INDIAN INSTITUTE OF TECHNOLOGY MADRAS

सिद्धिर्भवति कर्मजा

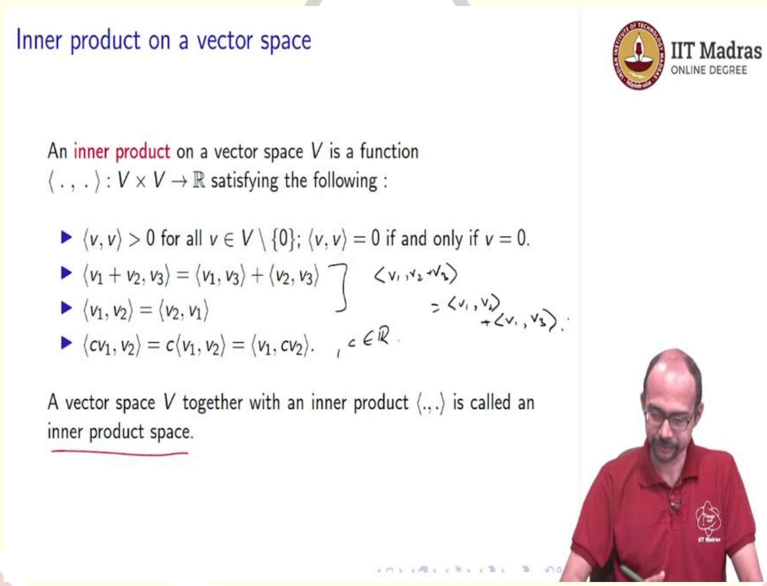
IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2
Professor Sarang S. Sane
Department of Mathematics
Indian Institute of Technology, Madras
Inner products and norms on a vector space

Hello, and welcome to the Maths 2 component of the online course on data science. In this video, we are going to talk about inner products and norms on a vector space. This is linked to our previous video which was on length, angles and dot product. So, we will see that the notion of an inner product generalizes what is a dot product and the notion of a norm generalizes what is the length. So, the point here is that we can do this for any vector space.

(Refer Slide Time: 00:47)



So, let us start by the, by defining what is an inner product. So, an inner product on a vector space V is a function, which is, which takes two vectors as input, so that is why the $\langle .,. \rangle$, so in place of the dots, you will have vectors. So, it is a function from $V \times V$ to $\mathbb{R}$. So, it produces a real number. For each pair of vectors, it produces a real number, and it satisfies the following. So, if you take into, if you apply this to $\langle v, v \rangle$, that means you take just one vector, and you think of it as the tuple $\langle v, v \rangle$, then the inner product of v with itself, that is how we will say $\langle v, v \rangle$ this angle brackets, this must be positive, strictly greater than 0 if v is non-zero, and it will be 0, precisely when $v = 0$.

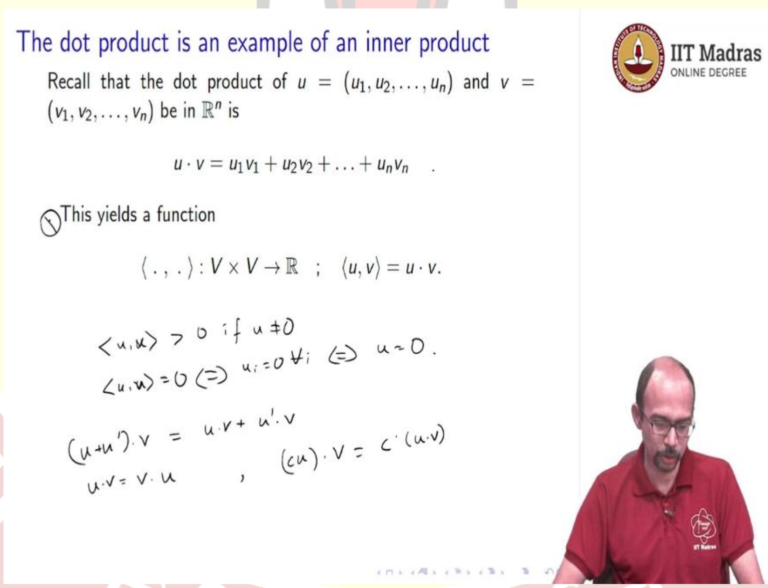
And then you have these three things, which are, which I have put together because they are called, this is saying that it is by linear. So, these three things say it is by linear. So, $\langle v_1 + v_2, \; v_3 \rangle =$

$\langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle$. And $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$. And $\langle cv_1, v_2 \rangle = c\langle v_1, v_2 \rangle = \langle v_1, cv_2 \rangle$, where c here is a real number, it is a scalar.

So, I will also point out that using these two, we can get that $\langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$ because $\langle v_1, v_2 + v_3 \rangle = \langle v_2 + v_3, v_1 \rangle$, which is then $\langle v_2, v_1 \rangle + \langle v_3, v_1 \rangle$, and then you can flip those again to the other side. So, now, these three together are what is called bilinear. So, this is a bilinear map, the first and third.

So, a vector space together with an inner product is called an inner product space. So, I will underline this. This is called an inner product space. This is a definition. And I maybe I will just put in the conclusions we do. So, the conclusions we do before where that, so first of all here, this is not a conclusion, this is a hypothesis, c is a scalar. And the conclusion out of this was that $\langle v_1, v_2 \rangle + v_3$ is $\langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$.

(Refer Slide Time: 03:55)

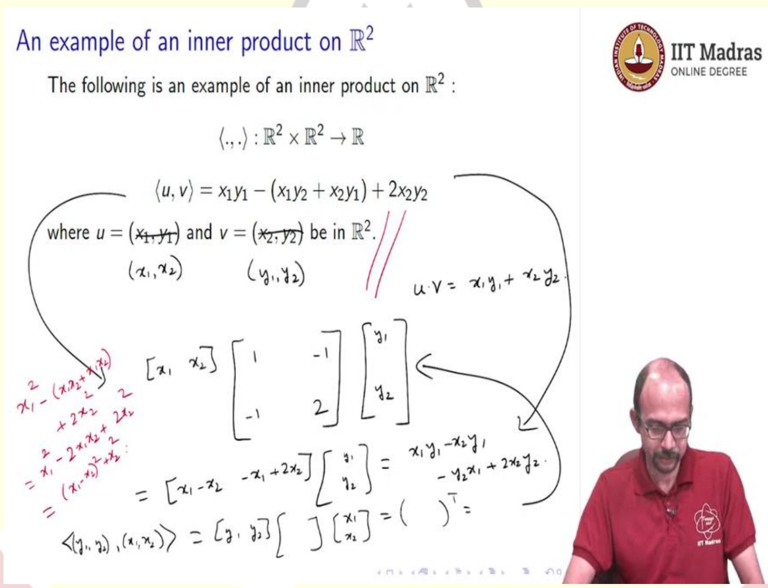


So, the dot product is an example of an inner product. This is what I started with. So, recall that the dot product of u and v in $\mathbb{R}^n$ is $u \, . \, v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. I apologize for this ugly looking full stop which should have been here. And this is a function which is given by $\langle u, v \rangle = u \, . \, v$. So, why is this inner product?

So, we have to verify the three axioms, the four axioms we had. So, we know already that if $\langle u, u \rangle$ is, so this is $u_1^2 + u_2^2 + \cdots + u_n^2$, for real numbers squares must be positive, sum of squares is positive. So, that means this is greater than 0 if $u \neq 0$ and this is equal to 0 exactly when each $u_i = 0$. So, $u_i = 0$ for all i, which happens precisely when $u = 0$. So, we approved the first one. And the other three just follow from the form of this equation.

So, if you have $(u, v').v$, you can check that this is actually $u.v + u'.v$. And similarly, if you have $u.v$, this is same as $u.v$ that is clear from the equation, because real numbers you can, they commute under multiplication. And then clearly if you have $(cu).v = c(u.v)$ because c multiplies into each component. So, for each component you will have it over there and then you can take it out. So, this completes the proof that this is indeed a inner product. So, the dot product is an inner product.

(Refer Slide Time: 05:58)



So, let us now look at an example of an inner product on $\mathbb{R}^2$, which is, this is not the standard inner product that we saw, the dot product that we saw earlier. This is slightly different example. So, here is the definition, $\langle u, v\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$, so where u is x_1. This is a typo here. So, $u = (x_1, x_2)$ I think and $v = (y_1, y_2)$. So, the standard inner product that we saw was, so if you do $u.v = x_1y_1 + x_2y_2$. So, sorry, in terms of, is $x_1y_1 + x_2y_2$. The inner product here is slightly different as you can see.

So, why is this an inner product, how do I prove that? So, this is proved by looking at the following. If you look at this expression, you can write down a matrix for this expression. So, you have x_1x_2

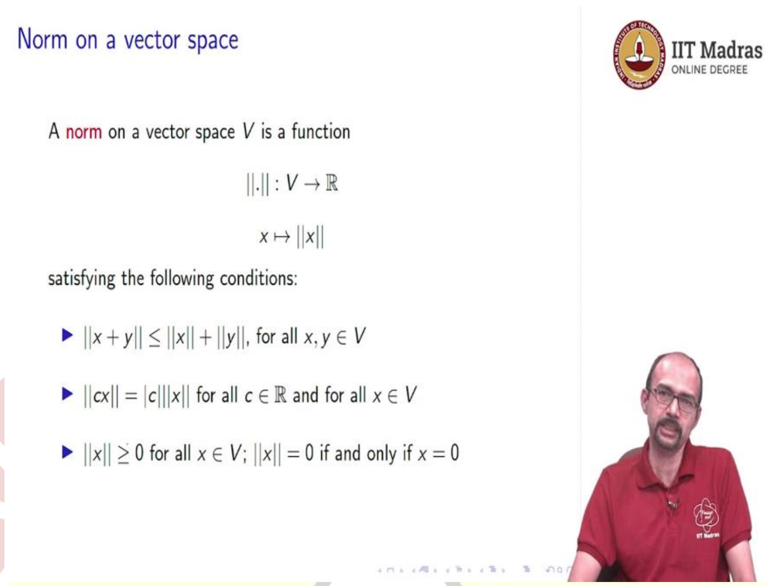
and then you have $y_1 y_2$. And then let us see. So, you have $x_1 y_1$ the coefficient is 1, $x_1 y_2$, the coefficient is -1, I believe, this should be -1 here. And then this is -1 and then this is 2. So, if you multiply this, let us see what we get. You get exactly, so the first term is $x_1 - x_2$, the second term is $-x_1 + 2x_2 y_1 y_2$ and if you work out what this is, this is $x_1 y_1 - x_2 y_1 - y_2 x_1 + 2x_2 y_2$. So, this is exactly what we have up here. This expression and this expression are the same.

So, this is a different expression from the inner product, which is $x_1 y_1 + x_2 y_2$. And now how do I use this matrix form to conclude that it is an inner product. So, the reason, so I can use a matrix form to calculate that this is an inner product because the bilinearity part follows directly. And the symmetry part follows because this is a symmetric matrix. So, if I want to do $y_1 y_2, x_1 x_2$, then you can see that this is exactly writing $y_1 y_2$ here, writing the same matrix here and $x_1 x_2$, but this is a real number. So, I can, so this is the same as its transpose. This is the same thing transpose. And if you do the transpose, you will get this, because the matrix in the middle is a symmetric matrix. So, its transpose is itself.

So, that is why it is symmetric. That is the second property. And then bilinearity is because matrix multiplication respects addition. So, if you have, so it distributes over addition. So, if you have $x_1 x_2$, let us say, $+x_1' x_2'$, then over here you will, that will distribute and it will give you what you want. That is the first thing. And then $cx_1 x_2$ again because constants come out of matrix multiplication, it will follow from there. So, I will urge you to check that fact, because these two expressions are the same. So, the key point here is these two expressions here are the same. So, maybe I will draw that better. These two expressions are the same. So, you can check it from there.

So, the only remaining thing is that this is always positive. So, if you put $x_1 x_2$ in place of $y_1 y_2$ what do you ge? So, you get $x_1^2 - (-x_1 x_2)$ or rather $+x_1 x_2$ so that is $2x_1 x_2 + 2x_2^2$, so this is $x_1^2 - 2x_1 x_2 + 2x_2^2$ and you can write this as $x_1 - x_2^2 + x_2^2$. So, this is a sum of square. Again, it must be positive. If it is equal to 0, then both the terms which are squares are 0, that means $x_1 - x_2$ is 0 and x_2 is 0. Once $x_2 = 0$, So $x_1 = 0$. So, both $x_1 = x_2 = 0$. So, that tells us that it is 0 exactly when u is 0. So, this proves that it is an inner product on $\mathbb{R}^2$.

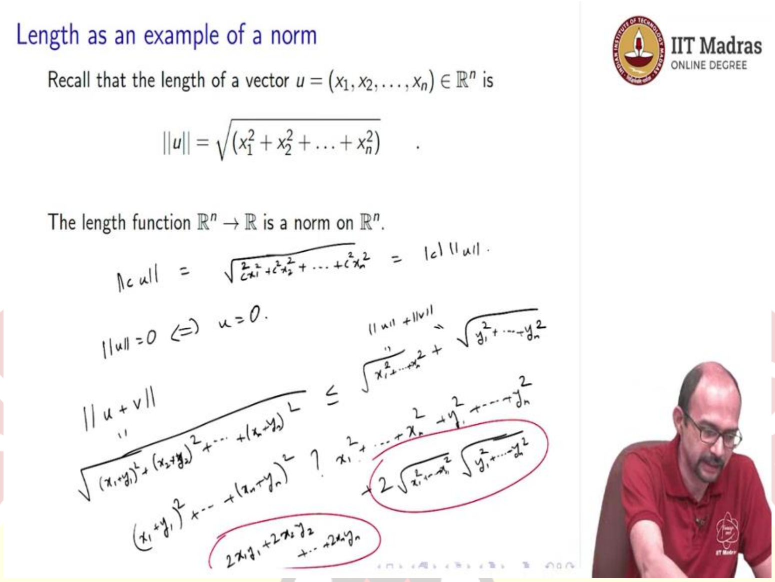
(Refer Slide Time: 11:16)



Let us define what is a norm. So, norm on a vector space is a function. It is indicated by that double lines. And we have seen these double lines in a previous, in the previous video, $x \mapsto \|x\|$, which satisfy the following conditions. So, $\|x + y\| \leq \|x\| + \|y\|$; $\|cx\| = |c|\|x\|$.

So, remember, in the previous video this norm was supposed to represent length. So, since it represents length, it better be positive. And so if you have, if you scale it, the length better remain positive. That is what is being reflected here. So, positivity is coming in the third axiom. $\|x\| \geq 0$. And it is 0 precisely when we are dealing with the 0 vector, if and only if the vector x is 0.

(Refer Slide Time: 12:16)



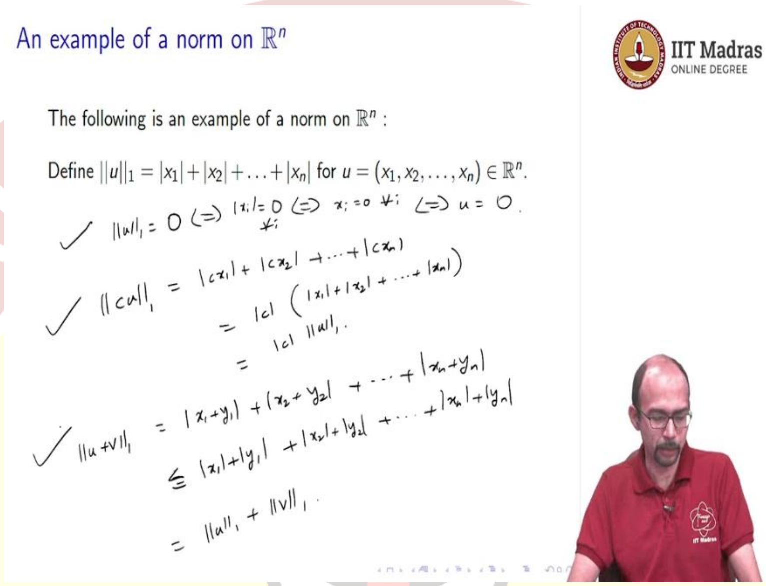
So, let us recall the length of a vector. This is the comment I made in the previous slide u. It is given by this $\|u\| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$. So, we have to say that the three previous conditions are satisfied. As I said, if you multiply this by c, this is clearly $\|cu\| = \sqrt{(c^2x_1^2 + c^2x_2^2 + \cdots + c^2x_n^2)}$. And the c comes out but with an absolute value i.e., $|c|\|u\|$.

Let us see when this is 0. This is 0, we have actually checked this, so the length is 0 exactly when u is 0. This is something we know which leaves us with the triangle inequality. Namely, if you take two vectors, $u + v$, then you compute the sum. So, this is $\sqrt{(x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2}$. And on the other hand, we have $\|u\| + \|v\|$, which is x_1, $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. This is $\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$. So, I want to say that these, this sum is less than or equal to that sum. And this is something that we know is true. And I will maybe not do a proof of this.

So, what you can do is you can square both sides. If you square both sides, you will get $(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2$. This side you get $x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2$ and then +2 times this root and then $y_1^2 + y_2^2 + \cdots + y_n^2$.

And then, so to compare these, so I want to compare these. So, to compare these you can expand the brackets. If you expand the brackets and you cancel off these square terms. You are going to have to compare $2x_1y_1 + 2x_2y_2 + \cdots + 2x_ny_n$ with this side. And now I will leave it at this point to. So, you can cancel out the 2s as well. So, you can, you have to compare this term with this term here. And I will may be not get into a proof that this is actually, this is less than or equal to that. It is a very, something you can do. So, the length function is a norm on $\mathbb{R}^n$.

(Refer Slide Time: 15:48)



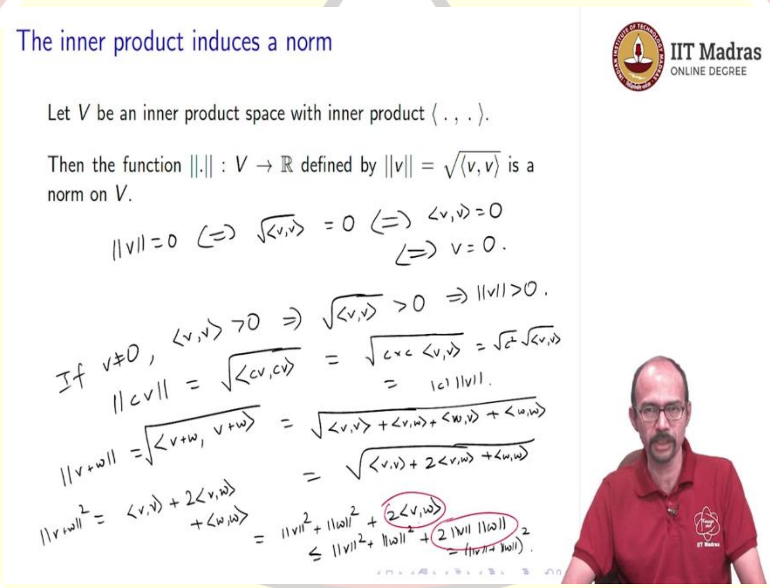
So, let us look at a norm on $\mathbb{R}^n$ which is different from the length function. Namely, you can take the absolute values of the coordinates. So, $\|u\|_1$, so this 1 is supposed to, I think this 1 has not come out the way I intended it to be. This should have been the subscript, but it did not come out as far as I wanted it. Anyway, so define $\|u\|_1 = |x_1| + |x_2| + \cdots + |x_n|$. So, let us check that this is a norm.

So, first of all, I want to check that if $\|u\|_1 = 0$, this is the third axiom, what happens. Well, this is a sum of non-negative integers. So, this happens exactly when each x_i is 0 in absolute value, which is exactly happening when x_i is 0 for all i, which is exactly saying that $u = 0$. So, if this $\|u\| = 0$, if this function is 0 on u, then u better be 0. And, of course, if the, if u is 0, then the function is 0. So, this is one thing that we have checked.

The other is, if you multiply this by a constant c, by a scalar, what happens. Well, each term is multiplied by c. So, I get this expression. Excuse me, I should put in the dots, this expression. And I can take my c out common, because remember that the absolute value of two numbers multiplied is the absolute value of each multiplied. So, you take c out, absolute value of c out common and you get $x_1 + x_2$, all these in absolute value, so $|x_1|$ and $|x_2|$ all the way up to n which is the definition of the norm here. So, c times this function. So, this gives you one more thing.

And then the last one that we have to do is the triangle inequality, which is saying that if you have $u + v$, you take this. Well, let us write that down, that is $x_1 + y_1 + x_2 + y_2 + x_n + y_n$, where x_i's are the coordinates of u, $y_i's$ are the coordinates of v. But we know that this is less than or equal to $x_1 + y_1$, this is less than or equal to $x_2 + y_2$. And then if you rearrange all these terms, you get that this is $u_1 + v_1$. So, this tells you that it is a norm. So, it is a fairly simple check that it is a norm.

(Refer Slide Time: 18:47)



So, finally, let us explore the relation between the inner product and the norm. So, suppose you have v which is an inner product space, so it has an inner product which we have represented by these angle brackets. Let us define this function. So, this function is $\|v\| = \sqrt{\langle v, v\rangle}$. Let us work this out, why this is a norm.

So, first of all, $\|v\| = 0$. This happens exactly when the square root of the inner product of v with itself is 0, which happens precisely when the inner product itself is 0. But remember, it is an inner product. This happens precisely when v is 0. So, I want to prove that this is always positive if v is non-zero, but remember that v, v is, so if $v \neq 0$, $\langle v, v\rangle > 0$, that means $\sqrt{\langle v, v\rangle}$ is strictly greater than, we are always looking at the positive square root, which means $\|v\|$ is strictly. So, the positive, non-negative and positive part is proved, that axiom is proved.

So, the other two axioms are the triangle inequality. And if you multiply a scalar, what happens? So, let us look at scalar. So, for the scalar, so we have $cv,$ by definition this is cv, cv. But as we know c comes out, so this is $\sqrt{c}$. So, it comes out of the first one and it comes out of the second one.

So, $c \times c$, excuse me, this is $\|v\|$, inner product of $\langle v, v\rangle$. So, this is $\sqrt{c^2}$, $\sqrt{\langle v, v\rangle}$. The second part is exactly the $\|v\|$ and the first part is mod of, meaning the absolute value of c, because you are taking $\sqrt{c^2}$. So, if c is a negative number, after taking square root, you always look at the positive part. So, it is the absolute value of $c\|v\|$. So, this is the another one of the axioms.

And finally, we have, if you have $v + w$, by definition this is $v + w, \sqrt{v + w}$ and this is, so you have four terms. You have v comma w plus v comma, w comma v plus w comma w which if you rewrite this is v, v and this is remember its inner product. So, it is, so the first, we got the first thing by bilinearity, and then, because it is symmetric, we get this is 2 times v comma w plus w comma w.

So, let us square this. So, if you square this, you get v comma v plus 2 times the inner product plus w comma w, but this is exactly $\|v\|^2 + \|w\|^2 + 2\langle v, w\rangle$ and this is less than or equal to $\|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$.

So, this needs a little bit of work. And then what you will get is that this is norm of v plus norm of w the whole square. So, we have to prove that $(\|v\| + \|w\|)^2$ is $\|v\| + \|w\|$ is less than or equal to $(\|v\| + \|w\|)^2$. The only thing missing here is why is, this thing less than or equal to this thing here. So, that needs a little bit of work which I will skip for now. So, I guess this tells you the relation between inner products and norms. So, the main point here is that once you have an inner product, it induces a norm. This is what you have to keep in mind. Thank you.