IIT Madras

ONLINE DEGREE

Mathematics for Data Science 2
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Lecture No. 11
The Gaussian Elimination Method

Hello and welcome to the maths 2 component of the online B.Sc. program. In this video we are going to talk about a process called the Gaussian elimination method. Using this method, we will be able to do a lot of things. So, in particular, we will be able to find the solutions to any system of linear equations, in a very fixed algorithmic manner. And we can also use it to find the determinant of a square matrix which we kind of saw last time. And we can also use it to find the inverse of a square matrix which is invertible. So, let us quickly recall, what was our, what we have seen before.

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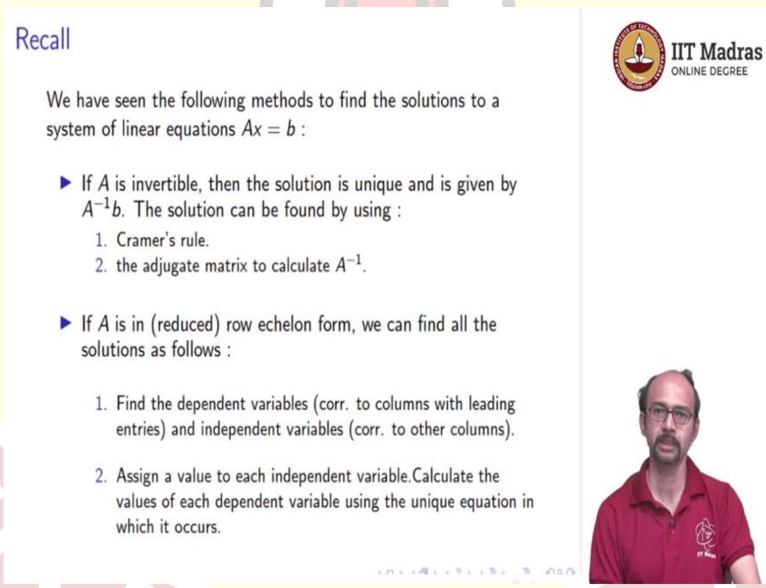

So, we have seen the following methods to find the solutions of a system of linear equations, $Ax = b$. Namely, we have seen if A is invertible, then the solution is unique. And it is given by $A^{-1}b$. And we can find the solution by either Cramer's rule, that was the first method we saw. And the other method we saw was to actually compute A^{-1}. To do this, we find the adjugate matrix and then we find $A^{-1}b$.

So, these are slightly cumbersome methods, because they involve a lot of determinant calculations. And finding determinants is a very resource intensive process. So, you have to do a lot of computation to find determinants. And here, we have to find not just one, but several determinants, and that is how these processes work. So, compared to this, remember that if A is in what we call the reduced row echelon form, then we can find the solution of, of such a system very easily.

So, if you, if $Ax = b$ is the system that we are solving, and if A is in reduced echelon form, then we have the following procedure to find the solutions and note that, there can be many solutions. Now, because it A is no longer a square matrix, it could be a rectangular matrix. So, you find the dependent variables. So, these are the ones which correspond to the leading entries. So, the there is a 1 in the nonzero rows, which is, which appears at the beginning of the row, the first entry of the, first nonzero entry of that row looked at from the left, that is a leading term.

So, the column corresponding to that, that column, the corresponding variable. So, if the ith column is a leading entry, then x_i is an, is a dependent variable. And if the ith column does not have a leading entry, that, that corresponding variable is an independent variable. So, all those columns which do not have leading ones, those correspond to independent variables. And then how do we find our solutions, you plug in any value that you want for the independent variables, and you back calculate to find the values of the dependent variable.

And note that each dependent variable will appears in a unique equation that is the advantage of the reduced row echelon form. And you can use that to back calculate and find the value of a dependent variable. And then all such values for every choice of values of the independent variables, we get a solution of our system $Ax = b$. Of course, you have to when before we do this, remember, this is very important that if there is a 0 row in your reduced row echelon form, and the corresponding entry in b is nonzero, then this system cannot have a solution.

This was one of the things we observed and we will use this as you go ahead. So, this tells you all the possible situations. So, it tells you whether there is a solution, if there is a solution, it enlists all of them. And we are going to make use of this technique. So, we saw this two videos ago and we are going to make use of technique along with what we saw in the previous video where we took an arbitrary matrix and we used row operations. So, elementary row operations to reduce it to reduced row echelon form. So, we are going to put both these two together. And we are going to solve an arbitrary system of equations $Ax = b$

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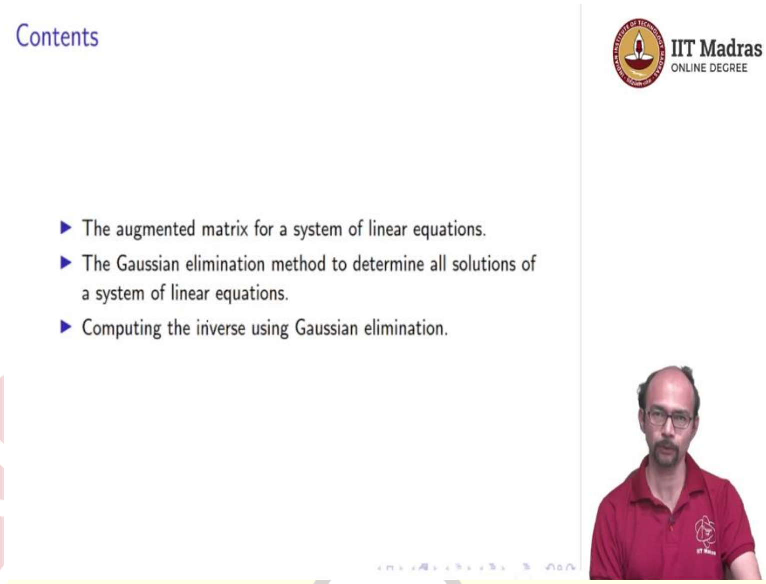

So, what are the contents of this video, we will study what is called the augmented system, augmented matrix for a system of linear equations, we will study the Gaussian elimination method to determine all solutions of a system of linear equations. And here, our system may not have the same number of variables as equations. So, in other words, your A could be a rectangular matrix, it need not be a square matrix. And finally, we will use this also, this is very brief to compute the inverse. So, we will use this technique also to compute the inverse of a, of an invertible matrix, of course, which is square.

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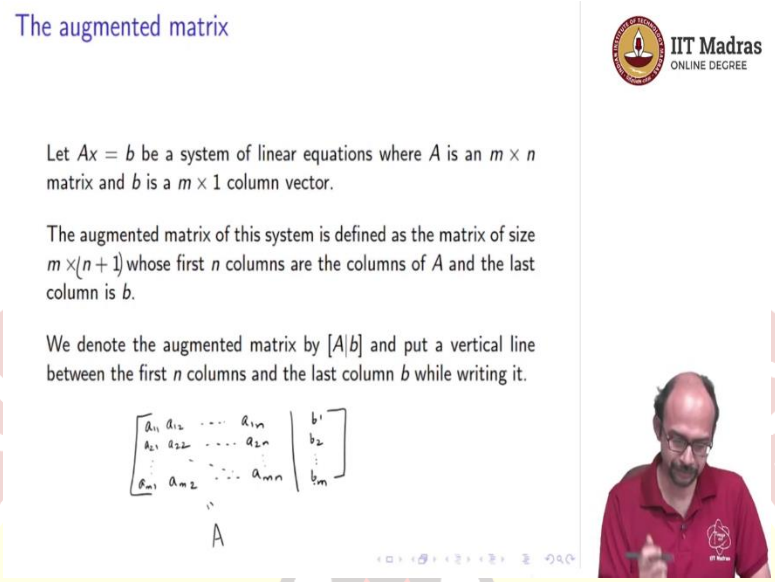

So, what is the augmented matrix? So, let $Ax = b$ be a linear system of equations. So, A is an m by n matrix. So, you have m rows and n columns, so remember that the n columns correspond to the n variables, and the m rows correspond to the m equations. And b is as a result, the m by one column vector, or column matrix of constants, which appears to the right. So, any such equation, we have these three matrices, $Ax = b$, A is m by n, x is a column vector a column matrix of size n by 1 and b is a column vector of size m by 1.

So, we form what is called the augmented matrix of the system. So, this is a new matrix, and it is of size m by $n + 1$. So, there is a small typo here, namely that our $n + 1$ should be in the brackets. So, m by $n + 1$. So, the first n columns are the columns of A, and the last column is b. So, this is, this is how I define the augmented matrix of this system. And we have a notation for the augmented matrix, it is $[A|b]$.

So, this is to remember that the first n columns are coming from A and the last column is coming from b, can we put this vertical line, so this is just notation. And how are we going to write this augmented matrix? So, we will write this augmented matrix as follows. So, if my coefficients for the equation are $\left(\begin{array}{c|c} a_{11}a_{12} \ldots \ldots . . a_{1n} & b_1 \\ a_{21}a_{22} \ldots \ldots . . a_{2n} & b_2 \\ \ldots & \vdots \\ a_{m1}a_{m2} \ldots \ldots . . a_{mn} & b_m \end{array}\right)$.

So now, this is my matrix A remember, so, A came in the first n columns, that is what the definition says. And in the last column, I have my vector b_1, b_2, b_m. So, this is the augmented matrix for the system. So, it is a matrix with m rows, same number of rows as number of equations, and $n+1$ columns, the n columns coming from the coefficients and the last column coming from the constants. So, what do we do with this augmented matrix?

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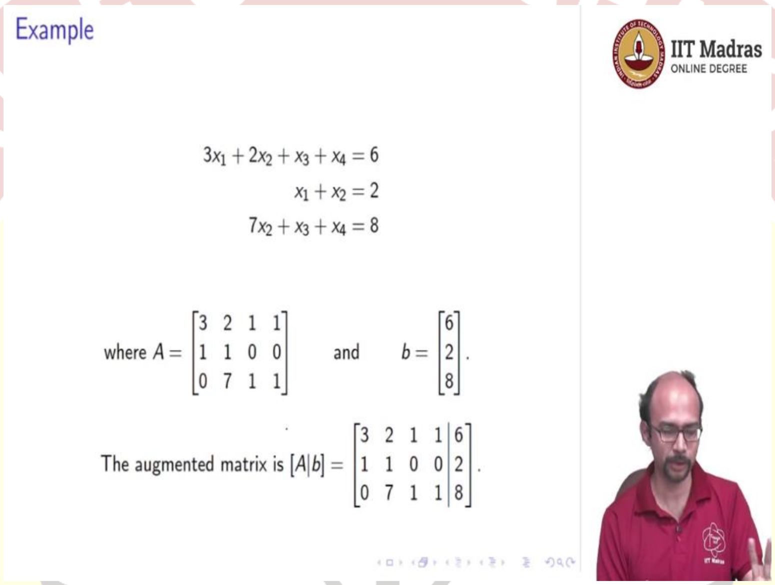

So, for the augmented matrix, let us, let us first do an example maybe, of an augmented matrix. So, here is my system of equations, $3x_1 + 2x_2 + x_3 + x_4 = 6$; $x_1 + x_2 = 2$; $7x_2 + x_3 + x_4 = 8$. So, A is the matrix $\begin{bmatrix} 3 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 \end{bmatrix}$. You might remember this from the previous video. So, we have already done the operation of row reduction on this matrix. And $b = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}$ is the column vector of the constants.

So, what is the augmented matrix? So, you just take these two, and you put them together, but you put a line in the middle. So, this line is just to be, it is not sort of part of the matrix, but it is something that you put to remember that on one side, you have coefficients and on the other side, you have the constants.

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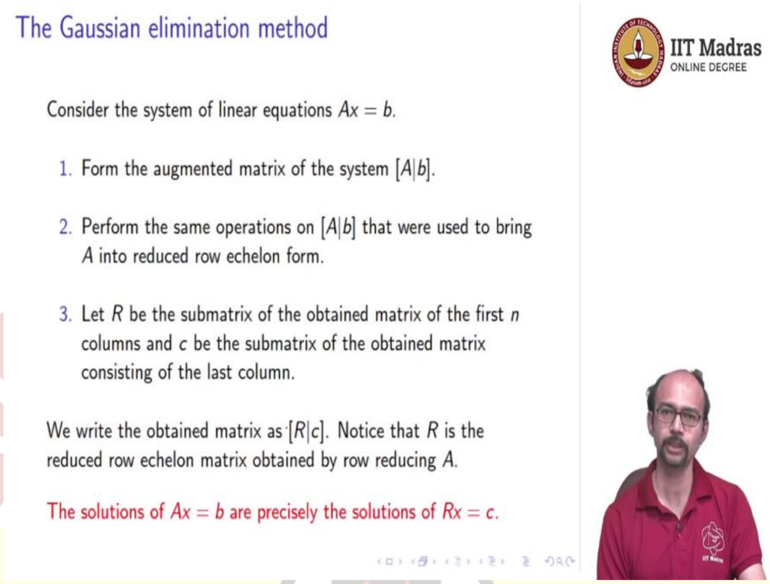

So, what do we do with the augmented matrix? So, this is what the Gaussian, where the Gaussian elimination method starts. So, you have a system of linear equations, $Ax = b$, and Gaussian elimination is going to tell you how to get all the solutions of this. So, it will use these two things. First, it will use the augmented matrix. And second, it will use the technique of reducing to the row echelon form that we have reduced row echelon form that we studied in the previous video.

And then it will, it will use that for, for matrix and reduced row echelon form. If I have something like $Rx = c$, then I can find all the solutions. This is exactly the technique. So, let us go through the steps. So, form the augmented matrix. So, we just saw what that was. Perform the same operations on this augmented matrix, that were used to bring A into reduced row echelon form.

So, we know how to bring any matrix into reduced row echelon form. So, if you take the matrix A, there is a series of steps and it is an algorithm, using which you can put it into reduced row echelon form, so you have to use elementary row transfer, row operations. So, you use the same operations on this new matrix that you used on A in order to put a into reduced row echelon form.

So, note that this new matrix that you have, need not be in reduced row echelon form. And indeed, that is the point, if it is not in reduced row echelon form at the end, then something special is going to happen. And if it is in reduced row echelon form at the end, then something else is going to happen. So, let R be the sub matrix of the obtained matrix of the first n columns. So, you look at the so after you do this process on $[A|b]$, so the augmented matrix, you apply all these elementary row operations.

So, in the end, you will get another new matrix, which is again with m rows and $n+1$ columns. So, you look at the first n columns, and that matrix you call R. And let c be the sub matrix of the obtain matrix consisting of the last column. So, the last column consists of c. And to keep track again of the first ten and the last column, we will put a vertical line between them. So, we write the obtained matrix as $[R|c]$.

And now this is the important part, notice that R is in the reduced row echelon form, it must be, because you took A and you applied all the row operations on A, which we used on $[A|b]$, that we used to put A into reduced row echelon form. So indeed, R is exactly the matrix that you got, after you applied all those operations to reduce it into reduced row echelon form. So, R is exactly the, that matrix.

So, what is next? So the, this is the main point, this is the crucial point, which is why the entire method works, these solutions of $Ax = b$ are precisely the solutions of $Rx = c$. So, what have we done, we have taken an arbitrary system of linear equations $Ax = b$, we have performed some operations on this, on the matrices occurring there, we have found two new matrices R, which is of size m by n and c, which is of size m by 1, where R is reduced row echelon form. And when we take the corresponding system of linear equations $Rx = c$, the set of solutions of $Rx = c$ is exactly the same as the set of solutions of a $Ax = b$.

Now, well, if you think a little about what is happening, you can see that we are already done here. So, from here, I can finish off how to find the solutions $Ax = b$.

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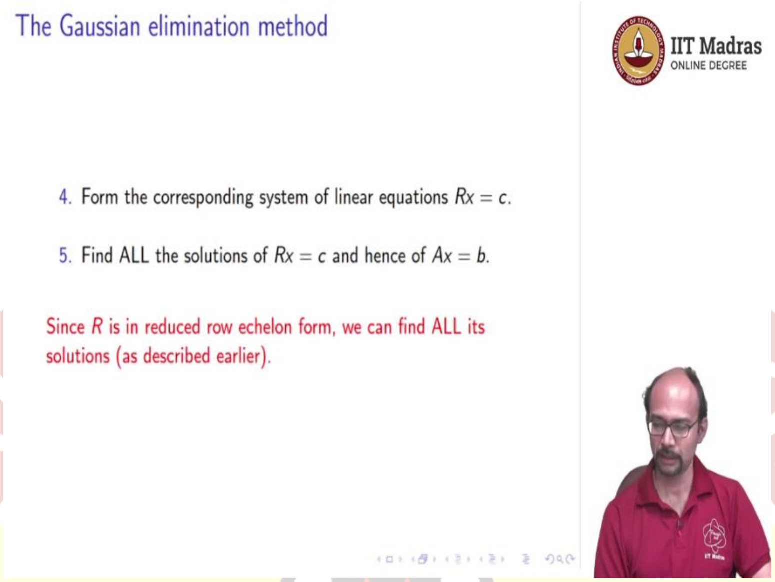

Let us list out the procedure entirely, form the corresponding system of linear equations $Rx = c$, find all the solutions of $Rx = c$ and hence of $Ax = b$. So, if I, as we observed, the solutions of $Ax = b$ are exactly the same as the solutions of $Rx = c$. That was the punchline of the previous slide. And now so if we find all the solutions of $Rx = c$, then I know all the solutions of $Ax = b$.

And how do I find the solutions of $Rx = c$? Well, R is in reduced row echelon form. So, I have a concrete procedure to find these. So, you find the independent variables and the dependent variables and then you put the independent variables to arbitrary values, and then you back calculate to find the values of the dependent variables. We have done examples of this as well. We recalled this right at the beginning of this video. So, we have a very concrete procedure now to, to find the solutions of $Ax = b$.

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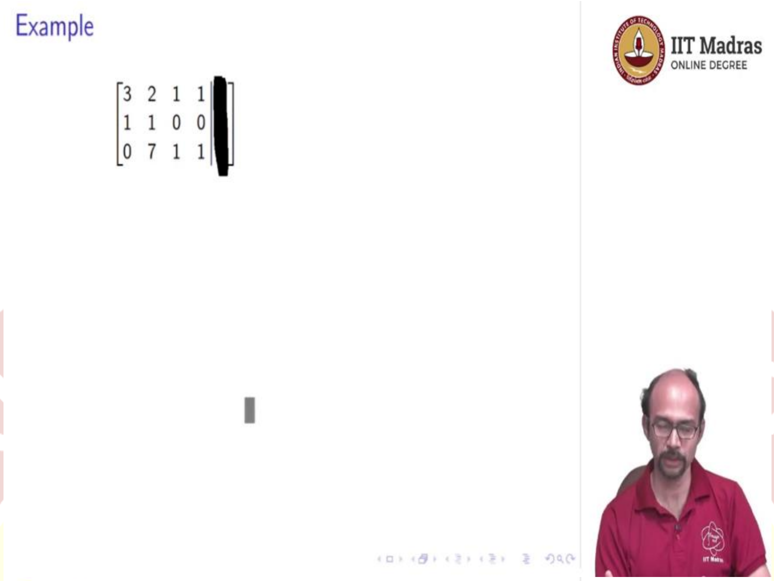

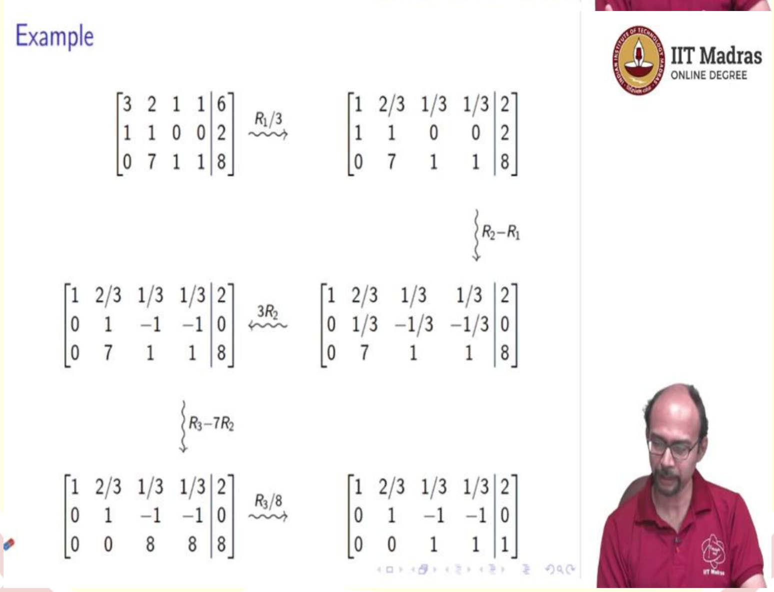

So, let us do an example. So, you might find this, this is the same example that we saw earlier, where we put the system of linear equations into the, we wrote down the augmented matrix of the system of linear equations. So, now let us perform the Gaussian elimination method on that same example. Now, if you look at this matrix on the left, just the matrix, just this part, the A part.

So, if you look at that A part, then that A part is exactly the matrix that we used in the previous video, which we use the which we completely computed, what was the reduced row echelon form for. And so you have to use those same steps over here. So, we can do, we will do a fresh, of course. So, we will do that a fresh, so the first step is, you get a 1 in the 1 1 place, because you start with the first column, and then you try to get a 1 in that 1 1 place.

So, you do $\frac{R_1}{3}$. And then you do $R_2 - R_1$, to cancel all the 0s below. So, this is what you get. And now notice what is happening here, we are not performing these operations only on the A part, we are performing it on the entire matrix. So, in that includes the b part, it includes the column b. So, when we went from the first step, in the first step, the $\frac{R_1}{3}$ was also applied on the column b. So, you have to apply it on the entire row, not just the row of A but the entire row.

In the second step, we did $R_2 - R_1$. And in $R_2 - R_1$, we got a 0 in the second place of the last column. So, your performance on the entire second row, not just the A part of the matrix. Let us continue. So, the next thing we do is we multiply the second row by 3 in order to bring a 1 over there. Then we cancel out, we cancel out the 0 below that 1, by doing $R_3 - 7R_2$. Keep noticing what is happening to the last column, you are playing all this on the last column as well. And then finally, we have to get a 1 in the third row now. So, we divide by 8. And notice that the 8 in the last column, also got divided by 8, and we got a 1.

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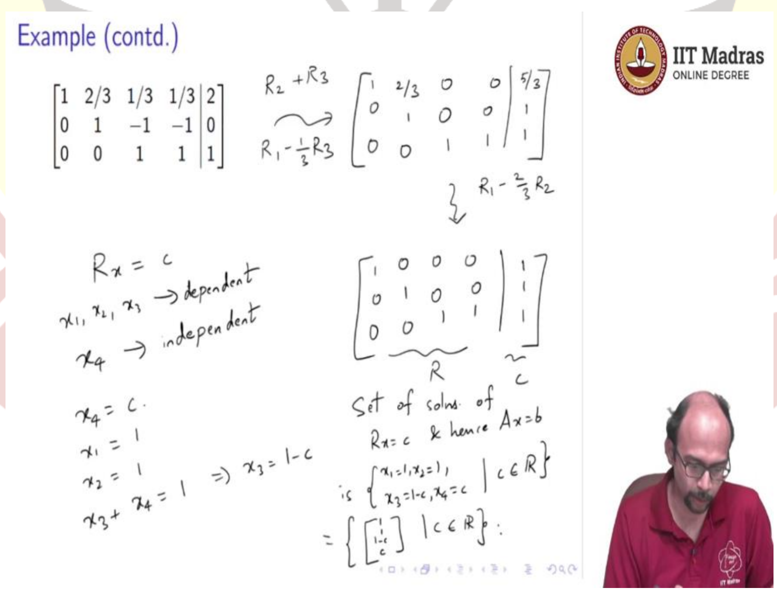

So, let us write down this matrix again. And maybe now I will finish this calculation. So, this is already, the matrix on the left is already in row echelon form. So, now I am going to reduce it to the reduced row echelon form. So, in order to do that, what do I do? So, I do $R_2 + R_3$, and $R_1 - \frac{R_3}{3}$.

So, what do I get? So, this is $\left[\begin{array}{cccc|c} 1 & \frac{2}{3} & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 1 & \end{array}\right]$. And, well, we have slightly lucky, we also get a 0 0 1 here. And then let us see what happens here.

So, $R_2 + R_3$ means I get a 1 in this place. So, I already have a 1 here. And then $R_1 - \frac{R_3}{3}$. So, $2 - \frac{1}{3}$. So, that is $\frac{6}{3} - \frac{1}{3} = \frac{5}{3}$. And in the next step, what do I get? I do the same thing for the second column. So, I do $R_1 - \frac{2R_2}{3}$. So, if I do that, I get $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 1 & \end{array}\right]$. But something may change here. So, $R_1 - \frac{2R_2}{3}$ means you would have $\frac{5}{3} - \frac{2}{3} = 1$, so $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$.

And this is my final matrix. So, in terms of our notation, this is what I call R, and this is what I call c. Now let us write down this corresponding equation. So, we have $Rx = c$. And now how do I find the solutions for this. So, we look at the independent and the dependent variables. So, here there is three leading ones, namely column one, column two, and column three. So, x_1 x_2 x_3 are independent, sorry dependent variables.

And x_4 is independent. So, let us put an arbitrary value for x_4. So, substitute $x_4 = c$. And now let us see what happens to the others. So, x_1, actually, there is no choice, x_1 is 1, x_2, again, there is no choice x_2 is 1 if you write down the equation, and in the third equation, we have $x_3 + x_4 = 1$, but we have put $x_4 = c$, that means $x_3 = 1 - c$. So, the set of solutions of $Rx = c$, and hence, $Ax = b$ is vectors of the form, so that is x_1.

So, you can either write it as a vector or in or explicitly, maybe let me write it explicitly. So, x_1 is 1, x_2 is 1, $x_3 = 1 - c$ and x_4 is c. And where c can take any choice, we can have any number of c, so any real number, I can write the same set as in terms of a vector as $\begin{bmatrix} 1 \\ 1 \\ 1-c \\ c \end{bmatrix}$, where $c \in \mathbb{R}$. So, we have determined all the solutions of the system of equations, $Ax = b$, where A was exactly that matrix that we started with, in our first example.

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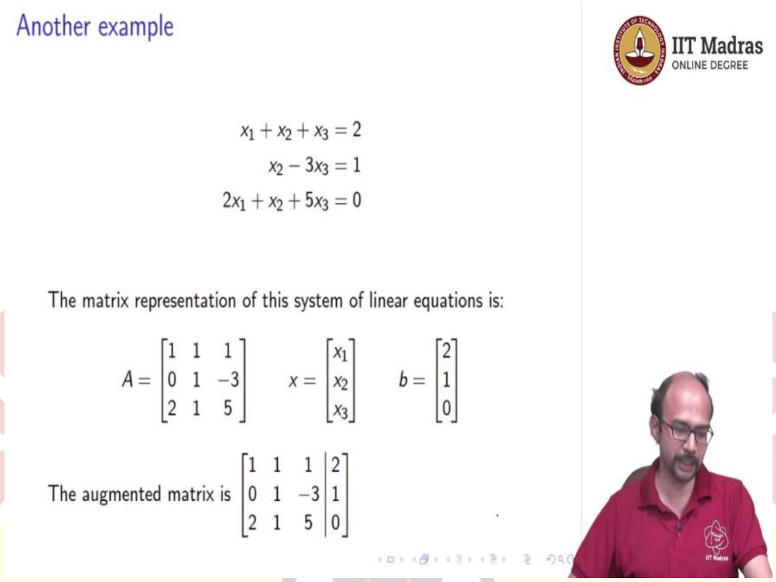

So, maybe let us do another example. So, here we have $x_1 + x_2 + x_3 = 2$; $x_2 - 3x_3 = 1$; $2x_1 + x_2 + 5x_3 = 0$. So, the matrix representation, so A is a square matrix, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 5 \end{bmatrix}$, and b is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. So, the augmented matrix, you put A and b beside each other, and put a vertical line across them. So, it is three rows and four columns, where the fourth column is b, and the first three rows are A.

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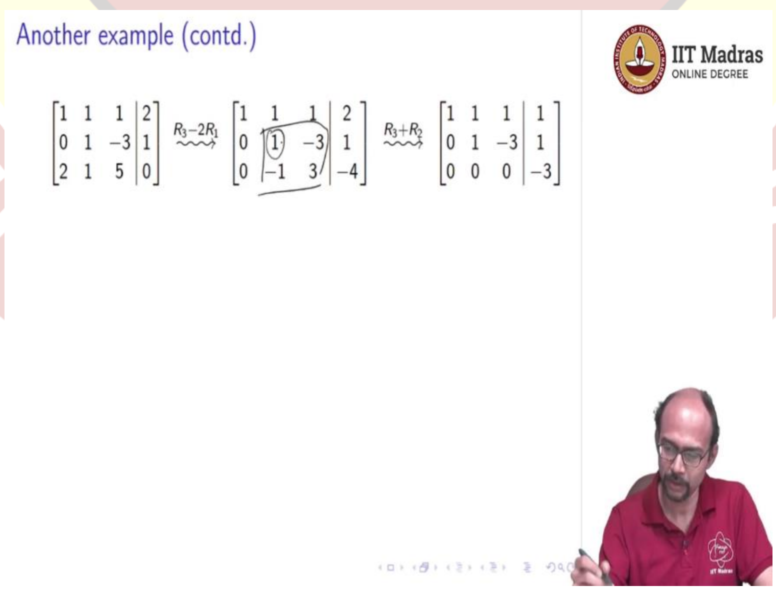

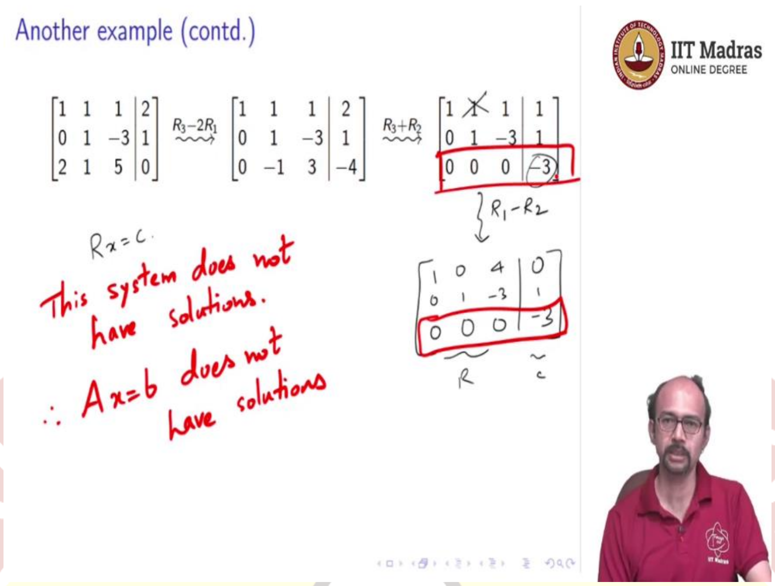

So, let us do this example, maybe explicitly. So, maybe let us write down the first step. So, this is $R_3 - 2R_1$, why $R_3 - 2R_1$, well, there is already a 1 in the 1 1-th position. So, $R_3 - 2R_1$ cancels at 2 at the, in the first column. And note that we have also applied this on the last column. So, the last column entry becomes -4. So, let us do one more step. So, you have $R_3 + R_2$.

Now, because once again, our so we want a 1 in the second column, in this part of the matrix over here, which we already have. So, so I do not have to do anything to make that 1. And now I want to use that 1, and clear off the -1 below that. So, to do that, we do $R_3 + R_2$. And again, the entry in the third, the third entry here changes. So, let us continue and finish off this example. So, well, this is actually already in row echelon form.

So, now let us put it into reduced row echelon form, to put it into reduced row echelon form, what do I have to do, I have to get rid of this 1 over here. So, let us do that, to do that I have to do R_2, sorry $R_1 - R_2$. So, I have $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$, we could actually have skipped this step but I am doing it just because I want to stick to the procedure.

So, this is in reduced row echelon form. So, this is our R, this is our c. So, let us write down $Rx = c$. So well, so R is is reduced row echelon form. So, I can read off all the solutions. And now you have to be slightly careful and carefully observe the following. Look at this last thing over here.

So, you have a 0 row in the, on the left meaning in R, you have a 0 row, but the corresponding constant is not 0.

So, as soon as that happens, what do we know? We know that this does not have solutions, meaning there is no solution to this equation, maybe that is how I should write it. So, there is no solution to this system. So, this system does not have solutions. And because this system does not have solutions, neither does $Ax = b$. So, therefore, $Ax = b$ does not have solutions. And of course, what does $Ax = b$. Now, we wrote down the explicit system in the previous slide. So, we have, we have shown an example where we can use this method to also check when a system does not have any solution.

So, I hope both these examples are, have cleared up any residual doubts, I will just make a comment before I move ahead, I need not have done this last step. Anytime in this procedure, when you hit this kind of situation where you have a zero row, and the corresponding entry in your, in your constants, on the other side of the vertical line is nonzero, we are saying that in your, you have an equation of the form 0 times $0x_1 + 0x_2 \dots + 0x_n =$ nonzero.

Of course, this cannot have a solution. And you can stop right there and say this does not have any solutions. But if you feel, sort of uncomfortable with stopping in the middle, and especially if you want to program this into a computer, then you might as well finish the entire method. So, this is the power of the Gaussian elimination method, you can find out all the solutions of a system of linear equations.

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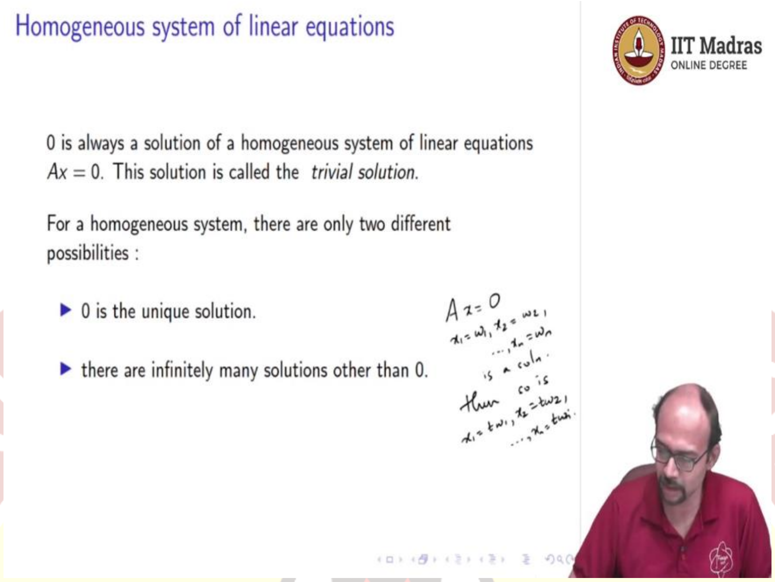

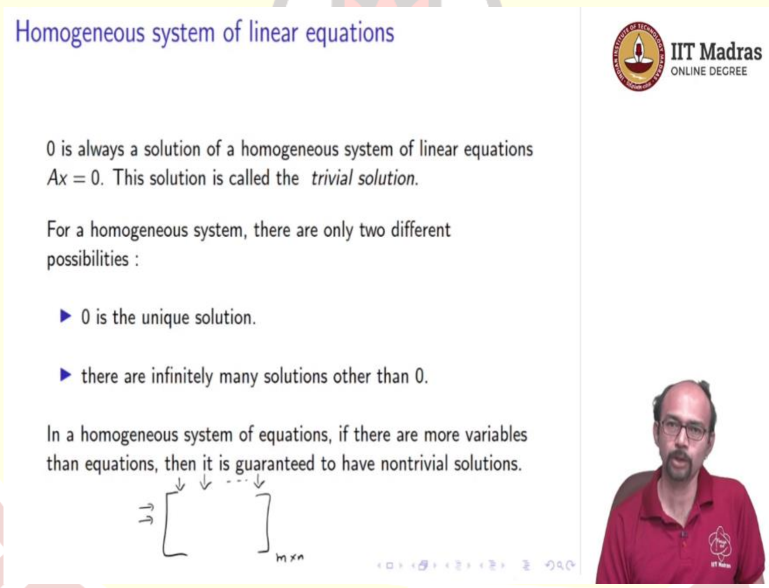

So, let us study a particular case, we have studied this in a previous video, namely the homogeneous system of, system of linear equations. That means $Ax = 0$. So, remember that for $Ax = 0$, we always have a solution, namely, $x = 0$. If you put each of your variables to be 0, then certainly this system is solved. So, 0 is always a solution. So, you cannot have a situation where there is no solution for such a system.

So, homogeneous system always has a solution. So this is called the trivial solution, the solution $x = 0$ is called the trivial solution. And of course, that also means that the other solutions, so if you have a solution, which is not 0, meaning there is some $x_i \neq 0$, then that is called a non trivial solution. So, for a homogeneous system, there are only two different possibilities. One is that the 0 is the unique solution. So, remember that 0 is always a solution, whatever happens. So, if there is a unique solution, then it must be 0. So, 0 is a unique solution. That is one possibility.

The other possibility is that there are infinitely many solutions other than 0. And how do we, how do we get this? Remember that if, if you have more than one solution, then you have infinitely many solutions. And why is that? That is because if you have more than one solution, then the only way to get that, that situation is if one of your variables is a, is an independent variable.

And as soon as you pick up an independent variable, you have an infinitely, you have infinitely many solutions. That is what they say. The other way of thinking about this is that suppose you have a solution, let us say, so you have $Ax = 0$ and $x_1 = w_1$, $x_2 = w_2$, and so on $x_n = w_n$ is a solution, then so is $x_1 = tw_1$, $x_2 = tw_2$, and $x_n = tw_n$.

Because if you substitute x_1, to be tw_1 etc., in Ax, then what will happen is the left hand side will be tA_{11}, let us take the first equation, so, I have $A_{11}x_1 + A_{12}x_2 \dots A_{1n}x_n = 0$. And I know that $A_{11}w_1 + A_{12}w_2 \dots A_{1n}w_n = 0$. But now, if you substitute tw_1w_2 and so on in that equation, then you will have $A_{11}tw_1 + A_{12}tw_2 \dots A_{1n}tw_n = t(A_{11}w_1 + A_{12}w_2 \dots A_{1n}w_n)$, but $A_{11}w_1 + A_{12}w_2 \dots A_{1n}w_n = 0$

So, that is $t0 = 0$. So, that is, that is the other way of thinking about this. So, if you have a homogeneous system, that is something special. So, in a homogeneous system of equations, if there are more variables than equations, then it is guaranteed to have non trivial solutions, why is this? So, what is what is being said here, so, you have n variables remember and m equations. So, you have a matrix of size m by n, where these rows are corresponding to equations and columns are corresponding to variables.

So, now, if you have more variables than equations, then what happens, then when you put it into reduced row echelon form, when you have any, only m rows, but you have n columns, and n this strictly greater than m, so, the leading coefficients, the leading ones can occur only in at most in many places. So, there will be some columns, which do not have leading ones, that means, there are some independent variables. And as soon as you have independent variables, you will have an infinite number of solutions. So, in particular, you will have nonzero solutions or non trivial solutions.

In this video, you have studied the Gaussian elimination method, where we put together the fact that we can find the solutions for a matrix in reduced row echelon form, the coefficient matrix is reduced row echelon form, along with the fact that any matrix can be reduced to reduces row echelon form, using elementary row operations. Those were the contents of the last two videos. And this is a much more powerful method, it is far more efficient than the previous methods, even when your matrix A is invertible.

So, this is a, this is a method that you will actually use in practice, and it is much more efficient. And even to compute the inverse of A rather than computing all the miners and so on this method is far more powerful also to compute the determinant which we did in the previous video. So, Gaussian elimination method is the most powerful method we know in these contexts. And we have seen explicitly using it, how to find all the solutions, we can check whether there is a solution at all and then we can take which, once we know there is at least one which of those are solutions.

And finally, we have seen for homogeneous equations. That is for a homogeneous system, 0 is always a solution. So, the question is, are there other solutions? And we also know how to find those, again using Gaussian elimination method. Thank you.