INDIAN INSTITUTE OF TECHNOLOGY MADRAS

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ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 10
Row Reduction

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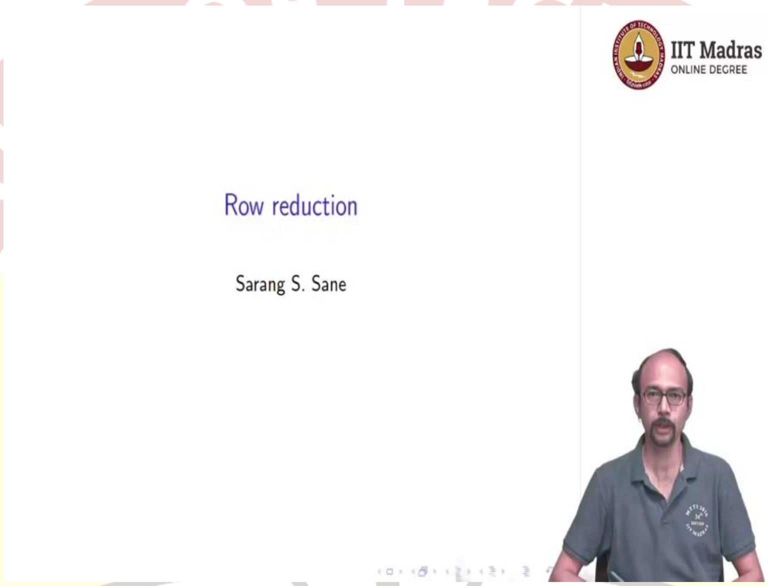

Hello and welcome to the Maths 2 component of the online BSc course on Data Science. In this video we are going to study something called Row Reduction. So if you remember what we did last video at the row echelon form. We looked at up when matrices are in row echelon form. And we saw that it is easy to solve equations when the coefficient matrix is in row echelon form. So what we are going to do in this video is to somehow see if we can reduce matrices into the reduced row echelon form.

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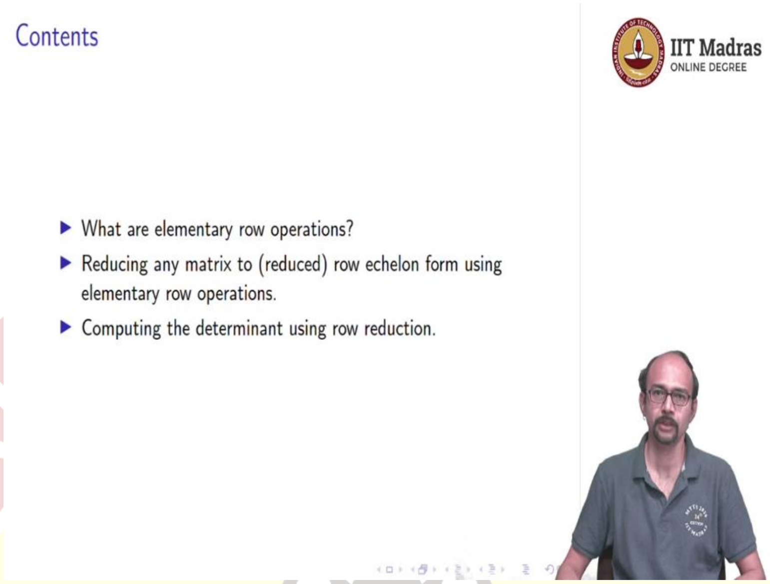

So what are the contents of this video; what are elementary row operations? We have actually seen this before when we studied determinants, but we are going to formalize them. Reducing any matrix to reduced row echelon form using elementary row operations. And then computing the determinant this is an application using row reduction. So this technique of reducing is called row reduction. And one application we will give in this video is we will compute the determinant.

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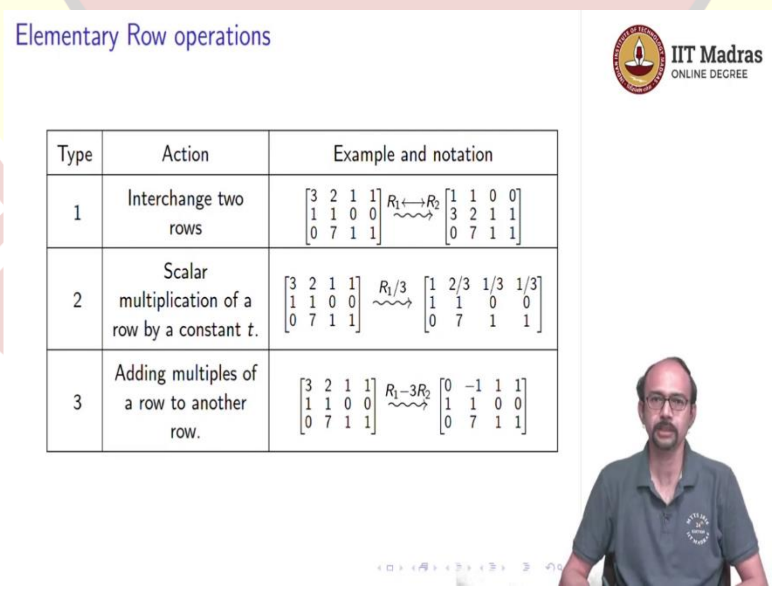

So let us start by asking what are elementary row operations. So I want to, this has a table and I am going to on the left I will describe the action which is the elementary row operation. And on the right we will have an example and you will also figure out what is the notation for the corresponding action. So the first elementary row operation, so this is called of type 1. So elementary row operation of type 1 is that when we interchange two rows.

So here is the example, so you look at this matrix $\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$ and we can interchange the first two rows. So the notation for this is R_1 and R_2 get interchange so we draw this two headed arrow or as the arrow on the both sides on the left hand and the right. And then we interchange the rows 1 and 2 and we obtain the matrix on the right. Then there is an elementary row operation of type 2.

So that is scalar multiplication of a row by a constant t. So for example in this the same example as before we divide R_1 by 3 or we can equivalently think of that as multiplying the first row by 1/3. So what happens then that means every entry is multiplied by 1/3. So every entry is on the, in the word divided by 3. So the other rows remain the same and the first row becomes $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$.

And then we have the elementary row operation of type 3 that is adding multiples of one row to another. So here is an example so here we subtract three times R_2 from R_1. So the important part in this notation is to remember that the first row meaning the row which occurs first in this notation is the one which is being changed. So this is written as $R_1 - 3R_2$ that is mean we have changed the first row by subtracting three times the entries of this second row to it.

So if you do that what happens the second and third row remain the same and the first row is changed to 0 minus 1. Why minus 1? Because we get 2 minus 3 which is 1 why 1 because we get 1 minus 3 times 0 and again 1 because again 1 minus 3 times 0 which is $\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$. This I hope this is clear. We have already seen this in determinants and we saw that something special happens for determinants we will recall that later. So let us see what these elementary row operations are useful for.

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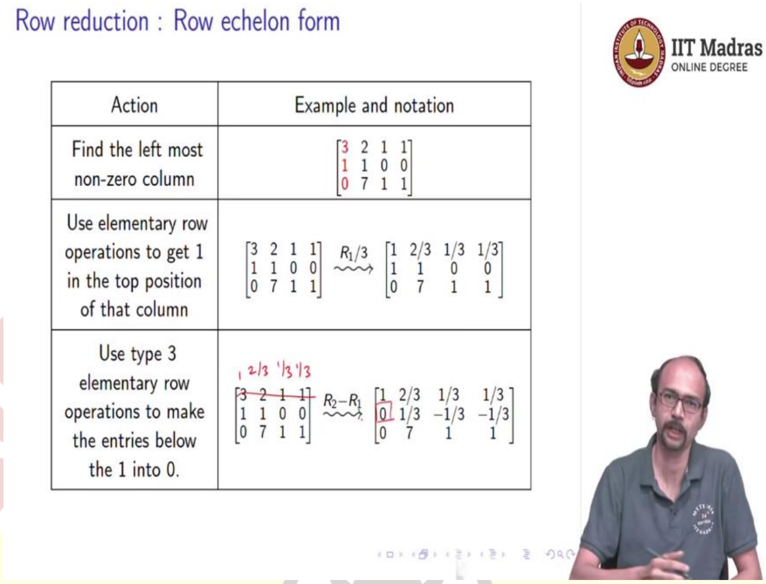

So there is a process by which we can use elementary row operations and reduce any matrix to the reduced row echelon form. And I mean there is a very particular way of doing this I am going to describe this algorithm and again once, again we have a on the left I will describe the action and on the right I will describe via an example and we will use the same notation that we used for elementary row operations.

So the first thing we do is find the left most non-zero column. So the left most non-zero column. Is the column 3 1 0 namely the first column of $\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$. So the next thing we do is use elementary row operations to get 1 in the top position of that column. So how do I do that, so in this case in the example I have 3 in the 1 1 place. So I can just divide the entire row by 3 I can divide the first row by 3 or multiply the first row by one third.

So this is an elementary row operation of type 2. So I can use elementary row operation of type 2 provided the 1 1 entry is non-zero. And get a 1 in that place. So here we if we do that this is exactly the example that we saw in the previous slide. Suppose now instead of this my first row happens to be 0 meaning the 1 1 entry happens to be 0 not the first row but the 1 1 entry happens to be 0. Then I cannot multiply by anything to get a 1 over there.

So then what I do is I will interchange the first row with some other row. Where that first entry in that row is going to be non-zero. And how do I know that there is an entry like that which is non-zero. That is because this was the left most non-zero column. So somewhere there must be a non-zero entry. So if the 1 1 entry is 0 you can interchange and a non-zero entry over there. So in now example we do not need to do that but otherwise you can get a non-zero entry.

And then you can multiply to make it 1. So use elementary row operations to get 1 in the top position of that column. So then once we get 1 in that column in that position use elementary row operations and of type 3. So you can use that 1 and subtract out some the correct multiples and get zeros in the entries which are below that 1. So in this case we can subtract so there is a typo here, this should have been 1 over here.

In fact, this entire row is not correct. This should have been the row 1 2/3 1/3 and 1/3 . So now in this $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$, we can do $R_2 - R_1$. So what does that mean in from the second row we subtract out the first row. So the first row remains same third row remains same the second row changes and doing this the second row this entry 0 over here. This entry 1 over here becomes 0 can then the other entries change correspondingly. So we can use type 3 elementary row operations to make all the entries below that 1 as 0. So in this example the next entry is already 0. So we do not have to do anything.

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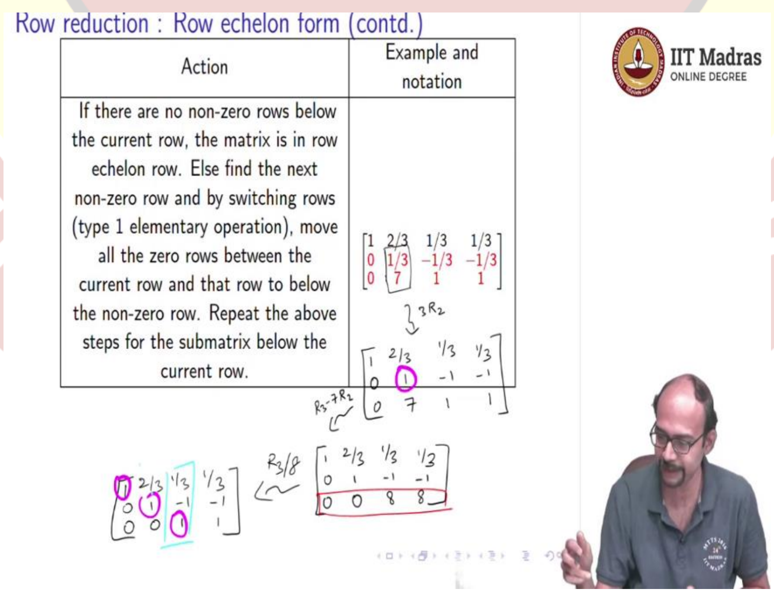

So what is the next thing we do? If there are non-zero entries below so that is mean all entries are 0 then we are done. Because the matrix is already in row echelon form. So what is our aim that the matrix should somehow come to row echelon form and then from row echelon form we will move to reduced echelon. So we want to put it in row echelon form, so if there are no non-zero rows then what that we mean is we have a row of which has which start with 1 and an everything below is 0 so we are done.

If there is indeed a non-zero row below then it may happen that there are some 0 rows and then there is non-zero. So then we can interchange rows and bring that non-zero row to just below the row that we have just finish dealing with. So once we do that what do we do? Now we look at the matrix meaning the submatrix which consist of all the entries the other than the row that we just dealt with.

So repeat the above steps for the submatrix below the current row. So what is that submatrix? That submatrix is in red here. So it is the submatrix consisting of the next two rows. So $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$. And what do we do now, we will repeat whatever process we did for the previous part we will do it all over again. So let us recall whatever process was, so process was to locate the left most non-zero column.

May be I should say the first non-zero column so here is a first non-zero column. So once you locate this you change it, so that the entry there is 1. So I want to make 1/3 into a 1. So the first row remains the same and what do I do I multiply the second row by 3. So I $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & -1 & -1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$and I do not do anything to the third row. So this is a elementary row operation of type 2.

And then what do I do next? I look at this 1 over here. So this is going to play the role of my leading coefficient and then below this 1 I want a 0. So I do $R_3 - 7R_2$. So what do I get? So again nothing happens to the first two rows and then for the third row I get 0 in this place which is what I was looking for. And then in the other two I have to compute.

So this is 1-7*(-1), so that is 1+7=8 and then this is again 1-7*(-1) which is again an 8. $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 8 & 8 \end{bmatrix}$So now what do I have to do? Well I am done dealing with this column. So I have got a 0 below so now I have to look for the next, I have to look at the rows below and ask is there a non-zero row, yes indeed there is a non-zero row. Then I have to move that row to just below this row that I have just dealt with.

So here there is no intermediate zero rows, so it is already there so I do not have to do any elementary row operations of type 1 to switch rows. So then within this I look for the first within this submatrix so what submatrix do I have now. So I have this submatrix, so within this submatrix I look for the first non-zero column. That is the third column. And then within that column I try get a 1 in the first possible place.

So in this case I have to multiply by 1/8 or that is divide by 8. So I do $R_3/8$ and what do I get? I get $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. So using elementary row operations I have changed my matrix into a matrix which is in row echelon form. This matrix is in row echelon form, why is it in row echelon form? Because if you look at each row then the leading entry the first non-zero entry is 1 sure that is the case 1 1 1.

Then if you look at the leading entry, so we look at the corresponding column, so the column is for the third row let say it is in this column. Then if I look at the previous row it is on the left, so that is fine. So we are going in a descending this way. We are going descending this way that is what we want when we have row echelon form. So and if there are 0 rows they have to be the non-zero rows. So here there is no 0 row, so we are good. So this is indeed in row echelon form.

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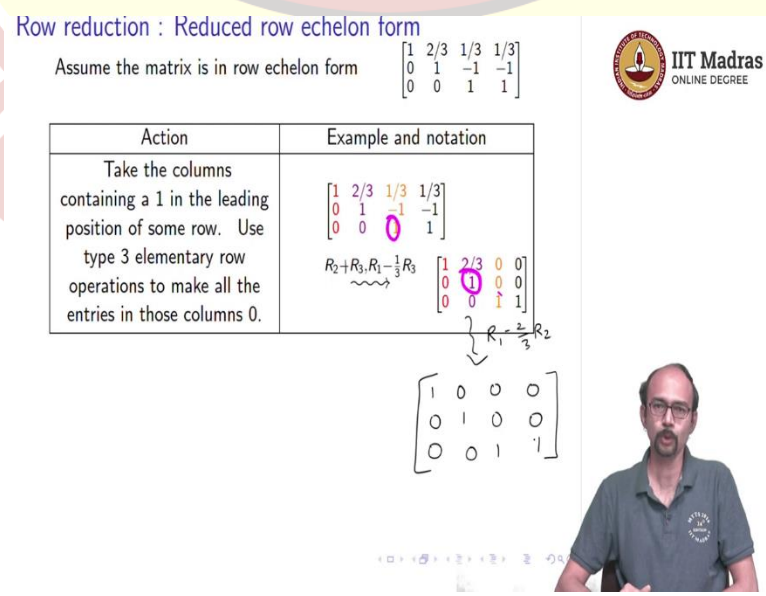

So now how do we reduce this further to reduced row echelon form? So here is the matrix we will assume that we have our matrix is already in row echelon form and then we will describe how to reduce it to reduced row echelon form so here is the example that we were just doing. So $\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. So what is action we have to take, take the columns containing 1 in the leading position of some row.

So they are the highlighted columns the first column the second column and the third column. And then using the ones the leading terms, the leading ones use elementary row operations of type 3 to make everything above that 1 into 0. So we have done it for below. So now we can do it for above, so here there is small trick you can really do it for in any for any column. But we have to choose which column to go with first and then which column to go with next.

So it is always best to start from the extreme right. So you start with the third column. So for the third column what do we do? We use this 1 over here this 1 over here, I should rather say you use this 1 over here and make the entries above that 1 into 0. How do I do that? So I do $R_2 + R_3$ so there is typo here this should have been $R_2 + R_3$. And I do $R_1 - \frac{1}{3}R_3$ and I will get zeros correspondingly so this is a matrix I get.

And now I can move 1 column I can move towards the left so if I do that let us complete this. So now I want to get a 1 I want to use this 1 over here to make the 2/3 above it and into 0 so how do I do that? I do $R_1 - \frac{2}{3}R_2$ and the point is since I have already taken care of the entries to the right it will not change what happens on the right. So it will only effect this column or it may affect, yeah it may only affect this column.

So it would not change anything any other column. So you do not have to worry about the other columns. So then in that case I get $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Yeah I should have been careful there. No I should not say it would not affect any other column what I should was it does not affect any other column which has leading coefficients which is what we really care about. So $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and now this matrix is indeed in reduced row echelon form.

So it was a leading row echelon form, so the leading entries here every row is 1 then for the ones if you look at the columns all the entries are 0. And of course there is no non-zero rows at all. So this is in reduced row echelon form now. So this is how we change a matrix to first into row echelon form and further into reduced row echelon form.

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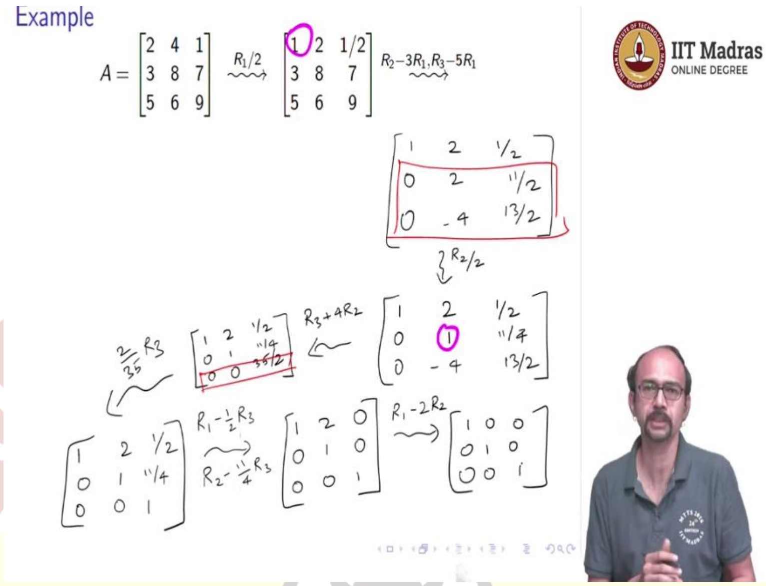

So let us do this example so for this example how do I go for doing this, let us do the first step. So the first step is you look at first non-zero column. So in this case the first column is non-zero and then you try to get 1 in the beginning place the 1 1 entry over here. So here I have 2 over there so I just divide by 2 and by dividing by 2 I get 1 2 and ½ in the first row and the other rows remain same.

Let us do one more step. So now using the 1 in this place this 1 1 place if we can sweep out meaning we can make the 3 and 5 into 0 how do I do that? I do $R_2 - 3R_1$ and $R_3 - 5R_1$. Then what do I get? So let me write this down, so what I get is well the first row remains the same so 1 2 and ½ . And then this entry become 0, so this is $R_2 - 3R_1$ that is $8 - 6 = 2$ and then $7 - \frac{3}{2} = \frac{11}{2}$.

And then for the third row I have $R_3 - 5R_1$. So this is 0 and then $6 - 10 = -4$ and then $9 - \frac{5}{2} = \frac{13}{2}$. So this is what I get after doing this step $\begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 2 & \frac{11}{2} \\ 0 & -4 & \frac{13}{2} \end{bmatrix}$. So I have finished dealing with my first column. So now I look for my submatrix. So here is I my submatrix over here and then we repeat this procedure, so if you repeat this procedure, what do we get? What we get is look at the first non-zero column so for the first non-zero column which one is that, that is the column the 2 minus 4 column.

So what do I do I try to get a 1 in the first place, how do I do that? I do that by doing $\frac{R_2}{2}$. So if I do $\frac{R_2}{2}$ I get 0 1 and $\frac{11}{4}$ and then the third row remains the same. And then what should I do? I should use this 1 and try to get a 0 below it. So I should use the 1 and get a 0 below so I should make that minus 4 into 0 so how do I do that? I do $R_3 + 4R_2$.

What do I get? So $R_3 + 4R_2$ will be so the row will remain same the second row remain the same and the third row becomes $\begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & \frac{35}{2} \end{bmatrix}$. And then the last thing so now this second column is done, so now I should move to the third to the submatrix, what is the submatrix? Well it is just this last row.

And I want to get 1 in the place of that $\frac{35}{2}$ so what do I do I just multiply by $\frac{2}{35}$ and that is the final step in this procedure. Sorry, so to do that I do $\frac{2}{33}R_3$and I get $\begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & 1 \end{bmatrix}$. So this is the row echelon form. So if I want to move from here to reduced row echelon form I should make the entries above the 1s into 0. So there is a as I said you move from the right towards the left.

So you take the last column, so you take the third column here. So I should do $R_1 - \frac{1}{2}R_3$ and $R_2 - \frac{11}{4}R_3$. So if I do that what do I do? What do I get? So I will get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So I should maybe so that is what we get. And then the final step is to get a 0 in the place of this 2 above which is above that 1 so to do that I should do $R_1 - 2R_2$ which gives me $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is familiar matrix as we know this is the identity matrix and very shortly we will see how we got the identity matrix. So why we got the identity matrix fine. So I hope the procedure is clear. So this is the second example that we have seen.

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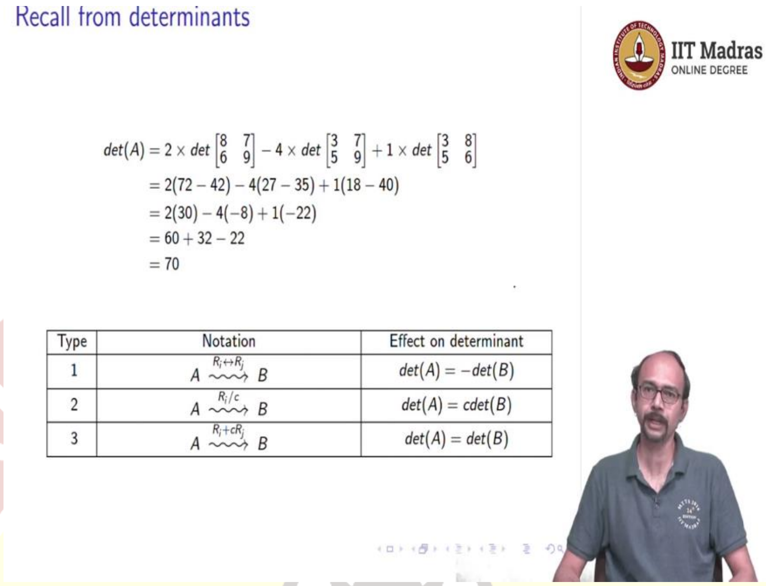

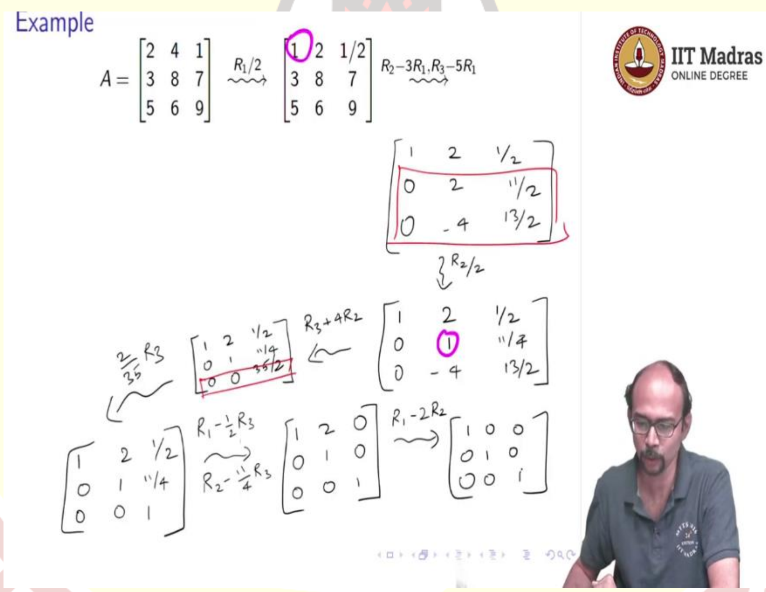

So now let us move to the determinants part. So let us in that same example that we had previously let us recall the determinant. So let us may be quickly look at that example again. So this example was $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$. Let me write that down $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$. And you can check in we have actually done this example in the determinants video. And you can check that if you use the definition of the determinant then you get a 70.

We did this example so I am not going over this in detail again but you can check the computation if you feel you want to recall what the determinant was. So keep this in mind, so now let us recall also from the determinants lecture that we studied what was the effect of elementary row operations on determinants. So if you perform a elementary row operation of type 1. So if I interchange two rows and the, from matrix A I get matrix B then the effect on the determinant is that determinant of A is minus determinant of B.

And then if I divide matrix particular row of the matrix A by c then determinant of A is c times determinant of B. So if you divide by c determinant of A is c times determinant of B. And then the type 3 elementary row operation was where we added a multiple of one row to another. And in that case the determinant remained unchanged, these are the properties we stated in the determinant's lecture. So let us see how we can use this method of row reduction to calculate the determinant.

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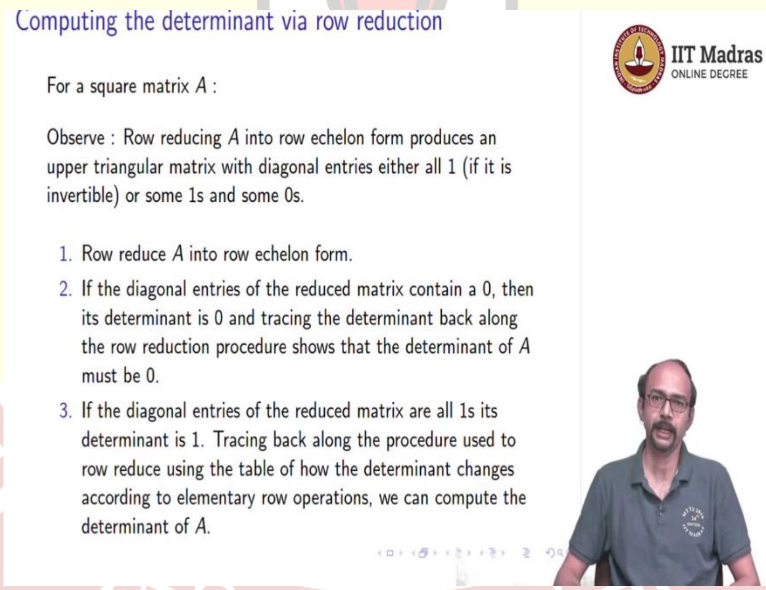

So for a square matrix A row reducing into row echelon form produces an upper triangular matrix, so if you remember the definition of the row echelon form. Remember the 1s are going like this, so that means if you have square matrix that the 1s must be either on the diagonal or above the diagonal. And everything below the 1s is 0 that means this matrix is in upper triangular matrix.

So and on the diagonal you have either 1s or 0s, either you have of the 1s or you have few 0s fine. So what is I mean why are we saying this? So the point of this statement is that remember that for an upper triangular matrix the determinant is given by the product of the diagonal entries. So this was again something we saw in the determinants video and now we will use this great effect by this row reduction technique.

So now let us determine how to find determinants via row reduction. So you have a square matrix is A row reduce it into row echelon form, if the diagonal entries of the reduce matrix is continuous 0, then its determinant is 0, determinant of A is 0. So you do not have to compute, do any computations at all, it is already 0 and how do I get this? Well you can trace back along the elementary row operations.

We know how the determinant changes and it either remains the same or it gets multiplied by some constant or it picks up a minus sign. So if you have final determinant is 0 then your beginning determinant is also going to be 0 that is why it must be 0 and if the other option is that the diagonal entries are all 1. And if they are all 1 then we know that determinant and then we can trace back through the operations that we have done and find the determinant.

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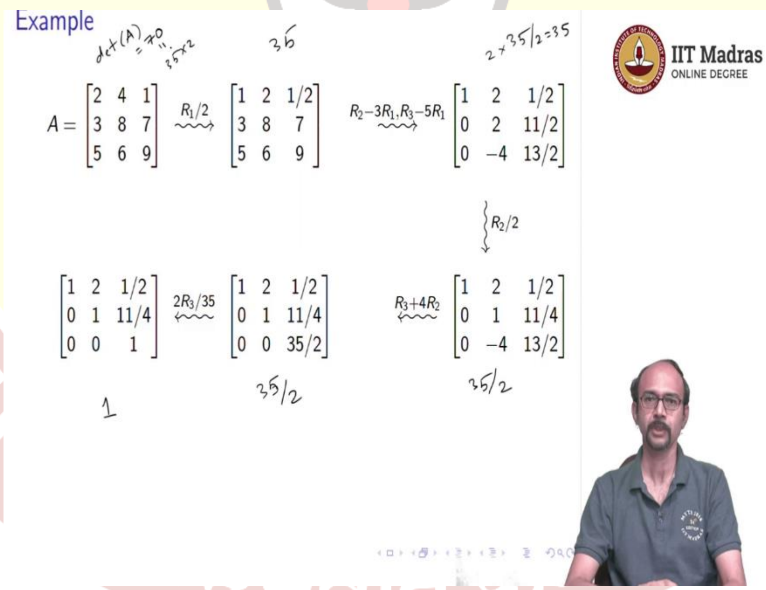

So let us see an example. Here is the example that we did earlier in the video. So here was our matrix A which is $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$. And remember that we said that this determinant is 70. So we know that determinant of A is actually 70. This was explicitly computed from the determinant video. Now let us use this procedure and see from here how do we get the determinant of A. So let us compute the let us start from here.

So the determinant over here is 1. Why? Product of diagonal entry, it is an upper triangular matrix. So notice that I do not have to go all the way till reduced row echelon form I can stop it the row echelon form. So how do I move to this, so I have divided this by 35/2. So this determinant is 35/2 we can also get this by multiplying by the diagonal entries because it is again upper triangular.

So what is this determinant well how did we go from here to here? We went by an elementary operation of type 3. So it does not change your determinant, so this is still 35/2. How did we go from here to here? Well we divide it by 2. So this is going to be $2 * 35/2 = 35$, so that determinant of this matrix is 35, you can check this by explicitly computing. How did we go from here to here elementary operations of type 3?

They do not change your determinant so this is still 35 and how did you go from here to here, we divide it by 2. So the determinant here must be 35 multiplied by 2 and that is indeed 70. That is what we describe on the previous slide and this is an example of the same. So today we have seen in this video we have seen that we can thus this wonderful procedure called row reduction that you can use to reduce any matrix into the reduced row echelon form. First to row echelon form and then into reduced row echelon form.

It is an algorithm we explicitly described it now in the previous video we have seen that if a matrix is in reduced row echelon form we can easily get solutions for $Ax = B$. When A is in reduced row echelon form. Now we are going to put these two together in the next video and describe a method very powerful method to obtain the solutions of any general system of linear equations $Ax = B$.

And in today video the other thing we did was we saw that this procedure of reducing a matrix to its reduced row echelon form or its row echelon form for a square matrix we can easily compute the determinant. In fact, if you go ahead and sort of try to work out what is more efficient, whether you can compute the determinant by our definition or by this procedure you will find this procedure is much, much easier. And this is very useful because if you are matrices are very large, then computing the determinant using the definition takes much, much more time, whereas this procedure gives you a far more efficient way of doing it. Thank you.