IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 01
Some Toπcs from Maths 1

Hello and welcome to the Maths 2 component of the online B.Sc. program on data science and programming. In this video, we are going to do a review of the, some of the ideas you have learnt in Maths 1. So, you have seen some ideas of calculus already in Maths 1. At least you have seen the ideas of functions and function of one variable and then the ideas of tangents and so on. So, you have seen some of the geometry and you have seen some examples of functions. So, the first couple of videos, including this one, we are going to do a little bit of a review of what you probably should have seen in Maths 1.

So, if you do not remember, this will help you to remember and also we may end up doing a little bit extra in each of those videos and then once we have refreshed ourselves with some of the basic concepts of, related to functions and the geometry corresponding to those, the graph and so on, we will go on from there and study the idea of, the idea, how to capture these ideas in terms of calculus.

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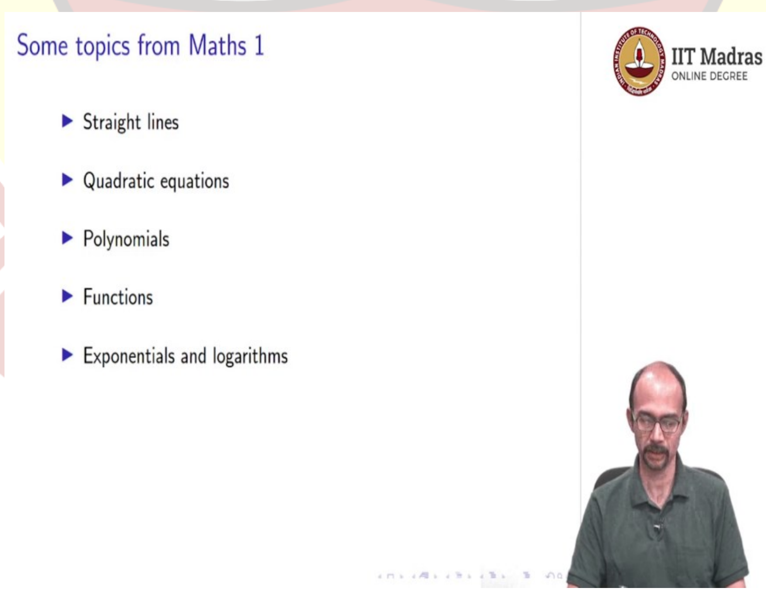

So, let us quickly look at some toπcs that you have seen in Maths 1. So, you have seen what is a straight line in Maths 1. We will be reviewing it. I am just sort of quoting the toπcs right now. You have seen what are quadratic equations, you have seen what are polynomials in general, you have seen what is the function. So, in particular, this includes functions from real numbers to real numbers. You have also seen other kinds of functions, but those will not show up in this particular course.

Well, I should take that back. What I meant was you have seen other functions, which are, which have, which are on natural numbers and so on. Those kinds of functions will not be studied here. But we will study other kinds of functions on Rn and so on and then I think you have also seen what are exponentials and logarithms. So, this is again something that we will review in this, in these couple of videos that we are going to do.

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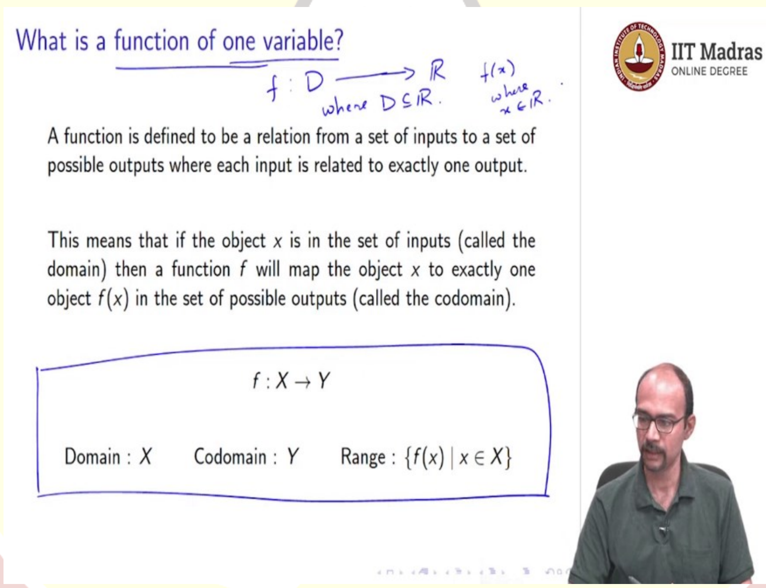

So, let us start with what is a function of one variable. So, a function is defined to be a relation between, a relation from a set of inputs to a set of possible outputs, where each input is related to exactly one output. This is a slightly cumbersome definition, but in the minute we will make more sense of that. In fact, this is not exactly a definition, but a heuristic about what a function is.

This means that if the object x is in the set of inputs, which is called the domain, this now you may remember, then a function f will map the object x or take the object x or change the object x or translate the object x, you can use any of these words or transform the object x to exactly one object $f(x)$ in the set of possible outputs, this is called the codomain. So, this is a very general idea of a function and so, maybe at this point, we should say not function of one variable, but what is a function. So, this is about what is a function.

So, if you have *f:* X to Y, so pictorially this set of sentences, these sentences are represented by this picture here, f X to Y, which says that f is a function from X to Y, X is the domain, Y is the codomain and then we have something called the range of a function. This means it is a set of values that f takes. In other words, it is a set of values, it is the set, the subset of Y of the codomain such that there is some X such that f of x is equal to that particular value.

So, now what is the function of one variable? We have said something about one variable. So, a function of one variable means we are thinking, so for the purpose of this video and next few videos, when we say function of one variable, we mean a function f from $\mathbb{R}$ or some subset of $\mathbb{R}$, need not be the entire $\mathbb{R}$, so a subset of $\mathbb{R}$ to $\mathbb{R}$. So, maybe I should draw this better. So, a subset D to $\mathbb{R}$, where D is a real, a subset of the real numbers. So, Y is this a function of one variable, because typically we represent this by $f(x)$, where x is a real number, hence one variable.

If you draw the function of, in, where D in $\mathbb{R}$2, then we would say it is a function in two variables and so on. So, those are things we will study in the next coming few weeks. But for now, we have the domain which is a subset of $\mathbb{R}$ and we have the core domain, which is again, the set $\mathbb{R}$. So, we know examples of such functions.

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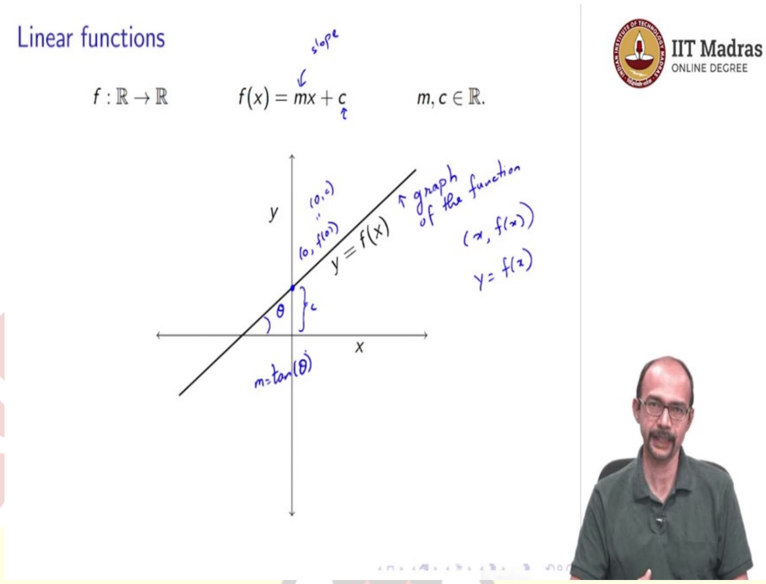

So, for example, we have seen in Maths 1, the idea of a linear function. So, a linear function looks like $mx + c$. So, m and c are real numbers and then when you evaluate this, for every x, it gives you some real number. So, for example, if you have $f(x)$ is $5x + 2$, then $f(0)$, when you evaluate this at 0, you get f of 0 = 5 ×0 + 2, so that is 2 and then $f(20)$ = 5 ×20 + 2. So, that is 100 + 2, so 102.

So, I hope you remember what is a linear function and there is a picture associated with it. This is the graph of that function. So, you plot the point x and you plot the corresponding point $f(x)$ on the y-axis. So, you plot the points x, $f(x)$ and in other words, this is the line $y = f(x)$. It is a line, the bold line in black which is drawn here. So, this is called the graph of the function. So, this is the graph of the function and one reason to call it a linear function is that the graph is a line.

So, linear has something to do with the fact that this is a line. So, for a linear function, we have fx is $f(x)$ is $mx + c$ and here m is called the slope of this line and it has something to do with this angle, which you may have seen in Maths 1 and this is supposed to represent the y intercept. So, if you if you put x =s 0, you get $f(0)$ = c. So, this is supposed to represent, so $f(0)$ is over here. This point here is 0, $f(0)$, and we just saw that $f(0)$ is 0. This is a point 0, c. So, c is exactly this length here.

So, this m is supposed to have something to do with the angle. So, this is theta and m is supposed to be tangent of θ and if you have not seen this before, we will be studying this in the next video. The point I wanted to make is that the m and the c have something to do with the geometry of this length.

So, one represents the slope, by the slope we mean the angle at which it is inclined to the x-axis, and the other determines the distance where it hits the y-axis from 0. So, when I say a distance, I mean, it could be a positive distance, which is in the positive side of the y-axis or it could be a negative distance, which is in the negative side of the y-axis. So, you have seen this idea of a linear function and this is the graph of such a function. It is a line.

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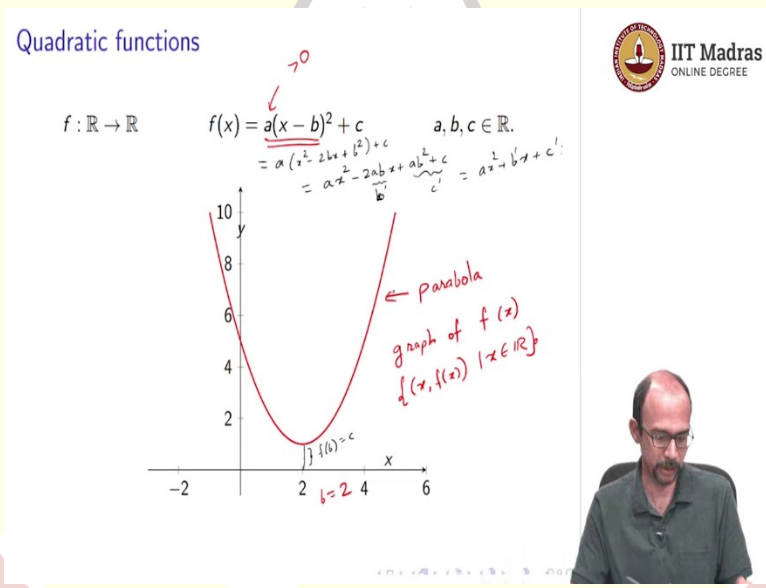

Let us recall what are quadratic functions. So, a quadratic function is a function of the type a ×x - b^2 + c. So, we could expand this and I will write it in a minute how to, how it looks like and here is an example of a quadratic function. So, the quadratic function typically looks like it has been drawn here. This is called a parabola and that is the finger in red. This is called a parabola, maybe I should use red.

And this is the graph of the function f, which means it is a set of values x, $f(x)$ as x runs over the real number. Of course, we have not plotted all the values. If you plot all the values or these two arms of the parabola will shoot off. So, this is an example of a curve. So, this is not a line. So, typically things which are not lines, but have this kind of shape of single, you can think of it as being drawn by without raising your pen or pencil, such things are called curves.

So, a line is a particular case of a curve. But if you have a line, then you usually do not use the word curve. You use the word line. So, when you say curve, typically you mean that it is something which is not a line, but which you can draw by a stroke of your pencil or pen and of course, when you draw such a thing, in principle, you could draw more complicated figures or curves are more general than, so graphs are particular kinds of curves. So, curves, this could also be a curve, but this is not a graph.

So, this is the graph of $f(x)$, with some particular values for a, b, and c. So, again, how do I read of, what the values of a, b and c are from the picture or can I say something about this picture. Where you can see that the, smallest value of this $f(x)$ is taken when $x = b$. That is when this part contributes the least which is 0 and I mean, I, so when you have drawn, so the first thing is when the graph is like this, this a is going to be positive. So, this a is positive if the graph is a parabola like this and a is negative if the graph is, a parabola like this.

And then further, so once it is greater than 0, then the smallest value of this function is taken when x = b. So, in this particular case here $b = 2$. And then once we know $b = 2$, you substitute x = b, then you get $f(b) = a \times b - b^2 + c$. So, b -b^2 is 0. So, that gives you $f(b) = c$. So, in other words, this distance here, maybe I will be drawn this in some other color.

This distance here is exactly $f(b)$ is c. So, in this case, it is $f(2)$. So, again, we see that these b and c and a have something to do with the geometry of the parabola, the geometry of the graph and we can go back and forth. From the graph, we can determine the a, b and c and from the a, b and c, once we know a, b, and c, we know the function and hence we can draw the graph.

So, there is an interplay here between the algebra, which is the functional form a ×x -b^2 + c and the geometry, which is the actual picture that you can see and what we are going to do more and more is try to understand how these two relate to each other. That is the sort of goal of this part of what we are doing.

So, along the way we are going to, so the way to do this or one way to do this is to study calculus, which is what we are going to do. So, let me expand this and also see what we get. So, this is the a $\times x^2$ -2bx + b^2 + c, which if you write as ax^2 -2abx + ab^2 .

So, I can use some other notations for a, b and $ab^2 + c$. Maybe I will call this a prime, b prime and maybe I will call this c prime. So, I can write it as ax^2+ b prime x + c prime and so the other way of thinking of this is that, this is a polynomial in x of degree 2. So, this is something that you may have seen earlier and we are going to see very shortly.

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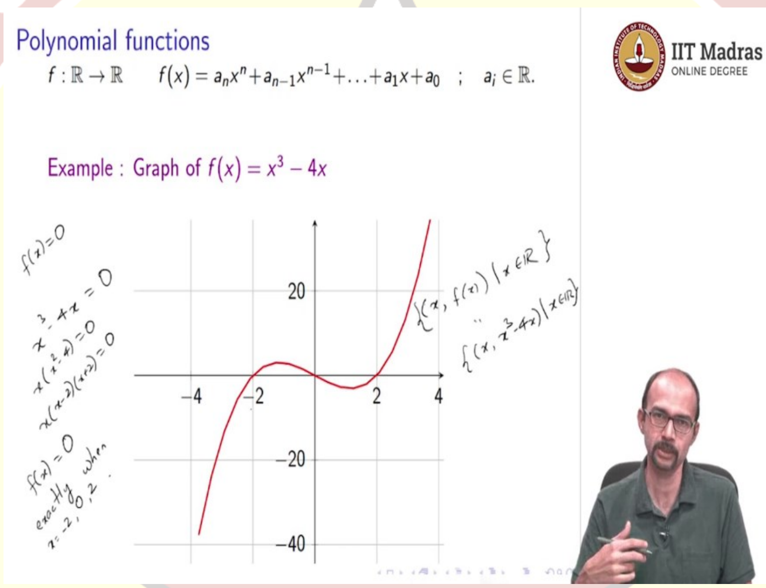

So, what are polynomial functions? So, a function f from ℝ to ℝ, you define $f(x)$to be an $\times x^n$+ an -1 $\times x^{n-1}$, all the way up to a_0. So, this is a polynomial of degree n and these ais are the coefficients of the polynomial and they belong to the real numbers. So, the example is, so the function here, you can look at x cube -4x and here is the graph. So, this is where you are plotting the graph x, $f(x)$. So, in other words, where x runs over ℝ. This is at set.

So, in this case, this example, this is x, x^3 $-4x$ as x runs over ℝ. That is what this red line, red curve is giving you. So, this is a function and it is a very nice symmetric function. I mean, not symmetric about the x or the y-axis, but symmetric in some axis and you can see that as x becomes large, this function is shooting off to ∞ and as x becomes small, the function is becoming smaller and smaller and shooting off towards what is called -infinity.

Well, what else do we get from the picture? We can see that there are three points where it becomes 0. So, how do we find those three points? Well, you solve this equation? So here is where the algebra kicks in. So, to get those points for which $f(x)$. is 0, so $f(x)$. is 0 that means x cube -4x is 0. It means x $\times x^2$-4 is 0. That means $x \times x - 2 \times x + 2$ is 0. So, this happens, so $f(x)$. is 0 exactly when x is -2, 2 or 0 or maybe I should just -2, 0 or 2 + 2 be just for the symmetry to sort of be clear.

So, -2, 0 and 2 give you the 0s of this polynomial. And so in general, of course, a polynomial may not have 0s and we will see this, such a polynomial later on, maybe not in this video, but in the next one and then we can use this, the form of the polynomial to get some interesting properties of the graph.

So, this is something that we saw for linear and quadratic functions that the coefficients had some role in the geometry of the graph and similarly, over here also the coefficients have a role in the geometry of the graph. So, such things are going to come on, come in and this is one of the starting points of calculus. We will study more as we go ahead.

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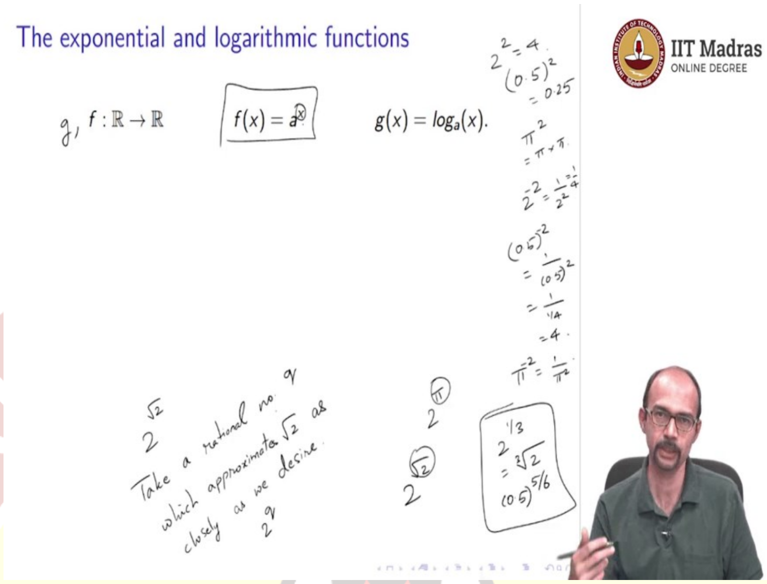

Well, let us look at the exponential and the logarithmic functions. So, this is maybe something that was touched on in Maths 1. So, I have missed a g. So, f is a function from $\mathbb{R}$ to $\mathbb{R}$ and g is also a function from $\mathbb{R}$ to $\mathbb{R}$. So, f is a function from $\mathbb{R}$ to $\mathbb{R}$ and g also a function from $\mathbb{R}$ to $\mathbb{R}$. And what is f, f is defined as a to the power x. So, what is the meaning of a^x? This is a question in itself and when a and x are both, well, at least when x is an integer, the value is clear.

If I say 2^2, then the value is clear. It is 4 or even if I say, 0.5^2, then the, we know how to compute this. So, this is 0.5 ×0.5 which is 5/ 10 multiplied by 5/ 10 or half multiplied by half, which is 1 fourth, which is 0.25. But I could, if I do something like π^2, well, this is just π^2. It means you take the number π and multiply by itself. So, this is $\pi \times \pi$ and we, if we want to know what it is, we can approximate it to as good a value as we want. So, we still have some understanding of what are numbers like π^2.

But if you change the exponent, so here this is the exponent. So, if you change the exponent to a number which is not an integer, so if you have something like 2 to the power -2, I still know what, this is $\frac{1}{2}^2$, which is 1 fourth or if I have 0.5^{-2}, then this is $\frac{1}{0.5}^2$ which we know is 1 by 1 fourth, which is 4 and similarly if I do π^{-2}, this is $\frac{1}{\pi^2}$, which again we can approximate to whatever degree of precision we would like.

So, we understand how to do this process when we have integers. What happens when we do not have integers? For example, if I take a rational number, so if I take 2 to the power 1 third, what does this mean? Well, so the idea here is that this is the cube root of 2. So, what this means is this is some number such that when you take the third power, you get 2 and does such a number exists, indeed it does, that is something that needs proof. We are not going to get into such things, but this is something I think you can believe.

Similarly, on similar grounds lines we can now answer things like what is 0.5 to the power 5 sixth, because we know what is 0.5 to the power 5 and then we have to evaluate that to the power 1 by 6, so it is that number such that when you take the sixth power you get 0.5 to the power 5. So, for rational numbers also, we know how to understand a to the power such a rational number. So, but now we are saying you look at the function $f(x)$ is a^x, where x is running over all real numbers. So, we also have to deal with things which are not rational.

For example, if you take $2^{\sqrt{2}}$, what is the meaning of such a thing? So, this is a little tricky. We have to really scratch our brains about what is the meaning of this and still I think with a little bit of thought, which I leave to you, you should be able to figure out what is 2 to the power root 2. What may be really not so easy is what is 2 to the power something which is even worse, something which is irrational. So, not just irrational, but not even of this form, like root 2, what to say 2 the power of π.

So, this you have to understand by understanding that when you have things like these in the exponent, what you do is, you take, so suppose I want to define $2^{\sqrt{2}}$, what I do is I take some numbers which are rational numbers. So, take rational numbers, which approximate $\sqrt{2}$ as closely as you would like or maybe let me change this to say, take a rational, take a rational which approximates $\sqrt{2}$ as closely as we desire. Let us call it n or maybe that is a bad notation, rational number cube, which approximates $\sqrt{2}$ as closely as we desire of.

So, for example, if you look up $\sqrt{2}$ in a table, you will be able to see it is one point something, something and then you can write it as approximately equal to something by 100. So, that is a rational number. So, one point something, something is a rational number and then you can do one, 2 to the power that rational number q, which we know how to do. This was exactly what we discussed over here.

So, we understand how to do 2^q. So, is very closely approximated by 2 to the power q and if you want a better approximation, you choose a rational number which is closer to it and that is how you define things like $2^{\sqrt{2}}$ or even worse things like 2^{π}. So, using this you can make sense of $f(x) = a^x$ that was the real point. How do you define for every x, what does a^x, you do it in this way.

If x is an integer, we know what it means. It just if x is 2 then it is a^2that is a ×a if x is 10. It is a^{10}, which means you multiply a to itself 10 times. If x is -5 that means it is $(\frac{1}{a})^5$ and so on and if now, if x is a rational number, you raise a to the power whatever there is in the numerator and in the denominator, you are taking that root, that a-th root. So, if x is m by n, where m and n are integers, then you are looking really at a to the power m which we know what it is, the nth root of that. So, it is the nth root of a to the power m. So, it makes sense.

And now for any other number, we can define it by getting rational numbers as close as we want to that. So, that is how we define this and what is the logarithm? Logarithm is the reverse process. When you reverse out the exponential function, you get logarithm. So, I am not defining these carefully in terms of the mathematics involved. But I want you to understand that even the definitions here, they are a little involved when you have to actually formally define them. So, there is some work to do.

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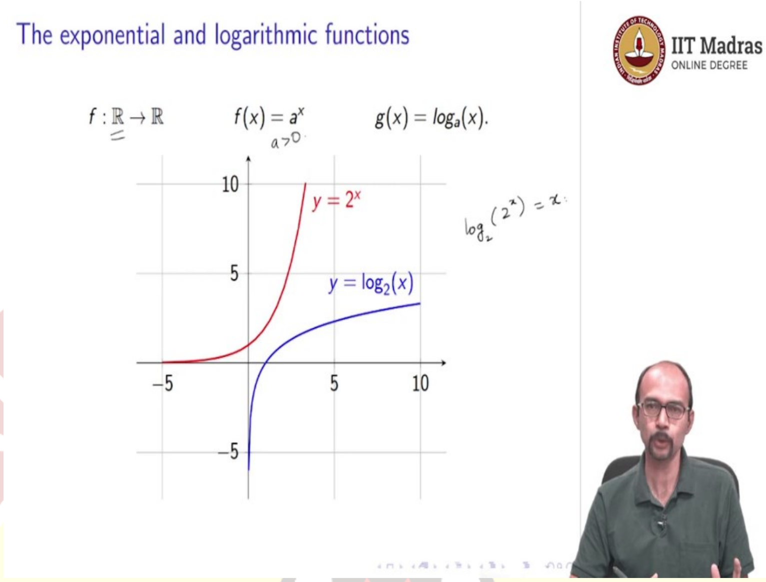

So, here is two examples, y is 2 to the power x. So, that is how this function looks like. So, the exponential is a very rapidly growing function. So, this is the exponential function with respect to 2, and this is the logarithm function. So, the logarithm function is also increasing, but it increases very slowly.

They are exactly sort of, they play the opposite role to each other. So, I hope these πctures shed some light on what these functions are and I have to point out one thing, a is in whatever I said, this a here better be a positive number, because if you have a negative number, then we are going to really struggle when we take roots and when we, when, if it is 0, then when you divide, you will have trouble.

So, for this to make sense on the entire domain R, a better be a positive number. So, here a is greater than 0. Similarly, the logarithm is defined only on that part, because it is the sort of reverse function of a to the power x. That means, and you can see a to the power x takes values which are only positive, a is positive. That means the logarithm takes positive values and gives you values across the real line. That is what it means.

So, here is the function $y = \log_2 x$. So, when we do it with the, with a particular base, when a is a particular number, that it is with respect to that base and what is the relation between these two. The relation between these two is that if you take logarithm of, with respect to 2 of 2^x then you exactly get back x. This is the relation between them. So, this is called composing the exponential and the logarithm and we will see what composing means in the next video. So, I hope this picture is again little helps you to understand what is happening.

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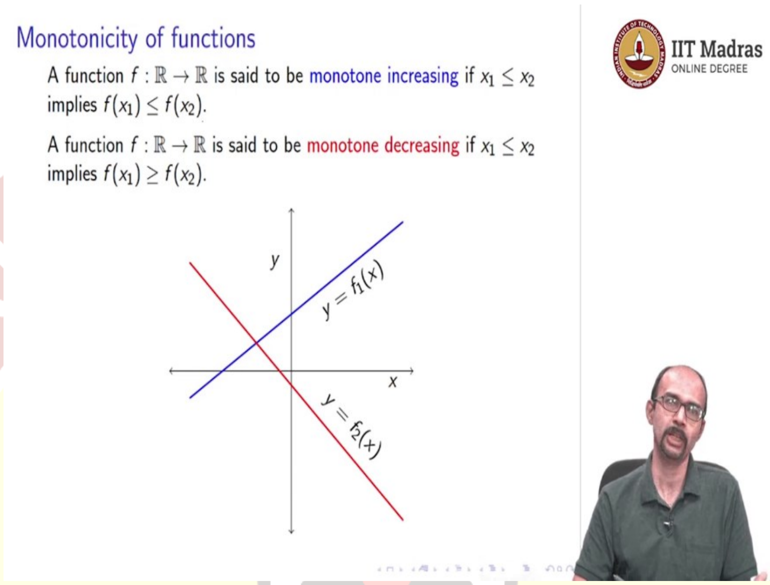

I am not defining the exponential and the logarithm in a technically sound manner. A function f is said to be monotone increasing if, well it is increasing. I mean, you, I hope increasing should make sense. As x increases, $f(x)$ increases. So, it says, if $x_1 \leq x_2$, then $f(x_1) \leq x_2$ and similarly, it is monotone decreasing, if the opposite happens, that as x increases, $f(x)$should decrease. So, sometimes, we also say monotonically increasing or monotonically decreasing. They mean the same thing.

So, here is an example of a function which is monotonically increasing. So, this is similar to the line that we saw in the one of the previous slides. So, this line is increasing, an increasing function, namely, as x is increasing, the value of $f(x)$ is increasing. That is what is demanded by this definition here, in this definition here and the line in red $f_2(x)$, that is a monotonically decreasing function.

So, here as x increases, the corresponding f value decreases. So, these are definitions that are worth keeping in mind, because eventually we are going to study maximum values and minimum values and so on. So, we should know where functions are increasing and decreasing. So, in particular, what are increasing and decreasing functions.

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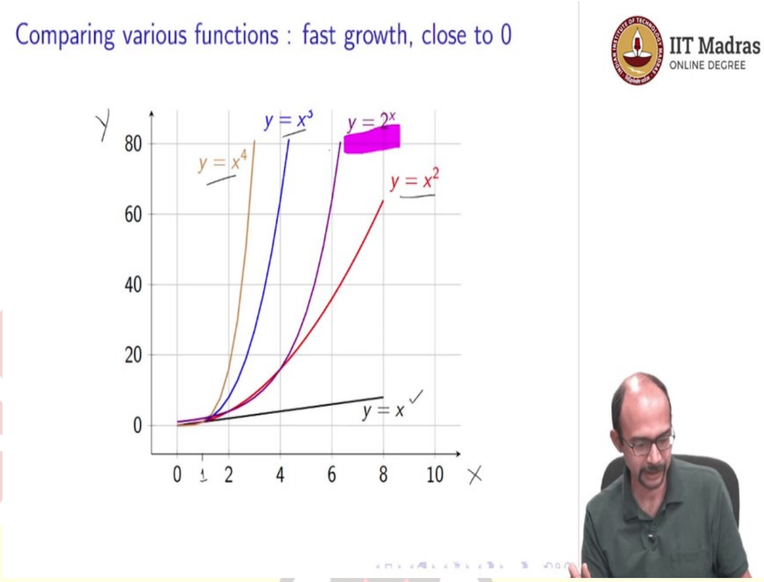

So, let us compare various functions. So, here is a comparison of functions, which are like powers an exponent function. So, I will point out that this function here is the exponential function and all the others are monomial functions, y is x, y is x^2, y is x^3, y is x^4 and as you can see, by the way, even before I go there, the axis here, the x-axis here and the y-axis, the labeling is a bit, I mean you should be careful that here the steps are 0 to 2, 2 to 4, 4 to 6 and so on, whereas here the steps are of order 20.

So, this is y is x should, I mean, if both sides were drawn in an = way, it should have been a line of with 45 degrees angle, but because of the, because we wanted more values on the y side, I have shrunk it and that is why, $y = x$ does not appear as 45 degrees, but appears like this, because it, as x increases, it increases. So, I have drawn 0 to 8 and y also goes from 0 to 8. But here we have 0 to 20, so and 20 to 40 and so on. So, keep this in mind.

And why did we do that, because the other functions are very rapidly increasing. So, the picture looks better like this. So, when we are, x is small, you can already see that as your monomials grow, so the exponent in the monomial grows. So, y is x. This is y is x and then this is y is $x^2$and then this is y is x cube, this is y is x to the power 4. So, you can see that $f(x)$ is increasing as the exponent grows after a point.

So, initially, of course, something else happens. Y is x is actually larger up till 1. So, up till 1, the others are below. But at 1 they change, they all become = 1 and then after that x^2starts going faster than x and x^3 even faster and x^4 even faster and then we have this exponential function 2^x, which is also going quite fast. As you can see, it started above 0 first of all and then it was caught by almost all the functions except y is x.

So, it always beats y is x. But then after some time, y is x^2 beat for a short time, but after that y is 2^x suddenly started taking off with respect to x^2 and you can see the way it is changing, it will change and change and change and what happens if you increase your scaling further.

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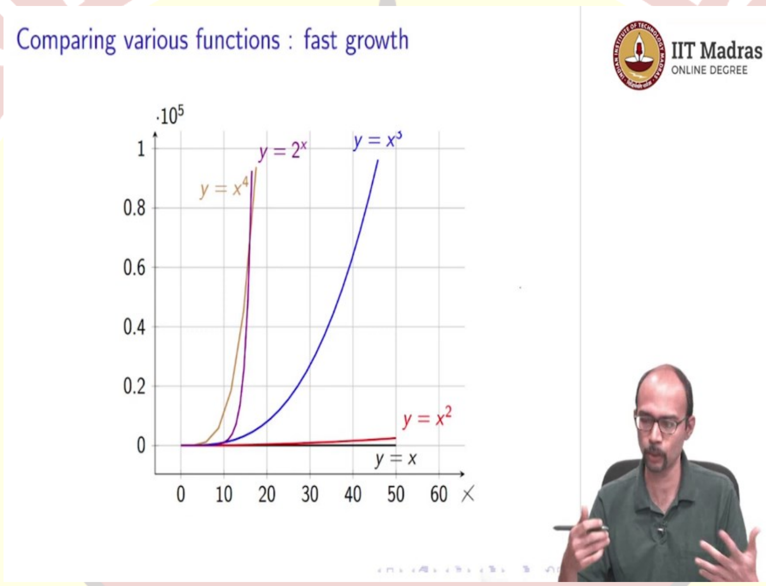

So, now we have 0 to 60 on the x-axis and this is 10^5, so which means 10 cube is 1000, so this is 1,00,000. So, here we have plotted values like 20,000, 40,000, 60,000, 80,000 and 1,00,000 and now you can see how raπd this function y is 2^x is. This is $y = x$. On this scaling it almost looks like a flat line and even y is x^2 looks somewhat flat, but it is not actually flat.

If you draw the scaling nicely, then it looks like our parabola. We have seen this parabola figure before in our previous slide and y is x cube is fairly fast, even in this scaling. And then you have y is x^4, which is this brown line. But now if you look at y is 2^x, you see that it further tilts and it beats y is x^4. So, it goes faster than y is x^4.

And the point I want to make here is that, if you have y is x or y is x^n, then as n increases these functions sort of go towards infinity more and more rapidly. But all of them are eventually overtaken by the exponential function. The exponential function is a terrifically fast growing function after something. After sometime it is always going to beat any polynomial. So, this is what this picture is trying to tell you. So, this is just intuition. This is not strictly required for our course, but it is good to know how these behave.

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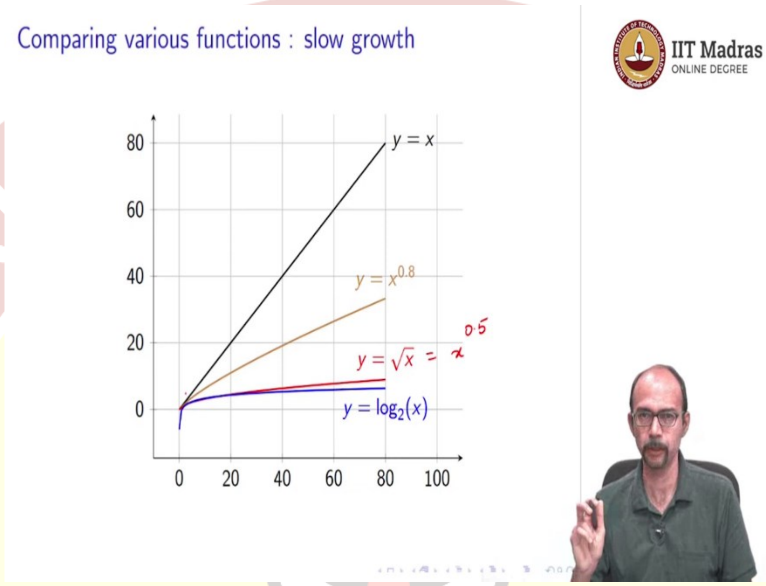

And then on the other side, we have slow growing functions. So, here is y is x, here is y is $x^{0.8}$, which is the line in brown, not the line, it is y is x is actually a line, but y is 0 point, x to the power 0.8 is actually a curve, but it is a very slow growing curve. So, it might look somehow what linear and then you have y is $\sqrt{x}$ which is, you can otherwise write as $x^{0.5}$

So, you can see that is below $x^{0.8}$ and then you have the logarithm function, which towards, I mean, it is defined only on the positive side and it starts sort of very, very far below, and then slowly it comes up and then it changes its direction and it starts going very slowly and the same thing that happened for the exponential happens here, but in the opposite direction, namely that the logarithm goes slower than any polynomial. So, that is something you should keep in mind just for your intuition.

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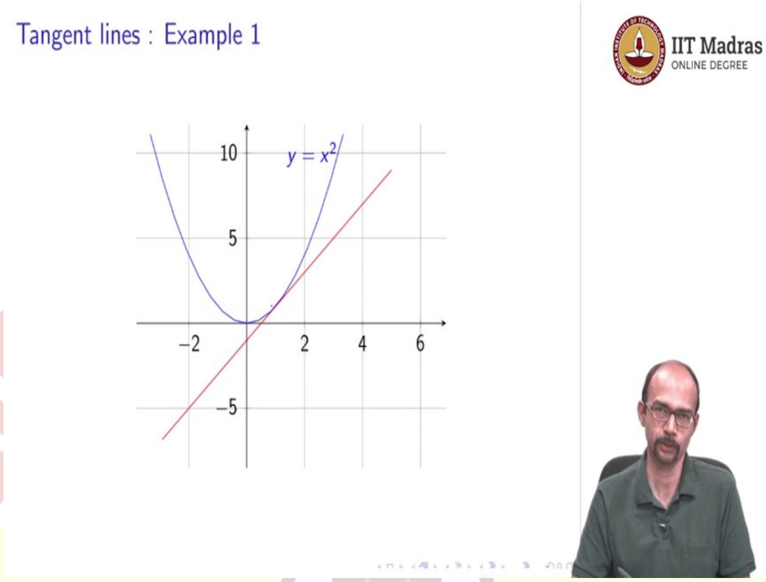

So, let us close with some final ideas of things you may have seen in Maths 1, namely what are tangent lines. So, tangent lines, if you remember, are lines which one obtains when you take a line which intersects the curve that you have. It may intersect in many points, but then you start moving it slowly parallely until maybe sometimes what happens is it intersects the points in which it intersects all sort of come together. So, it comes to a single point and so if a line intersects a curve at exactly one point, such a line is called a tangent line.

So, this is an example of a tangent line. So, this is the parabola y is x^2 and this is a tangent to that parabola at the point 1, I think and we will soon understand using calculus, how to find the equation of such a line. So, we know that such a line is given by $y = mx + c$. So, we will ask how do I find the equation of such a line and we will use calculus to study that.

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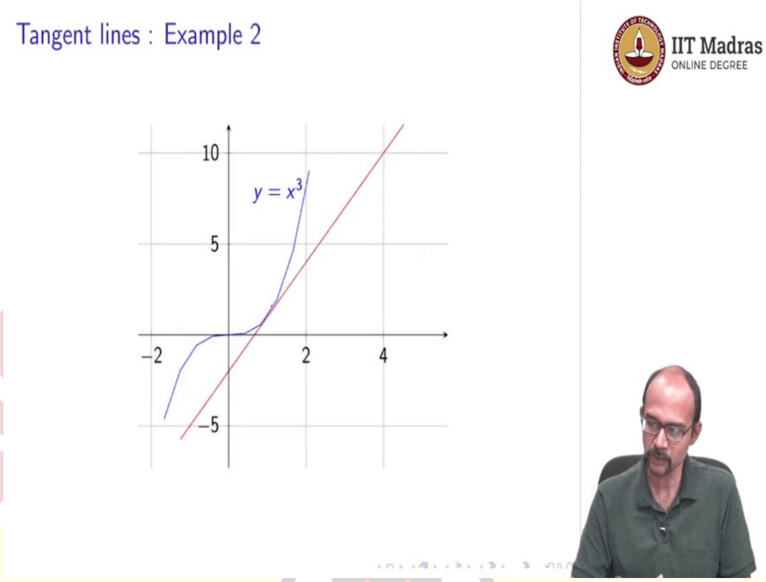

Here is an example of y is x^3 and tangent line to that.

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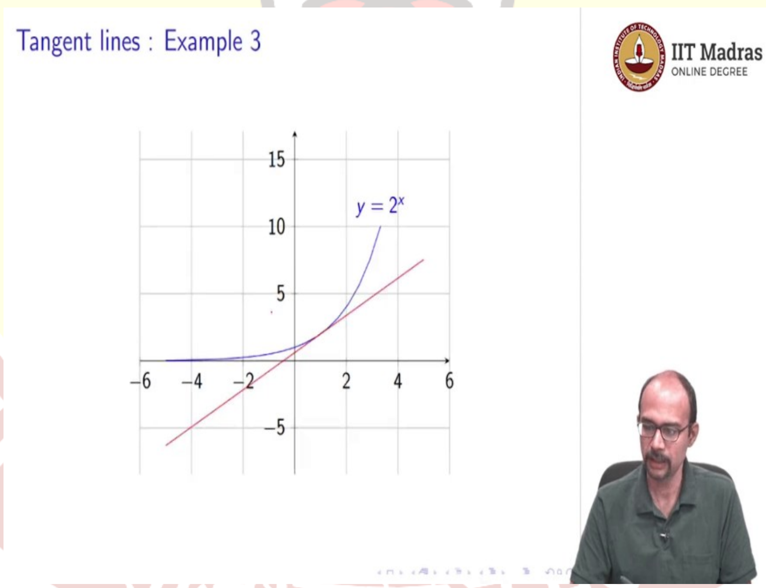

And here is an example of y is, the exponential function, y is 2^x and a tangent line to that. So, I, this was mainly to recall ideas that have been studied in some detail in Maths 1. So, if you feel somewhat uncomfortable, go, please do go back and check your Maths 1 videos or the tutorial problems. Thank you for now.