The null space of a matrix
finding nullity and a basis for the null space

Sarang S. Sane

Contents

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Recall : We can find the dimension and a basis for a vector space spanned by a set of vectors using Gaussian elimination.

Solution space of a homogeneous system of linear equations

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$x, y \in W \Rightarrow Ax = Ay = 0 \Rightarrow A(x+y) = Ax + Ay = 0 + 0 = 0.$

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$\lambda \in \mathbb{R}, x \in W \Rightarrow A(\lambda x) = \lambda(Ax) = \lambda 0 = 0.$

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The vectors obtained by substituting $t_i = 1$ and $t_j = 0 \forall j \neq i$ as i varies constitutes a basis of the null space of A (i.e. the solution space of $Ax = 0$).

Example : 3×3 matrix

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Hence, a basis for the null space is $(-1, 1, 0), (-1, 0, 1)$.

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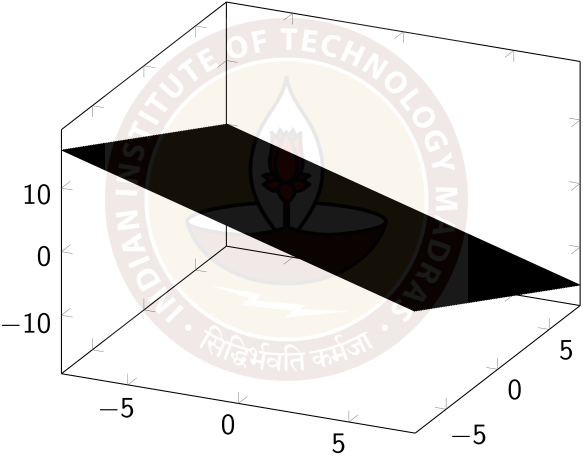

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So we will drop the 0 column augmented to the matrix while performing the row reduction computations and use it only for solving for the dependent variables.

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and hence we obtain that $x_1 = 3t, x_2 = -3t, x_3 = -2t$.

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Hence, a basis for the null space of $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ is

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Note that for a matrix R in the row echelon form, the number of non-zero rows = number of dependent variables for the corresponding homogeneous system $Rx = 0$.

The rank-nullity theorem (contd.)

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Theorem

For an $m \times n$ matrix A, $rank(A) + nullity(A) = n$.

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The standard basis $(1, 0), (0, 1)$ yields the identity matrix I with determinant 1.
The vectors $(1, -2), (5, -10)$ yields the matrix $\begin{bmatrix} 1 & 5 \\ -2 & -10 \end{bmatrix}$ with determinant 0. This is not a basis for $\mathbb{R}^2$.

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$det(A) = 1 \times (-6) - 0 \times (-9) + 1 \times (4-3) = -6+0+1 = -5 \neq 0.$

Example in $\mathbb{R}^3$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \shortparallel & \shortparallel & \shortparallel \end{array}$$

Is the set $\{(1,2,3),(0,1,2),(1,3,0)\}$ a basis for $\mathbb{R}^3$?

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Hence the given set of vectors forms a basis of $\mathbb{R}^3$.

Let $b \in \mathbb{R}^3$. Need: $a_1, a_2, a_3 \in \mathbb{R}$ s.t. $\sum a_i x_i = b$.

$A \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = b$. Unique soln. is $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A^{-1} b$.

Example in $\mathbb{R}^4$

Is $\{(1,2,3,0),(0,1,2,1),(1,3,0,2),(2,6,5,3)\}$ a basis for $\mathbb{R}^4$?

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$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 6 \\ 3 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

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$$det(A) = 1 \times det \begin{bmatrix} 1 & 3 & 6 \\ 2 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} - 0 + 1 \times det \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix} - 2 \times det \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Example in $\mathbb{R}^4$

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$$= 11 + 11 - 2 \times 11 = 0.$$

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$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

& solve.

$$det(A) = 1 \times det \begin{bmatrix} 1 & 3 & 6 \\ 2 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} - 0 + 1 \times det \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix} - 2 \times det \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= 11 + 11 - 2 \times 11 = 0.$$

Hence the given set of vectors does not form a basis of $\mathbb{R}^4$.

Thank you