INDIAN INSTITUTE OF TECHNOLOGY MADRAS

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IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 04
Limits for Sequences

Hello and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to talk about Limits for Sequences. In our previous video, we have seen the idea of tangents and we saw that it is a really tricky question as to what is the tangent and when does it exist and we saw some examples towards the end, which motivated that there is some more depth to that question. And indeed, this is a deeper question than it seems just from geometry. And for this reason, one has to start getting into what is called calculus.

So, we have seen calculus in some form presumably in high school or maybe you have not. But presumably, you may not have seen what the form that we are going to use it now. So, we are going to do a little bit of, we are going to develop a little bit of theory and we will then use that theory to get our handle on the idea of tangents much better. So, the, we are going to approach it from something called the notion of differentiability. So, without, I am already getting ahead of myself. So, without spending more time, let me start this video, which is about limits for sequences.

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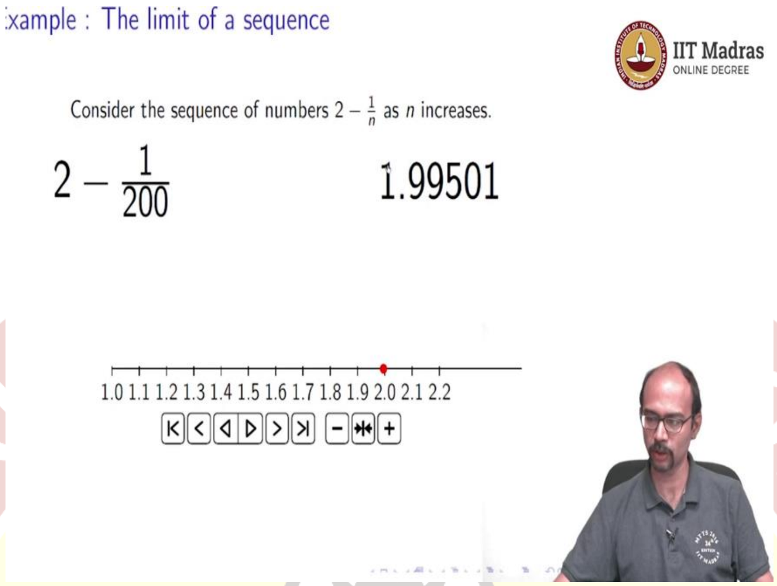

So, let me first start by looking at the following question. Let us look at the sequence of numbers $2 - \frac{1}{n}$ as n increases. So, before this, maybe I should first ask what is the sequence or I should first mention what is the sequence. So, a sequence is a set of, it is an assignment for each number n. There is some corresponding number, which we will call maybe a_n. So, in this case a_n is $2 - \frac{1}{n}$. So, as n increases your sequence of numbers a_n also changes. And the question is, as n increases what happens to this sequence of numbers?

So, one possible sequence of numbers is, let us say 1, 2, 3, 4, 5, 6. This is a sequence of numbers. Another possible sequence of numbers is, let us say, 1, -1, 1, -1, 1, -1 that goes on. So, you could write this in a more tangible fashion as $(-1)^n$. Or another possible sequence of numbers is, let us say, half, one fourth, one eighth, one sixteenth, 1/32, 1/64. If you know these numbers well, you will, you can tell that this is the sequence $\{\frac{1}{2^n}\}$. But of course, the sequence need not have a closed form. So, the sequence could just be for each n you associate some number, that is it. So, that is a sequence.

So, here is the sequence $2 - \frac{1}{n}$ and let us ask what happens as n increases. So, if you look at what happens as n increases, let us try to play this video and see what happens. So, as n increases, you can see on the left-hand side, you have $2 - \frac{1}{n}$. So, you can see what is n. As n increases, it will change. For example, the next term is $2-\frac{1}{2}$. The term after that is $2-\frac{1}{3}$, and its value is on the right-hand side. So, keep track of what happens to the value as your n changes.

Now, let me play this. So, as you can see, as your n is increasing, well, you have 1.98, 1.988, 1.989, 1.99, 1.993, 1.994. So, you can see this is increasing. And here below is the real line, the part between let us say 1 and 2.2. And if I play that again you can see how the geometry of the sequence, so the first term is $2-\frac{1}{2}$ which is 1.5.

The next term is $2-\frac{1}{3}$ so that is 1.66. And if I play it further, you can see that the red dot comes closer and closer and closer and closer to 2. And as your n increases, now it is 130. It is going to come very, very, very close as its 200, where I have stopped this sequence. It is really close to 2. It is 1.995.

So, you can tell that as n is going to become very, very, very large, this sequence is going to come very, very, very close to the number 2. So, in other words, this red dot is going to come even closer to this point 2. That is what this means geometrically and that is what it means in terms of your algebra or your numbers that it is going to come close to the value 2.

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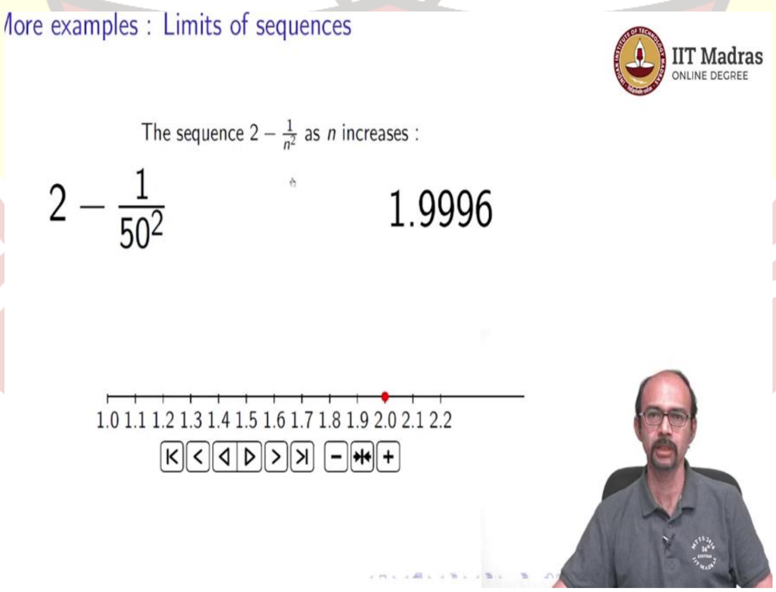

So, let us do the same thing for the sequence $2-\frac{1}{n^2}$. I want to do this because I want to see the speed with which this works. So, if you do it for $2-\frac{1}{n^2}$, well, you can see it went very, very, very fast. So, $2-\frac{1}{50^2}$, we are at 1.9996. Let me replay that. So, here is your first term $2-\frac{1}{1^2}$. So, the number is 1 and the red dot is at the position 1 on the real line.

And as you go ahead, you get $2-\frac{1}{2^2}$. That is $2-\frac{1}{4}$, $2-\frac{1}{3^2}$, $2-\frac{1}{9}$ and as you can see, it is already on the third itself, it is 1.8889. It took much longer for the sequence $2-\frac{1}{n}$. And another few terms and it is 1.97. So, it is very close. And if I play that, 29, 30, and look at what is happening. It is 1.999 already. So, this number, as you can see, it comes very close to 2.

In fact, the, if you look at the line, our points are not fine enough to distinguish between these numbers 2 and 1.9996. So, this red dot is already it seems as if it is sitting at 2, but it is not. It is actually very close, but not exactly 2. And you can see as n increases, this sequence is going to, the sequence of numbers is going to come closer and closer and closer to the number 2.

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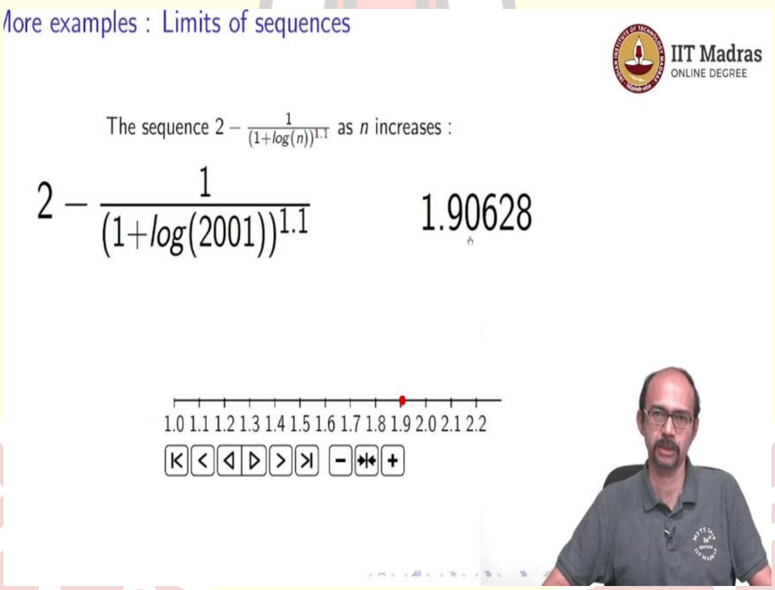

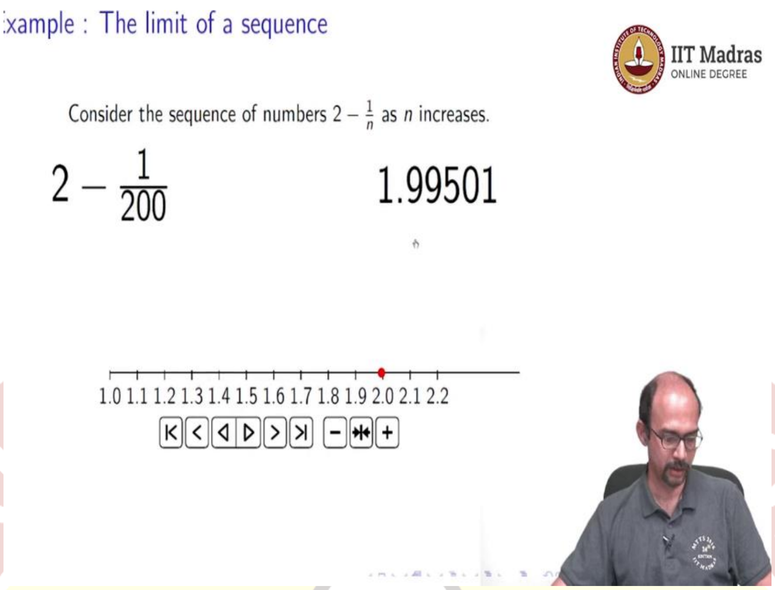

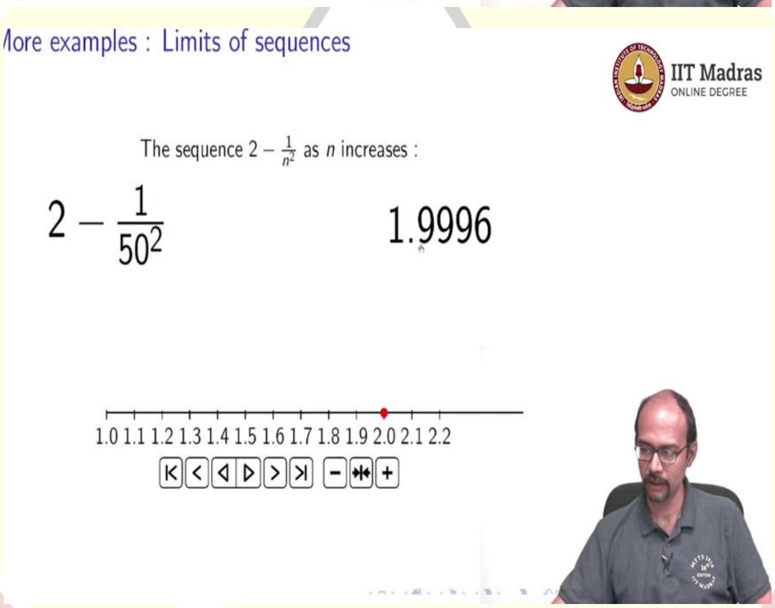

Let us do one more example just to motivate further what is happening. So, here is a slightly more complicated sequence. It is a sequence $2-\frac{1}{(1+\log n)^{1.1}}$. And to check what is happening, well, we again have the same counter. And this time I am going to play it for a much longer time. And you will see why. Now, it is 1.83, 1.84. Look at that red dot. It is moving very slowly. It is moving, moving, moving, but now it is very slow. And we are already well passed where we were in our previous sequences, we are now at closing on 300, still not passed 1.9.

In fact, at this point, if I stop it, it is at 367, you may feel, well, maybe it is not going to 2, maybe it is going to stop before 2. So, well, let us play it further. And now it is coming closer to 1.9, 1.88, but those numbers are moving very slowly. That is past 500 already and that red dot is closing on

1.9. So, I want you to have a visual feel of what convergence of sequences is, limits of sequences. So, here we are now. Where we are at 9, n is 800 or 900 and we are still not passed 1.9, but you can see it is coming close.

Now, we are going to come closer and closer and closer. And you probably feel that, very soon we will cross 1.9. Indeed, we are very close to doing that. There we are. We have crossed 1.9 and now the red dot is on the other side. And very, very, very slowly, you can see n is very large now. It is close, almost 1400. It is moving, but now it is very slow. Still at 1.90 and we can keep going. I will just take you till the end. And I have stopped at 2000, meaning when I say I have stopped, or rather 2001, I have stopped at the 2001^{th} element of this term of this sequence or element in the sequence.

So, what is that term? It is $2-\frac{1}{(1+\log 2001)^{1.1}}$ or rather strange looking. And where have we gone; we have come till 1.90628. So, this is a very slow-moving sequence. But my claim is and I will encourage you to check that this sequence indeed as your n becomes very, very, very large, this number will come very, very, very close to 2. So, this number is going to come very, very, very close to 2 or in other words is red dot, this red point which denotes that sequence is going to come very, very, very close to the number 2. It will keep moving, but it is going to come very slowly.

So, the first sequence $2-\frac{1}{n}$ that we had this was a reasonably fast-moving sequence. Within 200 terms, we came to 1.995. The second sequence which was $2-\frac{1}{n^2}$ within 50 terms we came till 1.9996. And the third sequence, well, we went till the 2001th term, and we still had 1.906. So, this is a, there is also an issue of how fast does it, does a limit, does it approach a limit, and the first one is reasonable, the second one is fairly fast and the third one is relatively slower.

So, you may see these ideas coming up again in some other course completely unrelated to what we are doing now. And in case you do, in the programming part maybe, and in case you do, you may want to think of this again. So, with that motivation for what is the limit of a sequence, let us define this explicitly.

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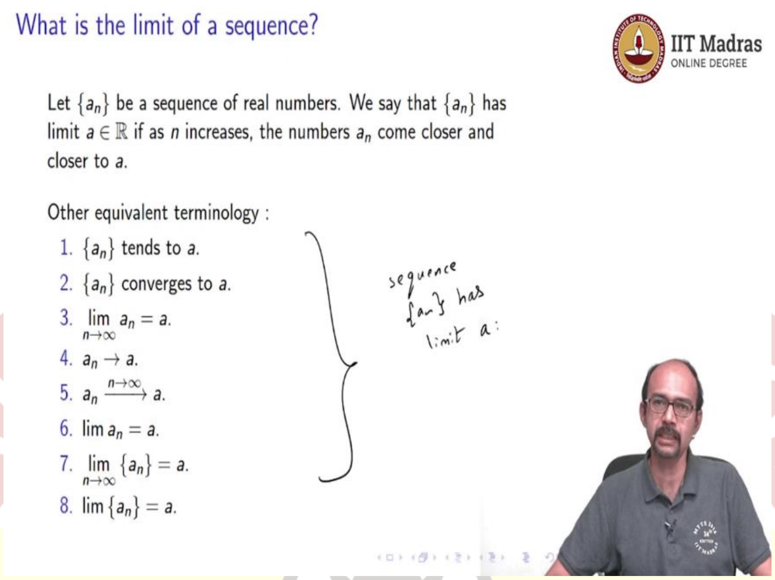

So, what is the limit of a sequence? So, let $\{a_n\}$ be a sequence of real numbers. We say that a_n has limit a, which is some other real number, or it could be one of the real numbers already appearing in your sequence, if as n increases, the numbers a_n come closer and closer to a. So, that is the intuitive definition of the limit of a sequence. Now, it is natural to ask what do we mean by closer and closer and that is something I am not going to get into. There is a more formal abstract definition, mathematical definition for what it means to come closer and closer and I am not going to get there.

So, we are going to just see enough to be able to make out what is going on. We would not be doing details here. What is, what are other ways of saying the same things? So, this, so the above statement was that the sequence has limit a if this happens. What is other equivalent terminology? $\{a_n\}$ tends to a, that means the same thing that the sequence $\{a_n\}$ has limit a, $\{a_n\}$ converges to a, again, same thing. $\lim_{n\to\infty} a_n = a$, that is a notation you may come across in if you read books or if you happen to look on the Internet, a_n, again, arrow a, again means that the sequence $\{a_n\}$ has limit a.

$a_n \to a$ and above that n tend to infinity or sometimes it is below that, again has the same meaning that the sequence $\{a_n\}$ has limit a or just simply $\lim_{n\to\infty} a_n = a$ that also means that the sequence as n tends to infinity the sequence has limit a or sometimes you may have the same thing with the bracket or with the n tends to infinity or without the n tends to. So, all of these are notations for the same statement. So, all of these mean the same thing. So, all of these mean that the sequence $\{a_n\}$ has limit a.

So, I may often use the first two, $\{a_n\}$ tends to a or $\{a_n\}$ converges to a in place of $\{a_n\}$ has limit a and do not be confused by that that is exactly why I had this slide up. So, what does this mean? Let us come back to that, which is the more important part. It means that as n becomes larger and larger, these numbers $\{a_n\}$ come closer and closer to a. That is what you have to take home from here.

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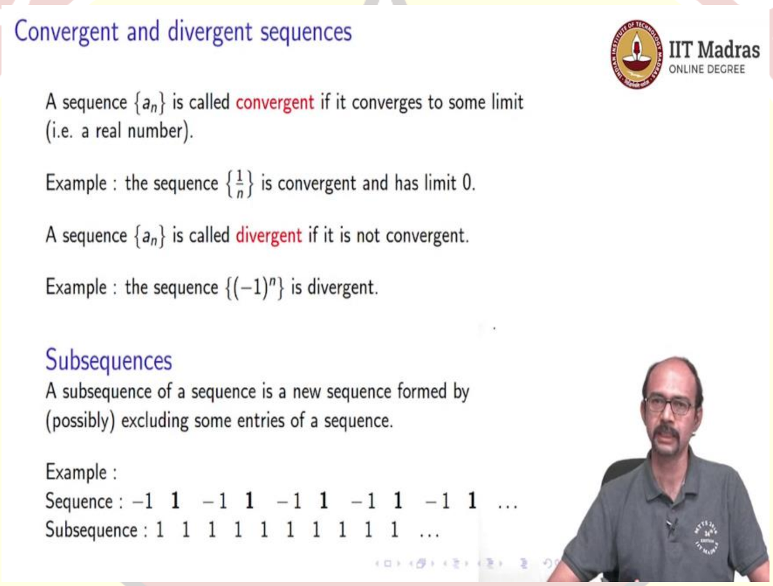

Let us do some more terminology. A sequence $\{a_n\}$ is called convergent if it converges to some limit. So, by a limit we mean some real number. So, for example, the sequence $\{\frac{1}{n}\}$ is convergent and it has limit 0. If you look at the sequence $\{\frac{1}{n}\}$, you can see that as n increases, n becomes very, very, very large, the numbers $\frac{1}{n}$ become very, very, very small. When I say small, I mean they are positive and are small. So, as n becomes very large, the numbers $\frac{1}{n}$ are going to come closer and closer to 0. So, this is an example of a convergent sequence.

A sequence an is called divergent if it is not convergent. If it is not convergent, it is divergent. That means these numbers do not converge to a limit which is a real number or do not converge at all. They may float around. So, here is an example. This is an example that we saw right at the beginning of this video. The sequence $(-1)^n$ is divergent. What is the sequence? Well, the first term is $(-1)^1$ that is -1. The second term is $(-1)^2$, which is 1. Third term is $(-1)^3$, which is minus 1. So, the sequence is -1, 1, -1, 1, -1, 1.

So, this sequence is what is called oscillating, it oscillates, -1, 1, -1, 1. So, clearly, the sequence does not have any limit. It is not coming close to any real number. So, this is an example of a divergent sequence. We will come to more examples. But let me define what is called a subsequence of a sequence.

So, a subsequence of a sequence is a new sequence formed by possibly excluding some entries of a sequence. So, if you have a sequence, so you have a_1, a_2, a_3, a_4, you have this sequence of real numbers and you drop some of those. You just take let us say a_1, a_4, a_{20}, a_{100}, $a_{1000000}$ and so on, some, in some prescribed order. So, whatever you get is a subsequence of the sequence.

So, some, often what you do is you drop some interesting terms which are given by some mathematical way. For example, you can say, let us take all the odd part of the sequence. So, that means you take the first term, third term, fifth term, seventh term and so on or you can say take the even part of the sequence, which is the second, fourth, sixth, eighth, and so on.

So, here is an example. So, here is your sequence, -1, 1, -1, 1 this is exactly the divergent sequence that we had above. And here is you have subsequence, which is the, one which is highlighted in that sequence. That is the even terms. So, that is the second, fourth, sixth, eighth, and so on. And if you do that, you get $(-1)^2$, $(-1)^4$, $(-1)^6$, and so on. So, those give you exactly once. So, your subsequence is 1,1,1,1,1,1,1.

I could, of course, take other kinds of subsequences which also have the same form. So, I can take the, let us say, the fourth term, the eighth term, the 12th term and so on, so $(-1)^{4n}$. So, if you do that, you still get the same numbers in your subsequence, but it is a different subsequence than the one written down because that is, this subsequence is the one obtained by taking the second, fourth, sixth, eight and so on. Whereas the 1 I just said right now is the 1 obtained by taking the fourth, eighth, 12th and so on.

So, as sequences these are the same, but as subsequences of this sequence they are different. So, some small quibbling here and maybe may not be important as we go ahead, but something to keep in mind for those of you who are more mathematically inclined. So, I hope you have a feeling for what is a sequence, what is the limit of a sequence, when is a sequence convergent, which is exactly saying it has a limit, when is a sequence divergent, which means it does not have a limit. So, we had this oscillating sequence. Let us write down another example of a sequence which is or maybe I think, next slide we have more examples.

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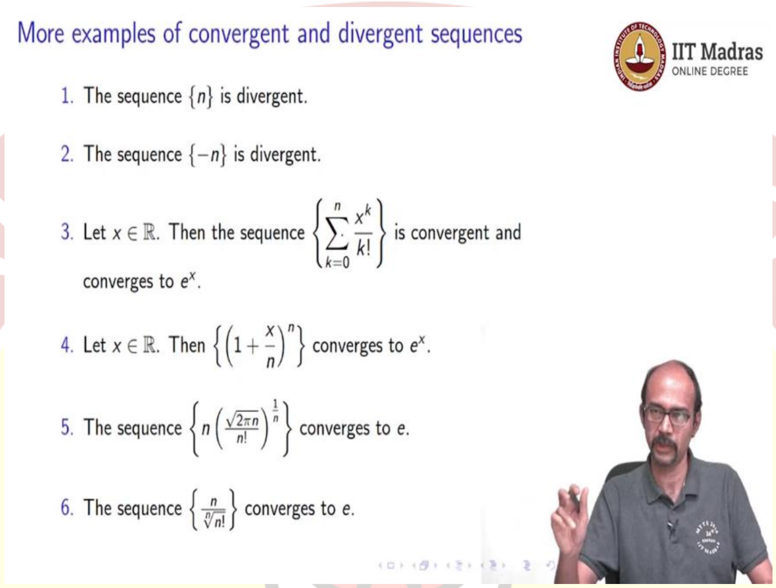

So, here is an example of a sequence which is divergent, but in a different way than the previous one. So, the sequence $\{n\}$ is divergent. What happens to this sequence $\{n\}$? So, that is the sequence 1, 2, 3, 4, 5, 6. So, this sequence does not have any limit. If you take any real number, let us say I take the real number 2 million, at some stage this sequence will go beyond this real number and it will keep going much, much, much beyond. It does not come close as n increases. So, there is no real number to which it comes close.

Now, some of you may of course have heard of this concept of infinity. In fact, I mentioned that concept previously also in the slide, where I define terminology. So, infinity we think of as something which is very, very, something larger than any number that we know, any real number that we know. So, you can say that this sequence converges to infinity.

So, in this case, we can actually say that this sequence converges to infinity. But as far as our notations are concerned, it is still a divergent sequence. So, you could say it converges to infinity, you could say it diverges to infinity or you could just say the sequence is divergent, which is correct.

So, the previous two things saying it converges to infinity or diverges to infinity say more than just saying it is divergent, but this sequence is divergent. So, that is what you have to keep in mind that it is divergent. Same thing for $\{-n\}$, the sequence $\{-n\}$, now what is the sequence, it is -1, -2, -3, -4, -5, -6, -7, and so on. So, as you go ahead, this sequence is going to become, the numbers in the sequence are going to become very, very, very small. And when I mean small, I do not mean positive and small. I mean, really small in the negative sense.

And if you take any real number, eventually, the sequence will cross that real number. So, if you take, let us say, -1003.89, then if you look at the 1000 terms, that is -1000, if you take the next term that is -1001, -1002, -1003, -1004 and you have gone beyond. You have gone smaller. And it keeps becoming smaller. So, it does not come close to this number and that is true of any number. So, it cannot converge to any real number. It cannot have limit any real number.

Now, again, if you have heard of this concept of minus infinity, which is sort of on the other side of the real axis, it is what is called what colloquially we say it is smaller than any number, any real number that we know. Then you can say that this sequence diverges to minus infinity or it converges to minus infinity, both are equally valid and depend on the context. Or you can just say the sequence $\{-n\}$ is divergent.

But I want to point out that these two sequences $\{n\}$ and $\{-n\}$ are different from the other, the previous example of the, of divergence that we saw, which was the sequence $(-1)^n$, which is -1, 1, -1, 1, -1, 1, -1, 1. So, that has no, it just jumps around. It has no concept of, it really does not converge in any sense. Whereas these sequences, they are really going off. But maybe they are going to something which is beyond our universe, whereas, which means to say that our universe is a real line and it is going beyond that.

So, there is a slight difference in these two types of divergence. But that is some pedantic point that we can ignore for this lecture, because those who are interested in mathematics can think more about this. Let us look at some more sequences. So, here is a first example of what is called a series which I have made into a sequence. So, you look at this sequence. What is this sequence? It is the sequence $\{\sum_k^n \frac{x^k}{k!}\}$. So, the nth term of the sequence is this sum.

So, for example, what is the second term? The second term is $\frac{x^0}{0!}+\frac{x^1}{1!}+\ \frac{x^2}{2!}$ that is a second term. So, if you have seen the exponential before, you can observe that if n tends to infinity, this is just $\sum_{k=0}^{\infty}\frac{x^k}{k!}$, which is exactly what we call e^x. This is how we define e^x. So, e^x is defined as whatever this number is.

Well, of course, you can ask, well, if I know what is e, then why not raise it to a power, certainly that is correct. But then what is e? And e is indeed defined in this way by taking x to be 1. So, if you take summation $\frac{1}{k!}$ that is exactly the number e. That is how you define the number e. And then you work out what the value is, which is what Euler did and so on. That is why it is one of the reasons it is named after him. So, keep in mind, this is e^x. That is what you need to remember.

Similarly, if you take the numbers, $\{\left(1+\frac{x}{n}\right)^n\}$, then this converges to e^x. So, the limit of this sequence is e^x. So, what is, what are the terms in the sequence? If you take x, if you take n to be 1, then this is $\left(1+\frac{x}{1}\right)^1$. If you take n to be 2, then this is $\left(1+\frac{x}{2}\right)^2$, and so on. And this sequence converges to e^x. Now, this is, you may, if you are seeing limits for the first time, I am sure this is appearing a little daunting. The point I am trying to make is do not try to figure out why this limit is e^x. So, you take these as a black box that this e^x.

And then similarly here is something that you may see in statistics which is why I have mentioned it. It comes about in some proofs in regarding the normal distribution and so on something called Stirling's formula. So, from there whatever that is you can tell that the sequence $\{n\left(\frac{\sqrt{2\pi n}}{n!}\right)^{\frac{1}{n}}\}$, very complicated looking sequence converges to e, something we know. Again, black box, keep this in mind.

And similarly, we have $\{\frac{n}{\sqrt[n]{n!}}\}$, again converges to e. So, you can see that these notions of sequence are very useful and important, because whatever this thing e is, the exponential which we have seen which is something which comes up in many, many, many places somehow it is very important in mathematics and statistics and science and technology at its very heart is this idea of sequences. So, you cannot really escape from sequences. So, every time you are doing an exponential, there is some sequence in the background. Only thing is we know how to deal with it. And that is why we do not keep repeating it all the time.

But I want you to understand that behind exponentials is this notion of sequences and series. And you do not need to understand it very well. What you need to understand is there are these sequences for which the limit is the exponential function or e and that is what you need to keep in mind.

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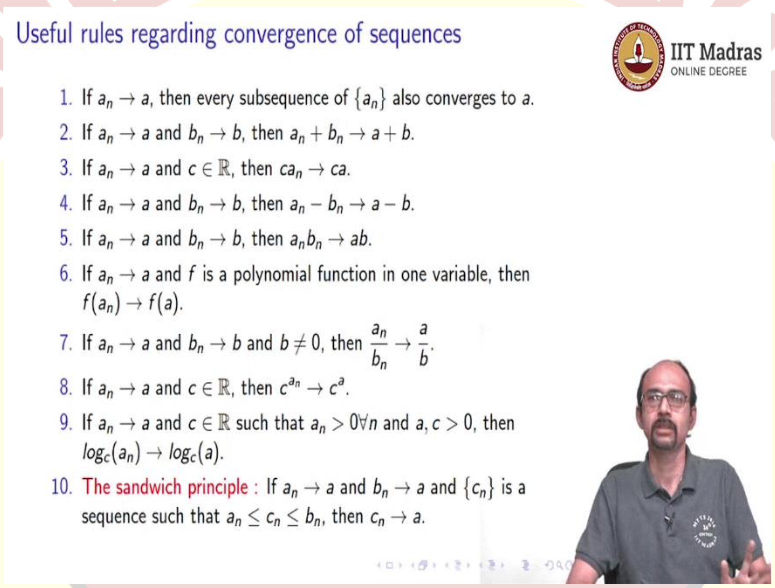

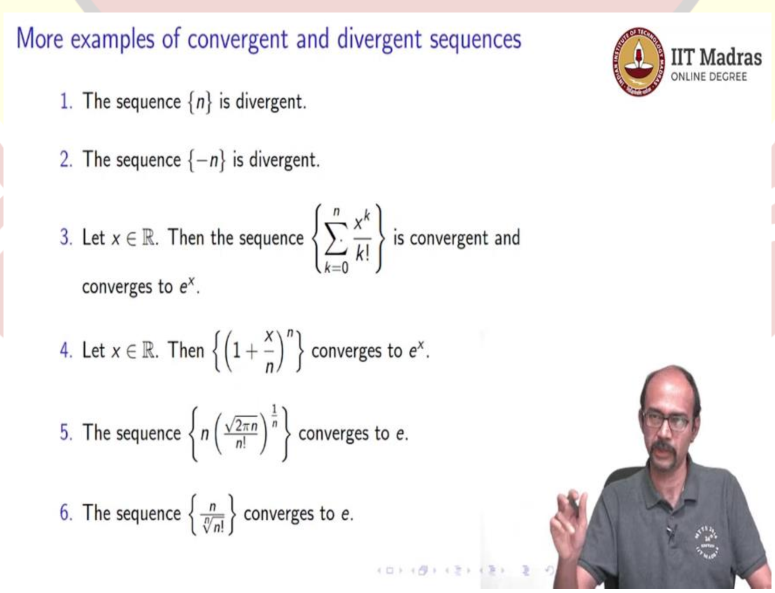

Let us write down some useful rules regarding convergence of sequences. This is what really helps to know when a sequence converges. So, often what will happen is you will have a sequence and you want to know whether it converges or not. And well you cannot keep evaluating. What we did in our first few slides was we took the sequence and we took large and we plotted it and we saw how did that red dot move, is it moving close to some things, is it not moving close to something, you cannot really do that.

Even in the third example, which was $2-\frac{1}{(1+\log n)^{1.1}}$, you saw it moved so slowly that you really do not know it moves to 2. How do I know it moves it converges to 2? So, we should have some theoretical way of seeing, and indeed, that is what we will do. We will have some sequences for which we know what the limits are. Those are exactly the sequences we have described before. And then we will use those sequences to study other sequences and these rules and try to determine when they converge or divergent.

So, here is one, some such, here are some such rules. So, if $a_n \rightarrow a$ then every subsequence of $\{a_n\}$ also converges to a. Seems very logical. If a sequence is going to a, if you throw out a few terms and take the subsequence given that will also converge to a. If $\{a_n\} \rightarrow a$ and $\{b_n\} \rightarrow b$ then the sum converges to the sum. So, $a_n + b_n \rightarrow a + b$.

Similarly, if you have $a_n \rightarrow a$ and you multiply each of these numbers a_n by c, then $ca_n \rightarrow ca$. $a_n \rightarrow a$ and $b_n \rightarrow b$, then $a_n - b_n \rightarrow a - b$. You can, in fact, derive this from the previous 2, 3 and 5. $a_n \rightarrow a$ and $b_n \rightarrow b$ then the product which means $a_n b_n \rightarrow ab$.

$a_n \rightarrow a$ and f is a polynomial function in one variable, very important, it is a polynomial function, then $f(a_n) \rightarrow \mathrm{f}(a)$. Again, you can derive this from your previous from two, three, four and five. If $a_n \rightarrow a$ and $b_n \rightarrow b$ and b is not 0, very important, then $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$. So, these are all rules. We have to prove these. And indeed, this is done in the first course on calculus, but we are not going to do it in this course.

If $a_n \rightarrow a$ and $c \in \mathbb{R}$, then $c^{a_n} \rightarrow c^a$. Again, something you can imagine being true. If $a_n \rightarrow a$ and $c \in \mathbb{R}$ such that a_n is positive for all n and you have a and $c > 0$, then $\log_c a_n \rightarrow \log_c a$. And finally, this is a very useful and important principle called the sandwich principle.

So, if $a_n \rightarrow a$ and $b_n \rightarrow a$, so you have two sequences tending to the same number, they have the same, these sequences have the same limits. So, a is the limit for a_n and a is also the limit for b_n. And $\{c_n\}$ is a sequence such that $a_n \leq c_n \leq b_n$. So, here is a_n and here is b_n and c_n is the middle.

Then as you can see, if a_n and b_n tend to something $c_n$will also tend to that thing. That is exactly why it is called the sandwich principle. So, c_n is sandwiched between a_n and b_n. And if you use these 10 rules and some known results, like the ones we described before, most of the sequences that you have to deal with are really can be dealt with using them, these rules.

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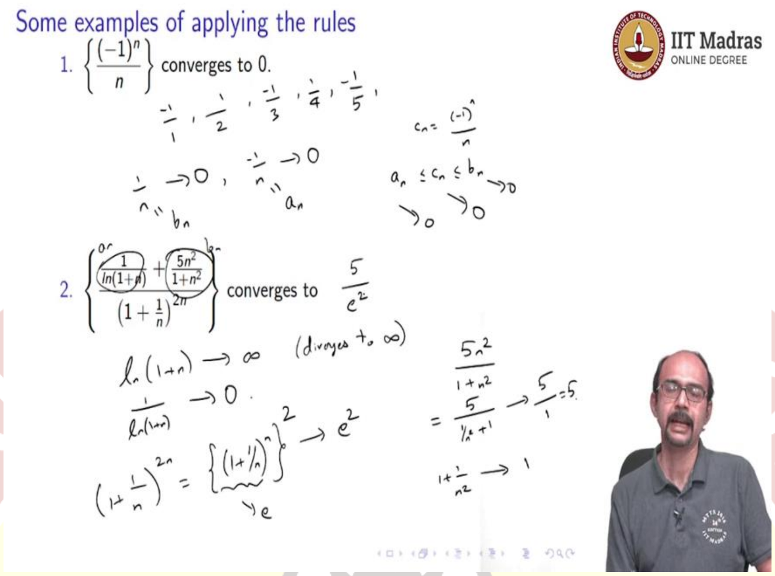

So, here is two examples that I am going to give you about applying these rules, $\frac{(-1)^n}{n}$ converges to 0 and the other is some complicated looking expression like this, $\{\frac{\frac{1}{\log 1+n}+\frac{5n^2}{1+n^2}}{(1+\frac{1}{n})^{2n}}\}$. What does it converged to. So, let us work out what these numbers are. Let us take the first one. So, this is an alternating thing again.

So, the first few terms here are -1/ 1, 1 / 2, -1 /3, so this minus, minus 1 is just plus. So, I just made a one half, then 1/4, -1/5, and so on. So, you can see as the numbers increase, as n increases, these numbers actually come, although they oscillate above 0, but they come closer and closer to 0. You can keep going.

How do I prove this? Well, you can apply the sandwich principle. First, let us look at $\{\frac{1}{n}\}$. We know that $\frac{1}{n} \to 0$ that was something that we described before. Not given a proof, because I have never defined what is a limit, but this is something we know and we can intuitively feel.

Well, now if you multiply the entire thing by some constant c, then it will go to c times the limit. So, if I multiply this by -1, this goes to 0. and now let us look at the sequence. So, you can call this a_n, you can call this b_n, both of them tend to 0. So, you call the sequence that you have as c_n and you can see that, maybe I should call this a_n, and I should call this b_n. And you can see that c_n is between a_n and b_n. And now if you apply the sandwich principle, this goes to 0, this goes to 0, so this also goes to 0. That is how we prove this. I am giving, and there are many ways of doing this, but I am giving you one way.

How about the second one? So, for the second one, just slightly harder and I, because it is not directly explained from the rules that I gave you, but I want to do it because I want to sort of explain what is going on. So, let us look at $\log(1+n)$. Well, n tends to infinity, what happens to $\log(1+n)$. It moves very slowly, but it goes to infinity. This, so this sequence diverges to infinity. This tends to infinity, so diverges to infinity.

Well, here is a fact. So, if that happens, 1 by this goes to 0. So, this also, this part, so I know something about the first term in the numerator. What about $\frac{5n^2}{1+n^2}$? This is interesting. So, here if you can, you can look at numerator and denominator both separately. $\frac{1}{1+(\frac{1}{n})^2} \rightarrow 1$. So, that means here is a sequence, so I have $\frac{5}{\frac{1}{n^2}+1}$. So, this tends to 5 /1 which is 5. And now I have, so the numerator I have kind of, I have understood what happens.

So, if I call this a_n, if I call this b_n, so I know what happens to a_n, $a_n \rightarrow 0$. I know what happens to b_n, $a_n \rightarrow 5$. So, $a_n + b_n \rightarrow 0 + 5$, which is 5. So, the numerator goes to 5. And let us look at the denominator. So, the denominator is $\frac{1}{(1+\frac{1}{n})^{2n}}$, looks very complicated. But I can rewrite this as $\{\frac{1}{\left(1+\frac{1}{n}\right)^n}\}^2$ .And now I know what term inside is. This term inside exactly goes to the number e. This was one of the things that we saw. So, this goes to e^2. So, in other words, this converges to $\frac{5}{e^2}$.

Again, I am applying that if you take two sequences $\frac{a_n}{b_n}$, a_n tends to something, $b_n$tends to something which is non-zero, then $\frac{a_n}{b_n}$ tends to the ratio of the corresponding limits. So, it goes to $\frac{5}{e^2}$. So, this is how you are going to apply these rules and known examples in actual limits. So, I hope this video gives you an idea of how, what is the limit and how to compute limits using these rules. So, thank you.