IIT Madras
ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 32: What is a Basis for a Vector Space?

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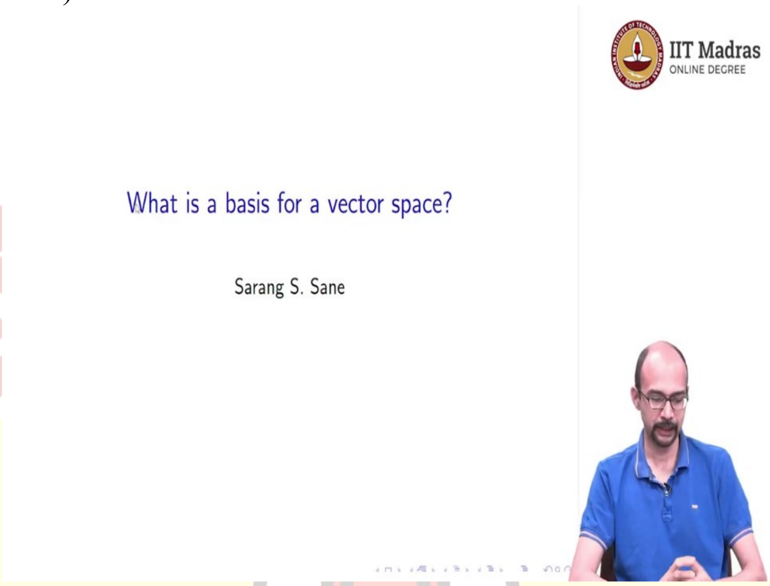

Hello, and welcome to the Maths 2 component of the online degree program on data science. So, in this video, we are going to study what is the basis for a vector space. So, just to recall what we have done as part of this module before, we have studied what are vector spaces, we have seen the notion of a linearly dependent set in a vector space. And we have seen the notion of a linearly independent set in a vector space. And, today's video about a basis builds on the fact that we know what is linear independence of a set of vectors.

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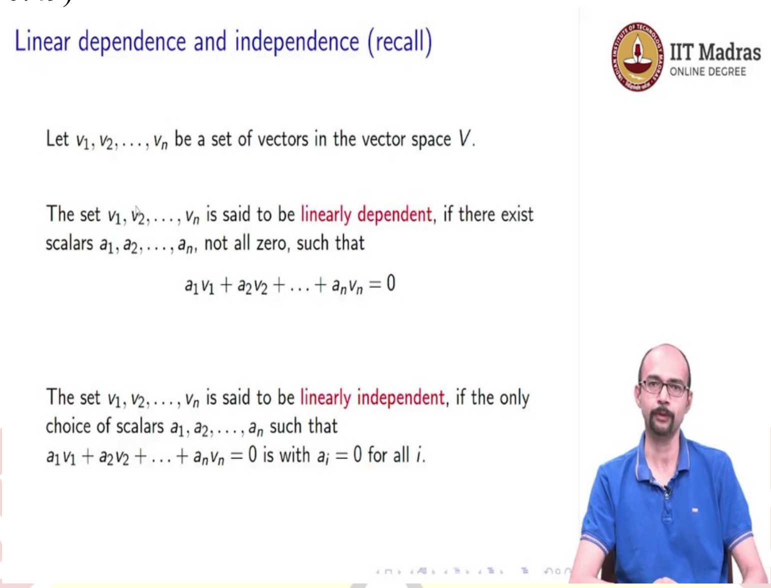

So, let us quickly recall this idea of linear dependence and linear independence. So, suppose we have a set of n vectors, $v_1, v_2, .., v_n$in a vector space V. This set is said to be linearly dependent, if there exists scalars $a_1, a_2, .., a_n$not all zero, such that $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$.

So, this means there is a linear combination of $v_i$which is a non-zero linear combination, which means that at least one of the coefficients is non-zero. Note that this does not mean that all the coefficients are non-zero, but it means that at least one of the coefficients is non-zero. So, we have a linear combination such that, with non-zero coefficients and that linear combination is 0.

And now, linear independence is saying it is not linearly dependent. So, explicitly what this means is that the set $v_1, v_2, .., v_n$is said to be linearly independent. If the only choice of scalars $a_1, a_2, .., a_n$such that the corresponding linear combination, $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$, is all of them are equal to zero.

So, if you solve the equation v times x is 0 where v is a matrix given by writing the vectors, this is an RN. If you look at that matrix vx is 0 where the columns of v are the vectors Vi then, that the only solution is x is 0. And saying that it is linearly dependent means that there are solutions which are not the 0 solution. So, this was linear dependence, and linear independence, particularly in the context of RN.

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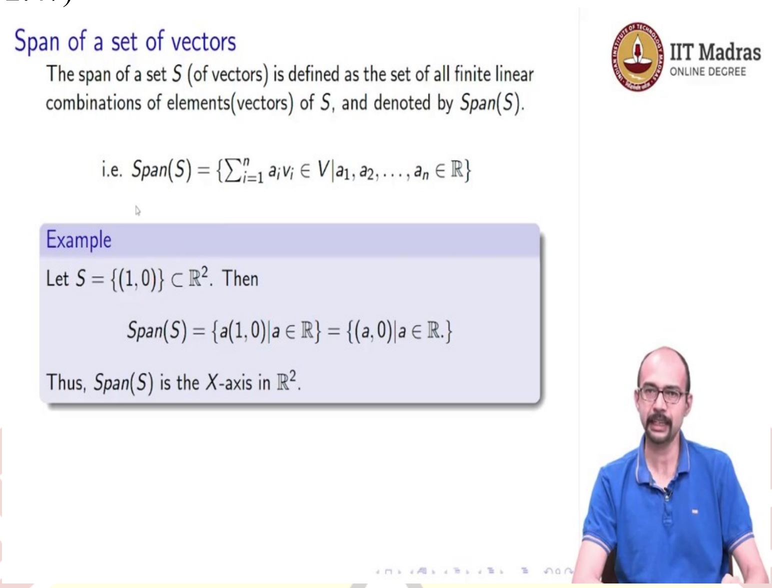

So, let us first talk about what is the span of a set of vectors? So, span is like an expanse. So how big can you get based on these vectors, that is the idea of span. So, the span of a set is, of vectors, is defined as the set of all finite linear combinations of vectors of S, and it is denoted by span S.

So, explicitly what that means is, you take all linear combinations where your coefficients run over all possible values in $\mathbb{R}$meaning each coefficient runs over all possible values in $\mathbb{R}$and the span is that set. Now, what is hidden here is that, the span of S is a vector space in its own rate, meaning it is what is called a subspace of V.

What that means is if you take a linear combination of elements within span S, then they are still in span S. And if you take and zero is also that, in particular by taking all the $a_i$to be zero. So, here is an example. So, let S, be the set, just the singleton set $(1,0)$. This is a vector in $\mathbb{R}^2$and you look at the set S, which consists of this particular vector.

So, this is a subset of $\mathbb{R}^2$, and we can ask what is the span of S. So, the span of S is, $a(1,0)$, where $a \in \mathbb{R}$which is the set of vectors, $(a, 0)$. And we know that this is exactly the x-axis in $\mathbb{R}^2$. So, the span of the vector $(1,0)$is the x-axis.

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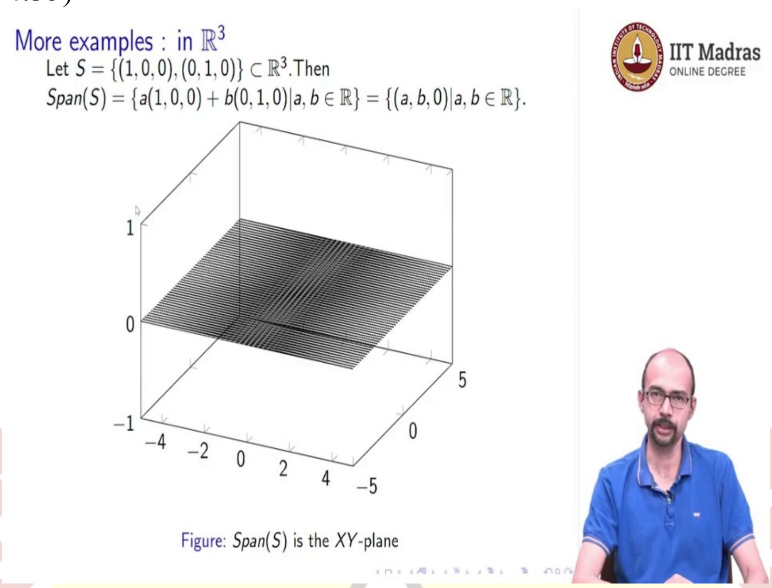

So, let us do a couple of other examples. Let us look at the set consisting of the singleton element $(1,1)$in $\mathbb{R}^2$. So, you can see that the span of S is $a(1,1)$which is all vectors of the form (a, a). So, looking at this in $\mathbb{R}^2$, this means you get the line, $y = x$? So, the blue line is the span of S.

Again, you can see that, if you add vectors on this line, you still remain on this line. So, let us see some more examples maybe in $\mathbb{R}^3$this time. So, let us look at the set $\{(1,0,0), (0,1,1)\}$in $\mathbb{R}^3$ and let us ask, what is the span of this set? So, if you take linear combinations, you will get $a(1,0,0) + b(0,1,0)$, which is all vectors of the form $(a, b, 0)$So, you can see, this is exactly the xy plane. So, span S in this case is the xy plane, which is drawn here in the picture.

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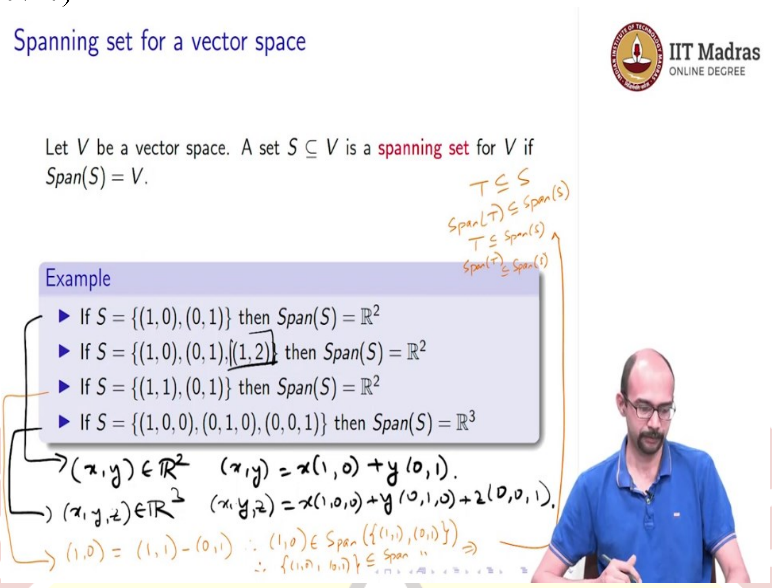

So, what are the spanning set for a vector space? So, let V be a vector space. So, a spanning set is a set such that $Span(S) = V$. So, that means we are looking at vectors so that if you take linear combinations of those vectors, you can produce any vector in V so, or the words, every vector of V is a linear combination of vectors, belonging to the set S, that is what we mean by saying that span S is V.

So, let us look at the $(1,0), (0,1)$, then span of S is the entire $\mathbb{R}^2$, so this is a spanning set for $\mathbb{R}^2$. Similarly, if you take $(1,0), (0,1), (1,2)$, so you add one more vector, clearly, still the span is going to be the entire $\mathbb{R}^2$. And you can change your elements a little. So, if you take the vectors $(1,1), (0,1)$then the span of S is still $\mathbb{R}^2$. So, how do I see that?

Well, one way of seeing it is that from $(1,1), (0,1)$, you can produce $(1,0)$by taking $(1,1) - (0,1)$. So, that means the span contains $(1,0), (0,1)$. So, then if the span contains some vectors, then this span of that set S, also contains a span of the vectors that you have produced in that span. I will make this more precise in a second.

And let us look at an example in $\mathbb{R}^3$. So, if you look at the set of vectors $\{(1,0,0), (0,0,1), (0,0,1)\}$, then span of S is $\mathbb{R}^3$. So, let me quickly specify why that is the case. So let us maybe look first at $(1,0), (0,1)$so this first example, so if you take (x, y)in $\mathbb{R}^2$ then you can write it as $x(1,0) + y(0,1)$. So, this means every vector in $\mathbb{R}^2$ can be written as a linear combination of these first two vectors. The same idea can be used for this if you take (x, y, z)in $\mathbb{R}^3$then $(x, y, z)$is $x(1,0,0) + y(0,1,0) + z(0,0,1)$. So, this tells you that the span of these three vectors is the entire $\mathbb{R}^3$. So, this is a spanning set for $\mathbb{R}^3$. Similar explanations are extended for the second and third examples.

So, the main point here is the following, so what I am trying to say is the following. If a set T is contained in a set S, then span of T is contained in span of S. So, this is in fact what is a sub space.

And if it so happens that T is a spanning set, that means S is also a spanning set. That is what we just said that if you have a superset, then we are good.

So, in particular one can also say the following that if T is contained in span of S, which is what I am using here, then span of T is contained in span of S. So, these are things that we will be doing as tutorial problems, so I will leave that for now. So, let us continue this. So, I hope we have understood what is a spanning set, so spanning set means a set such that every vector in V can be written as a linear combination of this some particular vectors from S.

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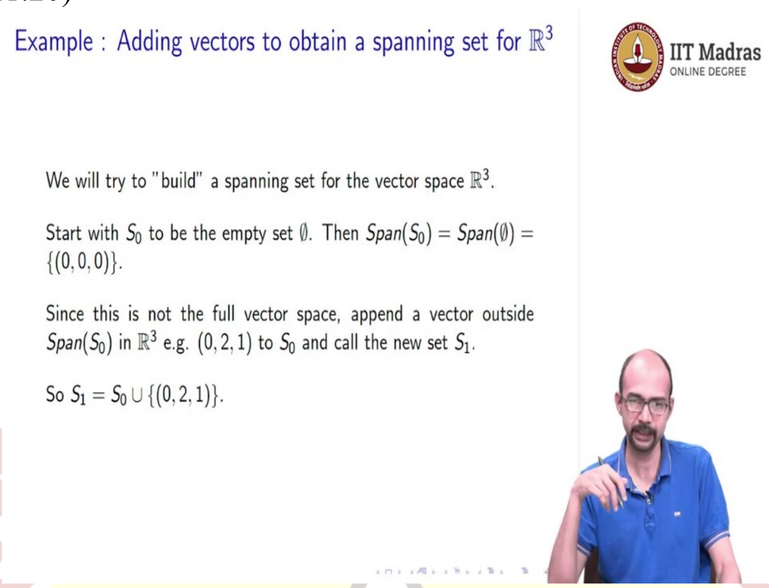

So how do I get a spanning set? So, the way to do this is, one way to do this is to add or maybe the right word here is ‘append’ vectors to obtain a spanning set for $\mathbb{R}^3$, I will use the word append instead of add, add may leave you confused by addition in the vector space that is not what we mean. We mean append, meaning take more and more vectors. So, we will try to build a spanning set for the vector space $\mathbb{R}^3$.

So, start with $S_0$to be the empty set $\emptyset$. So, then $Span(S_0) = Span(\emptyset) = \{(0,0,0)\}$. So, there is something here already to point out, namely, that the span of the empty set is $(0,0,0)$. We can think of this in two ways.

One is that this is a convention, the other is that this is the empty linear combination. So, the empty linear combination is always 0, that is again maybe a convention. So, from there it follows that $(0,0,0)$as the span of the empty set. So, this, if you are not, if you find this uncomfortable then just take this as a convention or a definition that span of the empty set is the zero vector.

So, this is not the full vector space. The full vector space is $\mathbb{R}^3$, so $(0,0,0)$, is not the entire vector space $\mathbb{R}^3$. You append a vector outside span of S_0in $\mathbb{R}^3$, and we will call this new set S_1. So, $S_1 = S_{(0)} \cup \{(0,2,1)\}$. So here, that is what I mean by append, you put this inside your, you create a new set with this vector appended to your original set. And we are going to keep continuing this process. This is how we will obtain a spanning set.

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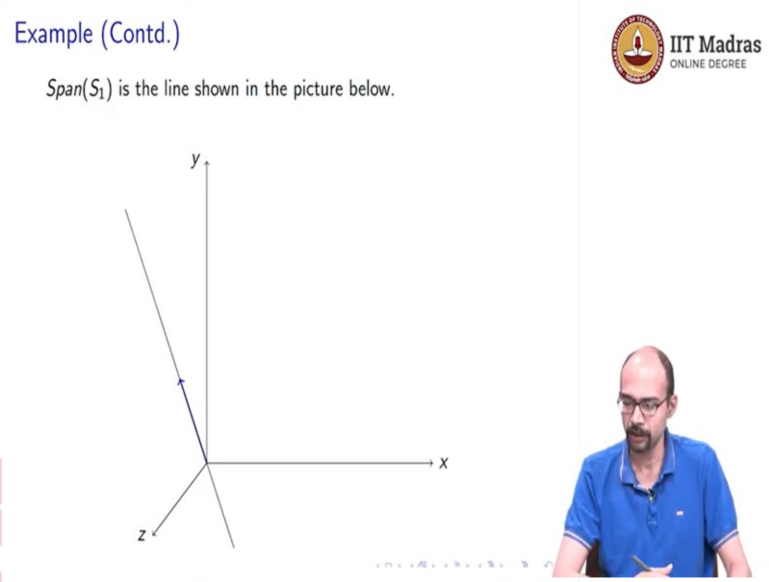

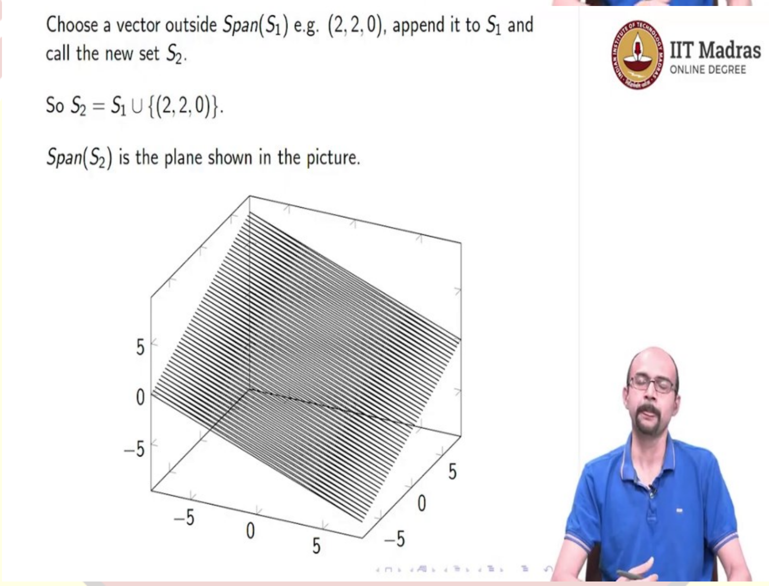

So, what is span of S_1? So, span of $S_1$is exactly this line shown here. So, the blue arrow is $(0,2,1)$, which is a vector that we had, and the span of $(0,2,1)$is the line containing that arrow. So, now choose a vector outside span of S_1. So, as we can see it is just the line, so that means it is not the entire $\mathbb{R}^3$. So, we have vectors which are outside this line, so choose a vector outside span of S_1. So, for example, $(2,2,0)$appended to $S_1$and call that new set S_2, so $S_2 = S_{(1)} \cup \{(2,2,0)\}$.

So, span of $S_2$is this plane that you get. So, you can write down the equations for the plane, which we will be doing eventually, in Maths 2. But if you can do it right away, maybe you have learned this already in Maths 1 or maybe you have done it as part of some other course, please do go ahead and write down what the equations for this plane are. In in any case, you get a plane and so this is not the entire $\mathbb{R}^2$, and so we can choose a vector outside this plane.

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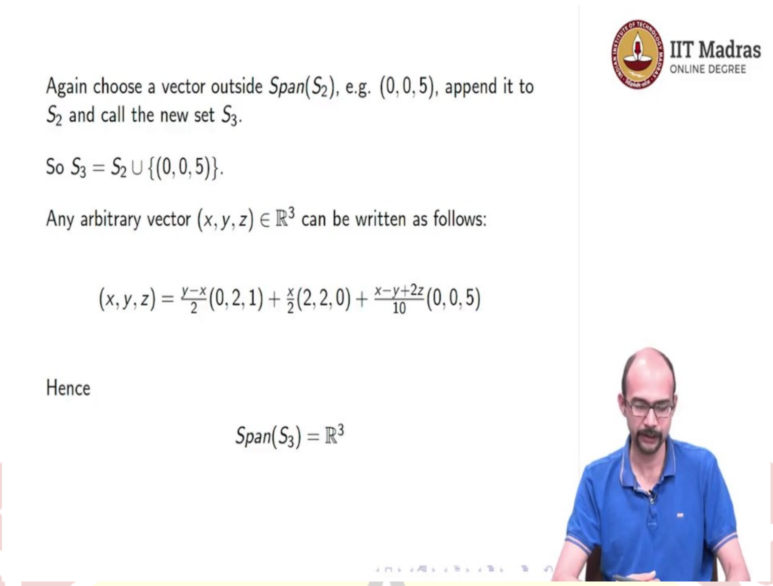

So, let us say choose a vector outside span of S_2say $(0,0,5)$and append it to $S_2$and we will call that new set S_3. So, $S_3 = S_{(2)} \cup \{(0,0,5)\}$. So, you can see, we are increasing the size of the set that we have. So, you keep adding more and more vectors, and hopefully as you keep doing this you will hit a spanning set.

So, now look at any arbitrary vector (x,y,z)in $\mathbb{R}^3$, and we can write it in the following way. So, this is something you have to check $(x,y,z) = \frac{y-x}{2}(0,2,1) + \frac{x}{2}(2,2,0) + \frac{x-y+2z}{10}(0,0,5)$.You might find this puzzling where did I suddenly come up with these coefficients from, but you can certainly check that this is correct. So, I will leave you to check it is correct.

How to actually obtain this coefficient is something we will see eventually or you can try to think of this as solving a system of linear equations, we have done this before, and I will ask you to try to think about how to do this. So, you can treat your coefficient as variables. We have done that in the previous video for linear independence, and you can try and rate (x,y,z)as a linear combination of these.

So, what is the point? The point is, span of $S_3$is the entire $\mathbb{R}^3$. How did I get that? Because (x,y,z) can be expressed as a linear combination of these three vectors. So that means the span is the entire $\mathbb{R}^3$.

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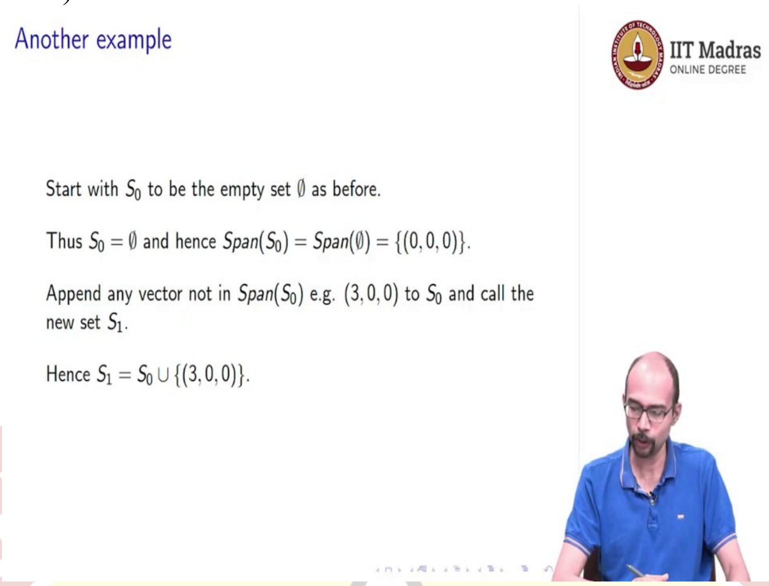

So, what did we do? We started with the empty set and then we kept adding vectors, but we added them in a particular way. We added them so that the new vector that we added was not in the span of the vectors that we had so far. And in this way, we created a spanning set. So, this is a general template to create a spanning set.

Let us do another example with this same idea. So, start with S_0 with the empty set as before, we know by convention or definition that the span of the empty set is $\{(0,0,0)\}$, append any vector which is not in the span of S_0, let us say in this case, we choose $(3,0,0)$ call this new set .

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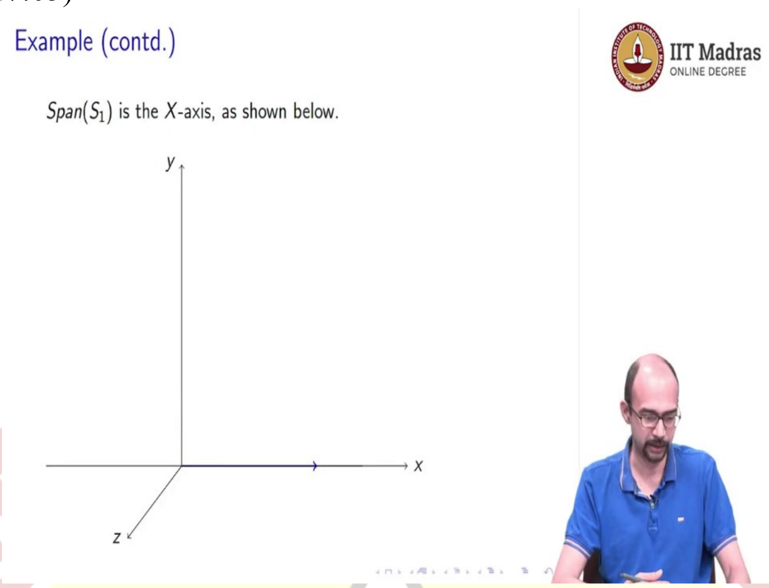

And what is the span of S_1? It is the x-axis. If you take $(3,0,0)$and take it spanned that means you will get vectors of the form $(3a, 0,0)$, which is exactly what the vectors on the X-axis look like. So, the blue arrow is the vector $(3,0,0)$. And the line of course is the X-axis.

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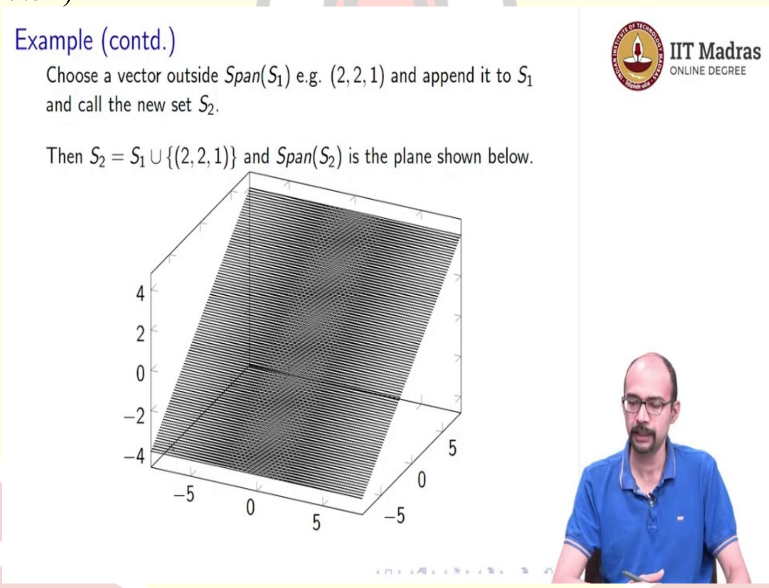

So, let us choose a vector, which is outside the X-axis, let us say $(2,2,1)$and let us append it to $S_1$and call the new set S_2. Let us look at what is span of S_2. Well, it is this plane, this plane certainly passes through the X-axis. You can see it intersects the X-axis somewhere, and it also contains the vector $(2,2,1)$. So, I will, again, encourage you to check what is the equation of this plane. In any case it is a plane, so it is not the entire $\mathbb{R}^3$.

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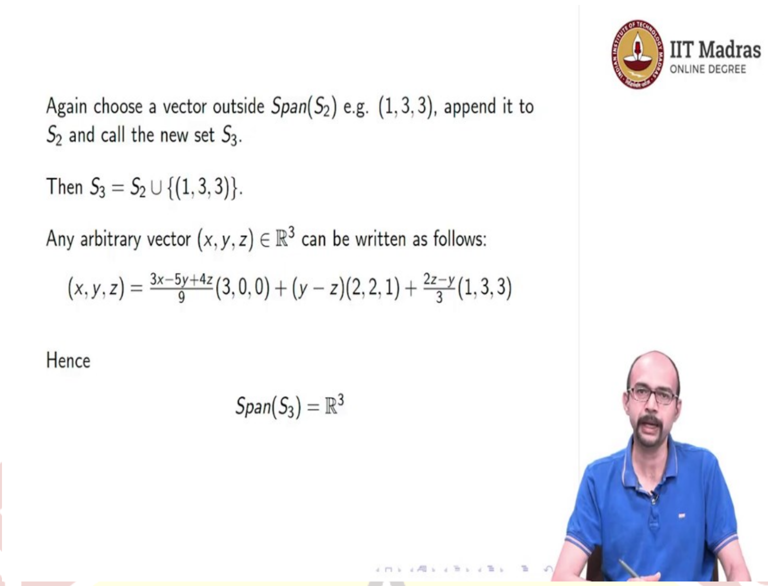

So, you can choose a vector, which is not in the span of S_2. So, let us say, we choose the vector $(1,3,3)$ and you appended to $S_2$and you call the new set S_3. And now once again, we will sort of say that every vector (x,y,z)can be written as a linear combination of these three vectors, and I will encourage you to work out what the coefficients are by solving the corresponding system of linear equations. If not, I have given the coefficients here and you can check that this indeed works out.

So, what is the point? The point is every vector in $\mathbb{R}^3$can be written as a linear combination of these three vectors. That means the span of $S_3$is the entire $\mathbb{R}^3$, which means $S_3$is a spanning set for $\mathbb{R}^3$, which now brings us to this idea of a basis.

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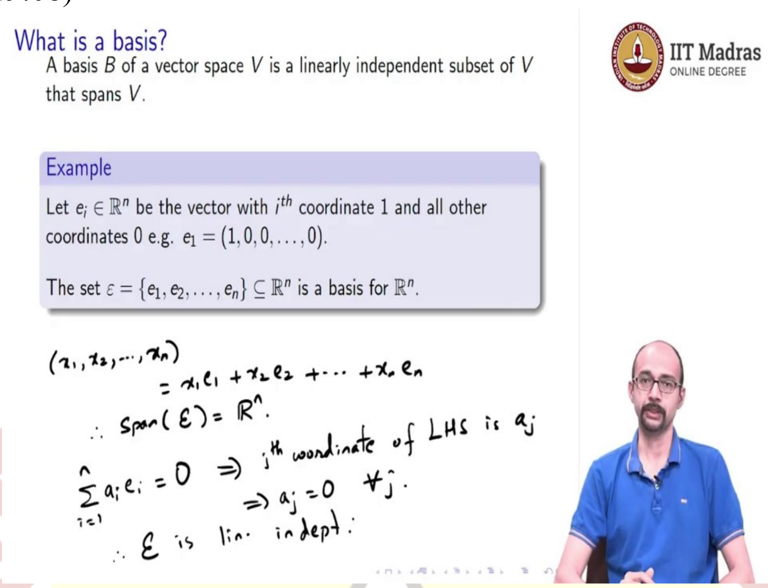

This is the main emphasis of this video, so what is a basis? So, basis B of a vector space V is a linearly independent subset of V that spans V, so it has two properties. One, that it spans V so it is a spanning set for V, and the other is that it is a linearly independent set.

Now, notice what is happening here. Span, if you want something to span V, the idea is you keep adding elements. So, you add as many elements as you like, you keep adding vectors and hopefully if you have enough vectors, then they will be able to span V.

Of course, we did this appending of vectors in a very particular way. We kept choosing vectors which are outside the span of the previous few vectors that we have chosen. So, because we did that, the new vector that we added was not a linear combination of the original vectors, which meant it was linearly independent, the sets that we keep obtaining are linearly independent sets.

So, in fact, what we did in our previous examples was a method to not just find a spanning set, but actually find a basis. So, there are two contradict, I mean, there are two sort of opposite things happening here. One is that you would like this set to be a spanning set, which means you would like it to be fairly large. On the other hand, you would like it to linearly independent, which means you do not want it to be too large.

Afterall, remember that, for example, if you have $\mathbb{R}^N$then any set which has a cardinality more than N, is already linearly dependent. So, in $\mathbb{R}^N$, if you want to have a basis, you would want it to, if you want it in particular to be linearly independent it can have size at most N. On the other hand, for a spanning set you would have you would want it to be very large.

So, the basis is sort of an optimal middle. It is both linearly independent and spanning. So, let us start with the most standard example. And in fact, this is called a standard basis. So let $e_i \in \mathbb{R}^N$.

$e_i = (1,0,0,\ldots,0)$. So, let us look at the set. $\epsilon = \{e_1, e_2, .., e_n\}$Is a basis of $\mathbb{R}^N$.

So, let us try to see why that is the case. If you take – this is the same idea that we have done for in the case of $\mathbb{R}^2, \mathbb{R}^3$which where we wrote it down. So, if you take anything of the form $(x_1, x_2, .., x_n)$we can write it in the form $x_1e_1 + x_2e_2 + .. + x_ne_n$.

So, it is a spanning set for $\mathbb{R}^N$, so what we have proved is that the span of this set $\epsilon$is$\mathbb{R}^N$. On the other hand, these are clearly linearly independent because if you take a linear combination, which is 0, but let us expand out this vector in terms of coordinates. That means it is linearly independent. So, therefore, this is linearly independent. So that proves it is a basis. So, this is called the standard basis for $\mathbb{R}^N$.

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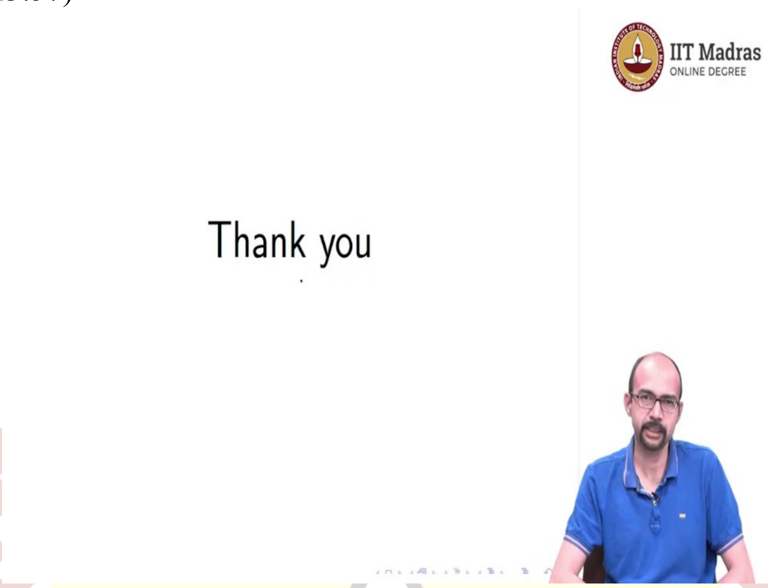

So, let us take a step back and ask, what have we done in this video. We have seen that there is a notion of a spanning set, namely, a set of vectors so that every vector can be written as a linear combination of these vectors, so that is a spanning set for the vector space V. And then we have seen that there is a notion called a basis. So, basis is a set of vectors, which is both linearly independent and spanning. Thank you.