What are vector spaces?

Sarang S. Sane

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A vector space is a set with two operations (called **addition** and **scalar multiplication** with the above properties (i)-(viii).

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The functions $+$ and $\cdot$ are required to satisfy the following rules :

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Then $M_{m \times n}(\mathbb{R})$ along with addition and scalar multiplication forms a vector space.

Example : Solutions of a homogeneous system

Consider the set of solutions V of of a homogeneous system $Ax = 0$ where $A \in M_{m\times n}(\mathbb{R})$ (i.e. this is a homogeneous system of m linear equations in n variables).

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This is an example of a subspace of a vector space.

Non-example

Let us define addition and scalar multiplication in $\mathbb{R}^2$ as follows:

- $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$
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Check that (i), (ii) and (viii) fail to hold.

(i) fails :

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2)$

$\neq$

$(y_1, y_2) + (x_1, x_2) = (y_1 + x_1, y_2 - x_2)$

$(0,0) + (1,1) = (1, -1)$

$\neq$

$(1,1) + (0,0) = (1, 1)$

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Check that (iii), (iv) and (v) fail to hold.

Cancellation law of vector addition

If $v_1, v_2, v_3 \in V$ such that $v_1 + v_3 = v_2 + v_3$, then $v_1 = v_2$.

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (z_1, z_2, z_3) + (y_1, y_2, y_3)$ } in $\mathbb{R}^3$ or gen. in $\mathbb{R}^n$

$\Rightarrow x_i + y_i = z_i + y_i \quad , i = 1, 2, 3$

Subtract y_i from both sides $\Rightarrow x_i = z_i \quad , i = 1, 2, 3$.

$v_1 + v_3 = v_2 + v_3 \Rightarrow (v_1 + v_3) + v_3' = (v_2 + v_3) + v_3'$

$\Rightarrow v_1 + (v_3 + v_3') = v_2 + (v_3 + v_3')$ $\quad \exists v_3'$ s.t. $v_3 + v_3' = 0$

$\Rightarrow v_1 + 0 = v_2 + 0$

$\Rightarrow v_1 = v_2$.

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Corollaries:

- The vector 0 described in (iii) is unique.

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Corollaries:

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  Suppose $\exists w \in V$ s.t. $v + w = v \ \forall v \in V$.

  $\cancel{v} + w = \cancel{v} + 0 \Rightarrow w = 0$.

- The vector v' described in (iv) is unique and it is standard to refer to it as $-v$.

  $v + v' = 0$ Suppose v'' also satisfies this.

  Then $v + v' = 0 = v + v''$ Cancel v $\therefore v' = v''$.

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- $0v = 0$ for each $v \in V$.

  $(0+0)v = 0v + 0v$

  $\|$

  $0v$

  $0v = 0v + 0v$

  $\Rightarrow 0v + 0 = 0v + 0v$

  Cancel $0v \Rightarrow 0 = 0v$.

- $(-c)v = -(cv) = c(-v)$ for each $c \in \mathbb{R}$ and for each $v \in V$.

  $(c + (-c))v = cv + (-c)v$

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  $0v = 0$

  $\Rightarrow cv + (-c)v = 0$.

  $\Rightarrow (-c)v = -(cv)$.

- $c0 = 0$ for each $c \in \mathbb{R}$.

  Check this!

Earlier Example : Stock taking

Stock taking in a grocery shop :

Items	In stock	Buyer A	Buyer B	Buyer C	New stock
Rice in kg	150	8	12	3	100
Dal in kg	50	8	5	2	75
Oil in Litres	35	4	7	5	30
Biscuits in packets	70	10	10	5	80
Soap Bars	25	4	2	1	30

Vector space : { (Quantity of rice in kg, quantity of dal in kg, the quantity of oil in litres, the no. of biscuit packets, no. of soap bars) }

IS Like $\mathbb{R}^5$.

−ve corresponds to demand

+ve " to supply.

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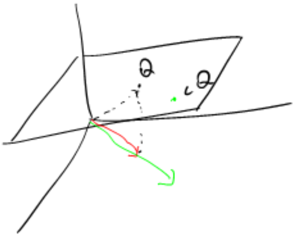
Scalar multiplication : Let $Q \in V$ and $c \in \mathbb{R}$. Project Q onto the XY-plane, scale the resulting vector by c and project the result back to V. Define cQ to be the tip of the obtained arrow.



Example : Affine flats (contd.)

Addition :

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Check that this is a vector space both geometrically (visualize!) and by writing down the algebra.

Thank you