Week-1
Mathematics for Data Science - 2
Some topics of Maths 1, Functions of one variable, Graphs and tangents,
Limits for sequence, Limits for functions of one variable
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. Let a_n and b_n be two sequences of real numbers. Consider the following statements.

   - **Statement 1:** If a_n and b_n both converge to some non-zero real number, then a_nb_n also converges to some non-zero real number.
   - **Statement 2:** a_nb_n may converge even if neither a_n nor b_n converge.
   - **Statement 3:** A constant sequence, i.e., $a_i = c$ for some real number c, for all $i \in \mathbb{N}$, is always convergent and it converges to c.

   Choose the correct option from the following.

   ◯ **Option 1:** All the three statements are true.
   ◯ Option 2: Statements 1 and 2 are true, but Statement 3 is false.
   ◯ Option 3: Statements 1 and 3 are true, but Statement 2 is false.
   ◯ Option 4: Only Statement 3 is true.
   ◯ Option 5: None of the statements is true.

   Solution:
   Statement 1:
   Assume the sequence a_n converges to the limit $\ell_1 \neq 0$ i.e., $\lim\limits_{n\to\infty} a_n = \ell_1$ and $\lim\limits_{n\to\infty} b_n = \ell_2 \neq 0$.
   Using multiplication rule of limits $\lim\limits_{n\to\infty} a_nb_n = \lim\limits_{n\to\infty} a_n. \lim\limits_{n\to\infty} b_n \implies \lim\limits_{n\to\infty} a_nb_n = \ell_1\ell_2$.
   Hence, statement 1 is true.
   Statement 2: Assume $a_n = b_n = (-1)^n$. As we know that $(-1)^n$ is not convergent.
   Observe, $a_nb_n = (-1)^{2n} = (1)^n = 1$ which is a constant sequence and so converging to 1.
   Hence, statement 2 is also true.
   Statement 3:
   Since, any terms of the sequence $a_n = c$, if we take n larger and larger, still $a_n = c$ i.e., as $n \to \infty, a_n = c$.
   Hence, $\lim\limits_{n\to\infty} a_n = c$ and so statement 3 is also true.
   Hence, option 1 is true.

2. Match the given functions in Column A with their types in column B and their graphs in Column C, given in Table M2W1P1.

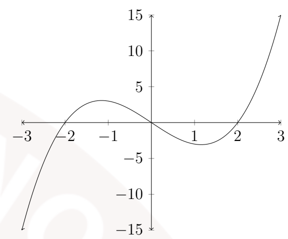
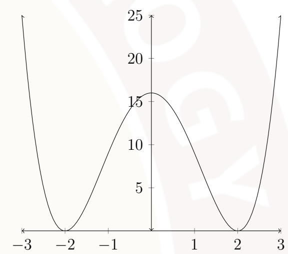
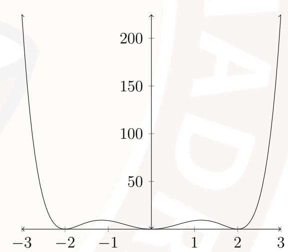
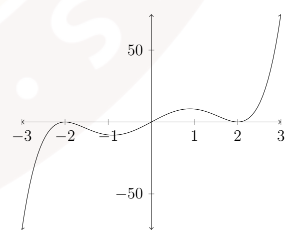
	Functions (Column A)		Types of functions (Column B)		Graphs (Column C)
i)	$f(x) = x^2(x-2)^2(x+2)^2$	a)	Polynomial of degree 3	1)	
ii)	$f(x) = x(x-2)(x+2)$	b)	Polynomial of degree 4	2)	
iii)	$f(x) = x(x-2)^2(x+2)^2$	c)	Polynomial of degree 5	3)	
iv)	$f(x) = (x-2)^2(x+2)^2$	d)	Polynomial of degree 6	4)	

Table: M2W1P1

◯ Option 1: i) → d) → 3), ii) → a) → 4), iii) → c) → 1), iv) → b) → 2).

◯ Option 2: i) $\to$ d) $\to$ 4), ii) $\to$ a) $\to$ 1), iii) $\to$ c) $\to$ 3), iv) $\to$ b) $\to$ 2).

◯ Option 3: i) $\to$ d) $\to$ 1), ii) $\to$ a) $\to$ 3), iii) $\to$ c) $\to$ 4), iv) $\to$ b) $\to$ 2).

◯ **Option 4:** i) $\to$ d) $\to$ 3), ii) $\to$ a) $\to$ 1), iii) $\to$ c) $\to$ 4), iv) $\to$ b) $\to$ 2).

Solution:
(i): $f(x) = x^2(x-2)^2(x+2)^2 = x^6 - 8x^4 + 16x^2$, which is of degree 6 polynomial and matches with (d) in column B. And having roots 0, 2 and -2 with all are even multiplicity. So the graph of the polynomial will touches the axis at 0, 2, and -2 and bounce back which follows only (3) in column C.
(ii): $f(x) = x(x-2)(x+2) = x^3 - 4x$, which is of degree 3 polynomial and matches with (a) in column B. And having roots 0, 2 and -2 with all are multiplicity 1. So the graph of the polynomial will cross the axis and look like straight line in small interval around 0, 2, and -2 which follows only (1) in column C.
(iii):$f(x) = x(x-2)^2(x+2)^2 = x^5 - 8x^3 + 16x^1$, which is of degree 5 polynomial and matches with (c) in column B. And having roots 0, 2 and -2 with 2 and -2 are even multiplicity and 0 is multiplicity 1. So the graph of the polynomial will cross the axis and look like straight line in small interval around 0 and will touches the axis at 2 and -2 and bounce back which follows only (4) in column C.
(iv): $f(x) = (x-2)^2(x+2)^2 = x^4 - 8x^2 + 16$, which is of degree 4 polynomial and matches with (b) in column B. And having roots 2 and -2 with all are even multiplicity. So the graph of the polynomial will touches the axis at 2 and -2 and bounce back which follows only (2) in column C.
Hence, option 4 is true.

2 Multiple Select Questions (MSQ)

3. Recall $n! = n.(n-1).(n-2)\ldots 3.2.1$. Define the sequence $\{a_n\}$ by $a_n = \frac{n}{(n!)^{\frac{1}{n}}}$. Note that $\lim a_n = e$. Which of the following option(s) is (are) true?

   ◯ Option 1: $\lim\limits_{n\to\infty} \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = 3e$

   ◯ Option 2: $\lim\limits_{n\to\infty} \dfrac{2n}{((2n)!)^{\frac{1}{2n}}} = 2e$

   ◯ **Option 3:** $\lim\limits_{n\to\infty} \dfrac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}} = e$

   ◯ **Option 4:** $\lim\limits_{n\to\infty} \ln \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = 1$

Solution:
Given $a_n = \frac{n}{(n!)^{\frac{1}{n}}}$ and $\lim a_n = e$.

Observe that, in option 1, in option 2 and in option 3, $\left\{\frac{3n}{((3n)!)^{\frac{1}{3n}}}\right\}, \left\{\frac{2n}{((2n)!)^{\frac{1}{2n}}}\right\}$, and $\left\{\frac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}}\right\}$ are the subsequences of the sequence $\{a_n\}$ and every subsequence of the sequence a_n converges to the same limit . Hence, $\lim\limits_{n\to\infty} \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = e$, $\lim\limits_{n\to\infty} \dfrac{2n}{((2n)!)^{\frac{1}{2n}}} = e$, and $\lim\limits_{n\to\infty} \dfrac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}} = e$

In option 4, as know we that if $a_n \to a$ and $a_n > 0$ for all $n \in \mathbb{N}$, and $a, c > 0$, then $log_c(a_n) \to log_c(a)$.

So, $\lim\limits_{n\to\infty} \ln \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = \lim\limits_{n\to\infty} \ln e = 1$.

4. The graph of some function is drawn below. Choose the set of correct statements about it.

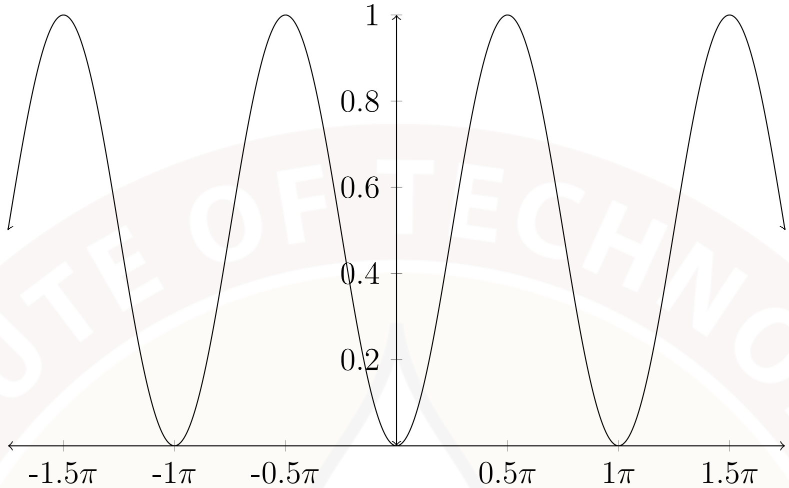


Figure: M2W1G1

- ◯ Option 1: Limit of the function as x tends to 0 is 1.
- ◯ **Option 2:** Limit of the function as x tends to 0 is 0.
- ◯ Option 3: Limit of the function as x tends to 0 is undefined.
- ◯ Option 4: There is a (unique) tangent at the point $x = \pi$, but not at $x = -\pi$.
- ◯ **Option 5:** There is a (unique) tangent at $x = \pi$, as well as at $x = -\pi$.
- ◯ Option 6: The given function is monotonically increasing in the interval $[-0.5\pi, 0]$.
- ◯ **Option 7:** The given function is monotonically decreasing in the interval $[-0.5\pi, 0]$.

Solution:
First, observe that as x approaches 0 from the left side or the right side, the value of the function in graph approaches 0. Hence limit at 0 exists and equal to 0. Hence, option 1 and option 3 are not true and option 2 is true.
Second, observer that x- axis touches the graph of the function at 0, 0.5π, -0.5π, π, $-\pi$, 1.5π, and -1.5π, so there is a unique tangent at the point $x = \pi$ as well as $x = -\pi$ which is x-axis. Hence, option 4 is not true and option 5 is true.
Third, observe that in the interval $[-0.5\pi, 0]$, the value of the function decreases from 1 to 0. Hence, option 6 is not true and option 7 is true.

5. Depending on the graphs given below, predict which do not have a (unique) tangent at the origin (i.e., $(0, 0)$) ?

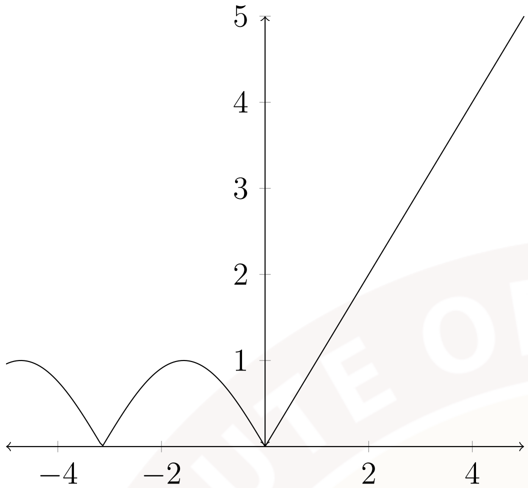


Figure: Curve 1

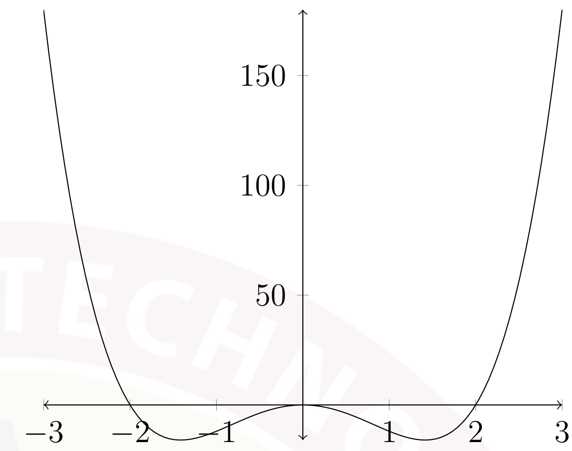


Figure: Curve 2

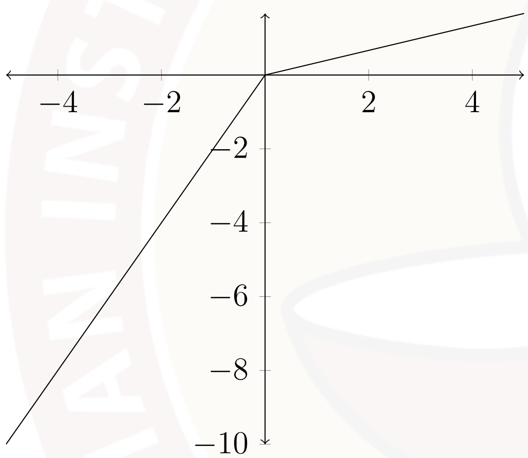


Figure: Curve 3

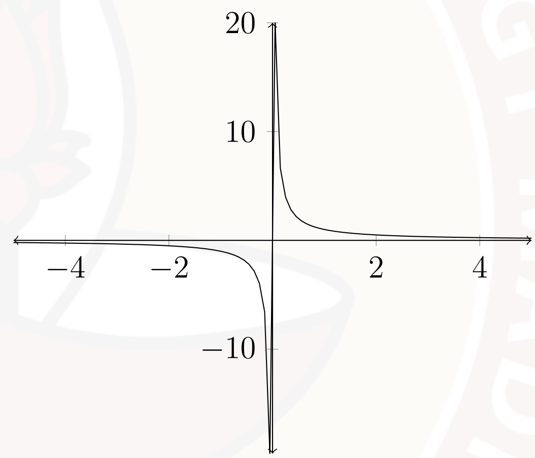


Figure: Curve 4

- ◯ **Option 1:** Curve 1
- ◯ Option 2: Curve 2
- ◯ **Option 3:** Curve 3
- ◯ **Option 4:** Curve 4

Solution:

Observe that in curve 1 and curve 3, at the origin if we approach from the left side, then there can be a tangent (the instantaneous direction of the curve) and if we approach from the right, then there can be another different tangent i.e., there is no unique tangent to the curve 1 and curve 3 at the origin. In curve 4, as x approaches the origin, the value of the function approaches to $-\infty$ from the left side and the value of the function approaches to ∞ from the right side i.e., function is not defined at the origin, so there is no need to talk about the instantaneous direction of the curve 4. Hence, there is no tangent to curve 4 at the origin. But in curve 2, $x-$ axis touches the curve 2 at the origin, so there is a (unique) tangent to the curve 2 at the origin.

3 Numerical Answer Type (NAT)

6. Consider the following sequences $\{a_n\}, \{b_n\},$ and $\{c_n\}$:

$$a_n = \frac{1}{n^2}$$
$$b_n = 0$$
$$c_n = 1 - \frac{1}{n}$$

How many among the three sequences given above are subsequences of the sequence $\{\frac{1}{n}\}$? (Answer 1)

Solution:

A sequence which is obtained from the main sequence by removing finite or infinite terms from the main sequence, is called a subsequence of the sequence.

Observe that 0 and $\frac{2}{3}$ are the terms in sequence $\{b_n\}$ and $\{c_n\}$ for all value of n in b_n and for $n = 3$ in c_n respectively which are not in the sequence $\left\{\frac{1}{n}\right\}$. Hence the sequences $\{b_n\}$ and $\{c_n\}$ are not subsequences of the sequence $\left\{\frac{1}{n}\right\}$.

But, the sequence $\{a_n\}$, $a_n = \frac{1}{n^2}$, observe that if n is a natural number then n^2 is also a natural number. And all terms of the sequence $\left\{\frac{1}{n}\right\}$ are $\frac{1}{n}$ for some $n \in \mathbb{N}$. So the sequence $\{a_n\}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}$. Hence, answer is 1.

7. Consider the sequence $\{a_n\}$ given by $a_n = n^{\frac{1}{n}}$, which is known to converge to 1. What will be the limit of the sequence $\{b_n\}$ given by $b_n = n^{\frac{2}{n}} + 2n^{\frac{1}{n}} - 1$? (Answer 2)

Solution:

We know that if $a_n \to a$, then $f(a_n) \to f(a)$, where f is any polynomial function.

Observe that $b_n = f(a_n)$, where $f(x) = x^2 + 2x - 1$. Given that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ so $\lim_{n\to\infty} b_n = f(1) = 1 + 2 - 1 = 1$.

Hence, answer is 2.

4 Comprehension Type Question:

Suppose there are three schemes available for renting a studio room given in the table below.

Schemes	Rent (in ₹) for t hours
Scheme A	$p_1(t) = 100\lfloor t \rfloor + 200$
Scheme B	$p_2(t) = 100\lceil t \rceil + 200$
Scheme C	$p_3(t) = 100t + 200$

Table: M2W1P2

Where $\lfloor t \rfloor$ denotes the largest integer lesser than or equal to t, and $\lceil t \rceil$ denotes the smallest integer greater than or equal to t. Answer questions 7,8, and 9 using the given information.

8. If Rana wants to book the studio for 2.6 hours, which Scheme should he choose to avail the studio in minimum cost? (MCQ)

   ◯ Option 1: For 2.6 hours, costs for Scheme A and Scheme C are the same, and that is the minimum.

   ◯ **Option 2:** For 2.6 hours, cost for Scheme A is the minimum.

   ◯ Option 3: For 2.6 hours, cost for Scheme B is the minimum.

   ◯ Option 4: For 2.6 hours, costs for Scheme B and Scheme C are the same, and that is the minimum.

   ◯ Option 5: For 2.6 hours, cost for Scheme C is the minimum.

   Solution : Substitute $t = 2.6$ in $p_1(t), p_2(t)$ and $p_3(t)$, we get,
   $p_1(2.6) = 100\lfloor(2.6)\rfloor + 200 = 100 \times 2 + 200 = 400$
   $p_2(2.6) = 100\lceil(2.6)\rceil + 200 = 100 \times 3 + 200 = 500$
   $p_3(2.6) = 100 \times 2.6 + 200 = 460$
   Observe that for 2.6 hours, cost for Scheme A is the minimum and none of the cost for Scheme A, B and C is same. Hence, option 2 is true.

9. Consider the following graphs where X-axis denotes the time in hours and Y-axis denotes the cost for rent of the studio :

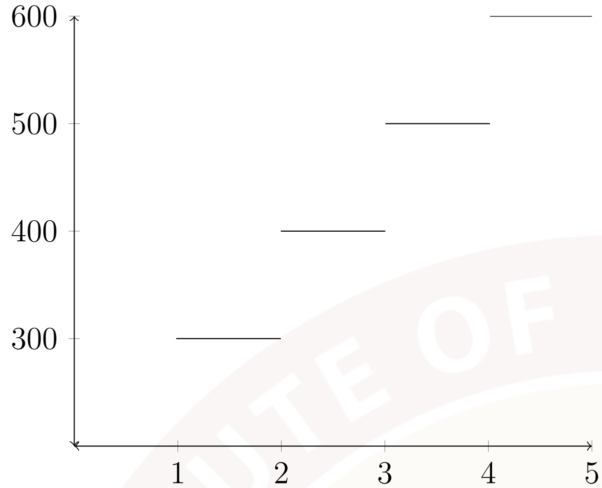


Figure: 1

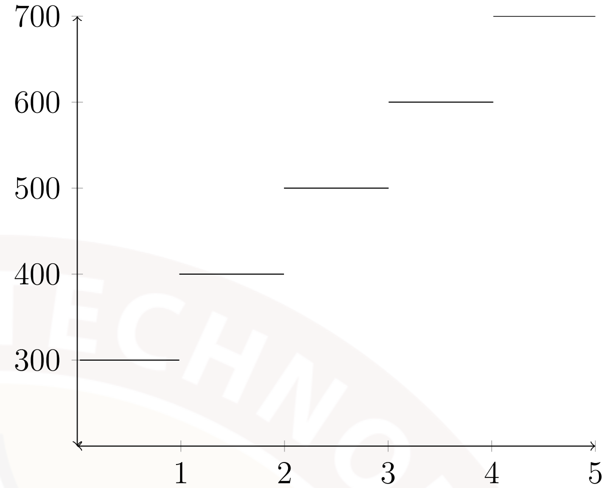


Figure: 2

Choose the correct option. (MCQ)

- ◯ **Option 1:** Figure 1 represents Scheme A, Figure 2 represents Scheme B.
- ◯ Option 2: Figure 2 represents Scheme A, Figure 1 represents Scheme B.
- ◯ Option 3: Figure 1 represents Scheme A, but Figure 2 does not represent Scheme B.
- ◯ Option 4: Figure 2 represents Scheme A, but Figure 1 does not represent Scheme B.

Solution:
As we know function $\lfloor t \rfloor$ denotes the largest integer lesser than or equal to t, and $\lceil t \rceil$ the smallest integer greater than or equal to t. From question 9, for Scheme A, $p_1(2.6) = 100\lfloor(2.6)\rfloor + 200 = 100 \times 2 + 200 = 400$, similarly we can check for other points. So the Figure 1 represents Scheme A.
For Scheme B, $p_2(2.6) = 100\lceil(2.6)\rceil + 200 = 100 \times 3 + 200 = 500$, similarly we can check for other points. So the Figure 2 represents Scheme B.
Hence, option 1 is true.

10. Choose the set of correct options. (MSQ)

- ◯ **Option 1:** The tangent to the function $p_1(t)$ which represents Scheme A does not exist at 1.
- ◯ **Option 2:** The tangent to the function $p_1(t)$ which represents Scheme A does not exist at a, for any natural number a.
- ◯ Option 3: The tangent to the function $p_3(t)$ which represents Scheme C does not exist at 1.
- ◯ **Option 4:** The tangent to the function $p_2(t)$ which represents Scheme B does not exist at a, for any natural number a.

Solution:
From question 10, the Figure M2W1PS1 represents the graph of p_1 and the Figure M2W1PS2 represents the graph of p_2. Observe that, p_3 represents a line with slope 100, so the Figure M2W1PS3 represents the graph of p_3.

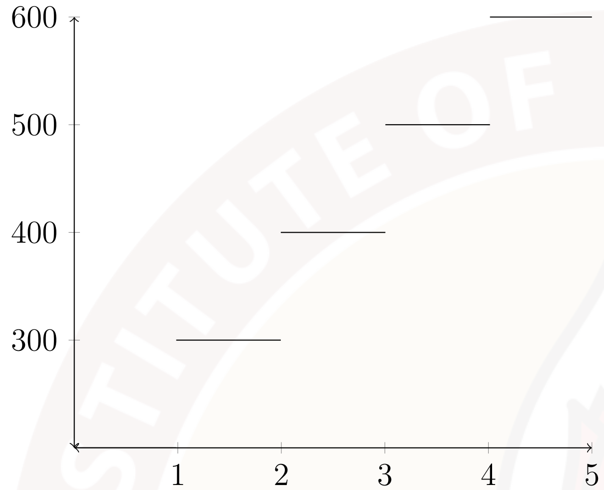


Figure: M2W1PS1

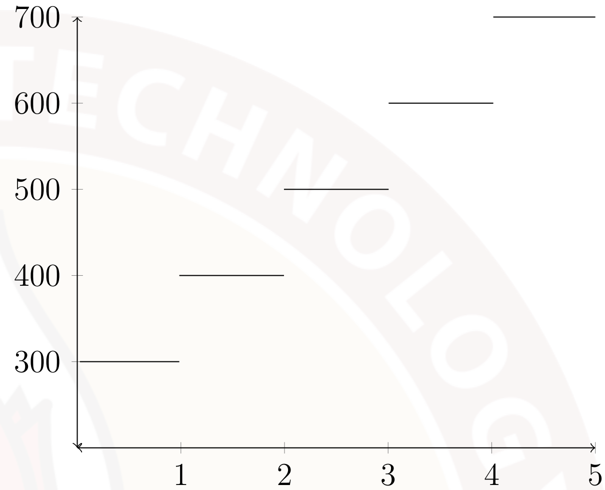


Figure: M2W1PS2

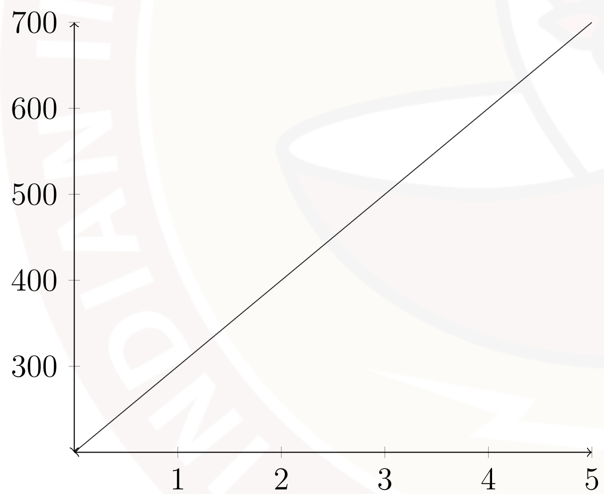


Figure: M2W1PS3

Now, for any natural number a, $p_1(a) = 100a + 200$ which is a natural number and the instantaneous direction of the curve (tangent) represented by the function $p_1(t)$) from left given by the line $y = 100a + 200$ and the instantaneous direction of the curve(tangent) from right given by the line $y = 100(a + 1) + 200$ i.e. there is no unique tangent line at a, i.e., there is no tangent to the function $p_1(t)$ at any natural number a.
for example, the instantaneous direction of the curve(tangent) at 1 from left given by

the line $y = 200$ and from right given by the line $y = 300$. So there is no tangent to the function $p_1(t)$ at 1.
Hence, option 1 and option 2 are true.
Similar things will happen with the function $p_2(t)$. So we can conclude that there is no tangent to the function $p_2(t)$ at any natural number a.
Now, the Figure M2W1PS3 represents the graph of the function $p_3(t)$. Observe that the instantaneous direction of the curve (tangent) represented by the function $p_3(t)$ exists at any point which is the line $y = 100t + 200$ itself. So the tangent to function $p_3(t)$ exists at 1 also.
Hence, option 3 is not true.