Q1 (a) Suppose X is normally distributed with mean 10 and variance 25. What is P(X<5)? Express your answer in terms of Fz, the CDF of the standard normal distribution with mean 0 and variance 1.

$\mu = 10, \sigma = 5 \qquad \frac{X-10}{5} \sim N(0,1)$

$(X<5) = \left(\frac{X-10}{5} < \frac{5-10}{5}\right) = \left(\frac{X-10}{5} < -1\right) \Rightarrow P(X<5) = F_Z(-1)$

Q1 (b) What is P(X>15)?

$(X>15) = \left(\frac{X-10}{5} > \frac{15-10}{5}\right) = \left(\frac{X-10}{5} > 1\right)$

$P(X>15) = 1 - F_Z(1)$

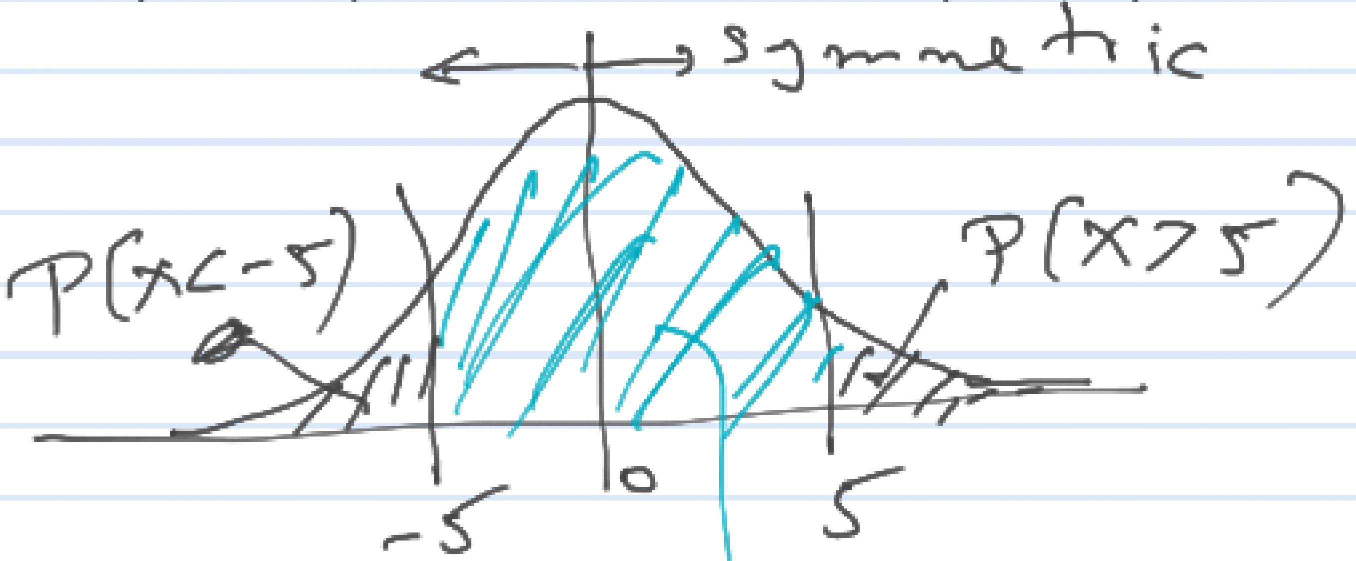
Q1 (c) What is P(8 < X < 12)?

$(8<X<12) = \left(\frac{8-10}{5} < \frac{X-10}{5} < \frac{12-10}{5}\right)$

$P(8<X<12) = F_Z\left(\frac{2}{5}\right) - F_Z\left(\frac{-2}{5}\right)$

Q2 (a) Suppose X is normally distributed with mean 0. Suppose P(X < -5) = 0.1. What is P(X > 5)?

$$P(X>5) = P(X<-5) = 0.1$$



Q2 (b) Suppose P(X < -5) = 0.1. What is P(-5 < X < 5)?

$$P(X<-5) + P(-5<X<5) + P(X>5) = 1$$

$$0.1 + P(-5<X<5) + 0.1 = 1$$

$$P(-5<X<5) = 1 - 0.2 = 0.8$$

Q2 (c) Suppose P(-a< X < a) = 0.95. What is P(X < -a)?

$$P(X<-a) + P(-a<X<a) + \underbrace{P(X>a)}_{=} = 1$$

$$P(X<-a) + 0.95 + P(X<-a) = 1$$

$$\Rightarrow P(X<-a) = \frac{1-0.95}{2} = 0.025$$

Q3 (a) Suppose X is normally distributed with mean 10 and variance 25. Find ‘a’ such that P(X > a) = 0.025. Express your answer in terms of the inverse CDF of the standard normal with mean 0 and variance 1.

$$(X > a) = \left(\underbrace{\frac{X-10}{5}}_{\sim Z} > \frac{a-10}{5}\right)$$

$$P(X<a) = 0.05 \Rightarrow P\left(\frac{X-10}{5} < \frac{a-10}{5}\right) = F_Z\left(\frac{a-10}{5}\right) = 0.05$$

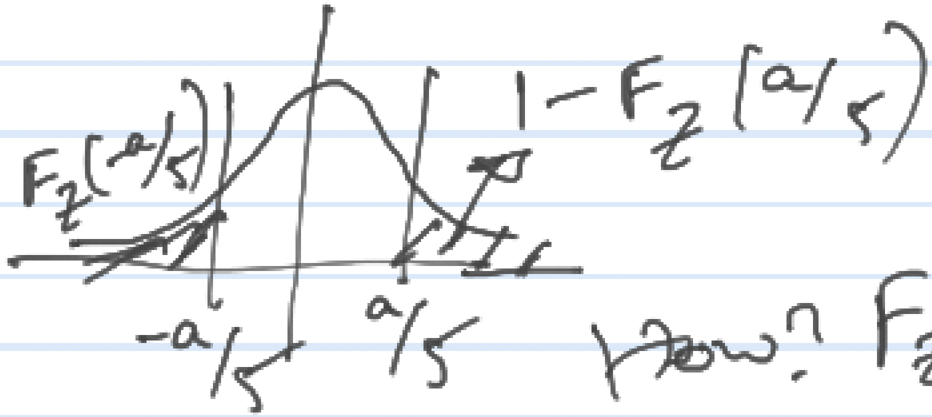
$$\frac{a-10}{5} = F_Z^{-1}(0.05) \Rightarrow a = 10 + 5F_Z^{-1}(0.05)$$

$$P(X>a) = P\left(Z > \frac{a-10}{5}\right) = 1 - F_Z\left(\frac{a-10}{5}\right) = 0.025$$

$$F_Z\left(\frac{a-10}{5}\right) = 0.975 \Rightarrow \frac{a-10}{5} = F_Z^{-1}(0.975) \Rightarrow a = 10 + 5F_Z^{-1}(0.975)$$

Q3 (b) Find ‘a’ such that P(|X - 10| < a) = 0.99.

$$(|X-10| < a) = \left(\underbrace{\frac{|X-10|}{5}}_{\sim |Z|} < \frac{a}{5}\right)$$



How? $F_Z\left(-\frac{a}{5}\right) = 1 - F_Z\left(\frac{a}{5}\right)$

$$P(|X-10|<a) = P\left(|Z| < \frac{a}{5}\right) = F_Z\left(\frac{a}{5}\right) - F_Z\left(-\frac{a}{5}\right) = 2F_Z\left(\frac{a}{5}\right) - 1 = 0.99$$

$$\Rightarrow F_Z\left(\frac{a}{5}\right) = 0.995 \Rightarrow a = 5F_Z^{-1}(0.995)$$

Q4 (a) Suppose X is exponentially distributed with lambda = 2. Find P(X > 5 | 2 < X < 8).

$$P(X>5 \mid 2<X<8) = \frac{P((X>5)\cap(2<X<8))}{P(2<X<8)} = \frac{P(5<X<8)}{P(2<X<8)}$$

$$P(5<X<8) = (1-e^{-2\times 8}) - (1-e^{-2\times 5}) = e^{-10} - e^{-16}$$

$$P(2<X<8) = (1-e^{-2\times 8}) - (1-e^{-2\times 2}) = e^{-4} - e^{-16}$$

$$\Rightarrow \text{Ans} = \frac{e^{-10}-e^{-16}}{e^{-4}-e^{-16}}$$

Q4 (b) Find P(|X-1/2| > 1/4).

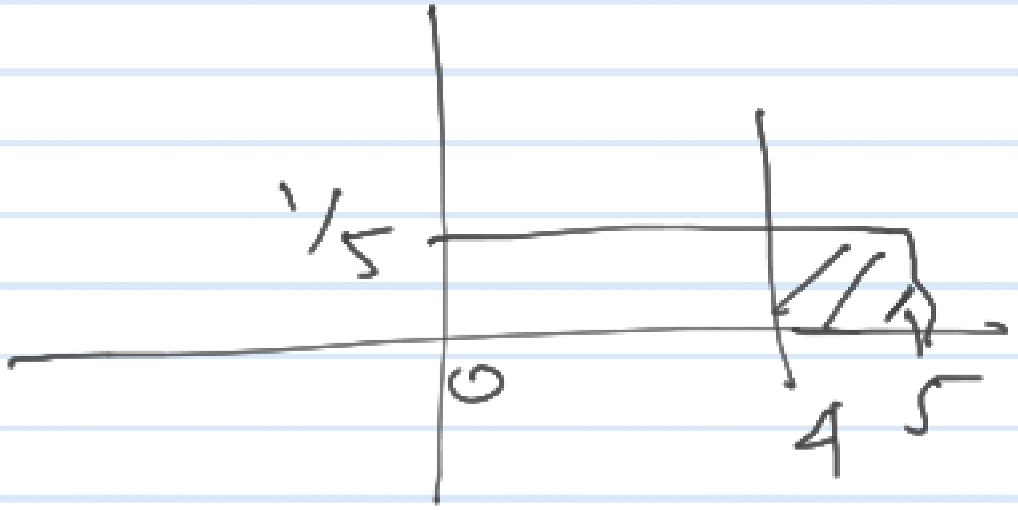
$$\left(\left|X-\tfrac{1}{2}\right| > \tfrac{1}{4}\right) = \left(X-\tfrac{1}{2} > \tfrac{1}{4}\right) \text{ OR } \left(X-\tfrac{1}{2} < -\tfrac{1}{4}\right) = \left(X>\tfrac{3}{4}\right) \text{ OR } \left(X<\tfrac{1}{4}\right)$$

$$P\left(X<\tfrac{1}{4}\right) = 1-e^{-2\times 1/4} = 1-e^{-1/2}$$

$$P\left(X>3/4\right) = 1-\underbrace{\left(1-e^{-2\times 3/4}\right)}_{P(X<3/4)} = e^{-3/2}$$

$$\Rightarrow \text{Ans: } 1-e^{-1/2}+e^{-3/2}$$

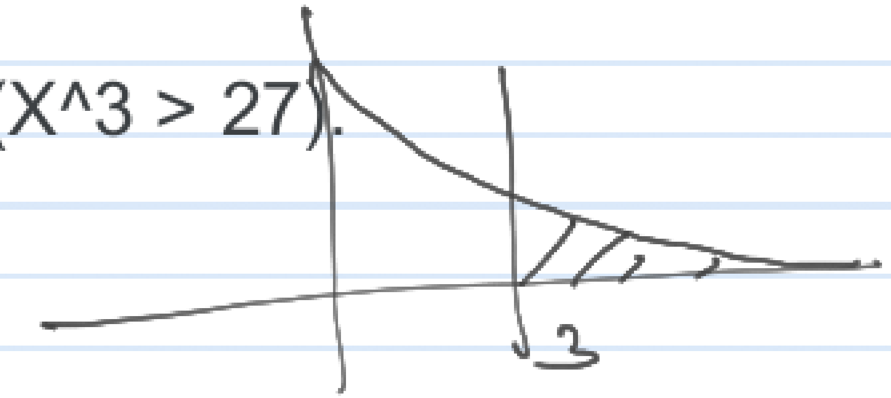
Q5 (a) Suppose X is uniformly distributed in [0, 5]. Find P(3X + 7 > 19).



$$(3x+7>19) = \left(x > \frac{19-7}{3} = 4\right)$$

$$P(3x+7>19) = P(x>4) = \frac{1}{5}\times 1 = \frac{1}{5}$$

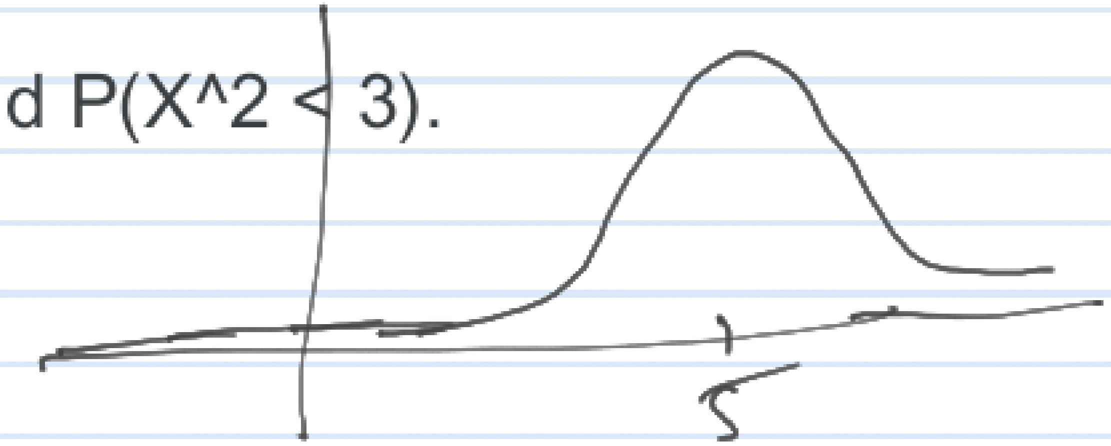
Q5 (b) Suppose X is exponentially distributed with lambda = 2. Find P(X^3 > 27).



$$(x^3>27) = (x>3)$$

$$P(x^3>27) = P(x>3) = 1 - (\underbrace{1-e^{-2\times 3}}_{P(x<3)}) = e^{-6}$$

Q5 (c) Suppose X is normally distributed with mean 5 and variance 1. Find P(X^2 < 3).

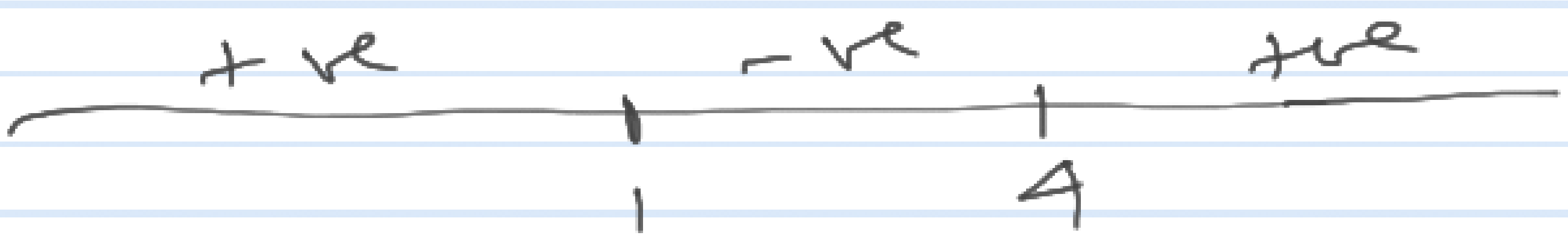


$$(x^2<3) = (-\sqrt{3} < x < \sqrt{3}) = \frac{-\sqrt{3}-5}{1} < \underbrace{\frac{x-5}{1}}_{\sim Z} < \frac{\sqrt{3}-5}{1}$$

$$P(x^2<3) = F_Z(\sqrt{3}-5) - F_Z(-\sqrt{3}-5)$$

Q6 Suppose X is normally distributed with mean 0 and variance 2. Find the following probability:
P(X^2 - 5X + 4 > 0)

$$(x^2-5x+4>0) = \left((x-1)(x-4)>0\right) = (x<1) \text{ or } (x>4)$$

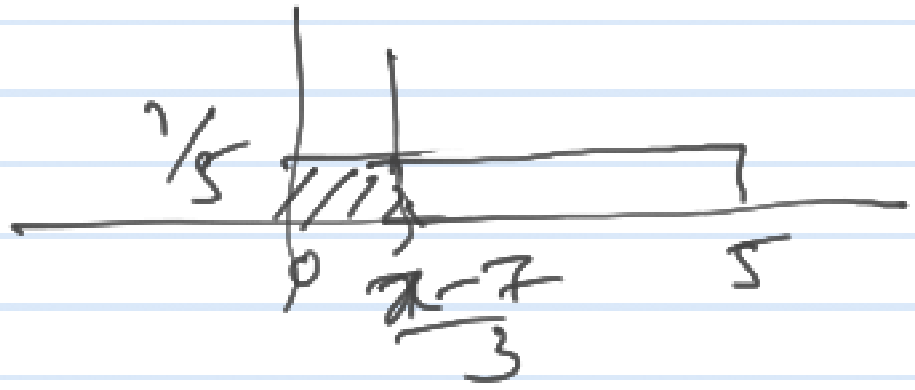


$$= \left(\underbrace{\frac{x}{\sqrt{2}}}_{\sim Z} < \frac{1}{\sqrt{2}}\right) \text{ or } \left(\frac{x}{\sqrt{2}} > \frac{4}{\sqrt{2}}\right)$$

$$P(x^2-5x+4>0) = P\left(Z < \frac{1}{\sqrt{2}}\right) + P\left(Z > 4/\sqrt{2}\right)$$

$$= F_Z\left(\frac{1}{\sqrt{2}}\right) + 1 - F_Z\left(4/\sqrt{2}\right)$$

Q7 (a) Suppose X is uniformly distributed in [0, 5]. Find CDF of 3X + 7.



$X \in [0,5] \Rightarrow 3X+7 \in [7,22]$

For $x<7$, CDF $=0$. For $x>22$, CDF $=1$

For $7<x<22$, CDF $= P(3X+7\le x) = P\left(X \le \frac{x-7}{3}\right) = \frac{1}{5}\left(\frac{x-7}{3}\right) = \frac{x-7}{15}$

PDF $= \frac{1}{15}$, $7<x<22$ and 0, otherwise

Q7 (b) Suppose X is exponentially distributed with lambda = 2. Find CDF of X^3.

$X\in[0,\infty] \Rightarrow X^3 \in [0,\infty]$

For $x\in[0,\infty]$, CDF $=P(X^3\le x) = P(X\le x^{1/3}) = 1-e^{-2x^{1/3}}$

PDF $= -e^{-2x^{1/3}} \times -2 \times \frac{1}{3} x^{-2/3} = \frac{2}{3} x^{2/3} e^{-2x^{1/3}}$, $x>0$.

Q7 (c) Suppose X is normally distributed with mean 5 and variance 1. Find CDF of X^2.

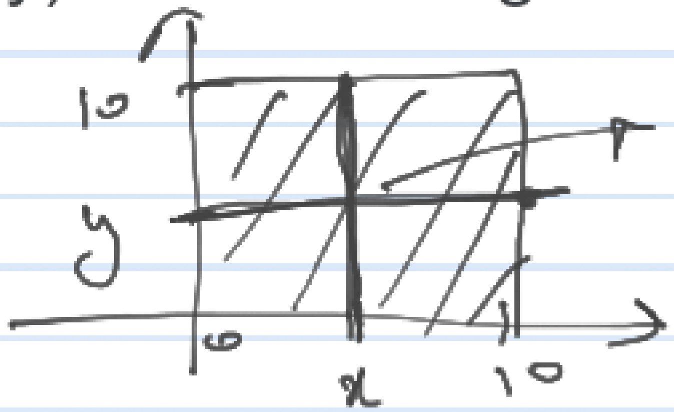
$X\in[-\infty,\infty] \Rightarrow X^2\in[0,\infty]$

For $x\in[0,\infty]$, CDF $= P(X^2<x) = P(-\sqrt{x}<X<\sqrt{x}) = P\left(\frac{-\sqrt{x}-5}{1} < \underbrace{\frac{X-5}{1}}_{\sim Z} < \frac{\sqrt{x}-5}{1}\right)$

$= F_Z(\sqrt{x}-5) - F_Z(-\sqrt{x}-5)$

PDF: $f_Z(\sqrt{x}-5)\cdot\frac{1}{2\sqrt{x}} - f_Z(-\sqrt{x}-5)\cdot\frac{-1}{2\sqrt{x}} = \left(f_Z(\sqrt{x}-5)+f_Z(-\sqrt{x}-5)\right)/2\sqrt{x}$

Q8 (a) Suppose X and Y are jointly uniform in the region {0 < x,y < 10}. Sketch the region. What is the value of f_xy(x,y) within the region of support?



$$f_{XY}(x,y) = \begin{cases} \frac{1}{100}, & 0 < x,y < 10 \\ 0, & \text{else} \end{cases}$$

Q8 (b) What is the range of values taken by X? What is the value of fx(5)?

$$X \in [0, 10]$$

$$f_X(x) = \int_{y=0}^{10} \frac{1}{100}\, dy = \frac{1}{100}\left(y\Big|_0^{10}\right) = \frac{1}{100}(10-0) = \frac{1}{10}$$

$$E[X] = 5$$

Q8 (c) Given X = 5, what is the range of values taken by Y? What is fY|X=5(5)?

$$(Y|X=5) \in [0, 10] \qquad f_X(5) = 1/10$$

$$f_{Y|X=5}(y) = \frac{f_{XY}(5,y)}{f_X(5)} = \frac{1/100}{1/10} = \frac{1}{10}$$

Q8 (d) Are X and Y independent?

$$f_Y(y) = \int_{x=0}^{10} \frac{1}{100}\, dx = \frac{1}{100}(10-0) = \frac{1}{10}$$

$$E[Y] = 5$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$\Rightarrow$ independent

Q8 (e) What are E[X], E[XY], E[X|Y=5]?

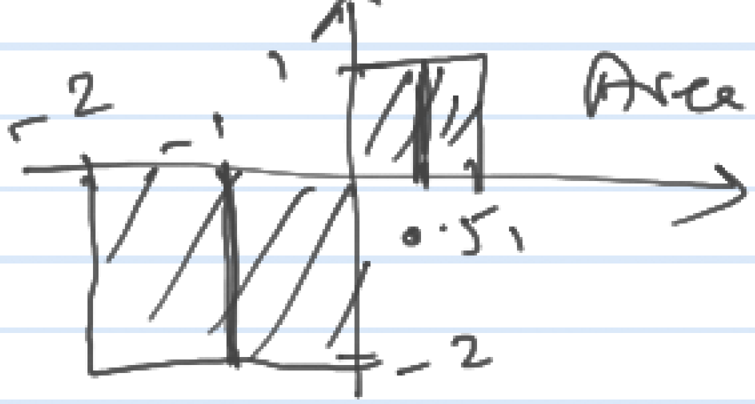
$$E[X] = 5$$

$$E[XY] = E[X]E[Y] \quad (\text{because } X \,\&\, Y \text{ are independent})$$

$$= 5 \times 5 = 25$$

$$E[X|Y=5] = E[X] = 5$$

Q9 (a) Suppose X and Y are jointly uniform in the region {-2 < x,y < 0} U {0 < x,y < 1}. Sketch the region. What is the value of f_xy(x,y) within the region of support?

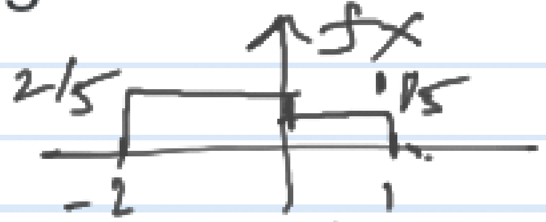


Area $= 2\times 2 + 1\times 1$

$= 5$

$f_{xy}(x,y) = \frac{1}{5}$ in region of support

Q9 (b) What is the range of values taken by X? Find fx(-1), fx(0.5).



$X \in [-2, 1]$

$f_x(-1) = \int_{y=-2}^{0} \frac{1}{5}\, dy = \frac{1}{5}\left(0-(-2)\right) = \frac{2}{5}$

$f_x(0.5) = \int_0^1 \frac{1}{5}\, dy = \frac{1}{5}$

Q9 (c) Given X = -1, what is the range of values taken by Y? Find f_Y|X=-1(-1), f_Y|X=-1(0.5).

$(Y|X=-1) \in [-2, 0]$

$f_{y|x=-1}(-1) = \frac{f_{xy}(-1,-1)}{f_x(-1)} = \frac{1/5}{2/5} = 1/2$

$f_{y|x=-1}(1/2) = \frac{f_{xy}(-1,1/2)}{f_x(-1)} = \frac{0}{2/5} = 0$

Q9 (d) Are X and Y independent?

$f_{xy}(-1, 0.5) = 0$

$f_x(-1) = \frac{2}{5}$ and $f_y(0.5) = \frac{1}{5}$

$f_x(-1)\cdot f_y(0.5) = \frac{2}{5}\cdot\frac{1}{5} \neq 0$

So, dependent.

Q9 (e) What are E[X], E[X|Y=-1], E[X|Y=0.5]?

$E[X] = \int_{-2}^{0}\int_{-2}^{0} x\cdot\frac{1}{5}\, dx\, dy + \int_0^1\int_0^1 x\cdot\frac{1}{5}\, dx\, dy$

$= \int_{-2}^{0} \frac{+2x}{5}\, dx + \int_0^1 \frac{x}{5}\, dx = \frac{1}{5}\left(x^2\right)\Big|_{-2}^{0} + \frac{1}{5}\left(\frac{x^2}{2}\Big|_0^1\right) = \frac{-4}{5} + \frac{1}{5}\cdot\frac{1}{2}$

$= -7/10$

$E[X|Y=-1] = -1$, $E[X|Y=0.5] = 0.5$

$\sim$ Unif[-2,0] $\qquad \sim$ Unif[0,1]

Q10 (a) Suppose X and Y have the following joint PMF:

f_XY(x,y) = k xy, 0 < x,y < 2, and f_XY(x,y) = 0, otherwise.

What is the value of k?

$$\int_0^2\int_0^2 kxy\,dx\,dy = k\int_0^2 y\left(\int_0^2 x\,dx\right)dy = k\underbrace{\int_0^2 2y\,dy}_{y^2\big|_0^2 = 4-0=4} = 4k = 1$$

$$\frac{x^2}{2}\Big|_0^2 = \frac{4}{2} - 0 = 2$$

$$\Rightarrow k = 1/4$$

Q10 (b) What is the range of values taken by X? Find fx(1).

$$X \in [0,2]$$

$$f_X(x) = \int_0^2 \frac{1}{4} \times x \times y\,dy = \frac{x}{4}\overbrace{\int_0^2 y\,dy}^{2} = \frac{x}{2}$$

$$f_X(1) = 1/2$$

Q10 (c) Given X = 1, what is the range of values taken by Y? Find f_Y|X=1(1).

$$(Y|X=1) \sim [0,2]$$

$$f_{Y|X=1}(y) = \frac{f_{X,Y}(1,y)}{f_X(1)} = \frac{1 \times y/4}{1/2} = y/2$$

Q10 (d) Are X and Y independent?

$$f_Y(y) = y/2, \quad f_X(x) = x/2$$

$$f_{XY}(x,y) = \frac{xy}{4} = f_X(x)f_Y(y)$$

$\Rightarrow$ Independent

Q10 (e) What are E[X], E[XY], E[X|Y=1]?2

$$E[X] = \int_0^2\int_0^2 \frac{x^2 y}{4}\,dx\,dy = \int_0^2 \frac{x^2}{4}\left(\int_0^2 y\,dy\right)dx = \int_0^2 \frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\Big|_0^2\right) = \frac{8}{2\times 3} = \frac{4}{3}$$

$$E[X|Y=1] = \int_0^2 x^2/2\,dx = \frac{1}{2}\cdot\left(\frac{x^3}{3}\Big|_0^2\right) = \frac{1}{2}\cdot\frac{8}{3} = 4/3$$

$$E[XY] = E[X]\,E[Y] = \frac{4}{3}\cdot\frac{4}{3} = \frac{16}{9}$$