Q1 (a) Let X be a r.v. such that P(X = -1) = 1/4, P(X = 0) = 1/2, P(X = 1) = 1/4. What is the MGF of X?

$$E[e^{tX}] = P(X=-1)\cdot e^{t(-1)} + P(X=0)\, e^{t(0)} + P(X=1)\, e^{t(1)}$$

$$= \frac{1}{4} e^{-t} + \frac{1}{2} + \frac{1}{4} e^{t}$$

Q1 (b) Let X1 and X2 be iid X. What are the MGF and PMF of X1+X2?

$$\text{MGF of } X_1+X_2: \left(\frac{1}{2} + \frac{1}{4} e^{-t} + \frac{1}{4} e^{t}\right)^2 = \frac{1}{4} + \frac{1}{16} e^{-2t} + \frac{1}{16} e^{2t} + 2\cdot\frac{1}{2}\cdot\frac{1}{4} e^{-t} + 2\cdot\frac{1}{2}\cdot\frac{1}{4} e^{t} + 2\cdot\frac{1}{4} \cancel{e^{-t}}\cdot\frac{1}{4}\cancel{e^{t}}$$

$$= \frac{3}{8} + \frac{1}{4} e^{t} + \frac{1}{16} e^{2t} + \frac{1}{4} e^{-t} + \frac{1}{16} e^{-2t}$$

$$\text{PMF}: \left\{ \overset{1/16}{-2}, \overset{1/4}{-1}, \overset{3/8}{0}, \overset{1/4}{1}, \overset{1/16}{2} \right\}$$

Q1 (c) Let X1 and X2 be iid X. What are the MGF and PMF of X1+2X2?

$$\text{MGF} = \left(\frac{1}{2} + \frac{1}{4} e^{-t} + \frac{1}{4} e^{t}\right)\left(\frac{1}{2} + \frac{1}{4} e^{-2t} + \frac{1}{4} e^{2t}\right) = \frac{1}{4} + \frac{3}{16} e^{t} + \frac{3}{16} e^{-t} + \frac{1}{8} e^{-2t} + \frac{1}{8} e^{2t} + \frac{1}{16} e^{-3t} + \frac{1}{16} e^{3t}$$

$$\text{PMF}: \left\{ \overset{1/16}{-3}, \overset{1/8}{-2}, \overset{3/16}{-1}, \overset{1/4}{0}, \overset{3/16}{1}, \overset{1/8}{2}, \overset{1/16}{3} \right\}$$

Q2 (a) Let X be a r.v. such that P(X = 1) = 1/2, P(X = 2) = 1/3, P(X = 3) = 1/6. Let X1, X2,..., Xn be iid X. Let T = X1 + … + Xn. What is E[T]? What is E[T/n]?

$$E[X] = \frac{1}{2}\times 1 + \frac{1}{3}\times 2 + \frac{1}{6}\times 3 = 5/3$$

$$E[T] = n\,E[X] = \frac{5n}{3}$$

$$E[T/n] = 5/3$$

Q2 (b) What is Var[T]? What is Var[T/n]?

$$E[X^2] = \frac{1}{2}\times 1 + \frac{1}{3}\times 4 + \frac{1}{\cancel{6}_{2}}\times \cancel{9}^{3} = \frac{10}{3}$$

$$Var(X) = \frac{10}{3} - \left(\frac{5}{3}\right)^2 = \frac{5}{9}$$

$$Var(T) = n\,Var(X) = \frac{5n}{9}$$

$$Var\left(\frac{T}{n}\right) = \frac{Var(X)}{n} = \frac{5}{9n}$$

Q3 (a) Let X be a r.v. such that P(X = -1) = 1/6, P(X = 0) = 2/3, P(X = 1) = 1/6. Let X1, X2,..., Xn be iid X. Let M2 = (X1)^2 + … + (Xn)^2. What is E[M2]? What is E[M2 / n]?

$$E[X^2] = \frac{1}{6} \times (-1)^2 + \frac{2}{3}(0)^2 + \frac{1}{6}(1)^2 = 1/3$$

$$E[M_2] = n\,E[X^2] = n/3$$

$$E\left[\frac{M_2}{n}\right] = 1/3$$

Q3 (b) What is Var[M2]? What is Var[M2 / n]?

$$E[X^4] = \frac{1}{6} \times (-1)^4 + \frac{2}{3}(0)^4 + \frac{1}{6}(1)^4 = 1/3, \quad Var(X^2) = E[X^4] - E[X^2]^2 = 1/3 - \frac{1}{9} = \frac{2}{9}$$

$$Var(M_2) = n\,Var(X^2) = \frac{2n}{9}$$

$$Var\left(\frac{M_2}{n}\right) = \frac{1}{n} Var(X^2) = \frac{2}{9n}$$

Q4 (a) Let X be a r.v. such that P(X = 1) = 1/6, P(X = 2) = 1/6, P(X = 3) = 1/3, P(X = 4) = 1/3.
Let X1, X2,..., Xn be iid X. Let Fi = number of ‘i’ in the samples. What is E[F1]? What is E[F3]?

Consider $X_{11}, X_{21}, \ldots, X_{n1} \sim \text{Bernoulli}(1/6)$ $\quad X_{i1} = \begin{cases} 1 & \text{if } X_i = 1 \\ 0 & \text{else.} \end{cases}$

$$F_1 = X_{11} + X_{21} + \cdots + X_{n1}$$

$$E[F_1] = n E[X_{11}] = n/6$$

Consider $X_{13}, X_{23}, \ldots, X_{n3} \sim \text{Bernoulli}(1/3)$ $\quad X_{i3} = \begin{cases} 1 & \text{if } X_i = 3 \\ 0 & \text{else} \end{cases}$

$F_3 = X_{13} + \cdots + X_{n3}$ and $E[F_3] = n E[X_{13}] = n/3$

Q4 (b) What is Var[F1]? What is Var[F3]?

$$\text{Var}(F_1) = n \,\text{Var}(X_{11}) = n \cdot \frac{1}{6} \times \left(1 - \frac{1}{6}\right) = \frac{5n}{36}$$

$$\text{Var}(F_3) = n \,\text{Var}(X_{13}) = n \cdot \frac{1}{3}\left(1 - \frac{1}{3}\right) = \frac{2n}{9}$$

Q5 (a) Let X be a r.v. such that P(X = 1) = 1/2, P(X = 2) = 1/3, P(X = 3) = 1/6. Let X1, X2,..., Xn be iid X. Let T = X1 + … + Xn. Using WLLN, find an upper bound for P(T > 2n).

$$E[T] = \frac{5n}{3}, \quad Var(T) = \frac{5n}{9}$$

$$T > 2n \iff T - \frac{5n}{3} > n/3 \subseteq \left|T - \frac{5n}{3}\right| > n/3$$

$$P(T > 2n) \le P\left(\left|T - \frac{5n}{3}\right| > n/3\right) \le \frac{5n/9}{(n/3)^2} = \frac{5}{n}$$

Q5 (b) Using CLT, find an estimate for P(T > 2n).

$$T \approx Normal\left(\frac{5n}{3}, \frac{5n}{9}\right)$$

$$T > 2n \iff \left(\frac{T - \frac{5n}{3}}{\sqrt{\frac{5n}{9}}}\right) > \frac{2n - \frac{5n}{3}}{\sqrt{\frac{5n}{9}}} = \frac{n/\cancel{3}}{\sqrt{5n}/\cancel{3}} = \sqrt{\frac{n}{5}}$$

$$\sim Z = N(0,1)$$

$$P(T > 2n) \approx P\left(Z > \sqrt{n/5}\right) = 1 - F_Z\left(\sqrt{\frac{n}{5}}\right)$$

Q6 (a) Let X be a continuous r.v. uniform in [-1,1]. Let X1, X2,..., Xn be iid X. Let T = X1 + … + Xn. Using WLLN, find an upper bound for P(|T| > n/2).

$$E[T]=0, \quad Var(T)=n\cdot\frac{(1-(-1))^2}{12}=n/3$$

$$P(|T|>n/2) \le \frac{Var(T)}{(n/2)^2} = \frac{n/3}{n^2/4} = \frac{4}{3n}$$

Q6 (b) Using CLT, find an estimate for P(|T| > n/2).

$$T \approx Normal(0, n/3)$$

$$|T|>n/2 \longleftrightarrow \frac{|T|}{\sqrt{n/3}} > \frac{n/2}{\sqrt{n/3}} = \frac{\sqrt{3}\,n}{2}$$

$$\approx Z = N(0,1)$$

$$P(|T|>n/2) \approx P\left(|Z|>\frac{\sqrt{3}n}{2}\right) = P\left(Z>\frac{\sqrt{3}n}{2} \text{ (or) } Z<-\frac{\sqrt{3}n}{2}\right) = 1-F_Z(\sqrt{3}n/2) + F_Z(-\sqrt{3}n/2)$$

$$= 2F_Z(-\sqrt{3}n/2)$$

Q7 (a) Let X be a r.v. such that P(X = 1) = 1/6, P(X = 2) = 1/6, P(X = 3) = 1/3, P(X = 4) = 1/3. Let X1, X2,..., Xn be iid X. Let Fi = number of 'i' in the samples. Using WLLN, find an upper bound for P(| F1 - n/6 | > 10 sqrt(n)).

$$E[F_1] = n/6, \quad Var(F_1) = \frac{5n}{36}$$

$$P\left(\left|F_1 - \frac{n}{6}\right| > 10\sqrt{n}\right) \le \frac{\cancel{5}n/36}{\underset{20}{\cancel{100}}\, n} = \frac{1}{720}$$

Q7 (b) Using CLT, find an estimate for P(| F3 - n/3 | > 10 sqrt(n)).

$$E[F_3] = \frac{n}{3}, \quad Var(F_3) = \frac{2n}{9}$$

$$F_3 \approx \text{Normal}\left(\frac{n}{3}, \frac{2n}{9}\right)$$

$$\left|F_3 - \frac{n}{3}\right| > 10\sqrt{n} \iff \underbrace{\frac{\left|F_3 - n/3\right|}{\sqrt{2n/9}}}_{\sim Z = N(0,1)} > \frac{10\sqrt{n}}{\sqrt{2n/9}} = \frac{30}{\sqrt{2}} = 15\sqrt{2}$$

$$P\left(\left|F_3 - \frac{n}{3}\right| > 10\sqrt{n}\right) \approx P(|Z| > 15\sqrt{2}) = 2 F_Z(-15\sqrt{2})$$

Q8 (a) Consider the following samples from Bernoulli(p): 1, 0, 0, 0, 1, 0, 0, 0, 0, 1
Find the sample mean. Find the MM estimate for p.

$n = 10, \quad m_1 = 3/10 = E[X] = p$

$\hat{p}_{MM} = 0.3$

Q8 (b) What is the likelihood function? Find the ML estimate for p.

$L = p^3 (1-p)^7$

$\log L = 3 \log p + 7 \log (1-p)$

$3/p + \frac{7}{1-p}(-1) = 0$

$\hat{p}_{ML} = 3/10$

Q9 (a) Consider the following samples from the discrete distribution P(X=1) = t / 3, P(X=2) = t / 6, P(X=3) = 1 - t / 2: 1, 2, 1, 3, 2, 3, 2, 1, 1, 2
Find the sample mean. Find the MM estimate for t.

$$m_1 = \frac{18}{10} = 1.8 = \frac{t}{3}\cdot 1 + \frac{t}{6}\cdot 2 + \left(1 - \frac{t}{2}\right)\cdot 3 = 3 - \frac{5t}{6}$$

$$\hat{t}_{mm} = \frac{6 \times 1.2}{5} = 1.45$$

Q9 (b) Find the likelihood function. Find the ML estimate for t.

$$L = \left(\frac{t}{3}\right)^4 \cdot \left(\frac{t}{6}\right)^4 \left(1 - t/2\right)^2 = \frac{t^8 (2-t)^2}{3^4 \cdot 6^4 \cdot 2^2}$$

$$\log L = (\text{constant}) + 8 \log t + 2 \log (2-t)$$

$$\cancel{8}^{4}/t + \frac{\cancel{2}}{2-t}(-1) = 0 \quad (\text{or}) \quad 8 - \cancel{4}t = t \quad (\text{or})$$

$$\hat{t}_{ML} = \frac{8}{5} = 1.6$$

Q10 (a) Consider the following samples from the Geometric(p) distribution: 4, 5, 7, 3, 6, 5, 4, 5
Find the sample mean. Find the MM estimate for p.

$$m_1 = \frac{39}{8} = 1/p$$

$$\hat{p}_{MM} = 8/39$$

Q10 (b) Find the likelihood function. Find the ML estimate for p.

$$L = (1-p)^3 p \, (1-p)^4 p \cdots (1-p)^4 p = (1-p)^{39-8} p^8$$

$$\log L = 31 \log(1-p) + 8 \log p$$

$$\frac{-31}{1-p} + \frac{8}{p} = 0 \quad \text{(or)} \quad \hat{p}_{ML} = \frac{8}{39}$$

Q11 (a) Consider the following samples from the Beta(2,b) distribution:
0.86, 0.76, 0.08, 0.24, 0.66
Find the sample mean. Find the MM estimate for b.

$$m_1 = \frac{2.6}{5} = 0.52 = \frac{2}{2+b} \quad (\text{or } 0.52b = 0.96)$$

$$\hat{b}_{mm} = \frac{24}{13}$$

Q11 (b) Consider the following samples from the Gamma(a, 5) distribution:
0.52, 0.25, 0.33, 0.87, 0.42
Find the sample mean. Find the MM estimate for a.

$$m_1 = \frac{2.39}{5} = 0.478 = a/5$$

$$\hat{a}_{ML} = 2.39$$

Q12 (a) Consider the following samples from the distribution with PDF (ax+1)/(2a+2), for 0 < x < 2: 0.1, 0.5, 0.2, 0.4, 1.1

$a > -0.5$

Find the sample mean. Find the MM estimate for 'a'.

$$m_1 = \frac{2.3}{5} = 0.46 = \int_0^2 \frac{x}{2a+2}(ax+1)\,dx = \frac{1}{\cancel{2}(a+1)}\left(a\frac{\cancel{8}\,4}{3} + \frac{\cancel{4}\,1}{\cancel{2}}\right) = \frac{4a+3}{3(a+1)}$$

$$= 1 + \frac{a}{3(a+1)}$$

$$a + 1.62a + 1.62 = 0 \Rightarrow \hat{a}_{MM} = \frac{-1.62}{2.62} = -0.618\,???$$

$$\hat{a}_{MM} = -0.5$$

Q12 (b) Consider the following samples from the distribution with PDF (ax+1)/(2a+2), for 0 < x < 2: 0.1, 0.5

Find the likelihood function. Find the ML estimate for 'a'.

$$L = \frac{0.1a+1}{(2a+2)} \cdot \frac{0.5a+1}{2a+2} = \frac{(0.1)(0.5)}{4} \cdot \frac{(a+10)(a+2)}{(a+1)^2}$$

$$\log L = (\text{constant}) + \log(a+10) + \log(a+2) - 2\log(a+1)$$

$$\frac{1}{a+10} + \frac{1}{a+2} - \frac{2}{a+1} = 0 \quad (\text{or}) \quad (2a+\cancel{12}\,6)(a+1) = 2(a+2)(a+10)$$

$$\cancel{a^2} + 7a + 6 = \cancel{a^2} + 12a + 20$$

$$\hat{a}_{ML} = \frac{-14}{5}\,??? \qquad \hat{a}_{ML} = -0.5$$