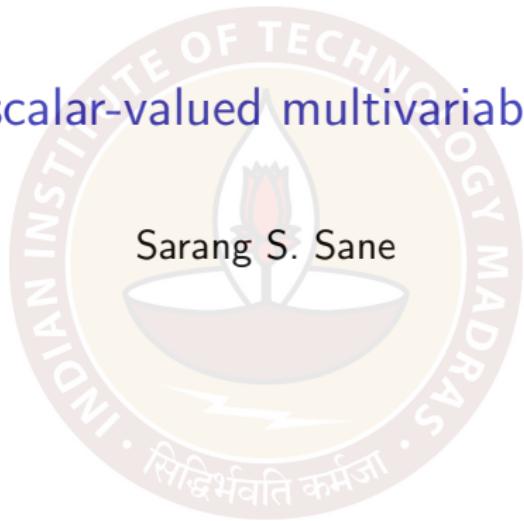


Limits for scalar-valued multivariable functions



Recall : limits of sequences of real numbers



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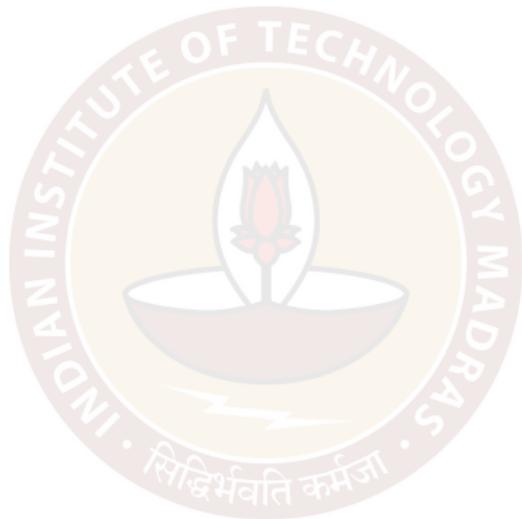
A subsequence of a sequence is a new sequence formed by (possibly) excluding some entries of a sequence.

Limits of sequences in \mathbb{R}^p



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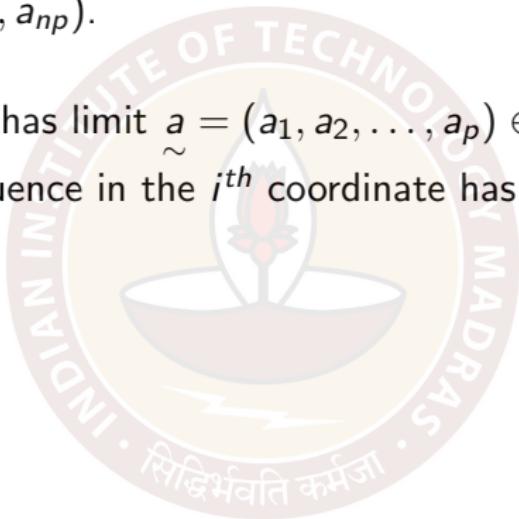
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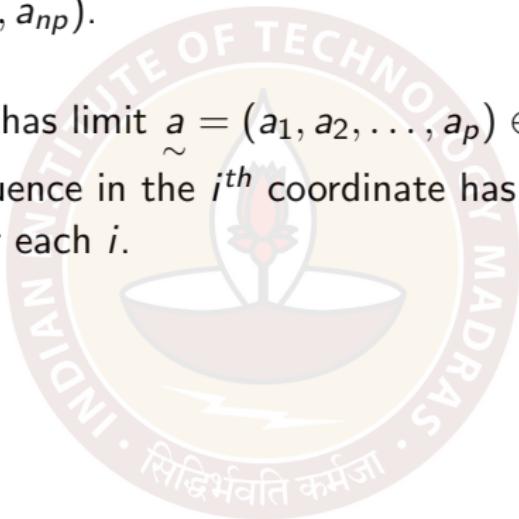
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Examples and visualization

$$\left\{ \left(\frac{1}{n}, n \sin \left(\frac{1}{n} \right) \right) \right\} \longrightarrow (0, 1)$$

$$\left\{ \left((-1)^n, n \sin \left(\frac{1}{n} \right) \right) \right\}$$

(i.e. Limit does not converge DNE)

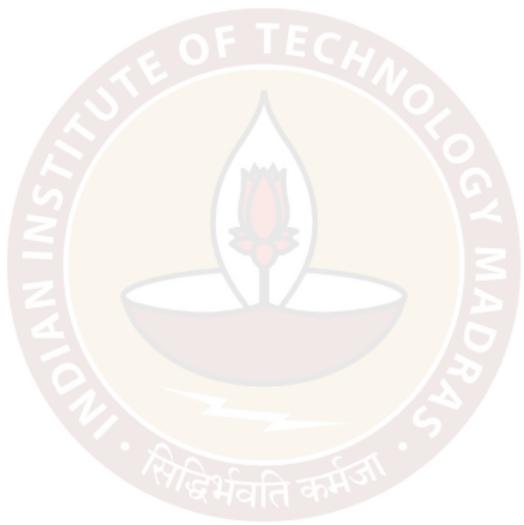
$$\left\{ \left(\frac{\cos(n)}{n}, \frac{\frac{1}{\ln(1+n)} + \frac{5n^2}{1+n^2}}{\left(1 + \frac{1}{n}\right)^{2n}}, \sum_{i=0}^n \frac{1}{i!}, n \cos \left(\frac{1}{n} \right) \right) \right\}$$

Limit DNE

\downarrow \downarrow \downarrow \searrow

0 ? e diverge to ∞

Limit of a scalar-valued multivariable function at a point



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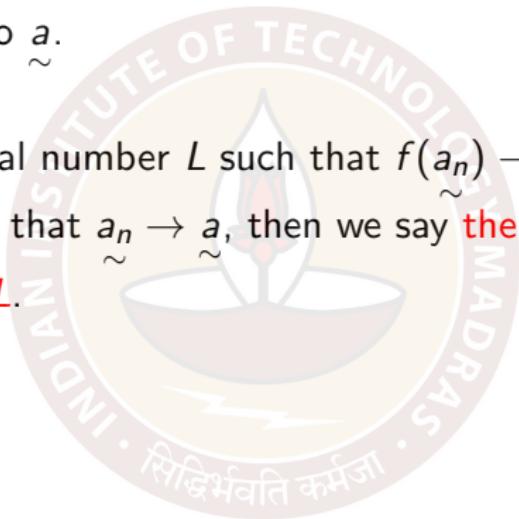
Let f be a scalar-valued multivariable function defined on a domain D in \mathbb{R}^k and \tilde{a} be a point such that there exists a sequence in D which converges to a .



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If there exists a real number L such that $f(\tilde{a}_n) \rightarrow L$ for all sequences \tilde{a}_n such that $\tilde{a}_n \rightarrow \tilde{a}$, then we say **the limit of f at \tilde{a} exists and equals L** .



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If there is no such number L then we say that the limit of f at \tilde{a} does not exist.

Some basic examples

$$1. \lim_{\substack{x \rightarrow a \\ \sim \\ \sim}} x_i^k; k \geq 0$$

a_i

$$2. \lim_{\substack{x \rightarrow a \\ \sim \\ \sim}} x_i^k; k < 0, a_i \neq 0$$

a_i^k

$$3. \lim_{\substack{x \rightarrow a \\ \sim \\ \sim}} e^{x_i}$$

e^{a_i}

$$5. \lim_{\substack{x \rightarrow a \\ \sim \\ \sim}} \sin(x_i)$$

$\sin(a_i)$

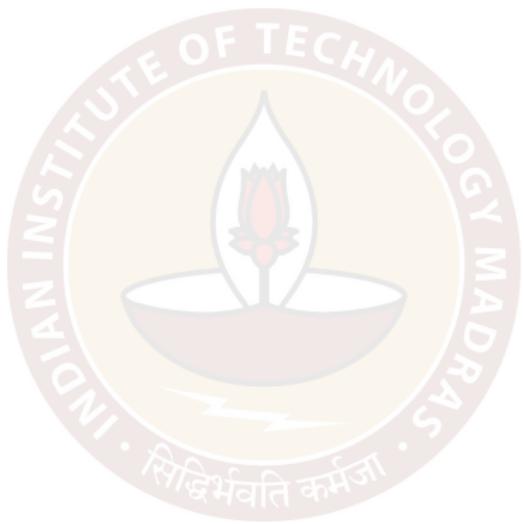
$$4. \lim_{\substack{x \rightarrow a \\ \sim \\ \sim}} \log_e(x_i); a_i > 0$$

$\log_e(a_i)$

$$6. \lim_{\substack{x \rightarrow a \\ \sim \\ \sim}} \tan(x_i); a_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\tan(a_i)$

Rules about limits of scalar-valued multivariable functions



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1. If $\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = F$, $\lim_{\substack{x \rightarrow a \\ \sim}} g(x) = G$ and $c \in \mathbb{R}$, then
$$\lim_{\substack{x \rightarrow a \\ \sim}} (cf + g)(x) = cF + G.$$

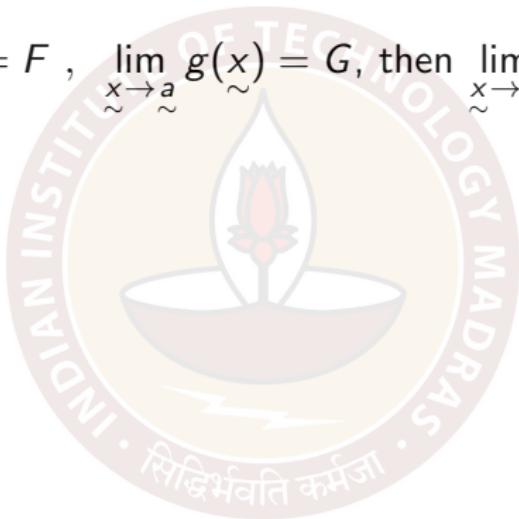


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3. If $\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = F$, $\lim_{\substack{x \rightarrow a \\ \sim}} g(x) = G \neq 0$, then the function $\frac{f}{g}$ is

defined in at least a small interval around a and

$$\lim_{\substack{x \rightarrow a \\ \sim}} \frac{f}{g}(x) = \frac{F}{G}.$$

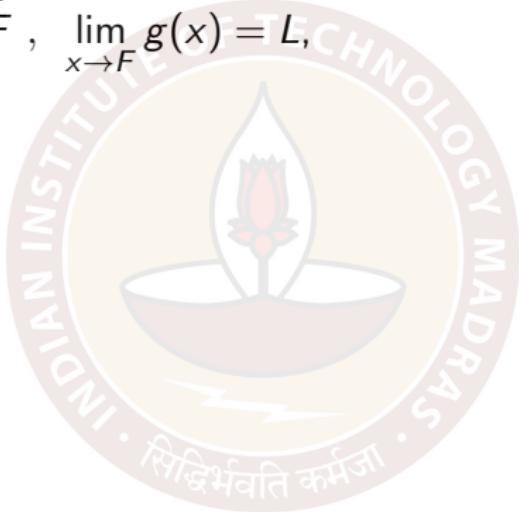
$$h(x, y, z) = x^2 y^3 + y^3 z^2 + x y z$$
$$\lim_{(x,y,z) \rightarrow (1,2,3)} h(x, y, z) = \lim_{(x,y,z) \rightarrow (1,2,3)} x^2 y^3 + \lim_{(x,y,z) \rightarrow (1,2,3)} y^3 z^2 + \lim_{(x,y,z) \rightarrow (1,2,3)} x y z$$
$$= 1^2 \times 2^3 + 2^3 \times 3^2 + 1 \times 2 \times 3$$
$$= 8 + 72 + 6 = 86.$$

$$\begin{matrix} x^2 & y^3 \\ \downarrow & \downarrow \\ 1^2 & 2^3 \\ 1^2 \times 2^3 & \\ 2^3 \times 3^2 & \\ 1 \times 2 \times 3 & \end{matrix}$$

Rules (contd.)

- 4 **Composition** : Suppose f is a scalar-valued multivariable function and g is a function of one variable such that the composition $g \circ f$ is well-defined. If

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$h(x,y,z) = e^{xyz}$. Want: $\lim_{(x,y,z) \rightarrow (1,2,3)} h(x,y,z) \rightarrow e^6$.

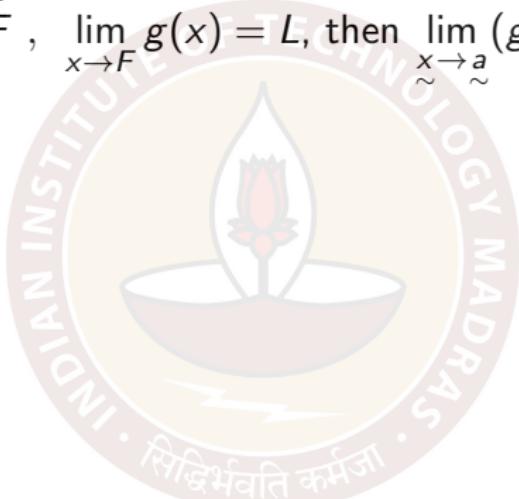
$f(x,y,z) = xyz$, $g(u) = e^u$

$\lim_{(x,y,z) \rightarrow (1,2,3)} f(x,y,z) = 1 \times 2 \times 3 = 6$, $\lim_{u \rightarrow 6} e^u = e^6$ ∴

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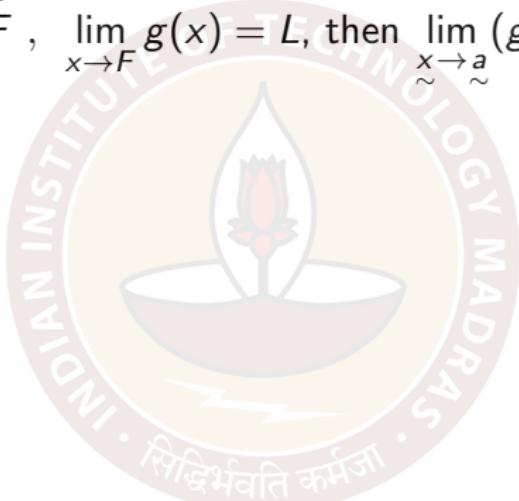


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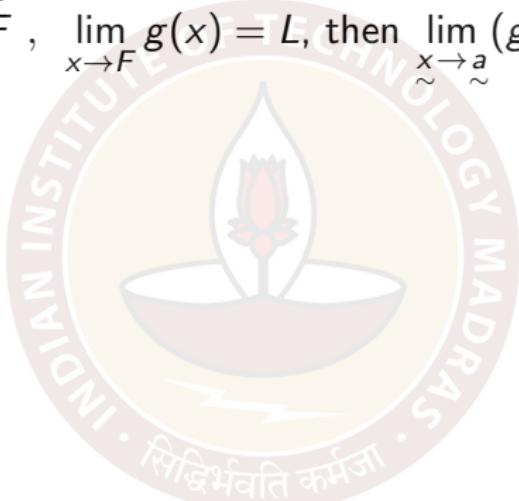


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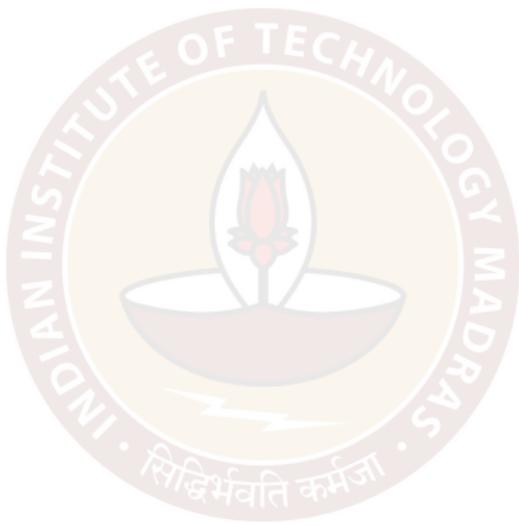
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Finding limits by substitution : beware



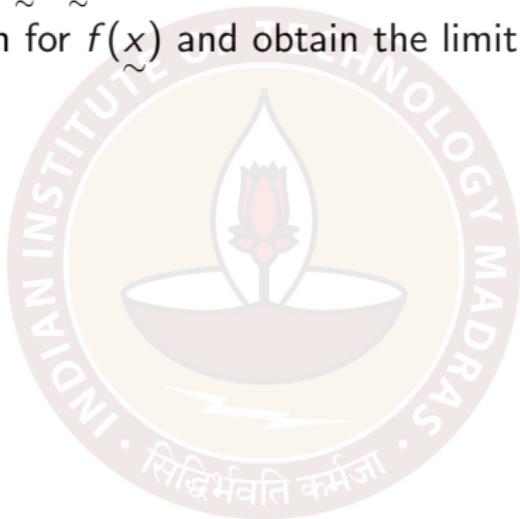
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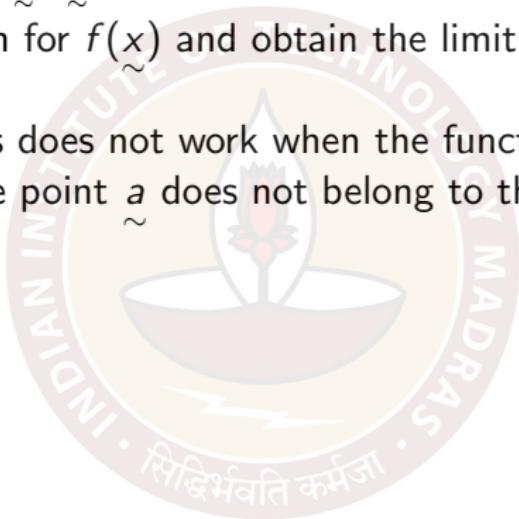
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Unfortunately, this does not work when the function gets slightly complicated or the point a does not belong to the domain of definition of $f(x)$.

Example :

$$\lim_{\tilde{x} \rightarrow (0,0)} \frac{x^3 - y^2x}{(x^2 + y^2)^2} \cdot \text{DNE}$$
$$a_n = \left(\frac{1}{n}, 0 \right) \cdot \begin{aligned} f(a_n) &= \frac{\left(\frac{1}{n} \right)^3 - 0^2 \times \frac{1}{n}}{\left(\left(\frac{1}{n} \right)^2 + 0^2 \right)^2} \\ &= \frac{\frac{1}{n^3} - 0}{\frac{1}{n^4}} = n. \end{aligned}$$
$$b_n = \left(0, \frac{1}{n} \right) \cdot \begin{aligned} f(b_n) &= \frac{0^3 - \frac{1}{n^2} \times 0}{\left(0^2 + \left(\frac{1}{n} \right)^2 \right)^2} = 0. \end{aligned}$$

Thank you

