IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2

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Lengths and angles

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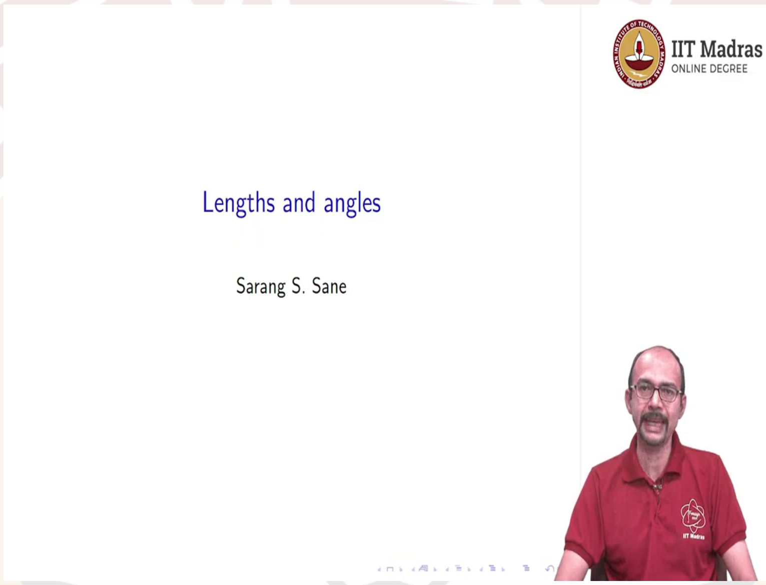

Hello, and welcome to the Maths 2 component of the online degree program on data science. In this video, we are going to talk about the notion of lengths and angles. So, some of this may have already been seen in Maths 1 in some form and some of this may be really easy and something you have seen before. But let us recall quickly what is the notion of lengths and angles.

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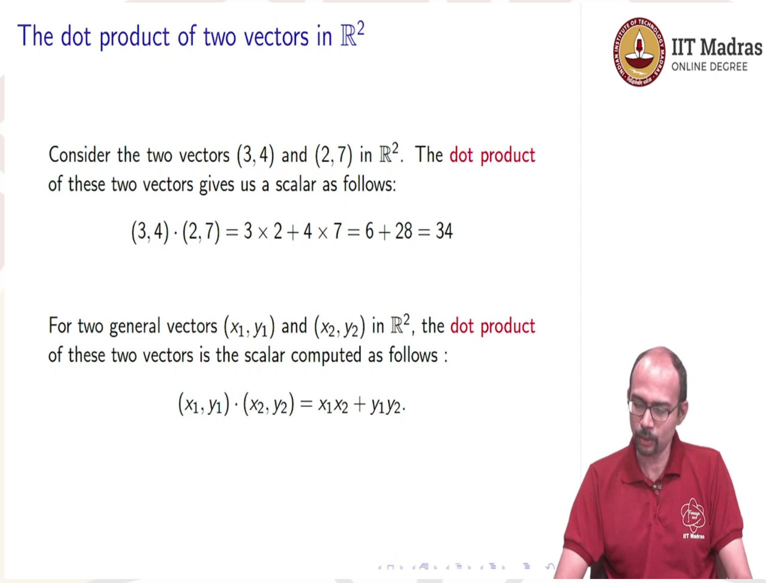

So, let us talk about the dot product of two vectors in R^2. So, consider the vectors, let us start with an example. So, consider the two vectors (3, 4) and (2, 7) in R^2. So, the dot product of these two vectors gives us a scalar as follows, scalar is a real number. So, you have (3, 4).(2, 7). This is, so you take the first components of each of these vectors, multiply them. You take the second components of each of these vectors, multiply them, and you add them. So, this is going to give you, 3 x 2 + 4 x 7 = 6 + 28 = 34.

Let us generalize this to any two vectors in R^2. So, if you have (x_1, y_1) and (x_2, y_2) in R^2, the dot product of these two vectors is the scalar real number computed as follows:

$(x_1, y_1).(x_2, y_2) = x_1.x_2 + y_1.y_2$. So, you multiply the first components, you multiply the second components, and then you add those.

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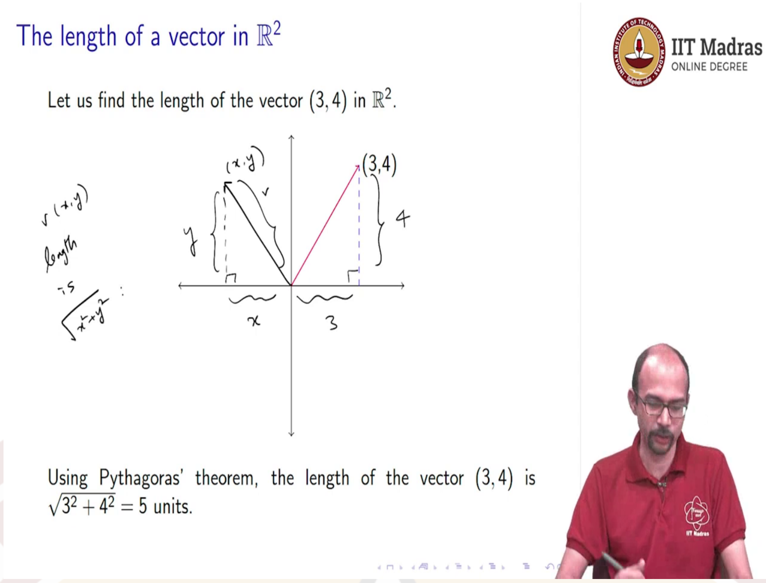

So, let us now ask what is the length of a vector in R^2? So, let us consider (3, 4) which we saw in the previous example while computing dot products. So, if you want to compute the length of this vector, how do we do it? Well, we use the Pythagoras theorem. So, you drop a perpendicular down to, let us say, the x-axis.

So, then if you do that, you get a right angled triangle. So, using this right angled triangle, this is the right angled triangle. And because this is a right angled triangle and the, since you have dropped a perpendicular, this length here is 3, this length here is 4, this is the x-coordinate, this part is a y-coordinate.

And so if you want to compute the length of the red line, that is the length of the vector, you use Pythagoras theorem, which says that the length of the vector (3,4) is the square root of, so the hypotenuse squared is equal to the sum of the squares of the sides that is a Pythagoras theorem. So, the length of the hypotenuse is the square root of the sum of the squares of the sides.

So, in this case, it is $\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25}$. So, the square root of 25 is 5. And we do not know here what units we are doing it in. So, whatever units we are doing it in. So, if this one is corresponding to centimeters, it is in centimeters, if one is corresponding to kilometer, it is in kilometers, if one is corresponding to miles, it is in miles and so on, so whatever units are being used for drawing this R^2.

So, the units are not important as far as this course is concerned. Of course, if you start doing physics and so on that matters or if you do real life problems, it matters. So, the main point here is that you have to remember that this, if you do a general x, y maybe we can do it right here, if I have (x,y) so if I have something like this, this you drop a perpendicular here, this represent, this is exactly x and this part is exactly y. So, it gives you a right-angled triangle. And so the length of this vector, if I call this vector v, so v which is (x,y), its length is $\sqrt{x^2+y^2}$. So, this is really what we are seeing here.

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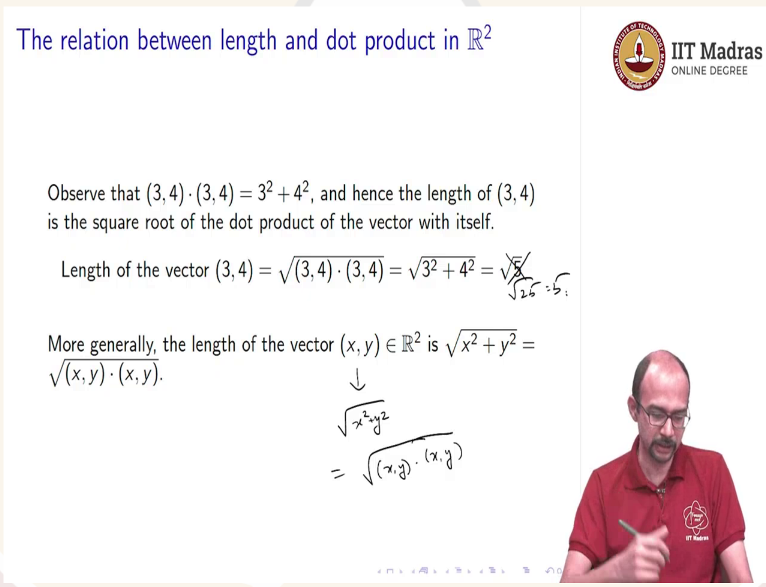

So, observe that, let us link this up with dot products. Observe that (3,4).(3,4) = $3^2 + 4^2$, which we saw was exactly what was in the square root part when we computed length. So, the length of (3,4) is the square root of the dot product of the vector with itself. So, the length of the vector (3, 4) = $\sqrt{(3,4).(3,4)} = \sqrt{3^2+4^2} = \sqrt{25} = 5$.

So, in general, of course, this is what I just said. The length of the vector is $\sqrt{x^2+y^2}$, which is exactly what you get by doing $\sqrt{(x,y).(x,y)}$. So, the length of the vector is given by taking the square root of the dot product with itself. This is the take home. So, the length and the dot product are related in a very intimate fashion.

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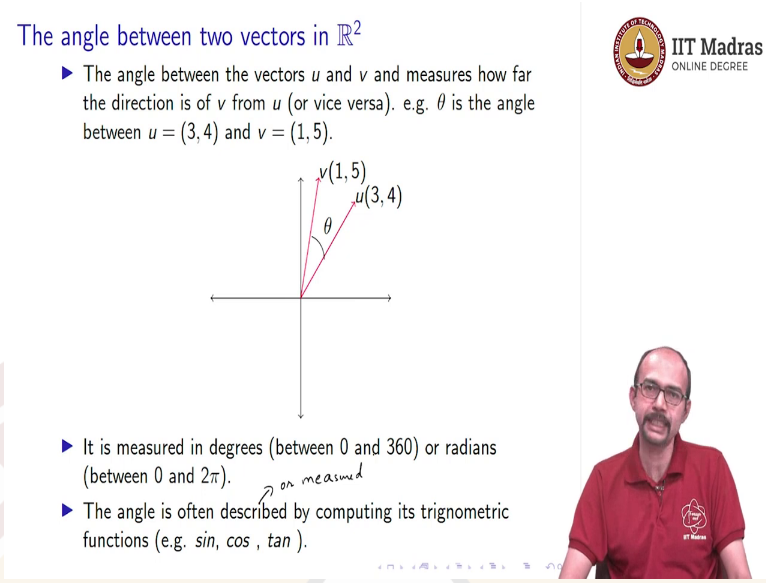

Let us look at the angle now. So, the other thing we want to study in this video is angles. So, the angle between two vectors u and v, so this is a measure of, sorry, this is not, there is no and here. The angle between the vectors u and v measures how far the direction is of v from u. So, if you have the angle, this will be a small angle, as compared to this, which is a large angle. So, in our previous picture or well, so there is a picture coming up.

So, we have our old friend, (3,4), and we have another vector (1,5). So, the angle is this thing θ here. So, it measures how far in terms of directions these two vectors are from each other, so how far the directions of these vectors are from each other. Now, we have seen this idea of directions and lengths which we called magnitudes in our very first video on vectors, so just recall that if you do not remember.

So, the angle is measured in degrees or radians. So, if it is in degrees, it is between 0 and 360. So, 0 is itself. And as you increase and go up to 360, it comes all the way back and then 360 is again itself. So, this is like a clock. If you have 9 am and 9 pm, the clock shows you the same thing, but 12 hours away lab, so that is what this 360 is saying. If, 360 means you have taken around and come back. But as far as the angle is concerned, it is 360 and 0 are telling you the same things.

Or it is, it can be measured in terms of radians. So, radians is more of a, it has something to do with the length of the arc. So, that is measured between 0 and 2π. So, again, 0 is like this, meaning they are the same. And 2π is when you go all round and come back. So, just π which is $\frac{2\pi}{2}$, so half of 2π is where they are like this. So, the two vectors are parallel to each other, but pointing in different directions, like the, from the center of the earth, the North Pole and the South Pole. They are in different directions. So, it is measured in degrees or radians.

So, this also tells you what is the relationship between degrees and radians, because 2π radians is equal to 360 degrees. So, from there you can say, in general, if you have an angle of θ degrees, what should be the corresponding radians. The angle is often described by computing or measured, maybe we should say, by computing its trigonometric functions described or I want to say or measured by computing its trigonometric functions, so which is sine, cosine, tan, etc.

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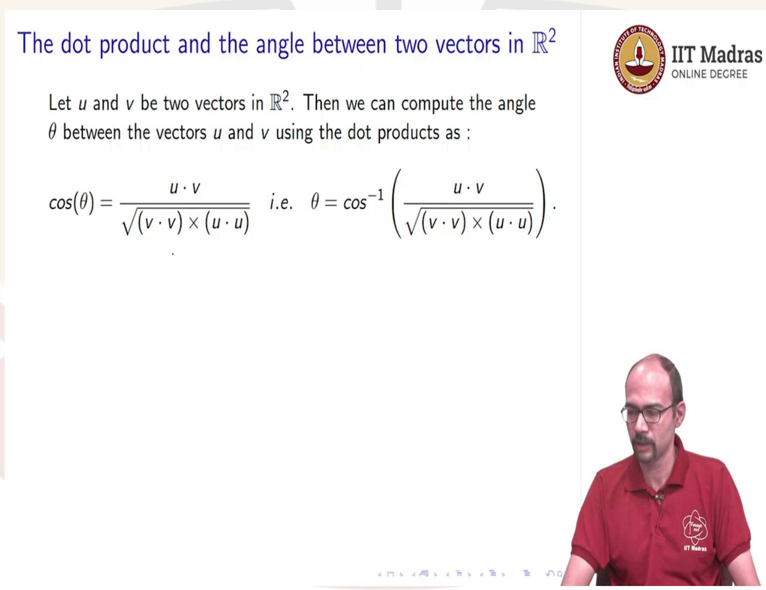

What is the relation between dot product and angles? So, if you have two vectors in R^2, we can compute the angle between which, let us call it θ, between the two vectors by using dot products as $\cos(\theta) = \frac{u.v}{\sqrt{(v.v) \times (u,u)}}$. So, this times is these are real numbers, remember. So, this is just multiplication in real numbers. So, and then you can take the inverse cosine. And we usua lly think of it as being between 0 and 2π again.

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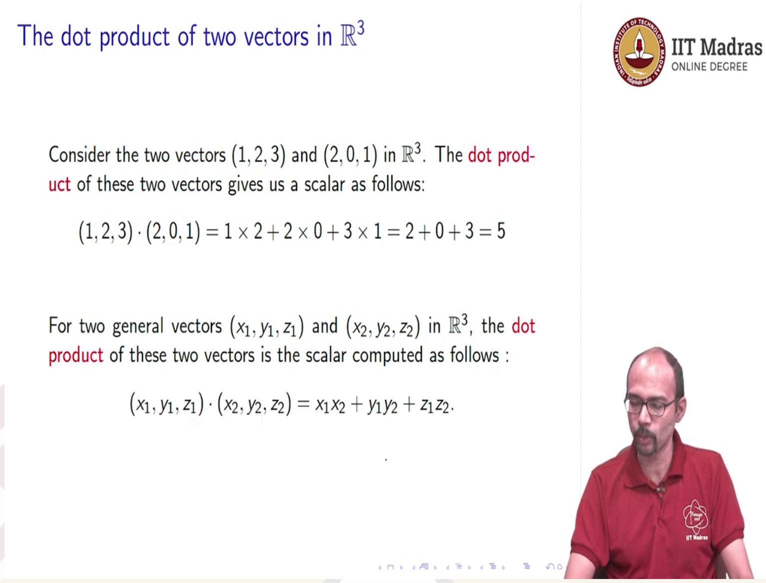

So, let us do an example. So, consider the two vectors. So, this is in R^3. So, now, we are talking about R^3. So, consider the two vectors (1, 2, 3) and (2, 0, 1) in R^3. So, the dot product of these two vectors gives us a scalar as follows. So, (1, 2, 3).(2, 0, 1) is you take component wise multiplication and then you add them. So, 1x2+2x0+3x1 = 2 + 3 = 5. So, for two general vectors, you do the same thing (x_1, y_1, z_1),(x_2, y_2, z_2) you do $(x_1, y_1, z_1).(x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2$.

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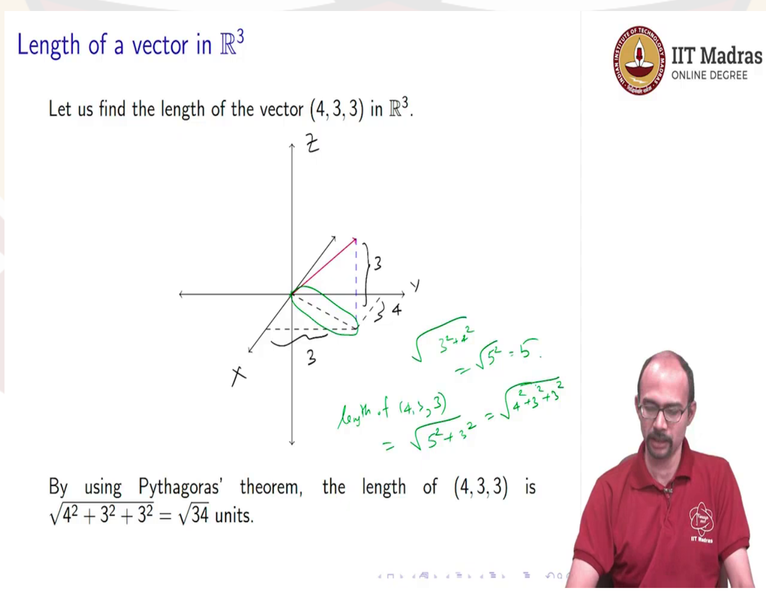

So, now, let us find the length of the vector, let us say, (4, 3, 3) in R^3. So, to do this we will first draw the xyz plane. So, let us say this is x, y and this z. This is our right handed coordinate system. This is not y, this is y and that is z. So, now, if you take x, y, z you drop a perpendicular, so this is 4, 3, 3.

So, we will drop a perpendicular down to the xy plane, then join that to the origin, and then drop further perpendiculars from this point to the x-axis and the y-axis, respectively. So, what you really have is that this is a perpendicular, this is exactly z. So, in this case, it is 3. This is 3. And then this is exactly the length of y which is again 3, the y-coordinate. And this is 4 or, because this is parallel to this, which is 4, this is parallel to this, which is 3, and this is parallel to this, which is 3.

So, now how do I compute the length? Again, you use Pythagoras theorem. We already know that this line in the middle here, this line you can compute by Pythagoras theorem as, $\sqrt{3^2+4^2}=\sqrt{5^2}=5$. And then again, this is the right angle triangle, the green thing, the blue line and the red vector itself. So, the red, blue and green form of right angle triangle and the red is the hypotenuse, which is exactly the length of the vector (4, 3, 3).

So, length of 4, 3, 3 is root of 5^2, which is the length of the line circled by the green line, plus 3^2, which is the length of the blue line. But 5^2, I can write as 3^2+4^2, which was the original x and y coordinates. So, this is $3^2+4^2+3^2$ or maybe we should write it as 4^2+3^2 to keep with the notation, and then you can compute what that is, in this case, it gives you $\sqrt{34}$ units, so 16+ 9+9.

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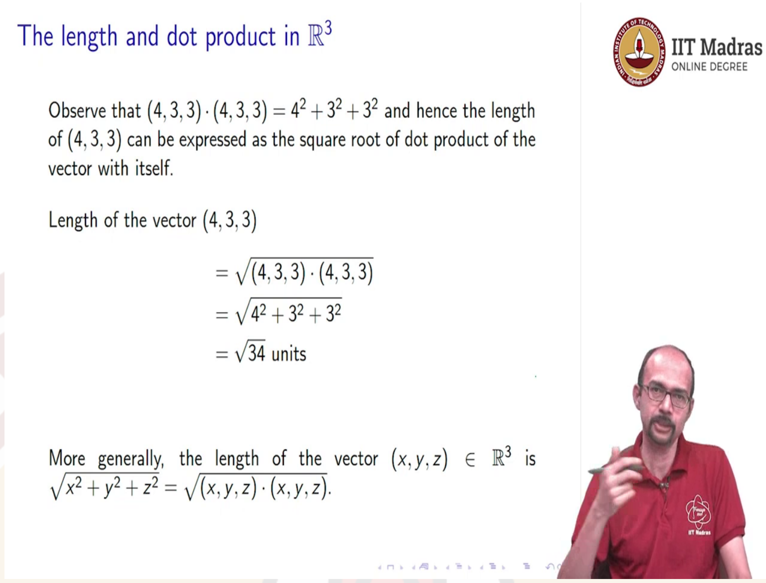

So, observe again that (4, 3, 3).(4, 3, 3) is exactly $4^2+3^2+3^2$. So, the length of (4, 3, 3) can be expressed as the square root of the dot product of the vector with itself. This is the same computation. And so the general situation is where you do x, y, z and the same reasoning tells you that the length of x, y, z is the square root of the dot product of x, y, z with itself. So this tells you the relationship between the dot product and lengths in, so we have seen the same relation in R^2, we have seen this in R^3 and certainly the idea is that this holds for any R^n, the proof is the same.

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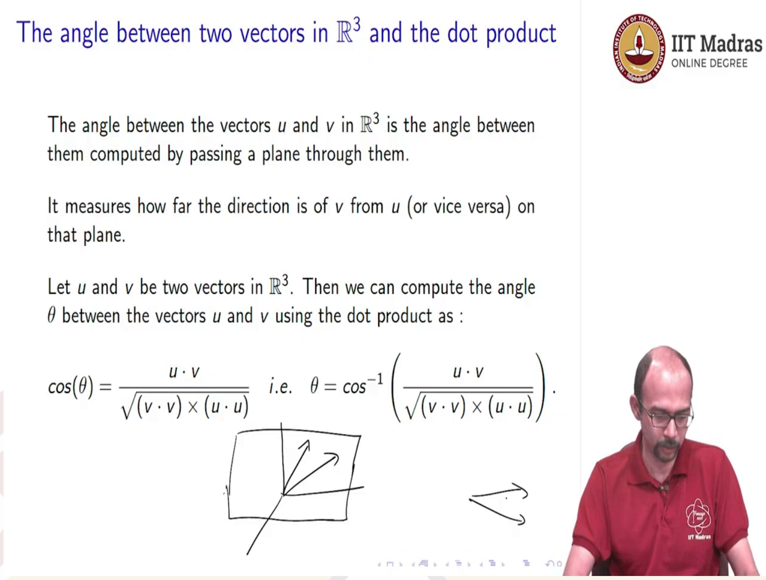

Let us now ask for what happens to the angle. So, the angle between two vectors u and v in R^3 is the angle between them computed by passing a plane through them. So, if you have two vectors u and v, you look at, there is a unique plane passing through those two vectors. Draw that plane. And now once you are in the plane, it is like you are in R^2. And in R^2 we know how to compute the angle between two vectors.

And it, so what does this angle do? It measures the direction, how far the direction is of v from u or u from v on that plane. So, we have seen this formula before in R^2 and the same idea holds for R^3. So, if u and v are two vectors in R^3, we can compute the angle θ between these vectors by using the dot product as follows. So, $\cos(\theta)= \frac{u.v}{\sqrt{(v.v) \times (u.u)}}$ or you can write the terms of the inverse cosine of this number that we have just seen.

So, let us maybe try to analyze why this is happening. So, the reason this is happening is because, if you have two vectors, so you pass some plane through them, I mean, there is a unique plane. So, you take this plane. And now on this plane, these vectors are like two ordinary vectors in R^2. So, if you want instead of looking at this plane you move this plane, rotate it, so that it becomes like the xy plane. And do not disturb those vectors. When you just do a rotation do not do any scaling or anything else, just do a rotation.

So, then these vectors are going to look like, on the xy plane they will maybe look like this. And you can compute the angle between them. And that is exactly what this angle is. And you can see that this will match with the formula that you have written down. That is the idea.

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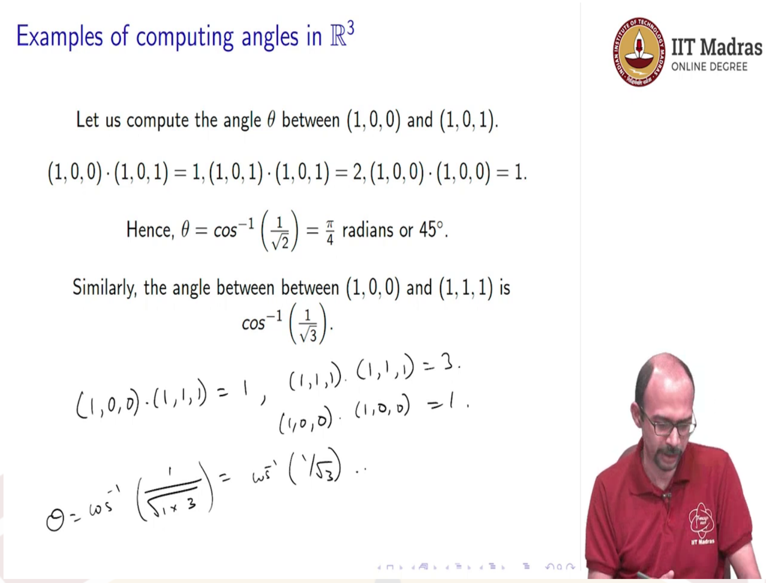

So, let us compute the angle θbetween these two vectors, (1, 0, 0) and (1, 0, 1). If our intuition is correct, we kind of know what the angle here is. Both of these are on the xz plane, y is 0, so xz plane. So, these are on the xz plane. And on the xz plane, so you just knock out the y coordinate, so on the xz plane, this is like (1, 0) and (1, 1). So, it is (1, 0) and (1, 1). So, if you think of the xz plane now as the xy plane this is (1, 0) and (1, 1) and we know how much the angle is. This angle is exactly 45^0 or $\frac{\pi}{4}$ radians, whichever you prefer.

So, let us see if the answer that we get by doing that computation is the same (1, 0, 0).(1, 0, 1)=1 , (1, 0, 1).(1, 0, 1)=2, (1, 0, 0).(1, 0, 0)=1. And so, we get

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2\times 1}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^{o}$$

or $\frac{\pi}{4}$ radians. So, this last thing is something we are getting from a cosine table. You can look those up.

So, let us look at this second example, the angle between (1, 0, 0) and (1, 1, 1). So, for (1, 0, 0) and (1, 1, 1) let us compute these numbers, (1, 0, 0).(1, 1, 1)=1 , (1, 1, 1).(1, 1, 1)=3, (1, 0, 0).(1, 0, 0)=1. So, we get $\cos^{-1}(\theta)$ = $\cos^{-1}\left(\frac{1}{\sqrt{1\times 3}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, and you can look up a table to get what the exact value is. It is something like 54 degrees, if I remember right.

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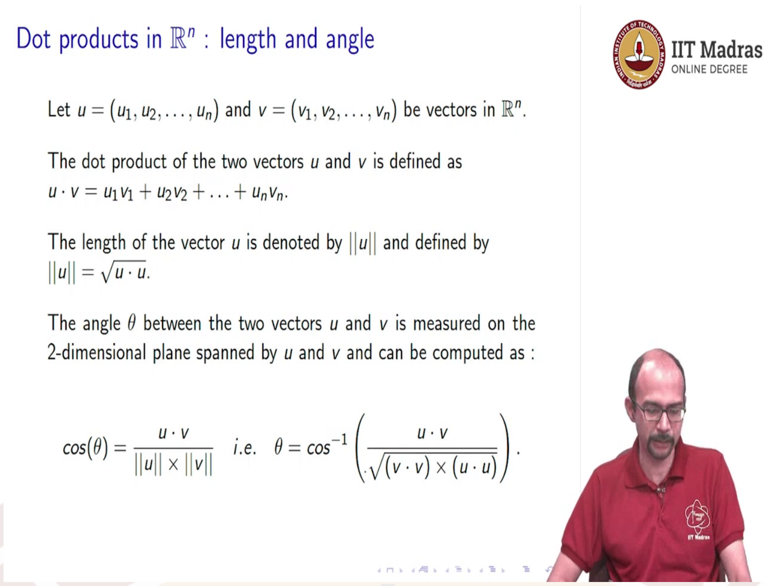

And now we can do the general idea of, make the general definition that is, if you have two vectors, u and v in R^n, the dot product is defined as component wise multiplication and then addition, so $u_1v_1 + u_2v_2 + \ldots + u_nv_n$. The length of the vector is denoted by these two bars. So, it is called a norm, which we are going to study in the next video.

So, it is given by the root of u dot product with itself. And the angle between two vectors, again, you pass a plane through those two vectors, there is a unique plane passing, and you can then rotate it so that it becomes parallel, it becomes the xy plane and then you can measure, and it turns out that this is exactly the same as doing $\cos(\theta) = \frac{u.v}{||u||.||v||}$ which, in other words, $\frac{u.v}{\sqrt{(v.v) \times (u.u)}}$. And then you can write it in terms of the inverse.

So, we actually have seen a little bit of this before lengths and angles in our video on vectors. But this is a reminder and it also is a reminder about what are dot products and how they help in computing the lengths and angles between vectors. Thank you.