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IIT Madras
ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 02
Matrices

Hello and welcome to the maths 2 component of the online BSC course on Data Science. In today's video, in this video we want to talk about matrices and as you can see up here. Now matrices, you may heard of matrices in a completely different context, namely there is this movie called the matrix and its sequence which you may have seen in one of these movie channels.

So, that has a much more difficult and intricate plot line and the use of the matrix there is much much much more in world. We will see that in actuality the matrices that we need, matrices is plural form matrix are much easier and how they deal with vectors and in linear algebra which we are studying now.

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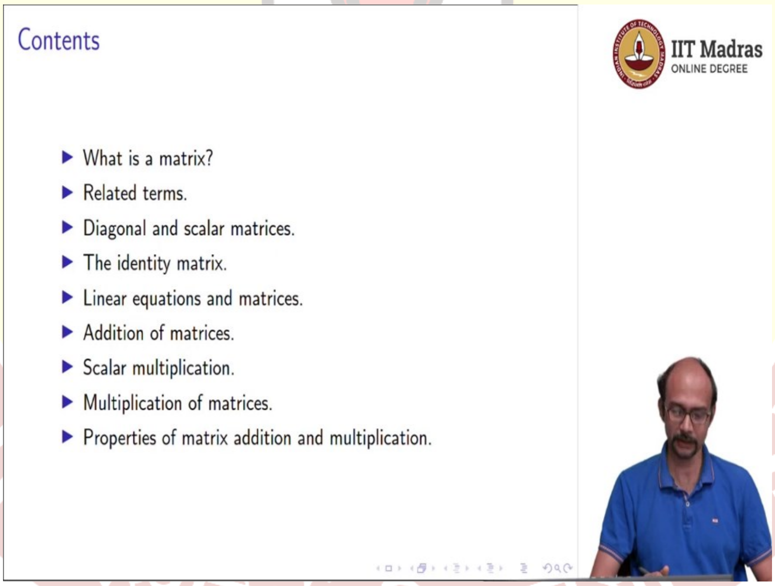

So, today's in this video we are going to study what is a matrix, we will study some related terms, we study diagonal and scalar matrices, we look at the identity matrix, we will talk about linear equations and matrices. So, this is the relationship with linear equations. This is really one of the main reason why we study matrices.

We look at the algebra of matrices. So, things like the addition of matrices, scalar multiplication, the multiplication of matrices, somewhat special. And then finally what properties do these may a matrices have with respect to addition and multiplication. So, let us start with what is a matrix.

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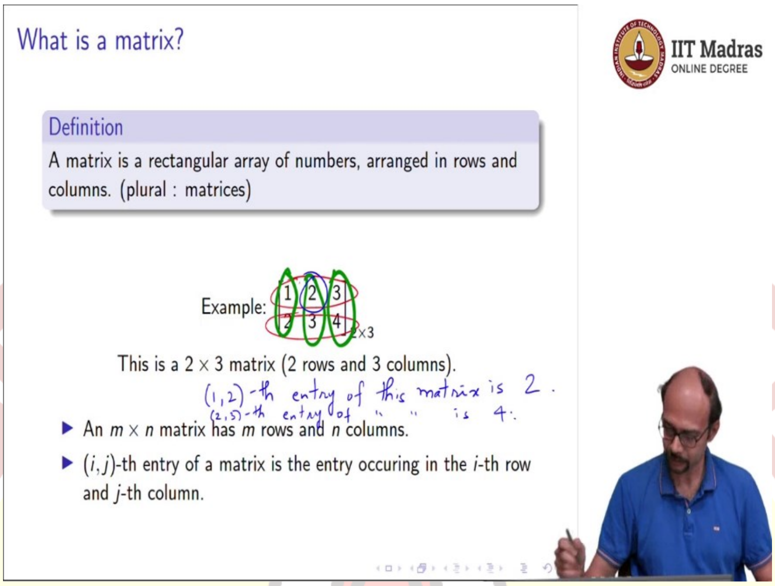

So, a matrix is really something very easy. It is a rectangular array of numbers arranged in rows and columns. So, it is like a table, very much a rectangle table which you are familiar with. And inside the table in various places you have numbers. So, here is an example. So, the example has the numbers 1, 2, 3 and 2, 3, 4 arranged in an array, rectangular array.

So, in your table if you want so, this is what is called the 2 by 3 matrix. So, the two refers to the first, so 2, how many rows there are in this matrix and the 3 refers to how many columns there are in this matrix. So, the 2 by 3 is written to the right of this matrix slightly below it, so 2 rows and 3 columns. So, in general, you may have an m *by* n matrix.

So, that means you have m rows in this matrix and n columns in this matrix. And we will talk about the entries of this matrix. So, the ijth entry of this matrix refers to the entry why is in the ith row and the jth column. So, there is a unique such entry and that will be the ijth entry. So, let us look at the example that we have. Here, I will look at the entries in this matrix. So, first of all this has two rows, how do the rows look like?

Here is the first row and here is the second row and it has 3 columns, so here is the first column, the second column and the third column. And now suppose I want to ask, what is the 1, 2th entry of this matrix? So, the one, two eth entry of this matrix, so it will be 1, 2th entry of this matrix means you look at first row, that is the first row over here and you look at the second column. So, that is, the first row is 1, 2, 3 and the second column is the 2, 3.

So, the entry that you have over there is 2. So, I will encourage you to think about what is the 2, 3th entry. So, the 2 3th entry of this matrix is, so by now you have probably figured out, you look at the second row and the third column. So, this is 4. So, I hope you understand what it means for a matrix to be an m by n matrix that means it has m rows and n columns and what it means for an entry to be the ij-th entry. So, let us move on.

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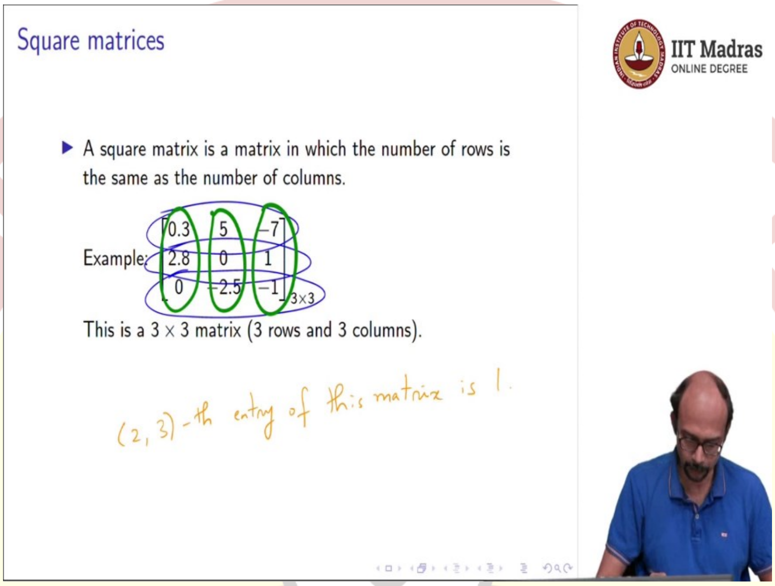

And look at some special kinds of matrices. So, as we saw matrices are rectangular arrays if numbers, so special shape within rectangles is the square. So, let us talk about square matrices. So, square matrix is a matrix in which the number of rows is the same as the number of columns. So, here is an example. So, here you have 3 rows and 3 columns.

So, just to jog your memory, here is the first row, here is the second row, here is the third row and about the columns, here is the first column, here is the second column and here is the third column. So, again just as a test, if you want to ask what is 2, 3th entry of this matrix, so the 2, 3th entry of this matrix, so you look at the second row and the third column that is 1. So, I hope you are comfortable with the notations.

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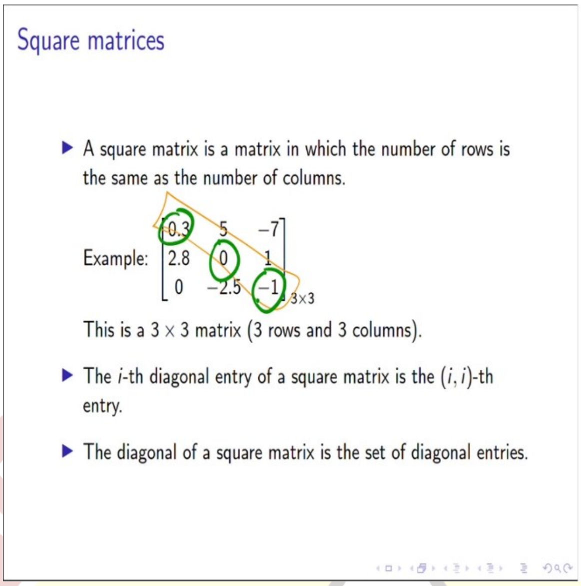

So, the ith diagonal entry of a square matrix is the i-th entry. So, the, so when you have a square, there is something called the diagonal of the square and we look at the ith entry on the diagonal and that is the i ith entry of the matrix. So, the diagonal of the square matrix is the set of diagonal matrix.

So, here is the set of diagonal entries and within this, this is the first diagonal entry. So, this is the second diagonal entry and this is the third diagonal entry. So, the diagonal entries are somewhat special because they occupy the i-th position. So, we generally talk about diagonals only for square matrices.

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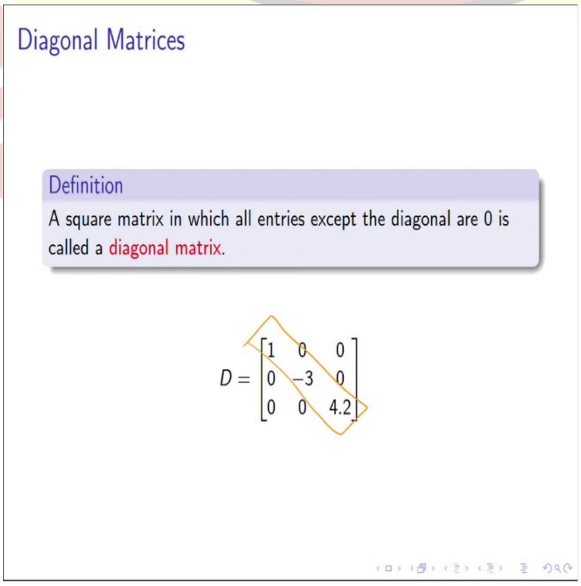

So, now some special further specialization that this square matrices, so diagonal matrix is one where the only non-zero entries are in the diagonal, everything else is 0. So, such a thing is called a diagonal matrix. Here is an example. So, you can see that the diagonal entry is here, so that is 1,-3 and 4.2. So, the first diagonal entry is 1, the second diagonal entry is-3 and the third diagonal entry is 4.2. So, everything else is 0. So, this is the diagonal matrix.

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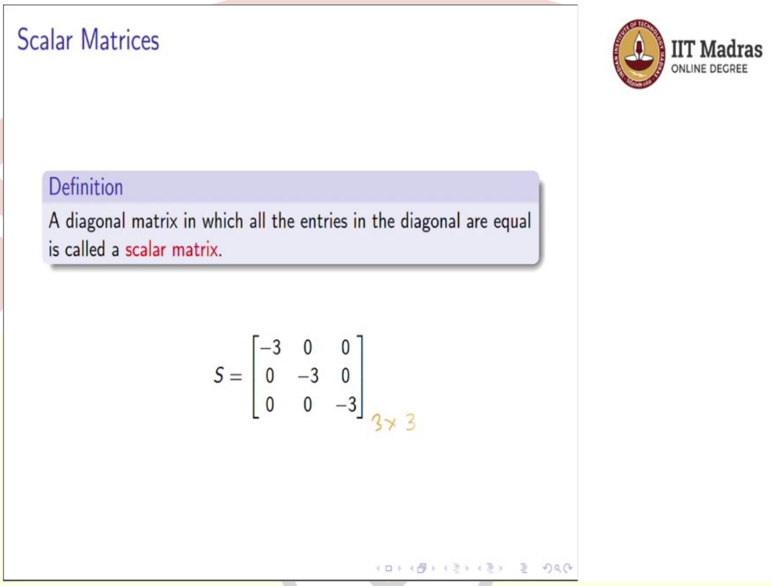

So, then an even more special case of this is what is called a scalar matrix. So, diagonal and matrix in which all the entries in the diagonal are the same, have the same value. So, it is the same number. So, this is called a scalar matrix. So, here is an example. So, here you can see that every entry here is-3, so the first entry is -3, the second entry is -3 and the third entry is -3.

They are all equal. So, this is an example of a scalar matrix. Why is this a scalar matrix? So, we will see later that this behaves like -3, this matrix in our mind should, we should think of it as-3. Again just to jog your memory, what is the order of this matrix, meaning what are the number of rows and columns? This is a 3 by 3 matrix.

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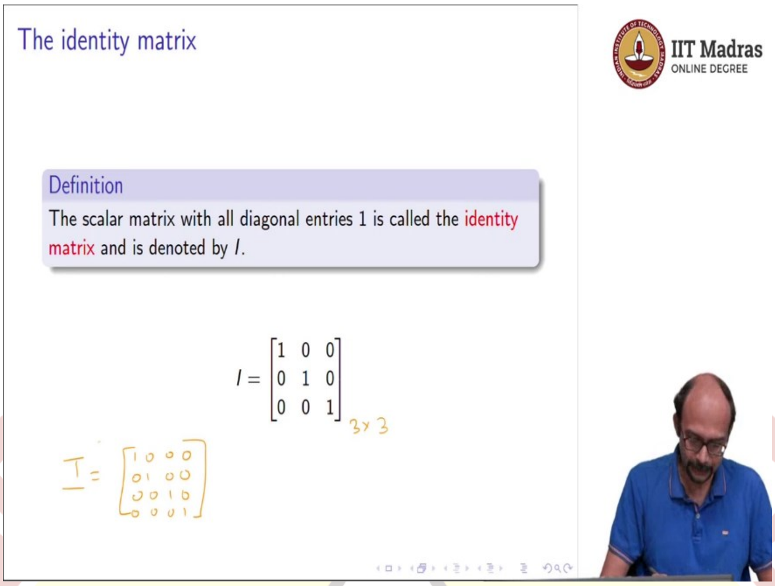

And even more special case is where the scale, you have a scalar matrix, but in the diagonal position, the entries are all 1. So, this is called the identity matrix, this is something very important and it has a special notation, it is denoted by I. So, this is capital I. So, here is what I is, so this is $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. So, again I will remind you this is 3 by 3.

So, I am not saying what the size of the matrix is, so if we are talking about a 4 by 4 matrix, then that will also be denoted by I and we will have $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$. So, it depends on context as to which I we are referring to. So, in the given context, it will be clear what we mean by I, meaning how many rows and columns it has.

But however many rows and columns it has, which of course, is the same number, the diagonal must consist of ones. So, the only choice is how many rows and columns it has, which will be clear from the context. So, these are the special kinds of matrices we wanted to look at.

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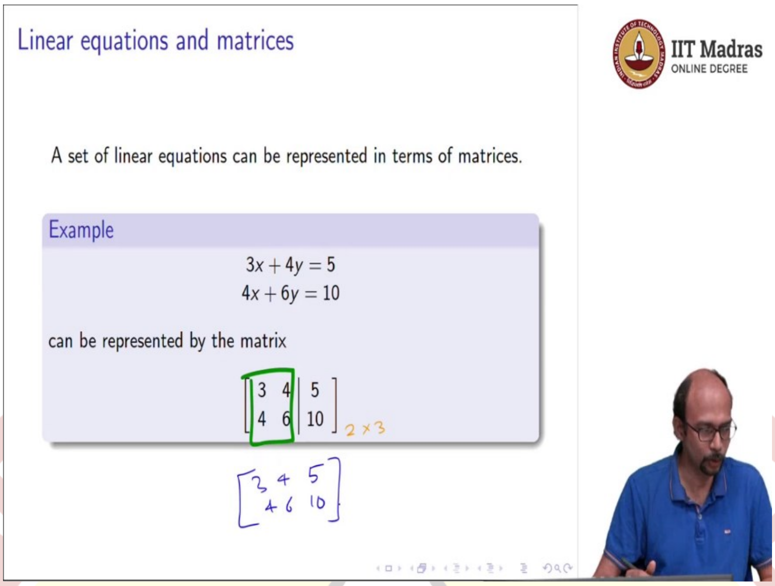

So, now let us look at the use of matrices. So, suppose we have a system of linear equations. So, we will study more about this in the forthcoming video. So, you have a set of linear equations. So, what does that mean? That means you have something like this, 3x+4y is 5 and 4x+6y is 10. So, we can think of this as in terms of matrices.

So, how is this being done? Again this will be explained in much more detail in the forthcoming video. So, the corresponding matrix is this matrix which has 2 rows and 3 columns. So, I will emphasize the 3 columns here. So, the two, the first two columns correspond to the coefficients of x and y.

So, 3, 4 and 4, 6 that is the part that is coming over here, that is this part and then the entries that are on the right beyond the equals that is on the last column, those are in the last column. So, this middle line is just for our own sake. We do not need to put it, so as a matrix this is the same matrix as what we have without the line.

So, this is $\begin{bmatrix} 3 & 4 & 5 \\ 4 & 6 & 10 \end{bmatrix}$. But we later see that putting that line helps us. So, for now this is the corresponding matrix. So, but we will see how the line helps us later on. So, that is how it is related to linear equations and this is really the main importance of matrices. So, now let us move on to how do we add matrices.

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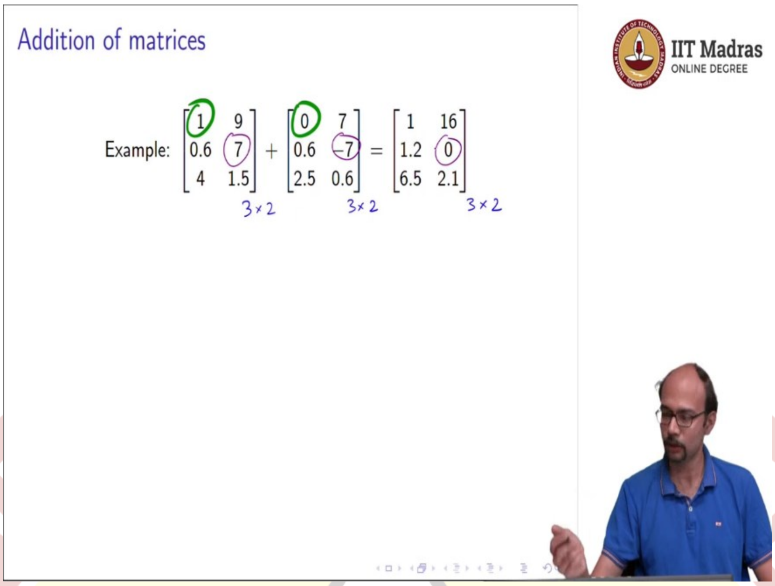

So, here is an example. So, we will add matrices by adding corresponding entries. So, the important part here is that, when we add two matrices, they must be of the same size. That means they have the same number of rows and the same number of columns. Both matrices have the same number of rows and both matrices have the same number of columns.

So, for example over here, this matrix has 3 rows and 2 columns, this matrix also has 3 rows and 2 columns, and hence we can add this. And the resulting matrix also has 3 rows and 2 columns. So, how do we add this? So, the addition the way it works is, we take 1 and add it to 0, so here is 1 and 0, so we add these and in the 1 1th place we get 1 again. We take maybe let us look at the 2, 2th entry. So, we have a 7 and over here we have a-7, so we add these and here we get a 0. So, you add corresponding entries. So, I hope this is clear.

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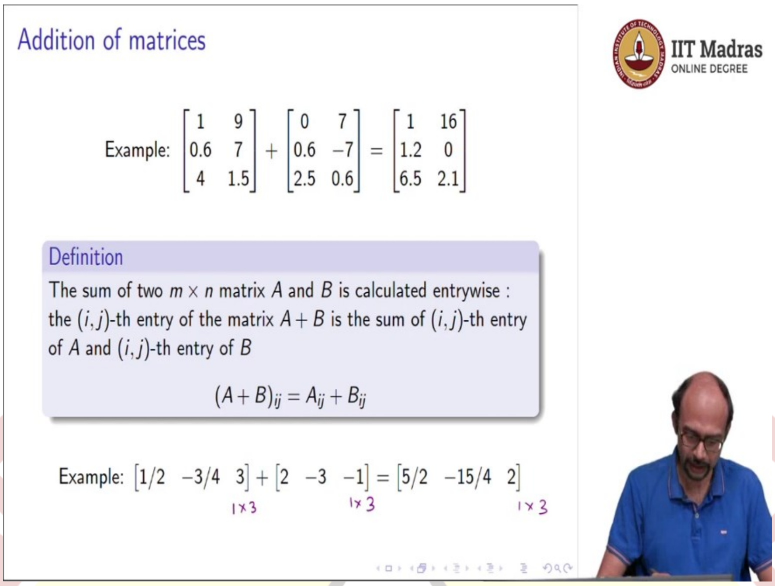

So, let us at this in a more formal way. So, the sum of two m by n matrices, apologies, this should be matrices. So, the sum of two m by n matrices, A and B is calculated entry wise. So, corresponding entries are added, that means you look at the i, jth entry of A and the i, jth entry of B and then you add those to get the i, j-th entry of A+B.

So, that is the formula; $A+B_{ij}$ that means the i jth entry of A+B that is what is in the left that is the same as the i jth entry of A+ the i jth entry of B. So, maybe one more example. So, suppose we have these two matrices, so this is, this has 1 row and 3 columns and this has 1 row and 3 columns.

And so the resulting matrix also has 1 row and 3 columns. And how do we add these? We add these entry wise. So, the first entry that is the 1 1th entry, we have half+2, so we get 5 halves. For the second entry we have-three fourth-3, so we get-15 by 4 and for the third entry we have 3-1 which is 2. So, I hope it is, so the addition of matrices is fairly easy. I hope you are getting the hang of this.

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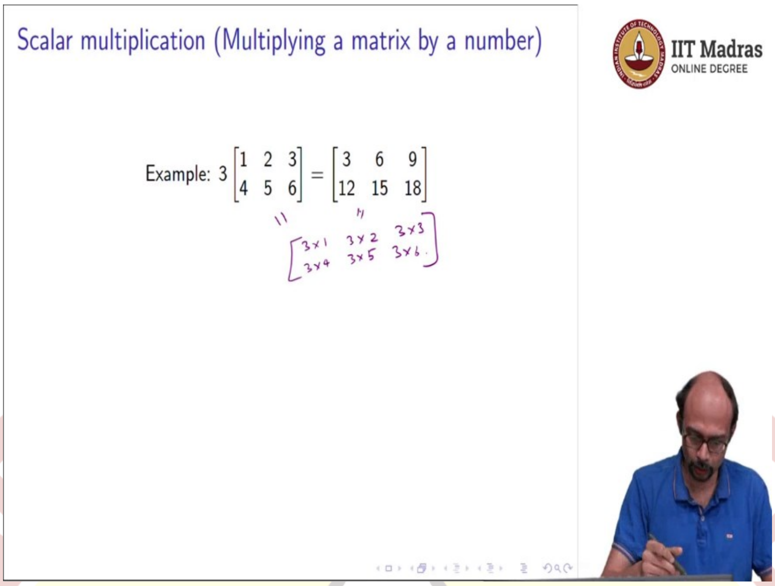

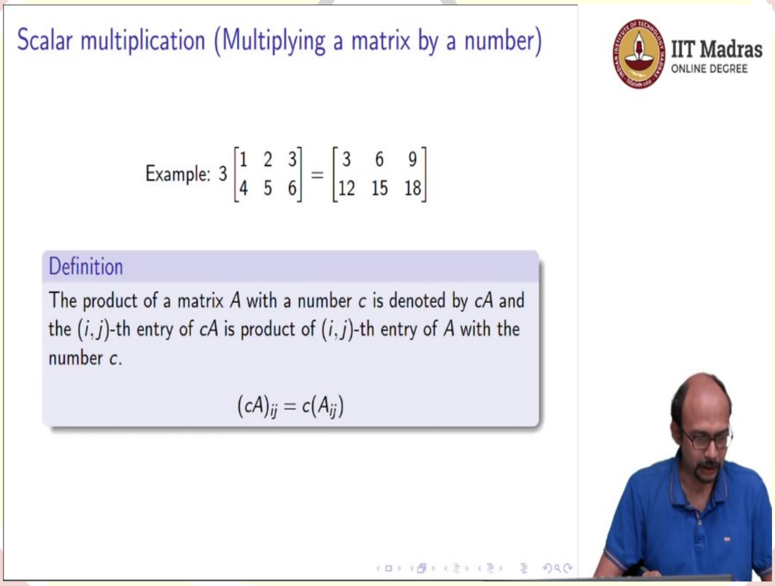

So, the next thing we want to look at is scalar multiplication. So, how do we multiply a matrix by a number? So, to multiply a matrix by a number, we multiply each entry of that matrix by that number. So, here we have 3 × the matrix $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$. 1, 2, 3 is in the first row; 4, 5, 6 is in the second row. So, what do we do?

We do this by multiplying each entry. So, this is 3 × 1, 3 × 2, 3 × 3 and then 3 × 4, 3 × 5, and 3 × 6, bracket complete and which is exactly what we get as the matrix 3 × 6 × 9; 3, 6, 9 in the first row and 12, 15, 18 in the second row.

So, let us make this formal. So, the product of a matrix A with a number c, so this is denoted by c × A and the i, jth entry of the product is c × the i, jth entry of A. So, c × A i, jth entry is c × A_{ij}. Then let us move on to matrix multiplication.

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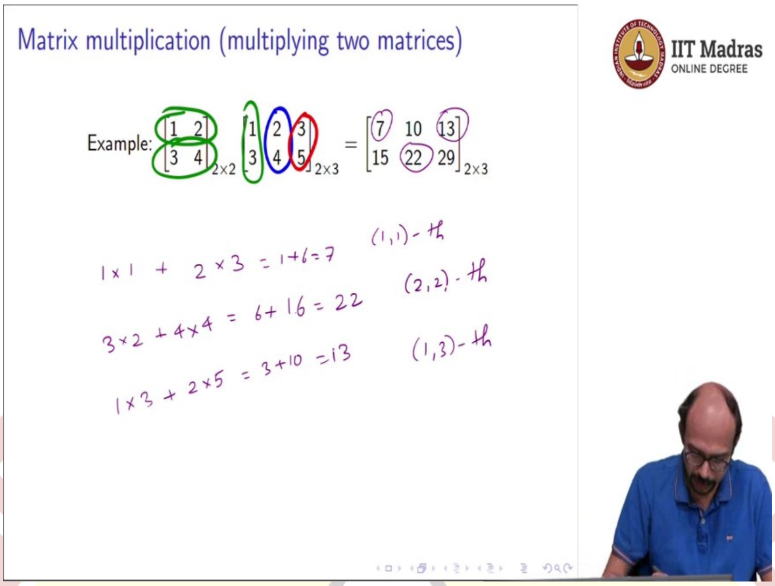

So, this is really the relatively harder thing to do. So, let us look at how to multiply matrices. So, here is an example. So, let me explain that example before we see how this equality comes about. So, let us explain this 7, how did I get 7? So, in order to get 7, so 7 is the which entry of this matrix is the 1, 1th entry of this matrix.

So, for the 1, 1th entry I look at the first row of the first matrix and then I look at the corresponding entries of the first column of the second matrix. So, the first 1 and 2 came from here and the second 1 and 3 came from here. So, how do I, so what is the sum? This is 1+6 which is indeed 7. So, that is how I got that 7.

Suppose I want to know what is the, how did I get this 22, let us ask for 22. So, first let us ask, 22 is which position? So, 22 in the is in the 2, 2th position. So, in order to get the 2, 2th positions, I should look at the second row and then I should look at the second column and then I should multiply corresponding entries and add them.

So, I get $3 \times 2+4 \times 4$. So, this is 6+24, 6+16 which is indeed 22. So, I get 6+16 which is indeed 22. So, let us maybe do one more entry. Let us look at the 1, 3th entry, 13, how do I get this entry. So, 13 is the 1, 3th entry. So, for the 1, 3th entry, I take the first row which is 1, 2 and I take the third column which is 3, 5.

So, I get $1 \times 3+2 \times 5$ which is 3+10 which is 13. So, I hope the process is clear of how do we get these entries in this matrix. So, I will encourage you to find the other entries, how do we get these 15, 10 and 29.

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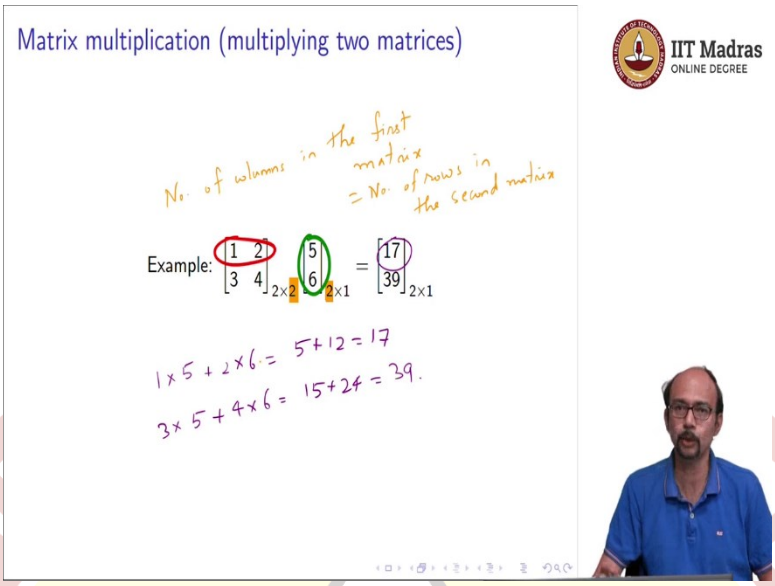

Maybe let us do one more example. So, here is another example, so we have the matrix 1, 2, 3, 4 × the matrix 5, 6 in columns, in a single column. So, I want to look at how did we get 17 and 39. So, for the 17, what do we do? It is a 1, 1th entry, so I look at the first row and I look at the first column.

So, of course, here is here we have only one column, so we have 1 time multiply the corresponding entries and add them, + 2 × 6, this is 5+12 and which is 17. So, for the 39, what do I do? I do 3 × 5+4 × 6 which is 15+24, namely this is 39. So, I hope it is clear how we are getting these entries.

So, now he do I abstractly define matrix multiplication. So, if I have two matrices, A and B, how do I multiply them? So, note here one of the important parts. Here, as in the previous example, the matrix sizes were very important. So, we had a 2 by 2 matrix, we multiply it to a 2 by 1 matrix. So, what was important about this?

The most important part was that these two things were the same and the 2 here. So, the number of columns in the first matrix must be the same as the number of rows in the second matrix. So, the number of columns in the first matrix must equal the number of rows in the second matrix. Why do we need that? Otherwise, there will be a mismatch. So, here I have a 1 and 2 that is a number of, so how many entries do I have?

I have 2 entries here that is corresponded to the number of columns. And how many entries do I have here? 5 and the 5 and 6, there are 2 entries which corresponds to the number of rows. If I had one more entry, then I would not know where to multiply it by. So, in order for this to make sense, the number of rows of the first matrix, the number of columns of the first matrix must be same as the number of rows of the second matrix.

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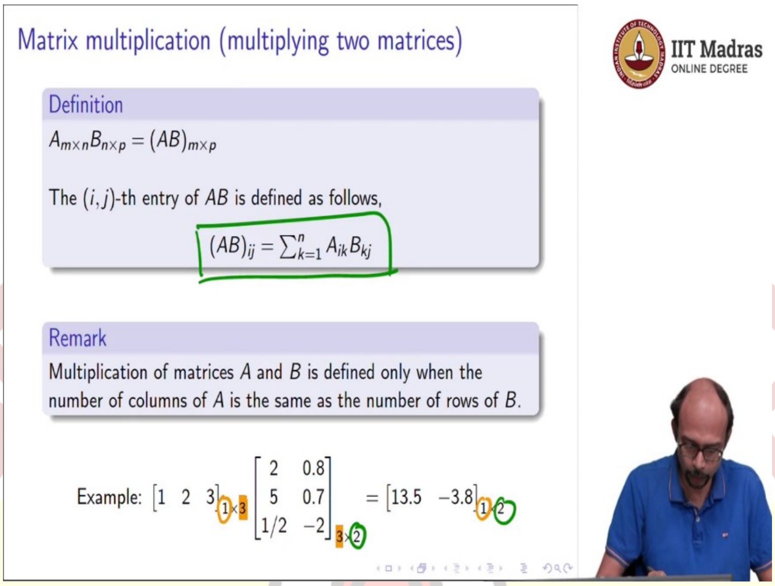

So, let us, with this caveat let us define matrix multiplication. So, here you have two matrices, the first has size m by n and the second one has size n by p, then the product has size m by p. So, this m is the number of rows of the first matrix, and the p is the number of columns of the second matrix. And how do we define the ij-th entry of A B?

So, for the ij-th entry you look at the ith row of A, look at the corresponding entries of the jth column of B, multiply those and add them up. So, abstractly this is summation $\sum_{k=1}^{n} A_{ik}B_{kj}$. So, if you find this formula cumbersome, go back to the examples, you will see it exactly matches with what we have done.

Now, the important remark which I want to reiterate it, I said this before, but I want to reiterate this, the multiplication of matrices A and B is defined only when the number of columns of A is the same as the number of rows of B. Very important, otherwise we do not defined multiplication of matrices. So, here is another example.

So, $[1 \quad 2 \quad 3] \times \begin{bmatrix} 2 & 0.8 \\ 5 & 0.7 \\ 1/2 & -2 \end{bmatrix} = [13.5 \quad -3.8]$. So, 1 row and 2 columns. Where did I get 1 row and 2 columns from? That is because over here, so first of all is it defined, can I multiply these at all? Yes. Because the number of columns of the first one is the same as the number of rows of the second one and then in the resulting matrix what was my, what was the size? So, this 1 is what came here and this 2 is what came here?

So, the resulting matrix has size 1 by 2, so 1 row and 2 columns. And how do we get the first entry? So, to get the first entry, we do 1 × 2+2 × 5+3 × half. To get the second entry, we do 1 × 0.8+2 × 0.7+3 ×-2 and this is what we will get. So, I will encourage you to check that what I just said is exactly this formula over here. So, that is matrix multiplication. So, let us look at a few special cases of matrix multiplication.

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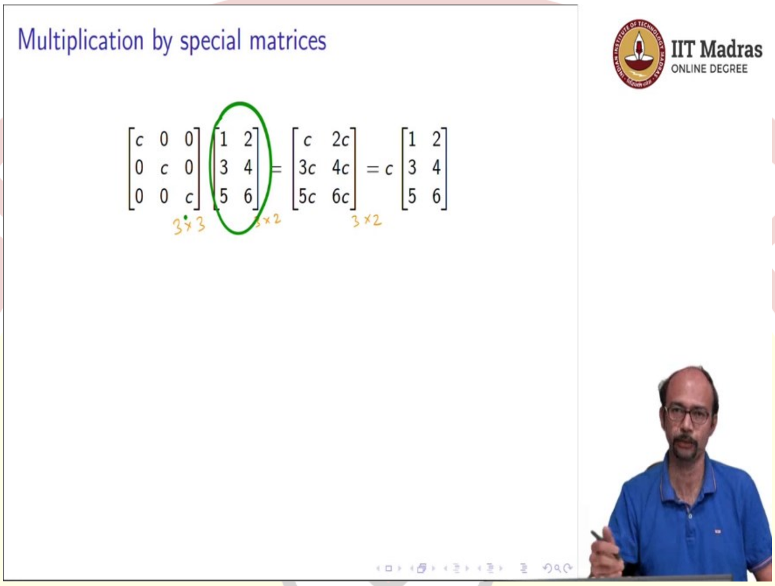

So, remember that we had scalar matrices, so scalar matrices were diagonal matrices in which all the diagonal terms were equal. So, they look like the matrix on the left, c 0 0, 0 c 0, 0 0 c. So, let us do this multiplication. So, the first term, so first of all is it well defined, so how many rows and columns does this have, this is a 3 by 3 matrix. How many rows does this have? 3. And how many columns does it have? 2. So, this is a 3 by 2 matrix.

So, indeed we can multiply this because the number of column here is 3 which is the same as the number of rows here. And when you multiply these, you will see that the first entry for example the 1, 1-th entry of this is c × 1+0 × 3+0 × 5. So, it is just c × 1 which is c. The 1, 2-th entry is c × 2+0 × 4+0 × 6 which is c × 2. So, 2 × c.

So, you can see what is happening. Each entry of this second matrix, each entry over here is getting multiplied by c. So, I can pull that c out and think of this as scalar multiplication by the constant c, multiplication by the scalar c. So, that is what special about scalar matrices. So, scalar multiplication by a scalar can also be thought of as multiplication by a matrix, namely the scalar matrix of the corresponding size.

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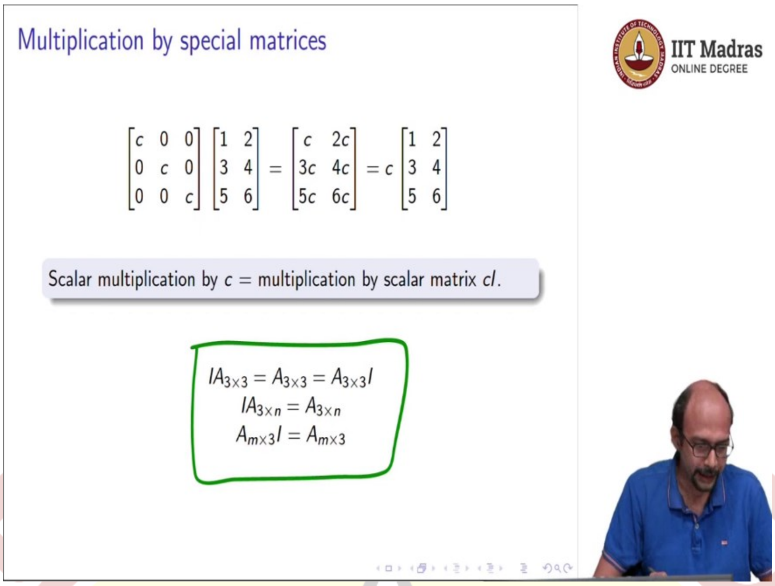

So, to reiterate what I said, scalar multiplication by c is multiplication by the scalar matrix c × the identity matrix. In particular, if we put c is 1, we get the identity matrix. So, from there what do we get? We get that if you have a let us say 3 by 3 matrix, then if you take the identity matrix of size 3 by 3, again as I said identity you will know from context which identity matrix we are using.

So, I × A is the same as A because you can pull out the 1, so nothing is happening. You are not multiplying by anything, meaning you are multiplying by 1 which is the same as A × I. So, I will encourage you to check the formulae. And more generally if you take instead of 3 by 3, if you take 3 be n, then if you multiply by a 3 by 3 matrix I on the left, you will get A again.

And if you take m by 3 and multiply by I, you will again get A. Again the 3 here is not relevant. You could have used any n instead of 3, but often our examples will be in small size that is why we have mentioned 3 here. So, there is something special about the identity matrix that is the point of this bunch of things here.

So, identity matrix acts like 1. So, when you multiply a number by 1, you get back the same matrix, the same number. The same thing happens with the identity matrix. When you multiply by the identity matrix, you do not change anything, you get back the same matrix. So, identity matrix acts like 1, it behaves like 1.

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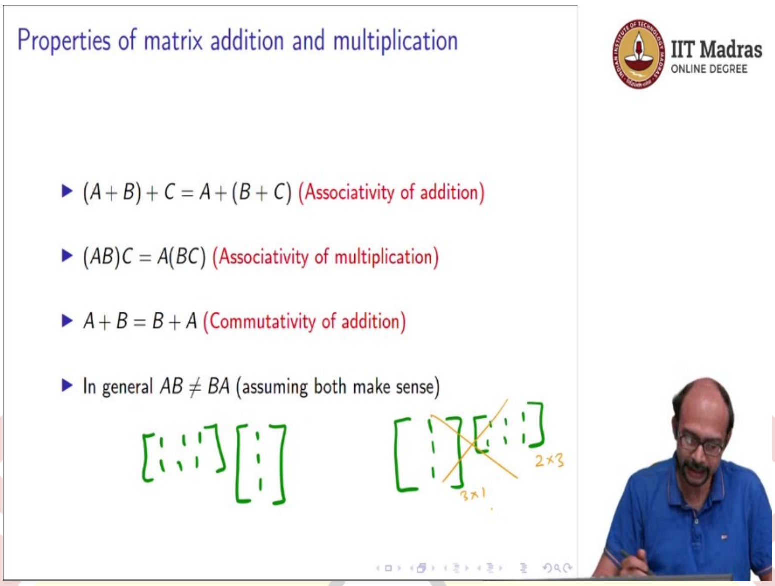

So, finally let us quickly summarize some properties of matrix addition and multiplication. So, if you add 3 matrices, it does not matter which order you add it in, so this is called associativity. If you multiply 3 matrices, it does not matter which ones you, which two you multiply first, but very importantly you have to multiply them in the same order. You cannot change the order.

If you do A+B then that is the same as B+A, so you, matrix addition is what is called commutative, does not matter which order you (multi) add them in. But it matters how you multiply them. A × B is in general not going to be equal to B × A. In fact, it may not even make sense to multiply B × A. Why is that? Because the number of A × B means that the number of rows of B is the same as a number of columns of A.

On the other hand, B × A means the number of columns of B is the same as the number of rows of A. This may not happen. So, for example if you take something like this and I can multiply by 1, but the other way it does not make sense, this does not make sense, this does not make sense, I hope it is clear.

This has 1 column and 3 rows; this has 2 rows and 3 columns. So, it does not, you cannot multiply these two. So, it may not even make sense to multiply them, they may be of, even if you can multiply them, they may end up being of completely different size and even if the sizes are same, they may not be equal. So, I encourage you to check such things.

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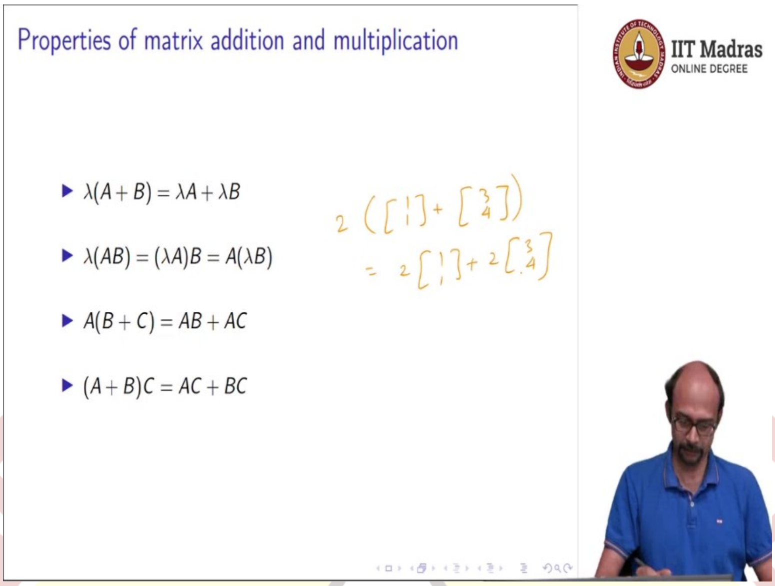

And then some more properties – So, if you multiply this by a scalar, if we have scalar multiplication remember, so if you do lambda × A+B, then that is the same as lambda × A+lambda × B. So, this is scalar (multi). So, lambda is a real number, so lambda is a scalar.

Similarly, if you do lambda × AB, then you can first do lambda × A and then multiply that to B and there will be no problems, it will be the same number, sorry the same matrix. And in fact you can go ahead and shift that lambda inside. So, scalars can be shifted inside. So, here you can change lambda to go inside.

So, A × B+C is a same as A × B+A × C. And then finally, A+B × C is A × C+B × C. So, these are identities, how do I check these? So, you, we have to check these based on the rules of addition and multiplication. So, even if you do not formally understand how to check them, in given examples I hope you will be able to actually compute, so I suggest that you actually compute things like this.

So, for example I will give you an example here, which I suggest that you compute and see what happens. So, see if for example this is the same as 2 × 1+2 × 3, 4 and so on. So, let us quickly recall what we have studied today. We have studied what is a matrix; we have studied special matrices, so diagonal, scalar and the identity matrix. We have studied what is the relationship of a matrix with linear equations.

And then we looked at matrix addition, we looked at scalar multiplication of matrices meaning, multiplying a scalar to a matrix and we looked at multiplying 2 matrices. Very important the number of columns of the first matrix must be the same as the number of rows of the second matrix, only then can we multiply. And then we looked at some properties of all these operations. So, thank you and see you in the next video.