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ONLINE DEGREE

Mathematics for Data Sciences 2
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Multivariable functions: Visualization

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Hello, and welcome to the Maths 2 component of the online BSc Program on Data Science and Programming. We are going to start our discussion on multivariable calculus. And to begin with, we will study multivariable functions. And we will do a little bit of visualizing of such functions. We will see some examples and we will end with by defining what are called curves.

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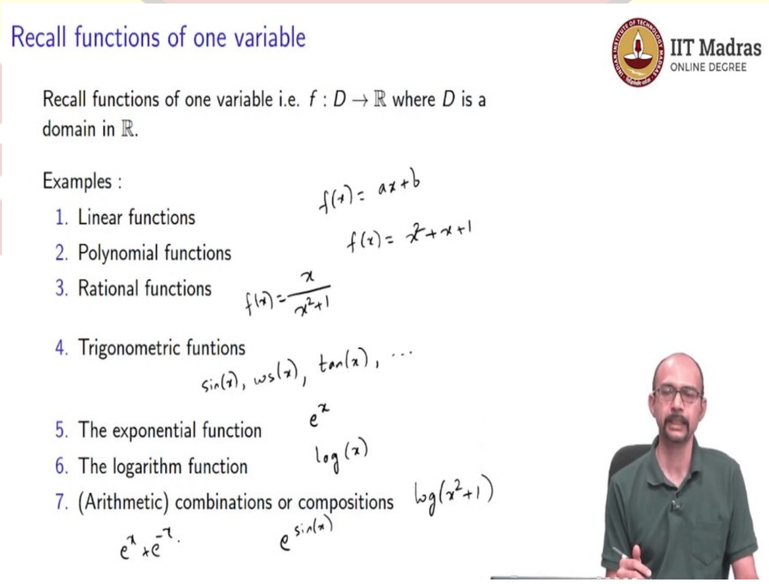

So, let us recall functions of one variable. So, we did this at the beginning of the course. So, recall that functions of one variable are functions from a domain D and R, to real numbers, so they are real valued functions. So, they are like f(x) is equal to some expression, which is a number, meaning, which evaluates to a number.

So, here are some examples, which we have seen earlier. So, linear functions, polynomial functions, rational functions and I will remind you of what that was. Trigonometric functions, the exponential function, the logarithmic function, and then finally, you can take arithmetic combinations of these. I mean, you can add them, subtract them. If you are lucky, you can even divide them, can multiply them or you could take compositions.

So, if you're functions are such that the range of one function is contained inside the domain of another function, then you can compose those two functions and you can produce a new function of one variable. Just to remind you for rational functions we meant things of the form say $\frac{x}{x^2+1}$, so this is what typically was a rational function. So, a division of polynomials.

And, of course, once you divide you have to be careful that the denominator is 1,0, if it is 0 somewhere then you have to say that it is not defined at that point define it separately or just say that the domain is a restricted part of the real length. So, polynomials was for example f(x) is $f(x) = x^2 + x + 1$, linear functions, of course, f(x) is $ax + b$, where a and b are real numbers and then trigonometric functions where sine of x, cosine of x, tan x, cotangent x, and so on, cosecant and secant x.

The exponential function was e^x, the logarithmic function was log of x, of course, you have to be careful because log is defined on the positive part. So, we do not define log for negatives. So here the domain is the positive side of the x axis. And then arithmetic combinations or compositions.

So, by this we mean things like $\log(x^2 + 1)$, so this is a composition because you have $x^2 + 1$, and then the range of $x^2 + 1$ is 1 to ∞, which is inside the domain of the logarithm function, so you do log $x^2 + 1$ that makes sense or you could do log $|x|$ and it is undefined at 0 or other examples are $e^{\sin x}$ let us say, so that is a composition or arithmetic combinations mean something like $e^x + e^{-x}$. So, these are all functions that we have seen in single variable calculus. And we studied various aspects of such functions. Things like how do you find maxima or minima, derivatives, continuity and so on.

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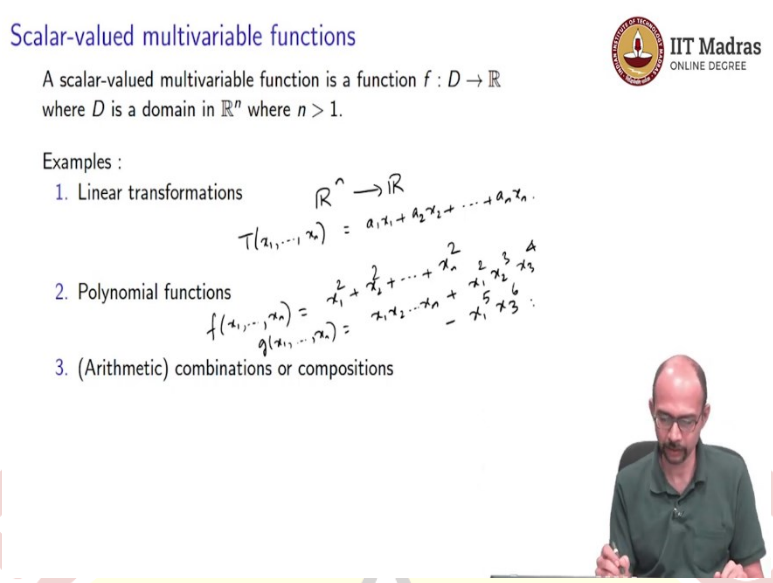

So, let us get to scalar-valued multivariable functions. So, a scalar-valued multivariable function is a function f from D to $\mathbb{R}^1$, again, the co-domain is $\mathbb{R}^1$, and that is why it is scalar-valued, meaning, it takes values in the real numbers. But now, D is a domain in $\mathbb{R}^n$, where n is greater than 1. So, this could be $\mathbb{R}^2$ or $\mathbb{R}^3$, $\mathbb{R}^5$, $\mathbb{R}^{20}$ whatever. So, such a thing is called a scalar-valued multivariable function. And this is something that we are going to study extensively in the next couple of weeks. And this will show up, possibly again in further courses. So, this is mainly, what we want to study going ahead.

So, what are examples? linear transformations, polynomial functions. And I will expand on that in a minute, and arithmetic combinations or compositions. So, just to be clear of what I mean by linear transformations, you could think of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^1$ or define it on some subset. So, for example, you could take something like $T(x_1, x_2, x_3, \dots x_n)$ $a_1 x_1 + a_2\, x_2 + a_3\, x_3 + \cdots + a_n x_n$. So, this will be a scalar-valued multivariable function.

So, we have been thinking of this in terms of linear algebra so far, but now we can think of this also, as a scalar-valued multivariable function. And, well, what is a polynomial function? Well, so you have $f(x_1, x_2, x_3, \dots x_n)$ and you can do something like ${x_1}^2 + {x_2}^2 + \dots + {x_n}^2$, maybe another polynomial function is $x_1 \times x_2 \times$ up to x_n. And then maybe let me add ${x_1}^2 {x_2}^3 {x_3}^4$, and then $-{x_1}^5$, $-{x_3}^6$. So, this is an example of a polynomial function in several variables. And then you could combine these or you could combine other ones as well. And we will do more examples as we go ahead.

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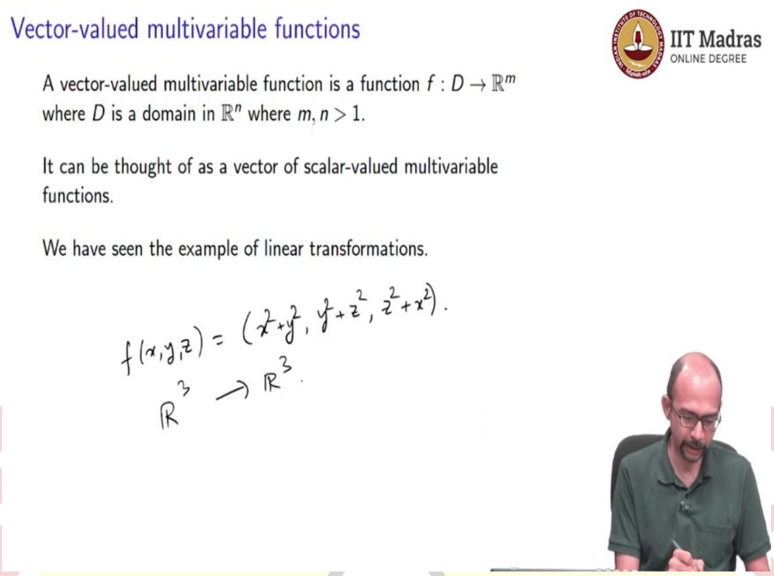

So, let us talk about vector-valued multivariable functions. So, vector-valued multivariable functions means that now instead of $\mathbb{R}^1$, the domain, sorry, the codomain can be $\mathbb{R}^m$, where m is also larger than 1. So, a vector-valued multivariable function is a function f from D to $\mathbb{R}^m$, where D is a domain in $\mathbb{R}^n$, and both m and n are strictly greater than 1. So, such a thing is called a vector-valued multivariable function.

So, we can think of this as a vector of scalar-valued multivariable functions. And we have seen the example of linear transformations, which means you can take a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. So that means, for each coordinate, you have a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^1$. So, you have linear combinations of your $x_1, x_2, x_3, \dots x_n$ and then you have m many of those, that was exactly how we got linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

And we could restrict those to some domain if we want. So, those will be vector-valued multivariable functions So, let me give another few examples. $f(x\ y, z)$ is $x^2 + y^2$, $y^2 + z^2$, $z^2 + x^2$. So, this is a vector-valued multivariable function, this is a function from $\mathbb{R}^3$ to $\mathbb{R}^3$. So, this is another example. And this is not a linear transformation. So, let us sort of combine these.

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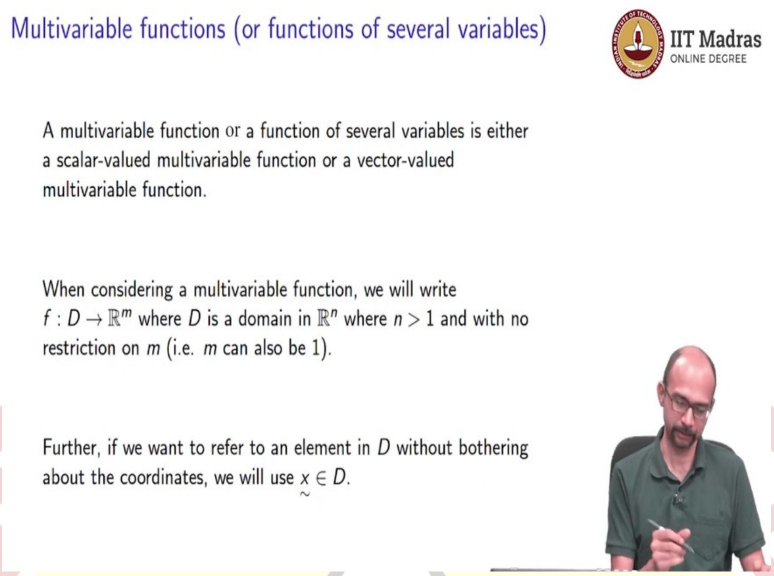

So, what is a multivariable function? Some× this is also called a function of several variables. So, a multivariable function or a function of several variables is either a scalar-valued multivariable function or a vector-valued multivariable function. So, that means you could have a function from Rn to Rm or some domain in Rn to Rm.

And here we insist that the n is bigger than 1, but there is no restriction on m. So, m could be 1 or m could be larger. If it is larger, then that will fall under the category of vector-valued multivariable functions if it is 1, it falls under the category of scalar-valued multivariable functions. So, we would not distinguish between these when we say just multivariable function. If there is a reason to distinguish, we will make it explicit. But much of what we are going to study is scalar-valued multivariable functions. Fine.

So, just as two set notations as we go ahead. If we want to refer to an element in D, what is D? D is this domain in $\mathbb{R}^n$, without bothering about the coordinate. So, remember that n, since we are in Rn, we have coordinates $x_1, x_2, x_3, \dots x_n$, but some× we do not want to explicitly talk about coordinates, we just want to, let us say, if I want to add two functions, so f + g of $g(x_1, x_2, x_3, \dots x_n)$. So, this, I do not want to refer to $x_1, x_2, x_3, \dots x_n$, so I will just refer to $\underset{\sim}{x}$ as $x_1, x_2, x_3, \dots x_n$. So, this is the vector $x_1, x_2, x_3, \dots x_n$ or the entuple $x_1, x_2, x_3, \dots x_n$, whichever is convenient for the purpose at head. So do not get confused if you see an $\underset{\sim}{x}$, it just means $x_1, x_2, x_3, \dots x_n$.

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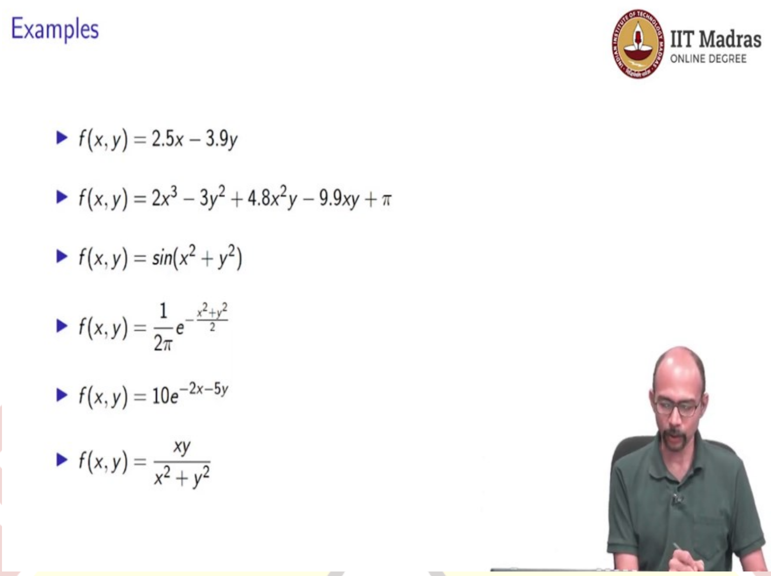

So, with that caveat, let us look at lots of examples. So, here is a linear function, so this is a scalar-valued function from R2 to R, 2.5x - 3.9y. Here is a polynomial, so $2x^3 - 3y^2 + 4.8x^2$ - × y, - 9.9xy + π. Here is a slightly more complicated function, sin of $x^2 + y^2$.

Here is an example that you may actually have seen in statistics. So, this is f(x), y is $1/2\pi$ e to the power - $x^2 + y^2/ 2$. So, this is the bivariate normal distribution in case that rings a bell. Another example maybe from statistics. So, this is f(x), y is 10 × e to the power - 2x - 5y, so you can think of this as bivariate exponential or you can think of this as two independent exponentials, and the joint density of that. Then here is a rational function x, y / $x^2 + y^2$. And do keep this in mind because we may come across this again, as we go ahead.

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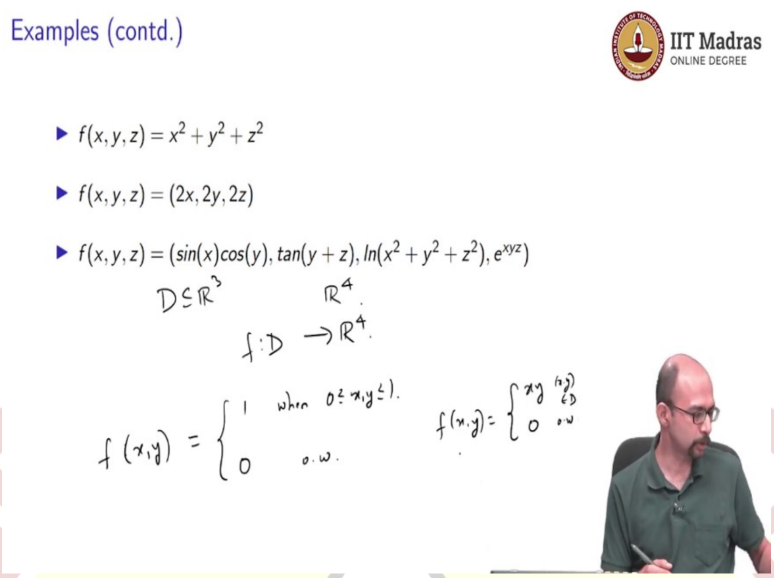

Let us continue with our examples. So, here is f(x), y, z is $x^2 + y^2 + z^2$. So again, all of these are scalar-valued function. So, they are taking values from R2 to R or R3 to R, this is from R3 to R. So, here is an example of something, which is not a scalar-valued multivariable function, this is from R3 to R3. So, f(x), y, z is 2x , 2y , 2z. And these two examples I have something to do with each other, and we will encounter them later on.

And just, as a more convoluted example here is f(x), y, z is sin of x × cosine of y , tangent of y + z , logarithm of $x^2 + y^2 + z^2$. Again, this can take values when x, y, z is 0, and all three are 0. So, we have to define this function a priori, on the domain, where x is not 0, y is not 0, z is not 0. Of course, we also have a tangent of y + z. So, we have to define this, when your tangent function there are, it does not exist everywhere, so you have to be careful that y + z is not multiples of odd multiples of π/ 2. And then, we have e to the power x, y, z in our last coordinate.

So, this is a function from some domain in R3. So, this is defined in some domain in R3, and its ranges in R4. So, its codomain is R4, so this is a function, this f has f is from D to R4. So just as an example. So, you can see here that we have got this by taking various products and compositions and so on of well-known functions, functions that we understand well. And I will also maybe mentioned a couple of other examples0 that so, some× you may be, we may be interested in defining functions piecewise. So, it may have some definition on some part and some definition somewhere else.

For example, you co^{2}uld have that f(x, y, z), x, y is 1 on the unit interval. So, x y or maybe I should write the size when x and y are both between 0 and 1. So, this is on the unit square and it is 0 otherwise, and this is something you may have seen again in the statistics course, this is the uniform density on the unit square. And there could be other piecewise definitions. For example, you may have f(x), y is let us say x × y, and 0 this is on some part x, y is in D and if it is not in D.

So, we have functions of this type also. So, they may be defined piecewise. So, of course, here, when we say piecewise, the piece refers to a piece of Rn, meaning the domain Rn. So, in this case, the domain is R2, so it refers to pieces of R2. So, now do not think of pieces as intervals in R. Instead, think of them as sets on the plane if you have R2. If you have Rn of course, the visualization is much harder, which leads us to an important point as to how do we visualize such things.

So, let us do examples of that. So of course, visualization for multivariable functions is much harder than it is for functions of one variable. And that is because to visualize, we have to think of the graph. And if you remember what the graph was, the graph meant, it is a subset of the domain cross the codomain. So, in here, the domain is Rn and the codomain is Rm, so it will be a subset of Rn cross Rm.

And, of course, our visualization is limited to three dimensions. So unfortunately, we cannot visualize functions, which for any n and m, but if n is 2, and m is 1, so meaning functions from R2 to R, such functions are subsets of R2 to R, such functions we can visualize. So, let us do some examples of those and some of the functions that we have seen, in fact.

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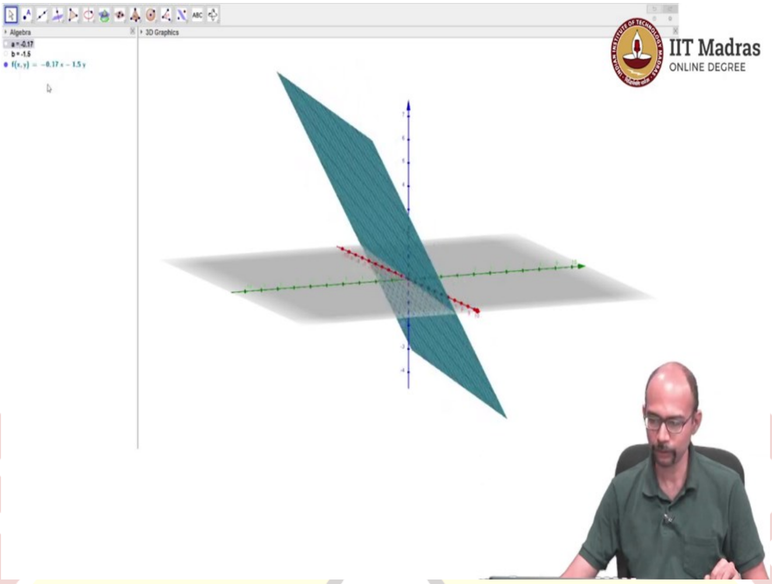

So, here is a linear function, the function f(x), y is - 0.9x - 1.5y. In fact, this function is f(x), y is a × x + b × y. And these a and b are varying in an interval. So, if I play the slider, the function will change. And this looks like a plane. So, this is a plane. So, linear functions, the graph looks like a plane. This is something, we understand because this is exactly the plane z is equal to ax + by. And then how does how does this plane move?

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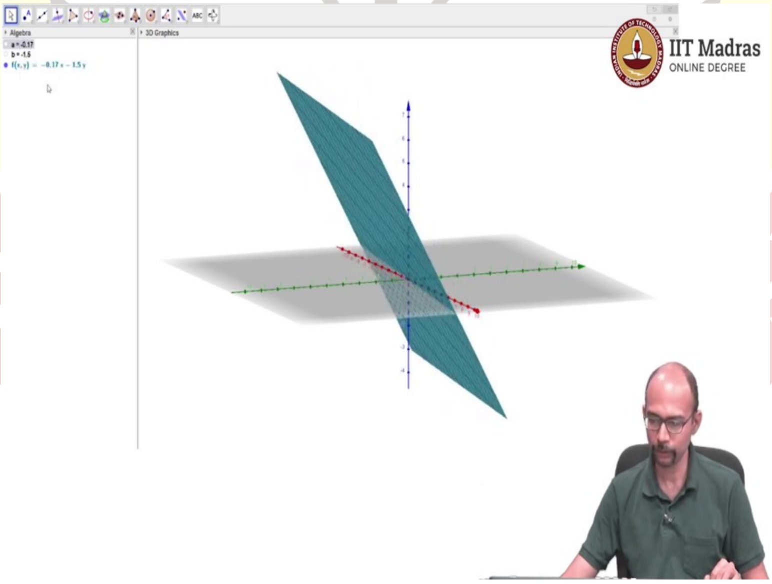

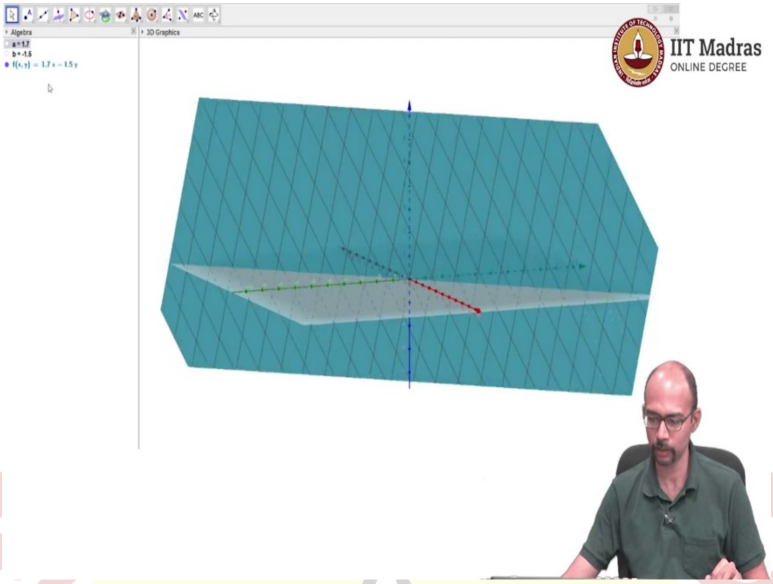
Algebra
a = 1.7
b = -1.5
f(x, y) = 1.7 x − 1.5 y
3D Graphics
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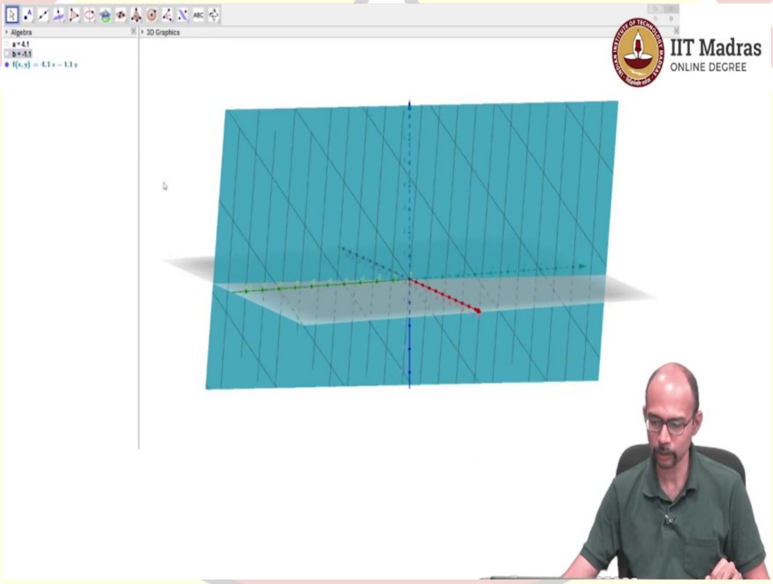
Algebra
a = 4.1
b = -1.1
f(x, y) = 4.1 x − 1.1 y
3D Graphics
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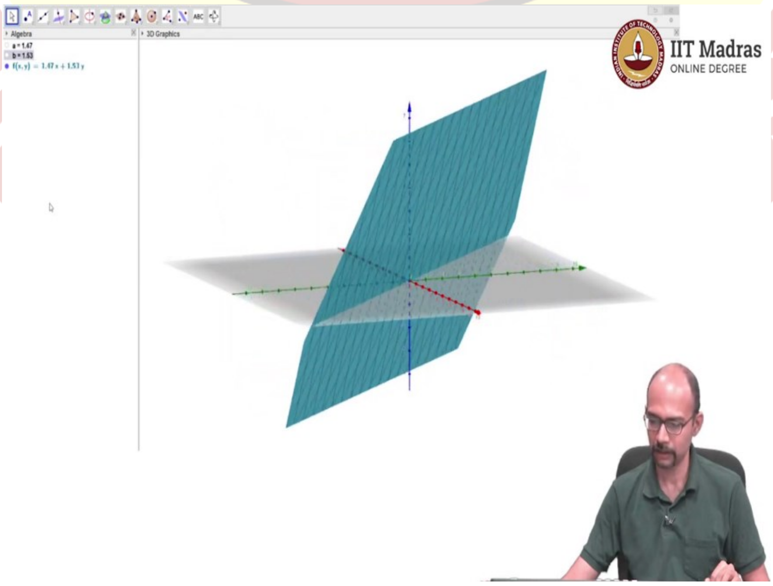
Algebra
a = 1.47
b = 1.53
f(x, y) = 1.47 x + 1.53 y
3D Graphics
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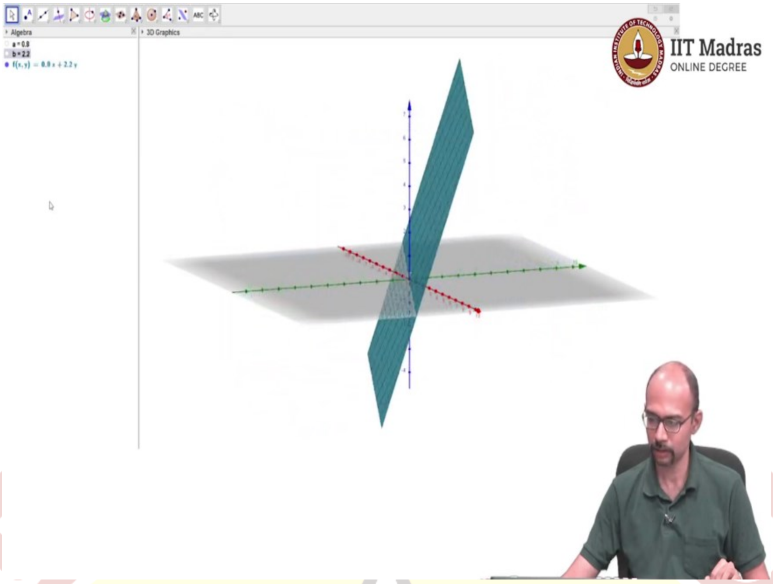
Algebra
a = 0.8
b = 2.2
f(x, y) = 0.8 x + 2.2 y
3D Graphics
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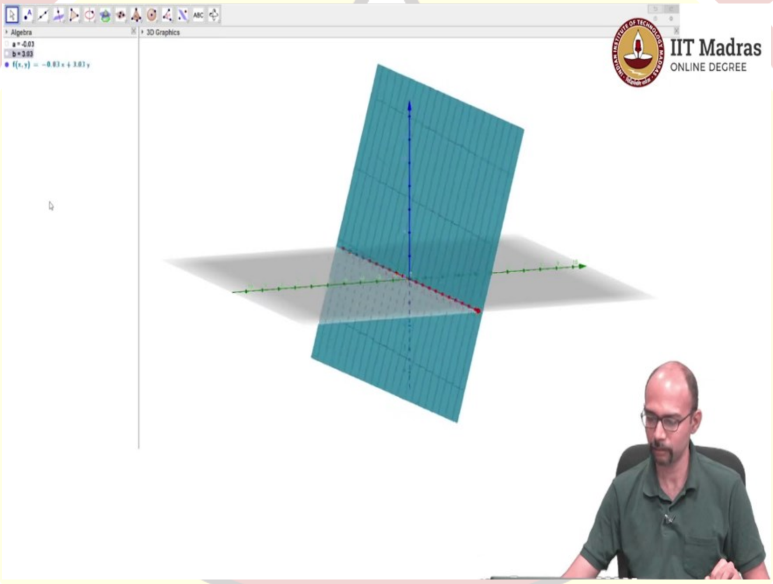
Algebra
a = -0.03
b = 3.03
f(x, y) = −0.03 x + 3.03 y
3D Graphics
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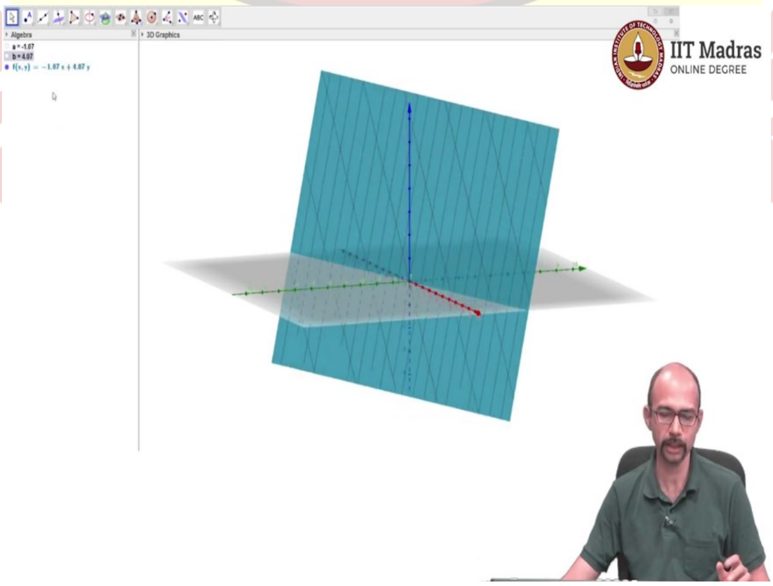
Algebra
a = -1.07
b = 4.07
f(x, y) = −1.07 x + 4.07 y
3D Graphics
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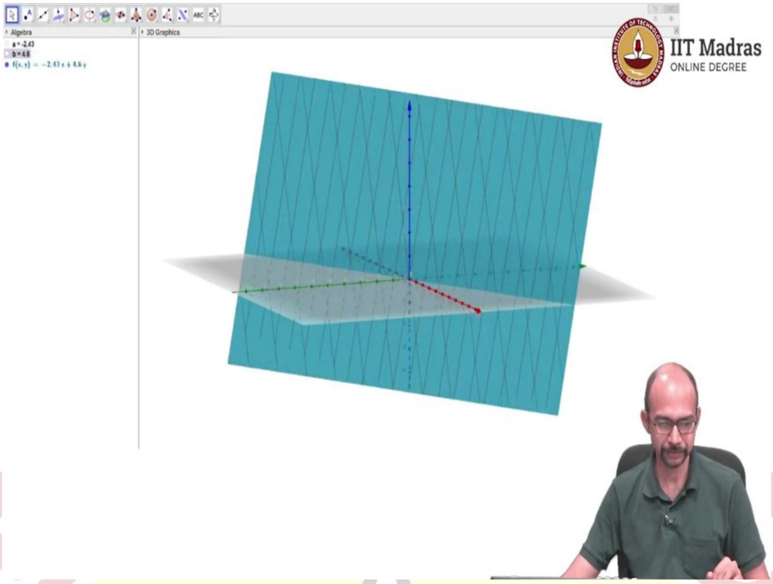

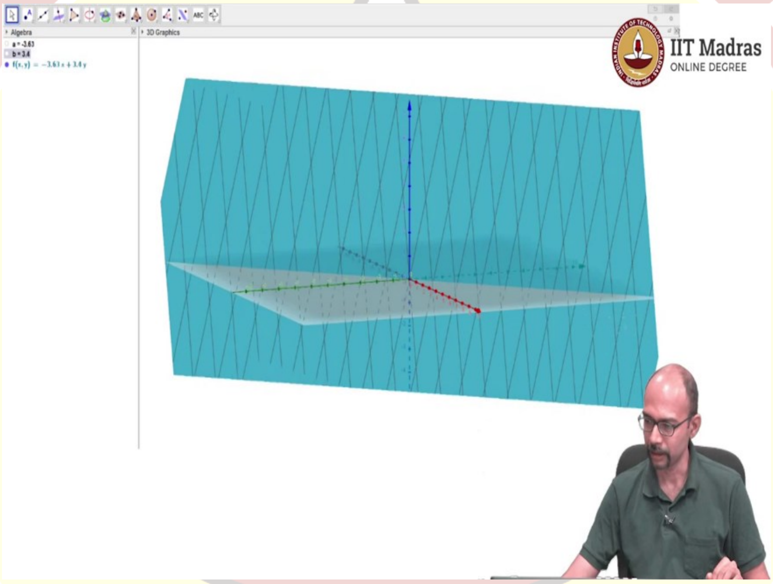

So, if we animate it, so this is what happens to the plane as a and well, we can animate b as well. So, if a and b both change, this is what happens. So, this plane moves around depending on the coefficients.

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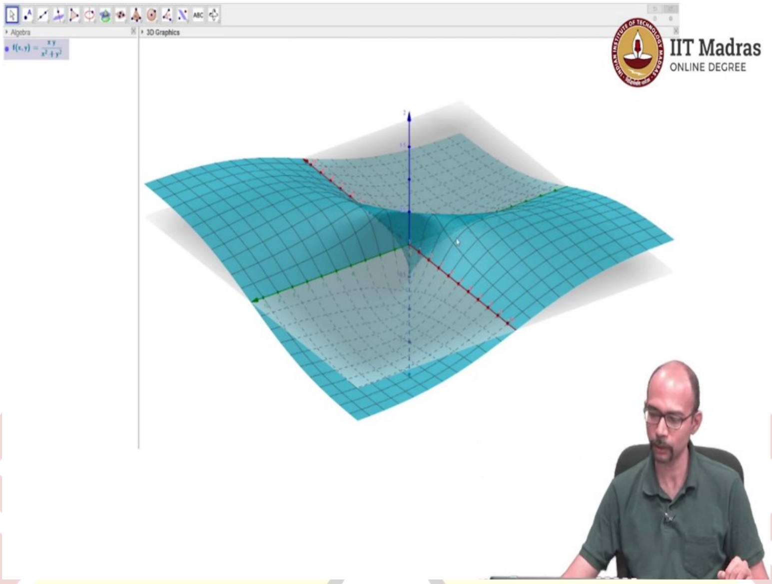

So, let us look at the example of the, of a rational functions. So, the rational function that we had, as an example, in our slides was f(x), y is x, y / x ^{2} + y ^{2}. And here is how the rational function looks like. So, this is, it is some kind of surface. And as you can see, there is some strange thing happening at 0, because at 0 this function is a priori not defined. So, but what is what noticing is what happens around 0.

So, if you look at this side, and this side, it is looking like you have hills, the function is like this. Whereas, if you have the other two sites, then the function is like you have troughs. So, such phenomenon is very interesting. And we will try to detect such phenomenon, as we go ahead.

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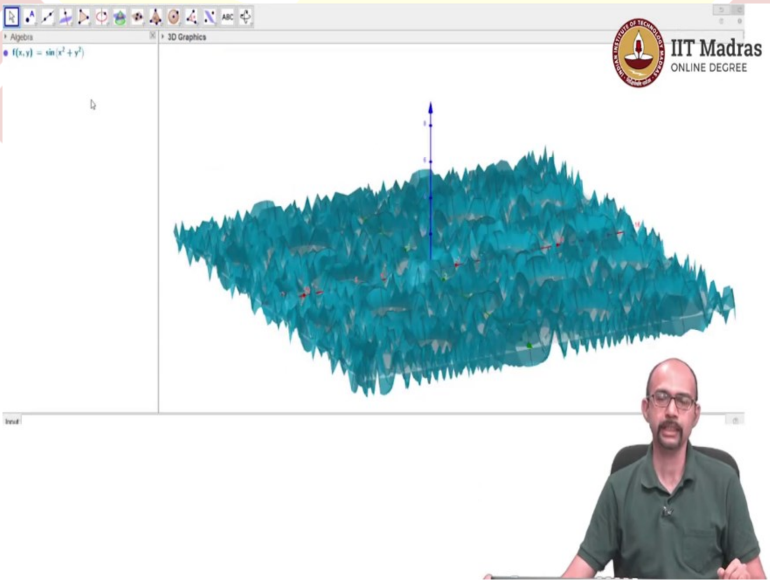

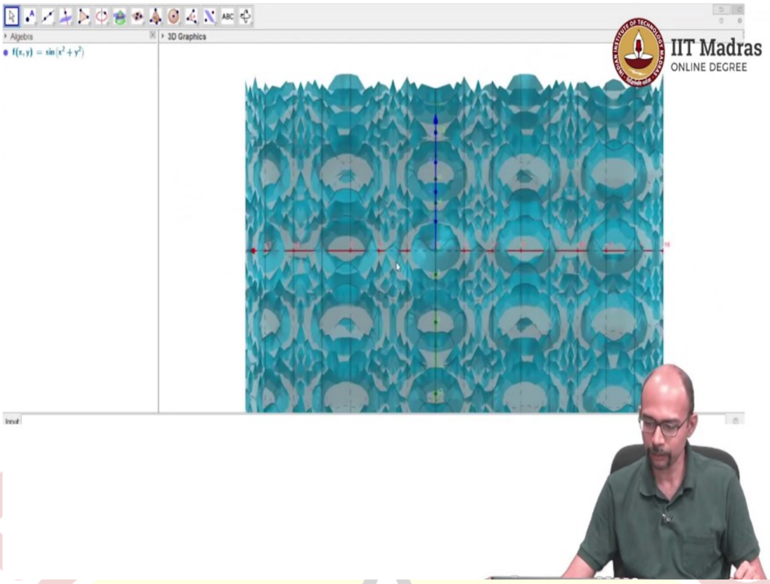
Algebra
$f(x, y) = \sin(x^2 + y^2)$
3D Graphics
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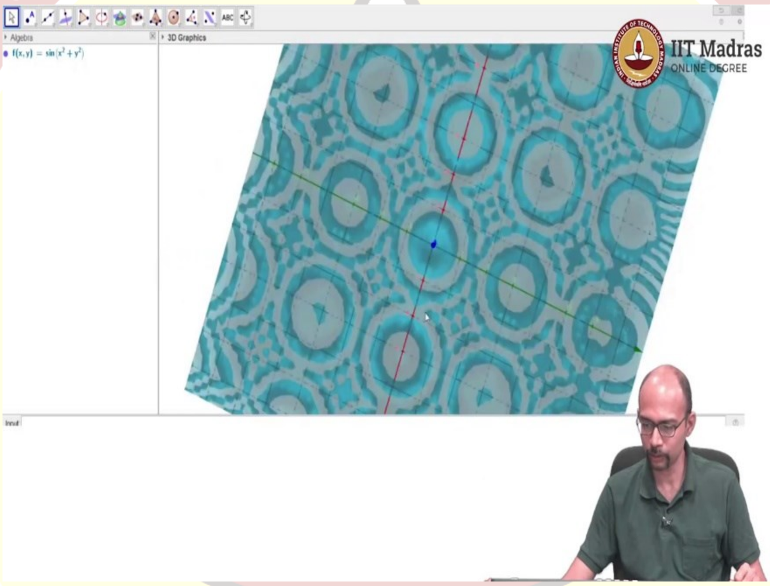
Input


Algebra
$f(x, y) = \sin(x^2 + y^2)$
3D Graphics
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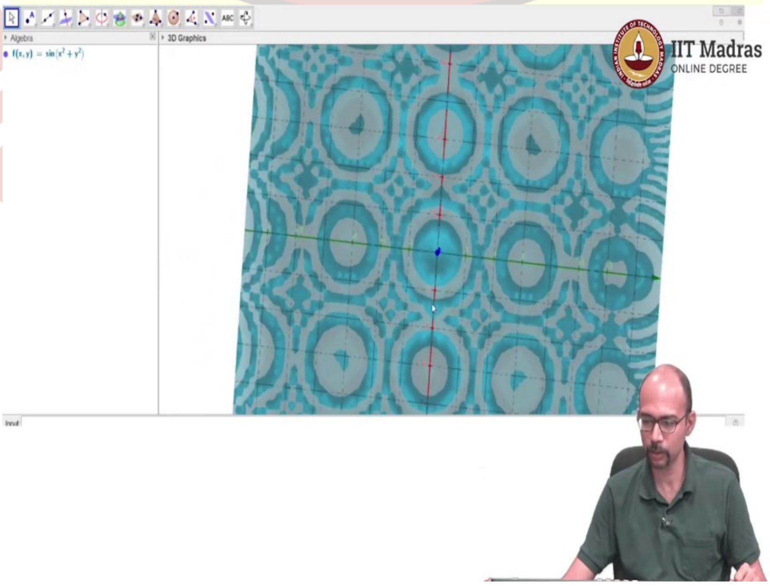
Input


Algebra
$f(x, y) = \sin(x^2 + y^2)$
3D Graphics
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Input

Let us look at another interesting function. So, this is a function, sin of $x^2 + y^2$. If we plot the graph, well, here is what it looks like. So, if this is the projection down to the down to R2, this is what happens. So, it is, you can see it is oscillating, because the sine function oscillates. So, depending on the value of $x^2 + y^2$, the sin function will take some value, so it will move in circles.

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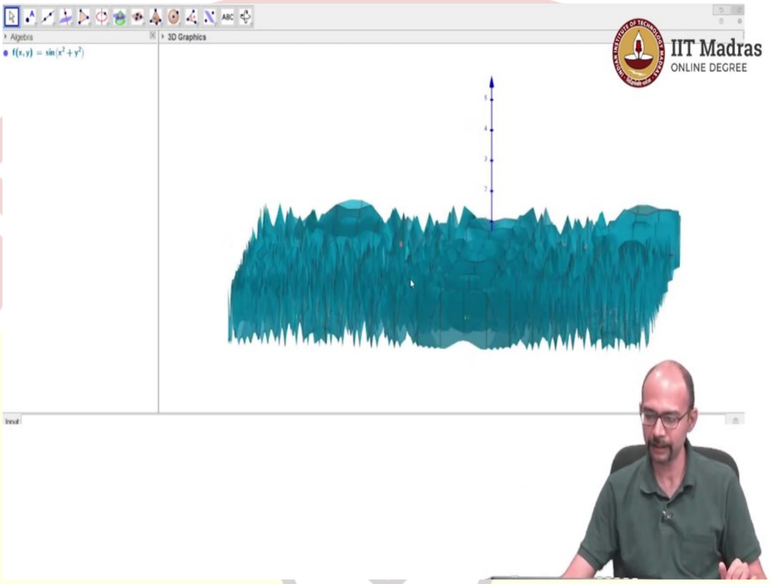

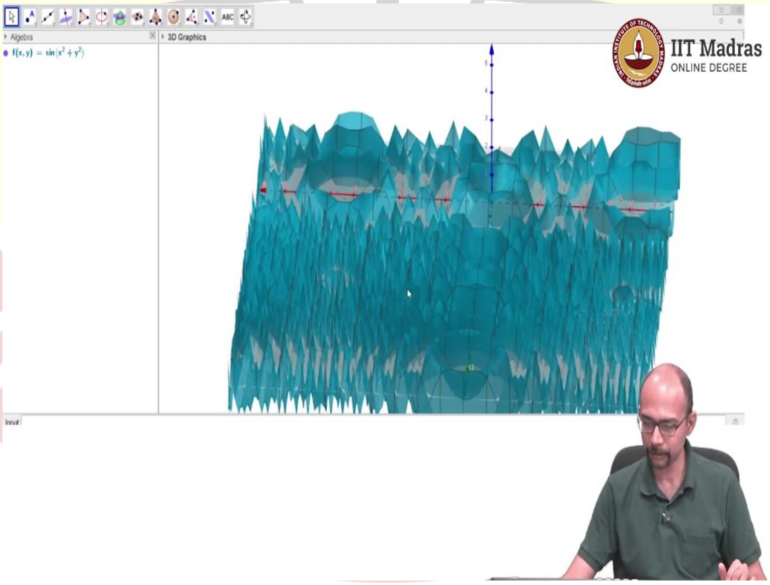

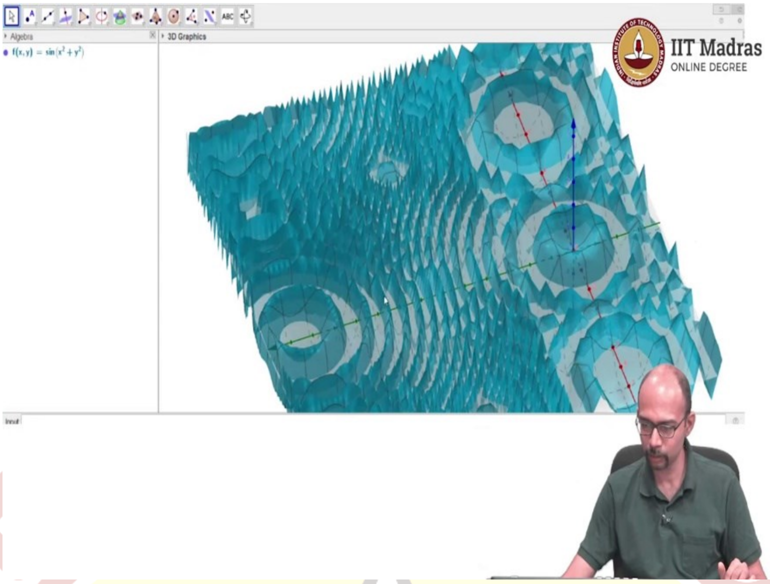
Algebra
f(x, y) = sin(x² + y²)
3D Graphics
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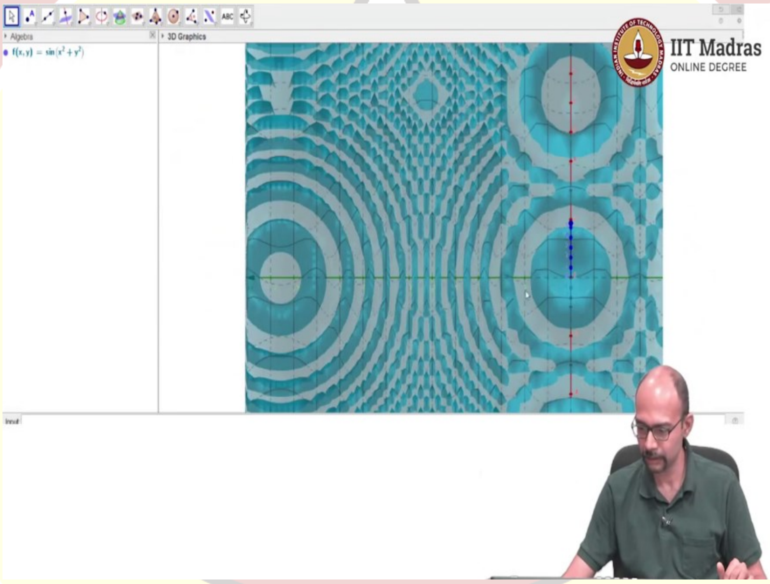
Algebra
f(x, y) = sin(x² + y²)
3D Graphics
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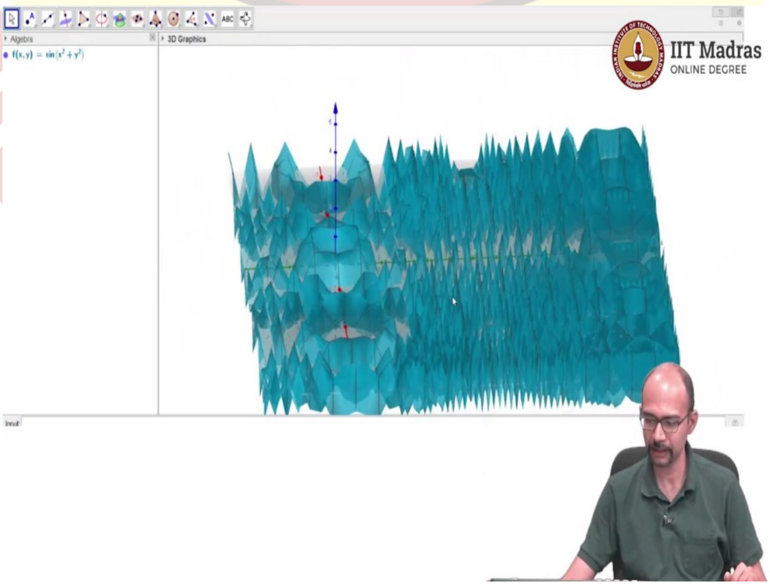
Algebra
f(x, y) = sin(x² + y²)
3D Graphics
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Input

So as your circles go out, that is how it looks. And this is how it looks. If you if you view it over the plain R2, so this is something in R3. And you can see it is moving in circles, as your circle, so this is the origin, so at 00 it is 0. And then as $x^2 + y^2$, the radius increases, you get that it increases.

And then as the radius is$\pi/0$, that is when sorry, root of $\pi/2$ so $x^2 + y^2$ is equal to $\pi/2$. That is when it attains 1 and then it goes down again. So, again atπit becomes 0 and that is how the behavior continues. So, this is a very interesting function. It looks very nice. So, as you can see graphs of functions can yield some very beautiful, visually appealing pictures.

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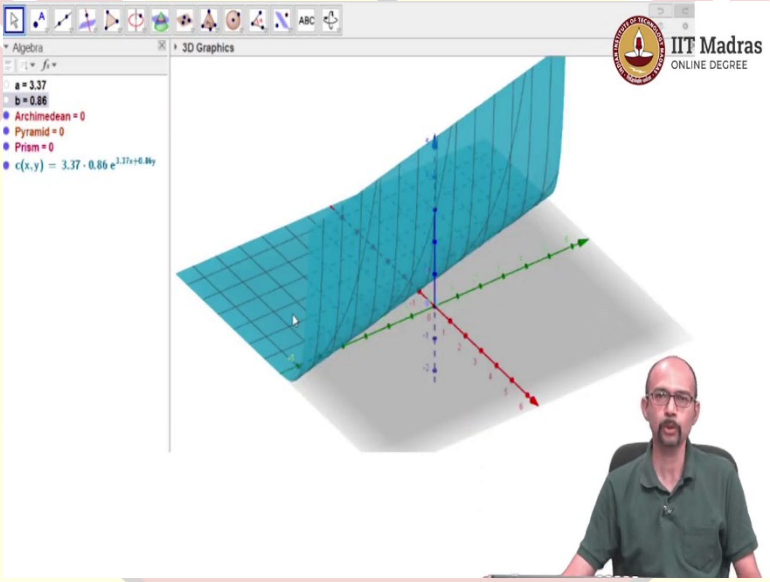

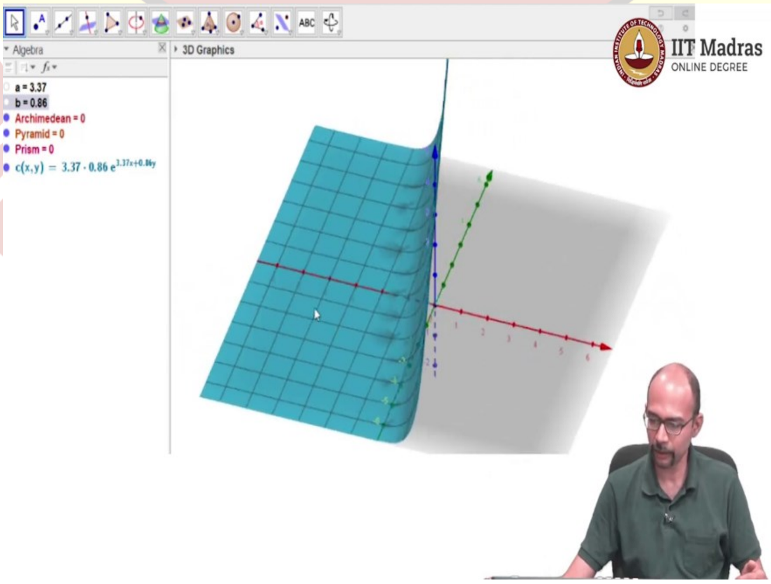

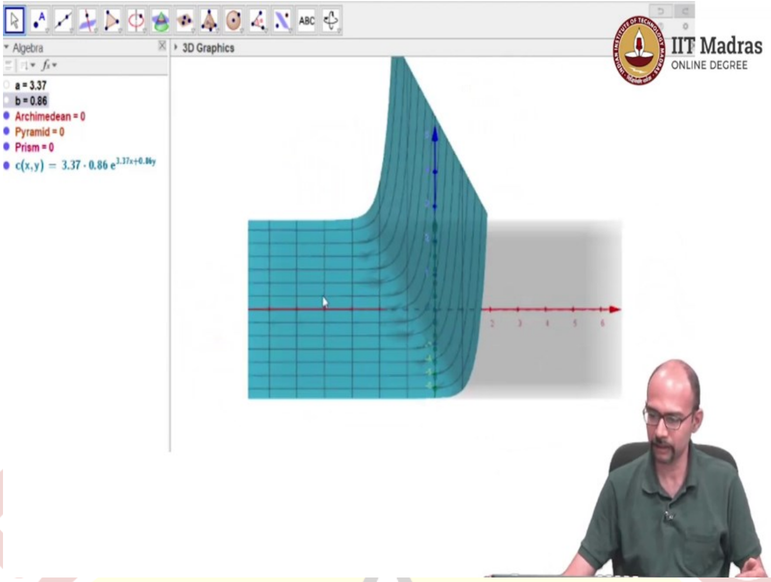
Algebra
a = 3.37
b = 0.86
Archimedean = 0
Pyramid = 0
Prism = 0
c(x,y) = 3.37 · 0.86 e^{3.37x+0.86y}
3D Graphics
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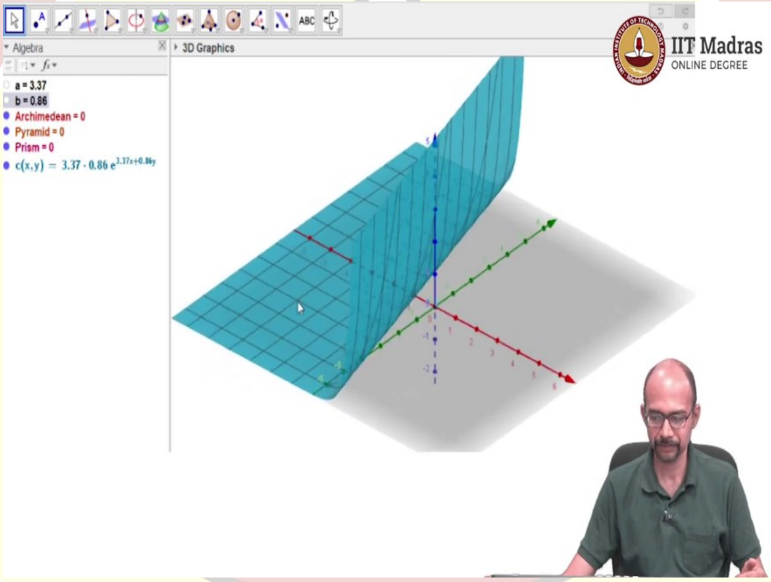
Algebra
a = 3.37
b = 0.86
Archimedean = 0
Pyramid = 0
Prism = 0
c(x,y) = 3.37 · 0.86 e^{3.37x+0.86y}
3D Graphics
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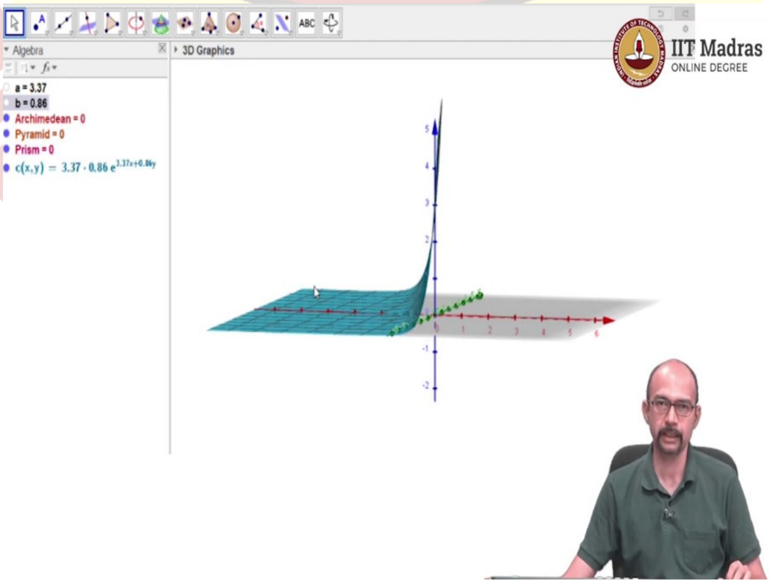
Algebra
a = 3.37
b = 0.86
Archimedean = 0
Pyramid = 0
Prism = 0
c(x,y) = 3.37 · 0.86 e^{3.37x+0.86y}
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Let us now look at the example of the exponential joint density of for two exponential, two independent exponential random variables. And so here, the means are 0.86 and 3.37. And as we change them, the picture will look different, and this is how it looks like. This is the exponential function. So, as you can see it, it is very sharp and then it goes down to 0 very fast. So, this is exactly what happens for the one variable exponential function.

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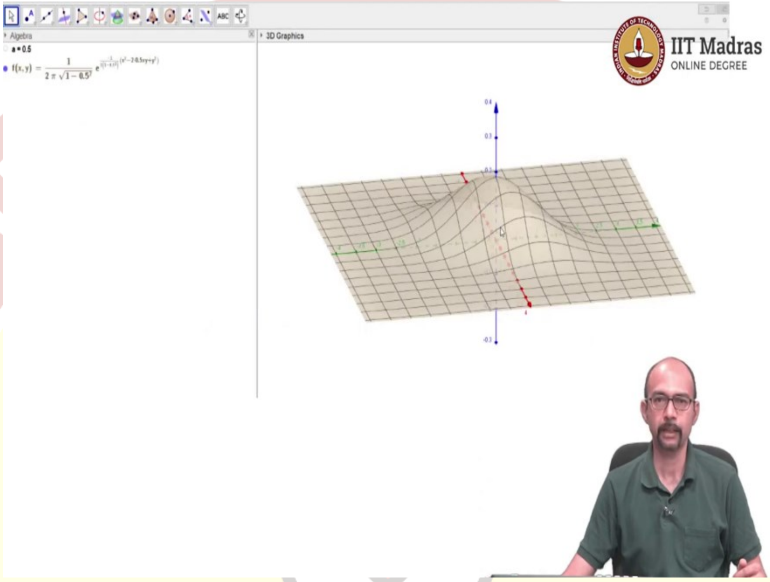

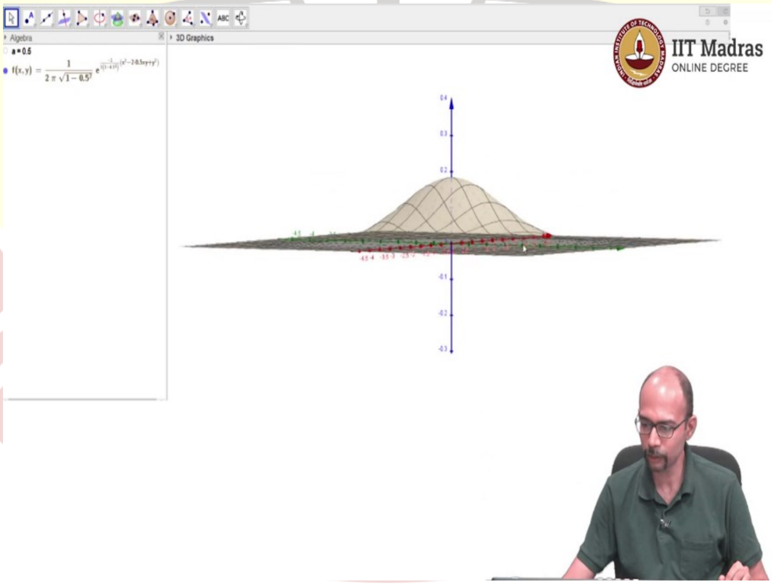

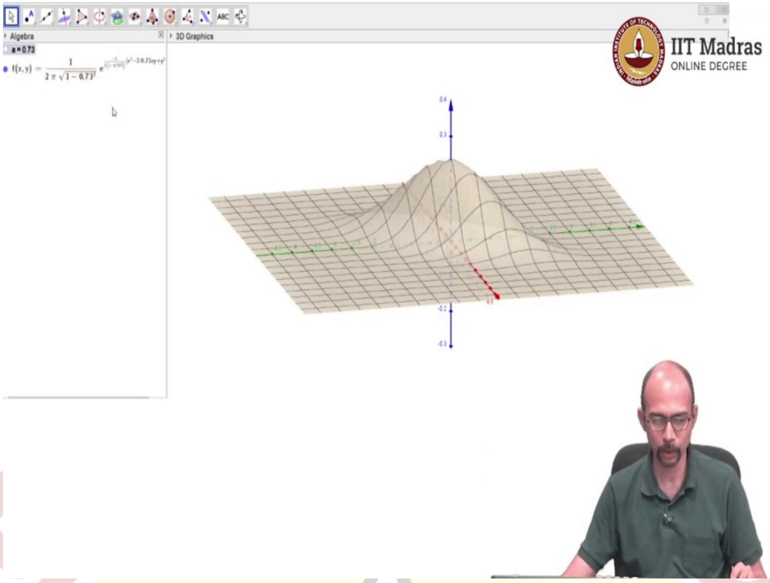
Algebra
a = 0.73
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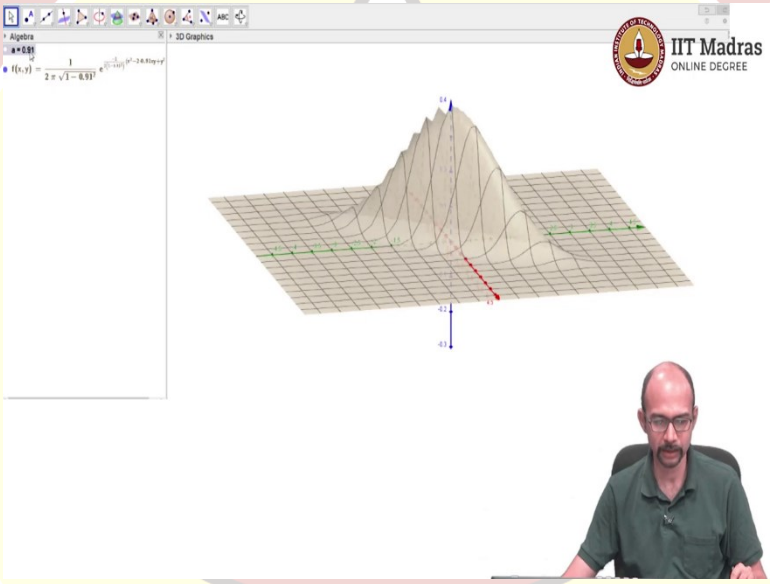
Algebra
a = 0.91
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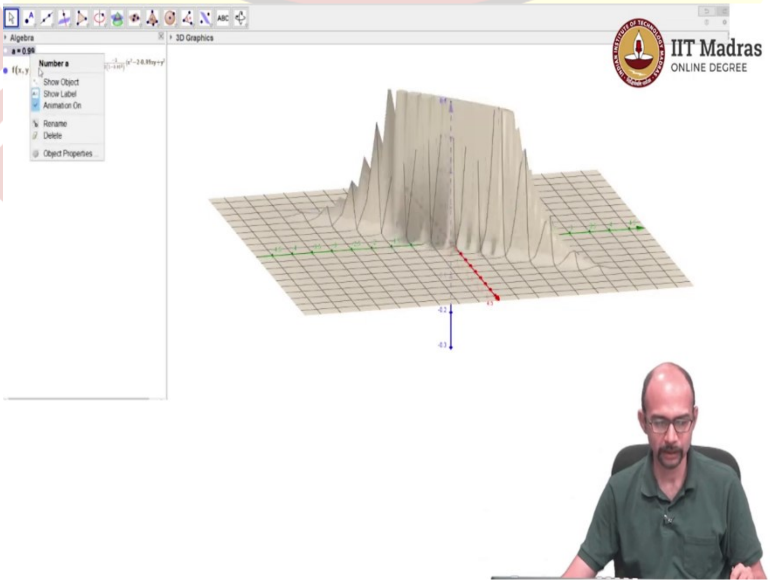
Algebra
a = 0.99
Number a
Show Object
Show Label
Animation On
Rename
Delete
Object Properties
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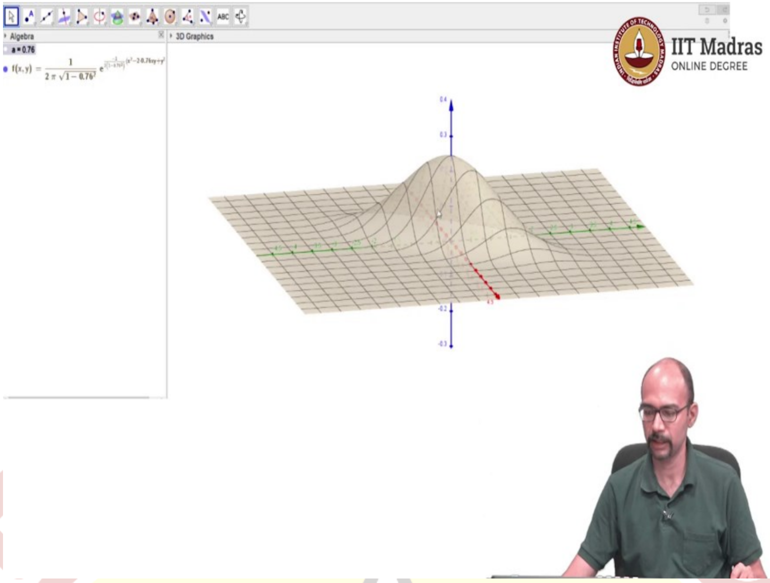

And finally, let us do the example of the bivariate normal. So, this is how the function looks like. So, it is an exponential function, but it goes down to 0 very fast. The only around the origin, there is a hill, and the size of the hill depends on some parameters like the means and so on. And so, this will show you how it changes. So, depending on how it couples or how the means are, it will be a very steep hill or it will be a much flatter hill, but it will always be a hill and for most of the plane, it will be very close to 0.

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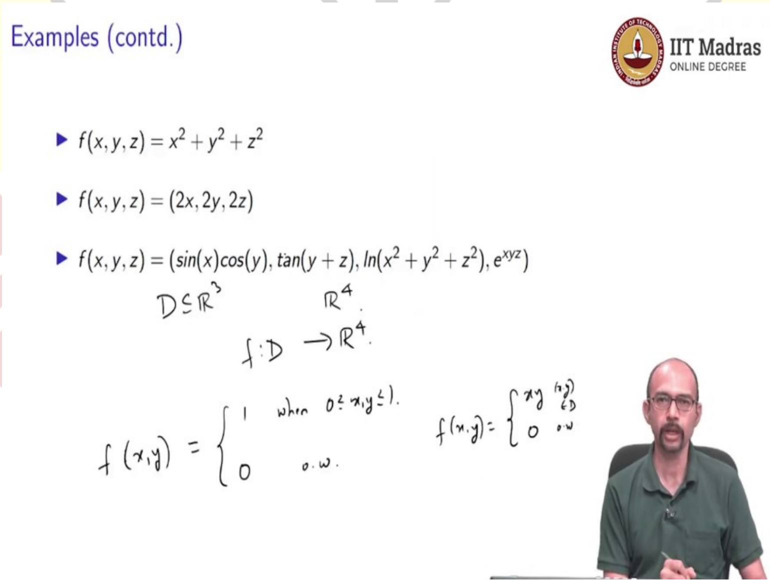

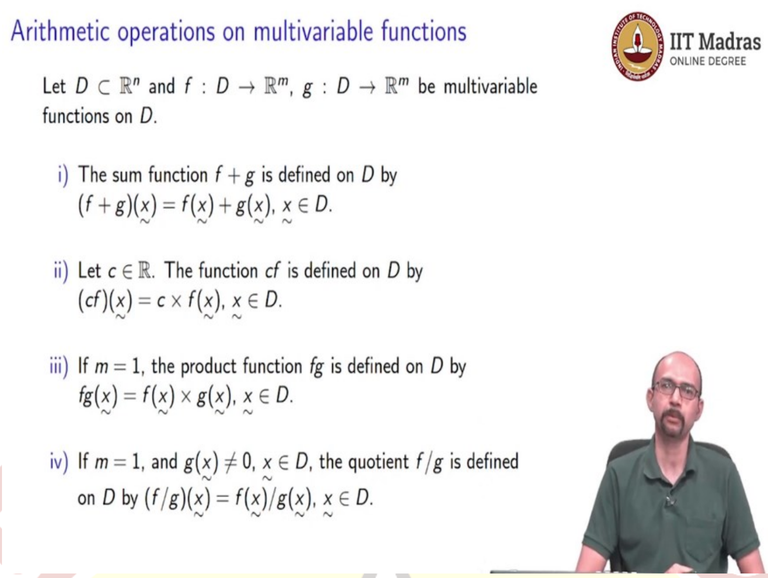

So, having seen some examples, let us return back to our slides. So, let us do the same things that we did for one variable functions. Namely, we can, let us note that we can perform arithmetic operations on these functions. So, if you have two functions on the same domain D, so you have f and g from D to Rm, and both of which are multivariable functions on D, then you could for example, add them. How do I add them? So, if you want f + $g(\underset{\sim}{x})$, then that is $f(\underset{\sim}{x}) + g(\underset{\sim}{x})$. This makes sense because in Rm you can add vectors. So, Rm is a vector space in particular. So, this is one of the reasons why we specifically studied linear algebra.

So, suppose you have a constant C, so a scaler, then the function C × f is defined on D by c, $f\left(\underset{\sim}{x}\right)$is C × $f\left(\underset{\sim}{x}\right)$. So, what does that mean? Again, this is scalar multiplication in Rm. So that means it multiplies each component in Rm. And if m is 1, so this is a case of scalar-valued multivariable functions, then we can even talk about the product. So, the product function fg is defined on D by $fg(\underset{\sim}{x})$ is $f(\underset{\sim}{x}) \times g(\underset{\sim}{x})$.

This makes sense because you can multiply real numbers. So, the same way as we did for single variable functions, you can do this for scalar-valued multivariable functions. And for the same reason, you can also divide, but provided the denominator is non-zero. So, if you have a scalar-valued if m is 1, meaning f and g are both scalar-valued multivariable functions and $g(\underset{\sim}{x})$ is non-zero, then the quotient f by g is defined. And $f/g(\underset{\sim}{x})$ is $f(\underset{\sim}{x})/g(\underset{\sim}{x})$. And so basically, it is defined on the domain D except the points where $g(\underset{\sim}{x})$ is 0, so that is a smaller domain, then it could be a smaller domain, then D.

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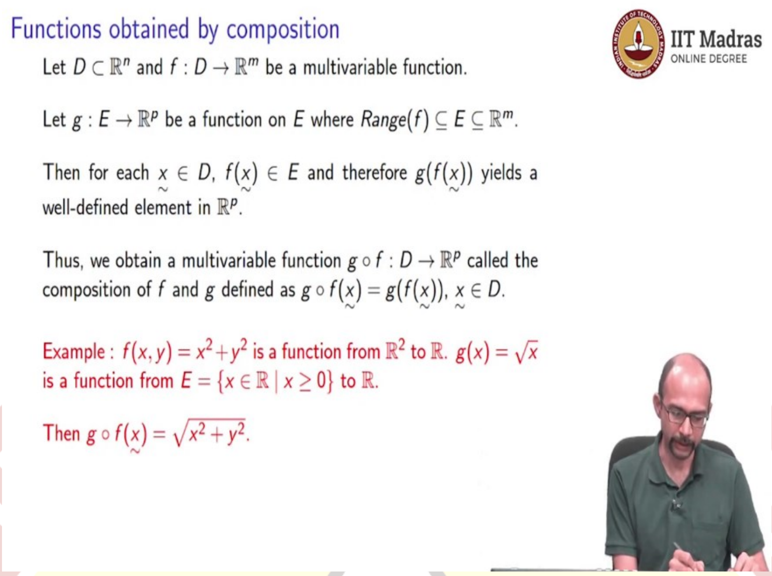

Let us continue and talk about composition. So, if you have functions, let us say a function f from D to Rm, and you have another function, g from E to Rp. What is E? E is sum domain, which contains the range of f. So, the range of f is contained in the domain E. Then we can talk about g composed f. So, for each $\underset{\sim}{x}$ in D, we can talk about g composed f by looking at $f(\underset{\sim}{x})$and then applying g on that. Why can we do that? Because a range of f is contained in E, so $f\left(\underset{\sim}{x}\right)$ is contained in E, so $g(f\left(\underset{\sim}{x}\right))$ makes sense.

So, this yields a well-defined element in $\mathbb{R}^p$. So, we can talk about g composed f, so this is called the composition So, we would not have a lot of use for this in when m and p are beyond 1. But for the cases where m and p are both 1, we have already seen examples of this. For example, we had sine of $x^2 + y^2$. So, $x^2 + y^2$ is a function from R2 two to R, and then we have a function from $\mathbb{R}^1$ to $\mathbb{R}^1$ again, which is sine of z. So, we can apply that on $x^2 + y^2$ or we had logarithm of $x^2 + y^2 + Z^2$. Of course, this works only for the positive side. So, $x^2 + y^2 + Z^2$, we have to restrict for nonzero x, y, z.

So, here is an example. f(x), y is $x^2 + y^2$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^1$, and then we could look at g of x is square root of x. So, this is a function from the non-negative part of the real line to $\mathbb{R}^1$. So, you can do g composed f and that gives us square root of $x^2 + y^2$. So, this is a function that we would like to understand, this is like the radius, if you think of $x^2 + y^2$ as being $\mathbb{R}^2$, where R is the radius.

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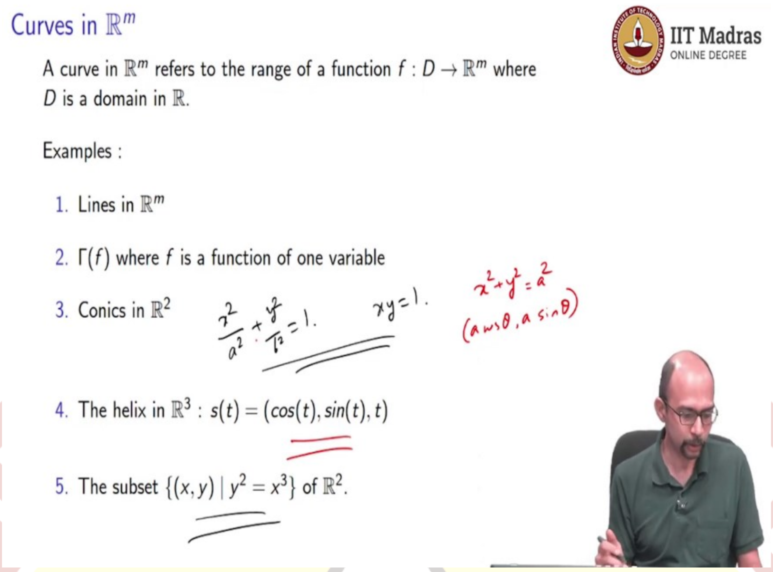

So, finally, I am going to end with something, which is slightly off topic, but this is one for the sake of completeness. And because we will have some use for this in what is coming, so this is curves in Rm. So, we have studied multivariable, we have studied single variable functions, which is $\mathbb{R}^1$ to $\mathbb{R}^1$, we have studied multivariable functions, which is Rn to Rm where n is strictly greater than 1 and m could be 1 or more.

So, that leaves us with one case, which is n = 1, and m > 1. That is exactly this case here called curves in Rm. So, a curve in $\mathbb{R}^m$ refers to the range of a function f D to $\mathbb{R}^m$ where D is a domain in $\mathbb{R}^1$. So, it is a function from some subset of $\mathbb{R}^1$ to $\mathbb{R}^m$. So, we have seen such examples before.

So just to recall, let us look at lines in $\mathbb{R}^m$. So, for example, if you have a line in $\mathbb{R}^2$, that is an example of a curve. Gamma of f where f is a function of one variable, so this is a curve. In fact, we did talk about general curves in at least in $\mathbb{R}^2$, when we were talking about tangents earlier. So, gamma of f, where f is a function of one variable, this is a curve in $\mathbb{R}^2$.

Well, actually, if it is a function of one variable with a range in something else, then gamma is then also gamma is a curve. So, conics in $\mathbb{R}^2$, this is very, a very important example. So, conics in R2, again, we have talked about these when we did tangents. So, this is, for example, ellipses, parabolas, hyperbolas, the circle. And then here is two more examples. This is one of my favorite examples, the helix in $\mathbb{R}^3$, so that is cosine t , sine t , t. So, this is a parametric equation.

So, as t varies, this vector cosine t sine t, t varies. And how does it vary? It varies like this. So, this is the helix. And as you may or as you may have heard, the helix is used in various branches of science, including biology. So, this is like the slinge, we have this thing called a slinge like this, and then you throw it, it is like a wire, then when you throw it, it kind of bounces from step to step, that is exactly how the helix looks like.

And finally, just for the sake of completeness, here is the subset x, y such that y is equal to x cube. So, this is a subset of $\mathbb{R}^2$, and I will encourage you to draw this subset and see how it looks like. So, conics in $\mathbb{R}^2$ let us give a couple of examples of that. So, one example is $x^2 / a^2 + y^2 / b^2$ is 1, so the ellipse. And a special case is when a is equal to b, that is a circle, so $x^2 + y^2 = a^2$ or we have things like x, y is.

So, these are examples of conics. And I want to make a point here namely that see this there is a slight difference in the representation that we have here. Whereas, as opposed to the representation we have here, this is called a parametric representation. S of t is equal to cosine t sine t, t. Because we are describing the curve in terms of a parameter t, whereas, these two are as sets. So, they are, we are giving the defining questions. Of course, it is quite often one can go between the two, so we can parameterize this.

For example, for if you have $x^2 + y^2$ is a^2, then we know that the parametric equation is a × cosine θ a × sine θ. More generally, if you have x^2 a ^{2} y ^{2} by v^2, you could have a cosine θ b sine θ. So, you can go between parameters and equations. And some× it is useful to do that.

So, let us just summarize what we have seen in this video. We have seen multivariable functions. In particular, we have seen scalar-valued multivariable functions and vector-valued multivariable functions. We saw that we could do arithmetic operations on these specifically addition and subtraction. And when they are scalar-valued, you can actually multiply and divide as well. And then if you have two multiple variable functions, such that there the range of one and the domain of another matches up, then you can compose them as well that gives you a new way, a multivariable function. Finally, we saw curves. We also saw a bunch of examples where we visualize the multivariable functions. Thank you