The Gram-Schmidt process

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An overview of the Gram-Schmidt process

In an inner product space

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Any basis

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$$\xrightarrow[\text{process}]{\text{Gram-Schmidt}}$$

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$$\boxed{\begin{array}{c}\text{Any basis}\\ x_1, x_2, \ldots, x_n\end{array}} \xrightarrow[\text{process}]{\text{Gram-Schmidt}} \boxed{\begin{array}{c}\text{Orthonormal basis}\\ v_1, v_2, \ldots, v_n\end{array}}$$

Example and intuition

Consider the basis $\beta = \{(1, 2, 2), (-1, 0, 2), (0, 0, 1)\}$ for $\mathbb{R}^3$.

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$$\text{Define } v_2 = (-1,0,2) - P_{v_1}((-1,0,2)) = \underbrace{(I - P_{v_1})(-1,0,2)}_{\in \langle v_1 \rangle^{\perp}}$$

$$= (-1,0,2) - \frac{\langle (-1,0,2),(1,2,2) \rangle}{\langle (1,2,2),(1,2,2) \rangle}(1,2,2)$$

$$= \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \quad .$$

$$\boxed{\langle v_2, v_1 \rangle = 0}$$

W $\quad W^{\perp} = \{ v \mid \langle v, w \rangle = 0 \ \forall w \in W \}$ = Null space of P_W.

$P_W(v) = 0 \iff w \in W^{\perp}$.

$W^{\perp} \ni \boxed{(I - P_W)(v)} = v - P_W(v)$ & $P_W(v - P_W(v)) = 0$.

Example and intuition (contd.)

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$(I - P_W)$

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Define $v_3 = (0, 0, 1) - P_{v_1}((0, 0, 1)) - P_{v_2}((0, 0, 1))$

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$$\begin{aligned}
\text{Define } v_3 &= (0,0,1) - P_{v_1}((0,0,1)) - P_{v_2}((0,0,1)) \\
&= (0,0,1) - \frac{\langle (0,0,1)(1,2,2) \rangle}{\langle (1,2,2),(1,2,2) \rangle}(1,2,2) \\
&\quad - \frac{\langle (0,0,1), (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}) \rangle}{\langle \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \rangle} \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)
\end{aligned}$$

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\begin{aligned}
\text{Define } v_3 &= (0,0,1) - P_{v_1}((0,0,1)) - P_{v_2}((0,0,1)) \\
&= (0,0,1) - \frac{\langle (0,0,1)(1,2,2) \rangle}{\langle (1,2,2),(1,2,2) \rangle}(1,2,2) \\
&\qquad - \quad \frac{\langle (0,0,1), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \rangle}{\langle \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right), \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \rangle} \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \\
&= \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right) \quad .
\end{aligned}
$$

Example and intuition (contd.)

We want a vector which is orthogonal to both v_1 and v_2, i.e. a vector in $Span(\{v_1, v_2\})^\perp$, so we use the projection $P_{Span(\{v_1,v_2\})}$ to $Span(\{v_1, v_2\})$.

Define
$$\begin{aligned} v_3 &= (0,0,1) - P_{v_1}((0,0,1)) - P_{v_2}((0,0,1)) \\ &= (0,0,1) - \frac{\langle (0,0,1)(1,2,2)\rangle}{\langle (1,2,2),(1,2,2)\rangle}(1,2,2) \\ &\quad - \frac{\langle (0,0,1), (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})\rangle}{\langle (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}), (-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3})\rangle} \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) \\ &= \left(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\right) . \end{aligned}$$

Check $\langle v_1, v_2 \rangle$
$= \langle v_1, v_3 \rangle$
$= \langle v_2, v_3 \rangle = 0$.

Thus $\{v_1, v_2, v_3\}$ is an orthogonal basis and dividing each vector by its norm yields an orthonormal basis

$$\left\{ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \right\}.$$

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$$v_1 = x_1 \ ; \ w_1 = \frac{v_1}{||v_1||}$$

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$$v_2 = x_2 - \langle x_2, w_1 \rangle w_1 \quad ; \quad w_2 = \frac{v_2}{||v_2||}$$

$$\vdots \qquad \vdots \qquad \vdots$$

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$$v_2 = x_2 - \langle x_2, w_1 \rangle w_1 \quad ; \quad w_2 = \frac{v_2}{||v_2||}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$v_i = x_i - \langle x_i, w_1 \rangle w_1 - \langle x_i, w_2 \rangle w_2 - \ldots - \langle x_i, w_{i-1} \rangle w_{i-1} \; ; \; w_i = \frac{v_i}{||v_i||}$$

$$\vdots \qquad \vdots \qquad \vdots$$

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$\mathrm{Span}(\{x_1, x_2, \ldots, x_{i-1}\})$
$= \mathrm{Span}(\{v_1, v_2, \ldots, v_{i-1}\})$
$= \mathrm{Span}(\{w_1, w_2, \ldots, w_{i-1}\})$

$$v_1 = x_1 \ ; \ w_1 = \frac{v_1}{||v_1||}$$

$$v_2 = x_2 - \langle x_2, w_1 \rangle w_1 \quad ; \quad w_2 = \frac{v_2}{||v_2||}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$v_i = x_i - \boxed{\langle x_i, w_1 \rangle w_1 - \langle x_i, w_2 \rangle w_2 - \ldots - \langle x_i, w_{i-1} \rangle w_{i-1}} \ ; \ w_i = \frac{v_i}{||v_i||}$$

$= P_{\mathrm{Span}(\{w_1, \ldots, w_{i-1}\})}(x_i)$
$= P_{\mathrm{Span}(\{x_1, \ldots, x_{i-1}\})}(x_i)$

$$\vdots \qquad \vdots \qquad \vdots$$

$$v_n = x_n - \langle x_n, w_1 \rangle w_1 - \langle x_n, w_2 \rangle w_2 - \ldots$$

$$\ldots - \langle x_n, w_{n-1} \rangle w_{n-1} \qquad ; \qquad w_n = \frac{v_n}{||v_n||}$$

Main take-homes

Theorem: Any finite-dimensional vector space with an inner product has an orthonormal basis.

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Any basis can be changed to an orthonormal basis using the Gram-Schmidt process.

Thank you