

Higher order partial derivatives and the Hessian matrix

Sarang S. Sane

Recall : Partial derivatives



Recall : Partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a scalar-valued multivariable function defined on a domain D in \mathbb{R}^n .



Recall : Partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a scalar-valued multivariable function defined on a domain D in \mathbb{R}^n .

The **partial derivative of f w.r.t. x_i** is the function denoted by $f_{x_i}(x)$ or $\frac{\partial f}{\partial x_i}(x)$ and defined as

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h} .$$

Recall : Partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a scalar-valued multivariable function defined on a domain D in \mathbb{R}^n .

The **partial derivative of f w.r.t. x_i** is the function denoted by $f_{x_i}(\tilde{x})$ or $\frac{\partial f}{\partial x_i}(\tilde{x})$ and defined as

$$\frac{\partial f}{\partial x_i}(\tilde{x}) = \lim_{h \rightarrow 0} \frac{f(\tilde{x} + he_i) - f(\tilde{x})}{h} .$$

Its domain consists of those points of D at which the limits exists.

Recall : Partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a scalar-valued multivariable function defined on a domain D in \mathbb{R}^n .

The **partial derivative of f w.r.t. x_i** is the function denoted by $f_{x_i}(\tilde{x})$ or $\frac{\partial f}{\partial x_i}(\tilde{x})$ and defined as

$$\frac{\partial f}{\partial x_i}(\tilde{x}) = \lim_{h \rightarrow 0} \frac{f(\tilde{x} + he_i) - f(\tilde{x})}{h} .$$

Its domain consists of those points of D at which the limits exists.

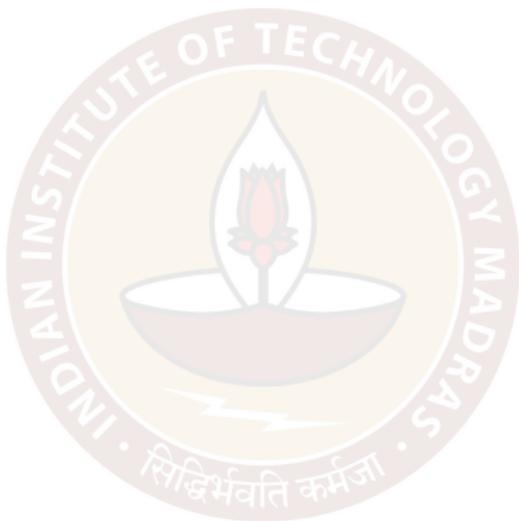
The partial derivative of f w.r.t. x_i at a point \tilde{a} measures the rate of change of f at \tilde{a} in the direction of the standard basis vector e_i (i.e. w.r.t. the variable x_i).

Second order partial derivatives for $f(x, y)$



Second order partial derivatives for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .



Second order partial derivatives for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .

Then the **second order partial derivatives** of f are the partial derivatives of the partial derivatives.



Second order partial derivatives for $f(x, y)$

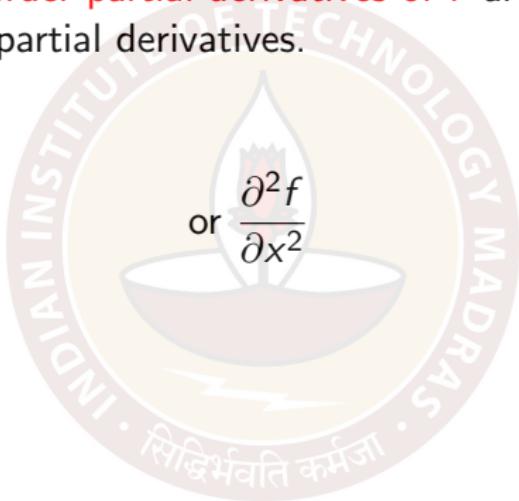
Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .

Then the **second order partial derivatives** of f are the partial derivatives of the partial derivatives.

Notation :

- ▶ f_{xx}

or
$$\frac{\partial^2 f}{\partial x^2}$$



Second order partial derivatives for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .

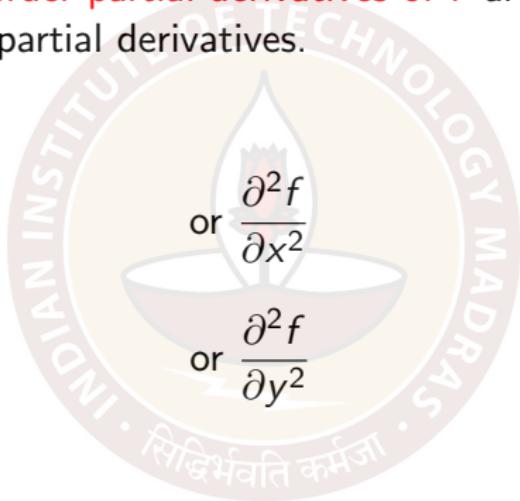
Then the **second order partial derivatives** of f are the partial derivatives of the partial derivatives.

Notation :

- ▶ f_{xx}
- ▶ f_{yy}

or $\frac{\partial^2 f}{\partial x^2}$

or $\frac{\partial^2 f}{\partial y^2}$



Second order partial derivatives for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .

Then the **second order partial derivatives** of f are the partial derivatives of the partial derivatives.

Notation :

- ▶ f_{xx}

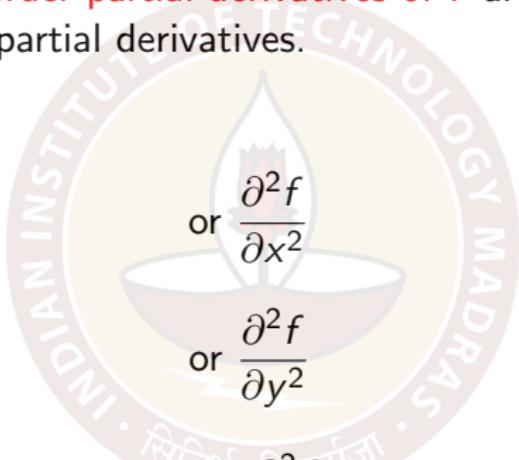
or $\frac{\partial^2 f}{\partial x^2}$

- ▶ f_{yy}

or $\frac{\partial^2 f}{\partial y^2}$

- ▶ f_{xy}

or $\frac{\partial^2 f}{\partial y \partial x}$



Second order partial derivatives for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .

Then the **second order partial derivatives** of f are the partial derivatives of the partial derivatives.

Notation :

$$\blacktriangleright f_{xx} = (f_x)_x \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right)$$

$$\blacktriangleright f_{yy} = (f_y)_y \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

$$\blacktriangleright f_{xy} = (f_x)_y \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

$$\blacktriangleright f_{yx} = (f_y)_x \quad \text{or} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

Mixed
partial
derivatives

Examples

$$f(x, y) = x + y$$

$$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = 1.$$

$$\frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y \partial x} = 0.$$

$$f(x, y) = \sin(xy)$$

$$\frac{\partial f}{\partial x} = y \cos(xy)$$

$$\frac{\partial f}{\partial y} = x \cos(xy).$$

$$\frac{\partial^2 f}{\partial x^2} = y \times \left\{ -\sin(xy) y \right\}$$

$$= -y^2 \sin(xy)$$

सिद्धिर्भवति कर्मजा

सिद्धिर्भवति कर्मजा

$$\frac{\partial^2 f}{\partial y^2} = -x^2 \sin(xy).$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1 \cdot \cos(xy) + x \times \left\{ y \sin(xy) \right\}$$

$$= \cos(xy) - xy \sin(xy).$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \times \cos(1 \cdot y) + y \times \left\{ -x \sin(xy) \right\}$$

$$= \cos(xy) - xy \sin(xy)$$

=

Clairaut's Theorem about mixed partials

Theorem (Clairaut's theorem)

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing a point \tilde{a} and an open ball around it.

Clairaut's Theorem about mixed partials

Theorem (Clairaut's theorem)

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 containing a point \tilde{a} and an open ball around it.

If the second order mixed partial derivatives f_{xy} and f_{yx} are continuous in an open ball around \tilde{a} , then $f_{xy}(\tilde{a}) = f_{yx}(\tilde{a})$.

Example advising caution

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

$$\frac{\partial f}{\partial y}(0, 0) = 0.$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{(x^2+y^2) \left\{ y(x^2-y^2) + xy(2x) \right\} - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{-y^5 + x^4y + 4x^2y^3}{(x^2+y^2)^2} \quad \text{if } (x, y) \neq (0, 0).$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{x^5 - xy^4 - 4x^3y^2}{(x^2+y^2)^2}$$

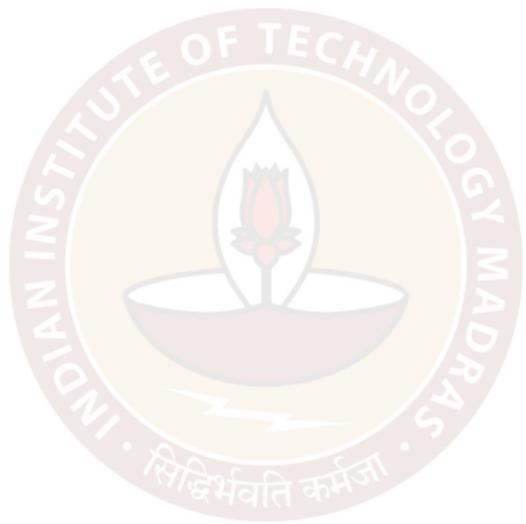
$$\frac{\partial^2 f}{\partial y^2}(0, 0) = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial y}(h, 0) - \frac{\partial f}{\partial y}(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial y}(h, 0) - 0}{h} = 0.$$

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(h, 0) - \frac{\partial f}{\partial x}(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(h, 0) - 0}{h} = 1.$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(0, h) - \frac{\partial f}{\partial x}(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(0, h) - 0}{h} = -1.$$

~~X~~

Second order partial derivatives



Second order partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .
Then the **second order partial derivatives of f** are defined
analogously as the partial derivatives of the partial derivatives.



Second order partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .
Then the **second order partial derivatives of f** are defined
analogously as the partial derivatives of the partial derivatives.

$$f_{x_i x_i} = (f_{x_i})_{x_i} \quad \text{or} \quad \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i} \right)$$

$$\underline{f_{x_i x_j}} = (f_{x_i})_{x_j} \quad \text{or} \quad \underline{\frac{\partial^2 f}{\partial x_j \partial x_i}} = \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) .$$

Example : $f(x, y) = xy + yz + zx$

$$\frac{\partial f}{\partial x} = y+z, \quad \frac{\partial f}{\partial y} = x+z, \quad \frac{\partial f}{\partial z} = x+y.$$

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y \partial x} = 1, \quad \frac{\partial^2 f}{\partial z \partial x} = 1.$$

$$\frac{\partial^2 f}{\partial x^2} = 1, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial z^2} = 1.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial y \partial z} = 1, \quad \frac{\partial^2 f}{\partial z \partial y} = 1.$$

$$\frac{\partial^2 f}{\partial x \partial z} = 1, \quad \frac{\partial^2 f}{\partial y \partial z} = 1, \quad \frac{\partial^2 f}{\partial z \partial x} = 0.$$

Higher order partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .



Higher order partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n . Then the **higher order partial derivatives of f** are defined analogously by taking successive partial derivatives.



Higher order partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n . Then the **higher order partial derivatives of f** are defined analogously by taking successive partial derivatives.

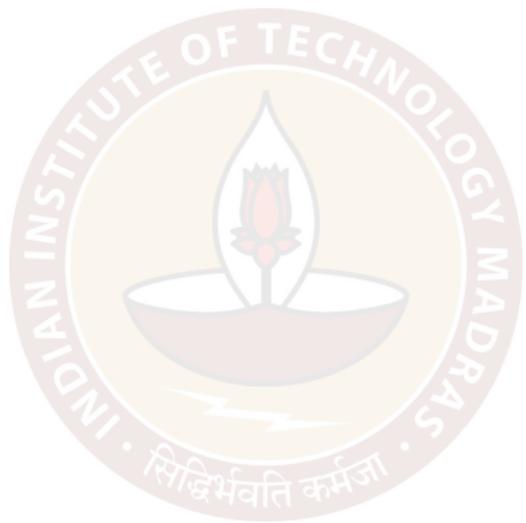
$$f_{x_{i_1} x_{i_2} \dots x_{i_k}} = \left(\left(f_{x_{i_1}} \right)_{x_{i_2}} \right)_{x_{i_3}} \dots \left(\dots \left(\frac{\partial}{\partial x_{i_1}} \right) \right) \quad \text{or} \quad \frac{\partial^k f}{\partial x_{i_k} \dots \partial x_{i_2} \partial x_{i_1}}$$

An appropriately modified statement of Clairaut's theorem holds.

Under suitable hypothesis

$$f_{x_{i_1} x_{i_2} \dots x_{i_k}} = f_{x_{i_k} x_{i_2} \dots x_{i_1}}.$$

The Hessian matrix



The Hessian matrix

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .



The Hessian matrix

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .

Then the **Hessian matrix of f** is defined as :

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_n} \\ \frac{\partial^2 f}{\partial x_i \partial x_1} & \frac{\partial^2 f}{\partial x_i \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

↓ ; ^{ith column}

← ; ^{ith row}

(The term $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is circled in red.)

The Hessian matrix

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in \mathbb{R}^n .

Then the **Hessian matrix of f** is defined as :

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_i \partial x_1} & \frac{\partial^2 f}{\partial x_i \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_i \partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_j} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Examples

$$f(x, y) = x + y$$

$$H_f = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

$$f(x, y) = \sin(xy)$$

$$H_f = \begin{bmatrix} -y^2 \sin(xy) & \cos(xy) - xy \sin(xy) \\ \cos(xy) - xy \sin(xy) & -x^2 \sin(xy) \end{bmatrix}$$

$$f(x, y, z) = xy + yz + zx$$

$$H_f = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \times 3}$$

Thank you

