INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
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Lecture No. 28
Some Properties of Vector Spaces

Hello, and welcome to the online B.Sc. program on data science and programming. In this video, we are going to continue from our previous video where we introduced vector spaces and study some properties of vector spaces. In particular, we will build on the axioms that we have seen in the definition of a vector space.

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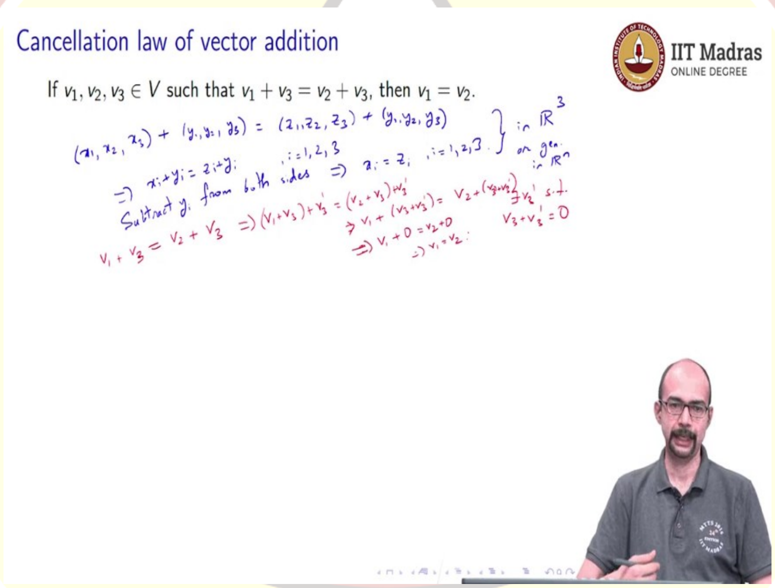

So, let us derive a consequence of the axioms, the conditions that are needed in order to, for something to be a vector space. So, this is called the cancellation law vector addition. So, if v_1 + v, you have three vectors, v_1, v_2, v_3 in the vector space V, and $v_1 + v_3$ is $v_2 + v_3$, then v_1 is v_2. Now, let us first see how we do this in $\mathbb{R}^n$. So, let us try to recall how we do this in $\mathbb{R}^n$.

So, in $\mathbb{R}^n$, what do we do? So, in $\mathbb{R}^n$ what we do is, let me do it for example in three $\mathbb{R}^3$, z1, z2, z3 + y1, y2, y3. So, how will I conclude that x1, x2, x3 is the same as z1, z2, z3? Well, what I will say is this means x1 + y1 is z1 + y1 or more generally xi + yi is zi + yi, where i is between 1 and 3, 1, 2, 3. And then from real numbers, I know that xi + yi is zi + implies xi = zi. How do I know that, because if you add - yi on both sides, then indeed what you get is xi, zi.

So, you can subtract yi from both sides. This was imply $x_i is\ z_i$ is zi. And from here we conclude that x_1, x_2, x_3 is the same as z_1, z_2, z_3. So, this is how we do it for real numbers, sorry, for $\mathbb{R}^3$ or in general for any $\mathbb{R}^n$. So, then let us see how to do it. I mean, we can use the same idea to do it for general vector spaces. So, $v_1 + v_3$ is $v_2 + v_3$. So, this was the argument in $\mathbb{R}^n$. So, now we are given that v1 + v3 is v2 + v3. This is argument for general vector spaces. V3 is v2 + v3. Well, one of the axioms told us that for v3 there exists some v3 ‘ such that there exists v3 ‘ such that v3 + v3 ‘ is 0.

So, what that tells me is $v_1 + v_2 + v_3$ ‘ is $v_2 + v_2 + v_3$ ‘. But now we know that addition is associative. So, I can write this as v1 + v3 + v3 ‘ is v2 + v3 + v3 ‘ which implies that v1 + 0 is v2 + 0 which implies v1 is v2. So, this proves a statement here. So, notice that all I used was the axioms. I did not use any other thing, any other information about the structure of v because really I do not know anything else. The only things I know are the axioms.

So, whenever you want to check something for a vector space, you have to use the axioms. But to get intuition about whether it is correct or wrong and how to go about a proof in case it is correct, you should look at $\mathbb{R}^n$, see how you do it in $\mathbb{R}^n$ and then sort of extrapolate that idea for general vector spaces.

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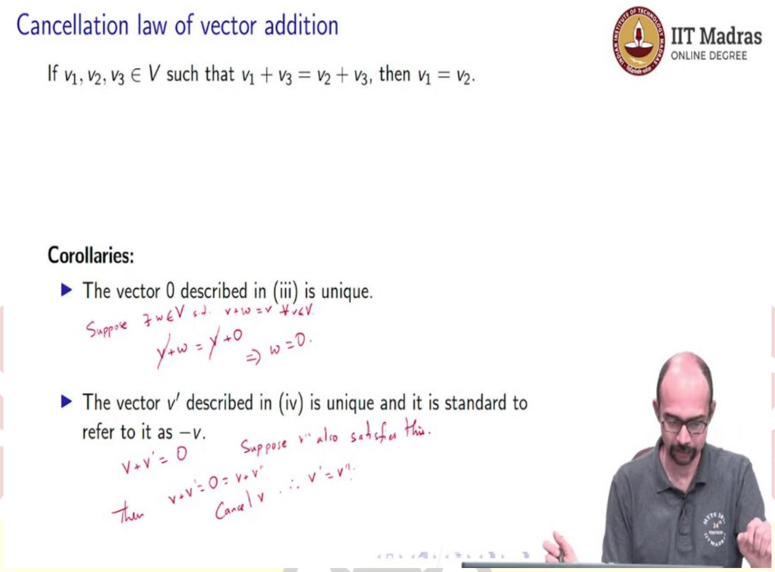

So, here is a corollary of the cancellation law. The vector 0 described in three is unit. So, remember, one of the axioms was that there is a vector called 0, but maybe there are many vectors satisfying that same property. So, this corollary says, you cannot have many vectors. And here is another corollary, namely that if you looked at the vector v ‘, this was in axiom 4 which, so this axiom 4 is what we used in order to conclude the cancellation law by looking at v_3 ‘. So, the vector v ‘ described in four is unique and it is standard to refer to it as - v. And even this reference to - v will be clarified in our upcoming slide.

So, maybe let us quickly do this proof from the cancellation law. So, suppose I have another vector called w, such that w also satisfies the same property that v does. So, suppose there exists w in v such that v + w is v for all v in V. This was what 0 satisfies. But then v + w is v means what, v + w is v + 0, because v + 0 is also v. So, I can write it like this. But then I can cancel. So, I am adding v on both sides, so I can cancel v. So, the idea is here to add v ‘. And so that leaves us with w 0.

So, cancellation means I can do this that leaves us with w 0. So, the vector 0 as described in the third axiom is unique. And similarly, the vector v ‘ described in 4 is unique. And so, $v + v$ ‘ is 0. So, now suppose v double ‘ also satisfies this. So, then v + v ‘ is 0 is $v + v''$. And now you can use cancellation. So, cancel v and therefore, $v' = v''$. So, that is the proof of this corollary from the cancellation law.

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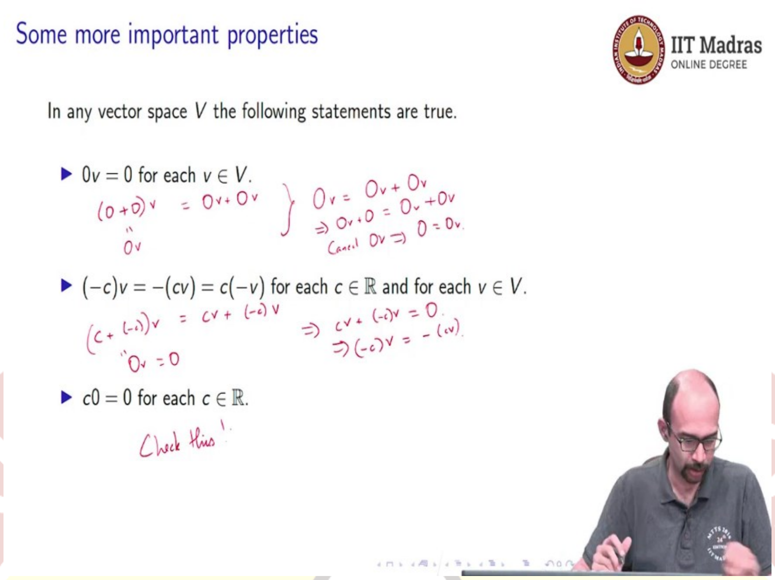

Let us look for some other properties. So, in any vector space V, the following statements hold true; 0×v is 0 for each v; - c×v is - of cv and that is the same as c×- v. So, what is - v, - v is exactly the vector v ‘, which was described in axiom 4 and our previous slide told us that it is unique. And hence, we can refer to it as - v, because we know that that is the vector that when you added to v you get 0. So, c×0 is 0 for each constant in our each scalar c in $\mathbb{R}^1$.

So, let us try and prove this. So, what I can do is I can look at 0 + 0×v. And one of the axioms tells me that this is 0×v + 0 × v. On the other hand, 0×v is, 0 + 0 is exactly 0, which is 0×v. So, what is the net result? The net result is that $0 \times v + 0 \times v$. So, you can write this as 0×v + 0 is $0 \times v + 0 \times v$, cancel 0×v. And that implies 0 = 0×v. So, that gives us the result.

And then we have $-c \times v$ is $-cv$ is $c \times -v$. So, how do I get this? The idea is exactly what we have done above. So, we have $c + -c \times v$ is $c \times v + -c \times v$. But on the other hand, I know that this is 0×v, which we have just proved 0, so $c \times v + -c \times v$ is 0. So, that means - c×v satisfies the condition required of the negative of cv.

So, this implies $-c \times v$ is - of cv. And then similarly, you can check for c×- v. Finally, we have c×0 is 0. And maybe I leave this to you because it is in a very similar spirit to what we have done earlier. So, check this.

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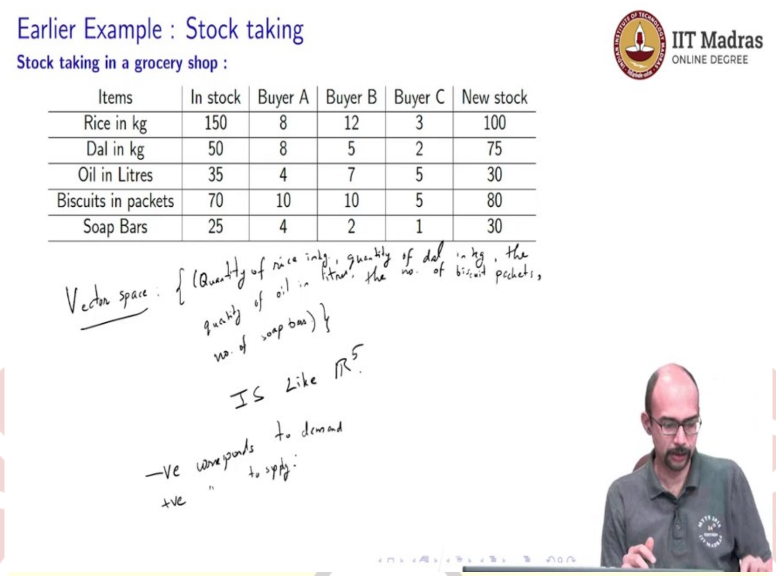

So, let us do an example sort of from so to say from real life. This is an example that we have seen in the previous video, the example of stock taking. So, what do we have here? We have stock taking in a grocery shop. So, we have five items; rice in kg, dal in kg, oil in liters, biscuits in packets and soap bars. And the first column describes how many of which quantity is in stock, the second column describes the demand for each product for buyer A, the second, third column describes the same thing for buyer B, the fourth one for buyer C, and the last one describes what is a new stock that has arrived.

So, what is, I mean, we did vectors in this context by looking at the corresponding column vectors and so on. So, but we can also think of this as a abstract vector space. So, what is the vector space showing up here? So, the vector space is the quantity of rice in kg, the quantity of dal in kg, the quantity of oil in liters, the number of biscuits or the number of biscuit packets, and finally, the number of soap bars. So, it is this set.

And so the claim is that this is a vector space. Why, because if you add, you add coordinate wise and what you get is, say if you have two vectors in this are added, you get the total quantity of rice within those two vectors, the total quantity of dal in kg, the total quantity of oil in liters, the number of, total number of biscuit packets and the number of soap bars, total number of soap bars that is how you do addition and scalar multiplication is coordinate wise.

So, it so happens that this vector space looks very much like $\mathbb{R}^5$. And why is that, because we have 5 quantities. And of course, here know once we write it like this, we have to also ask what does one mean by negative quantities and negative number of soap bars. So, this again we saw in that example what negative corresponded to. So, negative corresponded to demand and positive corresponded to supply. So, negative corresponds.

So, for example, if you have - 2 kgs of rice that means that there is a demand for 2 kgs of rice. And if we say that there are 3.5 liters of oil that means there is a supply of 3.5 liters of oil. And then we can use this + and - in order to take stock. This was exactly how we did it in that example. So, of course, one has to also interpret things like what is half a biscuit packet or what is one-eighth of a soap bar. So, one of the things that we do as a result is that we often express them in units for which we can have any real number instead of only natural numbers. So, then it seems more natural.

So, for example, if instead of biscuit packets we had written 100 grams of biscuits if we know that, let us say, one biscuit packet is 100 grams or if instead of soap bar we had written 400 grams of soup, then there is no issue with decimal places or what we add and subtract.

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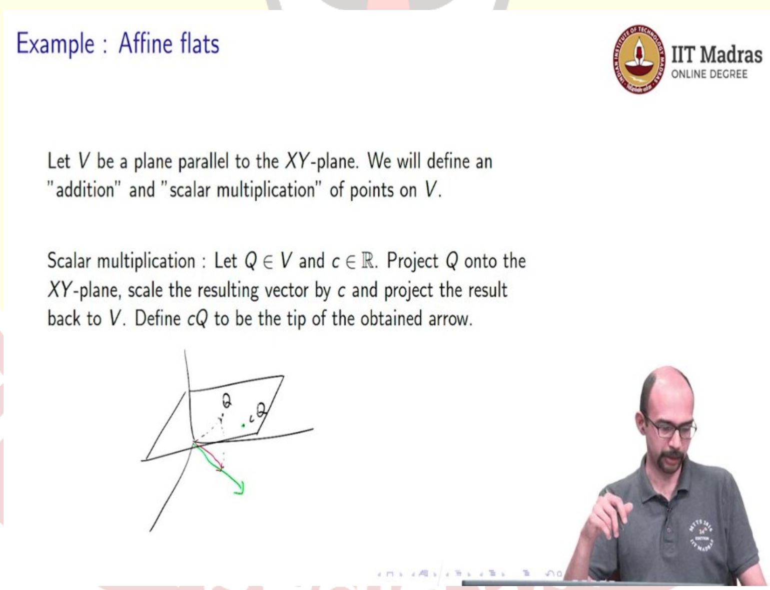

So, this is another example of a vector space. And maybe we will end with this example of affine flat, which you are going to see again later. So, suppose we have, so this is a slightly geometric view of vector spaces and deliberately convoluted one, so that I want you to sort of also be comfortable with the geometry involved here. So, maybe before I start the example, I should point out that much of what we will do is formal algebra. But behind the scenes, there is geometry which is guiding our algebra. Whenever we say $\mathbb{R}^3$ or $\mathbb{R}^2$ in our mind, we have geometry. We understand two dimensional space or three dimensional space.

So, suppose V is a plane parallel to the XY plane. So, we are going to define an addition and a scalar multiplication of points on V. So, scalar multiplication is going to be done as follows. So, let Q belong to this plane and let c be a constant. So, you project Q onto the XY plane, scale the resulting vector by c and project the result back to V. This is the procedure for doing scalar multiplication. So, what is $c \times Q$? $c \times Q$ is a tip of the obtained arrow. This is the procedure for scalar multiplication.

Maybe let me draw a quick picture here. So, here is your plane, here is your point. So, drop this perpendicular to the XY plane and then draw this line here and then if you want to scale this by c, so if you scale it by c, then that means you are going to get some new vectors, something like this, and then project this back up to the original space. So, you will get maybe something here. So, this is your point Q and this is your point $c \times Q$ that is scalar multiplication.

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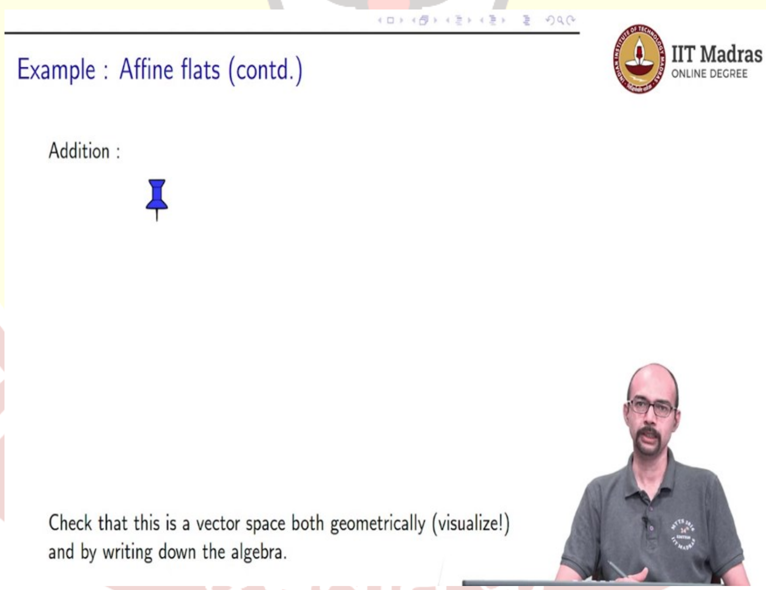

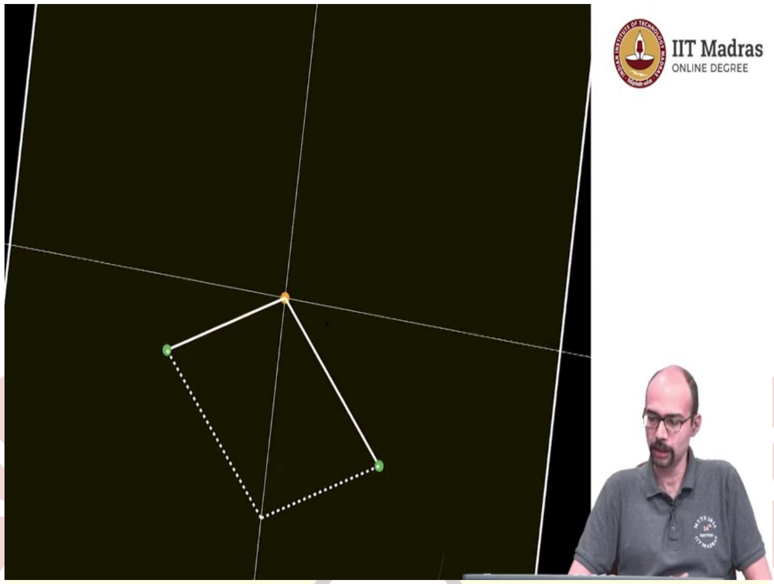

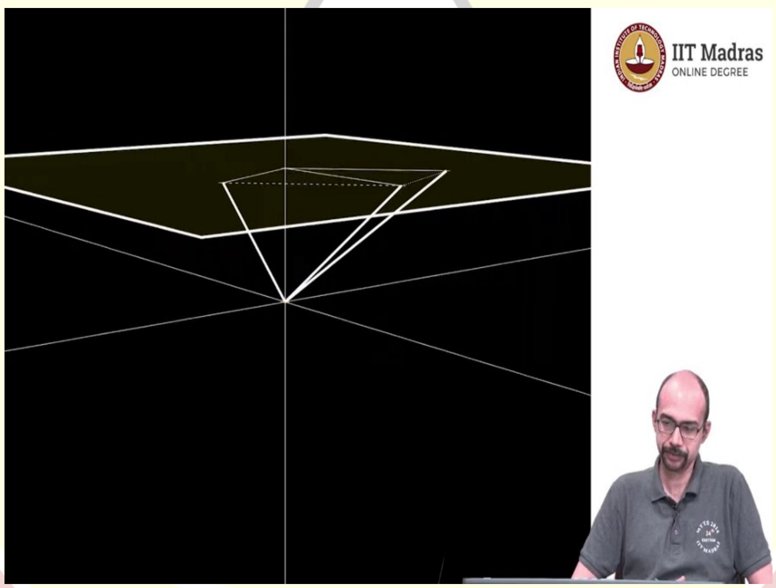

So, here is how you do addition. And we are going to watch this addition. Thanks to our support team. So, here is your plane which is parallel to the XY plane. And now, we take two points on this plane which we want to add. So, now we draw our usual vectors and then we project this down. And now, so now this is in the XY plane. So, in the XY plane we know how to add, because that is exactly $\mathbb{R}^2$. That is the parallelogram now.

So, use the parallelogram law, add them and then you project the entire thing back to your original plane that you are working with. And now the newly obtained point that you can see over there is the sum. This is the way you add.

So, I hope the video showed to you the geometry involved in things like addition. So, now the question is, so we have defined what is scalar multiplication and addition on this set. And now, I will leave you to check that this is a vector space. You can either do geometrically or better to do it both ways. So, try to think of geometrically by visualization why this is a vector space or you can just write down the algebra and work out that this is indeed a vector space.

So, in either case, what can help us the following. So, the idea behind the addition and scalar multiplication here is that really you are looking at vectors, we are taking points, drawing the corresponding vector, but then remember that you project down. So, this is essentially what we are doing is, we are looking at arrows which start at the point where the z-axis intersects this plane and where the tip is the point you are interested in.

So, you take those kinds of arrows and then you do addition and scalar multiplication for those arrows exactly the way you do it for $\mathbb{R}^2$. This is really at the heart of what is going on. So, I will leave that visualization to you and I hope you can see that this is indeed very similar to what we do in $\mathbb{R}^2$.

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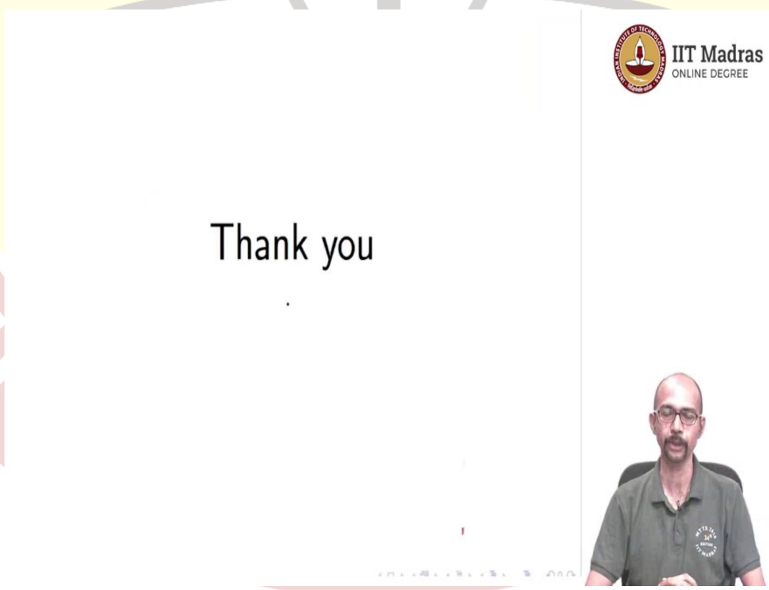

So, let me summarize what we have seen in this video. So, in this video, we saw, we began by seeing the some more properties of vector spaces, in particular, things like how the 0 vector behaves and how the negative of a vector behaves and so on. We saw somewhat real life application of vector spaces, namely, we took our good old grocery shop problem and saw how vector spaces fit into that context and then we have seen the example of affine flats. Thank you.