# MATHEMATICS FOR DATA SCIENCE - II
## WEEK 11 - PRACTICE ASSIGNMENT

1. Max. directional derivative is along the direction of $\nabla f$.

(1) $f(x,y) = x^2 - xy + y^2$

$f_x = 2x - y$ ; $f_y = -x + 2y^2$

$\nabla f|_{(0,0)} = (0,0)$ —— (iii)

(2) $f(x,y) = 3x^2 + 3x + y$

$f_x = 6x+3$ ; $f_y = 1$

$\nabla f|_{(0,0)} = (3,1)$ —— (iv)

(3) $f(x,y) = e^{x+y} - 4y$

$f_x = e^{x+y}$ ; $f_y = e^{x+y} - 4$

$\nabla f|_{(0,0)} = (1,-3)$ —— (ii)

(4) $f(x,y) = \sin y + \cos x$

$f_x = -\sin x$ ; $f_y = \cos y$

$\nabla f|_{(0,0)} = (0,1)$ —— (v)

2. Eqn of the tgt plane is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) = z.$$

The tgt plane should be parallel to

$$7x - 8y + 3z + 2 = 0$$

ie., $-\frac{7}{3}x + \frac{8}{3}y - \frac{2}{3} = z$

A plane parallel to this will be of the form $-\frac{7}{3}x + \frac{8}{3}y + k = z$ for some constant $k$.

$\therefore$ We want $f_x(x_0, y_0) = -\frac{7}{3}$, $f_y(x_0, y_0) = \frac{8}{3}$

$f_x(x_0, y_0) = -\frac{7}{3}x_0 \quad \left(\because f(x,y) = -\frac{7}{6}x^2 + \frac{y^2}{3} + \frac{11}{6}\right)$

$\Rightarrow \boxed{x_0 = 1}$

$f_y(x_0, y_0) = \frac{2}{3}y_0 \Rightarrow \boxed{y_0 = 4}$

$f(x_0, y_0) = z_0 \Rightarrow z_0 = -\frac{7}{6} + \frac{16}{3} + \frac{11}{6}$

$$\boxed{z_0 = 6}$$

$\therefore (1, 4, 6)$ is the pt. where the tgt plane is $\parallel^{e}$ to the given plane

3. Option 1: The directional derivatives does not exist at $(0,0)$.
In fact, $f_x$, $f_y$ do not exist at $(0,0)$.

$$f_x = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{|h|}{h} \text{ (does not exist)}$$

Option 2: This is the geometrical interpretation of the tgt. plane.

Option 3: Not necessary. The example in option 1 is such that its partial derivatives do not exist at $(0,0)$.

Option 4: Tgt. plane parallel to XY-plane implies $f_x = 0 = f_y$ (Check problem 6).

Thus at a point $P(a,b)$,

$$\begin{aligned} D_u f(a,b) &= \nabla f(a,b) \cdot u \quad \text{(where } u \text{ is a unit vector)} \\ &= (f_x(a,b), f_y(a,b)) \cdot u \\ &= (0,0) \cdot u = 0. \end{aligned}$$

4. $f(x,y) = 6x^2 - 2x^3 + 3y^2 - 6xy$

$f_x = 12x - 6x^2 - 6y$

$f_y = 6y - 6x$

We want $(x,y)$ s.t. $f_x = 0 = f_y$

$f_y = 0 \Rightarrow \boxed{y = x}$

$f_x = 0 \Rightarrow 12x - 6x^2 - 6x = 0 \quad (\because y = x)$

$\Rightarrow 6x^2 - 6x = 0$

$\Rightarrow x = 1, 0$

$\therefore$ The critical pts. are $(0,0)$ & $(1,1)$.

5. $f_1(x,y,z) = \ln(x^2+y^2+2z^2)$

$f_2(x,y,z) = \tan^{-1}(x^2+y^2+2z^2)$.

$$L_f = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

$$(f_1)_x = \frac{1}{x^2+y^2+2z^2}(2x) \;;\; (f_1)_x(1,-1,1) = \frac{1}{2}$$

$$(f_1)_y = \frac{1}{x^2+y^2+2z^2}(2y) \;;\; (f_1)_y(1,-1,1) = \frac{-1}{2}$$

$$(f_1)_z = \frac{1}{x^2+y^2+2z^2}(4z) \;;\; (f_1)_z(1,-1,1) = 1$$

$$L_{f_1} = \ln(4) + \frac{1}{2}(x-1) - \frac{1}{2}(y+1) + (z-1)$$
$$= \frac{x}{2} - \frac{y}{2} + z + (\ln 4 - 2)$$

$$\nabla L_{f_1} = \left(\frac{1}{2}, \frac{-1}{2}, 1\right).$$

$$f_2(x,y,z) = \tan^{-1}(x^2+y^2+2z^2)$$

$$(f_2)_x = \frac{1}{1+(x^2+y^2+2z^2)^2}(2x)\;;\;(f_2)_x(1,-1,1) = \frac{2}{17}$$

$$(f_2)_y = \frac{1}{1+(x^2+y^2+2z^2)^2}(2y)\;;\;(f_2)_y(1,-1,1) = \frac{-2}{17}$$

$$(f_2)_z = \frac{1}{1+(x^2+y^2+2z^2)^2}(4z)\;;\;(f_2)_z(1,-1,1) = \frac{4}{17}$$

$$L_{f_2} = \tan^{-1}(4) + \frac{2}{17}(x-1) - \frac{2}{17}(y+1) + \frac{4}{17}(z-1)$$

$= \frac{2}{17}(x$ [illegible]$) - \frac{2}{17}y + \frac{4}{17}z + \left(\tan^{-1}4 - \frac{8}{17}\right)$

$$\nabla L_{f_2} = \left(\frac{2}{17}, \frac{-2}{17}, \frac{4}{17}\right)$$

$$\nabla L_{f_1} \cdot \nabla L_{f_2} = \frac{1}{17} + \frac{1}{17} + \frac{4}{17} = \frac{6}{17}.$$

$$\therefore 17\left(\nabla L_{f_1} \cdot \nabla L_{f_2}\right) = 6.$$

6. Similar to problem 2.

Eqn. of XY-plane is $z=0$.

So we want $f_x = 0 = f_y$ at $(x_0, y_0)$.

$f(x,y) = x^3 + y^3 + 3x^2 - 9x - 12y + 20$.

$f_x = 3x^2 + 6x - 9$ ; $3x_0^2 + 6x_0 - 9 = 0$

$f_y = 3y^2 - 12$ ; $3y_0^2 - 12 = 0$

$\therefore$ There are 4 points where the tgt plane is $\parallel^{e}$ to the XY-plane.

$[(-3,2), (-3,-2), (1,2), (1,-2)]$

$f(x,y) = x^3 + 3y^3 + 9xy$.

Tgt. plane at $(-1,2)$ is

$f_x(-1,2)(x+1) + f_y(-1,2)(y-2) + f(-1,2) = z$

$f_x = 3x^2 + 9y \quad ; \quad f_y = 9y^2 + 9x$

$f_x(-1,2) = 21 \quad ; \quad f_y = 27$

$\therefore ax + by + cz = d_1$ is

$21(x+1) + 27(y-2) + 5 = z$.

$21x + 27y - z = 28$.

$\therefore a = 21, \; b = 27, \; c = -1, \; d_1 = 28$

7. Tgt. plane at $(\alpha, 1)$ is $ax + by + cz = d_2$

$\therefore f_y(\alpha, 1) = b \Rightarrow 9\alpha + 9 = 27 \Rightarrow \boxed{\alpha = 2}$

$\therefore$ Tgt. plane at $(2,1)$ is

$21(x-2) + 27(y-1) + 29 = z$

$\Rightarrow 21x + 27y - z = 40 \Rightarrow \boxed{d_2 = 40}$

$\therefore |d_1 - d_2| = |28 - 40| = 12$.

8. $\dfrac{2a+b+c}{d_1+d_2} = \dfrac{42+27-1}{68} = 1$ //

$f(x,y) = xe^y + \ln(x+y)$

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y)$

$f_x = e^y + \frac{1}{x+y}$ ; $f_x(1,0) = 2$

$f_y = xe^y + \frac{1}{x+y}$ ; $f_y(1,0) = 2$

$\therefore L(x,y) = 1 + 2(x-1) + 2y = 2x + 2y - 1$

9. Option 2

10. $L(1.1, 0.1) = 2(1.1) + 2(0.1) - 1 = 1.4$.