WEEK 7 GA SOLUTIONS

MATH 2

Q1. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $B = A^T$

$AB = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

The sequence of row operations : $\frac{R_1}{2}$, $R_3 - R_1$, $R_1 - R_3$ results in $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Hence, rank of AB is 3.

$BA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. The sequence of row operations $R_3 - R_1$, $R_1 - R_3$ results in $I_{3\times 3}$. Hence, rank of BA is 3.

rank(AB) = rank(BA) = 3. option 1 is correct.

Q2. a) $V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 2 & x \\ y & 0 & 0 \end{bmatrix} \mid x + 2y = z,\ x, y, z \in \mathbb{R} \right\}$

Basis for V : Any element in V is of the form

$$\begin{bmatrix} x & y & x+2y \\ 0 & x+2y & x \\ y & 0 & 0 \end{bmatrix} = x \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thus, the set $\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$ forms a basis for V. Dimension = No.. of elements in the basis = 2

$\therefore$ a $\to$ ii $\to$ 3

b) $V = \{A \mid A \in M_{3\times3}(\mathbb{R}),\ A$ is a symmetric matrix$\}$ Pg (2)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ since $a_{ij} = a_{ji}$ for a symmetric matrix.

$$= a_{11}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1} + a_{12}\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_2} + a_{13}\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{e_3}$$

$$+ a_{22}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_4} + a_{23}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{e_5} + a_{33}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{e_6}$$

The set $\{e_1, e_2, \ldots, e_6\}$ forms a basis for V.

Dimension = 6. Thus, $b \to iii \to 2(1)$

c) $V = \{A \mid A \in M_{3\times3}(\mathbb{R}),\ A$ is an upper triangular matrix$\}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$. It can be written as

$$a_{11}\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{e_1} + a_{12}\overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{e_2} + a_{13}\overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{e_3}$$

$$+ a_{22}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_4} + a_{23}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{e_5} + a_{33}\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{e_6}$$

Thet set $\{e_1, e_2, \ldots, e_6\}$ forms a basis.

Dimension = 6. Thus, $c \to i \to 1(2)$. Hence, $\boxed{\text{option 2}}$ is correct

Q3. Karthika : $T_1(x_1, x_2, x_3)$

Romy : $T_2(x_1, x_2, x_3)$

Farzana : $T_3(x_1, x_2, x_3)$

$$T_1(x_1, x_2, x_3) = \frac{1}{100}(40x_1 + 50x_2 + 60x_3)$$

$$T_2(x_1, x_2, x_3) = \frac{1}{100}(20x_1 + 50x_2 + 50x_3)$$

$$T_3(x_1, x_2, x_3) = \frac{1}{100}(30x_1 + 40x_2 + 70x_3)$$

Hence, option1 is correct while option 2 is incorrect.

If $x_1 = 20$, $x_2 = 20$, $x_3 = 60$, then

$$T_1(x_1, x_2, x_3) = \frac{1}{100}(800 + 1000 + 3600)$$

$$= 54 \quad \text{(option 4)}$$

$$T_2(20, 20, 60) = \frac{1}{100}(400 + 1000 + 3000)$$

$$= 44$$

$$T_3(20, 20, 60) = \frac{1}{100}(600 + 800 + 4200)$$

$= 56$. T_3 is the highest.

Hence, Farzana obtained the highest total marks. (option 5)

$T_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ since $T_i(x_1, x_2, x_3)$ is the $i = 1, 2, 3$

total marks obtained by an individual.

option 6 is also correct.

Q4.

	Molecule	# carbon atoms (C)	# oxygen atoms (O)	# Hydrogen atoms (H)
x_1	CO_2	1	2	0
x_2	H_2O	0	1	2
x_3	$C_6H_{12}O_6$	6	6	12
x_4	O_2	0	2	0

Q4 $x_1 CO_2 + x_2 H_2O \longrightarrow x_3 C_6H_{12}O_6 + x_4 O_2$ pg (4)

Balancing # Carbon atoms on both sides,

$x_1 = 6x_3 \Rightarrow x_1 - 6x_3 = 0$ — (1)

Balancing # oxygen atoms,

$2x_1 + x_2 = 6x_3 + 2x_4$

$2x_1 + x_2 - 6x_3 - 2x_4 = 0$ — (2)

Balancing # Hydrogen atoms,

$2x_2 = 12x_3 \Rightarrow x_2 - 6x_3 = 0$ — (3)

(1), (2), (3) together form the system $AX = 0$

$X = \cancel{(x_1, x_2, x_3)}\ (x_1, x_2, x_3, x_4)$

$$A = \begin{bmatrix} 1 & 0 & -6 & 0 \\ 2 & 1 & -6 & -2 \\ 0 & 1 & -6 & 0 \end{bmatrix}.$$ Let us reduce this into row echelon form.

$$\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 6 & -2 \\ 0 & 1 & -6 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & -12 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3/(-12)} \begin{bmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 6 & -2 \\ 0 & 0 & 1 & -1/6 \end{bmatrix} \xrightarrow[R_1 + 6R_3]{R_2 - 6R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/6 \end{bmatrix}}_{B}$$

Now, $BX = 0 \Rightarrow x_1 = x_4,\ x_2 = x_4,\ x_3 = \frac{x_4}{6}$.

Note that x_1, x_2, x_3 are dependent variables while x_4 is an independent (free) variable.

Thus, nullity of $A = 1$. (optional)

$\{(x_4, x_4, \frac{x_4}{6}, x_4)\}$. put $x_4 = 6$.

Thus, $\{(6, 6, 1, 6)\}$ is a basis of the null space.

Let $x_4 = 1$. $(x_1, x_2, x_3, x_4) = (1, 1, \frac{1}{6}, 1)$. Hence, option 4 is not correct. Similarly, options is not correct.

Let $x_4 = 0$. $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$. So, option 6 is not correct.

Since x_4 can take any positive real value, there are infinitely many ways to balance the equation with the solution $(x_4, x_4, \frac{x_4}{6}, x_4)$

Q 5. Note that a null matrix is also a scalar matrix with the diagonal elements all being equal to zero. For instance, Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Let $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Here, a vector in the null space, $X = (x_1, x_2)$ can take any set of values satisfying $AX = 0$. option 1 is not correct.

Let $A_{2\times 2} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for non-zero 'a' be a scalar matrix. If $AX = 0$, then $(x_1, x_2) = (0, 0)$.

Thus, option 2 is correct; Can be generalized for any $A_{n\times n}$

If $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, rank $= 2$, nullity $= 0$ if $a \neq 0$.

This is true for any $m\times m$ matrix A. Hence, option 3 is correct. Also, as we can see rank of $m\times m$ matrix A is m. Hence, option 4 is also correct.

Also, option 5 is incorrect for the same reason.

Q6. $V = \{(x,y,z) \mid x+y+z=0,\ z=0,\ x,y,z \in \mathbb{R}\}$ is a vector space with normal addition, scalar multiplication.

Let $u = (a,b,c) \in V$. Now, $a+b+c=0$, $c=0$

$\Rightarrow a+b=0$ i.e. $a=-b$.

$$u = \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$ Note that the basis for V is $\{(1,-1,0)\}$. Hence, dimension of $V = 1$.

Q7. Let $$A_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$ [since $a_{ij} = \min\{i,j\}$]

Now, using this sequence of row operations : $R_2 - R_1$, $R_3 - R_1$, $R_3 - R_2$, $R_1 - R_2$, $R_2 - R_3$

A is reduced into $I_{3\times3}$. Hence, rank $(A) = 3$.

Let $$A_{4\times4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & \cancel{3}4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow[R_3-R_1,\ R_4-R_1]{R_2-R_1,} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$ Note that the portion covered inside '$\boxed{\ }$' is nothing but $A_{3\times3}$ which can be reduced into $I_{3\times3}$.

Thus, we get $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1-R_3,\ R_1-R_4]{R_1-R_2,} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Hence, rank of $A_{4\times4}$ is 4.

Note that any $A_{m\times m}$ can be reduced in this way to $I_{m\times m}$. Thus, $A_{2021\times2021}$ with $a_{ij} = \min\{i,j\}$ has a rank $\underline{2021}$.

Q8. option 1: The matrix of ratings becomes

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_4-R_3]{R_2-R_1}$$

~~which can~~ be ~~expressed as~~ ~~the sum of linearly independent vectors as~~ :

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Thus, A has 3 linearly independent row vectors. So, dimension of the vector space spanned by the row vectors is 3.

option1 is correct.

option 2: $A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_5 - R_1} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A has 4 independent row vectors. Hence, dimension of the vector space spanned by the row vectors is 4.

option 2 is also correct.

Q8. option 3 : Note that the 5 persons gave distinct ratings. pg (8)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$ has 5 independent row

vectors. Hence, dimension of the vector space spanned by the row vectors is 5. option 3 is correct.

Q8. option 4 : $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ has 3 identical

rows with $(0,1,0,0,0)$ and 2 identical rows with $(1,0,0,0,0)$. A has 2 independent row vectors $(0,1,0,0,0)$, $(1,0,0,0,0)$. Hence, dimension of vector space spanned by the row vectors is 2. option 4 is not correct.

Q9. Given $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Let $AX = 0$

where $X = (x_1, x_2, x_3, x_4, x_5)$. we get, $x_2 = 0$, $x_1 = 0$, $x_5 = 0$. So, $X = (0, 0, x_3, x_4, 0)$

$X = x_3 (0,0,1,0,0) + x_4 (0,0,0,1,0)$

$\therefore$ Basis set for the nullspace of A is $\{(0,0,1,0,0), (0,0,0,1,0)\}$. option 2 is correct.

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Note that (0,0,1,0,0) or (0,0,0,1,0) alone cannot span the nullspace. Hence, options 3 & 4 are not correct.

Q10. Consider the matrix A from Q9.

A has 3 distinct rows: (0,1,0,0,0), (1,0,0,0,0), (0,0,0,0,1) being the distinct non-zero linearly independent vectors. Hence, rank(A) = 3. option 3 is correct.