Linear dependence

Sarang S. Sane

Vector addition (Recall)

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In $\mathbb{R}^n$ vector addition is defined by co-ordinate wise addition:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

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e.g.

$$\begin{aligned}(1.5, -3.3, 7.2, \frac{1}{2}, 1) + (-4, 5.8, 10, 5\frac{1}{2}, -3.4)\\ = (-2.5, 2.5, 17.2, 6, -2.4).\end{aligned}$$

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$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$$

e.g.

$$0.5(1.5, -3.3, 7.2, \frac{1}{2}, 1) = (0.75, -1.65, 3.6, \frac{1}{4}, 0.5).$$

Linear combination of vectors

Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$. The linear combination of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n \in \mathbb{R}$ is the vector $\sum_{i=1}^{n} a_i v_i \in V$.

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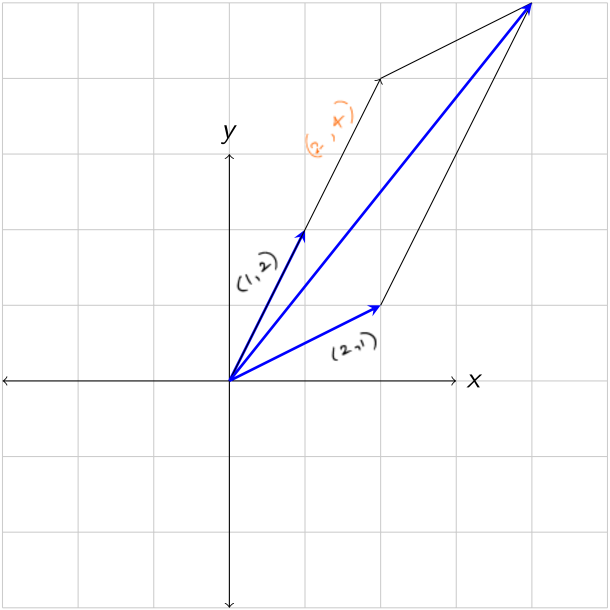
A vector $v \in V$ is a linear combination of $v_1, v_2, \ldots, v_n$ if there exist some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ so that $v = \sum_{i=1}^{n} a_i v_i$.

Example in $\mathbb{R}^2$: $\quad 2(1,2) + (2,1) = (4,5)$



In the previous example we see that $(4,5)$ is a linear combination of vectors $(1,2)$ and $(2,1)$, as follows:

$$2(1,2)+(2,1)=(2,4)+(2,1)=(4,5)$$

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Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

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$$\begin{gathered} \tfrac{1}{2}(4, 5) - \tfrac{1}{2}(2, 1) = (1, 2) \\ (4, 5) - 2(1, 2) = (2, 1) \end{gathered}$$

Note further that we can re-write these expressions as follows :

$$2(1, 2) + (2, 1) - (4, 5) = (0, 0)$$

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Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

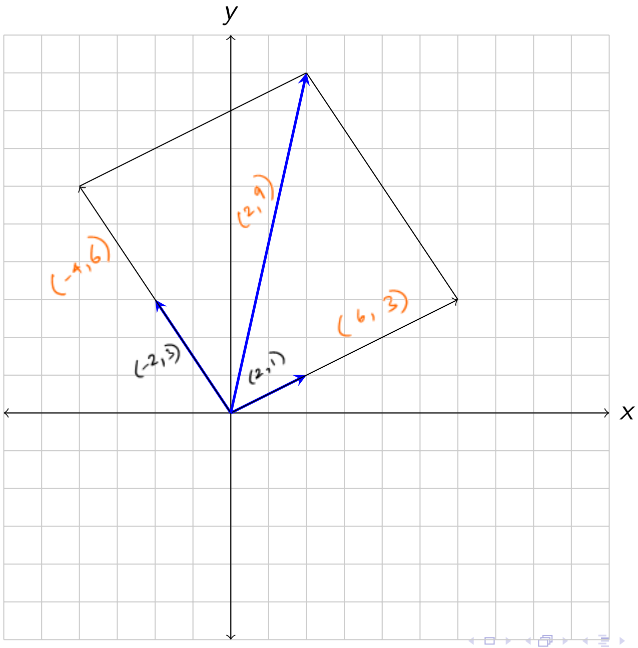
$$\begin{gathered}\tfrac{1}{2}(4,5)-\tfrac{1}{2}(2,1)=(1,2)\\ (4,5)-2(1,2)=(2,1)\end{gathered}$$

Note further that we can re-write these expressions as follows :

$$2(1,2)+(2,1)-(4,5)=(0,0)$$

Observe : the 0 vector is a linear combination of $(1,2),(2,1),(4,5)$ with non-zero coefficients.

Another example in $\mathbb{R}^2$: $\quad 3(2,1) + 2(-2,3) = (2,9)$



In the previous example we see that $(2, 9)$ is a linear combination of vectors $(2, 1)$ and $(-2, 3)$, as follows:

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In the previous example we see that $(2,9)$ is a linear combination of vectors $(2,1)$ and $(-2,3)$, as follows:

$$3(2,1)+2(-2,3)=(6,3)+(-4,6)=(2,9)$$

Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

$$\begin{array}{c} \frac{1}{3}(2,9)-\frac{2}{3}(-2,3)=(2,1) \\ \frac{1}{2}(2,9)-\frac{3}{2}(2,1)=(-2,3) \end{array}$$

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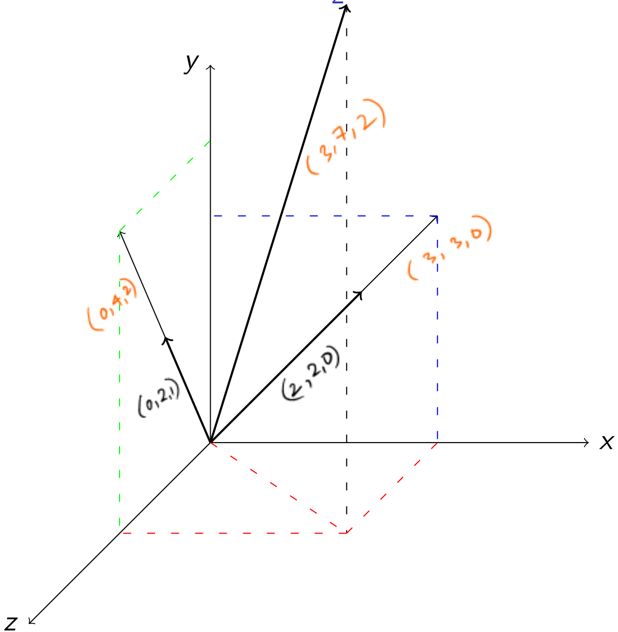
Note further that we can re-write these expressions as follows :

$$3(2,1)+2(-2,3)-(2,9)=(0,0)$$

Observe : the 0 vector is a linear combination of $(2,1),(-2,3),(2,9)$ with non-zero coefficients.

Example in $\mathbb{R}^3$: $\quad 2(0,2,1)+\frac{3}{2}(2,2,0)=(3,7,2)$



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Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

$$\begin{aligned} \tfrac{1}{2}(3, 7, 2) - \tfrac{3}{4}(2, 2, 0) &= (0, 2, 1) \\ \tfrac{2}{3}(3, 7, 2) - \tfrac{4}{3}(0, 2, 1) &= (2, 2, 0) \end{aligned}$$

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In the previous example we see that $(3,7,2)$ is a linear combination of vectors $(0,2,1)$ and $(2,2,0)$, as follows:

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Moreover, each of the vectors in the expression is a linear combination of the other two vectors.

$$\begin{array}{c}\frac{1}{2}(3,7,2)-\frac{3}{4}(2,2,0)=(0,2,1)\\ \frac{2}{3}(3,7,2)-\frac{4}{3}(0,2,1)=(2,2,0)\end{array}$$

Note further that we can re-write these expressions as follows :

$$2(0,2,1)+\frac{3}{2}(2,2,0)-(3,7,2)=(0,0,0)$$

Observe : the 0 vector is a linear combination of $(0,2,1),(2,2,0),(3,7,2)$ with non-zero coefficients.

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Let us choose a vector which is not on the plane, say $(1,2,0)$. We claim that, $(1,2,0)$ cannot be written as a linear combination of $(0,2,1)$ and $(2,2,0)$.

If possible let us assume we can write $(1, 2, 0)$ as a linear combination of the other two vectors as follows,

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which implies, $2b = 1$, $2a + 2b = 2$ and $a = 0$.

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We can use the discussion above to conclude that

$$a(0, 2, 1) + b(2, 2, 0) + c(1, 2, 0) = (0, 0, 0) \text{ if and only if } a = b = c = 0.$$

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We can use the discussion above to conclude that

$$a(0,2,1)+b(2,2,0)+c(1,2,0)=(0,0,0) \text{ if and only if } a=b=c=0.$$

i.e. the only way the 0 vector is a linear combination of $(0,2,1),(2,2,0),(1,2,0)$ is if the coefficients are 0.

Definition of Linear dependence

A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space V is said to be linearly dependent, if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

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$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$$

Equivalently, the 0 vector is a linear combination of $v_1, v_2, \ldots, v_n$ with non-zero coefficients.

More examples

Consider the following two vectors in $\mathbb{R}^3$,

$$(2, 3, 7) \text{ and } \left(\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{35}{6}\right).$$

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It is easy to check that

$$5(2,3,7) - 6(\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{35}{6}) = (0,0,0)$$

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Hence these two vectors are linearly dependent.

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$$5(2, 3, 7) - 6(\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{35}{6}) = (0, 0, 0)$$

Hence these two vectors are linearly dependent. Also observe that one is a scalar multiple of the other.

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Consider the following three vectors in $\mathbb{R}^3$,

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It is easy to check that

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It is easy to check that

$$2(2,1,2)-3(3,0,1)+\frac{1}{2}(10,-4,-2)=(0,0,0)$$

$$(4,2,4)-(9,0,3)+(5,-2,-1)$$

Hence these three vectors are linearly dependent.

More examples

Add one more vector $(2, 3, 7)$ to the set of vectors in the previous slide. Hence we have the following set of vectors in $\mathbb{R}^3$.

$$\{(2, 1, 2),\ (3, 0, 1),\ (10, -4, -2),\ (2, 3, 7)\}$$

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Add one more vector $(2,3,7)$ to the set of vectors in the previous slide. Hence we have the following set of vectors in $\mathbb{R}^3$.

$$\{(2,1,2),\ (3,0,1),\ (10,-4,-2),\ (2,3,7)\}$$

It is easy to check that

$$2(2,1,2) - 3(3,0,1) + \frac{1}{2}(10,-4,-2) + 0(2,3,7) = (0,0,0)$$

More examples

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It is easy to check that

$$2(2,1,2) - 3(3,0,1) + \tfrac{1}{2}(10,-4,-2) + 0(2,3,7) = (0,0,0)$$

It still satisfies the definition of linear dependence as all the scalars are not zero. Hence these four vectors are also linearly dependent.

Important remark

The previous example points to the following fact :

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If a set is linearly dependent, then so is every superset of it.

Thank you