Week-8
Mathematics for Data Science - 2
Rank of a matrix and Linear Transformation
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. A function $T : V \to W$ between two vector spaces V and W is said to be a linear transformation if the following conditions hold:

   - **Condition 1:** $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$.
   - **Condition 2:** $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{R}$.

   Consider the following function:

$$T : \mathbb{R}^2 \to \mathbb{R}$$

$$T(x, y) = \begin{cases} 3x & \text{if } y = 0 \\ 4y & \text{if } y \neq 0 \end{cases}$$

   Which of the following statements is true?

   ○ Option 1: Both the conditions hold.

   ○ Option 2: Condition 1 holds but Condition 2 does not.

   ○ **Option 3:** Condition 2 holds but Condition 1 does not.

   ○ Option 4: None of the conditions hold.

Solution: Consider two vectors $(1,0)$ & $(0,1)$ from $\mathbb{R}^2$ and suppose T is satisfying the first condition, then $T((1,0)+(0,1)) = T(1,1) = 4$

$$\Rightarrow T(1,0) + T(0,1) = 4$$

$$\Rightarrow 3 + 4 = 4$$

$$\Rightarrow 7 = 4$$

But this is not true. That means our assumption was wrong.

So, condition 1 is not satisfied by T.

Now, let $c \in \mathbb{R}$ & $(x,y) \in \mathbb{R}^2$

Consider $T(c(x,y)) = T(cx, cy)$

$$= \begin{cases} 3cx & \text{if } y = 0 \\ 4cy & \text{if } y \neq 0 \end{cases}$$

$$= cT(x,y)$$

$$\Rightarrow T(c(x,y)) = c\,T(x,y)$$

So, second condition is satisfied by T.

2. Suppose the matrix representation of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the basis $\beta = \{(1,0,1),(0,1,0),(0,0,1)\}$ for the domain and $\gamma = \{(1,0,0),(0,1,0),(1,0,1)\}$ for the range, is $I_{3\times3}$, i.e., the identity matrix of order 3. Which one is the correct matrix representation of the linear transformation T with respect to the standard ordered basis of $\mathbb{R}^3$ for both domain and range.

   - ○ Option 1: $I_{3\times3}$ i.e., identity matrix of order 3.
   - ○ Option 2: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
   - ○ Option 3: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$
   - ○ **Option 4:** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Solution :- Given $I_{3\times3}$ is is the matrix represenation of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. Ordered bases β & γ.

So,

$$T(1,0,1) = 1(1,0,0) + 0(0,1,0) + 0\cdot(1,0,1) = (1,0,0)$$

$$T(0,1,0) = 0(1,0,0) + (0,1,0) + 0(1,0,1) = (0,1,0)$$

$$T(0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(1,0,1) = (1,0,1)$$

Now, to get matrix represenation w.r.t standard Ordered bases of $\mathbb{R}^3$, we have to make each elements of Standard ordered basis need to expand along bases β

And expand along the basis β.

Let $(1,0,0) = a(1,0,1) + b(0,1,0) + c(0,0,1)$

$\Rightarrow (1,0,0) = (a, b, a+c)$

$\Rightarrow a = 1, b = 0, c = -1$

So $(1,0,0) = 1(1,0,1) + 0(0,1,0) - 1(0,0,1) = (1,0,1) - (0,0,1)$

Similarly, $(0,1,0) = 0(1,0,1) + 1(0,1,0) + 0(0,0,1) = (0,1,0)$

& $(0,0,1) = 0(1,0,1) + 0(0,1,0) + 1(0,0,1) = (0,0,1)$

Now, need to find linear transformation of each element of the standard ordered basis.

$T(1,0,0) = T(1,0,1) - T(0,0,1)$

$= (1,0,0) - (1,0,1)$

$\Rightarrow T(1,0,0) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) - 1(0,0,1)$

$$T(0,1,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

$$T(0,0,1) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

Hence, matrix representation of T w.r.t standard ordered bases is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

3. Match the linear transformations and sets of vectors in column A with the images of those sets under the linear transformation in column B and the geometric representations of both sets in column C.

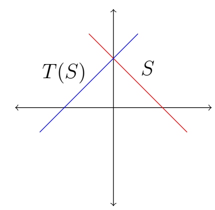
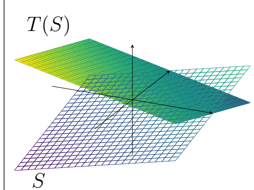
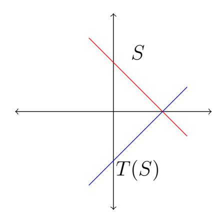
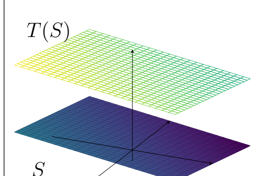
	Matrix form of linear transformation (Column A)		Image of the given set (Column B)		Geometric representations (Column C)
i)	$T: \mathbb{R}^3 \to \mathbb{R}^3$ $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ Set: $S = \{(x, y, z) \mid x + y + z = 1\}$	a)	$T(S) = \{(x, y) \mid x - y = 1\}$	1)	
ii)	$T: \mathbb{R}^3 \to \mathbb{R}^3$ $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Set: $S = \{(x, y, z) \mid x + y + z = 1\}$	b)	$T(S) = \{(x, y, z) \mid x + y + z = 3\}$	2)	
iii)	$T: \mathbb{R}^2 \to \mathbb{R}^2$ $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Set: $S = \{(x, y) \mid x + y = 1\}$	c)	$T(S) = \{(x, y) \mid x - y = -1\}$	3)	
iv)	$T: \mathbb{R}^2 \to \mathbb{R}^2$ $T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ Set: $S = \{(x, y) \mid x + y = 1\}$	d)	$T(S) = \{(x, y, z) \mid x + y - z = 1\}$	4)	

Table: M2W8G1

Choose the correct option from the following.

○ Option 1: i → d → 2, ii → b → 4, iii → a → 3, iv → c → 1

○ Option 2: i → b → 4, ii → d → 2, iii → c → 1, iv → a → 3

○ Option 3: i → b → 2, ii → d → 4, iii → a → 1, iv → c → 3

○ **Option 4:** i → b → 4, ii → d → 2, iii → a → 3, iv → c → 1

Solution:- i) Given $S = \{(x,y,z) \mid x+y+z=1\}$

$$= \{(x, y, 1-x-y) \mid x, y \in \mathbb{R}\}$$

let $v = (x, y, 1-x-y) \in S$.

So
$$T(v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 2x+2y+3-3x-3y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \\ 3-x-y \end{bmatrix}$$

Let $X = x$, $Y = y$, $Z = 3-x-y$

So $Z = 3 - X - Y \Rightarrow X + Y + Z = 3$

Hence, $\begin{bmatrix} X \\ Y \\ 3-X-Y \end{bmatrix}$ is a vector of $T(S)$

Or we can write. $T(S) = \{(x, y, z) \mid x+y+z=3\}$

Observe that elements of S & $T(S)$ are set of points from the plane $x+y+z=1$ & $x+y+z=3$ respectively.

If there are some x, y, z which satisfying both the equations, it yields $1 = 3$ which is not true.

That means there are no points of intersection of these two planes so both planes are parallel to each other.

Which is represented by 4th figure in column C.

ii) $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ & $S = \{(x,y,z) \mid x+y+z=1\}$

So $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1-x-y \end{bmatrix} = \begin{bmatrix} x \\ y \\ x+y-1 \end{bmatrix}$, here $z = x+y-1$

Hence $T(S) = \{(x,y,z) \mid x+y-z=1\}$

Observe that both planes $x+y-z=1$ & $x+y+z=1$ intersects at infinitely many points which is represented in the figure 2.

iii) $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $S = \{(x,y) \mid x+y=1\}$

So $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ x-1 \end{bmatrix}$, in the xy-plane $y = x-1$

So $T(S) = \{(x,y) \mid x-y=1\}$

Observe that lines $x-y=1$ & $x+y=1$

intersects at $(1,0)$ which represented in figure 3.

iv) $T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $S = \{(x, y) \mid x + y = 1\}$

So ,
$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 2x + 1 - x \end{bmatrix} = \begin{bmatrix} x \\ 1+x \end{bmatrix},$$

in the XY-plane $y = 1 + x \Rightarrow x - y = -1$

Hence $T(S) = \{(x, y) \mid x - y = -1\}$

which correspond to figure 1 in column C.

4. Consider two linear transformations T and S from $\mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (2x + y, x + y)$ and $S(x, y) = (x + cy, x + 2y)$. Let A and B be matrix representation of linear transformations T and S with respect to the standard bases of $\mathbb{R}^2$ respectively.
   Consider the following statements

   - **P:** If $c = 1$, then A and B are similar matrices.
   - **Q:** If $c = 2$, then A and B are similar matrices.
   - **R:** If $c = 1$ and $P^{-1}AP = B$, then $P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$
   - **S:** If $c = 1$ and $P^{-1}AP = B$, then $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.
   - **T:** If $c = 1$, then there are infinitely many P satisfying the equation $P^{-1}AP = B$.

   Which of the following options is true?

   ○ Option 1: Only P is true
   ○ Option 2: Both P and Q are true.
   ○ Option 3: Both R and S are true.
   ○ **Option 4:** Except Q and R, all the statements are true.

Solution :- Given $T(x,y) = (2x+y, x+y)$
& $S(x,y) = (x+cy, x+2y)$

Standard ordered basis of $\mathbb{R}^2$ is $\{(1,0), (0,1)\}$

So $T(1,0) = (2,1)$, $T(0,1) = (1,1)$

So matrix representation of T is $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Again $S(1,0) = (1,1)$, $S(0,1) = (c, 2)$

So matrix representation of S is $B = \begin{bmatrix} 1 & c \\ 1 & 2 \end{bmatrix}$

Now if $c = 1$ then

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

let A & B are similar matrices then $\exists\, P \ni$

$$P^{-1}AP = B$$

$$\Rightarrow \quad AP = PB$$

let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

so $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} a+b & a+2b \\ c+d & c+2d \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2a+c &= a+b \\ 2b+d &= a+2b \\ a+c &= c+d \\ b+d &= c+2d \end{aligned}\right\} \Rightarrow \begin{aligned} a-b+c &= 0 \quad \text{--- (1)} \\ a &= d \quad \text{--- (2)} \\ b-c-d &= 0 \quad \text{--- (3)} \end{aligned}$$

from equations (1) & (2)

$$d-b+c = 0 \Rightarrow b-c-d = 0$$

which is the exactly equation (3)

$\Rightarrow \quad b = c+d$

Now let $c=1, d=1 \Rightarrow b = 2$ & $a = 1$

so $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ Similarly we can obtain ifinitely many P which satisfying the equation $P^{-1}AP = B$

so statements P & T are true.

in statement S $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ which is the same which we obtained so statement S is also true.

Statement R: Observe that if $P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

then $AP \neq PB$ so $P^{-1}AP \neq B$.

Statement Q: if $c = 2$ then $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Now, let $\exists P \ni P^{-1}AP = B$ or $AP = PB$

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $AP = PB \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} a+b & 2a+2b \\ c+d & 2c+2d \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2a+c &= a+b \\ 2b+d &= 2a+2b \\ a+c &= c+d \\ b+d &= 2c+2d \end{aligned}\right\} \Rightarrow \begin{aligned} a-b+c=0 &\quad (1) \\ d = 2a &\quad (2) \\ a = d &\quad (3) \\ b-2c-d=0 &\quad (4) \end{aligned}$$

observe that in equations (2) & (3) $d = 2a$, $a = d$ can be possible only $a = 0$, $d = 0$.

Substitute these values of a & d in equation (1) & (2) we get $\left.\begin{aligned} -b+c=0 \\ b-2c=0 \end{aligned}\right\} \Rightarrow b = 0,\ c = 0$

So actually, $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is not invertible / that means if $c = 2$, then there is no $P \ni P^{-1}AP = B$, so statement Q is not true.

2 Multiple Select Questions (MSQ)

5. Consider a linear transformation $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ such that $S(A) = A^T$. Let B be the matrix representation of S with respect to the standard ordered bases:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $M_2(\mathbb{R})$. Choose the set of correct option

- ◯ Option 1: The order of the matrix B is 2×2.
- ◯ **Option 2:** The order of the matrix B is 4×4.
- ◯ **Option 3:** The dimension of the row space of the the matrix B is 4.
- ◯ Option 4: The dimension of the column space of the matrix B is 3.
- ◯ Option 5: The nullity of the the matrix B is 1.
- ◯ **Option 6:** The rank of the the matrix B is 4.
- ◯ **Option 7:** S is surjective.

<u>Solution</u>:- Given $S(A) = A^T$

Let $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$S(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = e_1$

$S(e_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e_3$

$S(e_3) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = e_2$

& $S(e_4) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = e_4$

Hence, matrix representation of S w.r.t given ordered basis is $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Row echlon form of matrix is identity matrix of order 4.

Hence, the dimension of the row space & the column space and also the rank of matrix of B is 4.

Use, the rank nullity theorem matrix nullity of the matrix is 0.

6. Let L and L' be affine subspaces of $\mathbb{R}^3$, where $L = (0,1,1)+U$ and $L' = (0,1,0)+U'$, for some vector subspaces U and U' of $\mathbb{R}^3$. Let a basis for U be given by $\{(1,1,0),(1,0,1)\}$ and a basis for U' be given by $\{(1,0,0)\}$. Suppose there is a linear transformation $T : U \to U'$ such that $(1,0,1) \in \ker(T)$ and $T(1,1,0) = (1,0,0)$. An affine mapping $f : L \to L'$ is obtained by defining $f((0,1,1)+u) = (0,1,0)+T(u)$, for all $u \in U$. Which of the following options are true?

- ◯ **Option 1:** $L = \{(x, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$.
- ◯ Option 2: $L' = \{(x, y+1, 0) \mid x, y \in \mathbb{R}\}$.
- ◯ Option 3: $L = \{(x-y, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$.
- ◯ Option 4: $L = \{(x, x+1, y+1) \mid x, y \in \mathbb{R}\}$.
- ◯ **Option 5:** $f(x, y+1, x-y+1) = (y, 1, 0)$
- ◯ Option 6: $f(x-y, y+1, x-y+1) = (x, y+1, 0)$
- ◯ Option 7: $f(x, x+1, y+1) = (y, 1, 0)$
- ◯ Option 8: $f(x, y+1, x-y+1) = (0, 1, y)$

Solution:- Let $\beta = \{(1,1,0), (1,0,1)\}$

So $U = \text{span}(\beta) = \{x(1,1,0) + y(1,0,1) \mid x, y \in \mathbb{R}\}$

$= \{(x+y, x, y) \mid x, y \in \mathbb{R}\}$

Let $X = x+y, \; Y = x, \; Z = y$

$\Rightarrow \quad Z = X - Y$

So, we can write, $U = \{(x, y, x-y) \mid x, y \in \mathbb{R}\}$

So, $L = (0,1,1) + U = \{(x, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$

Similarly, $U' = \{(x, 0, 0) \mid x \in \mathbb{R})$

So, $L' = \{(x, 1, 0) \mid x \in \mathbb{R}\}$

Given $T: V \to V'$, $(1,0,1) \in \ker(T)$

& $T(1,1,0) = (1,0,0)$

Let $(x,y,z) = a(1,1,0) + b(1,0,1)$

$= (a+b, a, b)$

$\Rightarrow \quad a = y, \; b = z$

So, $T(x,y,z) = y\,T(1,1,0) + z(1,0,1)$

$= y(1,0,0)$

Let $u = (x,y,z)$

So $f((0,1,1)+u) = (0,1,0) + T(x,y,z)$

$\Rightarrow \quad f(x, y+1, z+1) = (y,1,0)$

3 Numerical Answer Type (NAT):

7. Consider the following statements:

   - **Statement 1:** Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such T is not injective. Then $rank(T) < 3$.
   - **Statement 2:** If $T : V \to W$ is a linear transformation, whose matrix representation with respect to some ordered bases is given by the matrix $\begin{bmatrix} 0 & \alpha & \gamma \\ 1 & 0 & \gamma \\ 0 & \beta & \frac{\gamma\beta}{\alpha} \end{bmatrix}$, where $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$, then the rank of the linear transformation T is 3.
   - **Statement 3:** If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation such that $T(x, y, z) = (2x - z, 3y - 2z, z, 0)$, then {(-3, 1, 1, 0), (1, -5, 1, 0), (3, 5, -1, 0)} is a basis of the image space.
   - **Statement 4:** If $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a linear transformation such that $T(A) = PA$, where $A \in M_2(\mathbb{R})$ and $P = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, then $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis of the kernel.

   Write down the statement numbers corresponding to the correct statements in increasing order.

   [Note: Suppose Statement 1, Statement 2, and Statement 4 are correct then your answer should be 124. Similarly, if Statement 2 and Statement 3 are correct then your answer should be 23. In this list one or more than one statement can be correct. Do not add any space between the digits.] [Ans : 13]

Solution:- Statement 1:

Since T is not injective, so nullity of $T \geq 1$

{ Because if T is injective then $T(0,0,0) = (0,0,0)$ only possible, there is no other element $(x,y,z) \in \mathbb{R}^3 \ni T(x,y,z) = 0$ so null space of T will be $\{(0,0,0)\}$ which have dimension 0. But here, this is not the case)

Now, Use the rank nullity theorem

$\text{rank}(T) + \text{nullity}(T) = 3$

But $\text{Nullity}(T) \geq 1$

$\Rightarrow \quad \text{rank}(T) < 3$

Statement 2. Given matrix representation of T is the matrix $\begin{bmatrix} 0 & \alpha & \gamma \\ 1 & 0 & \gamma \\ 0 & \beta & \gamma\beta/\alpha \end{bmatrix} = A$ (Let)

then $\det(A) = -1(\gamma\beta - \gamma\beta) = 0$

that means, rows or columns of the matrix A are linearly dependent. So rank of A can not be 3

So $\text{rank}(T)$ can not be 3.

Statement 3 Given $T(x, y, z) = (2x - z, 3y - 2z, z, 0)$

Lets form the matrix using the vectors which are given in the set

$$\begin{bmatrix} -3 & 1 & 3 \\ 1 & -5 & 5 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -5 & 5 \\ -3 & 1 & 3 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 + 3R_1]{R_3 - R_1} \begin{bmatrix} 1 & -5 & 5 \\ 0 & -14 & 18 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{6}R_3]{-\frac{1}{14}R_2} \begin{bmatrix} 1 & -5 & 5 \\ 0 & 1 & -18/14 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -5 & 5 \\ 0 & 1 & -18/14 \\ 0 & 0 & 4/14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{14}{4}R_3} \begin{bmatrix} 1 & -5 & 5 \\ 0 & 1 & -18/14 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in row echelon form so these vectors are linearly indepedent & observe that these vectors are element of the image space because we have preimage for each element of the given set.

Also observe that nullity of T is 1

Rank of $(T) = 3$ (Using rank nullity theorem)

So $\{(-3,1,1,0), (1,-5,1,0), (3,5,-1,0)\}$ is basis of the image space.

Statement 4:

$$T(A) = PA, \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

So $$T\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

& $$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ not the element of Kernel(T)

So $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ can not be the basis of kernel.

4 Comprehension Type Question:

Suppose a bread-making machine B makes 6 breads from 2 eggs, 3 (in hundreds) grams of wheat, and 1 (in hundred) grams of sugar. B also makes 8 breads from 3 eggs, 4 (in hundreds) grams of wheat, and 2 (in hundreds) grams of sugar, and 10 breads from 5 eggs, 5 (in hundreds) grams of wheat, and 3 (in hundreds) grams of sugar. Suppose the production of breads is a linear function of the amount of eggs, wheat (in hundreds), and sugar (in hundreds) used as raw ingredients. Based on the above data answer the following questions. Suppose x eggs, y (in hundreds) grams of wheat, and z (in hundreds) grams of sugar are used as the raw materials to produce $ax + by + cz$ number of breads. We can express this as follows:

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$T(x, y, z) = ax + by + cz$$

where the co-ordinates in $\mathbb{R}^3$ denote the number of eggs, amount (in grams) of wheat (in hundreds), and amount (in grams) of sugar (in hundreds). Observe that T is a linear transformation.

8. Choose the correct set of options from the the following. (MSQ)

   - ◯ Option 1: $Nullity(T) = 1$
   - ◯ **Option 2:** $Rank(T) = 1$
   - ◯ **Option 3:** $Nullity(T) = 2$
   - ◯ Option 4: $Rank(T) = 2$
   - ◯ Option 5: $Nullity(T) = 3$
   - ◯ Option 6: $Rank(T) = 3$
   - ◯ Option 7: T is neither one to one nor onto.
   - ◯ Option 8: T is one to one but not onto.
   - ◯ **Option 9:** T is onto but not one to one.
   - ◯ Option 10: T is an isomorphism.

Solution:- Given $T(x, y, z) = ax + by + cz$ which is the number of breads produced.

Now 2 eggs, 3 (in hundred) gram wheat

& 1 (in hundred) grams sugar used to makes 6 breads

i.e $T(2,3,1) = 6$

$\Rightarrow \quad 2a + 3b + c = 6 \quad \text{--- (1)}$

Similarly we get other equations using the given information which is

$$3a + 4b + 2c = 8 \quad \text{--- (2)}$$

& $\quad 5a + 5b + 3c = 10 \quad \text{--- (3)}$

equations (1), (2) & (3) forms a system of linear equations.

After solving this system we get $a = 0,\ b = 2,\ c = 0$

So $T(x,y,z) = 2y$

which is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}$

Now, image space of T is $\{2y \mid y \in \mathbb{R}\}$

So Rank of T is 1

Now, use the rank nullity theorem

$$\text{Rank}(T) + \text{Nullity}(T) = \text{dimension of } \mathbb{R}^3$$

$$\Rightarrow \quad 1 + \text{Nullity}(T) = 3$$

$$\Rightarrow \quad \text{Nullity}(T) = 2$$

Since Nullity of T is 2, so T can not be one to one linear transformation but as we see image space of $T = \mathbb{R}$ (Range of T)

so T is onto.

Also T is onto but not one to one linear transformation so T can not be isomorphism.

9. Choose the set of correct statements. (MSQ)

- ○ Option 1: If 4 eggs and 2 (in hundreds) grams of sugar is used, and no wheat is used, then 9 breads are produced.
- ○ **Option 2:** If 4 eggs and 2 (in hundreds) grams of sugar is used, and no wheat is used, then no bread is produced.
- ○ **Option 3:** If only 3 (in hundreds) grams of wheat is used, then 6 breads are produced.
- ○ Option 4: If 3 (in hundreds) grams of wheat and 1 (in hundred) grams of sugar is used, and no egg is used, then no bread is produced.
- ○ **Option 5:** If 3 (in hundreds) grams of wheat and 1 (in hundred) grams of sugar is used, and no egg is used, then 6 breads are produced.

Solution: As we obtained

$$T(x, y, z) = 2y$$

So option 1: $T(4, 0, 2) = 0 \neq 9$

option 2: $T(4, 0, 2) = 0$

that means no bread is produced.

option 3: $T(0, 3, 0) = 2 \times 3 = 6$

So only 3 (in hundred) grams of wheat is used to produce 6 breads.

option 4: $T(0, 3, 1) = 6 \neq 0$

option 5: $(0, 3, 1) = 6$

So this option is also true

10. How many breads are produced by the machine from 6 eggs, 10 (in hundreds) grams of wheat, and 5 (in hundreds) grams of sugar?
(NAT) [Answer: 20]

Solution:- As we obtained $T(x,y,z) = 2y$

So $T(6,10,5) = 2\times 10$
$= 20$