Continuity for multivariable functions

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Define $\underset{\sim}{L} = (L_1, L_2, \ldots, L_m)$. Then $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = \underset{\sim}{L}$.

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This is equivalent to : as $\underset{\sim}{x}$ comes closer and closer to $\underset{\sim}{a}$, $f(\underset{\sim}{x})$ eventually comes closer and closer to $\underset{\sim}{L}$.

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If for some i, the limit f_i at $\underset{\sim}{a}$ does not exist, then then the limit of f at $\underset{\sim}{a}$ does not exist.

Examples

$$\lim_{\underset{\sim}{x}\to(1,2)} \left(x^2y + y^3, e^{xy}, \frac{x^2-1}{y^3-2} \right) = (10, e^2, 0).$$

$\lim_{(x,y)\to(1,2)} x^2y + y^3 = 1^2\times 2 + 2^3 = 10.$

$\lim_{(x,y)\to(1,2)} e^{xy} = e^{1\times 2} = e^2.$

$\lim_{(x,y)\to(1,2)} \frac{x^2-1}{y^3-2} = \frac{1^2-1}{2^3-2} = 0.$

$$\lim_{\underset{\sim}{x}\to(0,0)} \left(\frac{\sin(xy)}{xy}, \frac{x^3-y^2x}{(x^2+y^2)^2} \right)$$ DNE

$xy \to 0$, $\frac{\sin(u)}{u} \to 1$

$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = 1.$

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Limit of a function at a point along a curve

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Limit of a function at a point along a curve

Let f be a scalar-valued multivariable function defined on a domain D in $\mathbb{R}^k$ and $\underset{\sim}{a}$ be a point such that there exists a sequence in D which converges to $\underset{\sim}{a}$. Let C be a curve passing through the point $\underset{\sim}{a}$ belonging to the domain D.

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The limit of f at $\underset{\sim}{a}$ along the curve C exists and equals L if for every sequence $\underset{\sim}{a_n}$ contained in C which converges to $\underset{\sim}{a}$, the sequence $f(\underset{\sim}{a_n})$ converges to L.

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Example : $g(x, y) = \dfrac{x^3 - y^2 x}{(x^2 + y^2)^2}$

$\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=x}} g(x,y) = \lim\limits_{x\to 0} \dfrac{x^3 - x^3}{(2x^2)^2} = 0.$

Along the X-axis, the Y-coordinate is 0 & hence the fn. is $g(x,0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}.$

$g(0,y) = 0.$

$\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along X-axis}}} g(x,y) = \lim\limits_{x\to 0} \dfrac{1}{x}$ DNE.

$\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along Y-axis}}} g(x,y) = \lim\limits_{y\to 0} 0 = 0.$

Limits of a function along curves and limit of the function

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Important : This is often used to show that a limit at a point does not exist.

More examples

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

along the X-axis: $\lim_{x\to 0} \frac{x^2}{x^2} = 1.$ $\neq$

along " Y-axis: $\lim_{y\to 0} \frac{-y^2}{y^2} = -1.$

DNE

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

along the X-axis: $\lim_{x\to 0} \frac{0}{x^2+0} = 0.$

along the Y-axis: $\lim_{y\to 0} \frac{0}{0^2+y^2} = 0.$ $\neq$

along the line $y=x$: $\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$

DNE

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

along the line $y=mx$ $\lim_{x\to 0} \frac{x\times(mx)^2}{x^2+(mx)^4} = \frac{mx^3}{x^2(1+m^4x^2)}$

$= 0.$

along the line $x=0$ $\lim_{y\to 0} \frac{0}{y^4} = 0.$

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Note that continuity means "the limit at $\underset{\sim}{a}$ can be obtained by evaluating the function at $\underset{\sim}{a}$.

The function f is said to be continuous if it is continuous at all points in its domain D i.e. for all points $\underset{\sim}{a}$ for which $f(\underset{\sim}{a})$ is defined, $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = f(\underset{\sim}{a})$.

Example :

$$g(x,y) = \begin{cases} \frac{x^3 - y^2 x}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

If $\underset{\sim}{a} \neq (0,0)$ then $\lim_{(x,y)\to \underset{\sim}{a}} g(x,y) = g(\underset{\sim}{a})$.

$\underset{\sim}{a} = (a,b)$.

$g(x,y) = \frac{f(x,y)}{h(x,y)}$

$\lim_{(x,y)\to\underset{\sim}{a}} h(x,y) = h(\underset{\sim}{a}) = (a^2+b^2)^2 \neq 0$.

If $\underset{\sim}{a} = (0,0)$.

$\lim_{(x,y)\to(0,0)} g(x,y)$ DNE $\neq g(0,0) = 0$.

$\therefore$ The fn. $g(x,y)$ is continuous at all points except $(0,0)$.

Thank you