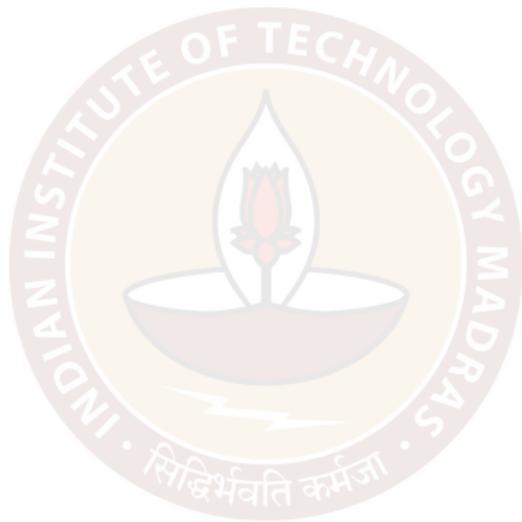


# The Hessian matrix and local extrema for $f(x, y)$



## Recall : the second derivative test

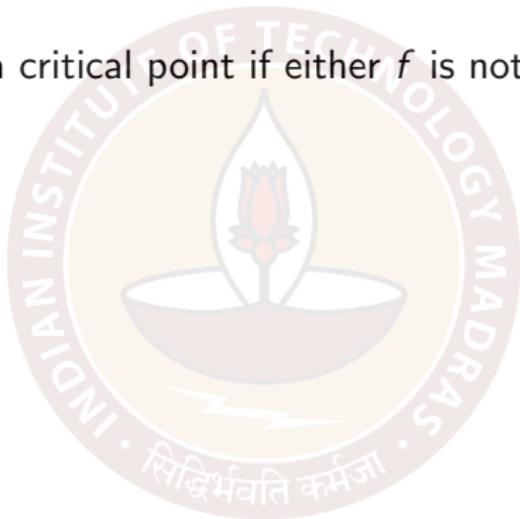
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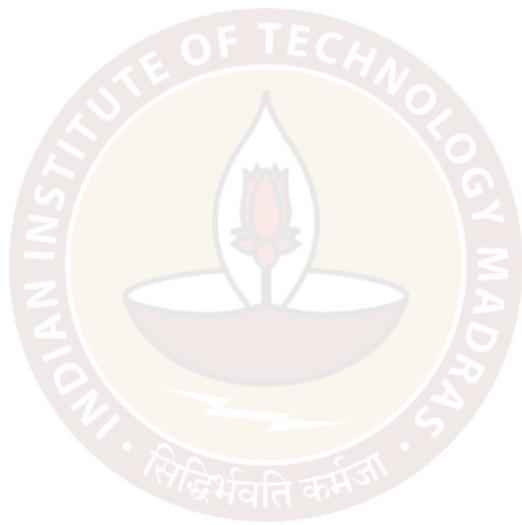
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# Recall : critical points for multivariable functions



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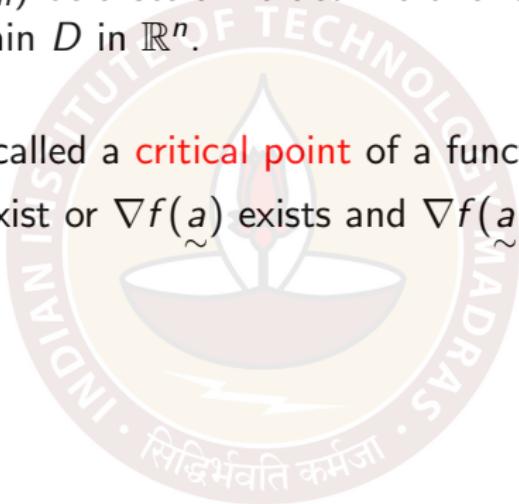
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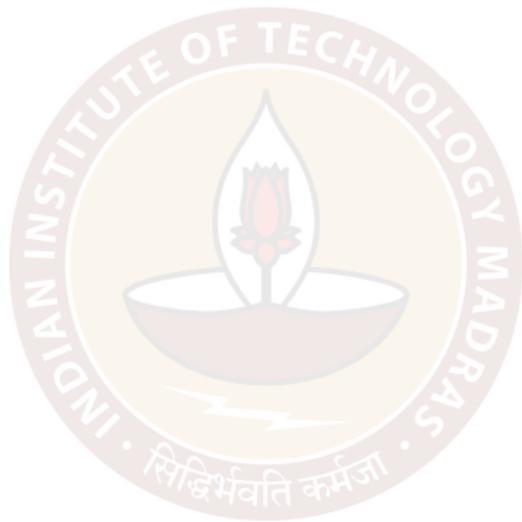
A **saddle point** is a critical point  $\tilde{a}$  such that  $\nabla f(\tilde{a})$  exists and  $\nabla f(\tilde{a}) = 0$  but  $\tilde{a}$  is not a local extremum.

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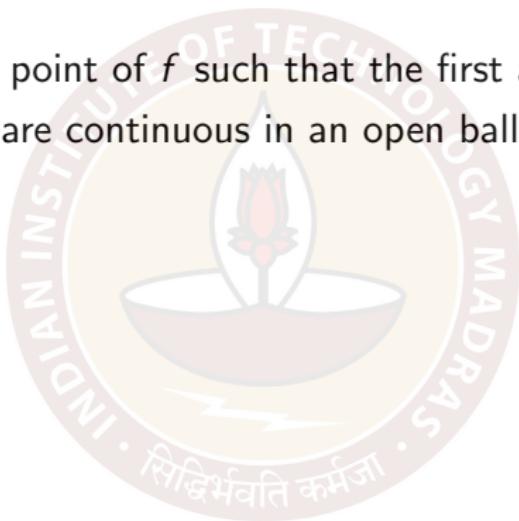
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4. If  $\det(Hf(\tilde{a})) = 0$  then the test is **inconclusive**.

## Examples

$$f(x, y) = x^2 + y^2$$

Critical pt. :  $(0, 0)$ .

$$\nabla f = (2x, 2y)$$

$$H_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(H_f(0,0)) = 4 > 0.$$

$$f_{xx}(0,0) = 2 > 0.$$

$\therefore (0, 0)$  is a local minimum.

$$f(x, y) = -x^2 - y^2$$

Critical pt. :  $(0, 0)$ .

$$\nabla f = (-2x, -2y)$$

$$H_f = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$H_f(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\det(H_f(0,0)) = 4 > 0$$

$$f_{xx}(0,0) = -2 < 0.$$

$\therefore (0, 0)$  is a local maximum.

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$$f(x, y) = x^4 + y^4$$

Critical pt. :  $(0, 0)$ .

$$\nabla f = (4x^3, 4y^3)$$

$$H_f = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

$$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\det(H_f(0,0)) = 0.$$

$\therefore$  The test is inconclusive.

## Examples (contd.)

$$f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$$

Equating to 0, we get

$$\nabla f = (2x + 6y + 2, 6x + 8y - 4)$$

the critical pt.  $(2, -1)$ .

$$H_f = \begin{bmatrix} 2 & 6 \\ 6 & 8 \end{bmatrix} = H_f(2, -1)$$

$\therefore (2, -1)$  is a saddle point of  $f(x, y)$ .

$$f(x, y) = xy - x^3 - y^2$$

Equating to 0, we get:

$$\nabla f = (y - 3x^2, x - 2y)$$

$$y = 3x^2, x = 2y \Rightarrow y = 3(2y)^2 = 12y^2 \Rightarrow y(1 - 12y) = 0.$$

Critical pts.:  $(0, 0), (\frac{1}{6}, \frac{1}{12})$

$$H_f(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \quad \det(H_f(0, 0)) = -1 < 0.$$

$\therefore (0, 0)$  is a saddle pt.

$$H_f\left(\frac{1}{6}, \frac{1}{12}\right) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det(H_f\left(\frac{1}{6}, \frac{1}{12}\right)) = 2 - 1 = 1 > 0.$$

$$f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) = -1 < 0.$$

$\therefore \left(\frac{1}{6}, \frac{1}{12}\right)$  is a local max.

## Examples (contd.)

$$f(x, y) = \sin(xy)$$

$$\nabla f = \begin{pmatrix} y \cos(xy), & x \cos(xy) \\ -y^2 \sin(xy) & \cos(xy) - xy \sin(xy) \end{pmatrix}$$

$$H_f = \begin{bmatrix} y \cos(xy) & -x^2 \sin(xy) \\ -x^2 \sin(xy) & \cos(xy) - xy \sin(xy) \end{bmatrix}$$

or ②  $x=y=0$ .

Equating  $\nabla f$  to 0, we get : ①  $\cos(xy) = 0$  or ②  $\sin(xy) = \pm 1$ .

$$H_f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det(H_f(0,0)) = -1 < 0.$$

$\Rightarrow (0,0)$  is a saddle pt. for  $f$ .

For pts. such that  $\cos(xy) = 0$

$$H_f(x,y) = \begin{bmatrix} -y^2 & -xy \\ -xy & -x^2 \end{bmatrix}$$

$\det(H_f(x,y)) = 1$

$\Rightarrow$  id  $\sin(xy) = \pm 1$

$\det(H_f(x,y)) = 0$

$$H_f(x,y) = \begin{bmatrix} y^2 & xy \\ xy & x^2 \end{bmatrix}$$

$$\det(H_f(x,y)) = 0.$$

# Thank you

