Examples of finding bases for the kernel and image of a linear transformation

Sarang S. Sane

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- For a linear transformation $f : V \to W$, let A be the matrix corresponding to f after choosing ordered bases $\beta = v_1, v_2, \ldots, v_n$ and $\gamma = w_1, w_2, \ldots, w_m$ for V and W respectively.

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\text{given by} & c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} & \mapsto & v = \sum_{j=1}^{n} c_j v_j \\
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- In particular, a basis of $\mathcal{N}(A)$ will yield a basis of $ker(f)$.

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Example

Consider $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4) =$
$(2x_1 + 4x_2 + 6x_3 + 8x_4, x_1 + 3x_2 + 5x_4, x_1 + x_2 + 6x_3 + 3x_4)$

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$$\overset{R_3 + R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 - 2R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Thus the reduced row echelon form is $\begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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Putting $X_3 = s$ and $X_4 = t$ and using the system of equations gives $X_1 = -9s - 2t$ and $X_2 = 3s - t$.

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Since we have chosen β to be the standard ordered basis, the basis for the $ker(T)$ is also the same, i.e.
$-9e_1 + 3e_2 + 1e_3 + 0e_4 = (-9, 3, 1, 0)$ and
$(-2e_1 - 1e_2 + 0e_3 + 1e_4 = (-2, -1, 0, 1)$.

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Since we have chosen γ to be the standard ordered basis, the basis for $im(T)$ is also the same, i.e. $2e_1 + 1e_2 + 1e_3 = (2, 1, 1)$ and $4e_1 + 3e_2 + 1e_3 = (4, 3, 1)$.

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Let $V = \mathbb{R}^2, W = \{(x,y,z) | x+y+z=0\}$. Let the respective ordered bases be $\beta = (1,1),(1,-1)$ and $\gamma = (-1,1,0),(-1,0,1)$. $T(x,y) = (0, x+2y, -x-2y)$ defines a linear transformation from V to W.

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The pivot column (dependent variable) is the first column while the non-pivot column (independent variable) is the second column.

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Therefore a basis for $ker(T)$ is the singleton set consisting of $3(-1, 1, 0) + (-3)(-1, 0, 1) = (0, 3, -3)$.

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$$rank(T) + nullity(T) = dim(V)$$

Thank you