Week 10

Solve with us (Graded)

Question 1

Suppose X ~ Normal (μ, 4). For n=100 i.i.d. Samples of X, the observed sample mean is 11.8. What conclusion would a z-test reach if the null hypothesis assumes μ =10

(against the alternative that

$$\mu \neq 10$$

Use alpha = 5% .

1. Reject H0.
2. Accept H0.

Q1. Given $X \sim \text{Normal}(\mu, 4)$

Let $X_1, X_2, \ldots, X_{100} \sim \text{i.i.d. } X$

$\bar{X} = 11.8$, $\alpha = 0.05$

$H_0 : \mu = 10$

$H_A : \mu \neq 10$

Solution: Test statistic : $\bar{X} - \mu$

Test : Reject H_0, if $|\bar{X} - \mu| > c$, c is the critical value.

Given, $X \sim \text{Normal}(\mu, 4)$

For 100 i.i.d samples of X, $\bar{X} \sim \text{Normal}(\mu, 4/100)$.

$$\alpha = P(|\bar{X} - \mu| > c \mid \mu = 10)$$

$$= P\left(\left| \frac{\bar{X} - \mu}{2/10} \right| > \frac{c}{2/10} \right) = P(|Z| > 5c)$$

$$\Rightarrow \quad 0.05 = 2P(Z < -5c)$$

$$\Rightarrow \quad 0.025 = F_Z(-5c)$$

$$\Rightarrow \quad -5c = -1.96 \Rightarrow \boxed{c = 0.392}$$

Test: $|\bar{X} - \mu| = |11.8 - 10| = 1.8 > 0.396 = c.$

$\Rightarrow$ Reject H_0.

Question 2.

It is claimed that the lifetimes of light bulbs are normally distributed with mean of 100 hours and a standard deviation of 2 hours. We wish to test the hypothesis that $\mu = 100$

against the hypothesis that $\mu \neq 100$ with a sample of size 9.

Q. Find the significance level if the acceptance region is $98.5 \leqslant \overline{X} \leqslant 101.5$

Q2. Let the random variable X denote the lifetime of electric bulbs.

Given, $X \sim \text{Normal}(100, 2^2)$

$n = 9$, $H_0 : \mu = 100$

$H_A : \mu \neq 100$

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ true})$$

$$= P((\bar{X} > 101.5 \text{ or } \bar{X} < 98.5) \mid \mu = 100)$$

$$= P(|\bar{X} - 100| > 1.5)$$

$$= P\left(\left|\frac{\bar{X}-100}{2/3}\right| > \frac{1.5}{(2/3)}\right) \quad \left\{ \text{Since, } n = 9 \Rightarrow \bar{X} \sim \text{Normal}(100, 4/9)\right.$$

$$= P(|Z| > 2.25)$$

$$= 2P(Z < -2.25) = 2F_Z(-2.25) = 2 \times (0.012) = \boxed{0.0244}$$

Question 2.

It is claimed that the lifetimes of light bulbs are normally distributed with mean of 100 hours and a standard deviation of 2 hours. We wish to test the hypothesis that $\mu = 100$

against the hypothesis that $\mu \neq 100$ with a sample of size 9.

Q. Find the power of the test against the alternative that the true mean life is 103 hours.

$$\text{Power} = 1-\beta = P(\text{Reject } H_0 \mid H_A \text{ true})$$

$$= P\left((\bar{X} > 101.5 \text{ or } \bar{X} < 98.5) \mid \mu = 103\right)$$

$$= P(\bar{X} > 101.5 \mid \mu = 103) + P(\bar{X} < 98.5 \mid \mu = 103)$$

$$= P\left(\frac{\bar{X} - 103}{2/3} > \frac{101.5 - 103}{2/3}\right) + P\left(\frac{\bar{X} - 103}{2/3} < \frac{98.5 - 103}{2/3}\right)$$

$$= P(Z > -2.25) + P(Z < -6.75)$$

$$= 1 - F_Z(-2.25) + F_Z(-6.75)$$

$$\approx 0.987$$

Question 3.

A commonly prescribed drug for relieving nervous tension is believed to be only 40% effective. To determine if a new drug is superior in providing relief, suppose that 100 people who were suffering with nervous tension are chosen at random and inoculated. If more than 50 are them are found to be relieved, we reject the null hypothesis that p=0.4 and the new drug will be considered superior to the one currently in use.

1. Find P(Type I error).
2. Find P(Type II error) for p=0.6

Use normal approximation to binomial.

Q3. To test if the new drug is superior.

$$H_0 : p = 0.4$$

$$H_A : p > 0.4$$

→ Critical value, $c = 50$

→ Test statistic, $T = \text{Binomial}(100, p)$

$$\approx \text{Normal}(100p, 100p(1-p))$$

→ Test : Reject H_0, if $T > c$, i.e., $T > 50$

$\rightarrow P(\text{Type I error}) = P(T > 50 \mid p = 0.4)$

$$= P\left(Z > \frac{50 - 100(0.4)}{\sqrt{100(0.4)(0.6)}}\right)$$

$$= 1 - F_Z(2.04) = 1 - 0.979 = 0.021$$

$\rightarrow P(\text{Type II error}) = P(T \leq 50 \mid p = 0.6)$

$$= P\left(Z \leq \frac{50 - 100(0.6)}{\sqrt{100(0.6)(0.4)}}\right)$$

$$= F_Z(-2.04) = 0.02$$

Question 4.

The proportion of adults living in a small town who are college graduates is estimated to be $p = 0.4$.

To test this hypothesis against the alternative $p > 0.4$, you decide to take a sample of adults from the town.

1. What sample size do you need for a test (against an alternative hypothesis $p = 0.6$) at a significance level of 5% and power 95%.

1. Find the critical value at a significance level of 5%.

Q4. $H_0 : p = 0.4$

$H_A : p > 0.4$

$\alpha = 5\%$, $1-\beta = 95\%$

Test statistic, $T = \text{Binomial}(n, p)$

$\approx \text{Normal}(np, np(1-p))$

Let $T = \bar{X} \sim \text{Normal}\left(p, \frac{p(1-p)}{n}\right)$

Test: Reject H_0, if $\bar{X} > c$.

$$\alpha = P(\bar{X} > c \mid p = 0.4)$$

$$= P\left(\frac{\bar{X} - p}{\sqrt{\frac{p(1-p)}{n}}} > \frac{c - p}{\sqrt{\frac{p(1-p)}{n}}} \;\middle|\; p = 0.4 \right)$$

$$= P\left(Z > \frac{c - 0.4}{\sqrt{0.24/n}} \right) = 1 - F_Z\left(\frac{c - 0.4}{\sqrt{0.24/n}} \right)$$

$$\Rightarrow \quad 0.95 = F_Z\left(\frac{c - 0.4}{\sqrt{0.24/n}} \right) \Rightarrow c = \sqrt{\frac{0.24}{n}}\, F_Z^{-1}(0.95) + 0.4 \quad \text{---} \; (1)$$

Given $1-\beta = 0.95$

$$1-\beta = P(\bar{X} > c \mid p = 0.6)$$

$$= P\left(\frac{\bar{X}-0.6}{\sqrt{\frac{0.24}{n}}} > \frac{c-0.6}{\sqrt{\frac{0.24}{n}}}\right)$$

$$= 1 - F_Z\left(\frac{c-0.6}{\sqrt{0.24/n}}\right)$$

$$\Rightarrow \quad 0.05 = F_Z\left(\frac{c-0.6}{\sqrt{\frac{0.24}{n}}}\right) \Rightarrow c = \sqrt{\frac{0.24}{n}}\, F_Z^{-1}(0.05) + 0.6 \quad \text{——} \quad 2$$

$F_Z(1.64) = 0.95$ and $F_Z(-1.64) = 0.05$

Solving (1) & (2), we have, $2 \times 1.64 \left(\sqrt{\frac{0.24}{n}} \right) = 0.2$

$$\Rightarrow \quad \frac{0.24}{n} = \left(\frac{0.1}{1.64} \right)^2 = \frac{25}{6784}$$

$$\Rightarrow \quad n = \frac{0.24 \times 6784}{25} \approx 65$$

Substitute $n = 65$ in (1), we have

$c = 0.49$

Question 5.

A survey of 200 randomly selected students from a school revealed that 70% of them participates in extra curricular activities in schools.

Can we conclude at 5% level of significance that 80% of the students participate in the extra curricular activities?

1. Yes
2. No

Q5. $n = 200$, $\alpha = 0.05$

$H_0 : p = 0.8$

$H_1 : p \neq 0.8$

Method-①

$$Z_{obs} = \frac{|0.70 - 0.80|}{\sqrt{\frac{0.80 \times 0.20}{200}}} = 3.53$$

$\alpha = 0.05$, $c = 1.96$

Since $Z_{obs} = 3.53 > 1.96$, reject H_0.

$\therefore$ We cannot conclude at 5% level of significance that 80% of the students participate in the extra-curricular activities.

Method ②:

$$\alpha = P\left(|\bar{X} - p| > c \mid p = 0.8\right)$$

$$= P\left(\left|\frac{\bar{X} - 0.8}{\sqrt{\frac{0.8 \times 0.2}{200}}}\right| > \frac{c}{\sqrt{\frac{0.8 \times 0.2}{200}}}\right)$$

$$= 2P\left(Z < -\frac{50c}{\sqrt{2}}\right)$$

$$\Rightarrow \frac{0.05}{2} = F_Z\left(-\frac{50c}{\sqrt{2}}\right)$$

$$\Rightarrow -1.96 = -\frac{50c}{\sqrt{2}} \Rightarrow \boxed{c = 0.05}$$

Since, $|\bar{X} - p| = |0.70 - 0.80| = 0.10 > 0.05 = c$

$\Rightarrow$ Reject H_0.