

# Continuity for multivariable functions

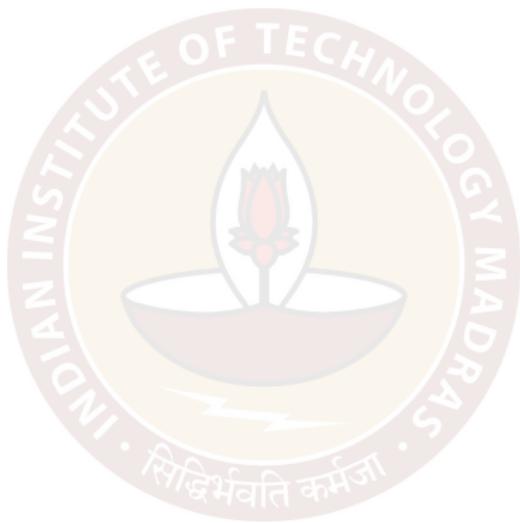
Sarang S. Sane

# Recall



## Recall

Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ .



## Recall

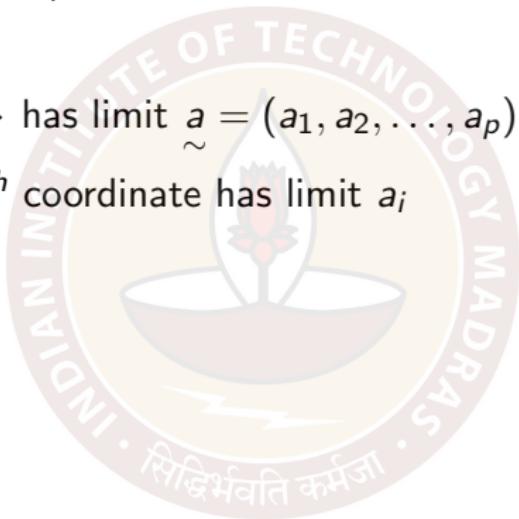
Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ . Denote the coordinates of  
 $\underset{\sim}{a_n} = (a_{n1}, a_{n2}, \dots, a_{np})$ .



## Recall

Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ . Denote the coordinates of  $\underset{\sim}{a_n} = (a_{n1}, a_{n2}, \dots, a_{np})$ .

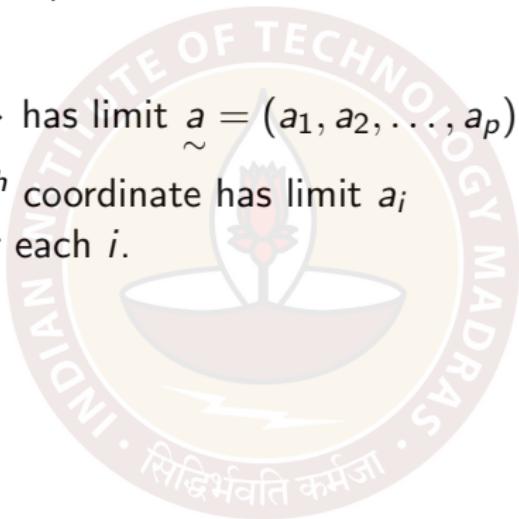
We say that  $\left\{ \underset{\sim}{a_n} \right\}$  has limit  $\underset{\sim}{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$  if the sequence in the  $i^{th}$  coordinate has limit  $a_i$ .



## Recall

Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ . Denote the coordinates of  $\underset{\sim}{a_n} = (a_{n1}, a_{n2}, \dots, a_{np})$ .

We say that  $\left\{ \underset{\sim}{a_n} \right\}$  has limit  $\underset{\sim}{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$  if the sequence in the  $i^{th}$  coordinate has limit  $a_i$ ; i.e.  $\{a_{ni}\} \rightarrow a_i$  for each  $i$ .



## Recall

Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ . Denote the coordinates of  $\underset{\sim}{a_n} = (a_{n1}, a_{n2}, \dots, a_{np})$ .

We say that  $\left\{ \underset{\sim}{a_n} \right\}$  has limit  $\underset{\sim}{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$  if the sequence in the  $i^{th}$  coordinate has limit  $a_i$ ; i.e.  $\{a_{ni}\} \rightarrow a_i$  for each  $i$ .

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

## Recall

Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ . Denote the coordinates of  $\underset{\sim}{a_n} = (a_{n1}, a_{n2}, \dots, a_{np})$ .

We say that  $\left\{ \underset{\sim}{a_n} \right\}$  has limit  $\underset{\sim}{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$  if the sequence in the  $i^{th}$  coordinate has limit  $a_i$ ; i.e.  $\{a_{ni}\} \rightarrow a_i$  for each  $i$ .

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

If there exists a real number  $L$  such that  $f(\underset{\sim}{a_n}) \rightarrow L$  for all sequences  $\underset{\sim}{a_n}$  such that  $\underset{\sim}{a_n} \rightarrow \underset{\sim}{a}$ , then we say **the limit of  $f$  at  $\underset{\sim}{a}$  exists and equals  $L$** .

## Recall

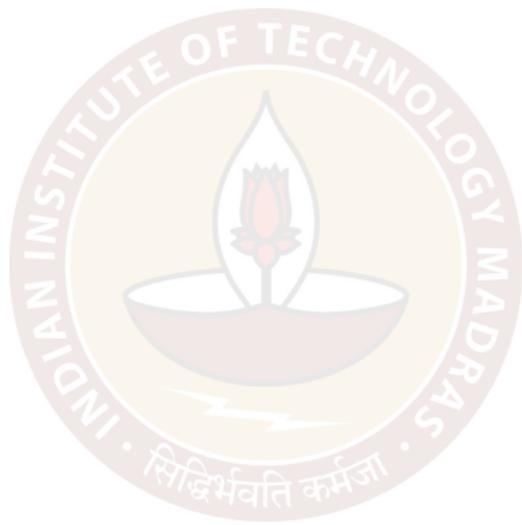
Let  $\left\{ \underset{\sim}{a_n} \right\}$  be a sequence in  $\mathbb{R}^p$ . Denote the coordinates of  $\underset{\sim}{a_n} = (a_{n1}, a_{n2}, \dots, a_{np})$ .

We say that  $\left\{ \underset{\sim}{a_n} \right\}$  has limit  $\underset{\sim}{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$  if the sequence in the  $i^{th}$  coordinate has limit  $a_i$ ; i.e.  $\{a_{ni}\} \rightarrow a_i$  for each  $i$ .

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

If there exists a real number  $L$  such that  $f(\underset{\sim}{a_n}) \rightarrow L$  for all sequences  $\underset{\sim}{a_n}$  such that  $\underset{\sim}{a_n} \rightarrow \underset{\sim}{a}$ , then we say **the limit of  $f$  at  $\underset{\sim}{a}$  exists and equals  $L$** . We denote this by  $\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = L$ .

# Limit of a vector-valued function at a point



## Limit of a vector-valued function at a point

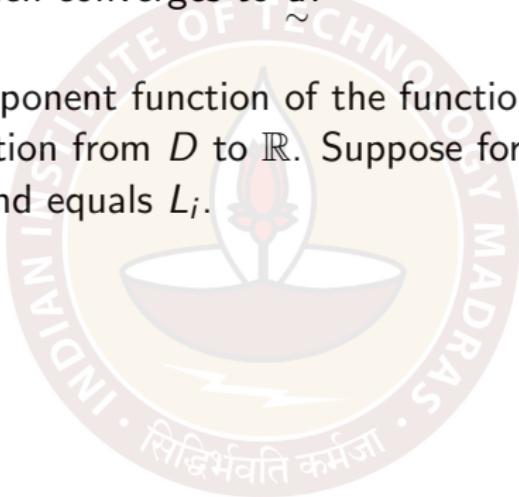
Let  $f : D \rightarrow \mathbb{R}^m$  be a vector-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\tilde{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\tilde{a}$ .



## Limit of a vector-valued function at a point

Let  $f : D \rightarrow \mathbb{R}^m$  be a vector-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

If  $f_i$  is the  $i^{th}$  component function of the function  $f$ , then  $f_i$  is a scalar-valued function from  $D$  to  $\mathbb{R}$ . Suppose for each  $i$ , the limit  $\lim_{\substack{x \rightarrow a \\ \sim}} f_i(x)$  exists and equals  $L_i$ .



## Limit of a vector-valued function at a point

Let  $f : D \rightarrow \mathbb{R}^m$  be a vector-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

If  $f_i$  is the  $i^{th}$  component function of the function  $f$ , then  $f_i$  is a scalar-valued function from  $D$  to  $\mathbb{R}$ . Suppose for each  $i$ , the limit  $\lim_{\substack{x \rightarrow a \\ \sim}} f_i(x)$  exists and equals  $L_i$ .

Define  $\underset{\sim}{L} = (L_1, L_2, \dots, L_m)$ . Then  $\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = \underset{\sim}{L}$ .

## Limit of a vector-valued function at a point

Let  $f : D \rightarrow \mathbb{R}^m$  be a vector-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

If  $f_i$  is the  $i^{th}$  component function of the function  $f$ , then  $f_i$  is a scalar-valued function from  $D$  to  $\mathbb{R}$ . Suppose for each  $i$ , the limit  $\lim_{\substack{x \rightarrow a \\ \sim}} f_i(x)$  exists and equals  $L_i$ .

Define  $\underset{\sim}{L} = (L_1, L_2, \dots, L_m)$ . Then  $\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = \underset{\sim}{L}$ .

This is equivalent to : as  $x$  comes closer and closer to  $\underset{\sim}{a}$ ,  $f(x)$  eventually comes closer and closer to  $\underset{\sim}{L}$ .

## Limit of a vector-valued function at a point

Let  $f : D \rightarrow \mathbb{R}^m$  be a vector-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\underset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\underset{\sim}{a}$ .

If  $f_i$  is the  $i^{th}$  component function of the function  $f$ , then  $f_i$  is a scalar-valued function from  $D$  to  $\mathbb{R}$ . Suppose for each  $i$ , the limit  $\lim_{\substack{x \rightarrow a \\ \sim}} f_i(x)$  exists and equals  $L_i$ .

Define  $\underset{\sim}{L} = (L_1, L_2, \dots, L_m)$ . Then  $\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = \underset{\sim}{L}$ .

This is equivalent to : as  $x$  comes closer and closer to  $\underset{\sim}{a}$ ,  $f(x)$  eventually comes closer and closer to  $\underset{\sim}{L}$ .

If for some  $i$ , the limit  $f_i$  at  $\underset{\sim}{a}$  does not exist, then the limit of  $f$  at  $\underset{\sim}{a}$  does not exist.

## Examples

$$\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \left( \underbrace{x^2y + y^3}_{=}, \underbrace{e^{xy}}_{=}, \underbrace{\frac{x^2 - 1}{y^3 - 2}}_{=} \right) = (10, e^2, 0).$$

$$\begin{aligned} & \lim_{(x,y) \rightarrow (1,2)} x^2y + y^3 = \underbrace{1^2 \times 2 + 2^3}_{= 10} = 10. \\ & \lim_{(x,y) \rightarrow (1,2)} e^{xy} = \underbrace{e^{1 \times 2}}_{= e^2} = e^2. \\ & \lim_{(x,y) \rightarrow (1,2)} \frac{x^2 - 1}{y^3 - 2} = \underbrace{\frac{1^2 - 1}{2^3 - 2}}_{= 0} = 0. \end{aligned}$$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left( \underbrace{\frac{\sin(xy)}{xy}}_{\sim}, \underbrace{\frac{x^3 - y^2x}{(x^2 + y^2)^2}}_{\sim} \right)$$

DNE

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{xy} = 1.$$

$$xy \downarrow 0, \quad \frac{\sin(u)}{u} \downarrow 1$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^2x}{(x^2 + y^2)^2} \quad \text{DNE}$$

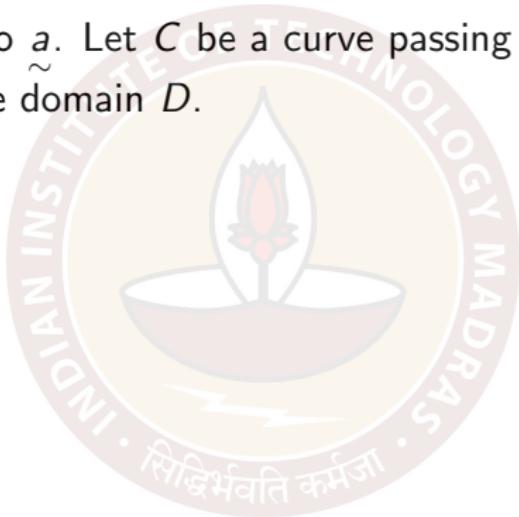
# Limit of a function at a point along a curve

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\tilde{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\tilde{a}$ .



## Limit of a function at a point along a curve

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\tilde{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\tilde{a}$ . Let  $C$  be a curve passing through the point  $\tilde{a}$  belonging to the domain  $D$ .



## Limit of a function at a point along a curve

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\tilde{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\tilde{a}$ . Let  $C$  be a curve passing through the point  $\tilde{a}$  belonging to the domain  $D$ .

The **limit of  $f$  at  $\tilde{a}$  along the curve  $C$**  exists and equals  $L$  if for every sequence  $a_n$  contained in  $C$  which converges to  $\tilde{a}$ , the sequence  $f(a_n)$  converges to  $L$ .

# Limit of a function at a point along a curve

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\overset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\overset{\sim}{a}$ . Let  $C$  be a curve passing through the point  $\overset{\sim}{a}$  belonging to the domain  $D$ .

The limit of  $f$  at  $\overset{\sim}{a}$  along the curve  $C$  exists and equals  $L$  if for every sequence  $a_n$  contained in  $C$  which converges to  $\overset{\sim}{a}$ , the sequence  $f(a_n)$  converges to  $L$ .

$$\text{Example : } g(x, y) = \frac{x^3 - y^2x}{(x^2 + y^2)^2}$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x}} g(x,y) = \lim_{x \rightarrow 0} \frac{x^3 - x^2x}{(x^2 + x^2)^2} = 0.$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=0}} g(x,y) = \lim_{x \rightarrow 0} \frac{x^3 - 0}{(x^2 + 0)^2} = \lim_{x \rightarrow 0} \frac{x^3}{x^4} = \lim_{x \rightarrow 0} \frac{1}{x} = \infty.$$

Along the X-axis, the y-coordinate is 0 & hence the fn. is

$$g(x,0) = \frac{x^3}{x^4} = \frac{1}{x}.$$

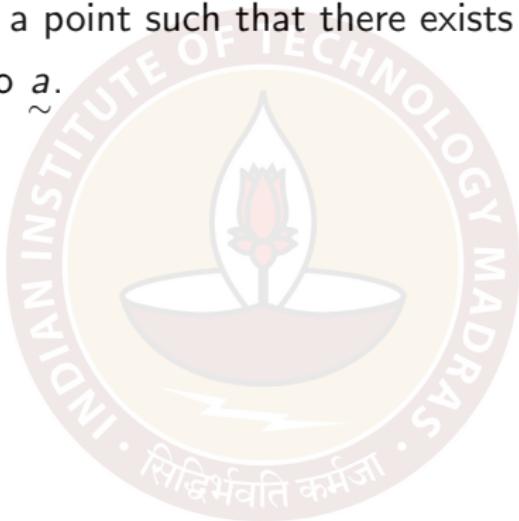
$$g(0,y) = 0.$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } x=y}} g(x,y) = \lim_{x \rightarrow 0} \frac{1}{x} \text{ DNE.}$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x}} g(x,y) = \lim_{y \rightarrow 0} 0 = 0.$$

# Limits of a function along curves and limit of the function

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\overset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\overset{\sim}{a}$ .

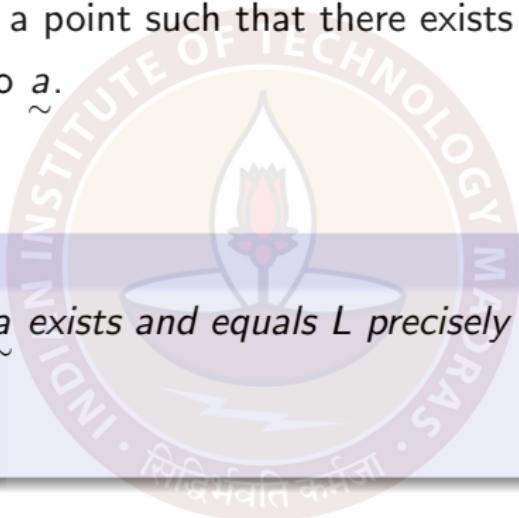


# Limits of a function along curves and limit of the function

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\overset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\overset{\sim}{a}$ .

## Theorem

*The limit of  $f$  at  $\overset{\sim}{a}$  exists and equals  $L$  precisely when*



# Limits of a function along curves and limit of the function

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\overset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\overset{\sim}{a}$ .

## Theorem

*The limit of  $f$  at  $\overset{\sim}{a}$  exists and equals  $L$  precisely when for every curve  $C$  in the domain  $D$  passing through  $\overset{\sim}{a}$*

# Limits of a function along curves and limit of the function

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\overset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\overset{\sim}{a}$ .

## Theorem

*The limit of  $f$  at  $\overset{\sim}{a}$  exists and equals  $L$  precisely when for every curve  $C$  in the domain  $D$  passing through  $\overset{\sim}{a}$  the limit of  $f$  at  $\overset{\sim}{a}$  along  $C$  exists and equals  $L$ .*

# Limits of a function along curves and limit of the function

Let  $f$  be a scalar-valued multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $\overset{\sim}{a}$  be a point such that there exists a sequence in  $D$  which converges to  $\overset{\sim}{a}$ .

## Theorem

*The limit of  $f$  at  $\overset{\sim}{a}$  exists and equals  $L$  precisely when for every curve  $C$  in the domain  $D$  passing through  $\overset{\sim}{a}$  the limit of  $f$  at  $\overset{\sim}{a}$  along  $C$  exists and equals  $L$ .*

Important : This is often used to show that a limit at a point does not exist.

## More examples

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

DNE

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$$

DNE

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$$

along the X-axis:  $\lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1.$  ~~X~~

along " Y- axis:  $\lim_{y \rightarrow 0} \frac{-y^2}{y^2} = -1.$

along the X-axis:  $\lim_{x \rightarrow 0} \frac{0}{x^2 + 0} = 0.$

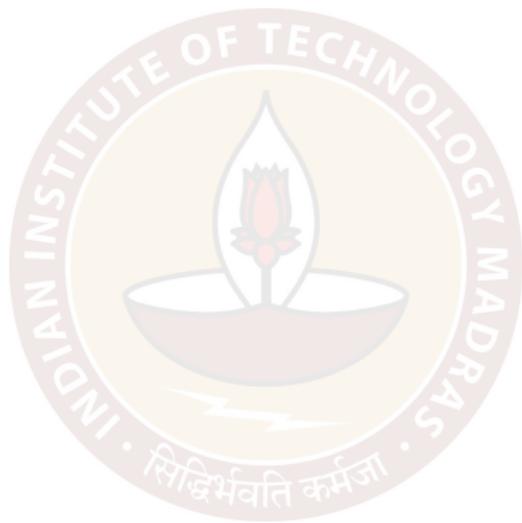
along the Y-axis:  $\lim_{y \rightarrow 0} \frac{0}{0^2 + y^2} = 0.$  ~~X~~

along the line  $y=x:$   $\lim_{x \rightarrow 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}.$

along the line  $y=mx:$   $\lim_{x \rightarrow 0} \frac{-x(m+m)^2}{x^2 + (mm)^2} = \frac{m^3}{1+m^2} = 0.$

along the line  $x=0:$   $\lim_{y \rightarrow 0} \frac{0}{y^4} = 0.$

# Continuity of a function



# Continuity of a function

Let  $f$  be a multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $a \in D$  be a point such that there exists a sequence in  $D$  which converges to  $a$ .



# Continuity of a function

Let  $f$  be a multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $a \in D$  be a point such that there exists a sequence in  $D$  which converges to  $a$ .

Definition :  $f$  is continuous at  $\underset{\sim}{a}$  if the limit of  $f$  at  $a$  exists and

$$\lim_{\substack{x \rightarrow a \\ \sim}} f(x) = f(\underset{\sim}{a}).$$



## Continuity of a function

Let  $f$  be a multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $a \in D$  be a point such that there exists a sequence in  $D$  which converges to  $a$ .

Definition :  $f$  is continuous at  $\tilde{a}$  if the limit of  $f$  at  $\tilde{a}$  exists and  $\lim_{\substack{x \rightarrow \tilde{x} \\ \sim \rightarrow \sim}} f(x) = f(\tilde{a})$ .  $f$  is continuous at  $\tilde{a}$  is equivalent to  $f(a_n) \rightarrow f(\tilde{a})$  whenever  $a_n \rightarrow \tilde{a}$ .

# Continuity of a function

Let  $f$  be a multivariable function defined on a domain  $D$  in  $\mathbb{R}^k$  and  $a \in D$  be a point such that there exists a sequence in  $D$  which converges to  $a$ .

Definition :  $f$  is continuous at  $\tilde{a}$  if the limit of  $f$  at  $\tilde{a}$  exists and  $\lim_{\substack{x \rightarrow \tilde{x} \\ \sim \rightarrow \sim}} f(x) = f(\tilde{a})$ .  $f$  is continuous at  $\tilde{a}$  is equivalent to  $f(a_n) \rightarrow f(\tilde{a})$  whenever  $a_n \rightarrow \tilde{a}$ .

Note that continuity means "the limit at  $\tilde{a}$  can be obtained by evaluating the function at  $\tilde{a}$ ".

The function  $f$  is said to be continuous if it is continuous at all points in its domain  $D$  i.e. for all points  $\tilde{a}$  for which  $f(\tilde{a})$  is defined,  $\lim_{\substack{x \rightarrow a \\ \sim \rightarrow \sim}} f(x) = f(\tilde{a})$ .

## Example :

$$g(x, y) = \begin{cases} \frac{x^3 - y^2 x}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

If  $\tilde{a} \neq (0, 0)$   
then  $\lim_{(x,y) \rightarrow \tilde{a}} g(x, y) = g(\tilde{a})$ .

If  $\tilde{a} = (0, 0)$   
 $\lim_{(x,y) \rightarrow (0,0)} g(x, y) \neq g(0, 0) = 0$ .  
∴ The fn.  $g(x, y)$  is continuous at all points except  $(0, 0)$ .

$$\tilde{a} = (a, b) .$$
$$g(x, y) = \frac{f(x, y)}{h(x, y)}$$

$$\lim_{(x,y) \rightarrow \tilde{a}} h(x, y) = h(\tilde{a}) \neq 0 .$$
$$= (a^2 + b^2)^2$$

# Thank you

