Affine subspaces and affine mappings

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However the vector v is not unique and in fact can be any vector in L.

$L = v + U$

$= v' + U'$

$v' \in L$

$\Rightarrow v' = v + u_1$ where $u_1 \in U$

$\Rightarrow v' - v = u_1 \in U$

$v - v' \in U'$

$\Rightarrow v' - v \in U'$

$v \in L$, $v' \in L$

Let $u \in U$

Then $v + u \in L = v' + U'$

$v + u = v' + u'$ for some $u' \in U'$

$\Rightarrow v + u = v' + u'$

$\Rightarrow u = \underbrace{(v' - v)}_{} + \underbrace{u'}_{} \in U'$

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$L = v + U$
$= v' + U'$, $\}$ $v - v' \in U$ & U'
$\Rightarrow U = U'$.

Affine subspaces are thus translates of a vector subspace of V.

Visual examples

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Affine subspaces in $\mathbb{R}^2$

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- Points

Affine subspaces in $\mathbb{R}^2$

- Points

- Lines

Affine subspaces in $\mathbb{R}^2$

- Points
- Lines
- the entire plane $\mathbb{R}^2$

Affine subspaces in $\mathbb{R}^2$

$L = \{(x,y)\} = (x,y) + \{(0,0)\}$

is a translate of $y = mx$

- Points

$L \equiv y = mx + c$

$(0,c) \in L$

$L = \underbrace{(0,c)}_{v} + \underbrace{\{(x,mx) \mid x \in \mathbb{R}\}}_{U}$

- Lines

- the entire plane $\mathbb{R}^2$

$\underbrace{\mathbb{R}^2}_{L} = \underbrace{(0,0)}_{v} + \underbrace{\mathbb{R}^2}_{U}$

A subset which is not an affine subspace : the parabola $y = x^2 + 1$ or the curve $y^2 = x^3$.

Affine subspaces in $\mathbb{R}^3$

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Affine subspaces in $\mathbb{R}^3$

$v + \lambda(0,0)$

$v + \lambda v_1$

$v + \lambda_1 v_1 + \lambda_2 v_2$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$.

- Points
- Lines
- Planes
- the entire space $\mathbb{R}^3$

Example: Two-dimensional affine subspaces in $\mathbb{R}^3$ can be expressed as

$$l = v + \boxed{\lambda_1 v_1 + \lambda_2 v_2}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and v, v_1, v_2 are vectors in $\mathbb{R}^3$.

$V = \{ \lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_1, \lambda_2 \in \mathbb{R} \}$

$= \mathrm{Span}(v_1, v_2)$

The solution set to a system of linear equations

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- $b \notin$ column space of A : In this case, $Ax = b$ does not have a solution, so the solution set is the empty set.
- $b \in$ column space of A : In this case, the solution set L is an affine subspace of $\mathbb{R}^n$. Specifically, it can be described as $L = v + \mathfrak{N}(A)$ where v is any solution of the equation $Ax = b$.

Affine mappings of affine subspaces

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For a linear transformation $T : U \to U'$ and fixed vectors $v \in L$ and $v' \in L'$, an affine mapping f can be obtained by defining $f(v + u) = v' + T(u)$, and in fact every affine mapping is obtained in this way.

$g(u) = f(u+v) - f(v)$

$g(u+u') \stackrel{?}{=} g(u) + g(u')$

$f(u+u'+v) - f(v) = f(u+v) - f(v) + f(u'+v) - f(v) \Rightarrow$

$g(u') = f(u'+v) - f(v)$

$g(u+u') = f(u+u'+v) - f(v)$

$f(u+u'+v) = f(u+v) + f(u'+v) - f(v)$

$\Rightarrow f(u+u'+v) + \boxed{f(v)} = f(u+v) + f(u'+v)$

An example and an important special case

Let $T(x, y, z) = (2x + 3y + 2, 4x - 5y + 3)$. Then this is an affine mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$T(x, y, z) = \underbrace{(2, 3)}_{v'} + \underbrace{(2x + 3y,\ 4x - 5y)}_{\text{Lin. trans.}}$$

An example and an important special case

Let $T'(x, y, z) = (2x + 3y + 2, 4x - 5y + 3)$. Then this is an affine mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$T'(x,y,z) = \underbrace{(2,3)}_{w} + \underbrace{(2x+3y,\ 4x-5y)}_{T(x,y,z)}$$

$T(x,y,z)$ is a lin. trans.

Let $T : V \to W$ be a linear transformation and $w \in W$, then the mapping

$$T' : V \to W$$
$$T'(v) = w + T(v)$$

is an affine mapping from V to W.

Thank you