Week-3
Mathematics for Data Science - 2
Critical points, Area under the curve, Integration
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. Suppose a wire of length m is cut into two pieces. One part is bent into a circle and other into a square. The minimum value of the combined area of the circle and the square is

   - ◯ Option 1: $\dfrac{m^2}{\pi+4}$
   - ◯ **Option 2:** $\dfrac{m^2}{4(\pi+4)}$
   - ◯ Option 3: $\dfrac{m^2}{\pi+2}$
   - ◯ Option 4: $\dfrac{m^2}{2(\pi+2})$

   Solution: Let the piece that is bent into a circle have length x and the remaining piece of wire that bent into a square have length $m-x$. Then radius of the circle is $r = \dfrac{x}{2\pi}$ and side of the square is $a = \dfrac{m-x}{4}$.

$$\begin{aligned}
\text{A} \;=\; \text{Total area} &= \text{Area of the circle} + \text{Area of the square} \\
&= \pi r^2 + a^2 \\
&= \pi \times \left(\frac{x}{2\pi}\right)^2 + \left(\frac{m-x}{4}\right)^2 \\
&= \frac{x^2}{4\pi} + \frac{(m-x)^2}{16}
\end{aligned}$$

To get the minima we can equate $\dfrac{dA}{dx}$ to 0.

$$\frac{dA}{dx} = \frac{x}{2\pi} - \frac{(m-x)}{8} = 0 \implies \frac{x}{2\pi} = \frac{(m-x)}{8} \implies x = \frac{m\pi}{4+\pi}$$

$\dfrac{d^2A}{dx^2} = \dfrac{1}{2\pi} + \dfrac{1}{8}$ is always greater than 0. Hence, $x = \dfrac{m\pi}{4+\pi}$ is a point of minimum.

Then the minimum value of the combined area is

$$\begin{aligned}
A_{min} &= \frac{(m\pi)^2}{4\pi(4+\pi)^2} + \frac{(4m+m\pi-m\pi)^2}{16(4+\pi)^2} \\
&= \frac{(m\pi)^2}{4\pi(4+\pi)^2} + \frac{(4m)^2}{16(4+\pi)^2} \\
&= \frac{(m\pi)^2}{4\pi(4+\pi)^2} + \frac{m^2}{(4+\pi)^2} \\
&= \frac{m^2\pi}{4(4+\pi)^2} + \frac{m^2}{(4+\pi)^2} \\
&= \frac{m^2\pi+4m^2}{4(4+\pi)^2} \\
&= \frac{m^2(\pi+4)}{4(4+\pi)^2} \\
&= \frac{m^2}{4(4+\pi)}
\end{aligned}$$

2. Match the given functions in Column A with the (signed) area between its graph and the interval $[-1, 1]$ on the X-axis in column B and the pictures of their graphs and the highlighted region corresponding to the area computation in Column C, given in Table M2W3P1.

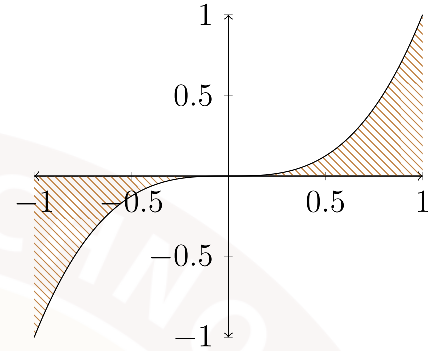
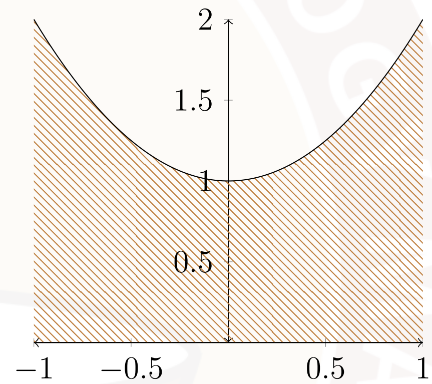
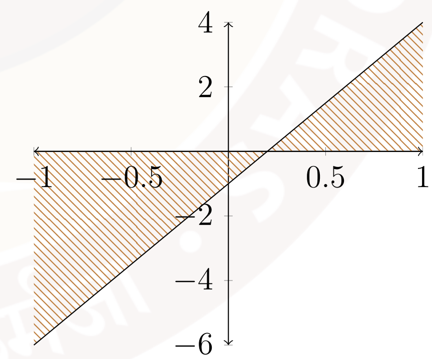
	Functions (Column A)		Area under the curve (Column B)		Graphs (Column C)
i)	$f(x) = 5x - 1$	a)	$\dfrac{\pi}{2}$	1)	
ii)	$f(x) = x^3$	b)	0	2)	
iii)	$f(x) = \dfrac{1}{x^2+1}$	c)	-2	3)	

Table: M2W3P1

◯ Option 1: i) $\to$ b) $\to$ 1), ii) $\to$ c) $\to$ 3), iii) $\to$ a) $\to$ 2).

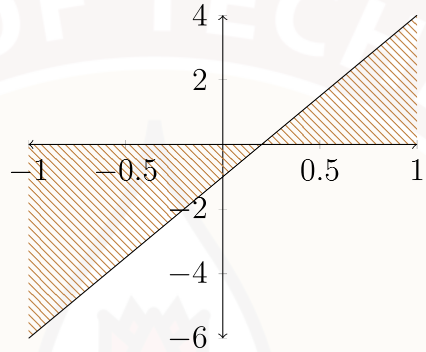
◯ Option 2: i) → b) → 3), ii) → a) → 1), iii) → c) → 2).

◯ **Option 3:** i) → c) → 3), ii) → b) → 1), iii) → a) → 2).

◯ Option 4: i) → b) → 3), ii) → c) → 1), iii) → a) → 2).

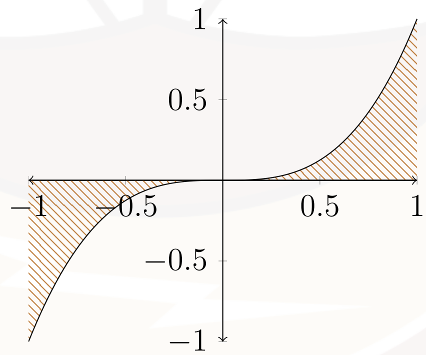
Solution:

- $f(x) = 5x - 1$



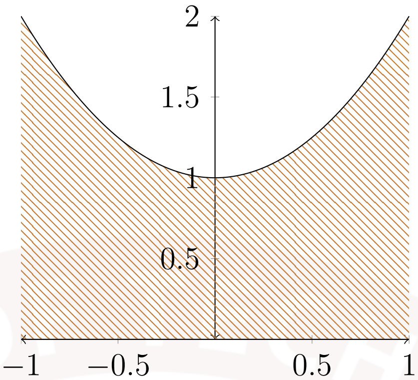
$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} (5x-1)\ dx = \int_{-1}^{1} 5x\ dx - \int_{-1}^{1} 1\ dx = (5 \times \frac{x^2}{2})\Big|_{-1}^{1} - x\Big|_{-1}^{1} = -2$$

- $f(x) = x^3$



$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} x^3\ dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

- $f(x) = \dfrac{1}{x^2+1}$



$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} \frac{1}{x^2+1}\ dx = \tan^{-1}(x)\Big|_{-1}^{1} = \tfrac{\pi}{2}$$

2 Multiple Select Questions (MSQ)

3. Suppose $\int x\ln(1+x)dx = f(x)\ln(x+1) - \frac{x^2}{4} + Ax + B$, where B is the constant of integration. Which of the following are correct?

   - ◯ **Option 1:** $f(x) = \frac{x^2-1}{2}$
   - ◯ Option 2: $f(x) = \frac{x^2-1}{4}$
   - ◯ Option 3: $A = \frac{1}{4}$
   - ◯ **Option 4:** $A = \frac{1}{2}$

Solution: By using integration by parts:

$$\begin{aligned}
\int x\ln(1+x)\ dx &= \ln(1+x)\int x\ dx - \int\left\{\frac{d(\ln(1+x))}{dx}\int x\ dx\right\}dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2+x-x}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2+x}{2(1+x)}\ dx + \int \frac{x}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1-1}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1}{2(1+x)}\ dx - \int \frac{1}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(1+x) + B \\
&= \frac{x^2-1}{2}\ln(1+x) - \frac{x^2}{4} + \frac{x}{2} + B
\end{aligned}$$

If we equate coefficients then $f(x) = \dfrac{x^2-1}{4}$, and $A = \dfrac{1}{2}$.

4. Consider the function $f(x) = x^3 - 6x$. Which of the following options are correct?

- ◯ Option 1: f has neither local maxima nor local minima.
- ◯ **Option 2**: $\sqrt{2}$ is a local minimum.
- ◯ Option 3: $\sqrt{2}$ is a local maximum.
- ◯ **Option 4:** $-\sqrt{2}$ is a local maximum.
- ◯ Option 5: $-\sqrt{2}$ is a local minimum.
- ◯ **Option 6:** f has two critical points.

Solution: Number of critical points will be same as the number of solutions of the following equation,

$$f^{'}(x) = 3x^2 - 6 = 0 \implies x^2 - 2 = 0 \implies (x - \sqrt{2})(x + \sqrt{2}) = 0$$

Hence, the number of critical points is 2.

Now, $f^{''}(\sqrt{2}) > 0$, and $f^{''}(-\sqrt{2}) < 0$. Therefore, $\sqrt{2}$ is a local minimum and $-\sqrt{2}$ is a local maximum.

5. Choose the set of correct options.

- ◯ **Option 1:** The left Riemann sum of the function $f(x) = x + 5$ on the interval $[1, 10]$ divided into three sub-intervals of equal length is 81.
- ◯ **Option 2:** The middle Riemann sum of the function $f(x) = x^2$ on the interval $[0, 8]$ divided into four sub-intervals of equal length is 168.
- ◯ **Option 3:** The left Riemann sum of the function $f(x) = x + 5$ on the interval $[3, 6]$ divided into n sub-intervals of equal length is $\frac{57}{2}$, as n tends to ∞.
- ◯ Option 4: The right Riemann sum of the function $f(x) = \frac{1}{x}$ on the interval $[1, 9]$ divided into four sub-intervals of equal length is $\frac{16}{15}$.

Solution:

Option 1: If we divide $[1, 10]$ in three sub-intervals of equal length, we get the partition: $\{1, 4, 7, 10\}$. The left Riemann sum of the function $f(x) = x + 5$ is:

$$(4-1)f(1)+(7-4)f(4)+(10-7)f(7) = 3\times(f(1))+f(4)+f(7)) = 3\times(6+9+12) = 81$$

Option 2: If we divide $[0, 8]$ in four sub-intervals of equal length, we get the partition: $\{0, 2, 4, 6, 8\}$. The middle Riemann sum of the function $f(x) = x^2$ is:

$$(2-0)f(1)+(4-2)f(3)+(6-4)f(5)+(8-6)f(7) = 2\times(f(1))+f(3)+f(5)+f(7)) = 2 \times (1 + 9 + 25 + 49) = 168$$

Option 3: If we divide $[3, 6]$ in n sub-intervals of equal length, we get the partition: $\{3, 3 + \dfrac{3}{n}, 3 + \dfrac{6}{n} \ldots 6 - \dfrac{3}{n}, 6\}$. The left Riemann sum of the function $f(x) = x + 5$ is:

$$\frac{3}{n}f(3)+\frac{3}{n}f\Big(3+\frac{3}{n}\Big)+\frac{3}{n}f\Big(3+\frac{6}{n}\Big)+\cdots+\frac{3}{n}f\Big(6-\frac{3}{n}\Big)$$
$$=\frac{3}{n}\Bigg[(3+5)+(3+\frac{3}{n}+5)+(3+\frac{6}{n}+5)+\cdots+(6-\frac{3}{n}+5)\Bigg]$$
$$=\Big(\frac{3}{n}\times 5n\Big)+\frac{3}{n}\Bigg[3+(3+\frac{3}{n})+(3+\frac{6}{n})+\cdots+(6-\frac{3}{n})\Bigg]$$
$$=15+\frac{3}{n}\Bigg[3+(3+\frac{3}{n})+(3+\frac{6}{n})+\cdots+\Big(3+(3-\frac{3}{n})\Big)\Bigg]$$
$$=15+\Big(\frac{3}{n}\times 3n\Big)+\frac{3}{n}\Bigg[\frac{3}{n}+\frac{6}{n}+\cdots+\frac{3n-3}{n}\Bigg]$$
$$=15+9+\frac{9}{n^2}\Bigg[1+2+\cdots+(n-1)\Bigg]$$
$$=24+\frac{9}{n^2}\frac{(n-1)n}{2}$$

As n tends to ∞, the above sum converges to $24+\frac{9}{2}=\frac{57}{2}$.

Option 4: If we divide $[1,9]$ in four sub-intervals of equal length, we get the partition: $\{1,3,5,7,9\}$. The right Riemann sum of the function $f(x)=\frac{1}{x}$ is:

$(3-1)f(3)+(5-3)f(5)+(7-5)f(7)+(9-7)f(9) = 2\times(f(3))+f(5)+f(7)+f(9)) = 2\times(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9})=\frac{496}{315}$

3 Numerical Answer Type (NAT)

6. The value of $\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\,dx$ is [Answer: 2]

Solution:

We make the substitution $t^2 = x \implies \frac{dx}{dt} = 2t \implies dx = 2tdt$, and the limits change to $t = \sqrt{0} = 0$ and $t = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$ The integral becomes

$$2\int_0^{\frac{\pi}{2}} t\sin t\;dt.$$

Now, $2\int_0^{\frac{\pi}{2}} t\sin t\;dt = \Big[-tcos(t) + sin(t)\Big]_0^{\frac{\pi}{2}} = 2$ [By using integration by parts].

7. Suppose $x + y = 16$. What is the value of xy when $x^3 + y^3$ is minimum? [Answer: 64]

Solution: It is given that $x + y = 16 \implies y = 16 - x$. so, $x^3 + y^3 = x^3 + (16 - x)^3$. Let $f(x) = x^3 + (16 - x)^3 = x^3 + 16^3 - x^3 - 768x + 48x^2 = 16^3 - 768x + 48x^2$. To get a minima we can equate $\frac{df}{dx}$ to 0.

$$\frac{df}{dx} = -768 + 96x = 0 \implies x = 8.$$

$\frac{d^2f}{dx^2} = 96 > 0$. Hence, $x = 8$, and $y = 16 - x = 8$. The value of xy is 64.

4 Comprehension Type Question:

A car manufacturer determines that in order to sell x number of cars, the price per car(in lakh) must be $f(x) = 1000 - x$, if $x \leq 800$, and the manufacturer also determines that the total cost(in lakh) of producing x number of cars is

$$g(x) = \begin{cases} 30000 + 300x & \text{if } x \leq 400, \\ 100x + 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

Although in the above context, x can take only integer values, assume that x is a continuous variable in the interval $[0, 800]$ and that the functions $f(x)$ and $g(x)$ are defined as above on this entire interval.
Answer Questions 8,9, and 10 using the data given above.

8. Suppose the company can produce a maximum of 400 cars due to a production issue. The number of cars the company should produce and sell in order to maximize profit is

   ◯ **Option 1:** 350
   ◯ Option 2: 250
   ◯ Option 3: 300
   ◯ Option 4: 200

   Solution: If the company sells x number of cars then the total income is $I(x) = x(1000 - x)$. Total profit of the company is:

   $$\text{Profit} = \text{Total income - Total cost}$$
   $$P(x) = I(x) - g(x)$$

   $$P(x) = \begin{cases} x(1000 - x) - (30000 + 300x) & \text{if } x \leq 400, \\ x(1000 - x) - (100x + 110000) & \text{if } 400 < x \leq 800, \end{cases}$$

   which is same as:

   $$P(x) = \begin{cases} -x^2 + 700x - 30000 & \text{if } x \leq 400, \\ -x^2 + 900x - 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

   It is given that the company can produce a maximum of 400 cars due to a production issue, i.e, $x \leq 400$. So, the total profit is: $P(x) = -x^2 + 700x - 30000$.

   To get a maxima we can equate $\frac{dP}{dx}$ to 0.

   $$\frac{dP}{dx} = -2x + 700 = 0 \implies x = 350.$$

$\frac{d^2P}{dx^2} = -2 < 0$. Hence $x = 350$ is a point of maximum. Therefore, company should produce and sell 350 numbers of cars in order to maximize its profit.

9. Suppose the company can produce a minimum of 401 cars and a maximum of 800 cars due to a production issue. The number of cars the company should produce and sell in order to maximize profit is

   - ◯ Option 1 750
   - ◯ Option 2: 650
   - ◯ Option 3: 550
   - ◯ **Option 4:** 450

   Solution: It is given that the company can produce a minimum of 401 cars and a maximum of 800 cars due to a production issue, i.e, $401 \leq x \leq 800$. So, the total profit is: $P(x) = -x^2 + 900x - 110000$.

   To get a maxima we can equate $\dfrac{dP}{dx}$ to 0.

$$\frac{dP}{dx} = -2x + 900 = 0 \implies x = 450.$$

   $\dfrac{d^2P}{dx^2} = -2 < 0$. Hence $x = 450$ is a point of maximum. Therefore, company should produce and sell 450 numbers of cars in order to maximize its profit.

10. Let $P(x)$ denotes the function representing the profit of the company. Choose the set of correct statements.

- ◯ **Option 1:** $P(x)$ is continuous in the interval $[0, 800]$
- ◯ **Option 2:** The function $P(x)$ has two local maxima in the interval $[0, 800]$.
- ◯ Option 3: All the global maxima of $P(x)$ lie in the interval $[0, 400]$.
- ◯ **Option 4:** All the global maxima of $P(x)$ lie in the interval $[300, 500]$.

Solution:

$$P(x) = \begin{cases} -x^2 + 700x - 30000 & \text{if } x \leq 400, \\ -x^2 + 900x - 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

Domain of P is $[0, 800]$. Clearly, P is continuous $[0, 400) \cup (400, 800]$. So, we need to check the continuity of the function only at $x = 400$.

$\mathbf{LHL\ of\ P(x)\ at\ x = 400}$:

$$\lim_{x \to 400^-} P(x) = \lim_{x \to 400^-} -x^2 + 700x - 30000 = -(400)^2 + 280000 - 30000 = 90000$$

$\mathbf{RHL\ of\ P(x)\ at\ x = 400}$:

$$\lim_{x \to 400^+} P(x) = \lim_{x \to 400^+} -x^2 + 900x - 110000 = -(400)^2 + 360000 - 110000 = 90000$$

Hence, $P(x)$ is continuous in the interval $[0, 800]$. From the solutions of Q8 and Q9, it is clear that the function $P(x)$ has two local maxima in the interval $[0, 800]$. Now,

$P(0) = -30000, P(350) = 92500, P(400) = 90000, P(450) = 92500$, and $P(800) = -30000$.

So, both $x = 350$ and $x = 450$ are global maxima of $P(x)$.