Differentiability and the derivative

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$$\text{Infinitesimal speed} = \lim_{\Delta t \to 0} \frac{\text{Distance travelled in time } \Delta t}{\Delta t}$$

(where Δt is measured in seconds and distance in km).

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$f(x) = x$ is differentiable at a point a.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{a+h-a}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1.$$

$f(x) = sin(x)$ is differentiable at the point 0.

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

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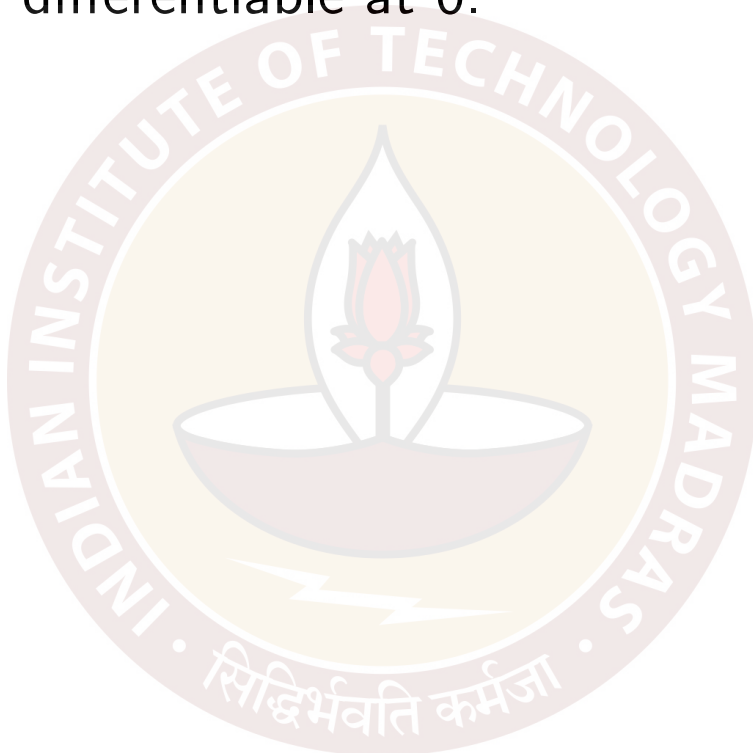
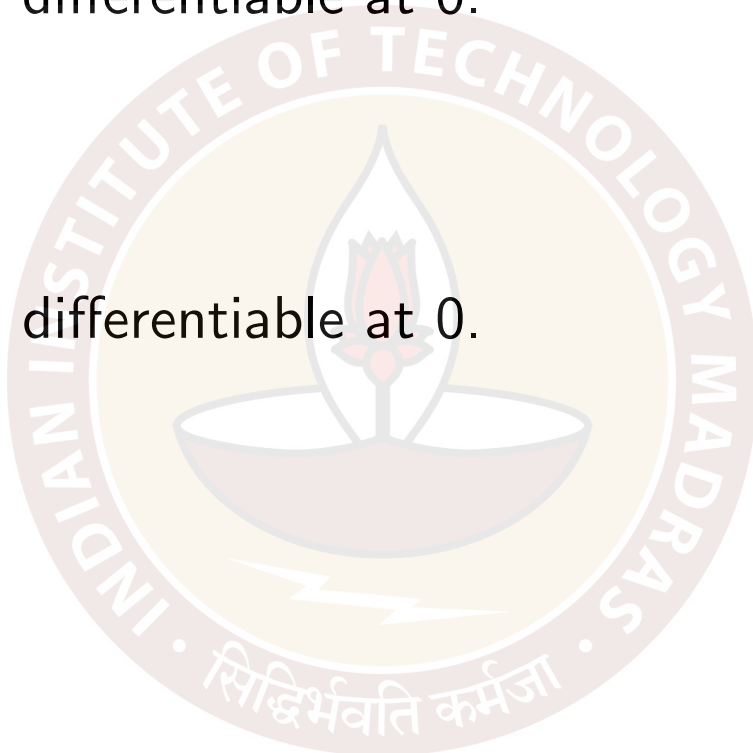


More examples

$f(x) = |x|$ is NOT differentiable at 0.

$f(x) = x^{\frac{1}{3}}$ is NOT differentiable at 0.



More examples

$f(x) = |x|$ is NOT differentiable at 0.

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$ DNE.

$\frac{|h|}{h} = \begin{cases} 1 & h>0 \\ -1 & h<0 \end{cases}$

$\lim_{h\to 0^-} \frac{|h|}{h} = -1.$

$\lim_{h\to 0^+} \frac{|h|}{h} = 1.$

$f(x) = x^{\frac{1}{3}}$ is NOT differentiable at 0.

$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3}-0}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} h^{-2/3} = \lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}}$

$\lim_{h\to 0^+} \frac{1}{\sqrt[3]{h^2}}$ DNE (diverges to ∞)

$\lim_{h\to 0^-} \frac{1}{\sqrt[3]{h^2}}$ DNE (diverges to $-\infty$)

$f(x) = \lfloor x \rfloor$ is NOT differentiable at any integer point.

$\lim_{h\to 0} \frac{\lfloor 0+h \rfloor - \lfloor 0 \rfloor}{h} = \lim_{h\to 0} \frac{\lfloor h \rfloor}{h}$

$\frac{\lfloor h \rfloor}{h} = \begin{cases} 0 & \text{if } 0<h<1 \\ -1/h & \text{if } -1<h<0 \end{cases}$

$\lim_{h\to 0^+} \frac{\lfloor h \rfloor}{h} = 0$ but $\lim_{h\to 0^-} \frac{-1}{h}$ DNE (diverges to $+\infty$)

Differentiability implies continuity

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Fact : If f is differentiable at a, then it is continuous at a.

$\lim_{x \to a} f(x)$

We know $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = L$.

$\lim_{h \to 0} h = 0$.

$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \cdot h \right)$

$= L \times 0 = 0$

$\Rightarrow \lim_{h \to 0} f(a+h) - f(a) = 0 \Rightarrow \lim_{h \to 0} f(a+h) = f(a)$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$.

The derivative function

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For a function $f(x)$ its derivative function, $f'(x)$ or $\frac{df}{dx}(x)$ is

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Examples

For $f(x) = x$, then $f'(x) =$ 1.

$f(x) = sin(x)$, then $f'(x) =$ $cos(x)$.

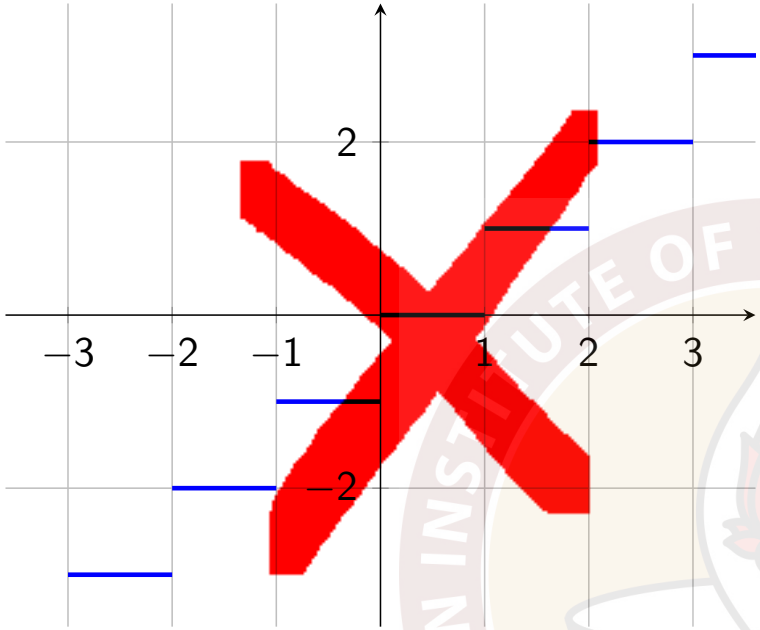
$$\lim_{h \to 0} \frac{\sin(a+h) - \sin(a)}{h} = \lim_{h \to 0} \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h}$$

$$= \lim_{h \to 0} \cos(a) \frac{\sin(h)}{h} + \lim_{h \to 0} \sin(a) \frac{\cos(h) - 1}{h}$$

$$= \cos(a) + 0 = \cos(a).$$

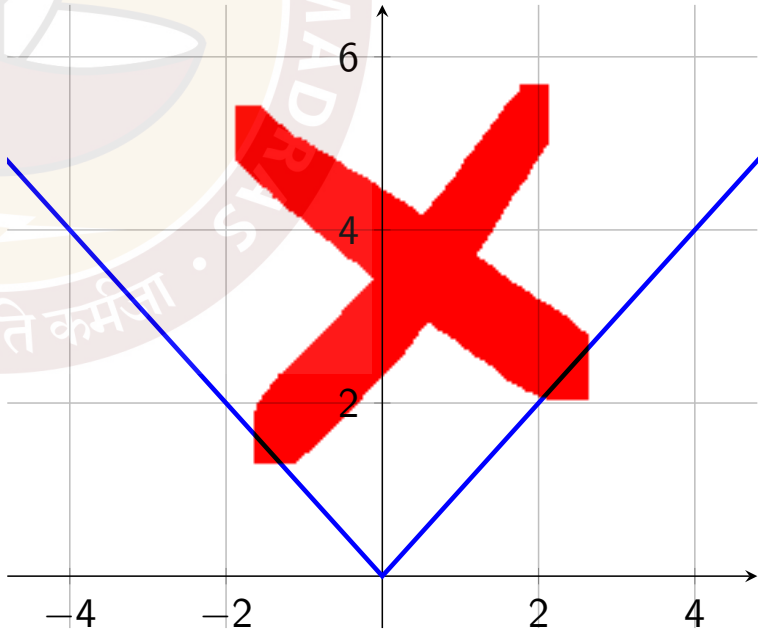
$f'(0) = 1.$

Tangents : Examples advising caution : where are we?



$\Gamma(\lfloor x \rfloor)$

Graph of $y = |x|$



Thank you