

Limits for sequences

Sarang S. Sane



Example : The limit of a sequence

Consider the sequence of numbers $2 - \frac{1}{n}$ as n increases.



More examples : Limits of sequences

The sequence $2 - \frac{1}{n^2}$ as n increases :



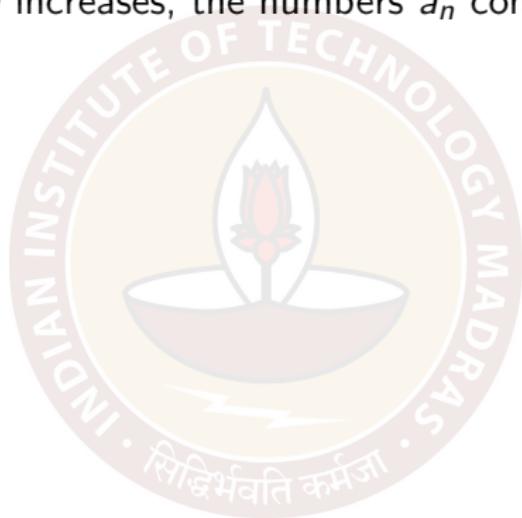
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The sequence $2 - \frac{1}{(1+\log(n))^{1.1}}$ as n increases :



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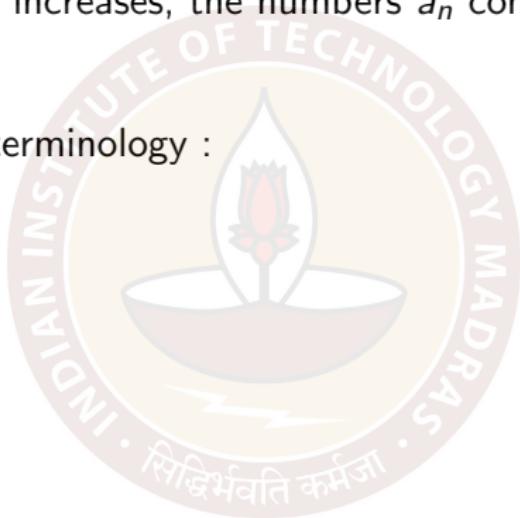
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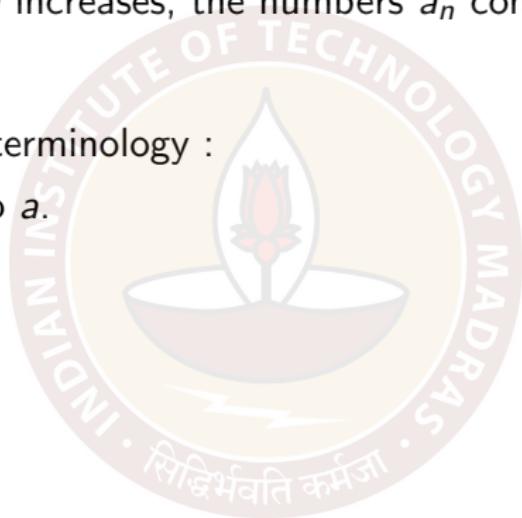


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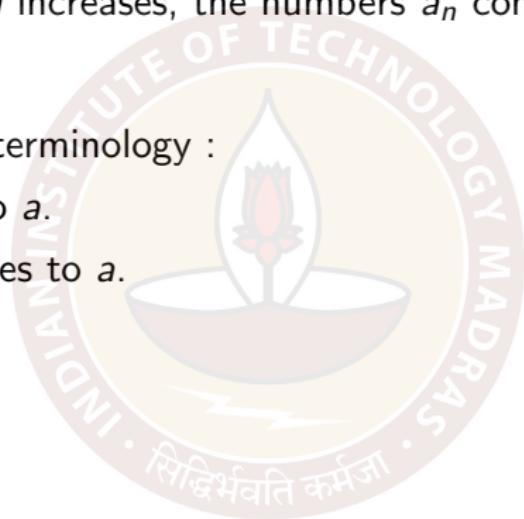


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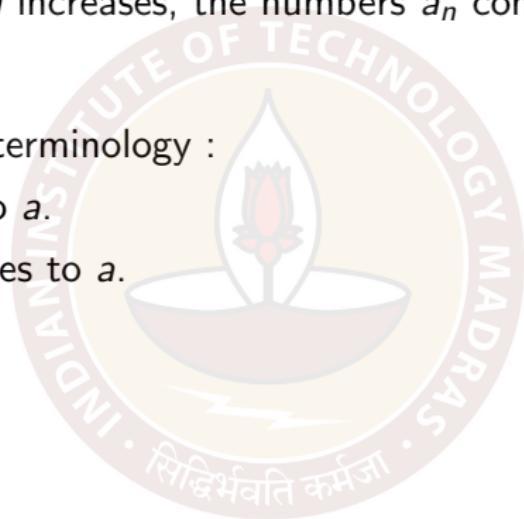


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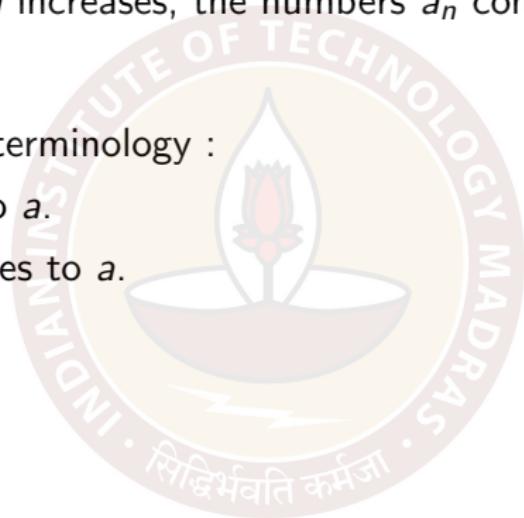


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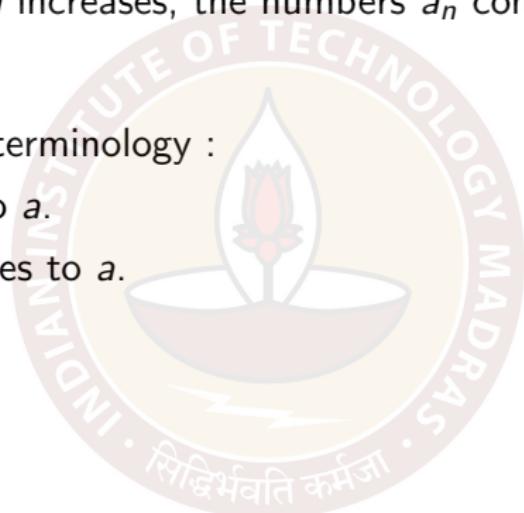


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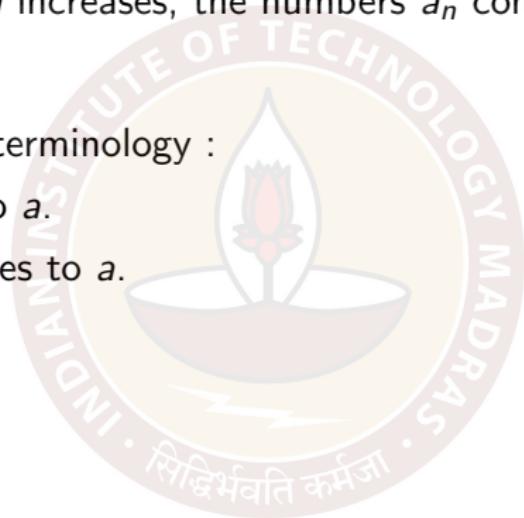


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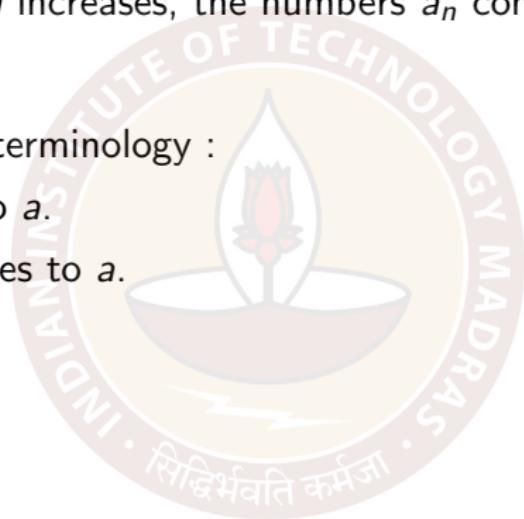


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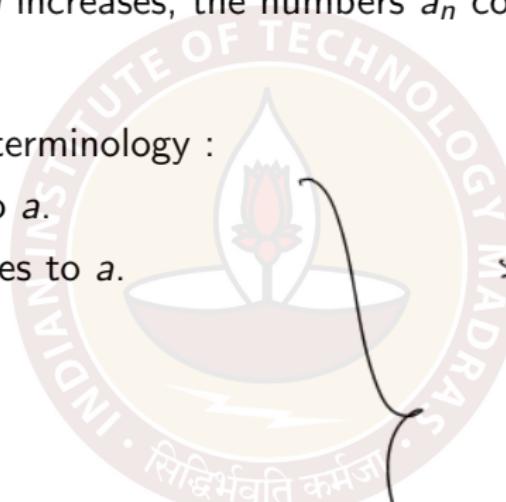
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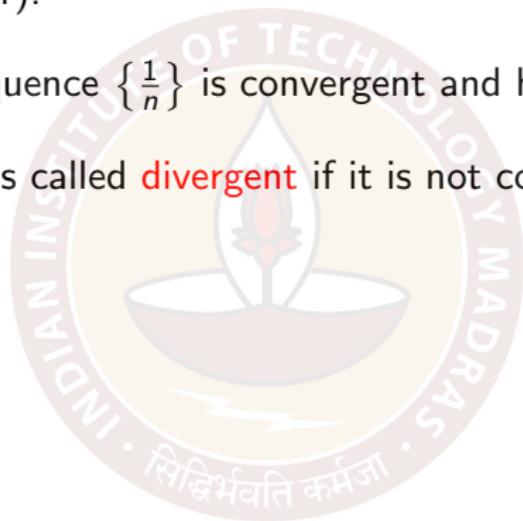


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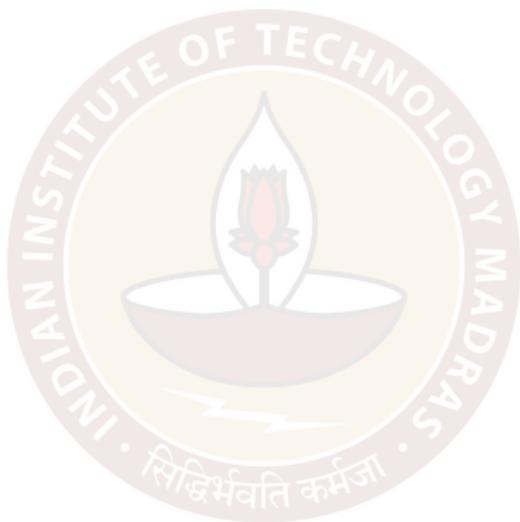
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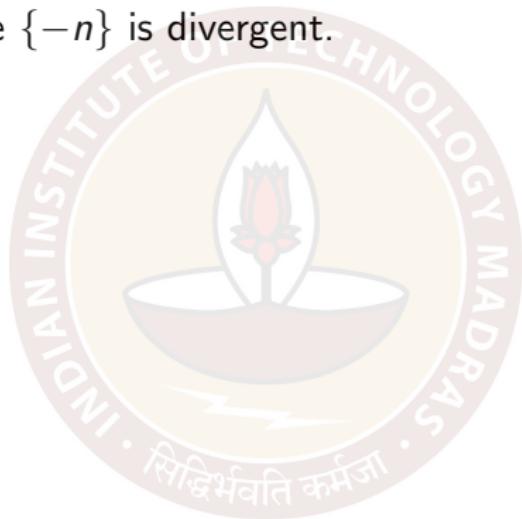
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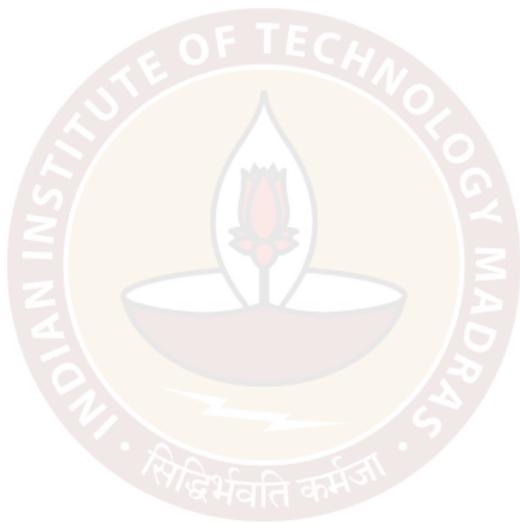
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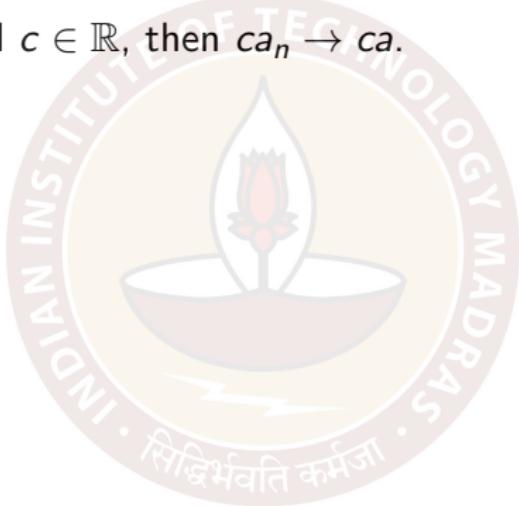
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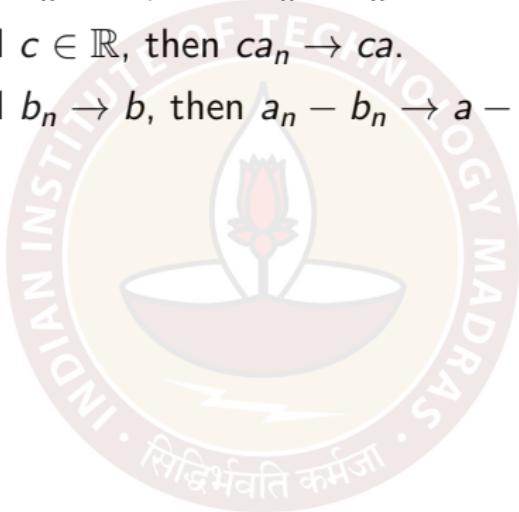
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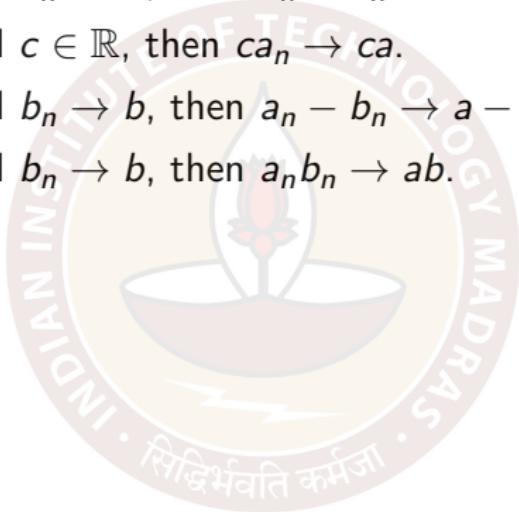
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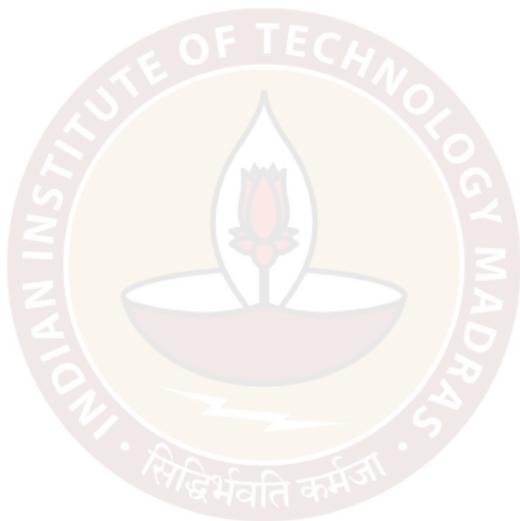
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$$\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5},$$

$$\frac{-1}{n} \rightarrow 0, \quad b_n$$

$$c_n = \frac{(-1)^n}{n}$$
$$a_n \leq c_n \leq b_n \rightarrow 0$$

2. $\left\{ \frac{\frac{1}{\ln(1+n)} + \frac{5n^2}{1+n^2}}{(1+\frac{1}{n})^{2n}} \right\}$ converges to $\frac{5}{e^2}$

$$\ln(1+n) \rightarrow \infty$$

(diverges $\rightarrow \infty$)

$$\frac{1}{\ln(1+n)} \rightarrow 0$$

$$\cdot \text{सिद्धि भवति कर्मजा संशोधनी कार्यालय}$$

$$\frac{5n^2}{1+n^2} \rightarrow \frac{5}{1+1} = 5$$

$$\left(1 + \frac{1}{n}\right)^{2n} = \left\{ \underbrace{\left(1 + \frac{1}{n}\right)}_e^n \right\}^2 \rightarrow e^2$$

$$1 + \frac{1}{n^2} \rightarrow 1$$

Thank you

