Maths−2 : Open session with problem solving

1. Find the rank of the matrix A, where $A = [a_{ij}]$ is of order 3×3 and $\boxed{a_{i,j} = \min\{i, j\}}$, $i, j = 1, 2, 3$.

Ans: $a_{ij} = \min\{i, j\}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{11} = \min\{1,1\} = 1$.

$a_{12} = \min\{1,2\} = 1$

$a_{13} = \min\{1,3\} = 1$

$a_{21} = \min\{1,2\} = 1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Rank(A) ?

$\det(A) = 1 \neq 0 \Rightarrow \text{Rank}(A) = 3$.

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\Rightarrow \text{Rank}(A) = \text{Rank}(I) = 3$.

If $\det(A) = 0 \Rightarrow \text{Rank}(A) < \underline{\underline{3}}$

$A_{m \times n} \xrightarrow{\text{RREF}} A^{1}$

no of non-zero rows of A^{1}

$= \text{rank of } A$

2. Let V be the vector space spanned by the vectors $(1, 2, 0)$ and $(0, 3, 1)$. Which one of the following is correct?

$\underset{\uparrow}{\underline{(1,2,0)}}$ $\mathbb{R}^3$ $\underset{\uparrow}{\underline{(0,3,1)}}$ $\mathbb{R}^3$.

- ◯ Option 1: $V = \{(a, 2a + 3b, b) \mid a, b \in \mathbb{R}\}$
- ◯ Option 2: $V = \{(0, 2a - 3b, b) \mid a, b \in \mathbb{R}\}$
- ◯ Option 3: $V = \{(-a, 2a + 3b, 0) \mid a, b \in \mathbb{R}\}$
- ◯ Option 4: $V = \{(0, 2a + 3b, 0) \mid a, b \in \mathbb{R}\}$

$V = \text{Span}\{\underline{(1,2,0), (0,3,1)}\}$.

$\underline{V}$ = all the linear combinations of of $\underline{(1,2,0)}$ & $\underline{(0,3,1)}$.

$a \in \mathbb{R}$, $b \in \mathbb{R}$. $a=1$, $b=1$

$\underline{a(1,2,0) + b(0,3,1)}$

$= (a, 2a, 0) + (0, 3b, b) = \underline{(a, 2a+3b, b)}$.

$V = \{\underline{(a, 2a+3b, b} \mid a, b \in \mathbb{R}\}$

$= \text{Span}\{\underline{(1,2,0), (0,3,1)}\}$.

$a, b \in \mathbb{R}$

$(1, 0, 0) \in V \Rightarrow (1, 0, 0) = (a, 2a+3b, b)$

$\Rightarrow \underline{\underline{a=1}}, \underline{\underline{b=0}} \quad \underline{\underline{2a+3b}} \neq 0$

$(1,0,0) \notin \underline{\underline{V}}$

Consider the coefficient matrix A of the following system of linear equations to answer questions 3 and 4:

$$\boxed{\begin{aligned} 3x_1 + 2x_2 + x_3 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}}$$

3. Which one of the following vector spaces represents the null space of A appropriately?
   - Option 1: $\{(-t, t, t) \mid t \in \mathbb{R}\}$. $= \{t(-1, 1, 1) \mid t \in \mathbb{R}\}$
   - Option 2: $\{(t_1, t_2, \frac{t_2 - t_1}{2}) \mid t_1, t_2 \in \mathbb{R}\}$.
   - Option 3: $\{(t, -t, t) \mid t \in \mathbb{R}\}$.
   - Option 4: $\{(t_1, t_2, \frac{t_1 + t_2}{2}) \mid t_1, t_2 \in \mathbb{R}\}$.

$= \text{Span}\{(-1, 1, 1)\}$

$$V = \{\underbrace{(x_1, x_2, x_3)}_{\in \mathbb{R}^3} \mid x_1 + x_3 = 0,\ 3x_1 + 2x_2 + x_3 = 0\}.$$

$$x_1 + x_3 = 0 \Rightarrow \boxed{x_1 = -x_3}$$

$$3x_1 + 2x_2 + x_3 = 0$$

$$\Rightarrow -3x_3 + 2x_2 + x_3 = 0 \Rightarrow 2x_2 = 2x_3$$
$$\Rightarrow x_2 = x_3$$

$V = \{(x_1, x_2, x_3) \mid$ two equations$\}$.

$= \{(-x_3, x_3, x_3) \mid x_3 \in \mathbb{R}\}$ — null space of A.

$= \{x_3(-1, 1, 1) \mid x_3 \in \mathbb{R}\}$.

$= \mathrm{Span}(-1, 1, 1)$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

no pivot

x_3 is an indep.

$A : \mathbb{R}^3 \to \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{matrix} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}}$$

2×3

$x_3 = t$

$x_2 = t$

$x_1 = -t$

$\mathrm{Rank}(A) = 2$

$\mathrm{Rank}(A) + \mathrm{Nu}(A) = 3.$

$\Rightarrow \mathrm{Nu}(A) = 1$

$\mathrm{Sol}(A) = \{(-t, t, t) \mid t \in \mathbb{R}\}.$

pivot x_2 x_4 — Independent variable

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

pivot

4. What will be the rank of A and nullity of A?

- ◯ Option 1: $rank(A) = 3, nullity(A) = 2$
- ◯ Option 2: $rank(A) = 2, nullity(A) = 1$
- ◯ Option 3: $rank(A) = 1, nullity(A) = 2$
- ◯ Option 4: $rank(A) = 2, nullity(A) = 0$

★ Null Space $= \{(-t, t, t) \mid t \in \mathbb{R}\}$.

$= \text{Span}\{\underline{(-1, 1, 1)}\} \subseteq \mathbb{R}^3$

$\text{Nul}(A) = 1$

$A_{2\times 3}$

$\text{rank}(A) + \text{Nullity}(A) = 3$.

$\Rightarrow \text{rank}(A) = 3 - 1 = 2$

$$x_1+x_2+x_3=0 \quad = \quad \begin{aligned} x_1+x_2+x_3&=0 \\ 2x_1+2x_2+2x_3&=0 \end{aligned}$$

$= \mathbb{R}^3$

5. If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation defined by $\boxed{T(x, y) = (2x + 3y, 5x - y, x + 6y)}$. Which of the following options is true?

$T : V_1 \to V_2$

$T(v+w) = T(v) + T(w)$

$T(\alpha v) = \alpha T(v)$

Line

- ◯ Option 1: T is both one to one and onto.
- ◯ Option 2: T is one to one, but not onto.
- ◯ Option 3: T is onto, but not one to one.
- ◯ Option 4: T is neither one to one nor onto.

☆ one-one

$\boxed{T(v) = 0 \text{ Iff } v = 0}$

$T(x, y) = (0, 0, 0) \text{ Iff } (x, y) = 0$

$\Rightarrow (2x+3y, 5x-y, x+6y) = (0, 0, 0)$

$v = (x, y)$

$T(v) = 0$ Iff $v = 0$

$$\Rightarrow \boxed{\begin{cases} 2x+3y = 0 \\ 5x - y = 0 \\ x + 6y = 0 \end{cases}} \Rightarrow \begin{bmatrix} 5 & -1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = y = 0$

If $T(x, y) = 0 \Rightarrow \boxed{(x, y) = 0}$ $(0, 0)$

$\Rightarrow T$ is one-one

$(a, b, c) \in \mathbb{R}^3$ $\quad T(x, y) = (x-y, y-x)$

$\exists\, (x, y) \in \mathbb{R}^2$ $\quad x - y = 0$

$T(x, y) = (a, b, c)$. $\quad x = y \quad (1, 1) \to (0, 0)$

$\Rightarrow (2x+3y, 5x-y, x+6y) = (a, b, c)$.

$$\left| \begin{array}{l} 2x+3y = 0 \\ 5x - y = 0 \\ x+6y = 0 \end{array} \right| \begin{array}{ll} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}$$

2

T-is on-to.

$\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$$\begin{cases} (1,0,0) \in \operatorname{Im}(T). \\ (0,1,0) \in \operatorname{Im}(T) \\ (0,0,1) \in \operatorname{Im}(T) \end{cases} \Rightarrow \mathbb{R}^3 \subseteq \operatorname{Im}(T)$$

$\Rightarrow$ T-is on-to

$T: \mathbb{R} \to \mathbb{R}^3$

$x \to (x, 2x, 3x)$.

$(1,0,0) \in \mathbb{R}^3$

$\exists\, x$ $\quad T(x) = (1,0,0)$

$\Rightarrow (x, 2x, 3x) = (1,0,0)$

$\Rightarrow$ $x = 1$ $x = 0$

6. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(v) = Av$, where $v = \begin{bmatrix} x \\ y \end{bmatrix}$, and $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$. Which of the following options is true?

- ◯ Option 1: T is an isomorphism. ✓
- ◯ ~~Option 2: T is one to one, but not onto.~~
- ◯ ~~Option 3: T is onto, but not one to one.~~
- ◯ Option 4: T is neither one to one nor onto.

$A \begin{bmatrix} x \\ y \end{bmatrix}$

Ans:- $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$T(x, y) = (3x + 2y, 4x + 5y)$.

$T(x, y) = (0, 0)$

$\Rightarrow$ $3x + 2y = 0$
$4x + 5y = 0$ $\Rightarrow$ $x = y = 0$

$\Rightarrow$ T-is one-one.

$\Rightarrow$ T-is on-to.

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (*, *, *)$

7. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, such that $T(x, y) = (x, 0)$. Which of the following matrices corresponds to T with respect to the standard ordered basis of $\mathbb{R}^2$, i.e., $\{(1,0),(0,1)\}$, for both the domain and co-domain?

- ◯ Option 1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- ◯ Option 2: $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
- ◯ ~~Option 3~~: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- ◯ Option 4: $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\{(1,0), (0,1)\}$
$\neq$
$\{(0,1), (1,0)\}$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(x, y) \longrightarrow (x, 0)$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\{(1,0), (0,1)\} - \mathbb{R}^2$

$\in \mathbb{R}^2$. a_{11} a_{21}

$T(1,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,1) = (0,0) = 0(1,0) + 0(0,1)$

/

8. If the matrix corresponding to a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to standard ordered basis of $\mathbb{R}^2$, i.e., $\{(1,0),(0,1)\}$, for both the domain and co-domain, is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Which of the following is the appropriate definition of T?

- ◯ Option 1: $T(x, y) = (2x + y, 3x + 4y)$
- ◯ Option 2: $T(x, y) = (2x + y, 3x - 4y)$
- ◯ Option 3: $T(x, y) = (x + 4y, 2x + 3y)$
- ◯ Option 4: $T(x, y) = (2x + 3y, x + 4y)$

$$\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$T(1,0) = \underline{x_1}(1,0) + \underline{x_2}(0,1)$$

$$T(0,1) = x_3(1,0) + x_4(0,1)$$

$$\underline{T(1,0)} = 2(1,0) + 1(0,1) = \underline{(2,1)}$$

$$\underline{T(0,1)} = \underline{(3,4)}$$

$$\underline{T(x,y)}$$

$$\underline{T(x)} \Rightarrow x \in \underline{\mathrm{dom}(T)}$$

$$\underline{(x,y)} = x(1,0) + y(0,1).$$

$$T(x,y) = x\,T(1,0) + y\,T(0,1)$$

$$= x(2,1) + y(3,4)$$

$$= (2x + 3y,\ x + 4y)$$

Consider the following linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$T(x, y, z) = (2x + 3z, 4y + z)$$

$\mathbb{R}^3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

Answer questions 9,10 and 11, using the information given above.

$\mathbb{R}^2 = \{(1,0),(0,1)\}$

9. Which of the following matrices corresponds to the given linear transformation T with respect to the standard ordered basis for $\mathbb{R}^3$ and the standard ordered basis for $\mathbb{R}^2$?

   ○ Option 1: $\begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 0 \end{bmatrix}$

   ○ Option 2: $\begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$

   ○ Option 3: $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

   ○ Option 4: $\begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

$T(1,0,0) = (2,0) = 2(1,0) + 0(0,1)$

$T(0,1,0) = (0,4) = 0(1,0) + 4(0,1)$

$T(0,0,1) = (3,1) = 3(1,0) + 1(0,1)$

10. Which of the following represents a basis of the kernel of T?

- ◯ Option 1: $\{(-\frac{3}{2}, 0, 1), (0, -\frac{1}{4}, 1)\}$.
- ◯ Option 2: $\{(-\frac{3}{2}, -\frac{1}{4}, 1)\}$.
- ◯ Option 3: $\{(-\frac{3}{2}, -\frac{1}{4}, 2)\}$.
- ◯ Option 4: $\{(2, 0, 3), (0, 4, 1)\}$.

$T(x,y,z) = (2x+3z, 4y+z)$.

$\underline{\text{Ker } T} = \{(x,y,z) \mid T(x,y,z) = 0\}$.

$$T(x,y,z) = 0 \Rightarrow 2x+3z=0, \quad 4y+z=0.$$

$$\Rightarrow 4y = -z \Rightarrow y = -z/4$$

$$x = -\tfrac{3}{2}z$$

$$\text{Ker}(T) = \{(-\tfrac{3}{2}z, -z/4, z) \mid z \in \mathbb{R}\}$$

$$= \text{Span}\{(-\tfrac{3}{2}, -\tfrac{1}{4}, 1)\}$$

$\dim(\text{Ker}(T)) = \text{Nullity}(T) = 1$

$\mathbb{R}^3 \to \mathbb{R}^2$

$\text{Rank}(T) = 2$

$T : V \to W$

$T(v_1),\ T(v_2) \cdots T(v_n)$

$T(3, 10, 8) = 3$ $\quad T(5, 20, 15) = 1.$

$T(14, 15, 10) = 2$

$T(x, y, z)$?

$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$

$T(x, y, z) = x\,T(1, 0, 0) + y\,T(1, 0, 0) + zT(0, 0, 1).$

1

$\text{Rank}(T) = \dim(\text{Image}(T))$

$V \qquad W$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

11. What will be the dimension of the subspace $Im(T)$?

Nullity of $T = 1$

$\text{Rank}(T) + \text{Nullity}(T) = 3$

$\Rightarrow \text{Rank}(T) = 3 - 1 = 2$

2

$T: V \to W$

$\text{Rank}(T) + \text{Null}(T) = \dim V$