Row reduction

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- What are elementary row operations?
- Reducing any matrix to (reduced) row echelon form using elementary row operations.

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- Computing the determinant using row reduction.

Elementary Row operations

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1	Interchange two rows	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \underset{\rightsquigarrow}{R_1 \longleftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

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2	Scalar multiplication of a row by a constant t.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1/3}{\rightsquigarrow} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

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3	Adding multiples of a row to another row.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1 - 3R_2}{\rightsquigarrow} \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

Row reduction : Row echelon form

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Row reduction : Row echelon form

Action	Example and notation
Find the left most non-zero column	$\begin{bmatrix} \color{red}{3} & 2 & 1 & 1 \\ \color{red}{1} & 1 & 0 & 0 \\ \color{red}{0} & 7 & 1 & 1 \end{bmatrix}$

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Find the left most non-zero column	$\begin{bmatrix} \color{red}{3} & 2 & 1 & 1 \\ \color{red}{1} & 1 & 0 & 0 \\ \color{red}{0} & 7 & 1 & 1 \end{bmatrix}$
Use elementary row operations to get 1 in the top position of that column	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_1/3}{\rightsquigarrow} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

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Use type 3 elementary row operations to make the entries below the 1 into 0.	$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix} \overset{R_2-R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

Row reduction : Row echelon form (contd.)

Action	Example and notation

Row reduction : Row echelon form (contd.)

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If there are no non-zero rows below the current row, the matrix is in row echelon row. Else find the next non-zero row and by switching rows (type 1 elementary operation), move all the zero rows between the current row and that row to below the non-zero row.	

Row reduction : Row echelon form (contd.)

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If there are no non-zero rows below the current row, the matrix is in row echelon row. Else find the next non-zero row and by switching rows (type 1 elementary operation), move all the zero rows between the current row and that row to below the non-zero row. Repeat the above steps for the submatrix below the current row.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1/3 & -1/3 & -1/3 \\ 0 & 7 & 1 & 1 \end{bmatrix}$

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Action	Example and notation
Take the columns containing a 1 in the leading position of some row. .	$\begin{bmatrix} \color{red}{1} & \color{purple}{2/3} & \color{orange}{1/3} & 1/3 \\ \color{red}{0} & \color{purple}{1} & \color{orange}{-1} & -1 \\ \color{red}{0} & \color{purple}{0} & \color{orange}{1} & 1 \end{bmatrix}$

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Action	Example and notation
Take the columns containing a 1 in the leading position of some row. Use type 3 elementary row operations to make all the entries in those columns 0.	$\begin{bmatrix} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\overset{R_2 - R_3, R_1 - \frac{1}{3} R_3}{\rightsquigarrow}$ $\begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Example

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

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Recall from determinants

$$\begin{aligned} det(A) &= 2 \times det \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} - 4 \times det \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix} + 1 \times det \begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix} \\ &= 2(72 - 42) - 4(27 - 35) + 1(18 - 40) \\ &= 2(30) - 4(-8) + 1(-22) \\ &= 60 + 32 - 22 \\ &= 70 \end{aligned}$$

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Type	Notation	Effect on determinant
1	$A \overset{R_i \leftrightarrow R_j}{\rightsquigarrow} B$	$det(A) = -det(B)$

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1	$A \overset{R_i \leftrightarrow R_j}{\rightsquigarrow} B$	$det(A) = -det(B)$
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Recall from determinants

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&= 2(72 - 42) - 4(27 - 35) + 1(18 - 40) \\
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1	$A \overset{R_i \leftrightarrow R_j}{\rightsquigarrow} B$	$det(A) = -det(B)$
2	$A \overset{R_i / c}{\rightsquigarrow} B$	$det(A) = c det(B)$
3	$A \overset{R_i + cR_j}{\rightsquigarrow} B$	$det(A) = det(B)$

Computing the determinant via row reduction

For a square matrix A :

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Observe : Row reducing A into row echelon form produces an upper triangular matrix with diagonal entries either all 1 (if it is invertible) or some 1s and some 0s.

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Observe : Row reducing A into row echelon form produces an upper triangular matrix with diagonal entries either all 1 (if it is invertible) or some 1s and some 0s.

1. Row reduce A into row echelon form.
2. If the diagonal entries of the reduced matrix contain a 0, then its determinant is 0 and tracing the determinant back along the row reduction procedure shows that the determinant of A must be 0.

Computing the determinant via row reduction

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1. Row reduce A into row echelon form.
2. If the diagonal entries of the reduced matrix contain a 0, then its determinant is 0 and tracing the determinant back along the row reduction procedure shows that the determinant of A must be 0.
3. If the diagonal entries of the reduced matrix are all 1s its determinant is 1. Tracing back along the procedure used to row reduce using the table of how the determinant changes according to elementary row operations, we can compute the determinant of A.

Example

$$
\begin{tikzcd}
A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \arrow[r, squiggly, "R_1/2"] & \begin{bmatrix} 1 & 2 & 1/2 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix} \arrow[r, squiggly, "R_2 - 3R_1, R_3 - 5R_1"] & \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 2 & 11/2 \\ 0 & -4 & 13/2 \end{bmatrix} \arrow[d, squiggly, "R_2/2"] \\
\begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & 0 & 35/2 \end{bmatrix} \arrow[l, squiggly, "2R_3/35"'] & \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 11/4 \\ 0 & -4 & 13/2 \end{bmatrix} \arrow[l, squiggly, "R_3 + 4R_2"']
\end{tikzcd}
$$

Thank you