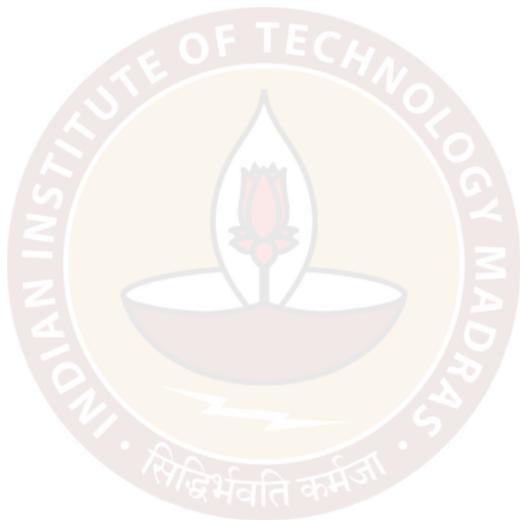


Image and kernel of linear transformations



Definitions of kernel and image

Let $f : V \rightarrow W$ be a linear transformation.

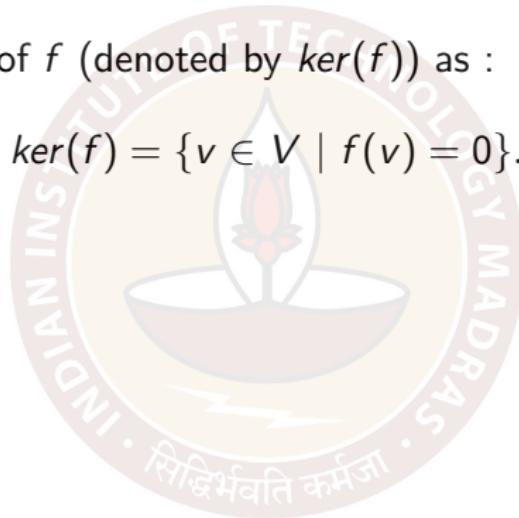


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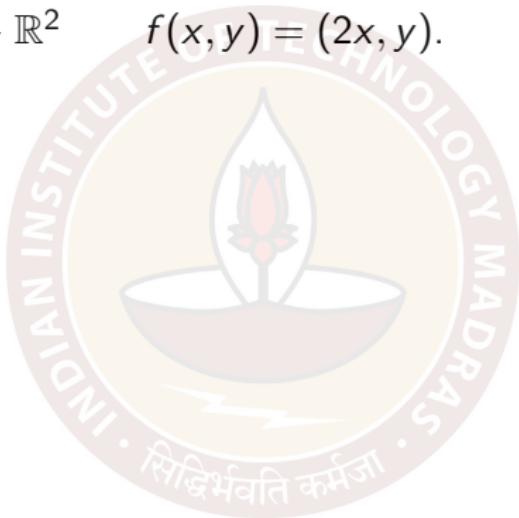
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$\text{Im}(f)$ is another name for the "range of the function f " which we have studied in Maths-1.

Examples

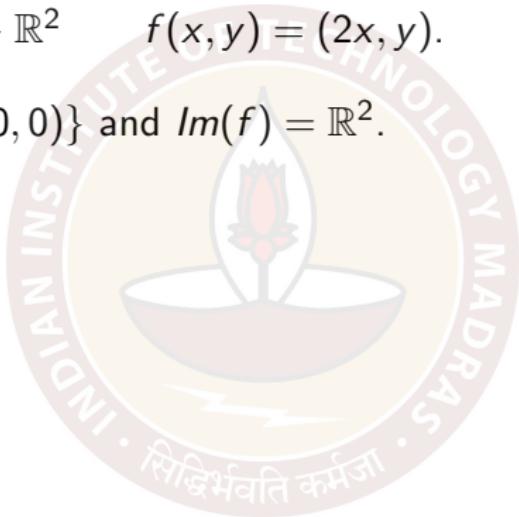
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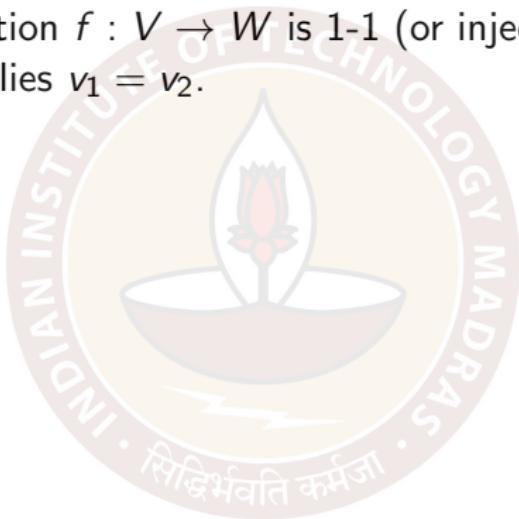
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Also $\text{Im}(f) = \{(x, 0) | x \in \mathbb{R}\}$ i.e. the X -axis.

The kernel and injectivity of a linear transformation

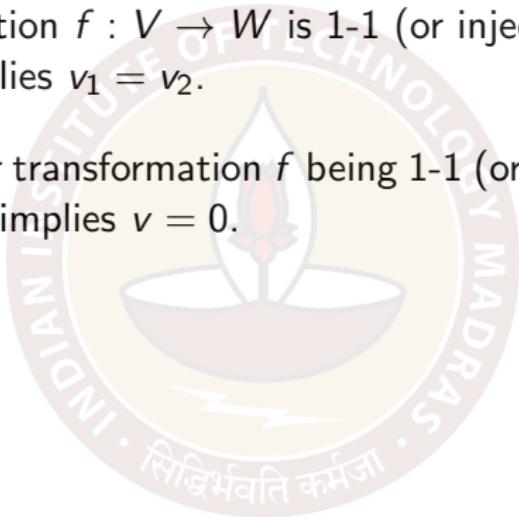
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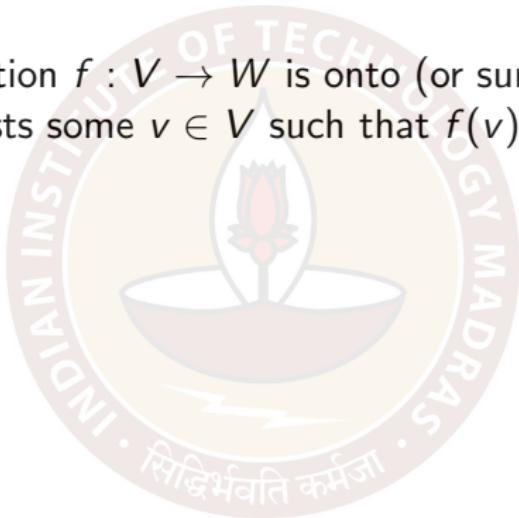
A linear transformation f is 1-1 if and only if $\ker(f) = 0$.

$$\ker(f) = \{0\}$$

$\ker(f)$ is the 0 subspace

The image and surjectivity of a linear transformation

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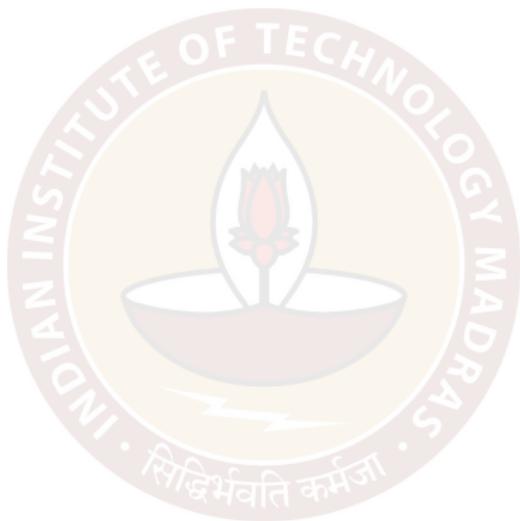
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Writing this out for linear transformations, we see that : a linear transformation $f : V \rightarrow W$ is onto if and only if $\text{Im}(f) = W$

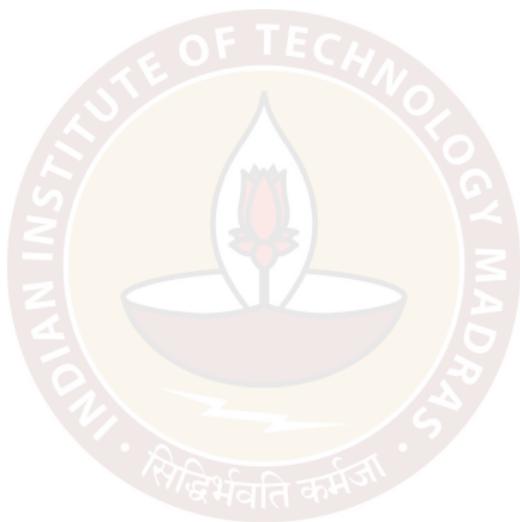
Kernels and null spaces

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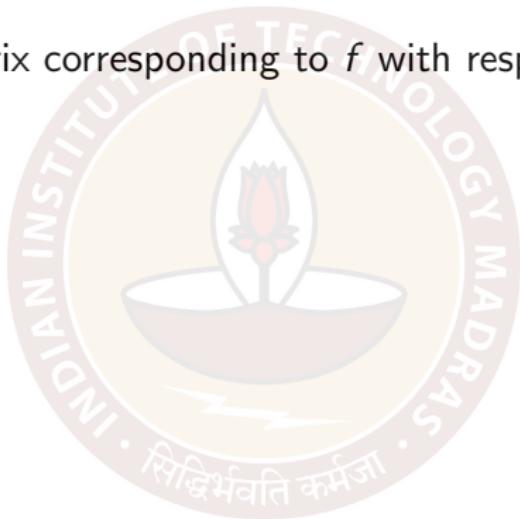
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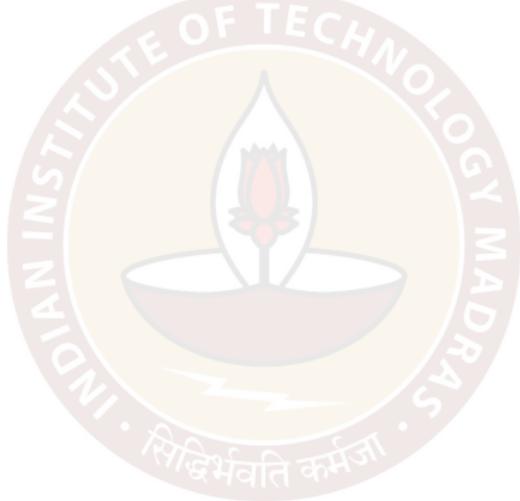
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Thus, $v = \sum_{j=1}^n c_j v_j \in \ker(f)$

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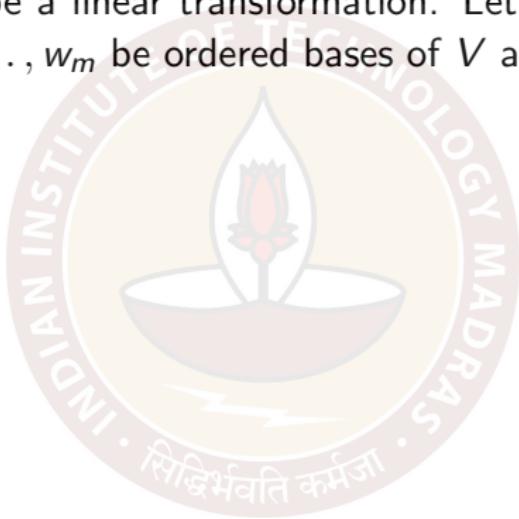
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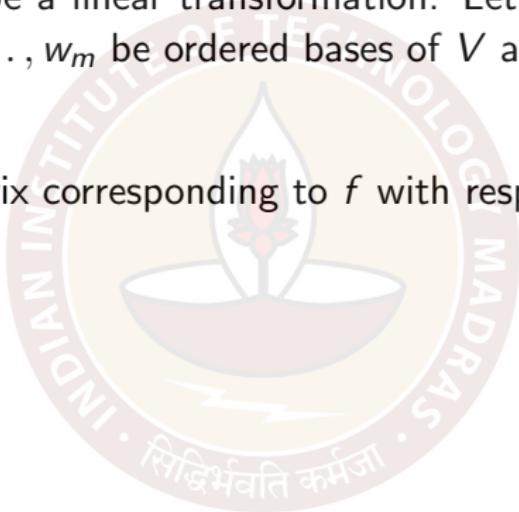
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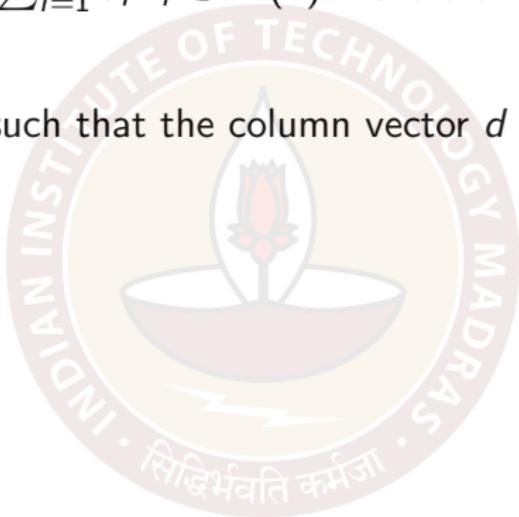
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Images and column spaces (contd.)

Equivalently $w = \sum_{i=1}^m d_i w_i \in Im(f)$ if there exists a column

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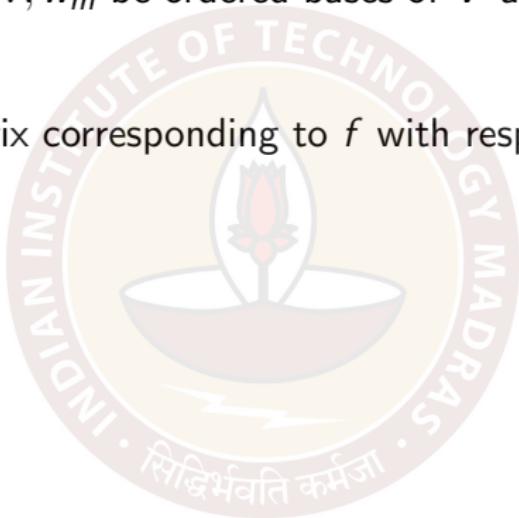
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We can thus use **row reduction** to obtain these bases.