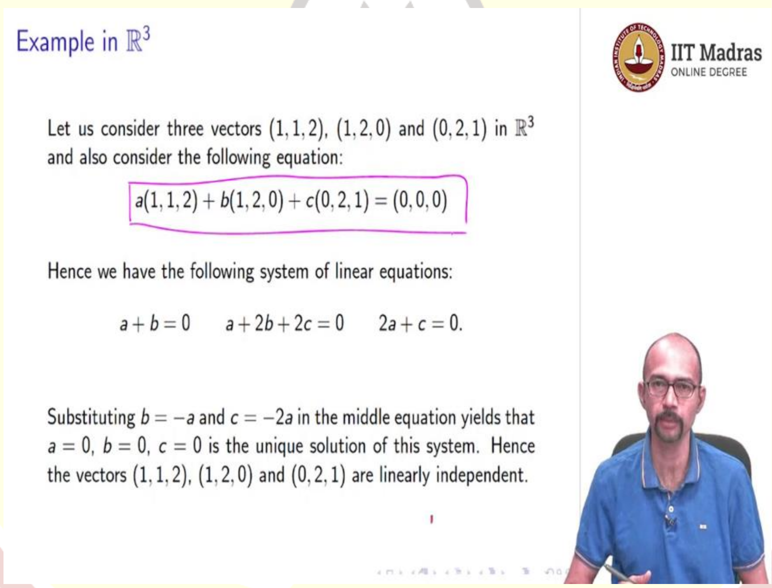
IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
Professor. Sarang S. Sane
Department of Mathematics
Indian Institute of Technology, Madras
Lecture No. 31
Linear Independence – Part 2

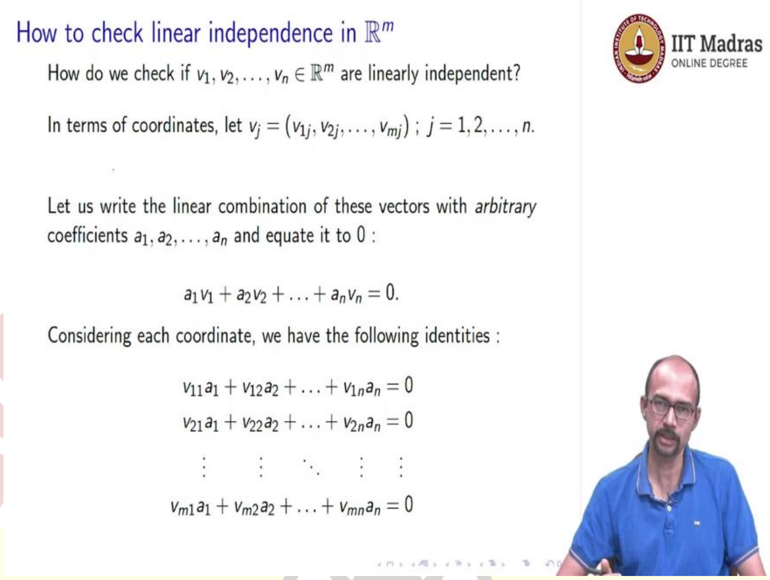
Hello, and welcome to the online B.Sc. program on data science and programming. In this video we are going to talk about linear independence. So, this continues from our previous video on the same topic. So, let us recall that linear independence means a set of vectors being linearly independent means that they are not linearly dependent, which is to say that if you take a linear combination which equals 0, then the only way that can happen is if the coefficients are 0.

(Refer Slide Time: 00:44)



So, just to recall, here is the last example that we did in our previous video. So, we have these three vectors, (1, 1, 2), (1, 2, 0) and (0, 2, 1) in $\mathbb{R}^3$ and we take unknown coefficients a, b and c for these vectors. So, $a(1,1,2) + b(1,2,0) + c(0,2,1) = (0,0,0)$. This is what we assume. And then we try to work out a, b and c. And indeed, if we equate the coefficients and from this equation here and write down the equations in a, b and c and solve them, we get $a = b = c = 0$.

(Refer Slide Time: 01:24)



So, we can now ask the same question in general, how do we check if you have a set of n vectors in $\mathbb{R}^m$ when are they linearly independent? So, just to, let us go back one step and ask what we did in $\mathbb{R}^3$. So, in $\mathbb{R}^3$ what we did is we took arbitrary coefficients. I will underline the word arbitrary or unknown. And then checked when, what are the possible solutions for these coefficients for this equation to be 0. This is a general template. So, in $\mathbb{R}^m$ you will do the same thing.

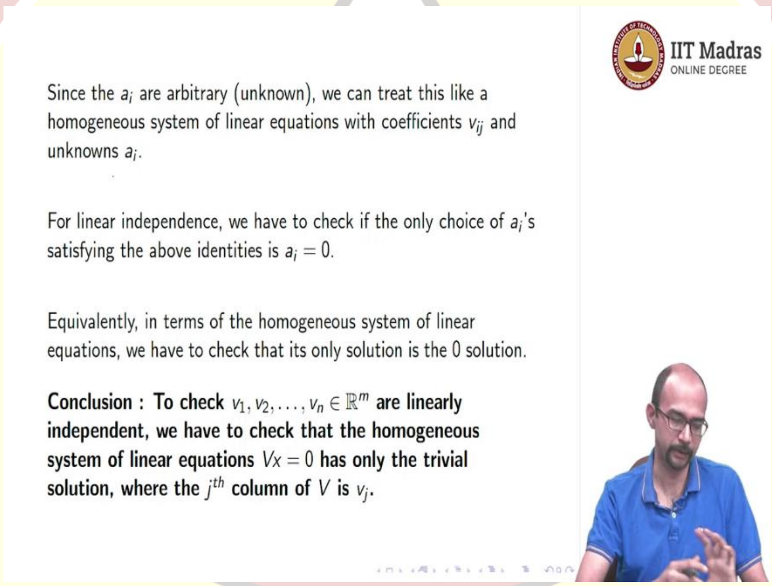
So, suppose you have coordinates $v_{1j}, v_{2j}, \ldots, v_{mj}$, it is in $\mathbb{R}^m$, so there are m coordinates, so you take the j^{th} vector, that is your, that is v_j and you write down the coordinates, so that is $v_{1j}, v_{2j}, \ldots, v_{mj}$ and you will do this for each of your vectors $v_1, v_2, \ldots, v_n$. So, let us write the linear combination of these vectors with arbitrary or unknown coefficients $a_1, a_2, \ldots, a_n$. So, here we want to determine the coefficients, which yields this equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. So, we equate this linear combination on the left to 0.

So, now, we can, this is in $\mathbb{R}^m$ remember, so both sides we can express in terms of coordinates. So, the first coordinate on the left is $v_{11}a_1 + v_{12}a_2 + \cdots + v_{1n}a_n$. The first coordinate on the right hand side is 0. That is a 0 vector. Similarly, for the second coordinate $v_{21}a_1 + v_{22}a_2 + \cdots + v_{2n}a_n$ on the left, on the right it is 0, and we can do this all the way up till m. So, remember, there are m coordinates.

So, now, you can notice that we have written this system in a different way. We have written the v_{ij}'s on the left and the a_j's on the right. Why do we do that? So, the reason is that here the unknowns are the coefficients $a_1, a_2, \ldots, a_n$ and we, our notation is that if you have a system $ax = b$, so you write $\sum a_{ij}x_j$. So, here $x_j's$ are $a_i's$ and $a_{ij}'s$ are $v_{ij}'s$. That is why we have written it in a slightly altered way.

And now, we know this is a system of linear equations. In fact, this is a homogeneous system of linear equations, because the right hand side is 0. And we know exactly how to solve this. This is something we have done in the previous weeks. So, the most general method is what we call Gaussian elimination. If you are in, there are other situations where you can do better by just looking at determinant and so on.

(Refer Slide Time: 04:26)

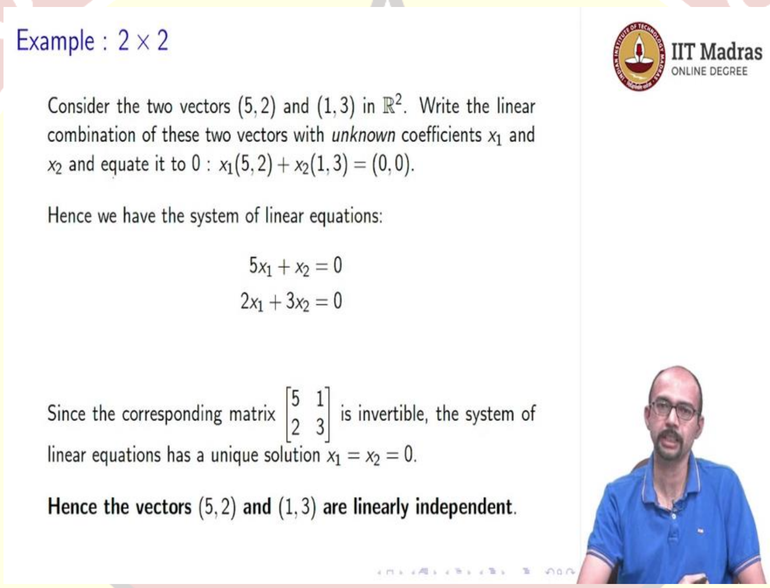


So, since the $a_i's$ are arbitrary or unknown, we can treat this like a homogeneous system of linear equations with coefficients v_{ij} and unknowns a_i. This is exactly what we discussed. So, for linear independence, we have to check if the only choice of a_i satisfying the above identities or equations is $a_i = 0$ for all i. So, $a_i = 0$ for each i.

So, remember, we are not, mean when we study linear independence, we are not really interested in asking what coefficients give you the linear combination 0. We do not want to expressly evaluate which a_i is, what is the set of solutions. All we want to know is whether the solution is only a set of 0's or whether non-zero solutions are, do exist, meaning at least one non-zero coefficient, whether such solutions exist. So, we need not solve the entire system. That is the point.

So, equivalently in terms of the homogeneous system of linear equations, we have to check if the only solution is the 0 solution, meaning each of the $a_i's$ is 0. So, the conclusion here is, this is the main point, to check if $v_1, v_2, \dots, v_n$ in $\mathbb{R}^m$ are linearly independent, we have to check that the homogeneous system of linear equations $Vx = 0$ has only the trivial solution, meaning the solution where x is 0, that is what we have to check. And what is this V, V is the matrix we can form out of these vectors, where the j^{th} column is the vector v_j, meaning you take the coordinates of the vector v_j and write that in as a column of V. That is how you get a matrix V.

(Refer Slide Time: 06:29)



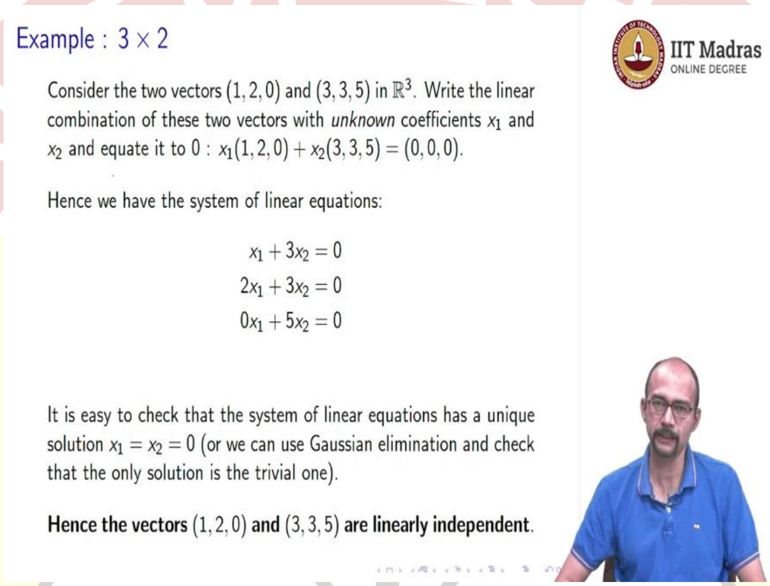
So, let us do this. Let us do a bunch of examples to set these ideas into place. Let us do a 2 by 2 example first. So, consider the two vectors (5, 2) and (1, 3) in $\mathbb{R}^2$, write the linear combination of these 2 vectors with unknown coefficients x_1 and x_2 and equate it to 0, $x_1(5,2) + x_2(1,3) = (0,0)$. So, now I am using x_1 and x_2 in place of a_1 and a_2.

So, we have the system of linear equations $5x_1 + x_2 = 0$ and $2x_1 + 3x_2 = 0$. And we want to know if the only solution for this is $x_1 = x_2 = 0$ or whether there are other solutions. Remember that this being a homogeneous system, the trivial solution always exists. This is something we have seen even in the previous week. We have discussed in the previous week.

So, since it is a 2 by 2 matrix, we can look at the corresponding determinant. So, here the determinant will be enough to check whether or not the only solution is 0, 0. So, in this case, the determinant is non-zero. It is an invertible matrix. So, the determinant is 13, which is non-zero.

So, since the determinant is non-zero, it is an invertible matrix. And once you note an invertible matrix, we know that the only solution is the trivial solution. So, it has a unique solution $x_1 = x_2 = 0$. So, the upshot is that the vectors (5, 2) and (1, 3) are linearly independent. So, I hope this is, you have understood how to solve this question of whether two vectors are linearly independent or not at least in this case.

(Refer Slide Time: 08:23)



We are going to do a bunch of examples. So, here is an example where you have a 3 by 2 matrix. So, consider the two vectors (1, 2, 0) and (3, 3, 5) in $\mathbb{R}^3$. So, let us also understand what is this 3 by 2? This means your vectors are in $\mathbb{R}^3$ and you have two vectors. So, your, this is saying you have a 3 by 2 matrix, which means you have two columns. So, remember, columns correspond to the number of vectors and the size of the vector is the number of rows. So, that tells you it is in $\mathbb{R}^3$.

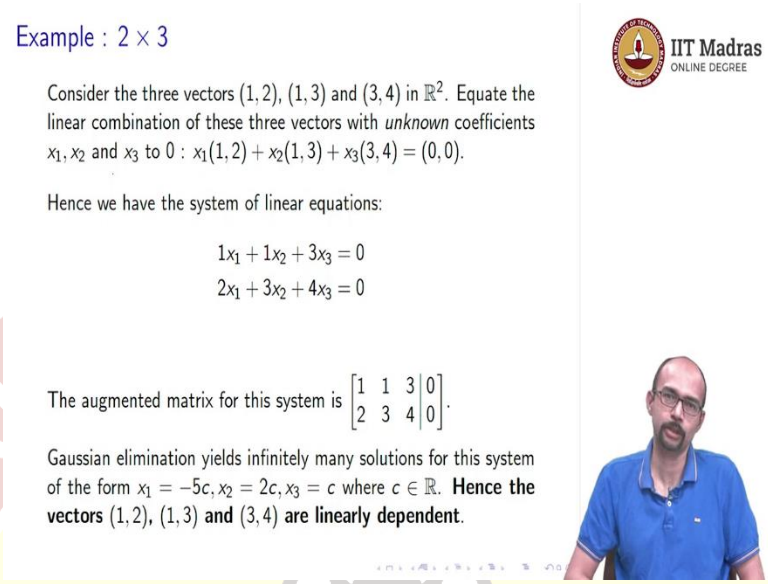
So, in general, if you have an m by n matrix, that means you have n columns which is corresponding to n vectors and they are, the size of the vectors is m, which means they are in $\mathbb{R}^m$. We studied this a few slides ago. So, you have these two vectors (1, 2, 0) and (3, 3, 5). So, before we go ahead, let us note first of all that we already know the linear independence of two vectors. Namely, two vectors are linearly independent means that they are not multiples of each other. They are two non-zero vectors. So, here you can see that these are not multiples of each other. That means they are linearly independent. So, you already know this fact. But let us do this from, in terms of our matrix.

So, write the linear combination of these two vectors with unknown coefficients x_1 and x_2 and equate it to 0. So, $x_1(1,2,0) + x_2(3,3,5) = (0,0,0)$. From here we get a set of system of linear equations by equating the corresponding coordinates. So, the first coordinate gives you $x_1 + 3x_2 = 0$. The second coordinate gives you $2x_1 + 3x_2 = 0$ And the third coordinate gives you $0x_1 + 5x_2 = 0$.

And now, in this case, it is actually easy to check that the only solution is where x_1 and x_2 are both 0, because the third equation gives you that $5x_2 = 0$ that means x_2 must be 0. And then you, if you put that into the first equation that tells you x_1 must be 0. And you need not cross check with the second equation, because we know, already know that x_1 and x_2 are 0 is a solution, because this is a homogeneous system.

So, the only solution here, we have a unique solution, namely that x_1 and x_2 are both 0. So, if you do not find this adhoc principal, this adhoc way of doing it agreeable, then you can actually do it the standard way. Namely, you can use Gaussian elimination and we will see soon that there are, we will see an example soon where we do actually have to use Gaussian elimination. So, what is the net result? The net result is what we observed right at the start, because they are not multiples of each other, namely that the vectors (1, 2, 0) and (3, 3, 5) are linearly independent.

(Refer Slide Time: 11:30)



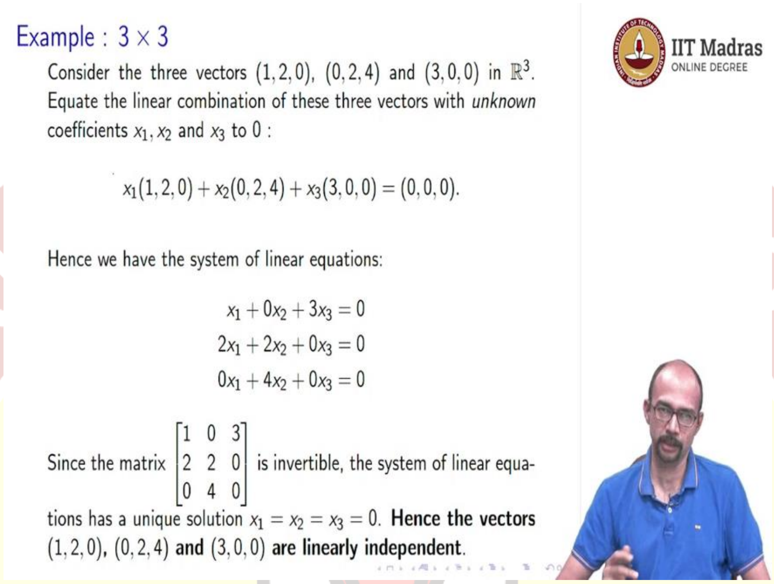
Let us do a 2 by 3 example. So, what does this mean? This means I have three vectors in $\mathbb{R}^2$. So, what, let us take the three vectors (1, 2), (1, 3) and (3, 4) in $\mathbb{R}^2$. And remember that for three vectors being linearly dependent is the same as saying that we can write some one of these vectors in as a linear combination of the others. So, one way of checking this would be to ask if one of these is a linear combination of the others. But a more direct method is to take unknown coefficients x_1, x_2 and x_3 and then write down the equation $x_1(1,2) + x_2(1,3) + x_3(3,4) = (0,0)$

Let us equate the coordinates. So, if you equate the coordinates, you get $1x_1 + 1x_2 + 3x_3 = 0$ and you get the $2x_1 + 3x_2 + 4x_3 = 0$. Those are your two equations. This is a system of linear equations. And now we can use Gaussian elimination. So, the augmented matrix for this system is the matrix $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 3 & 4 & 0 \end{array}\right]$.

Let us do Gaussian elimination. So, that means we have to row reduce. And if you row reduce, it is very easy to check that your solutions are of the form $x_1 = -5c$, $x_2 = 2c$ and $x_3 = c$, where $c \in \mathbb{R}$. So, the point is, as c varies, the solutions vary, which means there are infinitely many solutions. So, the net result is that the vectors (1, 2), (1, 3) and (3, 4) are linearly dependent.

So, here, let us observe what happened. You see, you had 3 unknowns and you had 2 equations. And we have already seen what happens in these cases in the previous week, that if you have more unknowns than the number of equations, then you always have lots of solutions. So, keep that in mind and we will make this precise later on.

(Refer Slide Time: 13:57)



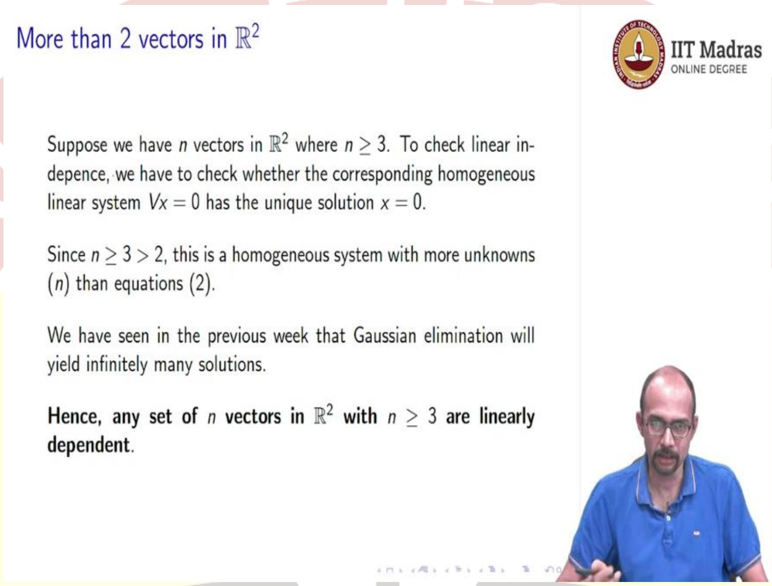
Finally, let us see a three by three example. So, that means you have three vectors in $\mathbb{R}^3$. So, let us take the vectors (1, 2, 0), (0, 2, 4) and (3, 0, 0). So, actually, right away, you can see that these are linearly independent, because if we can, if they are not that means one of them can be written as a linear combination of the other two. And just by observation you can see that is not possible. But let us go through our usual method.

So, let us take unknown coefficients, x_1, x_2 and x_3 and equate that linear combination to 0. So, you have $x_1(1,2,0) + x_2(0,2,4) + x_3(3,0,0) = (0,0,0)$. Let us take the corresponding coordinates on each side. So, doing that we get $1x_1 + 0x_2 + 3x_3 = 0$, then $2x_1 + 2x_2 + 0x_3 = 0$, and then $0x_1 + 4x_2 + 0x_3 = 0$.

So, now, you could either use Gaussian elimination or you can consider, because it is a 3 by 3 case, you can consider the determinant, which is what we do in this solution or you can just do it by observation, because in the last, if you look at the last equation, then this is $4x_2 = 0$ that means $x_2 = 0$. Then you put that, substitute that in the second equation that will give you $x_1 = 0$. And then substitute $x_1 = 0$ in the first equation that will give you $x_3 = 0$. Alternatively, you can look at this matrix. So, the corresponding matrix is (1, 0, 3), (2, 2, 0), (0, 4, 0). And note that this has non-zero determinant.

So, from here you can check that, so what is the determinant here, so, the determinant here is 24. And so this is an invertible matrix that tells us that this system has a unique solution 0, 0, 0. So, the upshot is that these vectors are linearly independent. This is, we sort of observed this at the start of the slide and indeed we have explicitly proved it.

(Refer Slide Time: 16:32)

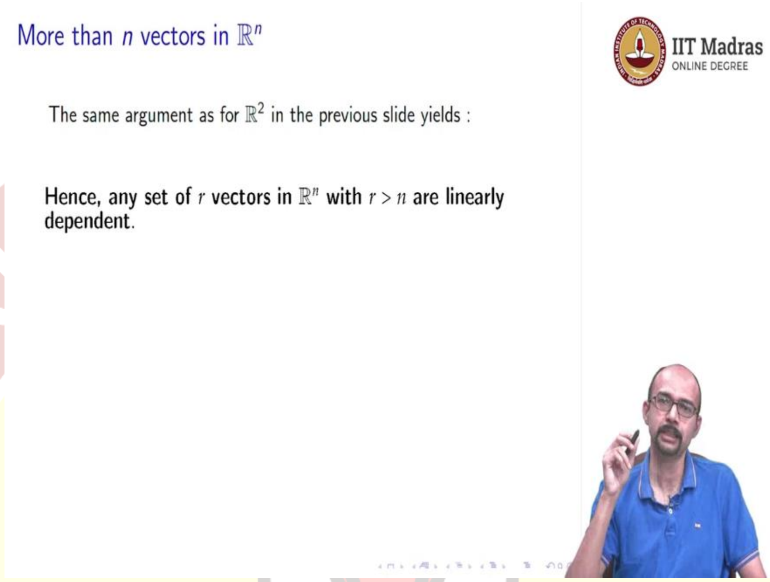


So, now let us address this question about, this comment that I made earlier that remember we had three vectors in $\mathbb{R}^2$ that means we had the 2 by 3 case and suppose now we have more than two vectors in $\mathbb{R}^2$, so suppose we have n vectors in $\mathbb{R}^2$, where n is at least 3. So, what happens? To check linear independence, we have to check whether the corresponding homogeneous linear system $Vx = 0$ has a unique solution $x = 0$.

But on the other hand, since n is bigger than 3, n is at least 3, which is bigger, strictly bigger than 2 and this is a homogeneous system with more unknowns than equations, so we have seen in the previous week that Gaussian elimination will yield infinitely many solutions. So, any set of n vectors in $\mathbb{R}^2$, where n is at least 3, so we have three vectors or four vectors or 20 vectors or 100 vectors that is going to be a linearly dependent set of vectors. That is the conclusion.

So, if you have either a single vector or two vectors in $\mathbb{R}^2$ only then do they have a chance of being linearly independent. More than two vectors, it is always going to be linearly dependent. So, you can see that these numbers somehow picks up the fact that we are in $\mathbb{R}^2$. So, we are in $\mathbb{R}^2$ and this fact is being picked up by linear independence. This is important and we will see this shortly in our next few videos.

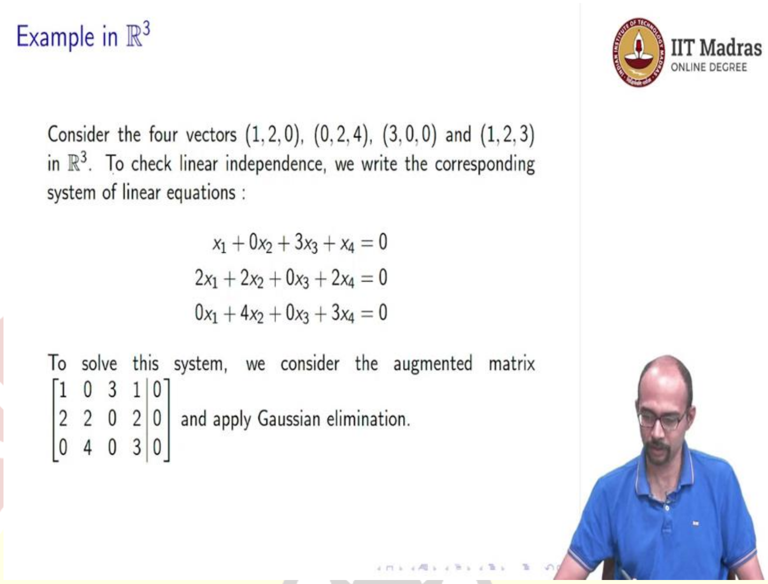
(Refer Slide Time: 18:12)



We can generalize this. So, suppose you are in $\mathbb{R}^n$ and you have more than n vectors. So, for example, if you are in $\mathbb{R}^3$ and you have 4 vectors or you are in $\mathbb{R}^4$ and you have 5 vectors or let us say you are in $\mathbb{R}^8$ and you have 20 vectors, you can make the same argument as in the previous slide for $\mathbb{R}^2$, namely that you will get a system of linear equations, homogeneous system of linear equations with more unknowns than number of equations, and hence, it always has a non-trivial solution that will tell you that these are linearly dependent.

So, if you have more vectors, then the n, I mean, what is in $\mathbb{R}^n$, you have more than n vectors in $\mathbb{R}^n$, then they are always going to be linearly dependent, important point. So, again, this is picking up that number n in some way.

(Refer Slide Time: 19:17)

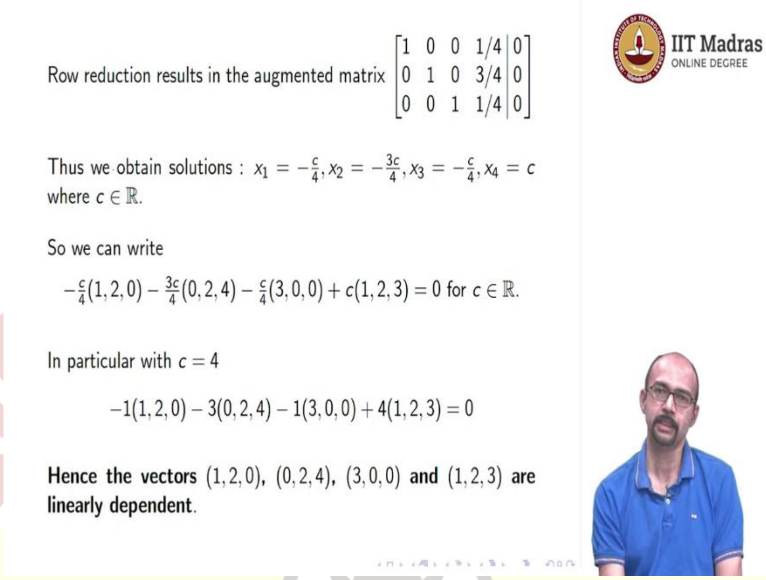


Let us do an example in $\mathbb{R}^3$. So, consider the four vectors (1, 2, 0), (0, 2, 4), (3, 0, 0) and (1, 2, 3) in $\mathbb{R}^3$, so we have already checked, remember that the first three are linearly independent. We checked that these three are linearly independent. So, now we are introducing a new vector (1, 2, 3). So, let us do our usual thing and write down three equations by equating the corresponding coordinates.

So, you have equations $x_1(1,2,0) + x_2(0,2,4) + x_3(3,0,0) + x_4(1,2,3) = (0,0,0)$. Look at the corresponding coordinates. So, we get $1x_1 + 0x_2 + 3x_3 + 1x_4 = 0$, $2x_1 + 2x_2 + 0x_3 + 2x_4 = 0$, $0x_1 + 4x_2 + 0x_3 + 3x_4 = 0$.

So, now, of course, there are lots of 0s here. And you may be able to manipulate and solve this by hand. But as we know the general and fastest and cleanest method of doing this is by Gaussian elimination. So, we will write on the augmented matrix that gives us $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 2 & 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 3 & 0 \end{array}\right]$.

(Refer Slide Time: 20:53)

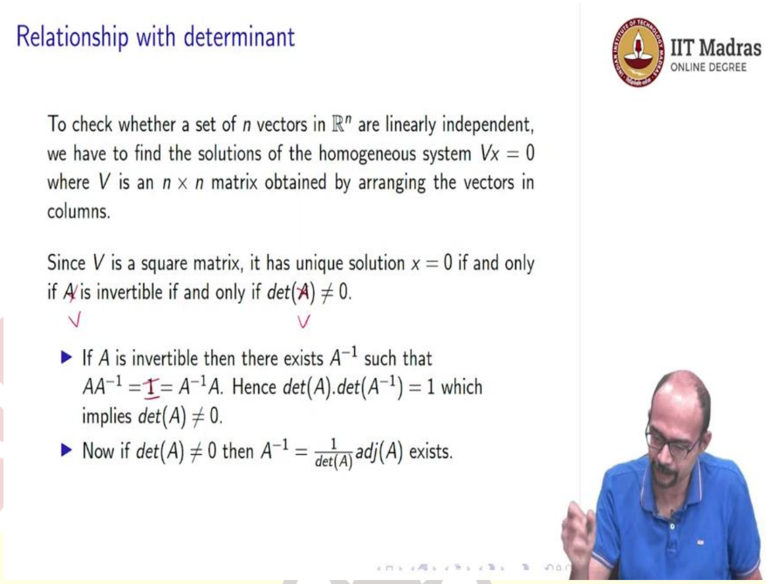


Let us do row reduction. So, if you do row reduction, we get the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & 3/4 & 0 \\ 0 & 0 & 1 & 1/4 & 0 \end{array}\right]$. So, what have we obtained? We obtained that the solutions to this are $x_1 = -\frac{c}{4}$. So, remember that we had this idea here of which variables are independent and which variables are dependent. So, x_4 is the independent variable and x_1, x_2, x_3 are dependent. So, you put $x_4 = c$ and once you put $x_4 = c$ from the first equation you will get that $x_1 = -\frac{c}{4}$, from the second you will get that $x_2 = -\frac{3c}{4}$, and from the third you will get that $x_3 = -\frac{c}{4}$.

So, as c varies, this is your set of solutions. So, just as an example you could take c to be, let us say 1 or maybe c to be 4. So, if you take c to be 4, you get $-1(1,2,0) - 3(0,2,4) - 1(3,0,0) + 4(1,2,3)$. And you can check that this is actually equal to 0. So, this gives us, so I have given you an explicit example of an equation or a linear combination of these four vectors, which yields 0, where the coefficients are not all 0. In fact, in this case, none of them are 0. But it is enough to have that not all of them are 0.

So, the net upshot is that these 3, these 4 vectors are linearly dependent. We already knew this. Because remember that we said, if you have more than three vectors in $\mathbb{R}^3$, meaning four or more vectors in $\mathbb{R}^3$, then they are going to be linearly dependent. And this example shows you why that is happening.

(Refer Slide Time: 23:08)



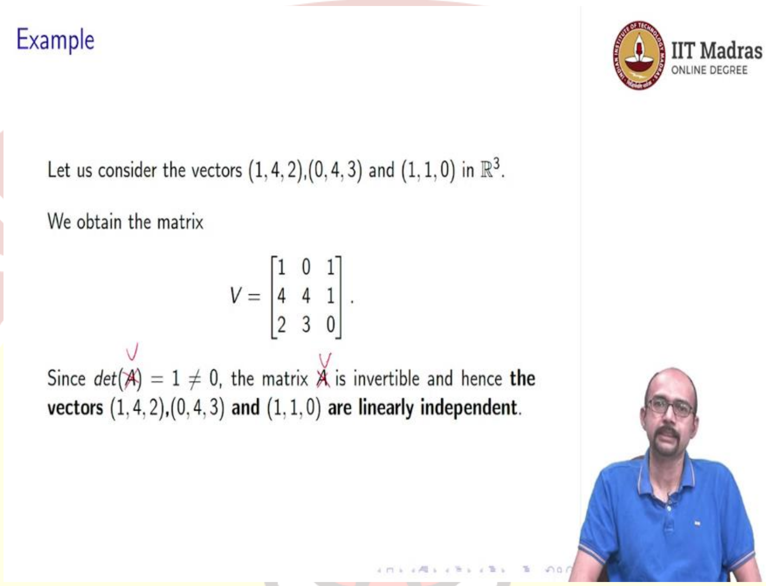
So, finally, let us talk about the relationship with the determinant. We have seen some examples with the determinant. So, let us talk about the relationship with the determinant. So, now, suppose you have a set of n vectors in $\mathbb{R}^n$, which are linearly independent, rather we want to check whether they are linearly independent. So, what do we do? We take those vectors, express them in terms of their coordinates, make the corresponding matrix V.

So, that is now an n by n matrix, where the j^{th} entry of, sorry, the j^{th} column of that matrix corresponds to the j^{th} vector v_j. And then you look at the corresponding homogeneous system of linear equations $Vx = 0$ and ask whether the only solution for this is x is 0. That will determine whether or not it is linearly independent.

So, if the only solution is 0, then it is linearly independent. If the only solution is, well, if there are solutions which are non-zero, then it is not linearly independent, meaning it is linearly dependent. So, I can check this by looking at whether or not V is invertible. And to check whether or not V is invertible I can check what is the determinant of V. So, if the determinant is 0, then these vectors are linearly dependent. If the determinant is non-zero, then these vectors are linearly independent.

So, this has a unique solution if and only if this vector V is, this matrix V is invertible, so this is not A but this is V, should have been V and if A is invertible let us recall that there exists A inverse such that A times A inverse is 1, meaning the identity, it should be identity is A inverse times A and so the determinant is non-zero. And we can reverse this, remember, by, if you recall how we went the other way, if the determinant is non-zero then you can look at the miners and look at the adjugate matrix. And if the determinant, so that is how you go the other way, this is V and this is V.

(Refer Slide Time: 25:40)



So, let us do an example. So, let us look at these three vectors in $\mathbb{R}^3$ (1, 4, 2), (0, 4, 3) and (1, 1, 0). So, the corresponding matrix is $\begin{bmatrix} 1 & 0 & 1 \\ 4 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix}$. Well, I should not say that way. I have always been saying it along the rows. So, the corresponding matrix is $\begin{bmatrix} 1 & 0 & 1 \\ 4 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix}$. So, what we do is, we put this 1, 4, 2 into the first column and 0, 4, 3 into the second column and 1, 1, 0 into the third column. And then, let us look at the determinant of V. Apologies, this should be V.

The determinant is 1. So, this matrix V is invertible. And so the vectors (1, 4, 2), (0, 4, 3) and (1, 1, 0) are linearly independent. So, this is an example of the previous idea where if you have n vectors in $\mathbb{R}^n$, you can deduce whether or not they are linearly independent just by putting them into a matrix, by putting the j^{th} vector into the j^{th} column and creating that matrix and then looking at the determinant. If the determinant is 0, then they are linearly independent, if the determinant is 0, they are linearly dependent. And if the determinant is non-zero, they are linearly independent.

So, let us summarize what we have seen in this video so far. So, we have seen in this video linear independence deduces to checking a system of linear equations where you take the coefficients to be the unknowns and you take the matrix to be, the matrix coming from the vectors by putting the j^{th} vector into the j^{th} column. So, all this is, of course, for $\mathbb{R}^m$. If you are in some other vector space, then what I am saying does not work, then you have to just do that by hand. So, we will see examples of that later on or in the, and in the tutorials.

So, and then you look at the system of linear equations $Vx = 0$. And if that system has a non-trivial solution is a homogeneous system. In fact, it has a non trivial solution, meaning if there is a solution where x is not 0, then the set of vectors is linearly dependent. If the only possible solution is a trivial solution, namely x is 0, then the set of vectors $v_1, v_2, \dots, v_n$ is linearly independent.

And note that if the number of vectors is bigger than the vector space in which one is working in, meaning the exponent of the vector space, meaning if you have $\mathbb{R}^n$ and if you have more than n vectors, so you have $n + 1$ or $n + 2$ and so on, if you have that many vectors, so bigger than or equal to $n + 1$ vectors, then they are always linearly dependent. And if you have exactly n vectors, so you have n vectors in $\mathbb{R}^n$, then you can check the linear dependence or independence by looking at the determinant of that corresponding matrix V formed by putting the j^{th} vector into the j^{th}column. So, this is more or less everything that we have discussed in this video. Thank you.