Week-3
Mathematics for Data Science - 2
Critical points, Area under the curve, Integration
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Match the given functions in Column A with the (signed) area between its graph and the interval $[-1, 1]$ on the X-axis in column B and the pictures of their graphs and the highlighted region corresponding to the area computation in Column C, given in Table M2W3G1.

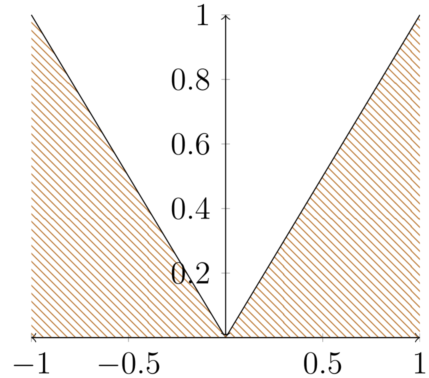
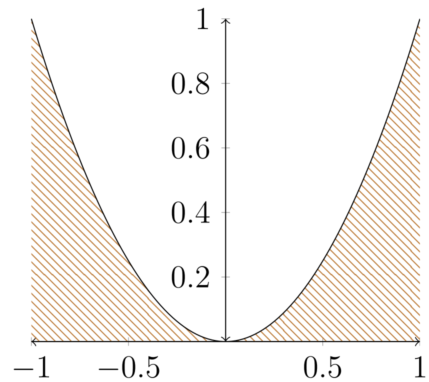
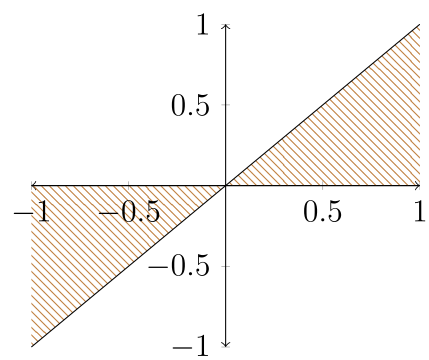
	Functions (Column A)		Area under the curve (Column B)		Graphs (Column C)
i)	$f(x) = x$	a)	$\frac{2}{3}$	1)	
ii)	$f(x) = \|x\|$	b)	0	2)	
iii)	$f(x) = x^2$	c)	1	3)	

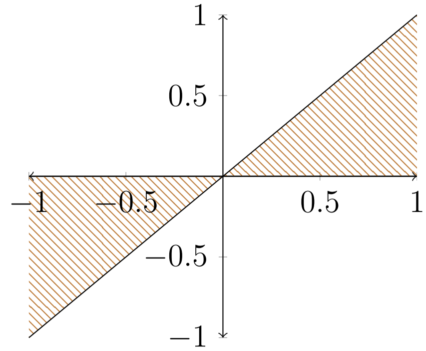
Table: M2W3G1

◯ Option 1: i) $\to$ b) $\to$ 1), ii) $\to$ c) $\to$ 3), iii) $\to$ a) $\to$ 2).

- ◯ Option 2: i) → b) → 3), ii) → a) → 1), iii) → c) → 2).
- ◯ Option 3: i) → c) → 3), ii) → b) → 1), iii) → a) → 2).
- ◯ **Option 4:** i) → b) → 3), ii) → c) → 1), iii) → a) → 2).

Solution:

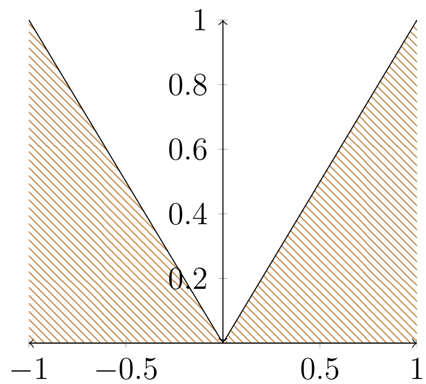
- $f(x) = x$



$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x\ dx = \frac{x^2}{2}\Big|_{-1}^{1} = \left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

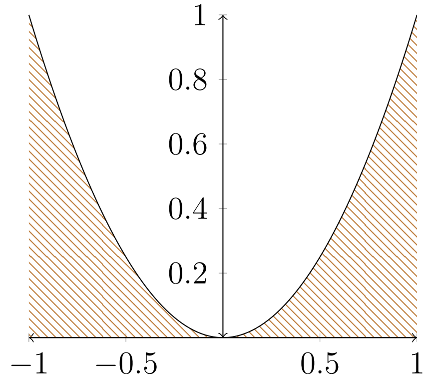
- $f(x) = |x|$

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$



$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} |x|\ dx = \int_{1}^{0} (-x)\ dx + \int_{0}^{1} x\ dx = -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1}$$
$$= -\left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) = 1.$$

- $f(x) = x^2$



$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2 \ dx = \frac{x^3}{3}\Big|_{-1}^{1} = \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3}.$$

2 Multiple Select Questions (MSQ)

2. A cylinder of radius x and height $2h$ is to be inscribed in a sphere of radius R centered at O as shown in Figure M2W3G1

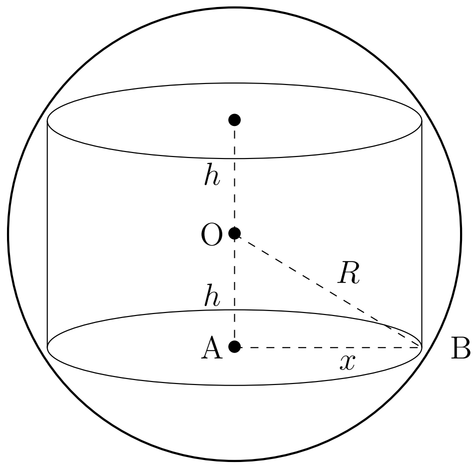


Figure M2W3G1

The volume of such a cylinder is given by $V = 2\pi x^2 h$ and the surface area of the outer curved surface is given by $S = 4\pi xh$. Choose the set of correct options.

- ◯ Option 1: The cylinder has maximum volume amongst all cylinders which can be inscribed when $h = R$.
- ◯ Option 2: The cylinder has maximum volume amongst all cylinders which can be inscribed when $h = \sqrt{3}R$.
- ◯ **Option 3:** The cylinder has maximum volume amongst all cylinders which can be inscribed when $h = \frac{R}{\sqrt{3}}$.
- ◯ Option 4: The cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed, when $h = 2R$.
- ◯ **Option 5:** The cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed, when $h = \frac{R}{\sqrt{2}}$.
- ◯ Option 6: The cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed, when $h = \sqrt{2}$.

Solution:
As the triangle ΔOAB is a right angle triangle, we have $h^2 + x^2 = R^2$, i.e., $x^2 = R^2 - h^2$.

For Volume:
The volume of the cylinder is $V = 2\pi(R^2 - h^2)h = 2\pi(R^2 h - h^3)$.
Hence, $\frac{dV}{dh} = 2\pi(R^2 - 3h^2)$.
To find the critical points, let us write the equation $\frac{dV}{dh} = 0$, i.e., $R^2 - 3h^2 = 0$, which

implies, $h = \frac{R}{\sqrt{3}}$ (as we can neglect the negative value of h).
Moreover we have, $\frac{d^2V}{dh^2} = 2\pi(-6h)$. For $h = \frac{R}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -12\pi\frac{R}{\sqrt{3}} < 0$.
So, $h = \frac{R}{\sqrt{3}}$ will give a maximum for V.
Therefore, the cylinder has maximum volume amongst all cylinders which can be inscribed when $h = \frac{R}{\sqrt{3}}$.

For Surface Area of the outer curved surface:
The surface area of the curved surface of the cylinder is $S = 4\pi xh = 4\pi h\sqrt{R^2 - h^2}$.
Hence, $\frac{dS}{dh} = 4\pi\left(\frac{1}{2}(-2h)(h)(R^2 - h^2)^{-\frac{1}{2}} + \sqrt{R^2 - h^2}\right)$
To find the critical points, let us write the equation $\frac{dS}{dh} = 0$, i.e.,

$$4\pi\left(-h^2(R^2 - h^2)^{-\frac{1}{2}} + \sqrt{R^2 - h^2}\right) = 0$$

So we have, $R^2 - h^2 = h^2$, i.e., $h = \frac{R}{\sqrt{2}}$.
Observe that, for $h = \frac{R}{\sqrt{2}}$, we have $\frac{d^2S}{dh^2} < 0$.
So, $h = \frac{R}{\sqrt{2}}$ will give a maximum for S.
Therefore, the cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed when $h = \frac{R}{\sqrt{2}}$.

3. Which of the following curves shown in the following figures enclose a negative area on the X axis in the interval $[0,1]$?

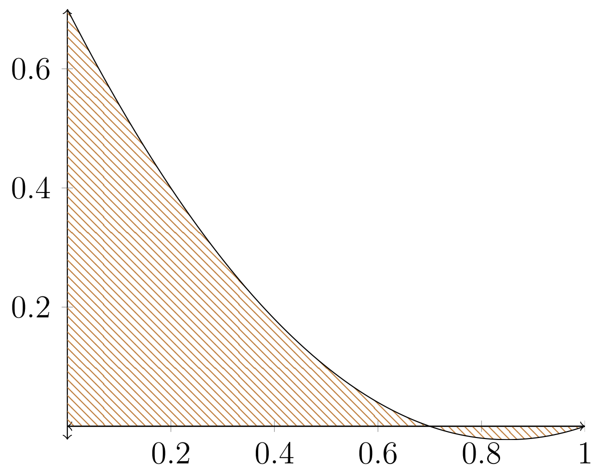


Figure: Curve 1

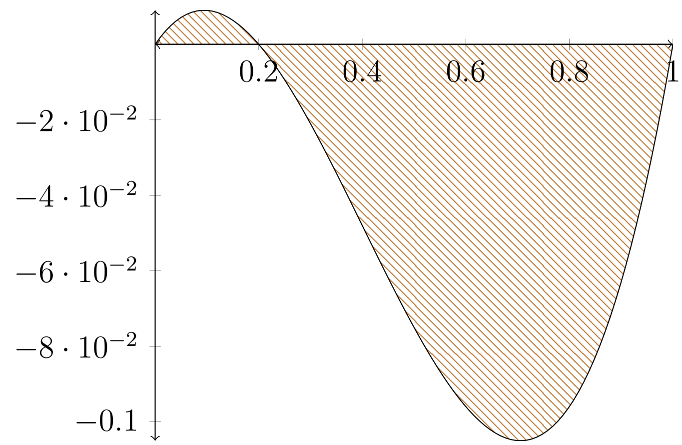


Figure: Curve 2

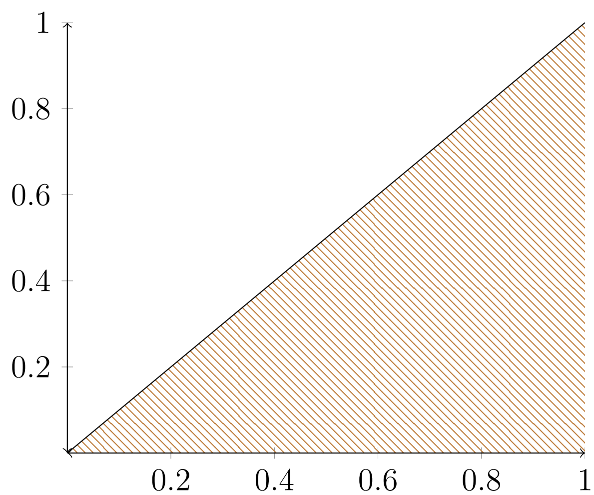


Figure: Curve 3

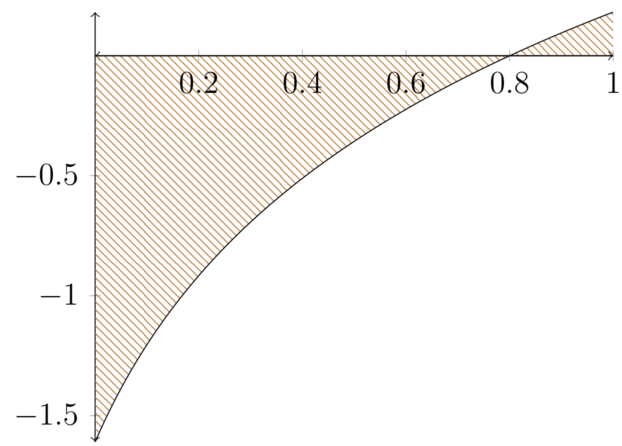


Figure: Curve 4

- ◯ Option 1: Curve 1
- ◯ **Option 2:** Curve 2
- ◯ Option 3: Curve 3
- ◯ **Option 4:** Curve 4

Solution:

The area enclosed above the X-axis, i.e., towards the positive direction of Y-axis, is positive, and the area enclosed below the X-axis, i.e., towards the negative direction of

Y-axis, is negative. So, if the portion of area enclosed by the curve above the X-axis is lesser than the portion of area enclosed by the curve below the X-axis, then the area enclosed by the curve as shown in the figure is negative. Hence, Curve 2 and Curve 4 enclose a negative area on the X-axis in the interval $[0, 1]$.

4. Suppose $\int x^2 sin\ 2x\ dx = Mx^2 cos\ 2x + Nxsin\ 2x + Pcos\ 2x + C$, where C is the constant of integration. Which of the following are correct?

   - ◯ Option 1: $M = N = \frac{1}{2}$
   - ◯ **Option 2:** $M = -N = -\frac{1}{2}$
   - ◯ **Option 3:** $P = \frac{1}{4}$
   - ◯ Option 4: $P = 0$

   Solution:
   Using integration by parts:

$$\int x^2 sin\ 2x\ dx = x^2 \int sin\ 2x\ dx - \int \left(\frac{d}{dx}(x^2) \int sin\ 2x\ dx \right) dx$$

$$= x^2 \left(\frac{-cos\ 2x}{2} \right) - \int 2x \left(\frac{-cos\ 2x}{2} \right) dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + \int x\ cos\ 2x\ dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + x \int cos\ 2x\ dx - \int \left(\frac{d}{dx}(x) \int cos\ 2x\ dx \right) dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + x \left(\frac{sin\ 2x}{2} \right) - \int 1 \left(\frac{sin\ 2x}{2} \right) dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + \frac{1}{2} xsin\ 2x + \frac{1}{4} cos\ 2x + C$$

   where C is the constant of integration.
   Hence, $M = -\frac{1}{2}$, $N = \frac{1}{2}$, and $P = \frac{1}{4}$.

5. Choose the set of correct options about estimating the area of the region bounded by the graph of function $f(x) = x^2 + 1$, above the interval [0,3] using Riemann sums.

   - ◯ **Option 1:** Estimated area will be 17 sq unit, by taking 3 subintervals of equal length and the right end points of the subintervals for the height of the rectangles.
   - ◯ Option 2: Estimated area will be 12 sq unit, by taking 3 subintervals of equal length and the left end points of the subintervals for the height of the rectangles.
   - ◯ **Option 3:** Estimated area will be $\frac{47}{4}$ sq unit, by taking 3 subintervals of equal length and the mid points points of the subintervals for the height of the rectangles.
   - ◯ **Option 4:** Estimated area will be 12 sq unit, by taking n subintervals of equal length and the right end points of the subintervals for the height of the rectangles, as n tends to ∞.

   Solution: If we divide $[0, 3]$ in 3 different sub-intervals of equal length, we get the partition: $\{0, 1, 2, 3\}$.

   - The estimated area by taking the right end points of the subintervals for the height of the rectangles is:
     $(1-0)f(1) + (2-1)f(2) + (3-2)f(3) = 1f(1) + 1(f(2) + 1f(3) = 2 + 5 + 10 = 17$ sq. units.
   - The estimated area by taking the left end points of the subintervals for the height of the rectangles is:
     $(1-0)f(0) + (2-1)f(1) + (3-2)f(2) = 1f(0) + 1(f(1) + 1f(2) = 1 + 2 + 5 = 8$ sq. units.
   - The estimated area by taking the mid points of the subintervals for the height of the rectangles is:
     $(1-0)f(\frac{1}{2})+(2-1)f(\frac{3}{2})+(3-2)f(\frac{5}{2}) = 1f(\frac{1}{2})+1(f(\frac{3}{2})+1f(\frac{5}{2}) = \frac{1}{4}+1+\frac{9}{4}+1+\frac{25}{4}+1 = \frac{47}{4}$ sq. units.

   If $[0, 3]$ is divided in n subintervals of equal length, then we get the partition:

   $$\left\{0, \frac{3}{n}, \frac{6}{n}, \frac{3(n-1)}{n}, \frac{3n}{n}\right\}$$

   The estimated area by taking the right end points of the subintervals for the height of the rectangles is:

   $$\frac{3}{n}f\left(\frac{3}{n}\right) + \frac{3}{n}f\left(\frac{6}{n}\right) + \ldots + \frac{3}{n}f\left(\frac{3(n-1)}{n}\right) + \frac{3}{n}f\left(\frac{3n}{n}\right)$$
   $$= \frac{3}{n}\left(\left(\frac{3}{n}\right)^2 1^2 + 1 + \left(\frac{3}{n}\right)^2 2^2 + 1 + \ldots + \left(\frac{3}{n}\right)^2 (n-1)^2 + 1 + \left(\frac{3}{n}\right)^2 n^2 + 1\right)$$

$$
\begin{aligned}
&= \frac{3}{n}\left(\frac{3}{n}\right)^2 \left(1^1 + 2^2 + \ldots + n^2\right) + \frac{3}{n}n \\
&= \frac{3^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 3 \\
&= \frac{3^3}{6}1(1+\frac{1}{n})(2+\frac{1}{n}) + 3
\end{aligned}
$$

As $n \to \infty$, this sum converges to $9 + 3 = 12$.

Hence the estimated area will be 12 sq unit, by taking n subintervals of equal length and the right end points of the subintervals for the height of the rectangles, as n tends to ∞.

Note: Observe that this value we obtained above is same as

$$\int_0^3 (x^2+1)\ dx$$

3 Numerical Answer Type (NAT)

6. Let $\int_0^{\frac{\pi}{2}} e^x sin\ x = \frac{1}{2}(e^{\frac{\pi}{2}} + a)$. What will be the value of a? [Answer: 1]

Solution: Let $I = \int_0^{\frac{\pi}{2}} e^x sin\ x\ dx$

By integrating by parts we get, $I = \int_0^{\frac{\pi}{2}} e^x sin\ x\ dx = -e^x cos\ x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x cos\ x\ dx$

$$= -e^x cos\ x \Big|_0^{\frac{\pi}{2}} + e^x sin\ x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x sin\ x\ dx$$

$$= -e^x cos\ x \Big|_0^{\frac{\pi}{2}} + e^x sin\ x \Big|_0^{\frac{\pi}{2}} - I = 1 + e^{\frac{\pi}{2}} - I$$

Hence, $2I = e^{\frac{\pi}{2}} + 1$, i.e., $I = \frac{1}{2}(e^{\frac{\pi}{2}} + 1)$.

Therefore, $a = 1$.

7. What will be the number of critical points of the function $f(x) = \dfrac{1}{6}(2x^3 + 3x^2 + 6)$? [Answer: 2]

   Solution:
   Number of critical points will be same as the number of solutions of the following equation,
   $$f'(x) = x^2 + x = 0$$
   The number of solutions of $x^2 + x$ is 2. Therefore, there are 2 critical points of the function $f(x) = \dfrac{1}{6}(2x^3 + 3x^2 + 6)$

4 Comprehension Type Question:

Suppose $f_1(x) = x^3$ and $f_2(x) = x$ denote the profits of Company A and Company B, respectively, throughout 1 year (the beginning of the year is denoted by $x = 0$ and the ending denoted by $x = 1$). The predicted profits of Company A and Company B of the same year are given by the functions $g_1(x) = \sqrt{x}$ and $g_2(x) = e^x$, respectively. The curves represented by the functions f_1 and g_1 are shown in Figure M2W3G2, and the curves represented by the functions f_2 and g_2 are shown in Figure M2W3G3.

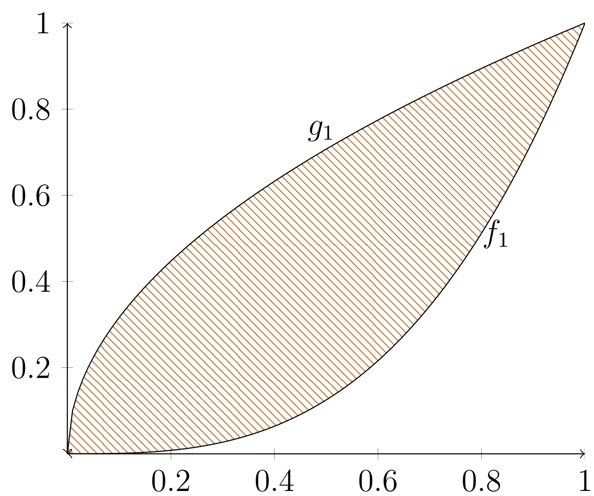


Figure: M2W3G2

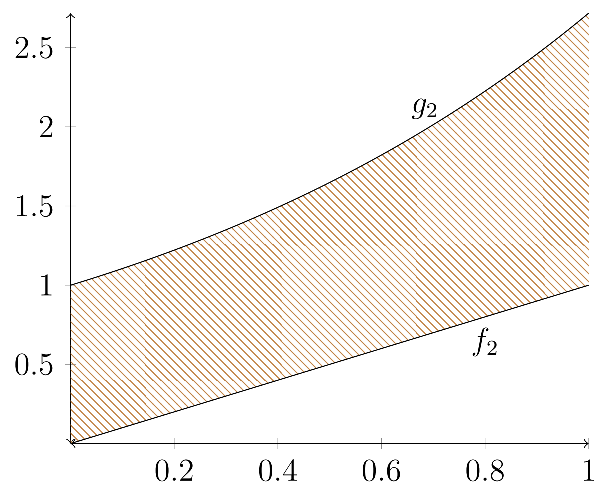


Figure: M2W3G3

Suppose the area of the region bounded by the two curves (the original curve and the predicted curve) in the interval $[0, 1]$ is defined to be the error in prediction.
Using the information above, answer the following questions.

8. What will be the difference between the minimum values of f_2 and g_2 in the interval $[0, 1]$?

   ◯ Option 1: 0

   ◯ **Option 2:** 1

   ◯ Option 3: $e - 1$

   ◯ Option 4: Cannot be determined from the given information.

   Solution: As both the functions f_2 and g_2 are increasing in the interval $[0, 1]$, the minimum values for both the functions will be at the origin, i.e. at $x = 0$. The minimum value of f_2 is 0 and the minimum value of g_2 is 1. Hence the difference between the minimum values of f_2 and g_2 will be 1, in the interval $[0, 1]$

9. What will be error in prediction for Company A?

- ◯ Option 1: $\frac{1}{4}$.
- ◯ Option 2: $\frac{2}{3}$
- ◯ **Option 3:** $\frac{5}{12}$
- ◯ Option 4: $\frac{11}{12}$

Solution: The error in prediction for Company A is the area enclosed by the functions f_1 and g_1 in the interval $[0, 1]$.
The area A_1 between the graph of f_1 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 f_1(x)\ dx = \int_0^1 x^3\ dx = \frac{x^4}{4}\Bigg|_0^1 = \frac{1}{4}.$$

The area A_2 between the graph of g_1 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 g_1(x)\ dx = \int_0^1 \sqrt{x}\ dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 = \frac{2}{3}.$$

The area enclosed by the functions f_1 and g_1 in the interval $[0, 1]$ is$= A_2 - A_1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$.

10. Choose the correct option from the following.
    - ◯ Option 1: The error in prediction for company A is more than that for company B.
    - ◯ **Option 2:** The error in prediction for company B is more than that for company A.
    - ◯ Option 3: The errors in prediction for both companies are equal.
    - ◯ Option 4: The error in prediction for Company A and Company B, cannot be compared using the given information.

Solution: The error in prediction for Company B is the area enclosed by the functions f_2 and g_2 in the interval $[0, 1]$.
The area A_3 between the graph of f_2 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 f_2(x)\ dx = \int_0^1 x\ dx = \frac{x^2}{2}\Bigg|_0^1 = \frac{1}{2}.$$

The area A_4 between the graph of g_2 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 g_2(x)\ dx = \int_0^1 e^x\ dx = e^x\Bigg|_0^1 = e - 1.$$

The area enclosed by the functions f_2 and g_2 in the interval $[0, 1]$ is$= A_4 - A_3 = e - 1 - \frac{1}{2} = e - \frac{3}{2}$.
Clearly, $e - \frac{3}{2} > \frac{5}{12}$.
Therefore, the error in prediction for company B is more than that for company A.