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Mathematics for Data Science 2
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Lecture 05
Limits for Functions of One Variable

Hello and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to talk about Limits for Functions of One Variable. So, we have seen the notion of limits for a sequence of real numbers. And in this video, we are going to build on that idea and talk about limits for a function.

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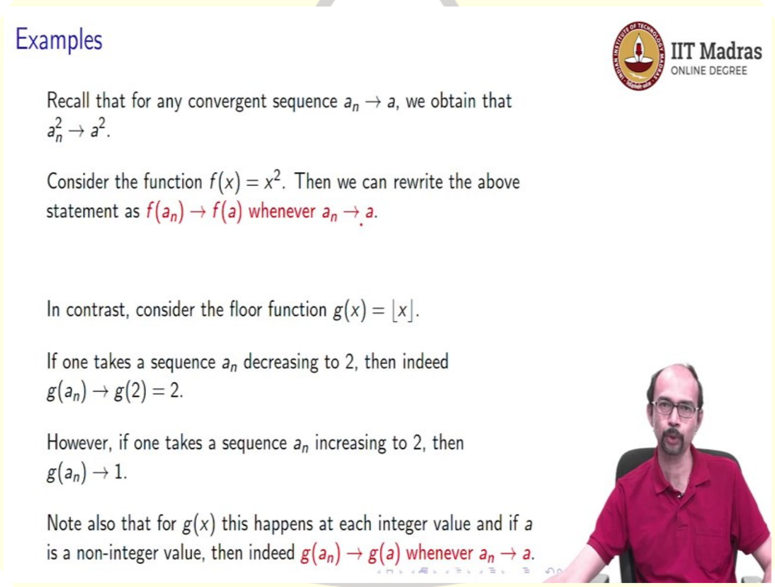

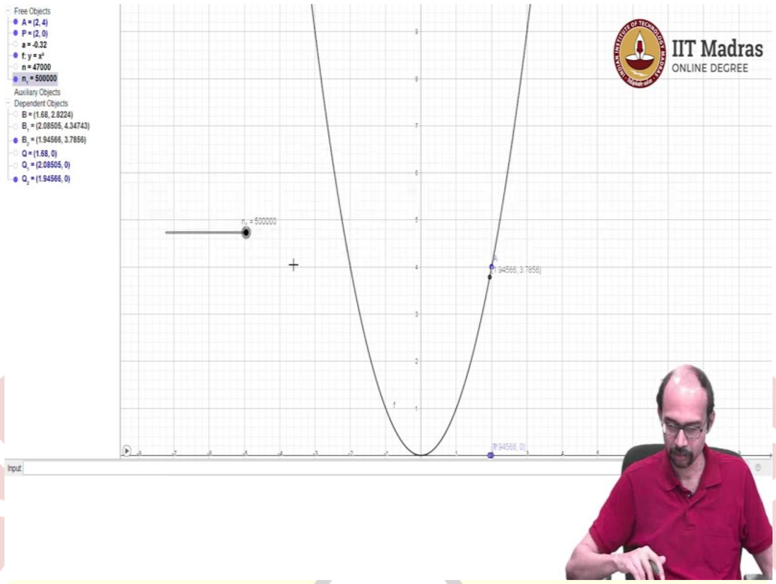
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Auxiliary Objects
Dependent Objects
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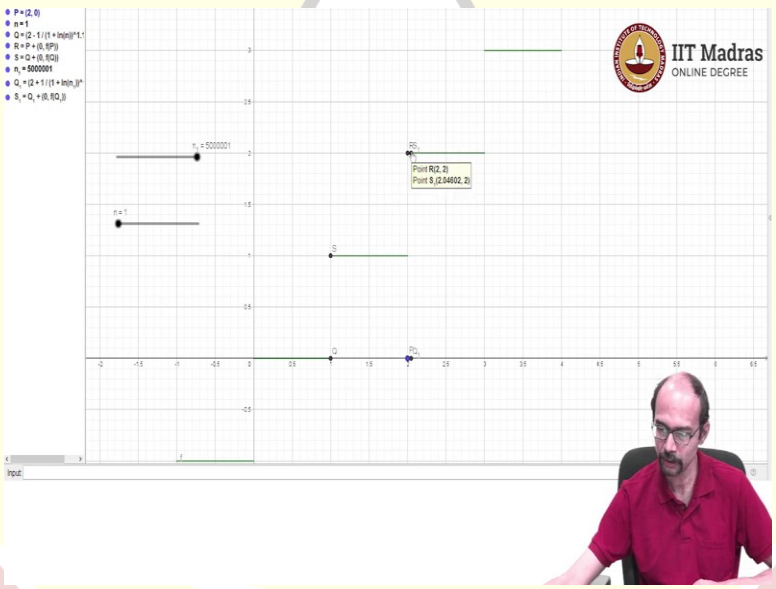
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Point R(2, 2)
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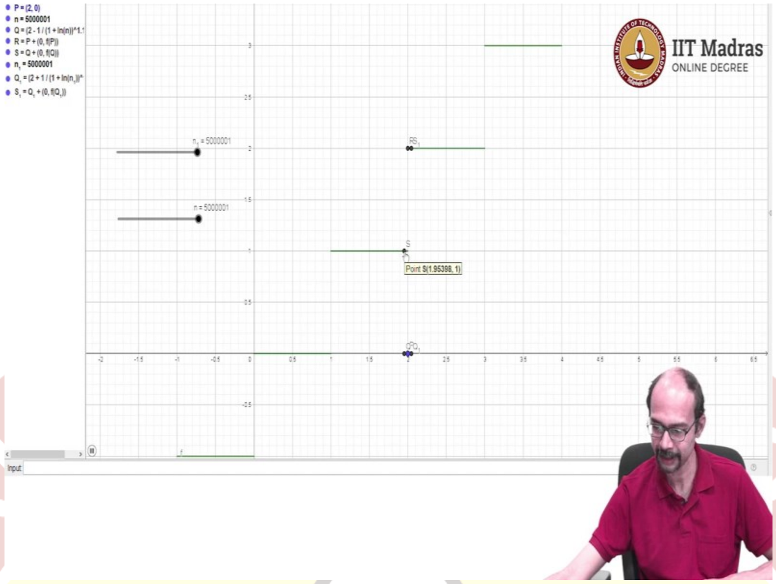

So, let us start with some examples. So, let us recall that if you have a convergent sequence a_n tending to a, we obtain that a_n^2 tends to a^2. This was one of the properties that we discussed that if you have two convergent sequences, a_n tends to a and b_n tends to b, then a_nb_n tends to ab. So, if you, in particular, that tells you that a_n^2 tends to a^2. And more generally, we use this to talk about polynomials and so on. So, a_n^2 tends to a^2.

So, we can think of this in a different way. Consider the function $f(x) = x^2$, then we can rewrite the statement as $f(a_n)$ tends to $f(a)$ whenever a_n tends to a, because what is $f(a_n)$?, $f(a_n) = a_n^2$ and what is f(a), it is a^2. So, essentially, this statement is saying that whenever a_n tends to a, then a_n^2 tends to a^2.

So, we can write, so this is an equivalent formulation. So, we can think of this as talking about the limit at ‘a’ of the function $f(x) =x^2$ and this video is going to talk about different kinds of functions and how to think about limits as you have a sequence of points approaching a particular point, how does the corresponding sequence of function values behave.

So, before we go ahead maybe let us qualify this statement over here. So, let’s look at this, the graph of the function $f(x) =x^2$. So, you can see that in the picture here. So, as n increases, we are going to have a sequence of points which will be from Q_1, which is (3, 0) and which should approach P. So, we are going to talk about the limit of this function at the value 2. So, P is (2, 0). And the function value which is on the graph, so that is (2, f (2)) which is 2^2, which is 4. So, this is the point A=(2, 4). And we will also have the function value of Q_1, the function value Q_1 of this point P, which if I decrease, so if I zoom out, you see this point (3, 9), that is a point B_1.

So, you will see that this, as I move this slider, it quickly starts moving towards this point. It is very fast in fact. So, you can see as the slider, as n is increasing, this point that we started with is coming closer and closer and closer to P. So, Q_1 is moving towards P, and the corresponding function value is moving towards a point A which is the corresponding point on the graph to the point (2, 0), to the point P.

So, what is this saying? This is saying that as this sequence comes closer and closer to the point 2 its function value approaches the function value of the point 2. So, that is a statement that we made, $f(a_n)$ tends to $f(a)$. so a_n^2 tends to a^2, where here $a = 2$. So, here we have a value coming from the right.

So, let us stop this for a second. And instead of this let us look at something which comes from the left. Q_2 is this point (1, 0) right now. So, that is when the value n_1 is 1. As n_1 changes, you will see that this point starts moving closer and closer. So, we will keep track of both these points. The point (1, 0) and how the point (1, 1) is going to move.

So, let me play this animation. So, you can see the values 1.93, 1.938, 1.939, 1.94 and the corresponding values for the function values are 3.77, 3.776, 3.777. So, they are coming closer and closer and closer to the point 4. So, if you go long, for long enough time, you will eventually come very, very, very close to 4. I stopped it at 50,000. But we could keep playing it for longer.

So, again, you can see that you have a function, you have a sequence coming from the left this time. And the same thing happens that if you take the function value, it approaches the value of the function at the point 2. This was the statement that we made in the slide. So, let us go back to our slides.

So, $f(a_n)$ tends to $f(a)$ whenever a_n tends to a and we saw this in two particular cases. We had two particular sequences tending to 2. And in both cases, the geometry clearly showed us that this is true. So, is this a more general phenomenon? So, instead, let us contrast the behavior of the function $f(x) = x^2$ with the behavior of the function which is called the floor function, $g(x) = \lfloor x \rfloor$.

So, what is floor of x, the largest integer which is less than the number that you have. So, if you have -6.5, the largest such integer which is less than that is -7. So, it is the integer part when your numbers are positive and it is one below the integer part when they are negative. So, let us look at what happens in this case. So, if one takes a sequence a_n decreasing to 2, then indeed $g(a_n)$ tends to g(2) which is 2. Let us see if this is indeed the case.

So, here is our step function. So, this is the function that we have, the floor function. So, you can see that for values between two integers, it takes the same value. It is, it looks like a bunch of steps. We have seen this example in a previous video. So, between 3 and 4, for example, the value is equal to 3. So, if you have 3.5, then the value of the function is 3. If you have 3.98, the value is 3. But at 4, it jumps. It jumps to this value 4. So, at every integer there is a jump and after that it is constant. So, it is like a step function.

So, what we are going to do is, let us check the statement that we made, that if you approach this for a sequence approach the value 2 from the right, then indeed it works out. So, let us look at this one here. So, once I start animating this, there is a sequence that I have written down. You can see it on the left. It is $2 + \frac{1}{(1+\ln(n_1))^{1.1}}$. So, the corresponding point is this point Q_1 and its function value meaning its point, the corresponding point in the graph is this point S_1. So, keep track of what is happening to Q_1 and S_1 as I play this animation.

So, as I play this animation, you can see it went very, very cleat towards P and now it is going further and further towards P. And it is come very, very, very close. So, you can see that as Q_1 comes close to P, S_1 comes close to this point R, which in other word means, other words means that the function value is going to be very close to, or in this case, actually, it is the same as the function value of P. So, that was the statement in the previous, in this slide. If one takes a sequence a_n decreasing to 2 then indeed $g(a_n)$ tends to g (2).

Now, let us look at what happens if you take a sequence increasing to 2. If you take a sequence increasing to 2, then $g(a_n)$ tends to 1. Why is that? Well, let us look at this point S and this point Q. So, Q is going to tend towards P from the left and the corresponding function value will be given by this whatever is happening on this line here. So, if I switch on my animation, you can see S is going towards the vertical, but it is on a different horizontal. So, the value of S meaning the function value is 1, see 1.95398, 1. Let us play that again.

So, if I play that again, you can keep track of the value, it is (1.94733, 1), (1.95086, 1), (1.95216, 1). So, the point is because it is increasing towards 2, it will never take the value 2 actually. So, it is going to remain between, after some time it will remain between 1 and 2. And so the function value will always continue to be 1. So, this function $g(a_n)$ will always be 1. So, it cannot be in the limit. It cannot be, it is not 2. So, this phenomenon of $f(a_n)$ tends to $f(a)$ whenever a_n tends to a is special.

And you can see in terms of the graphs, what is the difference? Here in the graph, there is a jump, whereas in our previous graph, which was x^2, it is a very nice function. So, the limit allows you to pick up things like this, whether there are breaks and so on. And that is really why we are studying limits.

So, remember that we started our study of limits after the video on tangents, because we said the tangent is a tricky idea, because you have to study things as they come close to something. When you come close to a point, how does the curve look like. And so we need this idea of limits. So, coming back here, this idea that $f(a_n)$ tends to $f(a)$ whenever a_n tends to 'a' gives you an idea of whether there is a jump at the point 'a' or not.

So, let us make this remark that for $g(x)$ this happens at each integer value. And if 'a' is a non-integer value, so suppose instead of 2, you, we did this for 1.5. So, if you do it for 1.5, then from both sides, you can see that the function value is going to be 1. So, if you take any non-integer point, there would not be any problem. So, this, and that reflects the fact that all the jumps are at the interior points. So, at the non-interior points $g(a_n)$) does tend to $g(a)$ whenever a_n tends to a.

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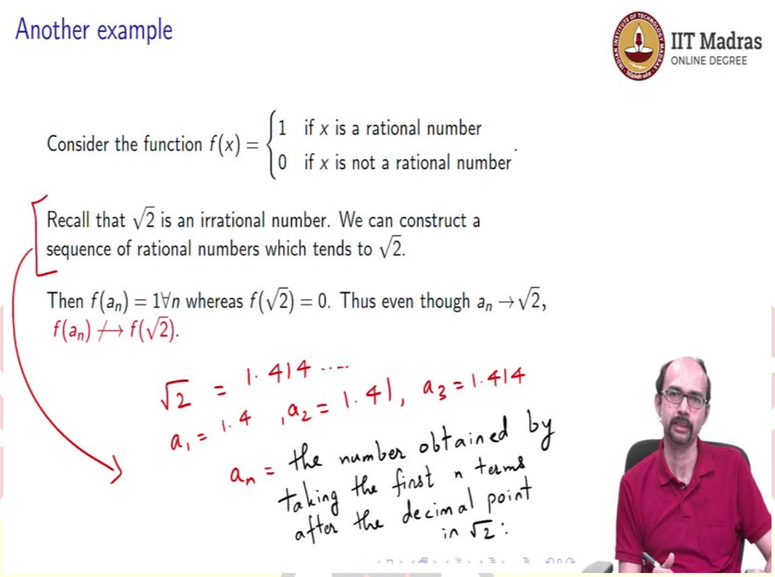

Let us look at some other examples. Some of these are kind of pathological cases. But it is important to keep this in mind, because it helps you to prevent making mistakes. It sorts of, whenever you want to make a statement, you can quickly run these examples past and see if you are making any mistake. So, here is a very strange looking function, maybe a function that we are not very used to, because for us functions are usually things given by nice looking expressions. So, here, there is no good expression for this function.

So, this function takes the value 1, if x is a rational number, and it takes the value 0, if x is not a rational number. So, you cannot draw this function. This function is very, it is not something that you can really graph and that is really the point of this function. But you can see that it has lots of jumps. In fact, it has jumps everywhere. So, the, you would expect that this limits there will be some problem and indeed there is. So, let us recall that root $\sqrt{2}$ is an irrational number. This is maybe something you have done before. If I remember, right, it is done in school. But if you do not remember, please go back and try to see why this is the case.

So, $\sqrt{2}$ is an irrational number, meaning you cannot write it as m/n, where m and n are integers. So, the value of f at $\sqrt{2}$ is going to be 0. It is not a rational number. So, we can construct a sequence of rational numbers, which tends to $\sqrt{2}$ and I will tell you in a minute how to do that. So, what is the point, once you construct such a sequence, $f(a_n)$ will be 1, because each of those a_n's will be rational number. So, for rational numbers, the function value is 1, but $f(\sqrt{2})$ is 0, because $\sqrt{2}$ is not a rational number. So, even though a_n tends to $\sqrt{2}$, $f(a_n)$ does not tend to $f(\sqrt{2})$.

So, this, of course, depends on the fact that there is a rational sequence, sequence of rational numbers, which approaches $\sqrt{2}$ or which tends to $\sqrt{2}$. So, why is that? Let me quickly qualify that statement. So, you see $\sqrt{2}$ has a decimal expansion. It is infinite by, we know that because it is not a rational number.

if it is finite, it would, it must be a rational number. And if you remember the first few digits it goes something like 1.414 and then so on. So, what you do is, you take let us say a_1 to be 1.4, then you take a_2 to be 1.41. And now both of these have only finite decimal expansions, which means they must be rational numbers.

So, for example, why is this a rational number 1.4, because you can write it as $\frac{14}{10}$. Why is 1.41 rational number, you can write it as $\frac{141}{100}$ and I hope the general idea is clear. So, once you have a finite, only a finite number of digits after your decimal place, they must be rational numbers. Take a_3 to be 1.414 and you get the general idea I think now. So, in general, if you want a_n, a_n is the number obtained by taking the first n terms after the decimal place or decimal point, in $\sqrt{2}$. That is a_n.

So, a_1 is 1.4, a_2 is 1.41, a_3 is 1.414 and you can clearly see that as a_n in the limit, a_n is going to tend to whatever decimal expansion you have for $\sqrt{2}$ and hence it will be $\sqrt{2}$. So, there is indeed a sequence of rational numbers which tends to $\sqrt{2}$ and because of that, we see that for this really strange function $f(a_n)$ does not tend to $f(\sqrt{2})$. And there is nothing special about $\sqrt{2}$, you can actually do this for any point.

So, I will suggest that you instead of $\sqrt{2}$ choose your favorite irrational number, let us say $\sqrt{3}$ or $\sqrt{5}$, anyone that you prefer and try to see if the same thing works. Slightly harder is try to construct a sequence of irrational numbers, which tends to a rational number, say try to get a sequence of irrational numbers which tends to 1.

See, if you can think about that or to 0. So, this function, there are plenty of sequences, which tend to a particular number, but 'f 'of those sequences do not tend to 'f' of that number. So, this phenomenon of sequence when you apply f, it has the same limit as f (a) is kind of special.

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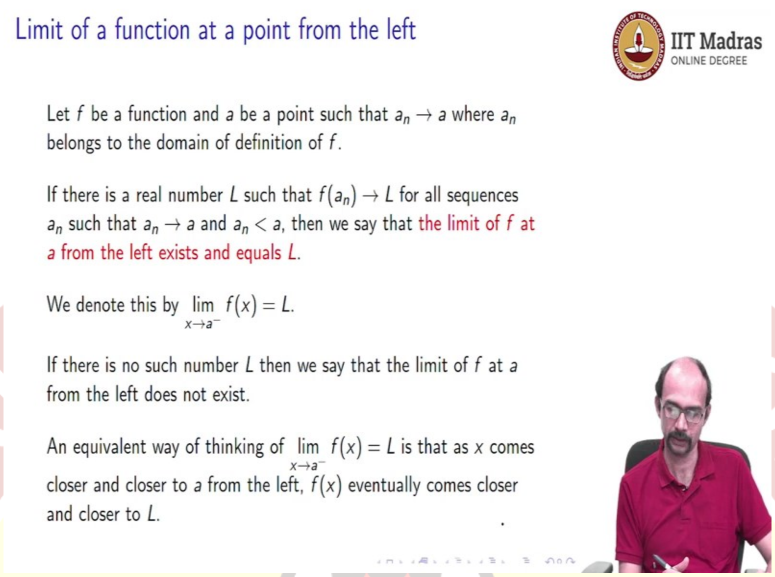

So, let us look two definitions limit of a function at a point from the left. So, we need to make these definitions because of what we saw happening for the step function. So, what is the limit of a function at a point from the left? So, let f be a function and 'a' be a point such that a_n tends to 'a', where a_n belongs to the domain of the definition of f. So, this is only to say that there are sequences which tend to this point. If there are no sequences which tend to this point, then we cannot really talk about whatever phenomenon we are interested in.

So, if there is a real number L such that $f(a_n)$ tends to L for all sequences a_n such that a_n tends to 'a' and a_n is strictly less than 'a', so which is why we are saying from the left. So, when a_n is less than 'a' that means it approaches 'a' from the left. Then we say that the limit of f at a from the left exists and equals L. So, we denote this by $\lim_{x \to a^-} f(x) = L$.

And if there is no such number L, then we say that the limit of f at a from the left does not exist. So, an equivalent way of thinking of $\lim_{x \to a^-} f(x) = L$, is that as x comes closer and closer to this point a from the left, f(x) eventually comes closer and closer to L. So, keep this equivalence in mind.

So, once we understand this equivalence, we can go beyond, we go, we can go pass talking about sequences and which we are going to do, I mean, in the next lectures we will stop thinking about sequences, we will think about it in this way that if you are, if you come closer and closer to this point 'a' from the left, then f (x) comes closer and closer eventually to the number L.

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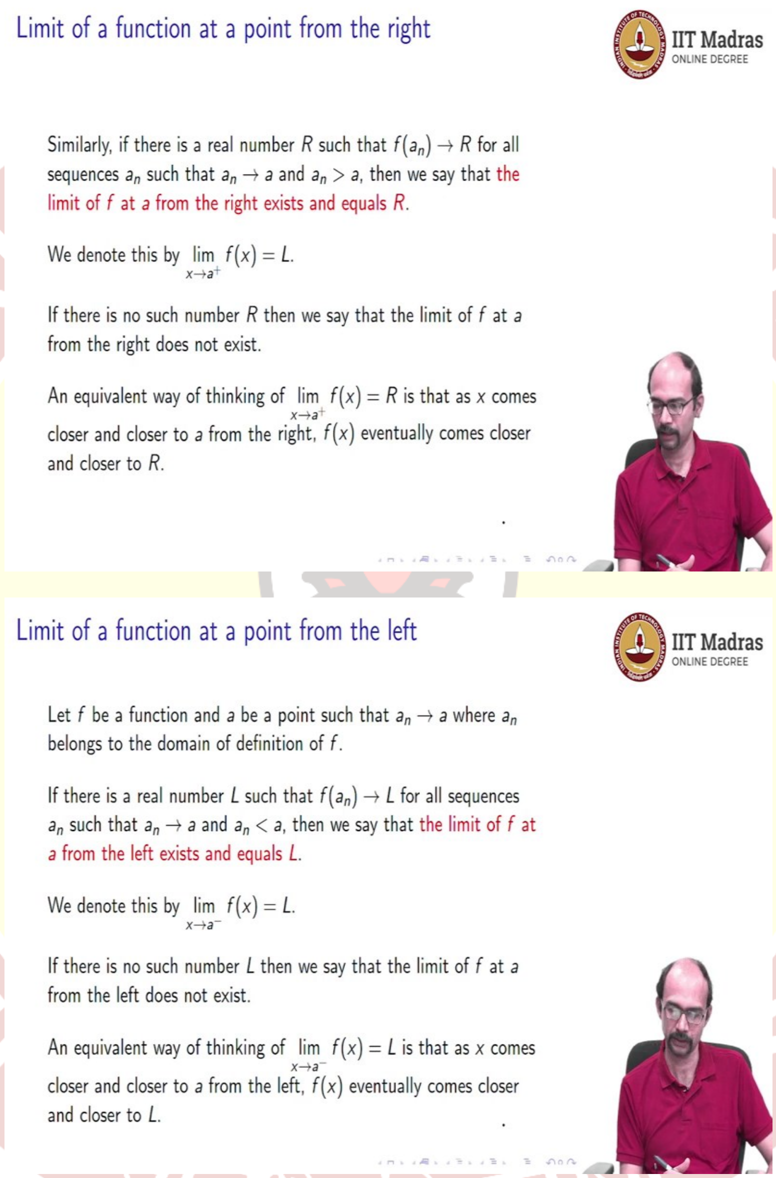

So, the same thing can be done from the right. So, I will just quickly go through this. Similarly, if there is a real number R such that $f(a_n)$ tends to R for all sequences a_n such that a_n tends to a and a_n greater than a, then we say that the limit of f at 'a' from the right exists and equals R. So, we denote this by $\lim_{x \to a^+} f(x) = R$. And if no such number exists, we say the limit of f at 'a' from the right does not exist. And an equivalent way of thinking of $\lim_{x \to a^+} f(x) = R$ is that as x comes closer and closer to 'a' from the right, f(x) eventually comes closer and closer to R.

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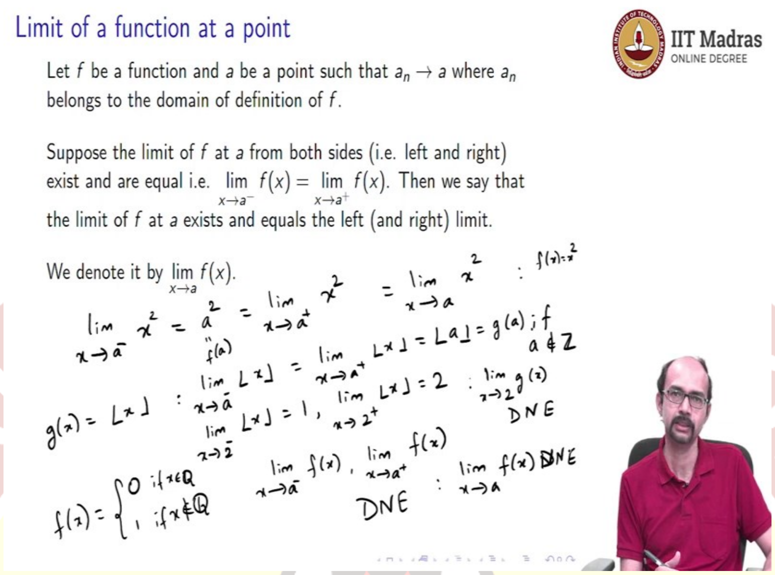

So, let us quickly put these two together to define what is the limit and then we look at plenty of examples. So, limit of a function at a point. So, let f be a function and 'a' be a point such that a_n tends to 'a' where a_n belongs to the domain of definition of f. I want to just point out that in the previous two definitions, we never assumed anything about 'a' being in the domain of definition. So, 'a' need not be, f need not be defined at 'a'. We will see examples of such things before. All we need is that there is a sequence a_n which tends to 'a' and f is defined on that sequence.

So, suppose the limit of f at 'a' from both sides exists and both of them are equal. So, which means $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. So, that L that we had and the R that we had both of them match. So, L=R. Then we say that the limit of f at a exists and equals left and right limits, so which is L or R. It is that number. So, we denote it by $\lim_{x\to a} f(x)$.

So, let us quickly recall our examples from before. So, the first example that we did was x^2. So, let us ask what is limit x tends to a^- of x^2, this is indeed a^2. We saw this for 2, meaning when 'a' is 2, it was 4. And this is also the limit when you approach it from the right. So, it is still x^2. So, this means this is equal to both of these exist and match. So, this is also equal to the limit. And in this case, it turns out that, so if $f(x) = x^2$, so this is equal to f(a). So, this is a very special function.

What about the other functions? What about the ceiling, sorry the floor function? This is how we denoted the floor function. So, if you take the floor function, if you take this limit, well, so now it depends on what is ‘a’. So, if your ‘a’ is not an integer, then these two limits actually match. That is what we saw. And they actually equal the floor function of ‘a’. So, this is g(a).

So, this is if ‘a’ is not an integer. But if ‘a’ is an integer, for example, if ‘a’ is 2, then the left limit, what is the left limit, the left limit was 1, and the right limit was 2. So, these two do not match, although both of them exist, but they do not match. And as a result, the limit does not exist, $\lim_{x \to 2} g(x)$ does not exist.

And let us look at the example that, of the pathological example that we had where we had, I think, 0 and 1 depending on whether x was rational or not. So, Q is the set of rationals and this is what the function was. So, in this case, both of these limits do not exist, do not exist. And one, both of these do not exist, the limit also, by definition, does not exist. So, DNE stands for does not exist.

So, I, in this slide, I should have qualified what happens if, when it does not exist. So, I said, when it exists, so if these properties do not, are not satisfied, which means that either one of the left limit, the limit from the left or the limit from the right do not exist or both of them exist, but they are not equal, in that case, we say that the limit does not exist without qualifying it with left or right. So, I hope the idea is clear. The idea of the limit is to pick up jumps, that is what limit of a function at a point is to become jumps. And that is why this somewhat complicated definition is being made.

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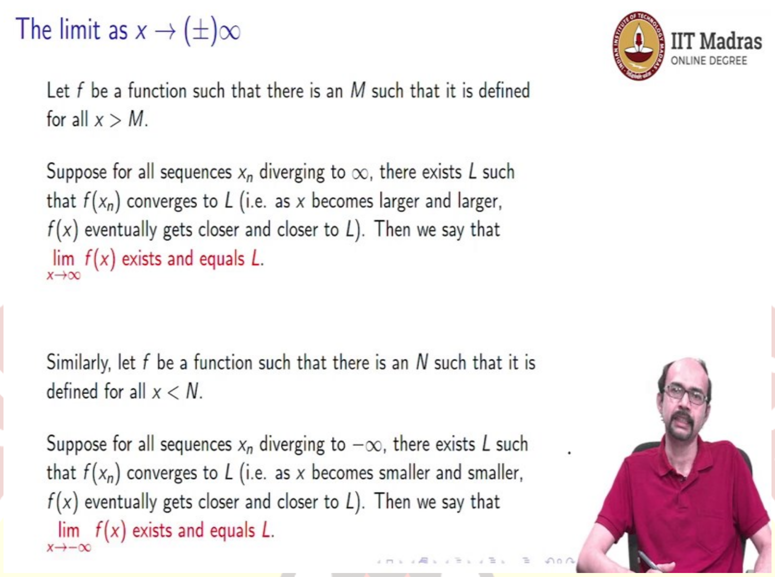

Let us do one more case, which is a case of what happens if x tends to $\pm\infty$. So, what does this mean, first of all. So, let f be a function such that there is an M such that the function is defined for all x greater than M. So, we want f to be defined for all large numbers, otherwise we cannot talk about x tends to ∞. What happens to f (x)?

So, suppose, for all sequences x_n diverging to ∞, there exists L such that $f(x_n)$ converges to L. So, that means as x becomes larger and larger, f(x) eventually gets closer and closer to L. Then we say that $\lim_{x \to \infty} f(x)$ exists and equals L. Similarly, let f be a function such that there is an 'n' such that it is defined for all 'x' less than 'n'. So, we want to talk about the limit as x tends to $-\infty$. So, we better have that the function is defined for all very, very small values. By small, we do not mean close to 0, we mean negative, large negatives.

So, suppose for all sequences x_n, which diverge to$-\infty$, we have talked about the notion of diverging to ∞ and diverging to $-\infty$. So, diverging to ∞ means they become larger and larger and larger, diverging to $-\infty$ be means they become larger and larger negatives or smaller and smaller. So, suppose for all such sequences, there exists L such that $f(x_n)$ converges to L. So, that means, x becomes smaller and smaller then f(x) eventually gets closer and closer to L, then we say that $\lim_{x \to \infty} f(x)$ exists, and equals L. So, you might wonder what, this looks strange. I mean, if, as you are going off, why would the limit exists?

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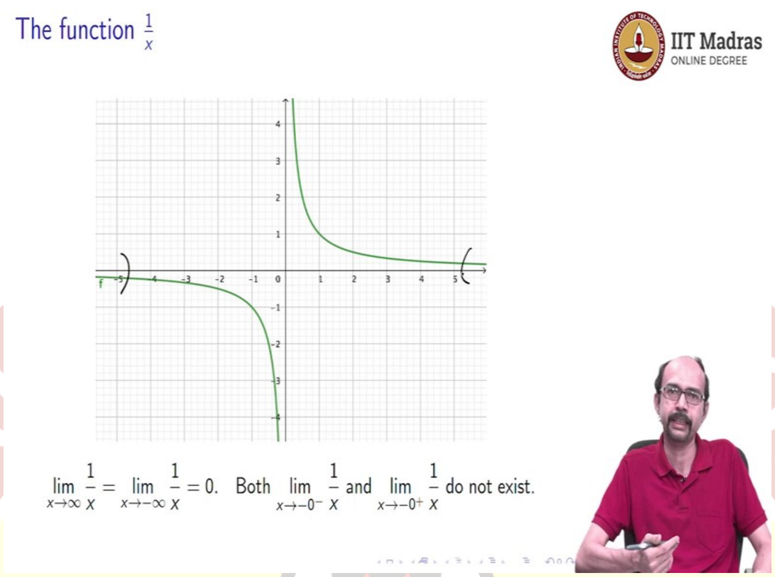

So, this example will tell you why. So, let us look at the function $\frac{1}{x}$. So, if you look at this function, f(x) = $\frac{1}{x}$, if you look at the graph, so this green curve is the graph of this function. And you can see as x increases towards, as the x-axis, you go along the x-axis towards the right, x increases.

The function comes closer and closer to the axis. What does that mean? That means a function value is going to become close to 0. Similarly, when x becomes small or rather it becomes large negatives, the same thing happens. The function value comes closer and closer to 0. Although they are negative numbers, but they are going to be very, very, very close to 0.

And in fact, the place where there is a real problem with the existence of a limit is at the point 0. You can see that at the point 0, the function does not, the limit does not exist. So, $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to-\infty} \frac{1}{x} = 0$. That is what we are saying when we say what happens on this side and what happens on this side. And in fact, both $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$ do not exist. On one side, you can think of the limit as diverging to, for 0^-, it diverges to $-\infty$ and for 0^+, it diverges to ∞. So, it does not converge. So, both of them do not converge.

And if you, even if you allow the infinities to sort of exist in your world, they still do not match. So, this is one of the reasons why we want to talk about what happens as x tends to ∞ or x tends to ∞. Now, this is very important in various, real life phenomena. For example, you will see things like the exponential distribution or the normal distribution and so on. And you will talk about tails of distributions. And there this thing about limit x tends to ∞ or x tends to $-\infty$ what happens is important. And if you find yourself getting worried about what does that mean? That is one of the things that this video is supposed to explain to you.

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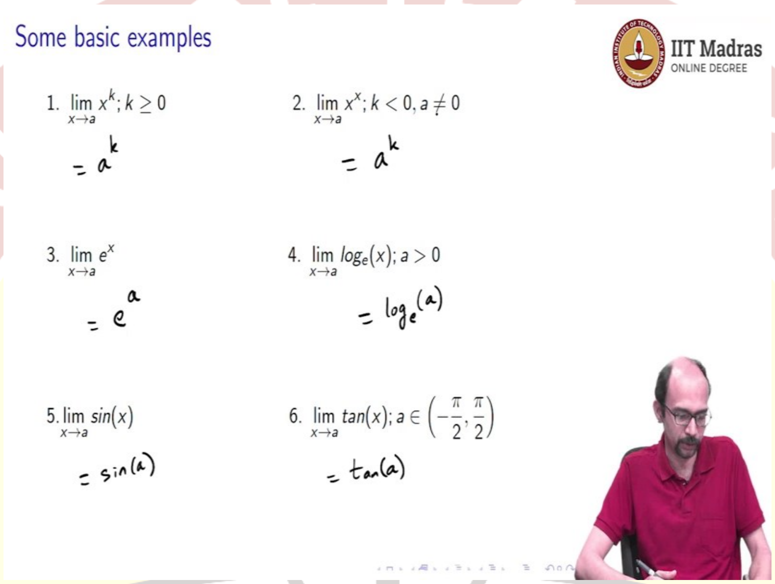

So, let us look at some basic examples. So, let us say k is integer which is positive integer and if you take limit x^k, well, this is just going to be a^k. So, this is, so if you take f(x)= x^k, this is f(a), the limit is f(a). So, in the second one, if you take $\lim_{x\to a} x^k$, where k is negative, but suppose a is not 0. We saw that if a is 0 then you could have this $\frac{1}{x}$ type situation. But suppose a is not 0 then this limit, no problem, you can just write it as a^k.

Similarly, if you have e^x, you take x tends to a, then this is e^a or if you take the logarithm, then this is log(a). So, you can just substitute the value of the number at which you want to evaluate the limit in your function. So, $\lim_{x\to a} \sin(x)$ this is sin(a) and $\lim_{x\to a} \tan(x)$, where 'a' is between -π/2 and π/2. This is tan(a), the tangent of a. So, these are very nice functions.

So, for these nice looking functions, the limit, taking limits is rather easy. And even the places where limits do not exist, for example if you take a=0 and you take k is negative in this second example, just by looking at the function, it is clear that the limit does not exist. So, those are the kinds of functions which are nice. We can really talk about jumps etc. just by looking at the graphs or plotting the functions.

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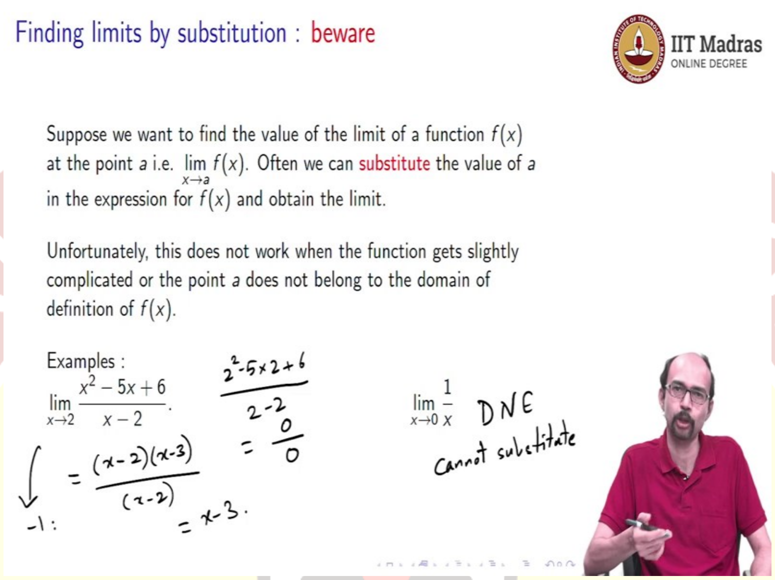

So, in general, can we find this by substitution, because that is what we did here. We looked at the expression and then we just substituted the value of 'a', which was the point at which we were taking the limit. So, this comes with a huge warning. That is why the beware is in red. Suppose we want to find the value of the limit of a function f(x) at the point 'a', so that is $\lim_{x \to a} f(x)$, we want to find this. And in the previous slide what we saw was that for many nice functions this is just f(a). So, you can substitute 'a' in the expression. So, often we can substitute the value of a in the expression for f (x) and obtain the limit.

Unfortunately, this does not work when the function gets slightly complicated or the point 'a' does not belong to the domain of definition of f(x). Remember that 'a' need not belong to the domain of definition. So, here is one example, $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$.

Here is another example $\lim_{x \to 0} \frac{1}{x}$. So, let us talk about what happens here. If you substitute x=0 in the second example, this obviously does not make sense. So, you cannot substitute this. So, this limit does not exist. Cannot substitute, does not make sense. This is our good old friend, division by 0, which we know we cannot do. And in this case, it actually does not exist.

Here, if you substitute, what happens? Let us substitute and see what happens. You get $\frac{2^2-(5\times2)+6}{2-2}$, already warning bells. So, this will give you the numerator is (4-10+6), 0 and the denominator is clearly 0. So, you get a $\frac{0}{0}$ situation. This is one of the most interesting situations.

Because if you get a $\frac{0}{0}$ situation, unlike this situation on the right, where you clearly see that you cannot substitute and the limit does not exist at all, $\frac{0}{0}$ situations, you cannot substitute, but maybe the limit exists. And indeed, here the limit exists, because this function is actually a function that you know very well, because you can solve this equation. You can factorize it as $\frac{(x-2)(x-3)}{(x-2)}$, which we know is x-3 .

So, actually, this function is the same as the function x-3. Well, when I say same, you, we have to be careful there. It is same for all values except if the value is 2. So, at 2 this expression, the original expression is not defined at all. But the expression x-3 is defined and it is -1. So, what happens to this expression as x approaches 2. So, as you can see, as x approaches 2, the expression here, the limit is going to approach -1. So, this number is going to be -1.

So, substitution does not work, but the limit does exist. So, you cannot blindly substitute. So, be careful about substitution. It does work for nice functions, special functions, sin(x), tan(x), e^x, polynomials, and so on. But, in general, even for rational functions, meaning polynomial by polynomial you have to be very, very careful of what is happening. And in a minute, we are going to see some very interesting examples.

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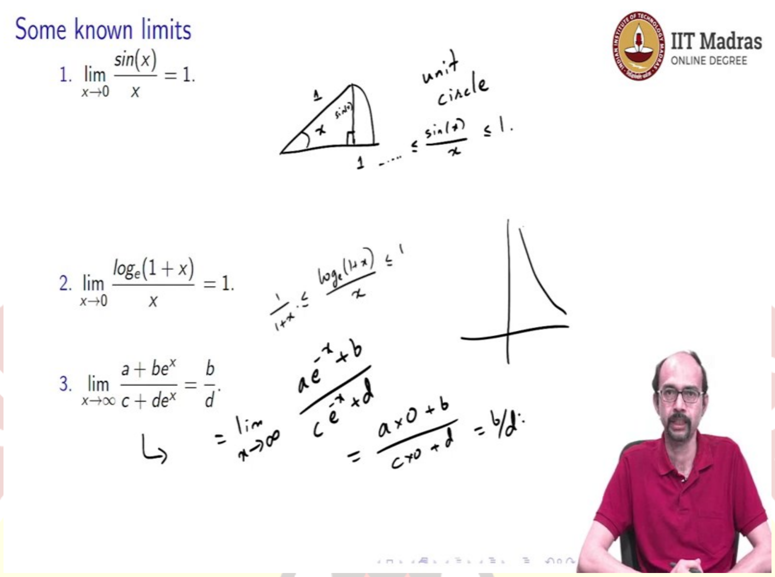

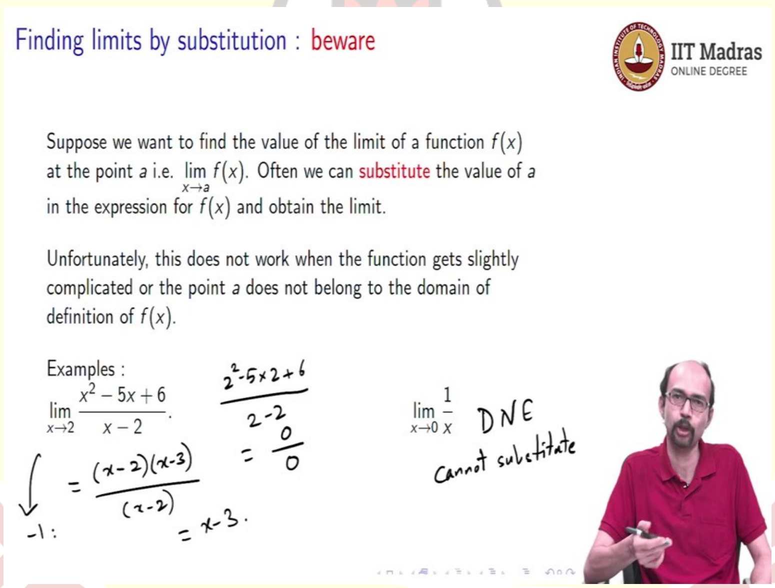

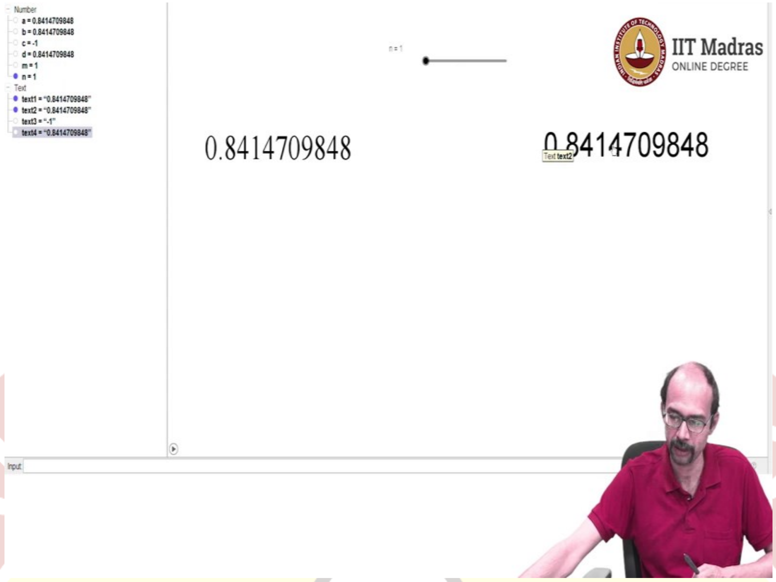
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IIT Madras
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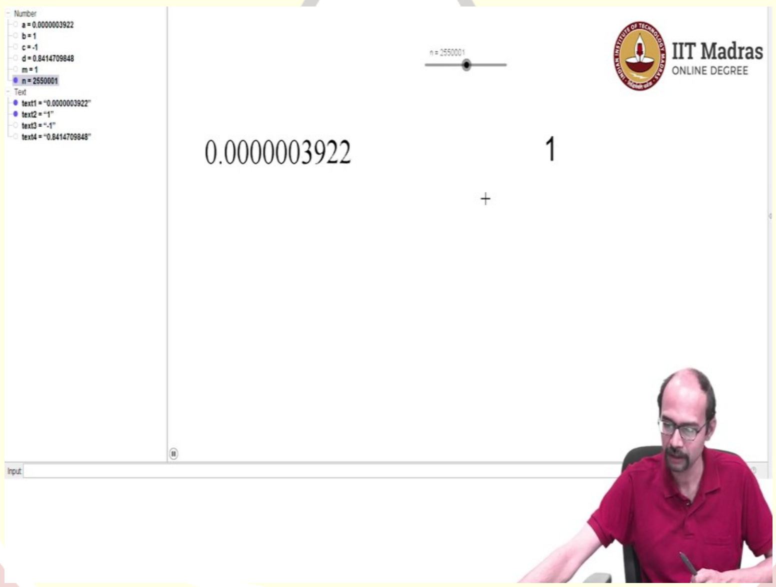
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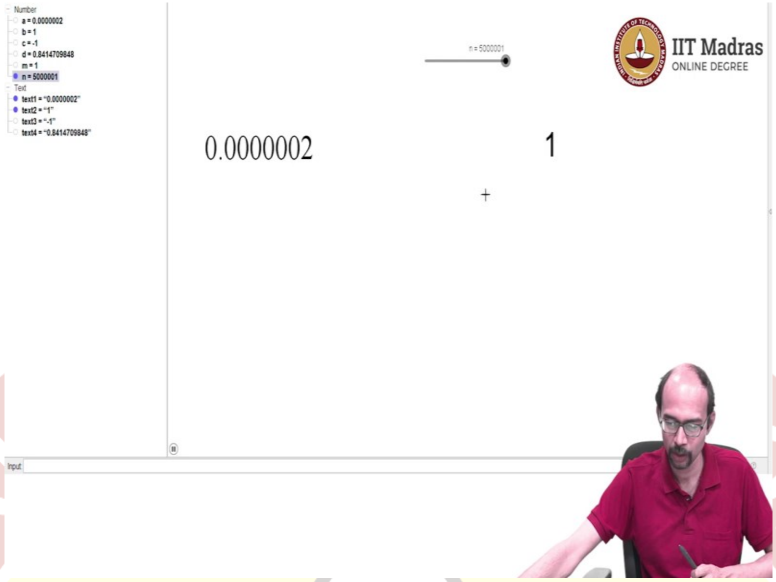
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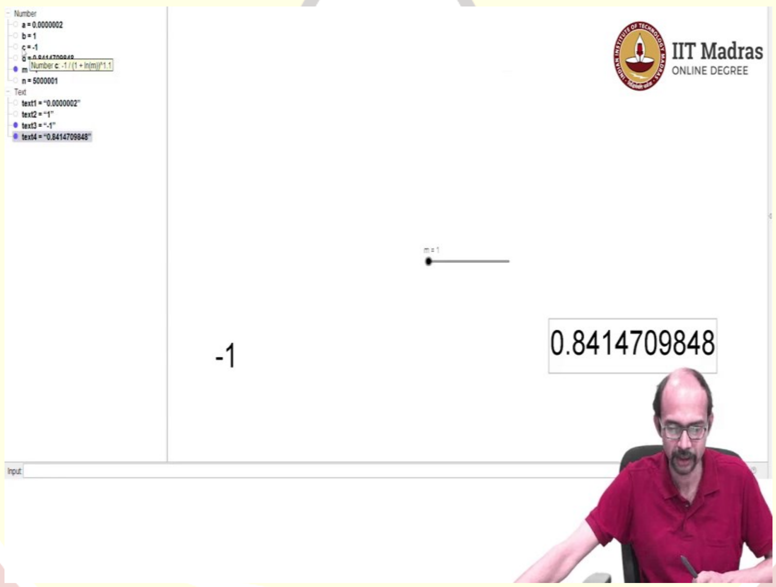
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IIT Madras
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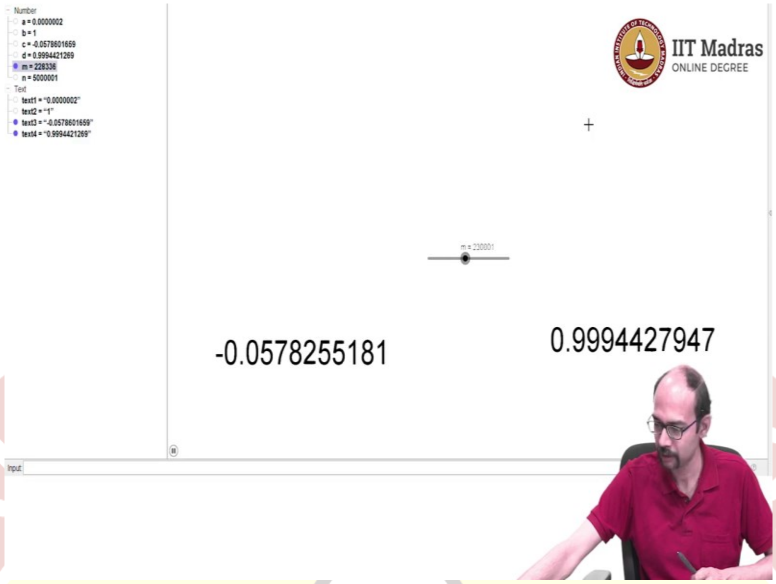

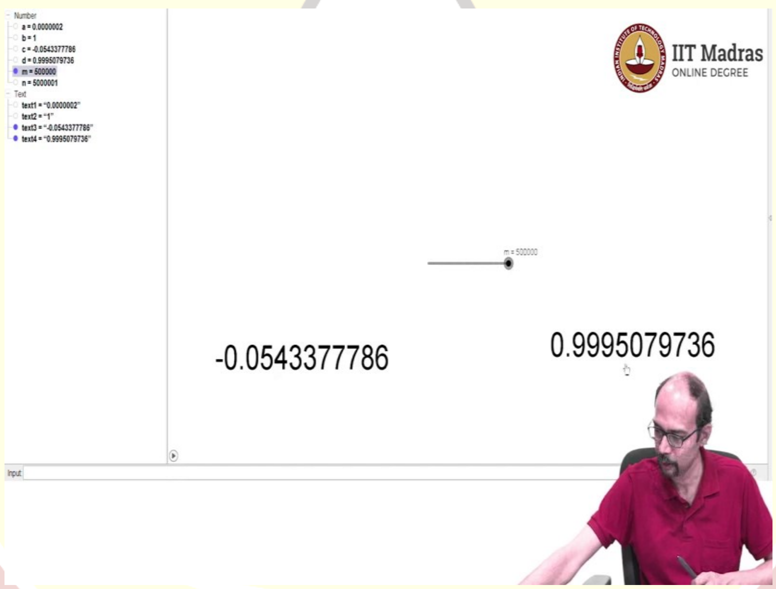

So, here is one, $\lim_{x\to 0}\frac{\sin(x)}{x}$=1. This is known. $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$, again, this is something known. And I want to warn you that, see, if you substitute here, then you get sin(0), which you know is 0, so it is a $\frac{0}{0}$ situation. It does not make sense. So, it is, but unlike the previous situation, where here you could quickly resolve this difficulty by factorizing and so on. Here, there is no really good way of doing it unless you know something more about these functions. And I will show you in a minute why this is true. Similarly, if you take $\frac{\log_e(1+x)}{x}$, where you get a $\frac{0}{0}$ situation. And then we have the third example, which is $\lim_{x\to\infty}\frac{a+be^x}{c+de^x}$. This is $\frac{b}{d}$. So, let us look at the first one to start with. So, we have a small animation here, which I hope will convey something to you. So, the, so what are these numbers? So, this number, so forget the, let me highlight the things that we need to start with. So, let us get rid of these three. So, these three, you have a, you have ‘n’, then this is the left hand side is a sequence, which tends to, so this is sin $(\frac{1}{n})$. And out here is $\frac{\sin(\frac{1}{n})}{\frac{1}{n}}$. So, the left hand side is $\sin(\frac{1}{n})$, and the right hand side is $\frac{\sin(\frac{1}{n})}{\frac{1}{n}}$. As we know $\frac{1}{n}$ tends to 0, so let us see what happens to the right hand side. So, you should to keep your eye on what happens to the right hand number. So, let us play this animation. It is already become ‘1’. You can see it is moving so fast that is already become 1. So, as it comes closer and closer it is not changing at all.

So, let us play the other one. So, this is another sequence, which is $\frac{-1}{(1+\ln(m))^{1.1}}$. So, this is a sequence which tends to ‘0’. And this is sin, so the D here is $\frac{\sin(c)}{c}$. So, let us see what happens to that. So, you can see what is happening 0.999, 0.99946, 0.99947, and we have stopped. So, let us see what happens in the end. So, at the end, you have 0.9995, so fairly close. And if you keep going, it is going to come closer and closer and closer.

So, this is at least a demonstration that $\frac{\sin(x)}{x}$. As you approach it, we saw two sequences which approach 0, and then we computed $\frac{\sin(x)}{x}$ for those two sequences. And we saw that for both of them it comes very close to 1. So, that is some heuristic, I mean, some demonstration that it is close to 1. But we can give a more sort of geometric argument, which is if you take $\frac{\sin(x)}{x}$, well, here is your arc, so you have to do this in radians of course. So, if this number is x, so this is x.

So, you take the unit circle. So, this is the unit circle. And you take this arc which is of x radian. So, these are both of length 1. So, if you drop this perpendicular, what you get is, so this is the hypotenuse which is 1, this is 90 degrees and so the opposite side is sin(x). So, this length is sin(x). And then if you compute $\frac{\sin(x)}{x}$ from this picture, you will see that it is bounded on both sides by 1 on one side.

So, clearly, it is less than 1. And you can also give something on the right which is going to tend to 1, which I am not elucidating right now. We will see this kind of argument again later on. and then both sides go to 1, so middle thing goes to 1. This is like a sandwich principle that we have seen in, for limits of sequences. So, once we do that, I will come back to this argument, but this is the idea. So, there is a way of doing this even geometrically.

So, let us look at the second example here. For the second example like the first example of $\frac{\sin(x)}{x}$, we have no direct way of computing this. So, the only way to do this as of now is to really check the limits based on some sequence like we did for $\frac{\sin(x)}{x}$ and convince ourselves that indeed it goes to 1.

And this proof will be more formally done in the next video and it will be based on the inequalities that if you take $\frac{\log_e(1+x)}{x}$, you can bound it on both sides. On one side by 1 and on the other side by $\frac{1}{1+x}$. So, we will see this in detail in the next video, how to use this fact to obtained that the limit is 1.

The third one I can give some justification. So, you can write this as limit, so what, can I substitute. Let us first ask that. So, if I substitute x is ∞, if you can see that both numerator and denominator are ∞, so that is not going to give me anything. So, instead what I can do is, I can multiply both sides by e^{-x}. So, take e^{x}common. So, if you do that, you get $\frac{ae^{-x}+b}{ce^{-x}+d}$. Now, indeed, I can take the limit, because if I substitute now as x tends to ∞, what happens to e^{-x}. Well, $e^{-x}$is very much like your $\frac{1}{x}$ function. We have this function actually before. So, it goes to 0 very, very, very fast. And so if you take x tends to ∞, e^{-x} is going to 0, so you can substitute, you get a times $\frac{a\times0+b}{c\times0+d}$ and indeed this is $\frac{b}{d}$. So, this is a justification.

So, what we are saying is that it is possible to substitute sometimes, but you cannot always do it. And in fact, for the, you have may have that. For a particular expression you cannot substitute, but then when you change your expression, you can substitute. And much of understanding limits is really understanding when you can do this. So, that is the gist of limits for a function.

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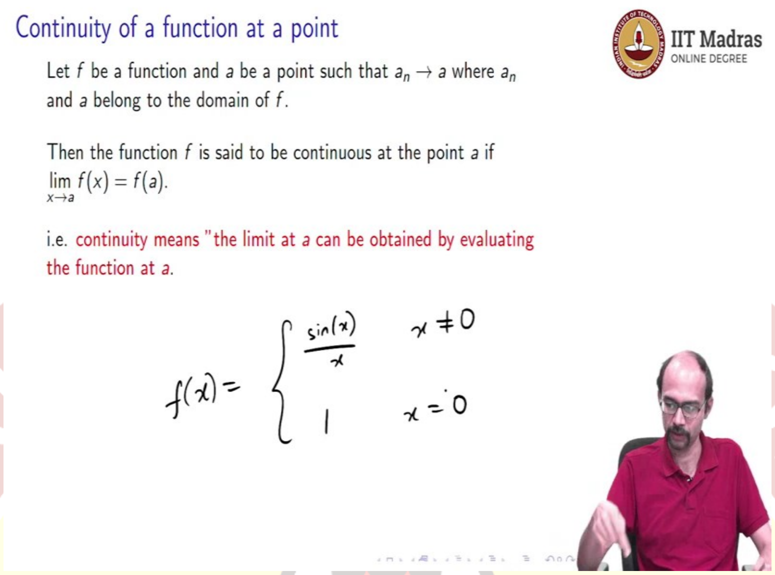

Let me end by talking about the continuity of a function at a point. This is a definition. Let f be a function and 'a' be a point such a_n tends to a, where a_n and a belong to the domain of f. So, now, remember that 'a' does belong to the domain of f. That is we are talking only at such point. Then the function f is said to be continuous at the point a, if $\lim_{x \to a} f(x) = f(a)$. So, continuity means, the limit at 'a' can be obtained by evaluating the function at 'a'. So, I will highlight the fact that I have written the word evaluating and not substituting. Why do I distinguish between these two? When we say substitute that means, you have some expression for that function and you are substituting in that expression. So, that may or may not be possible. That is what we have seen. But you can evaluate. So, the evaluation may make sense. Sometimes that is because you can change the expression and get a slightly different looking function, which is effectively the same function and then you can evaluate. Or sometimes the function may not have an expression, but you can still evaluate.

So, for example, if you look at the function, let us define the function f (x) which is $\frac{\sin(x)}{x}$. So, of course, this expression does not make sense at 0. So, at 0 you define it to be 1. Then indeed you can say that this function is continuous at 0 and you can obtain it, that is the same as saying that, you obtain the limit as x tends to 0 by evaluating it at 0, which is 1.

So, I guess, I just quickly recap what we have done in this video. We have studied the notion of the limit of a function at a point and we have used it, we have used the notion of the left limit and the right limit, meaning as sequences approach from the left and sequences approach from the right, so for the left limit we need that for all sequences which approach from the left. The, there is some number L such that the function value tends to that number L for all sequences, not for a few sequences, not for some particular sequences, for all sequences.

Similarly, for the right, $f(a_n)$ should tend to some number R, then the limit from the right exists. And if those two numbers match, then the limit exists. And if it actually equals the value of the function at that point, then we say that the function is continuous at that point. Thank you.