Linear independence

Sarang S. Sane

Linear dependence (recall)

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space V is said to be linearly dependent if there exists scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

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Equivalently : $v_1, v_2, \ldots, v_n$ are linearly dependent if the 0 vector can be expressed as a linear combination of $v_1, v_2, \ldots, v_n$ with non-zero coefficients (i.e. at least one coefficient is non-zero).

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Equivalently : A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space V is said to be linearly independent if the only linear combination of $v_1, v_2, \ldots, v_n$ which equals 0 is the linear combination with all coefficients 0.

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Hence $a = 0$, $b = 0$ is the unique solution of the system of linear equations, which implies that the vectors $(-1, 3)$ and $(2, 0)$ are linearly independent.

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Hence, a set of vectors $v_1, v_2, \ldots, v_n$ containing the 0 vector is always a linearly dependent set.

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Conclusion : Two non-zero vectors are linearly independent precisely when they are not multiples of each other .

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Conclusion : If three vectors are linearly independent then none of these vectors is a linear combination of the other two.

Example in $\mathbb{R}^3$

Let us consider three vectors $(1,1,2)$, $(1,2,0)$ and $(0,2,1)$ in $\mathbb{R}^3$ and also consider the following equation:

$$a(1,1,2) + b(1,2,0) + c(0,2,1) = (0,0,0)$$

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Hence we have the following system of linear equations:

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Substituting $b = -a$ and $c = -2a$ in the middle equation yields that $a = 0$, $b = 0$, $c = 0$ is the unique solution of this system. Hence the vectors $(1,1,2)$, $(1,2,0)$ and $(0,2,1)$ are linearly independent.

How to check linear independence in $\mathbb{R}^m$

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In terms of coordinates, let $v_j = (v_{1j}, v_{2j}, \ldots, v_{mj})$; $j = 1, 2, \ldots, n$.

Let us write the linear combination of these vectors with *arbitrary* coefficients $a_1, a_2, \ldots, a_n$ and equate it to 0 :

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Considering each coordinate, we have the following identities :

$$\begin{array}{ccccccccc}
v_{11}a_1 & + & v_{12}a_2 & + & \ldots & + & v_{1n}a_n & = & 0 \\
v_{21}a_1 & + & v_{22}a_2 & + & \ldots & + & v_{2n}a_n & = & 0 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
v_{m1}a_1 & + & v_{m2}a_2 & + & \ldots & + & v_{mn}a_n & = & 0
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Conclusion : To check $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ are linearly independent, we have to check that the homogeneous system of linear equations $Vx = 0$ has only the trivial solution, where the j^{th} column of V is v_j.

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Since the matrix $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}$ is invertible, the system of linear equations has a unique solution $x_1 = x_2 = x_3 = 0$. **Hence the vectors** $(1,2,0)$**,** $(0,2,4)$ **and** $(3,0,0)$ **are linearly independent**.

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Hence, any set of n vectors in $\mathbb{R}^2$ with $n \geq 3$ are linearly dependent.

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Consider the four vectors $(1, 2, 0)$, $(0, 2, 4)$, $(3, 0, 0)$ and $(1, 2, 3)$ in $\mathbb{R}^3$.

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To solve this system, we consider the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 2 & 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 3 & 0 \end{array}\right]$ and apply Gaussian elimination.

Row reduction results in the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & 3/4 & 0 \\ 0 & 0 & 1 & 1/4 & 0 \end{array}\right]$

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Relationship with determinant

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- Now if $det(A) \neq 0$ then $A^{-1} = \frac{1}{det(A)} adj(A)$ exists.

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Since $det(\overset{V}{\cancel{A}}) = 1 \neq 0$, the matrix $\overset{V}{\cancel{A}}$ is invertible and hence **the vectors** $(1,4,2)$**,**$(0,4,3)$ **and** $(1,1,0)$ **are linearly independent**.

Thank you