IIT Madras

ONLINE DEGREE

**Mathematics for Data Science 2**

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**Lecture 03**

**Graphs and Tangents**

Hello, and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to talk about Graphs and Tangents. So, we have briefly seen these ideas in the two videos back when we did a recall of the things that we have seen earlier in Maths 1, where you have probably seen a little bit of what is a tangent and also what is the graph of a function. But we will recall it in more details and we will add some new things in this video. Specifically, we will talk about things called curves.

So, we have seen in the previous video some particular functions. So, we saw the trigonometric functions and the exponential and polynomial functions and so on, some special families of functions and we are going to talk a little bit more about graphs to begin with.

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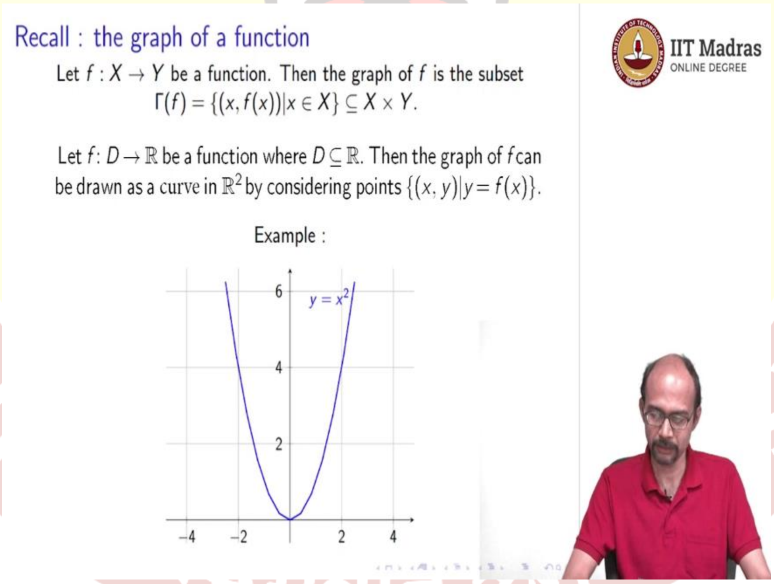


So, let us recall first, what is the graph of a function. So, if $f: X \to Y$ is a function, then the graph of *f* is the subset gamma *f*, which is all pairs *x*, *f(x)*, which belong to *X* cross *Y*, where the little *x* comes from *X*. This is the formal definition of the graph of a function. So, the graph of a function is a set. So, in principle, the graph does not have anything to do with geometry, but or calculus, but typically what we do is the sets that we have in mind are the *X* and the *Y* are in some *Rn*, so quite often in *R*. And hence, *f* is a function of one variable.

And in that case, we can give gamma of $f$ a geometric meaning by plotting it in $R_2$, which we will soon do. So, if $f: D \rightarrow R$ is a function where $D$ is a domain inside $R$. So, you think of $D$ for now as a set. But it is a slightly special kind of set, so that we can talk about some things that we want to. These what things will come later. So, it is a prior, it is just any set, any subset of $R$. Very often it will be the entire real line. But for some functions, you may want to restrict your $D$ to be the domain of the function. So, $f$ need not be defined on the entire $R$ and we have seen examples like this before.

So, if $f$ is a function from $D$ to $R$, where $D$ is a subset of the real numbers, then the graph of $f$ can be drawn as a curve in $R_2$ by considering points, $x$, or the set $x$, $y$ such that $y = f(x)$. And here is an example. Here is $y = x^2$ and this is how you draw that graph by plotting $x, x^2$. So, if you do that, you will get the parabola that is drawn here.

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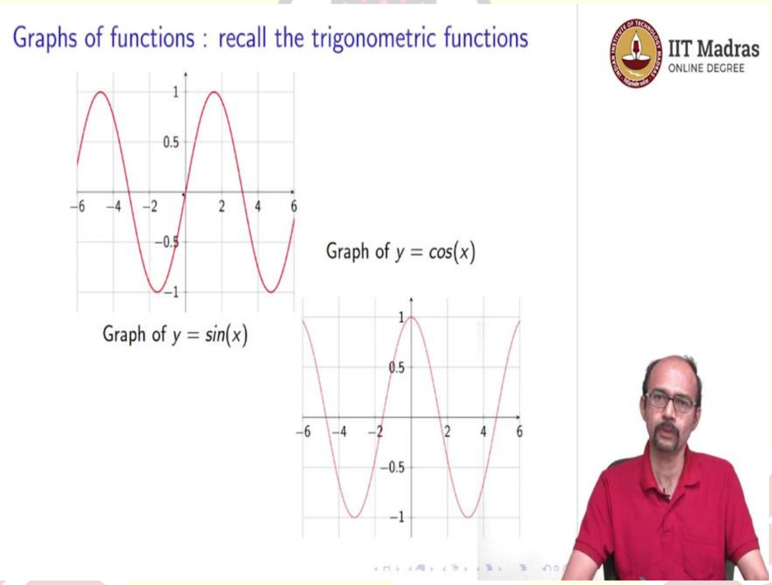


So, here is a couple of other functions, which we have seen before, some more graphs. So, here is the graph of $y = \sin x$. I think we saw this in the last video as well. So, this is, the red line is the or the red curve is the set $x$, $\sin x$ in $R_2$. So, for every $x$ in $R$, you plot $x$, $\sin x$. So, this is how it is going to look like.

And then we have graph of $y = \cos x$ and so both of these are, what I called, periodic or oscillating functions. And the reason is clear. You can see it right in the, from the graph, why they are called periodic or oscillating. So, if our graph of $y = \cos x$, we plot the red curve is the set $x$, $\cos x$, where $x$ is running over all of $R$. So, $x$ is a real number and we plot this over all the numbers.

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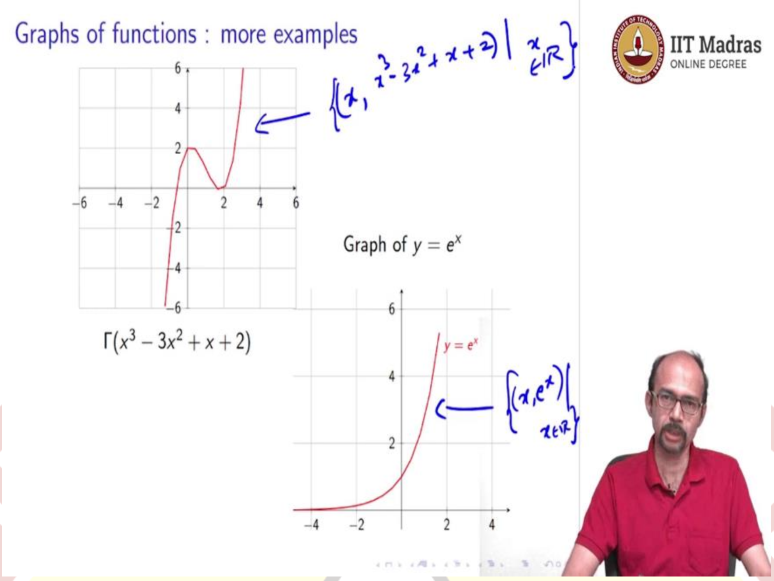


Here is a couple of other examples. Here is a slightly more complicated graph to plot. So, this is $x^3 - 3x^2 + x + 2$. So, you can see that this graph as *x* is between, let us say, [-1, 3], it is kind of bounded. It is not too, not increasing too fast. But beyond -1 or beyond 3, it rises very, very steeply. So, that is where as *x* increases, the $x^3$ sort of controls how fast the function increases. This is something that, this idea we saw also in the previous video when we talked about the rate of increase.

And here is the graph of $y = e^x$. Again, we saw this in the previous video. I am just plotting it again. So, here we have plotted the points *x*, $e^x$. So, this red line, red curve here corresponds to set *x*, $e^x$ as *x* belongs to *R*. And this red line here, this red curve here is the set of values *x*, $x^3 - 3x^2 + x + 2$, as *x* varies in *R*. So, this is what these graphs are. They are just, you are just plotting the *x*, *f(x)*, where *f* is a function.

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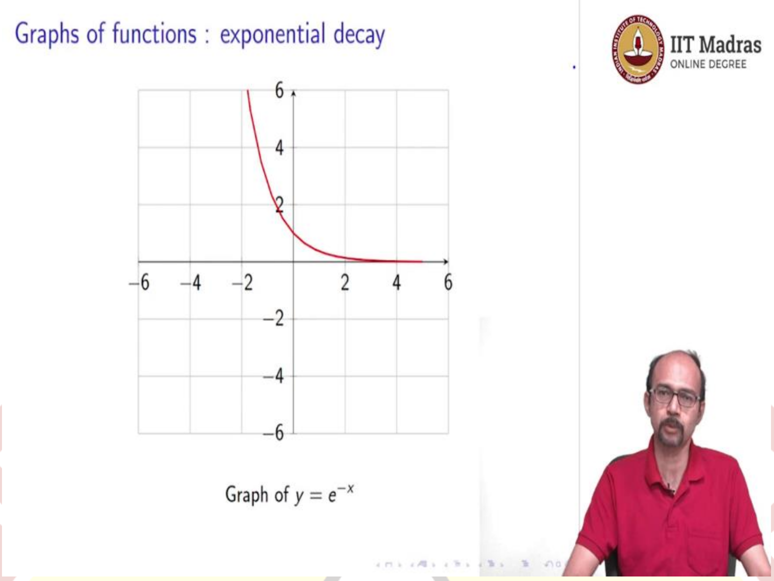


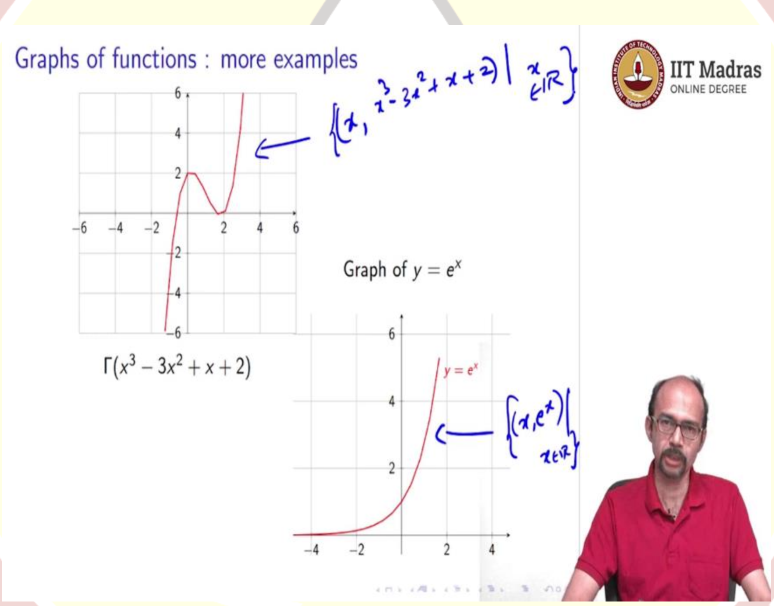


So, here is another example. This is, so in the previous slide, we saw $y = e^x$, this is $y = e^{-x}$. So, this is what is called exponential decay. You may have come across this term earlier. And you will use this particular function in, possibly in your statistics course.

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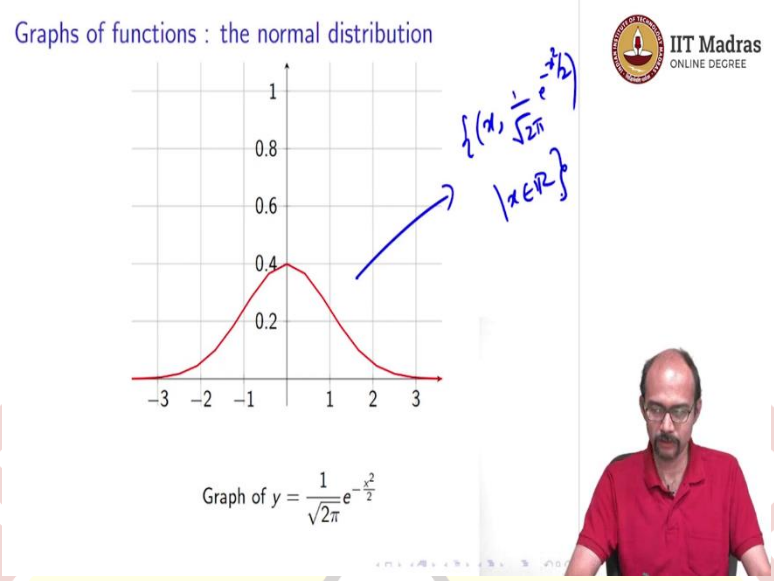


And this is another interesting function. So, this is the graph of $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. So, this is often called the bell curve, because it looks like a bell. And this corresponds to something called the normal distribution in statistics, which you will certainly come across in your other course. So, I hope it is clear what a graph of a function is. And here, for example, we have plotted the points *x*, this function here $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, *x* is in R.

So, the reason I started with plotting these functions is we want to have a very clear picture of what is the graph of a function. So, the graph is a curve and it looks like *x*, *f(x)*, where *f* is your function or it is equal to the set *x*, *f(x)*, where *f* is your function.

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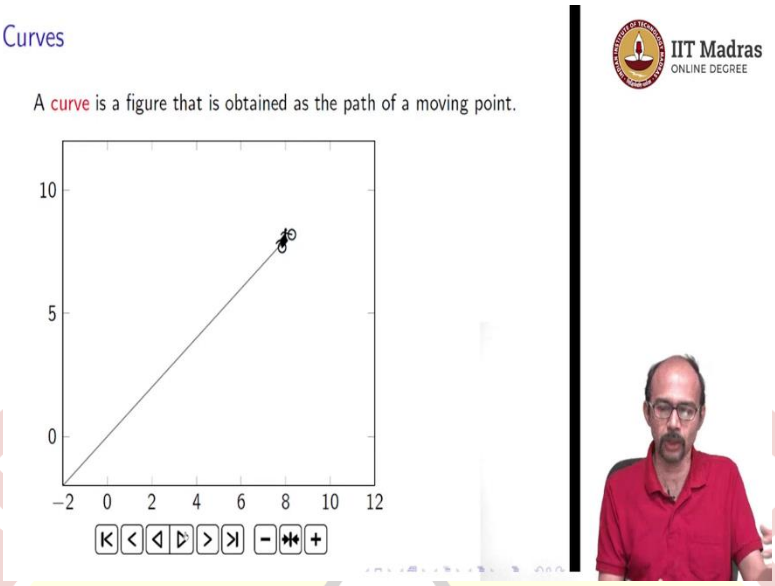


So, what is a curve? I kept saying this is a curve, this is a curve. So, what is a curve? So, I am not defining this explicitly. Again, the definition is a slightly, the mathematical definition is slightly more complicated than you think it is. So, I am going to give you a heuristic picture and probably most of you have a heuristic picture of what is a curve. So, a curve is a figure that is obtained as a part of a moving point.

So, if you have a point which is moving in either your, in your plane or your, in $R_3$ which is your space, so you think of an aeroplane flying or a bee flying or let us say the motion of a cricket ball when it is bowled by a bowler or when the ball is hit by the batsman or the movement of a figure skater so whatever movement you can think of so the path of that movement you can think of as a curve.

So, let us see an example. So, in this example, this is going to be a motorcycle moving. And here is the movement. So, this motorcycle moves along in a straight line. And in fact, it is moving with uniform speed. So, this is going to trace a line and probably is the line $y = x$, any case. It is a line. So, this is one possible curve. So, straight lines are also curves.

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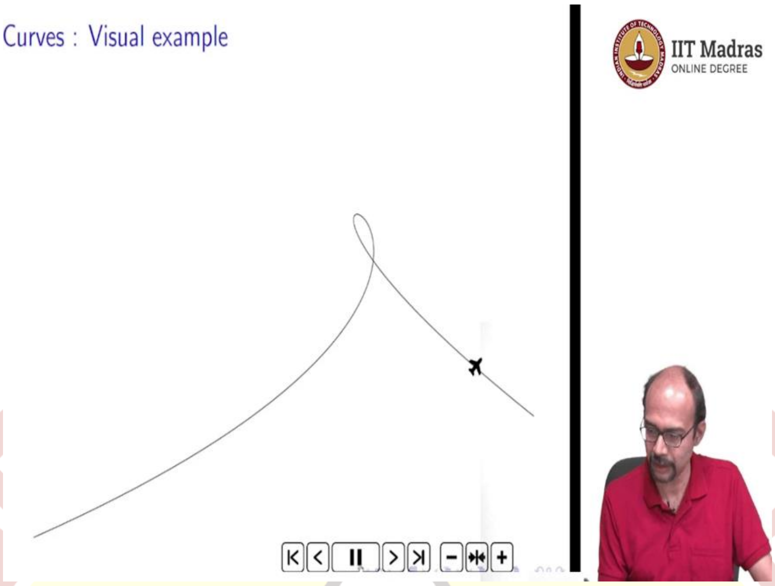


Let us look at another example. That is a more standard example, where you have a plane fly. So, here is your plane. Flying makes an arc and then it goes away. So, this is a possible curve. So, now this curve is drawn in a, I mean, when this plane flew, if it was a plane in one of those aeroplane shows that we have, then usually they go around in a loop which intersects itself. So, it goes around to the same position where it was. So, you can think of this an $R_2$. But if you have a more, I mean, the usual planes that you may have flown in, they, of course, do not curve around like this.

So, in that case, this, you can think of this as an $R_3$, where because of the, we are projecting it $R_2$ it may seem like it is intersecting itself, but it is, you can think of it as actually being separate. So, it goes like this and then comes out like this, so either way. It is a curve in $R_2$ or $R_3$, whichever you prefer. So, that is another example of a curve.

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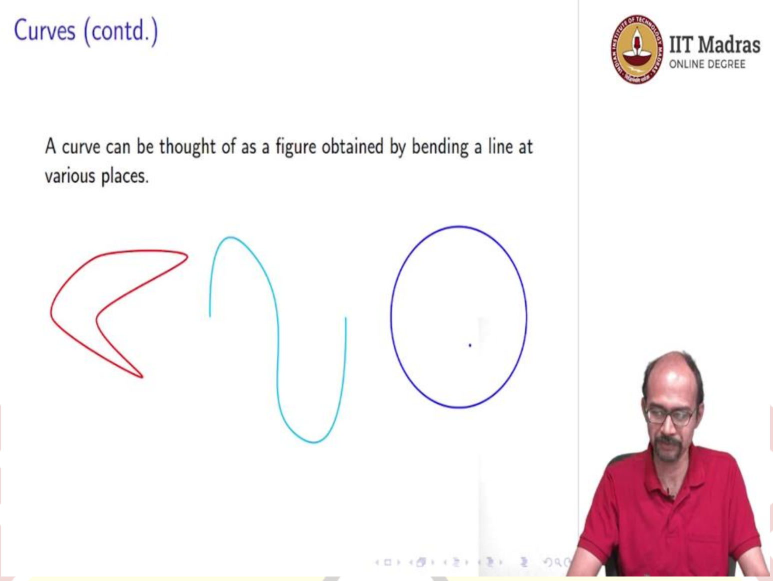


So, another way of thinking of a curve is that you can think of a curve as a figure obtained by bending a line at various places. So, you have a curve and then you have a line and then you sort of, you make it curved at various points. Sometimes the curves may be very, very, very steep. They may be like this or, and so on. And that is what you will get at the end is a curve. So, you do not, you allow, of course, that the line again meets itself. So, here is an example of such a thing.

So, here is a curve. So, this curve, it is what is called a closed curve, meaning it comes back end meets itself. So, if you want to make this from a line, the line is going to go around and come back and then keep going around that same thing. So, it will keep going. Of course, there is other ways of doing it in way that it does not have to keep going, but that is one way of thinking about.

Here is another one. This is not a closed curve. So, these, you have two ends, so you have one end on the left hand one end on the right. So, you can think of it as a piece of string. And here is a more familiar curve in the sense that this is a nice, it has some nice geometric property. It is a circle, so which we more often think of as a mathematical object. But all of these are curves. So, when we say curve, it need not be some very nice looking object like the circle. It could be some strange looking shape like we have over here or like the path of the, the trajectory of the aeroplane.

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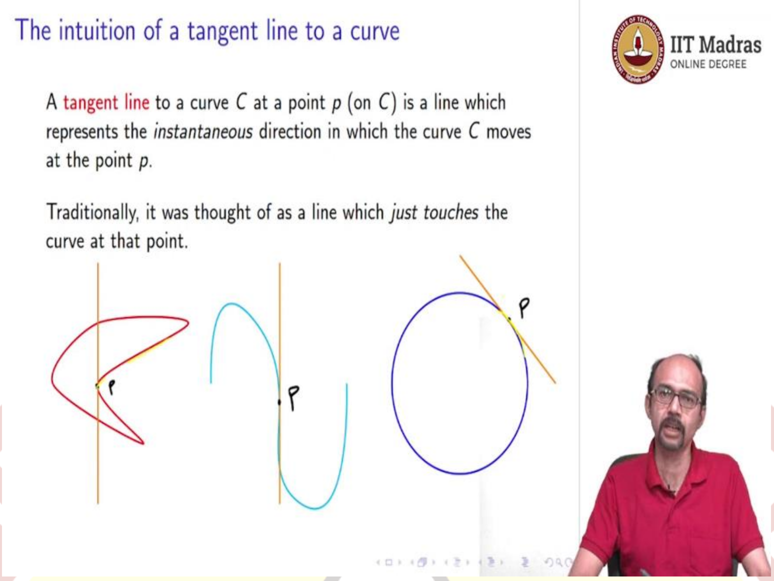


So, having understood what is a curve and that is what you should keep in mind that it is the motion of some particle. So, having understood that, let us get down to what is the tangent line to a curve. So, in the previous videos that we have seen, where we have drawn tangents, when we did a recall from the, from Maths 1, we drew tangents to the graphs of functions. Those were mainly quadratic functions. The tangent line can be talked about for any curve. And then of course we have to ask whether you can actually draw it or not, which we try to do.

So, what is the tangent line? So, the tangent line do a curve $C$ at a point $p$ which that point has to be on the curve. It is a line which represents the instantaneous direction in which the curve $C$ moves at the point $p$. So, this is some complicated looking definition. What it means is, if you think of your curve as the motion of a particle when you are, when the particle is at that point $p$. So, you think of the instant just before that and instant just after that.

So, in that time, what direction is it moving in? Is it moving southwest, southeast, northwest, northeast? And of course, these are only four directions or northeast, southwest, so eight directions. But you could think of it in terms of what angle. So, the direction is determined by the angle. So, what angle? So, can we precisely say that?

So, the tangent line, the idea of the tangent lane is that this line is the line at that point in the direction which, in which it is moving. That is the intuition of the tangent line. So, I will give you a more traditional way of thinking about it and this possibly how you may have seen it in school. Traditionally, it was thought of as a line which just touches the curve at that point. So, this is often the definition used in many textbooks at an elementary level. It is a line which just touches the curve at that point.

So, of course, it is not made clear what it means by just touches. But we will understand this by examples. So, here is the same figure, the same curve that we had in our previous example for curves. And here is the tangent line. The orange line is the tangent line at that internal point over there, which I will highlight in a minute.

Again, you have this orange line, which is a tangent line to a point in the middle. You can see that these are again intersecting the curve. So, it is not necessary that the line intersects the curve only once. So, often this is a mistake which is propagated and that is not, so I just want to warn you that that is not the case.

And here is the circle. So, for the circle, indeed, the line touches at only one point. It does not touches a circle anywhere else. So, where, what are the points $p$ here. So, the point $p$ here is, this a point $p$ here. Maybe I should use a different pen. Let us erase this. So, here is your point $p$. This is your point $p$. In this picture, here is your point $p$. And this picture, here is your point $p$. So, you can see in all these three cases that is the instantaneous direction in which the curve moves.

So, as you are traversing along this curve, as you move along that curve, so as you move along the curve, so you move like this, excuse me, you move like this. So, when you come close by, you are going to be, when you come close to that point, it is as if you are moving in a straight line and which straight line is that, it is this orange straight line. Similarly, here it is more clear, because the curve is less curved. So, the lesser the curvature, which we would not talk about at that point, but that means it is less curved, the closer the straight line is going to be to your curve.

So, here you can see it is very, very close to approximating your curve. So, over here, this is the instantaneous direction in which the curve is moving and similarly, for the circle, this is the instantaneous direction. So, if you come like this, you can see, now it is very, very close and again that is the case. So, it is the instantaneous direction in which the curve is moving at that point. And that the emphasis is on the word instantaneous, which is why it is highlighted.

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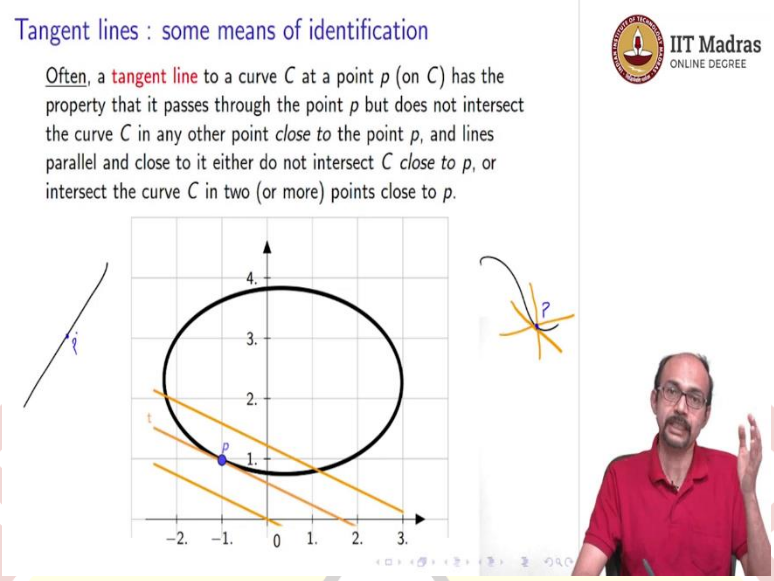


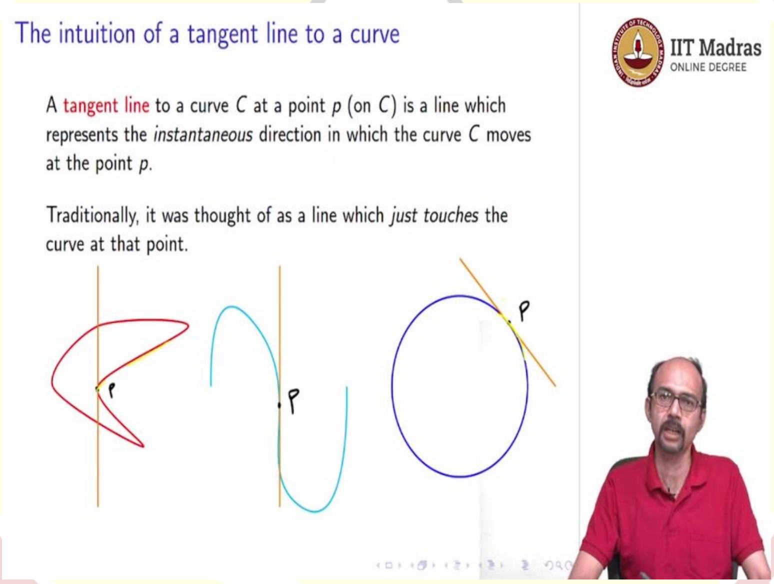


So, often, and this often has to be taken with a pinch of salt, but I making the statement anyway. So, often a tangent line to curve $C$ at a point $p$ has the property that it passes through the point $p$ but does not intersect the curve $C$ in any other point close to the point $p$, this is very important, close to the point $p$ and lines parallel and close to it either do not intersect $C$ close to $p$ or intersect the curve $C$ in two or more points close to $p$. So this is a lot of words.

What do I mean, here is my, the picture is probably a better explanation. So, here is your point $p$. This is a ellipse, an ellipse. And let us say we have chosen a point on that ellipse $p$. And this line $t$ is your tangent line. So, this line $t$ does not intersect the ellipse anywhere close to this point $p$. In fact, it does not intersect the ellipse at any other point. But if you move this line $t$ a little bit, if you move it a little bit parallely, then either it does not intersect this ellipse at all, which is what is happening to that line below or it intersects it in more than one point, so several points. And this kind of phenomenon is, you can also see in these pictures.

So, for the circle, if you draw a parallel line, it will either intersect in two points or it does not intersect at all. That is not a surprise. Circle is a special case of an ellipse. So, for this line here, if you move it a little bit, it will intersect either in, so a little bit is a very, very small amount, if you intersect, if you move too much, then it intersects at only one point. But if you move it a little very, very close by, then it does intersect in two points.

And similarly over here, if you move it until, it intersects in several points. Some, if you move it to the left, for the red curve, if you move to the left, the orange line to the left, it intersects in two points. If you move it to the right, it intersects in four points. So, I again want to emphasize this is not a definition or a general phenomenon. That is why I have written often. As we saw right here in the previous example, if you take the orange line and move it to the left for the light blue curve, it does intersect in only one point or to the right, considerably, it does intersect in only one point. But this is a, so that is why the often is given and it is underlying.

So, certainly, this is true for quadratics, so curves which come about as quadratics, where the $x$ term does not increase by, it is a polynomial of degree 2, which I think you have studied in Maths 1. So, this is a more, maybe I should say, a better heuristic. If you take this point $p$ and you have your line $t$ and it is a tangent line, so close to this point $p$ it does not intersect elsewhere and it determines the direction in which it moving.

And if you move it a little bit by fixing that point, so you move that. Maybe I draw something here. So, you move it a little. So, maybe something like this and you have a tangent line. I am drawing that with orange. So, let us go with orange, something like this. Now, if you fix this point $p$. So, here is my point $p$, excuse me, here is my point $p$. And you keep your line on this point and draw other lines, something like this.

So, then you can sort of control the angle made based on the curve. So, this is a little bit strange. I mean, what I am trying to say is, if you choose a certain amount that you want to wiggle it only this much. You want to rotate only by this much. Then you can choose the part of the curve for which it is going to intersect. This is some slightly vague statement I am making, but actually it is more precise than the previous things. But if you do not understand that, leave it for what it is.

And I also want to point out, one more thing that I want to point out, which is why this often is very is in quotes, this slide is not true to have a line. If you take the line, because remember the line is a curve. So, if you take this curve, what is a tangent line, it is the same line itself. But now if you move it a little bit, it would not intersect at all, so, on either side.

So, and as far as the curve, I mean, as far as the point is concerned, it intersects the point, intersects the curve everywhere. So, if your, this is your point p. Your line is going to intersect the entire, your tangent line is going to intersect the entire line. So, this idea that the tangent line intersects the curve in only that point is not correct. That is what I am trying to point out.

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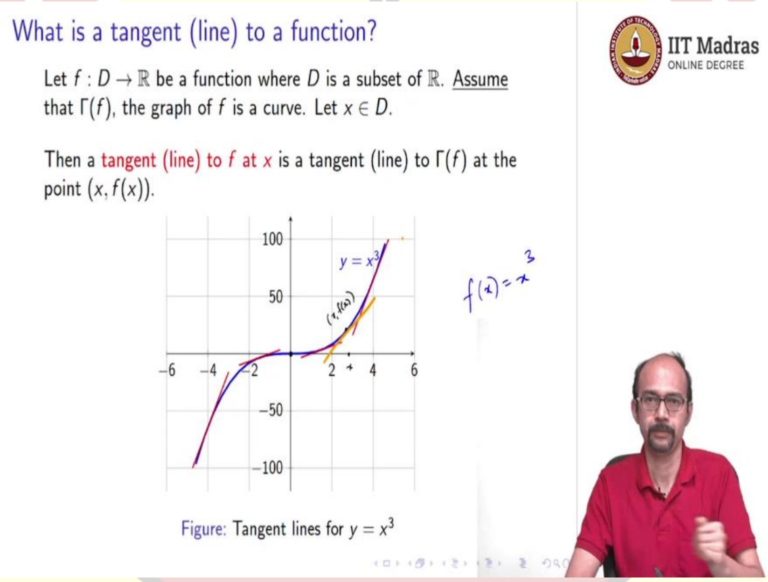


So, with that, let us move on. So, what is the tangent line to a function, which is what we are really interested in. So, we have seen that what is the tangent line to a curve? So it is a line which tells you the direction in which you are instantaneously traveling that is the way we are going to think about it. Of course, we have not yet talked about how to compute that direction or sort of algebraically describe that line, which we will do in subsequent videos.

So, what is the tangent line to a function? So, the tangent line to a function, first of all, let us start with the function *f*. So, assume that gamma *f* is a curve. So, I want to point out that, it need not always be a curve. So, we have seen examples where it is actually not a curve in the sense that we have talked about. So, if gamma *f* is a curve. So, let *x* be an element in your domain or a point in the domain, some real number on which *f(x)* is defined, then the tangent line to *f* at *x* is a tangent line to the graph of *f* at the point *x*, *f(x)*. This is a definition.

So, what are we saying? Here is an example. So, you have your function *f*, which is $y = x^3$. So, here your function is $f(x) = x^3$. So, $f(x) = x^3$. And now you draw the graph of your function. So, this is the graph here. And if you want to talk about the tangent line to *f* at say the point 0, so at the point 0, then you have to look at this point 0 and then look at the corresponding point on the graph. So, if you, so in this case, it is just the point (0, 0).

But if you want to do that for let us say this point here, then I have to look for this point which is the point corresponding to this. So, this is the point, if this is *x*, this is a point *x*, *f(x)*, and then asked for this curve, this curve meaning the graph, what does the tangent line? So, here the tangent line is going to be given by something like this. So, I have drawn these various tangent lines. So, you have this red line and this red line, red line, red line can run this orange line for the arbitrary point *x* that we chose. And you can see that the tangent lines are moving in a very, very nice way. So, they also keep changing.

So, you can see this picture is really reflective of the role of the tangent line. So, as I said, I will repeat what was the tangent line. So, the tangent line was supposed to tell you the direction in which the curve was moving. So, if you put all these tangent lines together, you should be able to sort of describe your curve and that is exactly what you see. So, if you draw enough tangent lines, then you can see your curve. That is what you can see in this picture.

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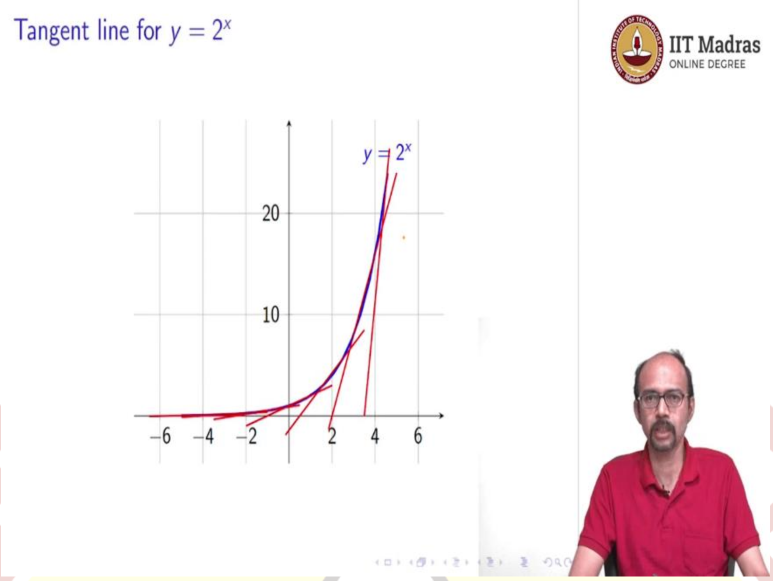


Let us do one more example. So, here is $y = 2^x$. And you can see the tangent lines are really describing this graph very, very, very well. Although, some of them may not have been as well drawn as they should have been, but nevertheless. So, they describe the graph of this function extremely well. So, they describe this curve very well. So, the, if you put together all your tangent lines and you can sort of make out the trajectory of your particle, which is your curve.

So, we have seen examples of tangent lines. We have seen, what is a curve, what a tangent lines to curves, what is the graph of a function, often it is going to be a curve and then we have seen that when it is a curve, we can talk about tangent lines to the function at a point. So, if I want to talk about the tangent line to, let us say, $f(x) = \sin x$ at the point 0, that means we are looking for the curve gamma of $\sin x$, which means the set $x$, $\sin x$. So, you plot that set. You look at the point 0, sin(0), which is 0, so 0, 0 on that set and that is a curve. So, you ask what is the tangent line to the curve at that point.

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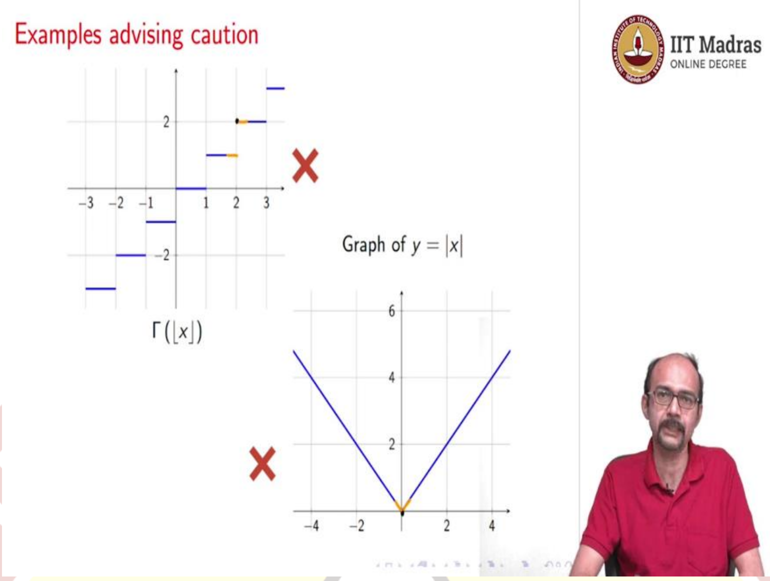


So, here is a couple of examples that we have to be cautious about. The first is the flow function. So, what is the flow function of *x*? For the flow function of *x*, we take the largest integer which is less than that number. So, for example, if you have the number 1000.23, you take the number 1000. That is the floor of 1000.23. If you have the number 6.829, you take 6. That is the floor of 6.829. If you have the number π, then floor of π is 3. If you have minus 2.896, the floor of minus 2.896 is minus 3. If you have -20, the floor of -20 is -20.

And if you draw the graph of this function, this is how it looks like. It looks like a step function. So, it is constant until you hit the next integer and then it jumps up by 1 and then it is constant again until you hit the next integer again it jumps up by 1. So, this s how the graph looks like. And now if I want to ask, so what does the tangent line to the graph of this function or to, of this function at the point 2, let us say. So, suppose I want to find it at the point 2. So, what is the procedure?

You look at the corresponding value. So, the corresponding value is (2, 2). And then you say how is my particle moving instantaneously at that point? But now we are in big trouble, because there is a huge jump here. So, when you are approaching to from below, so it is like you are on this blue line, which is here. So, over here, it will be something like this. And over here, it will be something like this. So, there is no line which really captures what is happening at that particular point. So, this is not, we cannot talk about a tangent line for this kind of function.

And similarly, let me redraw that. So, the main point here is that this is no tangent line. And what happens to the graph of $y = |x|$. So, if you look at graph of $y = |x|$, so here we do not have this problem of flow function of $x$, where the problem is really, really serious, because it is what is called, I mean, there are jumps, so it is not what is called continuous. So, here that kind of thing does not happen. It is indeed continuous. You can draw it nicely with the pen.

But at the point 0, so if we want to look at the graph of the tangent line at the point 0 to the function $|x|$, let us see what happen. So, here is your point (0, 0) on the graph. So, let us recall the definition of the tangent line. So, the tangent line definition was that it should reflect the direction in which you are moving instantaneously. So, now think of when you are coming, the particle is coming close to this point, so as you are coming close to this point from the left, as you are coming close from the left, you can see it is this link. So, it is the same line as the tangent line.

But on the other hand you can also think about what happens after you have left that point. So, if you if you think about that, then it is this line. So, there are two lines to this function. And one of them kind of gives you the instantaneous direction for one side. The other gives you the instantaneous direction for the other side. So, they do not really match up in any way. So, here again, we have a problem. So, this there is no tangent line here.

So, these are examples which tell you that the idea of a tangent line is more tricky than when you have things which are very nice and smoothly moving along and so on. So, if you have jumps or sudden zigzags like that tangent lines may not exist. So, we have to be careful and we have to understand in a more conceptual way or preferably in a more algebraic way, meaning by writing equations and so on, about how to describe the tangent line.

Unfortunately, it, for this to be sort of properly understood and theorized, it took the invention of calculus and then a considerable amount of time more before we could clearly understand how to this notion. So, the next few videos are going to be devoted to developing the theory necessary for or describing the tangent line and then we will talk about the tangent line. So, it will, talk about limits and then something called continuity and then we will talk about the notion of a derivative. Thank you.