What is a basis for a vector space?

Sarang S. Sane

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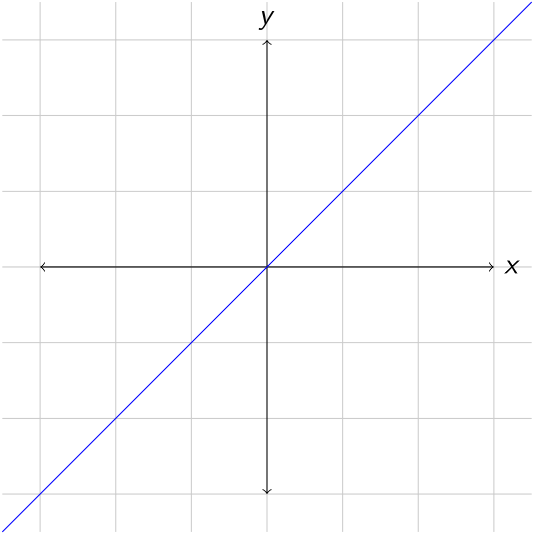

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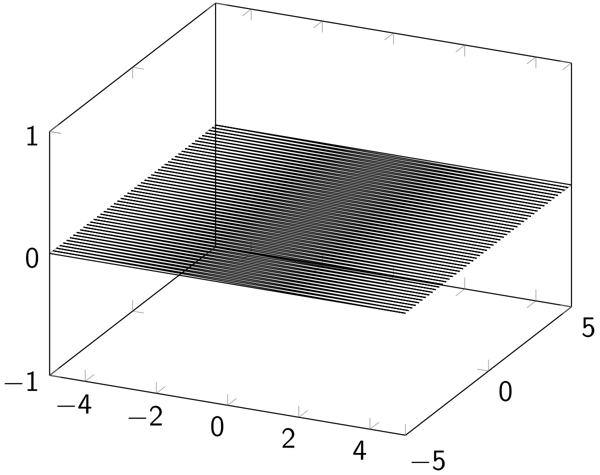

Figure: $Span(S)$ is the XY-plane

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$(x,y) \in \mathbb{R}^2 \quad (x,y) = x(1,0) + y(0,1).$

$(x,y,z) \in \mathbb{R}^3 \quad (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$

$(1,0) = (1,1) - (0,1) \quad \therefore (1,0) \in Span(\{(1,1),(0,1)\}) \Rightarrow$

$\therefore \{(1,0),(0,1)\} \subseteq Span$ "

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So $S_1 = S_0 \cup \{(0,2,1)\}$.

Example (Contd.)

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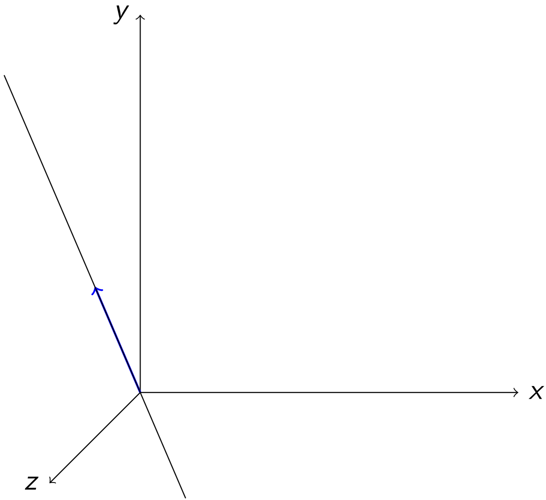
$Span(S_1)$ is the line shown in the picture below.



Choose a vector outside $Span(S_1)$ e.g. $(2, 2, 0)$, append it to S_1 and call the new set S_2.

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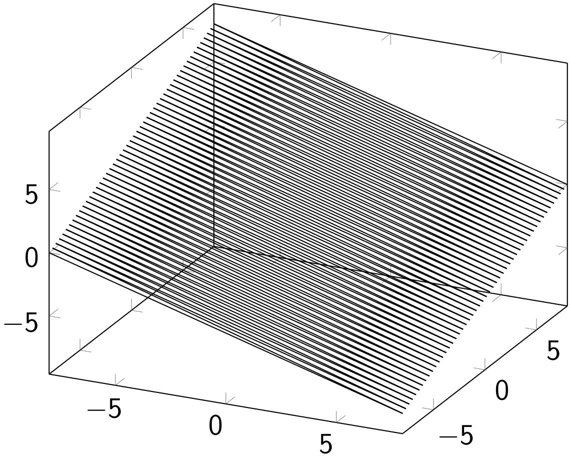

Again choose a vector outside $Span(S_2)$, e.g. $(0, 0, 5)$, append it to S_2 and call the new set S_3.

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$$(x,y,z) = \tfrac{y-x}{2}(0,2,1) + \tfrac{x}{2}(2,2,0) + \tfrac{x-y+2z}{10}(0,0,5)$$

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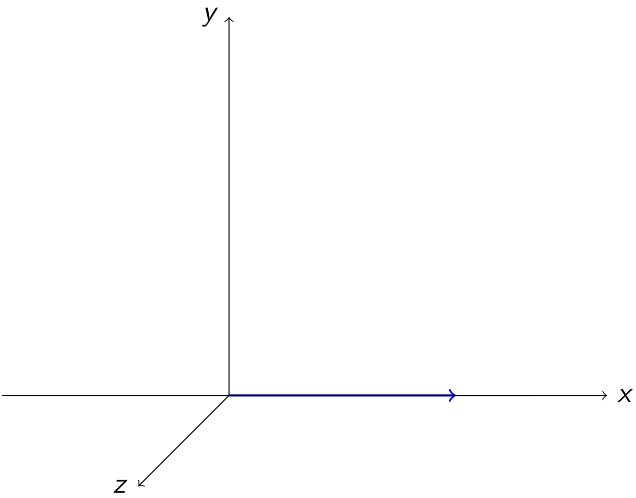
Example (contd.)

$Span(S_1)$ is the X-axis, as shown below.



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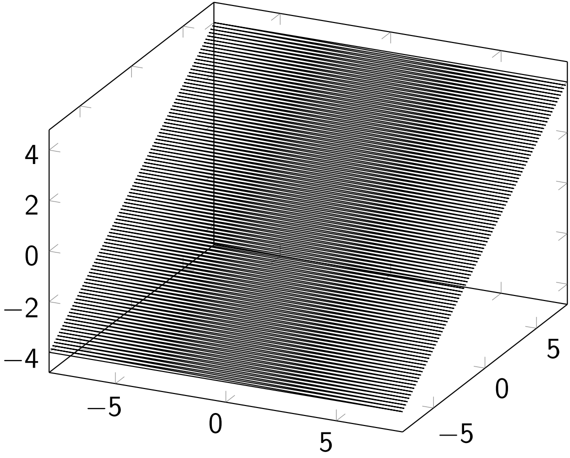

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Any arbitrary vector $(x,y,z) \in \mathbb{R}^3$ can be written as follows:

$$(x,y,z) = \tfrac{3x-5y+4z}{9}(3,0,0) + (y-z)(2,2,1) + \tfrac{2z-y}{3}(1,3,3)$$

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Let $e_i \in \mathbb{R}^n$ be the vector with i^{th} coordinate 1 and all other coordinates 0 e.g. $e_1 = (1, 0, 0, \ldots, 0)$.

The set $\varepsilon = \{e_1, e_2, \ldots, e_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$. ~~consisting of~~

$(x_1, x_2, \cdots, x_n)$
$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$\therefore \text{span}(\mathcal{E}) = \mathbb{R}^n$.

$\sum_{i=1}^{n} a_i e_i = 0 \Rightarrow j^{th}$ coordinate of LHS is a_j

$\Rightarrow a_j = 0 \quad \forall j$.

$\therefore \mathcal{E}$ is lin. indept.

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i) B is linearly independent and $Span(B) = V$.

ii) B is a maximal linearly independent set.

iii) B is a minimal spanning set.

Suppose B is a basis.
$\therefore$ B is lin. indept.
$B' = B \cup \{v\}$.
Suppose $v = \sum_{i=1}^{n} a_i v_i$ where $v_1, \ldots, v_n \in B$.
$\therefore$ B' is a lin. dep. set.
$\therefore$

maximal lin. indept. means
① it is lin. indept.
② appending any vector makes it lin. dep.

minimal spanning means
① it is spanning
② it is no longer spanning if we delete any vector

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Examples 1 & 2

ii) Take a spanning set and keep deleting vectors which are linear combinations of the other vectors, until the remaining vectors satisfy that they are not a linear combination of the other remaining ones.

Example : Method 1 : $V = \mathbb{R}^2$

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$Span(\{(1, 2), (2, 3)\}) = \mathbb{R}^2$.

Hence this set forms a basis for $\mathbb{R}^2$.

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Let us start with the set

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Check that $Span(S) = \mathbb{R}^3$.

Now observe that, $(0,4,2) = 2(1,2,0) + \frac{2}{3}(1,0,3) - \frac{8}{3}(1,0,0)$.

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$$S_1 = \{(1,0,0),(1,2,0),(1,0,3),(0,2,3)\}$$

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Hence S_2 forms a basis of R^3.

Thank you