Limits for scalar-valued multivariable functions

Sarang S. Sane

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Examples and visualization

$$\left\{ \left(\frac{1}{n} \, , \; n \, sin \left(\frac{1}{n} \right) \right) \right\} \longrightarrow (0, 1)$$

$$\left\{ \left((-1)^n \, , \; n \, sin \left(\frac{1}{n} \right) \right) \right\}$$

Does not converge (i.e. Limit DNE)

$$\left\{ \left(\underbrace{\frac{cos(n)}{n}}_{\downarrow \, 0} \, , \; \underbrace{\frac{\frac{1}{ln(1+n)} + \frac{5n^2}{1+n^2}}{\left(1 + \frac{1}{n}\right)^{2n}}}_{\downarrow \, ?} \, , \; \underbrace{\sum_{i=0}^{n} \frac{1}{i!}}_{\downarrow \, e} \, , \; n \, cos \left(\frac{1}{n} \right) \right) \right\}$$

diverge to ∞

Limit DNE

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If there exists a real number L such that $f(\underset{\sim}{a_n}) \to L$ for all sequences $\underset{\sim}{a_n}$ such that $\underset{\sim}{a_n} \to \underset{\sim}{a}$, then we say the limit of f at $\underset{\sim}{a}$ exists and equals L.

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$\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = L$ is equivalent to : as $\underset{\sim}{x}$ comes closer and closer to $\underset{\sim}{a}$, $f(\underset{\sim}{x})$ eventually comes closer and closer to L.

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$\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} f(x) = L$ is equivalent to : as $\underset{\sim}{x}$ comes closer and closer to $\underset{\sim}{a}$, $f(\underset{\sim}{x})$ eventually comes closer and closer to L.

If there is no such number L then we say that the limit of f at $\underset{\sim}{a}$ does not exist.

Some basic examples

1. $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} x_i^k; k \geq 0$

   a_i^k

2. $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} x_i^k; k < 0, a_i \neq 0$

   a_i^k

3. $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} e^{x_i}$

   e^{a_i}

4. $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} log_e(x_i); a_i > 0$

   $\log_e(a_i)$

5. $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} sin(x_i)$

   $\sin(a_i)$

6. $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} tan(x_i); a_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

   $\tan(a_i)$

Rules about limits of scalar-valued multivariable functions

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3. If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} g(\underset{\sim}{x}) = G \neq 0$, then the function $\dfrac{f}{g}$ is defined in at least a small interval around a and
   $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} \dfrac{f}{g}(\underset{\sim}{x}) = \dfrac{F}{G}$.

$$h(x,y,z) = x^2y^3 + y^3z^2 + xyz$$

$$\lim_{(x,y,z)\to(1,2,3)} h(x,y,z) = \lim_{(x,y,z)\to(1,2,3)} x^2y^3 + \lim_{(x,y,z)\to(1,2,3)} y^3z^2 + \lim_{(x,y,z)\to(1,2,3)} xyz$$

$$= 1^2\times 2^3 + 2^3\times 3^2 + 1\times 2\times 3$$

$$= 8 + 72 + 6 = 86.$$

$x^2 \downarrow 1^2$ $\quad$ $y^3 \downarrow 2^3$

$1^2\times 2^3$

$2^3\times 3^2$

$1\times 2\times 3$

Rules (contd.)

4 Composition : Suppose f is a scalar-valued multivariable function and g is a function of one variable such that the composition $g \circ f$ is well-defined. If $\lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = F$, $\lim_{x \to F} g(x) = L$,

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$$h(x,y,z) = e^{xyz}.$$

Want: $\lim_{(x,y,z) \to (1,2,3)} h(x,y,z) \to e^6.$

$$f(x,y,z) = xyz, \quad g(u) = e^u$$

$$\lim_{(x,y,z) \to (1,2,3)} f = 1 \times 2 \times 3 = 6, \quad \lim_{u \to 6} e^u = \boxed{e^6} \quad \therefore$$

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Example :

$$\lim_{\underset{\sim}{x} \to (0,0)} \frac{x^3 - y^2 x}{(x^2+y^2)^2}. \quad \text{DNE}$$

$\underset{\sim}{a_n} = \left(\frac{1}{n}, 0\right)$.

$$f(\underset{\sim}{a_n}) = \frac{\left(\frac{1}{n}\right)^3 - 0^2 \times \frac{1}{n}}{\left(\left(\frac{1}{n}\right)^2 + 0^2\right)^2} = \frac{1/n^3}{1/n^4} = n.$$

$\underset{\sim}{b_n} = \left(0, \frac{1}{n}\right)$.

$$f(\underset{\sim}{b_n}) = \frac{0^3 - \frac{1}{n}^2 \times 0}{\left(0^2 + \left(\frac{1}{n}\right)^2\right)^2} = \underline{\underline{0}}.$$

Thank you