Week-2
Mathematics for Data Science - 2
Limits, Continuity, Differentiability, and the derivative
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Match the given functions in Column A with the equations of their tangents at the origin $(0, 0)$ in column B and the plotted graphs and the tangents in Column C, given in Table M2W2G1.

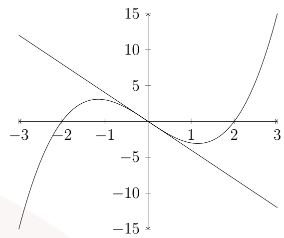
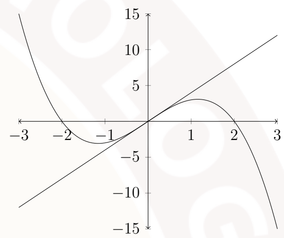
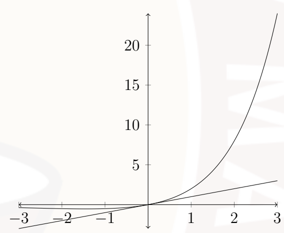
	Function (Column A)		It's tangent at (0,0) (Column B)		Graph (Column C)
i)	$f(x) = x2^x$	a)	$y = -4x$	1)	
ii)	$f(x) = x(x-2)(x+2)$	b)	$y = x$	2)	
iii)	$f(x) = -x(x-2)(x+2)$	c)	$y = 4x$	3)	

Table: M2W2G1

- ◯ Option 1: i) → b) → 3, ii) → c) → 1), iii) → a) → 2.
- ◯ **Option 2:** i) → b) → 3, ii) → a) → 1), iii) → c) → 2.
- ◯ Option 3: i) → b) → 3, ii) → a) → 2), iii) → c) → 1.
- ◯ Option 4: i) → c) → 3, ii) → a) → 1), iii) → b) → 2.

Solution:
i) Given $f(x) = x2^x \implies f'(x) = 2^x + x2^x \ln 2$.
So, $f(0) = 0$ and $f'(0) = 1$
Hence the equation of the tangent at the origin is

$$y - 0 = 1.(x - 0) \implies y = x.$$

In Column C, figure 3 has the line $y = x$ and exponential graph.
Hence i) $\rightarrow$ b) $\rightarrow$3).
ii) Given $f(x) = x(x-2)(x+2) = x^3 - 4x \implies f'(x) = 3x^2 - 4$.
So, $f(0) = 0$ and $f'(0) = -4$
Hence the equation of the tangent at the origin is

$$y - 0 = -4(x - 0) \implies y = -4x.$$

In Column C, figure 1 has the line $y = -4x$.
Hence ii) $\rightarrow$ a) $\rightarrow$1).
iii) Given $f(x) = -x(x-2)(x+2) = -x^3 + 4x \implies f'(x) = -3x^2 + 4$.
So, $f(0) = 0$ and $f'(0) = 4$
Hence the equation of the tangent at the origin is

$$y - 0 = 4(x - 0) \implies y = 4x$$

In Column C, figure 2 has the line $y = 4x$.
Hence iii) $\rightarrow$ c) $\rightarrow$2).

2 Multiple Select Questions (MSQ)

2. Consider the following two functions $f(x)$ and $g(x)$.

$$f(x) = \begin{cases} \frac{x^3-9x}{x(x-3)} & \text{if } x \neq 0, 3 \\ 3 & \text{if } x = 0 \\ 0 & \text{if } x = 3 \end{cases}$$

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \lfloor x \rfloor & \text{if } x > 2 \end{cases}$$

Choose the set of correct options.

- ◯ Option 1: $f(x)$ is discontinuous at both $x = 0$ and $x = 3$.
- ◯ Option 2: $f(x)$ is discontinuous only at $x = 0$.
- ◯ **Option 3:** $f(x)$ is discontinuous only at $x = 3$.
- ◯ Option 4: $g(x)$ is discontinuous at $x = 2$.
- ◯ **Option 5:** $g(x)$ is discontinuous at $x = 3$.

Solution:
(Options 1,2,3)
Given

$$f(x) = \begin{cases} \frac{x^3-9x}{x(x-3)} & \text{if } x \neq 0, 3 \\ 3 & \text{if } x = 0 \\ 0 & \text{if } x = 3 \end{cases}$$

Now, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{x^3-9x}{x(x-3)} = \lim\limits_{x\to 0} \frac{x(x-3)(x+3)}{x(x-3)} = \lim\limits_{x\to 0} x + 3 = 3 = f(0)$.
So $f(x)$ is continuous at $x = 0$.
Similarly, $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \frac{x^3-9x}{x(x-3)} = \lim\limits_{x\to 3} \frac{x(x-3)(x+3)}{x(x-3)} = \lim\limits_{x\to 3} x + 3 = 6 \neq f(3)$.
So $f(x)$ is not continuous at $x = 3$.
(Option 5)
Given

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \lfloor x \rfloor & \text{if } x > 2 \end{cases}$$

Observe that, as $x > 2$, $g(x) = \lfloor x \rfloor$. And $\lim\limits_{x\to 3^+} g(x) = 3 \neq 2 = \lim\limits_{x\to 3^-} g(x)$. i.e, $\lim\limits_{x\to 3} g(x)$ does not exist.
Hence $g(x)$ is discontinuous at $x = 3$.
(Option 4)
Observe that $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} \lfloor x \rfloor = 2$
and $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} |x| = 2$.

Hence, $\lim_{x \to 2^+} g(x) = 2 = \lim_{x \to 2^-} g(x)$
i.e., $\lim_{x \to 2} g(x) = 2 = g(2)$.
So $g(x)$ is continuous at $x = 2$.

3. Consider the graphs given below:

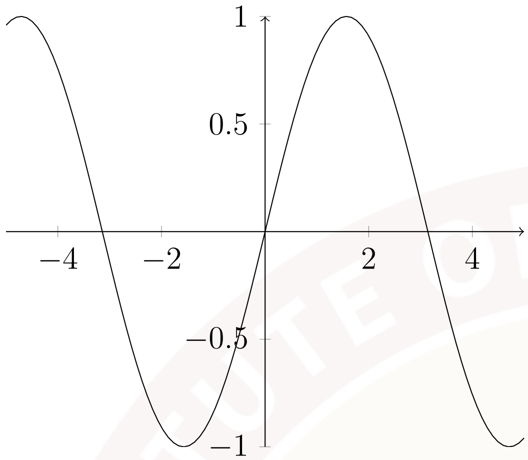


Figure: Curve 1

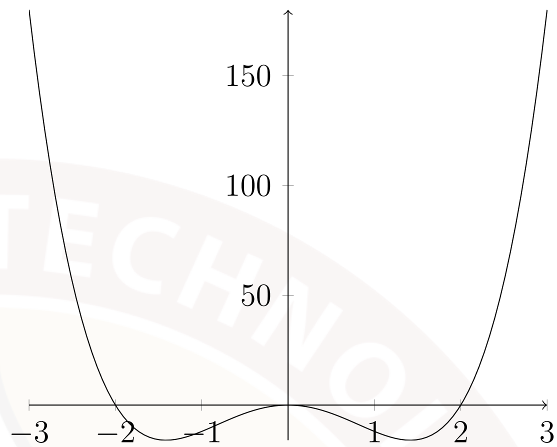


Figure: Curve 2

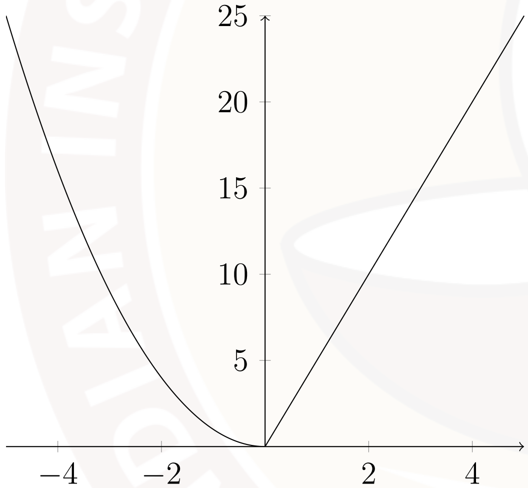


Figure: Curve 3

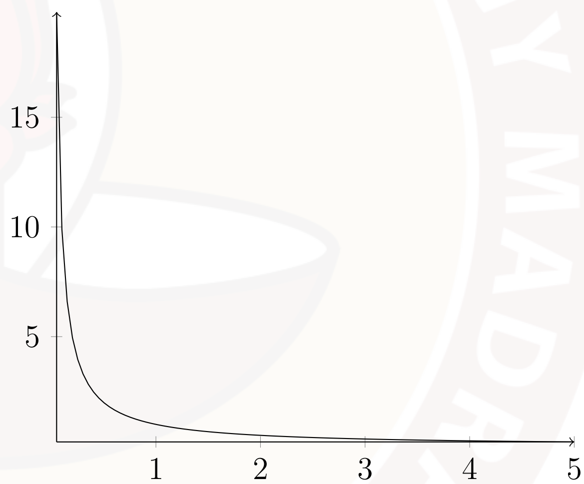


Figure: Curve 4

Choose the set of correct options.

- ◯ **Option 1:** Curve 1 is both continuous and differentiable at the origin.
- ◯ Option 2: Curve 2 is continuous but not differentiable at the origin.
- ◯ **Option 3:** Curve 2 has derivative 0 at $x = 0$.
- ◯ **Option 4:** Curve 3 is continuous but not differentiable at the origin.
- ◯ Option 5: Curve 4 is not differentiable anywhere.
- ◯ Option 6: Curve 4 has derivative 0 at $x = 0$.

Solution:
Option 1: Observe that if x approaches 0 from the left or from the right the value of the function represented by Curve 1 approaches 0. So, the limit of the function exists at $x = 0$ which is 0. And since the value of the function $f(x)$ is 0 at $x = 0$, the function represented by Curve 1 is continuous at $x = 0$.
And we can be draw a unique tangent to Curve 1 at the origin as shown in Figure M2W2GS (also observe that at $x = 0$, there does not exist any sharp corner).
Hence function is differentiable at the origin.

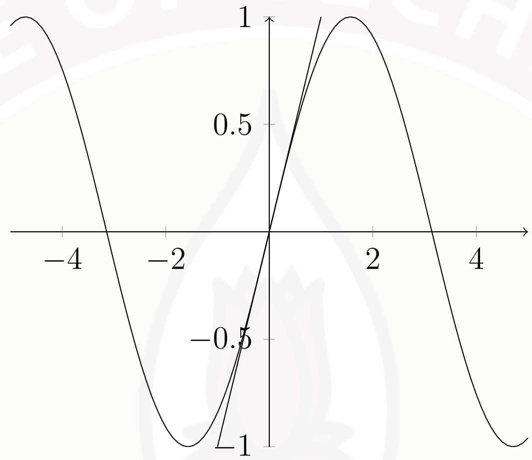


Figure M2W2GS

Options 2, 3: Observe that there is a unique tangent to the curve at the origin which is the $X-$axis itself and we know that slope of the $X-$axis is zero. Hence function represented by Curve 2 is differentiable with zero derivative at the origin.
And we know that a differentiable function is continuous.
Hence function represented by Curve 2 is continuous at the origin.

Option 4: Observe that there is sharp corner on Curve 3 at the origin. So function represented by Curve 3 is not differentiable at the origin.
But if x approaches 0 from the left or from the right the value of the function represented by Curve 3 approaches 0. So, the limit of the function exists at $x = 0$ which is 0. And since the value of the function $f(x)$ is 0 at $x = 0$, the function represented by Curve 3 is continuous at $x = 0$.
Option 6: If the derivative of the function represented by Curve 4 is 0 at the origin then at the origin the slope of of the tangent must be 0 i.e., the tangent must be parallel to the $X-$axis. For Curve 4, the tangent (if at all it exists) at the origin can never be parallel to the $X-$axis. Hence this statement is not true.

Option 5: Observe that at $x = 1$, there does not exist any sharp corner and at that

point, there exists a unique tangent (which is not vertical).
Hence function represented by Curve 4 is differentiable at $x = 1$.
Hence option 5 is not true.

4. Choose the set of correct options considering the function given below:

$$f(x) = \begin{cases} \frac{sin\ x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$$

- ◯ Option 1: $f(x)$ is not continuous at $x = 0$.
- ◯ **Option 2:** $f(x)$ is continuous at $x = 0$.
- ◯ Option 3: $f(x)$ is not differentiable at $x = 0$.
- ◯ **Option 4:** $f(x)$ is differentiable at $x = 0$.
- ◯ **Option 5:** The derivative of $f(x)$ at $x = 0$ (if exists) is 0.
- ◯ Option 6: The derivative of $f(x)$ at $x = 0$ (if exists) is 1.

Solution:
We know that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$. So $f(x)$ is continuous at $x = 0$.
Hence option 2 is true.
Now, $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{f(h)-1}{h} = \lim\limits_{h\to 0} \frac{\frac{\sin h}{h}-1}{h} = \lim\limits_{h\to 0} \frac{\sin h-h}{h^2} = \lim\limits_{h\to 0} \frac{\cos h-1}{2h} = \lim\limits_{h\to 0} \frac{-\sin h}{2} = 0$
(using L'Hopital's rule twice).
Hence the derivative of $f(x)$ at $x = 0$ is 0.
So options 4 and 5 are true.

5. Let f be a polynomial of degree 5, which is given by

$$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Let $f'(b)$ denote the derivative of f at $x = b$. Choose the set of correct options.

- ◯ **Option 1:** $a_1 = f'(0)$
- ◯ **Option 2:** $5a_5 + 3a_3 = \frac{1}{2}(f'(1) + f'(-1) - 2f'(0))$
- ◯ **Option 3:** $4a_4 + 2a_2 = \frac{1}{2}(f'(1) - f'(-1))$
- ◯ Option 4: None of the above.

Solution:
Given $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \implies f'(x) = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$
So $f'(0) = a_1$, $f'(1) = 5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1$, and $f'(-1) = 5a_5 - 4a_4 + 3a_3 - 2a_2 + a_1$
Hence $5a_5 + 3a_3 = \frac{1}{2}(f'(1) + f'(-1) - 2f'(0))$ and $4a_4 + 2a_2 = \frac{1}{2}(f'(1) - f'(-1))$

3 Numerical Answer Type (NAT)

6. Let f be a differentiable function at $x = 3$. The tangent line to the graph of the function f at the point $(3, 0)$, passes through the point $(5, 4)$. What will be the value of $f'(3)$?
[Answer: 2]
Solution: slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$.
So Slope of the tangent at $x = 3$ is, $\frac{4-0}{5-3} = 2$.
Since derivative of a function at a point equals the slope of the tangent at that point.
Hence $f'(3) = 2$

7. Let f and g be two functions which are differentiable at each $x \in \mathbb{R}$. Suppose that, $f(x) = g(x^2 + 5x)$, and $f'(0) = 10$. Find the value of $g'(0)$. [Answer: 2]
Solution:
Given $f(x) = g(x^2 + 5x) \implies f'(x) = g'(x^2 + 5x)(2x + 5)$
So $f'(0) = 5g'(0) \implies g'(0) = \frac{10}{5} = 2$

4 Comprehension Type Questions:

The population of a bacteria culture of type A in laboratory conditions is known to be a function of time of the form

$$p : \mathbb{R} \to \mathbb{R}$$

$$p(t) = \begin{cases} \frac{t^3-27}{t-3} & \text{if } 0 \leq t < 3, \\ 27 & t = 3 \\ \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) & \text{if } t > 3 \end{cases}$$

where $p(t)$ represents the population (in lakhs) and t represents the time (in minutes).
The population of a bacteria culture of type B in laboratory conditions is known to be a function of time of the form

$$q : \mathbb{R} \to \mathbb{R}$$

$$q(t) = \begin{cases} (5t-9)^{\frac{1}{t-2}} & \text{if } 0 \leq t < 2, \\ e^4 & t = 2 \\ \frac{e^{t+2}-e^4}{t-2} & \text{if } t > 2 \end{cases}$$

where $q(t)$ represents the population (in lakhs) and t represents the time (in minutes).
Using the above information, answer the questions 8,9, and 10.

8. Choose the correct option from the following (a function is said to be continuous if it is continuous at all the points in the domain of the function). (MCQ)

   ◯ Option 1: Both the functions $p(t)$ and $q(t)$ are continuous.

   ◯ **Option 2:** $p(t)$ is continuous, but $q(t)$ is not.

   ◯ Option 3: $q(t)$ is continuous, but $p(t)$ is not.

   ◯ Option 4: Neither $p(t)$ nor $q(t)$ is continuous.

   Solution:
   Given

$$p(t) = \begin{cases} \frac{t^3-27}{t-3} & \text{if } 0 \leq t < 3, \\ 27 & t = 3 \\ \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) & \text{if } t > 3 \end{cases}$$

   and

$$q(t) = \begin{cases} (5t-9)^{\frac{1}{t-2}} & \text{if } 0 \leq t < 2, \\ e^4 & t = 2 \\ \frac{e^{t+2}-e^4}{t-2} & \text{if } t > 2 \end{cases}$$

   It is enough to check the continuty of $p(t)$ at $t = 3$ and of $q(t)$ at $t = 2$.
   So right limit, $\lim\limits_{t\to 3^+} p(t) = \lim\limits_{t\to 3^+} \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) = \lim\limits_{t\to 3^+} \frac{27e^{27t}}{e^{81}} = 27$ (Using L'Hopital's rule).

Left limit, $\lim_{t\to 3^-} p(t) = \lim_{t\to 3^-} \frac{t^3-27}{t-3} = \lim_{t\to 3^-} 3t^2 = 27$
Hence, $\lim_{t\to 3^-} p(t) = \lim_{t\to 3^+} p(t) = 27 = p(3)$.
So $p(t)$ is continuous at $x = 3$.
Now right limit, $\lim_{t\to 2^+} q(t) = \lim_{t\to 2^+} \frac{e^{t+2}-e^4}{t-2} = \lim_{t\to 2^+} e^{t+2} = e^4$ (using L'Hopital's rule).
Left limit, $\lim_{t\to 2^-} q(t) = \lim_{t\to 2^-} (5t-9)^{\frac{1}{t-2}}$, to get the left limit,
let $y = (5t-9)^{\frac{1}{t-2}}$.
Taking *log* with base e on both sides and $t > \frac{9}{5}$,
we get, $\ln y = \frac{\ln(5t-9)}{t-2} \implies \lim_{t\to 2^-} \ln y = \lim_{t\to 2^-} \frac{\ln(5t-9)}{t-2} = \lim_{t\to 2^-} \frac{5}{5t-9} = 5$ (using L'Hopital's rule)
Hence, $\lim_{t\to 2^-} \ln y = 5 \implies \lim_{t\to 2^-} y = e^5$.
So $\lim_{t\to 2^-} (5t-9)^{\frac{1}{t-2}} = e^5$.
Since $\lim_{t\to 2^+} q(t) \neq \lim_{t\to 2^-} q(t)$ i.e., $\lim_{t\to 2} q(t)$ does not exists, $q(t)$ is not continuous at $t = 2$.
Hence option 2 true.

9. Which of the following linear functions denotes the best linear approximation $L_p(t)$ of the function $p(t)$ at the point $t = 1$? (MCQ)

   - ◯ Option 1: $L_p(t) = 3t + 10$
   - ◯ Option 2: $L_p(t) = 3t + 8$
   - ◯ **Option 3:** $L_p(t) = 5t + 8$
   - ◯ Option 4: $L_p(t) = 5t + 10$

   Solution:
   $p(t) = \frac{t^3-27}{t-3}$ if $0 \leq t < 3 \implies p(1) = 13$
   $p'(t) = \frac{(t-3)(3t^2)-(t^3-27)}{(t-3)^2} \implies p'(1) = 5.$
   Therefore the best linear approximation $L_p(t)$ of the function $p(t)$ at the point $t = 1$ is $L_p(t) = p(1) + p'(1)(t-1) = 13 + 5(t-1) = 5t + 8$

10. Which of the following linear functions denotes the best linear approximation $L_q(t)$ of the function $q(t)$ at the point $t = 3$? (MCQ)

- ◯ Option 1: $L_q(t) = e^5 t - 2e^5 - e^4$
- ◯ Option 2: $L_q(t) = e^5 t + e^5 - 4e^4$
- ◯ Option 3: $L_q(t) = e^4 t - 2e^5 - e^4$
- ◯ **Option 4:** $L_q(t) = e^4 t + e^5 - 4e^4$

Solution:

$q(t) = \frac{e^{t+2} - e^4}{t-2}$ if $t > 2 \implies q(3) = e^5 - e^4$

$q'(t) = \frac{(t-2)e^{t+2} - (e^{t+2} - e^4)}{(t-2)^2} \implies q(3) = e^4$

Therefore the best linear approximation $L_q(t)$ of the function $q(t)$ at the point $t = 3$ is $L_q(t) = q(3) + q'(3)(t-3) = e^5 - e^4 + e^4(t-3) = e^4 t + e^5 - 4e^4$