What is a linear mapping

Sarang S. Sane

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The cost of x_1 kg.of rice, x_2 kg. of dal and x_3 kg. of oil is :

$$x_1 \times 45 + x_2 \times 125 + x_3 \times 150 = 45x_1 + 125x_2 + 150x_3.$$

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Recall that linear combinations can also be expressed in terms of matrix multiplication e.g.

$$c_A(x_1, x_2, x_3) = 45x_1 + 125x_2 + 150x_3 = \begin{bmatrix} 45 & 125 & 150 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Example : Cost linearity

A caterer gets an order from Office 1 in Malgudi on Monday for 30 tiffins prepared in some prescribed way and buys 20 kg. of rice, 10 kg. of dal and 4 litres of oil from Shop A for the purpose.

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On Wednesday, they get orders from both offices for 15 and 50 tiffins respectively, each to be prepared in the respective prescribed ways. How much does the caterer spend at Shop A?

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On Wednesday, they get orders from both offices for 15 and 50 tiffins respectively, each to be prepared in the respective prescribed ways. How much does the caterer spend at Shop A?

Note that the required quantities of rice, dal and oil for preparing the 15 tiffins for office 1 on Wednesday will be half of the amounts on Monday i.e. 10 kg. of rice, 5 kg. of dal and 2 litres of oil.

Cost linearity (contd.)

Similarly, the required quantities of rice, dal and oil for preparing the 50 tiffins for office 2 on Wednesday will be $\frac{5}{4}$ times the amounts on Tuesday i.e. 37.5 kg. of rice, 15 kg. of dal and 2.5 litres of oil.

Cost linearity (contd.)

Similarly, the required quantities of rice, dal and oil for preparing the 50 tiffins for office 2 on Wednesday will be $\frac{5}{4}$ times the amounts on Tuesday i.e. 37.5 kg. of rice, 15 kg. of dal and 2.5 litres of oil.

Hence, the total required quantities of rice, dal and oil on Wednesday will be i.e. 47.5 kg. of rice, 20 kg. of dal and 4.5 litres of oil.

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$$45 \times 47.5 + 125 \times 15 + 150 \times 2.5 = 4387.5 \quad \text{rupees.}$$

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Cost to the caterer for office 1 on Monday : 2750 rupees
Cost to the caterer for office 2 on Tuesday : 3150 rupees

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$1/2 \times 2750 + 5/4 \times 3150 = 5312.5$ rupees.

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The first method can be summarized as calculating the vector for Wednesday as adding $1/2$ the vector for Monday and $5/4$ times the vector for Tuesday and the cost is computed by applying the linear combination on it.

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The second method as adding 1/2 the cost for Monday and 5/4 times the cost for Tuesday.

The following table summarizes this data : adding 1/2 the first row and 5/4 times the second row yields the third row.

	Kgs. of rice	Kgs. of dal	Litres of oil	Cost c_A
Monday	20	10	4	2750
Tuesday	30	12	2	3150
Wednesday	47.5	20	4.5	5312.5

Why does the second method work?

$$\text{Total Cost on Wed.} = 45 \times \text{rice reqd. on W.} + 125 \times \text{dal reqd. on W} + 150 \times \text{oil reqd. on W}$$

$$= 45 \times \left(\tfrac{1}{2} \times \text{rice on M} + \tfrac{5}{4} \times \text{rice on T}\right) + 125 \times \left(\tfrac{1}{2} \times \text{dal req. on M} + \tfrac{5}{4} \times \text{dal on T}\right) + 150 \times \left(\tfrac{1}{2} \times \text{oil on M} + \tfrac{5}{4} \times \text{oil on T}\right)$$

$$= \frac{1}{2} \times \left\{ (45 \times \text{rice on M}) + (125 \times \text{dal on M}) + (150 \times \text{oil on M}) \right\} + \frac{5}{4} \times \left\{ (45 \times \text{rice on T}) + (125 \times \text{dal on T}) + (150 \times \text{oil on T}) \right\}$$

$$= \frac{1}{2} \times \text{cost on Monday} + \frac{5}{4} \times \text{cost on Tuesday.}$$

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Write the expression for the total cost of buying x_1 kg.of rice, x_2 kg. of dal and x_3 kg. of oil for each of the three shops and try to compare them.

Example (contd.) : Expressions

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$$\begin{aligned} c_B(x_1, x_2, x_3) &= 40x_1 + 120x_2 + 170x_3 &= \begin{bmatrix} 40 & 120 & 170 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ c_C(x_1, x_2, x_3) &= 50x_1 + 130x_2 + 160x_3 &= \begin{bmatrix} 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Example(contd.) : Linear mappings

Comparing these expressions, it is clear that for any quantities x_1, x_2, x_3 that one would buy (i.e. when x_1, x_2, x_3 are positive), the third expression always yields larger values than the first one.

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A natural way to make this comparison would be to create a vector of costs i.e. $(c_A(x_1, x_2, x_3), c_B(x_1, x_2, x_3), c_C(x_1, x_2, x_3))$.

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We can then think of the cost vector as a function c from $\mathbb{R}^3$ to $\mathbb{R}^3$ by setting these expressions as the coordinates in $\mathbb{R}^3$ i.e.

$$c(x_1, x_2, x_3) = (c_A(x_1, x_2, x_3), c_B(x_1, x_2, x_3), c_C(x_1, x_2, x_3)) = (45x_1 + 125x_2 + 150x_3, 40x_1 + 120x_2 + 170x_3, 50x_1 + 130x_2 + 160x_3).$$

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e.g. the costs of buying 2 kg Rice, 1 kg Dal, and 2 litres oil are given by the cost vector $\begin{bmatrix} 45 & 125 & 150 \\ 40 & 120 & 170 \\ 50 & 130 & 160 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 515 \\ 540 \\ 550 \end{bmatrix}.$

Linearity

As in the case of the function c_A, the property of "linearity" of costs can now be extracted from the matrix form for the cost function c.

$$c\left(\tfrac{1}{2}\times \text{rice on } M + \tfrac{5}{4}\times \text{rice on } T,\ \tfrac{1}{2}\times \text{dal on } M + \tfrac{5}{4}\times \text{dal on } T,\ \tfrac{1}{2}\times \text{oil on } M + \tfrac{5}{4}\times \text{oil on } T\right)$$

$$= \left(c_A(\quad),\ c_B(\quad),\ c_c(\quad)\right)$$

$$= \left(\tfrac{1}{2}\times c_A(M) + \tfrac{5}{4}\times c_A(T),\ \tfrac{1}{2}\times c_B(M) + \tfrac{5}{4}\times c_B(T),\ \tfrac{1}{2}\times c_c(M) + \tfrac{5}{4}\times c_c(T)\right)$$

$$= \tfrac{1}{2}\times c(M) + \tfrac{5}{4}\times c(T)$$

$$M = (\text{rice on } M,\ \text{dal on } M,\ \text{oil on } M)$$

$$T = (\text{rice on } T,\ \text{dal on } T,\ \text{oil on } T)$$

$$c(\alpha(x_1,x_2,x_3) + (y_1,y_2,y_3)) = \alpha\, c(x_1,x_2,x_3) + c(y_1,y_2,y_3).$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\begin{bmatrix} & & \end{bmatrix}\left\{\alpha\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right\} = \alpha\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

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$$f(x_1, x_2, \ldots, x_n) \quad = \left(\sum_{j=1}^{n} a_{1j}x_j, \sum_{j=1}^{n} a_{2j}x_j, \ldots, \sum_{j=1}^{n} a_{mj}x_j \right).$$

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We can write the expressions on the RHS in matrix form as Ax

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Linearity of linear mappings

It follows that a linear mapping satisfies linearity, i.e. for any $c \in \mathbb{R}$ (scalar)

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = \\ f(x_1, x_2, \ldots, x_n) + cf(y_1, y_2, \ldots, y_n).$$

$$f(x_1+cy_1, x_2+cy_2, \ldots, x_n+cy_n) = A\begin{bmatrix} x_1+cy_1 \\ x_2+cy_2 \\ \vdots \\ x_n+cy_n \end{bmatrix}$$

$$= A\left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + c\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right\} = A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + cA\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= f(x_1, \ldots, x_n) + c\, f(y_1, \ldots, y_n).$$

Thank you