Projections using inner products

Sarang S. Sane

Shortest distances in $\mathbb{R}^2$

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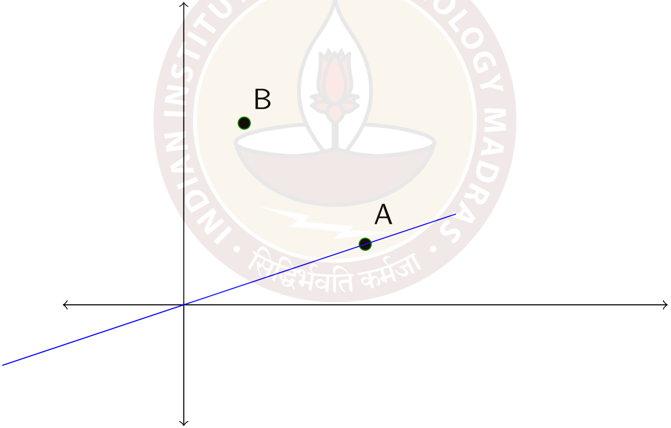

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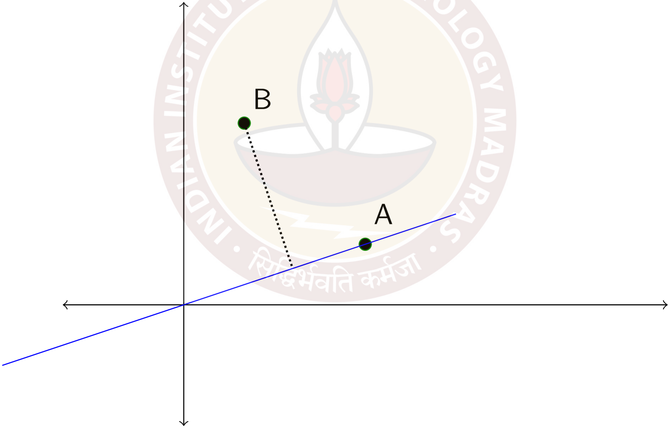

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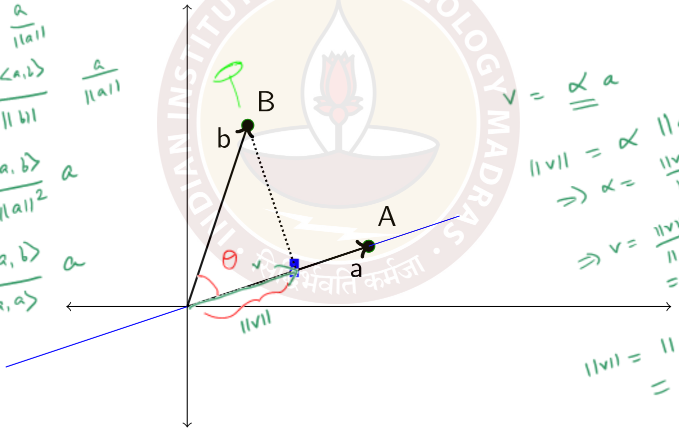


$$V = \|v\| \frac{a}{\|a\|}$$

$$= \frac{\|b\| \langle a,b\rangle}{\|a\| \|b\|} \frac{a}{\|a\|}$$

$$= \frac{\langle a,b\rangle}{\|a\|^2} a$$

$$= \frac{\langle a,b\rangle}{\langle a,a\rangle} a$$

$$v = \underline{\underline{\alpha}}\, a$$

$$\|v\| = \alpha \|a\|$$

$$\Rightarrow \alpha = \frac{\|v\|}{\|a\|}$$

$$\Rightarrow v = \frac{\|v\|}{\|a\|} a$$

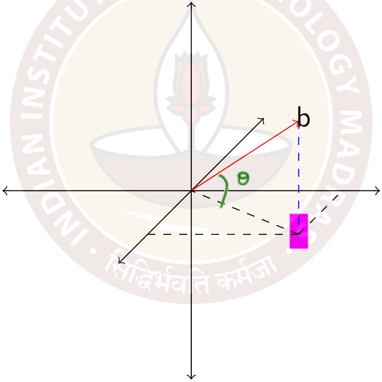
$$= \|v\| \frac{a}{\|a\|}$$

$$\|v\| = \|b\| \cos\theta$$

$$= \|b\| \frac{\langle a,b\rangle}{\|a\| \|b\|}$$

Shortest distances in $\mathbb{R}^3$

Suppose we have a vector b in $\mathbb{R}^3$, and we want to find out the nearest point ~~of~~ to b on the two dimensional plane generated by the vectors $(1,0,0)$ and $(0,1,0)$.



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$$\text{Define} \qquad proj_W(v) = \boxed{\sum_{i=1}^{n} \langle v, v_i \rangle v_i.}$$

Fact : The definition is independent of the chosen orthonormal basis (i.e. the expression on the RHS does not change even if you choose a different orthonormal basis).

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Find $w \in W$ s.t. $\|v - w\|$ is smallest.
Ans. : $w = proj_W(v)$.

The projection of v onto W is the vector in W closest to v. Note that "closest" is in terms of the distance based on the norm induced by the inner product.

Previous examples

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$\frac{1}{\sqrt{10}}(3,1)$ is an o.n. basis for W.

$proj_W(v) = \langle v, \frac{1}{\sqrt{10}}(3,1)\rangle \frac{1}{\sqrt{10}}(3,1) = \langle (1,3),(3,1)\rangle \frac{1}{10}(3,1)$

$= \frac{(1\times 3 + 3\times 1)}{10}(3,1) = \frac{6}{10}(3,1)$

$= (1.8, 0.6)$.

$V = \mathbb{R}^3$, $W = \langle(1,0,0),(0,1,0)\rangle$, $v = (2,3,5)$.
Then $proj_W(v) = (2,3,0)$.

o.n. basis $\langle (1,0,0), (0,1,0)\rangle$.

$proj_W(v) = \langle (2,3,5), (1,0,0)\rangle (1,0,0) + \langle (2,3,5), (0,1,0)\rangle (0,1,0)$

$= 2(1,0,0) + 3(0,1,0) = (2,3,0)$.

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Similarly, if $\{v_1, v_2, \ldots v_n\}$ is an orthogonal basis for a subspace W, then $\left\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \ldots \frac{v_n}{||v_n||} \right\}$ is an orthonormal basis for W

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$$proj_W(v) = \sum_{i=1}^{n} \left\langle v, \frac{v_i}{||v_i||} \right\rangle \frac{v_i}{||v_i||} = \sum_{i=1}^{n} \boxed{\frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}} v_i = \sum_{i=1}^{n} proj_{v_i}(v).$$

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$$proj_{v_1} v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = \frac{4}{6}(1,2,1) = \frac{2}{3}(1,2,1).$$

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$$\begin{aligned} \text{Hence} \quad proj_W(v) &= Proj_{v_1}(v) + Proj_{v_2}(v) \\ &= \frac{2}{3}(1, 2, 1) - \frac{2}{3}(1, -1, 1) \\ &= (0, 2, 0). \end{aligned}$$

Projection as a linear transformation

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Choose o.n. basis $\{w_1, \ldots, w_n\}$ for W.

$P_W(v) = \text{proj}_W(v)$.

$P_W(v_1+v_2) = P_W(v_1) + P_W(v_2)$ & $P_W(cv) = c\,P_W(v)$

$$\begin{aligned} P_W(v_1+v_2) &= \text{proj}_W(v_1+v_2) \\ &= \sum_{i=1}^{n} \text{proj}_{w_i}(v_1+v_2) \\ &= \sum_{i=1}^{n} \langle v_1+v_2, w_i\rangle w_i \\ &= \sum_{i=1}^{n} \langle v_1, w_i\rangle w_i + \sum_{i=1}^{n} \langle v_2, w_i\rangle w_i \end{aligned}$$

$$\begin{aligned} P_W(cv) &= \text{proj}_W(cv) \\ &= \sum_{i=1}^{n} \langle cv, w_i\rangle w_i \\ &= c\sum_{i=1}^{n} \langle v, w_i\rangle w_i \end{aligned}$$

Denote this linear transformation as P_W.

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ii) $Range(P_W) = W$.

iii) $W^{\perp} = \{v |\ v \in V,$ such that $\langle v, w\rangle = 0 \forall w \in W\}$ is the null space of P_W.

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iv) $P_W^2 = P_W$.

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$\{w_1, \ldots, w_n\}$ o.n. basis

$\sum_{i=1}^{n} \langle v, w_i \rangle w_i$ $=$ If $v \in W$ then $v = \sum c_i w_i$

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$P_W \circ P_W$ $\quad P_W(\underbrace{P_W(v)}_{\in W}) = P_W(v)$.

v) $||P_W(v)|| \leq ||v||$.

Thank you