

Multivariable functions : visualization

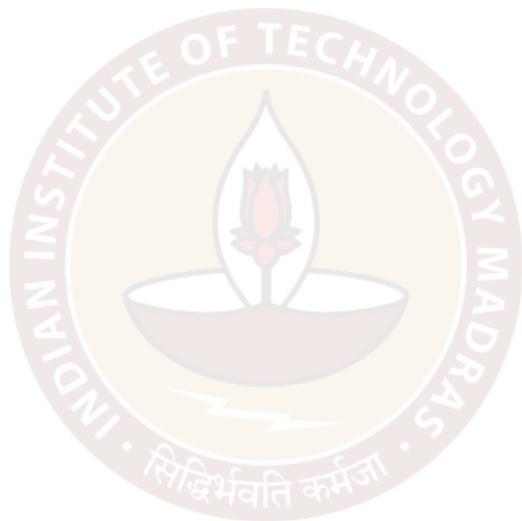
Sarang S. Sane

Recall functions of one variable



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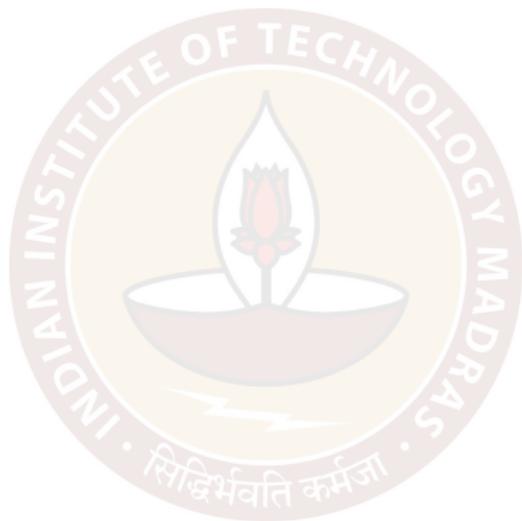
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1. Linear functions



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1. Linear functions
2. Polynomial functions

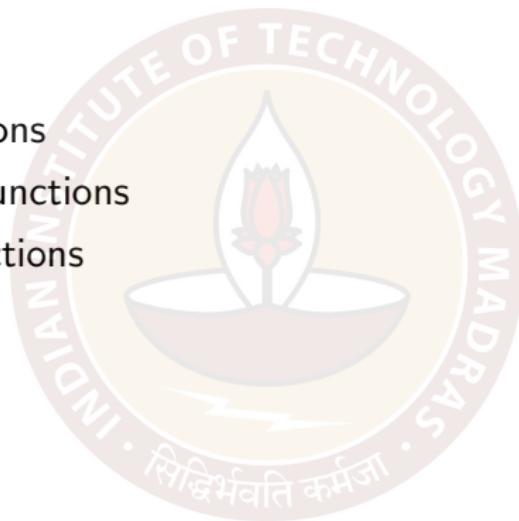


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3. Rational functions

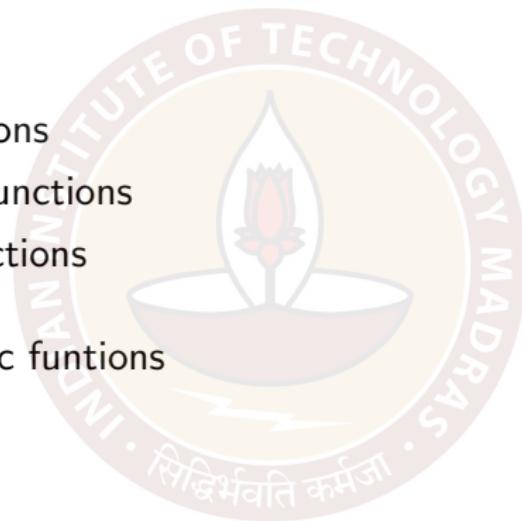


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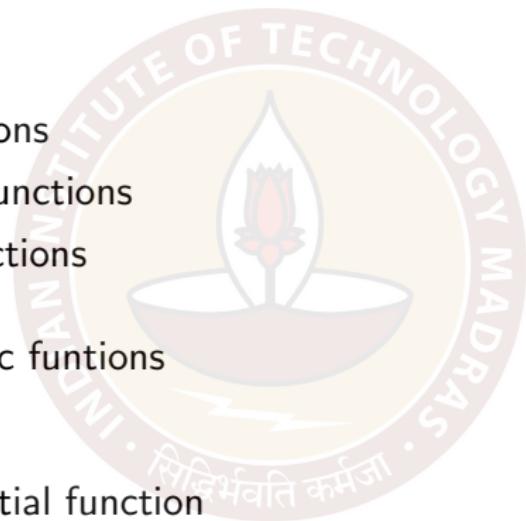


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5. The exponential function



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Examples :

1. Linear functions $f(x) = ax + b$
2. Polynomial functions $f(x) = x^2 + x + 1$
3. Rational functions $f(x) = \frac{x}{x^2 + 1}$
4. Trigonometric functions $\sin(x), \cos(x), \tan(x), \dots$
5. The exponential function e^x
6. The logarithm function $\log(x)$
7. (Arithmetic) combinations or compositions $\log(x^2 + 1)$

$$e^x \times e^{-x}$$

$$e^{\sin(x)}$$

Scalar-valued multivariable functions



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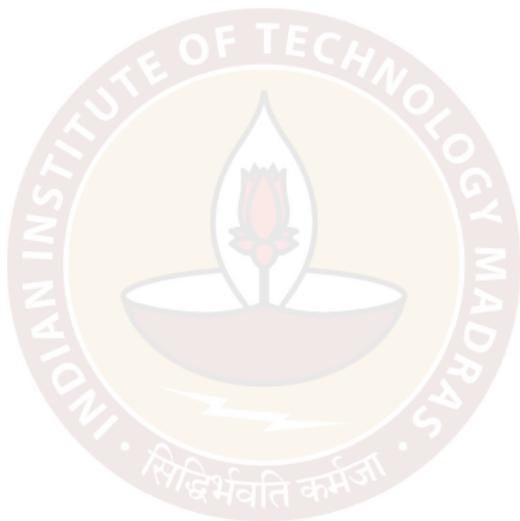
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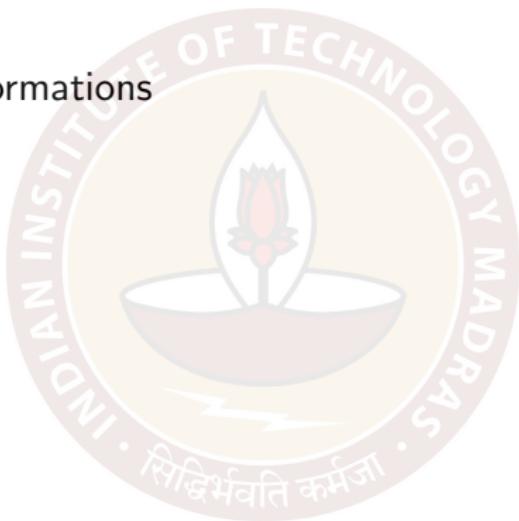


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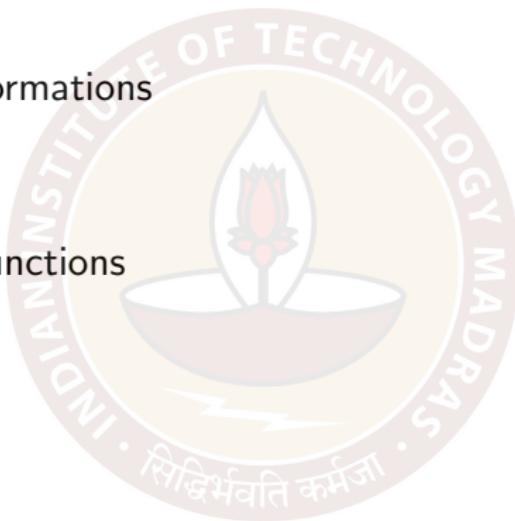


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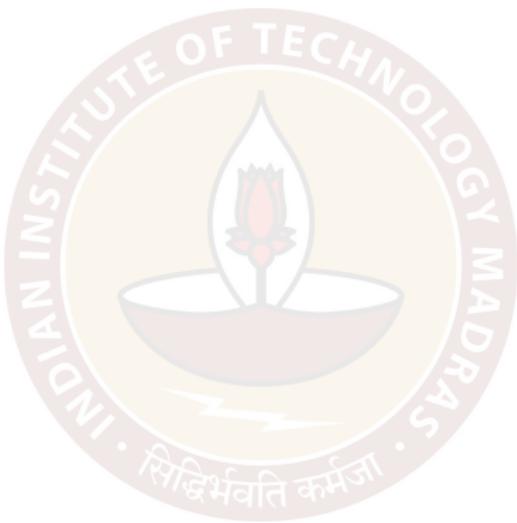
$$T(x_1, \dots, x_n) = \begin{matrix} \xrightarrow{\mathbb{R}} \\ a_1 x_1 + a_2 x_2 + \dots + a_n x_n \end{matrix}$$

2. Polynomial functions

$$f(x_1, \dots, x_n) = g(x_1, \dots, x_n) = \begin{matrix} x_1^2 + x_2^2 + \dots + x_n^2 \\ x_1 x_2 \dots x_n + x_1^2 x_2^3 x_3^4 \\ - x_1^5 x_2^6 \end{matrix}$$

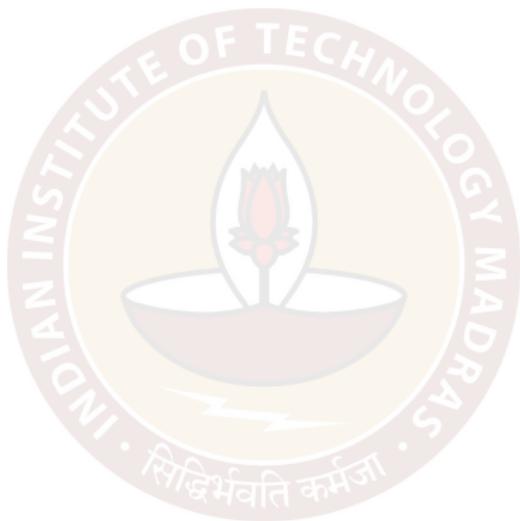
3. (Arithmetic) combinations or compositions

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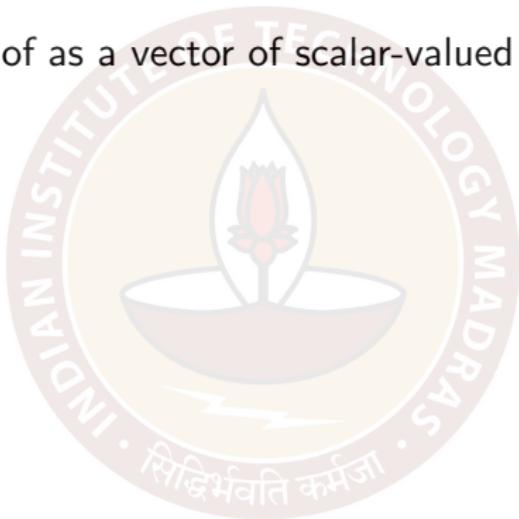
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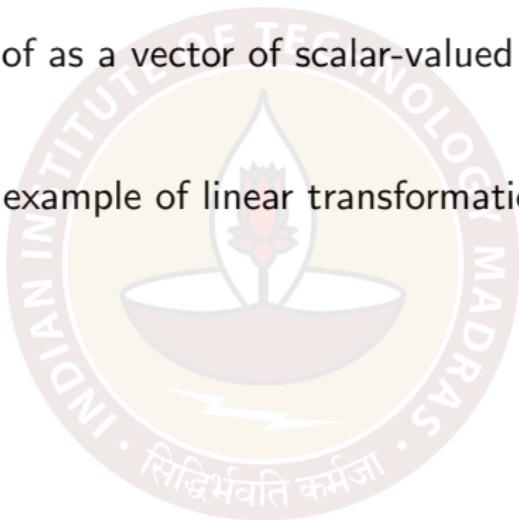


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We have seen the example of linear transformations.

$$f(x, y, z) = (x^2 + y^2, y^2 + z^2, z^2 + x^2).$$

$\mathbb{R}^3 \xrightarrow{\hspace{1cm}} \mathbb{R}^3$

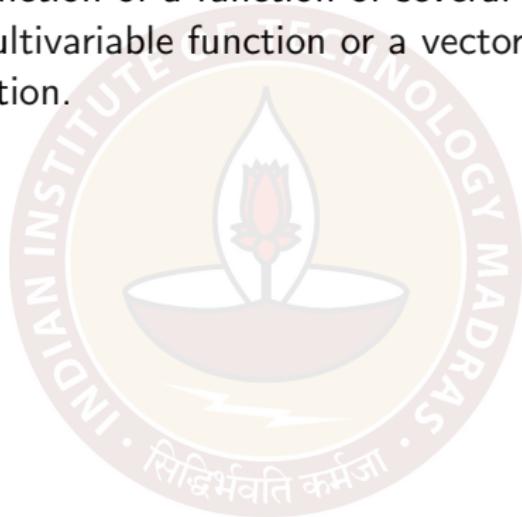
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Multivariable functions (or functions of several variables)



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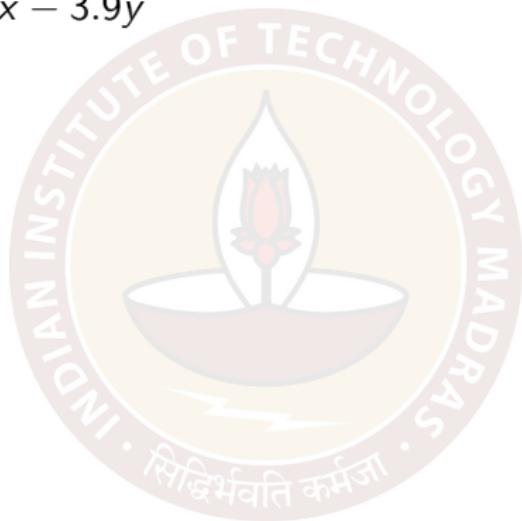
Further, if we want to refer to an element in D without bothering about the coordinates, we will use $x \in D$.

Examples



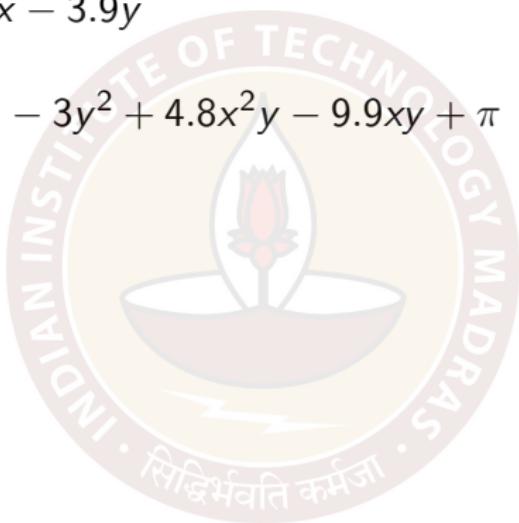
Examples

- ▶ $f(x, y) = 2.5x - 3.9y$



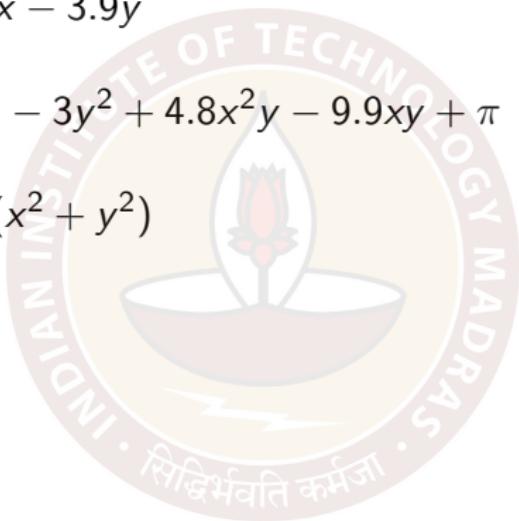
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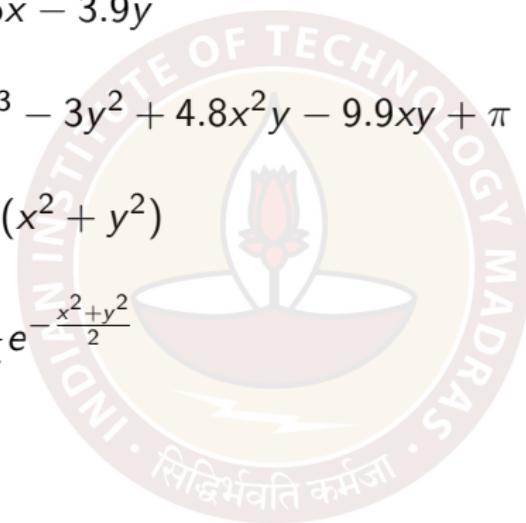
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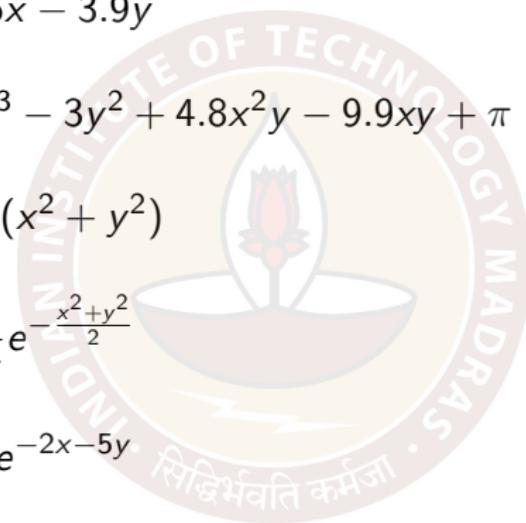
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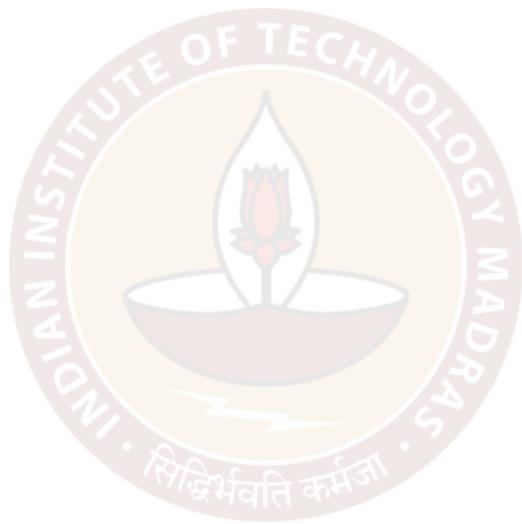
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Examples (contd.)



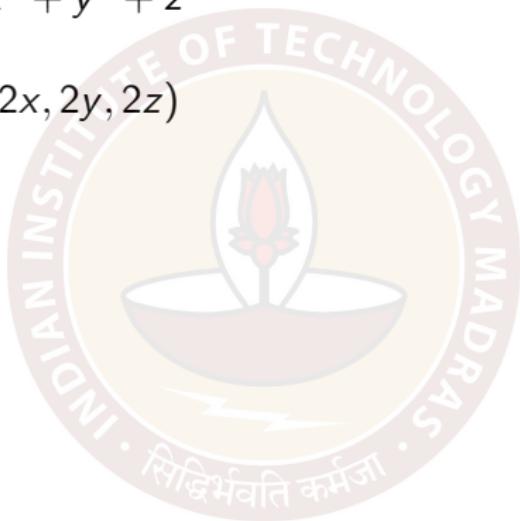
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- ▶ $f(x, y, z) = (2x, 2y, 2z)$
- ▶ $f(x, y, z) = (\sin(x)\cos(y), \tan(y+z), \ln(x^2 + y^2 + z^2), e^{xyz})$

$$D \subseteq \mathbb{R}^3 \quad \mathbb{R}^4 \\ f: D \rightarrow \mathbb{R}^4$$

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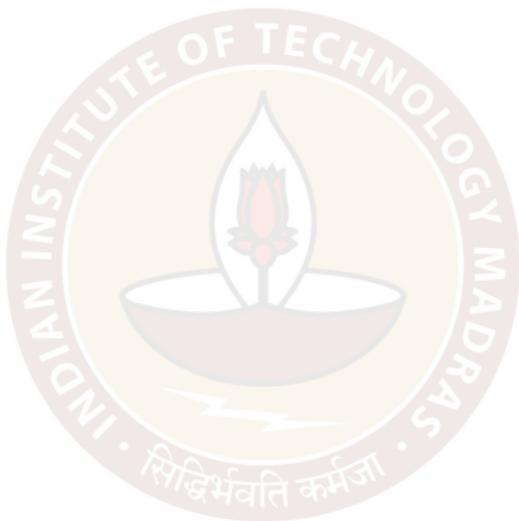
when $0 \leq x, y \leq 1$.

$$f(x, y) = \begin{cases} xy & \text{if } x, y \in [0, 1] \\ 0 & \text{o.w.} \end{cases}$$

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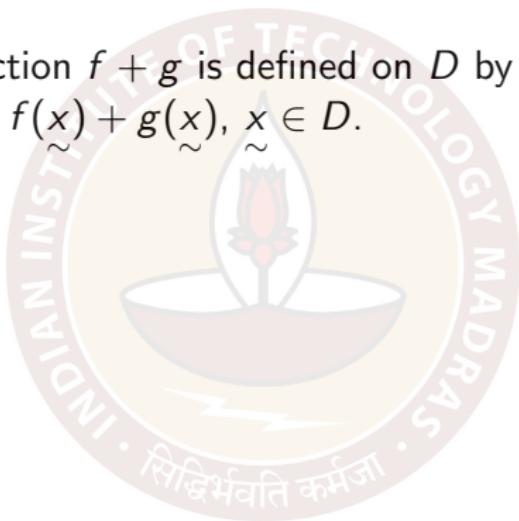


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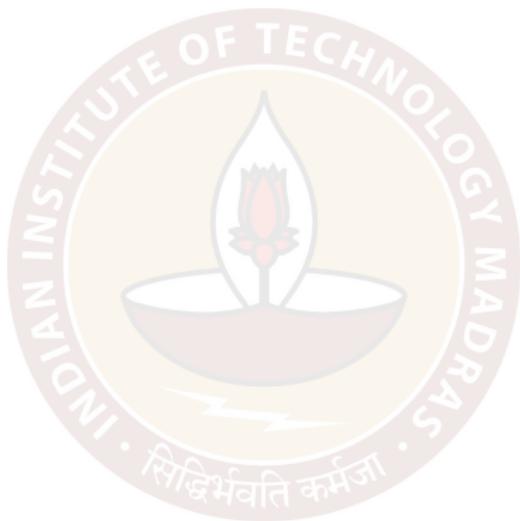
$$fg(\underset{\sim}{x}) = f(\underset{\sim}{x}) \times g(\underset{\sim}{x}), \quad x \in D.$$

- iv) If $m = 1$, and $\underset{\sim}{g}(x) \neq 0$, $x \in D$, the quotient f/g is defined

$$(f/g)(\underset{\sim}{x}) = f(\underset{\sim}{x}) / g(\underset{\sim}{x}), \quad x \in D.$$

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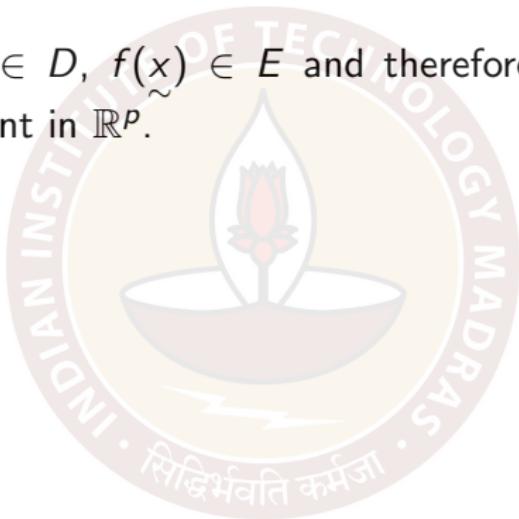


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Example : $f(x, y) = x^2 + y^2$ is a function from \mathbb{R}^2 to \mathbb{R} . $g(x) = \sqrt{x}$ is a function from $E = \{x \in \mathbb{R} \mid x \geq 0\}$ to \mathbb{R} .

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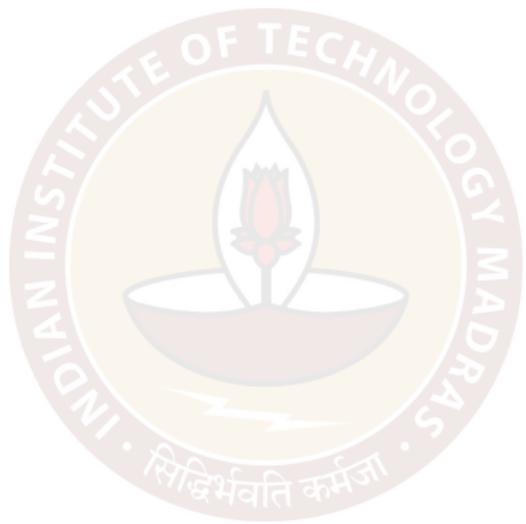
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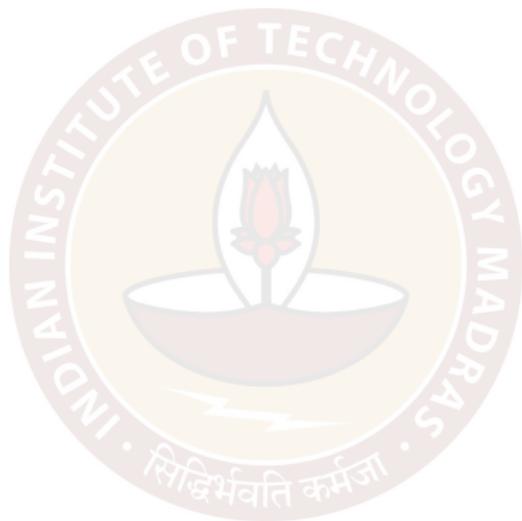
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Curves in \mathbb{R}^m



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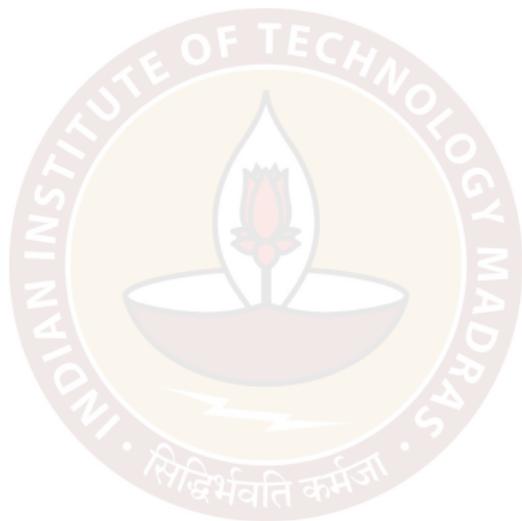
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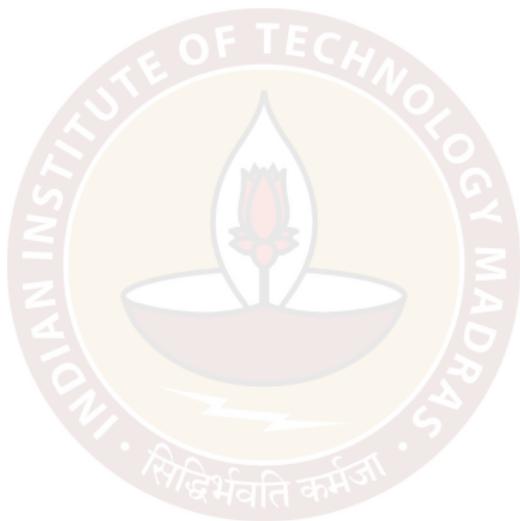


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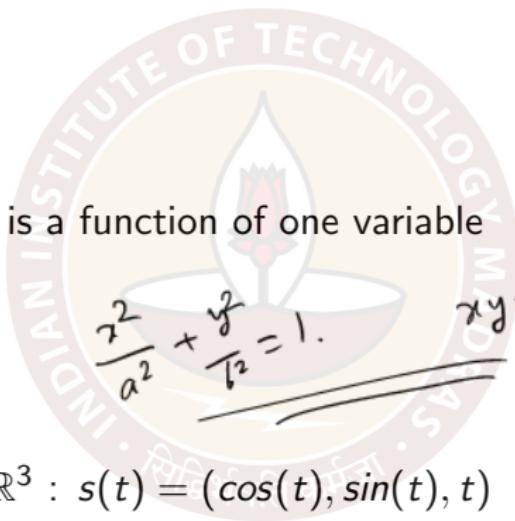


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3. Conics in \mathbb{R}^2
4. The helix in \mathbb{R}^3 : $s(t) = (\cos(t), \sin(t), t)$
5. The subset $\{(x, y) \mid y^2 = x^3\}$ of \mathbb{R}^2 .



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$xy = 1.$$

$$x^2 + y^2 = a^2$$
$$(a \cos \theta, a \sin \theta)$$



Thank you

