Week-7
Mathematics for Data Science - 2
Basis of a vector space, Rank and dimension of a matrix
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. A molecule is composed of atoms. A molecule of Ethylene, with the chemical formula C_2H_4, consists of two Carbon atoms and four Hydrogen atoms. A molecule of Oxygen, with the formula O_2, consists of two Oxygen atoms. Note that Carbon, Hydrogen and Oxygen are denoted by the letters C, H, and O respectively in the formula. When Ethylene comes in contact with Oxygen (O_2); Carbon dioxide (CO_2) and water (H_2O) are produced as the products of the chemical reaction . The equation corresponding to the chemical reaction (R) is given below

$$x_1C_2H_4 + x_2O_2 \longrightarrow x_3CO_2 + x_4H_2O. \qquad \ldots \text{(R)}$$

To balance the chemical equation we have to choose $x_1, x_2, x_3, \text{and} x_4$ such that both sides have the same number of carbon atoms on each side, the same number of hydrogen atoms on each side, and the same number of oxygen atoms on each side.
Note: An example to write the system of linear equations for balancing the chemical equation is the following :

$$x_1C + x_2O \longrightarrow x_3CO_2 \qquad \begin{aligned} x_1 &= x_3 \\ x_2 &= 2x_3 \end{aligned}$$

Consider the system of linear equations obtained for balancing the chemical equation (R) to answer the question.
Consider the following statements:

- **Statement 1:** The nullity of the matrix corresponding to this system is 1.
- **Statement 2:** $\{(1, 3, 1, 1)\}$ is a basis of the null space of the matrix corresponding to this system.
- **Statement 3:** There are an infinite number of ways to balance the chemical equation (R).

Which of the following statements is true?

- ◯ Option 1: Only Statement 1 is true.
- ◯ Option 2: Only Statement 2 is true.
- ◯ Option 3: Only Statement 3 is true.
- ◯ **Option 4:** Both, Statement 1 and Statement 3 are true.
- ◯ Option 5: Both Statement 2 and Statement 3 are true.
- ◯ Option 6: Both Statement 1 and Statement 2 are true.

Solution: The system of linear equations for balancing the chemical equation (R) is

$$\begin{aligned} 2x_1 &= x_3 \\ 4x_1 &= 2x_4 \\ 2x_2 &= 2x_3 + x_4 \end{aligned} \Rightarrow \begin{aligned} 2x_1 - x_3 &= 0 \\ 2x_1 - x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Matrix corresponding to the above system is

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Number of Independent variable is "1". Hence, Nullity of A is 1. [Statement 1 is true]

$\Rightarrow$ There are infinitely many solutions for the above system of linear equations.

Therefore, there are an infinite number of ways to balance the chemical equation "R".
(Statement-3 is correct)

Now, $A\begin{bmatrix}1\\3\\1\\1\end{bmatrix} = \begin{bmatrix}2 & 0 & -1 & 0\\ 2 & 0 & 0 & -1\\ 0 & 2 & -2 & -1\end{bmatrix}\begin{bmatrix}1\\3\\1\\1\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\Rightarrow$ $\{(1,3,1,1)\}$ can not be a basis of the null space of A. [Statement-2 is not correct].

2. Match the sets of vectors in column A with their properties of linear dependence or independence in column B and the dimension of the vector spaces in column C spanned by the sets.

	Set of vectors (Column A)		Linear dependence or independence (Column B)		Dimension of the vector space spanned by the set (Column C)
a)	$\{(1,0,0),(1,1,0),(1,1,1),(1,0,1)\}$	i)	Linearly independent	1)	1
b)	$\{(1,0,-1),(-1,2,0),(-2,0,0)\}$	ii)	Linearly dependent	2)	2
c)	$\{(1,-1,2),(-1,1,-2),(2,-2,4)\}$	iii)	Linearly dependent	3)	3
d)	$\{(1,0,1),(1,1,0),(0-1,1)\}$	iv)	Linearly dependent	4)	3

Table : M2W7G1

Choose the correct option.

- ○ Option 1: a → ii → 3, b → iii → 2, c → i → 4, d → iv → 1
- ○ Option 2: a → ii → 3, b → i → 4, c → iii → 2, d → iv → 1
- ○ **Option 3:** a → ii → 4, b → i → 3 , c → iv → 1, d → iii → 2
- ○ Option 4: a → ii → 4, b → i → 3 , c → iv → 2, d → iii → 1

Solution: a) $S_1 := \{(1,0,0),(1,1,0),(1,1,1),(1,0,1)\}$

Four vectors can't be linearly independent in $\mathbb{R}^3$.

$\Rightarrow S_1$ is Linearly dependent.

$(1,0,1)-(1,0,0)+(1,1,0)=(1,1,1)$

Let $S=\{(1,0,1),(1,0,0),(1,1,0)\} \Rightarrow \text{Span}(S) = \text{Span}(S_1)$

Now consider the matrix using the vectors of S

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \det(A) \neq 0 \Rightarrow S_1 \text{ is Linearly independent.}$$

$$\Rightarrow \dim(\text{span}(S)) = \dim(\text{span}(S_1)) = 3.$$

b) $S_2 = \{(1,0,-1), (-1,2,0), (-2,0,0)\}$

Consider the matrix using the vectors of S_2: $B = \begin{bmatrix} 1 & 0 & -1 \\ -1, & 2, & 0 \\ -2, & 0, & 0 \end{bmatrix}$

$\det(B) \neq 0 \Rightarrow S_2$ is linearly independent $\Rightarrow \dim(\text{span}(S_2)) = 3$.

c) $S_3 = \{(1,-1,2), (-1,1,-2), (2,-2,4)\}$

$$-1\,(1,-1,2) = (-1,1,-2) \text{ and } 2(1,-1,2) = (2,-2,4)$$

$$\Rightarrow \text{Span}(S_3) = \text{Span}\{(1,-1,2)\}$$

$$\Rightarrow \dim(\text{span}(S_3)) = \dim(\text{span}\{(1,-1,2)\}) = 1.$$

d) $S_4 = \{(1,0,1), (1,1,0), (0,-1,1)\}$

$$(1,0,1) - (1,1,0) = (0,-1,1)$$

$$\Rightarrow \text{Span}(S_4) = \text{Span}\{(1,0,1), (1,1,0)\}$$

Clearly, $(1,0,1) \neq \alpha(1,1,0)$ for any $\alpha \in \mathbb{R}$.

$\Rightarrow \{(1,0,1), (1,1,0)\}$ is Linearly independent.

Hence, $\dim(\text{Span}(S_4)) = \dim(\text{span}\{(1,0,1),(1,1,0)\}) = 2$.

3. Which of the following option(s) is (are) true?

- ○ Option 1: The number of linearly independent vectors in a vector space can be more than the dimension of the vector space.
- ○ Option 2: Dimension of the vector space spanned by the vectors $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$, $\begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}$, and $\begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$ is 2.
- ○ **Option 3:** Dimension of the vector space spanned by the vectors $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$, $\begin{bmatrix}-1 & 1\\-1 & -1\end{bmatrix}$, and $\begin{bmatrix}-2 & 2\\-2 & -2\end{bmatrix}$ is 1.
- ○ Option 4: Nullity of two different matrices cannot be equal.

Solution: Option 1:

No of linearly independent vectors in a vector space $V \leq \dim(V)$. [By the definition of the dimension of a vector space]

Option 2:

$$S := \left\{ \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix} \right\}$$

Suppose

$$\alpha \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} + \beta \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix} + \gamma \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\alpha-\beta-\gamma & -\alpha+\beta+\gamma\\ \alpha-\beta+\gamma & \alpha+\beta+\gamma\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{aligned} \alpha-\beta-\gamma &= 0 \quad — (1)\\ -\alpha+\beta+\gamma &= 0 \quad — (2)\\ -\alpha-\beta+\gamma &= 0 \quad — (3)\\ \alpha+\beta+\gamma &= 0 \quad — (4)\end{aligned}$$

eq 1 + eq 4 $\Rightarrow \alpha = 0$

Substitue the value of α in eq 2 and eq 3

$$\Rightarrow \quad \beta + \gamma = 0 \text{ and } -\beta + \gamma = 0$$

add the above two equations

$$\Rightarrow 2\gamma = 0 \Rightarrow \gamma = 0 \Rightarrow \beta = 0$$

Hence, S is a linearly independent set.

Therefore $\dim(\text{Span}(S)) = 3$.

Option 3:

$$S_1 := \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \right\}$$

$$-1 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } -2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$\Rightarrow \text{Span}(S_1) = \text{Span}\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(\text{Span}(S_1)) = \dim\left(\text{Span}\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}\right) = 1.$$

Option 4: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A \neq B$, but nullity $(A) = $ nullity $(B) = 0$.

[Because both A and B are Invertible matrix].

2 Multiple Select Questions (MSQ)

4. Which of the following option(s) is(are) true?

   ○ **Option 1:** If $A_{2\times 3}$ is a non zero matrix, then nullity of the matrix ≤ 2.

   ○ **Option 2:** If $A_{3\times 2}$ is a non zero matrix, then nullity of the matrix ≤ 1.

   ○ **Option 3:** Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix A is also zero.

   ○ **Option 4:** Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix B is also zero.

Solution: Option: 1

For an $m\times n$ matrix, $\text{rank}(A)+\text{nullity}(A)=n$

In particular, If A is a 2×3 matrix

then $\text{rank}(A)+\text{nullity}(A)=3$

Since A is a non-zero matrix $\Rightarrow \text{rank}(A)\geq 1$

$\Rightarrow \text{nullity}(A)\leq 2.$

Option 2: If A is a non-zero 3×2 matrix

then $\text{rank}(A)+\text{nullity}(A)=2$

Since A is non-zero $\Rightarrow \text{rank}(A)\geq 1$

$\Rightarrow \text{nullity}(A)\leq 1$

Option 3 & 4: Nullity of AB is zero

$\Rightarrow$ AB is an invertible matrix

$\Rightarrow \det(AB)\neq 0 \Rightarrow \det(A)\det(B)\neq 0$

$\Rightarrow \det(A) \neq 0$ and $\det(B) \neq 0$.

$\Rightarrow$ Both A and B are invertible.

$\Rightarrow \text{nullity}(A) = \text{nullity}(B) = 0$.

5. Let A be a nonzero 4×4 matrix. Which of the following options are true?

- ◯ **Option 1:** The rank of A must be at least 1.
- ◯ Option 2: The rank of A may be 0.
- ◯ **Option 3:** The rank of A must be less than 4.
- ◯ **Option 4:** If the rank of the matrix is 2, then the dimension of the vector space spanned by the vectors corresponding to each column of A, must be 2.
- ◯ **Option 5:** If the rank of the matrix is 3, then there exists one vector corresponding to one column of A, which can be expressed as a linear combination of the vectors corresponding to each of the remaining columns of A.

<u>Option 1</u>:- Since A is a non-zero matrix then at least one of the rows of A must be non-zero.

$\Rightarrow$ RowSpace of $A \neq \{(0,0,0,0)\}$

$\Rightarrow \text{Rank}(A) \geq 1$.

$\Rightarrow$ option 2 can not be correct.

<u>Option 3</u>: A is a 4×4 matrix

By Rank-Nullity theorem

$$\text{Rank}(A) + \text{Nullity}(A) = 4$$

$$\Rightarrow \text{Rank}(A) \leq 4$$

<u>Option 4</u>: For an $m \times n$ matrix A

Row rank(A) = Column rank(A)

Hence, option 4 is correct.

Option 5 :- If the rank of the matrix is 3, then the column rank is also 3.

A is a 4×4 matrix $\Rightarrow$ There are four columns in A and they can not be linearly independent (Column rank = 3)

Hence, one column can be expressed as a linear combination of the remaining three columns.

3 Numerical Answer Type (NAT)

6. Find the dimension of the vector space $V = M_{2\times 3}(\mathbb{R})$. [Answer: 6]

Solution:

General form of 2x3 matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$ where a_{ij}'s are real numbers.

Consider the set

$$S := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Clearly, the set S is linearly independent and $\text{span}(S) = V = M_{2\times 3}(\mathbb{R})$

Hence, $\dim(V) = |S| = 6$.

7. If rank of the matrix $\begin{bmatrix} 0 & -1 & a \\ 2 & 0 & -4 \\ 3 & -9 & -6 \end{bmatrix}$ is 2 then find the value of a. [Answer: 0]

Solution:

$$A := \begin{bmatrix} 0 & -1 & a \\ 2 & 0 & -4 \\ 3 & -9 & -6 \end{bmatrix}$$

$(2, 0, -4) \neq \alpha(3, -9, 6)$ for any $\alpha \in \mathbb{R}$

$\Rightarrow$ The set $\{(2,0,-4), (3,-9,6)\}$ is linearly independent.

$\Rightarrow \quad \text{Rank}(A) \geq 2$

Now, $\text{Rank}(A) = 2$ Iff $\det(A) = 0$.

$\det(A) = 1(-12+12) + a(-18)$

$\det(A) = 0 \Rightarrow -18a = 0 \Rightarrow \boxed{a = 0}$

4 Comprehension Type Question:

Ayesha, Pritha, Sabya, and Wang went on a trip to Manali and Kasol. Accommodation costs ₹1500 per day in Manali and ₹800 per day in Kasol. The total food cost is ₹2000 per day in Manali and ₹1200 per day in Kasol. They plan to spend 2 days in Manali and 2 days in Kasol. The first and second rows of the Table M2W7G2 shows the percentage of contribution by each of them for the accomodation at Manali and Kasol, respectively. Similarly, the first and the second row of the Table M2W7G3 shows the percentage of contribution by each of them for the food at Manali and Kasol, respectively.

Table for Accommodation cost:

	Ayesha	Pritha	Sabya	Wang
Manali	$x_1\%$	$x_2\%$	$x_3\%$	$x_4\%$
Kasol	$y_1\%$	$y_2\%$	$y_3\%$	$y_4\%$

Table: M2W7G2

Table for cost of food:

	Ayesha	Pritha	Sabya	Wang
Manali	$v_1\%$	$v_2\%$	$v_3\%$	$v_4\%$
Kasol	$w_1\%$	$w_2\%$	$w_3\%$	$w_4\%$

Table: M2W7G3

Suppose $T(x, y)$ denotes the contribution of a person for accommodation per day, where the first variable x denotes the percentage of contribution by that person for accommodation in Manali and the second variable y denotes the percentage of contribution by that person for accommodation in Kasol. (i.e., if a and b denote the costs for accommodation per day at Manali and Kasol, respectively, then $T(x, y) = \frac{1}{100}(ax + by)$). Similarly, $T'(v, w)$ denotes the contribution by a person for food per day, where the first variable v denotes the percentage of contribution for food in Manali and the second variable w denotes the percentage of contribution for food in Kasol. Answer the following questions based on the given information.

8. Choose the set of correct options. (MSQ)

   ◯ Option 1: $T(x, y) = 30x + 16y$

   ◯ **Option 2:** $T(x, y) = 15x + 8y$

   ◯ **Option 3:** $T'(v, w) = 20v + 12w$

○ Option 4: $T'(v, w) = 40v + 24w$

Solution: It is given that if a and b denote the Costs for accommodation per day at Manali and Kasol, respectively, then $T(x,y) = \frac{1}{100}(ax+by)$.

- Accommodation costs 1500 per day in Manali and 800 per day in Kasol.

$$\Rightarrow T(x,y) = \frac{1}{100}(1500x + 800y) = 15x + 8y.$$

- The total food cost is 2000 per day in Manali and 1200 per day in Kasol.

$$\Rightarrow T'(v,w) = \frac{1}{100}(2000v + 1200w)$$

$$= 20v + 12w$$

Clearly, Both T and T' are linear transformations.

9. Choose the set of correct options. (MSQ)

- ◯ **Option 1:** Suppose Ayesha contributes for herslef and also on behalf of Sabya. Then the contribution per day by Ayesha is given by $T(x_1 + x_3, y_1 + y_3)$ and $T'(v_1 + v_3, w_1 + w_3)$, for accommodation and food, respectively.
- ◯ **Option 2:** The total contribution (for the whole trip) for the accommodation by Pritha is given by $2T(x_2, y_2)$, which is equal to $T(2x_2, 2y_2)$.
- ◯ Option 3: The total contribution (for the whole trip) for the accommodation by Pritha is given by $2T(x_2, y_2)$, which is not equal to $T(2x_2, 2y_2)$.
- ◯ Option 4: Suppose Pritha contributes for herself and also on behalf of Wang for food. Then the contribution per day by Pritha for food is given by $T'(v_2, w_2) + T'(v_4, w_4)$, which is not equal to $T'(v_2 + v_4, w_2 + w_4)$.
- ◯ **Option 5:** Suppose Pritha contributes for herself and also on behalf of Wang for food. Then the contribution per day by Pritha for food is given by $T'(v_2, w_2) + T'(v_4, w_4)$, which is equal to $T'(v_2 + v_4, w_2 + w_4)$.

Solution:

Option 1: $T(x, y)$ denotes the contribution of a person for accommodation per day, where the first variable "x" denotes the percentage of contribution by the person for accommodation at Manali and "y" denotes the percentage of contribution by that person for accommodation in Kasol.

From Table: M2W7G2.

Accommodation contribution of Ayesha at Manali $= x_1\%$.
at Kasol $= y_1\%$.

" " " Sabya at Manali $= x_3\%$.
at Kasol $= y_3\%$.

If Ayesha contributes for herself and Sabya, Then the total accom: contribution by Ayesha at Manali $= (x_1+x_3)\%$

at Kasol $= (y_1+y_3)\%$

Total contribution by Ayesha $= T(x_1+x_3, y_1+y_3)$

Similarly, total contribution for food by Ayesha is $T'(v_1+v_3, w_1+w_3)$.

Option 2 & 3:

Total contribution for accomodation perday by Pritha is $T(x_2, y_2)$.

Contribution for whole trip (two days) is

$$2T(x_2, y_2) = T(2x_2, 2y_2)$$

(because T is a linear transformation).

Option 4 & 5.

Similarly we can check, If Pritha contributes for herself and wang for food then the total contribution by pritha is $T'(v_2+v_4, w_2+w_4)$

$$= T'(v_2, w_2) + T'(v_4, w_4)$$

[Because T' is a linear transformation].

10. The total contribution (accommodation and food) per day can be denoted by: (MCQ)

- ◯ Option 1: $f(x,y,v,w) = T(x,y) + T'(v,w)$ which is not a linear mapping.
- ◯ **Option 2:** $f(x,y,v,w) = T(x,y) + T'(v,w)$ which is a linear mapping.
- ◯ Option 3: $f(x,y,v,w) = T(x,y)T'(v,w)$ which is not a linear mapping.
- ◯ Option 4: $f(x,y,v,w) = T(x,y)T'(v,w)$ which is a linear mapping.

<u>Solution</u>: <u>option 1 & 2</u>:

$$f(x,y,v,w) = T(x,y) + T'(v,w)$$

☆ $f(x_1+x_2, y_1+y_2, v_1+v_2, w_1+w_2)$

$= T(x_1+x_2, y_1+y_2) + T'(v_1+v_2, w_1+w_2)$

$= T(x_1,y_1) + T(x_2,y_2) + T'(v_1,w_1) + T'(v_2,w_2)$

$= T(x_1,y_1) + T'(v_1,w_1) + T(x_2,y_2) + T'(v_2,w_2)$

$= f(x_1,y_1,v_1,w_1) + f(x_2,y_2,v_2,w_2).$

☆ $f(cx, cy, cv, cw)$

$= T(cx,cy) + T'(cv,cw)$

$= cT(x,y) + cT'(v,w)$

$$= c\,(T(x,y) + T'(v,w))$$

$$= c\,(F(x,y,v,w))$$

Option 3 & 4.

$$F(x,y,v,w) = T(x,y)\,T'(v,w)$$
$$= (15x+8y)(20v+12w).$$

$$F(1,0,0,0) = 15$$
$$F(0,0,1,0) = 20$$

$$F(1,0,1,0) = 300 \neq F(1,0,0,0) + F(0,0,1,0) = 35$$

So, F is not a linear transformation.