The Hessian matrix and local extrema for $f(x, y, z)$

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Understanding the terms better

The terms involved in the test are : f_{xx}, $\boxed{(f_{xx}f_{yy} - f_{xy}^2)}(\underset{\sim}{a})$ and $det(Hf(\underset{\sim}{a}))$.

$$Hf(\underset{\sim}{a}) \quad \begin{bmatrix} f_{xx}(\underset{\sim}{a}) & f_{xy}(\underset{\sim}{a}) & f_{xz}(\underset{\sim}{a}) \\ f_{xy}(\underset{\sim}{a}) & f_{yy}(\underset{\sim}{a}) & f_{yz}(\underset{\sim}{a}) \\ f_{xz}(\underset{\sim}{a}) & f_{yz}(\underset{\sim}{a}) & f_{zz}(\underset{\sim}{a}) \end{bmatrix}$$

1×1	2×2	3×3	
+	+	+	→ local min.
−	+	−	→ local max.

all other non-degenerate cases i.e. $det(Hf(\underset{\sim}{a})) \neq 0$ → saddle point

degenerate case $det(Hf(\underset{\sim}{a}))=0$ Inconclusive.

Examples

$$f(x,y,z) = x^2 + y^2 + z^2$$

$\nabla f = (2x, 2y, 2z)$ Critical pt. : $(0,0,0)$

$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Hf(0,0,0)$.

$\det(Hf(0,0,0)) = 8 > 0$

$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4 > 0$

$f_{xx}(0,0,0) = 2 > 0$.

$\therefore (0,0,0)$ is a local min.

$$f(x,y,z) = -x^2 - y^2 - z^2$$

$\nabla f = (-2x, -2y, -2z)$ Critical pt. $(0,0,0)$.

$Hf = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = Hf(0,0,0)$.

$\det(Hf(0,0,0)) = -8 < 0$. $-$

$\det\left(\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right) = 4 > 0$ $+$

$f_{xx}(0,0,0) = -2 < 0$ $-$

$\therefore (0,0,0)$ is a local max.

$$f(x,y,z) = x^2 - y^2 + z^2$$

$(0,0,0)$

$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(Hf) = -8$ $-$

$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\right) = -4$ $-$

$f_{xx}(0,0,0) = 2$ $+$

$\therefore (0,0,0)$ is a saddle pt.

$$f(x,y,z) = x^4 + y^4 + z^4$$

$\nabla f = (4x^3, 4y^3, 4z^3)$. Critical pt. $(0,0,0)$.

$Hf = \begin{bmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{bmatrix}$.

$Hf(0,0,0) = O_{3\times 3}$.

Inconclusive.

Example

$$f(x, y) = xy + yz + zx$$

$$\nabla f = (y+z,\ z+x,\ x+y).$$

Equating to 0, we get $\left.\begin{array}{l} x = -y = z \\ x = -z. \end{array}\right\} \Rightarrow x = y = z = 0.$

Critical pt. $(0,0,0)$.

$$Hf = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = Hf(0,0,0).$$

$$\det(Hf(0,0,0)) = 0 \times \det\begin{bmatrix} & \end{bmatrix} - 1 \times \det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 1 \times \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= 1 + 1 = 2 > 0.$$

$$\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0.$$

$\therefore$ $(0,0,0)$ is a saddle point.

Example

$$f(x,y) = x^4 + y^4 + z^4 + xyz$$

$\nabla f = (4x^3 + yz,\ 4y^3 + xz,\ 4z^3 + xy)$.

Equating to $\underset{\sim}{0}$, we get:

$4x^3 + yz = 4y^3 + xz = 4z^3 + xy = 0$.

Case 1 : $x = y = z = 0$.

Case 2 : All are non-zero. $4x^4 = -4xyz = 4z^4$.

$4x^4 + xyz = 0 \Rightarrow 4x^4 = -4xyz$

$4y^4 + xyz = 0 \Rightarrow 4y^4 = -4xyz$

$x^4 = y^4 = z^4 \Rightarrow x = \pm y = \pm z$.

Substitute in $4x^3 + yz = 0 \Rightarrow 4x^3 + x^2 = 0 \Rightarrow x = \frac{1}{4}$ or $x = -\frac{1}{4}$.

or $4x^3 - x^2 = 0$

Critical pts. : $(0,0,0)$, $\boxed{(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, -\frac{1}{4})}$, $\boxed{(-1/4, -1/4, -1/4)}$.

local min.

$(0,0,0) \rightarrow$ Inconclusive.

$$Hf = \begin{bmatrix} 12x^2 & z & y \\ z & 12y^2 & x \\ y & x & 12z^2 \end{bmatrix}. \quad Hf(-\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) = \begin{bmatrix} 3/4 & 1/4 & 1/4 \\ 1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & 3/4 \end{bmatrix}$$

$\det(Hf) > 0$. +

$\det[\ ] > 0$ +

$3/4 > 0$ +

$$Hf(-\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}) = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

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Thank you