Week 1-3
Mathematics for Data Science - 2
Practice Assignment

1. Let $\{a_n\}$ be a sequence defined as $a_n = \dfrac{\sin n\pi + \cos n\pi}{n^2}$, then which of the following option(s) is (are) true?

   [**Hint:** Use the sandwich theorem.]

   ○ Option 1: $\{a_n\}$ is divergent.
   ○ **Option 2:** $\{a_n\}$ is convergent.
   ○ **Option 3:** Limit of $\{a_n\}$ is 0.
   ○ Option 4: Limit of $\{a_n\}$ is $+\infty$.
   ○ Option 5: Limit of $\{a_n\}$ is π.

2. Consider a sequence $\{a_n\}$ defined as $a_1 = 1$ and $a_{n+1} = \sqrt{\alpha a_n}$, where $\alpha > 2$. Assume that $\lim\limits_{n\to\infty} a_n = \alpha$. Which of the following option(s) is (are) true?

   ○ **Option 1:** If $\alpha = 3$, then $\lim\limits_{n\to\infty}(3a_n^3 + 5a_n^2 - 1) = 125$.
   ○ **Option 2:** If $\alpha = 5$, then $\lim\limits_{n\to\infty}(2^{a_n} - 2a_n^2 - 5) = -23$.
   ○ Option 3: If $\alpha = 5$, then $\lim\limits_{n\to\infty}(2^{a_n} + 2a_n^2 - 5) = 23$.
   ○ **Option 4:** If $\alpha = 4$, then $\lim\limits_{n\to\infty}(\log_3(a_n + 4) - \log_{\sqrt{3}}(\frac{a_n}{2} + 2)) = -\log_3 2$.
   ○ Option 5: If $\alpha = 4$, then $\lim\limits_{n\to\infty}(\log_3(a_n + 4) - \log_{\sqrt{3}}(\frac{a_n}{2} + 2)) = \log_3 2$.

3. Which of the following option(s) is (are) true?

   [**Hint:** Use L'Hospital's rule]

   ○ Option 1: $\lim\limits_{x\to 0} \dfrac{e^{\frac{2}{x}}}{e^{\frac{2}{x}} + 2} = 0$.
   ○ Option 2: $\lim\limits_{x\to 0} \dfrac{e^{\frac{2}{x}}}{e^{\frac{2}{x}} + 2} = 1$.
   ○ **Option 3:** $\lim\limits_{x\to 0} \dfrac{e^{\frac{2}{x}} - e^{-\frac{2}{x}}}{e^{\frac{2}{x}} + e^{-\frac{2}{x}}}$ does not exist.
   ○ **Option 4:** $\lim\limits_{x\to\infty} x^{\frac{2}{x}} = 1$.
   ○ **Option 5:** $\lim\limits_{x\to 0} \left(\dfrac{2}{\sin x} - \dfrac{2}{x}\right) = 0$.

○ Option 6: $\lim_{x \to 0} \left(\frac{2}{\sin x} - \frac{2}{x} \right) = 1.$

4. Consider two functions $f(x)$ and $g(x)$ such that

$$f(x+y) = f(x)f(y) \text{ and } f(x) = 1 + g(x),$$

where $\lim_{x \to 0} g(x) = 0$. Assume that g is differentiable and $\lim_{x \to 0} g'(x) = 1$. Which of the following option(s) is (are) true?

[**Hint:** $\lim_{x \to a} f(x) = \lim_{h \to 0^+} f(a+h)$ (i.e., right limit) $= \lim_{h \to 0^-} f(a+h)$ (i.e., left limit) and use L'Hospital's rule.]

○ Option 1: $\lim_{x \to 0} f(x) = 0.$

○ **Option 2:** $\lim_{n \to \infty} f(\frac{1}{n}) = 1.$

○ **Option 3:** $f(x)$ is continuous for all $x \in \mathbb{R}$.

○ Option 4: $f(x)$ is not differentiable for all $x \in \mathbb{R}$.

○ Option 5: $f(x)$ is differentiable and $\lim_{x \to 0} f'(x) = 0.$

○ **Option 6:** $f(x)$ is differentiable and $f'(x) = f(x)$.

5. Consider a function $f(x)$ defined as $f(x) = |x(x-3)|$ in the domain [-4,4]. Then which of the following option(s) is (are) true?

○ **Option 1:** The number of critical points is 3.

○ Option 2: The number of discontinuities of $f(x)$ in the given domain is 2.

○ **Option 3:** The number of points where $f(x)$ is not differentiable in the given domain is 2.

○ **Option 4:** $x = 0$ is a critical point.

○ **Option 5:** A local maximum value of $f(x)$ is $\frac{9}{4}$.

○ Option 6: The global minimum value of $f(x)$ is -1.

○ Option 7: The number of points where $f(x)$ attains its global maximum value is 3.

6. Consider the following two functions f and g:

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ 2 & \text{if } x \neq 1. \end{cases}$$

$$g(x) = x + 1$$

Choose the set of correct options.

- ◯ Option 1: f is continuous on $\mathbb{R}$.
- ◯ **Option 2:** g is continuous on $\mathbb{R}$.
- ◯ **Option 3:** g is differentiable on $\mathbb{R}$.
- ◯ Option 4: $\lim_{x\to 0}(f\circ g)(x) = (f\circ g)(0)$
- ◯ Option 5: $(f\circ g)$ is continuous at $x=0$.

7. Consider the function defined as follows

$$f(x) = \begin{cases} x - \lceil x \rceil & \text{if } 0 < x \leq 1 \\ \lfloor x \rfloor - x & \text{if } 1 < x < 2. \end{cases}$$

- ◯ **Option 1:** $\lim_{x\to 1} f(x) = 0$
- ◯ Option 2: f is not continuous at $x=1$.
- ◯ **Option 3:** f is continuous at $x=1$.
- ◯ **Option 4:** f is not differentiable at $x=1$.
- ◯ Option 5: f is differentiable at $x=1$.

8. Consider a function $f(x)$ defined as

$$f(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Which of the following option(s) is(are) true?

[**Hint:** Use $\lim_{x\to\infty} \frac{\sin x}{x} = 0$]

- ◯ **Option 1:** $f(x)$ is continuous at $x=0$.
- ◯ Option 2: $f(x)$ is differentiable at $x=0$.
- ◯ **Option 3:** $\lim_{x\to 0} f(x) = 0$
- ◯ Option 4: $\lim_{x\to 0} f(x)$ does not exist.

Consider the following graph of a function $f(x)$ which contains only one saddle point as shown in Figure M2W1PS1-3 1, where the solid points denote the value of the function at the points, and the values denoted by the hollow points are not taken by the function. Use the above information to answer the questions 9 to 14.

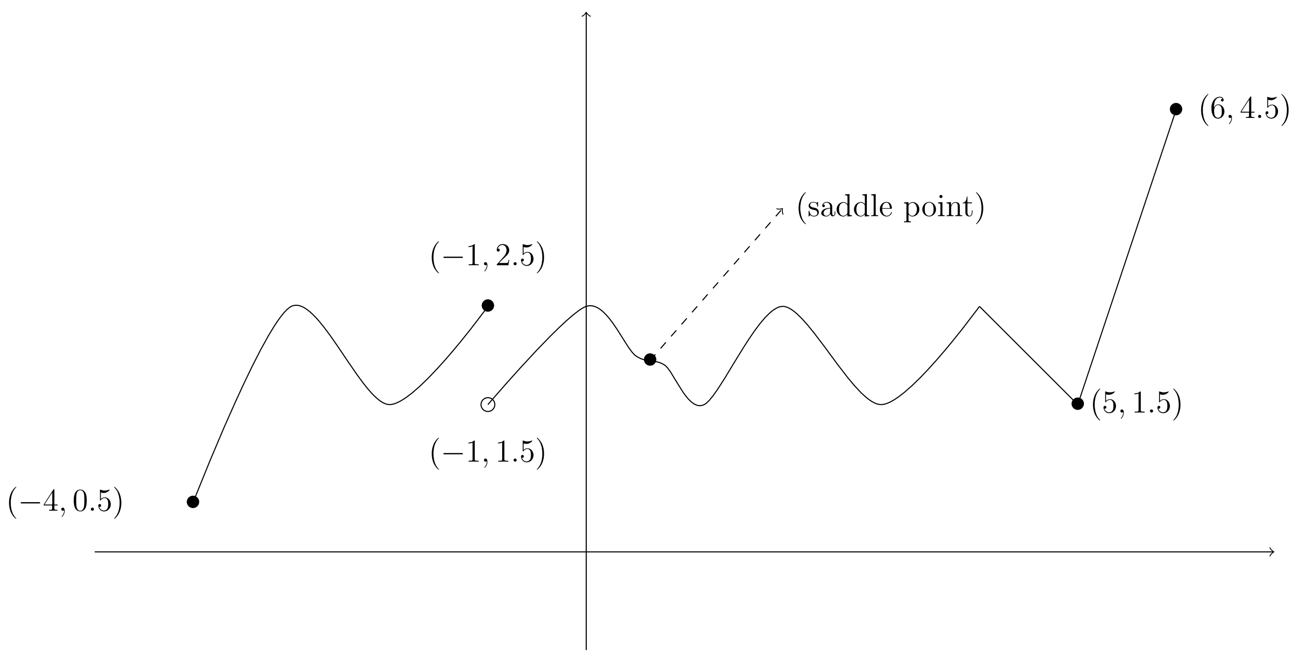

Figure M2W1PS1-3 1

9. Which of the following is(are) true?
   - ◯ **Option 1:** limit exists at $x = 5$.
   - ◯ Option 2: limit exists at $x = -1$.
   - ◯ **Option 3:** Limit exists at every point $x \in (-1, 6)$.
   - ◯ Option 4: The global minimum value of $f(x)$ is 1.5.
   - ◯ **Option 5:** $f(x)$ has the global maximum value at $x = 6$.
10. Find the number of critical points of $f(x)$ in the interval (-4, 6). [Ans: 10]
11. Find the number of discontinuities of $f(x)$ in the interval (-4, 6). [Ans: 1]
12. Find the number of points where $f(x)$ is not differentiable in the interval (-4, 6). [Ans: 3]
13. Find the number of points where $f(x)$ has local minimum in the interval [-4, 6] . [Ans: 5]
14. Find the number of points where $f(x)$ has local maximum in the interval [-4, 6] . [Ans: 6]

15. Let $f(x) = x(x-2)$ and $F(x)$ be the anti-derivative of $f(x)$ defined as $F(x) = \int_{-1}^{x} f(t)dt$. Choose the set of correct options about estimating the $\int_{-1}^{a} f(t)dt$ on the interval [-1, a] using Riemann sums, where a is a critical point of $F(x)$.

[**Hint:** Use $1^2 + 2^2 + \ldots + (n-1)^2 + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ and $1+2+3+4+\ldots+n = \dfrac{n(n+1)}{2}$]

- ◯ Option 1: If $F(x)$ has a local maximum at $x = a$, then the value of the left Riemann sum using a partition of $[-1, a]$ into two sub-intervals of equal length is $\frac{5}{8}$.
- ◯ **Option 2:** If $F(x)$ has a local minimum at $x = a$, then the value of the right Riemann sum using a partition of $[-1, a]$ into three sub-intervals of equal length is -1.
- ◯ **Option 3:** Suppose $F(x)$ has a local maximum at $x = a$. Then the limit as n tends to ∞ of the values of the left Riemann sums using a partition of $[-1, a]$ into n sub-intervals of equal length is $\frac{4}{3}$.
- ◯ Option 4: Suppose $F(x)$ has a local minimum at $x = a$. Then the limit as n tends to ∞ of the values of the right Riemann sums using a partition of $[-1, a]$ into n sub-intervals of equal length is 4.