Rank and dimension using Gaussian elimination

Sarang S. Sane

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$$\xrightarrow{-R_2/3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3-3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

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Hence, dimension of the vector space spanned by $\{(1,0,1),(-2,-3,1),(3,3,0)\}$ is 2 and a basis is given by $(1,0,1),(0,1,-1)$.

Example in $\mathbb{R}^4$

Apply the steps above to find the rank and a basis of the vector space spanned by the vectors
$\{(1, -2, 0, 4), (3, 1, 1, 0), (-1, -5, -1, 8), (3, 8, 2, -12)\}$.

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$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix}$$

Hence the dimension of this vector space is 2 and $\{(1, -2, 0, 4), (0, 1, 1/7, 12/7)\}$ is a basis.

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$$\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ \cancel{0} & \cancel{-7} & \cancel{-1} & \cancel{12} \\ \cancel{0} & \cancel{14} & \cancel{2} & \cancel{-24} \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 3 & 8 & 2 & -12 \end{bmatrix}$$

Hence the dimension of this vector space is 2 and $\{(1,-2,0,4),(0,1,1/7,12/7)\}$ is a basis.

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$$\xrightarrow{R_4-3R_1} \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix}$$

Hence the dimension of this vector space is 2 and $\{(1, -2, 0, 4), (0, 1, 1/7, 12/7)\}$ is a basis.

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$$
\begin{tikzcd}[column sep=large]
\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix} \arrow[r, squiggly, "R_2-3R_1"] &
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\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 3 & 8 & 2 & -12 \end{bmatrix} \arrow[d, squiggly, "R_4-3R_1"] \\
\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & 12/7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 1/7 & 12/7 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix} \arrow[l, squiggly, "R_3+7R_2"', "R_4-14R_3"] &
\begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & 12 \\ 0 & -7 & -1 & 12 \\ 0 & 14 & 2 & -24 \end{bmatrix} \arrow[l, squiggly, "R_2/7"']
\end{tikzcd}
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Thank you