IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Orthogonal transformation and rotations

Hello and welcome to the batch two component of the online BSc program and data science and programming. This is a video on orthogonal transformations and rotations. So, before I start the video, I want to sort of emphasize that this is maybe not an extremely important video from the perspective of the data science course.

But it is a useful video from a if you find mathematics interesting and certainly the idea of rotations and so on which we are very close to given the material that we have covered so far, are interesting even to people who do not particularly enjoy mathematics. So, in some sense this video is for a general view of what kind of mathematics you can study using linear algebra which we have studied so far.

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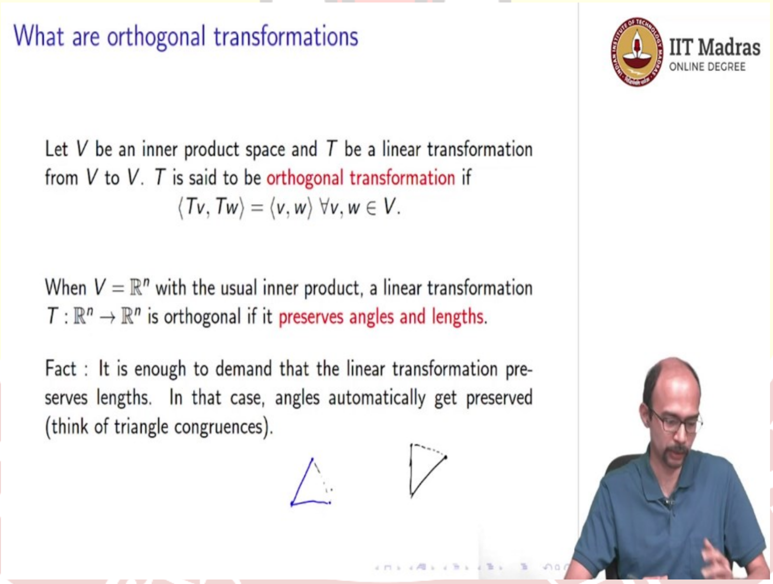

So, let us talk about orthogonal transformations and rotations. So, let V be an inner product space and T be are linear transformation from V to V. So, T said to be an orthogonal transformation, if the inner product of T_v and T_w is the same as the inner product of v and w for all v and w and in the vector space V. So, for any two vectors, you change them, you transform them using a linear transformation T, then the changed the vectors maintain the inner product.

So, the inner product does not change after the transformation, this is when you will say that this transformation is an orthogonal transformation. So, what does it mean for Rn, this is a main sort of intuition behind why we study orthogonal transformations. So, in these Rn with the usual inner product, a linear transformation T Rn to Rn is orthogonal, if it preserves angles and links.

So, let us understand where this is coming from. So, note that the inner product of two vectors in Rn, the usual inner product, meaning that dot product is the norm of v × the norm of w × cosine of the angle between these two vectors. So, now if you take these two sides, you get $T_v . T_w$ is inner product of T_v sorry, norm of T_v × norm of T_w or we can say the length of T_w × the length of T_w × cosine of the angle between T_v and Tw.

And on the other side, we have the length of v × the length of w × the cosine of the angle between v and w. So, now if the linear transformation preserves, if these two are equal, so first of all, and they are equal, remember for all v and w, so in particular, you can take v to be (θ) w in that case, the two sides will give you norm of ${T_v}^2$ is (θ) norm of v^2 norms are always positive, that is why norm of T_v is (θ) norm of T. So, in other words, the lengths are the same. After you apply T, the length of v remains unchanged. So, length of T_v is the same as length of V. So, that means it preserves length.

And now you apply it to the general situation of v and w. So, you have length of v × length of w × the cosine of the angle between them, the length of × length of T_w × cosine of the angle between but length of T_v is the same as length of v length of T_w is the same as length of w. And that is why on both sides, you get only that the angle between T_v and T_w is (θ) the angle between out cosine of the angle between T_v and T_w is the same as cosine of the angle between a v and w.

And that tells you that the angle between v and w is the same as the angle between our T_v and T_w. So, it also preserves angles. That is why we have this this comment, so in other words, orthogonal transformations are some very useful or interesting. They have some interesting geometry, when we consider the usual inner product, namely the dot product.

Here is a fact it is enough in the case of Rn to demand that the linear transformation preserves length, so in that case, angles automatically get preserved. So, what we are saying is, since this is for all v and w. So, if you take 3 points and then after transformation they become like this, so the relative lengths are the same. That means this length is also the same. And that will sort of force that the angle does not change. So, this is the idea. I am not proving this. But it is a fact that I think is very visually appealing. So, this is something you may be able to try and see.

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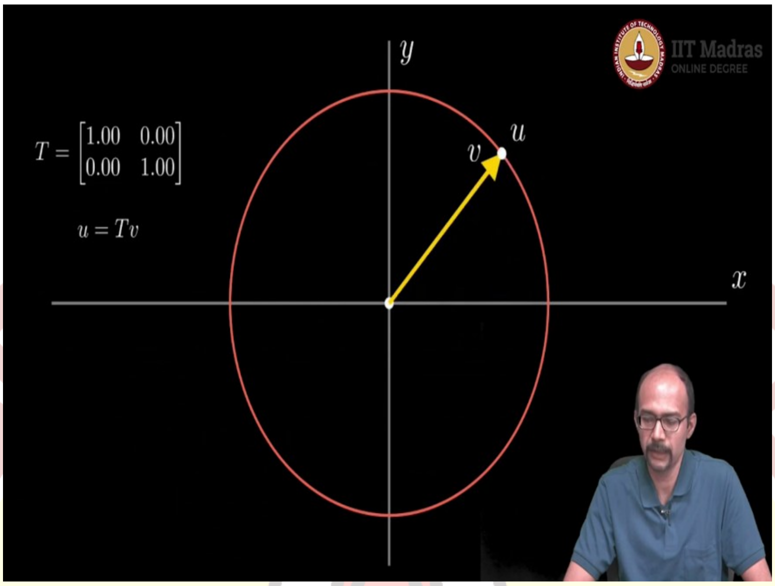

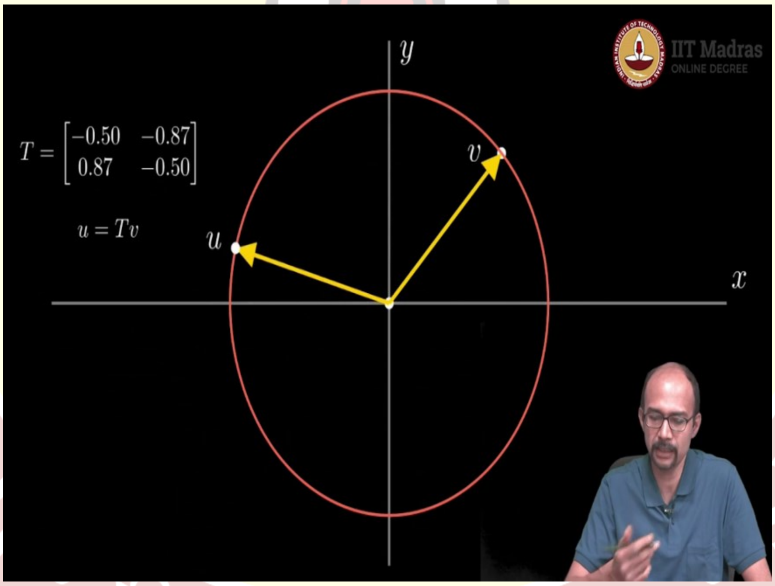

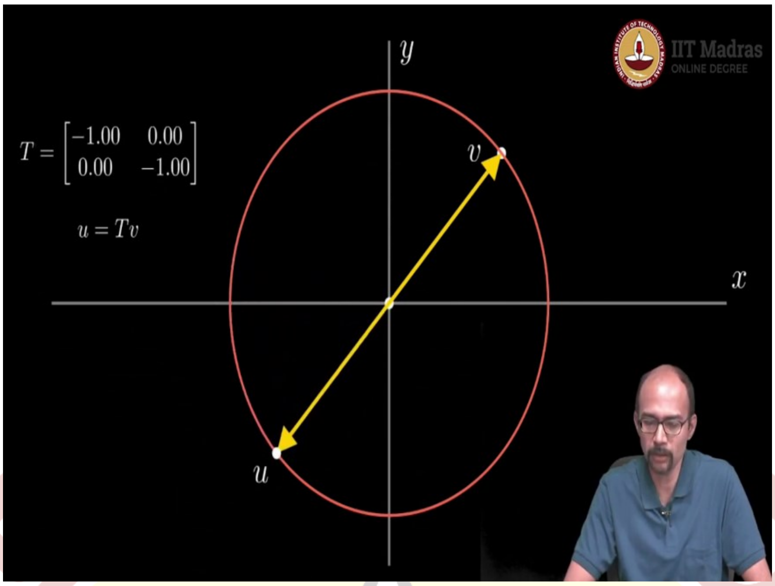
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$u = Tv$
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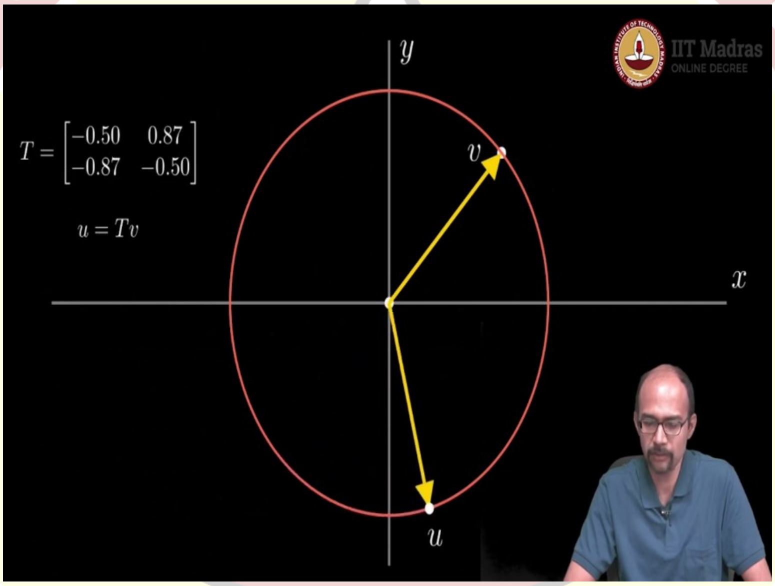
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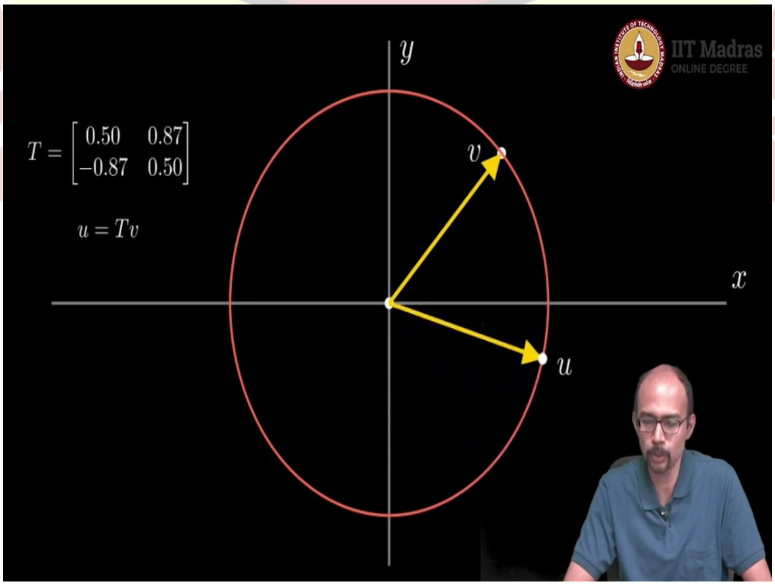
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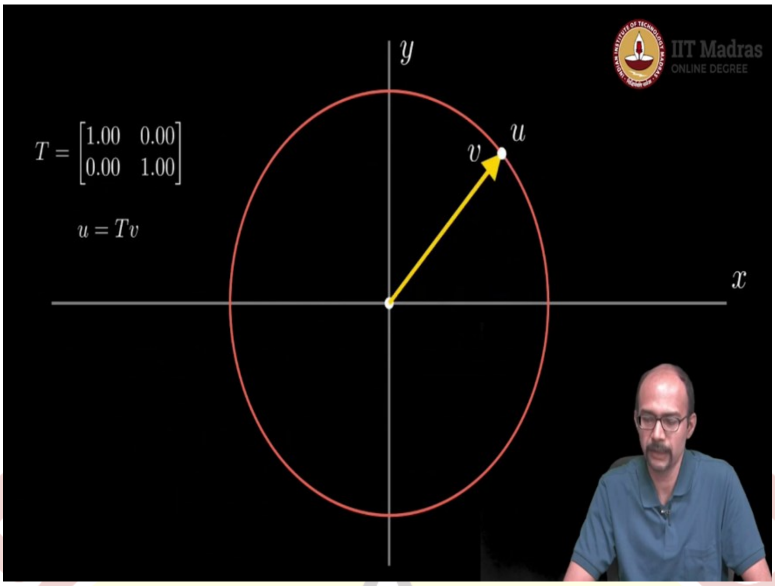

Let us see a very nice example of the, of this inaction. So, this is a video created by our team. So, you look at the xy plane. And let us look at this T. So, the T here is the identity. So, u is T × v, v is the vector in yellow. And if you apply T, let us see how it changes. So, if your T is 0.5 0.87 etc. So, look at how your T is changing, and the u is changing accordingly.

So, when it is - 1, on the diagonal, it is the opposite vector. And then it comes back closer to v. And now you are back. So, you can see that what happened here? These various different T's, maybe I let us play this again. So, if you play it again, let us look at the T's again. So, u is T × v, when T is the identity matrix, u is just (θ) v, when u changes to 0.5 - 0.87 0.87 0.5, you get changed. So, you can see that u is being rotated.

So, the length is remaining the same. So, it is rotating on the same circle that tells you that the length is remaining the same. So, the point here is that these T’s that we have all of them preserve length. And that is why they are orthogonal transformations. So, these are examples of orthogonal transformations.

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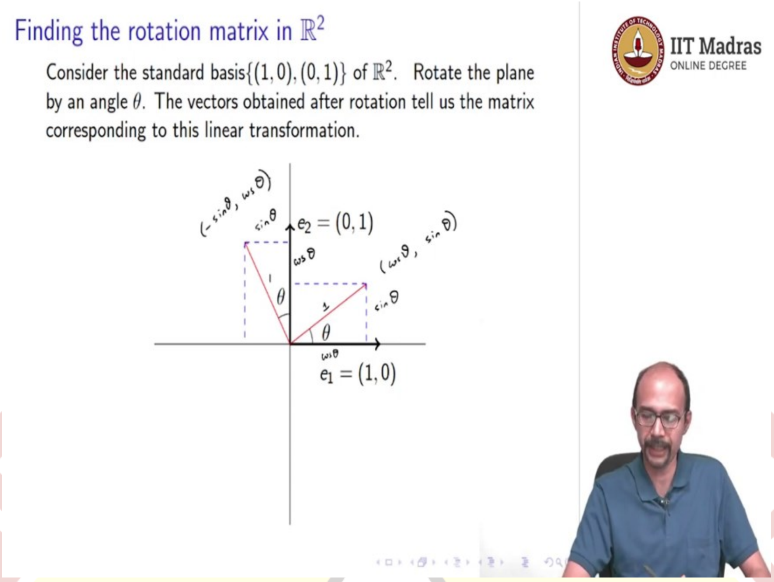

So, this is the next slide, finding the rotation matrix in the $\mathbb{R}^2$. So, what we saw were examples of what we call rotation matrices. So, they rotate vectors. So, how do we find rotation matrices in $\mathbb{R}^2$, so consider the standard basis, e_1 e_2 rotate the plane by an angle(θ), that is what we saw in the video. So, it was rotated by those various angles. And corresponding to that angle that you rotated, there was a T there.

So, we want to understand how to get the matrix corresponding to the rotation by(θ). So, if you rotate, then let us say you rotate by θ, then you get these vectors in red. So, you drop the you drop perpendiculars to these to the axis so that we can write down what are the coordinates of the vector. And what you can see is, so if this is(θ), then the opposite side, well, it is the hypotenuse. Since it is rotated, the hypotenuse is the vector in red. So, rotation will not change the length, so the hypotenuse has length 1, this red thing has length 1.

And this as a result has length sine(θ), and this as a result as length cosine(θ). So, the coordinate of this point, or the corresponding vector to this red line is cosine(θ), because that is the length that you get on the X axis when you drop the perpendicular or project and on the Y axis, its sine(θ). Similarly, if you do it for e_2 the coordinate you get are - sine(θ) and cosine of(θ). So, now we know how to write down the corresponding matrix.

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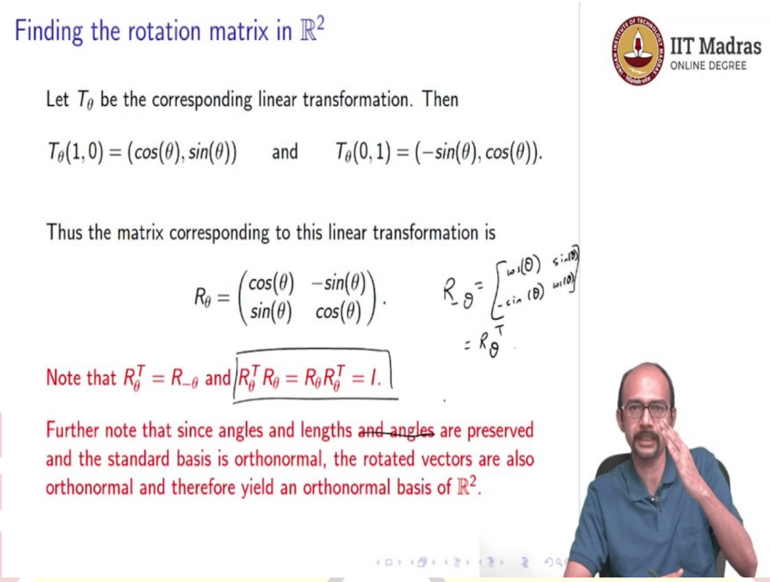

Let T(θ) we are corresponding linear transformation that is a linear transformation that you obtain by rotating by(θ), then we have seen T(θ) of 1, 0 is cosine(θ) sine(θ) and T(θ) of 0, 1 is - sine(θ) cosine(θ). So, now if you want to express this as a matrix meaning with respect to the standard ordered basis, then we just saw how to do it. Namely, you take these coordinates for 1, 0 and put it in your first column, take these coordinates for the image of 01 and put it in your second column. This we have seen in our previous videos.

And notice here that R(θ) transpose is R (-θ), and R(θ) transpose × R(θ) R(θ) × R(θ) transposes identity. And note which we have seen earlier that then that since angles and length. Now, there is a typo here, since angles and length are preserved, and the standard basis orthonormal, the rotated vectors are also all orthonormal. Why? Why is that the case? Because the angle between these two vectors cosine(θ) sine(θ) and - sine(θ), cosine(θ) is still 90 degrees.

Because we just rotated. So, they are orthogonal to each other, and the rotation will not change length. So, 1, 0 and 0, 1 have length 1. So, these two basis vectors also have length 1. So, there you learn orthonormal basis of $\mathbb{R}^2$. So, in other words, these columns that we have here, for R(θ) is an orthonormal basis to something what thing and it is worth observing that that fact is exactly borne out by this identity here, the R(θ) transpose × R(θ) is R(θ) × R(θ) transpose is identity. So, I encourage you to check that.

By the way, why is R(θ) transpose (θ) R (-θ). So, if you write down R (-θ), you get cosine of (-θ), but well, that same as cosine of(θ), sine of (-θ), which is - sine of(θ). And then - of sine(θ), with sine of, - of sine of (-θ), which is –(- sine(θ)) which is just sine(θ), and then again, cosine of(θ).

So, this is exactly R(θ) transpose. And now you can check that R(θ) transpose × R(θ) is R(θ) × R(θ) transpose identity. So, the point here is when you apply R(θ), you are rotating by(θ), when you apply R(θ) transpose, we just we have seen that that is the same as applying R (-θ), which is the same as saying that you are rotating back by(θ), you are rotating by (-θ). So, rotate by(θ) rotate backed by (-θ), you will exactly get the identity transformation. That is what this is seen.

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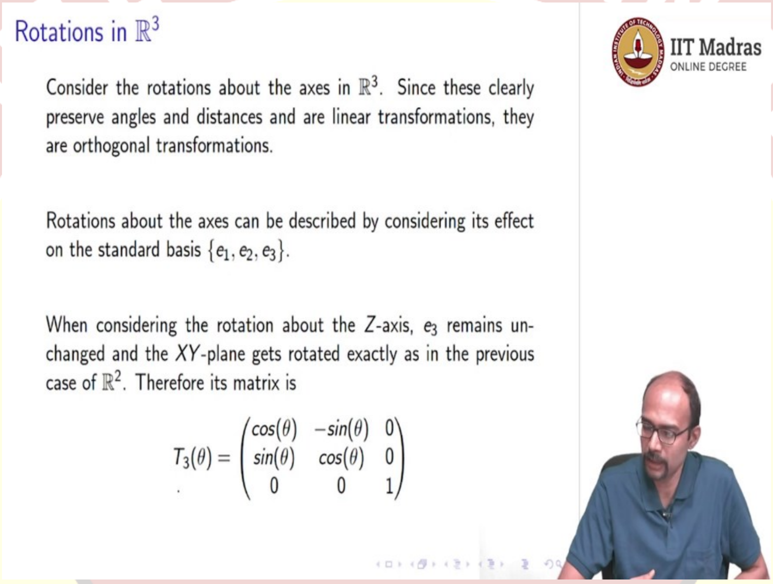

So, as I said, this video is largely a fun video more to do with the geometry behind what is all going on. So, let us look at rotations is $\mathbb{R}^3$. So, the rotations about the axes in $\mathbb{R}^3$ is the general rotations are a bit more to this difficult to describe. Since, these clearly preserves angles and distances and our linear transformations, if you rotate about the axis, then suddenly this is a linear transformation. And it is clearly want to preserve lengths and angles, this was what we also saw in $\mathbb{R}^2$.

So, these are orthogonal transformations. So, in the previous slide, we saw for $\mathbb{R}^2$, you could describe them by looking at the images of e1 and e2 here, you have to look at the images of e_1, e_2, e_3. So, if we consider the rotation about the Z axis, and then e3 is unchanged, because if you look at the Z axis that you are rotating about the Z axis, so the Z axis remains unchanged.

So e_3, which is a unit vector in the Z axis direction, remains unchanged. And the xy plane gets rotated exactly as in the previous slide. So, what we get is the corresponding matrix looks like cosine(θ) - sine(θ) 0 sine(θ) cosine(θ) 001. So, this is because we are basically rotating only the xy plane.

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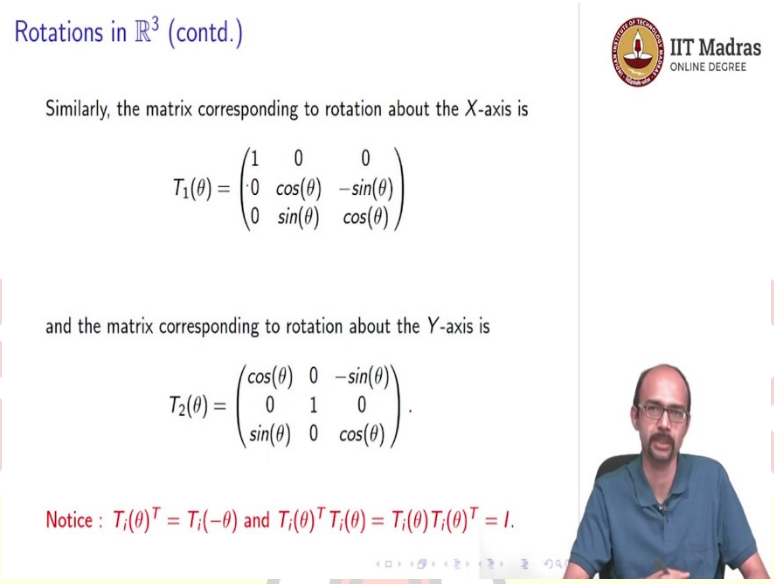

And if you followed that, then the same idea will tell you that if you have the X axis which is fixed, and you rotate about the X axis, so here is X Y Z, here is your X axis. So, you rotate like this fate. So, if you do that, then you think of the yz plane and on the yz plane, you will get a cosine(θ) - sine(θ) sine(θ) cosine(θ) and then it will be a block diagonal with one corresponding to the X Y X has been fixed.

And then for the Y axis, similarly, you have a one in the middle position and the minor corresponding that one or the matrix corresponding to that one is cosine(θ) - sine(θ) sine(θ) cosine(θ). So, I will encourage you to check this. And again, notice again, because of the previous slides, that Ti(θ) transpose is Ti (-θ) and Ti(θ) transpose Ti(θ) is Ti(θ) transpose is identity for the same reason that we gave in the, $\mathbb{R}^2$ example.

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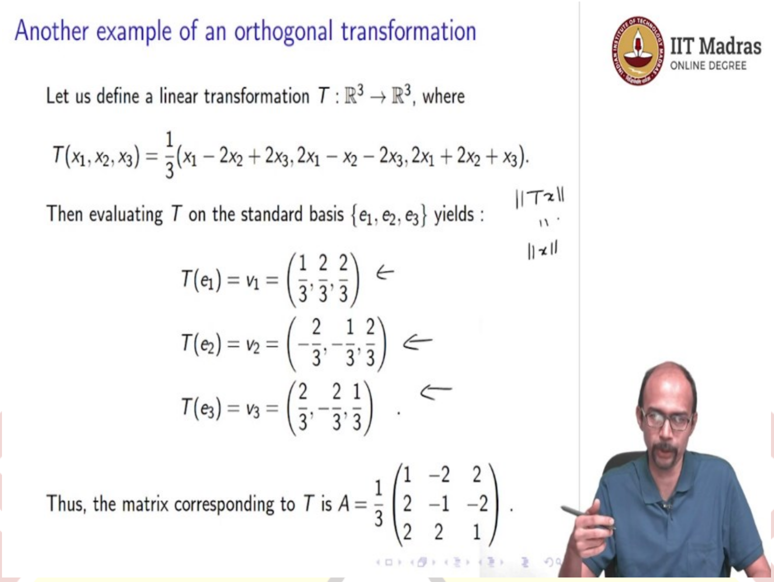

Let us do another example. So, this is a bit more involved to look at the linear transformation, T from $\mathbb{R}^3$ to $\mathbb{R}^3$, given by x_1, x_2, x_3 is $\frac{1}{3}(x_1 - 2x_2 + 2\,x_3, 2x_1 - x_2, -2x_3, 2x_1 + 2x_2 + \,x_3)$. Now, these coefficients should ring a bell in your mind. But if they do not, we will anyway study what happens to the standard bases vector and you will see that these are very familiar.

So, if you evaluate e_1, let us call that v_1, then you get one third, two thirds and two thirds. If you evaluate e_2 let us call that v_2, you get - $\frac{2}{3}$- $\frac{1}{3}$, two thirds. And if you evaluate e3, then you get v_3 which is two thirds - two thirds and one third. So, we have seen earlier that this is an ortho normal basis, this was exactly the an orthonormal basis we produced in the gram Schmidt process video.

So, the corresponding matrix is$1/3 \times \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$. So, where did this come from? So, the first column corresponds to v_1, the second column corresponds to v_2, and the third column corresponds to v_3, and I just taken$1/3$ common outside which I can do, because its scalar multiplication.

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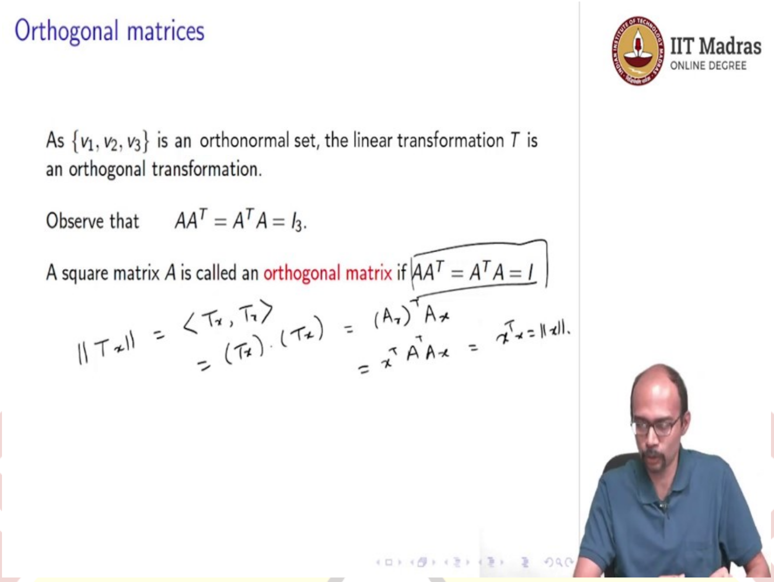

So, as v_1, v_2, v_3 is an orthonormal set the linear transformation T is an orthogonal transformation. Now, that needs a little bit of checking. So, what you have to do is, you have to take this T(x_1, x_2, x_3) and you have to check that if you apply the norm on both sides. So, if you take if you take norm of Tx, then this is the same as the norm of X. This is what you want to check. And the claim is that if, if I happen to know that v_1, v_2, v_3 is an orthonormal basis, which in this case I do, then then that will be true.

And the reason is this identity that A transpose is A transpose A is I_3 I will suggest you check this identity, actually it is a very easy check, because what happens is, when you do AA transpose, or A transpose A, really what you are getting is in each place is the inner product of two of these basis vectors. So, when they are not equal, it will be 0. And when they are equal, it will be 1. That is the main sort of reason we said $v_1, v_2, v_3$is an orthonormal set.

So, you obtain this and from here you can you can get that norm of Tx is (θ) norm of X, because when you apply norm, somewhere in there, this these inner products will play a role. So, a square matrix is called an orthogonal matrix. If AA transpose is A transpose A is identity, let me expand a little on that comment. So, the idea is that when you are looking at norm of Tx, well, you get in the product of Tx comma Tx.

But here we are looking at this in the product in terms of the usual dot product, so let us do it further dot product. So, Tx dot with Tx, this is the same in terms of matrices as looking at Tx transpose × Tx, which is the same as x transpose T transpose T × x, but T transpose T is identities. So, you get x transpose x, which is norm of x. This is the main point that is why we are demanding this thing here. I may have used bad notation and probably here, when I went to matrices, I should have used A and not T just A, every T of x when you evaluate these X as a vector and $\mathbb{R}^3$, as in terms of its coordinates you have to move to A. So, this is why it works. And so, an orthogonal matrix is AA transpose A transpose is identity that is the reason we call such matrices as orthogonal matrices, because when you use them to create linear transformations on Rn, then those linear transformations are going to be orthogonal linear transformations.

So, the take home from this video, I guess is that this notion of orthogonality is it is a very geometric notion, it studies essentially the angles and in particular right angles which, as we know since antiquity is something we have been very interested in then there are beautiful theorems related to that.

And this is some linear algebraic way of studying the changes that take place when you apply a linear transformation to angles and in particular, if they remain unchanged. Then these are some special linear transformations, and they are called orthogonal linear transformations. Of course, you have also maintained distances so both are inward. That is it for now. Thank you.