Week-9
Mathematics for Data Science - 2
Length of a vector, Inner products, Gram-Schmidt process, Orthogonal transformation
Graded Assignment-Solutions

1 Multiple Choice Questions (MCQ)

1. Consider the following functions $\langle ., . \rangle : V \times V \to \mathbb{R}$ defined as follows:

   i) $V = \mathbb{R}^2$ and the function defined as:

   $$\langle ., . \rangle : V \times V \to \mathbb{R}$$
   $$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2.$$

   ii) $V = M_{2\times 2}(\mathbb{R})$ and the function defined as:

   $$\langle ., . \rangle : V \times V \to \mathbb{R}$$
   $$\langle A, B \rangle = Tr(AB),$$

   where $Tr(M)$= denotes the trace of a matrix M, i.e., the sum of the diagonal elements of M.

   iii) $V = M_{2\times 1}(\mathbb{R})$ and the function defined as:

   $$\langle ., . \rangle : V \times V \to \mathbb{R}$$
   $$\langle A, B \rangle = Tr(AB^t),$$

   where $Tr(X)$ denotes the trace of a matrix X, i.e., the sum of the diagonal elements of X. Y^t denotes the transpose of matrix Y.

   iv) $V = \mathbb{R}^2$ and the function defined as:

   $$\langle ., . \rangle : V \times V \to \mathbb{R}$$
   $$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1.$$

   ○ Option 1: (i), (ii), (iii), (iv) all are inner products.

   ○ Option 2: (i) and (ii) are inner products, but (iii) and (iv) are not.

   ○ **Option 3:** (i) and (iii) are inner products, but (ii) and (iv) are not.

○ Option 4: (ii), (iii), and (iv) are inner products, but (i) is not.

○ Option 5: (ii) and (iii) are inner products, but (i) and (iv) are not.

Solution: Recall that $\langle ., . \rangle : V \times V \to \mathbb{R}$ is an inner product if

1. $\langle x, x \rangle \geq 0$ for all $x \in V$; $\langle x, x \rangle = 0$ iff $x = 0$.
2. $\langle x, y \rangle = \langle y, x \rangle$, $x, y \in V$.
3. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$, $x, y, z \in V$ and $\alpha \in \mathbb{R}$.

- Option 1: $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2$.

$$\begin{aligned}\langle (x_1, x_2), (x_1, x_2) \rangle &= x_1^2 - x_2x_1 - x_1x_2 + 2x_2^2 \\ &= x_1^2 - 2x_1x_2 + 2x_2^2 \\ &= (x_1 - x_2)^2 + x_2^2 \\ &\geq 0, \ \forall (x_1, x_2) \in \mathbb{R}^2.\end{aligned}$$

Also, $\langle (x_1, x_2), (x_1, x_2) \rangle = (x_1 - x_2)^2 + x_2^2 = 0$ iff $x_1 = x_2 = 0$.

$$\begin{aligned}\langle (x_1, x_2), (y_1, y_2) \rangle &= x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2 \\ &= y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 \\ &= \langle (y_1, y_2), (x_1, x_2) \rangle\end{aligned}$$

$$\begin{aligned}\langle (x_1, x_2) + (y_1, y_2), (z_1, z_2) \rangle &= \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle \\ &= (x_1 + y_1)z_1 - (x_2 + y_2)z_1 - (x_1 + y_1)z_2 + 2(x_2 + y_2)z_2 \\ &= x_1z_1 + y_1z_1 - x_2z_1 - y_2z_1 - x_1z_2 - y_1z_2 + 2(x_2z_2 + y_2z_2) \\ &= (x_1z_1 - x_2z_1 - x_1z_2 + 2x_2z_2) + (y_1z_1 - y_2z_1 - y_1z_2 + 2y_2z_2) \\ &= \langle (x_1, x_2), (z_1, z_2) \rangle + \langle (y_1, y_2), (z_1, z_2) \rangle\end{aligned}$$

$$\begin{aligned}\langle \alpha(x_1, x_2), (y_1, y_2) \rangle &= \langle (\alpha x_1, \alpha x_2), (y_1, y_2) \rangle \\ &= (\alpha x_1)y_1 - (\alpha x_2)y_1 - (\alpha x_1)y_2 + 2(\alpha x_2)y_2 \\ &= \alpha(x_1y_1 - x_2y_1 - x_1y_2 + 2x_2y_2) \\ &= \alpha \langle (x_1, x_2), (y_1, y_2) \rangle\end{aligned}$$

- Option 2: $V = M_{2\times2}(\mathbb{R})$, $\langle A, B \rangle = Tr(AB)$.
  Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then $\langle A, A \rangle = Tr(A^2) = Tr(\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}) = 0$. But $A \neq 0$. This contradicts $\langle x, x \rangle = 0$ if and only if $x = 0$.
- Option 3: $V = M_{2\times1}(\mathbb{R})$, $\langle A, B \rangle = Tr(AB^t)$.
  Note that $A \in M_{2\times1}(\mathbb{R})$ is of the form $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. For $A, B \in M_{2\times1}(\mathbb{R})$, $AB^t =$

$\begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}$. Thus $Tr(AB^t) = a_1b_1 + a_2b_2$. We can see that $\langle A, B \rangle = a_1b_1 + a_2b_2$. This looks like the standard inner product in $\mathbb{R}^2$ and the properties can be similarly verified.

- Option 4: $V = \mathbb{R}^2$, $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1$.
  For $x = (1, -1)$, $\langle x, x \rangle = -2 \not\geq 0$.

2. An inner product on $\mathbb{R}^3$ is defined as:

$$\langle ., . \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$
$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

Match the sets of vectors in column A with their properties of orthogonality or orthonormality in column B with respect to the above inner product.

	Set of vectors (Column A)		Properties (Column B)
a)	$\{(2,3,4), (-1,2,-1)\}$	i)	Forms a basis but not orthogonal
b)	$\{\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(-1,0,-1)\}$	ii)	Forms an orthogonal basis
c)	$\{(2,3,4), (-1,2,-1), (0,4,-3)\}$	iii)	Orthogonal but not orthonormal, and does not form a basis of $\mathbb{R}^3$
d)	$\{(2,3,4), (-1,2,-1), (11,2,-7)\}$	iv)	Orthonormal, but does not form a basis of $\mathbb{R}^3$

Table : M2W6G1

Choose the correct option.

- ○ Option 1: a → iv), b → iii), c → i, d → ii)
- ○ Option 2: a → iv), b → iii), c → ii), d → i)
- ○ **Option 3:** a → iii), b → iv), c → i), d → ii)
- ○ Option 4: a → iii), b → iv), c → ii), d → i)

Solution: Note that if $x, y \in V$ are orthogonal, then $\{x, y\}$ is a linearly independent set (Prove!). Also, since dimension of $\mathbb{R}^3$ is 3, any set of 3 linearly independent vectors forms a basis.

a) $\langle(2,3,4),(-1,2,-1)\rangle = 0$. Note $\langle(2,3,4),(2,3,4)\rangle \neq 1$ so the set is not orthonormal. Also, since there are only 2 vectors, it cannot form a basis for $\mathbb{R}^3$. This matches with iii).

b) $\langle\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(-1,0,1)\rangle = 0$. Also $\langle\frac{1}{\sqrt{2}}(1,0,-1), \frac{1}{\sqrt{2}}(1,0,-1)\rangle = 1$ and $\langle\frac{1}{\sqrt{2}}(-1,0,1), \frac{1}{\sqrt{2}}(-1,0,1)\rangle = 1$. Thus these vectors are orthonormal. But this does not form a basis for $\mathbb{R}^3$, since there are only two vectors.

c) $\langle(-1,2,-1),(0,4,-3)\rangle = 11 \neq 0$, so the set is not orthogonal. If we show that the set if linearly independent, then it forms a basis for $\mathbb{R}^3$ (this is because a linearly independent set with number of elements equal to $\dim(V)$ is a basis for V). To show that the set is linearly independent. show that the system $\alpha(2,3,4)+\beta(-1,2,-1)+\gamma(0,4,-3) = 0$ has only one solution $\alpha = \beta = \gamma = 0$.

d) $\langle(2,3,4),(-1,2,-1)\rangle = \langle(2,3,4),(11,2,-7)\rangle = \langle(-1,2,-1),(11,2,-7)\rangle = 0$. So the set is orthogonal and hence is linearly independent. Thus it forms a basis (since the linearly independent set has 3(=$\dim(\mathbb{R}^3)$) elements).

2 Multiple Select Questions (MSQ)

3. Let $\langle ., . \rangle$ denote the standard inner product on $\mathbb{R}^2$, i.e., $\langle(x_1, x_2),(y_1, y_2)\rangle = x_1y_1 + x_2y_2$. Which one of the following options is (are) true for the vector $\gamma \in \mathbb{R}^2$, such that $\langle\alpha, \gamma\rangle = 4$ and $\langle\beta, \gamma\rangle = 8$, where $\alpha = (3, 1)$ and $\beta = (6, 2)$.

   - ◯ Option 1: No such γ exists.
   - ◯ **Option 2:** There are infinitely many such vectors which satisfy the properties of γ.
   - ◯ Option 3: γ is unique in $\mathbb{R}^2$.
   - ◯ **Option 4:** Any vector in the set $\{(t, 4-3t) \mid t \in \mathbb{R}\}$ satisfies the properties of γ.
   - ◯ Option 5: $(1, 1)$ is the only vector which satisfies the properties of γ.

Solution : Let $\gamma = (\gamma_1, \gamma_2)$. Then we have $4 = \langle\alpha, \gamma\rangle = 3\gamma_1 + \gamma_2$. Also $8 = \langle\beta, \gamma\rangle = 6\gamma_1 + 2\gamma_2$. Thus we have the system of equations

$$3\gamma_1 + \gamma_2 = 4$$
$$6\gamma_1 + 2\gamma_2 = 8$$

Note that $6\gamma_1 + 2\gamma_2 = 8$ can be obtained by multiplying $3\gamma_1 + \gamma_2 = 4$ by 2. So, there are infinitely many solutions. Set $\gamma_1 = t$, then we get $\gamma_2 = 4 - 3t$. By varying t over $\mathbb{R}$, we get infinitely many solutions of the form $\{(t, 4-3t) | t \in \mathbb{R}\}$.

4. Choose the set of correct options.

   ◯ **Option 1:** Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis of an inner product space V. If there exists some $v \in V$, such that $\langle v, v_i \rangle = 0$ for all $i = 1, 2, \ldots, n$, then $v = 0$.

   ◯ **Option 2:** There exists an orthonormal basis for $\mathbb{R}^n$ with the standard inner product.

   ◯ **Option 3:** If P_W denotes the linear transformation which projects the vectors of an inner product space V to a subspace W of V, then $range(P_W) \cap null\ space(P_W) = \{0\}$, where 0 denotes the zero vector of V.

   ◯ **Option 4:** $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ cannot represent a matrix corresponding to some projection.

   Solution: Recall that a basis is a spanning set which is also linearly independent. A linear transformation is a projection if $P^2 = P$.

   - Option 1: Since β is a basis, $v \in V$ can be written as $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$. Now $\langle v, v_i \rangle = \alpha_i \langle v_i, v_i \rangle$ for all $i = 1, 2, ..., n$. Since it is given that v is such that $\langle v, v_i \rangle = 0$ for all i, we have $\alpha_i \|v_i\|^2 = 0$ for all i. But $\|v_i\| \neq 0$ (since $v_i \neq 0$). Thus $\alpha_i = 0$ for all i and hence $v = 0$.
   - Option 2: Suppose $\{v_1, v_2, ..., v_n\}$ is a basis for V. Using Gram-Schmidt orthonormalization process, we can construct an orthonormal basis for V.
   - Option 3: Since P_W is a projection, $P_W^2 = P_W$. Let $x \in range(P_W) \cap null\ space(P_W)$. Since $x \in range(P_W)$, $x = P_W y$ for some $y \in V$. But $x \in null\ space(P_W)$ implies that $P_W x = 0$, i.e., $P_W^2 y = 0$. But $P_W^2 = P_W$ and hence $x = P_W y = 0$. Thus $range(P_W)\ \cap\ null\ space(P_W) = \{0\}$.
   - Option 4: Note that $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. So the matrix cannot represent a projection.

5. Let $\langle ., . \rangle$ denote the standard inner product on $\mathbb{R}^2$, i.e., $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$T(u) = \langle u, v \rangle,$$

   where $v \in \mathbb{R}^2$. Which of the following options is (are) true for T?

   ◯ Option 1: T is one-one for all $v \neq 0 \in \mathbb{R}^2$.

   ◯ **Option 2:** T is onto for all $v \neq 0 \in \mathbb{R}^2$.

   ◯ Option 3: T is onto for all $v \in \mathbb{R}^2$.

   ◯ **Option 4:** T is not one-one for every $v \in \mathbb{R}^2$.

   ◯ **Option 5:** There exists a $v \in \mathbb{R}^2$ such that T is not onto.

Solution: First let us check if the map is one-one. If $v = 0$, then $T(u) = 0$ for all u, that is *null space*$(T) = \mathbb{R}^2$ and hence T is not one-one. If $v \neq 0$, then *null space*(T) is the set of all orthogonal vectors of v. Since there is at least one non-zero vector orthogonal to $v \neq 0$, T is not one-one. So T is not one-one for all v.

Next, let us check if T is onto. If $v = 0$, then as was seen earlier, T is the zero map and hence T is not onto. Thus there exists a $v \in \mathbb{R}^2$ such that T is not onto. For $v \neq 0$, consider the matrix A of the linear transformation T. Then A is a 1×2 matrix. If $v \neq 0$, then $T((1,0))$ or $T((0,1))$ is non-zero and hence A is not zero. Thus rank(A)=1 and hence T is onto.

3 Numerical Answer Type (NAT)

6. Let θ be the angle between the vectors $u = (4, 7, 3)$ and $v = (1, 2, -6)$, then what will be the value of $cos(\theta)$? [Answer: 0]
Solution: $cos(\theta) = \frac{\langle u,v\rangle}{\langle u,u\rangle\langle v,v\rangle}$. Since u, v are orthogonal, angle between them is $\theta = 90°$. Thus $cos(\theta) = cos(90°) = 0$.

7. Consider a basis
$$\{v_1 = (1, 2, 0), v_2 = (2, -1, 0), v_3 = (0, 0, 2)\}$$
of $\mathbb{R}^3$ with usual inner product. Suppose $v = (x, y, \frac{3x+y}{5}) \in V$ is written as $v = c_1v_1 + c_2v_2 + c_3v_3$, such that $c_1 + c_2 = 4$. What will be the value of c_3? [Answer: 2]
Solution: Given $v = c_1v_1 + c_2v_2 + c_3v_3$. Thus $v = (c_1 + 2c_2, 2c_1 - c_2, 2c_3)$. Also, v has the form $(x, y, \frac{3x+y}{5})$. This gives $2c_3 = \frac{1}{5}(3(c_1 + 2c_2) + (2c_1 - c_2) = c_1 + c_2$. It is given that $c_1 + c_2 = 4$. Thus $2c_3 = 4$ and so $c_3 = 2$.

4 Comprehension Type Question:

With a particular frame of reference (in $\mathbb{R}^3$), position of a target is given as the vector $(3, 4, 5)$. Three shooters S_1, S_2, and S_3 are moving along the lines $x = y$, $x = -y$, and $x = 2y$, on the XY-plane (i.e., $z = 0$) to shoot the target. Suppose that, there is another shooter S_4, who is moving on the plane $x + y + z = 0$. Suppose all of them shoot the target so that the target is at the closest distance from the respective path or plane on which they are travelling.
Answer questions 8,9 and 10 using the given information.

8. Choose the set of correct options.
   - ○ Option 1: S_1 will shoot the target from the point $(\frac{7}{2}, -\frac{7}{2}, 0)$.
   - ○ **Option 2:** S_1 will shoot the target from the point $(\frac{7}{2}, \frac{7}{2}, 0)$.
   - ○ Option 3: S_1 will shoot the target from the point $(1, 1, 0)$.

- ◯ Option 4: S_2 will shoot the target from the point $(-\frac{1}{2}, -\frac{1}{2}, 0)$.
- ◯ Option 5: S_2 will shoot the target from the point $(1, -1, 0)$.
- ◯ **Option 6:** S_2 will shoot the target from the point $(-\frac{1}{2}, \frac{1}{2}, 0)$.
- ◯ **Option 7:** S_3 will shoot the target from the point $(4, 2, 0)$.
- ◯ Option 8: S_3 will shoot the target from the point $(2, 1, 0)$.
- ◯ Option 9: S_3 will shoot the target from the point $(0, 0, 0)$.

9. From which point will the shooter S_4 shoot the target?
   - ◯ Option 1: $(1, 1, -2)$
   - ◯ Option 2: $(-1, -\frac{1}{2}, -\frac{3}{2})$
   - ◯ Option 3: $(-1, \frac{1}{2}, \frac{3}{2})$
   - ◯ **Option 4:** $(-1, 0, 1)$

10. let d_i be the distance of the target from the point where the shooter S_i shoots the target, for $i = 1, 2, 3, 4$. Choose the correct option from the following.
    - ◯ **Option 1:** d_1 will be the minimum.
    - ◯ Option 2: d_2 will be the minimum.
    - ◯ Option 3: d_3 will be the minimum.
    - ◯ Option 4: d_4 will be the minimum.

    Solution: Note that the closest point to shoot the target can be obtained by finding the orthogonal projection of (3,4,5) onto the subspace on which S_i's are moving.

    Equation of line of motion of S_1 is $x = y$ $(x - y = 0)$. The subspace $W_1 = \{(x, y, z) | x = y, z = 0\}$ is spanned by the vector $(1, 1, 0)$. The projection of $(3, 4, 5)$ onto W_1 is $\frac{1}{\langle(1,1,0),(1,1,0)\rangle}\langle(3, 4, 5), (1, 1, 0)\rangle(1, 1, 0) = (\frac{7}{2}, \frac{7}{2}, 0)$.

    Equation of line of motion of S_2 is $x = -y$ $(x + y = 0)$. The subspace $W_2 = \{(x, y, z) | x = -y, z = 0\}$ is spanned by the vector $(1, -1, 0)$. The projection of $(3, 4, 5)$ onto W_2 is $\frac{1}{\langle(1,-1,0),(1,-1,0)\rangle}\langle(3, 4, 5), (1, -1, 0)\rangle(1, -1, 0) = (-\frac{1}{2}, \frac{1}{2}, 0)$.

    Equation of line of motion of S_3 is $x = 2y$ $(x - 2y = 0)$. The subspace $W_3 = \{(x, y, z) | x = 2y, z = 0\}$ is spanned by the vector $(2, 1, 0)$. The projection of $(3, 4, 5)$ onto W_3 is $\frac{1}{\langle(2,1,0),(2,1,0)\rangle}\langle(3, 4, 5), (2, 1, 0)\rangle(2, 1, 0) = (4, 2, 0)$.

    Equation of line of motion of S_4 is $x + y + z = 0$. The subspace $W_4 = \{(x, y, z) | x + y + z = 0\}$ is spanned by the vectors $(1, 0, -1)$ and $(0, 1, -1)$. Using Gram Schmidt orthogonalization, we can get an orthonormal (orthogonal is sufficient) basis for W_4.

$\{(1,0,-1),(-\frac{1}{2},1,-\frac{1}{2})\}$ is an orthogonal basis for W_4. The projection of $(3,4,5)$ onto W_4 is $\frac{1}{\langle(1,0,-1),(1,0,-1)\rangle}\langle(3,4,5),(1,0,-1)\rangle(1,0,-1)+$
$\frac{1}{\langle(-\frac{1}{2},1,-\frac{1}{2}),(-\frac{1}{2},1,-\frac{1}{2})\rangle}\langle(3,4,5),(-\frac{1}{2},1,-\frac{1}{2})\rangle(-\frac{1}{2},1,-\frac{1}{2})=(-1,0,1)+(0,0,0)=(-1,0,1)$.

Distance between the points can be calculated using $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$. Thus we have $d_1=\sqrt{\frac{51}{2}}$, $d_2=\sqrt{\frac{99}{2}}$, $d_3=\sqrt{30}$ and $d_4=4\sqrt{3}$.