IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
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Finding the tangent hyper plane

Hello, and welcome to the maths 2 component of the online B.Sc. program on data science and programming. This video we are going to talk about how to find the tangent hyper plane. So, I will explain what this means in a few slides, but in terms of $\mathbb{R}^3$, meaning if you have a function of two variables, you can just think of this as finding the tangent plane.

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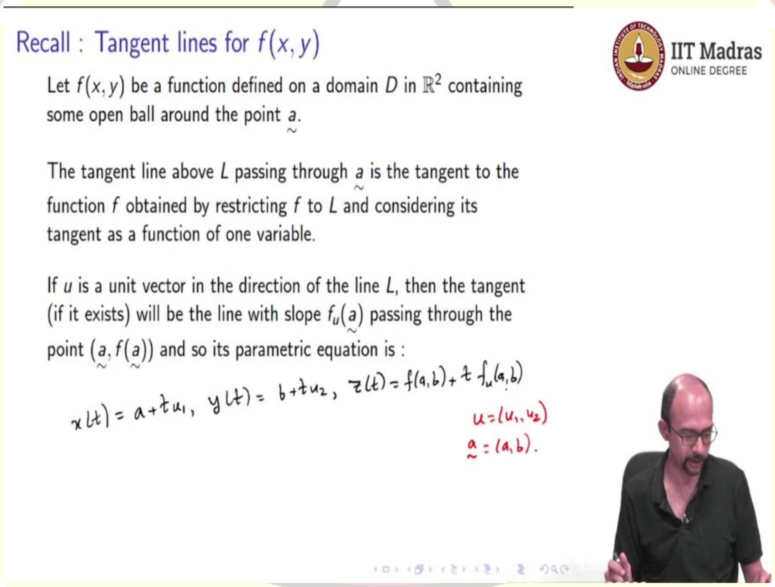

Let us recall first, we talked about tangent lines in a previous video. So, suppose you have a function of two variables x , y, then we have talked about the notion of tangent lines. So, the notion of a tangent line was, you take a line L passing through this point $\underset{\sim}{a}$ which is, which you can think of as a , b and you restrict your graph to only the plane above that line. So, once you do that, it becomes a function of one variable. And then for a function of one variable, we know exactly what it means for a tangent line to exist and how to write it down in terms of its algebraic equation.

And in the previous video, we did exactly that. So, if u is a unit vector in the direction of the line L, then the tangent, if it exists, will be the line slope f_u $(\underset{\sim}{a})$ passing through the point $\underset{\sim}{a}$,f($\underset{\sim}{a}$). And so its parametric equation is given by, so x(t) is a + t $\times u_1$, y(t) is b + t $\times$ u_2 and z(t) is f(a,b) + t $\times$

$f_u(a, b)$, where what are all the terms here. So, u is u_1, u_2. So, the vector u is u_1, u2. That is the unit vector in the direction of the line L, $\underset{\sim}{a}$ is the coordinates of $\underset{\sim}{a}$ are a , b in two dimensions.

So, if you want to think of it in $\mathbb{R}^3$, it is a , b , 0. And then $f_u(a, b)$ is the directional derivative of f in the direction u or in the direction of the unit vector u at the point a , b. So, this explains every(t)hing. And as you vary t, t is a variable, as you vary t, you get different points on this line. So, this was how the parametric equation was for the tangent line.

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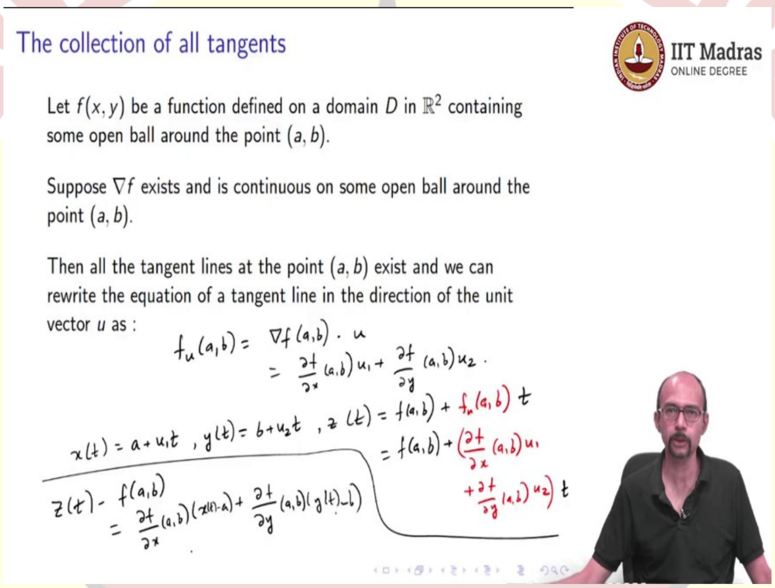

So, let us look at the collection of all the tangents. So, suppose f(x, y) is a function defined in the domain D in $\mathbb{R}^2$ containing some open ball around the point a , b. Suppose the ∇ f exists and is continuous on some open ball around the point a , b, then all the tangent lines at the point a , b exists. This was because once the ∇ is continuous in a small neighborhood in an open ball, then we know that the directional derivatives all exist. And once the directional derivatives all exist, all the tangent lines exist.

So, then we can write down the equation of the tangent line in the direction of the vector u as, which means we can, we are saying we can rewrite it. So, here, what we are saying is that I can rewrite $f_u(a, b)$ as $\nabla f(a, b)$. the vector u. In other words, I can rewrite this as $\frac{\partial f}{\partial x}(a, b)$ × u1 + $\frac{\partial f}{\partial y}(a, b)$ × u2.

So, now I can rewrite the parametric equations as x(t) is a + u_1t, y(t) is b + u_2t and z(t) is f(a , b) +, now here something has changed, because we had f_u(a , b)× t, but I know that f_u(a , b) is I can rewrite. So, what I get is, this is f(a , b) + $\frac{\partial f}{\partial x}(a, b)$ × u_1 + $\frac{\partial f}{\partial y}(a, b)$ × u_2 the whole thing × t. This is how the new equation looks like.

And so what we are saying you really is that somehow, if I know u_1 and u_2, this is actually a very easy equation in terms of u_1 and u_2 provided I know what are my partial derivatives. So, now this is how the parametric equations are for the tangent line in the direction of the unit vector u. So, now notice something interesting here. So, if I look at z(t), I can really write it as a linear combination of x(t) and y(t). How do I do that?

Well, with some constants, of course, so if I take z(t) - f(a,b), so this is a partial derivative of f with respect to x at a, b, × x - a + partial derivative of f with respect to y(a,b) × y - b. So, it satisfies this equation. I would write x(t) and y(t). So, this parametric equation satisfies this particular equation, which means that all of these lines are contained in this, whatever this shape is which satisfies this equation. So, let us look at some examples in GeoGebra.

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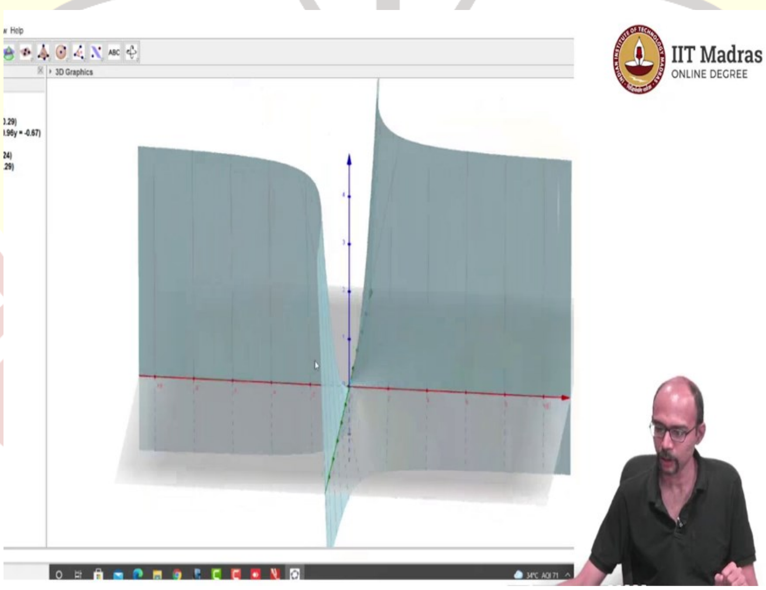

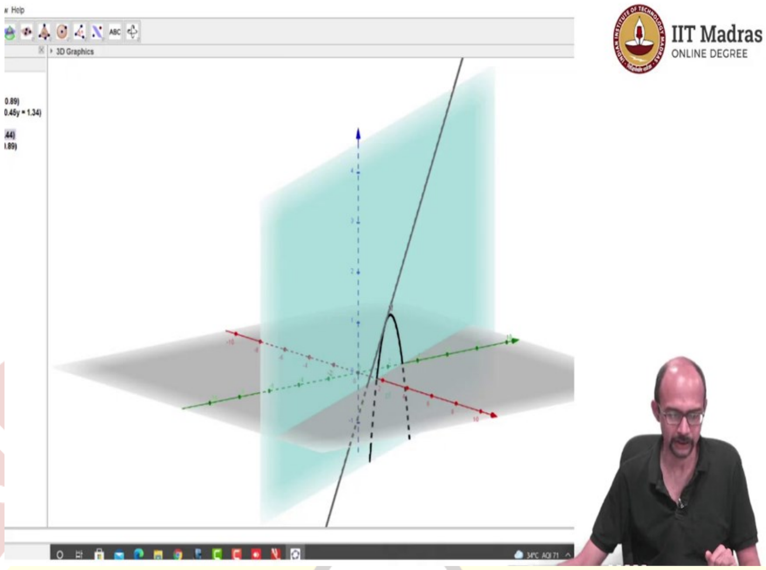
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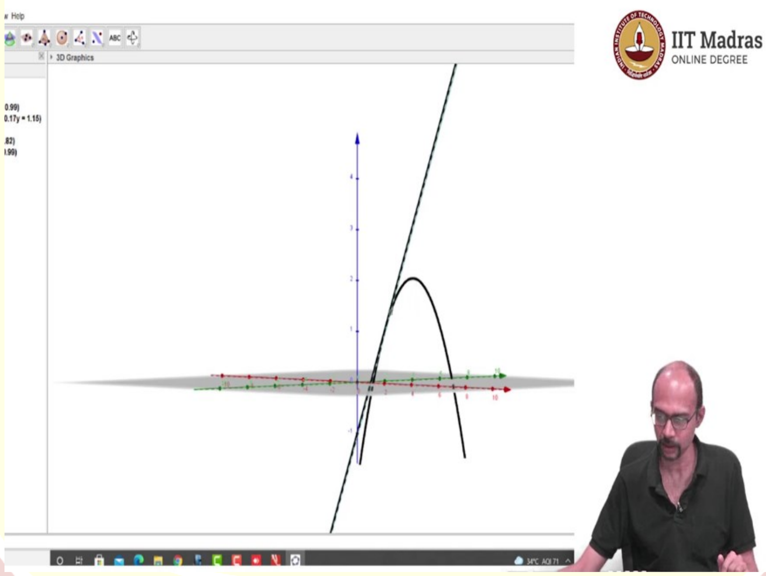
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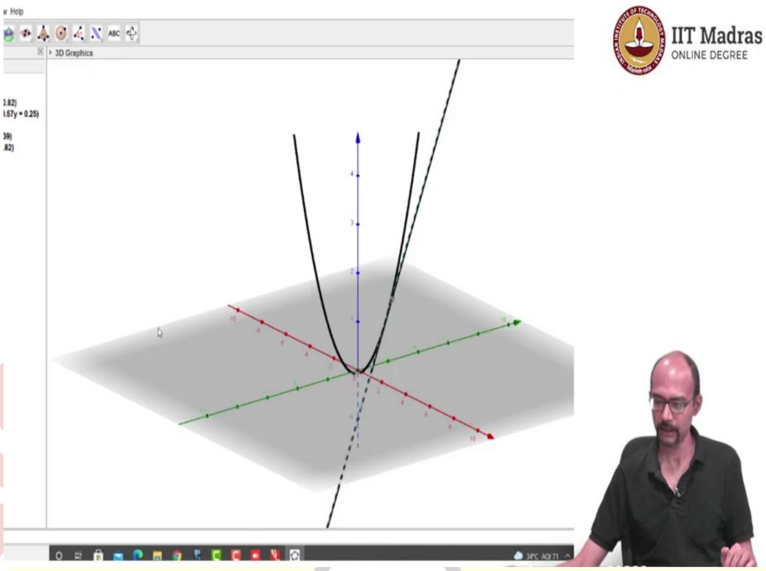

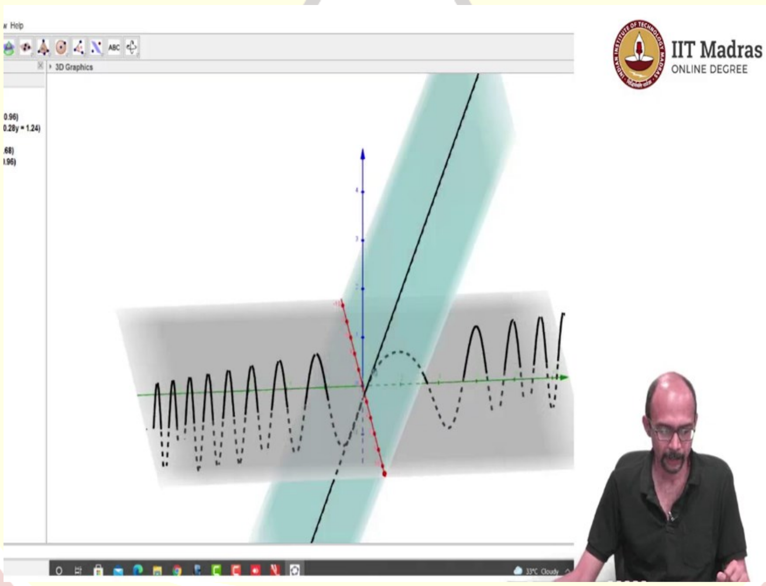

So, here is the function x, y. This is how the graph looks like. We have seen this function before. Let us look at what happens to the point 1 , 1. So, if you look at the point 1 , 1, let us take a plane passing through that point. So, this is how a plane passing through the point 1 , 1 looks like. And we can vary this plane. So, if you vary this unit vector, accordingly, the plane is going to vary. So, these are many of the planes passing through that point.

And now, let us restrict the function to that plane. So, if we do that, we get this curve. So, this is the graph. Let us remove this f(x) is x × y. This is how the graph looks like. Let us draw the tangent to that graph. So, here is how the tangent line looks like. Now, let us see what happens as we vary this plane. So, if we vary this plane, this is how the graph changes. Let us look at that equation that we just had, which was z - f(a,b) is $\frac{\partial f}{\partial x}(a, b)$ × x - a + $\frac{\partial f}{\partial y}(a, b)$ × y - b, so how does that look like? So, here is that plane.

And what we are saying is that, let us remove this plane in which we are cutting. So, what we are saying is that this line lies on this plane. So, I am going to keep the plane like this. And now let us again vary the tangent lines, if you vary the tangent lines, here us what happens. So, something interesting is happening. All these tangent lines are in fact lying on that plane. You can see they are not moving away from the plane.

So, if I change the perspective, you can see that they are all on that same plane, and which we have seen algebraically. So I am making that, so that we are viewing that plane like this. And you can see that the line is consistently on that plane. And here is how the line looks like. So, on that payment varies, but it remains on that plane.

So, let us do another example. So, let us look at the function sine of x, y. So, here is how that function looks like again. Let us take the point 1 , 0 and ask what happens to the tangents at that point. So, let us draw a plane through that point. So, here is a plane passing through that point. So, let us see what happens to the curve over there. So, here us the curve. So, again, these are pictures we have seen earlier. Let me remove the graph.

So, as I change my plane, the curve changes. So, here is how that looks like. And now let us see what happens to the tangent. Let me draw the tangent here. Here is the tangent. So, again, the tangent is changing. So, if we draw that plane that we just had or rather the solution of the equation that we just had, here is the plane that we get and we see that this tangent line is on that plane.

And if we change our line, which is the unit vector, here is how the tangent changes. But you can see it is always on this plane. So, if I play it like this, you can see it is always on the plane. It never leaves that plane. And again, this is something we have proved algebraically. So, this is just a demonstration to say that all of these tangent lines lie on a particular plane. And which plane is that that is exactly the equation of the plane that we have written down here.

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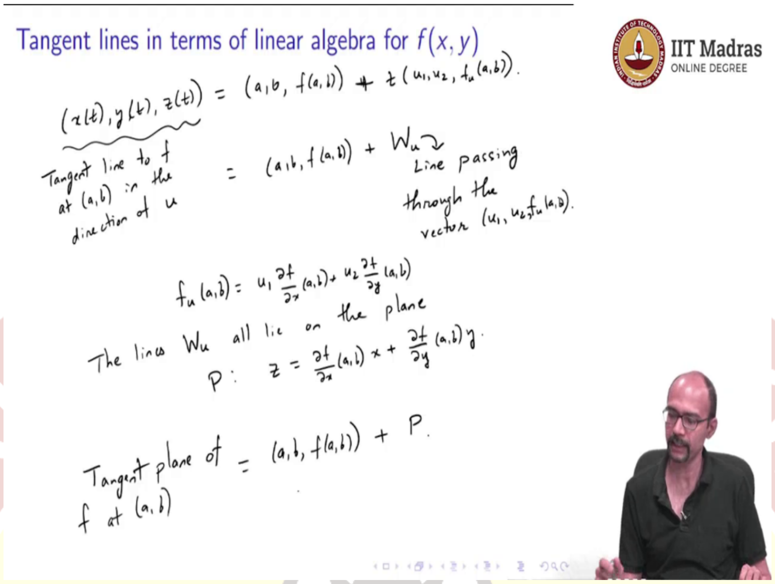

So, let us look at the tangent lines in terms of linear algebra. So, what one means by this is that we can view them as affine flats. This is an idea that we saw in the previous video as well. So, in terms of affine flats, x t, y t, z t is a , b , f of ab. This is the equation. So, here I will get the affine flat + t × the vector u1, u2, fu of a , b.

So, where is the affine flat coming from? So, the collection of these points is, if I call that the tangent line, so this is the tangent line to f at a, b in the direction of u. This line as an affine flat is given by this point a , b , f(a,b) + the line passing through the origin, so the subspace, which may be I will call W, and what is this W. So, W is the line passing through the vector u1, u2, fu of a , b.

So, of course, we know that if you have a line passing through a vector, then such a line, any vector on that is a scalar multiple of this vector. And as a result of that, W will be given by t × u1 , u2 , fu of a , b. So, this is how it looks like as an affine flat. And the point one is trying to make here is that according to how u varies this W vary. So, I will write this to emphasize that it depends on u.

And the vector u has an interesting property, namely, this is the property that we exploited in the previous slide as well. So, because $f_u(a, b) = u_1 \times \frac{\partial f}{\partial x}(a, b) + u_2 \times \frac{\partial f}{\partial y}(a, b)$, what we can see is that the lines Wu all lie on the plane $Z = \frac{\partial f}{\partial x}(a, b) \times x + \frac{\partial f}{\partial y}(a, b) \times y$. So, this is the plane on which they lie.

And now we can translate this plane to pass through the point a , b , f(a,b). So, if I call this plane, let us say, P. So, this is again a subspace of $\mathbb{R}^3$. And now I will translate this by a , b , f(a,b) to get another plane which I am going to call the tangent plane of f (a , b), this is a , b , f(a,b) + the plane P.

So, the net upshot of what I am saying is that, if you take the collection of all tangent lines, they lie on the particular plane. How do I get that plane, either by direct observation as we did in the previous slide, or else, you see that all these tangent lines are affine flats that means they are lines which have been moved from a line passing through the origin by this vector. You take all those lines.

And then you can see this very easy relation that that you are getting here. And then exploiting that relation, you can move it back to the point that you wanted to pass through. And so all these will pass through that line, whichever way you want, you will get that the tangent plane of f at a , b is an affine flat of this form.

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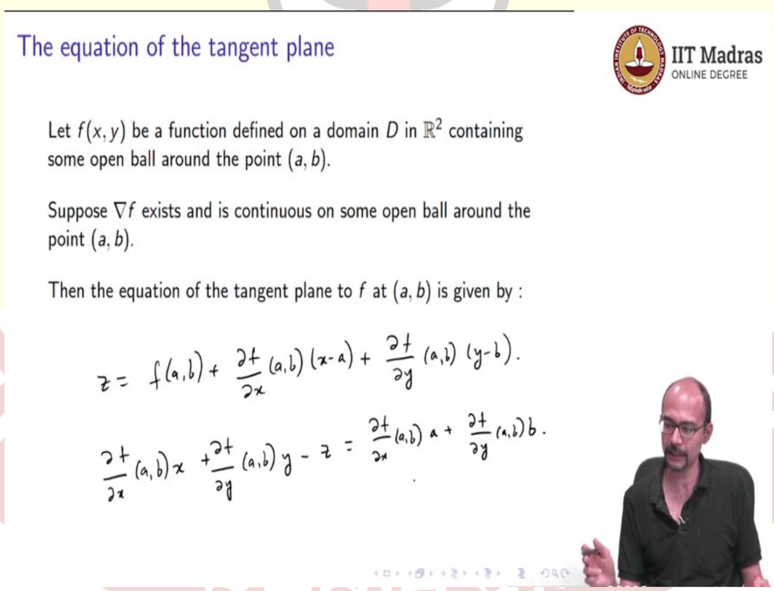

So, the net result is the following. If f(x), y is a function defined in the domain D in $\mathbb{R}^2$containing some open ball around the point a , b, suppose the ∇ exists and is continuous on an open ball, then the equation of the tangent plane to f at a , b is given by $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$. So, this is the question that of the tangent plane to $f(a,b)$.

So, this is indeed an affine flat because you have one linear equation. And but the equation, so you are looking for solutions of basically what we are saying is in terms of linear algebra, you are looking for solutions of $\frac{\partial f}{\partial x}(a,b)(x) + \frac{\partial f}{\partial y}(a,b)(y-z) = \frac{\partial f}{\partial x}(a,b)(a) + \frac{\partial f}{\partial y}(a,b)(b)$. And this is an equation in three variables. So, we know how solutions of this look like.

So, if it has even one solution, then it is basically you take that solution and translate the null space for this, meaning if you have 0 on the right hand side, you translate that null space by that solution. This is what we did in terms of linear algebra. That is how we get the tangent plane. So, whether the linear algebra aspect here is clear or not, what we are saying is this is the equation of the tangent plane and we have an interpretation in terms of what we have done in linear algebra.

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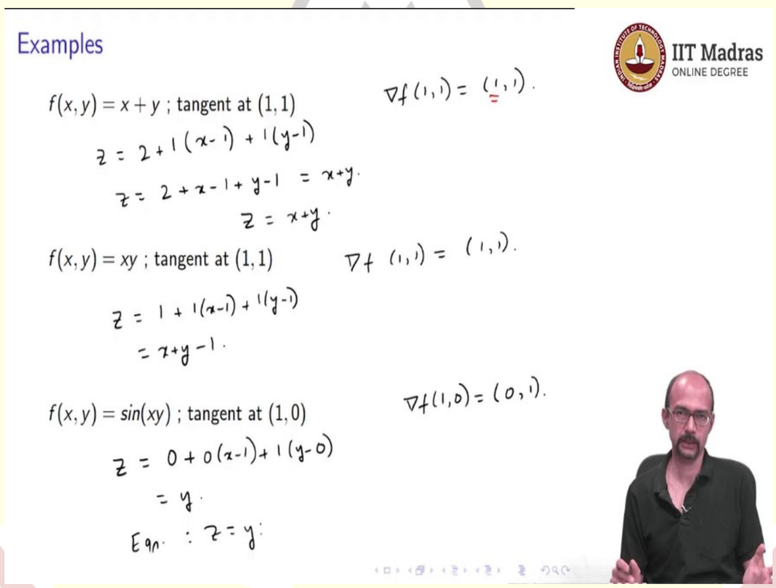

Let us do some examples to set ideas. So, here is f(x), y is x + y, what is the tangent plane that 1 , 1. So, here we computed the ∇. So, the ∇ at 1 , 1 was just 1 , 1 and so we can write down that $z = f(1)$, 1, which is 2 + 1. So, this 1 is coming from the first part here. So, $1 \times x - a$, which is again $1 + 1 \times y - b$, so b is again 1, and this 1 here is coming from the one in the second coordinate of the ∇.

So, this is an easy equation. So, this is saying that $z = 2 + x - 1 + y - 1$. In other words, this is 2. So, this is just x + y. So, the equation of the tangent plane at 1 , 1 is given by $z = x + y$. So, let us do f(x, y) is x × y. So, here we computed the ∇ at 1 , 1. And again, this ∇ was 1 , 1. So, here we get z =f(1 , 1), which is $1 + 1 \times x - 1 + 1 \times y - 1$. So, if you evaluate that, we are going to get $x + y - 1$. So, here the tangent plane is $x + y - 1$.

And finally, let us do f(x, y), sine x, y, let us maybe compute the tangent plane at 1 , 0. So, the ∇ at 1 , 0 is given by 0 , 1. This is a computation we have done earlier. And so the equation of the tangent plane is z =f(1 , 0), which is 0, + 0 × x - 1 + 1 × y - 0, which is just $z = y$. So, the equation of the tangent plane is $z = y$. So, I hope the, finding out the equations is really easy, because you have a formula and you can just substitute all the values in the formula.

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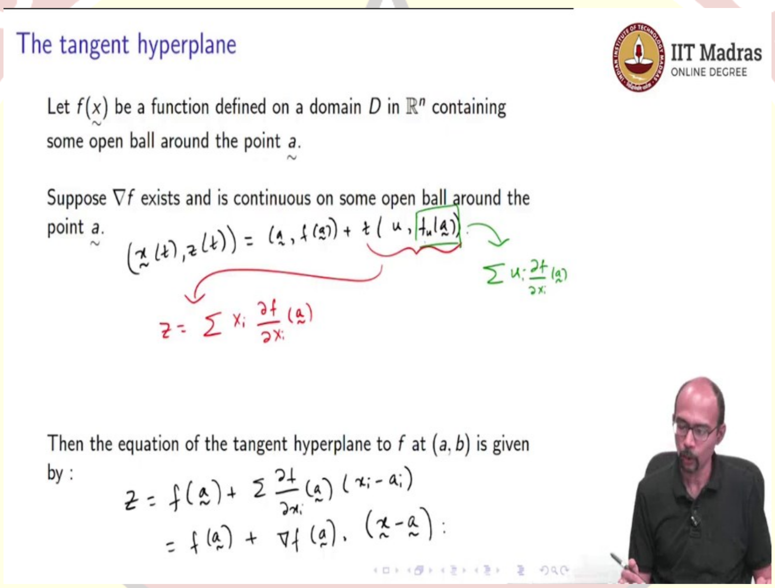

Let us now talk about tangent hyperplanes. So, if you do the same thing in n dimensions, unfortunately, what you get is not a plane, but it is, as we said, it is an affine flat, which is, for which the corresponding subspace is something which is the solution space of, for single equation. So, if you do that, that means you are going to get an affine flat of dimension n the solution space will be n dimensional. So, let us go through that.

So, if $f(\underset{\sim}{x})$ be a function defined domain D in $\mathbb{R}^n$ containing some ball around the point $\underset{\sim}{a}$, suppose the ∇ exists and is continuous on some open ball around the point a $\underset{\sim}{a}$. So, we can do the same things that we have done earlier and get the equation. So, for that, let us recall what is the equation of, the parametric equation of the tangent lines. So, in the direction of the vector u, we could write this down as the point, so $\underset{\sim}{a}$, $f(\underset{\sim}{a})$+ t × u , $f_u(\underset{\sim}{a})$. This was how we wrote it down in terms of the vector form.

So, notice that this part of the equation satisfies the following identity that the last coordinate which, if we call it z, last coordinate satisfies that z =$\sum u_i$ × or rather the i-th coordinate xi × ∇ fi with respect to xi at $\underset{\sim}{a}$× this is what it satisfies, because f_u ($\underset{\sim}{a}$)is, so just to emphasize, so this part. So, what is f_u ($\underset{\sim}{a}$), just to emphasize this part =submission u_i × ingredient f, $\frac{\partial f}{\partial x_i}(\underset{\sim}{a})$.

And these ui s are exactly what come in the first n coordinates. So, that means the $n + 1th$ coordinate which we called z is $\sum x_i \frac{\partial f}{\partial x_i}(\underset{\sim}{a})$. So, what that means is that each of these lines, so the lines which are translates of the tangent lines to the origin, which is the corresponding subspace, one dimension subspace, all of them lie on the, they are solutions of this equation.

So, if you have one equation that means you get a hyperplane that is exactly what is called a hyperplane. So, this is the null space of the single equation. It is an n dimensional subspace in $\mathbb{R}^{n+1}$. And so, it is a hyperplane in $\mathbb{R}^{n+1}$. So, it is of dimension in. And these lines all lie on that hyperplane. And now, if you take that hyperplane and shift it by this point, then you get the corresponding affine flat.

And that means that all the tangent lines lie on that affine flat or that hyperplane and so that is the tangent hyperplane of corresponding to the function f at the point $\underset{\sim}{a}$. And so its equation is going to be given by shifting this z $\sum x_i \frac{\partial f}{\partial x_i}(\underset{\sim}{a})$ by this point. So, if you do that, we will get the equation of the tangent hyperplane to f at $\underset{\sim}{a}$is given by z =$f(\underset{\sim}{a}) + \sum x_i \frac{\partial f}{\partial x_i}(\underset{\sim}{a}) \times (x_i - a_i)$. This is how it will look like. Or you can write this better in terms of the dot product. So, this is $\nabla f(\underset{\sim}{a})$, so now this is a vector and you dot this vector with $\underset{\sim}{x} - \underset{\sim}{a}$.

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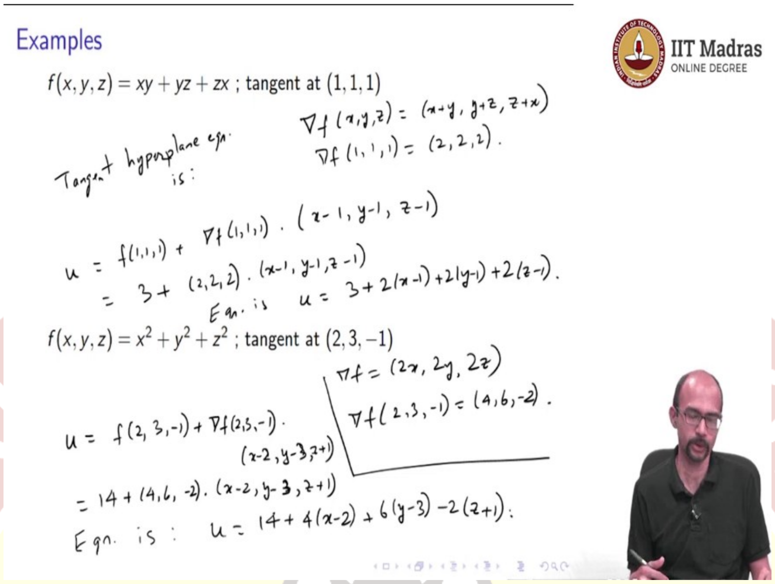

So, let us do some examples to set these ideas. So, if you have $f(x, y, z)$ is $xy + yz + zx$ let us look at the tangent hyperplane at 1 , 1 , 1, so for this, I need to first compute the ∇. So, the ∇ at x , y , z, well we know what this is, this is $x + y, y + z, z + x$. So, if you compute this at 1 , 1 , 1, this is 2 , 2 , 2.

And now, if we use this, what we will get is that the tangent hyperplane is z =f(1 , 1 , 1) + this ∇ (1 , 1 , 1) . x - 1, y - 1, z - 1. So, if you work that out, that is f(1, 1, 1) is 3, so 3 + 2 , 2 , 2 . x - 1, y - 1, z - 1, so I should not use said here. So, since I am using, so let me use some other variable u. So, that means, so the equation is u =3 + 2 × x - 1 + 2 × y - 1 + 2 × z - 1.

Similarly, if you have x square + y square + x square, so let us look at the ∇ it is going to be given by 2x , 2y , 2z. So, if you compute it at the point 2, 3 - 1 it will be 4 , 6 , - 2. And so the equation of the tangent hyperplane is u =f (2, 3, - 1) + ∇(2, 3 – 1) . x - 2 , y - 3 , z + 1.

So, the function value is 2^2 4 + 3^2, so 4 + 9, 13 + 1, so 14 + 4 , 6 , - 2 . x - 2 , y - 3 , z + 1. And if we evaluate what that is, we will get, so the equation is u =14 + 4 × x - 2 + 6 × y - 3 + or - 2 × z + 1. So, the computation is quite clear. The reason may be a little bit involved. But I hope I have convinced you that at least using the pictures that indeed, all of these do lie on a plane in, when you are in two dimensions or hyperplane when you are in higher dimensions.

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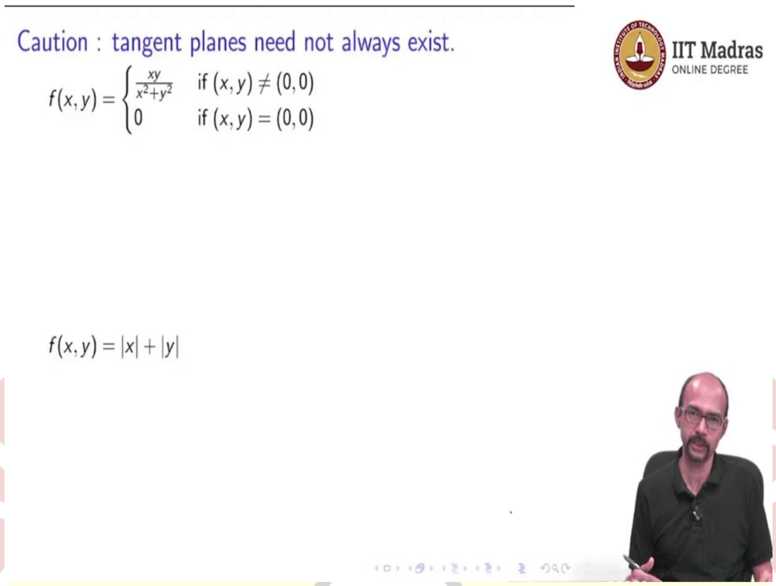

So, once again, same question, tangent planes need not always exist. So, why is that, because the tangent lines themselves do not exist in both of these cases. So, we have seen this in the previous video where we had xy by $x^2 + y^2$. There is no tangent lines in the directions other than the x and the y axis, positive and negative. And for f(x,y) is |x| +|y|, it is even worse. I will encourage you again to check what happens. So, you have to be careful.

And the hypothesis that the ∇ is continuous in the neighborhood is very, very important. And that often happens for polynomials or even for rational functions or other kinds of functions where it is easy to check for continuity. Even for this function, f(x, y) is $xy/(x^2 + y^2)$if x, y is not 0, 0. And other point is the tangent plane certainly exists. And I will encourage you to check what that is.

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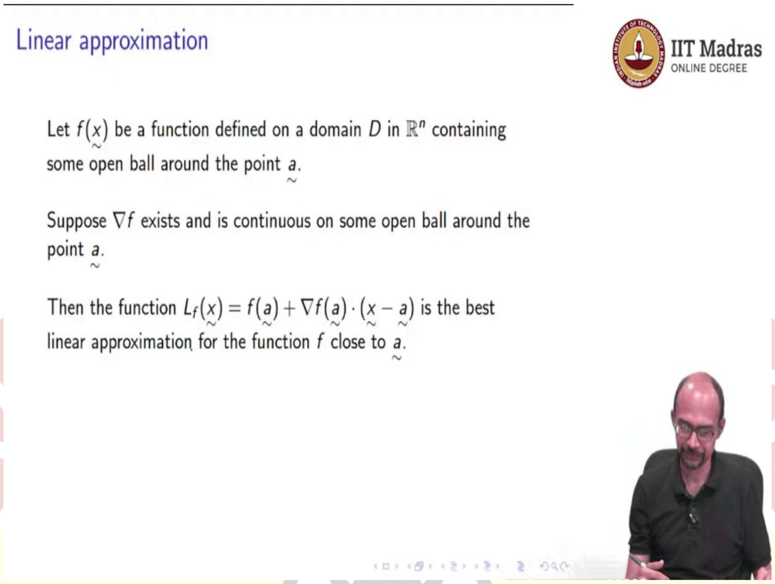

Let me end by with this idea of linear approximation. So, this is an idea that we have done in one variable calculus. So, if $f(\underset{\sim}{x})$ is a function defined in a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$, suppose ∇ of f exists and is continuous on some open ball around the point $\underset{\sim}{a}$, then we can take that expression and create a function out of that. So, that is a linear function.

So, this is a function $L_f(\underset{\sim}{x})$ is $f(\underset{\sim}{a}) + \nabla(\underset{\sim}{a}). \underset{\sim}{x} - \underset{\sim}{a}$. This is exactly what we, so the right hand side is exactly what we saw was the expression coming in the equation of the tangent plane, on the left, instead, we had z or u, meaning the $(n + 1)-$th variable.

So, then instead of that we will create a function and so this function is the best linear approximation for the function f close to $\underset{\sim}{a}$. This is exactly the idea of the tangent in the first place. For functions of one variable, if you have a curve, meaning the graph, it approximates the graph. For functions of several variables, for example, for two variables, if you have the graph of that function, the plane is going to approximate that graph close to the point.

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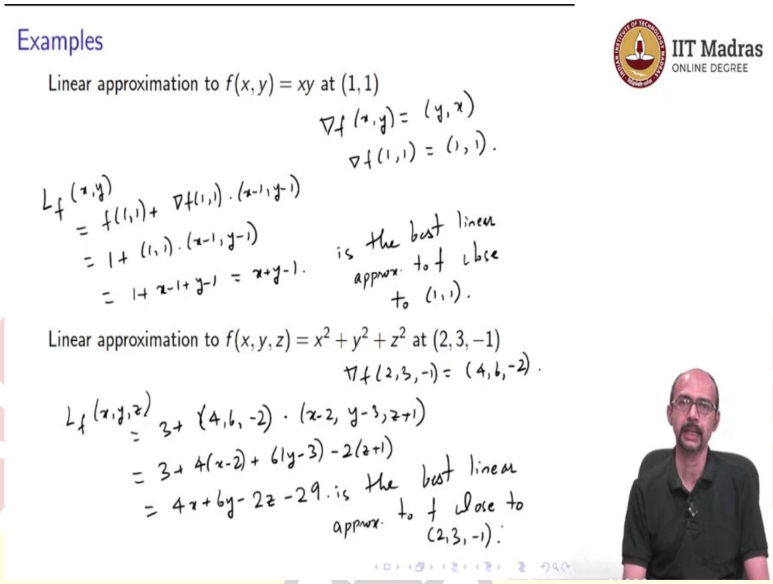

Let us do a couple of examples again. And this involves the same checking that we have done before. So, if you have f(x), y is xy at 1, 1, what is the linear approximation? Well, same idea, you have to compute the ∇ at 1 , 1. So, what is the ∇ of this function? So, it is y , x with the ∇ of f at 1 , 1. We have computed is 1 , 1.

So, what is the linear approximation, so Lf(x , y), so this is at the point 1 , 1, so I have to also sort of emphasize that this is the linear approximation close to 1 , 1. So, I will say that. Lf(x) , y is f(1) , 1 + the ∇ at 1 , 1 . x - 1 , y - 1, which is 1 + 1 , 1 . x - 1 , y - 1, which is 1 + x - 1 + y - 1. So, this is x + y - 1.

So, now we are thinking of this as a function. So, if you take this function, it approximates the function x, y. So, this is the function, which is the best linear approximation to f close to the point 1 , 1. So, if you change your point, the function will change, the linear function will change.

Let us do the same thing for in three variables. So, if you have f(x), y, z is x square + y square + x square, well, we know what is ∇ at this point. We have done couple of slides ago. This was 4, 6 - 2. So, Lf(x), y, z this is 3 + the ∇ which is 4, 6, - 2 . x - 2, y - 3, z + 1, which is 3 + 4 × x - 2 + 6 × y - 3 - 2 × z + 1.

If you rewrite this, this is $4x + 6y - 2z$ and then $3, -8 - 18 +, -2$, so - 18 - 2, - 20, so - 32 and then so this is - 29. So, I hope the numbers are correct. You can check. So, this is, what is this? This is the best linear approximation to f close to this point 2, 3, - 1.

So, let us quickly summarize what we have done so far in this video. We have seen that the collection of tangent lines lies on a plane when we have a function of two dimensions or two variables. And if it is off several variables it lies on the hyperplane and we characterize the, we obtained the equation of this hyperplane by looking at the corresponding subspace, which is exactly given by the solution to a single equation.

So, using that we can explicitly write down what is the equation of that hyperplane. And then we also saw that if you take the right hand side and make that into a function, then that is the best linear approximation to the function f at the given point $\underset{\sim}{a}$. Thank you.