Tangents for scalar-valued multivariable functions

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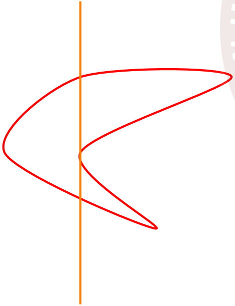

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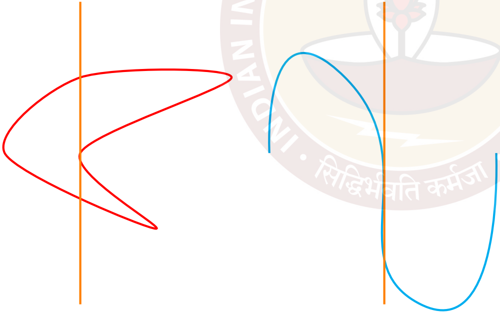

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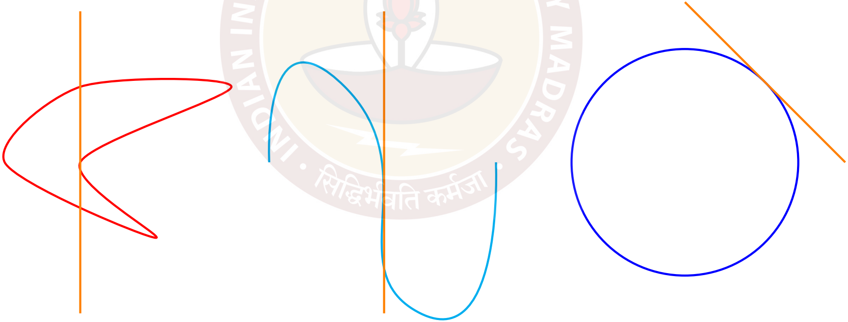

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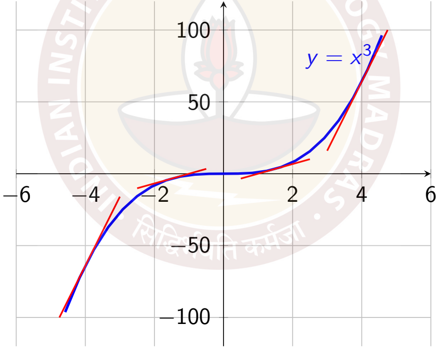

Figure: Tangent lines for $y = x^3$

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The equations of the tangent line

$$\left.\begin{aligned} x(t) &= a + tu_1 \\ y(t) &= b + tu_2, \\ z(t) &= f(a,b) + t f_u(a,b). \end{aligned}\right\} \text{Parametric eqns.}$$

$$\left.\frac{x-a}{u_1} = \frac{y-b}{u_2} = \frac{z-f(a,b)}{f_u(a,b)}\right\} \text{Symm. eqns.}$$

$$(x(t), y(t), z(t)) = (a, b, f(a,b)) + t\boxed{(u_1, u_2, f_u(a,b))}.$$

vector form.

$u = (u_1, u_2)$ unit vector on L.

$\underset{\sim}{a} = (a,b)$ is the point.

$f_u(a,b)$ is the directional derivative at (a,b).

$L: \; z = 0, \; u_1(y-b) = u_2(x-a).$

$P: \; u_1(y-b) = u_2(x-a).$

$$\begin{aligned} x(t) &= a + tu_1 \\ y(t) &= b + tu_2 \\ z(t) &= 0. \end{aligned}$$

$$(x(t), y(t), z(t)) = (a,b,0) + t\underline{(u_1, u_2, 0)}.$$

Examples

$f(x,y) = x + y$; tangent at $(1,1)$ in the direction of $(1,0)$

$f_u(1,1) = \frac{\partial f}{\partial x}(x,y) = 1.$

$(x(t), y(t), z(t)) = (1,1,2) + t(1,0,1).$

$x(t) = 1+t,\ y(t) = 1,\ z(t) = 2+t.$

$f(x,y) = xy$; tangent at $(1,1)$ in the direction of $(3,4)$

$u = \left(\frac{3}{5}, 4/5\right).$ $\quad f_u(1,1) = 1 \times \frac{3}{5} + 1 \times 4/5 = 7/5.$ $\quad \nabla f(x,y) = (y,x)$

$\nabla f(1,1) = (1,1).$

$(x(t), y(t), z(t)) = (1,1,1) + t(3/5, 4/5, 7/5)$

$= (1 + 3t/5,\ 1 + 4t/5,\ 1 + 7t/5).$

$f(x,y) = sin(xy)$; tangent at $(\pi,1)$ in the direction of $(1,2)$

$u = \frac{1}{\sqrt{5}}(1,2).$ $\quad \nabla f(x,y) = (y\cos(xy), x\cos(xy)).$ $\therefore \nabla f(\pi,1) = (-1,-\pi).$

$f_u(\pi,1) = (-1) \times \frac{1}{\sqrt{5}} + (-\pi) \times \frac{2}{\sqrt{5}} = \frac{-2\pi-1}{\sqrt{5}}.$

$(x(t), y(t), z(t)) = (\pi,1,0) + t\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{-2\pi-1}{\sqrt{5}}\right).$

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Parametric equations and an example

Similar to the two-variable case, we can deduce the equations of the tangent line as :

$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, z is the variable in which we are measuring the fn.

$\underset{\sim}{a} = (a_1, \ldots, a_n)$, $u = (u_1, \ldots, u_n)$.

Line through $\underset{\sim}{a}$ in the direction of u is

$$x_i(t) = a_i + t u_i$$
$$z = 0.$$
$$(\underset{\sim}{x}(t), z(t)) = (\underset{\sim}{a}, 0) + t(u, 0).$$

$\therefore$ The tangent line to f at $\underset{\sim}{a}$ above L is

$$(\underset{\sim}{x}(t), z(t)) = (\underset{\sim}{a}, f(\underset{\sim}{a})) + t(u, f_u(\underset{\sim}{a})).$$
$$x_i(t) = a_i + t u_i, \quad z(t) = f(\underset{\sim}{a}) + t f_u(\underset{\sim}{a}).$$

Example : $f(x, y) = xy + yz + zx$; tangent at $(1, 1, 1)$ in the direction $(-1, -2, 2)$

$u = \frac{1}{3}(-1, -2, 2)$.

$\nabla f(x, y, z) = (y+z, z+x, x+y)$; $\nabla f(1,1,1) = (2,2,2)$.

$f_u(1,1,1) = -2/3$

$(x(t), y(t), z(t), u(t)) = (1, 1, 1, 3) + t\left(\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}, -2/3\right)$

$x(t) = 1 - t/3$, $y(t) = 1 - 2t/3$, $z(t) = 1 + 2t/3$, $u(t) = 3 - 2t/3$.

Caution : tangents need not always exist.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$f_u(0,0) = \begin{cases} 0 & \text{if } (u_1,u_2) = \pm e_1 \text{ or } \pm e_2 \\ \text{DNE} & \text{o.w.} \end{cases}$$

The tangent lines in all directions other than along the x or y-axes at $(0,0)$ DNE.

$$f(x,y) = |x| + |y|$$

For many directions at many points, the tangent line will not exist.

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Conclusion : All the tangents at a point $\underset{\sim}{a}$ exist when ∇f exists and is continuous on some open ball around the point $\underset{\sim}{a}$.

Thank you