What is a linear transformation

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Linear mappings satisfy linearity, i.e. for any $c \in \mathbb{R}$ (scalar) :

$$f(x_1 + cy_1, x_2 + cy_2, \ldots, x_n + cy_n) = f(x_1, x_2, \ldots, x_n) + cf(y_1, y_2, \ldots, y_n).$$

Formal definition

A function $f : V \to W$ between two vector spaces V and W is said to be a linear transformation if for any two vectors v_1 and v_2 in the vector space V and for any $c \in \mathbb{R}$ (scalar) the following conditions hold :

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- $f(v_1 + v_2) = f(v_1) + f(v_2)$ ✓
- $f(cv_1) = cf(v_1)$ ✓

Equivalent to linearity : $f(v_1 + cv_2) = f(v_1) + cf(v_2)$.

$f(v_1 + cv_2) = f(v_1) + \underline{f(cv_2)} = f(v_1) + cf(v_2)$

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6. $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x, y) = x.$

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For a linear transformation, being 1-1 is equivalent to $f(v) = 0$ implies $v = 0$.

$f : V \to W$ is a lin. trans.

Assume f is 1-1. Then $f(v_1) = f(v_2) \Rightarrow v_1 = v_2$.

$f(0) = 0$. If $f(v) = 0 \Rightarrow f(v) = f(0)$

$\Rightarrow v = 0$.

$f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = f(0) + f(0)$

$\Rightarrow f(0) = 0$.

Conversely, assume $f(v) = 0 \Rightarrow v = 0$.

$f(v_1) = f(v_2) \Rightarrow f(v_1 - v_2) = 0$.

$\Rightarrow v_1 - v_2 = 0$

$\Rightarrow v_1 = v_2$.

$f(v_1) - f(v_2) = 0$.

$f(v_1) + f(-v_2) = 0$

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$f(v) + f(-v) = f(v + (-v)) = f(0) = 0$

$\Rightarrow f(-v) = -f(v)$

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A linear transformation $f : V \to W$ between two vector spaces V and W is said to be an isomorphism if it is a bijection.

Example 1 seen earlier is an isomorphism :

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(x, y) = (2x, y).$$

$f(x,y) = (0,0) \Rightarrow (2x,y) = (0,0) \Rightarrow 2x=0, y=0$
$\Rightarrow x=0, y=0$
$\Rightarrow (x,y) = (0,0)$ } f is 1-1

For $(u,v) \in \mathbb{R}^2$ consider $x = u/2$, $y = v$.
$\therefore f(x,y) = (2x,y) = (2 \cdot u/2, v) = (u,v)$ } onto

Isomorphisms : Non-examples

Example 2 seen earlier is not an isomorphism :

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(x, y) = (2x, 0).$$

There is no pre-image for the vector (u, v), where v is non-zero.

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Similarly, the fifth example $f : \mathbb{R} \to \mathbb{R}^3 \quad ; \quad f(t) = (t, 3t, \frac{23}{89}t)$ is 1-1 but not onto.

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Similarly, the fifth example $f : \mathbb{R} \to \mathbb{R}^3 \quad ; \quad f(t) = (t, 3t, \frac{23}{89}t)$
is 1-1 but not onto.

Also the sixth example $f : \mathbb{R}^2 \to \mathbb{R} \quad ; \quad f(x, y) = x$
is onto but not 1-1.

Bases determine linear transformations

Let V be a vector space with basis $\{v_1, v_2, \ldots, v_n\}$.

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Let V be a vector space with basis $\{v_1, v_2, \ldots, v_n\}$.
Let $f : V \to W$ be a linear transformation. Then the ordered vectors $f(v_1), f(v_2), \ldots f(v_n)$ uniquely determine f.

Let $v \in V$. $v = \sum_{i=1}^{n} c_i v_i$.

$$\boxed{f(v) = f\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i f(v_i)}$$

is determined by $c_1, \ldots, c_n$ & $f(v_1), f(v_2), \ldots, f(v_n)$.

Suppose $w_1, \ldots, w_n$ is a specified set of vectors in W

There is a unique lin. trans. f s.t. $f(v_i) = w_i$.

Define $f(v) = \sum_{i=1}^{n} c_i w_i$ where $v = \sum_{i=1}^{n} c_i v_i$.

$f(v_k) = w_k$.

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$$f((1,0)) = (2,0) = 2(1,0) \quad \leftarrow w_1$$
$$f((0,1)) = (0,1) \quad \leftarrow w_2$$

$(x,y) = x(1,0) + y(0,1).$

$f(x,y) = x(2,0) + y(0,1) = (2x,0) + (0,y).$

$= (2x,y).$

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Basis : $(1,0), (1,1)$
$w_1 = (2,0)$
$w_2 = (1,1)$

$f(x,y)$
$= (x-y)(2,0) + y(1,1)$
$= (2x-2y+y, y)$
$= (2x-y, y)$.

$$f(x,y) = (x-y)f(1,0) + y f(1,1)$$
$$= \boxed{(x-y)}(2,0) + \boxed{y}(0,1)$$
$$= (2(x-y), 0) + (0,y) = (2x-2y, y).$$

Thank you