Week - 5
Mathematics for Data Science - 2
Solutions of system of linear equations
Practice Assignment Solution.

1 Multiple Choice Questions (MCQ)

1. Consider the following systems of equations and choose the correct option.

System I:
$$
\begin{aligned}
-x + 2y - 2z &= 2\\
2x + z &= -1\\
x - 3y + z &= 3
\end{aligned}
$$

System II:
$$
\begin{aligned}
-2x + y + z &= 0\\
\frac{3}{2}x + 2y - z &= -2\\
3x + 4y - 2z &= 5
\end{aligned}
$$

System III:
$$
\begin{aligned}
x + 3z &= -5\\
-\frac{2}{5}x - \frac{1}{5}y - 2z &= 3\\
2x + y + 10z &= -15
\end{aligned}
$$

- ◯ Option 1: All the three systems have a unique solution.
- ◯ Option 2: System I has a unique solution, whereas, System II and System III have no solution.
- ◯ Option 3: System I has a unique solution, whereas, System II and System III have infinitely many solutions.
- ◯ **Option 4:** System I has a unique solution, System II has no solution, and System III has infinitely many solutions.
- ◯ Option 5: System I has no solution, System II and System III have infinitely many solutions.

Solution :- Matrix representation of System I is

$$\begin{bmatrix} -1 & 2 & -2 \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Augmented matrix is $\left[\begin{array}{ccc|c} -1 & 2 & -2 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & -3 & 1 & 3 \end{array}\right] \xrightarrow{(-1)R_1} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 2 & 0 & 1 & -1 \\ 1 & -3 & 1 & 3 \end{array}\right]$

$$\xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 4 & -3 & 3 \\ 0 & -1 & -1 & 5 \end{array}\right] \xrightarrow{\substack{\frac{1}{4}R_2 \\ (-1)R_3}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 1 & 1 & -5 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 7/4 & \frac{-23}{4} \end{array}\right] \xrightarrow{\frac{4}{7}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 1 & -23/7 \end{array}\right]$$

Observe that the row echelon form of Augmented matrix is $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 1 & -23/7 \end{array}\right]$, So in system form we can write

$$x - 2y + 2z = -2$$
$$y - \frac{3}{4}z = \frac{3}{4}$$
$$z = -23/7$$

So solution is $z = -23/7$

$$y - \frac{3}{4}z = 3/4$$
$$\Rightarrow y - \frac{3}{4} \times \left(-\frac{23}{7}\right) = \frac{3}{4} \Rightarrow y = -12/7$$

& $x - 2y + 2z = -2$

$$\Rightarrow x = 8/7$$

Hence the system has a unique solution.

<u>System II</u>: Matrix representation of the system II is

$$\begin{bmatrix} -2 & 1 & 1 \\ 3/2 & 2 & -1 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix of the system II is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 3/2 & 2 & -1 & -2 \\ 3 & 4 & -2 & 5 \end{array}\right]$$

Row echelon form of Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1/11 & -8/11 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

In system form we can write

$$x - \frac{1}{2}y - \frac{1}{2}z = 0,$$

$$y - \frac{z}{11} = -8/11$$

$$\& \quad 0 = 1$$

which is absured

So, the system II has no solution.

System III The matrix representation of system III is

$$\begin{bmatrix} 1 & 0 & 3 \\ -2/5 & -1/5 & -2 \\ 2 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -15 \end{bmatrix}$$

Augmented matrix of the system III is

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ -2/5 & -1/5 & -2 & 3 \\ 2 & 1 & 10 & -15 \end{array}\right]$$

Row echelon form of Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So, in system of linear equations form

$$x + 3z = -5$$
$$y + 4z = -5$$

So $y = -5 - 4z$

& $x = -5 - 3z$

Let $z = t$ any real number then

$$x = -5 - 3t$$
$$y = -5 - 4t$$
$$z = t$$

So system III has infinitely many solutions

Hence, option 4 is true.

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2. Match the matrices in Column A with their row operation steps (in the exact sequence given) in Column B, and their corresponding reduced row Echelon forms in Column C of Table M2W2PT1.

	Matrices (Column A)		Steps for row operation (Column B)		Reduced row Echelon form (Column C)
i)	$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	a)	$[\ \ ]$ $\rightsquigarrow$ $\frac{1}{2}R_1$ $[\ \ ]$ $\rightsquigarrow$ $R_2 + (-1)R_1$ $[\ \ ]$ $\rightsquigarrow$ $R_3 + R_1$ $[\ \ ]$ $\rightsquigarrow$ $R_3 + (-1)R_2$ $[\ \ ]$	1)	$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
ii)	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	b)	$[\ \ ]$ $\rightsquigarrow$ $R_2 + 2R_1$ $[\ \ ]$ $\rightsquigarrow$ $R_2 \leftrightarrow R_3$ $[\ \ ]$ $\rightsquigarrow$ $\frac{1}{3}R_3$ $[\ \ ]$ $\rightsquigarrow$ $R_1 + (-1)R_3$ $[\ \ ]$	2)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
iii)	$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$	c)	$[\ \ ]$ $\rightsquigarrow$ $R_1 \leftrightarrow R_2$ $[\ \ ]$ $\rightsquigarrow$ $\frac{1}{2}R_1$ $[\ \ ]$ $\rightsquigarrow$ $R_3 + (-1)R_2$ $[\ \ ]$ $\rightsquigarrow$ $R_1 + (-\frac{1}{2})R_2$ $[\ \ ]$	3)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table: M2W5PT1

Find the correct option.

- ◯ Option 1: i) → b) → 3); ii) → c) → 2); iii) → a) → 1)
- ◯ Option 2: i) → a) → 3); ii) → c) → 1); iii) → b) → 2)
- ◯ **Option 3**: i) → b) → 3); ii) → c) → 1); iii) → a) → 2)
- ◯ Option 4: i) → c) → 1); ii) → b) → 3); iii) → a) → 2)

Solution :

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3/3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now collect the row operation steps,

first $R_2 + 2R_1$ then $R_2 \leftrightarrow R_3$ then $R_3/3$ then $R_1 - R_3$

& Reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $(i) \to (b) \to 3$

Now, let $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now, collect the row operation steps,

first $R_2 \leftrightarrow R_1$ then $R_1/2$ then $R_3 - R_2$ then $R_1 - \frac{1}{2}R_2$

& Reduced row echelon form of B is $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $(ii) \to (c) \to 1$

Now, let $C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

collection of row operation steps,

first $\frac{1}{2}R_1$ then $R_2 - R_1$ then $R_3 + R_1$ then $R_3 - R_2$

& Reduced row echelon form of C is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $(iii) \to (a) \to 2$

Hence, option 3 is true.

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3. There are two laptop manufacturers, one is at Adyar and the other is at Tambaram. Suppose the production costs (in crore of ₹) at Adyar and Tambaram are represented by the equations $A(x) = a_1x^2 + b_1x + c_1$ and $T(x) = a_2x + c_2$, respectively, where x represents the number (in hundreds) of laptops produced. At Adyar, the initial investment is known to be ₹3 crore, and the production costs for manufacturing 100 (i.e., $x = 1$) and 300 laptops (i.e., $x = 3$) are ₹4 crore and ₹12 crore, respectively. At Tambaram, the production costs for manufacturing 100 and 200 laptops are ₹6 crore and ₹7 crore, respectively. Suppose, Parveena and Amenla need new laptops for their start up companies. Parveena needs 500 laptops and Amenla needs 150 laptops. Both of them want their laptops with minimum production cost. Choose the correct option from the given set of options below.

   - ○ Option 1: Parveena should place her order at Adyar and Amenla should place her order at Tambaram to avail the minimum production cost.
   - ○ **Option 2:** Parveena should place her order at Tambaram and Amenla should place her order at Adyar to avail the minimum production cost.
   - ○ Option 3: Both of them should place their order at Tambaram to avail the minimum production cost.
   - ○ Option 4: Both of them should place their order at Adyar to avail the minimum production cost.

Solution :- Given production cost at Adyar is $A(x) = a_1x^2 + b_1x + c_1$

& production cost at Tambaram is $T(x) = a_2x + c_2$

where x represents the number (in hundred) of laptops produced.

Given intial investment is ₹ 3 crore at Adyar

i.e if we substitute $x = 0$ in $A(x)$ then

$c_1 = 3$.

So $A(x) = a_1x^2 + b_1x + 3$

Given, at Adyar, production cost for manufacturing 100 (i.e $x = 1$) is ₹ 4 crore i.e $4 = a_1 + b_1 + 3 \Rightarrow a_1 + b_1 = 1$ — (1)

Production cost for manufacturing 300 laptops is ₹ 12 crore.

i.e $9a_1 + 3b_1 + 3 = 12$

$\Rightarrow 3a_1 + b_1 = 3$ —— ②

equation ① & ② form system of linear equations which having a unique solution which is, $a_1 = 1$ & $b_1 = 0$

So $A(x) = x^2 + 3$

Again, at Tambaram the production cost for manufacturing 100 laptops is ₹ 6 crore

i.e $a_2 + c_2 = 6$ —— ③

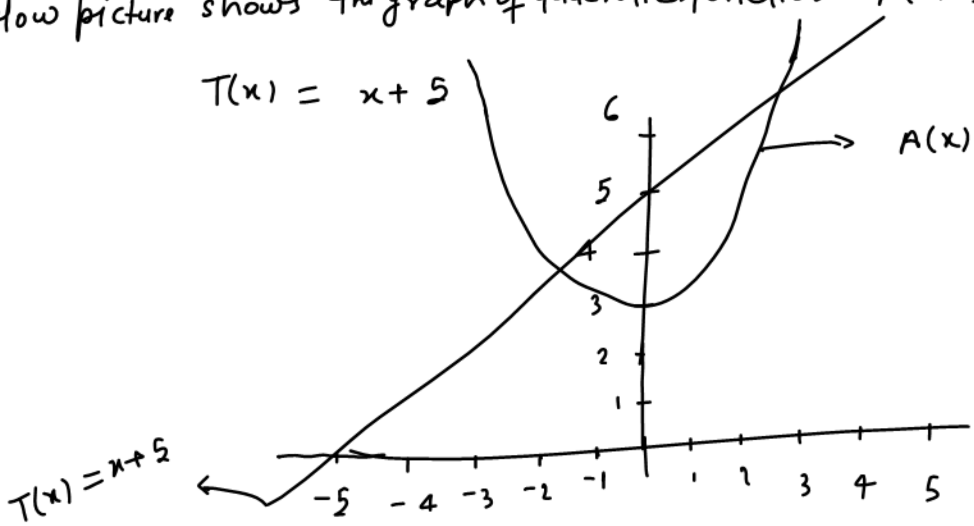
The production cost for manufacturing 200 laptops is ₹ 7 crore

i.e $2a_2 + c_2 = 7$ —— ④

After solving equations ③ & ④ we get $a_2 = 1$, $c_2 = 5$

So $T(x) = x + 5$

Below picture shows the graph of quadratic function $A(x) = x^2 + 3$ and line $T(x) = x + 5$



Now, production cost for manufacturing 500 laptops at Adyar is $A(5) = 5^2+3 = 28$ crore rupees,

and at Tambaram is $T(5) = 5+5 = 10$ crore rupees..

The production cost for manufacturing 150 laptops at

Adyar is $A(1\cdot5) = (1\cdot5)^2+3 = 2\cdot25+3$
$= 5\cdot25$ crore rupees,

and at Tamabaram is $T(1\cdot5) = 1\cdot5+5 = 6\cdot5$ crore rupees

So, the production cost for manufacturing 500 laptops at Tambaram makes minimum cost, and the production cost for manufacturing 150 laptops at Adyar makes minimum cost.

So, Praveena should place her order Tambaram and Amenla should place her order at Adyar to avail the minimum production cost.

Hence, option 2 is true.

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2 Multiple Select Questions (MSQ)

4. Choose the set of correct options.

   ○ **Option 1:** If A is an upper triangular 3×3 matrix, then the adjoint matrix of A is also an upper triangular matrix.

   ○ **Option 2:** If A is an invertible upper triangular 3×3 matrix, then the inverse matrix of A is also an upper triangular matrix.

   ○ Option 3: Let A is an arbitrary real 3×3 matrix. If C is the adjoint matrix of A, then C is also the adjoint matrix of A^T.

   ○ Option4: C_{jk} denotes the cofactor with respect to the j-th row and the k-th column of a 3×3 matrix A. If another matrix B is obtained from A by replacing the j-th row of A with $\begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$, then $det(B) = 3C_{jk}$

   ○ Option 5: If A is an invertible 3×3 matrix and $C = adj(adj(A))$, then $det(C) = det(A)^9$

Solution :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$$adj(A) = \begin{bmatrix} a_{22}a_{33} & -a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ 0 & a_{11}a_{33} & -a_{11}a_{23} \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$$

which is an upper triangular matrix

Hence, option 1 is true.

We know that $A^{-1} = \dfrac{adj(A)}{|A|}$ where $|A| = det(A)$

$$\text{So} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22}a_{33} & -a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ 0 & a_{11}a_{33} & -a_{11}a_{23} \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$$

which is also an upper triangular matrix.

Hence, option 2 is also true.

Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $adj(A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. then $adj(A^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Here $adj(A) \neq adj(A^T)$

Hence, option 3 is not true.

Now option 4: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

let $j = 2$ & $k = 2$, then

$$C_{22} = 1$$

As given, B is obtained from A by replacing the j^{th} row of A with $[3 \ 0 \ 0]$

so, for case $j = 2$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so, } \det(B) = 0 \neq 3\,C_{22}$$

So, option-4 is not true.

Option-5: first Method:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

then $adj(A) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

and $adj(adj(A)) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$

We know that determinant of a diagonal matrix is just multiplication of diagonal entries.

So, $det(A) = 6$

and $det(adj(adj(A))) = 6 \times 12 \times 18$

$\Rightarrow \quad det(adj(adj(A))) = 6^4 \neq det(A)^9 = 6^9$

Second Method:

Given $C = adj(adj(A))$, let $m = det(A) \Rightarrow \frac{1}{m} = det(A)$

$$det(C) = det(adj(adj(A)))$$

$$= det(adj(det(A) \cdot A^{-1})) , \left(\text{use, } A^{-1} = \frac{adj(A)}{det(A)} \right)$$

$$= det(adj(m \cdot A^{-1}))$$

$$= det(det(mA^{-1}) \cdot (mA^{-1})^{-1}) , \left(\text{use, } A^{-1} = \frac{adj(A)}{det(A)} \right)$$

$$= det\left(m^3 det(A^{-1}) \cdot \frac{1}{m}(A^{-1})^{-1}\right) \quad \left(\begin{array}{c}\text{Since } A \text{ is} \\ 3\times3 \text{ matrix} \\ \text{so } det(KA) = K^3 det(A)\end{array}\right)$$

$$= det\left(m^3 (det(A))^{-1} \cdot \frac{1}{m} \cdot A\right) , \left(\because (A^{-1})^{-1} = A \right)$$

$$= det\left(m^3 \cdot \frac{1}{m} \cdot \frac{1}{m} \cdot A\right)$$

$$= det(m \cdot A)$$

$$= m^3 \cdot det(A)$$

$$= m^3 \cdot m \quad (\because \text{we have assume } det(A) = m)$$

$$= m^4$$

$$\Rightarrow \quad det(C) = det(A)^4$$

Hence, Option 5 is not true.

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5. Choose the correct set of options based on the matrices given in Table M2W2PT2.

Column A	Column B
$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$B_2 = \begin{bmatrix} -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \\ -\frac{1}{3} & 0 & 0 \end{bmatrix}$
$A_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}$	$B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$A_4 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$	$B_4 = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

Table: M2W5PT2

- ◯ **Option 1:** A_1 and B_3 are inverses to each other.
- ◯ Option 2 : A_1 and B_1 are inverses to each other.
- ◯ **Option 3 :**A_2 and B_1 are inverses to each other.
- ◯ **Option 4:** A_3 and B_4 are inverses to each other.
- ◯ Option 5: A_2 and B_3 are inverses to each other.
- ◯ Option 6: A_3 and B_2 are inverses to each other.
- ◯ Option 7: A_4 and B_4 are inverses to each other.
- ◯ **Option 8:** A_4 and B_2 are inverses to each other.
- ◯ Option 9: A_2 and A_3 have different reduced row echelon form.
- ◯ Option 10: A_1 and A_2 have different reduced row echelon form.
- ◯ **Option 11:** All the matrices in column A have the same reduced row echelon form and that is the identity matrix of order 3.

○ Option 12: All the matrices in column A have the same reduced row echelon form but that is not the identity matrix of order 3.

Solution: Given $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$|A_1| = 1(-1) = -1$$

$$\text{adj}(A_1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\text{so} \quad A_1^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = B_3$$

similarly, we can calculate A_2^{-1} which is B_1, $A_3^{-1} = B_4$

& $A_4^{-1} = B_2$

Now $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Reduced row echelon form

$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Reduced row echelon form

$A_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix}$

$$\xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[R_3+3R_2]{R_1+2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Reduced row echelon form}$$

$$A_4 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Big\downarrow \left(-\tfrac{1}{3}\right) R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xleftarrow[R_3+R_2]{R_1+2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Reduced row echelon form}$$

Clearly, all matrices in column A have same reduced row echelon form which is Identity matrix of order 3.

Hence option 1, option 3, option 4, option 8

and option 11 are true.

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3 Numerical Answer Type (NAT):

6. A gym trainer suggested Pranjal to include banana, mozzarella cheese, and avocado in his daily diet, for his fitness. In 1 banana, there are 1 unit of protein, 20 units of carbohydrate, and 1 unit of fat. In $\frac{1}{2}$ cup mozzarella cheese, there are 10 units of protein, 50 units of carbohydrate and 0 unit of fat. In 1 avocado there are 3 units of protein, 10 units of carbohydrate, and 10 units of fat. Suppose the calories intake from 1 banana, $\frac{1}{2}$ cup mozzarella cheese, and 1 avocado are 105, 90 and 115, respectively. If the gym trainer suggested Pranjal to take 18 units of protein, 110 units of carbohydrate, and 22 units of fat by taking only these three items, then find out the calories intake by Pranjal each day from these three items only. [Answer: 530]

Solution: Given

	Protein	Carbohydrate	Fat	Calories
1 Banana	1	20	1	105
½ cup mozzarella Cheese	10	50	0	90
1 Avocado	3	10	10	115

Let, to take 18 units of protein, 110 units of carbohydrate & 22 units of fat, Pranjal takes x banana, y number of ½ cup mozzarella cheese & z avocado,

then system of linear equations is

$$x + 10y + 3z = 18$$

$$20x + 50y + 10z = 110 \Rightarrow 2x + 5y + z = 11$$

$$x + 10z = 22$$

Matrix representation of the above system of linear equations is

$$\begin{bmatrix} 1 & 10 & 3 \\ 2 & 5 & 1 \\ 1 & 0 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \\ 22 \end{bmatrix}$$

Augmented matrix of the above system is

$$\left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 2 & 5 & 1 & 11 \\ 1 & 0 & 10 & 22 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & -15 & -5 & -25 \\ 0 & -10 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{\left(\frac{-1}{15}\right) R_2} \left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & -10 & 7 & 4 \end{array}\right] \xrightarrow{R_3 + 10R_2} \left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 31/3 & 62/3 \end{array}\right]$$

$$\xrightarrow{\frac{3}{31} R_3} \left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Hence, the above system has a unique solution &

$$z = 2$$

$$y + \frac{z}{3} = 5/3 \Rightarrow y = \frac{5}{3} - \frac{2}{3} = 1$$

$$x + 10y + 3z = 18 \Rightarrow x + 10 + 6 = 18$$

$$\Rightarrow x = 2$$

So, Pranjal will take 2 banana, Only one $\frac{1}{2}$ cup mozzarella cheese & 2 avocado.

Calories intake from 1 banana, $\frac{1}{2}$ cup mozzarella & 1 avocado are 105, 90 & 115 respectively.

Hence, total calories taken by Pranjal is

$$2 \times 105 + 1 \times 90 + 115 \times 2 = 210 + 90 + 230$$
$$= 530$$

So Answer is 530.

.

7. Consider the system of linear equations $Ax = b$, where $A = \begin{bmatrix} 2 & a & 3 \\ a & -2 & -1 \\ -1 & a & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ 5/4 \\ x_3 \end{bmatrix}$, and $b = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$, and the solution for x is partially known.

   What is the value of a, if $a > 1$ is given? [Answer: 2]

Solution: Given $A = \begin{bmatrix} 2 & a & 3 \\ a & -2 & -1 \\ -1 & a & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ 5/4 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$

Observe, we have given value of x_2 which is $5/4$

$$|A| = \begin{vmatrix} 2 & a & 3 \\ a & -2 & -1 \\ -1 & a & 0 \end{vmatrix}, \quad \text{where } |A| = \det(A)$$

Expand along R_3

$$|A| = -1(-a+6) - a(-2-3a) + 0$$

$$= a - 6 + 3a^2 + 2a$$

$$|A| = 3a^2 + 3a - 6$$

replacing second column with b in A

we get

$$A_{x_2} = \begin{bmatrix} 2 & 1 & 3 \\ a & a & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Expand along R_3

$$\Rightarrow |A_{x_2}| = -1(-1-3a) - 1(-2-3a)$$

$$= 1 + 3a + 2 + 3a = 6a + 3$$

Now, using Cramer rule

$$x_2 = 5/4 = \frac{|A_{x_2}|}{|A|} = \frac{6a+3}{3a^2+3a-6}$$ this is well defined because $|A| \neq 0$ for $a > 1$

$$\Rightarrow 5(3a^2+3a-6) = 4(6a+3)$$

$$\Rightarrow 5a^2+5a-10 = 8a+4$$

$$\Rightarrow 5a^2-3a-14 = 0$$

$$\Rightarrow 5a^2-10a+7a-14 = 0$$ -14×5

$$\Rightarrow (a-2)(5a+7) = 0$$

$$\Rightarrow a = 2 \text{ or } a = -7/5$$

but $a > 1$

$$\Rightarrow a = 2$$

Hence, Answer is 2.

.

4 Comprehension Type Question:

In genetics, a classic example of dominance is the inheritance of seed shape (pea shape) in peas. Peas may be round (associated with genotype R) or wrinkled (associated with genotype r). In this case, three combinations of genotypes are possible: RR, rr, and Rr. The RR individuals have round peas and the rr individuals have wrinkled peas. In Rr individuals the R genotype masks the presence of the r genotype, so these individuals also have round peas. Thus, the genotype R is completely dominant to genotype r, and genotype r is recessive to genotype R. First, assume the crossing of RR with RR. This always gives the genotype RR, therefore the probabilities of an offspring to be RR, Rr, and rr respectively are equal to 1, 0, and 0. Second, assume crossing of Rr with RR. The offspring will have equal chances to be of genotype RR and genotype Rr, therefore the probabilities of RR, Rr, and rr repectively are 1/2, 1/2, and 0. Third, consider crossing of rr with RR. This always results in genotype Rr. Therefore, the probabilities of genotypes RR, Rr, and rr repectively are 0, 1, and 0, respectively.
This can be veiwed as the following table:

Parents' genotypes			Genotypes of offspring
RR-RR	RR-Rr	RR-rr	
1	1/2	0	RR
0	1/2	1	Rr
0	0	0	rr

Table: M2W5 PT 3

The matrix representing this observation is given by $P = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Let the probabilities of RR, Rr, and rr in the initial (i.e., at $t = 0$) sample space be $X_0^1, X_0^2,$ and X_0^3, respectively. This is represented by the initial distribution vector (3×1 matrix) is denoted by $X_0 = \begin{bmatrix} X_0^1 \\ X_0^2 \\ X_0^3 \end{bmatrix}$.

For any positive integer n, the distribution vector after n generations (i.e., at $t = n$) is denoted by X_n and given by the equation $PX_{n-1} = X_n$.
Using the above information answer the following questions.

8. Find out the correct set of options from the following. (MSQ)

   ◯ Option 1: The row reduced echelon form of P and P^2 are different in this case.

   ◯ **Option 2:** The row reduced echelon form of P and P^2 are same in this case.

◯ **Option 3:** The row reduced echelon form of P is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

◯ Option 4: The row reduced echelon form of P is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Given $P = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

This is the Reduced row echelon form of P

Now $P^2 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

So $P^2 = \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{4R_2} \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{R_1 - \frac{3}{4}R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Reduced row echelon form of P^2

Observe, Reduced row echelon form of P & P^2 are the same.

Hence, Option 2 & Option 3 are true.

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9. Suppose after 2 years the distribution vector i.e., X_2 is calculated to be $\begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}$, and the initial distribution of RR is $\frac{1}{3}$. Find out the initial distibution of Rr and rr. (MCQ)
   - ◯ Option 1: The initial distibution of Rr and rr : $\frac{2}{3}$, 0, respectively.
   - ◯ Option 2: The initial distibution of Rr and rr : 0, $\frac{2}{3}$, respectively.
   - ◯ **Option 3:** The initial distibution of Rr and rr : $\frac{1}{3}$, $\frac{1}{3}$, respectively.
   - ◯ Option 4: Cannot be determined from the given information.

Solution :- Given $PX_{n-1} = X_n$ & initial distribution of RR is $\frac{1}{3}$

Let initial distribution is $X_0 = \begin{bmatrix} 1/3 \\ y \\ z \end{bmatrix}$,

For simplicity of notations we are writing $X_0^1 = x$, $X_0^2 = y$, & $X_0^3 = z$.

Now, $PX_0 = X_1$ & $PX_1 = X_2$

$\Rightarrow \quad P \cdot P X_0 = PX_1 = X_2$

$\Rightarrow \quad P^2 X_0 = X_2$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

Now, this is a system of linear equations, where $x = \frac{1}{3}$

Observe, we can not apply cramer's rule because determinant of coefficient matrix is zero.

Now, Augmented matrix of the above system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 3/4 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{4R_2} \left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 3/4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 - \tfrac{3}{4} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, from reduced row echelon form of Augmented matrix

$$x - z = 0 \quad , \text{ but } x = 1/3$$

$$\Rightarrow \quad z = 1/3$$

again from above reduced row echelon form of Augmented matrix,

$$y + 2z = 1$$

$$\Rightarrow \quad y + \frac{2}{3} = 1$$

$$\Rightarrow \quad y = 1 - 2/3 = 1/3$$

But y & z denote the initial distribution of Rr & rr respectively

Hence initial distribution of Rr & rr are $1/3$, $1/3$ respectively.

Hence, the third option is correct.

.

10. Suppose after 3 generations the distribution vector i.e., X_3 is calculated to be $\begin{bmatrix}1\\0\\0\end{bmatrix}$, and recall that $0 \le X_0^1, X_0^2, X_0^3 \le 1$. Find out the correct set of options. (MSQ)

- ◯ **Option 1:** $X_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$
- ◯ Option 2: $X_0 = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$
- ◯ Option 3: $X_0 = \begin{bmatrix}0\\0\\1\end{bmatrix}$
- ◯ Option 4: X_0 cannot be determined from the given information.
- ◯ **Option 5:** $X_0 = X_n$ for all positive integer n.
- ◯ Option 6: There can be some positive integer n for which $X_0 \neq X_n$.

Solution:

Given $Px_{n-1} = x_n$ & $x_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\Rightarrow$ $Px_0 = x_1$ & $Px_1 = x_2$ & $Px_2 = x_3$

$\Rightarrow$ $P^2x_0 = Px_1 = x_2$

$\Rightarrow$ $P^3x_0 = Px_2 = x_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Now $P^3 = P^2 \cdot P = \begin{bmatrix}1 & 3/4 & 1/2\\0 & 1/4 & 1/2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1/2 & 0\\0 & 1/2 & 1\\0 & 0 & 0\end{bmatrix}$

$= \begin{bmatrix}1 & 7/8 & 3/4\\0 & 1/8 & 1/4\\0 & 0 & 0\end{bmatrix}$

For simplicity of notation we are writing $x_0^1 = x$, $x_0^2 = y$, $x_0^3 = z$

So, let initial distribution $X_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

then $P^3 X_0 = X_3$ form a system of linear equations

$$\begin{bmatrix} 1 & 7/8 & 3/4 \\ 0 & 1/8 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, Augmented matrix of the above system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 7/8 & 3/4 & 1 \\ 0 & 1/8 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{8R_2} \left[\begin{array}{ccc|c} 1 & 7/8 & 3/4 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 - \tfrac{7}{8}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is reduced row echelon from of Augmented matrix.

So,
$$x - z = 1$$
$$y + 2z = 0$$

$$\Rightarrow \quad x = 1 + z \quad \text{——} ①$$
$$y = -2z \quad \text{——} ②$$

Observe that, it is given that $0 \le x, y\ z \le 1$

From equation (1). we get $x \geq 1$ as $z \geq 0$

But we also have, $x \leq 1$

Hence, $x = 1$

Hence, from equation (1)

$$z = 0$$

and from equation (2) $z = 0 \Rightarrow y = 0$

So $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now again, $P x_{n-1} = x_n$

$$\Rightarrow \quad P x_0 = x_1 \Rightarrow x_1 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, $x_2 = P x_1 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vdots \qquad \vdots$$

$$x_n = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_0$$

Hence, option 1 & option 5 are true.