Week-2
Mathematics for Data Science - 2
Limits, Continuity, Differentiability, and the derivative
Practice Assignment

1 Multiple Choice Questions (MCQ)

1. Match the given functions in Column A with the equation of their tangents at the origin $(0, 0)$ in column B and the plotted graphs and tangents in Column C, given in Table M2W2P1.

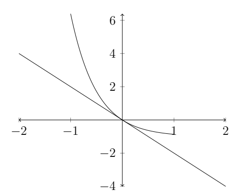
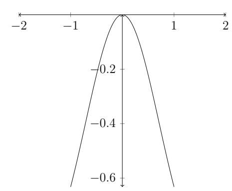
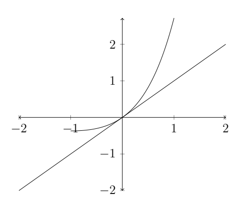
	Function (Column A)		It's tangent at (0,0) (Column B)		Graph (Column C)
i)	$f(x) = xe^x$	a)	$y = -2x$	1)	
ii)	$f(x) = e^{-2x} - 1$	b)	$y = x$	2)	
iii)	$f(x) = e^{-x^2} - 1$	c)	$y = 0$	3)	

Table: M2W2P1

- ◯ Option 1: i) → b) → 3, ii) → c) → 2), iii) → a) → 1.

- ◯ **Option 2:** i) $\to$ b) $\to$ 3, ii) $\to$ a) $\to$ 1), iii) $\to$ c) $\to$ 2.
- ◯ Option 3: i) $\to$ c) $\to$ 3, ii) $\to$ a) $\to$ 2), iii) $\to$ b) $\to$ 1.
- ◯ Option 4: i) $\to$ c) $\to$ 3, ii) $\to$ a) $\to$ 1), iii) $\to$ b) $\to$ 2.

Solution:

1. Given

$$f(x) = xe^x$$

$$f(1) = 1e^1 = e > 0$$

Only figure 3 has this property. Now differentiating the function,

$$f'(x) = 1e^x + xe^x$$

$$f'(0) = 1 + 0 = 1$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = 1$, then the equation of tangent

$$y = x$$

Which is b) in column B. Therefore, $i) \to b) \to 3$.

2. Given

$$f(x) = e^{-2x} - 1$$

$$f(-1) = e^2 - 1 > 0$$

Only figure 1 has this property. Now differentiating the function,

$$f'(x) = e^{-2x}(-2) = -2e^{-2x}$$

$$f'(0) = -2$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = -2$, then the equation of tangent

$$y = -2x$$

Which is a) in column B. Therefore, $ii) \to a) \to 1$.

3. Given

$$f(x) = e^{-x^2} - 1$$

$$f(-x) = e^{-x^2} - 1 = f(x)$$

The function is even and only figure 2 has this property. Now differentiating the function,

$$f'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$$

$$f'(0) = 0$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = 0$, then the equation of tangent

$$y = 0$$

Which is c) in column B. Therefore, $iii) \to c) \to 2$.

2 Multiple Select Questions (MSQ)

2. Consider the graphs given below:

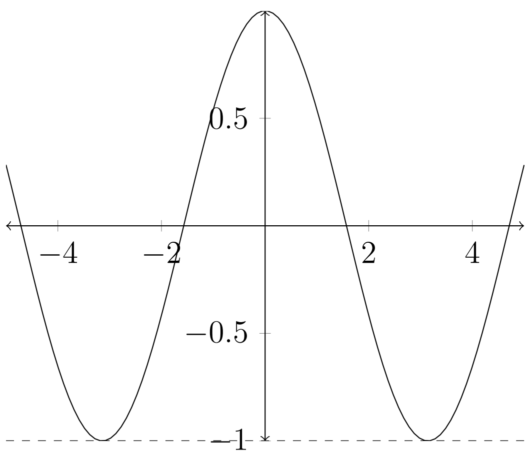


Figure: Curve 1

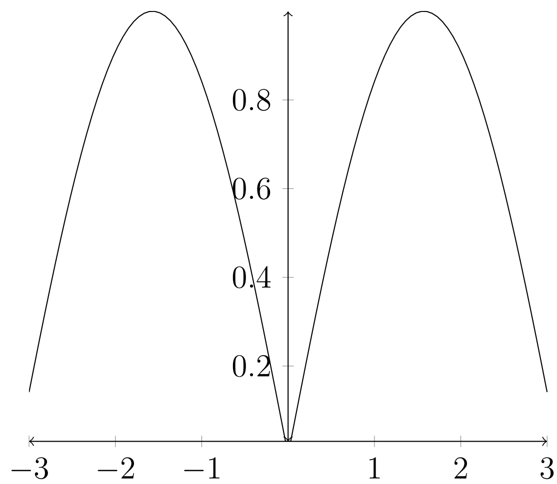


Figure: Curve 2

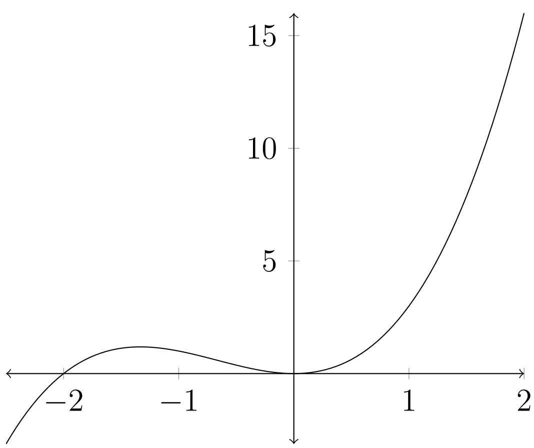


Figure: Curve 3

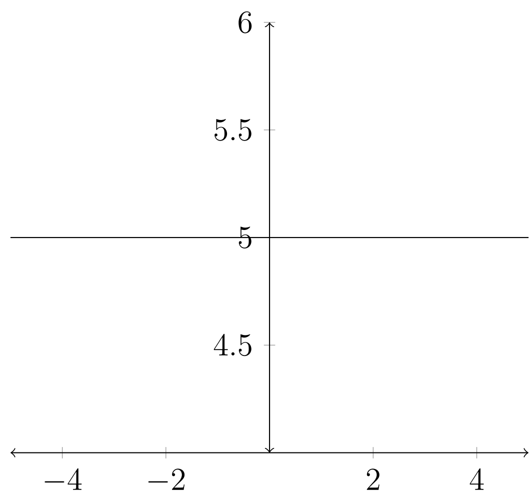


Figure: Curve 4

Choose the set of correct options:

- ◯ **Option 1:** There are at least two points on Curve 1, where the derivatives of the function corresponding to Curve 1, are equal.
- ◯ **Option 2:** At the origin the derivative of the function corresponding to Curve 2 does not exist.

◯ Option 3: The derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are the same.

◯ Option 4: The derivative of the function corresponding to Curve 4 does not exist at any point.

Solution:
Option 1: There are at least two points on Curve 1, where the derivatives of the function corresponding to Curve 1, are equal.

As it is shown in the figure, the straight line $y = -1$ is tangent at two point of the curve. So at those two points on Curve 1, the derivatives of the function corresponding to Curve 1, as slope of the tangents at those two points are the same.

Option 2: At the origin the derivative of the function corresponding to Curve 2 does not exist.

Curve 2 has a sharp corner at $x = 0$, which shows the derivative of the function corresponding to Curve 2 does not exist.That's why option 2 is correct.

Option 3: The derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are the same.

At origin the derivative of the function corresponding to Curve 3 is zero as the X-axis is the tangent of the curve at the origin. But at $x = -2$ the tangent is not parallel to the X-axis, hence the slope of the tangent at $x = -2$ must be different from 0. So the derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are not the same.

Option 4: The derivative of the function corresponding to Curve 4 does not exist at any point.

The function corresponding to Curve 4 is a constant function, therefore, the derivative of the function corresponding to Curve 4 always exists and is 0.

3. Let f be a function and the Figure M2W2P1 represent the graph of function f. The solid points denote the value of the function at the points, and the values denoted by the hollow points are not taken by the functions.

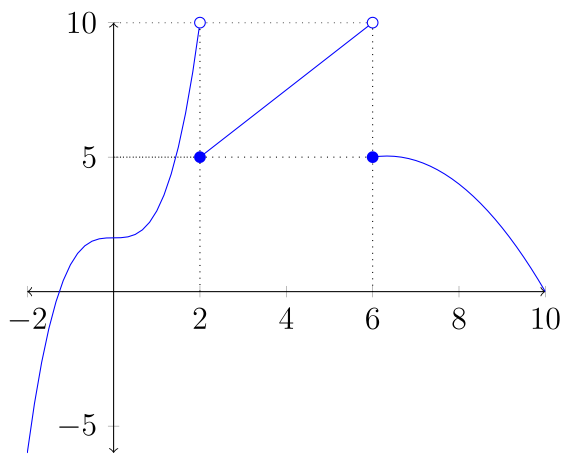


Figure: M2W2P1

Choose the set of correct options.

- ◯ **Option 1:** $\lim\limits_{t \to 2-} f(t) = 10$
- ◯ **Option 2:** $\lim\limits_{t \to 2+} f(t) = 5$
- ◯ **Option 3:** $\lim\limits_{t \to 6-} f(t) = 10$
- ◯ Option 4: $\lim\limits_{t \to 6+} f(t) = 10$
- ◯ Option 5: f is continuous at $x = 2$.
- ◯ **Option 6:** f is continuous at $x = 4$

Solution:

We can see that the curve is discontinuous at $t = 2$ and $t = 6$ only.

As t is approaching to 2 from left side, f is approaching to the value 10, which means the LHL (left hand limit) i.e., $t \to 2^-$ is 10.
As t is approaching to 2 from right side, f is approaching to the value 5, which means the RHL (right hand limit) i.e., $t \to 2^+$ is 5.
Hence limit at the function f does not exist at $t = 2$. So f is discontinuous at $t = 2$. Similar explanation can be given for $t = 6$.

4. Define a function f as follows:

$$f(x) = \begin{cases} x^3 & \text{if } x > 1, \\ x^2 & \text{if } 0 < x \leq 1 \\ x & \text{if } x < 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Choose the set of correct options.

- ◯ **Option 1:** f is continuous, but not differentiable at $x = 1$.
- ◯ Option 2: f is both continuous and differentiable at $x = 1$.
- ◯ **Option 3:** f is continuous, but not differentiable at $x = 0$.
- ◯ Option 4: f is both continuous and differentiable at $x = 0$.
- ◯ Option 5: f is not continuous at $x = 0$.
- ◯ Option 6: f is not continuous at $x = 1$.

Solution:

For $x = 1$:

Left Hand Limit:

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x^2 = 1$$

Right Hand Limit:

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x^3 = 1$$

Moreover $f(1) = 1$. Hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1$$

Therefore, the function is continuous at $x = 1$.

For differentiability at $x = 1$:

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^3 - 1}{h} = \lim_{h \to 0^+} \frac{(1 + h^3 + 3h^2 + 3h) - 1}{h}$$

$$\lim_{h \to 0^+} \frac{h^3 + 3h^2 + 3h}{h} = \lim_{h \to 0^+} (h^2 + 3h + 3) = 3$$

Similarly,

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^+} \frac{(1 + h^2 + 2h) - 1}{h}$$

$$\lim_{h\to 0^+} \frac{h^2+2h}{h} = \lim_{h\to 0^+} (h+2) = 2$$

Hence,

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \neq \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h}$$

So $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h}$ does not exist. Therefore the function f is not differentiable at $x=1$. Similar argument can be given for $x=0$.

5. Let f and g be two real valued functions defined as:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = e^x - 1$$

$$g : \mathbb{R} \to \mathbb{R}$$

$$g(x) = x$$

Choose the set of correct options.

- ◯ **Option 1:** The linear function $ex - 1$ is the best linear approximation of the function $f(x)$ at the point $x = 1$.
- ◯ Option 2: In this case, $\lim\limits_{x \to 0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to 0} f(x)}{\lim\limits_{x \to 0} g(x)}$.
- ◯ **Option 3:** In this case, $(f + g)$ (where, $(f + g)(x)$ is defined by $f(x) + g(x)$) is continuous at $x = 0$.
- ◯ **Option 4:** $\lim\limits_{x \to 0} f(x)g(x) = 0$.

Solution:

Given,

$$f(x) = e^x - 1$$

The linear approximation for this function at $x = 1$ would be

$$y = f'(1)(x - 1) + f(1) = e^1(x - 1) + e^1 - 1 = ex - 1$$

In option 2 given,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to 0} f(x)}{\lim\limits_{x \to 0} g(x)}$$

LHS:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{e^x - 1}{x}$$

This is 0 divided by 0 case, therefore, we can use L'Hôpital's rule,

$$LHS = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

Now

$$RHS = \frac{\lim\limits_{x \to 0} f(x)}{\lim\limits_{x \to 0} g(x)}$$

$RHS = \frac{\lim\limits_{x\to0} e^x - 1}{\lim\limits_{x\to0} x}$ is in indeterminate form, as both the numerator and denominator are 0.
Therefore,

$$LHS \neq RHS$$

In option 3,

$$(f+g)(x) = e^x - 1 + x$$

$$LHL = \lim_{x\to0^-}(f+g)(x) = \lim_{x\to0^-}(e^x - 1 + x) = 0$$

$$RHL = \lim_{x\to0^+}(f+g)(x) = \lim_{x\to0^+}(e^x - 1 + x) = 0$$

$$(f+g)(0) = 0$$

Therefore, $(f+g)(x)$ is continuous at $x = 0$.

In option 4,

$$\lim_{x\to0} f(x)g(x) = \lim_{x\to0}(e^x - 1)(x) = 0$$

3 Numerical Answer Type (NAT)

6. Let f be a differentiable function at $x = 0$. The tangent line to the curve represented by the function f at the point $(0, 5)$ passes through the point $(1, 5)$. What will be the value of $f'(0)$? [Answer: 0]

   Solution:

   Slope of the tangent line (as the line passes through (0,5) and (1,5)):

$$f'(0) = \frac{5-5}{1-0} = 0$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x - 5$. Find the value of $(fg)'(0) - (f \circ g)'(0)$, where $f \circ g(x) = f(g(x))$ and $fg(x) = f(x)g(x)$. [Answer: 10]

Solution:

Take,

$$(fg)(x) = f(x)g(x) = x^2(x-5) = x^3 - 5x^2$$
$$(fg)'(x) = 3x^2 - 10x$$
$$(fg)'(0) = 0$$

Now,

$$(f \circ g)(x) = (x-5)^2$$
$$(f \circ g)'(x) = 2(x-5)(1)$$
$$(f \circ g)'(0) = 2(0-5) = -10$$

Therefore,

$$(fg)'(0) - (f \circ g)'(0) = 10$$

4 Comprehension Type Question:

The profit of Company A with respect to time (in months) is given by the function $f(t)$ (in lakhs) as follows:

$$f(t) = \begin{cases} \frac{(t-2)^n-1}{t-3} & \text{if } 0 \leq t < 3, \\ \lfloor t \rfloor & \text{if } t \geq 3 \end{cases}$$

for some integer n.

The profit of Company B with respect to time (in months) is given by the function $g(t)$ (in lakhs) as follows:

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^m & \text{if } t \geq 3 \end{cases}$$

for some integer m.

Use the information given above answer Questions 8,9 and 10.

8. If the functions $f(t)$ and $g(t)$ denoting the profits of Company A and Company B, respectively, are known to be continuous at $t = 3$, then what will be the values of n and m? (MCQ)

   - ○ Option 1: $n = 2$, and $m = 2$
   - ○ Option 2: $n = 2$, and $m = 3$
   - ○ **Option 3:** $n = 3$, and $m = 2$
   - ○ Option 4: $n = 3$, and $m = 3$

   Solution:

   Given,

   $$f(t) = \begin{cases} \frac{(t-2)^n-1}{t-3} & \text{if } 0 \leq t < 3, \\ \lfloor t \rfloor & \text{if } t \geq 3 \end{cases}$$

   For $f(t)$ to be continuous at $t = 3$,

   $$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^+} f(t) = f(3)$$

   $$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} \frac{(t-2)^n - 1}{t-3}$$

   Using L'Hospital rule we get,

   $$\lim_{t \to 3^-} \frac{n(t-2)^{(n-1)}}{1} = n$$

   Further we have,

   $$\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} \lfloor t \rfloor = 3$$

Hence we have $n = 3 = f(3)$.

Now,

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^m & \text{if } t \geq 3 \end{cases}$$

For $g(t)$ to be continuous at $t = 3$,

$$\lim_{t \to 3^-} g(t) = \lim_{t \to 3^+} g(t) = g(3)$$

$$\lim_{t \to 3^-} g(t) = \lim_{t \to 3^-} \frac{t^3 - 3^3}{t - 3}$$

Using L'Hospital rule we get,

$$\lim_{t \to 3^-} \frac{3t^2}{1} = 27$$

Further we have,

$$\lim_{t \to 3^+} g(t) = \lim_{t \to 3^+} 3t^m = 3^{(m+1)}$$

Again we have $f(3) = 3^{(m+1)}$.
Therefore, $3^{(m+1)} = 27 \implies m + 1 = 3 \implies m = 2$.

9. Assuming g to be continuous at $t = 3$, choose the correct option from the following. (MCQ)

   - ○ Option 1: $\lim\limits_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 18$ and $\lim\limits_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 18$, hence g is differentiable at $t = 3$.
   - ○ Option 2: $\lim\limits_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 9$ and $\lim\limits_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 9$, hence g is differentiable at $t = 3$.
   - ○ Option 3: $\lim\limits_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 18$ and $\lim\limits_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 9$, hence g is not differentiable at $t = 3$.
   - ○ **Option 4:** $\lim\limits_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 9$ and $\lim\limits_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 18$, hence g is not differentiable at $t = 3$.

Solution:

As g is continuous at $t = 3$, we have

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^2 & \text{if } t \geq 3 \end{cases}$$

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{\frac{t^3-3^3}{t-3} - (3 \times 3^2)}{t-3}$$

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{t^3 - 27 - 27t + 81}{(t-3)^2}$$

Using L'Hospital rule two times consecutively,

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{3t^2-27}{2(t-3)} = \lim_{t\to 3^-} \frac{6t}{2} = 9$$

Similarly we can calculate,

$$\lim_{t\to 3^+} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^+} \frac{3t^2-27}{t-3} = \lim_{t\to 3^+} \frac{6t}{1} = 18$$

As,

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} \neq \lim_{t\to 3^+} \frac{g(t)-g(3)}{t-3},$$

g is not differentiable at $t = 3$.

10. Which of the following linear functions denotes the best linear approximation $L_f(t)$ of the function $f(t)$ at the point $t = 1$, assuming f to be continuous at $t = 3$? (MCQ)

- ◯ **Option 1:** $L_f(t) = 2 - t$
- ◯ Option 2: $L_f(t) = -t$
- ◯ Option 3: $L_f(t) = 2 + t$
- ◯ Option 4: $L_f(t) = -2 - t$

Solution:

For $t = 1 < 3$ and continuous at $t = 3$ means $n = 3$, then

$$f(t) = \frac{(t-2)^3 - 1}{t - 3}$$

$$L_f(t) = f'(1)(t-1) + f(1)$$

$$f'(t) = \frac{3(t-2)^2(t-3) - ((t-2)^3 - 1)}{(t-3)^2}$$

$$f'(1) = \frac{-6 - (-2)}{4} = -1$$

And

$$f(1) = 1$$

Therefore,

$$L_f(t) = -1(t-1) + f(1)$$

$$L_f(t) = 2 - t$$