# Lengths and angles

Sarang S. Sane

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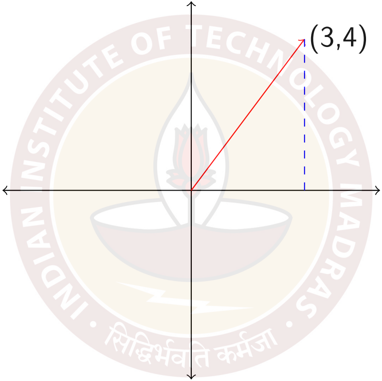

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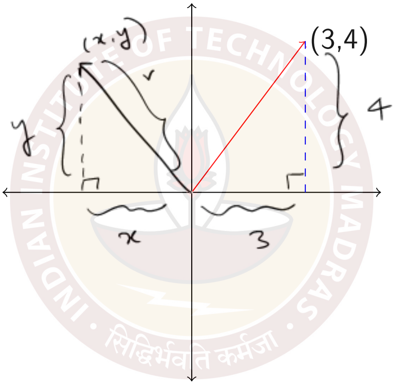


Using Pythagoras' theorem, the length of the vector $(3, 4)$ is $\sqrt{3^2 + 4^2} = 5$ units.

# The relation between length and dot product in $\mathbb{R}^2$

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More generally, the length of the vector $(x,y) \in \mathbb{R}^2$ is $\sqrt{x^2+y^2} = \sqrt{(x,y)\cdot(x,y)}$.

$$\downarrow$$

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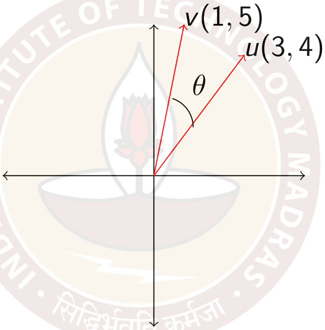

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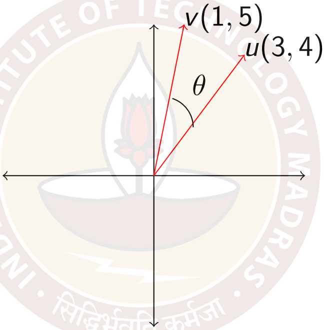


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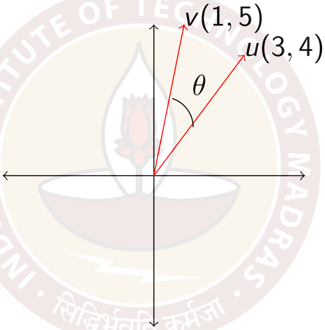




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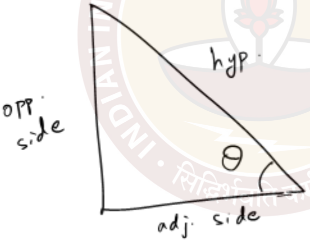
- The angle is often described by computing its trignometric functions (e.g. *sin*, *cos* , *tan* ).

# The dot product and the angle between two vectors in $\mathbb{R}^2$

Let $u$ and $v$ be two vectors in $\mathbb{R}^2$. Then we can compute the angle $\theta$ between the vectors $u$ and $v$ using the dot products as :

$$cos(\theta) = \frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}} \quad i.e. \quad \theta = cos^{-1}\left(\frac{u \cdot v}{\sqrt{(v \cdot v) \times (u \cdot u)}}\right).$$




$$\cos(\theta) = \frac{\text{length of adj. side}}{\text{length of hypotenuse}}$$

$$\sin(\theta) = \frac{\text{length of the opp. side}}{\text{length of hypotenuse}}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{length(opp.)}}{\text{length (adj.)}}.$$

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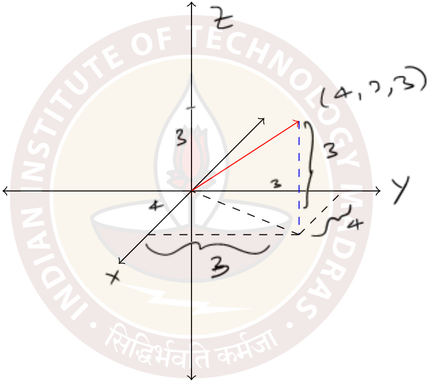

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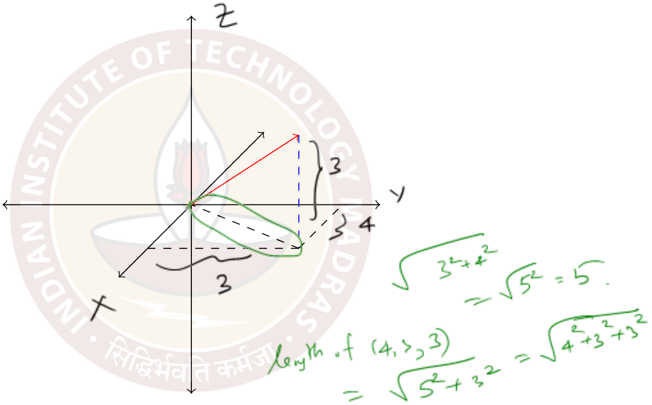


By using Pythagoras' theorem, the length of $(4, 3, 3)$ is $\sqrt{4^2 + 3^2 + 3^2} = \sqrt{34}$ units.

# The length and dot product in $\mathbb{R}^3$

Observe that $(4, 3, 3) \cdot (4, 3, 3) = 4^2 + 3^2 + 3^2$ and hence the length of $(4, 3, 3)$ can be expressed as the square root of dot product of the vector with itself.

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More generally, the length of the vector $(x,y,z) \in \mathbb{R}^3$ is $\sqrt{x^2 + y^2 + z^2} = \sqrt{(x,y,z) \cdot (x,y,z)}$.

# The angle between two vectors in $\mathbb{R}^3$ and the dot product

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Similarly, the angle between between $(1,0,0)$ and $(1,1,1)$ is $cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

$(1,0,0)\cdot(1,1,1) = 1$, $(1,1,1)\cdot(1,1,1) = 3$. $(1,0,0)\cdot(1,0,0) = 1$.

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{1\times 3}}\right) = \cos^{-1}(1/\sqrt{3})$.

# Dot products in $\mathbb{R}^n$ : length and angle

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Thank you