1) i) $f(x,y) = x+y$ It is a linear function of two variable. Hence it represents a plane.

i) $\rightarrow$ 3.

ii) $f(x,y) = xy$ If $y=0$, then, $f(x,y)=0$, again if $x=0$, then $f(x,y)=0$.
When $x<0$ and $y>0$, then $f(x,y)<0$.
Hence, there are some part of the graph which is at the negative direction of z-axis.

ii) $\rightarrow$ 4.

iii) $f(x,y) = x^2 y^2$

$f(x,y) \geqslant 0$ for all $x, y \in \mathbb{R}$.
If $y=0$, then $f(x,y)=0$.
If $x=0$, then $f(x,y)=0$.

iii) $\rightarrow$ 1.

iv) $f(x,y) = 5^{2x+4y}$

Exponent function.

$f(x,y) \geqslant 0$.

$f(0,0) = 5^0 = 1 \neq 0$.

iv) $\rightarrow$ 2.

2) $f_1(x,y) = xy + 2x$

$\frac{\partial f_1}{\partial x}(x,y) = y+2$ $\qquad$ $\frac{\partial f_1}{\partial y}(x,y) = x$

$\frac{\partial f_1}{\partial x}(1,0) = 2$ $\qquad$ $\frac{\partial f_1}{\partial y}(2,1) = 2$

$f_2(x,y) = x^2 + 2y + 3x^2 y$

$\frac{\partial f_2}{\partial x}(x,y) = 2x + 6xy$ $\qquad$ $\frac{\partial f_2}{\partial y}(x,y) = 2 + 3x^2$

$\frac{\partial f_2}{\partial x}(1,1) = 8$ $\qquad$ $\frac{\partial f_2}{\partial y}(\sqrt{2},2) = 8$

$$A = \begin{pmatrix} 2 & 8 \\ 2 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 8 \\ 2 & 8 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Rank}(A) = 1 \\ \det(A) = 0 \end{array}\right\}$$

3) Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

$$\lim_{x\to 0} \lim_{y \to 0} f(x,y) = \lim_{x\to 0} \lim_{y\to 0} \left(\frac{xy}{x^2+y^2} \right) = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Similarly, $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$.

If we approach $(0,0)$ along the line $y = mx$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2+m^2x^2} = \lim_{(x,y)\to(0,0)} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}.$$

For different values of m, there are different values of the $f(x,y)$ as it approaches to $(0,0)$.

Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, $f(x,y)$ is not cont. at the origin.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}(0-0) = 0$$

Similarly, $f_y(0,0) = 0$

Hence, $f_x(0,0)$ and $f_y(0,0)$ exist.

So, all the statements are not true.

4) $f(x,y) = \left(\sqrt{1-x}, \sqrt{y}\right)$

Domain of $f = \{(x,y) \mid 1-x \geq 0, y \geq 0\}$

$= \{(x,y) \mid 1 \geq x, y \geq 0\}$

$h(x,y) = \dfrac{x}{x^2+y^2}$

Domain of $h = \{(x,y) \mid x^2+y^2 \neq 0\}$

$= \mathbb{R}^2 \setminus \{(0,0)\}$

$g(x,y) = \sin\left((x^2-1)^2 + y^4\right)$

$g(f(x,y)) = g\left(\sqrt{1-x}, \sqrt{y}\right)$

$= \sin\left((1-x-1)^2 + (\sqrt{y})^4\right)$

$= \sin\left(x^2+y^2\right)$

$\left(\left(g \circ f\right) \times h\right)(x,y)$

$= \dfrac{x \sin(x^2+y^2)}{x^2+y^2}.$

$$((g \circ f) \times h + s)(x, y)$$

$$= \frac{x \sin(x^2 + y^2)}{x^2 + y^2} + x^2 e^{y^2}$$

$$\lim_{(x,y)\to(0,0)} x \cdot \frac{\sin(x^2+y^2)}{(x^2+y^2)}$$

$$= \lim_{(x,y)\to(0,0)} x \cdot \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)}$$

Let $x^2 + y^2 = m$.

Hence, $(x, y) \to (0,0)$

$\Rightarrow m \to 0$

$$= 0 \cdot \lim_{m \to 0} \frac{\sin m}{m}$$

$$= 0 \cdot 1 = 0.$$

$$\lim_{(x,y)\to(0,0)} x^2 e^{y^2} = 0 \cdot 1 = 0.$$

Hence, $\lim_{(x,y)\to(0,0)} ((g \circ f) \times h + s)(x, y) = 0.$

5) $$f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{2(x^2+y^2)}} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

$$x^2 \leq x^2+y^2 \quad \Rightarrow \quad |x| \leq \sqrt{x^2+y^2}$$

$$y^2 \leq x^2+y^2 \quad \Rightarrow \quad |y| \leq \sqrt{x^2+y^2}$$

$$0 \leq |x| \leq \sqrt{x^2+y^2}$$

$$\Rightarrow \; 0 \leq \frac{|x|}{\sqrt{x^2+y^2}} \leq 1$$

$$\Rightarrow \; 0 \leq \frac{2|x||y|}{\sqrt{2(x^2+y^2)}} \leq \sqrt{2}\,|y|$$

$$\lim_{(x,y)\to(0,0)} \sqrt{2}\,|y| = 0$$

Hence,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

So, $$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \neq 1 = f(0,0)$$

Hence, f is not cont. at the origin.

6) $f(x,y) = (x^2, y^2)$

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

If we approach the origin along $y = mx$,

we get, $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2(1-m^2)}{x^2(1+m^2)}$

$$= \lim_{(x,y)\to(0,0)} \frac{1-m^2}{1+m^2}$$

$$= \frac{1-m^2}{1+m^2}$$

So, for different value of m, we will get different values.

Hence, $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ does not exist.

$$g\circ f(x,y) = g(x^2, y^2) = \begin{cases} \dfrac{x^4 - y^4}{x^4 + y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

By the similar argument as above we can show that $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ does not exist.

$$\lim_{x\to 0}\lim_{y\to 0} g(x,y) = \lim_{x\to 0}\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

$$\lim_{y\to 0}\lim_{x\to 0}(g\circ f)(x,y) = \lim_{y\to 0}\lim_{x\to 0} \frac{x^4-y^4}{x^4+y^4}$$

$$= \lim_{y\to 0} \frac{-y^4}{y^4} = -1.$$

$$\lim_{(x,y)\to(1,1)} g(x,y) = \lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^2+y^2}$$

$$= \frac{0}{2} = 0$$

(It exists).

7) $f(x,y) = xy$

$f_x(x,y) = y$ $\qquad$ $f_y(x,y) = x$

$f_x(a,b) = b$ $\qquad$ $f_y(a,b) = a$

Directional derivative in the direction of $(1,-1)$

$$(b,a)\cdot\frac{1}{\sqrt{2}}(1,-1) = \frac{1}{\sqrt{2}}(b-a)$$

Directional derivative in the direction of $(1,1)$

$$(b, a) \cdot \frac{1}{\sqrt{2}}(1, 1) = \frac{1}{\sqrt{2}}(b+a)$$

It is given that, $\frac{1}{\sqrt{2}}(b-a) = 1$

$\frac{1}{\sqrt{2}}(b+a) = 5$

$$b - a = \sqrt{2}$$
$$b + a = 5\sqrt{2}$$

$2b = 6\sqrt{2}$ | $a = 2\sqrt{2}$

$\Rightarrow b = 3\sqrt{2}$ | $2a = 4\sqrt{2}$

$$2a + b = 7\sqrt{2}$$

$$\Rightarrow \frac{2a+b}{\sqrt{2}} = 7.$$

8) $T(x,y) = 2x^2 + 3xy + y^2$

$$\lim_{(x,y)\to(1,1)} T(x,y) = \lim_{(x,y)\to(1,1)} (2x^2+3xy+y^2)$$

$$= 6$$

$$\lim_{x\to1} \lim_{y\to1} T(x,y) = \lim_{x\to1} \lim_{y\to1} (2x^2+3xy+y^2)$$

$$= \lim_{x\to1} (2x^2+3x+1)$$

$$= 2+3+1 = 6$$

$T(1,1) = 6$.

$T(1,0) = 2$, $T(0,1) = 1$

$$T((1,0)+(0,1)) = T(1,1) = 6 \neq T(1,0) + T(0,1)$$

Hence, T is not linear function.

$$T(cx, cy) = 2(cx)^2 + 3(cx)(cy) + (cy)^2$$

$$= 2c^2x^2 + 3c^2xy + c^2y^2$$

$$= c^2(2x^2+3xy+y^2)$$

$$= c^2 T(x,y) \neq cT(x,y)$$

unless $c = 0$ or 1.

$$\lim_{(x,y)\to(1,2)} T(x,y) = \lim_{(x,y)\to(1,2)} \left(2x^2 + 3xy + y^2\right)$$

$$= 2 + 3(1)(2) + (2)^2$$

$$= 2 + 6 + 4 = 12.$$

9) $T(x,y) = 2x^2 + 3xy + y^2$

$T_x(x,y) = 4x + 3y$, $T_y(x,y) = 3x + 2y$

Hence gradient at the point (a,b) will be

$(4a+3b, 3a+2b)$

If $(4a+3b, 3a+2b) = (25, 18)$

then, $4a+3b = 25$ $\quad$ $3a+2b = 18$

Solving this we will get, $a = 4$, $b = 3$

$T(4,3) = 2(4)^2 + 3(4)(3) + (3)^2$

$= 32 + 36 + 9 = 77$

10) For option 1, we have to find the directional derivative at the point (a,b) in the direction of the unit vector $(1,0)$.

Which is,

$$(4a+3b,\ 3a+2b)\cdot(1,0) = 4a+3b.$$

For option 2, we have to find the directional derivative at the point (a,b) in the direction of the unit vector $(0,1)$

Which is,

$$(4a+3b,\ 3a+2b)\cdot(0,1) = 3a+2b.$$

For option 3, we have to find the directional derivative at the point $(1,1)$ in the direction of the unit vector $\frac{1}{\sqrt{13}}(2,3)$

Which is,

$$(7,5)\cdot\frac{1}{\sqrt{13}}(2,3) = \frac{29}{\sqrt{13}}.$$

For option 4, we have to find the directional derivative at the point $(1,2)$ in the direction of the unit vector $(0,1)$

which is

$$(10,7)\cdot(0,1) = 7$$