Week-2

Mathematics for Data Science - 2

Continuity and Differentiability

Activity Slides

1. Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(cx) = cf(x)$ for all $c, x \in \mathbb{R}$. Which of the following option(s) is(are) correct?

   ◯ Option 1: $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   ◯ Option 2: f is not continuous in $\mathbb{R}$.

   ◯ Option 3: f is continuous in $\mathbb{R}$.

   ◯ Option 4: $\lim\limits_{x \to a} f(x)$ exists for all $a \in R$, but f is not continuous in $\mathbb{R}$.

   Hint:

   - $f(cx) = cf(x)$ for all $c, x \in \mathbb{R} \implies f(x) = xf(1)$

   Solution:

   Step 1:
   $f(x + y) = (x + y)f(1) = xf(1) + yf(1) = f(x) + f(y)$

   Step 2:
   Let $a \in \mathbb{R}$, and consider a sequence $\{x_n\}$ such that $x_n \to a$.

   $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} x_n f(1) = f(1) \lim\limits_{n\to\infty} x_n = f(1)a = f(a)$

   Option 1 and 3 are correct.

2. Define a function f as follows:

$$f(x) = \begin{cases} \dfrac{1}{e^{\frac{1}{x}} + 1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Which of the following option(s) is(are) true?

- ◯ Option 1: $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$
- ◯ Option 2: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$
- ◯ Option 3: f is a bounded function on $\mathbb{R}$.
- ◯ Option 4: f is continuous at $x = 0$.

Solution:

Step 1: $e^{\frac{1}{x}} > 0 \implies e^{\frac{1}{x}} + 1 > 1 \implies 1 > \dfrac{1}{e^{\frac{1}{x}} + 1} > 0$

Step 2: Left hand limit

Consider a sequence $\{x_n\}$ such that $x_n < 0$ and $x_n \to 0$.

$\frac{1}{x_n} \to -\infty \implies e^{\frac{1}{x_n}} \to 0 \implies e^{\frac{1}{x_n}} + 1 \to 1 \implies \frac{1}{e^{\frac{1}{x_n}}+1} \to 1$

Step 3: Right hand limit

Consider a sequence $\{x_n\}$ such that $x_n > 0$ and $x_n \to 0$.

$\frac{1}{x_n} \to \infty \implies e^{\frac{1}{x_n}} \to \infty \implies e^{\frac{1}{x_n}} + 1 \to \infty \implies \frac{1}{e^{\frac{1}{x_n}}+1} \to 0$

LHL $\neq$ RHL

3. Which of the following options showing step wise solution to check whether a function is differentiable or not are true?

   ◯ Option 1: Checking whether a constant function $f(x) = c$ is differentiable at any real number a or not: $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \frac{c+h-c}{h} = \lim\limits_{h\to 0} \frac{h}{h} = 1$.

   ◯ Option 2: Checking whether $f(x) = x - c$ is differentiable at a for some real number a, or not: $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \frac{(a+h-c)-(a-c)}{h} = \lim\limits_{h\to 0} \frac{h}{h} = 1$.

   ◯ Option 3: Checking whether $f(x) = x^2$ is differentiable at any real number a or not: $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \frac{(a+h)^2-a^2}{h} = \lim\limits_{h\to 0} \frac{2ah+h^2}{h} = 0$

   ◯ Option 4: Checking whether $f(x) = e^x$ is differentiable at any real number a or not: $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \frac{e^{a+h}-e^a}{h} = \lim\limits_{h\to 0} \frac{e^a(e^h-1)}{h} = e^a \lim\limits_{h\to 0} \frac{e^h-1}{h} = e^a.1 = e^a$.

   Solution :

- if $f(x) = c$, then $f(a+h) = f(a) = c$. So, option 1 is wrong.

- if $f(x) = x^2$, then $f(a+h) - f(a) = (a+h)^2 - a^2 = 2ah + h^2$.

  $\lim\limits_{h\to 0} \dfrac{2ah+h^2}{h} = \lim\limits_{h\to 0} h \times \dfrac{2a+h}{h} = \lim\limits_{h\to 0} 2a + h = 2a$. So, option 3 is wrong.

4. Consider the function $f(x) = |\sin x|$. Then f is

   ◯ Option 1: periodic with period π.

   ◯ Option 2: everywhere continuous and differentiable.

   ◯ Option 3: everywhere continuous and not differentiable at $n\pi$, where $n \in \mathbb{Z}$.

   ◯ Option 4: neither continuous nor differentiable at $n\pi$, where $n \in \mathbb{Z}$.

   [**Hint:** Try to draw the graph of $|\sin x|$]

Solution :

Step 1:

$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

$\sin(x+\pi) = \sin(x)\cos(\pi) + \sin(\pi)\cos(x) = -\sin(x) \implies |\sin(x+\pi)| = |\sin(x)|$

f is periodic with period π.

Step 2:

$|\sin(x)| = |x| \circ \sin(x)$ (composition)

So, $|\sin(x)|$ is continuous and differentiable on $\mathbb{R}/\{n\pi \mid n \in \mathbb{Z}\}$

Step 3:

Let $a \in \{n\pi \mid n \in \mathbb{Z}\}$,

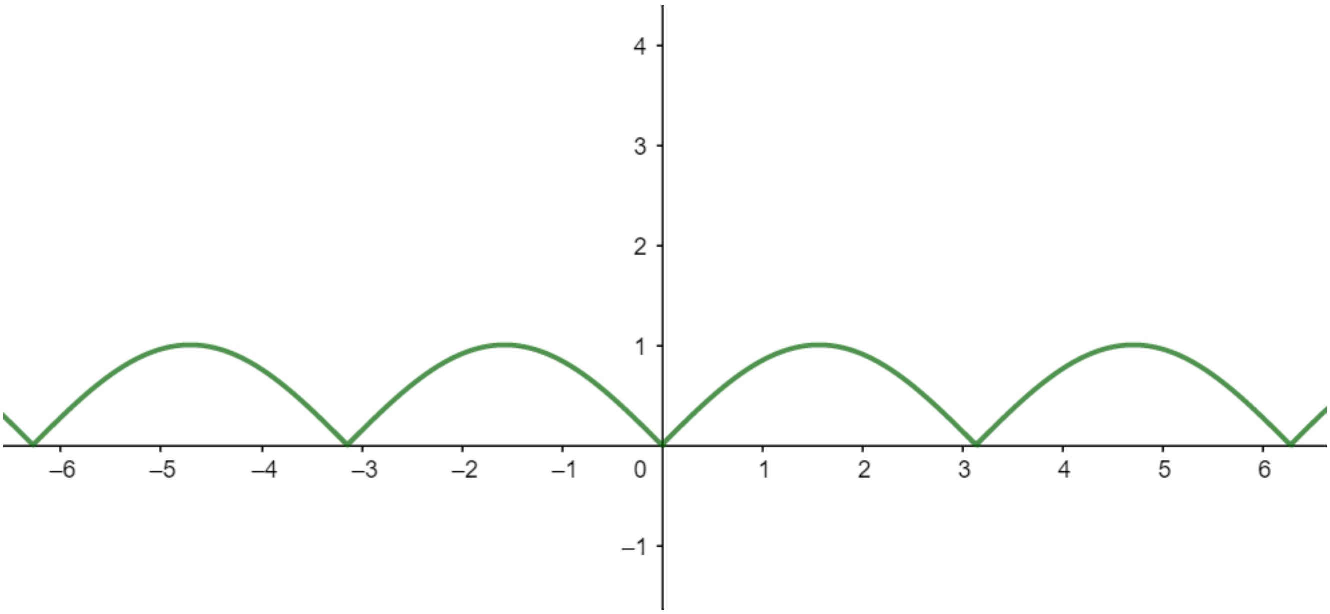
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{|\sin(x)| - |\sin(a)|}{x-a} = \lim_{x\to a} \frac{|\sin(x)| - |\sin(a)|}{x-a} \times \frac{\sin(x)-\sin(a)}{\sin(x)-\sin(a)} =$$

$$\lim_{x\to a} \frac{|\sin(x)| - |\sin(a)|}{\sin(x)-\sin(a)} \times \frac{\sin(x)-\sin(a)}{x-a} = \lim_{x\to a} \frac{|\sin(x)| - |\sin(a)|}{\sin(x)-\sin(a)} \times \lim_{x\to a} \frac{\sin(x)-\sin(a)}{x-a}$$

$$= \lim_{x\to a} \frac{|\sin(x)|}{\sin(x)} \times \cos(a)$$

$\displaystyle\lim_{x\to a} \frac{|\sin(x)|}{\sin(x)}$ does not exist.

Graph of $|\sin x|$:



At points $n\pi$, the graph has sharp "corners". At a sharp corner, there are many possible tangent lines. Hence, $|\sin x|$ is not differentiable on $\{n\pi \,|\, n \in \mathbb{Z}\}$.

5. If $f(x) = \sqrt{9 - x^2}$, then find out the value of $\lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$

- ◯ Option 1: $\dfrac{1}{\sqrt{8}}$
- ◯ Option 2: $-\dfrac{1}{\sqrt{8}}$
- ◯ Option 3: $\sqrt{8}$
- ◯ Option 4: Does not exist

[**Hint:** Use L'Hospital's rule]

Solution:

Step 1:

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\sqrt{9 - x^2} - \sqrt{8}}{x - 1} = \lim_{x \to 1} \frac{-x}{\sqrt{9 - x^2}} = -\frac{1}{\sqrt{8}}$$

6. Let f and g be two distinct functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following statements are true?

 ◯ Option 1: If fg is differentiable, then both f and g are differentiable.

 ◯ Option 2: Assume that $g(x) \neq 0$ for all $x \in \mathbb{R}$. If $\dfrac{f}{g}$ is differentiable, then both f and g are differentiable.

 ◯ Option 3: If f is an even differentiable function, then $f^{'}$ is an odd function.

 ◯ Option 4: If f is an odd differentiable function then, $f^{'}$ is an even function.

Solution :

Step 1: Consider two functions $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$, $g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$.

Step 2: Consider two functions $f(x) = \begin{cases} 4 & \text{if } x \geq 0 \\ 6 & \text{if } x < 0 \end{cases}$, $g(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ 3 & \text{if } x < 0 \end{cases}$.

Step 3: A function f is said to be even iff $f(x) = f(-x)$.

Differentiate both sides,

$$
\begin{aligned}
f'(x) &= f'(-x).\frac{d}{dx}(-x) \\
\implies f'(x) &= f'(-x).(-1) \\
\implies f'(x) &= -f'(-x) \\
\implies f'(-x) &= -f'(x)
\end{aligned}
$$

Step 4: A function f is said to be odd iff $f(-x) = -f(x)$.

Differentiate both sides,

$$
\begin{aligned}
f'(-x).\frac{d}{dx}(-x) &= -f'(x) \\
\implies f'(-x).(-1) &= -f'(x) \\
\implies -f'(-x) &= -f'(x) \\
\implies f'(-x) &= f'(x)
\end{aligned}
$$

7. Let f be a differentiable function at $x = 1$. The tangent line to the curve represented by the function f at the point $(1, 0)$ passes through the point $(5, 8)$. What will be the value of $f'(1)$?

   - ◯ Option 1: 1
   - ◯ Option 2: 2
   - ◯ Option 3: 3
   - ◯ Option 4: 4

   Solution:

   Step:1

   $f'(1)$ is equal to the slope of the line which passes through $(1, 0)$ and $(5, 8)$.