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IIT Madras
ONLINE DEGREE

Mathematics for Data Science - 2
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Examples of finding bases for the kernel and image of a linear transformation

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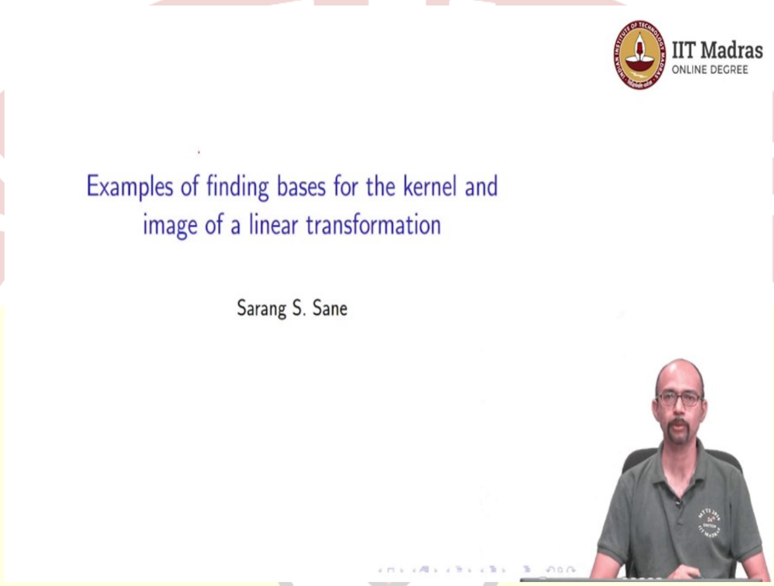

Hello, and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to continue from the previous video where we discussed the theory about the relationship between images and kernels for a linear transformation with the column space and null space of the matrix that we obtain after we fix ordered basis for the domain and the codomain.

So, we are going to do a bunch of examples, where we implement what we have seen earlier and we will find basis for the kernel and the image using the fact that we can find basis for the column space and the null space using row reduction.

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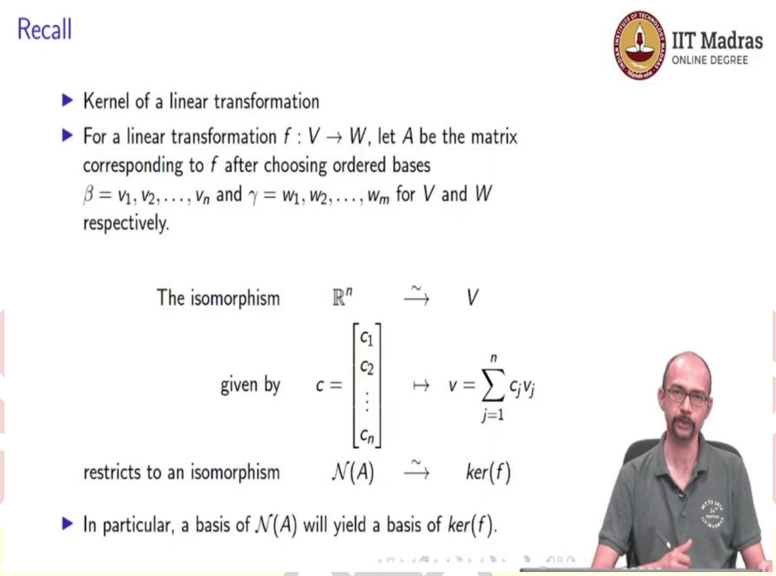

Let us just recall first what I just said. So, we have defined in the previous video what is the kernel of a linear transformation. For a linear transformation $f : V \to W$, let A be the matrix corresponding to f after choosing ordered basis, β which is $v_1, v_2, \ldots v_n$ and γ which is $w_1, w_2, \ldots . w_m$ for V and W, respectively. How do I get this matrix?

Well, you apply f on V and express that in terms of the $w_1, w_2, \ldots . w_m$ and then the coefficients that you get, you put all of them together and you get your m by n matrix. Once we choose the basis $v_1, v_2, \ldots v_n$ we have this isomorphism $\mathbb{R}^n$ to V. And what is the isomorphism. If you have a column vector in $\mathbb{R}^n$, $c_1, c_2, \ldots c_n$ then that corresponds to the vector summation $c_j v_j$. You take the corresponding linear combination that is the isomorphism from $\mathbb{R}^n$ to V.

And in the previous video, what we have discussed is that, if you consider the null space of the matrix A, which is a subspace of $\mathbb{R}^n$, then under this above isomorphism, that gives you an isomorphism with the kernel of f. This was the import of the first part of the previous video. So, in particular, if we can get a basis of the null space of A, then using this isomorphism, we can translate it to a basis of the kernel of *f*. That is one of the things we will do in this video for concrete examples.

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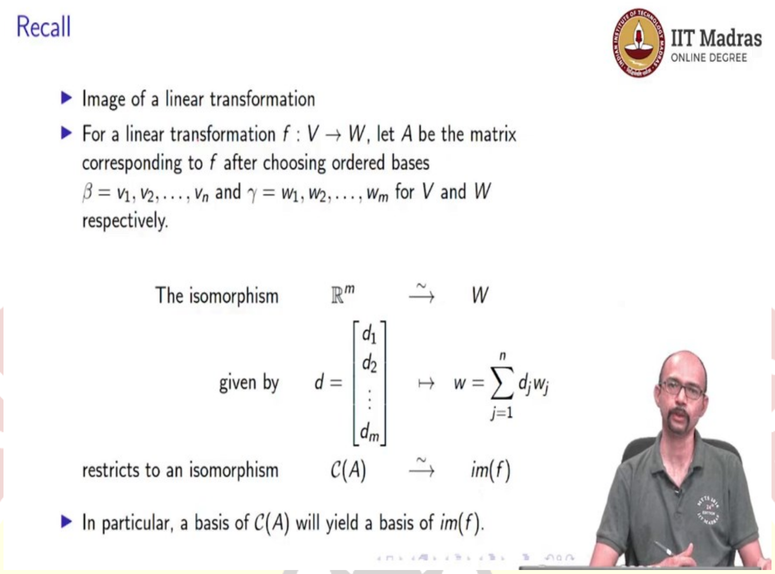

Similarly, we define the image of a linear transformation. And once again, we can choose ordered basis and then we get the matrix A. And now we have an isomorphism from $\mathbb{R}^m$ to W. This is because of the choice of ordered bases $w_1, w_2, \ldots. w_m$. So, if you have a vector $d_1, d_2, \ldots. d_m$, corresponding to that vector, you have the linear combination summation $d_j w_j$. The main point here is that, if you look at the column space of A, which notice is a subspace of Rm, then under this isomorphism, its image will be exactly the image of *f*. So, that is what we discussed in the previous video when we talked about the relation between image and column space.

And in particular, what this will do is, it will tell you that if you can get a basis of the column space, then by using this isomorphism, you can also get a basis for the image of *f* and we will implement this in some examples now.

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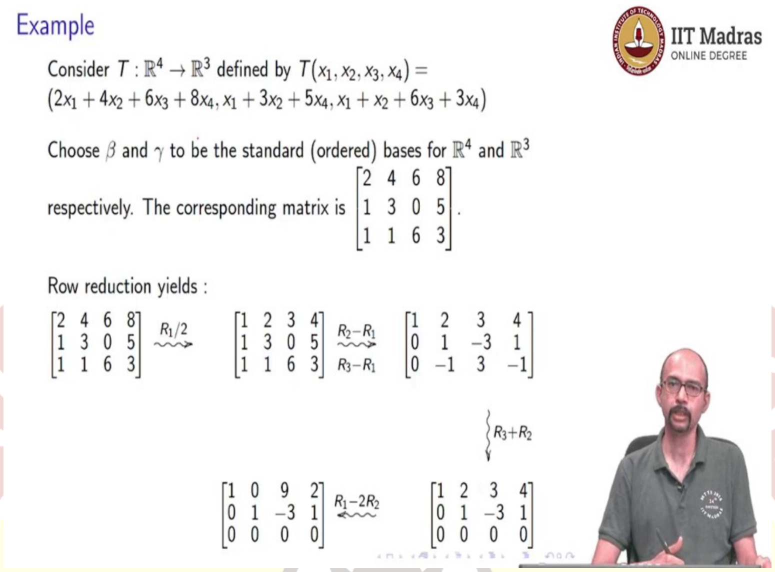

So, let us do this example. Consider the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by T of $x_1, x_2, \dots x_n$, which is $2 \times x_1 + 4 \times x_2 + 6 \times x_3 + 8x_4$, $x_1 + 3x_2$, $5x_4$ and $x_1 + x_2 + 6x_3 + 3x_1$. So, the first thing we have to do is choose ordered basis. So, let us choose the ordered basis to be the standard basis for $\mathbb{R}^4$ and $\mathbb{R}^3$. So, once we do that, what is a corresponding matrix where it is exactly the coefficients of, that are appearing this in this linear transformation. The first row is the coefficients appearing the first equation. So, those are 2, 4, 6, 8 and then a 1, 3, 0, 5 and then 1, 1, 6, 3.

So, let us row reduce. So, if you row reduce, well, what do I want to do? Maybe before I row reduce, let us ask, we want to get the basis for the image and the kernel of f of T and to do that, we have to get basis for the null space and the column space. And we know how to do that that is why row reduction. So, that is why we are row reducing. So, if you row reduce, well, we have a 2 in 1, 1 place, so we divide by 2. So, we get this matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix}$.

Sweep out the first column below the one. So, that is $\mathbb{R}^2 - \mathbb{R}^1$ and $\mathbb{R}^3 - \mathbb{R}^1$. So, if we do that, we get the first row is unchanged 1, 2, 3, 4 and in the second row, we will get 0, 1, - 3, 1 and in the third row, we will get 0, - 1, 3 and - 1. So, already you can see that the third row is a multiple of the second row. So, it is going to get knocked out in one more step.

Well, in the one, so we are done with the first column. So, now we look at the 2, 2th entry. It is conveniently 1. And so we can knock out the entry below that. So, that in fact knocks out the entire third row by doing $\mathbb{R}^3 + \mathbb{R}^2$. So, now this is already in pretty good shape. But let us do one more step and convert it into reduced row echelon form. So, to do that, we do $\mathbb{R}^1 - 2\ \mathbb{R}^2$ and then that gives us 1, 0, 9 and 2 in the first row, and then 0, 1, - 3, 1, 0, 0, 0, 0.

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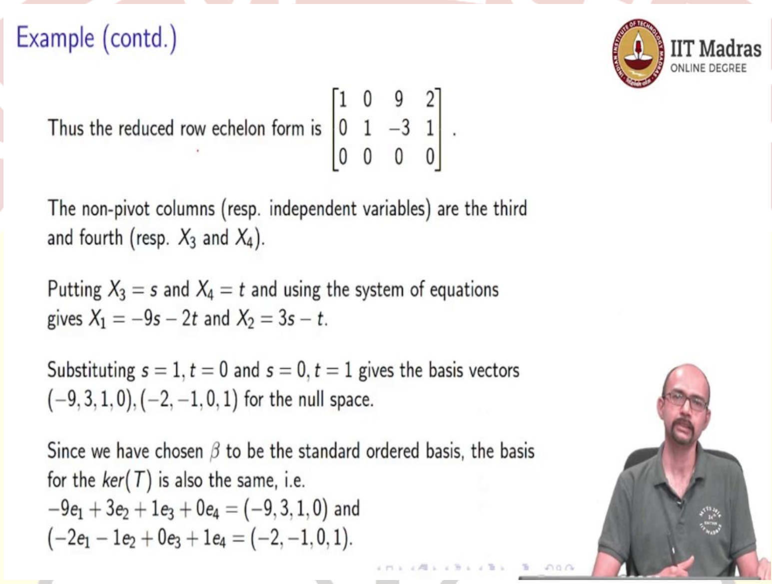

So, that is the reduced row echelon form, so $\begin{bmatrix}1 & 0 & 9 & 2\\0 & 1 & -3 & 1\\0 & 0 & 0 & 0\end{bmatrix}$. And from here, we can read off the basis for the null space and the column space. So, let us look at first what are the non-pivot and the pivot element elements. So, the pivot elements are in the 1, 1 and 2, 2 place. So, the non-pivot columns which will correspond to independent variables are in the third and fourth columns. And that tells us that x_3 and x_4 are the independent variables, x_1 and x_2 are dependent variables. And if we put x_3 is s and x_4 is t, then we can use the first row and the second row to get the corresponding equations. And from there, we will get that x_1 is $-9s - 2t$ and x_2 is $3s - t$.

So, just to be clear, this is in order to get the basis for the null space. So, for that, we have to look at the homogeneous equation. So, that is what, that is how we are getting these equations for x1 and x2. And what that means is, if you substitute s is 1 and t 0 or s is 0 and t is 1 this give you two basis vectors and that will contribute the basis for the null space. So, if we do that, what do we get? So, if s is 1 and t 0, we get - 9, 9, 1, 0. And if s is 0 and t is 1, we get - 2 , - 1 , 0 , 1. So, that is a basis for the null space.

And since we have chosen **β** to be the standard ordered basis, the basis for the kernel of t is exactly the same. So, it is - 9 , 3 , 1 , 0. How do I get that? That is because we take the linear combination of the corresponding ordered basis. And in this case, that is - 9 $\times e_1$ + 3 $\times$ e_2 + 1 $\times$ e_3 + 0 $\times$ e_4, which is the same as the vector you started with and similarly for the second one. So, we have got a basis for the kernel of t.

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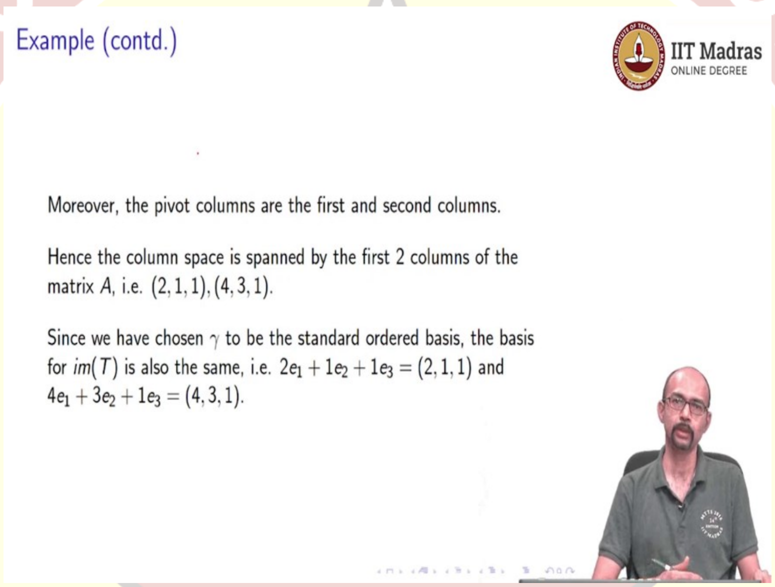

To get a basis for the column space, note that the pivot columns are the first and second columns. So, that means from the original matrix, if we look at the first two columns, those will be a basis for the column space. So, that is the vectors 2, 1, 1, and 4, 3, 1. So, just to remind you, here is the matrix. So, the first two columns are, in the first column we had 2, 1, 1, in the second column we had 4, 3, 1. That is a basis for the column space.

So, that tells us that again, that since we choose the standard basis, the standard ordered basis, the image of t also has the same basis. So, 2 $\times$ e_1 + 1 $\times$ e_2 + 1 $\times$ e_3 which is 2, 1, 1 and 4 $\times$ e_1 + 3 $\times$ e_2+ 1 $\times$ e_3 which is 4, 3, 1. So, I hope this example demonstrated how to implement what we studied in the previous video.

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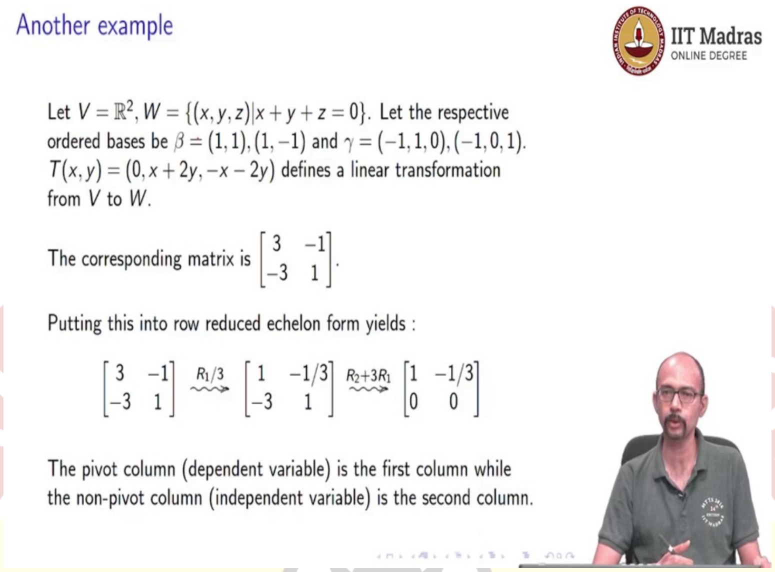

Let us do one more example, where we will use basis which are not the standard basis and so we will have to actually compute what the, what those linear combinations. So, let us look at V is R2 and W is this subspace of $\mathbb{R}^3$, x, y, z, such that $x + y + z$ is 0. So, let us look at respective ordered pieces. So, **β** is 1, 1 and 1, - 1 for R2. So, notice I am not using the standard ordered basis here, but the ordered basis 1, 1, 1, - 1. So, for **γ**, let us choose the ordered basis - 1, 1, 0 and - 1, 0, 1.

Let us look at the linear transformation, T of x, y is $0 , x + 2y , -x - 2y$. So, of course, we know this is a linear transformation from R2 to R3. How do I know it is a linear transformation to W? Well, if you look at the image, that is inside W, because the vectors $0 , x + 2y , -x - 2y$ satisfies that the sum of these three is 0. So, $0 + x + 2y + -x - 2y$ is indeed 0. So, this is a linear transformation from V to W.

So, now let us write down the corresponding matrix. So, here, we will have to do some work. So, here, we have to take the first vector in the basis, which is 1 , 1, and then apply T to that, if you apply T to that, you get, so T of 1, 1 is a 0 , 3 , - 3, and then we have to express that in terms of the basis **γ**. So, 0 , 3 , - 3 is $3 \times -1 , 1 , 0 + -3 \times -1 , 0 , 1$. This is something you can take. And that gives us our first column, which is 3, - 3.

Similarly, for the second basis vector, we have T of 1 , - 1, so that will give us 1 - 2, which is - 1 and then - 1 + 2, which is 1. So, we get 0 , - 1 , 1 and again it is a check that this leads to the column

- 1, 1 in the second column. So, this is our corresponding matrix. This is something we have done this in the previous videos. So, I will encourage you to check this.

So, now, we want to get the basis for the column space and the null space of this matrix. So, to do that, we will put it into reduced row echelon form. This is a particularly easy matrix. So, you divide the first row by 3. So, you get 1, $-\frac{1}{3}$, - 3, 1 and then sweep out this is actually clearly the second row is a multiple of the first. So, you will get 1, - $\frac{1}{3}$, 0, 0. So, this is in reduced row echelon form already.

So, now, let us check out what are the pivot and non-pivot columns. So, the pivot column is the first column. The non-pivot column is the second column. So, x_1 is the corresponding variables x_1 is the dependent variable, x_2 is the independent variable. And for the homogeneous equation, $x_1$satisfies, so x_1 - $\frac{1}{3}x_2$ is 0, so $x_1 = \frac{1}{3} \times x_2$.

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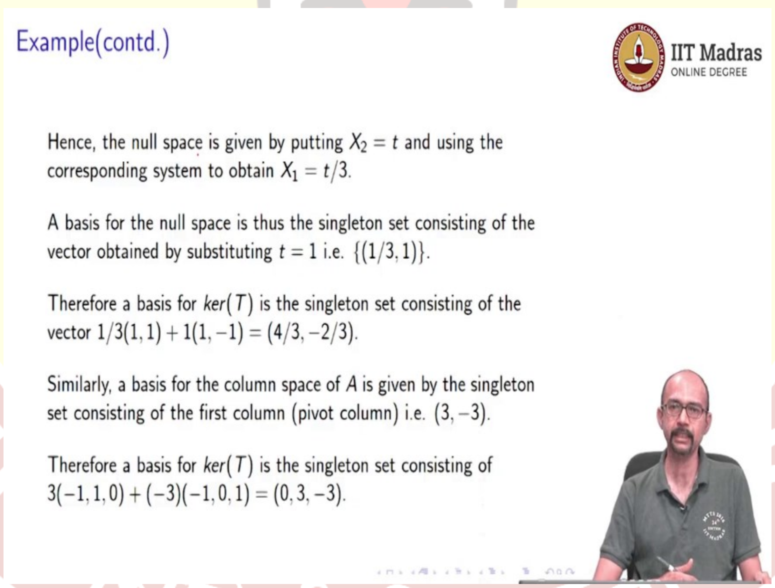

So, if you put x2 is t, then x1 is going to be given by t by 3. So, a basis for the null space is we can obtain that by putting t is 1. So, if you do that, we get the singleton set, which is a basis namely, one-third **,** 1. So, this is a basis for the null space of A. Now, from here, I want to get the basis for the kernel of t. So, remember, we have chosen the basis 1 **,** 1 and - 1 **,** 1, I just, sorry, 1 **,** - 1, **β** is 1 **,** 1, 1 **,** - 1. So, now, we have to take that linear combination one-third × 1 **,** 1 + 1 × 1 **,** - 1. So, if you evaluate that, that gives us 4 by 3 **,** $\frac{-2}{3}$. So, that is a basis for kernel of t.

Now, of course, you can simplify this basis. So, you can pull out the one-third or you can pull out the two-third if you want and instead of this basis you can write this as 2 , - 1, so that is another basis for kernel of t. So, similarly, let us go about what happens to the column space. Let us see what happens to the column space of A. So, that is, so to do that, we have to see what is the pivot column. So, the pivot column was the first column. So, that means the first column of the original matrix which was 3 - 3 is a basis for the column space.

So, now, we have 3 , - 3. And so what we should do is take the linear combination 3 × the first basis vector in **γ** + - 3 × the second basis vector in **,**. So, if you do that, we get 3 × - 1, 1, 0 + - 3 × - 1 , 0 , 1, and that gives us 0 , 3 , - 3. So, this is a basis for the image of the linear transformation t. So, I hope this example has again shed more light on the calculations that we, the theory that we saw in the previous video.

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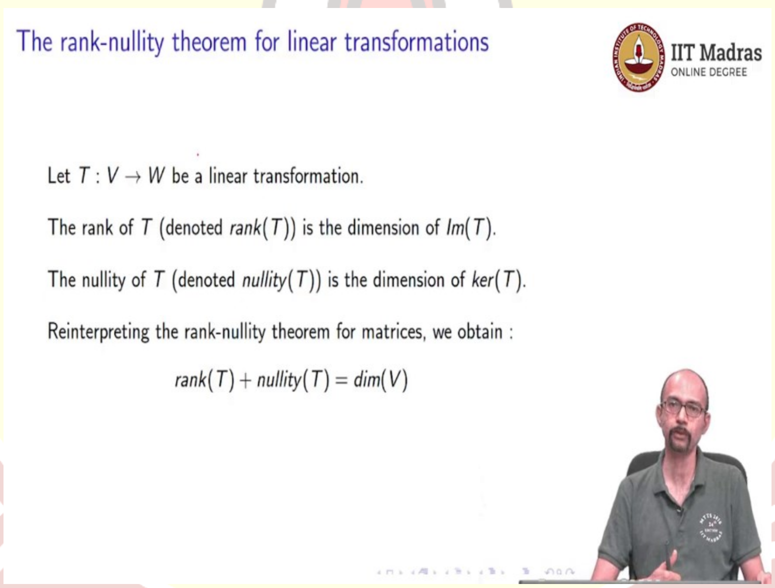

So, let us end this video with the rank nullity theorem for linear transformations. We have actually studied this theorem for matrices. So, now, because we have this very nice connection between the null spaces and the kernel and image and the column space, we can convert the rank nullity theorem for matrices into a theorem for linear transformations. So, suppose T V to W is a linear transformation then the rank of T is the dimension of the image of T. This is the definition. And the nullity of T is the dimension of kernel of T.

So, based on what we have studied earlier, notice that in order to compute the rank, we can choose ordered basis for the domain and codomain and using that we can get a matrix and the rank of T will be exactly the rank of that matrix. And similarly, the nullity of T will be exactly the nullity of that matrix.

So, if we reinterpret the rank nullity theorem for matrices, we will thus obtain rank of T + nullity of T is dimension of V. Why dimension of V, because remember that what we will get is rank of the matrix + nullity of the matrix = n. What was n? N was the number of columns in that matrix. But what is that number of columns? That number of columns is corresponding to the dimension of V. So, hence we get dimension of V here.

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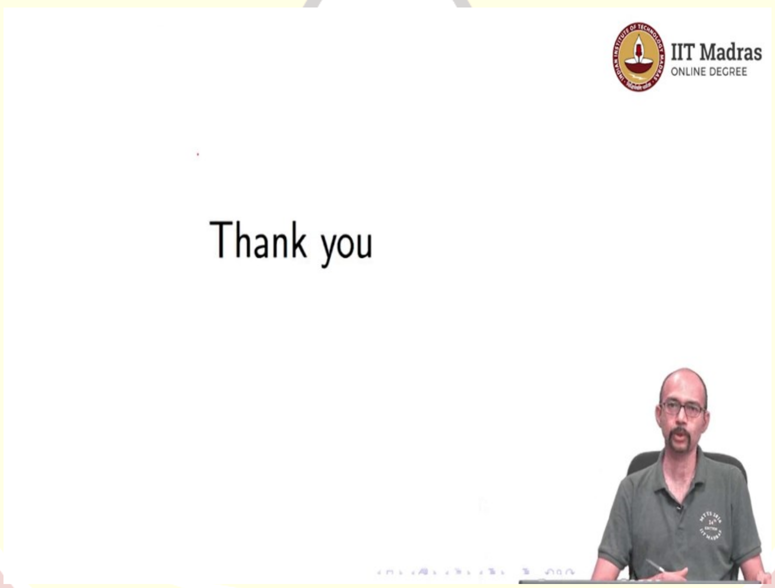

So, I guess that ends this video. Thank you.