IIT Madras
ONLINE DEGREE

Mathematics for Data Science - 2
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Lecture No. 29
Linear Dependence

Hello, and welcome to the Maths 2 component of the online B.Sc. program on data science. Today's video is about linear dependence. So, we have seen in the previous video what is the vector space, and in this video we are going to study the notion of linear dependence of vectors.

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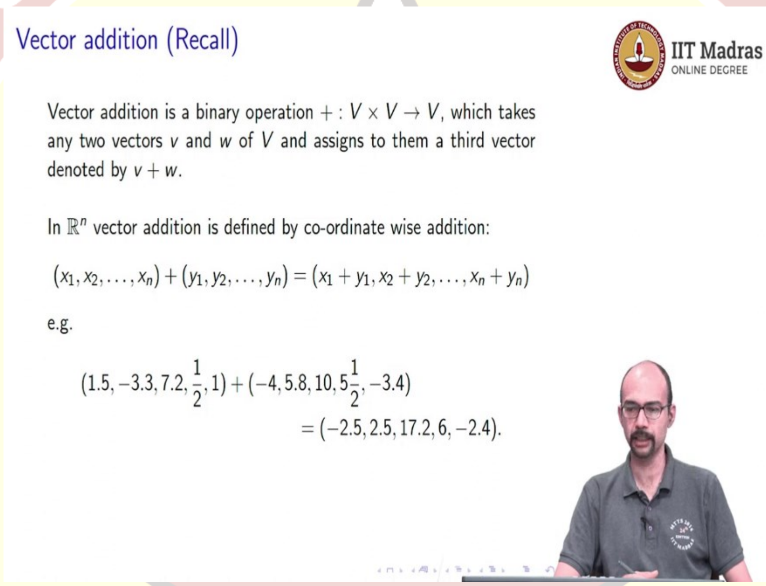

So, first of all, let us recall what is vector addition. So, vector addition is a binary operation $+: V \times V \to V$. So, what that means is we take two vectors, v and w, and we can talk about the vector v + w. So, in R^n we know that addition is defined coordinate wise. So, $(x_1, x_2, \ldots. x_n)+(y_1, y_2, \ldots y_n)=(x_1+y_1, x_2+y_2, \ldots., x_n+y_n)$

So, just as an example, if you are $(1.5,-3.3,7.2,\frac{1}{2,1})+(-4,5.8,10,5\frac{1}{2},-3.4)$ we have to add them coordinate wise. So, the first coordinate will be 1.5 - 4 that gives you -2.5. The second coordinate will be minus 3.3 + 5.8 that gives me 2.5. And I will leave you to check the rest.

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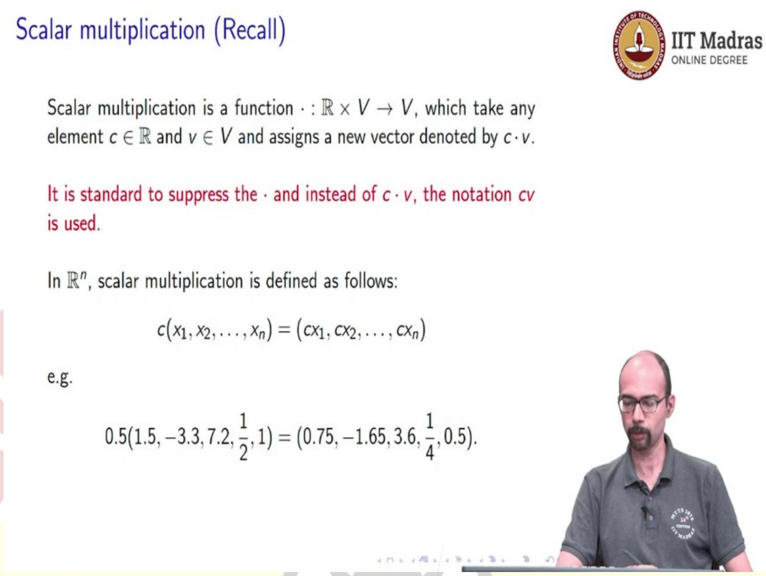

We also have something called scalar multiplication in vector spaces, which means you, it is a function from $. : R \times V \rightarrow V$ So, what that means is you take a scalar or a real number c and a vector v and then we can associate to that a new vector c.v. And so, remember this dot is corresponding to scalar multiplication. And it is standard to drop this dot, which is what we did in our previous video. So, we are going to just use c times v, cv and not c dot v.

And in R^n, scalar multiplication is coordinate wise again. So, $c(x_1, x_2, .., x_n) = (cx_1, cx_2, \ldots, cx_n)$

So, here is an example. If you do $0.5(1.5, -3.3, 7.2, \frac{1}{2}, 1)$ that gives me $(0.75, -1.65, 3.6, \frac{1}{4}, 0.55)$

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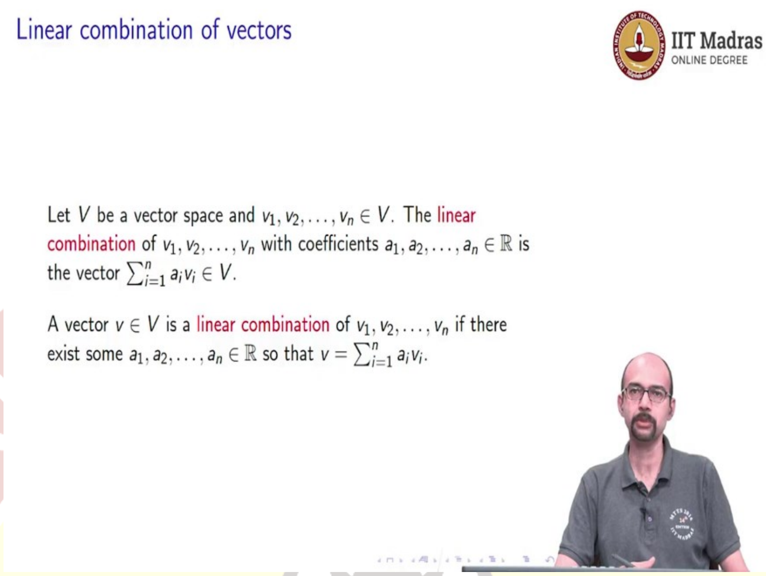

So, let us now look at what is the linear combination of vectors. So, this is a key, I mean, this is a key definition which is going to drive this video and several subsequent videos. So, let us pay careful attention to this. So, let V be a vector space and suppose we have a bunch of vectors $v_1, v_2, \ldots, v_n$ So, the linear combination of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n$ so these are real numbers, is the vector $\sum_{i=1}^{n} a_i v_i$

What does this mean? This means you take v_i you take the ith vector v_i you do scalar multiplication of a_i and v_i and that is a new vector. So, accordingly, we have n new vectors, $a_1 v_1, a_2 v_2, a_3 v_3, \ldots a_n v_n$ and then you add them up. And remember, since our axioms tell us that, it does not matter which two we add first what is called associativity of addition, you can just write it as $a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n$ I do not need to specify which one I do first.

So, that is what we write as $\sum_{i=1}^{n} a_i v_i$ So, this is a new vector. And this new vector $\sum_{i=1}^{n} a_i v_i$ is called the linear combination of $v_1, v_2, \ldots, v_n$ with coefficients $a_1, a_2, \ldots, a_n$ So, a vector v is a linear combination of $v_1, v_2, \ldots, v_n$ if there are some scalars $a_1, a_2, \ldots, a_n$ so that v is expressed as $\sum_{i=1}^{n} a_i v_i$

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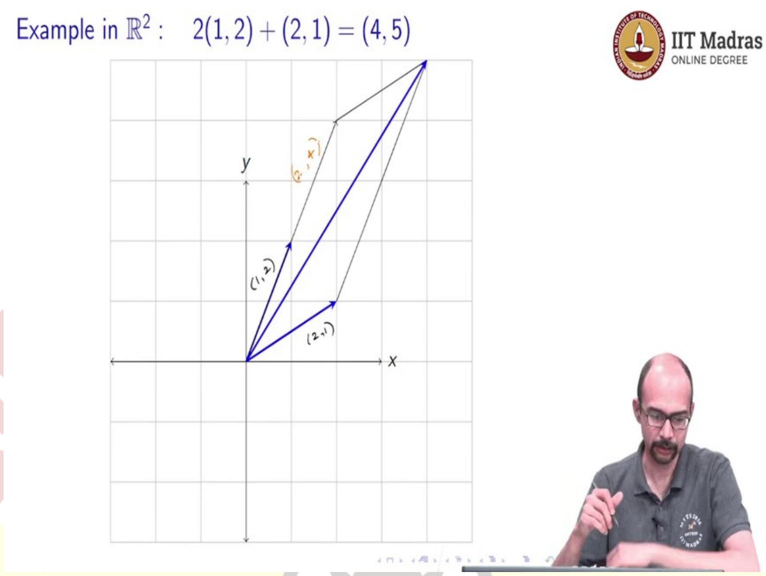

So, let us do a couple of examples. So, we are going to look at the geometry first and then we are going to look at the algebra. So, the example here is 2(1,2) + (2,1) = (4,5). So, what is (1,2)? So, (1,2) is over here. What is (2, 1)? (2, 1) is over here. And then you take 2 times (1, 2), you scale (1, 2) by 2. So, the new vector you get after scaling is this thing here which is (2, 4). So, you have (2, 4) + (2, 1), use the parallelogram law, and you get (4, 5).

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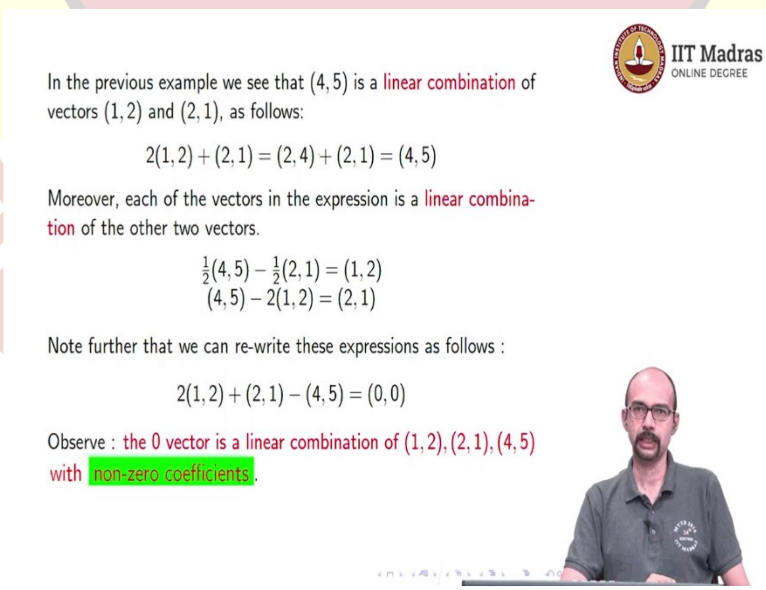

So, of course, you can do this algebraically. But that is not the point. The point we want to make here is that we have expressed (4, 5) as a linear combination of (1, 2) and (2, 1). And what is the linear combination? It is with the scalars 2 and 1. So, 2(1, 2) + 1(2, 1). So, if there is a 1 time something, we suppress it. So, 2(1, 2) + (2, 1) is (4, 5) and the working out is clear.

And we can say more. Each of the vectors in the expression is a linear combination of the other two vectors. So, what are the three vectors we are talking about? We are talking about (1, 2), (2, 1) and (4, 5). We have expressed (4, 5) as a linear combination of (2, 1) and (1, 2). So, instead, we could express (1, 2) as a linear combination of (4, 5) and (2, 1). This is clear, because you can take (2, 1) on the other side, and you can divide by half. That is what we have got here. So, the coefficients here are half and minus half.

And similarly, you can express (2, 1) as a linear combination of (4, 5) and (1, 2). So, that is, to do that you take (1, 2) on the other side, and then you exactly get the second expression written here. So, we can do other things. So, this is all some manipulation of expressions. But this manipulation is important, because it is going to lead us to our definition. So, you can rewrite this expression as 2(1, 2) + (2, 1) – (4, 5) is (0, 0).

So, we have taken that first expression on the top and moved (4, 5) to the left by subtracting it out. And so what we get is on the right hand side, we get the vector 0. So, we have a linear combination of these three vectors by some scalars and these are non-zero scalars. So, the coefficients here are nonzero. What are the coefficients, 2, 1 and -1. And with these coefficients, 0 is a linear combination of these three vectors, (1, 2), (2, 1) and (4, 5). So, I will ask you to remember that picture in the previous slide along with this observation.

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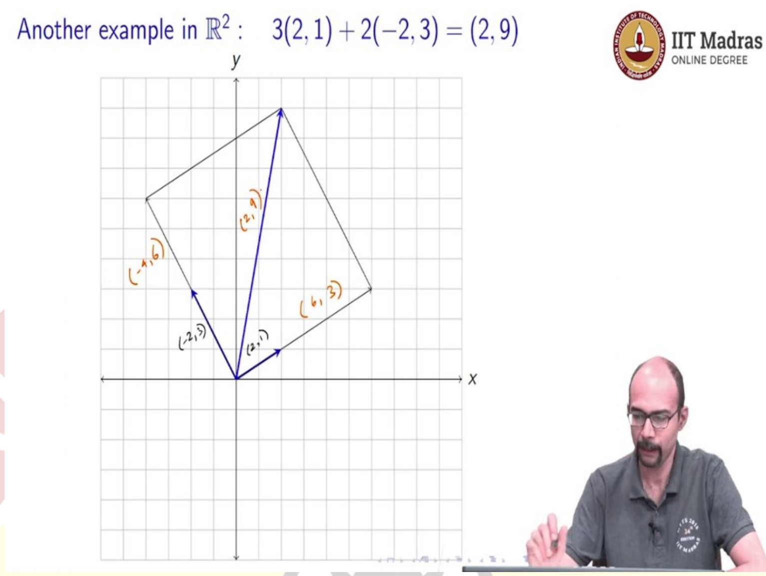

Let us do one more. So, here we have 3(2, 1) + 2(-2, 3) is (2, 9). So, what is (2, 1), here is (2, 1). And what is (-2, 3), here is (-2, 3). So, if we scale these, so 3 times (2, 1) is exactly this large thing we get here, which is (6, 3). And 2 times (-2, 3) is this larger vector we get here which is (-4, 6). And then when we add them, we use a parallelogram law and we get the vector (2, 9).

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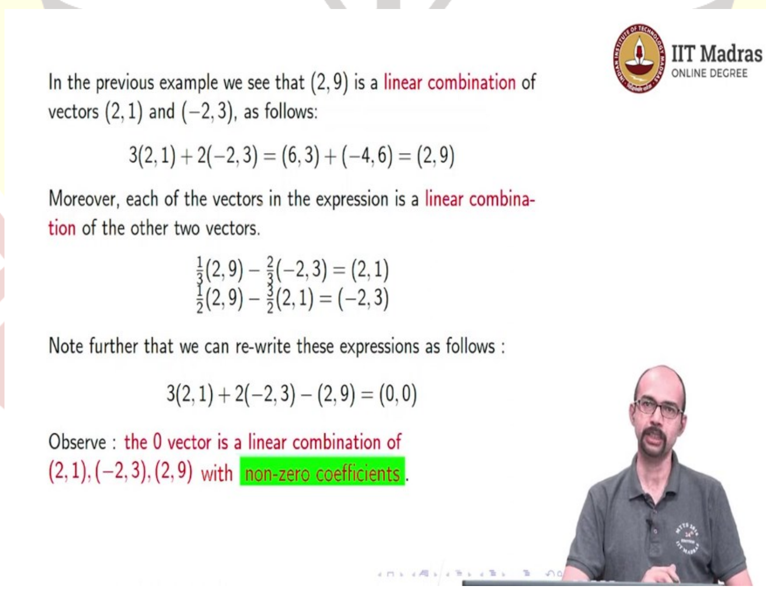

So, what have we done? We have expressed (2, 9) as a linear combination of the vectors (2, 1), and (-2, 3), the algebra is what we just did and the expression is written here. So, now, again, we can take these three vectors (2, 1), (-2, 3) and (2, 9) and we can write any one of these as a linear combination of the other two. So, already, you have written (2, 9) as a linear combination of the first two.

Now, here is how we write (2, 1) as a linear combination of (-2, 3) and (2, 9). Namely, we can take (-2, 3) on the other side and then we can divide by 3, and that is the expression we get here or we could write (-2, 3) as a linear combination of (2, 9) and (2, 1). How do we do that? We take (2, 1) on the other side and then we divide by half and indeed that is the expression we get. So, this is similar to what happened before.

And once again, instead of writing it this way, there is a third way we can write this expression namely by taking (2, 9) on the other side and that gives us 0, the zero vector on the right. So, that is 3(2, 1) + 2(-2, 3) – (2, 9) is (0, 0). Again, what do we observe? The 0 vector is a linear combination of (2, 1), (-2, 3) and (2, 9) with non-zero coefficients. So, this is the important part. What are the coefficients here? The coefficients are 3, 2 and -1.

So, I should point out I am emphasizing these non-zero coefficients. What happens if your coefficients are 0 if you take a linear combination of any vectors $v_1, v_2, \ldots, v_n$ with 0 efficient that is always going to give you 0, because the linear combination will be $0v_1, 0v_2, \ldots, 0v_n$ which of course is the 0 vector. So, the point is, here you can do it with non-zero coefficients. So, you can always do it with 0 coefficients, but it is not clear you can do it with non-zero coefficients. And if you can, then there is something special happening. So, remember, all these three vectors were on the same plane. This is the key point.

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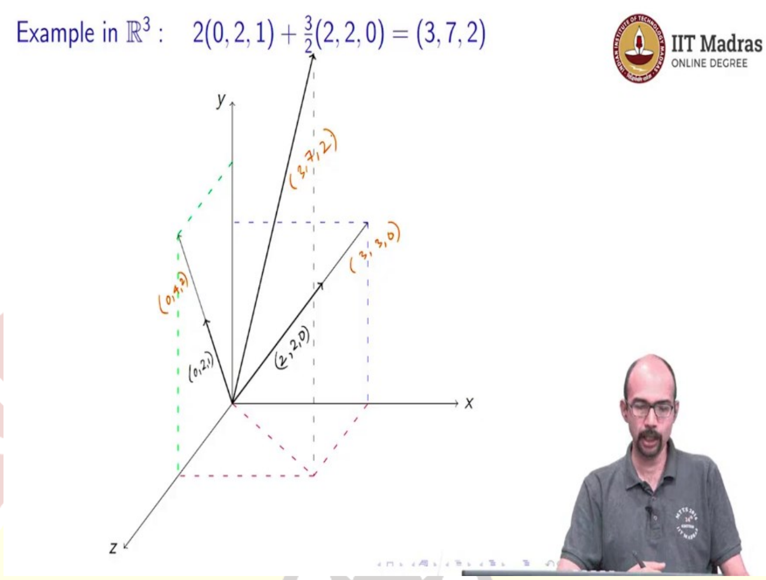

Let us do an example in R^3 So, in R^3 let us take these two vectors. So, we have (2, 2, 0). So, this is a vector (2, 2, 0) right here. So, this plane has been draw, I mean, this x, y, z axis has been drawn a bit differently than we usually draw it for the purpose of illustration. So, this is the XY plane. So, on this we have the vector (2, 2, 0).

On the left side, this is the YZ plane, on this we have the vector (0, 2, 1). And then we scale these. So, if you scale this, you get the vector (0, 4, 2). Here, we get the vector (3, 3, 0). And you use a parallelogram and you add them and what you get is (3, 7, 2). So, you can check that this is the, by projecting down to the exact plane what we get and by projecting to the y-axis and so on that is what these other dashed lines are indicating.

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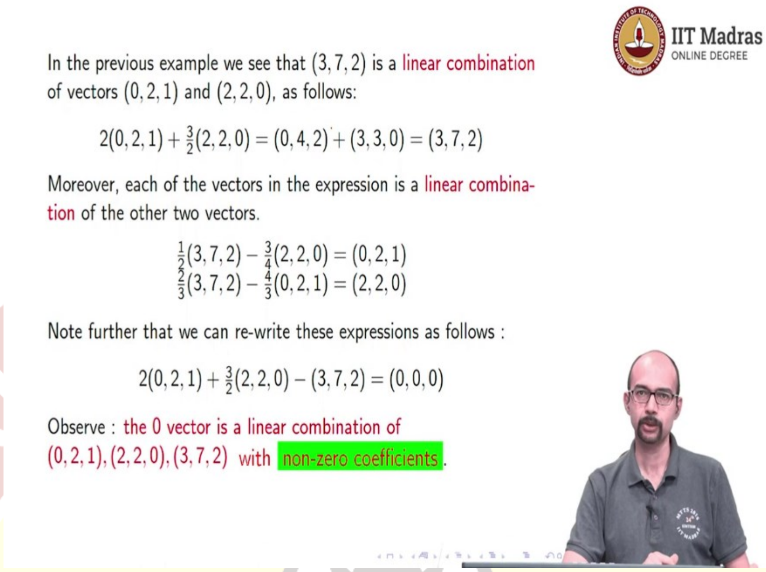

So, what does this mean in terms of linear combinations? So, in terms of linear combinations what we have seen is in R3 the vector (3, 7, 2) is a linear combination of the vectors (0, 2, 1) and (2, 2, 0) and the coefficients were 2 and 1.5. So, that is the expression. We just did that. So, we can again play the same game. So, instead of writing (3, 7, 2) as a linear combination of these two, we could instead write, let us say, (0, 2, 1) as a linear combination of the other two, (3, 7, 2) and (2, 2, 0). How do we do this?

Move the other term 1 to the right side and then divide by the corresponding coefficient or you could write (2, 2, 0) as a linear combination of (3, 7, 2) and (0, 2, 1) again same idea. So, I will suggest you check this. And then again I can do the same thing that I have done before. Namely, I can move the (3, 7, 2) in the original expression to the left hand side and get (0, 0, 0) on the right hand side. So, what is the net result that the zero vector is a linear combination of (0, 2, 1), (2, 2, 0), and (3, 7, 2) with non-zero coefficients. And again, I will encourage you to check that these three vectors lie on the same plane.

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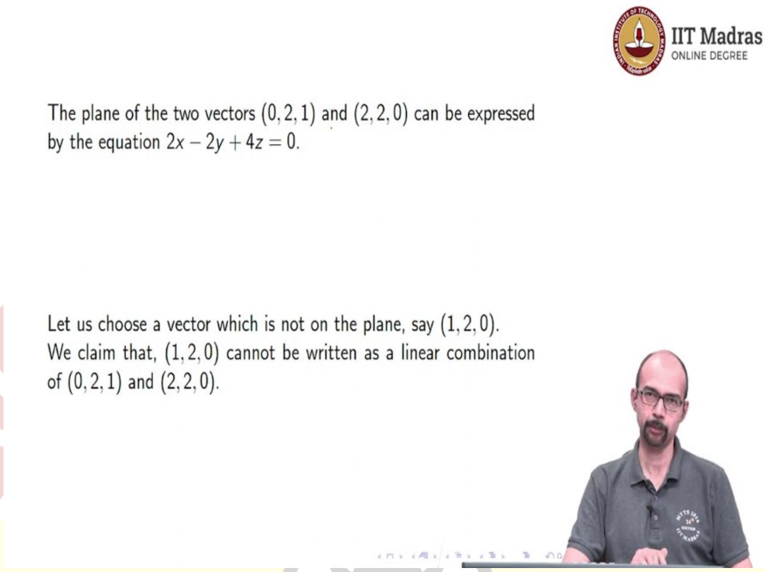

So, we can in fact get the equation of that plane. So, the equation of the plane containing the two vectors (0, 2, 1) and (2, 2, 0) is $2x-2y+4z=0$ And you can check that the third equation, the third point, indeed lies on this plane. So, these three lie on the same plane. And that is, so the geometry is somehow converting to this observation that if you take these three vectors, you can write 0 as a linear combination in a non trivial way, meaning with non-zero violations.

So, suppose we choose now a point or a vector which is not on the plane, say (1, 2, 0), so (1, 2, 0) is not on the plane. How do I check (1, 2, 0) is not on the plane, I plug in these values into the equation of the plane and get that it is not 0. So, $2\times1-2\times2+4\times0$ is not equal to 0. So, that is why it is not on the plane. So, we claim then that (1, 2, 0) cannot be written as a linear combination of (0, 2, 1) and (2, 2, 0). So, we saw the earlier example, that the third vector indeed was on the plane.

So, maybe we can go back and check what that was. So, it was (3, 2, 7) and (3, 7, 2) does satisfy this equation, because 2 times 3 is 6, plus 4 times 2 is 8. So, 6 plus 8 is 14, minus 2 times 7 is 0. So, let us look at (1, 2, 0). We saw already it is not, it does not satisfy that equation. So, it is not on the plane.

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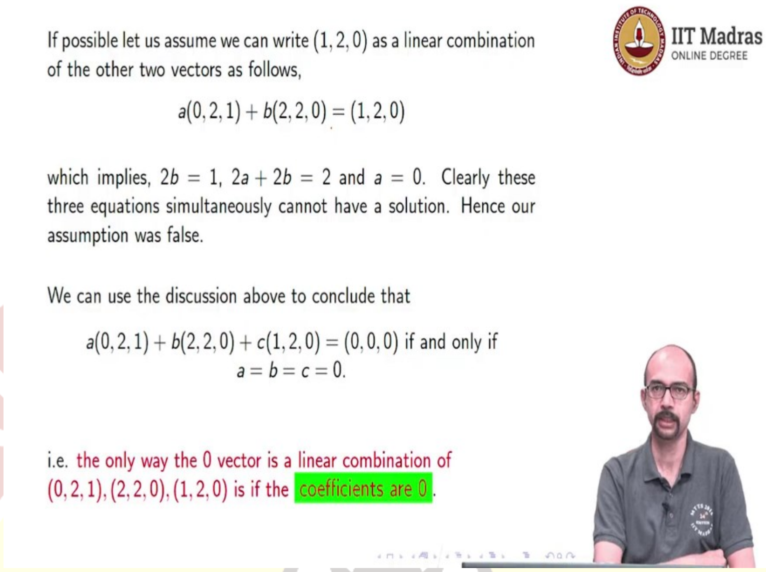

And now the claim is it cannot be written as a linear combination of (0, 2, 1) and (2, 2, 0). So, let us do this by contradiction. Suppose you can write this as a linear combination, so you have $a(0,2,1)+b(2,2,0)=(1,2,0)$. So, a and b are the coefficients. Then we can work out what are the expressions in each coordinate.

So, the expressions in the first coordinate is just 2b, the expression in the second coordinate is 2a+2b and the expression in the third coordinate is just a, and we equate it to the right hand side. So, if we do that, then we get $2b=1, 2a+2b=2,$ and a = 0. That means b is half and a is 0. That is what we get from the first and third equations. And if that happens, then 2a + 2b cannot be 2. So, that means there is no a and b which satisfies this equation. So, our assumption that there is an a and b is wrong. So, there is no such a and b.

So, what does that imply? That implies that this vector (1, 2, 0) cannot be written as a linear combination of (0, 2, 1) and (2, 2, 0). And we can equally well conclude from here and I will suggest that you check this that if you do $a(0,2,1)+b(2,2,0)+c(1,2,0)=(0,0,0)$ Suppose I asked, can I get a, b and c such that it is (0, 0, 0), so what the above discussion tells us is that the only possible choices for a, b, c are 0, 0 and 0, meaning all three of a, b and c must be 0.

So, what does this mean? So, this is now the other, well, the negation of the observation we did earlier. So, the only way the zero vector is a linear combination of these three vectors is if the coefficients are 0. So, in the previous examples, where those vectors were on the same plane, we could write 0 as a linear combination of those three vectors with non-zero coefficients. But here, there is no way of doing that. The only way of writing 0 is if your coefficients are 0 and that is a trivial here, because if the coefficient are 0 we certainly have, we get the 0 vector.

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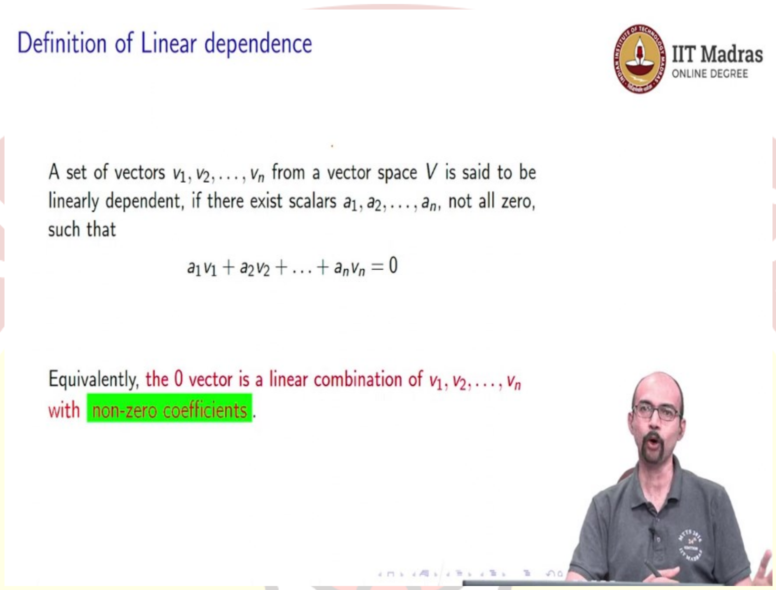

So, now we can define linear dependence. Keep this geometry picture in mind and keep the algebraic expressions in mind. A set of vectors $v_1, v_2, \ldots, v_n$ from a vector space V is said to be linearly dependent, if there exists scalars $a_1, a_2, .., a_n$ not all 0 such that $a_1 \times v_1 + a_2 \times v_2 + \ldots + a_n \times v_n = 0.$ So, the key point here is that these $a_1, a_2, .., a_n$ are not all 0. So, there is a way of writing 0 as a linear combination where the coefficients are not 0. At least some of them are not 0.

So, the equivalent way of saying this is that the zero vector is a linear combination of $v_1, v_2, \ldots, v_n$ with non-zero coefficients. Meaning, when I say with non-zero coefficients that means some of the coefficients are not 0. It is not, we do not need to say that all coefficients are non-zero, some non-zero coefficients. Fine.

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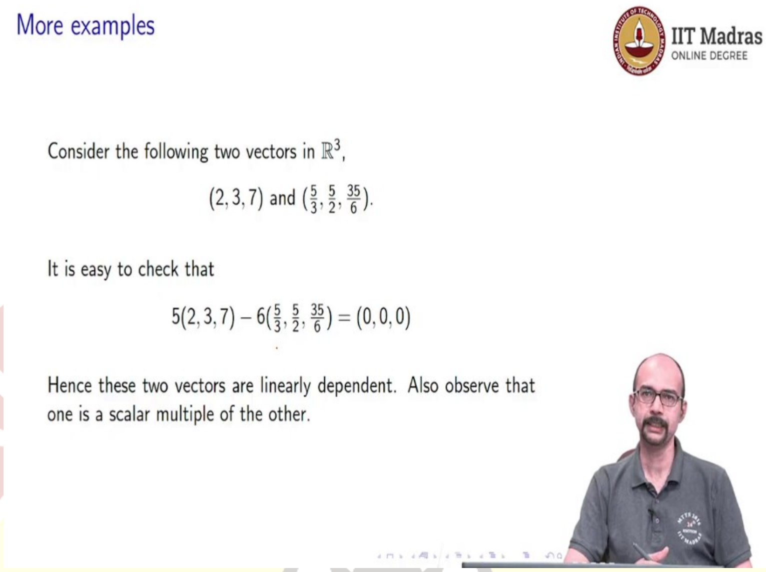

So, just to jog our memory, the first three examples we did, the two examples in R^2 and the third one in R^3 the three vectors we had were linearly dependent. That is what we saw, because we could write 0 as a linear combination of these three vectors where the coefficients were non-zero, meaning not all of them were 0. The fourth example meaning, where we had the vector (1, 2, 0) that example, the three vectors in question, are not linearly dependent. So, next video, we will see that they are what are called linearly independent.

So, consider the vectors (2, 3, 7) and $(\frac{5}{3},\frac{5}{2},\frac{35}{6})$ in R^3 So, you can check that if you multiply this second vector by 6 and you multiply the first vector by 5, then you get the same vector. So, in other words, if you do $5(2,3,7)-6(\frac{5}{3},\frac{5}{2},\frac{35}{6})=(0,0,0)$ So, at least one of these coefficients is non-zero. In fact, in this case, both of them are non-zero. And so these two vectors are linearly dependent. These two vectors (2, 3, 7) and $(\frac{5}{3},\frac{5}{2},\frac{35}{6})$ are linearly dependent.

And note that the way this worked is that they are scalars, I mean, they are scalar multiples of one another. If you draw $(\frac{5}{3},\frac{5}{2},\frac{35}{6})$ and if you draw (2, 3, 7), they will appear on the same line. That is clear, because you scale one of them by 6 and the other by 5. They give you the same vector.

So, in other words, when you have two vectors, what we are saying is if two vectors are linearly dependent, then essentially one is a multiple of the other.

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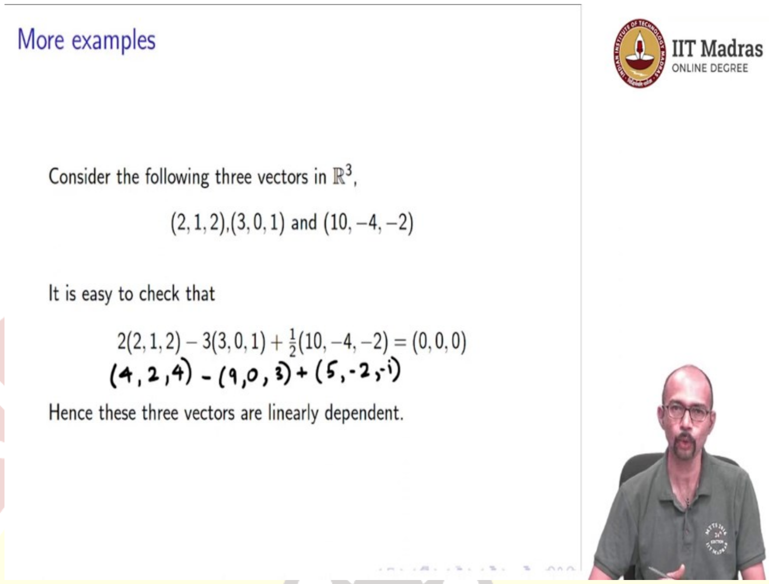

So, let us continue and study one more example. So, consider the following three vectors in R^3, (2, 1, 2), (3, 0, 1), and (10, -4, -2). So, here is an equation that we can check easily, $2(2,1,2)-3(3,0,1)+\frac{1}{2}(10,-4,-2)$ is the 0 vector. So, from here it follows that these three vectors are linearly dependent. Let us do this checking just to be sure. So, the checking here is $(4,2,4)-(9,0,3)+(5,-2,-1)$ Well, the first component is 4 - 9 + 5 that is indeed 0.

The second component is 2 - 0- 2, which is 0. And then the third component is 4 - 3 - 1, which is indeed 0. So, this equation is indeed correct. And this tells us that there is a linear equation satisfied by these three vectors with non-zero coefficients which equates to 0. So, they are linearly dependent.

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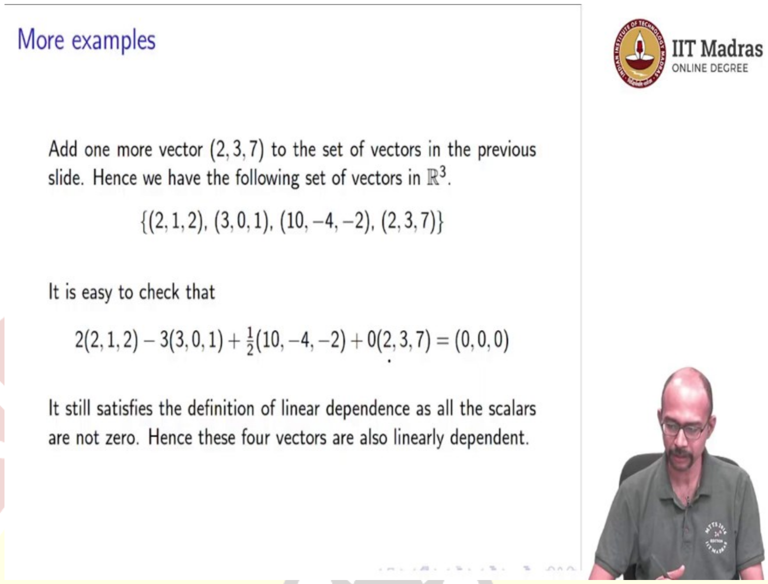

Let us continue and append a vector to this set. So, the set is now (2, 1, 2), (3, 0, 1), (10, -4, -2), and (2, 3, 7). So, we have appended one more vector to this set. And we can ask, is this set now linearly dependent? Is this a set of linearly dependent vectors? And the answer is, well, certainly it is. Because since the original set, meaning the set with three of them was linearly dependent, we can write down the same equation, but we also have to include the new vector (2, 3, 7).

So, what do we do, we put the 0 coefficient for that. And we of course, still get (0, 0, 0). So, you can, if you are not convinced, you can do this checking by yourself. And from here it is clear that this set is linearly dependent.

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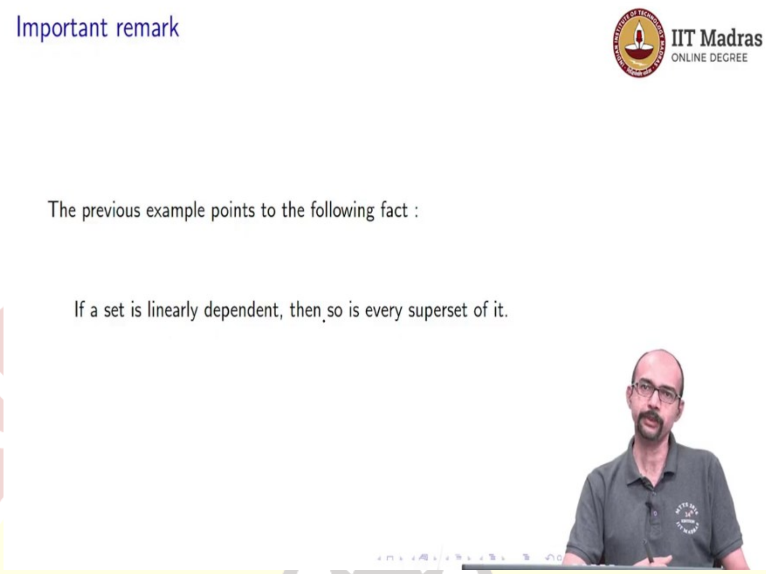

So, what is the point? The point is this we get the following fact. I mean, this is by observation, but you can clearly see because the proof is the same. If a set of vectors is linearly dependent, then so is every super set of that set. So, $(v_1, v_2, \ldots, v_n)$ is linearly dependent, then $(v_1, v_2, \ldots, v_n)$ and you add one more vector w, then $(v_1, v_2, \ldots, v_n, w)$ is also linearly dependent. You can add as many vectors as you want. They are all linearly dependent.

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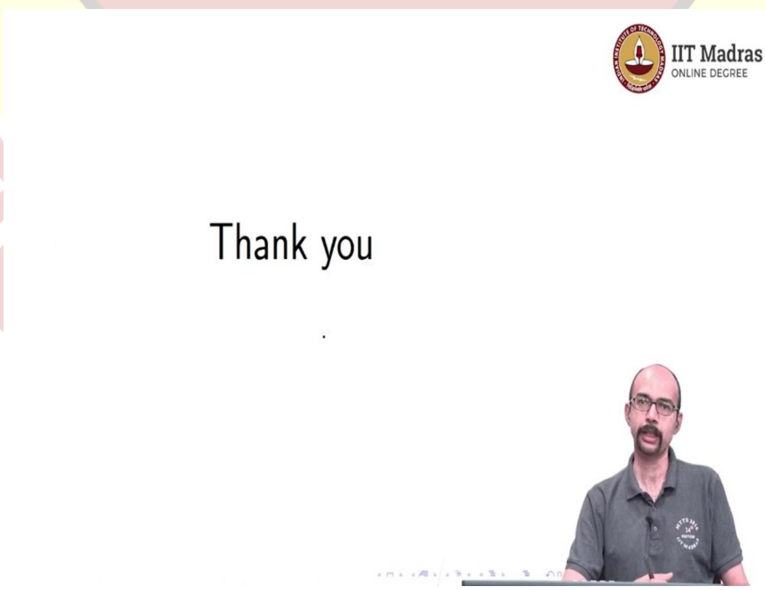

So, I guess that finishes what, the material in this video. Let us quickly recall that we have seen in this video that when a set of vectors in a vector space is linearly dependent, and importantly, the, in the case of R^2 or R^3 what this means is that if you have two vectors are linearly dependent, they must be on the same line. If you have three vectors are linearly dependent, they must be on the same plane. This is the geometry of what is going on.

And finally, to check these things, we have to solve some equations. So, finally, this kind of checking is also related to solving systems of linear equations. So, thank you.