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ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 05
Determinants (Part 2)

Hello and welcome to the Maths 2 component of the online BSC program on data science. Today's video is on determinants, this is the second part of the videos on determinants. So, let us recall first what we did in part 1.

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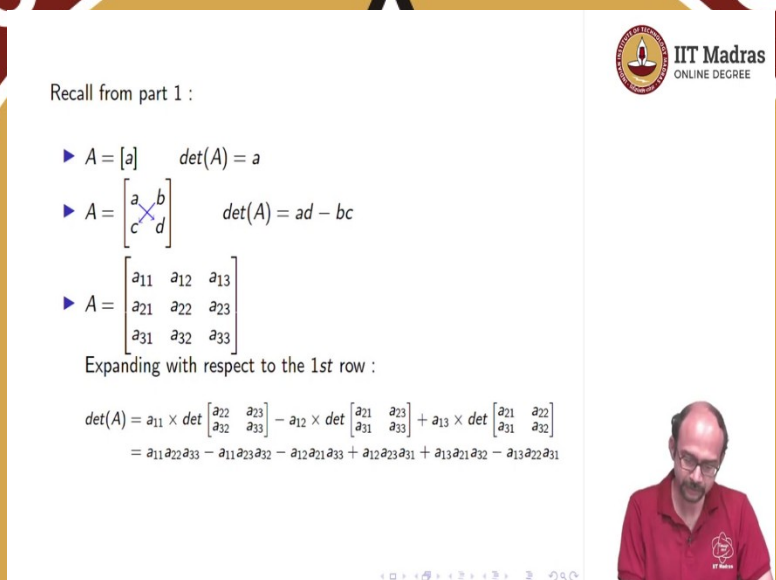

So, recall from part 1, that we defined the determinant of a 1 by 1 matrix so if a is singleton a then determinant of A is just that number a. If A is the 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is $ad-bc$ you multiply the diagonal terms and subtract out the off diagonal terms and if you have a 3 by 3 matrix, then we defined the determinant by what we call expanding with respect to the first row.

So, what that meant was you take the first the 1 1 entry of the first row, the first entry of the first row and then disregard everything in the first row and the first column you get a 2 by 2 matrix remaining take its determinant and multiply that with a 1 1 that is the first term. For the second term you look at a 1 2 and do the same process disregard everything in the first row and the second column, multiply the determinant of the resulting 2 by 2 matrix with a 1 2 and for the third term you look at a 1 3 disregard everything in the first row and the third column, take the determinant of the resulting 2 by 2 matrix and multiply it with a 1 3.

And then you alternately add these meaning take the first expression - the second expression + the third expression. So, that was the determinant of a. So, the determinant of the 3 by 3 matrix was defined in terms of determinants for 2 by 2 matrices and then we can expand this out we can work out explicitly what the expression is and you get some rather long and nasty 6 term expression.

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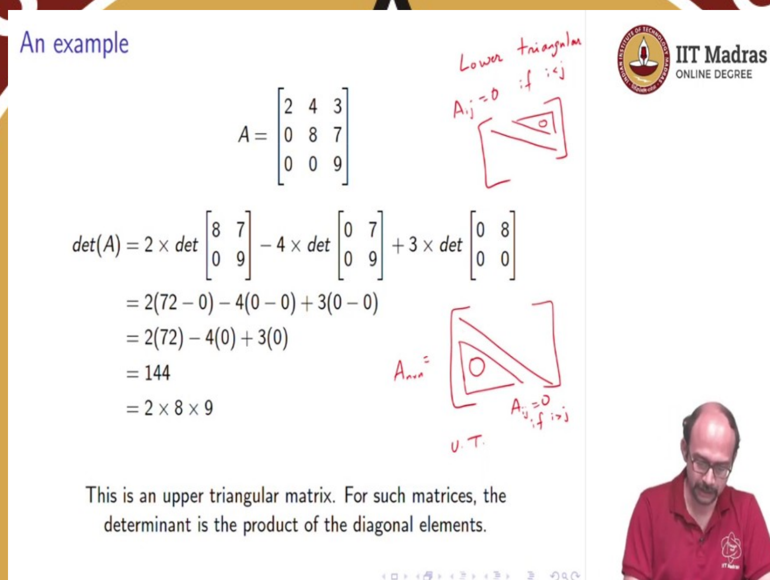

So, let us do an example just to refresh ourselves maybe a 3 by 3 example. So, here is a 3 by 3 matrix $\begin{bmatrix} 2 & 4 & 3 \\ 0 & 8 & 7 \\ 0 & 0 & 9 \end{bmatrix}$ so maybe I should mention that such a matrix is called an upper triangular matrix because it has zeros below the diagonal.

So, let us compute its determinant, so determinant of A is 2×the matrix $\begin{bmatrix} 8 & 7 \\ 0 & 9 \end{bmatrix}$ which is what is remaining if you disregard everything in the first row and column - 4×determinant of $\begin{bmatrix} 0 & 7 \\ 0 & 9 \end{bmatrix}$, + 3×determinant of $\begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}$ and then if we work out what these are using the determinants of 2 by 2 matrices, we will get $2 \times 8 \times 9 - 7 \times 0$ which is $72 - 0 - 4 \times 0 - 0 + 3 \times 0 - 0$, so this is just $2 \times 72 - 4 \times 0 + 3 \times 0$ which is just 2×72 which is 144 and I will ask you to notice something important which is that this is 2×8×9 which is exactly what we have on the diagonal.

So, these are the diagonal entries over here and that is what we got, so the determinant of this matrix A is the product of the diagonal entries. So, I hope you followed both the computation and the fact that it is the product of the diagonal entries. So, this is not some fact specific to this example, this is a general fact for what we call any upper triangular matrix or any lower triangular matrix. So, for such matrices the determinant is the product of the diagonal elements.

So, I will just quickly recall what is an upper triangular matrix as I said in the beginning. So, an upper triangular matrix means if you so this is a general matrix of size n by n, so this is called upper triangular if you look at the diagonal and everything below the diagonal is 0. So, all these entries are zeros. So, we are going to write this as $a_{ij} = 0$ if $i > j$. So, if the matrix satisfies this property it is an upper triangular matrix and this is how it looks like.

Similarly, so then so this is upper triangular. So, similarly we have lower triangular, so a lower triangular matrix. So, this is again n by n, that is if $a_{ij} = 0$ if now this time $i < j$. So, how does that look like, not a table but a matrix, so this part must be 0 and these elements can be anything that any numbers. So, for such matrices the determinant is the product of the diagonal elements. So, of course we know this only for 3 by 3 and I will suggest that you check this for 3 by 3 and 2 by 2 matrices.

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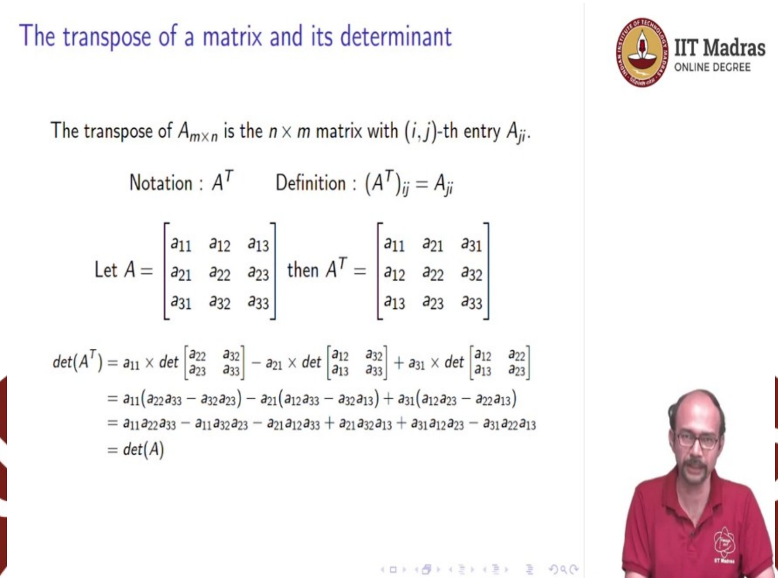

So, let us move on to a special kind of matrix namely if you have a matrix then its transpose, so let us define the transpose and compute its determinant. So, in general if you have an m by n matrix, the transpose of this matrix is a new matrix of size n by m so it has n rows and m columns. So, remember the first one had m rows and n columns a and m rows in columns, so a transpose has n rows and m columns and its *ij*-th entry is a_{ji}. So, the ij-th entry of a transpose is whatever was the jith entry of A. So just, this is the notation it is called A^T, so T is for transpose and its definition as we noted is A transpose i j is A_{ji} j i.

So, just for example if A is a 3 by 3 matrix then a transpose is again 3 by 3. So, square matrices the transpose has the same size it is also a square matrix of the same size and what happens to the entries, so if A is like this $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ a 1 1 a 1 2 a 1 3 and so on then $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$. So, what happened here these two entries got interchanged, these two entries got interchanged, so the diagonal remains the same.

So, it reflects about the diagonal if you prefer a more geometric way of thinking about it. So, that is what happens in the transpose. So, what do we want to do now? We want to compute what is the determinant of the transpose. So, let us work this out. So, determinant of a transpose, so we have a clear formula for the determinant so let us use that. So, we look at the 1 1 term of A transpose and we multiply by the determinant of what is left when we drop the first row and column and - the 1 2 term of the transpose× determinant of whatever is left when we drop the first row and second column of A transpose and + the 1 3 term of A transpose× determinant of whatever is left when you drop the first row and the third column of A transpose.

So, if we do that then this is exactly the expression we get and now we can work out explicitly what happens and you will see that the final expression you get is just some is just the same as the expression for determinant of A except that the terms have changed places but it is the same expression. So, this is just determinant of A. So, what is the upshot? If you take the transpose of A square matrix, then its determinant remains unchanged and we have in fact proved this in the 3 by 3 case I will request that you do it for the 2 by 2 as well, 1 by 1 there is of course nothing to do.

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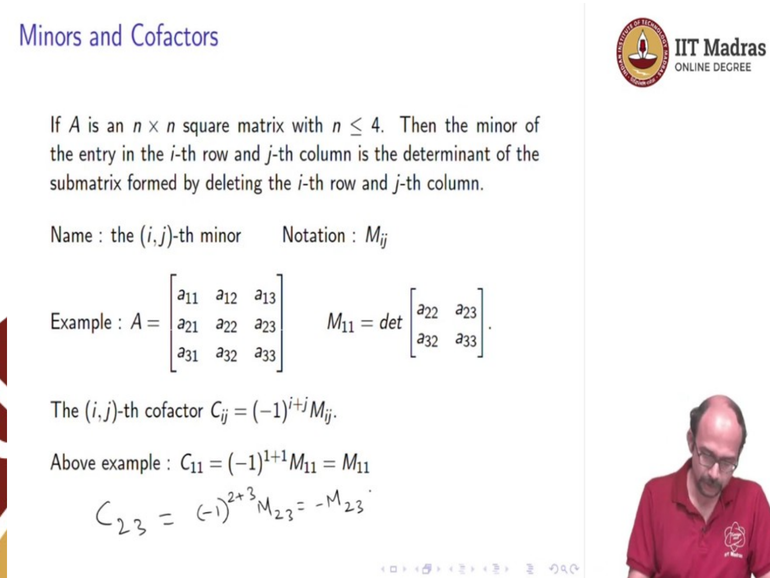

So, now let us move on to minors and cofactors. So, this is a very important idea and this is what is going to help us to generalize the determinant to the n by n situation meaning where the size of the square matrix is n by n. So, now suppose A is an n by n square matrix where n is at most 4 we will do it in this particular case so n is so either it is a 2 by 2 matrix or a 3 by 3 matrix or a 4 by 4 matrix.

Then the minor of the entry in the ith row and jth column is the determinant of the sub matrix formed by deleting the ith row and the jth column. So, we have been doing this process in computing the determinant, so to compute the determinant of the 3 by 3 matrix for the first expression in the determinant we take a 1 1×determinant of the 2 by 2 matrix which is obtained by deleting the first row and first column. So, now we are talking about just that part, just that operation.

So, you look at the ith row and the jth column and you delete those and then you get a matrix of size n - 1 by n - 1 and then you look at its determinant. So, now this makes sense because we have defined determinants for 2 by 2, 1 by 1, 2 by 2 and 3 by 3 matrices. So, if you take a matrix of size 4 by 4 and you delete the ith row and the jth column you will get matrix of size 3 by 3 and then you take its determinant. So, this is called the ij-th minor and its denoted by M_{ij}.

So, here is an example, I said a lot of words so here is a explicit example. So if a is a 3 by 3 matrix as written here $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the 1 1 minor the 1 1th minor is nothing but determinant of $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$. So, this is what you obtain by dropping the first row and first column, this is exactly the kind of operation we have been using in computing determinants by expansion along the first row and let us go ahead and first define what is the ij-th cofactor.

So, the ij-th cofactor is where you take the ij-th minor and multiply it by $(a-1)^{(i+j)}$a - 1 to the power i + j. Why is the sign coming, so as we saw in the expression for the determinant there was an alternating sign, the first term was with a + then you subtracted the second term then you added the third term, so it was alternating. So, somehow this - 1 to the i + j is supposed to take care of that sign that is why we have the sign over here. So, it is the same number but it could be with a - sign depending on what is i and what is j.

So, for example C 1 1 is just - $1(-1)^{1+1}=(-1)^2$, so - 1 squared is $1\times M_{11}$, so it is $(-1)^2\times M_{11}$ which is just M_{11}. So, as a similar in similar light if we have C_{23}, then this is - 1 to the 2 + $3\times M_{23}$, but now 2 + 3 is 5 which is odd so$(-1)^5$ is - 1. So, this is $-M_{23}$. So, it depends on what is i and j accordingly we will pick up a - sign. So, I hope it is clear what the cofactors and minors are. So, let us now try to rewrite the determinant in terms of - and cofactors.

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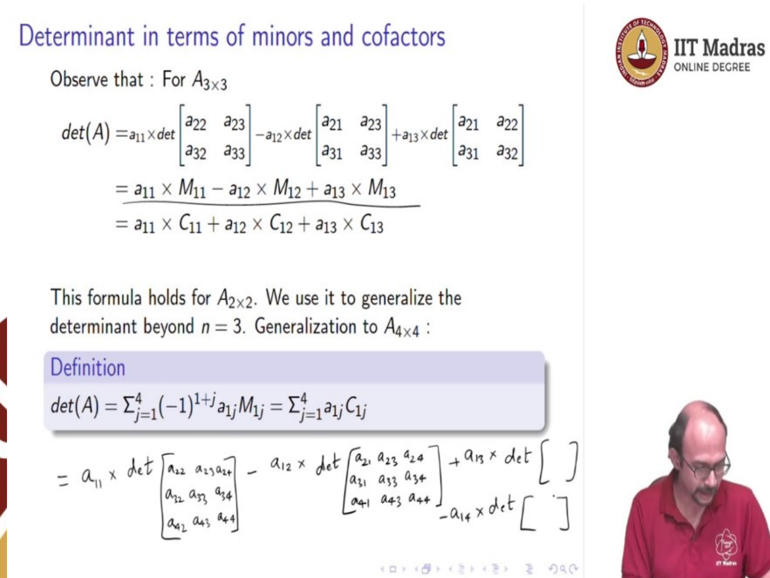

So, for a 3 by 3 matrix so we have determinant of AA is this is expansion along the first row, this is the definition a 1 1×determinant of the matrix obtained by deleting the first row and column and then you take its determinant but that is exactly what we called the minor and then $-a_{12}\times\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, so you deleted the first row and second column and then took the determinant but that is exactly what we call the 1 2th minor and then similarly for the third term.

So, we can rewrite it now in terms of our - as $a_{11}\times M_{11} - a_{12}\times M_{12} + a_{13}\times M_{13}$. So, we are just rewriting these determinants as - this was the definition if you remember from the previous slide and then if I want to take care of this sign, there is a - sign here and I want to write everything as a + so then I can rewrite this asa_{11}×the 1 1th cofactor + a_{12}×the 1 2th cofactor + a_{13}×the 1 3th cofactor. So, that is why we introduce cofactors if we want to not remember the signs, then we will take care of those signs using the cofactors.

So, this is the expression for the determinant of a 3 by 3 matrix in terms of the minors and remember the minors meant we or the cofactors the minors meant that you compute determinants of 2 by 2 matrices. So, now we can go beyond, so maybe first we should remark that this formula works even for a 2 by 2 matrix, so if you take a 2 by 2 matrix say $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant is

$ad-bc$ so we can rewrite that as $A\times M_{11}-B\times M_{12}$, so C is exactly M_{12} and d is exactly M_{11}or if you want a + sign then we can write it as $a\times C_{11}+b\times C_{12}$, so check this.

So, now we can use this to generalize beyond n is 3. So, let us generalize this to A 4 by 4 so suppose you have a 4 by 4 matrix now then how do we define the determinant. So, the determinant is this is expansion along the first row so the determinant is you take $(-1)^{(i+j)}$ because we have the first row here $a_{ij}\times M_{1j}$, this is exactly the expression that we have up here this is exactly this expression here, except that this is for 3 by 3 I am writing the same expression for 4 by 4.

And if you want to take care of the signs not bother about the signs then you can write it just as a sum of a_{1j} × the corresponding cofactor for the 1 jth term. So, let us maybe write this expression down explicitly. So, what does this mean? This means that the determinant of so for a 4 by 4 matrix so a_{11} ×determinant of whatever you get after dropping the first row and the first column

so $\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$, that is your first expression, the first term in this sum and then $-a_{12}$ – a 1 2 and I will maybe write one more term there should be a determinant here determinant of so now we drop the first row and the second column.

So, now we get $\begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}$. That is the second term in this expression and note we have a - sign here and then + a_{13} and I hope now you can write down these two determinants. So, check out what this is and then -a_{14} ×determinant of whatever we get here. So, check these two terms see if you can write down these two terms that will mean you have understood what is going on.

So, what is the point here? The point is we could write the determinant of a we defined it this is a 4 by 4 matrix in terms of determinants for 3 by 3 matrices. So, in other words if I know how to define the determinant for a matrix of smaller size then I can sort of use this process inductively and generalize and that is exactly what we do.

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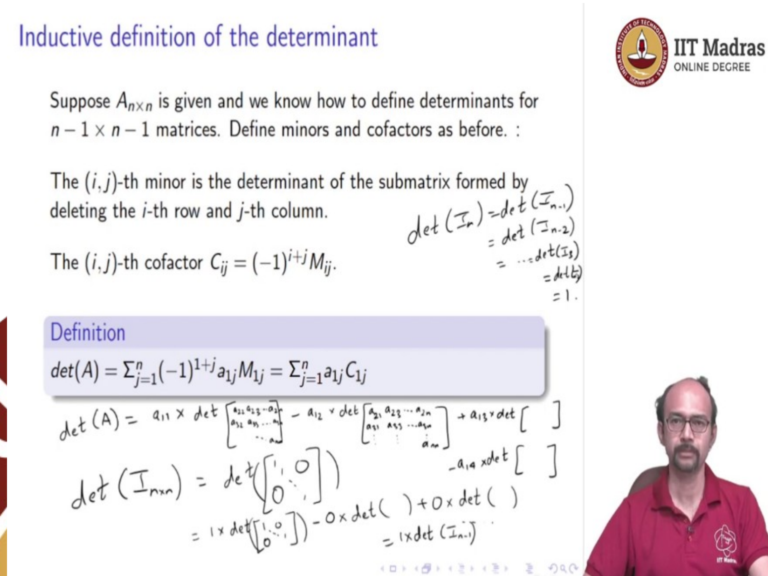

So, here is the inductive definition of the determinant. So, suppose you have a matrix of size n by n and we know how to define determinants for size n - 1, so n - 1 by n - 1 matrices we know how to define the determinant. So, then we will define minors and cofactors as before, so what does that mean? The ij-th minor is the determinant of the sub matrix formed by deleting the ith row in the jth column. So, we delete the ith row and the jth column we get a matrix of size n - 1 by n - 1 and we look at its determinant that is called the ij-th minor.

This makes sense because we have assumed that we know how to define matrices, determinants for n - 1 by n - 1 matrices and then if you want to again we want to do something similar to what we did earlier, so if you do not want to bother about signs going ahead you define the ij-th cofactor as $(-1)^{(i+j)}$ ×the ij-th minor and then we define the determinant of A as the again this is by expansion along the first row as $(-1)^{(i+j)}$.

Now the sum runs from 1 through n because remember there are n entries in the first row, so $(-1)^{(1+j)} \times a_{1j} \times M_{ij}$ and again if you do not want to bother about signs and you just want a simple sum then you absorb that sign into the minor and you can write it in terms of cofactor, so $\sum_{j=1}^{n} a_{1j} \times C_{1j}$.

So, maybe let us write down this expression let me write down the first term, so now suppose let us say I have a n by n matrix so this determinant of A is a_{11}×determinant of whatever you get by deleting the first row and the first column. So, you will get a_{22}, $a_{23}$and it will run all the way till a to n and then below you will have a_{32}, a_{33} and all the way up to a three n and if you complete this you get up go all the way up to a n n and then the second term is going to be $-a_{12}$×determinant of a_{21}, a_{23}.

And then you keep going up till a to n and then you have a_{31}, a_{33} going up till $a_{3n}$and you keep doing this here you will get the first column, here you will get the third column meaning all the entries remaining in third column and here you will get the nth column and then you keep going, you have a_{12}, then you have a_{13} ×determinant see if you can write down this term and then $-a_{14}$ ×some determinant and note what is happening. These determinants are of smaller size.

So, now if I want to do this for a 5 by 5 matrix, I will need to compute of the determinant of a 4 by 4 matrix but I know how to do that which means I now know how to compute the determinant of a 5 by 5 matrix but then I can use the same inductive definition and go for a 6 by 6 matrix and then using 6 by 6 to 7 by 7 and then so on to 8 by 8 and then using that to 9 by 9 and you can see that inductively for any number I can do this process. So, that is the idea of the determinant. So, this is a very complicated looking definition and it is kind of a quirk that it is indeed very useful I in fact I cannot emphasize how useful you will see it later when we solve equations and it is used for other things as well you will see this in calculus too.

So, maybe as an example let us compute the determinant of the identity matrix of size n. So, remember the identity matrix had 1 on the diagonal so and everything else is 0. So, I am this is shorthand to say that everything else is 0. So, let us use the definition, so by definition this is 1×determinant of whatever the 1 1th minor, 1×the 1 1th minor. So determinant of you drop the first row and first column so you get ones on the diagonal and zeros on the off diagonal.

So, this is again the identity matrix of size n - 1 by n - 1 and then - 0×the 1 2th minor but I do not want I do not really care about computing it because it is 0×something it is going to be 0 anyway + 0×the 1 3th minor and so on. So, all these terms are going to be 0 because in the rest of the first row all the terms you have are 0, so what do you get, you are left only with 1×determinant of the identity matrix of smaller size.

So, determinant of I_n is determinant of I_{n-1} that is what we obtained in by this process and then I can keep doing this. So, this is determinant of I_{n-2} and so on. So, the correct way of doing this is what is called induction and doing this we will reach all the way down till maybe I3 which in or I_2 whichever you prefer and both of these we computed in the previous video to be 1.

So, let us recall what we have done in this video we started by defining recalling the definition of the determinant for the 1 by 1, 2 by 2, and 3 by 3 cases and we saw that for the 4 by 4 matrix we can define the determinant by using the 3 by 3 definition because we can do that for the 3 by 3 using the 2 by 2 and for the 2 by 2 using the 1 by 1. So, it follows that for the 4 by 4 we can use the 3 by 3 definition and then we generalize this to the n by n matrix, so we said that if we can define the determinant for n - 1 by n - 1 matrix, then we can define it for an n by n matrix so, this is called the inductive definition of the determinant.

And finally, we have computed we did one example which was we computed that the determinant of the identity matrix of any size 3 by 3, 4 by 4, 5 by 5, 10 by 10; any n by n is 1, thank you.