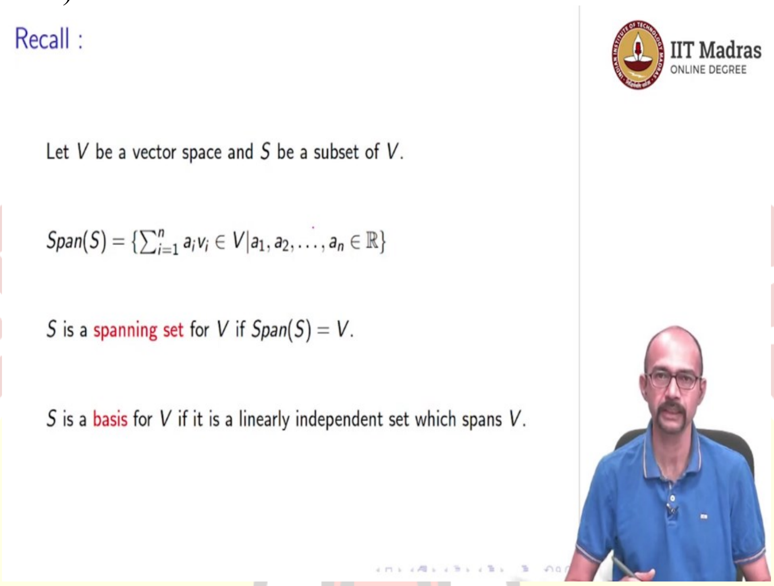
IIT Madras

ONLINE DEGREE

Mathematics of Data Science 2
Professor Sarang Sane
Department of Mathematics
Indian Institute of Technology Madras
Lecture 33: Finding Bases for Vector Spaces

(Refer Slide Time: 00:14)

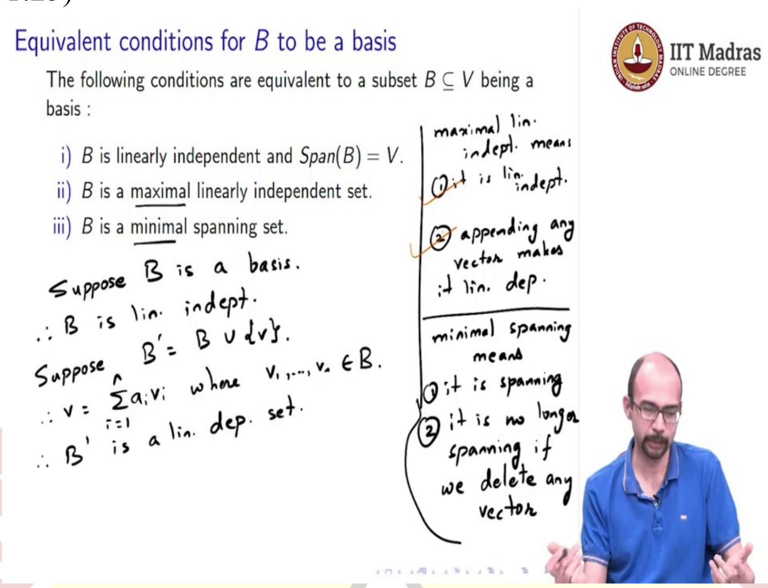


Hello, and welcome to the online BSC Program on Data Science and Programming. In this video, we are going to talk about finding bases for vector spaces. This is a continuation of our previous video, where we defined a basis. So just to recall, suppose V is a vector space and S is a subset of V, we define the span of S to be or the set of linear combinations of vectors in V. So, you take some mission ai vi where ai are real numbers and vis are vectors in S.

So, S is said to be spanning so it is a spanning set for V, if the span of S is V, which means that the set of linear combinations exhausts V. So, any vector in V can be written as a linear combination of vectors from S.

And then finally we defined a basis for V to be a linearly independent set with spans V. So, it is both spanning and linearly independent. Just a small point that, the title we had bases, which is a plural of basis.

(Refer Slide Time: 01:25)



So, let us do a slightly theoretical statement. So, the following conditions are equivalent to a subset being a basis. So, if B is a basis, then these conditions are going to hold. On the other hand, if any of these conditions hold for a set, subset B, then it must be a basis. What are the conditions? B is linearly independent and the span of B is V. This is exactly saying that it is a basis. B is a maximal linearly independent set and B is a minimal spanning set.

So, let us maybe quickly see why these are equivalent. So, the first one is just the definition of a basis. So, let us look at, let us say, let us look at why the second and the first are the same. So, suppose I know that B is a basis. Suppose B is a basis. Let us, let me say why it is a maximal linearly independence set.

Before that, maybe I should even first explain these terms. What is the maximal linearly independent set? And what is a minimal spanning set? So, maximal linearly independent set means it is linearly independent, first of all. And if you add any vector to this, then it is no longer a linearly independent set. Adding any vector or maybe I should say, use the word appending, appending any vector makes it linearly dependent. That is what we mean by maximal linearly independent.

And let us now ask what is minimal spanning? So, a minimal spanning set means it is spanning, first of all, and if you would remove any vector, it is no longer spanning. If you delete a vector, so deleting a vector, it is no longer spanning if we delete any vector. So that is what we mean by a minimal spanning.

So, let us ask, suppose it is a basis wise it is a maximal linearly independent set? So, maybe this let me write down. So, since it is a basis, we already know it is a linearly independent set. So, therefore B is linearly independent. So, maximal linearly independent means it is linearly independent and appending any vector makes it linearly dependent. So, the first part we know that it is linearly independent, so we have to just check that if you append any vector, then that makes it linearly dependent.

So, well suppose you append V, suppose $B' = B \cup v$. Well, but then v is already in the span of B, because remember B is a basis, that means B is a spanning set. So, therefore $v = \sum_{i=1}^{n} \quad a_i v_i$. Because B is already a spanning set.

So therefore, V is a linear combination of some elements of v_i. So, in particular it is a linear combination of the other elements in B'. So, B'cannot be linearly independent. So, $B'$is linearly dependent set. So, we have checked that it is maximal linearly independence. We have checked one then two here, what we have here. So, both of these are true for a basis.

Similarly, you can check that it is minimal spanning. We already know it is spanning. So, to check it is a minimal spanning you remove a vector, you delete a vector and then you try to see if it is spanning. Well, you can take that same vector and ask is that vector in the span of the other things, if it were, then B would not be linearly independent. So that means the rector that you deleted cannot be a linear combination of the other vectors. So that means if you remove it, if you delete it, the set that you obtain will no longer be spanning, that is how you get that it is a minimum spanning set.

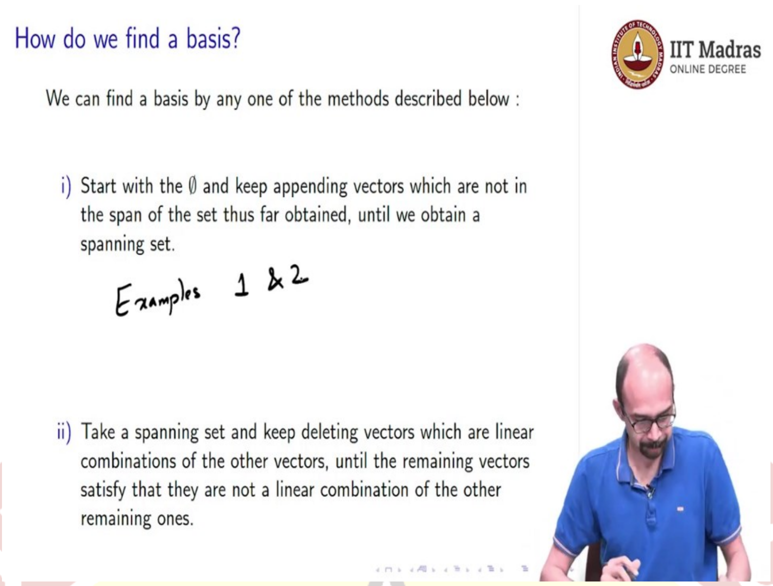
So, this means that statement 1 gives you statements 2 and 3. You can go the other way as well. Let me just talk out how let us say statement 2 implies statement 1. So, suppose it is a maximal linearly independent set, so that means we already know that it is linearly independent. So, we have to only prove that it is a spanning set.

So, in order to prove it is a spanning set you can take any arbitrary vector and you have to express it as a linear combination of these vectors. Well, now, you know that if you append that vector to this set it makes it linearly dependent. So that means, if you take that vector that you appended is a linear combination of the other vectors, because if this one was not a linear combination of the other vectors, then that means the other vectors would have already been linearly dependent, which is not true.

So, the only way that you can introduce linear dependence is if that new vector that you added is a linear combination of the vectors that you have, which means that you can express this v as a linear combination of the vectors in B and that tells you that span of B is the entire v, that means B is linearly independent and spanning, so that means it is a basis.

So, these are some theoretical juggleries you have to do, and some of this will be done in a tutorial and via problems. And I will encourage you to check this for yourselves. Not hard at all, but you have to know little bit of how to write proofs. This is not strictly essential for the data science program this is just something that is good to know as part of the Maths 2 component.

(Refer Slide Time: 09:12)



So how do we find a basis? Let us take a step back and understand what we have done so far. We have defined something called a basis. By a basis we mean something, a set which is linearly independent and spanning linearly. Linearly independent means that the only linear combination, which gives you zero is where all the coefficients are zero. Spanning means if you have any vector you can write it as a linear combination of this set of vectors.

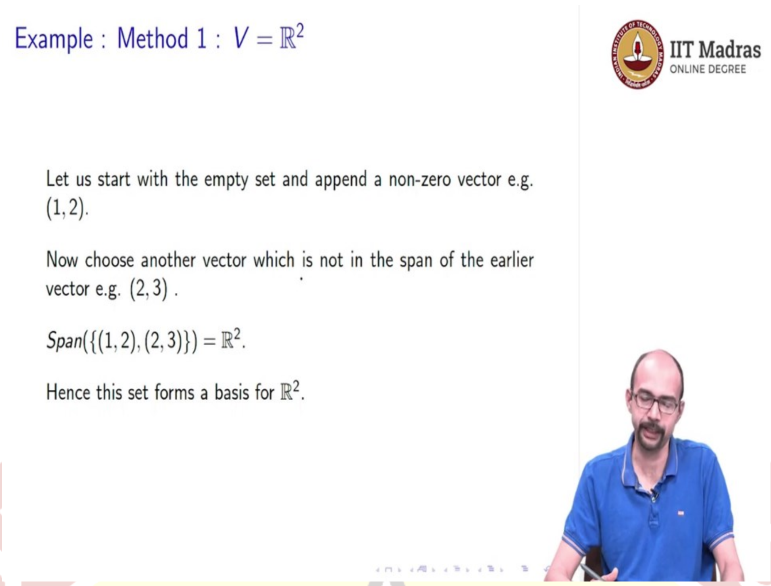
So, now let us use that, these equivalent conditions that we had before to ask, to answer this question, how do we find a basis? So, we can find a basis by any one of the methods described below. So, you start with the empty set and keep appending vectors, which are not in the span of the set thus far obtained. This is exactly, what we did in our two examples before, until we obtain a spanning set.

So, you keep going, you keep adding vectors to your spending set, but you add them in a way so that the intermediate sets that you get are all linearly independent. So, what you will get will be a maximal linearly, independent set, and that will ensure that it is a basis. Similarly, you can take a spanning set that means you take a very large set. You take lots and lots of vectors, then it is very likely that it is a spanning set. Every vector can be written as a linear combination of these.

So, then what you do is, within this set that you have chosen, you see which vectors can be written as linear combinations of the other vectors. If there is a vector like that, you delete it and you keep doing this process. Keep checking for a vector that you can write as a linear combination with the others. If you find such a vector, keep deleting it, so slowly your set will grow smaller. And once you reach a place where you cannot delete any vector, meaning there is no vector which linear combination of the others, you have reached a situation where your set as linearly independent it is already spanning.

So, that means it is a basis. So, these are the two ways that you can find a basis. So, let us, let me just note here that this process we have already done, these were the examples done before.

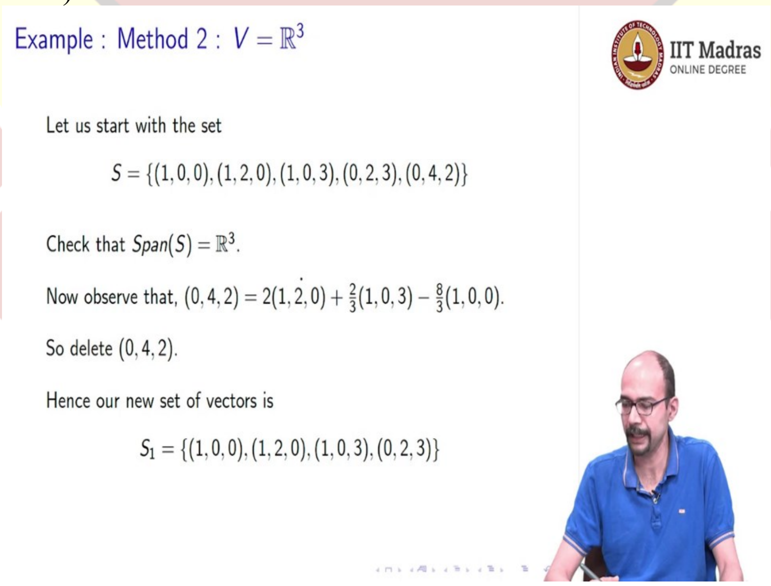
(Refer Slide Time: 11:47)



So maybe let us do another quick example. So, let us start with the empty set append a non-zero vector $(1,2)$. Clearly, the set containing $(1,2)$does not span $\mathbb{R}^2$, because think of what line it will give you. So, the Y coordinate is two times the X coordinate, so that is the line it will give you. So, let us choose something, which is not on that line.

So, let us say you choose the vector $(2,3)$and well, you can check. Now I leave this to you that span of $(1,2)$and $(2,3)$is $\mathbb{R}^2$. So, since you have a vector, which is not on that line, the span will actually give you a plane and $\mathbb{R}^2$is itself a plane, so it will be the plane $\mathbb{R}^2$. So, this set forms a basis for $\mathbb{R}^2$. So, the fact that it is linearly independent is clear from the way we have chosen it. And its span is $\mathbb{R}^2$. The other way of doing this is to say that it is maximal linearly independent. Meaning, if you choose any other vector, then they become linearly dependent. This is the same idea.

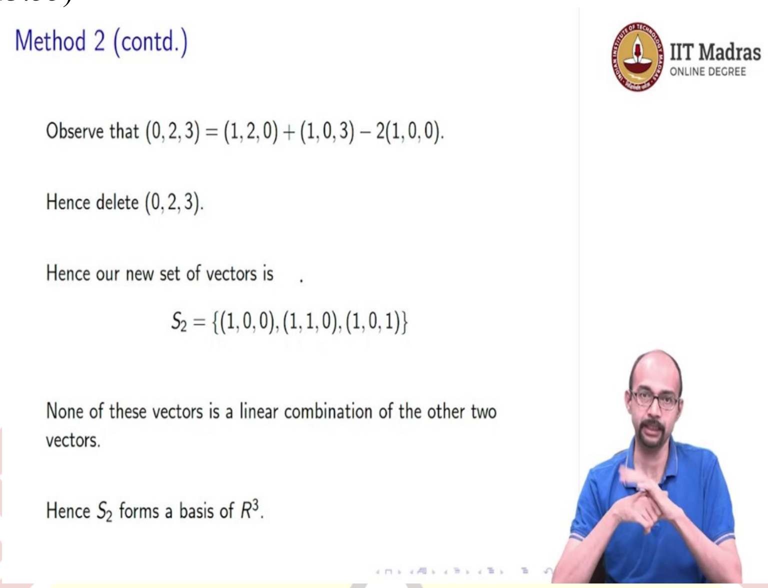
(Refer Slide Time: 13:01)



So, let us look at method 2. We have already done method 1 before, so I am not going into lot of detail on that. Let us look at method 2. So, method 2 involves choosing a set, which is spanning and then deleting vectors from that set. So, here is a set which is spanning. I will encourage you to check that it is a spanning set. So, check that span of S is a $\mathbb{R}^3$.

Now observe that the last vector $(0,4,2) = 2(1,2,0) + \frac{2}{3}(1,0,3) - \frac{8}{3}(1,0,0)$. This is by observation. You could have done this by solving equations. So, delete $(0,4,2)$. So your new set of vectors is $\{(1,0,0), (1,2,0), (1,0,3), (0,2,3)\}$.

(Refer Slide Time: 13:55)



Observe that $(0,2,3) = (1,2,0) + (1,0,3) - 2(1,0,0)$. What are you left with? You are left with the set $S_2 = \{(1,0,0), (1,1,0), (1,0,1)\}$. None of these vectors is a linear combination of the other two vectors. This is another thing you can check. So, S_2forms a basis of $\mathbb{R}^3$. This is the second method that we have seen.

Okay. So, let us take a step back and ask what have we done in this video? We have seen various techniques of finding a basis, namely by either appending vectors, so that what the sets that you get are linearly independent or by deleting vectors from a spanning set. We saw plenty of examples of this. And I will also, now at this point, ask you to go back and check what were the sizes of the bases that you got.

So, for example, here, the size of the basis that we have for $\mathbb{R}^3$is 3. Now notice what we know. We know that if you have a linearly independent set in $\mathbb{R}^3$, then it must be of size at most 3, you cannot have four elements in $\mathbb{R}^3$, which are linearly independent. So, your basis here has size exactly 3.

So, this is what I meant by saying, that you reach some kind of optimality. Similarly, if you take your standard basis, for $\mathbb{R}^N$your standard basis has a size N. So, we will study this phenomenon in the next video. Thank you.