Week-1
Mathematics for Data Science - 2
Some topics of Maths 1, Functions of one variable, Graphs and tangents, Limits for sequence, Limits for functions of one variable
Activity Slides

Keywords:

- Let $f : D_1 \to \mathbb{R}$, and $g : D_2 \to \mathbb{R}$ be two functions, such that, $range(f) \subseteq D_2$, where $D_1, D_2 \subseteq \mathbb{R}$. The composition of two functions is defined as:

$$g \circ f : D_1 \to \mathbb{R}$$

$$(g \circ f)(x) = g(f(x))$$

- Some rules to find limits: Let $a_n \to a$ and $b_n \to b$, then

  i) $a_n + b_n \to a + b$
  ii) $ca_n \to ca$
  iii) $a_n - b_n \to a - b$
  iv) $a_n b_n \to ab$
  v) $f(a_n) \to f(a)$, where f is any polynomial function.
  vi) $\frac{a_n}{b_n} \to \frac{a}{b}$, if $b \neq 0$
  vii) $c^{a_n} \to c^a$, for any real number c, where c^{a_n} is real number for each $n \in \mathbb{N}$.
  viii) $log_c(a_n) \to log_c(a)$, if $a_n > 0$ for all $n \in \mathbb{N}$, and $a, c > 0$.

- (**Sandwich Theorem**) Let $a_n \to a$ and $b_n \to a$ also. If $a_n \leq c_n \leq b_n$, for all $n \in \mathbb{N}$, then $c_n \to a$.

1. If $f(x) = x^2 + 2$ and $g(x) = 5x$, then $f(g(x)) =$
   - ◯ Option 1: $5(x^2 + 2)$
   - ◯ Option 2: $25x^2$
   - ◯ Option 3: $25x^2 + 2$
   - ◯ Option 4: $5x + 2$

Solution:
What is composition $f(g((x))$ of two functions f and g?

Step 1:
$f(g((x)) = (5x)^2 + 2 = 25x^2 + 2$

Hence option 3 is true.

2. Let $f(x) = \frac{x}{x+a}$, where $x > 0$ and $a > 0$. If $f(f(x)) = \frac{x}{3x+4}$, then find the value of a.

[Hint: $f(f(x)) = f(\frac{x}{x+a}) = \frac{\frac{x}{x+a}}{\frac{x}{x+a}+a}$]

Solution:
Step 1: Given $f(x) = \frac{x}{x+a}$
Think $f(f(x)) = f(\frac{x}{x+a}) = \frac{\frac{x}{x+a}}{\frac{x}{x+a}+a}$

What will be the expression after rearranging the terms?

Step 2:
$f(f(x)) = f(\frac{x}{x+a}) = \frac{\frac{x}{x+a}}{\frac{x}{x+a}+a} = \frac{x}{a^2+ax+x} = \frac{x}{a^2+(a+1)x}$

What is use of this expression?

Step 3:

Compare $f(f(x)) = \frac{x}{a^2+(a+1)x}$ with the given $f(f(x)) = \frac{x}{3x+4}$.
we get $a^2 = 4$ and $a + 1 = 3$,
which gives $a = 2$.
Hence, Answer is 2.

3. Let $\{a_n\}$ be a sequence defined as $a_n = \frac{3}{2n} + \frac{2}{2n+1} + 1$. The limit of the sequence $\{a_n\}$ is
[Hint: It is known that $\frac{1}{n} \to 0$, as $n \to \infty$]
   - ◯ Option 1: $\frac{3}{2}$
   - ◯ Option 2: 1
   - ◯ Option 3: 0
   - ◯ Option 4: 2

Solution :
Why $\mathbf{\frac{1}{n} \to 0}$ as $\mathbf{n \to \infty}$?

Step 1:
Since, $\frac{1}{n} \to 0$ as $n \to \infty \implies \frac{1}{2n} \to 0$ and $\frac{1}{2n+1} \to 0$ as $n \to \infty$ **Why?**

Step 2:
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{3}{2n}) + \lim_{n\to\infty} (\frac{2}{2n+1}) + \lim_{n\to\infty} 1 = 0 + 0 + 1 = 1$

4. Consider the sequence $\{f_n\}$, defined as $f_n = \sqrt{2n+1} - \sqrt{2n}$. Which of the following option(s) is(are) true?

[**Hint:** Do rationalization and use $a^2 - b^2 = (a+b)(a-b)$]

- ◯ Option 1: $\{f_n\}$ is convergent.
- ◯ Option 2: Limit of $\{f_n\}$ is 0.
- ◯ Option 3: Limit of $\{f_n\}$ is 1.
- ◯ Option 4: Limit of $\{f_n\}$ is $+\infty$.

Solution :

Step 1:

$f_n = \sqrt{2n+1} - \sqrt{2n} = (\sqrt{2n+1} - \sqrt{2n}) \times \frac{\sqrt{2n+1}+\sqrt{2n}}{\sqrt{2n+1}+\sqrt{2n}} = \frac{2n+1-2n}{\sqrt{2n+1}+\sqrt{2n}} = \frac{1}{\sqrt{2n+1}+\sqrt{2n}}$

What is use of getting this expression?

Since, $\frac{1}{n} \to 0$ as $n \to \infty \implies \frac{1}{\sqrt{2n+1}+\sqrt{2n}} \to 0$ as $n \to \infty$. **Why?**

Step 2:

Since $\frac{1}{\sqrt{2n+1}+\sqrt{2n}} < \frac{1}{2\sqrt{2n}}$

And so $\lim\limits_{n\to\infty} \frac{1}{2\sqrt{2n}} = 0$

Hence option 1 and option 2 are true.

5. If $\{f_n\}$ and $\{g_n\}$ be two sequences such that $|f_n| \leq |g_n|$ for all $n \geq m$, where $m \in \mathbb{N}$ and $\lim_{n\to\infty} g_n = 0$, then find the limit of f_n.

   [**Hint:** Use sandwich theorem.]

   Solution:
   How we can use sandwich theorem?

   Step 1:
   $|f_n| \leq |g_n| \implies f_n \leq |g_n| \text{ and } -f_n \leq |g_n| \implies -|g_n| \leq f_n \leq |g_n|$

   Step 2:
   Given $\lim_{n\to\infty} g_n = 0 \implies \lim_{n\to\infty} |g_n| = 0$ and $\lim_{n\to\infty} (-|g_n|) = 0$ **Why?**

   Now use sandwich theorem to conclude, limit of f_n is 0.

6. Find the value $\lim_{x \to 2}(x^3 + 4x^2 - 6x - 7)$.

Solution :
Why the substitution of value $x = 2$ is enough to find the limit value of the function at $x = 2$?

Step 1:
$\lim_{x \to 2}(x^3 + 4x^2 - 6x - 7) = 5 = 2^3 + 4.2^2 - 6.2 - 7 = 5$

7. A function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{1+x+x^2}{5x^2+1}$, then $\lim\limits_{x\to\infty} \frac{1+x+x^2}{5x^2+1} =$

- ◯ Option 1: $\frac{2}{5}$
- ◯ Option 2: 1
- ◯ Option 3: 0
- ◯ Option 4: $\frac{1}{5}$

Solution:

Here, will direct substitution work?, If yes then why? If not then why?

Step 1:

$f(x) = \frac{1+x+x^2}{5x^2+1} = \frac{x^2(\frac{1}{x^2}+\frac{1}{x}+1)}{x^2(5+\frac{1}{x^2})} = \frac{(\frac{1}{x^2}+\frac{1}{x}+1)}{(5+\frac{1}{x^2})}$

Now substitution of value of x will work? If yes, then what we need to substitute the value of x?

Step:2

We know that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$, Hence $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$ **Why?**

$\lim\limits_{x\to\infty} \frac{(\frac{1}{x^2}+\frac{1}{x}+1)}{(5+\frac{1}{x^2})} = \frac{0+0+1}{5+0} = \frac{1}{5}$