Critical points for multivariable functions

Sarang S. Sane

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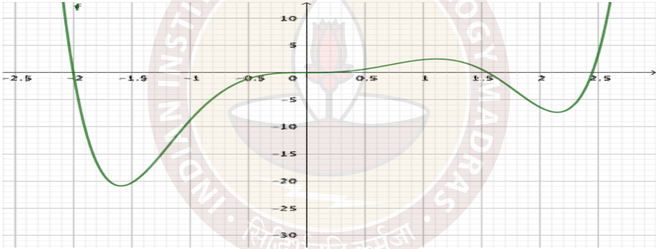

Figure: $f(x) = (x^2 - 4x + 3.8)(x+2)x^3$

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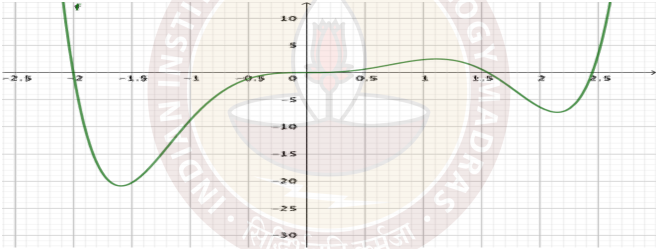

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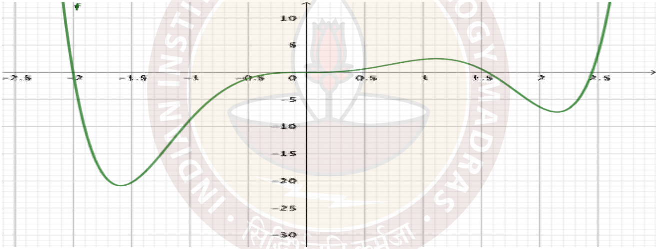

Figure: $f(x) = (x^2 - 4x + 3.8)(x + 2)x^3$

Not every critical point is a point of local extremum. A saddle point is a critical point which is not a point of local extremum.

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A local extremum (or point of local extremum) of f is either a local maximum or a local minimum of f.

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If $\nabla f(\underset{\sim}{a})$ exists for a local extremum $\underset{\sim}{a}$, then $\nabla f(\underset{\sim}{a}) = 0$. vector

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Example : Critical points of $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

$$\frac{\partial f}{\partial x} = 2x + 6y + 2. \qquad \frac{\partial f}{\partial y} = 6x + 8y - 4.$$

$$\nabla f(x,y) = (2x + 6y + 2, 6x + 8y - 4).$$

Set $\nabla f = 0$ i.e. $(2x + 6y + 2, 6x + 8y - 4) = (0, 0)$.

$$2x + 6y + 2 = 0$$
$$6x + 8y - 4 = 0$$

$$\left[\begin{array}{cc|c} 2 & 6 & -2 \\ 6 & 8 & 4 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 6 & 8 & 4 \end{array}\right] \xrightarrow{R_2 - 6R_1} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & -10 & 10 \end{array}\right] \xrightarrow{R_2/-10} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$\Rightarrow x = 2, y = -1$.

$\therefore$ Critical pt. of f is $(2, -1)$.

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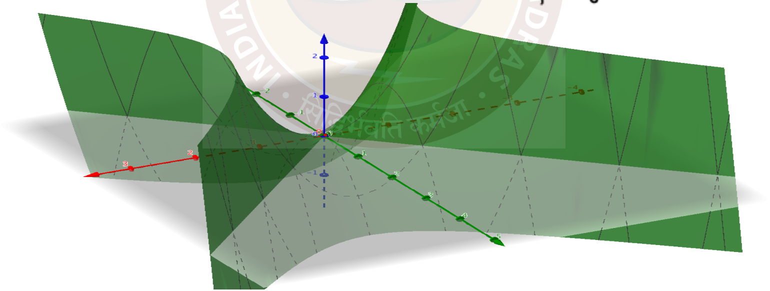
$f(x,y) = x^2 - y^2$



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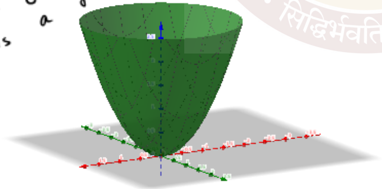
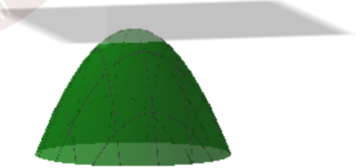
x^2+y^2 (0,0) is a global minimum



$-x^2-y^2$ (0,0) is a global maximum.



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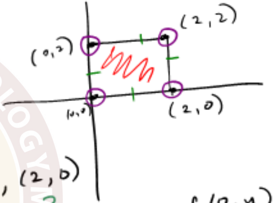
Thus to find the global maximum and minimum, we find the critical points

- inside the domain D
- on the boundary of D
- on the boundary of the boundary of D

... and check the value of f on all of them.

Example

Find the absolute maximum and minimum of the function $f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ over the square with vertices $(0,0), (2,0), (2,2), (0,2)$.



$\nabla f = (3x^2 - 3,\ 3y^2 - 6y)$.

Set to 0. $3x^2 - 3 = 0$, $3y^2 - 6y = 0$.

$x^2 = 1$, $y(y-2) = 0$.

$x = \pm 1$, $y = 0$ or 2.

$(1,0)$, $(1,2)$, $(0,0)$, $(0,2)$, $(2,2)$, $(2,0)$

-1 $\quad -5$ $\quad 1$ $\quad -3$ $\quad -1$ $\quad 3$

$f(x,0) = x^3 - 3x + 1.$ $\quad : \quad 0 < x < 2$

$f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1 \Rightarrow (1,0).$

$f(x,2) = x^3 + 8 - 3x - 12 + 1 = x^3 - 3x - 3.$

$f'(x) = 0 \Rightarrow \times$

Abs. max. occurs at $(2,0)$

" min. " " $(1,2)$

$0 \le y \le 2$

$f(0,y) = y^3 - 3y^2 + 1.$

$f'(y) = 0$

$3y^2 - 6y = 0$

$y(y-2) = 0.$

$(0,0)$, $(0,2)$.

$f(2,y) = y^3 - 3y^2 + c.$

Thank you