Inner products and norms on a vector space

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- $\langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$ $\to$ $\langle v_1, v_2 + v_3\rangle = \langle v_1, v_2\rangle + \langle v_1, v_3\rangle$
- $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$
- $\langle cv_1, v_2\rangle = c\langle v_1, v_2\rangle = \langle v_1, cv_2\rangle$. where $c \in \mathbb{R}$.

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- $\langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle = \langle v_1, cv_2 \rangle$. , $c \in \mathbb{R}$.

$\langle v_1, v_2 + v_3 \rangle$
$= \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$.

A vector space V together with an inner product $\langle ., . \rangle$ is called an inner product space.

The dot product is an example of an inner product

Recall that the dot product of $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ be in $\mathbb{R}^n$ is

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This yields a function

$$\langle \ . \ , \ . \ \rangle : V \times V \to \mathbb{R} \quad ; \quad \langle u, v \rangle = u \cdot v.$$

$\langle u, u \rangle > 0$ if $u \neq 0$

$\langle u, u \rangle = 0 \iff u_i = 0 \ \forall i \iff u = 0.$

$(u + u') \cdot v = u \cdot v + u' \cdot v$

$u \cdot v = v \cdot u \quad , \quad (cu) \cdot v = c \, (u \cdot v)$

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An example of an inner product on $\mathbb{R}^2$

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$$\langle ., . \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$\langle u, v \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2$$

where $u = (\cancel{x_1, y_1})$ and $v = (\cancel{x_2, y_2})$ be in $\mathbb{R}^2$.

(x_1, x_2) $\quad$ (y_1, y_2)

$u \cdot v = x_1 y_1 + x_2 y_2$.

$$[x_1 \quad x_2] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [x_1 - x_2 \quad -x_1 + 2x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{array}{l} x_1 y_1 - x_2 y_1 \\ - y_2 x_1 + 2 x_2 y_2 . \end{array}$$

$$\langle (y_1, y_2), (x_1, x_2) \rangle = [y_1 \quad y_2] [\quad] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\quad)^T =$$

$x_1^2 - (x_1 x_2 + x_2 x_1) + 2 x_2^2$

$= x_1^2 - 2 x_1 x_2 + 2 x_2^2$

$= (x_1 - x_2)^2 + x_2^2$.

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$$||.|| : V \to \mathbb{R}$$

$$x \mapsto ||x||$$

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  v + w v w

- $||cx|| = |c|||x||$ for all $c \in \mathbb{R}$ and for all $x \in V$

  cv |c| ||v||

- $||x|| \geq 0$ for all $x \in V$; $||x|| = 0$ if and only if $x = 0$

  ||v|| v ||v|| = 0 iff v = 0.

Length as an example of a norm

Recall that the length of a vector $u = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is

$$||u|| = \sqrt{(x_1^2 + x_2^2 + \ldots + x_n^2)} \qquad .$$

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The length function $\mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$.

$$||cu|| = \sqrt{c^2x_1^2 + c^2x_2^2 + \cdots + c^2x_n^2} = |c|\,||u||.$$

$$||u|| = 0 \iff u = 0.$$

$$||u+v|| \le ||u|| + ||v||$$

$$||u+v|| = \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2 + \cdots + (x_n+y_n)^2}$$

$$||u|| + ||v|| = \sqrt{x_1^2 + \cdots + x_n^2} + \sqrt{y_1^2 + \cdots + y_n^2}$$

$$(x_1+y_1)^2 + \cdots + (x_n+y_n)^2 \;?\; x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 + 2\sqrt{x_1^2 + \cdots + x_n^2}\sqrt{y_1^2 + \cdots + y_n^2}$$

$$2x_1y_1 + 2x_2y_2 + \cdots + 2x_ny_n$$

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Define $||u||_1 = |x_1| + |x_2| + \ldots + |x_n|$ for $u = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

✓ $\|u\|_1 = 0 \iff |x_i| = 0 \;\forall i \iff x_i = 0 \;\forall i \iff u = 0.$

✓ $\|cu\|_1 = |cx_1| + |cx_2| + \cdots + |cx_n|$

$= |c| \left(|x_1| + |x_2| + \cdots + |x_n| \right)$

$= |c| \, \|u\|_1.$

✓ $\|u+v\|_1 = |x_1+y_1| + |x_2+y_2| + \cdots + |x_n+y_n|$

$\leq |x_1| + |y_1| + |x_2| + |y_2| + \cdots + |x_n| + |y_n|$

$= \|u\|_1 + \|v\|_1.$

The inner product induces a norm

Let V be an inner product space with inner product $\langle\ .\ ,\ .\ \rangle$.

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Let V be an inner product space with inner product $\langle \, . \, , \, . \, \rangle$.

Then the function $||.|| : V \to \mathbb{R}$ defined by $||v|| = \sqrt{\langle v, v \rangle}$ is a norm on V.

$$||v|| = 0 \iff \sqrt{\langle v,v \rangle} = 0 \iff \langle v,v \rangle = 0 \iff v = 0.$$

If $v \neq 0$, $\langle v,v \rangle > 0 \Rightarrow \sqrt{\langle v,v \rangle} > 0 \Rightarrow ||v|| > 0$.

$$||cv|| = \sqrt{\langle cv, cv \rangle} = \sqrt{c \times c \langle v,v \rangle} = \sqrt{c^2}\sqrt{\langle v,v \rangle} = |c| \, ||v||.$$

$$||v+w|| = \sqrt{\langle v+w, v+w \rangle} = \sqrt{\langle v,v \rangle + \langle v,w \rangle + \langle w,v \rangle + \langle w,w \rangle} = \sqrt{\langle v,v \rangle + 2\langle v,w \rangle + \langle w,w \rangle}$$

$$||v+w||^2 = \langle v,v \rangle + 2\langle v,w \rangle + \langle w,w \rangle = ||v||^2 + ||w||^2 + 2\langle v,w \rangle \leq ||v||^2 + ||w||^2 + 2||v|| \, ||w|| = (||v|| + ||w||)^2.$$

Thank you