Week-3
Mathematics for Data Science - 2
Introduction to Vector Space
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Which of the following sets with the given addition and scalar multiplication operations (scalars are real numbers in every case) form vector spaces?

$$V_1 = \{(x, y) | x, y \in \mathbb{R}\}$$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1);\ (x_1, y_1), (x_2, y_2) \in V_1$
Scalar multiplication: $c(x, y) = (cx, 1);\ (x, y) \in V_1,\ c \in \mathbb{R}$

$$V_2 = \{(x, y) | x, y \in \mathbb{R}\}$$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);\ (x_1, y_1), (x_2, y_2) \in V_2$
Scalar multiplication: $c(x, y) = (cx, 0);\ (x, y) \in V_2,\ c \in \mathbb{R}$

$$V_3 = \{(x, y) | x, y \in \mathbb{R}\}$$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2);$
$(x_1, y_1), (x_2, y_2) \in V_3$
Scalar multiplication: $c(x, y) = (cx, cy);\ (x, y) \in V_3,\ c \in \mathbb{R}$

$$V_4 = \{(x, y, z) | x, y, z \in \mathbb{R}, x + y = z\}$$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_4$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_4,\ c \in \mathbb{R}$

- ○ Option 1: V_1 is a vector space, but others are not.
- ○ Option 2: V_2 is a vector space but others are not.
- ○ Option 3: V_3 is a vector space but others are not.

○ **Option 4:** V_4 is a vector space but others are not.

○ Option 5: V_2 and V_4 are vector spaces, V_1 and V_3 are not.

○ Option 6: V_2 and V_3 are vector spaces, V_1 and V_4 are not.

Solution:

1. Given V_1,

$$V_1 = \{(x, y)|x, y \in \mathbb{R}\}$$
$$\textit{Addition: } (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1);\ (x_1, y_1), (x_2, y_2) \in V_1$$
$$\textit{Scalar multiplication: } c(x, y) = (cx, 1);\ (x, y) \in V_1,\ c \in \mathbb{R}$$

Given addition,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1)$$

Let $(x_1, y_1) = v_1$ and (a, b) be a zero vector in the vector space V_1, then

$$0 + v_1 = (a, b) + (x_1, y_1) = (a + x_1, 1) \neq v_1$$

Which shows V_1 does not have any zero vector.
Therefore, V_1 is not a vector space.

2. Given V_2,

$$V_2 = \{(x, y)|x, y \in \mathbb{R}\}$$
$$\textit{Addition: } (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);\ (x_1, y_1), (x_2, y_2) \in V_2$$
$$\textit{Scalar multiplication: } c(x, y) = (cx, 0);\ (x, y) \in V_2,\ c \in \mathbb{R}$$

Here we can see that addition follows all the four laws of addition for a vector space.
But given multiplication,

$$c(x, y) = (cx, 0)$$

Let $c = 1$, then

$$1(x, y) = (1x, 0) = (x, 0)$$

If we write $(x, y) = v$, where $v \in V_2$, then

$$1v \neq v$$

Which fails the unitary law of multiplication in a vector space i.e.,

$$1.v = v$$

Therefore, V_2 is not a vector space.

3. Given V_3,

$$V_3 = \{(x, y)|x, y \in \mathbb{R}\}$$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2);$
$(x_1, y_1), (x_2, y_2) \in V_3$
Scalar multiplication: $c(x, y) = (cx, cy);\ (x, y) \in V_3,\ c \in \mathbb{R}$

Given addition,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2)$$

Let $(x_1, y_1) = v_1$ and (a, b) be a zero vector in the vector space V_3, then

$$0 + v_1 = (a, b) + (x_1, y_1) = (a + x_1 + b + y_1, a + x_1 + b + y_1) \neq v_1$$

Which shows V_1 does not have any zero vector.
Therefore, V_3 is not a vector space.

4. Given V_4,

$$V_4 = \{(x, y, z)|x, y, z \in \mathbb{R}, x + y = z\}$$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_4$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_4,\ c \in \mathbb{R}$

Clearly, V_4 follows all the laws of Addition and Multiplication for a vector space.
Therefore, V_4 is a vector space.

2. Choose the set of correct options

- ◯ Option 1: If V is a real vector space, then $(\alpha+\beta)(x+y) = \alpha x + \beta y$, for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$.
- ◯ Option 2: A vector space can have more than one zero vector.
- ◯ **Option 3:** (-1, 0 ,0), (-1, 1, -1) and (0, 2, 3) are linearly independent vectors in $\mathbb{R}^3$.
- ◯ Option 4: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are linearly dependent vectors in $M_{2\times 2}(\mathbb{R})$.

Solution:

Option 1: Let us assume $\alpha = 2, \beta = 5$, then according to option 1

$$(\alpha+\beta)(x+y) = \alpha x + \beta y$$

$$\implies (2+3)(x+y) = 2x + 3y$$

But we know that if V is a real vector space then,

$$a(x+y) = ax + by, a \in \mathbb{R}$$

Therefore,

$$(\alpha+\beta)(x+y) = (\alpha+\beta)x + (\alpha+\beta)y$$

$$\implies (2+3)(x+y) = (2+3)x + (2+3)y = 5x + 5y \neq 2x + 3y$$

Therefore, V is not a vector space.

Option 2: A vector space can have more than one zero vector.

Incorrect, as the zero vector is unique in a vector space. Let v and v' are two zero vectors in a vector space then,

$$v = v + v' = v' + v = v'$$

both are the same i.e, the zero vector is unique in a vector space.

Option 3: (-1, 0 ,0), (-1, 1, -1) and (0, 2, 3) are linearly independent vectors in $\mathbb{R}^3$.

Let $a, b, c \in R$ and

$$a \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -a & -b & 0 \\ 0 & b & 2c \\ 0 & b & 3c \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies Ax = B$$

Where,

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}, x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$det(A) = -1(3-2) \neq 0$$

It means the system has unique solution (trivial solution only) i.e., $a = b = c = 0$.
Therefore, the vectors are linearly independent.

Option 4: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are linearly dependent vectors in $M_{2\times 2}(\mathbb{R})$.

Take $a, b, c \in \mathbb{R}$ and

$$a \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} -a-c & -2b+c \\ b+c & a-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Which means all the element in the matrix should be zero. i.e.,

$$-a-c = 0 \implies a = -c$$

$$a - c = 0 \implies a = c$$
$$\implies a = c = 0$$

and

$$-2b + c = 0 \implies b = 0$$

As **we can see** $a = b = c = 0$**, therefore, the vectors are linearly independent.**

3. A healthy juice consist of 30 units of protein, 11 units of carbohydrate (in tens), 53 units of fat, and 213 units of calcium (in average). A juice maker makes two types of juice, Type A and Type B. Type A consists of banana, milk, and almond, where as Type B consists of apple, milk, and almond. Table M2W3G1 shows the amount of protein, carbohydrate (in tens), fat, and calcium present in each banana, apple, and almond, and in 100 ml of milk.

Items	Protein	Carbohydrate (in tens)	Fat	Calcium
Banana (1 piece)	2	3	1	5
Apple (1 piece)	1	2	1	6
Almond (1 piece)	6	1	15	1
Milk (100 ml)	4	1	3	100

Table: M2W3G1

Use the above information to choose the correct option.

- ○ **Option 1:** With the right quantities of ingredients of Type A, it can be a healthy juice, and those amounts are unique.
- ○ Option 2: With the right quantities of ingredients of Type B, it can be a healthy juice, and those amounts are unique.
- ○ Option 3: With the right quantities of ingredients of Type A, it can be a healthy juice, and those amounts are not unique.
- ○ Option 4: With the right quantities of ingredients of Type B, it can be a healthy juice, and those amounts are not unique.

Solution:

Let Type A juice is made by mixing a unit of banana, b unit of milk, and c unit of almond and it is a healthy juice, then

$$a(2,3,1,5) + b(4,1,3,100) + c(6,1,15,1) = (30,11,53,213)$$

$$\implies \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 1 \\ 1 & 3 & 15 \\ 5 & 100 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 30 \\ 11 \\ 53 \\ 213 \end{bmatrix}$$

For solving the linear system take $A|b$,

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 30 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{array}\right]$$

$$\downarrow \frac{R_1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{array}\right]$$

$$\downarrow R_4 - 5R_1$$

$$\downarrow R_3 - R_1$$

$$\downarrow R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & -5 & -8 & -34 \\ 0 & 1 & 12 & 38 \\ 0 & 90 & -14 & 138 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & -5 & -8 & -34 \\ 0 & 90 & -14 & 138 \end{array}\right]$$

$$\downarrow R_3 + 5R_2$$

$$\downarrow R_4 - 90R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & 0 & 52 & 156 \\ 0 & 0 & -1094 & -3282 \end{array}\right]$$

$$\downarrow R_3/52$$

$$\downarrow R_4/2094$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$\downarrow R_4 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can see the rank of matrix is 3, which means unique solution.
Therefore, option 1 is correct.

Let Type B juice is made by mixing a unit of apple, b unit of milk, and c unit of almond and it is a healthy juice, then

$$a(1,2,1,6) + b(4,1,3,100) + c(6,1,15,1) = (30,11,53,213)$$

$$\implies \begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 1 \\ 1 & 3 & 15 \\ 6 & 100 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 30 \\ 11 \\ 53 \\ 213 \end{bmatrix}$$

For solving the linear system take $A|b$,

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 2 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 6 & 100 & 1 & 213 \end{array}\right]$$

$$\downarrow R_4 - 6R_1$$

$$\downarrow R_3 - R_1$$

$$\downarrow R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & -7 & -11 & -59 \\ 0 & -1 & 9 & 23 \\ 0 & 76 & -35 & 33 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\downarrow -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & -7 & -11 & -59 \\ 0 & 76 & -35 & 33 \end{array}\right]$$

$$\downarrow R_3 + 7R_2$$

$$\downarrow R_4 - 76R_2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & 0 & -74 & -172 \\ 0 & 0 & 649 & 1781 \end{array}\right]$$

$$\downarrow -R_3/74$$

$$\downarrow R_3/649$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & 0 & 1 & \frac{86}{37} \\ 0 & 0 & 1 & \frac{1781}{649} \end{array}\right]$$

$$\downarrow R_4 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & 0 & 1 & \frac{86}{37} \\ 0 & 0 & 0 & k \end{array}\right]$$

here $k = \frac{1781}{649} - \frac{86}{37} \neq 0$ Which means no solution therefore, options 2,3, and 4 are incorrect.

4. Match the vector spaces (with the usual scalar multiplication and vector addition as in $M_{3\times 3}(\mathbb{R})$) in column A with their bases in column B in Table : M2W3G2.

	Vector space (Column A)		Basis (Column B)
a)	$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid x+y+z=0, \text{ and } x, y, z \in \mathbb{R} \right\}$	i)	$\left\{ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$
b)	$V = \{A \mid A \in M_{3\times 3}(\mathbb{R})$, A is a diagonal matrix, with sum of the elements in the diagonal is zero $\}$	ii)	$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$
c)	$V = \left\{ \begin{bmatrix} 0 & z & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \mid x+y+z=0, \text{ and } x, y, z \in \mathbb{R} \right\}$	iii)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

Table : M2W3G2

Choose the correct option.

- ◯ Option 1: a → ii, b → i, c → iii.
- ◯ **Option 2:** a → ii, b → iii, c → i.
- ◯ Option 3: a → i, b → ii, c → iii.
- ◯ Option 4: a → iii, b → ii, c → i.

Solution:

a) Given,

$$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid x+y+z=0, x,y,z \in \mathbb{R} \right\}$$

$$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid z=-x-y, x,y,z \in \mathbb{R} \right\}$$

$$V = \left\{ \begin{bmatrix} x & y & -x-y \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid z=-x-y, x,y,z \in \mathbb{R} \right\}$$

Putting $x=0, y=1$, we get $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and putting $x=1, y=0$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
Therefore, the following set,

$$\left\{ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V . Hence it forms a basis of V.

b) Given, $V = \{A \mid A \in M_{3\times3}(\mathbb{R})$, A is a diagonal matrix with sum of the elements in the diagonal is zero.$\}$
Let $x, y, z \in \mathbb{R}$ and

$$V = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, x+y+z=0 \textit{ or } z=-x-y \right\}$$

$$\implies V = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -x-y \end{bmatrix} \right\}$$

Putting $x=0, y=1$, we get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and Putting $x=1, y=0$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Therefore, the following set,

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V . Hence it forms a basis of V.

Given in iii) in B column.

c) If $x, y, z \in R$, Given,

$$V = \left\{ \begin{bmatrix} 0 & z & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}, x + y + z = 0 \ or \ x = -z - y \right\}$$

$$\implies V = \left\{ \begin{bmatrix} 0 & z & -z-y \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

Putting $z = 1, y = 0$, we get $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and Putting $0 = 1, y = 1$, we get $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore, the following set,

$$\left\{ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V . Hence it forms a basis of V.

2 Multiple Select Questions (MSQ)

5. Consider the set of vectors $V = \{(-1, 1, 5), (2, 1, 3), (2, 1, 2), (1, -1, 7), (-1, 3, -5)\}$ from $\mathbb{R}^3$, choose the set of correct options

   ◯ Option 1: The singleton set of vector $\{(-1, 1, 5)\}$ is linearly dependent.

   ◯ **Option 2:** If $\alpha, \beta \in V$ and α, β are distinct then $\{\alpha, \beta\}$ is a linearly independent set of vectors.

   ◯ Option 3: The set {(-1, 1, 5), (2, 1, 3),(-2, 2, 10)} is a linearly dependent set of vectors.

   ◯ Option 4: The set V is a linearly independent set of vectors.

   ◯ Option 5: The set $\{\alpha, \beta, \gamma\}$ is a linearly dependent set of vectors for any $\alpha, \beta, \gamma \in V$, where all the three are distinct vectors.

   ◯ Option 6: The set $\{\alpha, \beta, \gamma, \delta\}$ is a linearly independent set of vectors for any $\alpha, \beta, \gamma, \delta \in V$, where all the four are distinct vectors.

   ◯ **Option 7:** The system $AX = b$, where $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 7 \\ -1 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have infinitely many solutions.

   ◯ **Option 8:** The system $AX = b$, where $A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have unique solution.

   ◯ Option 9: The system $AX = b$ where $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -5 \\ -1 & 1 & 5 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have infinitely many solutions.

Solution:

Option 1 is incorrect as singleton set is always independent.

Given,

$$V = \{(-1, 1, 5), (2, 1, 3), (2, 1, 2), (1, -1, 7), (-1, 3, -5)\}$$

Here no vector is multiple of any other vector which means any set of vectors $V' \subset V$ two vectors is a linearly independent set of vectors. Therefore, option 2 is correct.

In option 3 given set is $\{(-1, 1, 5), (2, 1, 3), (-2, 2, 10)\}$. Visualize,

$$(-2, 2, 10) = 2(-1, 1, 5) + 0(2, 1, 3)$$

Hence linearly dependent.

We know that m vectors in $\mathbb{R}^n$ are always linearly dependent for $m > n$. Therefore, maximum three vectors can only be linearly independent. That's why option 4 and 6 are incorrect.

Now to find the maximum number of linearly independent vectors,

$$a(-1,1,5) + b(2,1,3) + c(2,1,2) + d(1,-1,7) + e(-1,3,-5) = 0$$

$$\implies \begin{bmatrix} -1 & 2 & 2 & 1 & -1 \\ 1 & 1 & 1 & -1 & 3 \\ 5 & 3 & 2 & 7 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A|b$

$$\left[\begin{array}{ccccc|c} -1 & 2 & 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 3 & 0 \\ 5 & 3 & 2 & 7 & -5 & 0 \end{array}\right]$$

$$\downarrow -R_1$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 3 & 0 \\ 5 & 3 & 2 & 7 & -5 & 0 \end{array}\right]$$

$$\downarrow R_2 - R_1$$
$$\downarrow R_3 - 5R_1$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 & 2 & 0 \\ 0 & 13 & 12 & 12 & -10 & 0 \end{array}\right]$$

$$\downarrow R_2/3$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2/3 & 0 \\ 0 & 13 & 12 & 12 & -10 & 0 \end{array}\right]$$

$$\downarrow R_3 - 13R_2$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2/3 & 0 \\ 0 & 0 & -1 & 12 & -32/3 & 0 \end{array}\right]$$

$$\downarrow R1 + 2R_2$$

$$\downarrow R_2 + R_3$$

$$\Longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 7/4 & 0 \\ 0 & 1 & 0 & 12 & -10 & 0 \\ 0 & 0 & -1 & +12 & -32/3 & 0 \end{array}\right]$$

$$\downarrow -R_3$$

$$\Longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 7/4 & 0 \\ 0 & 1 & 0 & 12 & -10 & 0 \\ 0 & 0 & 1 & -12 & 32/3 & 0 \end{array}\right]$$

Clearly we can see that the rank is 3, which means there are atleast three linearly independent vectors. Therefore, option 5 are incorrect.

In option 7, four vectors of V are used. We saw in the above explanation that the rank is 3, which means the system will have one independent variable with the three dependent variable, depending on the value of fourth variable, therefore infinite solutions.

In option 8,

$$A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix}^t = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ -5 & 3 \end{bmatrix}$$

Which means

$$\begin{bmatrix} -1 & 2 \\ 3 & 1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix,

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & 1 & 0 \\ -5 & 3 & 0 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 7 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore unique solution.

In option 9 given,

$$A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -5 \\ -1 & 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 3 & 1 \\ 2 & -5 & 5 \end{bmatrix}$$

$$det(A) = 2(15+5) - 1(-5-5) + 2(-1+3) = 40 + 10 + 4 \neq 0$$

Therefore, unique solution. So, the option is incorrect.

3 Numerical Answer Type (NAT):

6. Find the value of c for which the vector $(3, 2, c)$ will be in the spanning set of the vectors $(1, 0, 1)$ and $(0, 1, -1)$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. [Answer: 1]

   Solution:

   If vector $(3, 2, c)$ is in the spanning set of the vectors $(1, 0, 1)$ and $(0, 1, -1)$, then we get $(3, 2, c)$ as linear combination of the other two vectors.

$$x(1, 0, 1) + y(0, 1, -1) = (3, 2, c)$$

$$\implies x = 3, y = 2$$

$$\implies x - y = c \implies c = 1$$

7. Consider the set of three vectors $S = \{(c, -1, -2), (1, 0, -1), (-1, -3, c)\}$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. For which value of c, the above set S will be linearly dependent? [Answer: 2.5]

Solution:

If set $S = \{(c, -1, -2), (1, 0, -1), (-1, -3, c)\}$ is linearly dependent, then the determinant of the matrix created by S will be 0.

$$S = \begin{bmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & c \end{bmatrix}$$

$$det(S) = c(0-3) - 1(-c-6) - 1(1-0) = 0$$

$$-3c + c + 6 - 1 = 0 \implies c = 2.5$$

4 Comprehension Type Question:

In genetics, a classic example of dominance is the inheritance of shape of seeds in peas. Peas may be round (associated with genotype R) or wrinkled (associated with genotype r). In this case, three combinations of genotypes are possible: RR, rr, and Rr. The RR individuals have round peas and the rr individuals have wrinkled peas. In Rr individuals the R genotype masks the presence of the r genotype, so these individuals also have round peas. Thus, the genotype R is completely dominant to genotype r, and genotype r is recessive to genotype R. First, assume the crossing of RR with RR. This always gives the genotype RR, therefore the probabilities of an offspring to be RR, Rr, and rr respectively are equal to 1, 0, and 0. Second, assume crossing of Rr with RR. The offspring will have equal chances to be of genotype RR and genotype Rr, therefore the probabilities of RR, Rr, and rr repectively are 1/2, 1/2, and 0. Third, consider crossing of rr with RR. This always results in genotype Rr. Therefore, the probabilities of genotypes RR, Rr, and rr are 0, 1, and 0, respectively.
This can be veiwed as the following table:

Parents' genotypes			Genotypes of offspring
RR-RR	RR-Rr	RR-rr	
1	$\frac{1}{2}$	0	RR
0	$\frac{1}{2}$	1	Rr
0	0	0	rr

Table: M2W3G3

The matrix representing this observation is given by $P = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and the initial distribution vector (3×1 matrix) is denoted by $X_0 = \begin{bmatrix} X_0^1 \\ X_0^2 \\ X_0^3 \end{bmatrix}$, where X_0^1 denotes the distribution of RR, X_0^2 denotes the distribution of Rr, and X_0^3 denotes the distribution of rr. For any positive integer n, the distribution vector after n generations (i.e., at $t = n$) is denoted by X_n and given by the equation $PX_{n-1} = X_n$.
Using the above information, answer the following questions.

8. Suppose, in an experiment, 100 pairs of parents with genotype combinations RR-RR, 100 pairs of parents with genotype combinations RR-Rr, and 200 pairs of parents with genotype combination RR-rr are taken to observe the genotypes of their offspring. Suppose from crossing of each pair of parents a single offspring is produced. Find the set of correct options from the following. (MSQ)

   ◯ Option 1: There will be at least 200 offspring with wrinkled peas.

   ◯ **Option 2:** There will be no offspring with wrinkled peas.

◯ Option 3: There will be no offspring with round peas.

◯ **Option 4:** All the offspring will have round peas.

◯ Option 5: All the offspring will have wrinkled peas.

◯ **Option 6:** There will be at least 100 offspring with combination of genotypes RR.

◯ **Option 7:** There will be at least 200 offspring with combination of genotypes Rr.

Solution:

The offspring result would be

$$100(1, 0, 0) + 100(1/2, 1/2, 0) + 200(0, 1, 0) = (150, 250, 0)$$

Which means $RR = 150(round-shaped), Rr = 250(round-shaped), rr = 0(wrinkled)$
Using this information, therefore, option 4 is correct.

As it is given that the probability of RR-RR Parents' genotypes is 1 if Genotypes of offspring is RR. In the experiment 100 pairs of parents with genotype combinations RR-RR are used therefore, there will be at least $100 \times 1 = 100$ offspring with combination of genotypes RR.

Similarly, probabilty of RR-rr with Rr is 1. In the experiment 200 pairs of parents with genotype combinations RR-Rr are used therefore, there will be at least $200 \times 1 = 200$ offspring with combination of genotypes Rr.

9. Suppose the vector space X spanned by the column vectors of P (i.e., the probability vectors) with the usual addition and scalar multiplication, is known as the probability space. Which of the following options are correct? (MSQ)

- ○ Option 1: $\{(\frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0), (0, 0, \frac{1}{2})$ is a linearly independent set in X.
- ○ **Option 2:** $\{(\frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0)$ is a linearly independent set in X.
- ○ **Option 3:** The vector space X can be expressed as the set $\{(a, b, 0) | a, b \in \mathbb{R}\}$ with the usual addition and scalar multiplication.
- ○ Option 4: The vector space X can be expressed as the set $\{(a, 0, b) | a, b \in \mathbb{R}\}$ with the usual addition and scalar multiplication.
- ○ Option 5: The vector space X can be expressed as the set $\{(a, b, c) | a, b, c \in \mathbb{R}\}$ with the usual addition and scalar multiplication.

Solution:

Given vector space X is spanned by the column vectors P. Therefore, a vector X_s in X for $x, y, z \in \mathbb{R}$ would be

$$X_s = x(1, 0, 0) + y(1/2, 1/2, 0) + z(0, 1, 0)$$

$$X_s = (x + \frac{y}{2}, \frac{y}{2} + z, 0)$$

Here the third element of vector should always be zero.Therefore, (0, 0, 1/2) is not in X as X. So, option 1 is incorrect.

Let us take $x + \frac{y}{2} = a$, $\frac{y}{2} + z = b$, then $X_s = (a, b, 0)$. So, option 3 is correct.

Take $a = 1/2$ and $b = 1/2$ then $X_s = (1/2, 1/2, 0)$. Therefore, option 2 is correct.

10. Supoose $X_0 = \begin{bmatrix} \frac{1}{3} \\ \\ \frac{1}{3} \\ \\ \frac{1}{3} \end{bmatrix}$. Find out the correct set of correct options. (MSQ)

◯ Option 1: X_0 and X_1 are linearly dependent.

◯ **Option 2:** X_0 and X_1 are linearly independent.

◯ Option 3: The set $\{X_0, X_1, X_2\}$ is a linearly dependent set.

◯ **Option 4:** The set $\{X_0, X_1, X_2\}$ is a linearly independent set.

Solution:

Given,

$$PX_{n-1} = X_n$$

$$\implies X_1 = PX_0$$

$$X_1 = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\implies X_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

Clearly X_0 ans X_1 are linearly independent.

Now,

$$\implies X_2 = PX_1$$

$$X_1 = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\implies X_2 = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

Now the set $X = \{X_0, X_1, X_2\} = \{(1/3, 1/3, 1/3), (1/2, 1/2, 0), (3/4, 1/4, 0)\}$. Let matrix A be the matrix made by the elements of X as column vectors then,

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

$$det(A) = \frac{1}{3}(0) + \frac{1}{3}(0) + \frac{1}{3}(\frac{1}{8} - \frac{3}{8}) \neq 0$$

Which means columns of A are linearly independent. Therefore, set X is a linearly independent set.