Directional derivatives

Sarang S. Sane

Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a

scalar-valued multivariable function.

Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a

scalar-valued multivariable function.

If $\underset{\sim}{a}$ is a point in $\mathbb{R}^n$, then an open ball of radius r around $\underset{\sim}{a}$ is the set defined as

$$\{\underset{\sim}{x} \in \mathbb{R}^n \mid ||\underset{\sim}{x} - \underset{\sim}{a}|| < r\}.$$

Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a

scalar-valued multivariable function.

If $\underset{\sim}{a}$ is a point in $\mathbb{R}^n$, then an open ball of radius r around $\underset{\sim}{a}$ is the set defined as

$$\{\underset{\sim}{x} \in \mathbb{R}^n \mid ||\underset{\sim}{x} - \underset{\sim}{a}|| < r\}.$$

$e_1, e_2, \ldots, e_n$ is the standard ordered basis of $\mathbb{R}^n$.

Recall : partial derivatives

Recall : partial derivatives

Let $f(x_1, x_2, \dots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$. The partial derivative of f w.r.t. x_i is the function denoted by $f_{x_i}\left(\underset{\sim}{x}\right)$ or $\dfrac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right)$ and defined as

$$\frac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right) = \lim_{h \to 0} \frac{f\left(\underset{\sim}{x} + he_i\right) - f\left(\underset{\sim}{x}\right)}{h} \quad .$$

Recall : partial derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$. The partial derivative of f w.r.t. x_i is the function denoted by $f_{x_i}\left(\underset{\sim}{x}\right)$ or $\frac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right)$ and defined as

$$\frac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right) = \lim_{h \to 0} \frac{f\left(\underset{\sim}{x} + he_i\right) - f\left(\underset{\sim}{x}\right)}{h} \quad .$$

The function f_{x_i} computes the rate of change of the function f w.r.t. the variable x_i.

Recall : partial derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$. The partial derivative of f w.r.t. x_i is the function denoted by $f_{x_i}\left(\underset{\sim}{x}\right)$ or $\dfrac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right)$ and defined as

$$\frac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right) = \lim_{h \to 0} \frac{f\left(\underset{\sim}{x} + he_i\right) - f\left(\underset{\sim}{x}\right)}{h} .$$

The function f_{x_i} computes the rate of change of the function f w.r.t. the variable x_i. Another way of thinking about this is that f_{x_i} computes the rate of change of the function f in the direction of the unit vector e_i or in the direction of the x_i-axis.

$$\lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots x_n) - f(x_1, \ldots, x_n)}{h}$$

$$f(x_1 + h\cdot 0, x_2 + h\cdot 0, \ldots, x_{i-1} + h\cdot 0, x_i + h\cdot 1, x_{i+1} + h\cdot 0, \ldots, x_n + h\cdot 0)$$

Rate of change in a particular direction at a point

Rate of change in a particular direction at a point

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing a point $\underset{\sim}{a}$ and an open ball around it.

Rate of change in a particular direction at a point

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing a point $\underset{\sim}{a}$ and an open ball around it.

Suppose instead of in the direction of axes, we are interested in the rate of change of the function f at $\underset{\sim}{a}$ in some other direction.

Rate of change in a particular direction at a point

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing a point $\underset{\sim}{a}$ and an open ball around it.

Suppose instead of in the direction of axes, we are interested in the rate of change of the function f at $\underset{\sim}{a}$ in some other direction.

We can use the same idea as for partial derivatives and choose a unit vector $u \in \mathbb{R}^n$ in the direction we want and compute :

Rate of change in a particular direction at a point

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing a point $\underset{\sim}{a}$ and an open ball around it.

Suppose instead of in the direction of axes, we are interested in the rate of change of the function f at $\underset{\sim}{a}$ in some other direction.

We can use the same idea as for partial derivatives and choose a unit vector $u \in \mathbb{R}^n$ in the direction we want and compute :

$$\lim_{h\to 0} \frac{f\left(\underset{\sim}{a} + hu\right) - f\left(\underset{\sim}{a}\right)}{h}.$$

$g(h) = f(a_1+hu_1, \ldots a_n+hu_n)$

$\lim_{h\to 0} \frac{g(h) - g(0)}{h}$

$\underset{\sim}{a} = (a_1, \ldots, a_n)$, $u = (u_1, \ldots, u_n)$

$$\lim_{h\to 0} \frac{f(a_1+hu_1, a_2+hu_2, \ldots, a_n+hu_n) - f(a_1, \ldots, a_n)}{h}$$

Examples

The rate of change of $f(x,y) = x + y$ at $(0,0)$ in the direction of the $y = x$ line. $u = (1/\sqrt{2}, 1/\sqrt{2})$

$$\lim_{h\to 0} \frac{f(0+h\frac{1}{\sqrt{2}}, 0+h\frac{1}{\sqrt{2}}) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h}{\sqrt{2}} + \frac{h}{\sqrt{2}} - (0+0)}{h} = \lim_{h\to 0} \frac{\sqrt{2}h}{h} = \sqrt{2}.$$

The rate of change of $f(x,y,z) = xy + yz + zx$ at $(1,2,3)$ in the direction of the vector $(4,3,0)$ $= v$ $\quad u = \frac{v}{\|v\|} = \frac{1}{5}(4,3,0)$.

$\|v\| = \sqrt{4^2+3^2+0^2} = \sqrt{25} = 5.$

$$\lim_{h\to 0} \frac{f(1+h4/5, 2+h3/5, 3+h\cdot 0/5) - f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\, 12/25 + 2h4/5 + 1h3/5 + 3h3/5 + 3h4/5}{h} = \frac{8}{5} + \frac{3}{5} + \frac{9}{5} + \frac{12}{5}$$

$= 32/5$

The rate of change of $f(x,y) = sin(xy)$ at $(1,0)$ in the direction 60° (from the X-axis). $u = (\cos 60, \sin 60) = (\frac{1}{2}, \sqrt{3}/2)$

$$\lim_{h\to 0} \frac{f(1+h1/2, 0+h\sqrt{3}/2) - f(1,0)}{h} = \lim_{h\to 0} \frac{\sin((1+h/2)\sqrt{3}/2\, h) - \sin(1,0)}{h}$$

$g(h) = \sin((1+\frac{h}{2})\frac{\sqrt{3}}{2}h)$

$= g'(0) = \cos(0) \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}/2.$

$g'(h) = \cos(\ ) \times \{\frac{\sqrt{3}}{2}h \times \frac{1}{2} + (1+\frac{h}{2})\sqrt{3}/2\}$

Directional derivatives

Directional derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$.
The directional derivative of f in the direction of the unit vector u is the function denoted by $f_u\left(\underset{\sim}{x}\right)$ and defined as

$$f_u\left(\underset{\sim}{x}\right) = \lim_{h\to 0} \frac{f\left(\underset{\sim}{x} + hu\right) - f\left(\underset{\sim}{x}\right)}{h} \quad .$$

Its domain consists of those points of D at which the limits exists.

$$f_{e_i}(\underset{\sim}{x}) = f_{x_i}(\underset{\sim}{x}) = \frac{\partial f}{\partial x_i}(\underset{\sim}{x}).$$

Properties : Linearity and products

Properties : Linearity and products

Linearity :

Properties : Linearity and products

Linearity : Let $c \in \mathbb{R}$. If the directional derivative at the point $\underset{\sim}{a}$ in the direction of the unit vector u exists for both the functions $f\left(\underset{\sim}{x}\right)$ and $g\left(\underset{\sim}{x}\right)$, then it also exists for $(cf+g)\left(\underset{\sim}{x}\right)$ and

$$(cf+g)_u\left(\underset{\sim}{a}\right) = cf_u\left(\underset{\sim}{a}\right) + g_u\left(\underset{\sim}{a}\right) \quad .$$

Properties : Linearity and products

Linearity : Let $c \in \mathbb{R}$. If the directional derivative at the point $\underset{\sim}{a}$ in the direction of the unit vector u exists for both the functions $f\left(\underset{\sim}{x}\right)$ and $g\left(\underset{\sim}{x}\right)$, then it also exists for $(cf+g)\left(\underset{\sim}{x}\right)$ and

$$(cf+g)_u\left(\underset{\sim}{a}\right) = cf_u\left(\underset{\sim}{a}\right) + g_u\left(\underset{\sim}{a}\right) \quad .$$

The product rule

Properties : Linearity and products

Linearity : Let $c \in \mathbb{R}$. If the directional derivative at the point $\underset{\sim}{a}$ in the direction of the unit vector u exists for both the functions $f\left(\underset{\sim}{x}\right)$ and $g\left(\underset{\sim}{x}\right)$, then it also exists for $(cf+g)\left(\underset{\sim}{x}\right)$ and

$$(cf+g)_u\left(\underset{\sim}{a}\right) = cf_u\left(\underset{\sim}{a}\right) + g_u\left(\underset{\sim}{a}\right) \quad .$$

The product rule

If the directional derivative at the point $\underset{\sim}{a}$ in the direction of the unit vector u exists for both the functions $f\left(\underset{\sim}{x}\right)$ and $g\left(\underset{\sim}{x}\right)$, then it also exists for $(fg)\left(\underset{\sim}{x}\right)$ and

$$(fg)_u\left(\underset{\sim}{a}\right) = f_u\left(\underset{\sim}{a}\right) g\left(\underset{\sim}{a}\right) + f\left(\underset{\sim}{a}\right) g_u\left(\underset{\sim}{a}\right) \quad .$$

Properties (contd.) : Quotients

The quotient rule

Properties (contd.) : Quotients

The quotient rule

If the directional derivative at the point $\underset{\sim}{a}$ in the direction of the unit vector u exists for both the functions $f\left(\underset{\sim}{x}\right)$ and $g\left(\underset{\sim}{x}\right)$, and $g\left(\underset{\sim}{a}\right) \neq 0$, then it also exists for $\frac{f}{g}\left(\underset{\sim}{x}\right)$ and

$$(f/g)_u\left(\underset{\sim}{a}\right) = \frac{f_u\left(\underset{\sim}{a}\right) g\left(\underset{\sim}{a}\right) - f\left(\underset{\sim}{a}\right) g_u\left(\underset{\sim}{a}\right)}{g\left(\underset{\sim}{a}\right)^2} \quad .$$

Examples

- $f(x,y) = x + y$

$u = (u_1, u_2)$

$$f_u(x,y) = \lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h} = \lim_{h\to 0} \frac{x+hu_1+y+hu_2 - (x+y)}{h}$$

$$= \lim_{h\to 0} \frac{hu_1+hu_2}{h} = u_1+u_2 = u_1\times 1 + u_2 \times 1.$$

- $f(x,y,z) = xy + yz + zx$

$u = (u_1, u_2, u_3)$

$$f_u(x,y,z) = \lim_{h\to 0} \frac{(x+hu_1)(y+hu_2) + (y+hu_2)(z+hu_3) + (z+hu_3)(x+hu_1) - (xy+yz+zx)}{h}$$

$$= \lim_{h\to 0} \frac{h^2(u_1u_2+u_2u_3+u_3u_1) + h(u_1y + xu_2 + u_3y + zu_2 + zu_1 + xu_3)}{h}$$

$$= u_1y + u_2x + u_3y + u_2z + u_1z + u_3x$$

$$= u_1(y+z) + u_2(x+z) + u_3(x+y).$$

- $f(x,y,z) = sin(xy)$

$$\lim_{h\to 0} \frac{\sin((x+hu_1)(y+hu_2)) - \sin(xy)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(xy)\cos(xhu_2 + yhu_1 + h^2u_1u_2) + \cos(xy)\cdot\sin(\_) - \sin(xy)}{h}$$

Another example :

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Directional derivative at $(0,0)$ in the direction of the unit vector u.

$$\lim_{h\to 0} \frac{f(0+hu_1, 0+hu_2) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(hu_1, hu_2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{hu_1 hu_2}{h^2u_1^2 + h^2u_2^2}}{h} = \lim_{h\to 0} \frac{h^2 u_1 u_2}{h(h^2u_1^2 + h^2u_2^2)}$$

$$= \lim_{h\to 0} \frac{u_1 u_2}{h(u_1^2+u_2^2)} = \lim_{h\to 0} \frac{u_1 u_2}{h} \quad \text{DNE}$$

assuming both u_1, u_2 are not 0.

Thank you