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IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2

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Continuity for multivariable functions

Hello and welcome to the Maths 2 component of the online BSc programme on data science and programming. In this video we are going to talk about continuity for multivariable functions.

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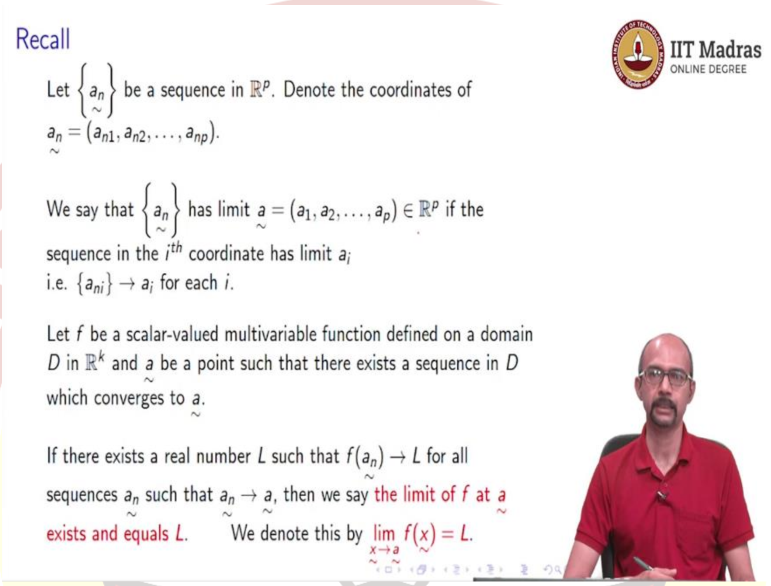

Let us recall that in the previous video, we have studied the notion of limits for sequences and limits for scalar valued multivariable functions. So, these were sequences in higher dimensions. So, let $\{\underset{\sim}{a_n}\}$ be a sequence in R^p denote the coordinates of $\underset{\sim}{a_n}$ as a_{n1}, a_{n2} up to a_{np}. We say $\{\underset{\sim}{a_n}\}$ is limit $\underset{\sim}{a}$ = (a_1,a_2,..,a_n) if the sequence in the i^{th} coordinate has limit a_i.

So, which means that if you take the sequences given by each coordinate that converges to some number a_i, and you put those numbers into this vector, and then we say $\{\underset{\sim}{a_n}\}$ has that element. So, a_{ni} in particular has to converge to a_i for each i. So, if each of those coordinate limits exists, then the sequence $\{\underset{\sim}{a_n}\}$ as a limit, if even one of those fails to exist, it does not have a limit, so limit will not exist.

So, we use this idea to define limits for functions at a point. So, let f be a scalar valued multivariable function defined on a domain D in our case is that you have a sequence converging to $\underset{\sim}{a}$. So, that sequence must belong to D. So, if there exists a real number L such that $f\left(\underset{\sim}{a}\right) \to L$ for all sequences $\left\{\underset{\sim}{a_n}\right\}$ and this is the important part for all sequences $\left\{\underset{\sim}{a_n}\right\}$ such that $\underset{\sim}{a_n} \to \underset{\sim}{a}$, then we say that the limit $f\left(\underset{\sim}{a}\right)$ exists and equals L.

And we have seen examples of these things. So, in the previous video, we also studied some properties, and using them how we can find these limits. So, for example, in the case of polynomials, rational functions and so on. We also saw an example where substitution directly does not work. And so, we have to be careful when we find these limits.

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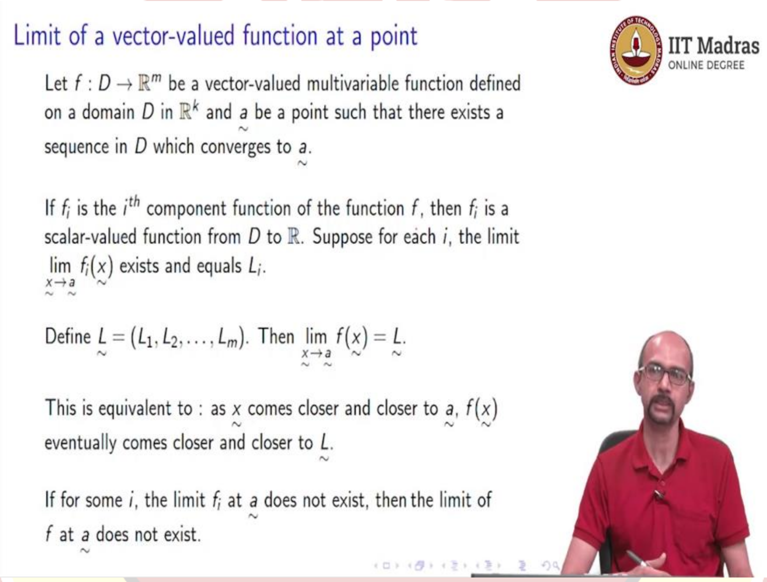

Let us first define what is the limit of a vector valued function at a point. So, we have seen for a scalar valued function what happens, so vector valued function is a vector of functions. And so, what we basically will do is demand that for each of them, we study what happens.

So, let $f: D \to R^m$ be a vector valued multivariable function defined on a domain D in R^k, $\underset{\sim}{a}$will be a point such that there exists a sequence in D which converges to $\underset{\sim}{a}$ if f_i is the i^{th} component function of the function f, and in a minute, we will see examples of what I mean. Then f_i is a scalar valued function from D to R. And then suppose for each i the $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} f\left(\underset{\sim}{x}\right) = \underset{\sim}{L}$. Then we define $\underset{\sim}{L}$ to be $(L_1, L_2, .., L_m)$. So, this is not the definition, but this is what I would call a working definition. And it is equivalent to the original definition, which we would not talk about. So, this is equivalent to as $\underset{\sim}{x}$ comes closer and closer to $\underset{\sim}{a}$ f($\underset{\sim}{x}$)comes closer and closer to this vector $\underset{\sim}{L}$. Remember now, that f is a vector valued function, so, it will take vectors as values. And so it has to come closer and closer to some vector $\underset{\sim}{L}$.

And if for some i, the limit $\underset{\sim}{a}$ for f_i does not exist, then the limit of f at $\underset{\sim}{a}$ will not exist. So, we have seen this idea already for sequences. For sequences, what did we do, we took each coordinate sequence, and then check for the limit for that. The same thing is happening here for functions. If you have vector valued functions, you take each component and check for the limit.

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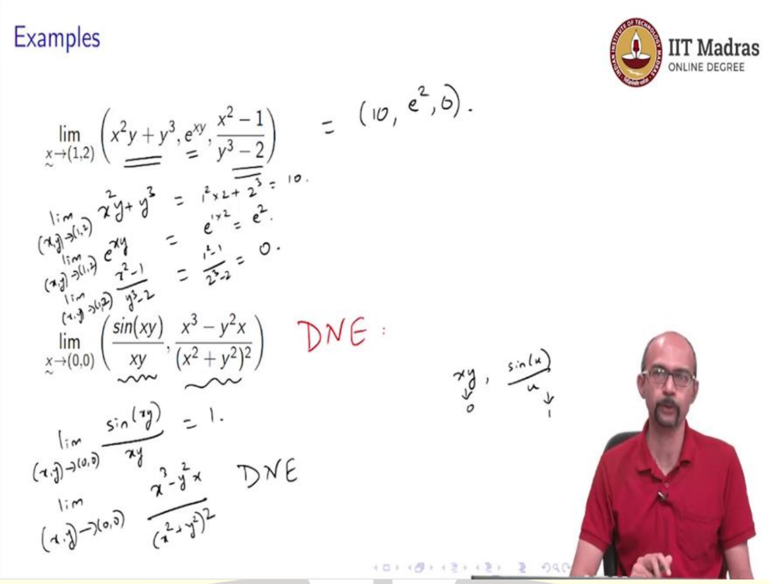

So, let us do a couple of examples. So, here is limit $\lim_{\underset{\sim}{x}\to(1,2)}(x^2y+y^3, e^{xy}, \frac{x^2-1}{y^3-2})$. So, recall from a previous video that if we had polynomials, we can just substitute. So, this first one exists. So, to check this limit, what do I have to do I have to take each coordinate function, what are the coordinate functions, this is the first coordinate function f_1, this is the second coordinate function f_2, this is the third coordinate function f_3.

And then I have asked, let us look at these three functions, what happens for each of these limits? Each of these exists, then you just put that vector together. And that is what this limit is. Even one of them does not exist, this one will not exist. So, for the first one, we have a polynomial. So, we can just substitute, that is what you saw on the in the previous video. So, this is 10. For the second one, we saw that e^{4xy} also you can substitute by using that xy is a polynomial and then using the fact that you are composing with the exponential function.

So, this follows from the composition property. So, this is e^2. And for the third one, you use the quotient. So, let us check that the quotient is. Well, the main problem comes from the denominator. So, the denominator here, the limit is 2^3-2, which is non-zero. So, no problem we can just substitute. This is 0.

So, each of these limits exists. And now you just take these components and put them in their yproper place. So, this is the limit. So, I hope it is clear what now the previous slide said, you take each component function check for its limit, once it exists, the function for this and the limit for this entire function exists.

Let us look at the second one. So, this is the first component function. This is the second component function. So, for the first component function, you have $\frac{\sin xy}{xy}$. So, I want to check what happens to this limit. Well, as we noted in the previous video, if you have things which are not like polynomial rational functions, the way to do them is to try and use the composition law.

So here, this is a composition of two functions. The first function is xy. The second function is $\frac{\sin u}{u}$. And as u tends to 0 $\frac{\sin u}{u}$ tends to 1, so this limit is going to be 1, let us see what happens to the second coordinate. So, it is $\frac{x^3-y^2x}{(x^2+y^2)^2}$.

But in the previous video, we have seen at the end that this limit actually does not exist. So, I suggest you recall this. And so, one of these coordinate functions does not have a limit that means this also does not exist. So, I hope it is very clear what it means for vector valued functions to have limits, basically it boils down to asking for scalar valued functions, what the limits are, which are the individual components.

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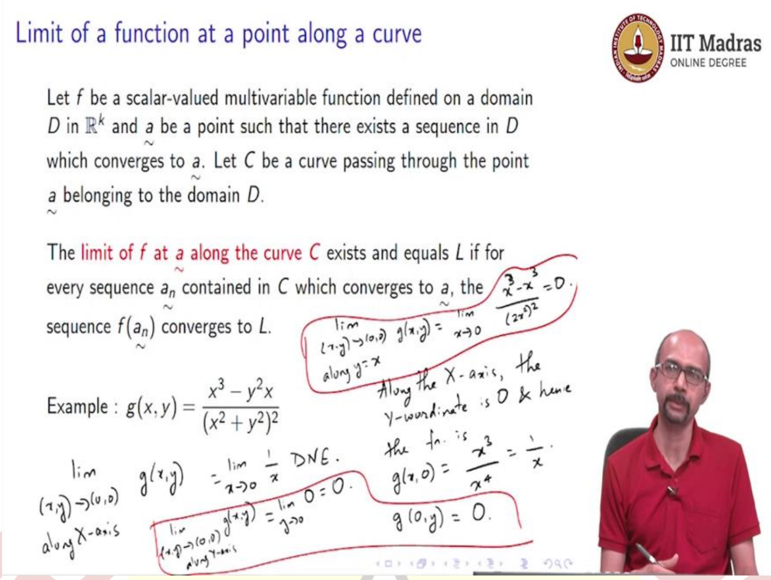

So, what is the limit of a function at a point along a curve? So, now we have seen examples of or rules about how to compute the limits of functions at a point that is for scalar valued functions that are at a point if you can write them in terms of other functions that you already know have limits. So, addition, subtraction, products composition quotients if the denominator is nice, so these are these are ways we know of how to prove that a limit actually does exist. How do we know that the limit does not exist?

So, we have seen the idea of how to do this in the previous video at the end, when we looked at an example that we just saw in the previous slide. And now we are going to refine that idea. And to do that we are going to talk about the limit of a function at a point along a curve. And this will really generalise what happens in one variable calculus.

So, let f be a scalar valued multivariable function defined on a domain D in R^k and $\underset{\sim}{a}$ will be a point such that there exists a sequence in D which converges to $\underset{\sim}{a}$. This is our standard hypothesis. Because without this, we do not know how to make sense of limits at that point $\underset{\sim}{a}$. Let C be a curve passing through the point $\underset{\sim}{a}$ belonging to the domain D.

So, recall that we have studied curves in one variable calculus, so a curve is nothing but something like a wavy line may be something, it could be a circle, straight lines are special cases of curves, but curves can have curvature, they can be curve. So, we want to look at curves, which are passing through that point a tilde and which also belong to the domain D. Except for that point $\underset{\sim}{a}$ may belong or may not belong.

The limit of f at $\underset{\sim}{a}$ along the curve C exists and equals L if for every sequence $\underset{\sim}{a_n}$ contained in C, which converges to $\underset{\sim}{a}$, the sequence $f(\underset{\sim}{a_n})$ converges to L. So, now what we are doing is, we are saying well, I want to only check for what happens along this curve. I do not for now, I do not care about the function may be defined on the entire, let us say R^k, but I only want to see what happens to this function on this particular curve.

We have seen this kind of idea, before, when we talked about directional derivatives or partial derivatives, we restricted our attention to a particular line, and then said what happens to the function along that line. Of course, there we talked about rate of change, here, we just want to talk about the function itself, not we are not yet going to rate of change. So, what happens to the function when we restrict it to a curve? So, on that, as you come closer to the point $\underset{\sim}{a}$, do the function values come close somewhere.

And if they do, and that is the number L, then we say that the limit of $f(\underset{\sim}{a})$ along the curve C exists and is L. So, let us again look at that same example from two slides ago. So, let us look at $\frac{x^3-y^2x}{(x^2+y^2)^2}$. So, limit (x,y) tends to (0, 0) along the x axis, let us say. So, along the x axis, the y is 0. So, this reduces to a function of one variable. So, I will solve this in a minute.

So, along the x axis, the y coordinate is 0. And the hence the function is g(x,0). So, it is a function of one variable. And that is really the point of defining this notion of long ago. So, $g(x,0) = \frac{1}{x}$. And now we know what happens. So, this limit does not exist.

So, along the curve, y is equal to 0, which is the x axis, this limit does not exist, we could ask the same question about what happens along the y axis. So, you can see that $g(0,y) = 0$. And so, this is limit as y tends to 0 of the function 0, so this is 0.

So, for g along the curve, which is the x axis, the limit does not exist along the y axis, it is 0, we could do some other curve, you can choose your favourite curve, let us say I choose the curvey=x. So, what happens along that line, so limit (x, y) tends to (0, 0) along y is equal to x.

So, in this case, we are asking for a limit as x tends to 0, where you put y = x in the in this expression, so you get 0. So, this is like limit of x tends to zero of the function 0 is again 0. So, I hope it is clear what we what we mean by the limit along a curve with this example in mind. So, here are the 3 curves for which we computed the limit. So, along two of them it is 0 and along one of them it does not exist.

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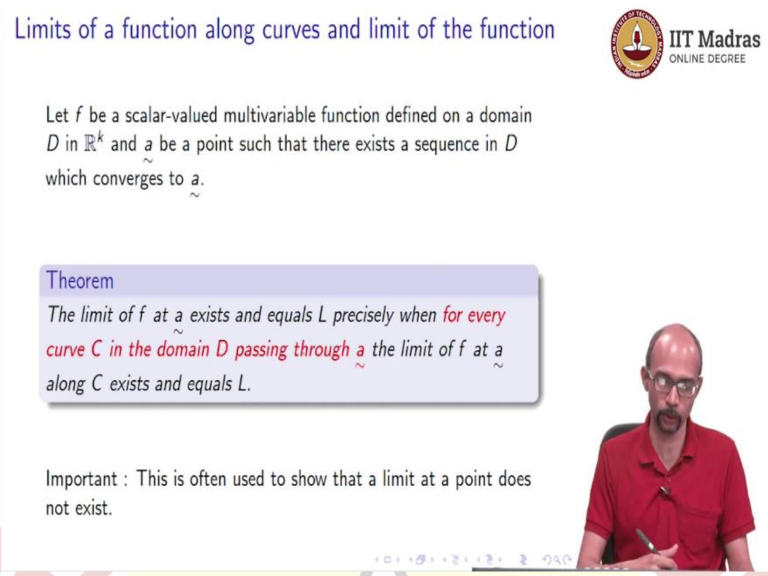

So, now what is the connection, this is a real point and this is the most important slide in this video, what is the connection between the limits of a function along curves and the limit of a function? So, the limit of a function we saw was you take any sequence tending to that point $\underset{\sim}{a}$ and look at $f(\underset{\sim}{a})$ does it converge somewhere.

So, whereas for a curve we are saying you take only sequences on that curve. So, now again let f be a scalar valued multivariable function or usual hypothesis. So, the limit of f at $\underset{\sim}{a}$ will exist and equals L precisely when which means, if this happens then whatever is next happens and if that happens then this limit exists and equals L, precisely when for every curve C in the domain D passing through $\underset{\sim}{a}$ the limit of f at $\underset{\sim}{a}$ along C exists and equals L.

So, what are we saying? We are saying that if the limit at a point is L, then whatever curve you choose the limit along that curve for that function will also be L the other side if for every curve C, as you come close to this point $\underset{\sim}{a}$ along that curve the function value comes close to L, the limit is L and this happens for every curve is the same, then the limit of f not just along the curve, but globally is L.

So, this is a very important statement. And the spirit of this statement is for one variable calculus, we did the same thing when we defined limits. In fact, that was the definition we said you come along the curve from the right, you come from the left and if they match that was left limit and right limit, then that is, then whatever value that is that is the limit. So, here we are we have many directions. In fact more than the number of directions, we have many curves.

So, you look at all possible curves, via which you can come to that point. So, you trace some curve come to this point ask does the function value tend somewhere. You trace some other curve come to that point asking does the function value tend to that same something, if for any one of those it does not exist, that is it, you that means the global function value, that limit is not defined. If for all of them, it is some number, it is the same number, then the global limit value is that same number.

So, important point is, this is often used to show that a limit at a point does not exist. So, we have seen an example already when we did that example, $\frac{x^3-y^2x}{(x^2+y^2)^2}$.

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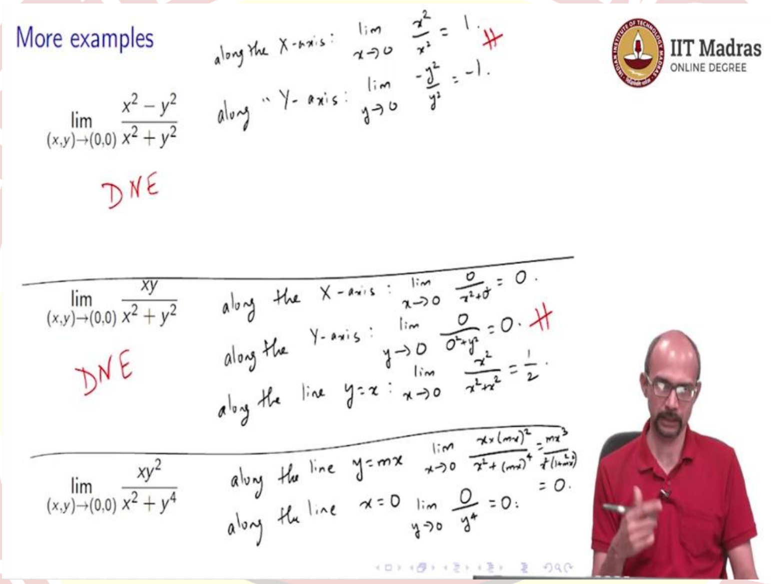

Let us do some more examples. So, this is $\frac{x^2-y^2}{x^2+y^2}$. As you can see, these are all rational functions, wherein you cannot substitute. If you can substitute and get away, go ahead and do that. That is the best possible scenario. Unfortunately, here, you cannot do that. So, we have to really sit and compute. So, what we will do is we will again find what is the limit along the x axis and the y axis.

So, along the x axis, so along the x axis means y is 0, then this becomes limit as x tends to 0, function value is 1. Along the y axis, this becomes limit as y tends to 0, so, here x is 0. So, you get -1 and already these two values do not match. These values do not match and because these two values do not match, this limit, does not exist.

How do I know that? Because had the limit existed, then along every curve, you would have had the same limit. And then that that means in particular along the x axis, you would have had that limit along the y axis also, you would have had that limit, but then these values do not match. So, I hope it is clear how we use the theorem on the previous slide.

Let us look at this example, where you have $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$. Let us do along the x axis, what happens. So, along the x axis, when here you have to substitute you have y=0, so that means you get 0, so as you can see about the y axis also the same thing will happen. So, limit as y tends to 0, this is 0.

So, unlike our previous case, where we had, we took the two curves, x axis and y axis, and they already gave different limits. Here, for the axis they are matching. So, there is still a chance that this limit exists. So, let us look along the line, y= x. So, along the line, this is you substitute y=x and then take limit x tends to 0. So, this is $\frac{1}{2}$. And these do not match. So, this does not match with either of these two. So, this limit does not exist.

So, I hope it is clear what I am doing, I am somehow choosing curves. So, my first possible choices will be lines, so easiest curves or lines. So, I am choosing lines along which these limits do not match. Sometimes what will happen is that for every line, the limit will match. But then you can somehow be smart and choose some other curve.

So, if you want to take the general line, you should take, for example, in this example, let us look along the line $y = mx$. Of course, $y = mx$ covers everything other than the line x = 0, so that we have to make a special case. So, let us see what happens for those, along the line $y = mx$, well you substitute $y = mx$. So, this is a $\frac{mx^3}{x^2(1+mx^2)}$. And after cancelling, you can see that this limit is 0. So, this limit does exist and is 0, what happens along the line x is 0. So, along the line x is 0, you have to substitute x is 0. So, this limit clearly is 0. So, you can see that for all the lines, you are getting the same limit. So, this limit really has a very strong chance of existing.

And in the tutorial, you will be discussing this in more detail. So, I leave this for now, fine. So, I hope it is clear how to use that previous theorem about the limit of a function along a curve. It is a very important idea. And, and it is very useful in the examples like these.

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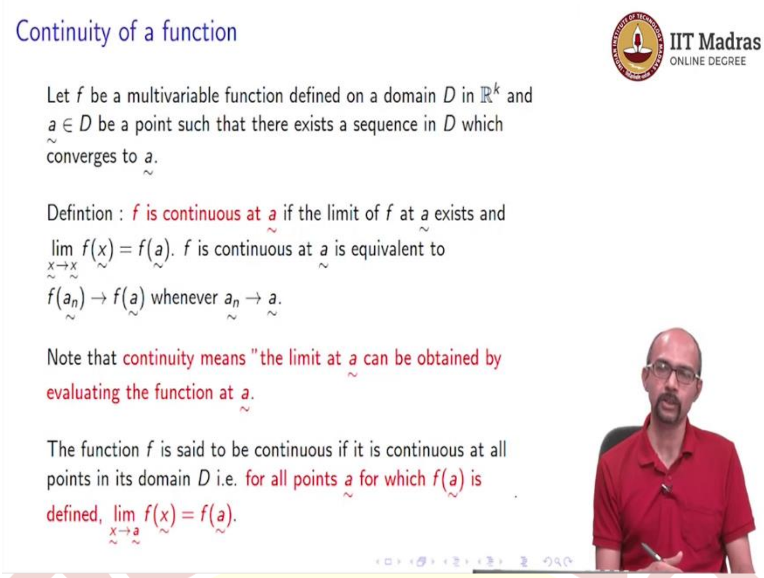

So, let us go on to talk about the continuity of a function. So, we have talked about the continuity for one variable functions using limits of functions. And now we are going to do the exactly the same thing for multivariable functions. Let f be a multivariable function defined in our domain D in R^k. And $\underset{\smile}{a}$ be a point in D. So, now we want a tilde to be a point in D because we want f to be defined on $\underset{\smile}{a}$ such that there exists a sequence in D which converges to $\underset{\smile}{a}$.

We still want that condition; f is continuous at $\underset{\smile}{a}$ if the limit of f at $\underset{\smile}{a}$ exist. And not does it exist, the value of that limit is $f(\underset{\smile}{a})$ this is what the definition was even when we add a one variable function. So, f is continuous at $\underset{\smile}{a}$ is equivalent to saying that, if you take a sequence which is convergent to $\underset{\smile}{a}$, then $f(\underset{\smile}{a_n})$ is convergent to $f(\underset{\smile}{a})$. That is exactly what it means for f to be continuous at $\underset{\smile}{a}$.

And note what this means note that continuity means the limit at $\underset{\smile}{a}$ can be obtained by evaluating the function at $\underset{\smile}{a}$. So, you can, you can just say that limit is equal to f of $\underset{\smile}{a}$. So, you can evaluate the function in $\underset{\smile}{a}$. So, the function f is said to be continuous. So, this is all talking about the when the function is continuous at a point $\underset{\smile}{a}$. So, now we say that the function f the entire function is continuous if it is continuous at all points in the domain. So, that means $\lim\limits_{\underset{\smile}{x} \to \underset{\smile}{a}} f(\underset{\smile}{x})$ is $\underset{\smile}{a}$ for all points $\underset{\smile}{a}$ in the domain.

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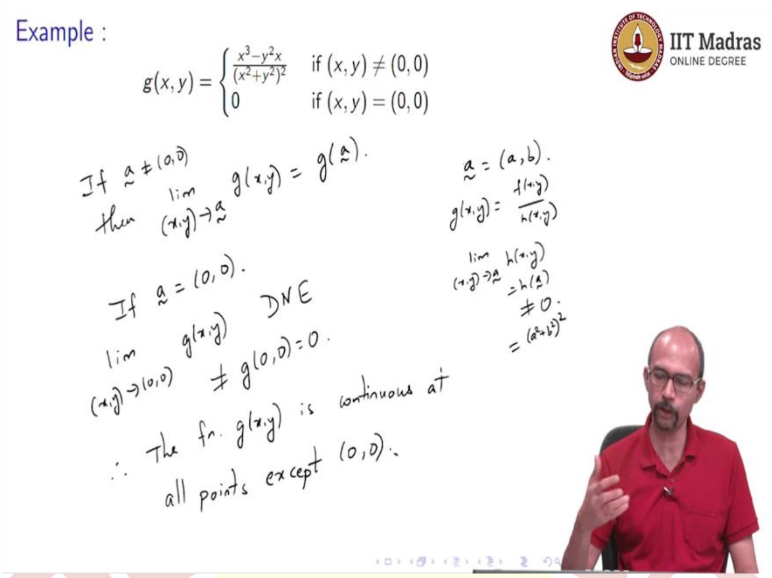

So, let us end with this example of our good old friend $\frac{x^3-y^2x}{(x^2+y^2)^2}$ if (x, y) ≠ (0, 0) and it is 0 if (x,y) = (0,0). So, I think you already know what is coming, but let us complete this. So, the question is this continuous at (0, 0) or is this a continuous function in general? So, if $\underset{\sim}{a}$ is not equal to (0, 0), then limit (x,y) tends to $\underset{\sim}{a}$. So, if $\underset{\sim}{a} \neq$ (0, 0) that means at least one of its coordinates is non-zero say $\underset{\sim}{a}$ = (a,b). And that means, if it is not (0, 0) that means at least one of them is not 0.

So, I can write g(x, y) as a quotient. So, let us say I can write it as $\frac{f(x,y)}{h(x,y)}$ both of these are polynomials. So, for polynomials to get the limit I just substitute and in this case limit as (x,y) tends to $\underset{\sim}{a}$. This is $h(\underset{\sim}{a})$ and is not 0. Why is it non-zero, because this is going to be equal to $(a^2 + b^2)^2$ and if (a,b) is not (0,0) then at least one of them is not 0 that means, this number is non-zero. So, this is not 0.

So, I can use my quotient rule and I can just substitute. So, this is just $g(\underset{\sim}{a})$. So, this limit exists. So, if $\underset{\sim}{a}$ is not (0, 0) the function is continuous at $\underset{\sim}{a}$. So, the only question is what happens at (0, 0). And we have actually seen what happens, but I just recall for you. So, $\lim_{(x,y)\to(0,0)} g(x,y)$ we checked in a slide a few minutes ago that this does not exist. So, it just does not exist. Why, because we came along the x axis and the y axis and along one of the axes, we found that the limit does not exist.

So, therefore in particular, it does not equal g(0, 0), which is 0 which does not exist, so there is no question of this. So, therefore, the function $g(x,y)$ is not continuous, or maybe we will see is continuous at all points except (0, 0). So, I hope this example also clarifies the notion of continuity. This is really the same as we have done for one variable calculus, except that you have more variables now. Thank you.