IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture 06
Determinants (Part 3)

Hello, and welcome to the maths 2 component of the online BSc course on data science. Today's video is a continuation of the determinants topic that we have been studying. So, this is part three of that same topic.

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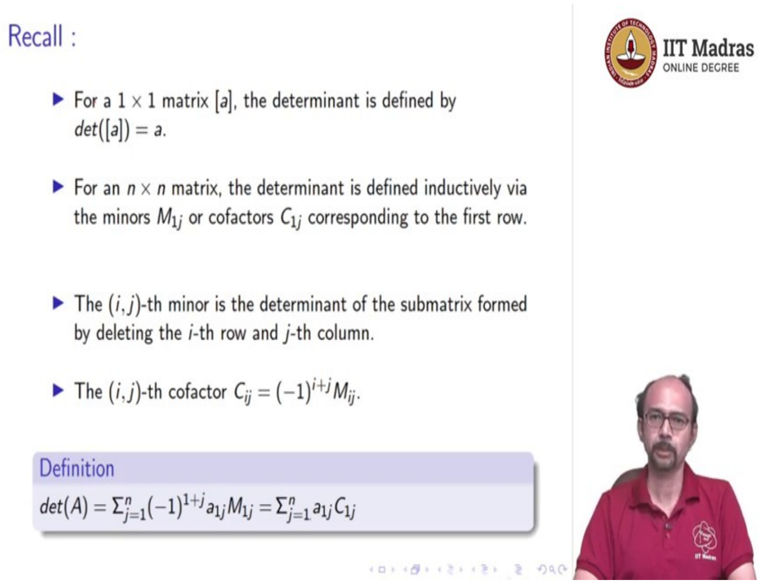

So, let us recall quickly what we did in the previous so far in this topic. So, we have defined for a 1×1 matrix, that the determinant of the matrix a is just the number a. And then we have seen that for an n n matrix, the determinant is defined inductively. So, we did the definition explicitly for 2×2 and 3×3.

And then I showed by for 4 4 how I can use a 3 3 definition and then you we said that, in general, we use the minors or co-factors, corresponding to the first row in order to define the determinant. So, that means if so the idea is if you know how to determine to define the determinant for (n-1) (n-1) matrix, then you can use this procedure to get it for the n n matrix.

So, just to recall, the $(i, j)^{th}$ minor is the determinant of the submatrix formed by deleting the i^{th} row and j^{th} columns, so you have n rows and n columns, you delete the i^{th} row and the j^{th} column.

And then you are left with an (n -1) × (n – 1) matrix. And then you take the determinant of that. And the (i, j)th cofactor is minus $(-1)^{i+j}$ times the (i, j)th minor. And just to recall again, this was the final thing that we did in the previous video, we have seen that the determinant of a is $(-1)^{1+j}$ a_{1j} M_{1j}.

So, this was given by what I called expansion along the first row and then you can take this $(-1)^{1+j}$, inside along with M_{1j} and then you can replace it by C_{1j} that is why we define the cofactors where we can get rid of the science coming here. So, this is what we have studied so far. So, why did we study the (i, j)th minor or the (i, j)th cofactor, because it is not occurred so far. So, that is one of the things we will see in this video and we will also see some other properties of the determinant.

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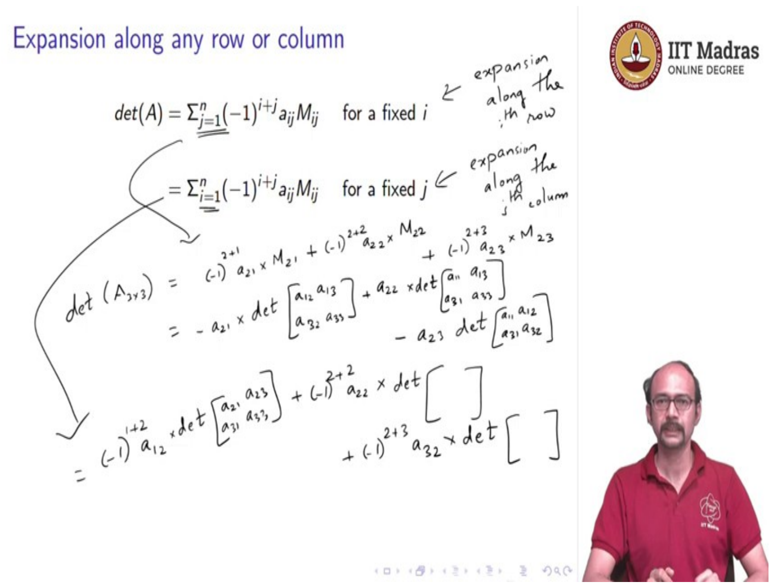


So, why did we define the (i, j)th minor or the (i, j)th cofactor? So, the reason is because we had a formula for the determinant in fact the definition in terms of expanding for any row or any sorry with respect to the first row. And now what we are going to see is that you can actually do this with any row or any column.

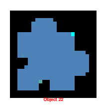
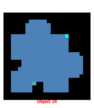
And there is nothing very special about the first row. So, this is called expansion along any row or column. So, here is what happens when we do expansion along the ith row. So, in that case we can write the expression for the determinant as the sum of the (i, j)th entry of a times the (i, j)th minor multiplied by $(-1)^{i+j}$.

So, remember, when i was 1 that was the definition. So, when i was 1, we had $(-1)^{1+j} a_{1j} M_{1j}$. So, now what we are saying is we can replace that 1 by i and the formula works equally well. So, the expansion along the first row was the definition that we are going to use and what we are saying is, it does not matter whether you use 1 there or any other row, you can expand along any other row.

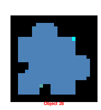
And the fact is, you can even use a column to do the same thing. So, you can expand along any column. So, here, notice that the j was what was varying and the i was fixed. So, this is expansion along the i^{th} row and this is the expansion. So, now the j is fixed and the i is varying. Here i is varying and j is fixed.

So, that is expansion along the j^{th} column. So, maybe let us do an explicit expression, let us write down one explicit expression here. So, suppose I have a 3 × 3 matrix, and I want to expand along the second row. So, what is this term telling me? So, this is expansion along second row.

So, this is saying what you have to do is, you have to take (-1) to the power. So, now i is 2 and now j varies, so minus $(-1)^{2+1}$ times a_{21} times the $(2, 1)^{th}$ cofactor sorry the $(2, 1)^{th}$ minor plus $(-1)^{2+2}$, a_{22} times the $(2,2)^{th}$ minor $(-1)^{2+3}$ a_{23} times the $(2,3)^{th}$ minor. So that is the expression.

So, if you write it out, concretely, this is $-a_{21}$ times determinant of so you have to drop the second row and first column. So, you get $det\begin{bmatrix} a_{12} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$ plus a_{22} times  minus a_{23} times det  . So, I hope it is clear what we mean by expanding along the i^{th} row this is an example of how to do this, you can do this for an n  n matrix also. It is correct.

The proof is slightly involved I mean, it is not hard, but it involves manipulation of a lot of expressions. So, if you like your algebra, I will suggest you go ahead. And let me do maybe a similar example for expansion along the second row sorry second column. So, this is expansion along second column.

So, what this is saying is that this is$(-1)^{1+2}$ a_{12} times determinant of  plus $(-1)^{2+2}$ a_{22} times similar determinant, I will encourage you to write this down plus $(-1)^{2+3}$ sorry this was what the second column, so my bad, this is a_{32} times the determinant of the corresponding matrix you obtain namely so that is a minor.

So, this is what we mean by expansion along any row or column. And it is a fact that the determinant can be computed in any of these ways. And it remains the same, so there is no nothing special about the first row of the matrix. That is how we define it. So, now, that we have seen that the determinant can be computed in terms of any row or column meaning by expanding along any row or column, we see that in particular, there is nothing special about the first row.

And we see that the expression that we are getting, there is some kind of symmetry about them. But of course, with some negative signs thrown in and so on. So, keeping that in mind, let us, now look at some properties of the determinant. So, we already saw in the first video, that the determinant of a product is the product of determinants and we could do various operations on matrices and the determinant either remain the same or change sign. So, we need all of those 2 2 and 3 3 matrices. So, let us see now what happens for the general case. So, I will make these statements but I will not really prove too many of these.

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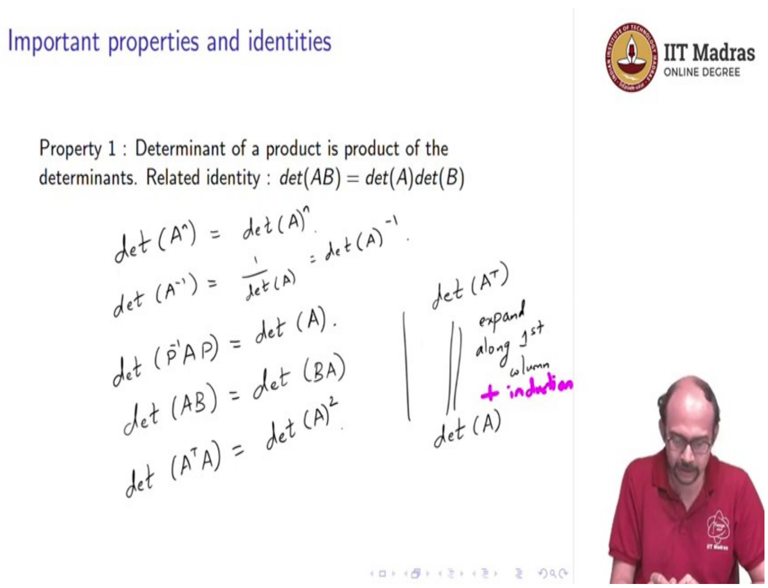

So, the first property that we want to look at is the product. So, the same thing that was earlier that we saw earlier still holds, namely that a determinant of a product is the product of determinants. So, what we mean by that is determinant of (AB) is determinant of A times determinant of B.

So, we explicitly check this, when they were of size 2×2 and I said you can try to check this for 3 3. If you did, you must have seen that the expression starts becoming rather complicated, it is not difficult, but you have to be careful with your algebra. And so you can imagine that for 3 3, sorry, beyond 3 3, if you work it out in terms of the explicit expressions, it going to be slightly hard.

So, I will not, we will not really get into the proof of the statement. But we have seen it for 2 2, and hopefully we have taken for 3 3. So, this is believable. So, let us use this instead and derive some interesting formulae. So, the first thing that we can do is we can ask, if you take the power of a matrix of a square matrix of course, namely you do A^2 A^3 A^4, A^n in general, then what happens to the determinant.

And what this identity tells you is that you can take this power out, so determinant of (A^n) is the $\det(A)^n$. The second thing that you get out of this identity is if you take the determinant of the inverse, then that is just the reciprocal of the determinant of A. So, I can take this more fancy language as $\det(A)^{-1}$.

So, the proof of this is exactly the same as we did for the 2 2 and 3 3 case, I will suggest that you go back and look at it. Another very useful formula is that when you get determinant of P inverse AP, so you have a matrix P, you take its inverse. So, A and P are both of size square matrices of size n by n, and P has an inverse.

So, take determinant of $P^{-1}AP$, then you can use this formula and what we derive just above to get that the determinant is the same as the determinant of A. And then another thing to note very easy fact. Remember that we saw that the product of two matrices may not be even defined, if you do not, if you take the opposite order and even if it is, they may be of completely different size and even if they are of the same size, AB and BA may not be the same.

But the determinant must be the same because the determinant is the product of determinants. This is of course remember if A and B are of square matrices of size n n and the final identity maybe one can write down. So, determinant of A^TA. So, maybe before I even do this, I would ask, what is determinant of A transpose? So, determinant of A transpose. So, now given that, we can expand along any row or column, so for A transpose instead of expanding along the first row, we expand along the first column.

So expand along the first column, so I say expand along first column. So, I strongly suggest you do this and if you do that, the expression you are going to get will allow you to conclude that this is determinant of A. So, I will leave this to you so along with so I will have I should use this through this plus what I will call induction.

Namely, assume that this formula is correct for (n – 1) (n – 1) matrices, that the transpose of a matrix has the same determinant, then expand along the first column and in the minors, remember there are determinants of (n – 1) (n – 1) matrices. So, think of those as transpose of something, and you will see that you can equate these two expressions. So, once we know this determinant of A transpose is determinant of A, so determinant of A^TA is $\det(A)^2$, this might not be very seem very interesting right now, but maybe at some later point, you might find expressions of this type coming up.

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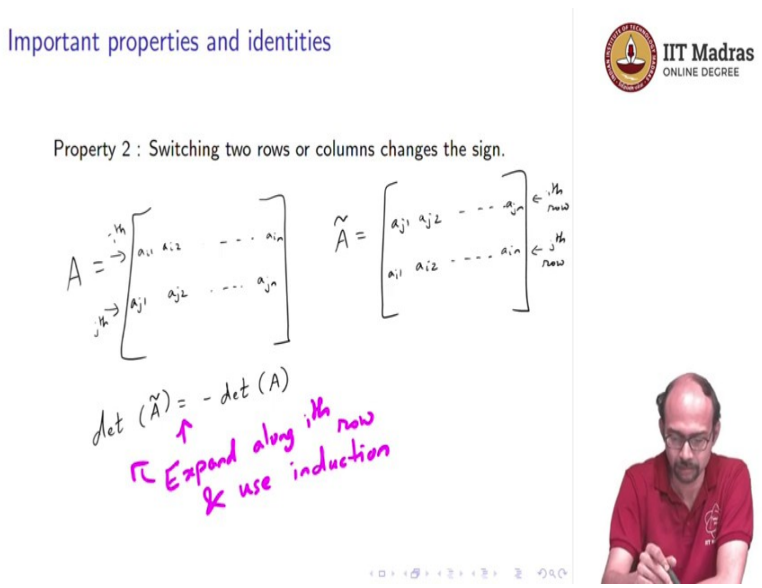

So, this long list of interesting observations, let us go on to another property. So, again, we have check this property for 2 2 and 3 × 3 matrices or other I have checked it for 2 × 2 and I hope you have solved 3 × 3. So, switching two rows or columns changes the sign of the determinant. So, what do we mean by switching two rows or columns, let me recall on that was, so if A is like this, this is the i^{th} row, this is the j^{th} row.

So, this is $\begin{bmatrix} a_{i1} & a_{i2}---a_{\text{¿}} \\ a_{j1} & a_{j2}---a_{jn} \end{bmatrix}$. So, we will switch these two rows to get this new matrix A . So, now it is ith row will be a_{j1}, a_{j2}, a_{jn} this is i^{th} row of $\widetilde{A}$ and the j^{th} row will be a_{i1}, a_{i2} up to a_{in} this is the j^{th} row. So, we have switched these two column rows, and everything else in $\widetilde{A}$ is exactly the same as it was for A, all other entries are exactly the same. So, then determinant of $\widetilde{A}$ is minus determinant of A, how do we get this? So, the again the idea is to expand along the i^{th} row and then use induction. So, expand along i^{th} row and use induction. So, if you do that, you will see a minus sign coming up.

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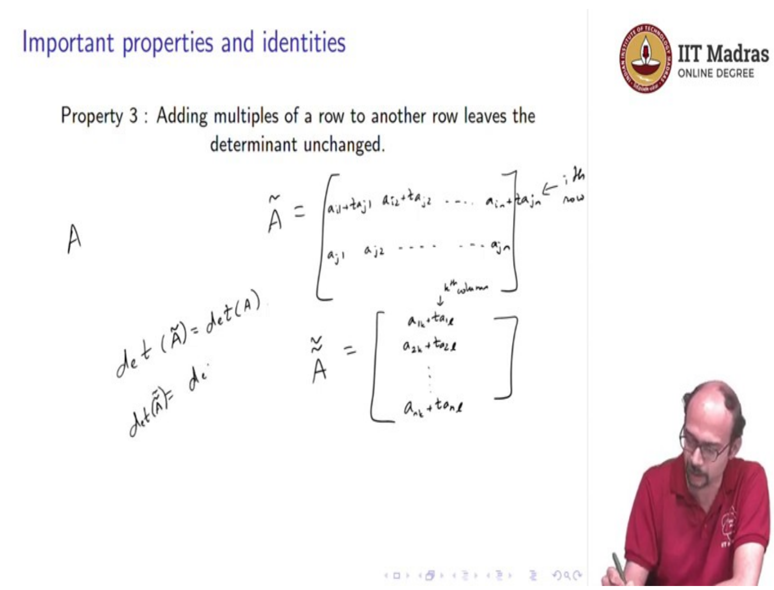

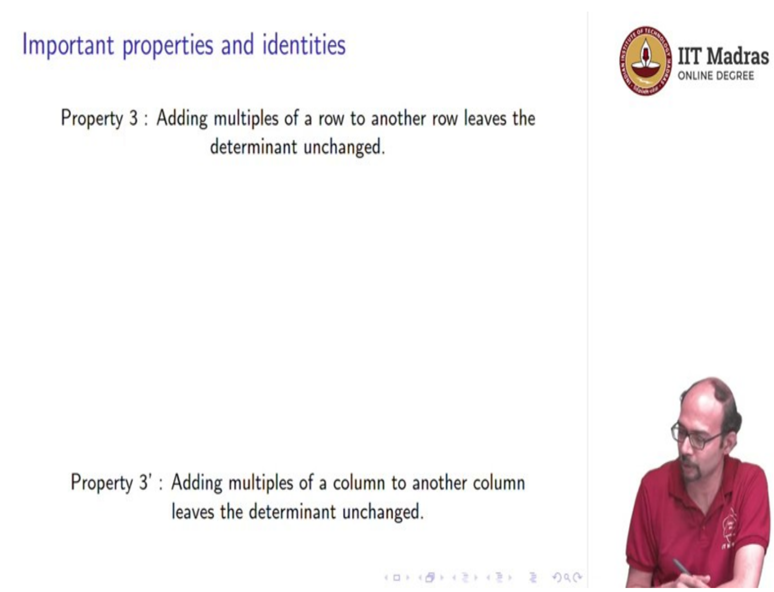

Let us, move on to the next property that we want to discuss, I should have said in the previous slide that similarly, if you interchange columns you get the same result. So, adding multiples of a row to another row leaves the determinant unchanged. So, what does this mean? So, here is A, which was like in the previous slide and what is $\widetilde{A}$. So, let us say I add t times the j^{th} row to the i^{th} row. So, then here I get the $\begin{bmatrix} a_{i1}+t\,a_{j1} & a_{i2}+t\,a_{j2}---a_{\Im}+t\,a_{jn} \\ a_{j1} & a_{j2}---a_{jn} \end{bmatrix}$ and the j^{th} row remains the same.

So, the only thing that has changed is the is the i^{th} row and what happens in that case, so the claim is the determinant of $\widetilde{A}$ is the same as the determinant of A. So, again, one has to prove this. So, the proof is so one way of proving this is to expand along the i^{th} row and then you will get two different expressions one as this determinant away and the other is determinant of a matrix, which has two rows which are the same and then one will have to prove that is 0. So, that is how you get this.

So, one can do this also for columns. So, for columns, let us say if I want to add the k^{th} column to the l^{th}, sorry t times l^{th} column to the k^{th} column, then we will get we want a_{1k} + ta_{11}, a_{2k} + ta_{21} and so on a_{nk} + t a_{nl} this is the new k^{th} column and the same thing will have to work, the determinant of $\widetilde{\widetilde{A}}$ is determinant of A. So, adding multiples of a columns to another column leaves the determinant unchanged.

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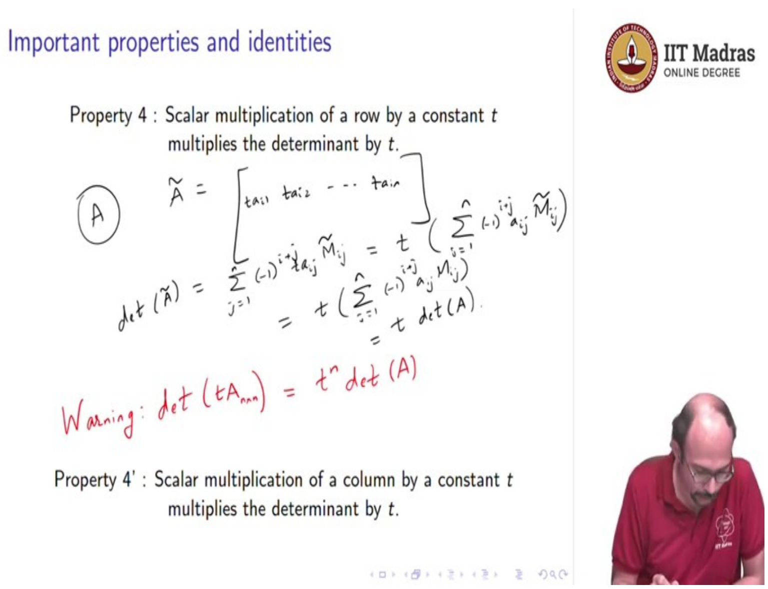

So, then another property. So, again, we have seen this for 2 2 and I leave it to you to check for 3 3, if you multiply a particular row by a constant t, then the determinant gets multiplied by t. So, the same thing holds for columns, if you multiply a column by t, then the determinant gets multiplied by t. So, what is this thing, so you have your matrix A and let us say in you multiply the i^{th} row by t.

So, we have $[t a_{i1} t a_{i2} --- t a_{¿}]$ everything else is the same. So, then what you do is you expand using the i^{th} row so, if you expand using i^{th} row you get t times a_{i1}, maybe for that there is a minus sign which I should have included, anyways sign I forget. So, determinant of $\widetilde{A}$, so expansion along the i^{th} row gives us $\sum_{j=1}^{n} (-1)^{i+j}$. So, maybe let me do this proof so that you get an idea of how to prove such things.

So, you can take t out and so you get ¿ . So, now so you have to be very careful here because when you write down the minus sign with respect to a particular matrix, so let me put this tilde here. So, this is the $(i,j)^{th}$ of the matrix $\widetilde{A}$ but if you delete i^{th} row and whatever column then the remaining matrix remember it may have the same matrix. So, for this particular row i is the same as M_{ij}.

So, this is just t and now, this is exactly the expression for the determinant of A. So, we have actually proved this. So, what is upshot? The upshot is that if you multiply a particular row by t then in the determinant it comes out, the same thing holds for columns. So, I will suggest that you check this so there you will have to expand along columns. So, I will also put in a warning here because this is something that often students make mistakes.

If you multiply the entire matrix by a constant, remember that we have studied multiplying matrices by scalars. So, scalar multiplication of matrices that means you multiply each entry of that matrix by that scalar. So, if you do that, and this is an n × n matrix, so, just to keep track of that this is an n n matrix, then this is like multiplying not just one particular row or one particular column by t, but every row or every column by t.

So, you will pick up a t in the determinant from each row. So, what you will get this $t^n det(A)$. What is this n? This n is exactly the size of A, the number of rows or number of columns in A.

So, $t^n det(A)$, you will not get $tdet(A)$. So, be careful of this property, it says you multiply a row by a constant t, then that t comes out in the determinant. If you multiply the entire matrix by t, then t^n comes out of the determinant. This is something students often make mistakes, so be careful with this.

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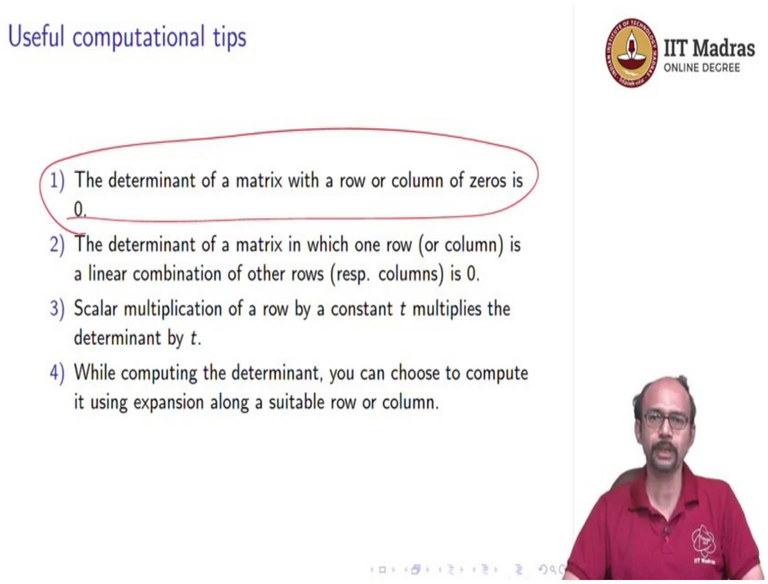

So finally, let me end with some computational tips. So, if you have a zero row or a zero column in a matrix, then the determinant is 0. How do you get this? We expand along that particular row or column. The determinant of a matrix in which one row or column is a linear combination of the other rows, respectively columns is 0. Why is that? Because you can remember, we can add multiples of a row to another row and the determinant remains unchanged.

So, you keep adding the correct expressions of the other rows to that particular row and you make it 0. So, once you make it 0, remember, the determinant remained unchanged and then you have a zero row or column. And that is how you get the determinant to be 0. So, the second property we will be using later on. So, we will come back to this if you did not understand this.

The third one is what we just saw scalar multiplication of a row by a constant t multiplies the determinant by t. And the fourth is a general sort of tip, while computing the determinant, you can choose to compute it using whatever row or column is most convenient. For example, as we saw in the first thing here, we used that row or column which is 0.

So, this is a general statement or it may be of use so what have we done in this video, we have seen how to compute determinants. First of all, we define determinants for square matrices and then we saw that you can compute them using expansion along any row or column in terms of minors and then we use that to look at some other properties. So, most importantly, determinant of a product is product of determinants. So, keep this in mind.

And then we also saw that if you do some particular operations, namely adding two rows or adding a multiple of one row to another, or multiplying a particular row by a constant, or swapping rows, we saw how the determinant behaves. So, that is it for now. In the next video, we will see how to use determinants to compute solutions to systems of linear equations.