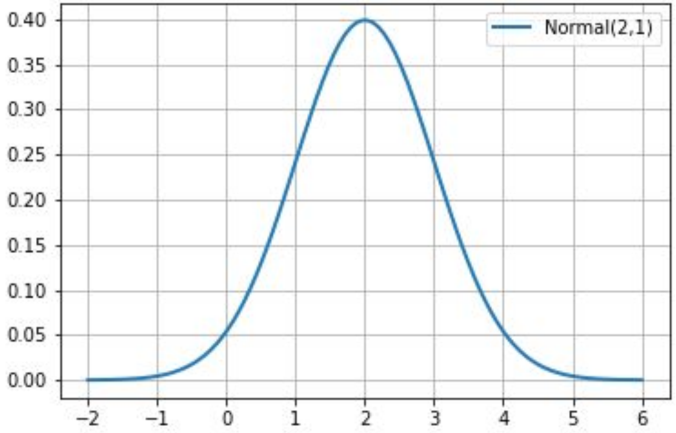
Statistics 2 Live Session

Jul 5, 2021

Normal distribution

PDF: Mean μ and variance σ^2

$$\frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma^2}), -\infty < x < \infty$$



For Z ~ Normal(0,1), CDF denoted $F_Z(x)$

X ~ Normal(μ , σ^2), CDF is $F_Z((x-\mu)/\sigma)$

$$P(X < x) = F_Z(\frac{x-\mu}{\sigma}) \qquad P(X > x) = 1 - F_Z(\frac{x-\mu}{\sigma})$$

$$P(a < X < b) = F_Z(\frac{b-\mu}{\sigma}) - F_Z(\frac{a-\mu}{\sigma})$$

Q1 (a) Normal

Suppose X is normally distributed with mean 10 and variance 25. What is P(X<5)? Express your answer in terms of F_Z, the CDF of the standard normal distribution with mean 0 and variance 1.

Q1 (b) Normal

Suppose X is normally distributed with mean 10 and variance 25. What is P(X>15)? Express your answer in terms of F_Z, the CDF of the standard normal distribution with mean 0 and variance 1.

Q1 (c) Normal

Suppose X is normally distributed with mean 10 and variance 25. What is $P(8<X<12)$? Express your answer in terms of F_Z, the CDF of the standard normal distribution with mean 0 and variance 1.

Q2 (a) Symmetry in Normal distribution

Suppose X is normally distributed with mean 0. Suppose $P(X < -5) = 0.1$. What is $P(X > 5)$?

Q2 (b) Symmetry in Normal distribution

Suppose X is normally distributed with mean 0. Suppose $P(X < -5) = 0.1$. What is $P(-5 < X < 5)$?

Q2 (c) Symmetry in Normal distribution

Suppose X is normally distributed with mean 0. Suppose $P(-a < X < a) = 0.95$. What is $P(X < -a)$?

Q3 (a) Normal inverse CDF

Suppose X is normally distributed with mean 10 and variance 25. Find ‘a’ such that $P(X < a) = 0.05$. Express your answer in terms of F_Z^{-1} the inverse CDF of the standard normal with mean 0 and variance 1.

Q3 (b) Normal inverse CDF

Suppose X is normally distributed with mean 10 and variance 25. Find ‘a’ such that $P(X > a) = 0.025$. Express your answer in terms of F_Z^{-1} the inverse CDF of the standard normal with mean 0 and variance 1.

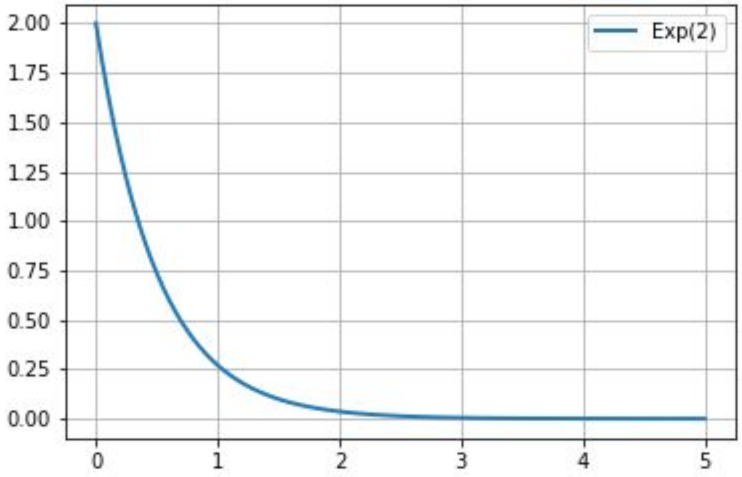
Q3 (b) Normal inverse CDF

Suppose X is normally distributed with mean 10 and variance 25. Find ‘a’ such that $P(|X - 10| < a) = 0.99$. Express your answer in terms of F_Z^{-1} the inverse CDF of the standard normal with mean 0 and variance 1.

Exponential distribution

PDF: Mean $1/\lambda$ and variance $1/\lambda^2$

$$\lambda \exp(-\lambda x), x > 0$$



CDF: $1 - \exp(-\lambda x), x > 0$

$$P(X < x) = 1 - \exp(-\lambda x) \qquad P(X > x) = \exp(-\lambda x)$$

$$P(a < X < b) = \exp(-\lambda a) - \exp(-\lambda b)$$

Q4 (a) Exponential

Suppose X is exponentially distributed with lambda = 2. Find $P(X > 5 \mid 2 < X < 8)$.

Q4 (b) Exponential

Suppose X is exponentially distributed with lambda = 2. Find P(|X-1/2| > 1/4).

Functions of a random variable

- Main things to remember
    - Function of a random variable is another random variable
        - Distribution of a function of a random variable will be different from the original distribution
    - For any function, how to compute $P(f(X) < a)$?
        - Invert the function: Find $S = \{x: f(x) < a\}$
        - Find P(X belongs to S)
        - CDF of $f(X)$: $P(F(X) < x)$, PDF: derivative of CDF
            - If function is monotonic, there is a simple formula
- Why functions of a random variable?
    - Data science is about finding functional relationships between factors
    - Observed value of a factor = True value + Random noise = random variable
    - f(observed value) = f(random variable)

Q5 (a) Probability of functions of random variables

Suppose X is uniformly distributed in [0, 5]. Find P(3X + 7 > 19).

Q5 (b) Probability of functions of random variables

Suppose X is exponentially distributed with lambda = 2. Find $P(X^3 > 27)$.

Q5 (c) Probability of functions of a random variable

Suppose X is normally distributed with mean 5 and variance 1. Find $P(X^2 < 3)$.

Q6 Quadratic function of normal*

Suppose X is normally distributed with mean 0 and variance 2. Find the following probability: $P(X^2 - 5X + 4 > 0)$

Q7 (a) CDF of functions of random variables

Suppose X is uniformly distributed in [0, 5]. Find CDF of 3X + 7.

Bonus: Find the PDF.

Q7 (b) CDF of functions of random variables

Suppose X is exponentially distributed with lambda = 2. Find CDF of X^3.

Bonus: Find the PDF.

Q7 (c) CDF of functions of a random variable

Suppose X is normally distributed with mean 5 and variance 1. Find CDF of X^2.
Bonus: Find the PDF.

Q8 (a) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{0 < x,y < 10\}$. Sketch the region. What is the value of $f_{XY}(x,y)$ within the region of support?

Q8 (b) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{0 < x,y < 10\}$. What is the range of values taken by X? What is the value of $f_X(5)$?

Q8 (c) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{0 < x,y < 10\}$. Given $X = 5$, what is the range of values taken by Y? What is $f_{Y|X=5}(5)$?

Q8 (d) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{0 < x,y < 10\}$. Are X and Y independent?

Q8 (e) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{0 < x,y < 10\}$. What is $E[X]$? What is $E[XY]$? What is $E[X|Y=5]$?

Q9 (a) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{-2 < x,y < 0\} \cup \{0 < x,y < 1\}$. Sketch the region. What is the value of $f_{XY}(x,y)$ within the region of support?

Q9 (b) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{-2 < x,y < 0\} \cup \{0 < x,y < 1\}$. What is the range of values taken by X? What is $f_X(-1)$? What is $f_X(0.5)$?

Q9 (c) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{-2 < x,y < 0\} \cup \{0 < x,y < 1\}$. Given X = -1, what is the range of values taken by Y? What is $f_{Y|X=-1}(-1)$? What is $f_{Y|X=-1}(0.5)$?

Q9 (d) Two random variables: uniform distribution

Suppose X and Y are jointly uniform in the region $\{-2 < x,y < 0\} \cup \{0 < x,y < 1\}$. Are X and Y independent?

Q9 (e) Two random variables: uniform distribution*

Suppose X and Y are jointly uniform in the region $\{-2 < x,y < 0\}$ U $\{0 < x,y < 1\}$. What is $E[X]$? What is $E[X|Y=-1]$? What is $E[X|Y=0.5]$?

Q10 (a) Two random variables

Suppose X and Y have the following joint PMF:

$$f_{XY}(x,y) = k\,xy,\ \ 0 < x,y < 2, \text{ and } f_{XY}(x,y) = 0, \text{ otherwise.}$$

What is the value of k?

Q10 (b) Two random variables

Suppose X and Y have the following joint PMF:

$$f_{XY}(x,y) = k\,xy,\quad 0 < x,y < 2, \text{ and } f_{XY}(x,y) = 0, \text{ otherwise.}$$

What is the range of values taken by X? What is $f_X(1)$?

Q10 (c) Two random variables

Suppose X and Y have the following joint PMF:

$$f_{XY}(x,y) = k\,xy,\ \ 0 < x,y < 2, \text{ and } f_{XY}(x,y) = 0, \text{ otherwise.}$$

Given X = 1, what is the range of values taken by Y? What is $f_{Y|X=1}(1)$?

Q10 (d) Two random variables

Suppose X and Y have the following joint PMF:

$$f_{XY}(x,y) = k\,xy,\ \ 0 < x,y < 2, \text{ and } f_{XY}(x,y) = 0, \text{ otherwise.}$$

Are X and Y independent?

Q10 (e) Two random variables*

Suppose X and Y have the following joint PMF:

$$f_{XY}(x,y) = k\,xy,\ \ 0 < x,y < 2, \text{ and } f_{XY}(x,y) = 0, \text{ otherwise.}$$

What is E[X]? What is E[XY]? What is E[X|Y=1]?