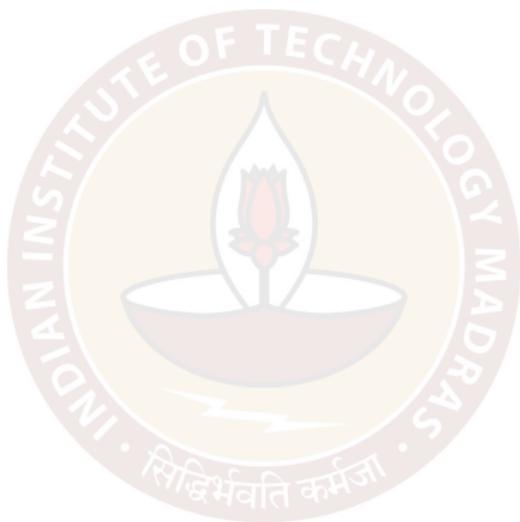


What is a linear transformation

Sarang S. Sane

Recall : linear mappings

A linear mapping f from \mathbb{R}^n to \mathbb{R}^m is :

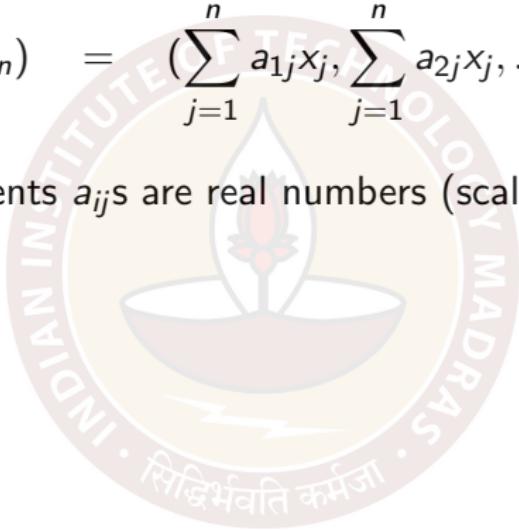


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$$f(x_1, x_2, \dots, x_n) = \left(\sum_{j=1}^n a_{1j}x_j, \sum_{j=1}^n a_{2j}x_j, \dots, \sum_{j=1}^n a_{mj}x_j \right).$$

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Linear mappings satisfy linearity, i.e. for any $c \in \mathbb{R}$ (scalar) :

$$\begin{aligned} f(x_1 + cy_1, x_2 + cy_2, \dots, x_n + cy_n) &= \\ f(x_1, x_2, \dots, x_n) + cf(y_1, y_2, \dots, y_n). \end{aligned}$$

Formal definition

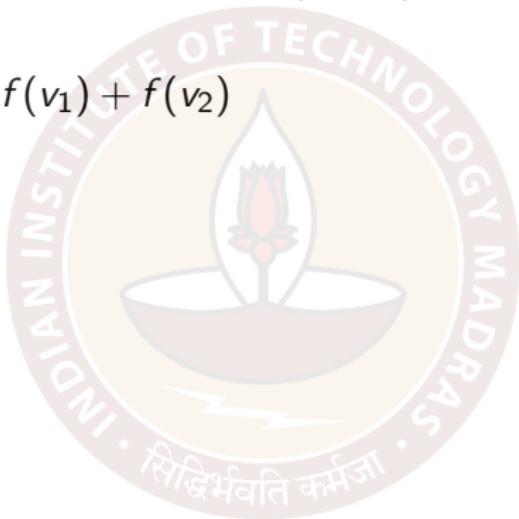
A function $f : V \rightarrow W$ between two vector spaces V and W is said to be a linear transformation if for any two vectors v_1 and v_2 in the vector space V and for any $c \in \mathbb{R}$ (scalar) the following conditions hold :



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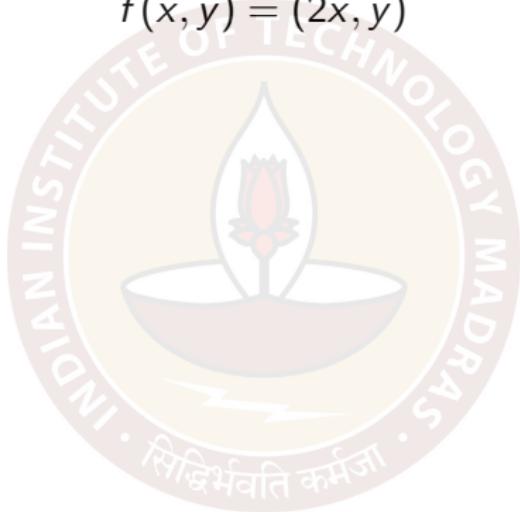
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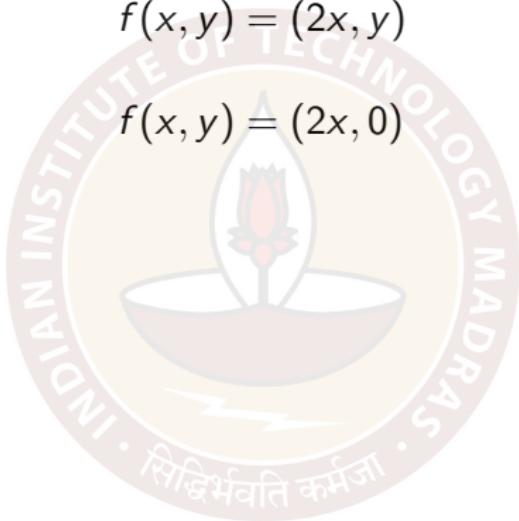
Examples

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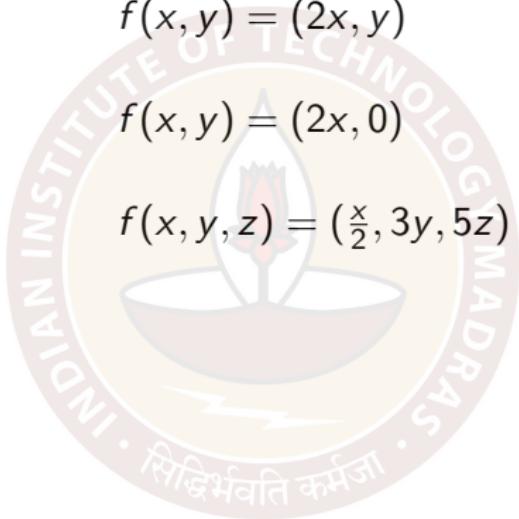
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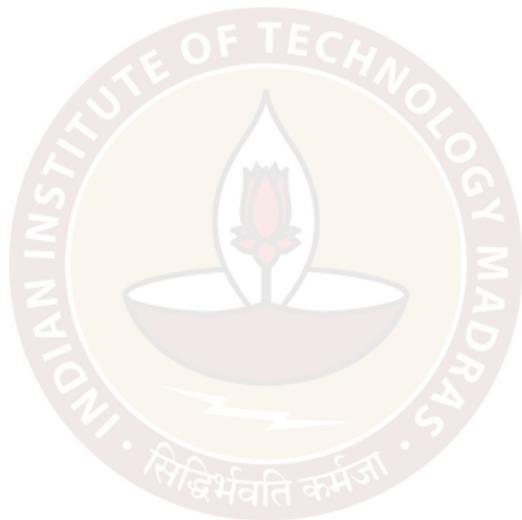
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1-1 and onto functions



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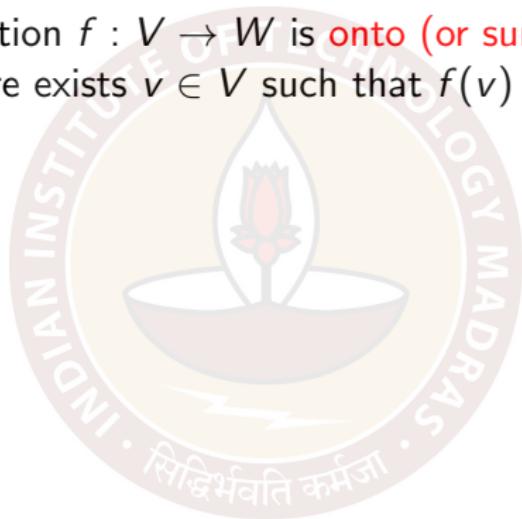
Recall that a function $f : V \rightarrow W$ is **1-1 (or injective)** if
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For a linear transformation, being 1-1 is equivalent to $f(v) = 0$ implies $v = 0$.

$f : V \rightarrow W$ is a lin. trans.
Assume f is 1-1. Then $f(v_1) = f(v_2) \Rightarrow v_1 = v_2$.
 $f(0) = 0$. If $f(v) = 0 \Rightarrow f(v) = f(0)$
 $\Rightarrow v = 0$.

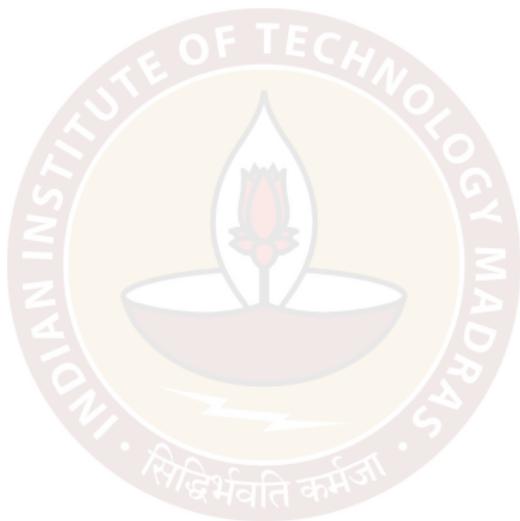
Conversely, assume $f(v) = 0 \Rightarrow v = 0$.

$$\begin{aligned}f(v_1) = f(v_2) &\Rightarrow f(v_1 - v_2) = 0 \\&\Rightarrow v_1 - v_2 = 0 \\&\Rightarrow v_1 = v_2.\end{aligned}$$

$$\left| \begin{array}{l} f(0+0) \\ = f(0) + f(0) \\ \Rightarrow f(0) = f(0) + f(0) \\ \Rightarrow f(0) = 0 \\ \\ \frac{f(v) + f(-v)}{1} \\ = f(0) = 0 \\ \Rightarrow f(-v) = -f(v) \\ \\ f(v_1) - f(v_2) = 0 \\ f(v_1) + f(-v_2) = 0 \\ \Rightarrow f(v_1 - v_2) = 0 \end{array} \right.$$

What is an isomorphism

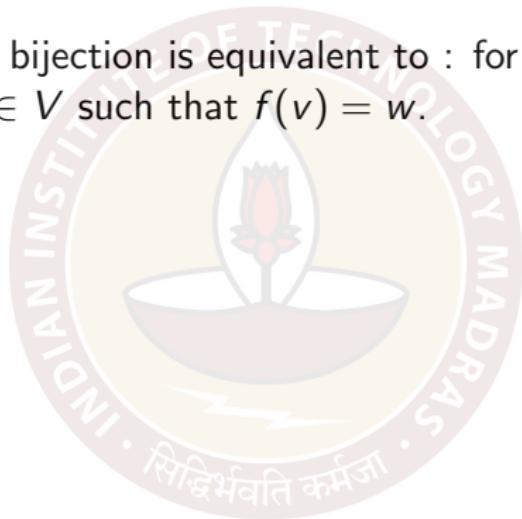
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Note that being a bijection is equivalent to : for any $w \in W$ there exists a **unique** $v \in V$ such that $f(v) = w$.



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$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad f(x, y) = (2x, y).$$

$$f(x, y) = (0, 0) \Rightarrow (2x, y) = (0, 0) \Rightarrow \begin{cases} 2x = 0, y = 0 \\ \Rightarrow x = 0, y = 0 \end{cases} \Rightarrow (x, y) = (0, 0)$$

For $(u, v) \in \mathbb{R}^2$ consider $x = u, y = v$.
 $\therefore f(x, y) = (2x, y) = (2u, v) = (u, v)$

onto

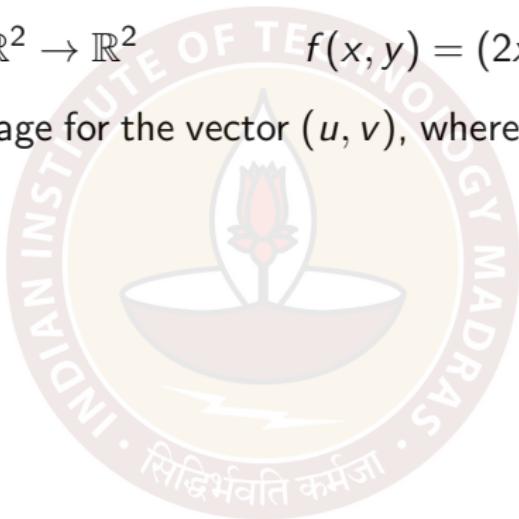
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Isomorphisms : Non-examples

Example 2 seen earlier is not an isomorphism :

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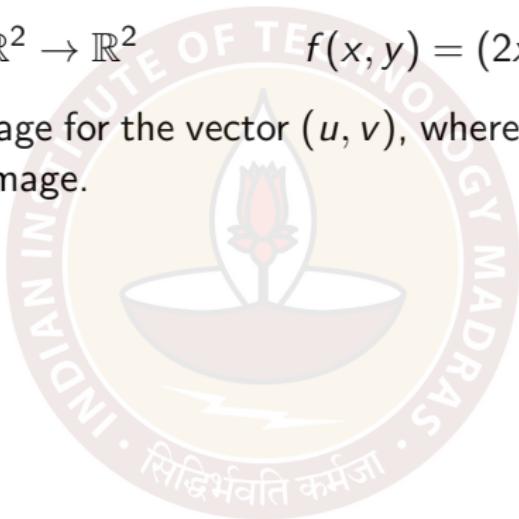


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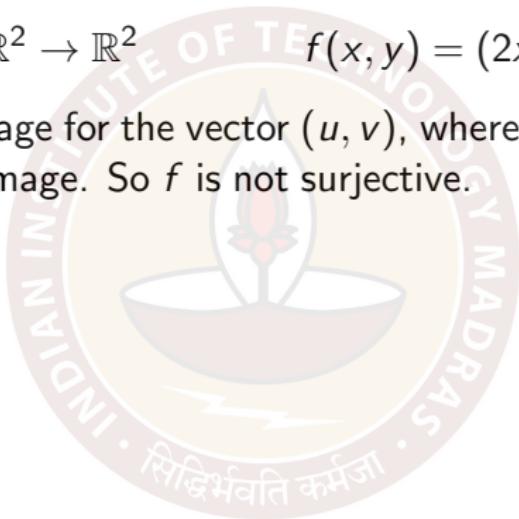


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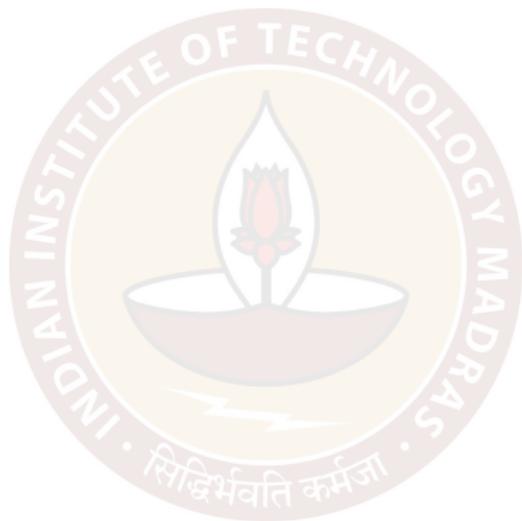
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Bases determine linear transformations

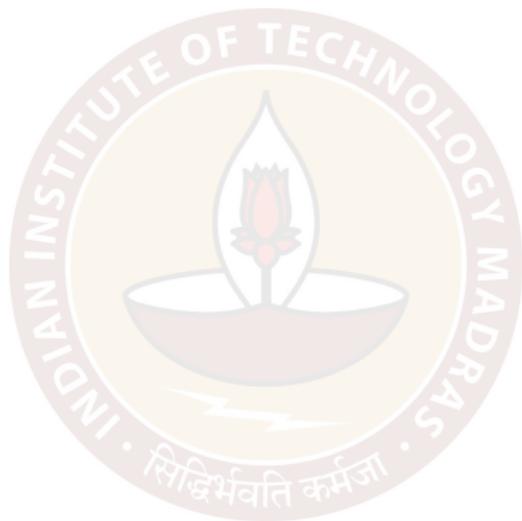
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Let V be a vector space with basis $\{v_1, v_2, \dots, v_n\}$.

Let $f : V \rightarrow W$ be a linear transformation. Then the ordered vectors $f(v_1), f(v_2), \dots, f(v_n)$ uniquely determine f .

Let $v \in V$. $v = \sum_{i=1}^n c_i v_i$.

$$f(v) = f\left(\sum_{i=1}^n c_i v_i\right) = \sum_{i=1}^n c_i f(v_i)$$

is determined by
 c_1, \dots, c_n & $f(v_1), f(v_2), \dots, f(v_n)$.

Suppose w_1, \dots, w_n is a specified set of vectors in W is a unique lin. trans. f s.t. $f(v_i) = w_i$.

There is a unique $\sum_{i=1}^n c_i v_i$ where $v = \sum_{i=1}^n c_i v_i$.

Define $f(v) = \sum_{i=1}^n c_i w_i$ where $v = \sum_{i=1}^n c_i v_i$.

$$f(v_k) = w_k.$$

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Consider the standard basis $\{(1, 0), (0, 1)\}$ of \mathbb{R}^2 .



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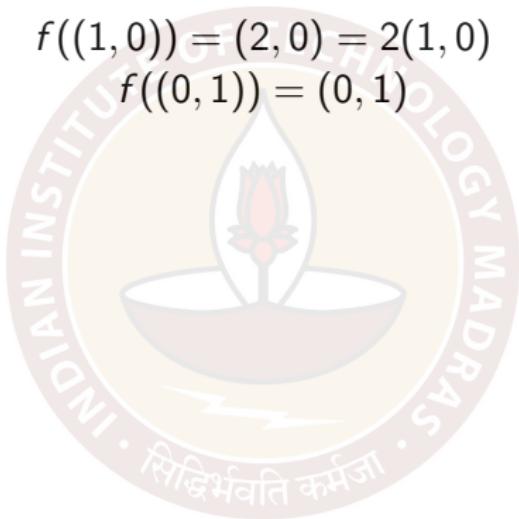
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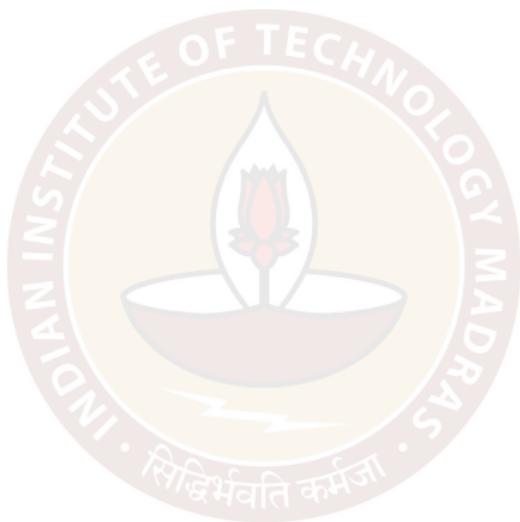
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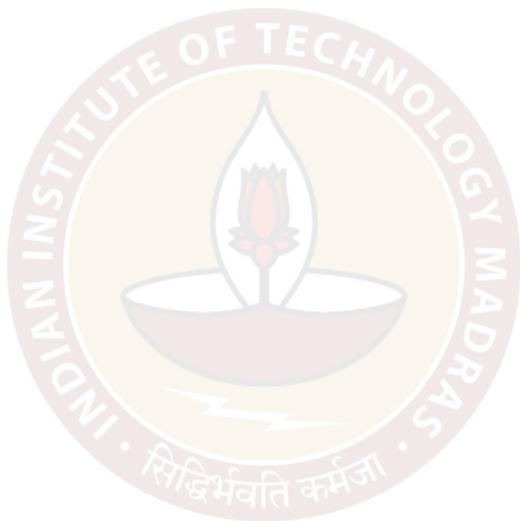
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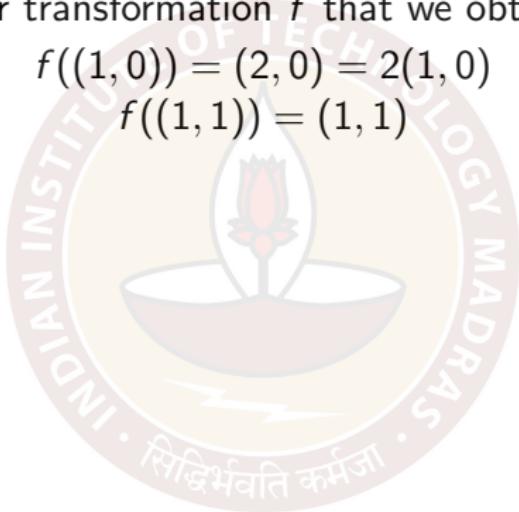


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$$f((1, 1)) = (0, 1) \xleftarrow{\omega_2} \omega_2$$

Note that every element (x, y) is uniquely represented in terms of this basis as $(x, y) = (x - y)(1, 0) + y(1, 1)$.

Basis : $(1, 0), (1, 1)$
 $w_1 = (2, 0)$
 $w_2 = (0, 1)$

$$\begin{aligned} f(x, y) &= (x-y)f(1, 0) + yf(1, 1) \\ &= \boxed{(x-y)}(2, 0) + \boxed{y}(0, 1) \\ &= (2(x-y), 0) + (0, y) = (2x-2y, y). \end{aligned}$$
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Thank you

