Q1 (a) Suppose the average marks in an exam in a class of 100 students is 50. What is the maximum number of students who could have got more than 90 marks?

X : marks $\quad E[X] = 50$

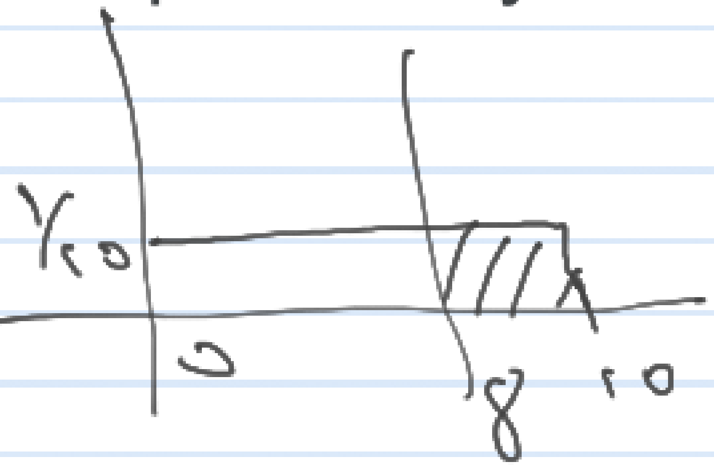
$$P(X > 90) \leq \frac{E[X]}{90} = \frac{50}{90} = \frac{5}{9}$$

$$\frac{\text{No. of students} > 90}{100} \leq \frac{5}{9} \Rightarrow \text{No. of students} > 90 \leq \frac{500}{9} = 55.5\ldots$$

Q1 (b) Suppose X is a continuous random variable uniformly distributed in [0,10]. Find an upper bound on P(X > 8) using Markov inequality. Compare with the actual probability.

$E[X] = 5$

$$P(X > 8) \leq \frac{E[X]}{8} = \frac{5}{8} = 0.625$$



$$P(X > 8) = \frac{1}{10} \times (10-8) = \frac{2}{10} = \frac{1}{5} = 0.2$$

Q2 (a) Suppose a fair coin is tossed 200 times. Find an upper bound (using Markov's inequality) for the probability that more than 150 heads are seen.

$X =$ No. of heads $\sim$ Binomial$(200, 1/2)$, $E[X] = 200 \times \frac{1}{2} = 100$

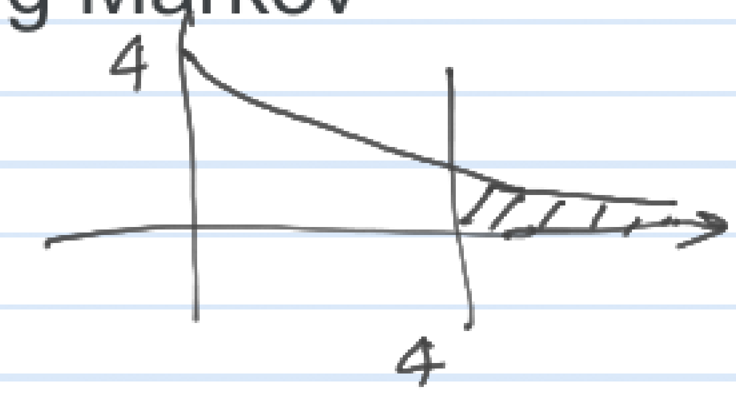
$$P(X > 150) \leq \frac{E[X]}{150} = \frac{100}{150} = \frac{2}{3}$$

Q2 (b) A biased coin with probability of heads equal to 1/3 is tossed two hundred times. Find an upper bound (using Markov's inequality) for the probability that more than 150 heads are seen.

$X \sim$ Binomial$(200, 1/3)$, $E[X] = 200 \times 1/3 = \frac{200}{3}$

$$P(X > 150) \leq \frac{E[X]}{150} = \frac{\overset{4}{\cancel{200}}}{\underset{3}{\cancel{150}} \times 3} = \frac{4}{9}$$

Q3 (a) Suppose X is Exponential(4). Find an upper bound on P(X > 4) using Markov inequality. Compare with the actual probability.



$E[X] = 1/4$

PDF: $f_X(x) = 4e^{-4x}, \quad x > 0$

Markov: $P(X>4) \le \frac{1/4}{4} = \frac{1}{16} = 0.0625$

CDF: $F_X(x) = 1 - e^{-4x}, \quad x > 0$

$P(X>4) = 1 - P(X \le 4) = 1 - F_X(4) = 1 - (1 - e^{-4\times 4})$

$= e^{-16} = 1.125 \times 10^{-7}$

Q3 (b) Suppose X is Poisson(4). Find an upper bound on P(X > 12) using Markov inequality.

$E[X] = 4$

$P(X>12) \le \frac{E[X]}{12} = \frac{4}{12} = 1/3 = 0.33\ldots$

Q3 (c) Suppose X is Geometric(1/4). Find an upper bound on P(X > 8) using Markov inequality. Compare with the actual probability.

$E[X] = 4$

CDF: $F_X(k) = 1 - (1-p)^k = 1 - \left(\frac{3}{4}\right)^k$

Markov: $P(X>8) \le \frac{E[X]}{8} = \frac{4}{8} = 0.5$

$P(X>8) = 1 - F_X(8) = 1 - \left(1 - (3/4)^8\right)$

$= (3/4)^8 = 0.1001\ldots$

Q4 (a) Let X1, X2,..., X5 be iid Uniform[0,100]. Let X = X1 + X2 +...+ X5. Find an upper bound for P(X>450) using Markov's inequality.

$$E[X_i]=50, \quad E[X]=5\times E[X_i]=5\times 50=250$$

$$P(X>450)\le \frac{E[X]}{450}=\frac{250}{450}=\frac{5}{9}$$

Q4 (b) Let X1, X2,..., X50 be iid X, where X has the following distribution:
P(X = -3) = 0.1, P(X = 0) = 0.3, P(X = 0.5) = 0.1, P(X = 1) = 0.3, P(X = 2) = 0.2
Let S = X1 + X2 +...+ X50. Find an upper bound for P(X>80) using Markov's inequality.

$$E[X_i]=-3\times 0.1+0\times 0.3+0.5\times 0.1+1\times 0.3+2\times 0.2=0.45$$

$$E[X]=50\times E[X_i]=50\times 0.45=22.5$$

$$P(X>80)\le \frac{E[X]}{80}=\frac{22.5}{80}=0.28125$$

Q4 (c) Let X1, X2,..., X100 be iid Beta(3,10). Let Y = (X1 + X2 +...+ X100)/100. Find an upper bound for P(X > 0.9) using Markov's inequality.

$$E[X_i]=\frac{3}{3+10}=\frac{3}{13}, \quad E[Y]=100\times\frac{1}{100}\times E[X_i]=3/13$$

$$P(Y>0.9)\le\frac{3/13}{0.9}=\frac{10}{39}=0.2564\ldots$$

Q5 (a) 10 balls are thrown into 10 bins independently and uniformly at random.
Let Xi = 1 if Bin i is empty and 0, otherwise. What is P(Xi = 1)? In other words, what is the probability that the i-th bin is empty? What is E[Xi]?

$(X_i = 1) =$ No ball lands in Bin i = (Ball 1 not in Bin i) AND (Ball 2 not in Bin i) AND ... AND (Ball 10 not in Bin i)

$$P(X_i = 1) = \frac{9}{10}\cdot\frac{9}{10}\cdots\frac{9}{10} = \left(\frac{9}{10}\right)^{10}$$

$$E[X_i] = 1\cdot P(X_i = 1) + 0\cdot P(X_i = 0) = \left(\frac{9}{10}\right)^{10}$$

Q5 (b) Let X = X1 + … + X10 be the number of empty bins. What is E[X]? Can you comment on the distribution of X?

$$E[X] = 10\cdot E[X_i] = 10\cdot\left(\frac{9}{10}\right)^{10} = \frac{9^{10}}{10^{9}} = 3.48678\ldots$$

Since X_i are dependent, distribution of X is complicated ....

Q5 (c) Using Markov's inequality, find an upper bound for P(X > 5).

$$P(X > 5) \leq \frac{3.48\ldots}{5} = 0.697\ldots$$

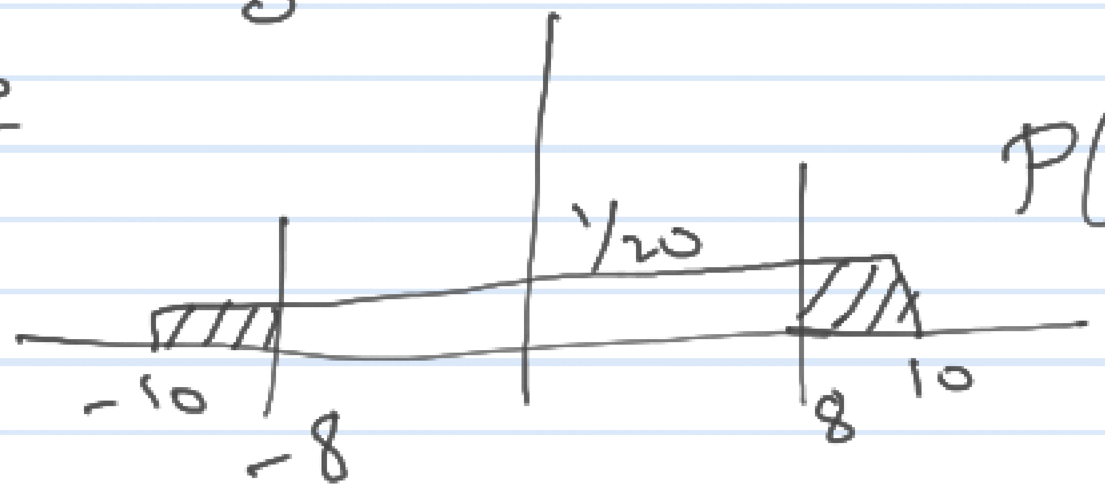
Q6 (a) Suppose X is a continuous random variable uniformly distributed in [-10,10]. Find an upper bound on P(|X| > 8) using Chebyshev inequality. Compare with the actual probability.

$E[X] = \frac{10+(-10)}{2} = 0$

Chebyshev:

$P(|X - \underbrace{E[X]}_{=0}| > 8) \leq \frac{Var(X)}{8^2} = \frac{\cancel{100}\,25}{\cancel{64}\,16 \times 3} = \frac{25}{48} = 0.52\ldots$

$Var(X) = \frac{(10-(-10))^2}{12} = \frac{100}{3}$



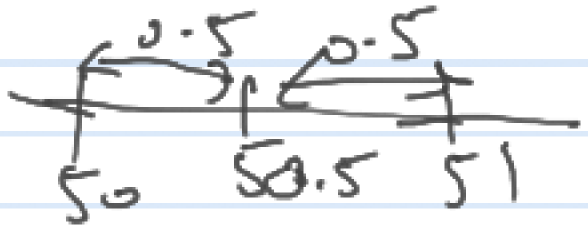
$P(|X|>8) = \frac{1}{20}(2(10-8)) = 0.2$

Q6 (b) Suppose X is a discrete random variable uniformly distributed in {1,...,100}. Find a lower bound on P(X = 50 or 51) using Chebyshev inequality. Compare with the actual probability.

$E[X] = \frac{1+100}{2} = 50.5$

Chebyshev: $P(|X - E[X]| > 0.5) \leq \frac{Var(X)}{(0.5)^2}$

$Var(X) = \frac{100^2 - 1}{12}$



very large . . .

$(X = 50 \text{ (or) } 51) \Leftrightarrow |X - E[X]| \leq 0.5$

$P(|X-E[X]| \leq 0.5) = 1 - P(|X - E[X]| > 0.5) \geq 1 - \frac{Var(X)}{0.5^2}$

Actual:

$P(X = 50 \text{ (or) } 51) = 2 \times \frac{1}{100} = \frac{1}{50}$

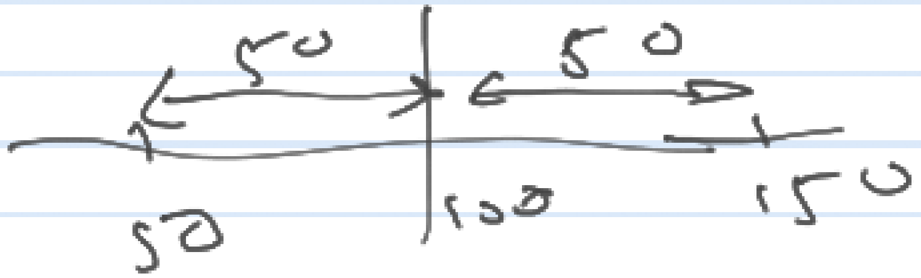
Q7 (a) Suppose a fair coin is tossed 200 times. Find an upper bound (using Chebyshev inequality) for the probability that more than 150 heads or fewer than 50 heads are seen.

X = No. of heads $\sim$ Binomial$(200, \frac{1}{2})$, $E[X] = 200 \times \frac{1}{2} = 100$

$$\mathrm{Var}(X) = 200 \times \frac{1}{2}\left(1 - \frac{1}{2}\right) = 50$$

(Markov: $2/3$ (>150))

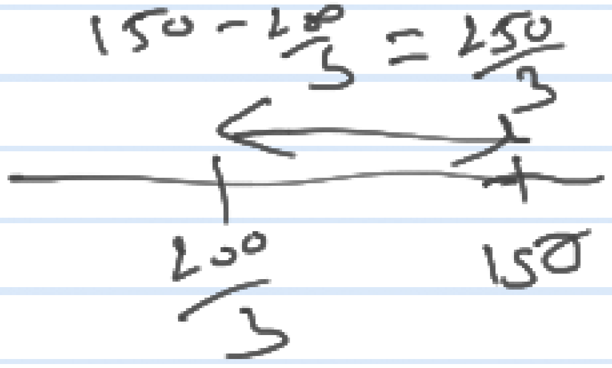
$$(X > 150 \text{ (or) } X < 50) \Leftrightarrow |X - 100| > 50$$



$$P(|X - 100| > 50) \le \frac{50}{50^2} = \frac{1}{50} = 0.02$$

Q7 (b) A biased coin with probability of heads equal to 1/3 is tossed two hundred times. Find an upper bound (using Chebyshev's inequality) for the probability that more than 150 heads are seen.

X = No. of heads $\sim$ Binomial$(200, \frac{1}{3})$, $E[X] = \frac{200}{3}$, $\mathrm{Var}(X) = 200\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{400}{9}$



$$P(X > 150) \le P\left(\left|X - \frac{200}{3}\right| > \frac{250}{3}\right) \le \frac{400/9}{250^2/3^2} = \frac{4}{625} = 0.0064$$

(Markov: $4/9$)

Q8 (a) Suppose X is Normal(0, 2). Find an upper bound on P(|X| > 8) using Chebyshev inequality.

$$E[X] = 0, \quad Var(X) = 2$$

$$P(|X| > 8) = P(|X - E[X]| > 8) \le \frac{2}{8^2} = \frac{1}{32}$$

Q8 (b) Suppose X is Normal(100, 10). Find an upper bound on P(|X-100| > 80) using Chebyshev inequality.

$$E[X] = 100, \quad Var(X) = 10$$

$$P(|X - 100| > 80) \le \frac{10}{80^2} = \frac{1}{640}$$

Q9 (a) Suppose X is Exponential(4). Find an upper bound on P(X > 4) using Chebyshev inequality. Compare with Markov.

$E[X] = \frac{1}{4}$, $Var(X) = \frac{1}{4^2} = \frac{1}{16}$

Markov: $1/16$

$P(X > 4) = P\left(|X - 1/4| > 4 - \frac{1}{4} = \frac{15}{4}\right) \leq \frac{1/16}{(15/4)^2} = \frac{1}{225}$

Q9 (b) Suppose X is Poisson(4). Find an upper bound on P(X > 12) using Chebyshev inequality. Compare with Markov.

$E[X] = 4$, $Var(X) = 4$

Markov: 0.33....

$P(X > 12) \leq P(|X - 4| > 8) \leq \frac{4}{8^2} = \frac{1}{16} = 0.0625$

Q9 (c) Suppose X is Geometric(1/4). Find an upper bound on P(X > 8) using Chebyshev inequality. Compare with Markov.

$E[X] = 4$, $Var(X) = \frac{1 - 1/4}{(1/4)^2} = 12$

Markov: 0.5

$P(X > 8) \leq P(|X - 4| > 4) \leq \frac{12}{16} = 0.75$

Q10 (a) Let X1, X2,..., X5 be iid Uniform[0,100]. Let X = X1 + X2 +...+ X5. Find an upper bound for P(|X - 250| > 200) using Chebyshev's inequality.

$E[X_i] = 50, \ Var(X_i) = \frac{(100-0)^2}{12}, \ E[X] = 5 \times E[X_i] = 250, \ Var(X) = 5 \times Var(X_i) = \frac{5 \times 10^4}{12}$

$P(|X-250| > 200) \leq \frac{5\times 10^4/12}{(200)^2} = \frac{5}{48} = 0.104$

Q10 (b) Let X1, X2,..., X50 be iid X, where X has the following distribution:
P(X = -3) = 0.1, P(X = 0) = 0.3, P(X = 0.5) = 0.1, P(X = 1) = 0.3, P(X = 2) = 0.2
Let S = X1 + X2 +...+ X50. Find an upper bound for P(X>80) using Chebyshev's inequality.

$E[X_i] = 0.45, \ Var(X_i) = 1.6125, \ E[X] = 50 \times 0.45 = 22.5, \ Var(X) = 50 \times 1.6125 = 80.625$

$P(X > 80) = P(X - 22.5 > 80 - 22.5 = 57.5) \leq P(|X - 22.5| > 57.5) \leq \frac{80.625}{(57.5)^2} = 0.024\ldots$

(Markov: 0.28...)

Q10 (c) Let X1, X2,..., X100 be iid Beta(3,10). Let Y = (X1 + X2 +...+ X100)/100. Find an upper bound for ~~P~~(X > 0.9) using Chebyshev's inequality. (Markov: 0.2564)

$E[X_i] = \frac{3}{13}, \ Var(X_i) = \frac{3 \times 10}{13^2(14)} = 0.0127\ldots, \ E[X] = 100 \times \frac{1}{100} \times \frac{3}{13} = \frac{3}{13} = 0.23\ldots, \ Var(X) = 100 \times \frac{1}{100^2} \times 0.0127\ldots = 0.000127\ldots$

$P(Y > 0.9) = P(Y - 0.23 > \underset{-0.23}{0.9}) \leq P(|Y - 0.23| > 0.67) \leq \frac{0.000127\ldots}{(0.67)^2} = 0.00028\ldots$

Q11 (a) 10 balls are thrown into 10 bins independently and uniformly at random.
Let Xi = 1 if Bin i is empty and 0, otherwise. Let X = X1 + … + X10 be the number of empty bins. Write an expanded form for $E[X^2]$.

Q11 (b,c) What is $E[X_i^2]$? What is $E[X_i X_j]$?

Q11 (d,e) What is $E[X^2]$? What is Var(X)? Using Chebyshev inequality, find an upper bound for P(X > 5).