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IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Limits for Scalar-Valued Multivariable Functions

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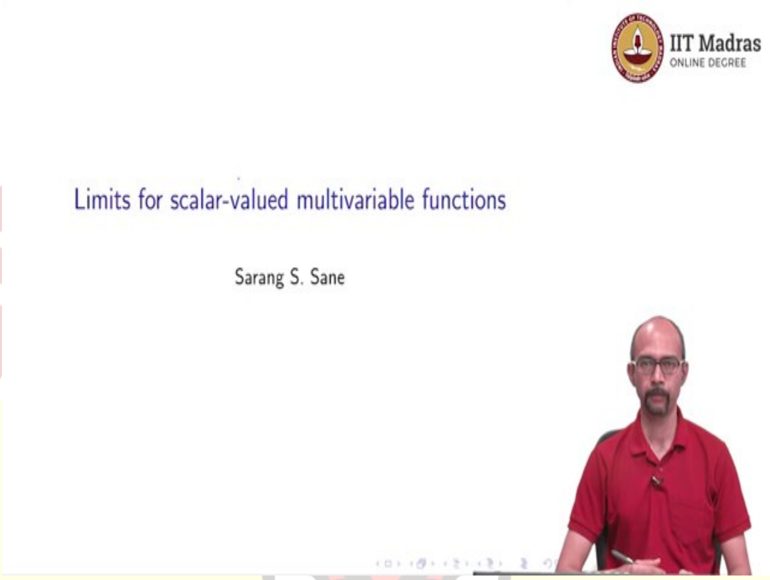

Hello, and welcome to the Maths 2 component of the online BSc program on Data Science and programming. We have seen multivariable functions and today we are going to discuss Limits for Scalar Valued Multivariable Functions. So, let us recall first that we have seen the notion of limits for sequences of real numbers.

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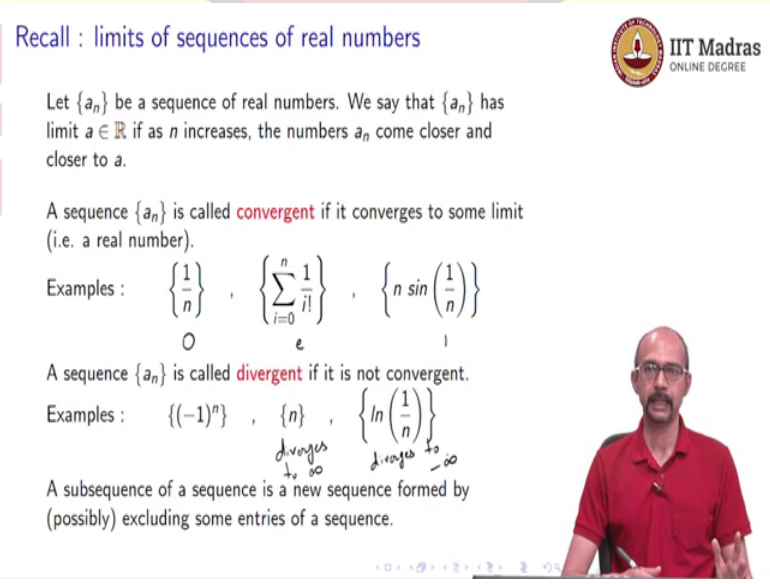

So, if a $\{a_n\}$ is a sequence of real numbers, we say that the $\{a_n\}$ has limit a if as n increases the numbers $\{a_n\}$ come closer and closer to the number a. So, a sequence $\{a_n\}$ is called convergent if it converges to some limit, so that means there is some a, such that $\{a_n\}$ tends to $\{a_n\}$ that is $\{a_n\}$ comes closer and closer to a

Here is a couple of examples. So, $\{\frac{1}{n}\},\{\sum_{i=0}^{n}\frac{1}{i!}\},\{n\sin(\frac{1}{n})\}$ And in a minute, we will quickly recall what these limits are. A sequence an is called divergent, if it is not convergent.

And examples of such sequences are $\{(-1)^n\}$ just n, and then $\{\ln(\frac{1}{n})\}$

So, in a minute, we will see, we will recall what happens to these as well. And also, we defined the notion of a subsequence. So, a subsequence of a sequence is a new sequence, which we formed by possibly excluding some entries of the original sequence. So, you choose only some of the entries of the original sequence.

So now, the goal is to move from real numbers to higher dimensional space, so R^p And in a minute, we will talk about sequences there and convergence over there. So let us recall first that what happens to these sequences. So, for these sequences, $\frac{1}{n}$ for example, tends to 0, $\{\sum_{i=0}^{n}\frac{1}{i!}\}$ if you remember this one, it tends to e, and then $\{n\sin(\frac{1}{n})\}$ you can think of this as sin of 1 by n divided by 1 by n.

And as n tends to infinity, 1 by n tends to 0. And we know that $\frac{\sin x}{x}$ tends to 1. So, this tends to 1. So, this was one of the non-trivial limits that we actually computed in, back when we did one variable calculus. And then we had, these are divergent $\{(-1)^n\}$ is divergent, because it keeps oscillating. So, $\{(-1)^1\}=-1$ then squared is 1, then again – 1, 1, so it keeps oscillating, it does not come close to any number.

So, this just diverges because it oscillates. The number {n} diverges, but it diverges in a different way, than $\{(-1)^n\}$ because we say diverges to infinity. So, what happens is, it keeps increasing, and it becomes larger and larger on the positive side, and it goes to infinity, so we say that this diverges to infinity.

And then similarly, if you take $\{\ln(\frac{1}{n})\}$ if you recall the log function, then this diverges to minus infinity. So, in one variable calculus, we had these special things called infinity and minus infinity. And we had these sort of three different notions of divergence, one was just where it does not come close to anything. And the other two were where it diverges, but it diverges to infinity, and the third was where it diverges to minus infinity. So, now, we are going to use this as a basis for exploring what happens in higher dimensional space.

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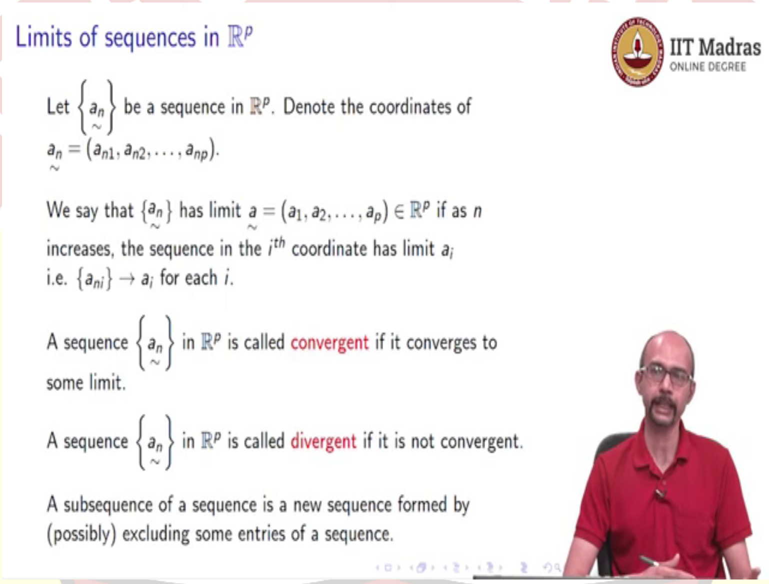

So, limits of sequences in $\mathbb{R}^p$ So, let $\{a_n\}$ tilde be a sequence in $\mathbb{R}^p$ Let us recall that we have been using tilde to emphasise when we are in higher dimension. So $\{a_n\}$ tilde means it is a vector or an element of $\mathbb{R}^p$ and it has p components. So, there are p real numbers. So, those coordinates are denoted by $(a_{n1}, a_{n2}, \ldots\ldots, a_{np})$ for each $\{a_n\}$ tilde. So, you have a sequence of such things. And for each one you have p many coordinates.

So, you can look at the coordinate sequences in particular. So, for each coordinate, suppose I decide to fix the first coordinate, so for the first coordinate, I have $(a_{11}, a_{21}, \ldots\ldots, a_{n1})$ and so on. So, that is the first coordinate sequence. Then you have a second coordinate sequence, $(a_{12}, a_{22}, \ldots\ldots, a_{n2})$ and so on and then all the way up to the pth coordinate sequence. So, that is $a_{1p}, a_{2p},$ and so on.

So, we have such a sequence and, in the sequence, consists of vectors, or elements of $\mathbb{R}^p$ And each of those have p coordinates, that is the main thing you have to remember. And each of those coordinates forms a sequence in R, in the sense that we studied on the previous slide. So, we say that $\{a_n\}$ has limit a tilde, which is some other vector or element of $\mathbb{R}^p$ where the coordinates are $(a_1, a_2, ..., a_p)$ if, as n increases, the sequence in the ith coordinate has limit a_i

So, as I said, what you do is, let us say you take the first coordinate, so from this sequence $\{a_n\}$ tilde, for each of these vectors, you extract the first coordinate. So, then you get $a_{11}, a_{21},$ and so on, that forms a sequence in R and you ask, does it have a limit? Well, if it has a limit, let us say suppose the limit is a1. So, you keep that in store, then you go to the second one, you do $a_{12}, a_{22}, a_{32}, a_{42}$ and so on. Does that have a limit? Well, if that has a limit, you call that limit a2, and then you go on.

And you do this for all p coordinate sequences. And if all of them have limits, then you put that together into a vector, which we have called a tilde. And then we say that this $\{a_n\}$ tilde. So, we say that $\{a_n\}$ tilde has limit a tilde, which is this which has coordinates $(a_1, a_2, ..., a_p)$ if for each coordinate, the sequence tends to the corresponding coordinate of a tilde. So, the sequence $\{a_{ni}\}$ tends to a_i

So basically, the idea of convergence of sequence says, so limits of sequence says in $\mathbb{R}^p$ is just borrowed from R, it is nothing very special, what you do is you look at each coordinate, ask if the sequence over there converges, and if it does converge, you put them all together, that is all that the limit in $\mathbb{R}^p$ does. So, a sequence $\{a_n\}$ tilde in $\mathbb{R}^p$ is called convergent if it converges to some limits. So now, beyond this, the definitions are exactly the same as we have for R.

So, a sequence $\{a_n\}$ tilde in $\mathbb{R}^p$ is called divergent if it is not convergent. And then we will define something called a subsequence of a sequence, which is given by deleting some terms, so excluding some entries in that sequence, or you could take all of them or you could delete some of them, exclude some of them. So, this is exactly in parallel with what happens in R. And in fact, to check limits in $\mathbb{R}^p$ you have to check limits in R, that is what this is saying.

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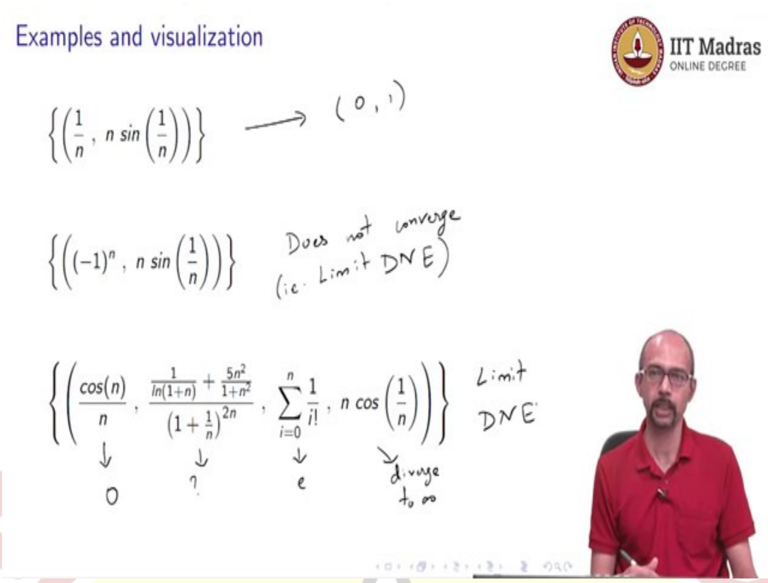

So here is a couple of examples. So, the first example is $\{(\frac{1}{n}, n\sin(\frac{1}{n}))\}$ The second example is $\{((-1)^n, n\sin(\frac{1}{n}))\}$ The third example is a bit bigger, so this is $\{(\cos\frac{(n)}{n}, \frac{\frac{1}{\ln(1+n)}+\frac{5n^2}{1+n^2}}{(1+\frac{1}{n})^{2n}}, \sum_{i=0}^{n}\frac{1}{i!}, n\cos(\frac{1}{n}))\}$ So let us ask what happens to each of these and visualise the first two.

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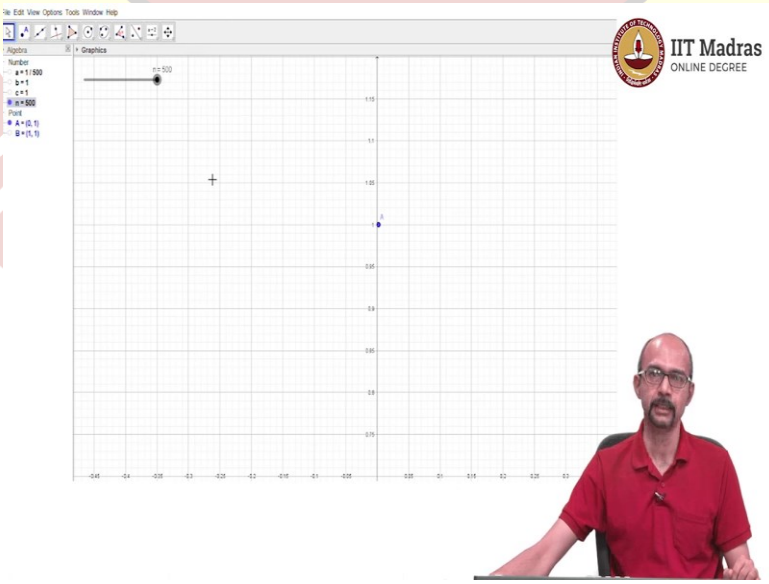

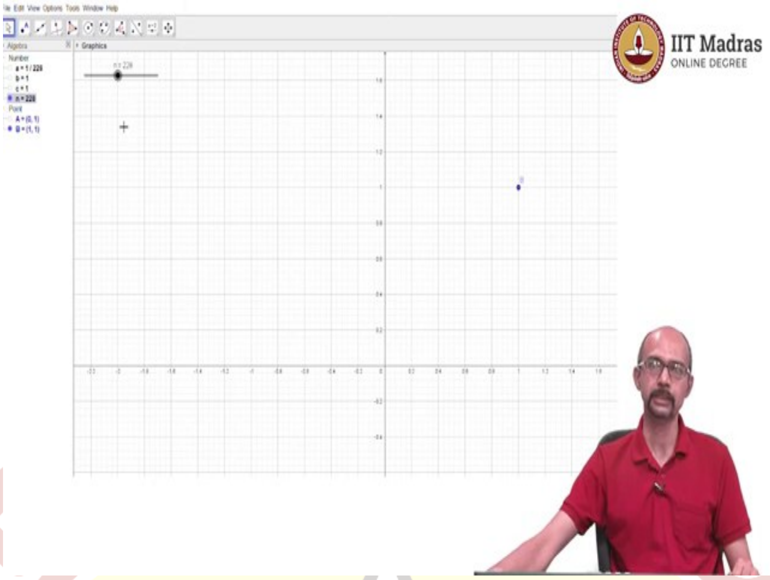

So, the first sequence was $\{(\frac{1}{n}, n\sin(\frac{1}{n}))\}$ So, let us see how this behaves. So, I am plotting this over the range of n as n goes from 1 to 500. And we will keep track of this point a. So, when n is 1, this point is (1, 0.84). So, 1, because 1 by 1 is 1, and, 1 times sin of 1 by 1. So, 1 times sin 1. So, this is (1, sin 1), so sin 1 is 0.84.

And as you increase in, you can see what is happening to this point, it started coming closer and closer and closer and closer to the point (0, 1). And this is at 156, it is already quite close. And you can see it is closing in further and further as we increase it. So, if we make this larger, so it is still not reached, but it is coming very close. And you can see it is moving, moving, moving, moving, and it is coming to 500, where it is almost at (0, 1).

So, if we make it still larger, you can see it is not exactly at (0,1). But, it is very, very, very close. So that says that as n tends to infinity, this sequence, so each coordinate, we know what happens. So, for 1/n tends to 0 as n tends to infinity. So that is the first coordinate. So, that is why in the first coordinate, we have 0.

And in the second coordinate, we have $n\sin(\frac{1}{n})$ which we know because $\frac{\sin x}{x}$ as x tends to 0 tends to 1, we know that the second coordinate, the sequence tends to 1. So, the limit is (0, 1). And we have seen a geometric explanation for that. So, for the second sequence, we have $\{((-1)^n, n\sin(\frac{1}{n}))\}$

Now we know from from what we just did that $(-1)^n$ oscillates, and $n\sin(\frac{1}{n})$ converges to 1. So, this function is, this sequence is also going to oscillate. Let us see the behaviour of of the sequence. So, as it increases, you can see that the point b is changing from, changing value from one side of the axis to the other.

So, as I am at 1, this is (-1, 0.84), then we increase it to 3, it is (-1, 0.98) then we have n is even, say n is 8, this is (1, 1). And after that, it just oscillates between (-1, 1) and (1, 1). And so we know that this sequence is not going to converge, it does not come close to any number in rather any vector in $\mathbb{R}^2$ So, this sequence will not converge.

So, if I play this one, you will see it is still oscillating between (-1, 1) and (1, 1). And we knew that already, as I explained, because $(-1)^n$ oscillates. So, in other words, this does converge and this converges to (0, 1) we saw this graphically also, this does not converge or this limit does not exist. And now we have the last one. So, let us look at these terms.

So, the first term is $\frac{\cos n}{n}$ So, cosine n, as we know, varies between -1 and 1 and, but when you divide it by n, so that means in absolute value, it is between 0 and 1/n. And then by sandwich, we know that this tends to 0. The second one we have done in our previous video on limits, when we did limits for sequences. So, I encourage you to solve it yourself and if you do not remember, go back to that video and ask, see what happens. It is five times something, I will leave that to you.

The third one, we saw in the previous slide goes to e. And what about the fourth one, so the fourth one is $n\cos(\frac{1}{n})$ And unfortunately for this, we know that $\cos(\frac{1}{n})$ as n tends to infinity, so 1/n tends to 0, so $\cos(\frac{1}{n})$ tends to cosine of 0, which is 1, but n shoots off to infinity. So, this is going to go to, this is going to diverge to infinity.

And so, what does this mean? That means that in this sequence as a whole, as a sequence in $\mathbb{R}^4$ three of these coordinate sequences do indeed have limits, but the fourth one does not. So, this limit also does not exist, limit does not exist. So, I hope it is clear what we mean by sequences of limits in higher dimensions. So, we have to just check individual limits for each coordinate.

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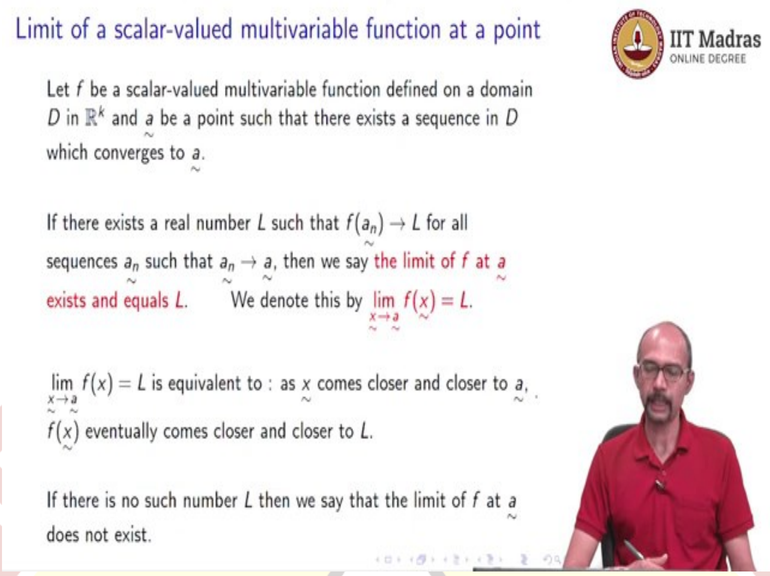

So, let us now talk about the limit of a scalar valued multivariable function at a point. So, this is going to generalise what we saw earlier about one variable calculus, limits of functions of one variable at a point. So, let f be a scalar valued multivariable function defined on a domain D in $\mathbb{R}^k$ and "a tilde" be a point such that there exists a sequence in D which converges to "a tilde".

So, this is some technical condition that we need in order to define the notion of the limit. So, if there exists a real number L, remember this is a scalar valued function. So, f takes values in R, which means real numbers. So, if there exists a real number L such that f of a_n tilde tends to L for all sequences a_n tilde such that a_n tilde tends to a tilde. Then we say that the limit of a at a tilde exists and equals L.

So, this is again, direct generalisation of what we have seen in one variable calculus. We denote this by a limit extends to a tilde f(x) tilde is equal to L. So, this is the same as saying that as x tilde comes closer and closer to the point at a tilde, then f(x) tilde eventually comes closer and closer to the number L. This is what what it means for this limit to exist. And if there is no such number L, we say that the limit of f at "a tilde" does not exist.

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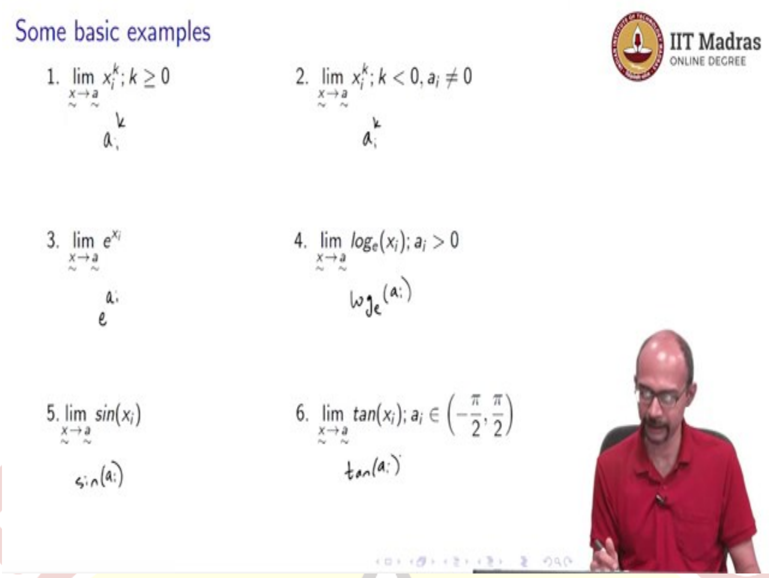

Let us do some basic examples. So, this first limit is a_i^k What is a_i ? a_i is the ith coordinate of the vector a tilde. So, you look at that and raise it to the power k and it is fairly obvious that that is what happens. Because as x tilde tends to a tilde you want x_i^k well, x tilde tends to a tilde means x_i is coming close to a_i that means x_i^k is coming close to a_i^k

Similarly, for the second one, you will get a_i^k because here we have assumed a_i is non-zero. So even if your power is negative, meaning this k is negative, the same argument will hold, how would e^{x_i} well this will go to e^{a_i} How would $\log_e(x_i)$ well again, if a_i is positive, this is going to be $\log_e(a_i)$ And then $\sin(x_i)$ will go to $\sin(a_i)$

Please check this, whatever I am, I am doing this fast, because I think this is very doable from what we have done before. But you should check this for yourself, it is very important that you check these and understand what is being said here based on the definitions in the previous slides, fine.

And then finally, if you have $\tan(x_i)$ and $a_i \in (\frac{-\pi}{2}, \frac{\pi}{2})$ then this is $\tan(a_i)$ So, this is not, this is nothing very deep, it just follows from one variable calculus. And what I am going to try and say, tell you now is that most of the things that we do for easy limits are from one variable calculus. There are difficult limits, of course, and that is what we will be studying as we go along. Those need more refined techniques.

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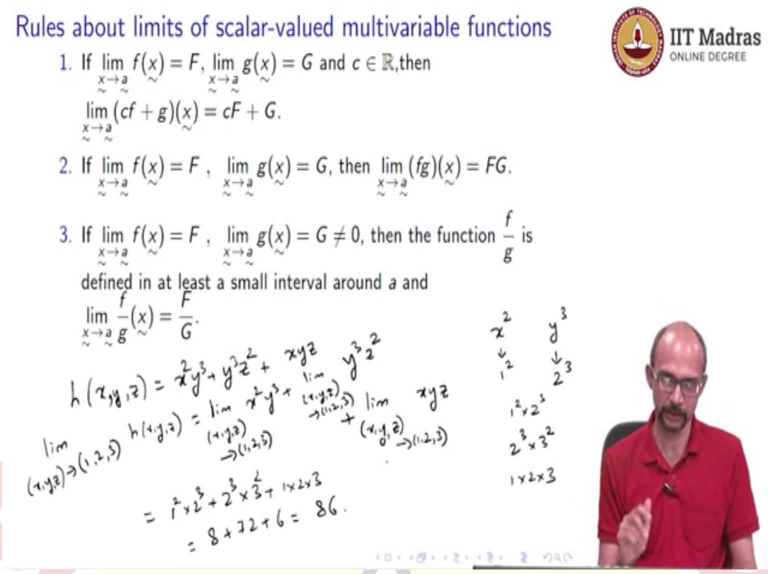

So, let us look at some rules about limits of scalar valued multivariable functions. These are again in the same spirit as the ones for one variable calculus. So, if you have two functions such that, for both of them, the limits as x tilde tends to a tilde exists and equal F and G respectively, and you have some scalar c, then if you take (cf + g), then that limit is cF+G. So, you can push the limit inside and take the scalar out, that is what it says.

And of course, special cases are, of this are where c=1. So, in that case, it will say that the sum of the limits is the limit of the sums. Similarly, you can take c is -1 and that will say that if you take the difference of the functions and take the limit, that is just taking the difference of the limits, you can just take the function g is 0, and in that case, this is saying c times f limit, you can take the c out.

So, the next one is for product. So, if you take the multiplication of two scalar valued functions. And this makes sense because these are scalar valued multivariable functions, then if you take the limit, and both of these limits exist as x tilde tends to a tilde, then the limit is just the product. So, it is FG. Similarly, if you take the quotient $\frac{f}{g}$ here, of course you need the caveat that as x tilde comes close to a tilde this the limit is nonzero. So, this capital G is non-zero. So once that happens, the quotient also has limit and it is exactly $\frac{F}{G}$

So, let us quickly write down what this means based on what we have seen. So, for example, if I have a function like like this, so if I have $h(x,y,z)=x^2y^3+y^3z^2+e^{xyz}$ And I want to ask what is the limit of let us say $(x,y,z)\to(1,2,3)$ Well, let us let us see if I can apply whatever I have above, let me not take e^{xyz} instead let me take just xyz. We will come to e^{xyz} in a minute.

So, if you take this limit, well, first let us look at $x^2 y^3$ We saw on the previous slide that both of these individually exists. And this limit is going to be 1 squared the substitution and this limit is going to be 2 cubed, then you can use the second one and say that for the, for $x^2 y^3$ the limit is $1^2 2^3$ Similarly, for $y^3 z^2$ it will be $2^3 3^2$

And for the third term, it will be $1 \times 2 \times 3$ using 2, and then using 1 that you can, the limit of a sum of functions is the sum of the limits, if all of them exist, you can just say that this is the limit of each of these individually. And of course, limit $(x, y, z) \rightarrow (1, 2, 3)$ and we have computed all those using again these rules. So, this is $1^2 \times 2^3 + 2^3 \times 3^2 + 1 \times 2 \times 3$ essentially, what we have got is you can just substitute h(1, 2, 3) inside this function. So, you can, this limit is exactly h(1, 2, 3), which is, so 8 + 72 + 6, so whatever that is, that is I think 86.

So, I hope it is, it is clear how these rules are useful. Similarly, you could have something like limit let us say as $(x, y) \rightarrow (1, 1)$ then that would be just 1, because you can substitute for the numerators, substitute for the denominator, and for the denominator it works because the limit is non-zero. So, this is very, very, very similar to what we have done in one variable calculus in many, many, many places you are, you can just substitute the values.

Not always though, remember even in one variable calculus, we had trouble with substitution, for example, in things like $\frac{\sin x}{x}$ But I hope it is clear how these rules can be used. So, for polynomials, for example, it is very easy to find limits.

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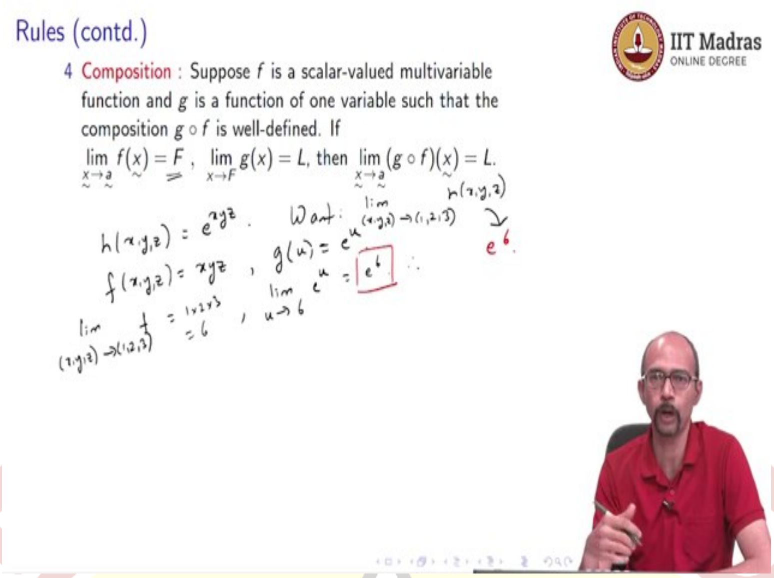

The next thing that we want to study is composition. So, suppose f is a scalar valued multivariable function, and g is a function of one variable such that the composition g composed f is well defined. Then, if limit f(x) is F and limit g(x) is L, then limit $gof(x)$ is also L. This may take a little time to digest, what we are saying is this.

So, suppose I have the function $h(x,y,z)=e^{xyz}$ Then I can write this function as a composition of two functions. One is $h(x,y,z)=xyz$ This is a very nice function that we understand, not h, but $f(x,y,z)=xyz$ and the other function is that is a $g(u)=e^{u}$ again, a very nice function, both of these are nice functions.

Let us see what happens to limit, let us say I want to compute limit (x, y, z). So, want limit $(x,y,z)\rightarrow(1,2,3)$ now h(x, y, z) this is what I want. Now, whatever we did on the previous page does not unfortunately help us directly. But what we can do is we can use the fact that this is com, this is a composition. So, let us first ask what happens to f? Well, this, of course, we know what happens this is $1\times2\times3$

So, in terms of what is your, this is F and, and then if we want u tends to this number which is 6, so $\lim_{u\rightarrow6} g(u)$ which is e^{u} this is e^{6} So, what is the net result? The net result is that this limit that we wanted is exactly this number here, e^{6} because I can compose. So, that is the answer. And this is again, eventually what we are saying is you can substitute.

So, this is a useful rule, then we have functions beyond just polynomials. For polynomials the previous page, whatever we had or rational functions, the previous page sufficed. But if you have exponentials, trigonometric functions, logarithms then we can use this composition rule and we can find the limits.

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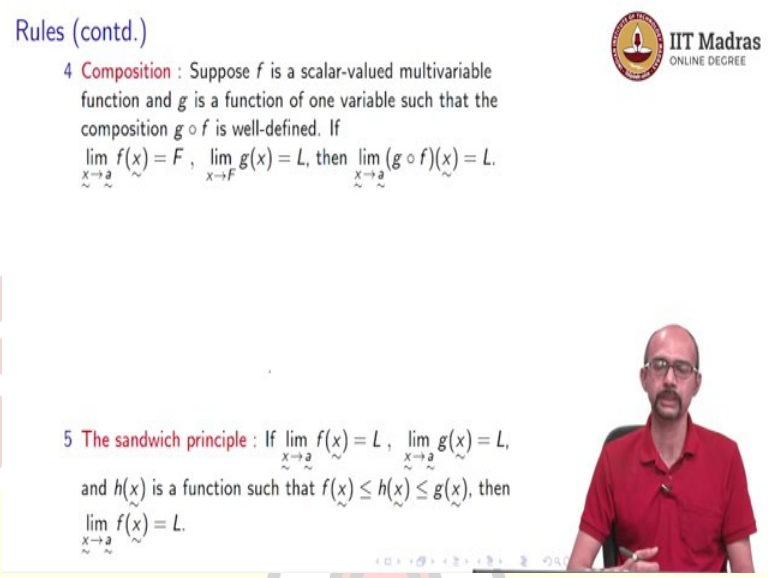

And finally, we have the sandwich principle, which again, we have studied in one variable calculus. So, if you have two functions f and g such that for both of them, the limit as x tilde tends to a tilde is L. And then if you have some function which is caught between them, sandwiched between them, then the limit as x tends to a tilde h(x tilde) is also equal to L. So, you will see an example of this in the tutorial. So, I will not expound more on this. But we have seen examples already in one variable calculus.

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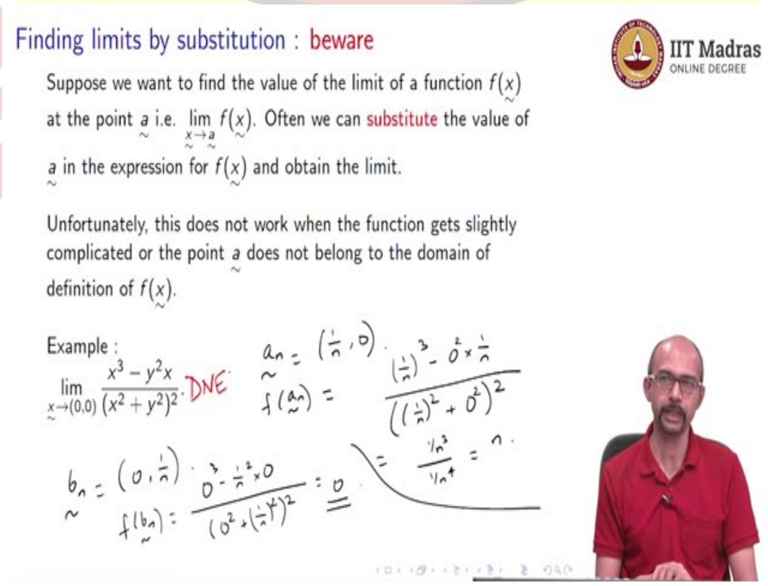

So let us come to finding limits by substitution. And this is where you have to be fairly careful. So, suppose we want to find the value of the limit of a function f(x) at the point a tilde so that is limit x tilde tends to a tilde, f(x tilde). So, often we can substitute the value of a tilde in the expression of f(x tilde) and obtain the limit. This is what many of the examples that we have seen, we could do.

So, unfortunately, this does not work, when the function gets slightly complicated, or the point a tilde does not belong to the domain of the definition of f (x tilde). It may happen that f is not defined a tilde at all, then what do you do? So, let us look at this example, you may actually remember this example from somewhere else.

In any case, let us try and understand what is $\lim_{xtilde \to (0,0)} \frac{x^3 - y^2 x}{(x^2+y^2)^2}$ So, what we want to ask is, what happens as, to this limit as x tilde comes close to (0, 0), which means (x, y) comes close to (0, 0). Now, we cannot use any of the previous things because if you look at this expression, it is a 0 by 0 type expression, so you cannot, so the rational function where you had F by G, unfortunately, the denominator here is becoming 0 in the limit, so you cannot use that rule.

So, there is no other rule that we can really use. So, how do I try to attempt this? So, we have to sort of try and do this by first principles, ahead we will also see other ways. So, we will try and look at what happens to sequences as they come close to (0, 0). So let us first try and look at sequences of the form, let us say I look at a sequence of the form a_n tilde is equal to $(\frac{1}{n}, 0)$

So, along the x axis as I come close to this point (0, 0), what happens to this function? So, if you look at f(a_n tilde) well, I can substitute $(\frac{1}{n}, 0)$ in this expression, and what do I get? So, the numerator is $(\frac{1}{n})^3 - \frac{0^2 \times 1}{n}$ And the denominator is $((\frac{1}{n})^2 + 0^2)^2$

So, this is, if we compute this, this is $\frac{\frac{1}{n^3}}{\frac{1}{n^4}}$ which is just n. And what that means is, if you take f (a_n tilde) that is exactly n, so f (a_n tilde) as n increases, diverges to infinity. So, this function in terms of what we have seen, there is a sequence for which this function diverges to infinity.

So in particular, what this means is that the sequence does not have a limit. So, this limit x tilde tends to (0, 0) does not exist of this function. Let us just for practice, take another sequence was so $(0, \frac{1}{n})$ So let us look at what is f (b_n tilde)? so for b_n tilde if we substitute that, so the numerator is $0^3 - (\frac{1}{n})^2 \times 0$ that is 0 and the denominator is $(0^2 + (\frac{1}{n})^2)^2$ So, which is 0 by something non-zero which is just 0.

So, for this, this sequence b_n tilde f (b_n tilde) does actually have a limit. So, along the y axis, this function does have a limit. So, notice what is happening here along the x axis as we take a sequence along the x axis which was $(\frac{1}{n}, 0)$ the f (a_n tilde) diverged to infinity along the y axis as we took a sequence that actually converges to 0.

So, we have two sequences which are giving us two different limits. So, there is no way in fact, one of them diverges. So, this limit does not exist. So, the main point is you cannot substitute, this for example, in this kind of function, you cannot substitute and this limit actually does not exist. So, this idea of taking sequences from different directions is very important and we will see more about this ahead. Thank you.