

The Gram-Schmidt process

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An overview of the Gram-Schmidt process

In an inner product space



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Any basis

x_1, x_2, \dots, x_n



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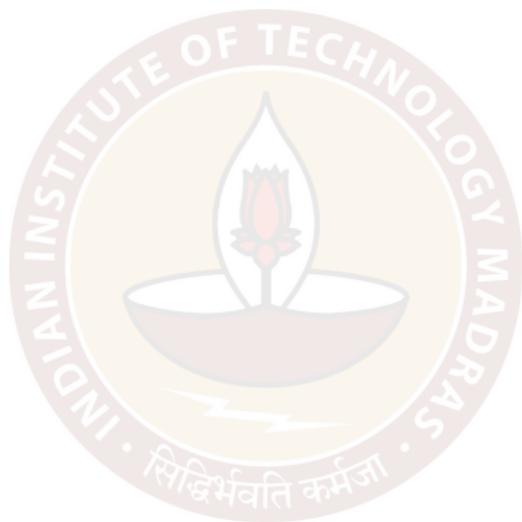
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Orthonormal basis

v_1, v_2, \dots, v_n

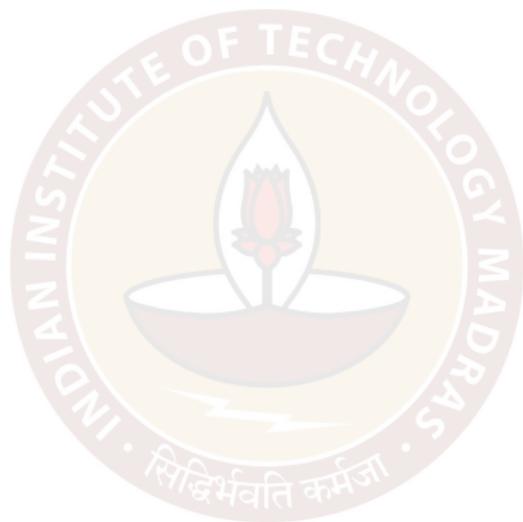
Example and intuition

Consider the basis $\beta = \{(1, 2, 2), (-1, 0, 2), (0, 0, 1)\}$ for \mathbb{R}^3 .



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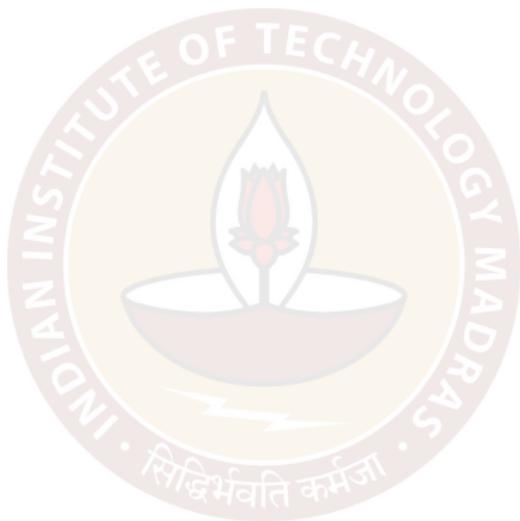
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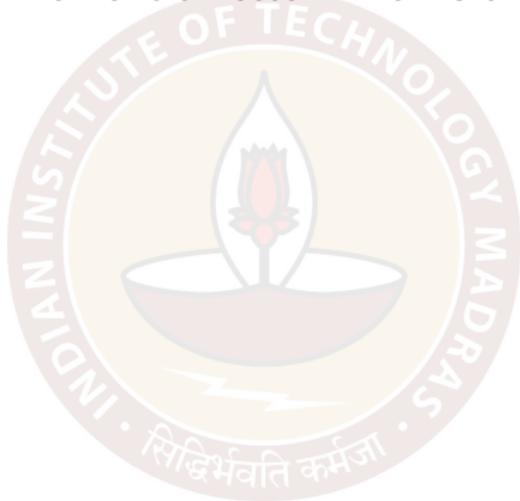
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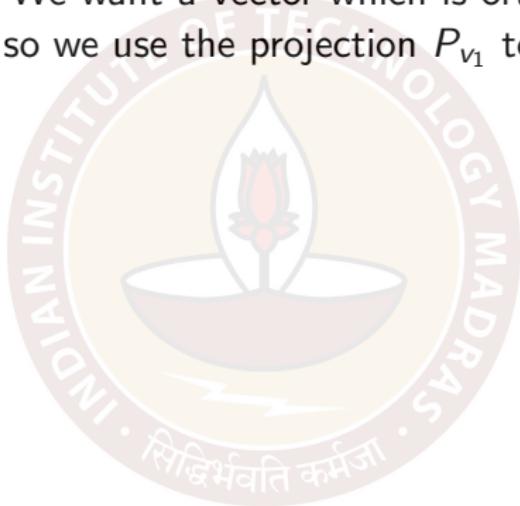
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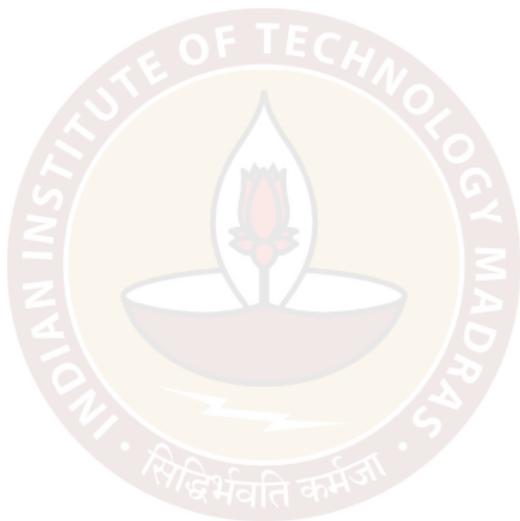
$$\begin{aligned} \text{Define } v_2 &= (-1, 0, 2) - P_{v_1}((-1, 0, 2)) = \underbrace{(-1, 0, 2)}_{\in \langle v_1 \rangle^\perp} - \underbrace{P_{v_1}((-1, 0, 2))}_{(\mathbb{I} - P_{v_1})(-1, 0, 2)} \\ &= (-1, 0, 2) - \frac{\langle (-1, 0, 2), (1, 2, 2) \rangle}{\langle (1, 2, 2), (1, 2, 2) \rangle} (1, 2, 2) \\ &= \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3} \right) . \quad \boxed{\langle v_2, v_1 \rangle = 0} \end{aligned}$$

$$W \quad W^\perp = \{v \mid \langle v, w \rangle = 0 \ \forall w \in W\} = \text{Null space of } P_W.$$

$$W^\perp \ni P_W(v) = 0 \Leftrightarrow w \in W^\perp. \quad \Rightarrow \boxed{(\mathbb{I} - P_W)(v) = v - P_W(v) \ \& \ P_W(v - P_W(v)) = 0}.$$

Example and intuition (contd.)

We want a vector which is orthogonal to both v_1 and v_2 , i.e. a vector in $\text{Span}(\{v_1, v_2\})^\perp$,



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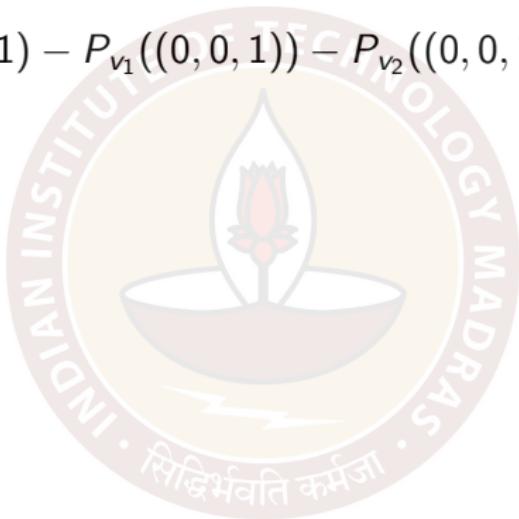
$$(\mathbb{I} - P_W)$$



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Define $v_3 = (0, 0, 1) - P_{v_1}((0, 0, 1)) - P_{v_2}((0, 0, 1))$



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$$\begin{aligned} \text{Define } v_3 &= (0, 0, 1) - P_{v_1}((0, 0, 1)) - P_{v_2}((0, 0, 1)) \\ &= (0, 0, 1) - \frac{\langle (0, 0, 1), (1, 2, 2) \rangle}{\langle (1, 2, 2), (1, 2, 2) \rangle} (1, 2, 2) \\ &\quad - \frac{\langle (0, 0, 1), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \rangle}{\langle \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \rangle} \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \end{aligned}$$

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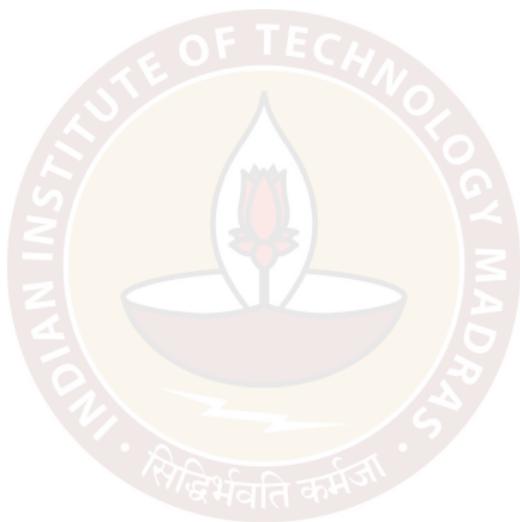
Check $\begin{aligned} &\langle v_1, v_2 \rangle \\ &= \langle v_1, v_3 \rangle \\ &= \langle v_2, v_3 \rangle = 0 \end{aligned}$

Thus $\{v_1, v_2, v_3\}$ is an orthogonal basis and dividing each vector by its norm yields an orthonormal basis

$$\left\{ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \right\}.$$

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Let V be an inner product space with a basis $\{x_1, x_2, \dots, x_n\}$.



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Define the orthogonal basis $\{v_1, v_2, \dots, v_n\}$ and the corresponding orthonormal basis $\{w_1, w_2, \dots, w_n\}$ as follows :



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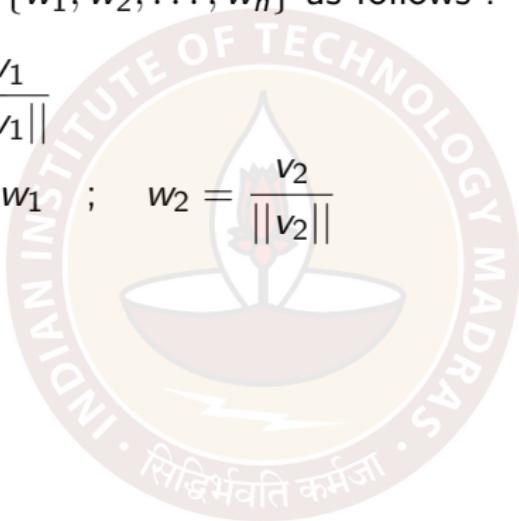
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$$v_2 = x_2 - \langle x_2, w_1 \rangle w_1 ; \quad w_2 = \frac{v_2}{\|v_2\|}$$

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$$v_i = x_i - \langle x_i, w_1 \rangle w_1 - \langle x_i, w_2 \rangle w_2 - \dots - \langle x_i, w_{i-1} \rangle w_{i-1} ; \quad w_i = \frac{v_i}{\|v_i\|}$$

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$\vdots \quad \vdots \quad \vdots$

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$\vdots \quad \vdots \quad \vdots$

$$v_n = x_n - \langle x_n, w_1 \rangle w_1 - \langle x_n, w_2 \rangle w_2 - \dots$$

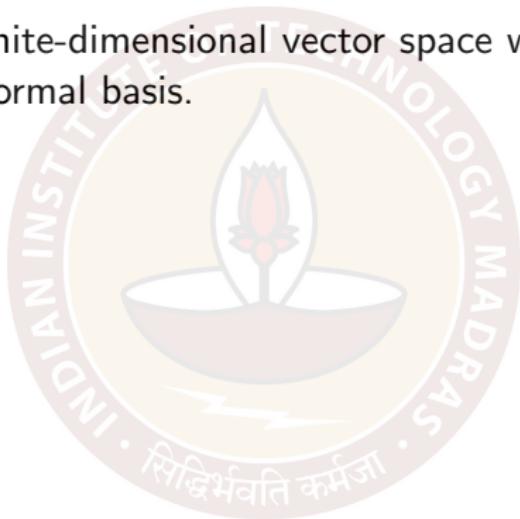
$$\dots - \langle x_n, w_{n-1} \rangle w_{n-1} ; \quad w_n = \frac{v_n}{\|v_n\|}$$

$$\begin{aligned} & \text{Span}(\{x_1, x_2, \dots, x_{i-1}\}) \\ &= \text{Span}(\{v_1, v_2, \dots, v_{i-1}\}) \\ &= \text{Span}(\{w_1, w_2, \dots, w_{i-1}\}) \end{aligned}$$

$$\begin{aligned} & \in P_{\text{Span}(\{w_1, \dots, w_{i-1}\})}(x_i) \\ & \in P_{\text{Span}(\{x_1, \dots, x_{i-1}\})}(x_i) \end{aligned}$$

Main take-homes

Theorem: Any finite-dimensional vector space with an inner product has an orthonormal basis.



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Theorem: Any finite-dimensional vector space with an inner product has an orthonormal basis.

Any basis can be changed to an orthonormal basis using the Gram-Schmidt process.

Thank you

