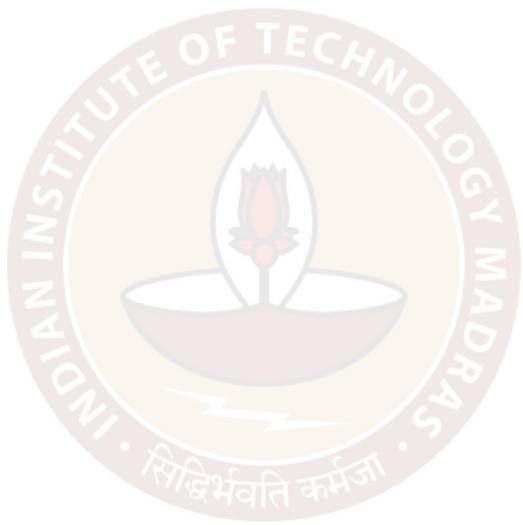


Linear transformations, ordered bases and matrices

Sarang S. Sane

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- ▶ What is an isomorphism of vector spaces?

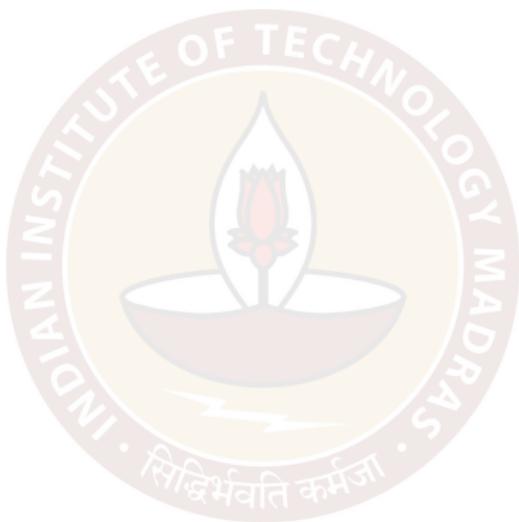


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- ▶ What is a linear transformation?
- ▶ What is an isomorphism of vector spaces?
- ▶ How does a basis of V determine a linear transformation from V (extending maps on bases to linear transformations)?

An important property of finite dimensional vector spaces

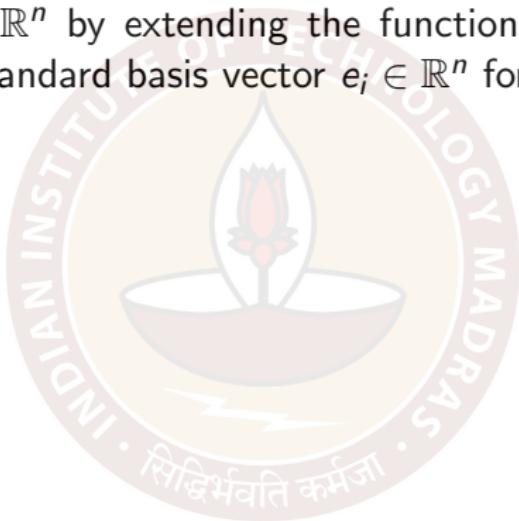
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Then f is an isomorphism.

$$v = \sum c_i v_i \\ f(v) = \sum c_i e_i \Rightarrow f(v_i) = e_i$$

On to : $(x_1, \dots, x_n) \in \mathbb{R}^n$. Let $v = \sum x_i v_i$

Then $f(v) = \sum x_i e_i = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$v = \sum c_i v_i$$

1 : $f(v) = 0 \Rightarrow \sum_{i=1}^n c_i e_i = (0, \dots, 0)$

$$\Rightarrow (c_1, \dots, c_n) = (0, \dots, 0)$$
$$\Rightarrow c_i = 0 \Rightarrow v = 0 v_1 + 0 v_2 + \dots + 0 v_n = 0$$

Example

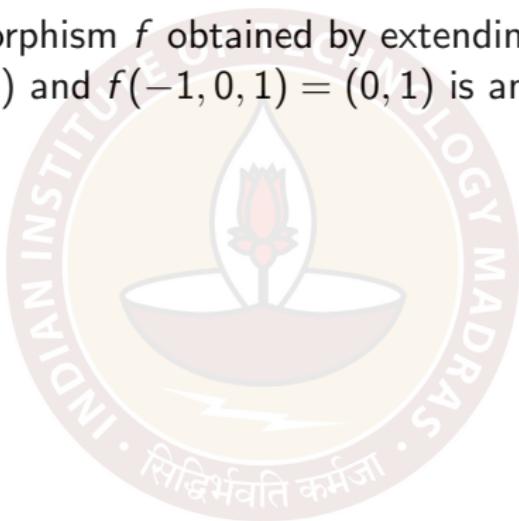
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Note that $(x, y, z) \in W$ can be uniquely expressed as $(x, y, z) = y(-1, 1, 0) + z(-1, 0, 1)$.

Hence, $f : W \rightarrow \mathbb{R}^2$ is $f(x, y, z) = y(1, 0) + z(0, 1) = (y, z)$.

onto: $(y, z) \in \mathbb{R}^2$. Choose $x = -y - z$ & consider $(x, y, z) \in W$.

$$\therefore f(x, y, z) = (y, z).$$

$$f(x, y, z) = (y, z) = (0, 0) \Rightarrow y = 0, z = 0 \\ \Rightarrow x = -y - z = 0.$$

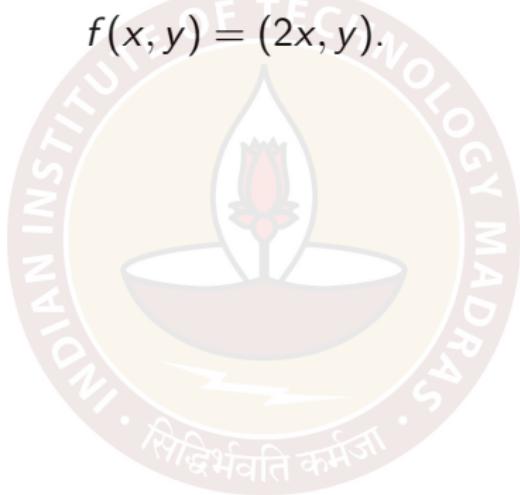
1-1: $f(x, y, z) = \emptyset$ for $(x, y, z) \in W$

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Example : Linear transformation in matrix form

Consider the linear transformation

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 ; \quad f(x, y) = (2x, y).$$



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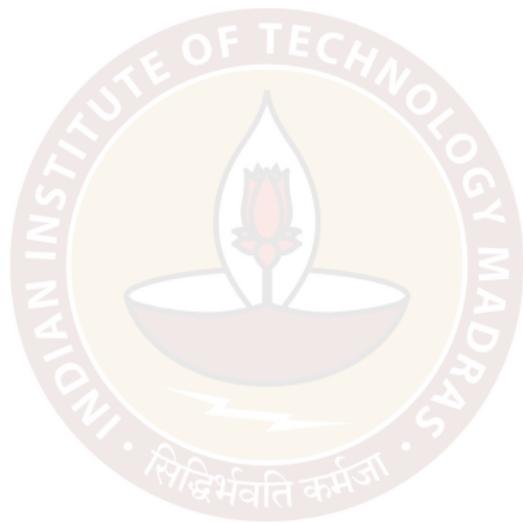
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The matrix corresponding to a linear transformation with respect to ordered bases

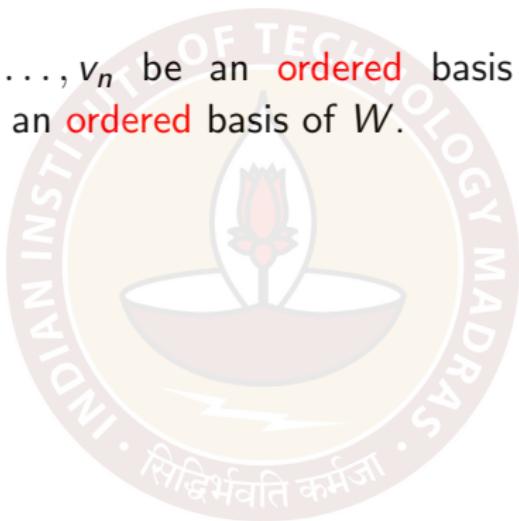
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Let $f : V \rightarrow W$ be a linear transformation.

Let $\beta = v_1, v_2, \dots, v_n$ be an **ordered** basis of V and $\gamma = w_1, w_2, \dots, w_m$ be an **ordered** basis of W .



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⋮

$$f(v_n) = a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m$$

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e.g. Let $V = W = \mathbb{R}^2$, $\beta = \gamma = (1, 0), (1, 1)$ and $f(x, y) = (2x, y)$.

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Hence the matrix corresponding to f w.r.t. the ordered bases

$$\{(1, 0), (1, 1)\}$$
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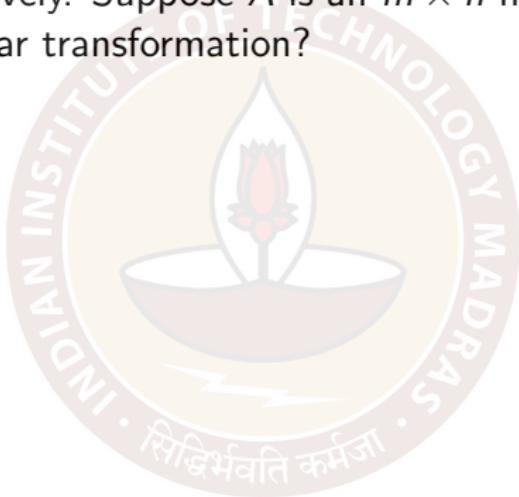
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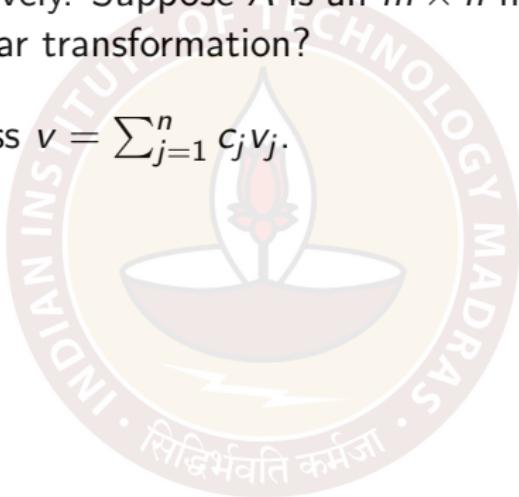
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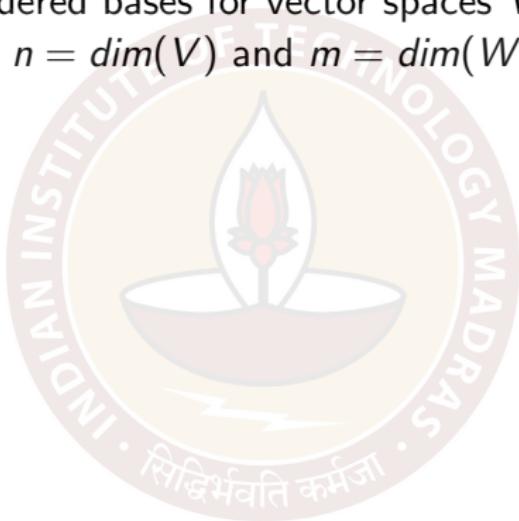
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Hence the matrix corresponding to f is indeed A .

Fixed ordered bases : Linear transformations \leftrightarrow matrices

Let β and γ be ordered bases for vector spaces V and W respectively where $n = \dim(V)$ and $m = \dim(W)$.



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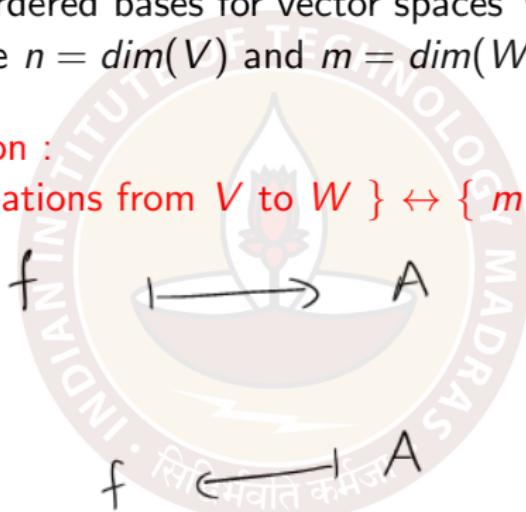
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There is a bijection :

{ linear transformations from V to W } \leftrightarrow { $m \times n$ matrices } .

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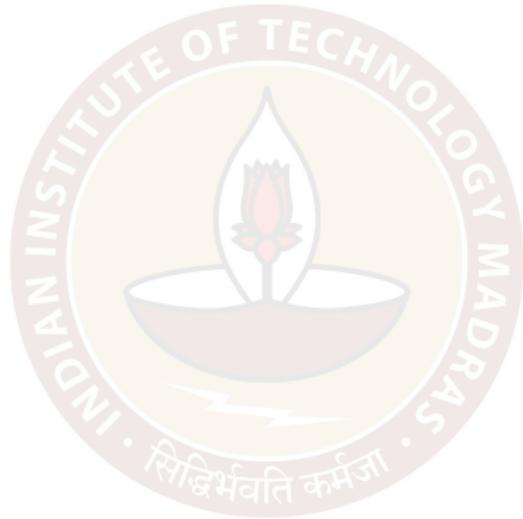
Extend f
to a lin.
trans.



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Another example

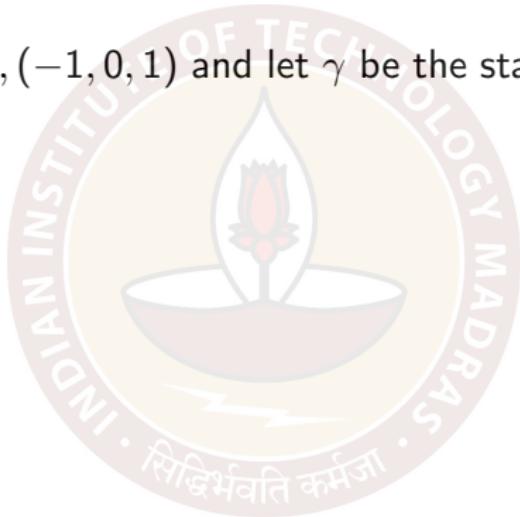
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Recall that the isomorphism $f(x, y, z) = (y, z)$ from W to \mathbb{R}^2 was obtained by extending $f(-1, 1, 0) = (1, 0)$ and $f(-1, 0, 1) = (0, 1)$.

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॥ ज्ञानवति कर्मजा ॥

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Example contd. (changed basis)

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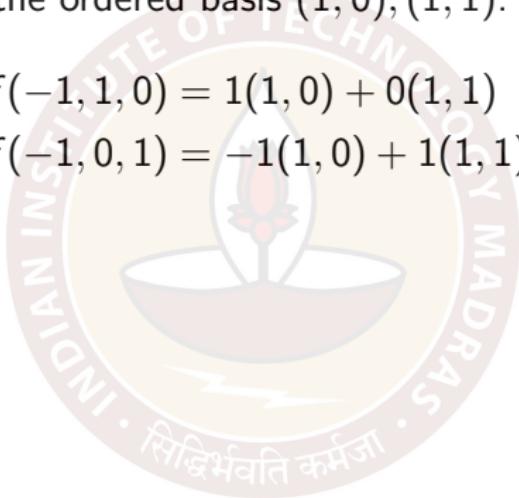


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Thus, changing the ordered bases gives us different matrices corresponding to the same linear transformation.

Thank you

