Note: Suppose W_1 and W_2 are two subspaces of a vector space "V". Then the sum of two vector subspaces is defined as

$$W_1+W_2 = \{x_1+x_2 \mid x_1 \in W_1,\ x_2 \in W_2\}.$$

Claim: W_1+W_2 is a subspace of V.

So we need to show the following conditions:

1) If $u, v \in W_1+W_2 \Rightarrow u+v \in W_1+W_2$
2) For any $r \in \mathbb{R}$ and $u \in W_1+W_2$, we have $ru \in W_1+W_2$

Cond 1: $u, v \in W_1+W_2$

$\Rightarrow u = u_1+u_2$ where $u_1 \in W_1, u_2 \in W_2$

$v = v_1+v_2$ where $v_1 \in W_1, v_2 \in W_2$.

Now, $u+v = u_1+u_2 + v_1+v_2$

$= (u_1+v_1)+(u_2+v_2) \in W_1+W_2$

Because: $u_1 \in W_1, v_1 \in W_1 \Rightarrow u_1+v_1 \in W_1$ \
$u_2 \in W_2, v_2 \in W_2 \Rightarrow u_2+v_2 \in W_2$

[W_1, W_2 are closed under addition]

Hence, If $u, v \in W_1+W_2 \Rightarrow u+v \in W_1+W_2$

i.e W_1+W_2 is closed under addition.

<u>Cond 2</u>: If $r \in \mathbb{R}$, $u = u_1+u_2 \in W_1+W_2$

$$ru = r(u_1+u_2) = ru_1 + ru_2 \in W_1+W_2$$

Because: $r \in \mathbb{R}, u_1 \in W_1 \Rightarrow ru_1 \in W_1$
$r \in \mathbb{R}, u_2 \in W_2 \Rightarrow ru_2 \in W_2$
[Both W_1, W_2 are closed under scalar multiplication]

Hence, If $r \in \mathbb{R}, u \in W_1+W_2 \Rightarrow ru \in W_1+W_2$

i.e W_1+W_2 is closed under scalar multiplic-
-ation.

Therefore, W_1+W_2 is a subspace of V.

Q1: a) $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = \sum_{i,j=1}^{2} x_i y_j$

$$= x_1y_1 + x_2y_2 + x_2y_1 + x_1y_2$$

$(1,-1) \neq (0,0) \in \mathbb{R}^2$, but

$\langle (1,-1), (1,-1) \rangle = 1-1-1+1 = 0$

But $(1,-1) \neq (0,0)$.

Hence, this can not be an inner product.

b) $V = M_{2\times 2}(\mathbb{R})$ and If $\langle \cdot , \cdot \rangle$ and $\langle \cdot , \cdot \rangle_2$ are two inner products on the vector space V.

<u>Claim</u>: $\langle \ \rangle = \langle \ \rangle_1 + \langle \ \rangle_2$ is also an inner-product.

★ Let $x, y, z \in M_{2\times 2}(\mathbb{R})$ and $r \in \mathbb{R}$.

$$\langle x+y, z\rangle = \langle x+y, z\rangle_1 + \langle x+y, z\rangle_2$$

$$= \langle x, z\rangle_1 + \langle y, z\rangle_1 + \langle x, z\rangle_2 + \langle x, z\rangle_2$$

$$= \langle x, z\rangle_1 + \langle x, z\rangle_2 + \langle y, z\rangle_1 + \langle y, z\rangle_2$$

$$= \langle x, z\rangle + \langle y, z\rangle$$

★ $\langle rx, y\rangle = \langle rx, y\rangle_1 + \langle rx, y\rangle_2$

$$= r\langle x, y\rangle_1 + r\langle x, y\rangle_2$$

$$= r\left(\langle x, y\rangle_1 + \langle x, y\rangle_2\right)$$

$$= r\langle x, y\rangle.$$

Similarly, we can check the other conditions for an innerproduct space.

c) Take $\langle \ \rangle_1 = \langle \ \rangle_2$. Then

$\langle \ \rangle = \langle \ \rangle_1 - \langle \ \rangle_2$ cannot be an inner product on $V = M_{3\times 3}(\mathbb{R})$, because

$$\langle x, y\rangle = 0 \quad \text{For all } x, y \in V.$$

d) $V=\mathbb{R}^2$ and $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_2 - x_2y_1$

$\langle (1,1),(1,1)\rangle = 0$, but $(1,1)\neq(0,0)$.

This can not be an innerproduct on V.

Q2 $\langle \cdot,\cdot\rangle : \mathbb{R}^3\times\mathbb{R}^3 \longrightarrow \mathbb{R}$

$\langle (x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = x_1y_1+x_2y_2+x_3y_3$

a) $\{(-2,1,3)\}=S$

$\langle (-2,1,3),(1,-1,1)\rangle = -2-1+3=0.$

$\langle (-2,1,3),(2,1,1)\rangle = -4+1+3=0.$

Hence, $T=\{(1,-1,1),(2,1,1)\}\subset(\mathrm{Span}(S))^{\perp}$

$\langle (1,-1,1),(2,1,1)\rangle \neq 0$, Hence T is not orthogonal.

$a \Rightarrow 1) \Rightarrow 3)$.

b) $\{(1,-2,1)\}=S$

$\langle (1,-2,1), \frac{1}{\sqrt{3}}(1,1,1)\rangle = \frac{1}{\sqrt{3}}(1-2+1)=0.$

$\langle (1,-2,1), \frac{1}{\sqrt{2}}(-1,0,1)\rangle = \frac{1}{\sqrt{2}}(-1+0+1)=0.$

Hence, $T=\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,0,1)\}\subseteq(\mathrm{Span}(S))^{\perp}$

Now, $\langle \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,0,1)\rangle = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}(-1+1)=0.$

$\left\|\frac{1}{\sqrt{3}}(1,1,1)\right\| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$, $\left\|\frac{1}{\sqrt{2}}(-1,0,1)\right\| = \sqrt{1/2+1/2} = 1.$

T - is an orthonormal set and $\dim(\operatorname{span}(T)) = 2$.

$$b \Rightarrow iv) \Rightarrow i)$$

Similarly, you can try for other two options.

Q3: <u>Option 1</u>: $P_{W_1}+P_{W_2}$ is a projection from V to W_1+W_2 iff $(P_{W_1}+P_{W_2})\circ(P_{W_1}+P_{W_2}) = P_{W_1}+P_{W_2}$

$$\Leftrightarrow P_{W_1}\circ P_{W_1} + P_{W_1}\circ P_{W_2} + P_{W_2}\circ P_{W_1} + P_{W_2}\circ P_{W_2} = P_{W_1}+P_{W_2}$$

$$\Leftrightarrow P_{W_1} + P_{W_1}\circ P_{W_2} + P_{W_2}\circ P_{W_1} + P_{W_2} = P_{W_1}+P_{W_2}$$

[P_{W_1}, P_{W_2} are projection maps $\Rightarrow P_{W_1}\circ P_{W_1} = P_{W_1}$ & $P_{W_2}\circ P_{W_2} = P_{W_2}$]

$$\Leftrightarrow P_{W_1}\circ P_{W_2} + P_{W_2}\circ P_{W_1} = 0.$$

<u>Option 2</u>: $P_{W_1}^2 = P_{W_1}\circ P_{W_1} = P_{W_1}$ (because P_{W_1} is a projection).

Hence $P_{W_1} = P_{W_1}^2$ is a projection.

<u>Option 3</u>: Take $V=\mathbb{R}^2$, $W_1 = x$-axis $W_2 = y$-axis

Now $P_{W_2}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Take $B = \{(1,0),(0,1)\}$ as a basis for both domain and codomain.

Now, $(1,0) = (1,0) + (0,0)$

$$P_{W_2}(1,0) = (0,0) = 0(1,0) + 0(0,1)$$

$$(0,1) = (0,0) + (0,1)$$

$$P_{W_2}(0,1) = (0,1) = 0(1,0) + 1(0,1)$$

Matrix of P_{W_2} w.r.t B is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, which is a symmetric matrix.

Option 4:- Take P_{W_1}, P_{W_2} as given in option (3).

$$\text{Now, } (P_{W_1} - P_{W_2})(0,1) = P_{W_1}(0,1) - P_{W_2}(0,1)$$
$$= -(0,1)$$

which is not a projection on $W_1 + W_2 := \mathbb{R}^2$.

Q4:

Option 1:⇒ Take $V = \mathbb{R}^2$ & $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$

Now;

$$\langle (1,-1), (1,1) \rangle = \langle (1,-1), (0,0) \rangle = 0$$

but $(1,1) \neq (0,0)$.

Option 2:- If $\langle u, v \rangle = 0$ for all $v \in V$, then

take $v = u$

$\Rightarrow \langle v, v \rangle = 0 \Rightarrow \underline{\underline{v = 0}}$ [one of the condition for inner product].

Option 3: $\langle u+v, u-v\rangle = 0$

$$\Rightarrow \langle u,u\rangle + \langle u,-v\rangle + \langle v,u\rangle + \langle v,-v\rangle = 0$$

$$\Rightarrow \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle = 0$$

$$\Rightarrow \langle u,u\rangle - \langle v,v\rangle = 0$$

$$\Rightarrow \langle u,u\rangle = \langle v,v\rangle \Rightarrow \|u\|^2 = \|v\|^2$$

Option 4. $S \subset V$, claim: $S^\perp = \text{Span}(S)^\perp$.

Suppose $S = \{x_1, x_2, \dots, x_n\}$.

Let $v \in S^\perp \Rightarrow \langle v, x_i\rangle = 0 \ \forall i = 1 \sim n$.

Let $a \in \text{Span}(S)$

$$\Rightarrow a = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

Now, $\langle v, a\rangle = \langle v, \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\rangle$

$$= \alpha_1\langle v, x_1\rangle + \alpha_2\langle v, x_2\rangle + \dots + \alpha_n\langle v, x_n\rangle$$

$$= 0$$

$$\Rightarrow v \in \text{Span}(S)^\perp.$$

Hence, $S^\perp \subset \text{Span}(S)^\perp$. —— 1

If $v \in \text{Span}(S)^\perp \Rightarrow v \in S^\perp$ (because $S \subset \text{Span}(S)$)

Hence, $\text{Span}(S)^{\perp} \subset S^{\perp}$ — ②

From ① + ② $\Rightarrow$ $\text{Span}(S)^{\perp} = S^{\perp}$.

Option 5:- $\{u_1, u_2, --, u_m\}$ orthogonal set of vectors, i.e $\langle u_i\ u_j\rangle = 0$ if $i \neq j$.

$\|u_1+u_2+...+u_m\| = \langle u_1+u_2+...+u_m, u_1+u_2+...+u_m\rangle$

$= \langle u_1, u_1+u_2+--+u_m\rangle + \langle u_2, u_1+u_2+-+u_m\rangle + ...$

$+ \langle u_m, u_1+u_2+-+u_m\rangle$

$= \langle u_1, u_1\rangle + \langle u_2, u_2\rangle + ... + \langle u_m, u_m\rangle$

$= \|u_1\|^2 + \|u_2\|^2 + ... + \|u_m\|^2$

$(\because \langle u_i, u_1+u_2+...+u_i+...+u_m\rangle$

$= \langle u_i, u_1\rangle + ... + \langle u_i, u_i\rangle + ... + \langle u_i, u_m\rangle$

$= \langle u_i, u_i\rangle)$.

Q5:- Option 1:- $\|(x, y, z)\|_0 = x+y+z$

$\|(1, -1, 0)\|_0 = 1 + -1 + 0 = 0$ but $(1, -1, 0) \neq (0, 0, 0)$

this can not be a norm on $\mathbb{R}^3$.

Option-6: $V_2 = M_{2\times 2}(\mathbb{R})$ and $A \in V_2$

☆ $\|A\|_5 = \max\{|a_{11}|+|a_{21}|,\ |a_{12}|+|a_{22}|\}$.

Suppose $\|A\|_5 = 0 \Rightarrow \max\{|a_{11}|+|a_{21}|,\ |a_{12}|+|a_{22}|\} = 0$

$\Rightarrow |a_{11}|+|a_{21}| = 0,\ |a_{12}|+|a_{22}| = 0$

$\Rightarrow a_{11} = 0,\ a_{21} = 0,\ a_{12} = 0,\ a_{22} = 0$.

Hence $A = 0$.

☆ Let $r \in \mathbb{R}$. $rA = \begin{pmatrix} ra_{11}, & ra_{12} \\ ra_{21}, & ra_{22} \end{pmatrix}$

$\|rA\|_5 = \max\{|ra_{11}|+|ra_{21}|,\ |ra_{12}|+|ra_{22}|\}$

$= \max\{|r|(|a_{11}|+|a_{21}|),\ |r|(|a_{12}|+|a_{22}|)\}$.

$= |r| \max\{|a_{11}|+|a_{21}|,\ |a_{12}|+|a_{22}|\}$.

$= |r|\ \|A\|_5$

☆ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+a_{22} & b_{21}+b_{22} \end{pmatrix}$

Now

$$\|A+B\|_5 = \max\{|a_{11}+b_{11}|+|a_{21}+b_{21}|,\ |a_{12}+b_{12}|+|a_{22}+b_{22}|\}$$

$$\leq \max\{|a_{11}|+|b_{11}|+|a_{21}|+|b_{21}|,\ |a_{12}|+|b_{12}|+|a_{22}| + |b_{22}|\}.$$

$$\leq \max\{|a_{11}|+|a_{21}|,\ |a_{12}|+|a_{22}|\}$$

$$+ \max\{|b_{11}|+|b_{21}|,\ |b_{12}|+|b_{22}|\}$$

$$= \|A\| + \|B\|.$$

Similarly, you can try for other options.

Q6: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$T(u) = \langle u, v \rangle$ where $v \in \mathbb{R}^3$.

option 1 &

option 3 :- T is a linear map from $\mathbb{R}^3$ to $\mathbb{R}$ and $\dim(\mathbb{R}^3) > \dim(\mathbb{R})$. So it can not be injective.

option 2 :- If $v \neq 0 \in \mathbb{R}^3$, Then

$$T(v) = \langle v, v \rangle \neq 0 \in \mathbb{R}$$

$\Rightarrow$ Image of T is non zero

$\Rightarrow$ $\text{Rank}(T) \geq 1$.

Since $\dim(\mathbb{R}) = 1 \Rightarrow \text{Img}(T) = \mathbb{R}$

Hence, T is onto.

Option 4 :- T is not one-one $\Rightarrow$ T is not an isomorphism.

Q7: Given that $v_1 = (1, 0, -1)$, $v_2 = (a, b, c)$

$\star$ $\langle v_2, (1, -1, 1) \rangle = -3$

$\Rightarrow a - b + c = -3$ —— ①

$\star$ $\langle v_2, (-1, 2, 1) \rangle = 1$

$\Rightarrow -a + 2b + c = 1$. —— ②

$\star$ angle between v_1 and v_2 is 45° and length of the vector v_2 is 3.

$$\cos 45^\circ \, \|v_1\| \, \|v_2\| = \langle v_1, v_2 \rangle$$

$$\Rightarrow a-c = \frac{1}{\sqrt{2}} \cdot \sqrt{2} \cdot 3 = 3$$

So, we have

$$a-b+c=-3$$
$$-a+2b+c=1$$
$$a-c=3$$

If we solve the above system, then we have

$$a=1 \quad b=2 \quad c=-2$$

$$\Rightarrow a+b+c=1 \quad \boxed{\text{Ans}}.$$

Q8:

* The closest Point from Rahul's home to the Road(1) is the orthogonal projection of the Point (8, 1) on the line $y=2x$.

The line $y=2x$ is equal to $V_1=\{(x,2x) \mid x\in\mathbb{R}\}$
$= \text{span}\{(1,2)\}$.

The projection of (8, 1) on to V_1 is

$$\frac{1}{\langle(1,2),(1,2)\rangle} \langle(8,1),(1,2)\rangle (1,2) = \frac{10}{5}(1,2)=(2,4)$$

* For road-2: equation is $y=-2x$

$$V_2=\{(x,-2x) \mid x\in\mathbb{R}\}=\text{Span}\{(1,-2)\}$$

orthogonal projection is:

$$\frac{1}{\langle(1,-2),(1,-2)\rangle}\langle(8,1),(1,-2)\rangle(1,-2)=\frac{6}{5}(1,-2).$$

Q9: Since θ is an actue angle there $\cos\theta$ must be non-negative.

$$\cos\theta=\left|\frac{\langle 1,2\rangle\langle 1,-2\rangle}{\sqrt{5}\ \sqrt{5}}\right|=\frac{3}{5}$$

$$\Rightarrow 5\cos\theta=3\ \underline{\underline{[\text{Ans}]}}.$$

Q10: $\star\ W_1=\{(x,2x)\mid x\in\mathbb{R}\}=\text{Span}\{(1,2)\}.$

If $v=(x,y)\in W_1^{\perp}$

$$\Rightarrow \langle(x,y),(1,2)\rangle=\cos 90=0$$

$$\Rightarrow x+2y=0\Rightarrow x=-2y$$

Hence, $W_1^{\perp}=\{(x,y)\mid x=-2y\}.$

$\star\ W_2=\{(x,-2x)\mid x\in\mathbb{R}\}=\text{Span}\{(1,-2)\}.$

If $v=(x,y) \in W_2^{\perp}$

$\Rightarrow \langle (x,y),(1,-2)\rangle = 0$

$\Rightarrow x-2y=0 \Rightarrow x=2y$

So $W_2^{\perp} = \{(x,y) \mid x=2y\}$.