Week-6
Mathematics for Data Science - 2
Introduction to Vector Space
Practice Assignment

1 Multiple Choice Questions (MCQ)

1. Consider the set $V = \{(-1, x, -y) \mid x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with the usual addition and scalar multiplication as in $\mathbb{R}^3$ and associated statements given below.

- **P:** V is not closed under addition.
- **Q:** V is not closed under scalar multiplication.
- **R:** V has zero element with respect to addition. i.e., there exists some element 0 such that $v + 0 = v$, for all $v \in V$.
- **S:** V is a vector space.

Which of the following statements is true?

○ Option 1: Only P is true.
○ Option 2: Only Q is true.
○ Option 3: Both P and R are true.
○ Option 4: Both R and S are true.
○ **Option 5:** Both P and Q are true.

Solution:

- Let $v_1 = (-1, x_1, -y_1)$ and $v_2 = (-1, x_2, -y_2)$ be in V.
$v_1 + v_2 = (-2, x_1 + x_2, -y_1 - y_2)$, which is clearly not in V as the first coordinate is -2. Therefore, V is not closed under addition.
- Now for any $c(\neq 1) \in \mathbb{R}$ and any $v = (-1, x, -y) \in V$, $cv = (-c, x, -y) \in V$ as the first coordinate is not -1. Hence V is not closed under scalar multiplication.
- Let assume that there exists a zero element $(-1, a, -b)$ in V with respect to addition.
$v + 0 = (-1, x, -y) + (-1, a, -b) = (-2, x + a, -y - b)$, which can never be same as v for any $v = (-1, x, -y) \in V$, as the first coordinate is -2. So V does not have any zero element with respect to addition.
- As we have already proved that V is not closed under addition and scalar multiplication, we need not have to check the other conditions of vector space. The given set V is not a vector space.

2. Match the vector spaces (with the usual scalar multiplication and vector addition as in $M_{3\times 3}(\mathbb{R})$) in column A with their bases in column B in Table : M2W6P1.

	Vector space (Column A)		Basis (Column B)
a)	$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \mid x+y+z=0, \text{ and } x,y,z \in \mathbb{R} \right\}$	i)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$
b)	$V = \{A \mid A \in M_{3\times 3}(\mathbb{R}),$ A is a diagonal matrix $\}$	ii)	$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$
c)	$V = \{A \mid A \in M_{3\times 3}(\mathbb{R}),$ A is a scalar matrix $\}$	iii)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Table : M2W6P1

Choose the correct option.

- ○ Option 1: a → ii, b → i, c → iii.
- ○ **Option 2:** a → ii, b → iii, c → i.
- ○ Option 3: a → i, b → ii, c → iii.
- ○ Option 4: a → iii, b → ii, c → i.

Solution:

- $$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \;\middle|\; x+y+z=0, \text{ and } x,y,z \in \mathbb{R} \right\}$$

  $$= \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \;\middle|\; z=-x-y, \text{ and } x,y,z \in \mathbb{R} \right\}$$

  $$= \left\{ \begin{bmatrix} x & y & -x-y \\ 0 & -x-y & x \\ y & 0 & 0 \end{bmatrix} \;\middle|\; x,y \in \mathbb{R} \right\}$$

  Putting $x=1$ and $y=0$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and putting $x=0$ and $y=1$, we get $\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Therefore the following set,

  $$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

  is linearly independent set and it also spans the given vector space V. Hence it forms a basis of V.

- $$V = \{A \mid A \in M_{3\times 3}(\mathbb{R}), A \text{ is a diagonal matrix } \}$$

  $$= \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \;\middle|\; x,y,z \in \mathbb{R} \right\}$$

  Putting $x=1$ and $y=z=0$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

  Putting $y=1$ and $x=z=0$, we get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

  Putting $z=1$ and $x=y=0$, we get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore the following set,

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V. Hence it forms a basis of V.

- $$V = \{A \mid A \in M_{3\times 3}(\mathbb{R}), A \text{ is a scalar matrix } \}$$

$$= \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} \middle| \, x \in \mathbb{R} \right\}$$

Therefore the following set,

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V. Hence it forms a basis of V.

2 Multiple Select Questions (MSQ)

3. The marks obtained by Karthika, Romy and Farzana in Quiz 1, Quiz 2 and End sem (with the maximum marks for each exam being 100) are shown in Table M2W6P2.

	Quiz 1	Quiz 2	End sem
Karthika	20	10	50
Romy	30	20	70
Farzana	40	30	c

Table: M2W6P2

Let Quiz 1, Quiz 2 and End sem marks obatined by Karthika, Romy and Farzana be represented by vectors. Use the above information, to choose the correct option(s).

◯ Option 1: If Farzana obtained 90 marks in End sem (i.e $c = 90$), then the marks in Quiz 1, Quiz 2 and End sem represent linearly independent vectors.

◯ **Option 2:** If Farzana obtained 80 marks in End sem (i.e $c = 80$), then the marks in Quiz 1, Quiz 2 and End sem represents linearly independent vectors.

- ◯ **Option 3:** If 20% of Quiz 1, 30% of Quiz 2, and 50% of End sem are used to obtain total marks, then the vector representing the total marks is linear combination of vectors representing the marks of Quiz 1, Quiz 2 and End sem.
- ◯ Option 4: If 0% of Quiz 1, 50% of Quiz 2 and 50% of End sem are used to obtain total marks, then the vector representing the total marks is not a linear combination of vectors representing the marks of Quiz 1, Quiz 2 and End sem.

Solution: The marks obtained by Karthika, Romy, and Farzana in Quiz 1 can be expressed as the vector $Q_1 = (20, 30, 40)$, the marks obtained by Karthika, Romy, and Farzana in Quiz 2 can be expressed as the vector $Q_2 = (10, 20, 30)$, and the marks obtained by Karthika, Romy, and Farzana in End sem can be expressed as the vector $S = (50, 70, c)$.

- **For Option 1:** If $c = 90$, then $S = (50, 70, 90)$. We have,

$$3Q_1 - Q_2 - S = 0$$

  Hence, the set $\{Q_1, Q_2, S\}$ is linearly dependent.

- **For Option 2:** If $c = 90$, then $S = (50, 70, 80)$. We can write the vectors Q_1, Q_2 and S as columns of a matrix as follows:

$$A = \begin{bmatrix} 20 & 10 & 50 \\ 30 & 20 & 70 \\ 40 & 30 & 80 \end{bmatrix}$$

  $det(A) \neq 0$. Hence, the vectors are linearly independent.

- **For Option 3:** Let the vector representing the total marks be v.

$$v = \frac{20}{100}Q_1 + \frac{30}{100}Q_2 + \frac{50}{100}S = \frac{1}{5}Q_1 + \frac{3}{10}Q_2 + \frac{1}{2}S$$

  Hence, v is a linear combination of the vectors Q_1, Q_2, and S.

- **For Option 4:** Let the vector representing the total marks be v.

$$v = \frac{0}{100}Q_1 + \frac{50}{100}Q_2 + \frac{50}{100}S = 0\ Q_1 + \frac{1}{2}Q_2 + \frac{1}{2}S$$

  Hence, v is a linear combination of the vectors Q_1, Q_2, and S.

4. Which of the following sets with the given addition and scalar multiplication (scalars are real numbers in every case) form vector spaces?

$V_1 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_1$
Scalar multiplication:

$$c(x, y, z) = \begin{cases} (0, 0, 0) & c = 0 \\ (cx, cy, \frac{z}{c}) & c \neq 0 \end{cases} \qquad (x, y, z) \in V_1,\ c \in \mathbb{R}$$

$V_2 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_2$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_2,\ c \in \mathbb{R}$

$V_3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_3$
Scalar multiplication: $c(x, y, z) = (cx, cy, z);\ (x, y, z) \in V_3,\ c \in \mathbb{R}$

$V_4 = \{(x, y, z) \mid x, y, z \in \mathbb{R}, x + y + z = 1\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_4$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_4,\ c \in \mathbb{R}$

$V_5 = \{(x, y) \mid x, y \in \mathbb{R}, x + 2y = 0\}$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_5$
Scalar multiplication: $c(x, y, z) = (cx, cy);\ (x, y) \in V_5,\ c \in \mathbb{R}$

[**Hint:** Check the axioms related to scalar multiplication in V_1.]

- ○ Option 1: V_1 is a vector space.
- ○ Option 2: V_2 is a vector space.
- ○ **Option 3:** V_3 is not a vector space.
- ○ Option 4: V_4 is a vector space.
- ○ **Option 5:** V_5 is a vector space.

Solution:

- **For V_1:**

$$5(x, y, z) = \left(5x, 5y, \frac{z}{5}\right)$$

and

$$(2+3)(x, y, z) = 2(x, y, z)+3(x+y, z) = \left(2x, 2y, \frac{z}{2}\right)+\left(3x, 3y, \frac{z}{3}\right) = \left(5x, 5y, \frac{z}{2} + \frac{z}{3}\right)$$

Observe that, $\frac{z}{5} \neq \frac{z}{2} + \frac{z}{3}$ is not in general.
Hence,

$$(2 + 3)(x, y, z) \neq 2(x, y, z) + 3(x, y, z)$$

Therefore V_1 is not a vector space.

- **For V_2:**

$$2(x, y, z) = (2x, 2y, 2z)$$

and

$$3(x, y, z) = (3x, 3y, 3z)$$

.
Again we have, $(2 + 3)(x, y, z) = 5(x, y, z) = (5x, 5y, 5z)$

$2(x, y, z)+3(x, y, z) = (2x, 2y, 2z)+(3x, 3y, 3z) = (2x+3x, 2y+3y, 2z\times 3z) = (5x, 5y, 6z^2)$

$5z$ and $6z^2$ are not equal in general, as for example if we choose $z = 1$, then $5z = 5$ and $6z^2 = 6$. So we have,

$$(2 + 3)(x, y, 1) \neq 2(x, y, 1) + 3(x, y, 1)$$

Therefore V_2 is not a vector space.

- **For V_3:**

$$2(x, y, z) = (2x, 2y, z)$$

and

$$3(x, y, z) = (3x, 3y, z)$$

Again we have $(2 + 3)(x, y, z) = 5(x, y, z) = (5x, 5y, z)$

$2(x, y, z)+3(x, y, z) = (2x, 2y, z)+(3x, 3y, z) = (2x+3x, 2y+3y, z+z) = (5x, 5y, 2z)$

$2z = z$ is true only for $z = 0$. Hence for all $z \in \mathbb{R} \setminus \{0\}$,

$$(2 + 3)(x, y, z) \neq 2(x, y, z) + 3(x, y, z)$$

Therefore V_3 is not a vector space.

- **For V_4:** Observe that both the addition and scalar multiplication are the usual addition and scalar multiplication. Hence the zero respect to the addition should be $(0,0,0)$ as usual. But, $(0,0,0) \notin V_4$, as $0+0+0 = 0 \neq 1$.
  Therefore V_4 is not a vector space.
- **For V_5:** Observe that both the addition and scalar multiplication are the usual addition and scalar multiplication. Hence the zero respect to the addition should be $(0,0,0)$ as usual and $(0,0) \in V_5$. Any vector $v \in V_5$ is of the form $v = (-2y, y)$ where $y \in \mathbb{R}$. Now it is easy to check that V_5 is closed under vector addition and scalar multiplication. All the other axioms of vector space are satisfied as the addition and scalar multiplication are the usual ones.
  Therefore V_5 is a vector space.

5. Choose the set of correct options.

- ○ **Option 1:** Any subset of any linearly independent set of vectors is linearly independent.
- ○ Option 2: Any subset of any linearly dependent set of vectors is linearly dependent.
- ○ Option 3: Any subset of any linearly dependent set of vectors is linearly independent.
- ○ **Option 4:** {(1,1,0), (0,1,0), (0,-1,1)} is a basis of vector space $\mathbb{R}^3$ with usual addition and scalar multiplication.
- ○ **Option 5:** {(1,4), (7,2)} is a basis of vector space $\mathbb{R}^2$ with usual addition and scalar multiplication.

Solution:

- **For Option 1:** Let S_1 be a subset of linearly independent set of vectors (say, S). Now if we take any finite collection of vectors $v_1, v_2, \ldots, v_n$ from S_1, then they also belongs to S. So $v_1, v_2, \ldots, v_n$ are linearly independent vectors. It implies that any finite collection of set of vectors from S is linearly independent. So S is linearly independent set. As we have chosen S_1 and S arbitrarily, we can conclude that, Any subset of any linearly independent set of vectors is linearly independent.
- **For Option 2:** Observe that, the set $\{(1,0),(0,1),(1,1)\}$ is a linearly dependent set in $\mathbb{R}^2$. But it's subset $\{(1,0),(0,1)\}$ is linearly independent.
  Moreover, any singleton subset of a set is linearly independent. So if we choose any singleton subset of a linearly dependent set, then it must be linearly independent.
- **For Option 3:** Observe that, the set $S = \{(1,0),(0,1),(1,1),(2,2)\}$ is a linearly dependent set in $\mathbb{R}^2$. $\{(1,0),(0,1),(1,1)\}$ is a subset of S which is again a linearly dependent set.
- **For Option 4:** Consider the following equality:

$$a(1,1,0) + b(0,1,0) + c(0,-1,1) = (a, a+b-c, c) = (0,0,0)$$

  Hence we have $a = 0, a + b - c = 0$, and $c = 0$, which gives us $a = b = c = 0$. Hence the given set is linearly independent. Moreover, let (x, y, z) be any arbitrary vector in $\mathbb{R}^3$, then we can express (x, y, z) in linear combination of the vectors given in the set as follows:

$$(x, y, z) = x(1,1,0) + (y + z - x)(0,1,0) + z(0,-1,1)$$

  Therefore the given set spans $\mathbb{R}^3$. So {(1,1,0), (0,1,0), (0,-1,1)} is a basis of vector space $\mathbb{R}^3$ with usual addition and scalar multiplication.

- **For Option 5:** Consider the following equality:

$$a(1,4) + b(7,2) = (a + 7b, 4a + 2b) = (0,0)$$

Hence we have $a + 7b = 0, 4a + 2b = 0$, which gives us $a = b = 0$. Hence the given set is linearly independent. Moreover, let (x, y) be any arbitrary vector in $\mathbb{R}^2$, then we can express (x, y) in linear combination of the vectors given in the set as follows:

$$(x, y) = \frac{7y - 2x}{26}(1,4) + \frac{4x - y}{26}(7,2)$$

Therefore the given set spans $\mathbb{R}^2$. So {(1,4), (7,2)} is a basis of vector space $\mathbb{R}^2$ with usual addition and scalar multiplication.

3 Numerical Answer Type (NAT):

6. Consider the set of three vectors $S = \{(1, c, -1), (-1, 0, c), (c, c, 2c)\}$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. Find the number of real values of c, so that the above set S will be linearly dependent? [Answer: 1]

   Solution: We can write these vectors as the columns of a matrix as follows:

$$A = \begin{bmatrix} 1 & -1 & c \\ c & 0 & c \\ -1 & c & 2c \end{bmatrix}$$

   The given set S is linearly dependent if $det(A) = 0$
   $det(A) = 1(-c^2) - (-1)(2c^2 + c) + c(c^2) = -c^2 + 2c^2 + c + c^3 = c^3 + c^2 + c = c(c^2 + c + 1) = 0$
   The only real solution of this equation is 1. Hence, the number of real values of c so that the set S will be linearly dependent is 1.

7. Find out the value of a for which the matrix $\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix}$ will be in the spanning set of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$ with usual matrix addition and scalar multiplication. [Answer: 5]

Solution: If the matrix $\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix}$ will be in the spanning set of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$ with usual matrix addition and scalar multiplication, then the matrix $\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix}$ is linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}$$

Hence $x = a$, $y = 3$, and $-x = -5$. Therefore $a = x = 5$.

4 Comprehension Type Question:

Suppose in a village there are four farmers A, B, C and D, each owning 1 acre of land. They cultivate paddy, pulses and/or sugarcane in their lands as follows: Farmer A uses 50% of his land for paddy, 30% for pulses and the remaining for sugarcane. Farmer B uses 40% of her land for paddy and she divides her remaining land equally for pulses and sugarcane. Farmer C uses the whole land for paddy only, and Farmer D uses 30% for paddy, 30% for pulses and the remaining for sugarcane. Using the above information to answer the following questions.

8. Suppose the area used by a farmer for different crops is denoted by a row vector. Let S be the span of the resulting four row vectors . Choose the correct set of options. (MSQ)

   ◯ **Option 1:** The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmers A, B, and C.

   ◯ **Option 2:** The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

   ◯ Option 3: The row vectors corresponding to the area used for different crops by Farmer C can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

   ◯ Option 4: The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer C.

Solution:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{2}{10}$
B	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{3}{10}$
C	$\frac{10}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
D	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Table: M2W6PS1

For Option 1: The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmers A, B, and C, as follows:

$$\left(\frac{3}{10}, \frac{3}{10}, \frac{4}{10}\right) = -1\left(\frac{5}{10}, \frac{3}{10}, \frac{2}{10}\right) + 2\left(\frac{4}{10}, \frac{3}{10}, \frac{3}{10}\right) + 0\left(\frac{10}{10}, \frac{0}{10}, \frac{0}{10}\right)$$

For Option 2: The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B, as we have already found.

$$\left(\frac{3}{10}, \frac{3}{10}, \frac{4}{10}\right) = -1\left(\frac{5}{10}, \frac{3}{10}, \frac{2}{10}\right) + 2\left(\frac{4}{10}, \frac{3}{10}, \frac{3}{10}\right)$$

For Option 3: Let the row vectors corresponding to the area used for different crops by Farmer A, Farmer B, and Farmer C be written as the columns of a matrix (say A).

$$A = \begin{bmatrix} \frac{5}{10} & \frac{4}{10} & \frac{10}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{0}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{0}{10} \end{bmatrix}$$

$$det(A) \neq 0$$

Hence, the row vectors corresponding to the area used for different crops by Farmer A, Farmer B, and Farmer C are linearly independent. Therefore, the row vectors corresponding to the area used for different crops by Farmer C cannot be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

For Option 4: Let the row vectors corresponding to the area used for different crops by Farmer A, Farmer C, and Farmer D be written as the columns of a matrix (say A).

$$A = \begin{bmatrix} \frac{5}{10} & \frac{10}{10} & \frac{3}{10} \\ \\ \frac{3}{10} & \frac{0}{10} & \frac{3}{10} \\ \\ \frac{2}{10} & \frac{0}{10} & \frac{4}{10} \end{bmatrix}$$

$$det(A) \neq 0$$

Hence, the row vectors corresponding to the area used for different crops by Farmer A, Farmer C, and Farmer D are linearly independent. Therefore, the row vectors corresponding to the area used for different crops by Farmer D cannot be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer C.

9. Let S be the vectors space defined in the previous question, with the usual addition and scalar multiplication on $\mathbb{R}^3$. Which of the following sets will be a basis of S? (MCQ)

   - ◯ Option 1: $\{(5, 3, 2)\}$
   - ◯ Option 2: $\{(5, 3, 2), (4, 3, 3)\}$
   - ◯ **Option 3:** $\{(5, 3, 2), (4, 3, 3), (10, 0, 0)\}$
   - ◯ Option 4: $\{(5, 3, 2), (4, 3, 3), (10, 0, 0), (3, 3, 4)\}$

Solution: In the previous question we have already proved that, the row vectors corresponding to the area used for different crops by Farmer A, Farmer B, and Farmer C are linearly independent and the row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmers A, B, and C. But the row vectors corresponding to the area used for different crops by Farmer C cannot be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

Hence the set $\{\frac{1}{10}(5, 3, 2), \frac{1}{10}(4, 3, 3)\}$ does not span S, which also implies that the set $\{\frac{1}{10}(5, 3, 2)\}$ does not span S. Hence the set given in Option 1 and 2 does not span S.
The set $\{\frac{1}{10}(5, 3, 2), \frac{1}{10}(4, 3, 3), \frac{1}{10}(10, 0, 0), \frac{1}{10}(3, 3, 4)\}$ is linearly dependent. Hence the set given in Option 4 does not span S.
The set $\{\frac{1}{10}(5, 3, 2), \frac{1}{10}(4, 3, 3), \frac{1}{10}(10, 0, 0)\}$ is linearly independent and it also spans S. Hence the set given in Option 3 is a basis of S.

10. Suppose Farmer B buys the same amount of land (1 acre) and uses it in the same ratio for different crops as she was using for her land earlier. Choose the correct set of options. (MCQ)

- ◯ Option 1: Farmer A sells his whole land to Farmer D, and Farmer D uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B is using more amount of area of land for pulses than Farmer D.
- ◯ **Option 2:** Farmer A sells his whole land to Farmer D, and Farmer D uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B and Farmer D are using the same amount of area of land for paddy.
- ◯ Option 3: Farmer A sells his whole land to Farmer C, and Farmer C uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B and Farmer C are using the same amount of area of land for paddy.
- ◯ Option 4: Farmer A sells his whole land to Farmer C, and Farmer C uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B is using more amount of area of land for paddy than Farmer C.

Solution:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{2}{10}$
B	$\frac{4}{10} + \frac{4}{10} = \frac{8}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$
C	$\frac{10}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
D	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Table: M2W6PS2

For Option 1 and 2:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	0	0	0
B	$\frac{4}{10} + \frac{4}{10} = \frac{8}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$
C	$\frac{10}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
D	$\frac{3}{10} + \frac{5}{10} = \frac{8}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$	$\frac{4}{10} + \frac{2}{10} = \frac{6}{10}$

Table: M2W6PS2

Hence Option 1 is not true, but Option 2 is true.
For Option 3 and 4:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	0	0	0
B	$\frac{4}{10} + \frac{4}{10} = \frac{8}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$	$\frac{3}{10} + \frac{3}{10} = \frac{6}{10}$
C	$\frac{10}{10} + \frac{5}{10} = \frac{15}{10}$	$\frac{0}{10} + \frac{3}{10} = \frac{3}{10}$	$\frac{0}{10} + \frac{2}{10} = \frac{2}{10}$
D	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Table: M2W6PS3

Hence both Option 3 and Option 4 are false.