What is the rank/dimension of a vector space?

Sarang S. Sane

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Equivalently : A basis B of a vector space V is a subset $B \subseteq V$ such that every element of V can be uniquely written as a linear combination of elements of B.

i.e. if $B = \{v_1, v_2, \ldots, v_n\}$ then for every $v \in V$, there is a unique set of scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $v = \sum_{i=1}^{n} a_i v_i$.

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For every vector space there exists a basis,and all bases of a vector space have the same number of elements (or cardinality) ; hence, the dimension (or rank) of a vector space (say V) is uniquely defined and denoted by $dim(V)$ (or $rank(V)$) respectively.

Dimension of $\mathbb{R}^n$

Recall the i^{th} **standard basis vector** in $\mathbb{R}^n$.

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Hence the dimension of $\mathbb{R}^n$ is n.

Example

Let use calculate the dimension of the subspace W of $\mathbb{R}^3$ spanned by $\{(1, 0, 0), (0, 1, 0), (3, 5, 0)\}$.

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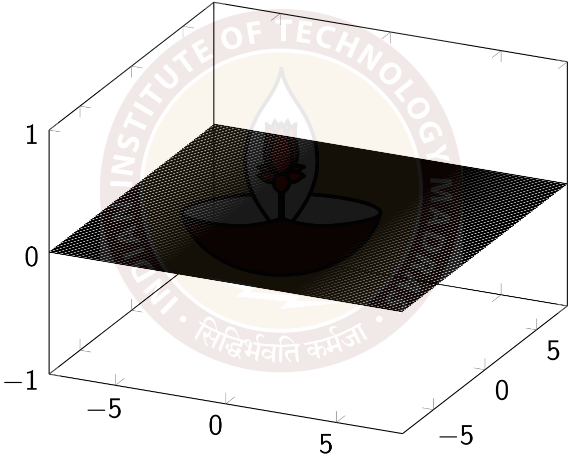
Hence dimension of the subspace W is 2.

Example contd.

Geometrically the subspace W is the XY-plane.



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The final matrix is the row echelon form of the original matrix and its rows form a basis of the subspace W.

In particular, the number of non-zero rows is $2 = dim(W)$.

Rank of a matrix

Let A be an $m \times n$ matrix.

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Fact : **Column rank= Row rank** and this number is called the **rank** of A.

Example

Let us find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$.

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$rk(A) = rk(\text{Row space}(A))$

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$A \rightsquigarrow R$ via row operations then rank is unchanged.

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{-3} & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\Big\downarrow -R_2/3$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3-3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

There are two non-zero rows. Hence $rank(A) = 2$.

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