Linear transformations, ordered bases and matrices

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- What is an isomorphism of vector spaces?
- How does a basis of V determine a linear transformation from V (extending maps on bases to linear transformations)?

An important property of finite dimensional vector spaces

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Then f is an isomorphism.

$v = \sum c_i v_i$

$f(v) = \sum c_i e_i$ $\Rightarrow f(v_i) = e_i$.

Onto : $(x_1, \dots, x_n) \in \mathbb{R}^n$. Let $v = \sum x_i v_i$.

Then $f(v) = \sum x_i e_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, \dots, x_n)$

$v = \sum c_i v_i$

1-1 : $f(v) = 0 \Rightarrow \sum_{i=1}^{n} c_i e_i = (0, \dots, 0)$

$\Rightarrow (c_1, \dots, c_n) = (0, \dots, 0)$

$\Rightarrow c_i = 0 \Rightarrow v = 0v_1 + 0v_2 + \dots + 0v_n$

$= 0$.

Example

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Note that $(x, y, z) \in W$ can be uniquely expressed as $(x, y, z) = y(-1, 1, 0) + z(-1, 0, 1)$.

Hence, $f : W \to \mathbb{R}^2$ is $f(x, y, z) = y(1, 0) + z(0, 1) = (y, z)$.

onto : $(y,z) \in \mathbb{R}^2$. Choose $x = -y-z$ & consider $(x,y,z) \in W$.

$\therefore f(x,y,z) = (y,z)$.

1-1 : $f(x,y,z) = 0 \Rightarrow (y,z) = (0,0) \Rightarrow y=0, z=0$

for $(x,y,z) \in W$ $\Rightarrow x = -y-z = 0$.

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Example : Linear transformation in matrix form

Consider the linear transformation
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The matrix corresponding to a linear transformation with respect to ordered bases

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$$
\begin{aligned}
f(v_1) &= a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \\
f(v_2) &= a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \\
&\vdots \\
f(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m
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e.g. Let $V = W = \mathbb{R}^2, \beta = \gamma = (1,0),(1,1)$ and $f(x,y) = (2x,y)$.

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Hence the matrix corresponding to f w.r.t. the ordered bases $\{(1,0),(1,1)\}$ is $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

Recovering the linear transformation

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Let $v \in V$. Express $v = \sum_{j=1}^{n} c_j v_j$. Define

$$f(v) = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} A_{ij} w_i.$$

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Letting $c_k = 1$ and $c_j = 0$ for all $j \neq k$, we get that

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$$f(v_k) = A_{1k} w_1 + A_{2k} w_2 + \ldots + A_{mk} w_m.$$

Hence the matrix corresponding to f is indeed A.

Fixed ordered bases : Linear transformations $\leftrightarrow$ matrices

Let β and γ be ordered bases for vector spaces V and W respectively where $n = dim(V)$ and $m = dim(W)$.

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Let β and γ be ordered bases for vector spaces V and W respectively where $n = dim(V)$ and $m = dim(W)$.

There is a bijection :
$\{$ linear transformations from V to W $\} \leftrightarrow \{$ $m \times n$ matrices $\}$.

$$f \longmapsto A$$

$$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

$$f \longmapsfrom A$$

$$f(v_j) = \sum_{i=1}^{m} A_{ij} w_i$$

Extend f to a lin. trans.

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Recall that the isomorphism $f(x, y, z) = (y, z)$ from W to $\mathbb{R}^2$ was obtained by extending $f(-1, 1, 0) = (1, 0)$ and $f(-1, 0, 1) = (0, 1)$.

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Hence, the matrix corresponding to the linear transformation f with respect to the ordered bases β and γ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example contd. (changed basis)

Suppose in the previous example, we change the basis γ from the standard basis to the ordered basis $(1,0),(1,1)$.

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Hence, the matrix corresponding to the linear transformation f with respect to the new ordered bases $\beta = (-1,1,0),(-1,0,1)$ and $\gamma = (1,0),(1,1)$ is $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1}$.

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Thus, changing the ordered bases gives us different matrices corresponding to the same linear transformation.

Thank you