The echelon form

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System of linear equations

A general system of m linear equations with n unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Matrix Representation

The matrix representation of this system of linear equations is $Ax = b$ where :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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A solution is an assignment of values for x so that the equations are satisfied (i.e. hold true).

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

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- Rows with all zero elements, if any, are below rows having a non-zero element.
- For a non-zero row, the leading entry in the row is the only non-zero entry in its column.

Examples

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Since $b_i \neq 0$ this equation cannot be satisfied.

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- If the i-th column has the leading entry of some row, we call x_i a dependent variable.
- If the i-th column does not have the leading entry of some row, we call x_i an independent variable.

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Conclusion : If A is in reduced row echelon form, this easy procedure provides us with ALL the solutions of $Ax = b$.

Thank you