What is an orthonormal basis?

Sarang S. Sane

Recall :

- An orthogonal set of vectors $\{v_1, v_2, \ldots, v_k\}$ of an inner product space V is a set of vectors whose elements are mutually orthogonal.

Recall :

- An orthogonal set of vectors $\{v_1, v_2, \ldots, v_k\}$ of an inner product space V is a set of vectors whose elements are mutually orthogonal. i.e.

$$\langle v_i, v_j \rangle = 0 \text{ for } i, j \in \{1, 2, \ldots, k\} \text{ and } i \neq j.$$

Recall :

- An orthogonal set of vectors $\{v_1, v_2, \ldots, v_k\}$ of an inner product space V is a set of vectors whose elements are mutually orthogonal. i.e.

$$\langle v_i, v_j \rangle = 0 \text{ for } i, j \in \{1, 2, \ldots, k\} \text{ and } i \neq j.$$

- An orthogonal set of vectors is linearly independent.

Recall :

- An orthogonal set of vectors $\{v_1, v_2, \ldots, v_k\}$ of an inner product space V is a set of vectors whose elements are mutually orthogonal. i.e.

$$\langle v_i, v_j \rangle = 0 \text{ for } i, j \in \{1, 2, \ldots, k\} \text{ and } i \neq j.$$

- An orthogonal set of vectors is linearly independent.
- A maximal orthogonal set is a basis and is called an orthogonal basis.

What is an orthonormal set?

What is an orthonormal set?

An orthonormal set of vectors of an inner product space V is an orthogonal set of vectors such that the norm of each vector of the set is 1.

What is an orthonormal set?

An orthonormal set of vectors of an inner product space V is an orthogonal set of vectors such that the norm of each vector of the set is 1.

Explicitly, if $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$, then S is an orthonormal set of vectors if

$$\langle v_i, v_j \rangle = 0 \qquad \text{for } i, j \in \{1, 2, \ldots, k\} \text{ and } i \neq j$$

$$\text{and} \qquad ||v_i|| = 1 \qquad \forall i \in \{1, 2, \ldots, k\}$$

What is an orthonormal set?

An orthonormal set of vectors of an inner product space V is an orthogonal set of vectors such that the norm of each vector of the set is 1.

Explicitly, if $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$, then S is an orthonormal set of vectors if

$$\langle v_i, v_j \rangle = 0 \qquad \text{for } i, j \in \{1, 2, \ldots, k\} \text{ and } i \neq j$$

$$\text{and} \qquad ||v_i|| = 1 \qquad \forall i \in \{1, 2, \ldots, k\}$$

e.g. consider $\mathbb{R}^4$ with the usual inner product i.e. the dot product.

What is an orthonormal set?

An orthonormal set of vectors of an inner product space V is an orthogonal set of vectors such that the norm of each vector of the set is 1.

Explicitly, if $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$, then S is an orthonormal set of vectors if

$$\langle v_i, v_j \rangle = 0 \qquad \text{for } i, j \in \{1, 2, \ldots, k\} \text{ and } i \neq j$$

$$\text{and} \qquad ||v_i|| = 1 \qquad \forall i \in \{1, 2, \ldots, k\}$$

e.g. consider $\mathbb{R}^4$ with the usual inner product i.e. the dot product. Then the set

$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$ $\quad$ $\frac{4}{42}+\frac{1}{42}+\frac{1}{42}+\frac{36}{42}=1.$ $\quad$ $\frac{4}{9}+\frac{4}{9}+\frac{1}{9}=1.$

$$\left\{ \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0 \right), \left(\frac{2}{\sqrt{42}}, \frac{1}{\sqrt{42}}, \frac{1}{\sqrt{42}}, \frac{6}{\sqrt{42}} \right), \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{-1}{3} \right) \right\}$$

is an orthonormal set of vectors.

What is an orthonormal basis?

What is an orthonormal basis?

An orthonormal basis is an orthonormal set of vectors which forms a basis.

What is an orthonormal basis?

An orthonormal basis is an orthonormal set of vectors which forms a basis.

Equivalently : An orthonormal basis is an orthogonal basis where the norm of each vector is 1.

What is an orthonormal basis?

An orthonormal basis is an orthonormal set of vectors which forms a basis.

Equivalently : An orthonormal basis is an orthogonal basis where the norm of each vector is 1.

Equivalently : An orthonormal basis is a maximal orthonormal set.

What is an orthonormal basis?

An orthonormal basis is an orthonormal set of vectors which forms a basis.

Equivalently : An orthonormal basis is an orthogonal basis where the norm of each vector is 1.

Equivalently : An orthonormal basis is a maximal orthonormal set.

Example : The standard basis w.r.t. the usual inner product forms an orthonormal basis.

$$\langle e_i, e_j \rangle = (0, 0, \ldots, 0, \overset{i^{th}}{1}, 0, \ldots, 0) \cdot (0, 0, \ldots, \overset{j^{th}}{1}, 0, \ldots, 0) \quad i \neq j$$

$$= 0 \times 0 + \cdots + 1 \times 0 + 0 \cdots + 0 + 0 \times 1 + 0 + \cdots + 0 = 0.$$

$$\|e_i\| = \sqrt{\langle e_i, e_i \rangle} = \sqrt{0 \times 0 + \cdots + 1 \times 1 + 0 + \cdots + 0} = \sqrt{1} = 1.$$

Another example

Consider $\mathbb{R}^3$ with the usual inner product and the set $\beta = \left\{\frac{1}{3}(1,2,2), \frac{1}{3}(-2,-1,2), \frac{1}{3}(2,-2,1)\right\}$. Then β forms an orthonormal basis of $\mathbb{R}^3$.

v_1, v_2, v_3

$\|v_1\|^2 = \langle v_1, v_1\rangle = \left\langle \frac{1}{3}(1,2,2), \frac{1}{3}(1,2,2)\right\rangle = \frac{1}{9}(1\times1+2\times2+2\times2) = 9/9 = 1.$

$\|v_2\|^2 = \langle v_2, v_2\rangle = \left\langle \frac{1}{3}(-2,-1,2), \frac{1}{3}(-2,-1,2)\right\rangle = \frac{1}{9}\{(-2)\times(-2) + (-1)\times(-1) + 2\times 2\} = 9/9 = 1.$

$\|v_3\|^2 = 1.$

$\langle v_1, v_2\rangle = \left\langle \frac{1}{3}(1,2,2), \frac{1}{3}(-2,-1,2)\right\rangle = \frac{1}{9}\langle(1,2,2),(-2,-1,2)\rangle = \frac{1}{9}(1\times(-2) + 2\times(-1) + 2\times2) = 0.$

$\langle v_1, v_3\rangle = \langle v_2, v_3\rangle = 0$

$\vdots$

$|\beta| = 3$ & β is lin. indept. & $\dim(\mathbb{R}^3) = 3$

$\Rightarrow \beta$ is an o.n. basis.

Obtaining orthonormal sets from orthogonal sets

Let V be an inner product space.

Obtaining orthonormal sets from orthogonal sets

Let V be an inner product space. If $\Gamma = \{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of vectors, then we can obtain an orthonormal set of vectors β from Γ by

Obtaining orthonormal sets from orthogonal sets

Let V be an inner product space. If $\Gamma = \{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of vectors, then we can obtain an orthonormal set of vectors β from Γ by

$$\beta = \left\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \ldots, \frac{v_k}{||v_k||} \right\} \qquad .$$

Obtaining orthonormal sets from orthogonal sets

Let V be an inner product space. If $\Gamma = \{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of vectors, then we can obtain an orthonormal set of vectors β from Γ by

$$\beta = \left\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \ldots, \frac{v_k}{||v_k||} \right\} .$$

$\langle v_i, v_j \rangle = 0$

$\Rightarrow \langle \frac{v_i}{||v_i||}, \frac{v_j}{||v_j||} \rangle = 0$

$|| \frac{v_i}{||v_i||} || = \frac{1}{||v_i||} ||v_i|| = 1.$

Example : Consider $\mathbb{R}^2$ with the usual inner product and the orthogonal basis $\Gamma = \{(1, 3), (-3, 1)\}$

Obtaining orthonormal sets from orthogonal sets

Let V be an inner product space. If $\Gamma = \{v_1, v_2, \ldots, v_k\}$ is an orthogonal set of vectors, then we can obtain an orthonormal set of vectors β from Γ by

$$\beta = \left\{ \frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \ldots, \frac{v_k}{||v_k||} \right\} .$$

Example : Consider $\mathbb{R}^2$ with the usual inner product and the orthogonal basis $\Gamma = \{(1,3), (-3,1)\}$

Then $\beta = \left\{ \frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1) \right\}$ is an orthonormal basis of $\mathbb{R}^2$.

$\left|\left| \frac{v_i}{||v_i||} \right|\right| = \frac{1}{||v_i||} ||v_i|| = 1 .$

$\langle v_i, v_j \rangle = 0$

$\Rightarrow \left\langle \frac{v_i}{||v_i||}, \frac{v_j}{||v_j||} \right\rangle = \frac{1}{||v_i|| \, ||v_j||} \langle v_i, v_j \rangle = 0 .$

Why are orthonormal bases important?

Suppose $\Gamma = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Why are orthonormal bases important?

Suppose $\Gamma = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Then v can be written as

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$

Why are orthonormal bases important?

Suppose $\Gamma = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Then v can be written as

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

How do we find $c_1, c_2, \ldots, c_n$?

Why are orthonormal bases important?

Suppose $\Gamma = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Then v can be written as

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

How do we find $c_1, c_2, \ldots, c_n$? For any basis, this means writing a system of linear equations and solving it.

Why are orthonormal bases important?

Suppose $\Gamma = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of an inner product space V and let $v \in V$.

Then v can be written as

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

How do we find $c_1, c_2, \ldots, c_n$? For any basis, this means writing a system of linear equations and solving it.

But since Γ is orthonormal, we can use the inner product and compute $c_i = \langle v, v_i \rangle$.

$$\langle v, v_i \rangle = \langle c_1 v_1 + c_2 v_2 + \cdots + c_i v_i + \cdots + c_n v_n, v_i \rangle$$

$$= c_1 \cancel{\langle v_1, v_i \rangle} + c_2 \cancel{\langle v_2, v_i \rangle} + \cdots + c_i \boxed{\langle v_i, v_i \rangle} + \cdots + c_n \cancel{\langle v_n, v_i \rangle}$$

$$= c_i \langle v_i, v_i \rangle = c_i \|v_i\|^2 = c_i.$$

Example

$\left\{\frac{1}{\sqrt{10}}(1,3), \frac{1}{\sqrt{10}}(-3,1)\right\}$ is an orthonormal basis of $\mathbb{R}^2$. Write $(2,5)$ as a linear combination in terms of these basis vectors.

$$(2,5) = c_1 \frac{1}{\sqrt{10}}(1,3) + c_2 \frac{1}{\sqrt{10}}(-3,1)$$

$$c_1 = \left\langle (2,5), \frac{1}{\sqrt{10}}(1,3) \right\rangle = \frac{1}{\sqrt{10}}(2\times 1 + 5\times 3) = \frac{1}{\sqrt{10}} 17.$$

$$c_2 = \left\langle (2,5), \frac{1}{\sqrt{10}}(-3,1) \right\rangle = \frac{1}{\sqrt{10}}(2\times(-3) + 5\times 1) = \frac{-1}{\sqrt{10}}.$$

$$(2,5) = \frac{17}{\sqrt{10}} v_1 + \frac{-1}{\sqrt{10}} v_2 = \frac{17}{\sqrt{10}} \times \frac{1}{\sqrt{10}}(1,3) - \frac{1}{\sqrt{10}} \times \frac{1}{\sqrt{10}}(-3,1)$$

$$= \frac{17}{10}(1,3) - \frac{1}{10}(-3,1).$$

Thank you