line with that slope, passing through the point $(a\ tilde, f(a\ tilde))$ so this uniquely determines the line. And let us see a couple of examples of this in GeoGebra.

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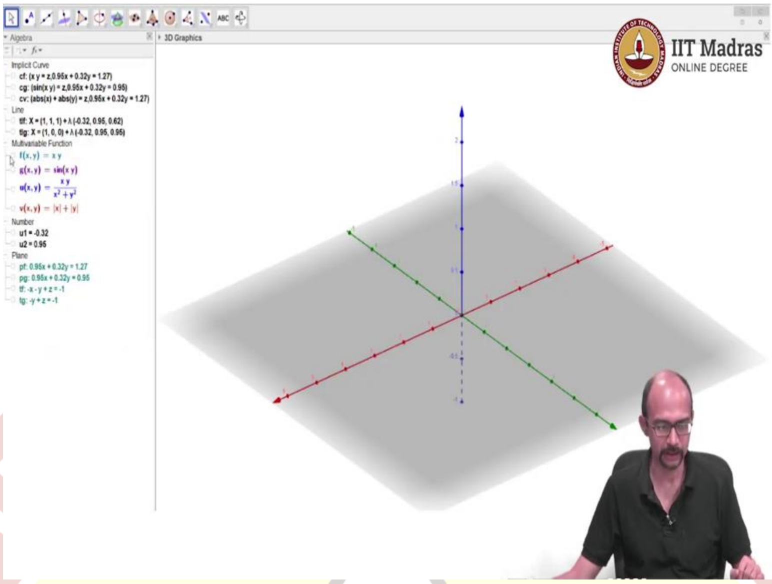

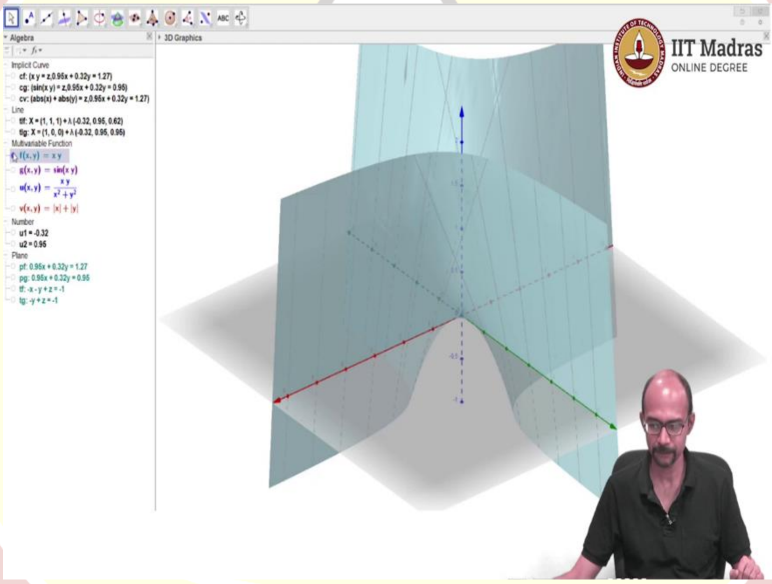

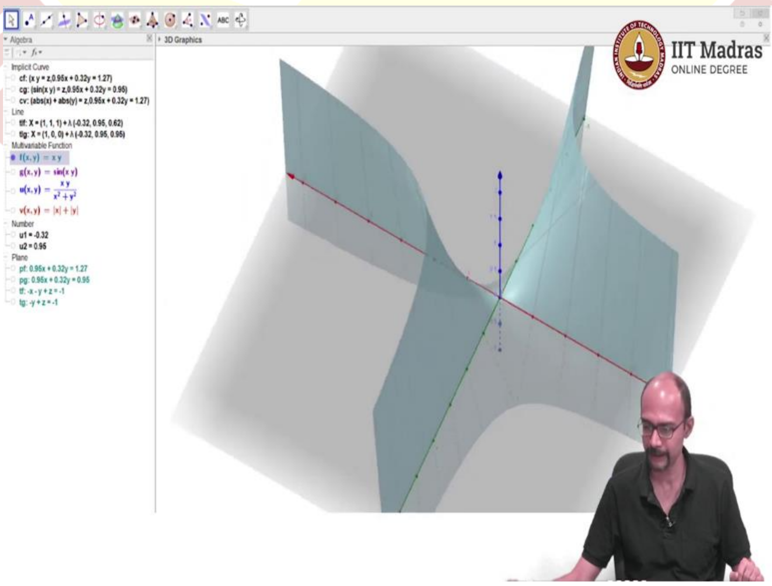

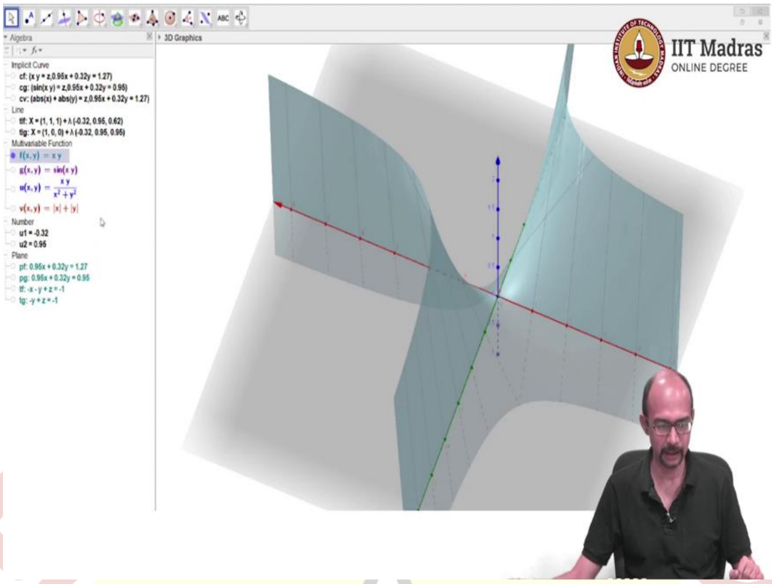

So, let us look at the function, $f(x,y) = xy$. So, here is the function $f(x,y) = xy$, so rather interesting looking function. Let us look at the point, let us say (1, 1). So, if you look at the point (1, 1). Let us take a plane passing through that point, according to this unit vector. So, I have some unit vector, which I am going to change now as this changes.

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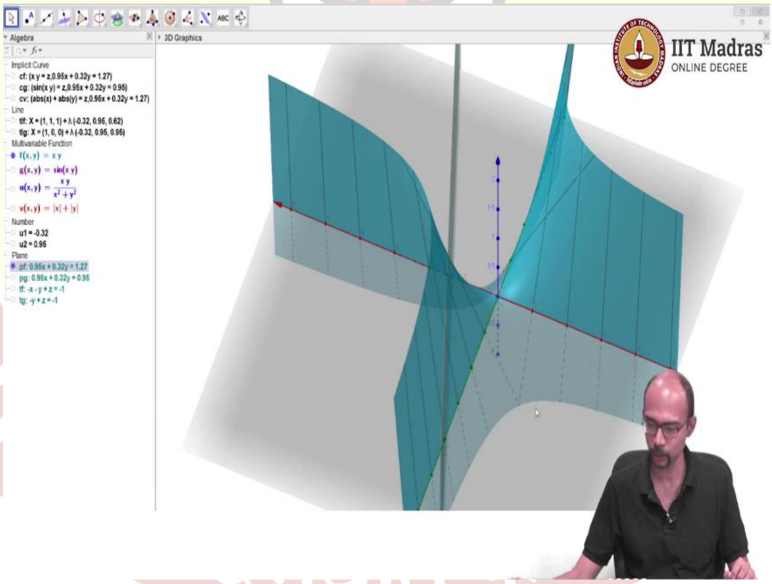

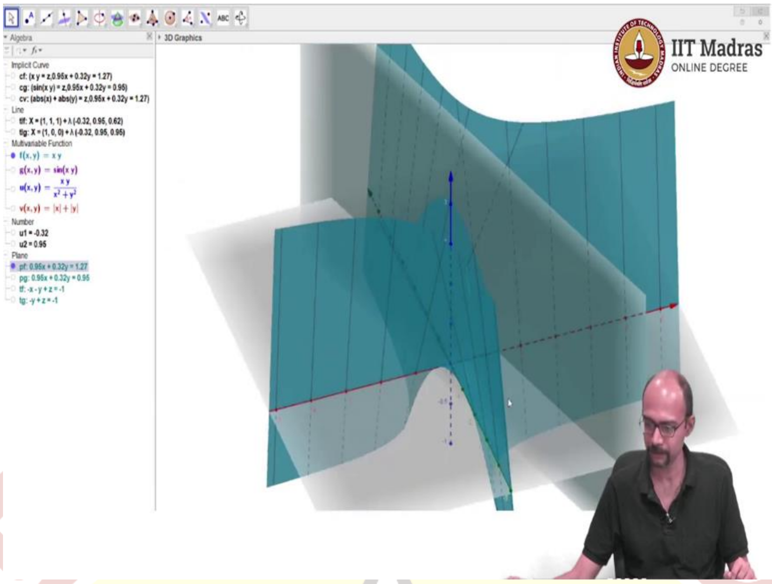
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3D Graphics
Implicit Curve
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cg: (sin(x y) = z,0.95x + 0.32y = 0.95)
cv: (abs(x) + abs(y) = z,0.95x + 0.32y = 1.27)
Line
tlf: X = (1, 1, 1) + λ(-0.32, 0.95, 0.62)
tlg: X = (1, 0, 0) + λ(-0.32, 0.95, 0.95)
Multivariable Function
f(x, y) = x y
g(x, y) = sin(x y)
u(x, y) = x y / (x² + y²)
v(x, y) = |x| + |y|
Number
u1 = -0.32
u2 = 0.95
Plane
pf: 0.95x + 0.32y = 1.27
pg: 0.95x + 0.32y = 0.95
tf: -x - y + z = -1
tg: -y + z = -1
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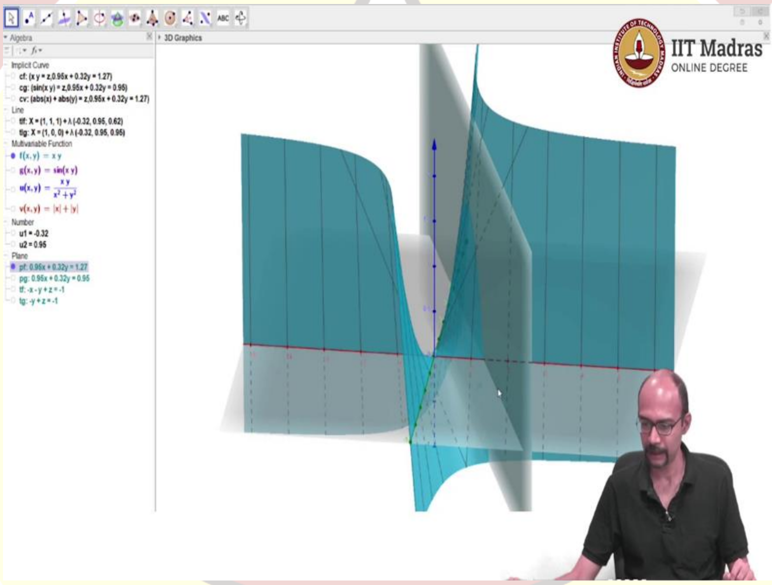
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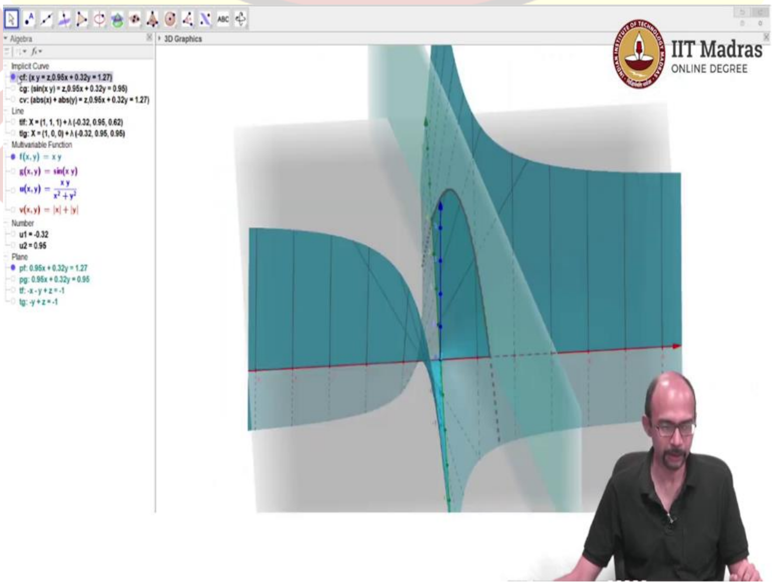
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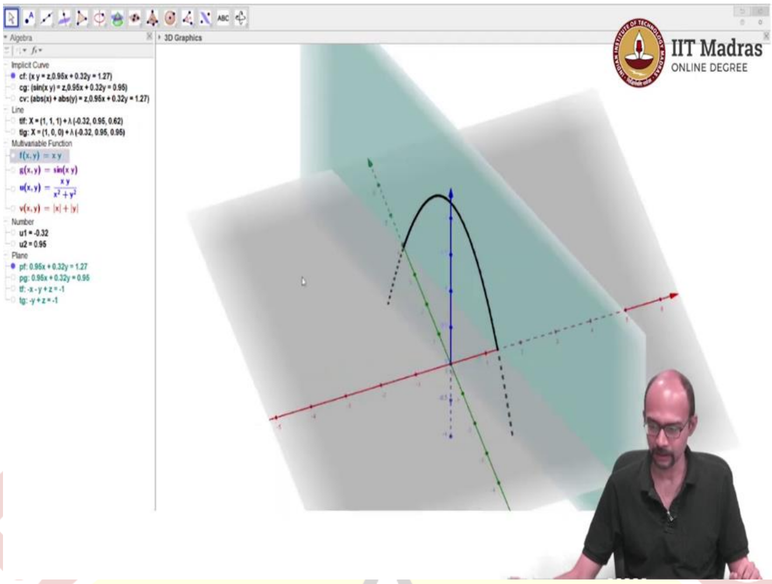
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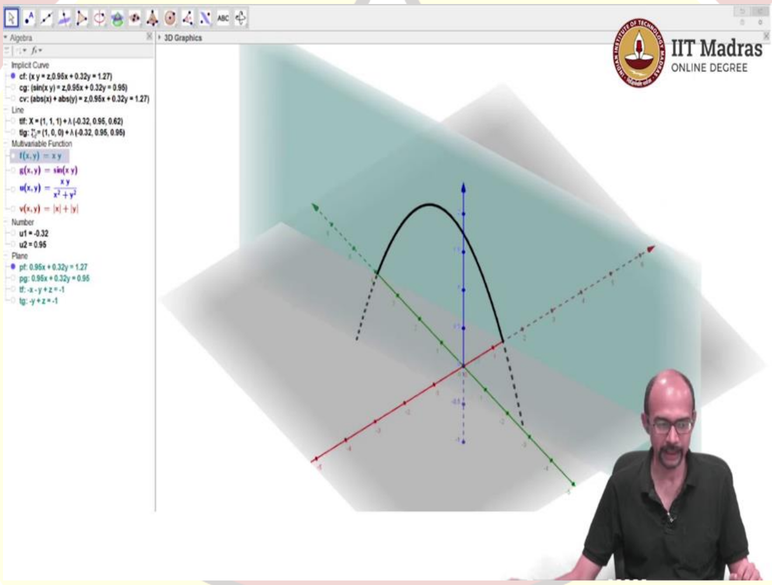
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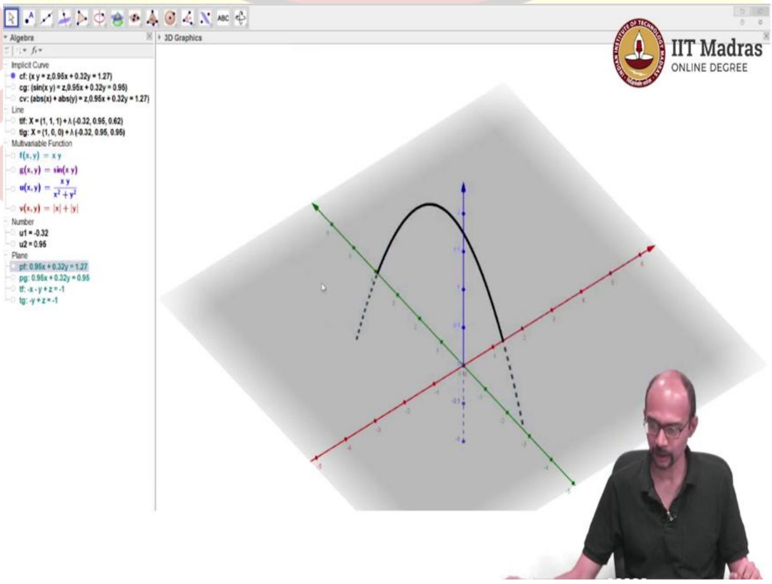
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tg: -y + z = -1
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So, here is a plane passing through the point $(1,1)$. So, I hope you can see the plane. And now let us restrict the graph of this function to that plane. So, if we do that, then we get this curve. So, let me let me remove the, so I hope you can see the curve on the plane. So let me remove the graph. So, here is the graph, which is 1 and here is how the how it looks like on that plane. And now if I remove that plane, you will see further that this is how it looks like on the line below that, which is passing through the point $(1,1)$.

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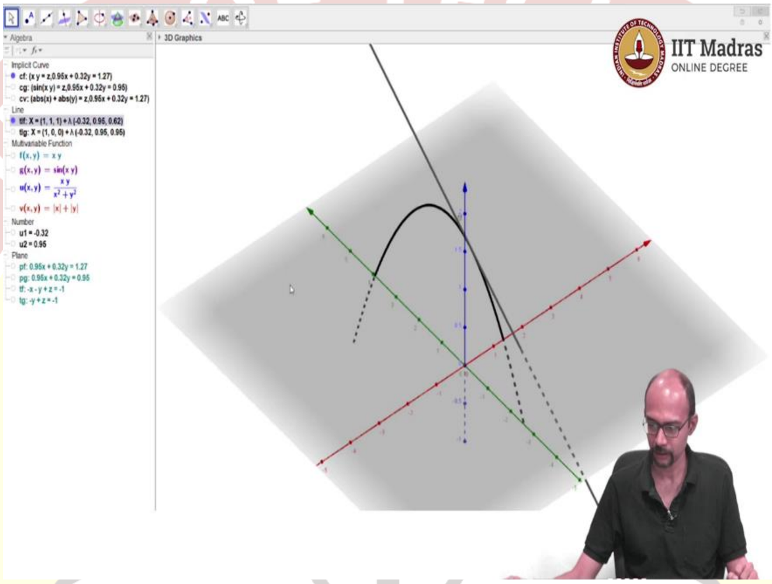

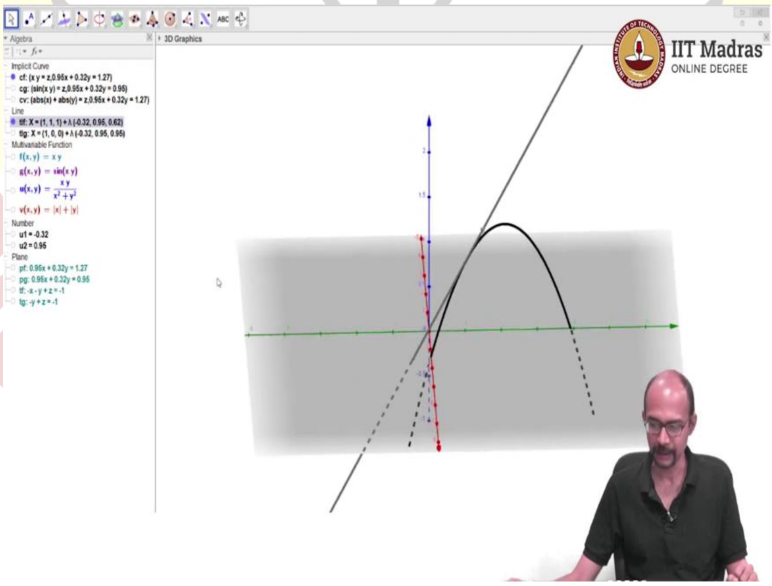

And now, we can ask what is the tangent to this add $(1,1)$. So, if we compute what that is, well, here is how that tangled looks like. So, here is the tangent. So, I hope you can see the tangent. And we can vary this as the unit vector u varies, which is the same as saying as the plane vary.

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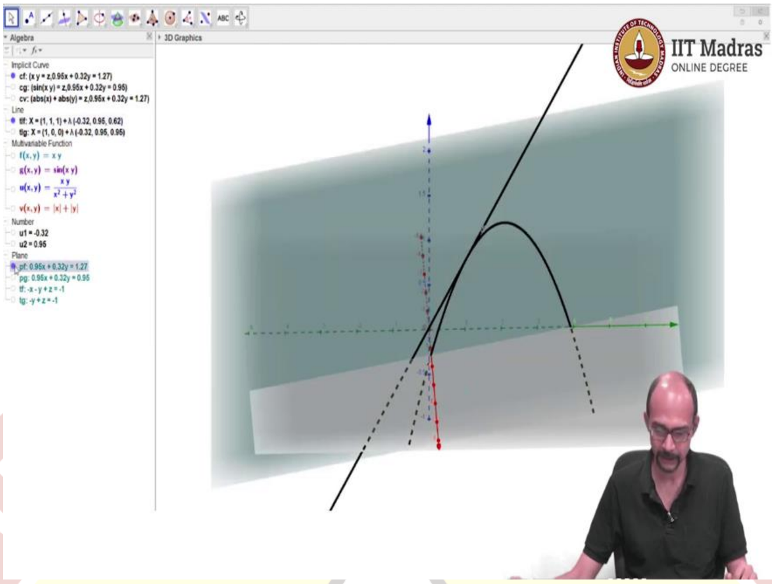

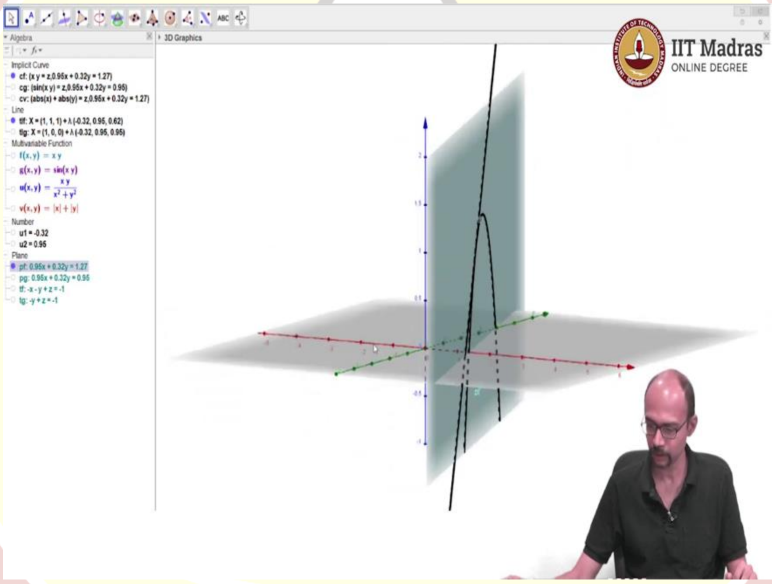

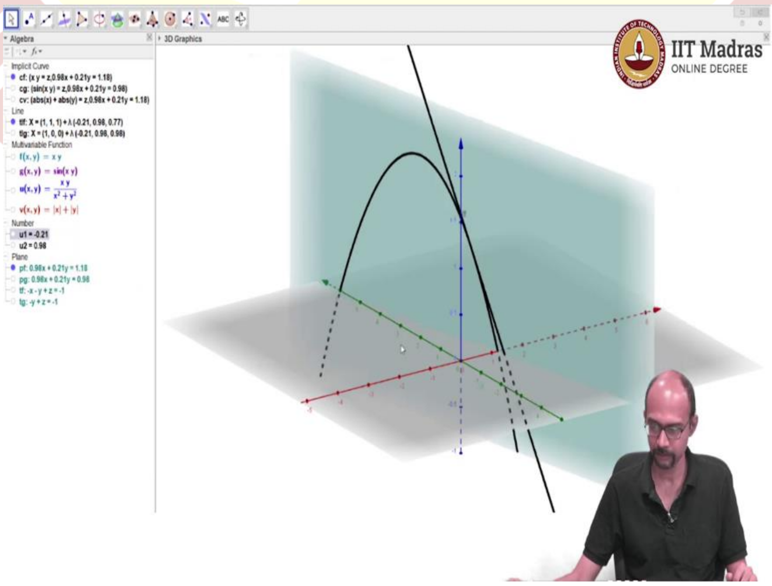

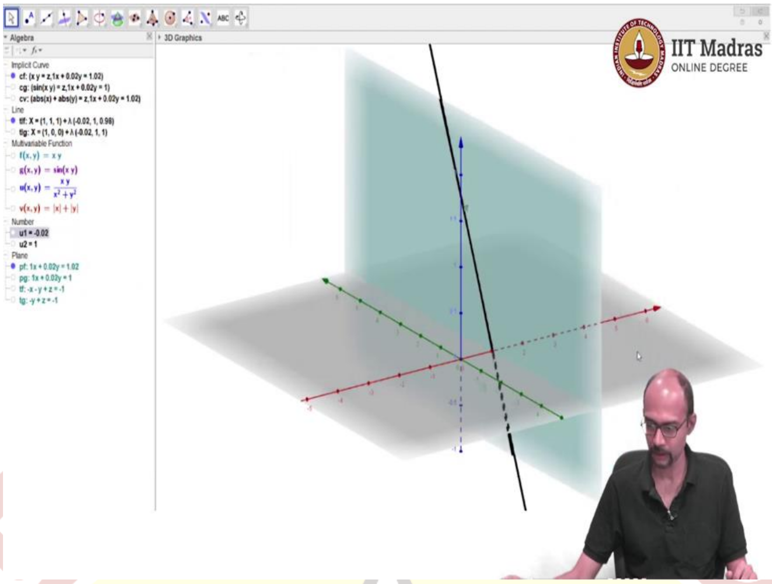
Algebra
3D Graphics
Implicit Curve
cf: (x y = z,1x + 0.02y = 1.02)
cg: (sin(x y) = z,1x + 0.02y = 1)
cv: (abs(x) + abs(y) = z,1x + 0.02y = 1.02)
Line
tf: X = (1, 1, 1) + λ (-0.02, 1, 0.98)
tg: X = (1, 0, 0) + λ (-0.02, 1, 1)
Multivariable Function
f(x, y) = x y
g(x, y) = sin(x y)
u(x, y) = x y / (x² + y²)
v(x, y) = |x| + |y|
Number
u1 = -0.02
u2 = 1
Plane
pf: 1x + 0.02y = 1.02
pg: 1x + 0.02y = 1
tf: -x - y + z = -1
tg: -y + z = -1
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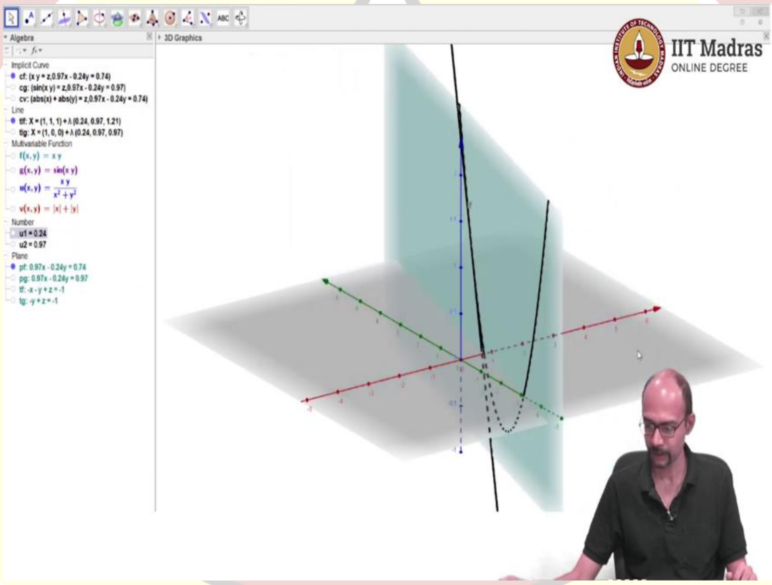
Algebra
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Implicit Curve
cf: (x y = z,0.97x - 0.24y = 0.74)
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Line
tf: X = (1, 1, 1) + λ (0.24, 0.97, 1.21)
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Multivariable Function
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g(x, y) = sin(x y)
u(x, y) = x y / (x² + y²)
v(x, y) = |x| + |y|
Number
u1 = 0.24
u2 = 0.97
Plane
pf: 0.97x - 0.24y = 0.74
pg: 0.97x - 0.24y = 0.97
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tg: -y + z = -1
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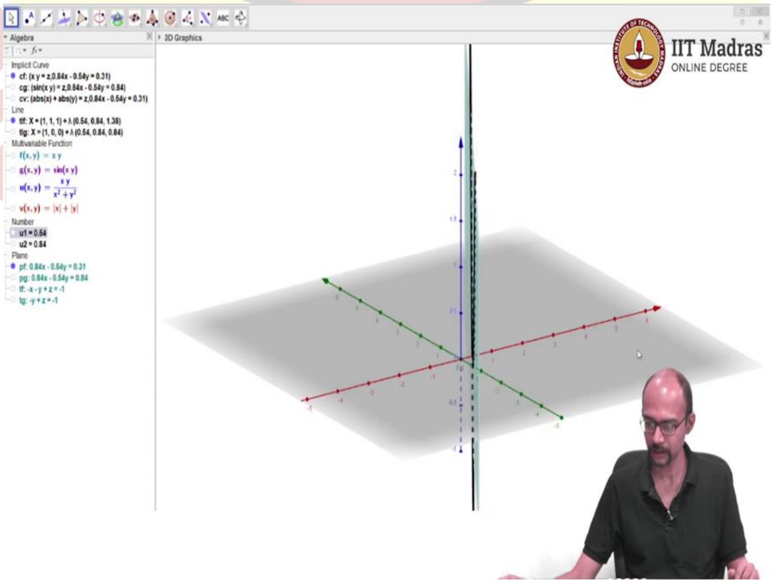
Algebra
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Implicit Curve
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Line
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tg: X = (1, 0, 0) + λ (0.54, 0.84, 0.84)
Multivariable Function
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g(x, y) = sin(x y)
u(x, y) = x y / (x² + y²)
v(x, y) = |x| + |y|
Number
u1 = 0.54
u2 = 0.84
Plane
pf: 0.84x - 0.54y = 0.31
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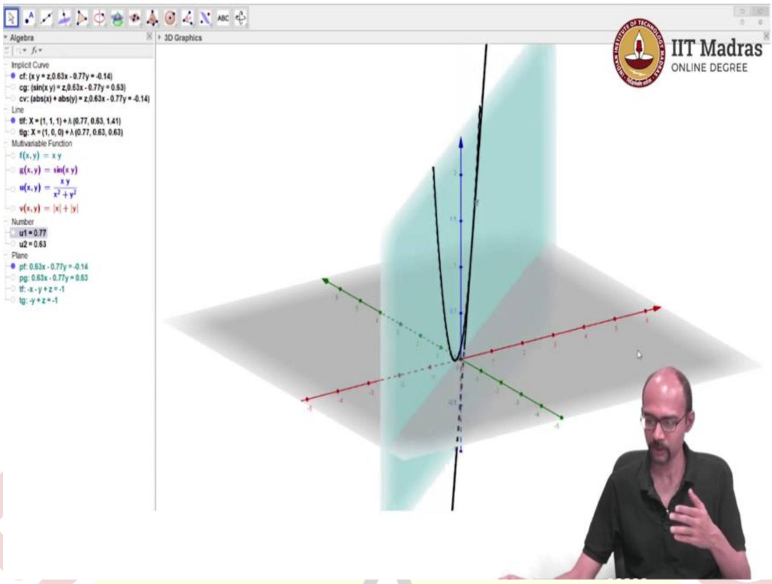
Algebra
3D Graphics
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Implicit Curve
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cv: (abs(x) + abs(y) = z,0.63x - 0.77y = -0.14)
Line
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tlg: X = (1, 0, 0) + λ (0.77, 0.63, 0.63)
Multivariable Function
f(x, y) = x y
g(x, y) = sin(x y)
u(x, y) = $\frac{x y}{x^2 + y^2}$
v(x, y) = |x| + |y|
Number
u1 = 0.77
u2 = 0.63
Plane
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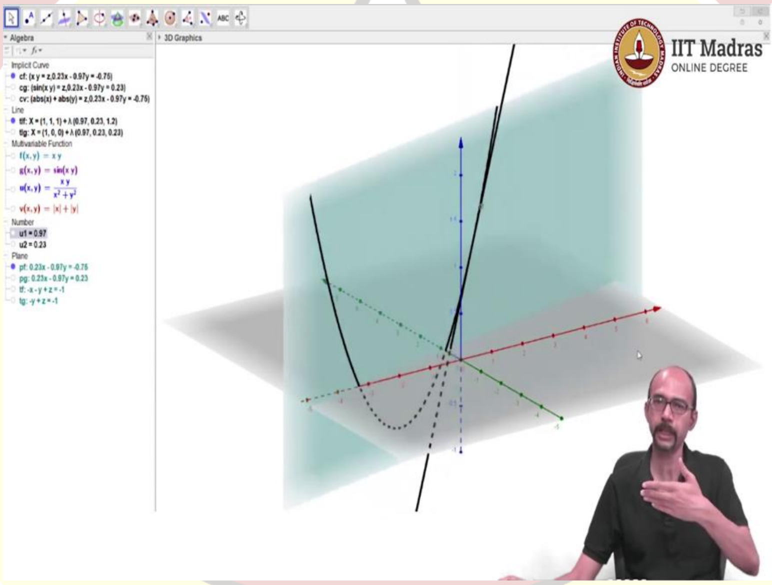
Algebra
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Implicit Curve
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Multivariable Function
f(x, y) = x y
g(x, y) = sin(x y)
u(x, y) = $\frac{x y}{x^2 + y^2}$
v(x, y) = |x| + |y|
Number
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u2 = 0.23
Plane
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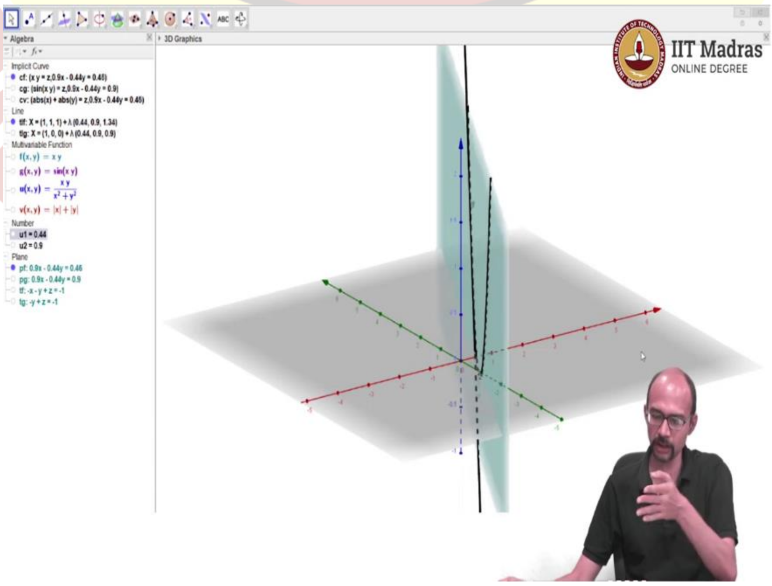
Algebra
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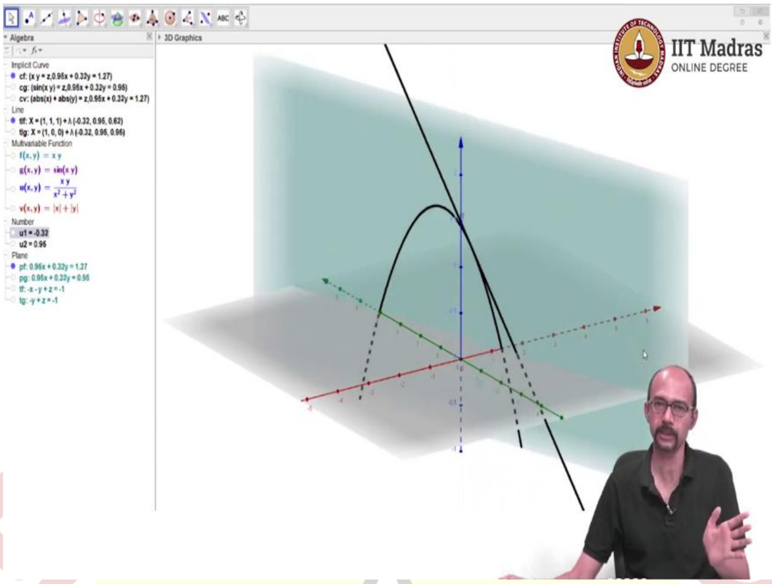
Implicit Curve
cf: (x y = z,0.9x - 0.44y = 0.46)
cg: (sin(x y) = z,0.9x - 0.44y = 0.9)
cv: (abs(x) + abs(y) = z,0.9x - 0.44y = 0.46)
Line
tlf: X = (1, 1, 1) + λ (0.44, 0.9, 1.34)
tlg: X = (1, 0, 0) + λ (0.44, 0.9, 0.9)
Multivariable Function
f(x, y) = x y
g(x, y) = sin(x y)
u(x, y) = $\frac{x y}{x^2 + y^2}$
v(x, y) = |x| + |y|
Number
u1 = 0.44
u2 = 0.9
Plane
pf: 0.9x - 0.44y = 0.46
pg: 0.9x - 0.44y = 0.9
tf: -x - y + z = -1
tg: -y + z = -1


Algebra
3D Graphics
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Line
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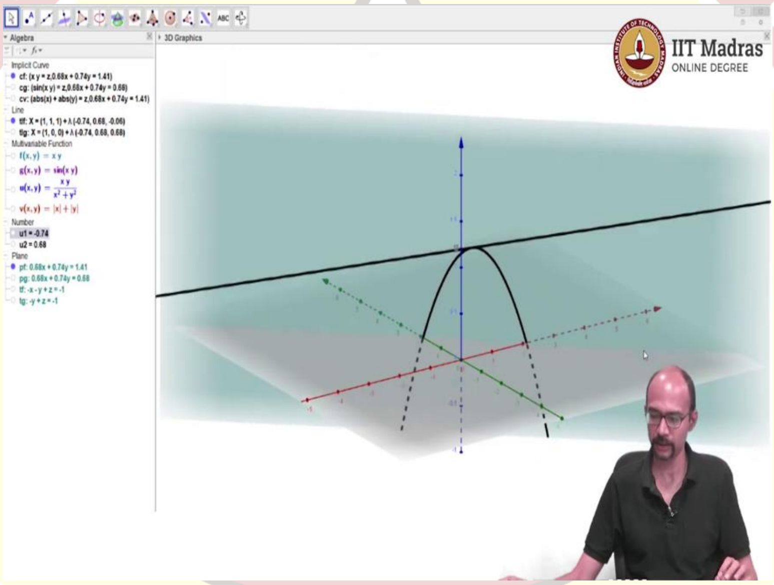
IIT Madras
ONLINE DEGREE


Algebra
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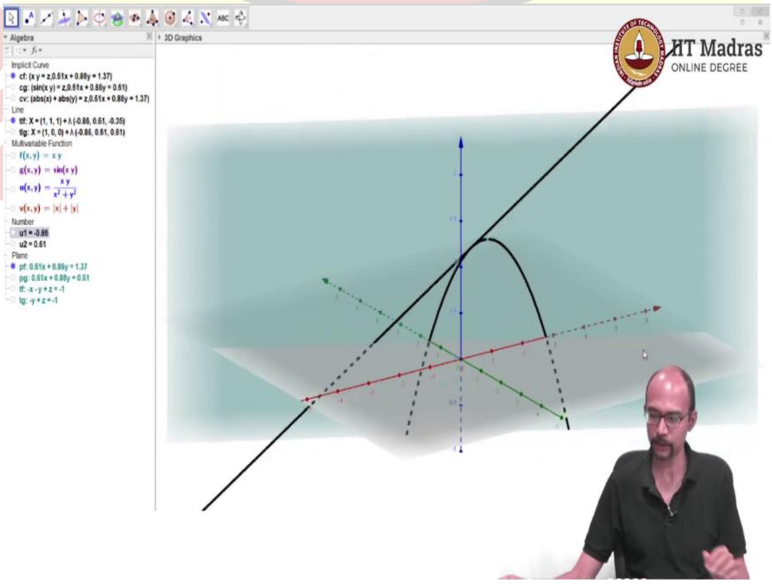
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Implicit Curve
Line
Multivariable Function
Number
Plane
IIT Madras
ONLINE DEGREE

So, I will bring back the plane. So, this is all happening on this plane. It is clear I hope from the picture. And as I vary my plane or my unit vector, rather, you will see how this changes. So, as the plane changes the corresponding, so these are all planes passing through that same point as the unit vector changes, the corresponding part of the graph changes that you have changes and so the tangent changes. So, this is how it looks like.

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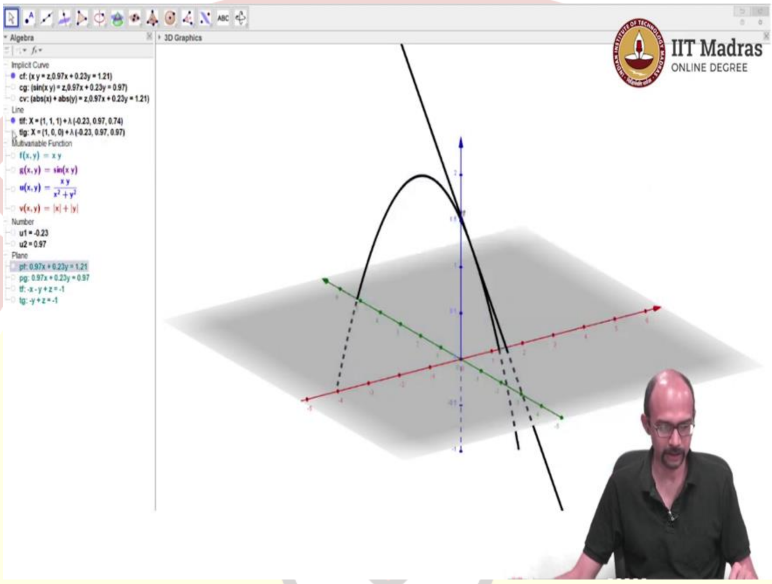
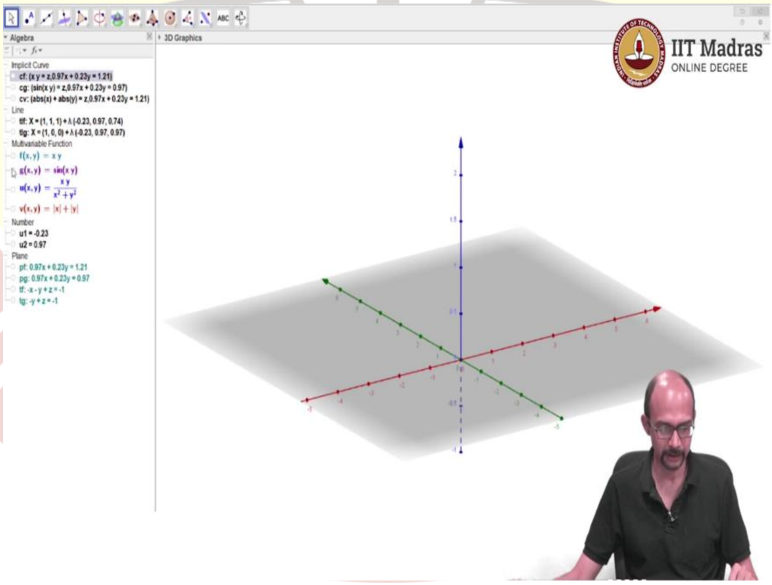
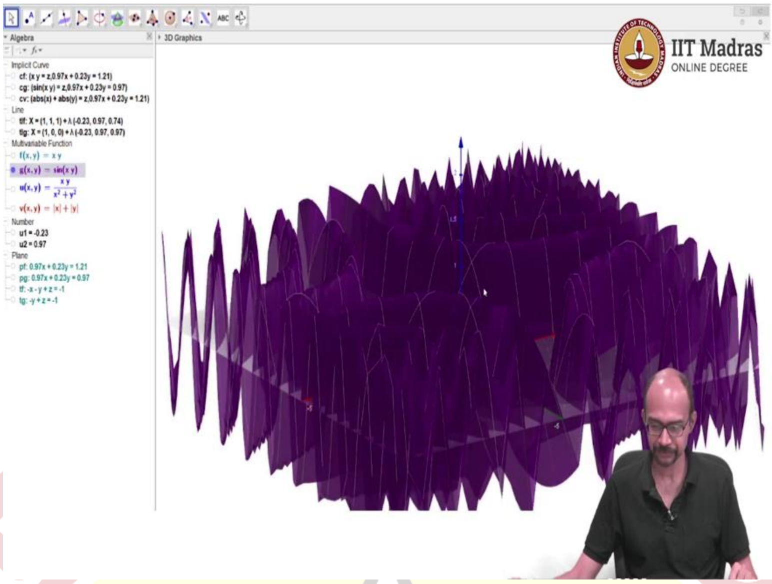
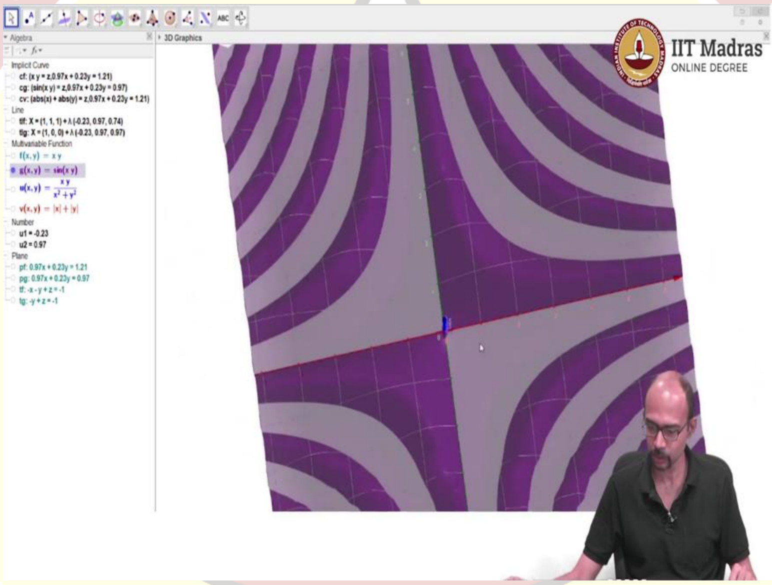






Algebra
3D Graphics
IIT Madras
ONLINE DEGREE
Implicit Curve
cf: (x y = z,0.97x + 0.23y = 1.21)
cg: (sin(x y) = z,0.97x + 0.23y = 0.97)
cv: (abs(x) + abs(y) = z,0.97x + 0.23y = 1.21)
Line
tlf: X = (1, 1, 1) + λ (-0.23, 0.97, 0.74)
tlg: X = (1, 0, 0) + λ (-0.23, 0.97, 0.97)
Multivariable Function
f(x, y) = x y
g(x, y) = sin(x y)
u(x, y) = x y / (x² + y²)
v(x, y) = |x| + |y|
Number
u1 = -0.23
u2 = 0.97
Plane
pf: 0.97x + 0.23y = 1.21
pg: 0.97x + 0.23y = 0.97
tf: -x - y + z = -1
tg: -y + z = -1



Algebra
3D Graphics
IIT Madras
ONLINE DEGREE
Implicit Curve
cf: (x y = z,0.97x + 0.23y = 1.21)
cg: (sin(x y) = z,0.97x + 0.23y = 0.97)
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pg: 0.97x + 0.23y = 0.97
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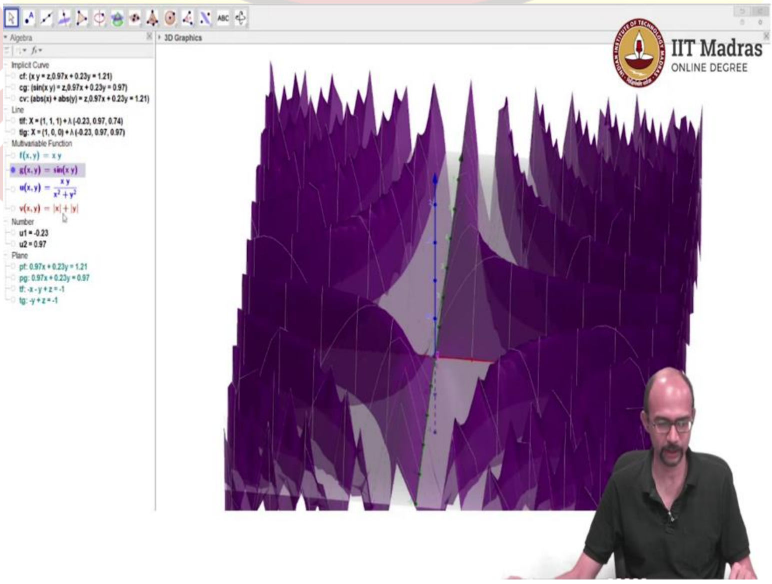

Algebra
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IIT Madras
ONLINE DEGREE
Implicit Curve
cf: (x y = z,0.97x + 0.23y = 1.21)
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Plane
pf: 0.97x + 0.23y = 1.21
pg: 0.97x + 0.23y = 0.97
tf: -x - y + z = -1
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Let us do another example. So instead of these, suppose now I have the graph of the function, $\sin(x,\ y)$. So, this was again a function that we have seen before. So, here is $\sin(x,\ y)$. Let us look at the point maybe $(1, 0)$.

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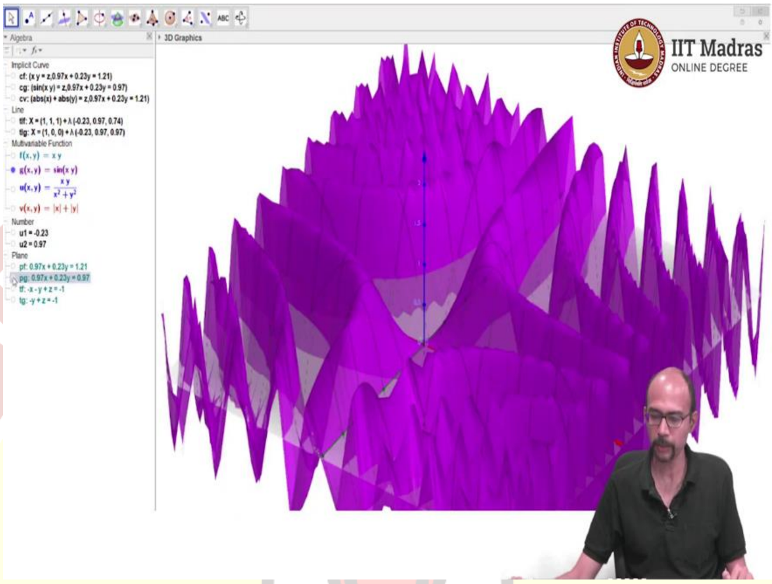

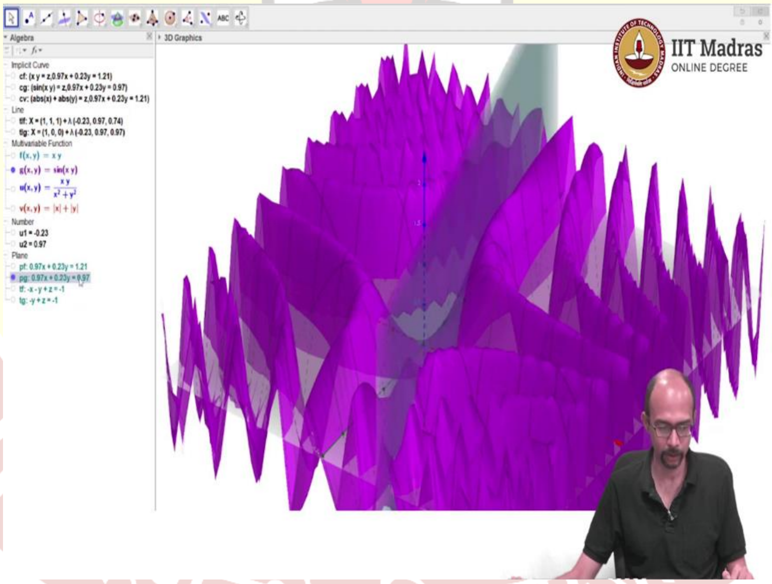

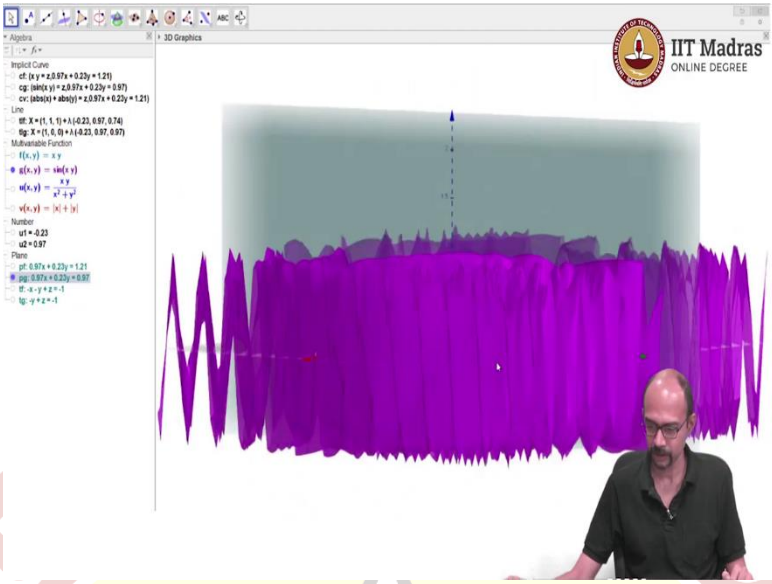
Algebra
3D Graphics
IIT Madras
ONLINE DEGREE
Implicit Curve
cf: (x y = z,0.97x + 0.23y = 1.21)
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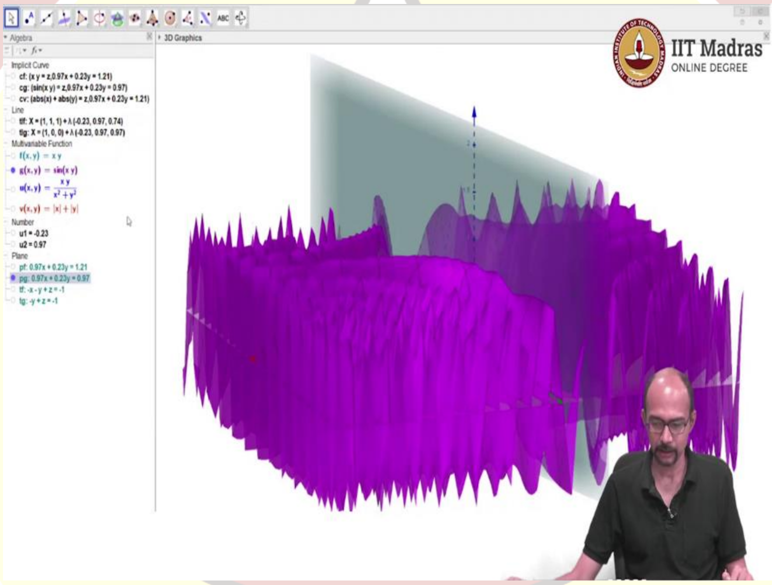
u2 = 0.97
Plane
pf: 0.97x + 0.23y = 1.21
pg: 0.97x + 0.23y = 0.97
tf: -x - y + z = -1
tg: -y + z = -1


Algebra
3D Graphics
IIT Madras
ONLINE DEGREE
Implicit Curve
cf: (x y = z,0.97x + 0.23y = 1.21)
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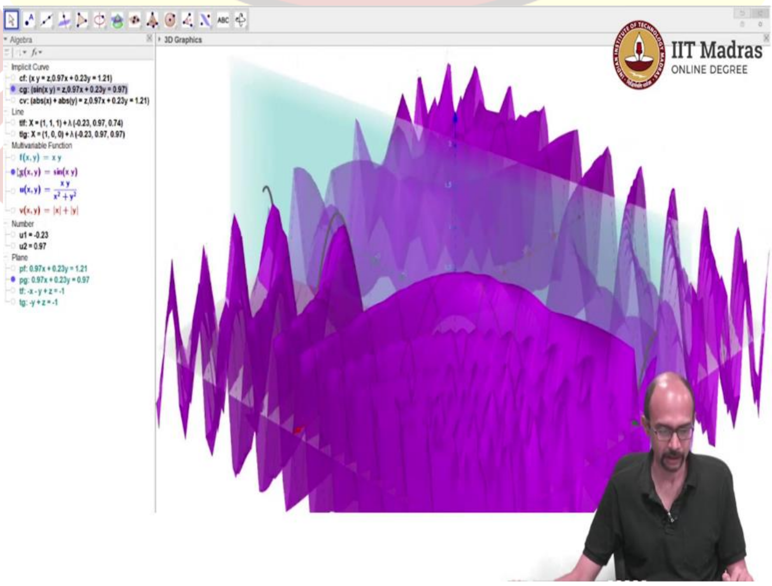
pg: 0.97x + 0.23y = 0.97
tf: -x - y + z = -1
tg: -y + z = -1


Algebra
3D Graphics
IIT Madras
ONLINE DEGREE
Implicit Curve
cf: (x y = z,0.97x + 0.23y = 1.21)
cg: (sin(x y) = z,0.97x + 0.23y = 0.97)
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tlf: X = (1, 1, 1) + λ (-0.23, 0.97, 0.74)
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f(x, y) = x y
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Number
u1 = -0.23
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Plane
pf: 0.97x + 0.23y = 1.21
pg: 0.97x + 0.23y = 0.97
tf: -x - y + z = -1
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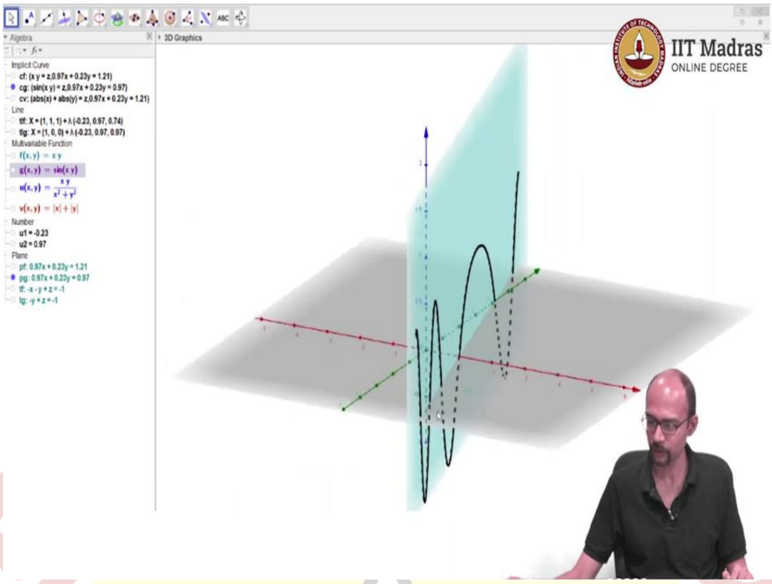

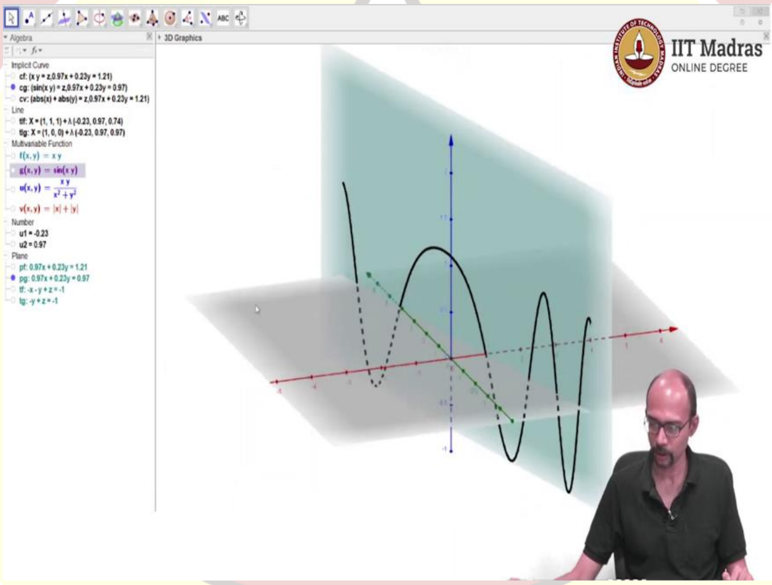

So, here is a plane passing the point $(1, 0)$. I hope you can see it. And now let us see what the intersection of these two is that gives you the following curve, and I will remove this graph so you can see the curve. So, we have seen this curve before, as well. So, this is the curve passing through $(1, 0)$.

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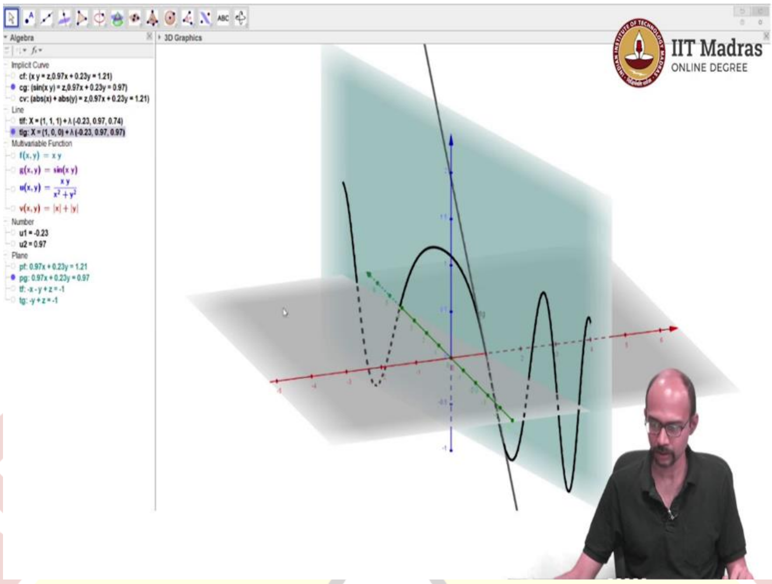

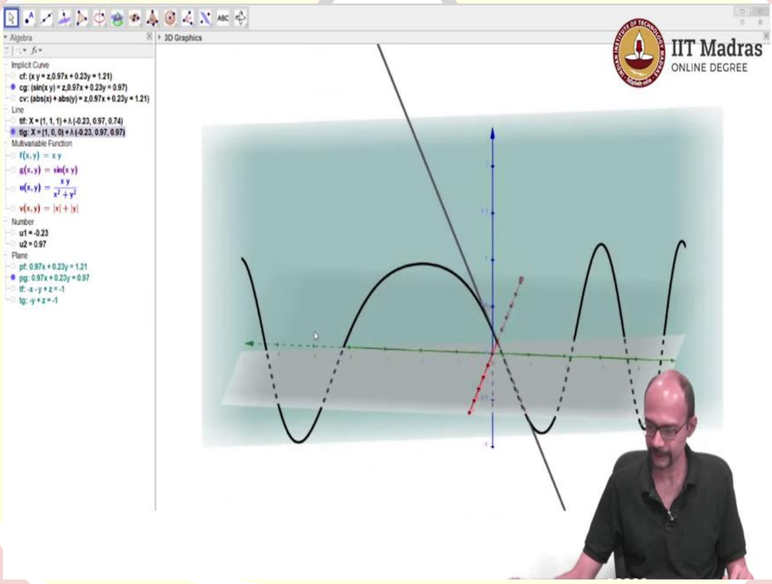

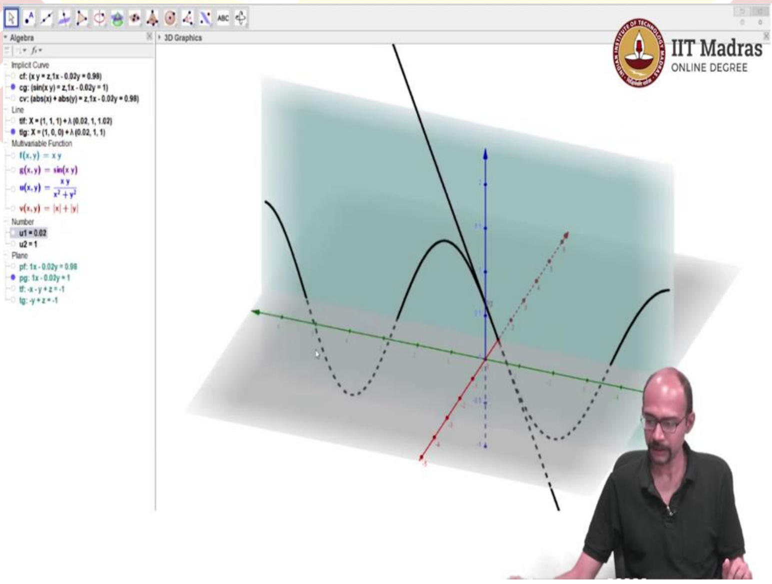

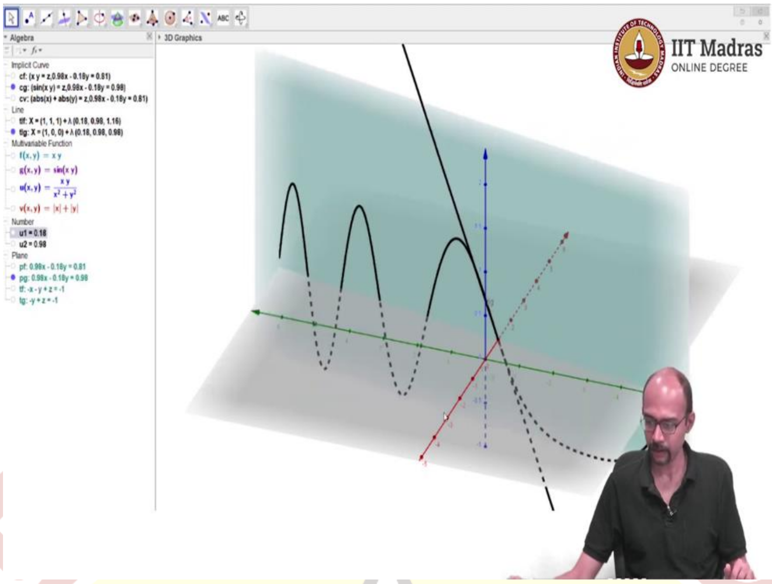

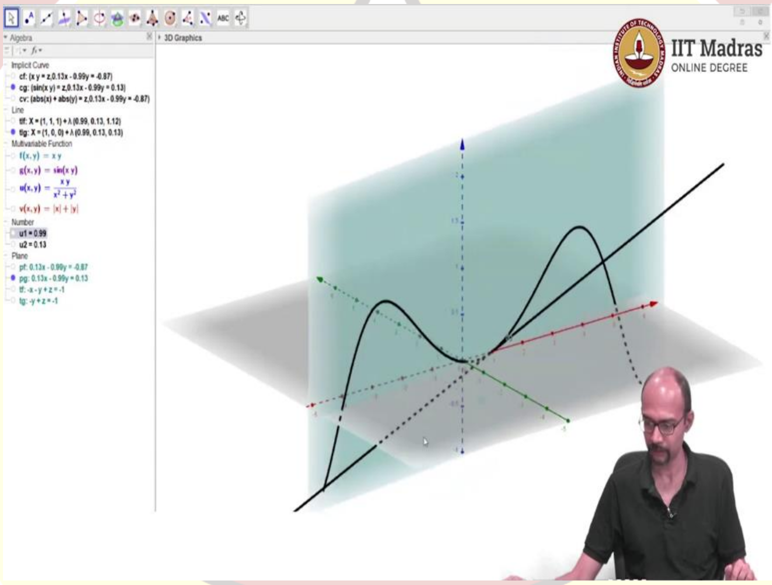

And now we can ask what is the tangent to that. So, if you look at the tangent, here is how the tangent looks like. So, this is at the point $(1, 0)$, and I can vary my unit vector. And as I vary my unit vector, the plane changes accordingly, the part of the graph which is intersected with the plane changes, and accordingly the tangent line changes. So, I hope you can see what is happening. So, this is just a visual demonstration of what we said in that previous slide.

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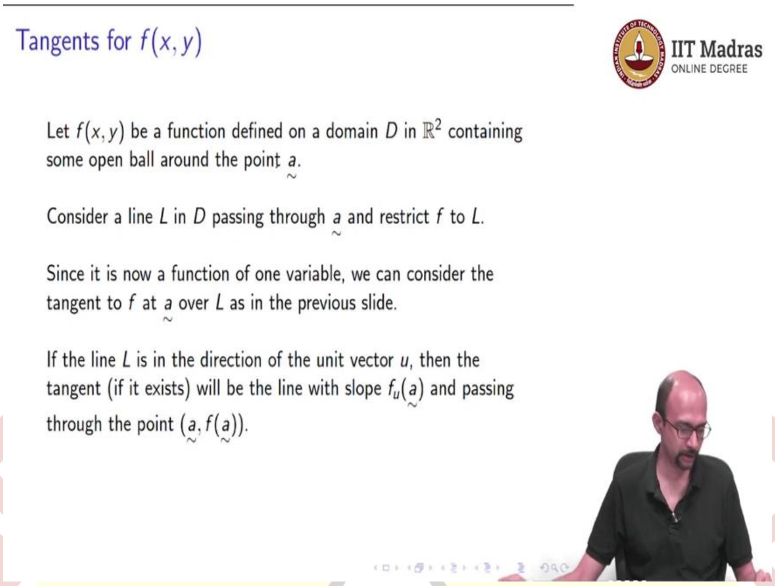

So, the main point here is that, if you have this line L, you get the corresponding plane above that, and on that plane you have this line, which has slope $f_u(atilde)$, and which passes to the point $(a\,tilde, f(a\,tilde))$ tilde. So, using these facts we can write down the equation of this line.

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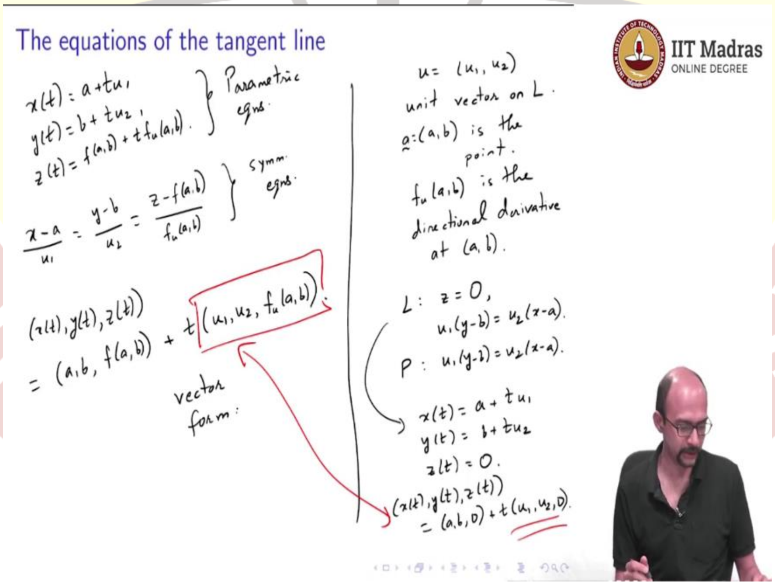

And let us try to write down what that equation is. So, we can write this equation down in several ways. So, first of all, what is the line below? So, I am going to take $u = (u_1, u_2)$. So, this is the unit vector on that line. So, unit vector on L and then (a, b) is my point. So, (a, b) so a tilde $= (a, b)$ is the point where I want to compute the equation of the tangent line. So, $f_u(a, b)$ is the directional derivative at that point.

So, these are the, these are the, this is everything that I know. So first of all, what is the equation of the line? So, for the line, so first of all, we know that it is on the x, y plane, so that means z = 0, this is one of the defining factors. And the other is that it passes through the point (a, b), and it is in the direction of the unit vector, (u_1, u_2).

So, since that is the case, we can write down the equation as so $u_1(y - b) = u_2(x - a)$. And how do I check that this is indeed correct? Well, you can see that the slope is $\frac{u_2}{u_1}$ provided of course that you u_1 is non-zero, and if $u_1 = 0$ you can see that it is the correct line which is $x = a$.

And if you plug-in the values of x and y to a and b respectively then this equation is satisfied because you have $0 = 0$ that means it passes through (a, b). So, it is the intersection of these two planes. So $z = 0$ is a plane that is x, y plane, and $u_1(y - b) = u_2(x - a)$ is another plane, so it passes through that plane.

And in fact, if I remove this restriction that $z = 0$, then that is a plane above the line L. So, the corresponding plane which I will denote by $P: u_1(y - b) = u_2(x - a)$. So now, this is what I know. So, instead of writing L like this, I can write it as a parametric equation and this is often very useful, so that we know what is happening.

So, the other way of writing L is as $x(t) = a + tu_1$, $y(t) = b + tu_2$, and $z(t) = 0$ because it is on the x, y plane, so this is how you write L. And how did I get this?

Well, this is if you remember from linear algebra, lines are meaning any arbitrary line is an affine flat, and how do I get those by translating some line which passes through the origin, which means a subspace of dimension one. And with subspace is that? That is the line which is passing through the vector u_1, u_2. So, based on that what I can say is that $x(t), y(t), z(t)$ is exactly just to reiterate this, $(a, b, 0) + t(u_1, u_2, 0)$

So, this is saying that if you have you are shifting the line passing through $(u_1, u_2, 0)$, because it is on the x, y axis, x, y plane, so that it passes through (a, b). This was exactly the idea of affine flats. And we are going to use the same idea now to write down the equation of the tangent line as well.

So, what happens for the tangent line. So, we have identified the line L and the plane P. So, the tangent line is actually on this plane P, and it is at an angle of a f_u meaning the tangent of the angle, the tan of the angle, which it makes with L is $f_u(a, b)$. So, parsing that out, what it means is, if on that line you move a unit distance, which means that you move by that unit vector then the z coordinate will move by $f_u(a,b)$.

So, what that means is, just to put that in perspective, what that means is that the parametric equations will be given by $x(t) = a + tu_1,\ y(t) = d + tu_2$. So, this you expected because when you project that line down the tangent line down you will get this line L, and we have written down the parametric equations for that. So, the question is what happens to $z(t)$.

And $z(t)$, as we just saw, if you move a unit distance, which means if t is 1 so that means you're moved by that unit vector then the z coordinate will move by $f_u(a,b)$. So, then I can write this, as $f(a,b) + tf_u(a,b)$, so these are the parametric equations for the tangent line.

Now you can rewrite this in various ways. For example, you can rewrite this in the following form $\frac{x-a}{u_1} = \frac{y-b}{u_2} = \frac{z-f(a,b)}{f_u(a,b)}$. Of course, you assume that these are non-zero, if they are 0 there is a way of interpreting this, etc. So, this is called the symmetric equations for the line. Not very important, but I am just mentioning it. And the other way of writing it is the way we have viewed this as an affine flat and that is really the very interesting.

So, what we are saying is that, this line is a line passing through $(a, b, f(a,b))$, so $x(t), y(t), z(t)$ just to rewrite this is $\big(a, b, f(a,b)\big) + t(u_1, u_2, f_u(a,b))$. And what is this saying? This is exactly corresponding to the equation here.

So, again this is, so this line is in affine flat, it passes through $\big(a, b, f(a,b)\big)$ that is why I have translated it by this point. And if you translate it back to the origin, what line do you get, you get the line passing through $(u_1, u_2, f_u(a,b))$. Why is that? Because the line here was $(u_1, u_2, 0)$, it was a line passing through $(u_1, u_2, 0)$.

And the line that you have your makes an angle of θ where $\tan\theta$ is $f_u(a,b)$. So, if you translate that you get this same line here and that is what it say that it is the line passing through this vector here, that is exactly what we mean by the derivative. So, this explains why we get this line. So again, this is the vector form of you can often called the vector form.

So, these are the equations of the tangent line. So, I hope this explanation shed some light on how we obtain these tangent line equations. So, let us see a couple of examples so that we can make this concrete.

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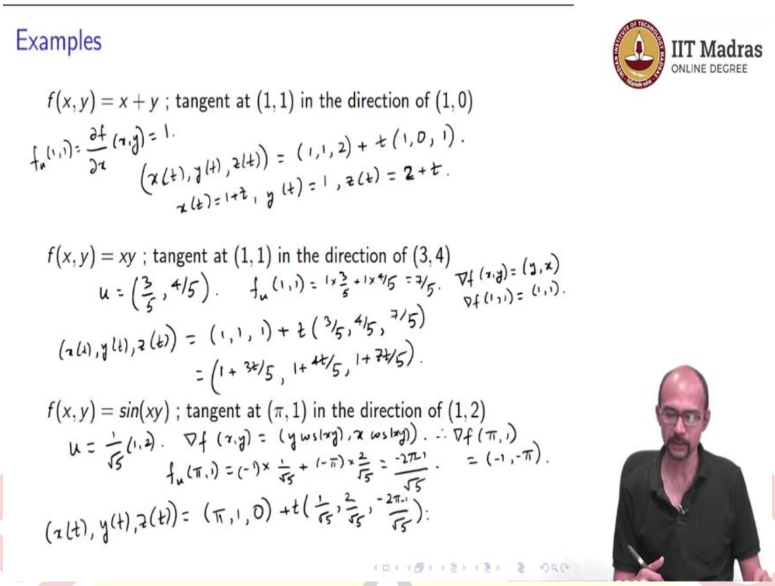

So, let us compute some of these things in the at the mentioned points and in the mentioned directions. So, let us look at the function $f(x, y) = x + y$ and the tangent at $(1, 1)$ in the direction of $(1, 0)$. So, how do I get this? So, for this, I have to first find the directional derivative.

So, let us find the directional derivative. So, the directional derivative, so first I should find $\frac{\partial f}{\partial x}(x, y) = 1$. So, her and then, so what I want here is, so this is exactly f_u because $u = (u, 0)$, so this is f_u at any point so I am particular at $(1, 1)$. So $f_u(1, 1) = 1$ and now we can write down the equations.

So, maybe, let me do it in the vector form that is the easiest. So, in the vector form it is $\big(x(t), y(t), z(t)\big) = (1, 1, 2) + (1, 0, 1)$. So, the other way of writing this is to say that $x(t) = 1 + t,\ y(t) = 1,\ z(t) = z + t$.

So, and now you can try and visualize this in GeoGebra. I will encourage you to do that. Let us do the next one, which is $f(x, y) = xy$. We actually saw this example a few minutes ago. So here, we have to compute this in the direction of $(3, 4)$, so first, we need a unit vector in that direction.

So, what is a unit vector in that direction? Well, you can compute it is $u = \left(\frac{3}{5}, \frac{4}{5}\right)$, then we need to compute what is $f_u = (1, 1)$. So, to do that let us use since it is a nice function, I can use my gradient formula. So, what is the $\nabla f(x, y) = (y, x)$. So $\nabla f(1, 1) = (1,1)$

And then $f_u = (1,1) = 1 \times \frac{3}{5} + 1 \times \frac{4}{5} = \frac{7}{5}$. So, now I can write down my equations. So, in the parametric form or in the vector form I get $\big(x(t), y(t), z(t)\big) = (1,1,1) + t\left(\frac{3}{5}, \frac{4}{5}, \frac{7}{5}\right)$.

So, which gives us $\left(1 + \frac{3t}{5}, 1 + \frac{4t}{5}, 1 + \frac{7t}{5}\right)$. And from here, you can get the parametric form. So, I hope you can see it is not at all difficult to compute these once we know how to do a directional-derivatives, fine. The final example is you have $f(x,y) = \sin(xy)$. I think, we have computed this before.

So, in the direction of $(1,2)$. So first what is a unit vector? So, this is $u = \frac{1}{\sqrt{5}}(1,2)$. And then what is the gradient? So, the $\nabla f(x,y) = (y\cos(xy)\ , x\cos(xy))$. So, if we compute this at$(\pi, 1)$, we get a $\nabla f(\pi,1) = (-1\ , -\pi)$..

So, $f_u(\pi, 1) = (-1) \times \frac{1}{\sqrt{5}} + (-\pi) \times \frac{2}{\sqrt{5}} = -\frac{2\pi}{\sqrt{5}}$ and now we can easily write down what is the vector form of this tangent line.

So, $\big(x(t), y(t), z(t)\big) = (\pi, 1, 0) + t\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{-2\pi-1}{\sqrt{5}}\right)$, and now you can get the parametric form as well. So, I hope these examples are somewhat illustrative of how to compute these tangent lines.

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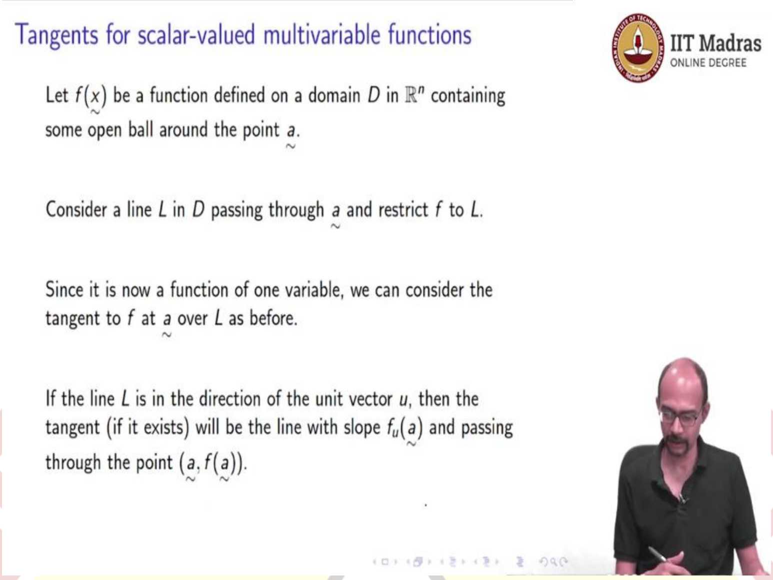

So, let us now go ahead and do the same thing for general functions of n variables. So, we did this in two variables because it is easy to visualize this in terms of pictures, but now we can do this for n variables.

Now, we cannot really visualize this any longer, but the ideas are exactly the same. So, suppose you have a function $f(x\ tilde)$, which is defined on a domain D in $\mathbb{R}^n$ containing some open ball around your point a tilde you take a line restrictive f to that line passing a tilde that is, and since it is a function of one variable, we can consider the tangent to f at a tilde over L as before.

So, if the line L is in the direction of the unit vector u, then the tangent if it exists will be the line with slope $f_u(a\ tilde)$ and passing through the point $(a\ tilde, f(a)\ tilde)$. And from here we can find out the equation of the line.

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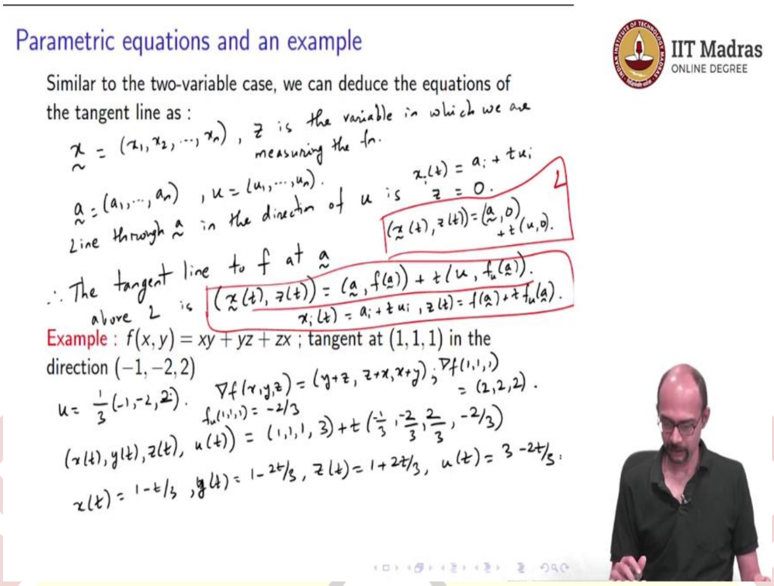

So, how do we do that? So, this is very similar to the two-variable case. And what we can do is now suppose, I have n variables. So, the $(n+1)^{th}$ variable, let me denote by x_{n+1}. So now, I have or let me denote the $n+1$ variable by z. So, the first $n+1$ variables are $x_1,\ x_2, \dots, x_n$ and the $(n+1)^{th}$ variable, which we are using to measure the function, let us call that z. So, z is the variable in which we are measuring the function.

So, then, exactly the same way as we have done for the previous case, we can look at the line passing through the point a tilde. So, a tilde $= a_1, a_2, \dots, a_n$ and which is in the direction of the unit vector u. So, $u = u_1, u_2, \dots, u_n$. So, the line through a tilde in the direction of u is for the parametric equations are $x_i(t) = a_i + tu_i$. So again, a line passing through $u_1, u_2, \dots, u_n$ it will be $t(u_1, u_2, \dots, u_n)$ and now you are translating it by the vector a.

So, that is for i that this happens. So, in terms of the vector form. And of course, I have to also say what happens for the coordinates z, so $z = 0$. So, in other words, in the coordinate form this is $\big(x\ tilde\ (t), z(t)\big) = (a\ tilde, 0) + t(u, 0)$.

So, z is not playing any role, because everything is right now it is still the line in the $x_1, x_2, \dots, x_n$ space. So, now, this line, I want to take the line which is at an angle of θ , where θ is $\tan\theta = f_u a\ tild$. And once again, what that means is, so therefore, the tangent line to f at a tilde above L, so if I call this above line L, so this this line I am calling L.

So, above L is, so the first n coordinates remain the same and then for z we know how much it jumps. So, if x the first n coordinates move by a unit vector then z moves by f_u , this is exactly what we mean by rate of change. So, that means what we will get is it $(x\ tilde\ (t), z(t)) = (a\ tilde, f(a\ tilde)) + t(u, f_u(a\ tilde))$.

And now you can write down the parametric form. So, this is the vector form. So, for the parametric form you will have $x_i(t) = a_i + tu_i,\ z(t) = f(a\ tilde) + tf_u(a\ tilde)$. So, we can write it in either which way, the vector form or the parametric form.

Let us do an example. So, this might make it easier to view. So, what you do remember is that these are the two forms for the tangent line. So, either this which is the vector form or the second one, which is the parametric form. So, you have $f(x, y) = xy + yz + zx$ and let us see what is the tangent at (1,1,1) in the direction of $(-1, -2, 2)$. So first, we will compute the unit vector. So, if you compute the unit vector $u = \frac{1}{3}(-1, -2, 2)$.

So now, we need to know what is the directional derivative in this direction, at the point (1,1,1). So, for that we need the gradient. So, let us compute what is the ∇ of x, y, z. So, the $\nabla f(x, y, z) = (y + z, z + x, x + y)$. So, at (1,1,1). This is (2,2,2). So, from here, we can compute the vector equation so that is going to be $x(t), y(t), z(t)$. I am using $z(t)$ here, already.

So, the function, let us view it in terms of u, so that is a last variable. So, $u(t)$, this is the point which is $(1,1,1,3) + t\left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ as $f_u(1,1,1) = -\frac{2}{3}$.

So, here I will get $-\frac{2}{3}$. So, the vector equation will be $x(t) = 1 - \frac{t}{3}, y(t) = 1 - \frac{2t}{3}, z(t) = 1 + \frac{2t}{3},\ u(t) = 3 - \frac{2t}{3}$. So, this is very easy. Once we know what to compute, it is very easy to actually compute it. Of course, we have implicitly used here that we know that the gradient is continuous, which is why we can compute the directional derivative fairly easily.

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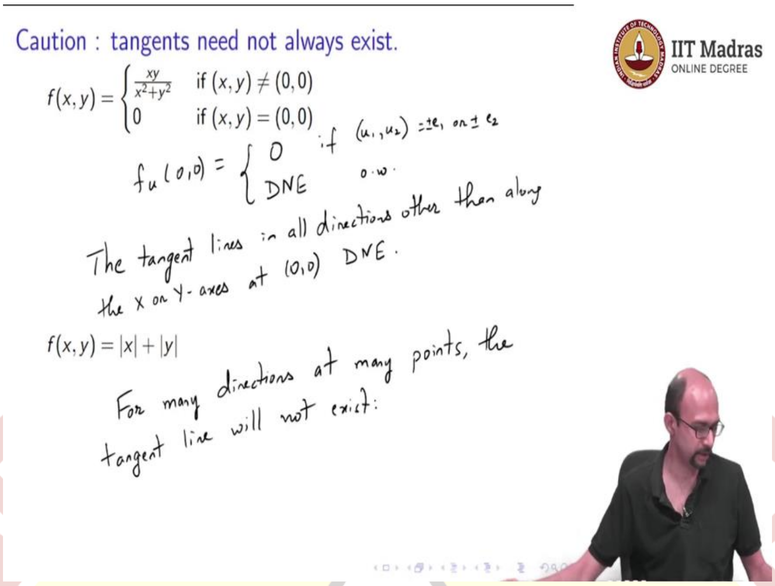

So, let us now add a word of caution, because from here, it may seem that all is hunky dory. So, let us remember that tangents need not always exist. So, we have seen this even in one dimension. So, if you take $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$,we know that at the point $(0,0)$ there is some problem.

So, we have seen in this example that the partial derivatives exist, so $f_u(0,0) = \begin{cases} 0 & \text{if } (u_1,u_2) = \pm e_1 \text{ or } \pm e_2 \\ \text{DNE} & \text{otherwise} \end{cases}$.

And so, once we know that the partial derivative does not. The directional derivative in the particular direction does not exist, we know from our one variable calculus that the derivative of a one variable function exists is precisely the same as saying that the tangent line exists otherwise, there is no notion of a tangent line.

There is no line, which captures the behavior of the instantaneous direction. So, that means the tangent lines in all directions other than along the x or y axis at (0, 0) do not exist, that is exactly what it means. Let us look at the function $|x| + |y|$, this is maybe even crazier. And here, it is not going to exist for many, many directions.

So, for most directions or let us say at for many directions at many points, the tangent link will not exist. I will suggest you do the algebra yourself, but instead, we look at a picture of this graph and I think you will appreciate what I mean.

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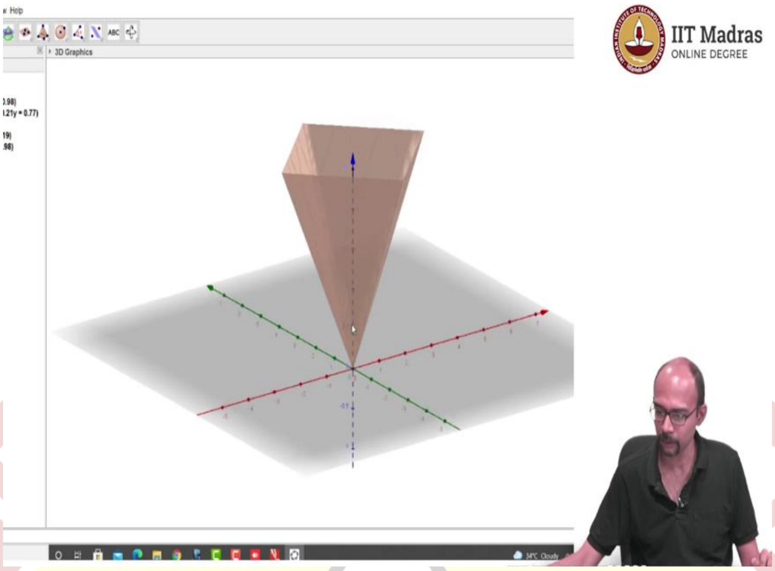

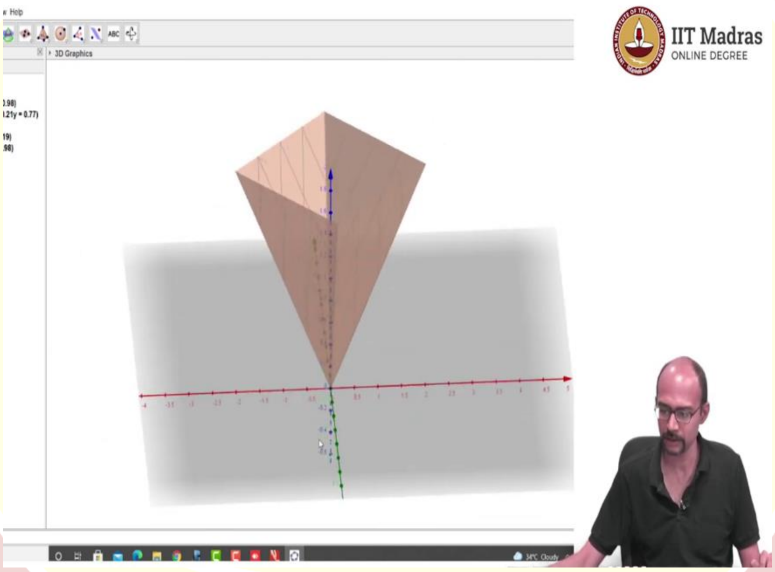

So, here is a picture of the graph of this function. This is the function $|x| + |y|$, and you can see it has lots and lots of corners, it has faces, and then it has corners. So, if you are along a face then then maybe you have a chance that a tangent does exist because that will be like some kind of a constant function, but if you move your plane even a little that will be a lot of trouble because then you will have lots of corners and then the tangent may not exist. So just to give you an idea of what I am saying here let us intersect this with some plane.

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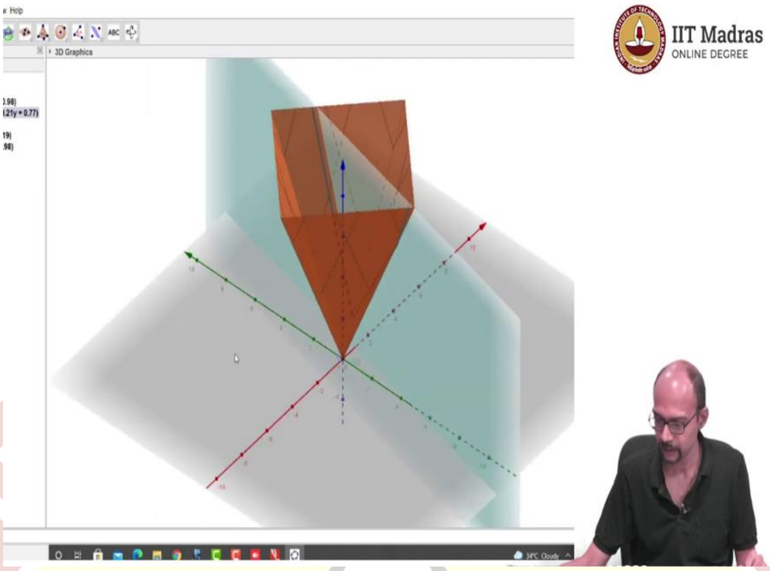

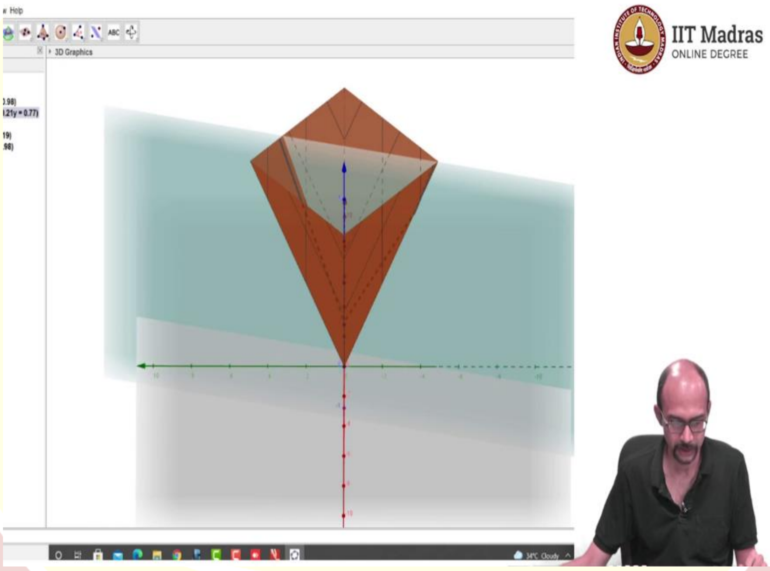

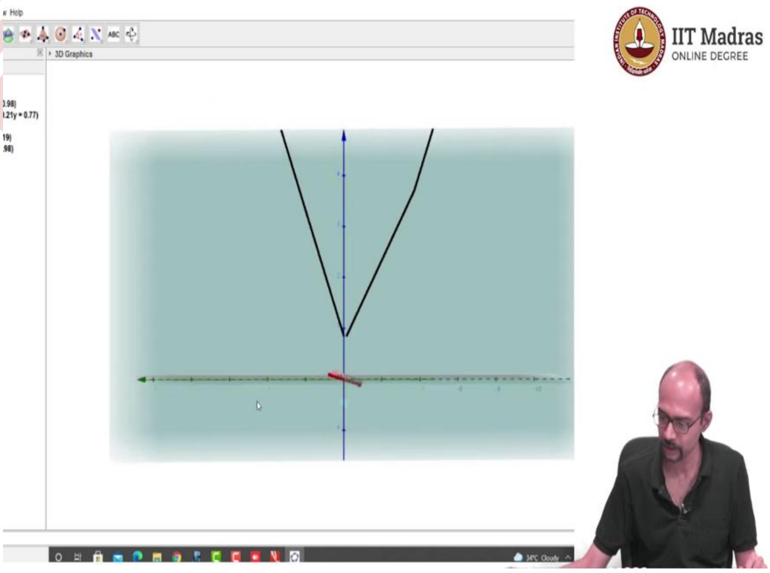

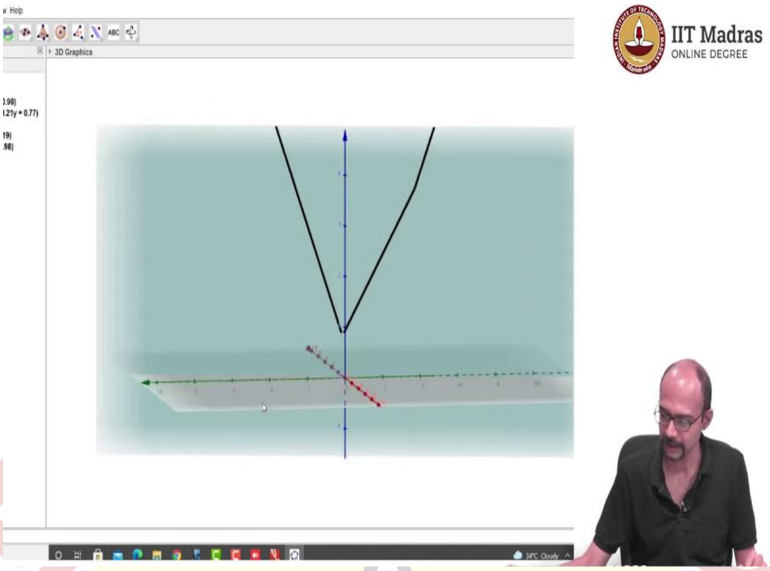

So, if I look at the corresponding graph, you can see how it looks like. I will remove the graph of the function, here is how it looks like. You can see there is lots of edges, and it is not even at some point it is even becoming, it is not even touching each other and there are corners at various places. And as we move the plane, you will see that the graph has lots of problems.

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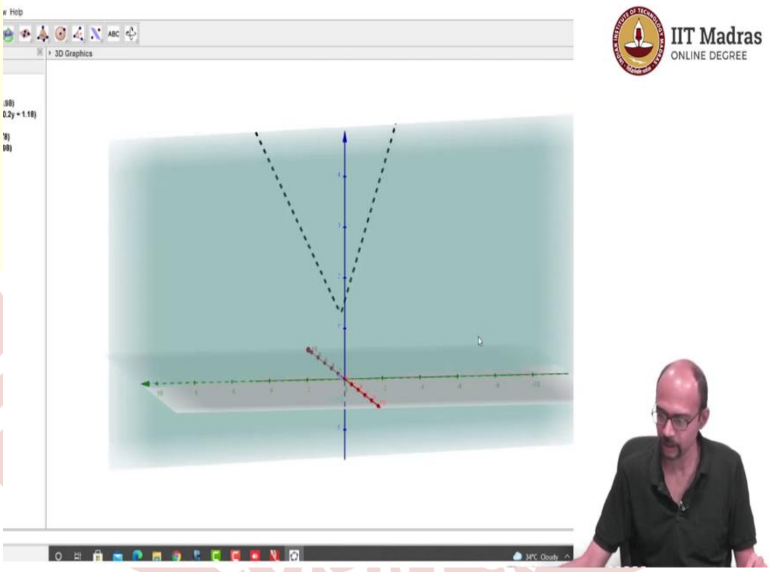

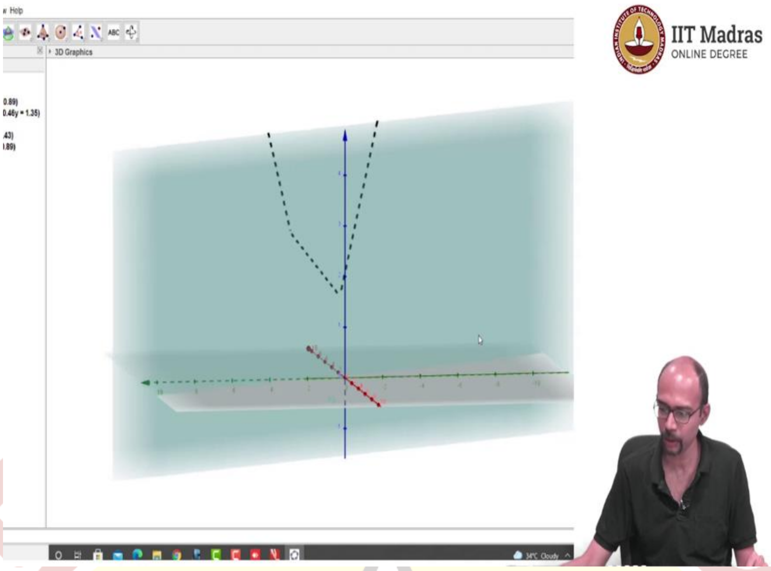
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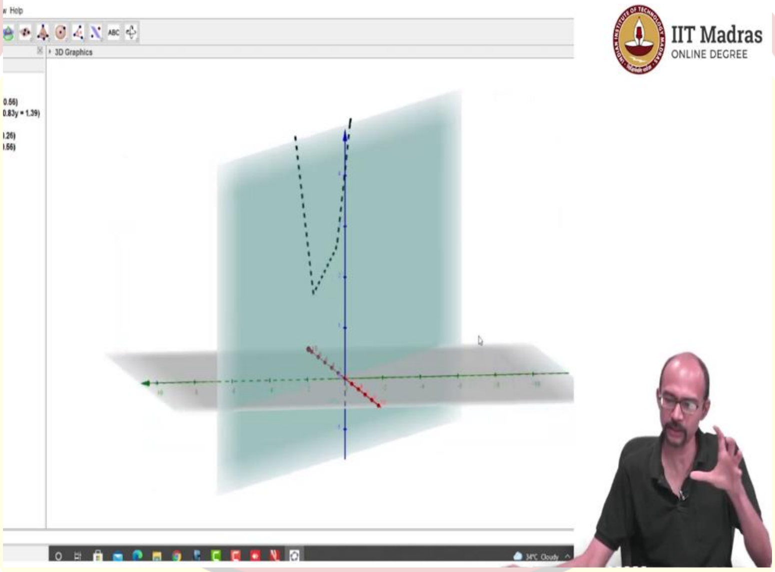
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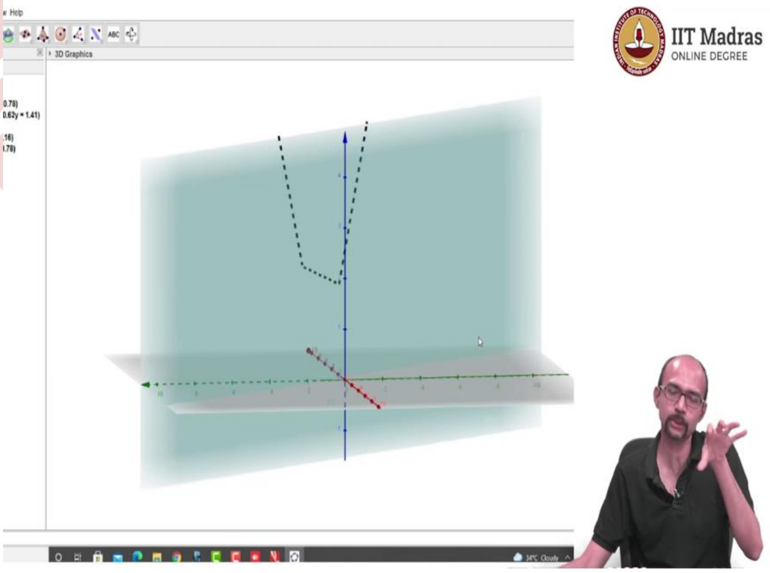
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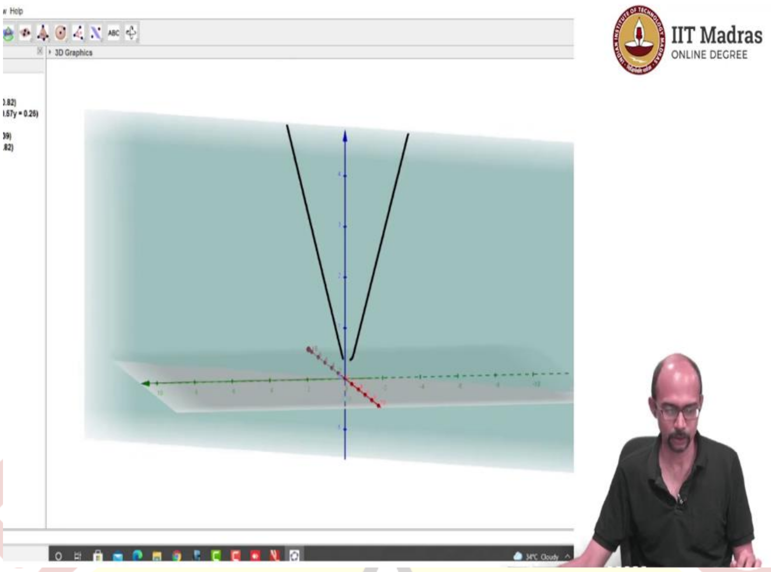

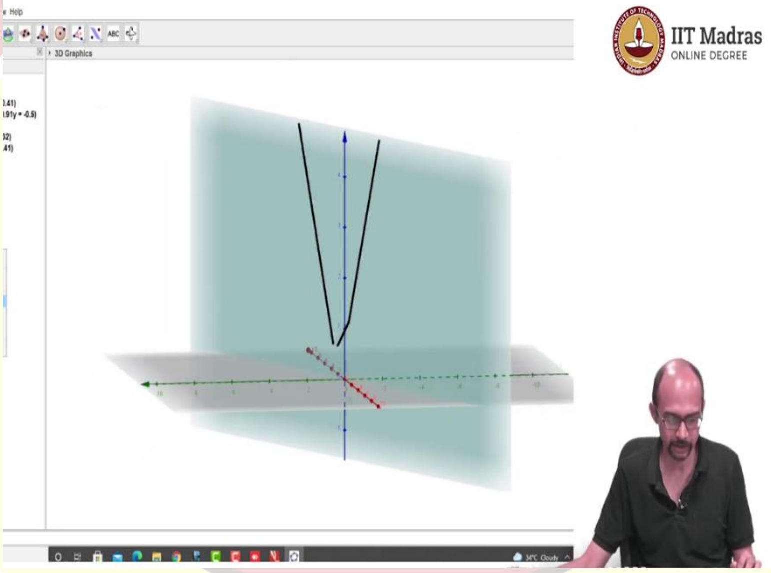

So, let us switch this animation on, and you can see, it looks very much like the $|x| + |y|$. See, it always has some jagged part. So, if you are at those points you will have trouble. So, I hope this animation convinces you that there is a lot of places where the tangent and there are a lot of directions that in which the tangent will not exist.

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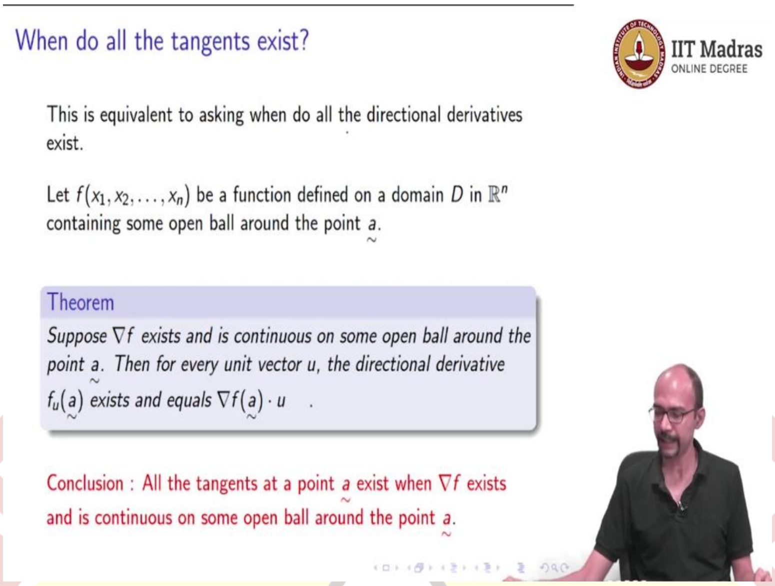

So, we can ask when do all the tangents exist? And so, this is equivalent to asking when do all the directional derivatives exist? And this is a question that we actually know the answer for. So, if $f(x_1, x_2, \ldots, x_n)$ is a function defined in our domain D in $\mathbb{R}^n$ containing some open ball around the point a tilde, then we have this wonderful theorem, that if the gradient exists and is continuous on some open ball around the point a tilde then for every unit vector u, the directional derivative $f_u(a\ tilde)$ exists and equals gradient $\nabla f(a\ tilde) \cdot u$. And this says that the directional derivative exists, and we have a formula to compute it, and that exactly means that the tangent exists in that direction, and that we can write down its equation, which we have done a few slides ago.

So, conclusion is that all the tangents at a point a tilde exist when ∇f exists and is continuous on some open ball around the point a tilde. And once that happens, we know how to write down the equation of the tangent line. Thank you.