

# Directional derivatives

Sarang S. Sane



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**scalar-valued multivariable function.**



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If  $\tilde{a}$  is a point in  $\mathbb{R}^n$ , then an open ball of radius  $r$  around  $\tilde{a}$  is the set defined as

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$e_1, e_2, \dots, e_n$  is the standard ordered basis of  $\mathbb{R}^n$ .

# Recall : partial derivatives



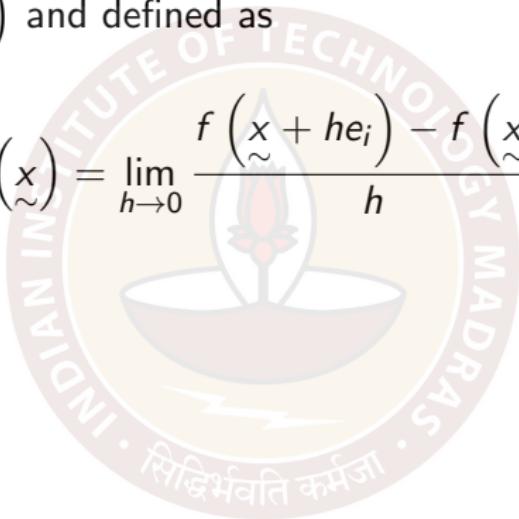
## Recall : partial derivatives

Let  $f(x_1, x_2, \dots, x_n)$  be a function defined on a domain  $D$  in  $\mathbb{R}^n$ .

The **partial derivative of  $f$  w.r.t.  $x_i$**  is the function denoted by

$f_{x_i}(x) \underset{\sim}{\sim}$  or  $\frac{\partial f}{\partial x_i}(x) \underset{\sim}{\sim}$  and defined as

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{f(x + h\mathbf{e}_i) - f(x)}{h}.$$



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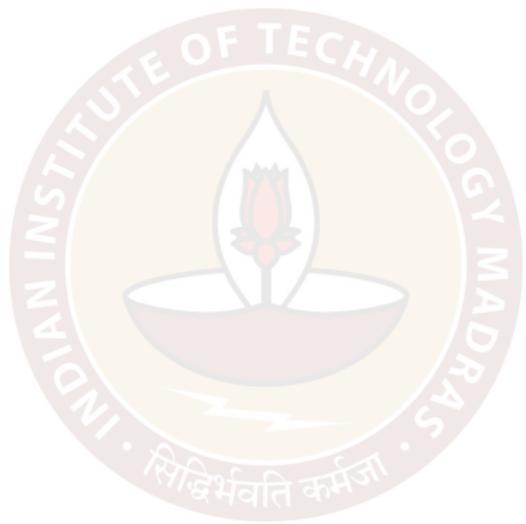
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The function  $f_{x_i}$  computes the rate of change of the function  $f$  w.r.t. the variable  $x_i$ . Another way of thinking about this is that  $f_{x_i}$  computes the rate of change of the function  $f$  in the direction of the unit vector  $e_i$  or in the direction of the  $x_i$ -axis.

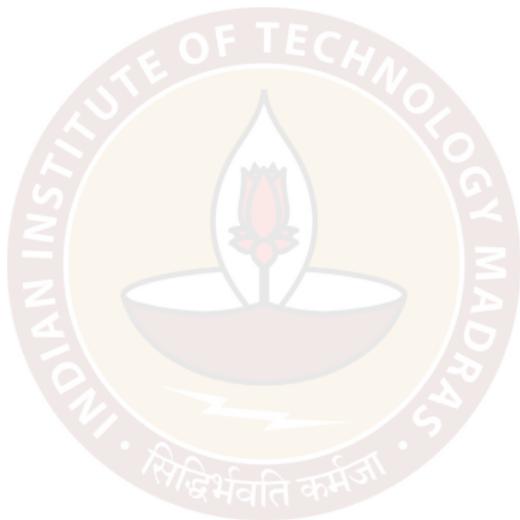
$$\lim_{h \rightarrow 0} \frac{f(x_1, x_2, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

Rate of change in a particular direction at a point



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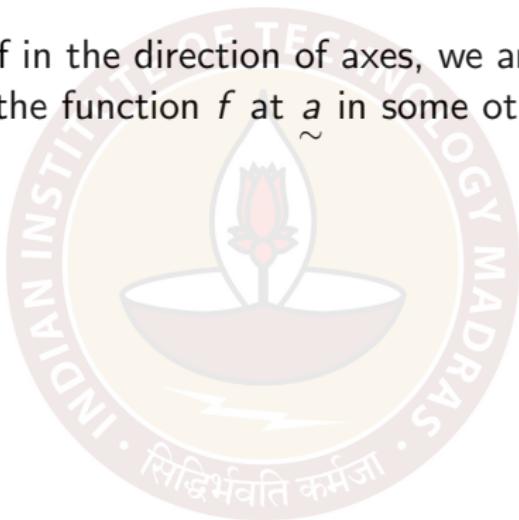
Let  $f(x_1, x_2, \dots, x_n)$  be a function defined on a domain  $D$  in  $\mathbb{R}^n$  containing a point  $a$  and an open ball around it.



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We can use the same idea as for partial derivatives and choose a unit vector  $u \in \mathbb{R}^n$  in the direction we want and compute :

$$\lim_{h \rightarrow 0} \frac{f(\underset{\sim}{a} + hu) - f(\underset{\sim}{a})}{h}$$

$$\underset{\sim}{a} = (a_1, \dots, a_n)$$

$$\lim_{h \rightarrow 0}$$

$$u = (u_1, \dots, u_n)$$

$$\frac{f(a_1 + hu_1, a_2 + hu_2, \dots, a_n + hu_n) - f(a_1, \dots, a_n)}{h}$$

$$g(h) = f(a_1 + hu_1, \dots, a_n + hu_n)$$
$$\lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h}$$

## Examples

The rate of change of  $f(x, y) = x + y$  at  $(0, 0)$  in the direction of the  $y = x$  line.  $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\lim_{h \rightarrow 0} \frac{f(0+h\frac{1}{\sqrt{2}}, 0+h\frac{1}{\sqrt{2}}) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h}{\sqrt{2}} + \frac{h}{\sqrt{2}} - (0+0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2}h}{h} = \sqrt{2}.$$

The rate of change of  $f(x, y, z) = xy + yz + zx$  at  $(1, 2, 3)$  in the direction of the vector  $(4, 3, 0)$ .  $v = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$

$$\lim_{h \rightarrow 0} \frac{f(1+h\frac{4}{\sqrt{17}}, 2+h\frac{3}{\sqrt{17}}, 3+h\cdot 0) - f(1,2,3)}{h} = \lim_{h \rightarrow 0} \frac{\frac{8}{\sqrt{17}} + \frac{3}{\sqrt{17}} + \frac{9}{\sqrt{17}} + \frac{12}{\sqrt{17}}}{h} = \frac{32}{5}.$$

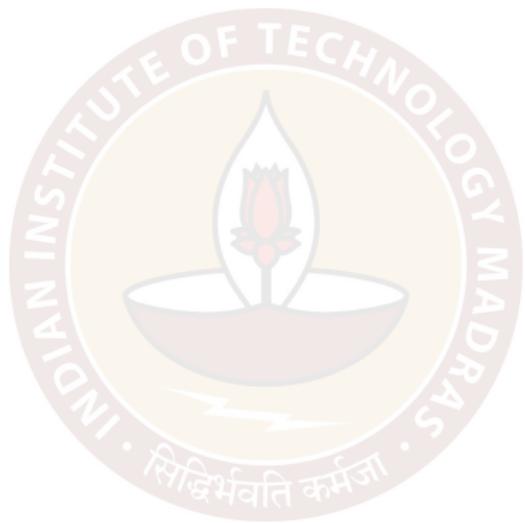
$\sqrt{4^2 + 3^2 + 0^2} = \sqrt{25} = 5.$

The rate of change of  $f(x, y) = \sin(xy)$  at  $(1, 0)$  in the direction  $60^\circ$  (from the  $X$ -axis).  $u = (\cos 60, \sin 60) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$$\lim_{h \rightarrow 0} \frac{f(1+h\frac{1}{2}, 0+\frac{\sqrt{3}}{2}h) - f(1,0)}{h} = \lim_{h \rightarrow 0} \frac{\sin((1+h\frac{1}{2})\frac{\sqrt{3}}{2}h) - \sin(1,0)}{h} = g'(0) = \cos(0) \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

$g'(h) = \cos(\ ) \times \left\{ \frac{\sqrt{3}}{2}h \times \frac{1}{2} + \left( \frac{1}{2} \right)^2 \right\}$

# Directional derivatives



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Let  $f(x_1, x_2, \dots, x_n)$  be a function defined on a domain  $D$  in  $\mathbb{R}^n$ .

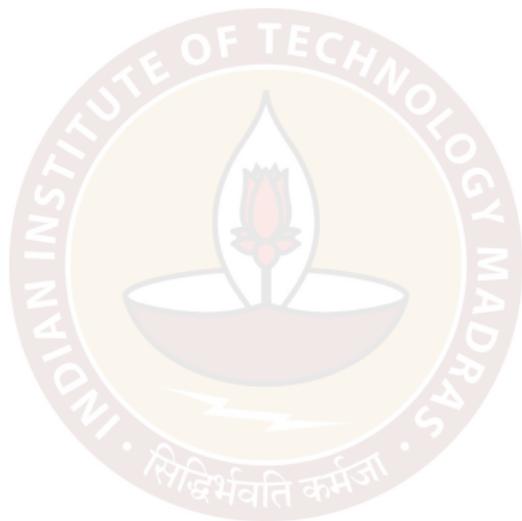
The **directional derivative of  $f$  in the direction of the unit vector  $u$**  is the function denoted by  $f_u(\tilde{x})$  and defined as

$$f_u(\tilde{x}) = \lim_{h \rightarrow 0} \frac{f(\tilde{x} + hu) - f(\tilde{x})}{h}.$$

Its domain consists of those points of  $D$  at which the limits exists.

$$f_{e_i}(x) = f_{x_i}(\tilde{x}) = \frac{\partial f}{\partial x_i}(x).$$

# Properties : Linearity and products



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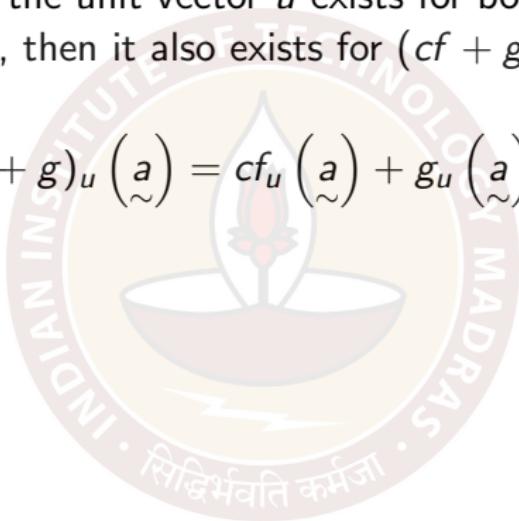
## Linearity :



## Properties : Linearity and products

**Linearity :** Let  $c \in \mathbb{R}$ . If the directional derivative at the point  $\tilde{a}$  in the direction of the unit vector  $u$  exists for both the functions  $f(\tilde{x})$  and  $g(\tilde{x})$ , then it also exists for  $(cf + g)(\tilde{x})$  and

$$(cf + g)_u(\tilde{a}) = cf_u(\tilde{a}) + g_u(\tilde{a}) \quad .$$



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### The product rule

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$$(fg)_u(\tilde{a}) = f_u(\tilde{a})g(\tilde{a}) + f(\tilde{a})g_u(\tilde{a}) \quad .$$

# Properties (contd.) : Quotients

## The quotient rule



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### The quotient rule

If the directional derivative at the point  $\tilde{a}$  in the direction of the unit vector  $u$  exists for both the functions  $f(\tilde{x})$  and  $g(\tilde{x})$ , and  $g(\tilde{a}) \neq 0$ , then it also exists for  $\frac{f}{g}(\tilde{x})$  and

$$(f/g)_u(\tilde{a}) = \frac{f_u(\tilde{a})g(\tilde{a}) - f(\tilde{a})g_u(\tilde{a})}{g(\tilde{a})^2} .$$

## Examples

$$\blacktriangleright f(x, y) = x + y$$

$u = (u_1, u_2)$

$$f_u(x, y) = \lim_{h \rightarrow 0} \frac{f(x+hu_1, y+hu_2) - f(x, y)}{h} = \lim_{h \rightarrow 0} \frac{x+hu_1 + y + hu_2 - (x+y)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{hu_1 + hu_2}{h} = u_1 + u_2 = u_1 \times 1 + u_2 \times 1.$$

$$\blacktriangleright f(x, y, z) = xy + yz + zx$$

$u = (u_1, u_2, u_3)$

$$f_u(x, y, z) = \lim_{h \rightarrow 0} \frac{(x+hu_1)(y+hu_2) + (y+hu_2)(z+hu_3) + (z+hu_3)(x+hu_1) - (xy + yz + zx)}{h^2(u_1u_2 + u_2u_3 + u_3u_1)}$$

$$= \lim_{h \rightarrow 0} \frac{h(u_1y + xu_2 + u_3y + zu_2 + u_1z + xu_3 + u_2z + u_3x)}{h^2(u_1u_2 + u_2u_3 + u_3u_1)} = \frac{u_1y + u_2x + u_3y + u_2z + u_1z + u_3x}{h^2(u_1u_2 + u_2u_3 + u_3u_1)}.$$

$$\blacktriangleright f(x, y, z) = \sin(xy)$$

$$\lim_{h \rightarrow 0} \frac{\sin((x+hu_1)(y+hu_2)) - \sin(xy)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(xy)\cos(xhu_2 + yhu_1 + h^2u_1u_2) + \cos(xy)\cdot \sin(\underline{\hspace{2cm}}) - \sin(xy)}{h}$$

## Another example :

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Directional derivative at  $(0, 0)$  in the direction of  
the unit vector  $u$ .

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(0+hu_1, 0+hu_2) - f(0, 0)}{h} &= \lim_{h \rightarrow 0} \frac{f(hu_1, hu_2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{hu_1 hu_2}{h(u_1^2 + u_2^2)} = \lim_{h \rightarrow 0} \frac{u_1 u_2}{h(u_1^2 + u_2^2)} \\ &= \lim_{h \rightarrow 0} \frac{u_1 u_2}{h(u_1^2 + u_2^2)} = \lim_{h \rightarrow 0} \frac{u_1 u_2}{h} \quad DNE \\ &= h \rightarrow 0 \end{aligned}$$

assuming both  
 $u_1, u_2$  are not 0.

# Thank you

