Statistics for Data Science - 2

Week 10 practice Assignment

Hypothesis testing

1. Consider nine samples from Normal$(100, 2^2)$. Let we wish to test $H_0 : \mu = 100$ against $H_A : \mu \neq 100$.

   (i) If the acceptance region is defined as $98.5 \leq \overline{X} \leq 101.5$, find the significance level. Write your answer correct to two decimal places.
   (Use $P(-2.25 < Z < 2.25) = 0.975$)

   Solution:
   Given that

   $$H_0 : \mu = 100, \qquad H_A : \mu \neq 100$$

   The acceptance region is defined as $98.5 \leq \overline{X} \leq 101.5$.
   Now,

   $$\begin{aligned}
   \alpha &= P(\text{reject } H_0 | H_0 \text{ is true}) \\
   &= P((\overline{X} > 101.5 \text{ or } \overline{X} < 98.5) | \mu = 100) \\
   &= P(|\overline{X} - 100| > 1.5) \\
   &= P\left(\left|\frac{\overline{X} - 100}{2/3}\right| > \frac{1.5}{2/3}\right) \\
   &= P(|Z| > 2.25) \\
   &= 1 - P(-2.25 < Z < 2.25) \\
   &= 1 - 0.975 = 0.02
   \end{aligned}$$

   (ii) Find the power of the test against an alternative that the mean is 103. Write your answer correct to two decimal places.
   (Use $P(-6.75 < Z < -2.25) = 0.012$)

   Solution:

$$\begin{aligned}
1-\beta &= P(\text{reject } H_0|H_A \text{ is true})\\
&= P((\overline{X} > 101.5 \text{ or } \overline{X} < 98.5)|\mu = 103)\\
&= P(\overline{X} < 98.5) + P(\overline{X} > 101.5)\\
&= P(\overline{X} - 103 < -4.5) + P(\overline{X} - 103 > -1.5)\\
&= P\left(\frac{\overline{X} - 103}{2/3} < \frac{-4.5}{2/3}\right) + P\left(\frac{\overline{X} - 103}{2/3} > \frac{-1.5}{2/3}\right)\\
&= P(Z < -6.75) + P(Z > -2.25)\\
&= 1 - P(-6.75 < Z < -2.25)\\
&= 1 - 0.012 = 0.98
\end{aligned}$$

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2. Air crew escape systems are powered by a solid propellant. The mean burning rate of this propellant must be 50 centimeters per second. We know that the standard deviation of burning rate is $\sigma = 2$ centimeters per second. An engineer suspects that the mean burning rate is greater than 50. The engineer decides to test at a significance level of 0.05 and selects a random sample of $n = 25$ and obtains a sample average burning rate of 51.3 centimeters per second.

   (i) Define null hypothesis and alternative hypothesis.

   (a) $H_0 : \mu = 50,\ H_A : \mu \neq 50$
   (b) $H_0 : \mu = 50,\ H_A : \mu < 50$
   (c) $H_0 : \mu = 50,\ H_A : \mu > 50$
   (d) $H_0 : \overline{X} = 50,\ H_A : \overline{X} > 50$

   Solution:
   Since, the mean burning rate of the propellant must be 50 centimeters per second and engineer suspects that the mean burning rate is greater than 50. Therefore, null and alternative hypothesis will be

$$H_0 : \mu = 50, \quad H_A : \mu > 50$$

   (ii) What is the critical value (c) if the acceptance region is $\overline{X} \leq c$? Write your answer correct to two decimal places.
   (use: $F_Z(1.64) = 0.95$)

   Solution:
   If the significance level of the test is 0.05, then

$$
\begin{aligned}
&P(\text{reject } H_0 | H_0 \text{ is true}) = 0.05\\
\Rightarrow &P(\overline{X} > c | \mu = 50) = 0.05\\
\Rightarrow &P(\overline{X} - 50 > c - 50) = 0.05\\
\Rightarrow &P\left(\frac{\overline{X} - 50}{2/5} > \frac{c - 50}{2/5}\right) = 0.05\\
\Rightarrow &P\left(Z > \frac{c - 50}{2/5}\right) = 0.05\\
\Rightarrow &1 - F_Z(\frac{c - 50}{2/5}) = 0.05\\
\Rightarrow &F_Z(\frac{c - 50}{2/5}) = 0.95\\
\Rightarrow &\frac{c - 50}{2/5} = 1.64\\
\Rightarrow &c = 50 + \frac{2}{5}(1.64)\\
\Rightarrow &c = 50.65
\end{aligned}
$$

(iii) What conclusions should be drawn from the selected sample?

(a) The mean burning rate of the propellant is 50.

(b) The mean burning rate of the propellant is greater than 50.

(c) The mean burning rate of the propellant is lesser than 50.

(d) No conclusion can be drawn from the given sample.

Solution:
Given that $\overline{X} = 51.3$
We will reject H_0 if $\overline{X} > 50.65$ and $\overline{X} = 51.3 > 50.65$, we will reject the null hypothesis.
It implies that the mean burning rate of the propellant is greater than 50.

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3. Suppose a manufacturer of memory chips observes that the probability of chip failure is $p = 0.05$. A new procedure is introduced to improve the design of chips and lower the probability of chip failure. To test this new procedure, 200 chips are produced using this new procedure and tested. We would accept the new procedure if the total number of failed chips is less than 5 out of 200. Find the significance level of the test. Use the normal approximation. Write your answer correct to three decimal places.
(Use $P(Z < -1.62) = 0.052$)
Solution:
A new procedure is introduced to improve the design of chips and lower the probability of chip failure. Therefore, null and alternative hypothesis will be

$$H_0 : p = 0.05, \quad H_A : p < 0.05$$

Define a test statistic T as $T =$ number of failed chips out of 200.

Given that: We would accept the new procedure if the total number of failed chips is less than 5 out of 200.

It implies that we will reject the null hypothesis if $T < 5$.

Notice that $T \sim \text{Binomial}(200, p)$.
When the null hypothesis is true, $E[T] = 200p = 200(0.05) = 10$ and
$\text{Var}(T) = 200p(1-p) = 200(0.05)(0.95) = 9.5$

By CLT, we can say that

$$\frac{T-10}{\sqrt{9.5}} \sim \text{normal}(0, 1)$$

Now, significance level is given by

$$\begin{aligned}
\alpha &= P(\text{reject } H_0 | H_0 \text{ is true}) \\
&= P(T < 5) \\
&= P\left(\frac{T-10}{\sqrt{9.5}} < \frac{5-10}{\sqrt{9.5}}\right) \\
&= P(Z < -1.62) \\
&= F_Z(-1.62) \\
&= 0.052
\end{aligned}$$

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4. The mean lifetime of a sample of 100 light bulbs produced by a company is computed to be 1570 hours with a standard deviation of 120 hours. μ is the mean lifetime of all the bulbs produced by the company,

   (i) Test the hypothesis $\mu = 1600$ against the alternative hypothesis $\mu \neq 1600$ at a level of significance of 0.05.

   (a) Reject the null hypothesis

   (b) Accept the null hypothesis

   Solution:
   Given that

   $$H_0 : \mu = 1600, \quad H_A : \mu \neq 1600$$

   Define a test statistic T as $T = \overline{X}$.
   Test: reject H_0 if $|\overline{X} - 1600| > c$ Notice that when null hypothesis is true, we have

   $$\frac{\overline{X} - 1600}{{}^{120}\!/_{10}} \sim \text{Normal}(0, 1)$$

Now,

$$\begin{aligned}
&\alpha = P(\text{reject } H_0 | H_0 \text{ is true})\\
\Rightarrow &P(|\overline{X} - 1600| > c) = 0.05\\
\Rightarrow &P\left(\left|\frac{\overline{X} - 1600}{120/10}\right| > \frac{c}{120/10}\right) = 0.05\\
\Rightarrow &P\left(|Z| > \frac{c}{12}\right) = 0.05\\
\Rightarrow &2P\left(Z < \frac{-c}{12}\right) = 0.05\\
\Rightarrow &F_Z\left(\frac{-c}{12}\right) = 0.025\\
\Rightarrow &\frac{-c}{12} = -1.96\\
\Rightarrow &c = 12(1.96) = 23.52
\end{aligned}$$

It implies that we will reject the null hypothesis if $|\overline{X} - 1600| > 23.52$

Given that $\overline{X} = 1570$
$\Rightarrow |\overline{X} - 1600| = |1570 - 1600| = 30 > 23.52$
Therefore, we will reject the null hypothesis.

(ii) Find the P-value. Write your answer correct to three decimal places.
Solution:
P-value is the minimum significance level at which null hypothesis is rejected for the observed test statistic value.
Therefore, P-value is given by

$$\begin{aligned}
\alpha &= P(|\overline{X} - 1600| > |1570 - 1600|)\\
&= P(|\overline{X} - 1600| > 30)\\
&= P\left(\left|\frac{\overline{X} - 1600}{120/10}\right| > \frac{30}{120/10}\right)\\
&= P(|Z| > 2.5)\\
&= 2P(Z < -2.5)\\
&= 2(0.0062) = 0.012
\end{aligned}$$

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5. The average IQ of the students of a school is reported to be 107 with a standard deviation of 4. You suspect that the average may be higher, possibly 110, and decide to sample students to find their IQs. What sample size do you need for a test at the significance level 0.05 and power 0.95?
(Use: $F_Z(1.64) = 0.95$ and $F_Z(-1.64) = 0.05$)

Solution:
According to the question, we have

$$H_0 : \mu = 107, \quad H_A : \mu > 107$$

Define a test statistic T as $T = \overline{X}$.

Test: reject H_0 if $\overline{X} > c$.
Notice that when null hypothesis is true, we have

$$\frac{\overline{X} - 107}{4/\sqrt{n}} \sim \text{Normal}(0, 1)$$

Now, the significance level of the test is given to be 0.05. It implies that

$$\begin{aligned}
&P(\text{reject } H_0 | H_0 \text{ is true}) = 0.05 \\
\Rightarrow &P(\overline{X} > c) = 0.05 \\
\Rightarrow &p\left(\frac{\overline{X} - 107}{4/\sqrt{n}} > \frac{c - 107}{4/\sqrt{n}}\right) = 0.05 \\
\Rightarrow &P\left(Z > \frac{c - 107}{4/\sqrt{n}}\right) = 0.05 \\
\Rightarrow &1 - P\left(Z \leq \frac{c - 107}{4/\sqrt{n}}\right) = 0.05 \\
\Rightarrow &P\left(Z \leq \frac{c - 107}{4/\sqrt{n}}\right) = 0.95 \\
\Rightarrow &\frac{c - 107}{4/\sqrt{n}} = 1.64 \\
\Rightarrow &c = 107 + (1.64)\frac{4}{\sqrt{n}} \qquad \text{...(1)}
\end{aligned}$$

Again, when alternative hypothesis is true, we have

$$\frac{\overline{X} - 110}{4/\sqrt{n}} \sim \text{Normal}(0, 1)$$

Now, the power of the test is given to be 0.95. It implies that

$$
\begin{aligned}
1-\beta =& P(\text{reject } H_0|H_A \text{ is true}) = 0.95\\
\Rightarrow & P(\overline{X} > c) = 0.95\\
\Rightarrow & p\left(\frac{\overline{X}-110}{4/\sqrt{n}} > \frac{c-110}{4/\sqrt{n}}\right) = 0.95\\
\Rightarrow & P\left(Z > \frac{c-110}{4/\sqrt{n}}\right) = 0.95\\
\Rightarrow & 1-P\left(Z \leq \frac{c-110}{4/\sqrt{n}}\right) = 0.95\\
\Rightarrow & P\left(Z \leq \frac{c-110}{4/\sqrt{n}}\right) = 0.05\\
\Rightarrow & \frac{c-110}{4/\sqrt{n}} = -1.64\\
\Rightarrow & c = 110-(1.64)\frac{4}{\sqrt{n}} \qquad\qquad ...(2)
\end{aligned}
$$

From equation (1) and (2), we have

$$
\begin{aligned}
& 107+(1.64)\frac{4}{\sqrt{n}} = 110-(1.64)\frac{4}{\sqrt{n}}\\
\Rightarrow & 2(1.64)\frac{4}{\sqrt{n}} = 3\\
\Rightarrow & \sqrt{n} = \frac{2\times 1.64\times 4}{3} = 4.37\\
\Rightarrow & n = 19.12\\
\Rightarrow & n = 20
\end{aligned}
$$

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6. An instructor gives a quiz involving 10 true-false questions. To test the hypothesis that the student is guessing, the following decision rule is decided: (i) If 7 or more are correct, the student is not guessing; (ii) if fewer than 7 are correct, the student is guessing. Find the significance level of the test. Write your answer correct to two decimal places.
(Hint: If student is guessing then, probability of getting a question correct is $p = 0.5$)
Solution:
If a student is guessing the answer then, each question is equally likely to get corrected that is $p = 0.5$ but if student is not guessing the answer then, probability of getting the question correct is more than 0.5 that is $p > 0.5$.
It implies that

$$H_0 : p = 0.5, \quad H_A : p > 0.5$$

Define a test statistic T as $T=$ number of correct answers out of ten.

As per the given information, we will reject the null hypothesis if $T \geq 7$

Notice that if null hypothesis is true then, $T \sim \text{Binomial}(10, 0.5)$.

Now,

$$
\begin{aligned}
\alpha &= P(\text{reject } H_0 | H_0 \text{ is true}) \\
&= P(T \geq 7) \\
&= \sum_{i=7}^{10} {}^{10}C_i (0.5)^{10} \\
&= ({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10})(0.5)^{10} \\
&= (120 + 45 + 10 + 1)(0.00097) \\
&= 0.17
\end{aligned}
$$

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7. A cricket ball production line must produce of balls weights 163 g with a standard deviation of 4 g in order to get top rating. To test the hypothesis of mean weights of the balls to be 163, a sample of 16 balls are considered. If we want 0.01 level of significance, what will be the acceptance region?
(Use $F_Z(2.57) = 0.995$)

   (a) $[162.43, 164.57]$
   (b) $[158.13, 166.57]$
   (c) $[160.43, 165.57]$
   (d) $[162.13, 164.98]$

   Solution:
   Since, a cricket ball production line must produce of balls weights 163 g, null and alternative hypothesis are given by

   $$H_0 : \mu = 163, \quad H_A : \mu \neq 163$$

   Define test statistic T as $T = \overline{X}$.

   Test: reject the null hypothesis if $|\overline{X} - 163| > c$.

   Notice that when null hypothesis is true, $\dfrac{\overline{X} - 163}{4/4} = \overline{X} - 163 \sim \text{Normal}(0, 1)$

   Now, the significance level of the test is given to be 0.01. It implies that

$$
\begin{aligned}
&P(\text{reject } H_0|H_0 \text{ is true}) = 0.01\\
\Rightarrow &P(|\overline{X} - 163| > c) = 0.01\\
\Rightarrow &P(|Z| > c) = 0.01\\
\Rightarrow &2P(Z < -c) = 0.01\\
\Rightarrow &F_Z(-c) = 0.005\\
\Rightarrow &-c = -2.57\\
\Rightarrow &c = 2.57
\end{aligned}
$$

Therefore, acceptance region will be $[163 - 2.57,\ 163 + 2.57] = [160.43, 165.57]$.

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8. A researcher has recently come into contact with a number of left-handed artists and wonders whether artists are more likely to be left-handed than peoples in the general population. She selects a random sample of 150 members of the Artists and asks each whether they are left-handed or not. The sample proportion (who are left-handed) is 0.15. Suppose that 10% of people are left-handed in the general population.

   (i) Does the data provide strong evidence that artists are more likely than the general public to be left-handed if she decides a significance level of 0.05?

   (a) Yes

   (b) No

   Solution:
   10% of people are left-handed in the general population but a researcher wonders whether artists are more likely to be left-handed. So, probability of an artist being left-handed will be more than 0.1. Therefore, null and alternative hypothesis are given by

   $$H_0 : p = 0.1, \quad H_A : p > 0.1$$

   Define a test statistic T as $T = \overline{X} = \dfrac{X_1 + X_2 + \ldots + X_{150}}{150}$, where each $X_i \sim$ Bernoulli(0.1) (If null hypothesis is true).

   Therefore, $E[X] = p = 0.1$ and $\text{Var}(X) = \dfrac{p(p-1)}{n} = \dfrac{(0.1)(0.9)}{150} = \dfrac{0.09}{150}$

   Then, by CLT $\dfrac{\overline{X} - 0.1}{\sqrt{0.09/150}} \sim \text{Normal}(0, 1)$.

   Test: reject H_0 if $\overline{X} > c$.

   Now, the significance level of the test is given to be 0.05. It implies that

$$
\begin{aligned}
&P(\text{reject } H_0|H_0 \text{ is true}) = 0.05\\
\Rightarrow &P(\overline{X} > c) = 0.05\\
\Rightarrow &P\left(\frac{\overline{X}-0.1}{\sqrt{0.09/150}} > \frac{c-0.1}{\sqrt{0.09/150}}\right) = 0.05\\
\Rightarrow &P\left(Z > \frac{c-0.1}{\sqrt{0.09/150}}\right) = 0.05\\
\Rightarrow &1 - P\left(Z \leq \frac{c-0.1}{\sqrt{0.09/150}}\right) = 0.05\\
\Rightarrow &F_Z\left(\frac{c-0.1}{\sqrt{0.09/150}}\right) = 0.95\\
\Rightarrow &c = 0.1 + (1.64)\frac{0.3}{\sqrt{150}}\\
\Rightarrow &c = 0.14
\end{aligned}
$$

Since, $\overline{X} = 0.15 > 0.14$, we will reject the null hypothesis. It implies that artists are more likely than the general public to be left-handed if she decides a significance level of 0.05.

(ii) Find the P-value. Write your answer correct to three decimal places.

Solution:
P-value is the minimum significance level at which null hypothesis is rejected for the observed test statistic value.
Therefore, P-value is given by

$$
\begin{aligned}
\alpha &= P(\overline{X} > 0.15\\
&= P(\overline{X} - 0.1 > 0.15 - 0.1)\\
&= P\left(\frac{\overline{X}-0.1}{\sqrt{0.09/150}} > \frac{0.05}{\sqrt{0.09/150}}\right)\\
&= P(Z > 2.04)\\
&= P(Z < -2.04)\\
&= 0.02
\end{aligned}
$$

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9. A cereal manufacturer tests its equipment weekly to be assured that the correct weight of cereal is in each box. The company wants to test if the weight differs from the expected weight. The weight of each box is expected to be 500g with a standard deviation of 100g. The manufacturer takes a random sample of 100 boxes and finds that the average

weight is 520g. What is the sample's P-value? Write your answer correct to two decimal places.
(Use $F_Z(-2) = 0.022$)

Solution:
The company wants to test if the weight differs from the expected weight and the weight of each box is expected to be 500g. So, null and alternative hypothesis are given by

$$H_0 : \mu = 500, \quad \mu \neq 500$$

Define a test statistic T as $T = \overline{X}$.

Test: reject the null hypothesis if $|\overline{X} - 500| > c$.

By CLT, we can say that $\dfrac{\overline{X} - 500}{100/\sqrt{100}} = \dfrac{\overline{X} - 500}{10} \sim \text{Normal}(0, 1)$.

P-value is the minimum significance level at which null hypothesis is rejected for the observed test statistic value.
Therefore, P-value is given by

$$\begin{aligned}
\alpha &= P(|\overline{X} - 500| > |500 - 520|) \\
&= P(|\overline{X} - 500| > 20) \\
&= P\left(\left|\frac{\overline{X} - 500}{10}\right| > 2\right) \\
&= P(|Z| > 2) \\
&= 2P(Z < -2) \\
&= 2(0.022) = 0.04
\end{aligned}$$

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10. A machine produces iron rods of mean weight 12kg with a standard deviation of 2kg. An engineer suspects that average weight is less than 12kg, probably 10kg. So, he collects the weights of n iron rods. He wants the significance level to be less than 10^{-4} and probability of type two error to be less than 10^{-8}.
(use $F_Z(-3.74) = 10^{-4}$ and $F_Z(5.61) = 1 - 10^{-8}$)

    (i) Find the required sample size.

    Solution:
    According to the question, we have

    $$H_0 : \mu = 12, \quad H_A : \mu < 12$$

    Define a test statistic T as $T = \overline{X}$.

Test: reject H_0 if $\overline{X} < c$.
Notice that when null hypothesis is true, we have

$$\frac{\overline{X} - 12}{2/\sqrt{n}} \sim \text{Normal}(0, 1)$$

Now, the significance level of the test is given to be less than 10^{-4}. It implies that

$$\begin{aligned}
&P(\text{reject } H_0 | H_0 \text{ is true}) \leq 10^{-4} \\
\Rightarrow &P(\overline{X} < c) \leq 10^{-4} \\
\Rightarrow &p\left(\frac{\overline{X} - 12}{2/\sqrt{n}} < \frac{c - 12}{2/\sqrt{n}}\right) \leq 10^{-4} \\
\Rightarrow &P\left(Z < \frac{c - 12}{2/\sqrt{n}}\right) \leq 10^{-4} \\
\Rightarrow &\frac{c - 12}{2/\sqrt{n}} \leq -3.74 \\
\Rightarrow &c \leq 12 - (3.74)\frac{2}{\sqrt{n}} \qquad \text{...(1)}
\end{aligned}$$

Again, when alternative hypothesis is true, we have

$$\frac{\overline{X} - 10}{2/\sqrt{n}} \sim \text{Normal}(0, 1)$$

Now, probability of type two error to be less than 10^{-8}. It implies that

$$\begin{aligned}
\beta = &P(\text{accept } H_0 | H_A \text{ is true}) \leq 10^{-8} \\
\Rightarrow &P(\overline{X} \geq c) \leq 10^{-8} \\
\Rightarrow &p\left(\frac{\overline{X} - 10}{2/\sqrt{n}} \geq \frac{c - 10}{2/\sqrt{n}}\right) \leq 10^{-8} \\
\Rightarrow &P\left(Z \geq \frac{c - 10}{2/\sqrt{n}}\right) \leq 10^{-8} \\
\Rightarrow &1 - P\left(Z < \frac{c - 10}{2/\sqrt{n}}\right) \leq 10^{-8} \\
\Rightarrow &P\left(Z < \frac{c - 10}{2/\sqrt{n}}\right) \geq 1 - 10^{-8} \\
\Rightarrow &\frac{c - 10}{2/\sqrt{n}} \geq 5.61 \\
\Rightarrow &c \geq 10 + (5.61)\frac{2}{\sqrt{n}} \qquad \text{...(2)}
\end{aligned}$$

From equation (1) and (2), we have

$$
\begin{aligned}
12 - (3.74)\frac{2}{\sqrt{n}} &= 10 + (5.61)\frac{2}{\sqrt{n}} \\
\Rightarrow &(5.61 + 3.74)\frac{2}{\sqrt{n}} = 2 \\
\Rightarrow &\sqrt{n} = 9.35 \\
\Rightarrow &n = 87.42 \\
\Rightarrow &n = 88
\end{aligned}
$$

(ii) Find the critical value (for the acceptance region to be defined as $\overline{X} \geq c$, where $\overline{X}$ is the mean weight of the rods). Write your answer correct to two decimal places.

Solution:
Putting the value of n in the equation (1), we have

$$
\begin{aligned}
&c \leq 12 - (3.74)\frac{2}{\sqrt{88}} \\
\Rightarrow &c \leq 11.20 \qquad \text{...(3)}
\end{aligned}
$$

Putting the value of n in the equation (2), we have

$$
\begin{aligned}
&c \geq 10 + (5.61)\frac{2}{\sqrt{88}} \\
\Rightarrow &c \geq 11.19 \qquad \text{...(4)}
\end{aligned}
$$

From the equation (3) and (4), we have $c = 11.19$
