Cramer's Rule

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JUNE 2020

An example of using Cramer's rule

Consider the following system of linear equations

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Matrix representation : $Ax = b$ where
the matrix A is given by $A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 11 \\ 7 \end{bmatrix}.$

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Consider the following system of linear equations

$$4x_1 - 3x_2 = 11$$
$$6x_1 + 5x_2 = 7$$

$12x_1 - 9x_2 = 33$
$12x_1 + 10x_2 = 14$
$19x_2 = 14 - 33 = -19.$
$\Rightarrow x_2 = -1.$
$\Rightarrow x_1 = 2.$

Matrix representation : $Ax = b$ where

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Unique solution : $x_1 = 2, x_2 = -1$

Example (Contd.) : Steps to apply Cramer's rule

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- Calculate $det(A)$. $= 4 \times 5 - (-3) \times 6 = 20 + 18 = 38.$

Example (Contd.) : Steps to apply Cramer's rule

- Coefficient matrix $A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$
- Calculate $det(A)$.
- Replace the first column of A by the column vector b and call it A_{x_1} . $\quad A_{x_1} = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}$

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- Coefficient matrix $A = \begin{bmatrix} 4 & -3 \\ 6 & 5 \end{bmatrix}$
- Calculate $det(A)$.
- Replace the first column of A by the column vector b and call it A_{x_1} . $A_{x_1} = \begin{bmatrix} 11 & -3 \\ 7 & 5 \end{bmatrix}$
- Replace the second column of A by the column vector b and call it A_{x_2}. $A_{x_2} = \begin{bmatrix} 4 & 11 \\ 6 & 7 \end{bmatrix}$
- Calculate $det(A_{x_1}) = 76$.

Example (Contd.) : Steps to apply Cramer's rule

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- Calculate $det(A_{x_1}) = 76$.
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Example (Contd.) : Cramer's Rule

Calculate $\dfrac{det(A_{x_1})}{det(A)} = \dfrac{76}{38}$ $\dfrac{det(A_{x_2})}{det(A)} = \dfrac{-38}{38}$

The solutions are :

Example (Contd.) : Cramer's Rule

Calculate $\dfrac{det(A_{x_1})}{det(A)}$ $\dfrac{det(A_{x_2})}{det(A)}$

The solutions are :

$$x_1 = \frac{det(A_{x_1})}{det(A)} = \frac{76}{38} = 2. \qquad x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{-38}{38} = -1.$$

Cramer's rule for invertible 2×2 matrices

Consider the following system of linear equations of two variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

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Define $A_{x_1} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$ and $A_{x_2} = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$.

Cramer's rule for invertible 2×2 matrices

The solution of the system of equations in 2 variables is:

$$x_1 = \frac{det(A_{x_1})}{det(A)} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}$$

$$x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

Cramer's rule for invertible 3×3 matrices

Consider the following system of linear equations in 3 variables :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

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Matrix representation : $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

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The solution of the system of equations of 3 variables is:

$$x_1 = \frac{det(A_{x_1})}{det(A)} \qquad x_2 = \frac{det(A_{x_2})}{det(A)} \qquad x_3 = \frac{det(A_{x_3})}{det(A)}$$

Example of Cramer's rule for a 3×3 invertible matrix

Consider the system of linear equations $Ax = b$ where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 3 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

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$$A_{x_1} = \begin{bmatrix} 0 & 0 & 3 \\ 2 & 2 & 5 \\ 1 & 3 & 1 \end{bmatrix} \qquad A_{x_2} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 4 & 1 & 1 \end{bmatrix} \qquad A_{x_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

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$$det(A_{x_1}) = 12 \qquad det(A_{x_2}) = -27 \qquad det(A_{x_3}) = 4.$$

Example (Contd.) :

Applying Cramer's rule, the solution of the system of equations is :

$$x_1 = \frac{det(A_{x_1})}{det(A)} = -\frac{12}{37}$$
$$x_2 = \frac{det(A_{x_2})}{det(A)} = \frac{27}{37}$$
$$x_3 = \frac{det(A_{x_3})}{det(A)} = \frac{4}{37}$$

Cramer's rule for invertible $n \times n$ matrices

Consider the system of linear equations $Ax = b$ where A is an $n \times n$ invertible matrix and b is a column vector with n entries.

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$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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Define A_{x_i} to be the matrix obtained by replacing the $i-th$ column of A by the column vector b.

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Define A_{x_i} to be the matrix obtained by replacing the $i-th$ column of A by the column vector b. Cramer's rule states that the (unique) solution is :

$$x_i = \frac{det(A_{x_i})}{det(A)}.$$

Thank you