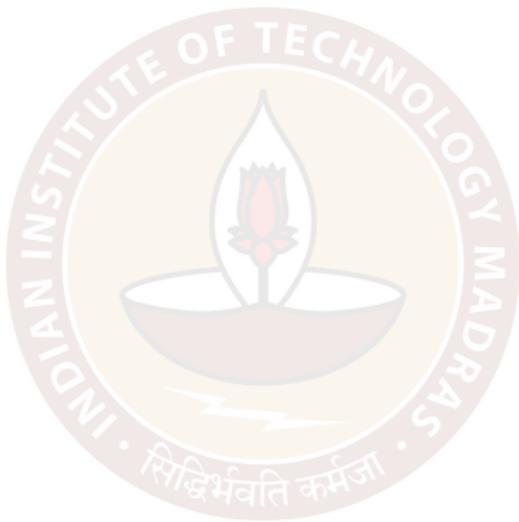


The Hessian matrix and local extrema for $f(x, y, z)$

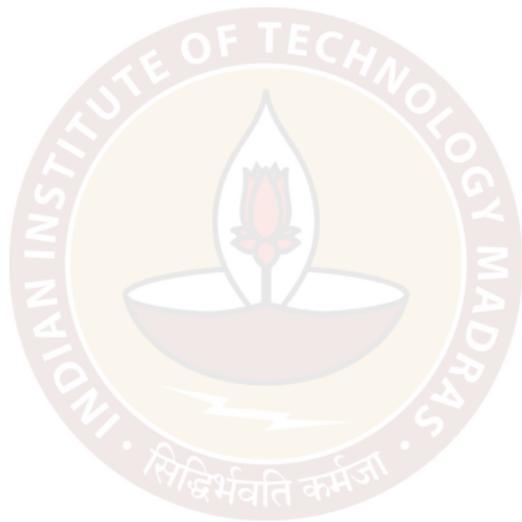
Sarang S. Sane

Recall : The Hessian test for $f(x, y)$



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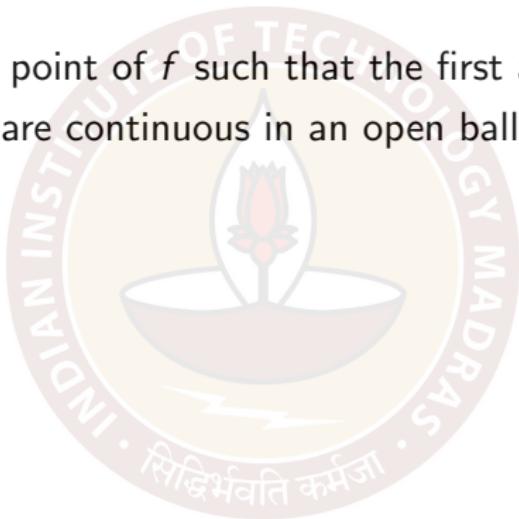
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Recall : The Hessian test for $f(x, y)$

Let $f(x, y)$ be a function defined on a domain D in \mathbb{R}^2 .

Let $\underset{\sim}{a}$ be a critical point of f such that the first and second order partial derivatives are continuous in an open ball around $\underset{\sim}{a}$.



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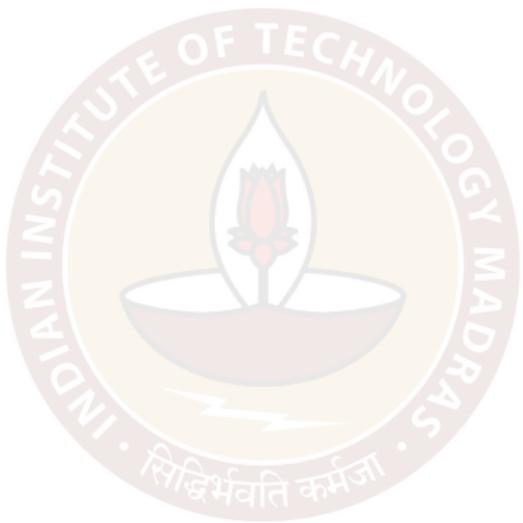
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The Hessian test : Classifying critical points of $f(x, y, z)$



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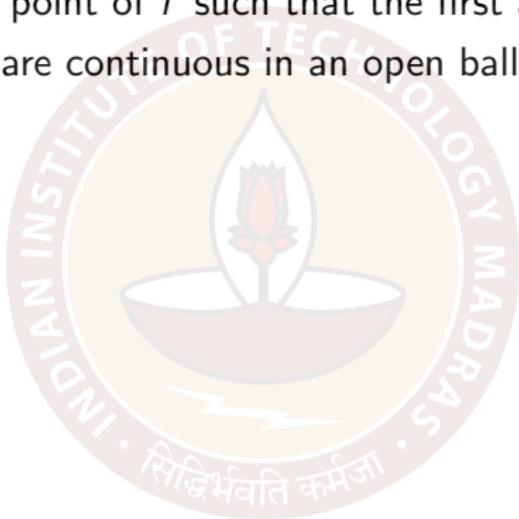
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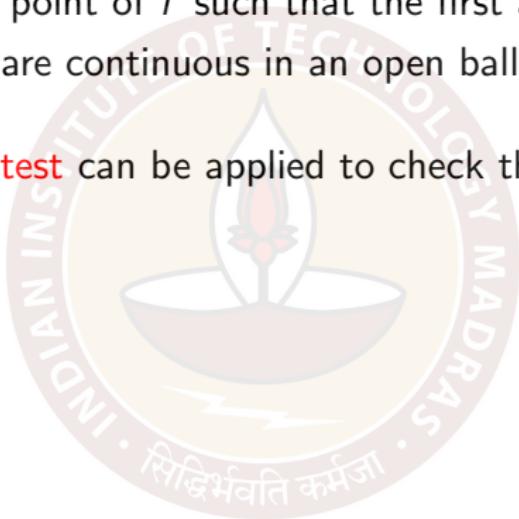


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Understanding the terms better

The terms involved in the test are : f_{xx} , $\boxed{(f_{xx}f_{yy} - f_{xy}^2)} \underset{\sim}{(a)}$ and $\det(Hf(a))$.

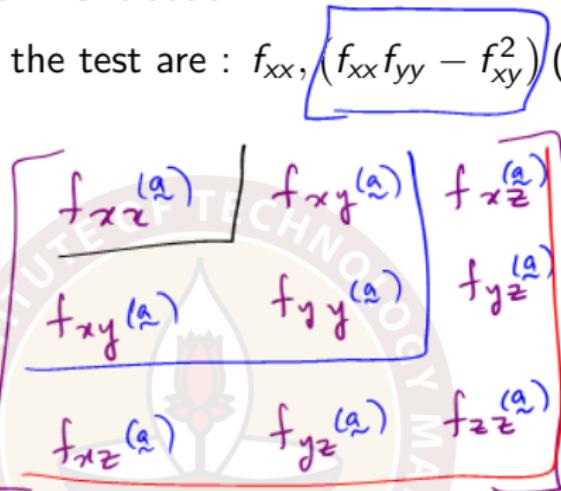
$Hf(\tilde{a})$

$\begin{matrix} \sqrt{1} \\ 1 \times 1 \\ 2 \times 2 \\ 3 \times 3 \end{matrix}$

$+ \quad + \quad +$
 $- \quad + \quad -$

all other
non-degenerate cases
i.e. $\det(Hf(a)) \neq 0$

degenerate case $\det(Hf(a)) = 0$


$$\begin{bmatrix} f_{xx}(a) & f_{xy}(a) & f_{xz}(a) \\ f_{yx}(a) & f_{yy}(a) & f_{yz}(a) \\ f_{zx}(a) & f_{zy}(a) & f_{zz}(a) \end{bmatrix}$$

→ local min.

→ local max.

→ saddle point

Inconclusive.

Examples

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla f = (2x, 2y, 2z) \quad \text{critical pt. : } (0, 0, 0)$$

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = Hf(0, 0, 0) \quad \therefore (0, 0, 0) \text{ is a}$$

$$\det(Hf(0, 0, 0)) = 8 > 0$$

$$\det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 > 0$$

$$f_{xx}(0, 0, 0) = 2 > 0$$

$\therefore (0, 0, 0)$ is a local min.

$$f(x, y, z) = -x^2 - y^2 - z^2$$

$$\nabla f = (-2x, -2y, -2z) \quad \text{critical pt. : } (0, 0, 0)$$

$$Hf = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = Hf(0, 0, 0) \quad \therefore (0, 0, 0)$$

$$\det(Hf(0, 0, 0)) = -8 < 0 \quad -$$

$$\det \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = 4 > 0 \quad +$$

$$f_{xx}(0, 0, 0) = -2 < 0 \quad -$$

$\therefore (0, 0, 0)$ is a local max.

$$f(x, y, z) = x^2 - y^2 + z^2$$

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (0, 0, 0)$$

$$\det(Hf) = -8 \quad -$$

$$\det \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = -4 \quad -$$

$$f_{xx}(0, 0, 0) = 2 \quad +$$

$\therefore (0, 0, 0)$ is a saddle pt.

$$f(x, y, z) = x^4 + y^4 + z^4$$

$$\nabla f = (4x^3, 4y^3, 4z^3) \quad \text{critical pt. : } (0, 0, 0)$$

$$Hf = \begin{bmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{bmatrix} .$$

$$Hf(0, 0, 0) = \mathbb{O}_{3 \times 3}$$

Inconclusive.

Example

$$f(x, y) = xy + yz + zx$$

$$\nabla f = (y+z, z+x, x+y).$$

Equating to 0, we get $\begin{cases} x = -y = z \\ x = -z \end{cases} \Rightarrow x = y = z = 0.$

Critical pt.

$$H_f = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = H_f(0, 0, 0)$$

$$\det(H_f(0, 0, 0)) = 0 \times \det \begin{bmatrix} \end{bmatrix} - 1 \times \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 1 \times \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

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$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0.$$

$(0, 0, 0)$ is a saddle point.

Example

$$f(x, y) = x^4 + y^4 + z^4 + xyz$$

$$\nabla f = (4x^3 + yz, 4y^3 + xz, 4z^3 + xy).$$

Equating to 0, we get:
 $4x^3 + yz = 4y^3 + xz = 4z^3 + xy = 0.$

Case 1: $x = y = z = 0$.

All are non-zero.

$$4x^4 + xyz = 0 \Rightarrow 4x^4 = -xyz = 4z^4.$$

$$4y^4 + xyz = 0 \Rightarrow 4y^4 = -xyz = \pm z.$$

$$x^4 = y^4 = z^4 \Rightarrow x = \pm y = \pm z.$$

$$x^4 = y^4 = z^4 \Rightarrow 4x^4 + z^2 = 0 \Rightarrow x = \frac{1}{4} \text{ or } x = -\frac{1}{4}$$

Substitute in $4x^3 + yz = 0 \Rightarrow 4x^3 - z^2 = 0$

Critical pts.: $(0, 0, 0), \left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right), \left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$

$$Hf = \begin{bmatrix} 12x^2 & z & y \\ z & 12y^2 & x \\ y & x & 12z^2 \end{bmatrix}, Hf\left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \begin{bmatrix} 3/4 & 1/4 & 1/4 \\ 1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & 3/4 \end{bmatrix} \rightarrow \text{Inconclusive.}$$

$$Hf\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right) = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ 1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix} \quad \det(H+) > 0 \quad \det(L) > 0 \quad \det(L') > 0$$

+++

Thank you

