IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
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Lecture No. 30
Linear Independence – Part 1

Hello, and welcome to the Maths 2 component of this online B.Sc. program in data science. Today, we are going to study the topic of Linear Independence. In the last couple of videos, we have studied the idea of linear dependence. So, in particular, we saw the notion of linear combinations of vectors. So, let me remind you that now we are studying vectors in the sense of elements of a vector space, which we defined our two videos before this.

So, we defined the notion of linear combination which is nothing but a $\sum a_i v_i$, where v_i's are vectors in a vector space and $a_i{}'$s are real numbers. So, this is a finite sum $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$. And we talked about linear dependence.

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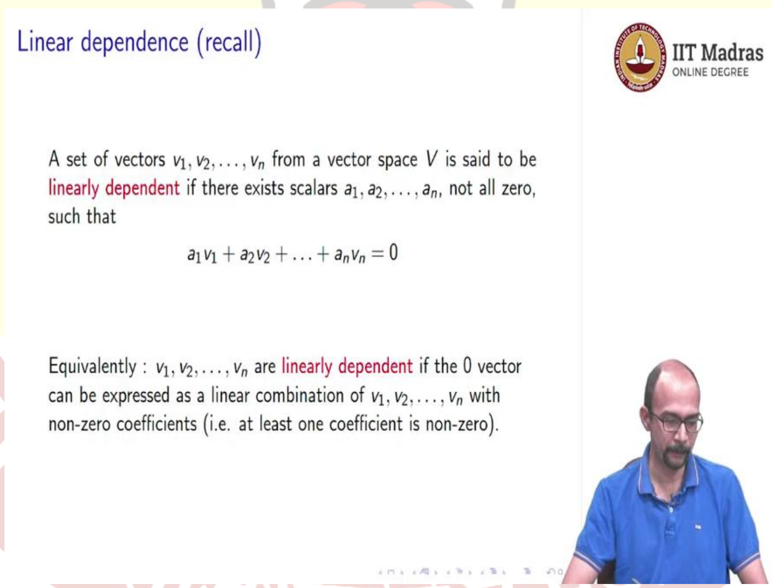

So, today, in this video, we are going to talk about linear independence. Let us quickly recall what is linear dependence. So, we say that a set of vectors $v_1, v_2, \ldots, v_n$, from a vector space V we call them linearly dependent if there exists scalars $a_1, a_2, .. a_n$, such that $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$. Now, of course, there is a crucial condition extra which is that not all of the coefficients should be 0. So, at least some coefficient must be non-zero. If all the coefficients are 0, of course, we know that $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$.

So, the point is that there is some linear combination with non-zero coefficients. So, some coefficients are non-zero so that the sum is still 0. So, if there exist such scalars then we say that $v_1, v_2, \dots, v_n$ is linearly dependent. So, equivalently in terms of linear combinations which is something I defined last time and we just talked about, if we can express the 0 vector as a linear combination of $v_1, v_2, \dots, v_n$ with non-zero coefficients, which means that there are some coefficients, meaning at least one coefficient which is non-zero, then we say that $v_1, v_2, \dots, v_n$ are linearly dependent. So, we have studied this notion in the previous video.

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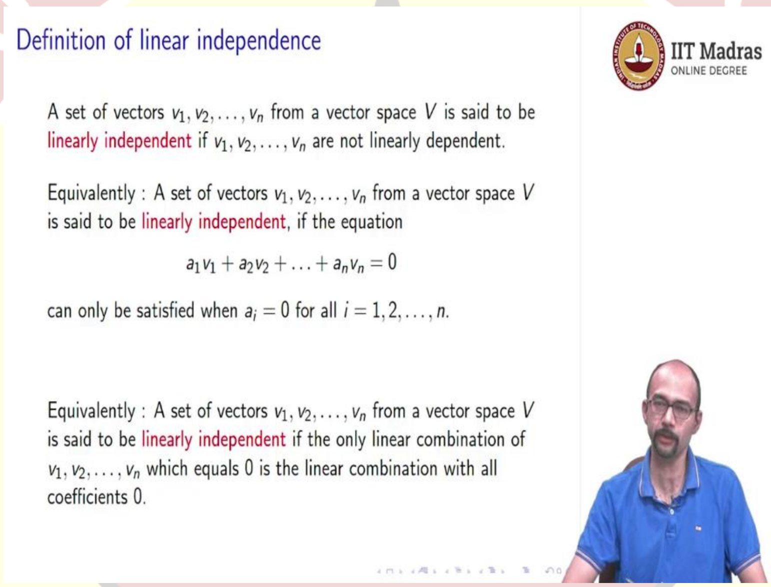

So, in this video, we are going to talk about linear independence. So, we say that a set of vectors $v_1, v_2, \dots, v_n$ from a vector space V we call them linearly independent if they are not linearly dependent. So, this is a very easy definition. We have already studied what is linear dependence. So, we will say they are linearly independent if they are not linearly dependent. So, what does this mean in terms of linear combinations and summing up to 0?

So, the equivalent formulation is a set of vectors $v_1, v_2, \dots, v_n$ from a vector space V is said to be linearly independent if the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ can only be satisfied when a_i is 0. So, just to go back to dependence, dependence said there is some linear combination, there is some, there are some scalars $a_1, a_2, .. a_n$, not all of which are 0, so that when you take the sum $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ you get 0. Linear independence is saying, if there is a linear combination, meaning if there is a sum $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ which is 0, then the only way in which this can happen is if the $a_i's$ are all 0.

So, equivalently in terms of linear combinations, we are saying that a set of vectors $v_1, v_2, \ldots, v_n$ is linearly independent if the only linear combination of these vectors $v_1, v_2, \ldots, v_n$ which equals 0 is the linear combination with all coefficients 0. So, let us take a pause and understand what we are saying here. If the coefficients are 0, certainly $\sum a_i v_i = 0$. If all the a_i's are 0, then $\sum a_i v_i$ is going to be 0.

So, what we are saying is that if the sum is 0, then the coefficients must be 0. In other words, the only way of getting 0 as a linear combination is if the coefficients are 0. So, I want to emphasize that this means we have to check something is linearly independent, which we will do at the end of this video. We have to check that all the a_i's are 0.

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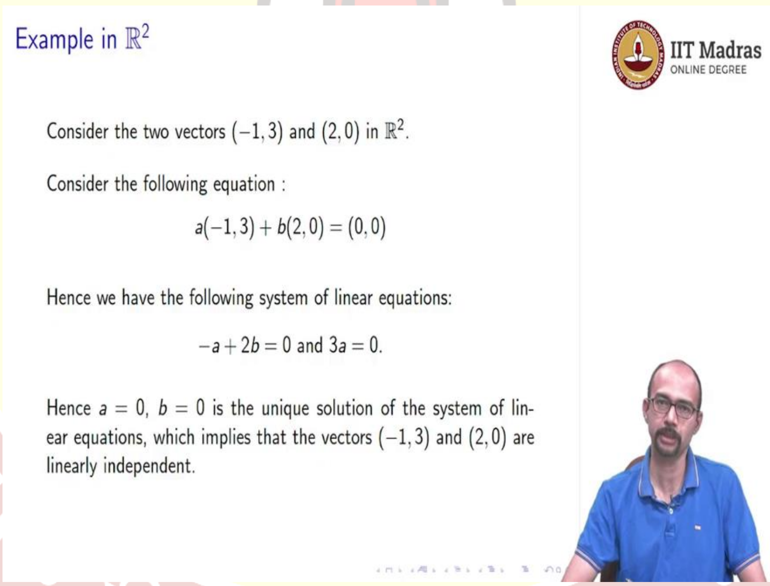

So, let us do some examples. So, let us first check out this example in $\mathbb{R}^2$. So, look at the two vectors $(-1, 3)$ and $(2, 0)$. So, we want to see if which coefficients give us that the sum is 0. So, you write this equation, $a(-1, 3) + b(2, 0) = (0, 0)$ and then you try to solve for a and b. So, if you do that, you equate the corresponding coordinates.

So, the first coordinate for the left hand side is $-a + 2b$, the first coordinate for the right hand side is 0. They are equal so that means these two must be the same. So, $-a + 2b = 0$. And the second coordinate is $3a + 0b$. And the second coordinate on the right is 0. So, you get $3a = 0$. So, we have a system of linear equations, $-a + 2b = 0$ and $3a = 0$.

Now, we know how to solve a system of linear equations. We in fact know very general method for that. And indeed, towards the end of the video, we will start using that method. But for now, you can see this is very easy to solve. Namely, $3a = 0$, so $a = 0$. And once $a = 0$, you put that into the first equation and you get that $2b = 0$. So, $b = 0$.

So, the unique solution for this system is $a = 0$ and $b = 0$, which means that if $a(-1,3) + b(2,0) = (0,0)$, the only solutions of, the only coefficients which yield this identity to be true are $a = 0$ and $b = 0$. That means, the vectors $(-1,3)$ and $(2,0)$ are linearly independent.

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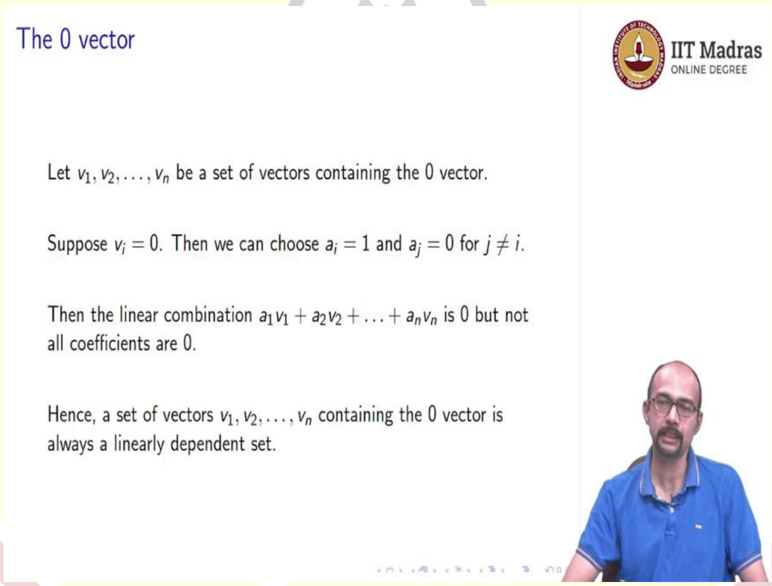

Let us look at the 0 vector. This is a very special vector in our vector space. So, suppose I have a set of vectors $v_1, v_2, \ldots, v_n$ and one of the vectors here is the 0 vector. So, suppose v_i is 0. One of them is 0. Let us say that one is v_i. So, then what you do is you choose the ith coordinate, sorry, you choose the ith coefficient to be 1 and you choose the jth coefficient, where $j \neq i$ to be 0. So, if you do that, then look at this linear combination, $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ now for all the vectors where $j \neq i$, a_jv_j is going to be 0, because $a_j = 0$.

On the other hand for the ith vector, you have a_iv_i, $v_i = 0$, so $a_iv_i = 0$. So, for each of these terms, a_1v_1, a_2v_2 etc. up till a_nv_n each term is 0. Hence, the sum is also 0. So, the linear combination is 0. But of course, all the coefficients are not 0, because $a_i = 1$. So, one of the coefficients is 1. So, not all these coefficients are 0. So, what does that mean? That means a set of vectors which contains the 0 vector is always a linearly dependent set. So, this is not linearly independent. So, if your $v_1, v_2, \ldots, v_n$ happens to contain 0, then this is linearly dependent. It is not linearly independent. So, let us keep this in mind.

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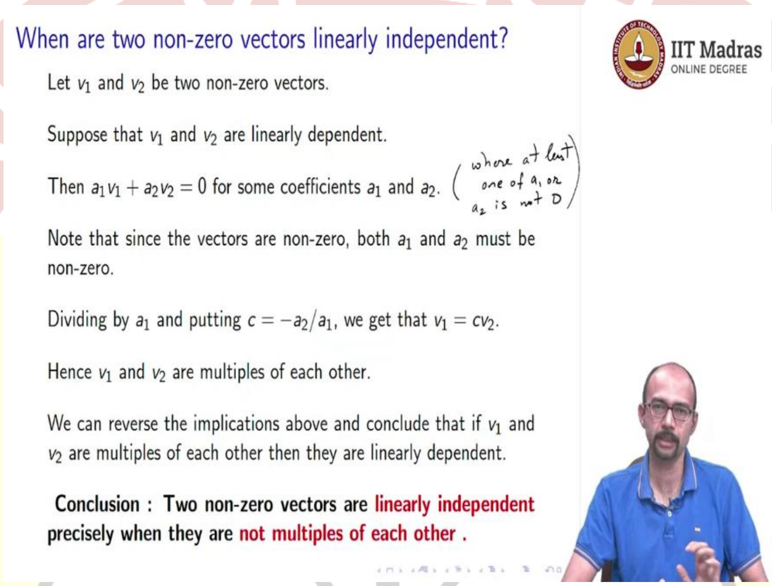

So, now let us ask the question when a two non-zero vectors linearly independent. So, we have already seen that if a vector is 0, you take that in your collection of vectors, then that is a linearly dependent set. So, now we can take two non-zero vectors. That is your first starting point for our discussion. So, let us take two non-zero vectors and ask when are they linearly independent? When is the set of vectors linearly independent?

So, let v_1 and v_2 be two non-zero vectors. So, suppose v_1 and v_2 are linearly dependent. So, we want to study when they are linearly independent instead we will study when they are linearly dependent and from there we will conclude when they are linearly independent. So, suppose v_1 and v_2 are linearly dependent that means $a_1v_1 + a_2v_2 = 0$ for some coefficients a_1 and a_2, and these coefficients, both of these coefficients are not 0. So, we are saying $a_1v_1 + a_2v_2 = 0$ for some coefficients a_1 and a_2, where both of them, I mean, at least one of them is non-zero. So, at least one of a_1 or $a_2$is not 0. That is the definition of a linear dependence.

But the point is here, since the vectors are non-zero, if one of them, one of these coefficients is non-zero, then the other better be non-zero, because otherwise you will have, let us say, a_1 is non-zero, but $a_2 = 0$, then you will have $a_1 v_1$ which is a non-zero vector equates to 0 that cannot be possible. So, if one of them is not zero, then both of them are not 0. And really, that is what this statement here means. So, keep in mind that one of a_1 or a_2 is non-zero and so both of them must be non-zero. That was the import of this statement here.

So, dividing by a_1 and putting $c = -\frac{a_2}{a_1}$, we get that $v_1 = cv_2$. And so v_1 and v_2 are multiples of each other. So, what are we saying? We are saying that if v_1 and v_2 are linearly dependent, then v_1 and v_2 are multiples of each other. And we can go backwards. We can reverse this set of statements. So, we can reverse the implications above and conclude that if v_1 and v_2 are multiples of each other, then they are linearly independent. If v_1 and v_2 are multiples, then $v_1 = cv_2$ for some c.

And then note that c is non-zero, because both v_1 and v_2 are non-zero vectors and then you can write this as $v_1 - cv_2 = 0$ and so the coefficients are 1 and $-c$, and in fact, both of them are non-zero. Actually, it is enough for them, one of them to be non-zero. But in this case, both of them are non-zero. And so we get that they are linearly dependent. So, that is what I mean by we can reverse the implications.

So, the conclusion is, if two, you have two non-zero vectors, then they are linearly independent precisely when they are not multiples of each other. We have seen that linear dependence of two non-zero vectors is equivalent to them being multiples of each other. So, linear independence is exactly same as saying that they are not multiples of each other. Let me qualify this coefficient a_1 and a_2 so, where at least one of a_1 or a_2 is not 0. And as I said, because both the vectors are non-zero and you have only two vectors in this case both of them must be non-zero. That is what we are saying.

So, the take home from here is, if you have two non-zero vectors, then linear independence means that they are not multiples of each other. So, the point one is trying to make here is that linear independence as a notion is a very useful and important notion and it is saying something very important about these vectors, namely that they are not multiples.

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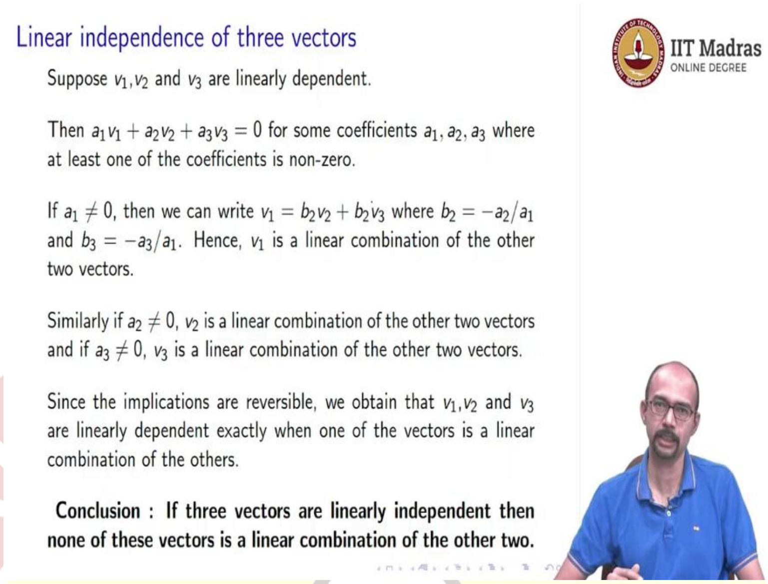

So, let us ask the same of what happens to three vectors? Let us ask the same question for three vectors. When the three vectors linearly independent what does that say? So, we will do the same thing. We will first study when they are dependent and then from there we will try to draw conclusions for when they are independent. So, suppose v_1, v_2 and v_3 are linearly dependent then we have an equation of the form $a_1v_1 + a_2v_2 + a_2v_3 = 0$ where for some coefficients a_1, a_2, a_3 where at least one of the coefficients is non-zero, which is exactly what I wanted to say on the previous slide as well. At least one of these is non-zero.

So, let us assume that a1 is non-zero. We will, in a minute we will also study the other cases, but this is a prototype case. Let us study a_1 is not 0. So, if $a_1 \neq 0$, I can divide by a_1 and I can take the terms for v_2 and v_3 on the other side and I can write $v_1 = b_2v_2 + b_3v_3$, where $b_2 = -\frac{a_2}{a_1}$ and $b_3 = -\frac{a_3}{a_1}$. So, v_1 is a linear combination of the other two vectors that is the main point.

So, now we can make the same argument of a_2 is non-zero or a_3 is non-zero. Remember that we know that one of them is non-zero. If it is not a_1, maybe a_2 is non-zero, maybe a_3 is non-zero. And you can see that you can make the same argument and express. So, if a_2 is non-zero, you can express v_2 as a linear combination of the other two. And if a_3 is non-zero, you can express v_3 as a linear combination of the other two vectors. And I again note that this is b_3.

So, once again these implications are reversible. So, if, let us say if v_1 is $b_2v_2 + b_3v_3$, so then I can write, I can take the v_2 and v_3 terms on the other side and write $v_1 - b_2v_2 - b_3v_3$ is 0, but remember that v_1 has coefficient 1. So, I get $1v_1 - b_2v_2 - b_3v_3$ is 0, which tells me that I have a linear combination where at least one of the coefficients is non-zero, which one, the coefficient for v_1, because that is 1. And so this is linear, they are linearly dependent. And you can argue the same way if v_2 is a linear combination of v_1 and v_3 or v_3 is a linear combination of v_1 and v_2. So, these implications are reversible.

So, the upshot is that, if you have three vectors v_1, v_2, v_3, which are linearly dependent, this is exactly the same as saying that one of these vectors is a linear combination of the other two vectors. So, now we can talk about linear independence. We have studied when they are linearly dependent. So, we can now answer the question about linearly independent.

So, if three vectors are linearly independent, then none of these vectors is a linear combination of the other two. That is what we are seeing. So, you can think of this geometrically. I will, we will do that in a few minutes. But already you can start thinking of what this means in terms of geometry. So, the conclusion here is the take home. If three vectors are linearly independent, this is exactly the same as saying that none of them is a linear combination of the other two.

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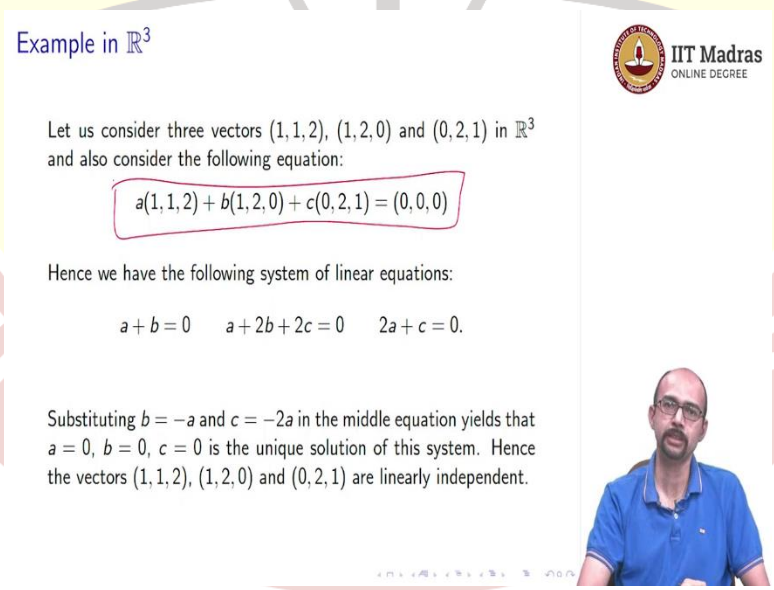

Let us look at an example in $\mathbb{R}^3$. So, let us take these three vectors (1, 1, 2), (1, 2, 0) and (0, 2, 1). So, we want to ask whether or not they are linearly independent. So, let us take some arbitrary coefficients a, b and c. So, $a(1,1,2) + b(1,2,0) + c(0,2,1) = (0,0,0)$. And then we ask, can we get from here what are the choices for $a, b, c,$ in particular, is there a choice where at least one of a, b or c is non-zero.

So, we have the following system of linear equations. How do we get this? By equating the corresponding coordinates. So, the first coordinate on the left hand side is $a * 1 + b * 1 + c * 0 = a + b = 0$. The first coordinate on the right hand side is 0, so this is $a + b = 0$. Similarly, if you equate the second coordinates, you get $a + 2b + 2c = 0$. And the third coordinate then you get $2a + 0b + 1c = 2a + c = 0$. So, that is how you get the system of equations.

Now, let us solve for a, b and c and see if we can get solutions which are non-zero. So, from the first equation we get that $b = -a$, from the third equation we get $c = -2a$, if you substitute into the middle equation, you get some combination of a is not 0. So, specifically, you get, $a - 2a - 4a$. So, $a - 6a$, so $-5a = 0$, which tells you that $a = 0$. And then once $a = 0$, $b = -a$, so $b = 0$, and $c = -2a$, so $c = 0$. So, the only equation for this system is a is, b is, c is 0.

So, what have we achieved? We have achieved that if you have an equation like this with coefficients $a, b,$ and $c,$ then the only way this equation is satisfied, when I say this equation, I mean, this equation here, the only way this equation is satisfied is when a is 0, b is 0 and c is 0. That tells us that these three vectors (1, 1, 2), (1, 2, 0) and (0, 2, 1) are linearly independent.

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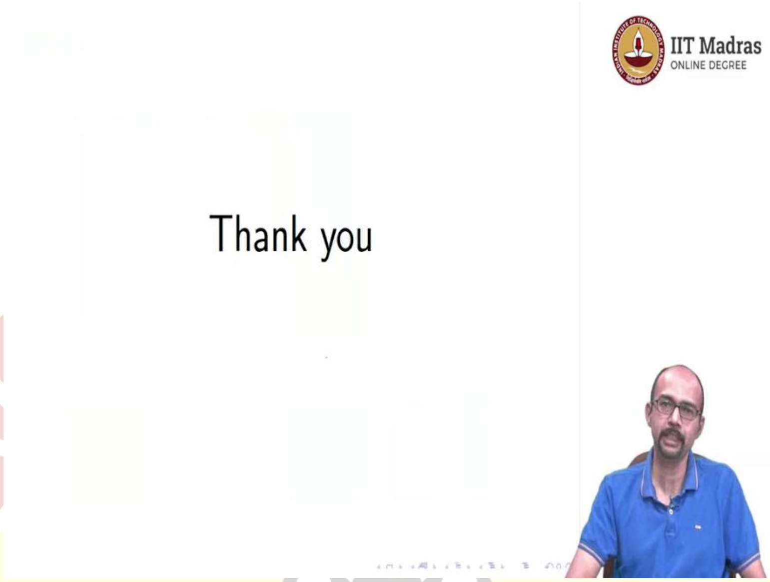

So, we have seen in this video the notion of linearly, a linearly independent set of vectors. So, we say that a set of vectors is linearly independent if they are not linearly dependent, which really means that if you take a linear combination of these vectors and equate it to 0, the only way that that is possible is if all the coefficient are 0. Thank you.