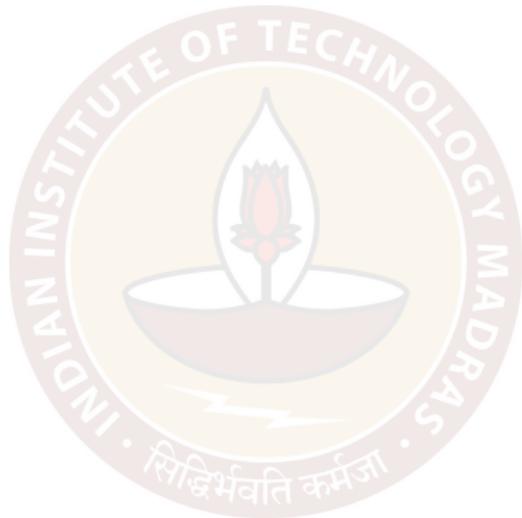


# Affine subspaces and affine mappings



# Affine Subspaces

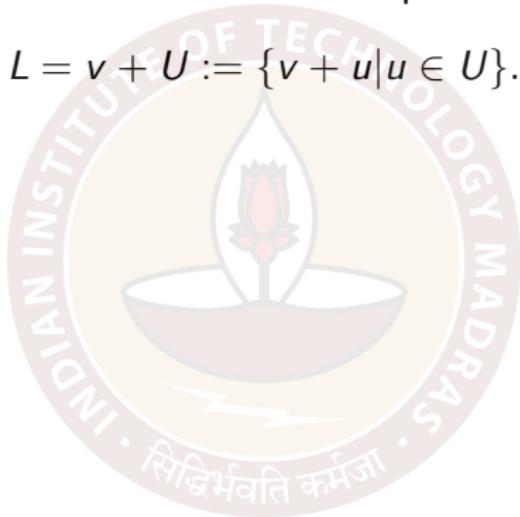
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$$\begin{aligned} & \begin{array}{l} v' \in L \\ \Rightarrow v' = v + u \\ \Rightarrow v' - v = u \\ \Rightarrow v' - v \in U \end{array} \quad \text{where } u \in U \\ & L = v + U \\ & = v' + U' \end{aligned}$$

$$\begin{aligned} & v \in L, v' \in L \\ & \text{Let } u \in U \\ & \text{Then } v + u \in L = v' + U' \quad \text{for some } u' \in U' \\ & \Rightarrow v + u = v' + u' \\ & \Rightarrow u = (v' - v) + u' \\ & \Rightarrow u \in U' \end{aligned}$$

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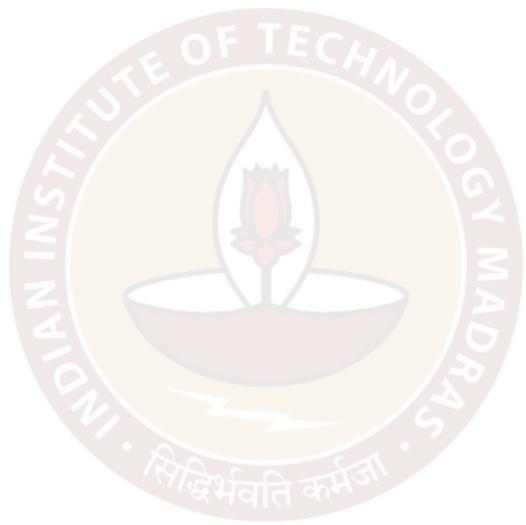
$$\begin{aligned} L &= v + U \\ &= v' + U \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \begin{array}{l} v - v' \in U \wedge v' \\ \Rightarrow v = v' \end{array}$$

Affine subspaces are thus **translates** of a vector subspace of  $V$ .

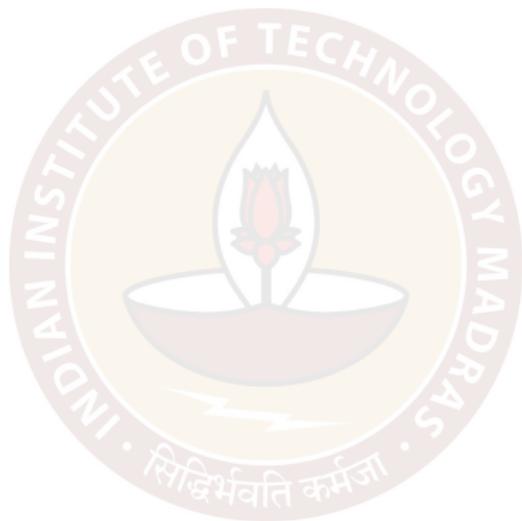
# Visual examples



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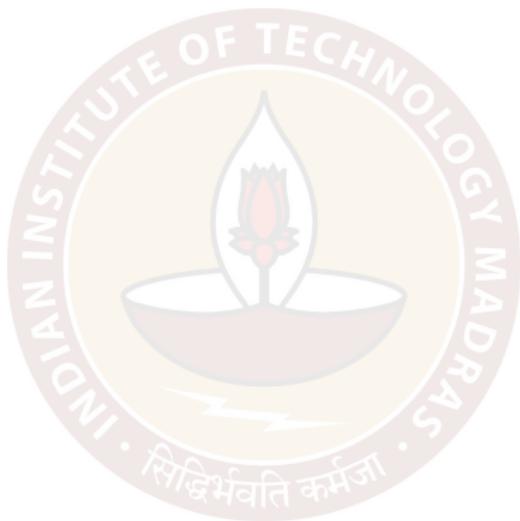


# Affine subspaces in $\mathbb{R}^2$



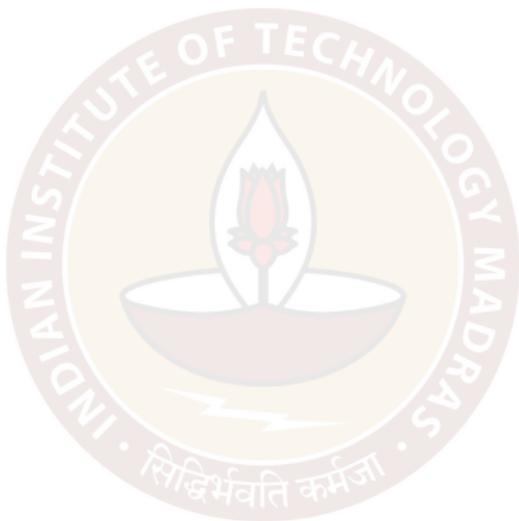
# Affine subspaces in $\mathbb{R}^2$

- ▶ Points



# Affine subspaces in $\mathbb{R}^2$

- ▶ Points
- ▶ Lines



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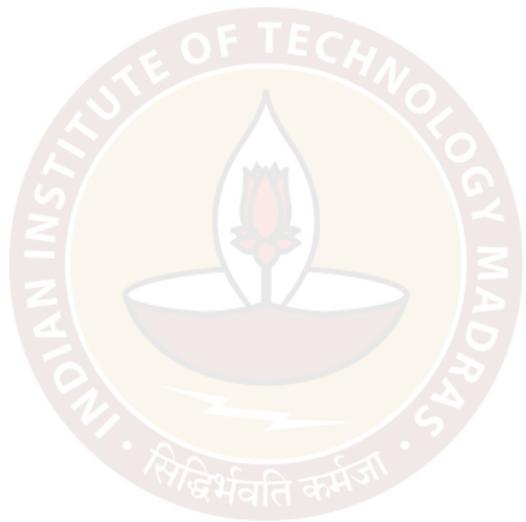
$$L = \{(x, y)\} = (x, y) + \{(0, 0)\}$$

is a translate of  
 $y = mx$

$$L = \{(x, y) \mid y = mx + c\}$$
$$L = \{(0, c)\} + \{(x, mx) \mid x \in \mathbb{R}\}$$
$$\mathbb{R}^2 = (0, 0) + \mathbb{R}^2$$

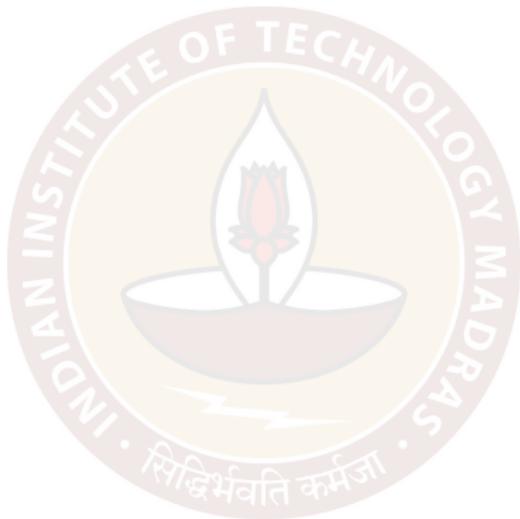
**A subset which is not an affine subspace :** the parabola  $y = x^2 + 1$  or the curve  $y^2 = x^3$ .

# Affine subspaces in $\mathbb{R}^3$



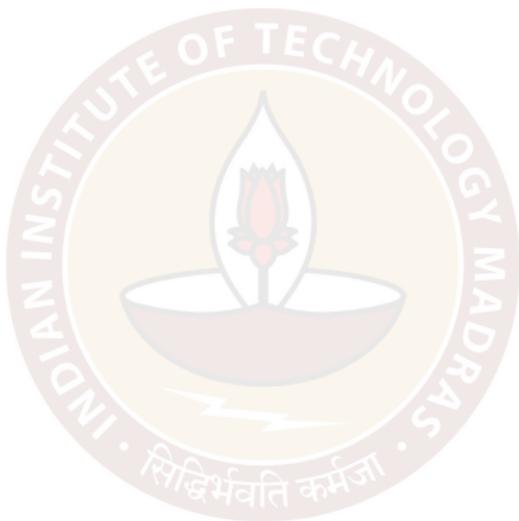
# Affine subspaces in $\mathbb{R}^3$

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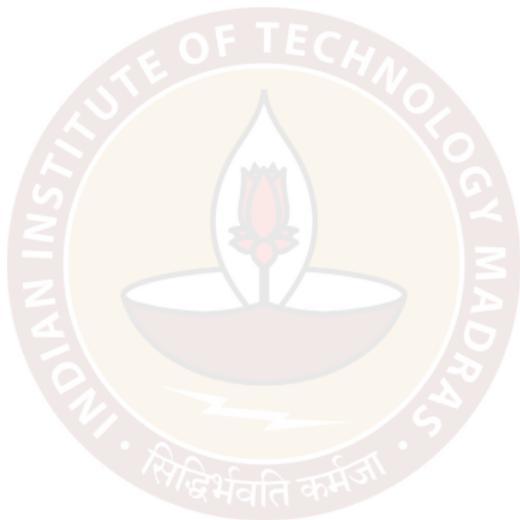
# Affine subspaces in $\mathbb{R}^3$

- ▶ Points
- ▶ Lines



# Affine subspaces in $\mathbb{R}^3$

- ▶ Points
- ▶ Lines
- ▶ Planes



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$$\begin{aligned}v + \lambda(0,0) \\v + \lambda v_1 \\v + \lambda_1 v_1 + \lambda_2 v_2 \\v + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3.\end{aligned}$$

**Example:** Two-dimensional affine subspaces in  $\mathbb{R}^3$  can be expressed as

$$l = v + \boxed{\lambda_1 v_1 + \lambda_2 v_2}$$

where  $\lambda_1, \lambda_2 \in \mathbb{R}$  and  $v, v_1, v_2$  are vectors in  $\mathbb{R}^3$ .

$$\begin{aligned}V &= \left\{ \lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_1, \lambda_2 \in \mathbb{R} \right\} \\&= \text{Span}(v_1, v_2)\end{aligned}$$

# The solution set to a system of linear equations

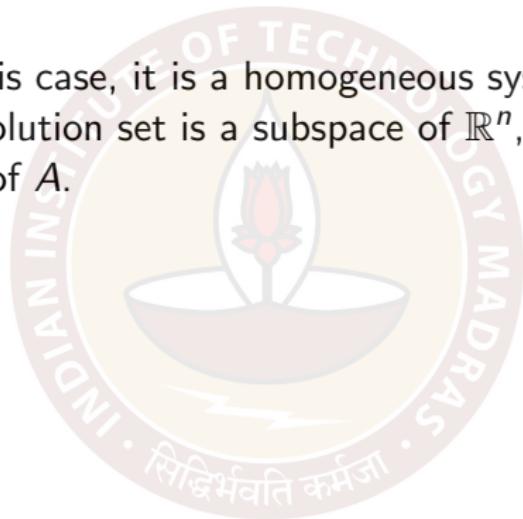
Let  $Ax = b$  be a linear system of equations.



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- ▶  $b \in$  column space of  $A$  : In this case, the solution set  $L$  is an **affine subspace** of  $\mathbb{R}^n$ . Specifically, it can be described as  $L = v + \mathfrak{N}(A)$  where  $v$  is **any** solution  $\textcolor{red}{v}$  of the equation  $Ax = b$ .

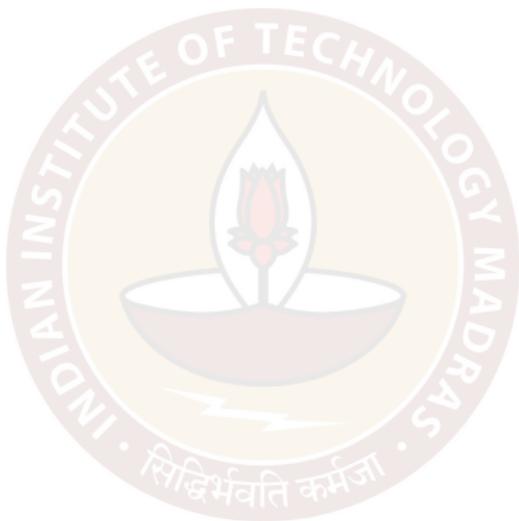
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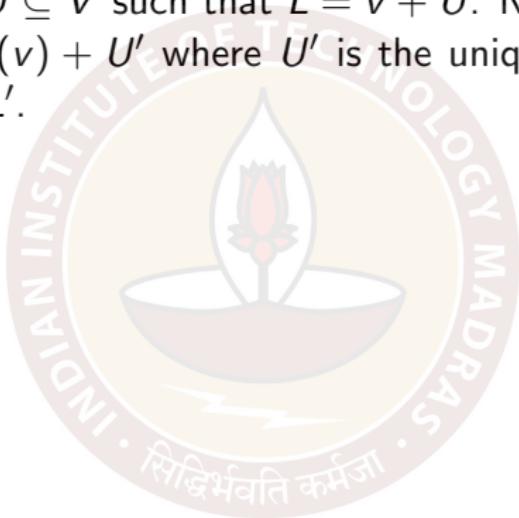
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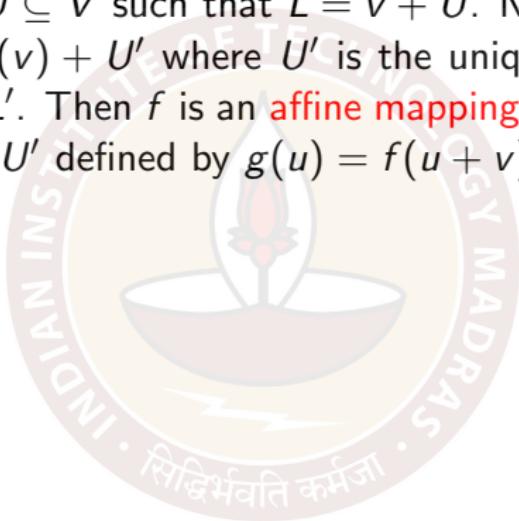
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For a linear transformation  $T : U \rightarrow U'$  and fixed vectors  $v \in L$  and  $v' \in L'$ , an affine mapping  $f$  can be obtained by defining  $f(v + u) = v' + T(u)$ , and in fact every affine mapping is obtained in this way.

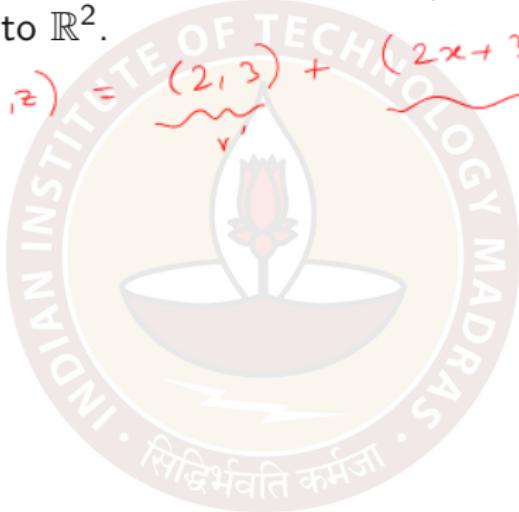
$$\begin{aligned} g(u) &= f(u+v) - f(v) & g(u') &= f(u'+v) - f(v) \\ g(u+u') &= g(u) + g(u') & g(u+u') &= f(u+u'+v) - f(v) \\ f(u+u'+v) - f(v) &= f(u+v) - f(v) + f(u'+v) - f(v) & f(u+u'+v) &= f(u+v) + f(u'+v) - f(v) \\ &\quad + f(u'+v) - f(v) \Rightarrow & \Rightarrow f(u+u'+v) + \boxed{f(v)} &= f(u+v) + f(u'+v) \end{aligned}$$

## An example and an important special case

Let  $T(x, y, z) = (2x + 3y + 2, 4x - 5y + 3)$ . Then this is an affine mapping from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

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$(2, 3)$  +  
 $y'$   
Lin. trans.



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$$T'(x, y, z) \stackrel{\sim}{=} w + (2x + 3y, 4x - 5y)$$

(T(x, y, z) is a lin. trans.)

Let  $T : V \rightarrow W$  be a linear transformation and  $w \in W$ , then the mapping

$$\begin{aligned} T' : V &\rightarrow W \\ T'(v) &= w + T(v) \end{aligned}$$

is an affine mapping from  $V$  to  $W$ .

# Thank you

