Statistics for Data Science-2

Week 6 Solve with us

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1. The joint pdf of two continuous random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} ke^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of k.

(a) $\frac{1}{2}$

(b) 1

(c) 2

(d) 3

Solution:

We know that

$$\int\int_{\text{Supp}(X,Y)} f_{XY}\, dxdy = 1$$

Therefore,

$$\int_{y=0}^{\infty}\int_{x=0}^{\infty}(ke^{-(x+y)})dxdy = 1$$

$$\Rightarrow k\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-y}e^{-x}dxdy = 1$$

$$\Rightarrow k\int_{y=0}^{\infty} e^{-y}(-e^{-x})\Big|_{0}^{\infty} dy = 1$$

$$\Rightarrow k\int_{y=0}^{\infty} e^{-y}(0+1)dy = 1$$

$$\Rightarrow k \int_{y=0}^{\infty} e^{-y} dy = 1$$

$$\Rightarrow k(-e^{-y})\Big|_{0}^{\infty} = 1$$

$$\Rightarrow k(0+1) = 1$$

$$\Rightarrow k = 1$$

2. The joint pdf of two continuous random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} ke^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(X \geq 4, Y \leq 4)$.

(a) $e^4(1 - e^4)$

(b) $e^{-4}(1 - e^{-4})$

(c) $e^4(1 + e^4)$

(d) $e^{-4}(1 + e^{-4})$

Solution:

From the previous question, we have $k = 1$. So, the joint PDF of X and Y will be

$$f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

To find: $P(X \geq 4, Y \leq 4)$

Now,

$$
\begin{aligned}
P(X \geq 4, Y \leq 4) &= \int_{y=0}^{4} \int_{x=4}^{\infty} (e^{-(x+y)}) dxdy \\
&= \int_{y=0}^{4} \int_{x=4}^{\infty} e^{-y} e^{-x} dxdy \\
&= \int_{y=0}^{4} e^{-y}(-e^{-x}) \Big|_{4}^{\infty} dy \\
&= \int_{y=0}^{4} e^{-y}(0 + e^{-4}) dy \\
&= (e^{-4}) \int_{y=0}^{4} e^{-y} dy
\end{aligned}
$$

$$
\begin{aligned}
P(x \geq 4, Y \leq 4) &= (e^{-4})(-e^{-y})\Big|_{0}^{4} \\
&= (e^{-4})(-e^{-4} + 1) \\
&= (e^{-4})(1 - e^{-4})
\end{aligned}
$$

3. The joint pdf of two random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 3xy(1-x) & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $P(X > \frac{3}{4} | Y = \frac{1}{2})$.

(a) $\frac{7}{64}$
(b) $\frac{5}{32}$
(c) $\frac{7}{32}$
(d) $\frac{5}{64}$

Solution:

We know that

$$P(a < X < b|Y = y) = \frac{f_{XY}(a < X < b, y)}{f_Y(y)}$$

Now,

$$\begin{aligned} f_Y(y) &= \int_0^1 3xy(1-x)dx \\ &= \int_0^1 (3xy - 3x^2y)dx \\ &= \left(\frac{3x^2y}{2} - x^3y\right)\Bigg|_0^1 \\ &= \frac{3y}{2} - y = \frac{y}{2} \end{aligned}$$

Therefore, $f_Y(\frac{1}{2}) = \frac{1}{4}$

Now,

$$\begin{aligned}
P(X > \frac{3}{4} | Y = \frac{1}{2}) &= \frac{f_{XY}(X > \frac{3}{4}, Y = \frac{1}{2})}{f_Y(\frac{1}{2})} \\
&= 4 f_{XY}(X > \frac{3}{4}, Y = \frac{1}{2}) \\
&= \int_{x=\frac{3}{4}}^{1} (4\frac{3x}{2}(1-x))dx \\
&= 2\int_{\frac{3}{4}}^{1} (3x - 3x^2)dx \\
&= 2\left(\frac{3x^2}{2} - x^3\right)\Bigg|_{\frac{3}{4}}^{1} \\
&= 2\left(\frac{3}{2} - 1\right) - 2\left(\frac{27}{32} - \frac{27}{64}\right) = \frac{5}{32}
\end{aligned}$$

Question: 4

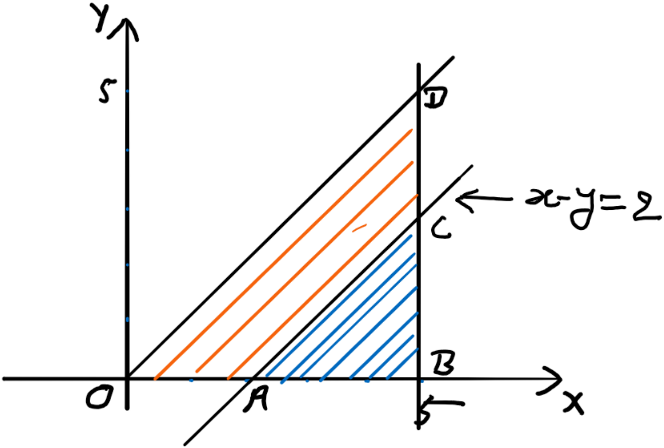
The amount of milk (in litres) in a shop at the beginning of any day is a random amount X from which a random amount Y (in litres) is sold during that day. Assume that the joint density function of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} \frac{2}{25} & 0 \leq x \leq 5, 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that amount of milk left at the end of day is less than 2 litres.

(a) $\frac{8}{25}$
(b) $\frac{15}{25}$
(c) $\frac{16}{25}$
(d) $\frac{1}{5}$

Solution:



In the above figure, support of (X, Y) will be region inside the triangle OBD.
Area of triangle OBD $= \frac{1}{2} \times 5 \times 5 = \frac{25}{2}$.
Therefore, given joint PDF is uniform in its support.
To find: $X - Y < 2$
The region $X - Y < 2$ will be the region OACDO(orange region).

$$\begin{aligned}\text{Area of region OACDO} &= \text{Area of triangle OBD} - \text{area of triangle ABC} \\ &= \frac{1}{2} \times 5 \times 5 - \frac{1}{2} \times 3 \times 3 \\ &= \frac{16}{2} = 8\end{aligned}$$

$$P(X - Y < 2) = \frac{\text{area of region OACDO}}{\text{area of region OBD}}$$
$$= \frac{8}{25/2} = \frac{16}{25}$$

Question: 5

The joint pdf of two random variables X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \frac{3}{2}xy & 0 \le x \le 2, 0 \le y \le 2, x + y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

(a) Yes

(b) No

Solution:

$$
\begin{aligned}
f_X(x) &= \int_0^{2-x} (\frac{3}{2}xy)dy \\
&= \left(\frac{3}{4}xy^2\right)\Bigg|_0^{2-x} \\
&= \frac{3}{4}x(2-x)^2
\end{aligned}
$$

Similarly,

$$f_Y(y) = \int_0^{2-y} (\frac{3}{2}xy)dx$$
$$= \left(\frac{3}{4}x^2y\right)\Big|_0^{2-y}$$
$$= \frac{3}{4}y(2-y)^2$$

Clearly, $f_{XY}(x,y) \neq f_X(x)f_Y(y)$.
Hence, X and Y are not independent.

Question: 6

A person randomly chooses a battery from a store which has 30 batteries of type A and 70 batteries of type B. Battery life of type A and type B batteries are exponentially distributed with average life of 3 years and 7 years, respectively. If the chosen battery lasts for 5 years, what is the probability that the battery is of type A?

(a) $\dfrac{1}{1+e^{\frac{5}{7}}}$

(b) $\dfrac{1}{1+e^{\frac{-5}{7}}}$

(c) $\dfrac{e^{\frac{-5}{3}}}{1+e^{\frac{-5}{7}}}$

(d) $\dfrac{1}{1+e^{\frac{20}{21}}}$

Solution:

Define a event X as follows:

$$X = \begin{cases} 1 & \text{If the chosen battery is of type A} \\ 0 & \text{If the chosen battery is of type B} \end{cases}$$

Let Y denote the battery life of the chosen battery.
By the given information, we have
$Y|X = 1 \sim \text{Exp}(\frac{1}{3})$ and
$Y|X = 0 \sim \text{Exp}(\frac{1}{7})$
It implies that
$f_{Y|X=1}(y) = \frac{1}{3}e^{\frac{-y}{3}}; y > 0$ and
$f_{Y|X=0}(y) = \frac{1}{7}e^{\frac{-y}{7}}; y > 0$

Also given that

$$P(X=1)=\frac{30}{100}=\frac{3}{10} \text{ and}$$

$$P(X=0)=\frac{70}{100}=\frac{7}{10}$$

To find: $f_{X|Y=5}(1)$. Now,

$$\begin{aligned}
f_{X|Y=5}(1) &= \frac{f_{Y|X=1}(5).P(X=1)}{f_Y(5)} \\
&= \frac{f_{Y|X=1}(5).P(X=1)}{f_{Y|X=1}(5).P(X=1)+f_{Y|X=0}(5).P(X=0)} \\
&= \frac{\frac{1}{3}e^{\frac{-5}{3}}.\frac{3}{10}}{\frac{1}{3}e^{\frac{-5}{3}}.\frac{3}{10}+\frac{1}{7}e^{\frac{-5}{7}}.\frac{7}{10}}
\end{aligned}$$

$$= \frac{\frac{1}{10}e^{\frac{-5}{3}}}{\frac{1}{10}e^{\frac{-5}{3}} + \frac{1}{10}e^{\frac{-5}{7}}}$$

$$= \frac{e^{\frac{-5}{3}}}{e^{\frac{-5}{3}} + e^{\frac{-5}{7}}}$$

$$= \frac{1}{1 + e^{\frac{20}{21}}}$$