Week-11
Mathematics for Data Science - 2
Graded Assignment solutions

1 Multiple Choice Questions (MCQ)

1. Match the functions of two variables in Column A with their maximum directional derivatives at (0,0) given in Column B.

	Functions of two variables (Column A)		maximum directional derivative at (0,0) (Column B)
a)	$f(x, y) = y^2 e^{2x}$	i)	3
b)	$f(x, y) = 5 - x^2 + 3x - 2y^2$	ii)	$\sqrt{2}$
c)	$f(x, y) = x + y - 2xy$	iii)	1
d)	$f(x, y) = x\sin(x) + y\cos(y)$	iv)	0

Table : M2W6G1

Choose the correct option.

- ◯ Option 1: a → iv), b → iii), c → i, d → ii)
- ◯ **Option 2:** a → iv), b → i), c → ii), d → iii)
- ◯ Option 3: a → iii), b → iv), c → i), d → ii)
- ◯ Option 4: a → iii), b → iv), c → ii), d → i)

Solution:

The maximum directional derivative of a function $f(x, y)$ at (0,0) will be $||\nabla f||_{(0,0)} = ||(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})||_{(0,0} = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\ |_{(0,0)}$.

(a) $f(x, y) = y^2e^{2x}$
At point (0,0), $\frac{\partial f}{\partial x} = 2y^2e^{2x} = 2(0)^2.e^{2(0)} = 0$ and $\frac{\partial f}{\partial y} = 2ye^{2x} = 2(0)e^{2(0)} = 0$
Therefore, the maximum directional derivative of $f(x, y)$ at (0,0) $= \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\ |_{(0,0)}$
$= \sqrt{0^2 + 0^2} = 0$.

(b) $f(x, y) = 5 - x^2 + 3x - 2y^2$
At point (0,0), $\frac{\partial f}{\partial x} = -2x + 3 = -2(0) + 3 = 3$ and $\frac{\partial f}{\partial y} = -4y = -4(0) = 0$
Therefore, the maximum directional derivative of $f(x, y)$ at (0,0) $= \sqrt{3^2 + 0^2} = 3$.

(c) $f(x, y) = x + y - 2xy$
At point (0,0), $\frac{\partial f}{\partial x} = 1 - 2y = 1 - 2(0) = 1$ and $\frac{\partial f}{\partial y} = 1 - 2x = 1 - 2(0) = 1$
Therefore, the maximum directional derivative of $f(x, y)$ at (0,0) $= \sqrt{1^2 + 1^2} = \sqrt{2}$.

(d) $f(x, y) = xsin(x) + ycos(y)$
At point (0,0), $\frac{\partial f}{\partial x} = sin(x) + xcos(x) = sin(0) + 0cos(0) = 0$ and $\frac{\partial f}{\partial y} = cos(y) - ysin(y) = cos(0) - 0sin(0) = 1$
Therefore, the maximum directional derivative of $f(x, y)$ at (0,0) $= \sqrt{0^2 + 1^2} = 1$.

Hence, a $\longrightarrow$ iv), b $\longrightarrow$ i), c $\longrightarrow$ ii), d $\longrightarrow$ iii). So, option 2 is correct.

2. The equation of the tangent plane to the surface $z = 3 - x^2 - y^2$ at the point $(2, 1, -2)$ is

   - ○ Option 1: $z = 4x - 2y + 8$
   - ○ Option 2: $z = 4x + 2y + 8$
   - ○ **Option 3:** $z = -4x - 2y + 8$
   - ○ Option 4: $z = -4x - 2y - 8$

Solution:

Given surface, $f(x, y) = z = 3 - x^2 - y^2$
The tangent plane to the given surface at (2,1,-2) will be

$$\begin{aligned} z =& f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) + f(2, 1) \\ =& (-2(2))(x - 2) + (-2(1))(y - 1) + (3 - 2^2 - 1^2) \\ =& -4(x - 2) - 2(y - 1) + (3 - 1 - 4) \\ =& -4x + 8 - 2y + 2 - 2 \\ =& -4x - 2y + 8 \end{aligned}$$

Hence, the equation of the tangent plane to the given surface at (2,1,-2) is
$z = -4x - 2y + 8$.

3. Let $L_f(x, y)$ be the linear approximation to the function $f(x, y) = ye^x - \frac{1}{4}(x^2 + y^2)$ at $(0, 1)$. Then the equation of $L_f(x, y)$ is

   ◯ Option 1: $x - \frac{y}{2} + \frac{1}{4}$
   ◯ Option 2: $y - \frac{x}{2} + \frac{1}{4}$
   ◯ **Option 3:** $x + \frac{y}{2} + \frac{1}{4}$
   ◯ Option 4: $y + \frac{x}{2} + \frac{1}{4}$

   Solution:

   Given function $f(x, y) = ye^x - \frac{1}{4}(x^2 + y^2)$
   The linear approximation to the given function at (0,1) will be

$$\begin{aligned}
L_f(x, y) =& f(0, 1) + f_x(0, 1)(x - 0) + f_y(0, 1)(y - 1)\\
=&(1e^0 - \frac{1}{4}(0^2 + 1^2)) + (1e^0 - \frac{2}{4}(0))(x - 0) + (e^0 - \frac{2}{4}(1))(y - 1)\\
=&(1 - \frac{1}{4}) + 1(x - 0) + (1 - \frac{1}{2})(y - 1)\\
=&\frac{3}{4} + x + \frac{y}{2} - \frac{1}{2}\\
=&x + \frac{y}{2} + \frac{1}{4}
\end{aligned}$$

   Hence, the equation of $L_f(x, y)$ is $x + \frac{y}{2} + \frac{1}{4}$.

2 Multiple Select Questions (MSQ)

4. Which of the following is (are) the critical points of the scalar valued function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$?

   ◯ **Option 1:** $(0, 0)$
   ◯ **Option 2:** $(0, 2)$.
   ◯ **Option 3:** $(1, 1)$
   ◯ Option 4: $(1, 2)$
   ◯ Option 5: $(0, 1)$

   Solution:

   Given $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$
   Now,

$$\begin{aligned}
f_x =& 6xy - 6x = 6x(y - 1) &(1)\\
f_y =& 3x^2 + 3y^2 - 6y = 3x^2 + 3y(y - 2) &(2)
\end{aligned}$$

Equating the equations (1) and (2) to zero and find the x and y values.
$f_x = 6x(y-1) = 0$
$\implies x = 0$ or $y - 1 = 0$
$\implies x = 0$ or $y = 1$
Now, substitute $\boldsymbol{x = 0}$ in the equation $f_y = 0$, we get
$f_y = 3x^2 + 3y(y-2) = 0$
$\implies 3(0)^2 + 3y(y-2) = 0$
$\implies \boldsymbol{y = 0}$ or $\boldsymbol{y = 2}$
and substitute $\boldsymbol{y = 1}$ in the equation $f_y = 0$, we get
$f_y = 3x^2 + 3y(y-2) = 0$
$\implies 3x^2 + 3(1)(1-2) = 0$
$\implies 3x^2 = 3$
$\implies x^2 = 1$
$\implies \boldsymbol{x = 1}$ or $\boldsymbol{x = -1}$
Hence, **(0,0), (0,2), (1,1) and (-1,1)** are the critical points of the given scalar valued function.
So, options (1),(2),(3) are correct

5. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

Which of the following is (are) true about $f(x,y)$?

- ○ Option 1: The directional derivative at $(0,0)$ in the direction of a unit vector $u = (u_1, u_2)$ is 1.
- ○ **Option 2:** The directional derivative at $(0,0)$ in the direction of a unit vector $u = (u_1, u_2)$ is $\frac{u_2^2}{u_1}$, where u_1 is non-zero.
- ○ Option 3: Amongst all directional derivatives at $(0,0)$, the maximum occurs in the direction of the vector $(5,5)$.
- ○ **Option 4:** There is no plane which contains all the tangent lines at $(0,0)$ and hence the tangent plane at $(0,0)$ does not exist.

Solution:

Given, $f(x,y)$ a piece wise multivariable function.

So the directional derivative at (0,0) in the direction of a unit vector $u = (u_1, u_2)$ is

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h} &= \lim_{h\to 0} \frac{f(0+hu_1, 0+hu_2) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{\frac{(hu_1)(hu_2)^2}{(hu_1)^2+(hu_2)^4} - 0}{h} \\
&= \lim_{h\to 0} \frac{h^3 u_1 u_2^2}{h^3(u_1^2 + h^2 u_2^4)} \\
&= \lim_{h\to 0} \frac{u_1 u_2^2}{(u_1^2 + h^2 u_2^4)} \\
&= \frac{u_1 u_2^2}{u_1^2} \\
&= \frac{u_2^2}{u_1}
\end{aligned}$$

Hence, the directional derivative at (0,0) in the direction of a unit vector (u_1, u_2) will be $\frac{u_2^2}{u_1}$, where $u_1 \neq 0$.
So, option (2) is correct and option (1) is not correct.
The directional derivative at (0,0) in the direction of the vector $(1, \sqrt{3})$ will be $\frac{3}{2}$ which is greater than $\frac{1}{\sqrt{2}}$ in the direction of the vector (5,5) implies it is not maximum.
So, option (3) is incorrect.

Now, $f_x(x,y) = \begin{cases} \frac{y^2(y^4-2x^2)}{(x^2+y^4)^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$ and $f_y(x,y) = \begin{cases} \frac{2xy(x^2-y^4)}{(x^2+y^4)^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$

Observe that these partial derivatives are not continuous at (0,0) (use the definition of continuity) . so there will be no plane which contains all the tangent lines at (0,0) and hence the tangent plane will not exist at (0,0).
So, option (4) is correct.

3 Numerical Answer Type (NAT)

6. Consider a function $f(x,y) = 2\sqrt{x^2+4y}$. Let S denote the set of unit vectors u for which the directional derivative of f at $(-2,3)$ in the direction of u is 0. Find the cardinality of the set S.

   Solution:

   Given, $f(x,y) = 2\sqrt{x^2+4y}$
   So, $\boldsymbol{\nabla} f = (f_x, f_y) = (\frac{2(2x)}{2\sqrt{x^2+4y}}, \frac{2(4)}{2\sqrt{x^2+4y}}) = (\frac{2x}{\sqrt{x^2+4y}}, \frac{4}{\sqrt{x^2+4y}})$
   Also given that, S be the set of unit vectors $u = (u_1, u_2)$ such that $\boldsymbol{\nabla} f_{(-2,3)}.u = 0$

Now,

$$\begin{aligned}
&\boldsymbol{\nabla} f_{(-2,3)}.u = 0\\
&\implies (\frac{2(-2)}{\sqrt{(-2)^2+4(3)}}, \frac{4}{\sqrt{(-2)^2+4(3)}}).(u_1,u_2) = 0\\
&\implies (\frac{-4}{16}, \frac{4}{16}).(u_1,u_2) = 0\\
&\implies \frac{-1}{4}u_1 + \frac{1}{4}u_2 = 0\\
&\implies -u_1 + u_2 = 0\\
&\implies u_1 = u_2......(1)
\end{aligned}$$

and also u is a unit vector. So, $\sqrt{u_1^2+u_2^2} = 1.....(2)$
Substituting equation (1) in equation (2), we get
$\sqrt{u_1^2+u_1^2} = 1 \implies \sqrt{2u_1^2} = 1 \implies u_1^2 = \frac{1}{2} \implies u_1 = \frac{\pm 1}{\sqrt{2}}$
Hence, the unit vector u can be $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$
Therefore, $S = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$. So the cardinality of the set S is **2**.

7. Suppose $f\,(x,y) = xye^x$ be a scalar valued multivariable function. Then using the linear approximation $L_f(x,y)$ of the function f at $(1,1)$, the estimate value of $f(1.2, 0.9)$ is found to be βe, where β is a real number. The value of β is

   Solution:

   Given scalar valued multivariable function $f(x,y) = xye^x$
   So, $f_x = xye^x + ye^x$ and $f_y = xe^x$
   Now, the linear approximation to the given function at $(1,1)$ will be

$$\begin{aligned}
L_f(x,y) =& f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)\\
=&(1)(1)e^1 + ((1)(1)e^1 + ((1)e^1)(x-1) + ((1)e^1)(y-1)\\
=&e + 2e(x-1) + e(y-1)\\
=&e + 2ex - 2e + ey - e\\
=&2ex + ey - 2e
\end{aligned}$$

   So, the linear approximation of the given function at (1,1) is $L_f(x,y) = 2ex + ey - 2e$
   Hence, the estimate value of $f(1.2, 0.9)$ will be $L_f(1.2, 0.9) = 2e(1.2) + e(0.9) - 2e = 1.3e$
   Therefore, the value of $\beta = \mathbf{1.3}$

4 Comprehension Type Question:

The temperature T (in degree centigrade, ^0C) in a solid metal sphere is given by the function $e^{-(x^2+y^2+z^2)}$. Answer Questions 8,9 and 10 from the given information.

8. Choose the set of correct options.

   - ◯ **Option 1:** The rate of change of temperature in the direction of X-axis is continuous at every point.
   - ◯ Option 2: The rate of change of temperature in the direction of Z-axis is not continuous at the origin.
   - ◯ **Option 3:** The rate of change of temperature at the origin from any direction is constant and that is 0.
   - ◯ Option 4: The rate of change of temperature at the origin from any direction is constant and that is e.
   - ◯ Option 5: The rate of change of temperature at the origin from any direction is not constant.

   <u>Solution:</u>

   Given, $T = e^{-(x^2+y^2+z^2)}$ is the temperature in a solid metal sphere at a point (x, y, z)
   So,
   $\nabla T = (T_x, T_y, T_z) = (-2xe^{-(x^2+y^2+z^2)}, -2ye^{-(x^2+y^2+z^2)}, -2ze^{-(x^2+y^2+z^2)}) = -2e^{-(x^2+y^2+z^2)}(x, y, z)$
   Observe that each component of ∇T is continuous and therefore the rate of change of temperature in the direction of X-axis is continuous at every point.
   So, option (1) is correct.
   Also, the rate of change of temperature in the direction of Z-axis is continuous at every point.
   So, option (2) is not correct.

   Now, the rate of change of temperature at the origin (which is at (0,0,0)) in a direction u will be $\nabla T_{(0,0,0)}.\frac{u}{||u||}$

   $$\begin{aligned}\nabla T_{(0,0,0)}.\frac{u}{||u||} &= -\,2e^{-(x^2+y^2+z^2)}(x, y, z).\frac{u}{||u||}\\ &= -\,2e^{-(0^2+0^2+0^2)}(0, 0, 0).\frac{u}{||u||}\\ &= (0, 0, 0).\frac{u}{||u||}\\ &= 0\end{aligned}$$

   Therefore, The rate of change of temperature at the origin in any direction u is constant and that is 0.
   So, option (3) is correct and options (4) & (5) are incorrect.

9. Find the rate of change of the temperature at point $(1, 0, 0)$ in the direction toward point $(8, 6, 0)$.

   - ◯ Option 1: $\frac{1.6}{e}$.
   - ◯ **Option 2:** $-\frac{1.6}{e}$.

◯ Option 3: $\frac{2.8}{e}$.

◯ Option 4: $-\frac{2.8}{e}$.

Solution:

The rate of change of the temperature at point (1,0,0) in the direction toward point (8,6,0) will be

$$\begin{aligned}(\boldsymbol{\nabla} T|_{(1,0,0)}).\frac{(8,6,0)}{||(8,6,0)||} =&(-2(1)e^{-(1^2+0^2+0^2)}, -2(0)e^{-(1^2+0^2+0^2)}, -2(0)e^{-(1^2+0^2+0^2)}).\frac{(8,6,0)}{\sqrt{8^2+6^2+0^2}}\\ =&(-2e^{-1},0,0).\frac{(8,6,0)}{10}\\ =&\frac{-1.6}{e}+0+0\\ =&\boldsymbol{\frac{-1.6}{e}}\end{aligned}$$

10. Which of the following statements are true?

◯ **Option 1:** At a point (a,b,c) on the sphere the maximum rate of change in temperature is given by $2e^{-(a^2+b^2+c^2)}\ \sqrt{a^2+b^2+c^2}$.

◯ Option 2: At a point (a,b,c) on the sphere the maximum rate of change in temperature is given by $-2e^{-(a^2+b^2+c^2)}\ \sqrt{a^2+b^2+c^2}$.

◯ Option 3: At a point (a,b,c) on the sphere the maximum rate of in temperature is in the direction of the unit vector $\left(-\frac{a}{a^2+b^2+c^2}, -\frac{b}{a^2+b^2+c^2}, -\frac{c}{a^2+b^2+c^2}\right)$.

◯ Option 4: At a point (a,b,c) on the sphere the maximum rate of change in temperature is in the direction of the unit vector $\left(-\frac{2a}{e^{a^2+b^2+c^2}}, -\frac{2b}{e^{a^2+b^2+c^2}}, -\frac{2c}{e^{a^2+b^2+c^2}}\right)$.

Solution:

The maximum rate of change in temperature at a point (a,b,c) will be in the direction of the unit vector $\frac{\boldsymbol{\nabla} T}{||\boldsymbol{\nabla} T||}$ at (a,b,c)

Now,

$$\begin{aligned}\frac{\boldsymbol{\nabla} T}{||\boldsymbol{\nabla} T||} =&\frac{(T_x,T_y,T_z)}{\sqrt{T_x^2+T_y^2+T_z^2}}\\ =&\frac{(-2xe^{-(x^2+y^2+z^2)}, -2ye^{-(x^2+y^2+z^2)}, -2ze^{-(x^2+y^2+z^2)})}{\sqrt{(-2xe^{-(x^2+y^2+z^2)})^2+(-2ye^{-(x^2+y^2+z^2)})^2+(-2ze^{-(x^2+y^2+z^2)})^2}}\\ =&\frac{2e^{-(x^2+y^2+z^2)}(-x,-y,-z)}{2e^{-(x^2+y^2+z^2)}\sqrt{x^2+y^2+z^2}}\\ =&\frac{(-x,-y,-z)}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

Therefore, at a point (a,b,c) on the sphere the maximum rate of in temperature is in the direction of the unit vector $\frac{(-a,-b,-c)}{\sqrt{a^2+b^2+c^2}}$.
So, options (3) & (4) are not true.
Now, the maximum rate of change in temperature at a point (a, b, c) will be $||\boldsymbol{\nabla} T||_{(a,b,c)} = \sqrt{T_x^2 + T_y^2 + T_z^2}|_{(a,b,c)} = 2e^{-(x^2+y^2+z^2)}\sqrt{x^2 + y^2 + z^2}|_{(a,b,c)} = 2e^{-(a^2+b^2+c^2)}\sqrt{a^2 + b^2 + c^2}$
So, option 1 is true and option 2 is not true.