# IIT Madras

ONLINE DEGREE

**Mathematics for Data Science - 2**
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**What is an orthonormal basis?**

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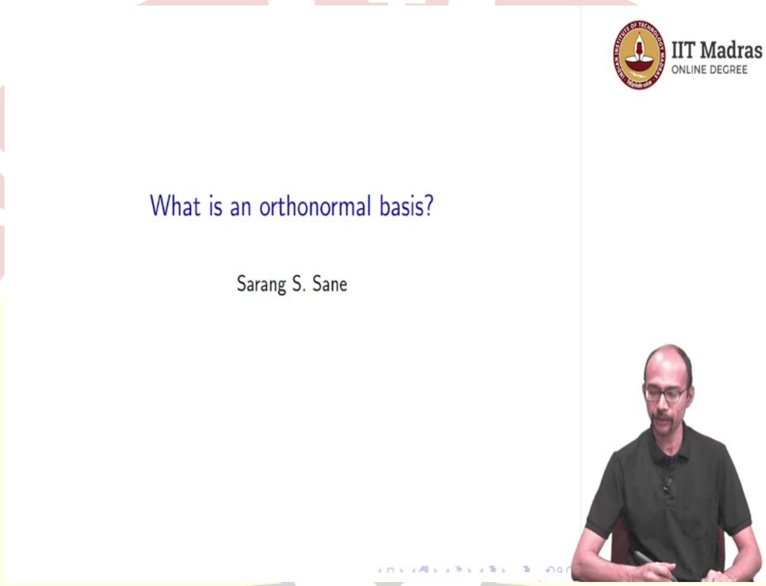


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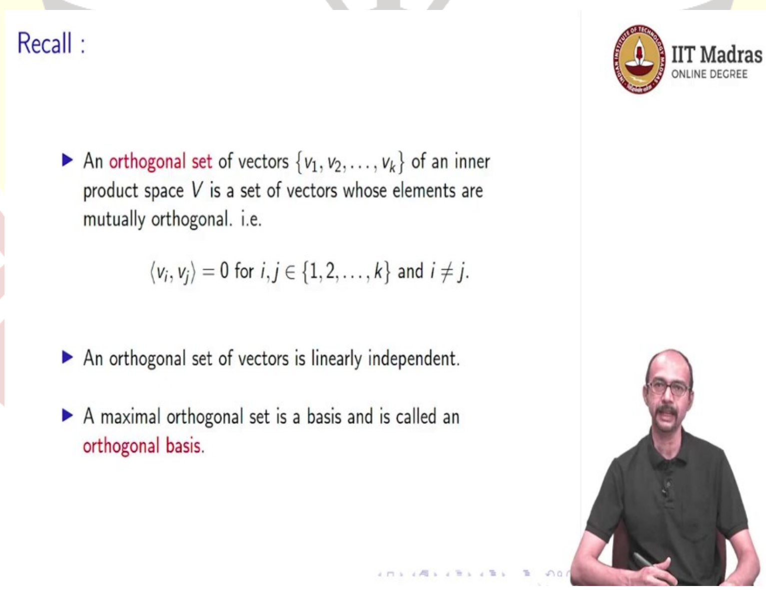


Hello, and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to talk about what is an orthonormal basis. So, let us just recall first that we have defined what is an orthogonal set. So, an orthogonal set of vectors

$v_1, v_2, \dots v_k$ in an inner product space is a set of vectors whose elements are mutually orthogonal. That means, if you take the inner product, $v_i, v_j$, where i $\neq$ j, then this is 0.

So, we checked in the previous video that an orthogonal set of vectors is linearly independent. And that is why if you take a maximal orthogonal set of vectors, then it is a maximal linearly independent set. And we have seen before that a maximal linearly independent set is a basis. So, this is one way of getting a basis for an inner product space. So, this is somewhat special. If you have an inner product on your vector space, this is a slightly enhanced way of getting a basis. So, such a basis was called an orthogonal basis.

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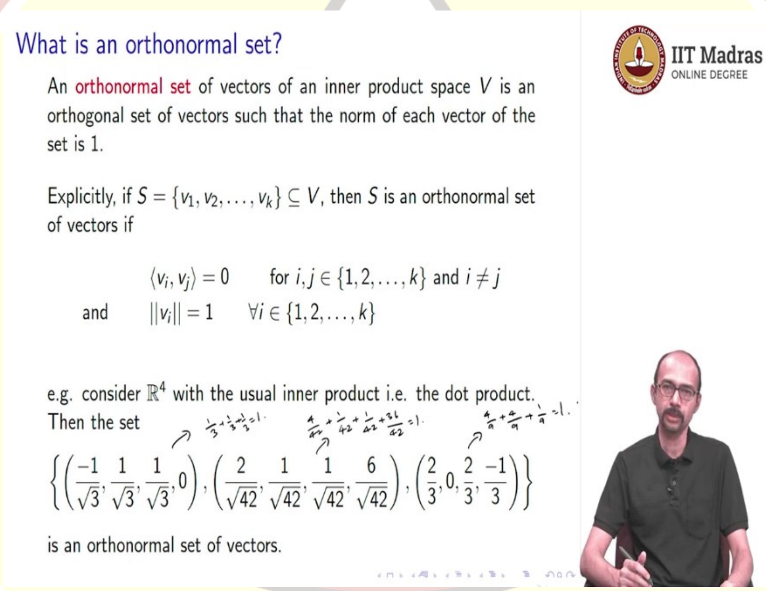


So, what is an orthonormal set? So, we are going to use these two terms, an orthonormal set and a basis and put them together and we will get an orthogonal, orthonormal basis. So, an orthonormal set of vectors in an inner product space is an orthogonal set of vectors. So, that means the inner product of $v_i, v_j$ is 0 for all i $\neq$ j, such that the norm of each vector of the set is 1.

So, let is recall that, if you have an inner product, it automatically gives you a norm that is if you have a vector v and you take the inner product of $v, v$, then that is a positive number, non-negative number and if you take its square root, that gives you the norm. So, that is defined as the norm of the vector.

So, explicitly, what this means is if you have a set v1, v2, vk then S is an orthonormal set of vectors if the inner product of $v_i, v_j$ is 0 for i and j in 1 through k and i ≠ j, and the norm of vi is 1. So, just to explicitly say that norm of $v_i$ is 1 is the same as saying that inner product of $v_i, v_i = 1$. So, here, of course, we do not take square root because a square root of 1 is 1. So, we know that if the norm is 1, this is the same as saying the inner product is 1.

So, as an example, let us consider R4 with the usual inner product, that is the dot product. So, then the set - 1/√3, 1/√3, 1/√3, 0, 2/√42, 1/√42, 1/√42, 6/√42 and 2/3, 0, 2/3s, - 1/3 is an orthonormal set of vectors. So, this example, we have sort of seen a similar example before.

And the idea is here we have made them all into a norm, we have made all of them to have norm 1. So, if you take the norm, so norm of the first vector is, so this is the usual inner product, so the norm is just given/taking each component, squaring it, and adding it up. So, that gives us 1/3 + 1/3 + 1/3, which is 1.

So, similarly here, if you take the norm, that is going to give you 4/42 + 1/42 + 1/42 + 36/42, which is indeed 1. And similarly, over here, the norm is going to be given/2/3 $^2$ + 2/3 $^2$ + - 1/3 $^2$, that is 4/9 + 4/9 + 1/9, which is 1. So, all these have norm 1. And you can check that the inner product is 0 if you take two different vectors.

So, for that, since all of them have the same denominator, you can ignore the denominator and take the inner product and you can see clearly that the norm is indeed 0, sorry, the inner product is indeed 0. So, I hope it is clear what is an orthonormal set. It is just an orthogonal set with the additional property that the norms of each of the vectors in that set is 1.

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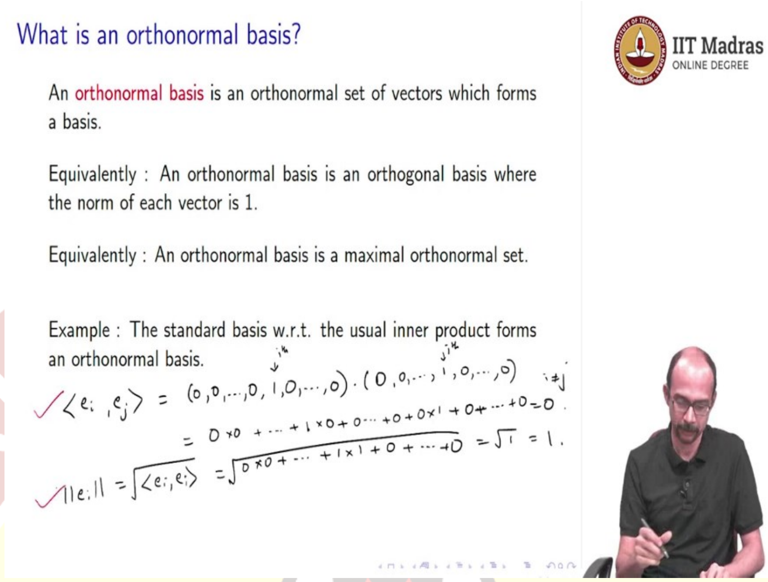


So, now, what is an orthonormal basis? So, we know what is an orthonormal set. So, now, an orthonormal set which forms a basis is an orthonormal basis. So, this is similar to what we saw for the orthogonal basis, namely, an orthogonal basis was one where it was an orthogonal set and it was a basis. So, here an orthonormal basis is one where it is a basis and it is an orthonormal set.

So, equivalently an orthonormal basis is an orthogonal basis where the norm of each vector is 1. So, because an orthogonal set is, orthonormal set is nothing but an orthogonal set where each vector has norm 1 and orthonormal basis in particular is an orthogonal basis where each vector has norm 1.

So, equivalently, an orthonormal basis is a maximal orthonormal set. So, just to make it clear what we mean/that, that means this set is an orthonormal set and there is no set which is bigger than this, which properly contains this set and which is also an orthonormal set. So, this is the largest possible orthonormal set you can get. You cannot expand this set further and retain the property of being orthonormal.

So, here is an example. The standard basis with respect to the usual inner product forms an orthonormal basis, maybe let us check that quickly before going ahead. So, I have the vectors ei. So, we already know that if you take ei,ej, then this is and $i \neq j$, then this inner product is going to be, so if you work this out, $0 \times 0 + 0 \times 0$ all the way, so, there is a lot of 0s.

And then when you come to the ith place, you will get a 0, sorry, a 1 time 0. And when you come to the jth place, you will get 0 × 1, this is if i is larger than, sorry, less than j then you get this. If i is larger than j then the 0 × 1 comes first and the 1 × 0 comes next, either way the point is you get this to be 0.

So, this inner product is 0 and the norm of $e_i$, so that is $e_i$,$e_i$, so, well, root of this, so then if you do the same computation, you get 0 × 0 + bunch of 0s. And then in the ith place there is one in each component, so 1 × 1 and then again 0s. So, this gives you, I should have √of this, so which is √1.

And of course, when we say √and we are talking about norms, we always take the positive square root, so, this is again 1. So, this is an example of an orthonormal basis. It is an orthogonal basis. Well, we already know it is a basis. It is an orthogonal set that is what we checked over here. So, here we checked it is orthogonal and here we checked it is orthonormal.

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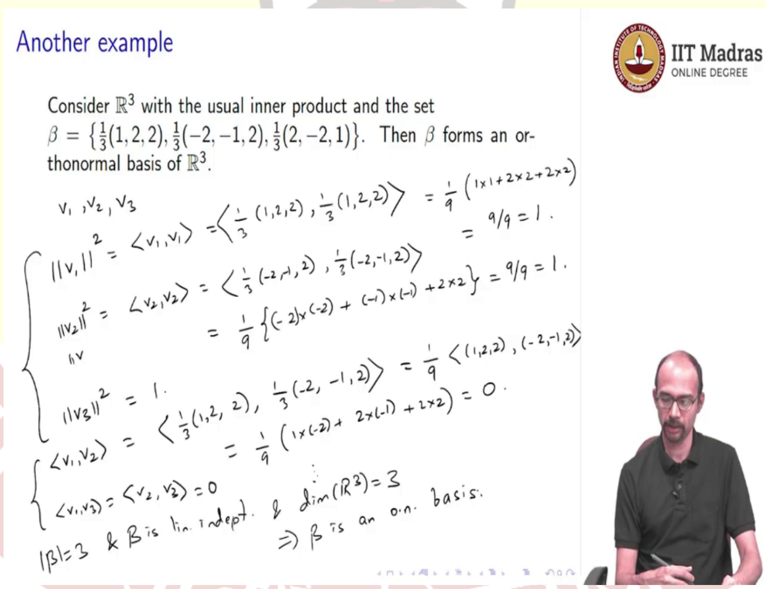


Let us do another example. So, consider R3 with the usual inner product and the set 1/3 1, 2, 2, 1/3 - 2, - 1, 2 and 1/3 2, - 2, 1, then this set which we have called β is an orthonormal basis. So, let us check this. So, let us look at the norms first. So, let us give these names. So, let us call this, let us call the first vector $v_1$. So, we have $v_1$, $v_2$ and $v_3$. So, the norm of $v_1$ is root of, so, typically, if you want to, if you want to check that something has norm 1, then it is enough to check that its square is 1.

So, instead of checking that norm v1 is 1, I will be checking norm of v1 squared is 1. So, norm of v1 squared is just the inner product of v1 itself, which we can compute easily. So, this is 1/3 1, 2, 2, 1/3 1, 2, 2, and that is 1/9 × 1 × 1 + 2 × 2 + 2 × 2 which is 1 + 4 + 4 so 9/9 which is 1.

The same, the same thing is going to happen if you take v2, norm of v2 which is 1/3 - 2, - 1, 2, 1/3 - 2, - 1, 2. So, this is giving us 1/9 × - 2 × - 2 + - 1 × - 1 + 2 × 2, which is 9/9, which is 1. Maybe I will leave norm v3 square to you. So, check that this is 1.

And then we are left with these three inner products. So, let us check v1, v2 what is the inner product. So, we have 1/3 1, 2, 2, and 1/3 - 2, - 1, 2. So, again, the tip here is, if you want to check that the inner product is 0, then these constants you can remove out and check for the term inside, which if it is better term, so in this case, for example, you have integers, then you would rather check that.

So, in this case, you get 1/9 × 1 × - 2 + 2 × - 1 + 2 × 2. So, that gives us - 2 - 2 + 4. So, that is 0. I will again encourage you to check the other terms. So, this shows that this is an orthonormal basis. Why is it a basis? We have checked here that this is an orthonormal set, but the reason it is a basis is because it is a maximal orthonormal set.

Meaning, this is a set of size 3 and you already know it is a linearly independent set, because if it is an orthonormal set, it is in particular an orthogonal set, which we have checked is linearly independent. So, this is a, so I should end this/saying once you finish this checking, this will so, this part will show and along with v3 will show that the norms are 1.

This part if you finish, so check also that $v_1$, $v_3$ is $v_2$, $v_3$ is 0, this will, altogether will show that it is orthogonal. And then the cardinality of $\beta$ is 3 and β is linearly independent, because it is orthogonal and we know that dimension of $\mathbb{R}^3$ is 3. So, that implies that β is an orthonormal basis, so because the sizes match. So, you have a linearly independent set of size which = the dimension of the vector space that is why it is a basis. So, in particular, it is an orthonormal basis in this case.

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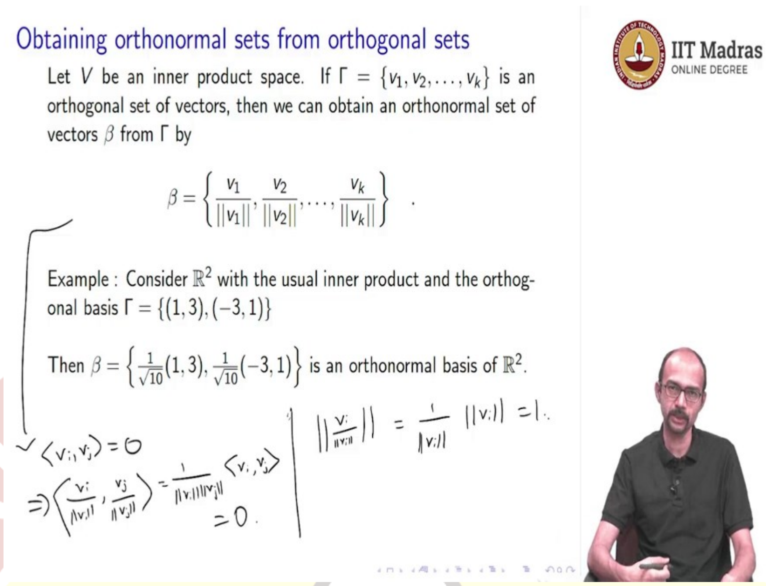


So, we have, I hope this gives you a window into how to check something is a basis. Some× you happen to not know an explicit basis, but you know the dimension of the space in consideration, so you can check that something is a basis. So, let us talk about obtaining orthonormal sets from orthogonal sets.

So, suppose you have inner product space, an inner product space and if γ is a set v1, v2, vk, which is orthogonal, so it is an orthogonal set of vectors. So, then we can obtain an orthonormal set of vectors which let us call it β from γ/the following procedure. You take each of these vectors and divide it/its norm then we are claiming that this is an orthonormal set.

So, why is it an orthonormal set? So, the reason is because this is exactly what we did in the previous example. Each of those vectors was divided/its norm which made it orthonormal and the orthogonality was evident already from just the integer part. So, here also that is what is happening. So, here we have that v1, v2 is, so let us say vivj is 0, so this will imply that vi/norm vi, vj/norm vj is 0.

And so, this is orthogonal. So, this new set is orthogonal β. So, the only thing that is remaining to check is what is a norm. So, if you take the norm of vi/norm vi, well, constants come out of the norm. So, you get 1/norm vi × norm vi, which is 1. So, that is why this is an orthonormal set.

So, let us consider R2 with the usual inner product and the orthogonal basis γ 1/3, - 3, 1. So, we can make this into an orthonormal basis/looking at the norms of each of these vectors and then dividing/the term. So, the norm of both of these vectors turns out to be√10. So, if you do 1/√10 × 1, 3 and then 1/√10, 1/√10 × - 3, 1 this is an orthonormal basis for R2.

So, the fact that it is an orthogonal set is already because γ was orthogonal. The fact that it is orthonormal set is because we divided/the norms. And the fact that it is a basis, well, γ was already a basis. So, from there it follows this is a basis or the fact that $\mathbb{R}^2$ is of dimension 2 and this is a orthonormal set of size 2 so it must be linearly independent, so linearly independent of size 2, which tells you it is a basis.

So, maybe let me just write that again here, since it got lost out. So, here I was just saying that $v_i, v_j$ is 0 implies $v_i$, norm $v_i, v_j$ /norm $v_j$. So, this is 1/norm $v_i$ × × norm $v_j$ × the inner product of $v_i, v_j$, which is 0. So, from here, we get that the existing set is orthogonal, sorry, the new set is orthogonal.

And norm of vi/norm vi we can take the norm out. So, this gives us 1. So, that is what, that is why this statement holds true. And the example tells you, if you do not understand what I have written here, tells you why these things work. So, work out the example for yourself.

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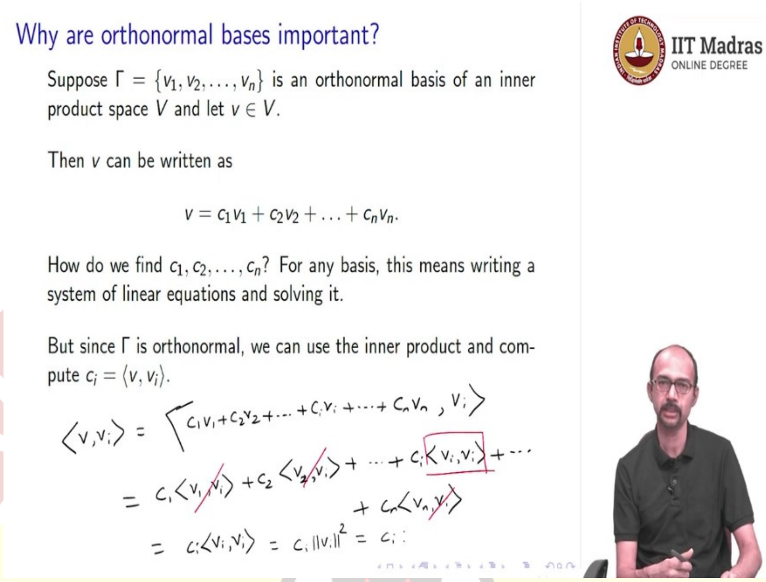


So, why are orthonormal bases important? This is a sort of the punch line of what we are doing. So, suppose γ is $v_1, v_2, \ldots v_n$ and this is an orthonormal basis of an inner product space V and suppose you have a vector v inside capital V. Well, we already know because it is a basis that you can write little v as $c_1v_1$+$c_1v_1 + c_3v_3 + \cdots + c_nv_n$. Why is that, because remember that a basis is in particular a spanning set, a basis is a spanning set, which means every vector is a linear combination of the basis vectors.

And in fact, for a basis, it is a unique linear combination. So, there are a unique $c_1, c_2, \ldots c_n$such that little v $c_1v_1$+$c_1v_1 + c_3v_3 + \cdots + c_nv_n$. So, this is a general statement for any basis. Now, what is the importance of it, being an inner product space and with an orthonormal basis? So, how do we find $c_1, c_2, \ldots c_n$, for any basis this means writing a system of linear equations and solving.

So, that is how I typically solve for $c_1, c_2, \ldots c_n$. You write your v and then you write your $v_1, v_2, \ldots v_n$ and then you solve these equations. But since $\gamma$ is orthonormal, we can use the inner product and compute ci is v, inner product of v, $v_i$. So, how do I do that? Let me quickly show you why that is true.

So, let us compute what is inner product of v, $v_i$. So, if you compute what is inner product of v,$_{vi}$, I substitute $c_1v_1 + c_2v_2 \ldots . + c_iv_i + \cdots . c_nv_n$ and then inner product with respect with vi, well, we know that the inner product is a linear in each variable, which means that I can write this as c1 ×

inner product of $v_1 v_i$+ c2 × inner product of $v_1 v_2 v_i$all the way up to, then we have + ci × inner product of $v_1 v_i + c_n$+ cn inner product of $v_n v_i$. This is what we get.

But now, we know that this is an orthonormal basis. So, because it is an orthonormal basis, first of all, it is orthogonal. So, other than the $v_i v_i$term, every other term is going to be 0. So, this term will remain and all these terms are 0. So, this is, I can just write this as ci × $v_1 v_i$. So, this, we have used here that it is orthogonal, but now we also know it is orthonormal, because it is orthonormal this is, we can rewrite this as ci × norm vi squared, and norm $v_i$ we know is 1.

So, this is just $c_i$. This is where we are using the fact that we have an orthonormal basis. So, what did we get? We got that the inner product of v and $v_i$ = $c_i$. So, this is a very easy way of saying what are the coefficients which come into this linear combination, which gives you v. How do we find $c_1, c_2, \ldots c_n$, the answer is each $c_i$ is take the inner product of v with $v_i$ and that is your $c_i$.

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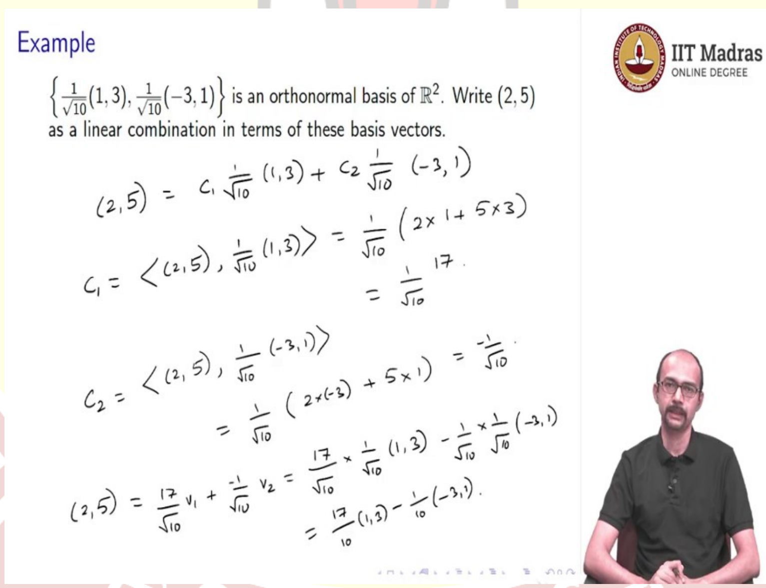


Let us do an example. We saw this orthonormal basis for $\mathbb{R}^2$ earlier. So, write 2,5 as a linear combination in terms of these basis vectors. So, if you take 2,5 let us write this as $c_1$ × 1/√10 1,3 + $c_2$ × 1/√10 - 3,1. Well, we just saw, what is $c_1$. $c_1$ is the inner product of v,$v_1$. So, here $v_1$ is 1/√10 × 1,3, which is, you can pull out the constant, so 1/√10 and then the remaining things, take the inner product, so this is 2 × 1 + 5 × 3.

So, what does that give us, 1/√10 × 2 + 15, so that is 17. And then we get c2, which is v,v2, so 1/√10 × - 3,1 which gives us, again, the 1/√10 comes out, and we get 2 × - 3 + 5 × 1, which is - 6 + 5, so - 1/√10. So, this is what we obtained as the coefficients.

So, now, if we write 2,5, in terms of these vectors we get 2,5 is 17/√10 × v1 + - 1/√10 × v2, so which if you write out completely 17 ×√10 × 1/√10 × 1/3, sorry, 1,3 + or rather - 1/√10 ×, 1/√10× - 3,1. And you can check actually that this is indeed going to work.

If you want, we can do that quickly. This is 17/10 1,3 - 1/10 - 3,1. And if you work this out, the first entry is 17/10 – (- 3/10). So, 17/10 + 3/10 so 20/10, which is giving you 2. And then the second entry is 51/10 - 1/10, which is giving you 5. So, this shows that, indeed, what the equation we wrote down is correct. This would have been maybe slightly harder if we did not have this knowledge that this is an orthonormal basis.

So, I hope in this video you have figured out what, I mean, you have understood the main point. The main point is that we define something called an orthonormal basis, that is an orthogonal basis in which each vector has norm 1. And the crux of the video is that every vector which you can write as a linear combination of the orthonormal basis, the coefficients of, in that linear combination $c_1, c_2, \dots c_n$are essentially are equal to the inner product of the vector v with the corresponding basis vector $v_i$. That is the main point. So, I guess that finishes this video. Thank you.

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