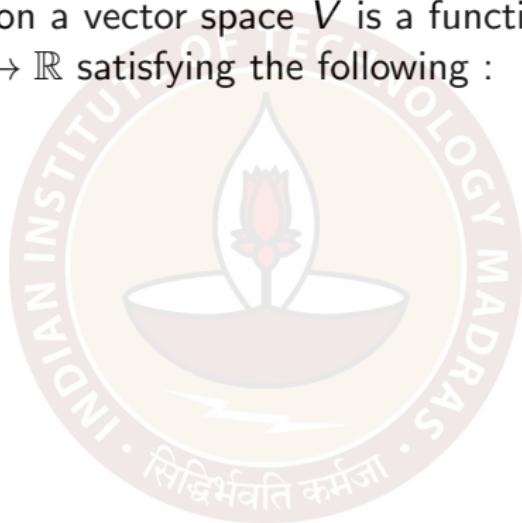


# Inner products and norms on a vector space

Sarang S. Sane

# Inner product on a vector space

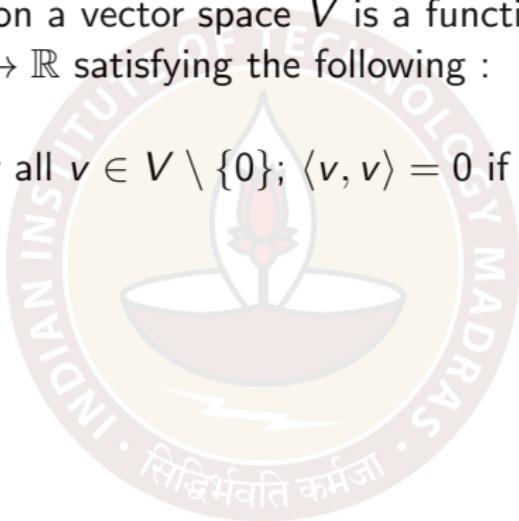
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- ▶  $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$  ]  $\rightarrow \langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$
- ▶  $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
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- ▶  $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$
- ▶  $\langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle = \langle v_1, cv_2 \rangle$ .       $c \in \mathbb{R}$ .

A vector space  $V$  together with an inner product  $\langle \cdot, \cdot \rangle$  is called an inner product space.

# The dot product is an example of an inner product

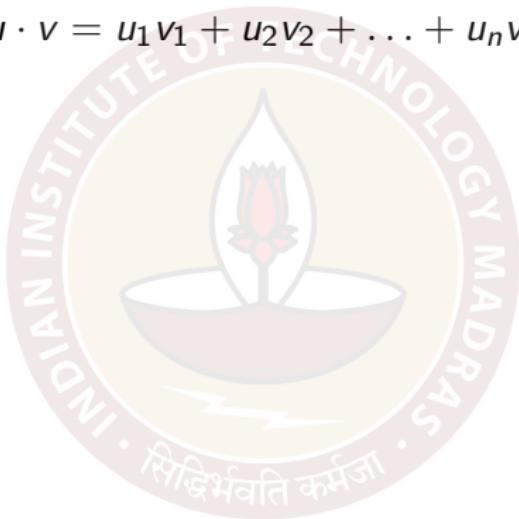
Recall that the dot product of  $u = (u_1, u_2, \dots, u_n)$  and  $v = (v_1, v_2, \dots, v_n)$  be in  $\mathbb{R}^n$  is



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⌚ This yields a function

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R} ; \quad \langle u, v \rangle = u \cdot v.$$

$$\langle u, u \rangle > 0 \text{ if } u \neq 0$$

$$\langle u, u \rangle = 0 \Leftrightarrow u_i = 0 \forall i \Leftrightarrow u = 0 .$$

$$(u+u') \cdot v = u \cdot v + u' \cdot v , \quad (cu) \cdot v = c(u \cdot v)$$
$$u \cdot v = v \cdot u$$

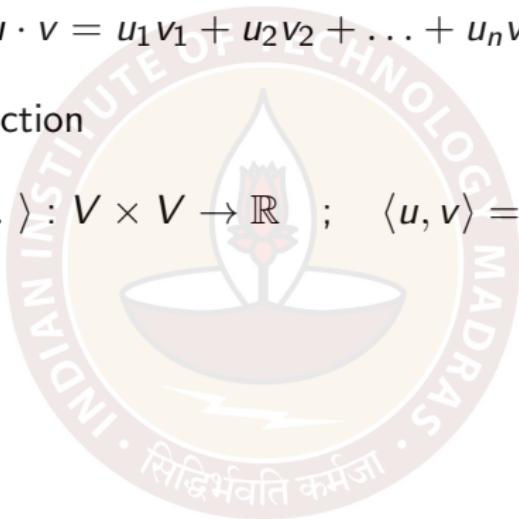
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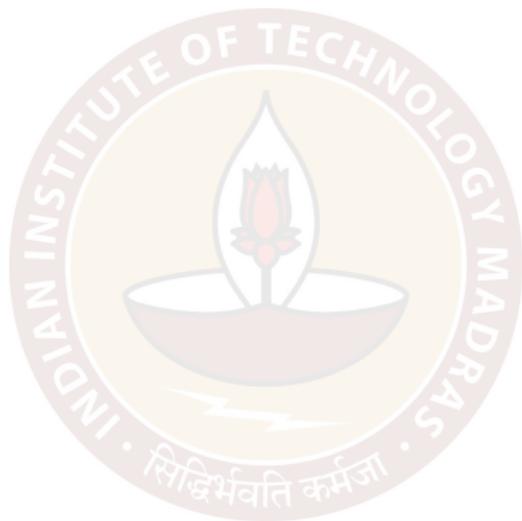
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$$\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\langle u, v \rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$$

where  $u = (x_1, y_1)$  and  $v = (x_2, y_2)$  be in  $\mathbb{R}^2$ .

$$(x_1, x_2)$$

$$(y_1, y_2)$$

$$u \cdot v = x_1y_1 + x_2y_2$$

$$\begin{aligned} & x_1^2 - (x_1x_2 + x_2x_1) \\ & x_2x_2 + x_2x_2 + x_2x_2 \\ & = x_1^2 - 2x_1x_2 + x_2x_2 \\ & = (x_1 - x_2)^2 + x_2x_2. \end{aligned}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_2 & -x_1 + 2x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 - x_2y_1 \\ -y_2x_1 + 2x_2y_2 \end{bmatrix}$$
$$\langle (y_1, y_2), (x_1, x_2) \rangle = [y_1 \ y_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\quad )^\top =$$

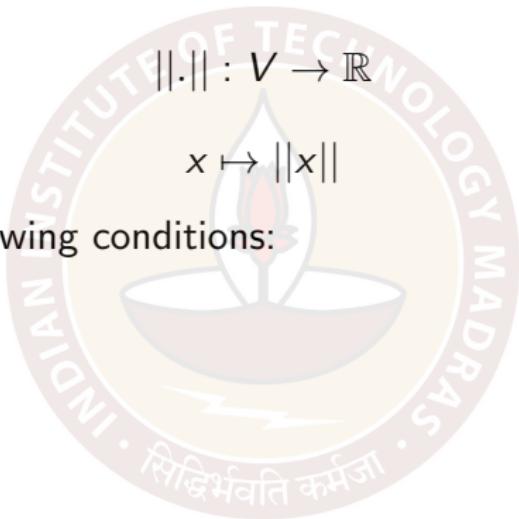
# Norm on a vector space

A **norm** on a vector space  $V$  is a function

$$\| \cdot \| : V \rightarrow \mathbb{R}$$

$$x \mapsto \|x\|$$

satisfying the following conditions:



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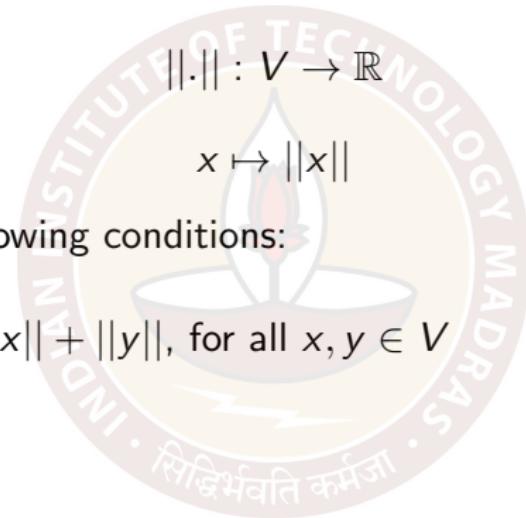
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 $x+y$        $x$        $y$
- ▶  $\|cx\| = |c|\|x\|$  for all  $c \in \mathbb{R}$  and for all  $x \in V$   
 $c$        $|c|$        $x$        $\|x\|$
- ▶  $\|x\| \geq 0$  for all  $x \in V$ ;  $\|x\| = 0$  if and only if  $x = 0$   
 $\|x\|$        $x$        $\|x\| = 0$       iff       $x = 0$ .

## Length as an example of a norm

Recall that the length of a vector  $u = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  is

$$\|u\| = \sqrt{(x_1^2 + x_2^2 + \dots + x_n^2)} .$$



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$$\|u\| = \sqrt{(x_1^2 + x_2^2 + \dots + x_n^2)} .$$

The length function  $\mathbb{R}^n \rightarrow \mathbb{R}$  is a norm on  $\mathbb{R}^n$ .

$$\|cu\| = \sqrt{c^2x_1^2 + c^2x_2^2 + \dots + c^2x_n^2} = |c| \|u\| .$$

$\|u\| = 0 \Leftrightarrow u = 0.$

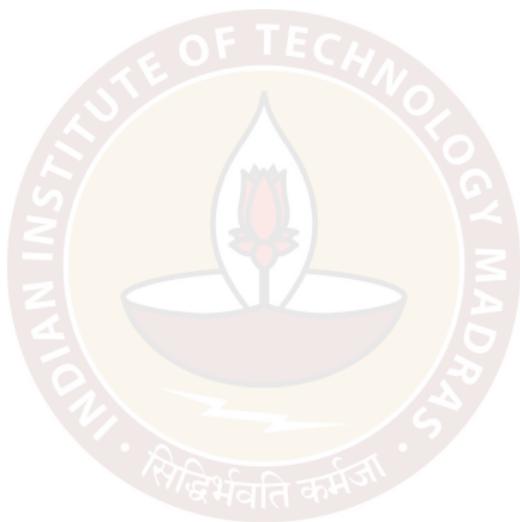
$$\begin{aligned} \|u+v\| &= \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2 + \dots + (x_n+y_n)^2} \\ &\leq \sqrt{x_1^2 + \dots + x_n^2} + \sqrt{y_1^2 + \dots + y_n^2} \\ &\leq \sqrt{x_1^2 + \dots + x_n^2} + \sqrt{2(y_1^2 + \dots + y_n^2)} \\ &= \sqrt{x_1^2 + \dots + x_n^2} + \sqrt{2(y_1^2 + \dots + y_n^2)} \end{aligned}$$

$(x_1+y_1)^2 + (x_2+y_2)^2 + \dots + (x_n+y_n)^2$

$2(x_1^2 + x_2^2 + \dots + x_n^2)$

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The following is an example of a norm on  $\mathbb{R}^n$ :

Define  $\|u\|_1 = |x_1| + |x_2| + \dots + |x_n|$  for  $u = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ .

✓  $\|u\|_1 = 0 \iff |x_i| = 0 \forall i \iff x_i = 0 \forall i \iff u = 0$ .

✓  $\|cu\|_1 = |cx_1| + |cx_2| + \dots + |cx_n| = |c| (|x_1| + |x_2| + \dots + |x_n|) = |c| \|u\|_1$ .

✓  $\|u+v\|_1 = |x_1+y_1| + |x_2+y_2| + \dots + |x_n+y_n| \leq |x_1|+|y_1| + |x_2|+|y_2| + \dots + |x_n|+|y_n| = \|u\|_1 + \|v\|_1$ .

# The inner product induces a norm

Let  $V$  be an inner product space with inner product  $\langle \cdot, \cdot \rangle$ .



## The inner product induces a norm

Let  $V$  be an inner product space with inner product  $\langle \cdot, \cdot \rangle$ .

Then the function  $\| \cdot \| : V \rightarrow \mathbb{R}$  defined by  $\|v\| = \sqrt{\langle v, v \rangle}$  is a norm on  $V$ .

$$\|v\| = 0 \iff \sqrt{\langle v, v \rangle} = 0 \iff \langle v, v \rangle = 0 \iff v = 0.$$

If  $v \neq 0$ ,  $\langle v, v \rangle > 0 \Rightarrow \sqrt{\langle v, v \rangle} > 0 \Rightarrow \|v\| > 0$ .

$$\|cv\| = \sqrt{\langle cv, cv \rangle} = \sqrt{c \times c \langle v, v \rangle} = \sqrt{c^2} \sqrt{\langle v, v \rangle} = |c| \|v\|.$$

$$\begin{aligned}\|v+w\|^2 &= \langle v+w, v+w \rangle \\&= \sqrt{\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle} \\&= \sqrt{\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle} \\&= \sqrt{\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle} \\&= \|v\|^2 + \|w\|^2 + 2\langle v, w \rangle \\&\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| \\&= (\|v\| + \|w\|)^2.\end{aligned}$$

# Thank you

