

Partial derivatives

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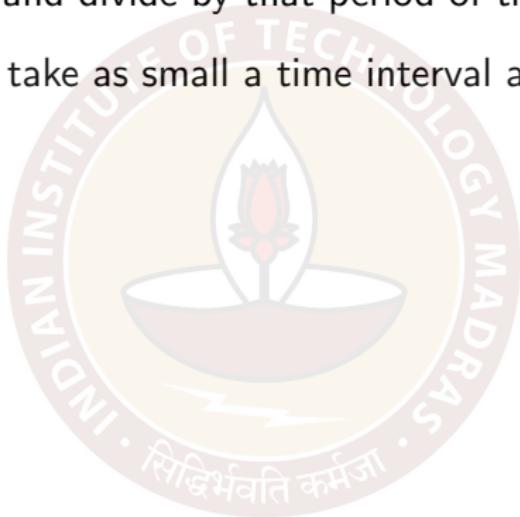
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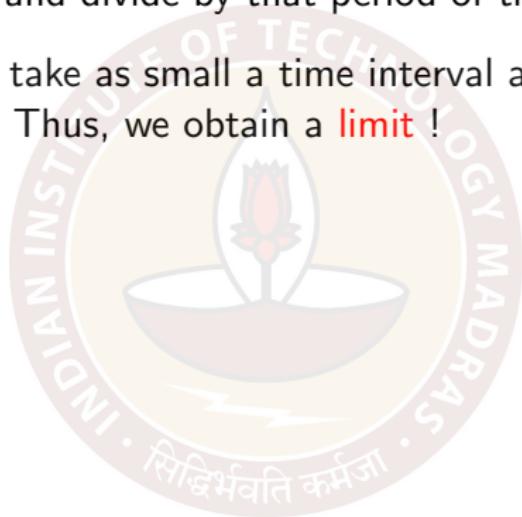
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Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a
scalar-valued multivariable function.



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If \tilde{a} is a point in \mathbb{R}^n , then an open ball of radius r around \tilde{a} is the set defined as

$$\{\tilde{x} \in \mathbb{R}^n \mid \|\tilde{x} - \tilde{a}\| < r\}.$$

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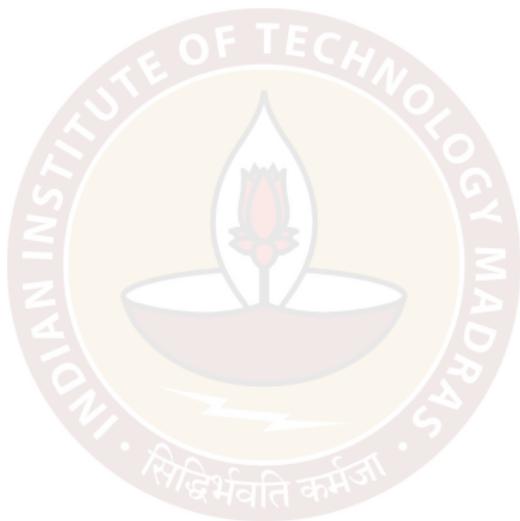
e_1, e_2, \dots, e_n is the standard ordered basis of \mathbb{R}^n .

Rate of change w.r.t. a particular variable at a point



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Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n containing a point \tilde{a} and an open ball around it.

Then the rate of change of f at \tilde{a} w.r.t. the variable x_i is

$$\lim_{h \rightarrow 0} \frac{f(\tilde{a} + he_i) - f(\tilde{a})}{h}$$

$$\tilde{a} = (a_1, a_2, \dots, a_n)$$

$$e_i = (0, 0, \dots, 0, \overset{i\text{th place}}{1}, 0, \dots, 0)$$

$$f(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + h(f(a_1, \dots, a_n) - f(\tilde{a}_1, \dots, \tilde{a}_n))$$

$$\lim_{h \rightarrow 0}$$

$$= \lim_{h \rightarrow 0} \frac{f(a_1, a_2, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_n)}{h}$$

$$g(h) = f(\tilde{a} + he_i)$$

g is a fn. at 1-variable

$$\lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h}$$

Examples

The rate of change of $f(x, y) = x + y$ at $(0, 0)$ w.r.t. x

$$\lim_{h \rightarrow 0} \frac{f((0,0) + h(1,0)) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{h - 0}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1.$$

The rate of change of $f(x, y, z) = xy + yz + zx$ at $(1, 2, 3)$ w.r.t. y

$$\lim_{h \rightarrow 0} \frac{f((1,2,3) + h(0,1,0)) - f(1,2,3)}{h} = \lim_{h \rightarrow 0} \frac{f(1, 2+h, 3) - f(1,2,3)}{h}$$
$$= \lim_{h \rightarrow 0} \frac{1 \times (2+h) + (2+h) \times 3 + 1 \times 3 - 1 \times 2 - 2 \times 3 - 1 \times 3}{h} = \lim_{h \rightarrow 0} \frac{h + 3h}{h} = 4.$$
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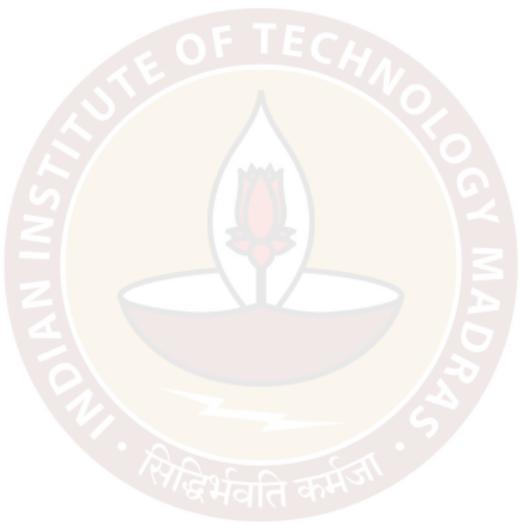
The rate of change of $f(x, y) = \sin(xy)$ at $(1, 0)$ w.r.t. x

$$\lim_{h \rightarrow 0} \frac{f((1,0) + h(1,0)) - f(1,0)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h,0) - f(1,0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

w.r.t. y :

$$\lim_{h \rightarrow 0} \frac{f((1,0) + h(0,1)) - f(1,0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h) - 0}{h} = \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1.$$

Partial derivatives



Partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .

The **partial derivative of f w.r.t. x_i** is the function denoted by

$f_{x_i}(\tilde{x})$ or $\frac{\partial f}{\partial x_i}(\tilde{x})$ and defined as

$$\frac{\partial f}{\partial x_i}(\tilde{x}) = \lim_{h \rightarrow 0} \frac{f(\tilde{x} + he_i) - f(\tilde{x})}{h} .$$

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$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{f(\underset{\sim}{x} + h e_i) - f(\underset{\sim}{x})}{h}.$$

Its domain consists of those points of D at which the limits exists.

Example

$$f(x, y) = x + y$$

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(x, y + h) - (x + y)}{h} \\&= \lim_{h \rightarrow 0} \frac{x + y + h - (x + y)}{h} = 1.\end{aligned}$$
$$\begin{aligned}\frac{\partial f}{\partial y}(x, y) &= \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h} \\&= \lim_{h \rightarrow 0} \frac{f(x, y + h) - (x + y)}{h} \\&= \lim_{h \rightarrow 0} \frac{x + y + h - (x + y)}{h} = 1.\end{aligned}$$

Calculating partial derivatives



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To calculate the partial derivative w.r.t. x_i , think of f only as a function of x_i while treating all other variables as constants. Then calculate it as the derivative of a function of one variable.



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Examples :

► $f(x, y, z) = xy + yz + zx$

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= y + 0 + z = y + z \\ \frac{\partial f}{\partial y}(x, y, z) &= x + z \\ \frac{\partial f}{\partial z}(x, y, z) &= x + y.\end{aligned}$$

► $f(x, y, z) = \sin(xy)$

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \omega_s(xy) \cdot y = y \omega_s(xy) \\ \frac{\partial f}{\partial y}(x, y) &= x \omega_s(xy).\end{aligned}$$

$$\begin{aligned}x^3 + 20 + 18x \\ 3 + 0 + 18\end{aligned}$$

$$\begin{aligned}\sin(3x) \\ \omega_s(3x) \times 3\end{aligned}$$

Another example :

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$(x, y) \neq (0, 0)$

What is $\frac{\partial f}{\partial x}(x, y)$ & $\frac{\partial f}{\partial y}(x, y)$

$$\frac{\partial f}{\partial x}(x, y) = \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{x^3 - y^2x}{(x^2+y^2)^2}$$

$$\frac{3x}{x^2+4}$$

$$\begin{aligned} (x, y) &= (0, 0) & f(h, 0) - f(0, 0) & \xrightarrow{\text{lim}} 0 \\ \frac{\partial f}{\partial x}(0, 0) &\stackrel{\text{defn}}{=} \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0. \end{aligned}$$

$$\frac{\partial f}{\partial y}(0, 0) = 0.$$

