Week 1-3
Mathematics for Data Science - 2
Practice Assignment

1. Let $\{a_n\}$ be a sequence defined as $a_n = \dfrac{\sin n\pi + \cos n\pi}{n^2}$, then which of the following option(s) is (are) true?

   [**Hint:** Use the sandwich theorem.]

   ○ Option 1: $\{a_n\}$ is divergent.

   ○ **Option 2:** $\{a_n\}$ is convergent.

   ○ **Option 3:** Limit of $\{a_n\}$ is 0.

   ○ Option 4: Limit of $\{a_n\}$ is $+\infty$.

   ○ Option 5: Limit of $\{a_n\}$ is π.

2. Consider a sequence $\{a_n\}$ defined as $a_1 = 1$ and $a_{n+1} = \sqrt{\alpha a_n}$, where $\alpha > 2$. Assume that $\lim\limits_{n\to\infty} a_n = \alpha$. Which of the following option(s) is (are) true?

   ○ **Option 1:** If $\alpha = 3$, then $\lim\limits_{n\to\infty}(3a_n^3 + 5a_n^2 - 1) = 125$.

   ○ **Option 2:** If $\alpha = 5$, then $\lim\limits_{n\to\infty}(2^{a_n} - 2a_n^2 - 5) = -23$.

   ○ Option 3: If $\alpha = 5$, then $\lim\limits_{n\to\infty}(2^{a_n} + 2a_n^2 - 5) = 23$.

   ○ **Option 4:** If $\alpha = 4$, then $\lim\limits_{n\to\infty}(\log_3(a_n + 4) - \log_{\sqrt{3}}(\frac{a_n}{2} + 2)) = -\log_3 2$.

   ○ Option 5: If $\alpha = 4$, then $\lim\limits_{n\to\infty}(\log_3(a_n + 4) - \log_{\sqrt{3}}(\frac{a_n}{2} + 2)) = \log_3 2$.

3. Which of the following option(s) is (are) true?

   [**Hint:** Use L'Hospital's rule]

   ○ Option 1: $\lim\limits_{x\to 0} \dfrac{e^{\frac{2}{x}}}{e^{\frac{2}{x}} + 2} = 0$.

   ○ Option 2: $\lim\limits_{x\to 0} \dfrac{e^{\frac{2}{x}}}{e^{\frac{2}{x}} + 2} = 1$.

   ○ **Option 3:** $\lim\limits_{x\to 0} \dfrac{e^{\frac{2}{x}} - e^{-\frac{2}{x}}}{e^{\frac{2}{x}} + e^{-\frac{2}{x}}}$ does not exist.

   ○ **Option 4:** $\lim\limits_{x\to\infty} x^{\frac{2}{x}} = 1$.

   ○ **Option 5:** $\lim\limits_{x\to 0} \left(\dfrac{2}{\sin x} - \dfrac{2}{x}\right) = 0$.

- ◯ Option 6: $\lim_{x\to 0}\left(\frac{2}{\sin x}-\frac{2}{x}\right)=1$.

4. Consider two functions $f(x)$ and $g(x)$ such that

$$f(x+y)=f(x)f(y) \text{ and } f(x)=1+g(x),$$

where $\lim_{x\to 0} g(x)=0$. Assume that g is differentiable and $\lim_{x\to 0} g'(x)=1$. Which of the following option(s) is (are) true?

[**Hint:** $\lim_{x\to a} f(x)=\lim_{h\to 0^+} f(a+h)$ (i.e., right limit) $=\lim_{h\to 0^-} f(a+h)$ (i.e., left limit) and use L'Hospital's rule.]

- ◯ Option 1: $\lim_{x\to 0} f(x)=0$.
- ◯ **Option 2:** $\lim_{n\to\infty} f(\frac{1}{n})=1$.
- ◯ **Option 3:** $f(x)$ is continuous for all $x\in\mathbb{R}$.
- ◯ Option 4: $f(x)$ is not differentiable for all $x\in\mathbb{R}$.
- ◯ Option 5: $f(x)$ is differentiable and $\lim_{x\to 0} f'(x)=0$.
- ◯ **Option 6:** $f(x)$ is differentiable and $f'(x)=f(x)$.

5. Consider a function $f(x)$ defined as $f(x)=|x(x-3)|$ in the domain [-4,4]. Then which of the following option(s) is (are) true?

- ◯ **Option 1:** The number of critical points is 3.
- ◯ Option 2: The number of discontinuities of $f(x)$ in the given domain is 2.
- ◯ **Option 3:** The number of points where $f(x)$ is not differentiable in the given domain is 2.
- ◯ **Option 4:** $x=0$ is a critical point.
- ◯ **Option 5:** A local maximum value of $f(x)$ is $\frac{9}{4}$.
- ◯ Option 6: The global minimum value of $f(x)$ is -1.
- ◯ Option 7: The number of points where $f(x)$ attains its global maximum value is 3.

6. Consider the following two functions f and g:

$$f(x)=\begin{cases}0 & \text{if } x=1\\ 2 & \text{if } x\neq 1.\end{cases}$$

$$g(x)=x+1$$

Choose the set of correct options.

- ◯ Option 1: f is continuous on $\mathbb{R}$.
- ◯ **Option 2:** g is continuous on $\mathbb{R}$.
- ◯ **Option 3:** g is differentiable on $\mathbb{R}$.
- ◯ Option 4: $\lim_{x\to 0}(f\circ g)(x) = (f\circ g)(0)$
- ◯ Option 5: $(f\circ g)$ is continuous at $x = 0$.

7. Consider the function defined as follows

$$f(x) = \begin{cases} x - \lceil x \rceil & \text{if } 0 < x \leq 1 \\ \lfloor x \rfloor - x & \text{if } 1 < x < 2. \end{cases}$$

- ◯ **Option 1:** $\lim_{x\to 1} f(x) = 0$
- ◯ Option 2: f is not continuous at $x = 1$.
- ◯ **Option 3:** f is continuous at $x = 1$.
- ◯ **Option 4:** f is not differentiable at $x = 1$.
- ◯ Option 5: f is differentiable at $x = 1$.

8. Consider a function $f(x)$ defined as

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Which of the following option(s) is(are) true?

[**Hint:** Use $\lim_{x\to\infty} \frac{\sin x}{x} = 0$]

- ◯ **Option 1:** $f(x)$ is continuous at $x = 0$.
- ◯ Option 2: $f(x)$ is differentiable at $x = 0$.
- ◯ **Option 3:** $\lim_{x\to 0} f(x) = 0$
- ◯ Option 4: $\lim_{x\to 0} f(x)$ does not exist.

Consider the following graph of a function $f(x)$ which contains only one saddle point as shown in Figure M2W1PS1-3 1, where the solid points denote the value of the function at the points, and the values denoted by the hollow points are not taken by the function. Use the above information to answer the questions 9 to 14.

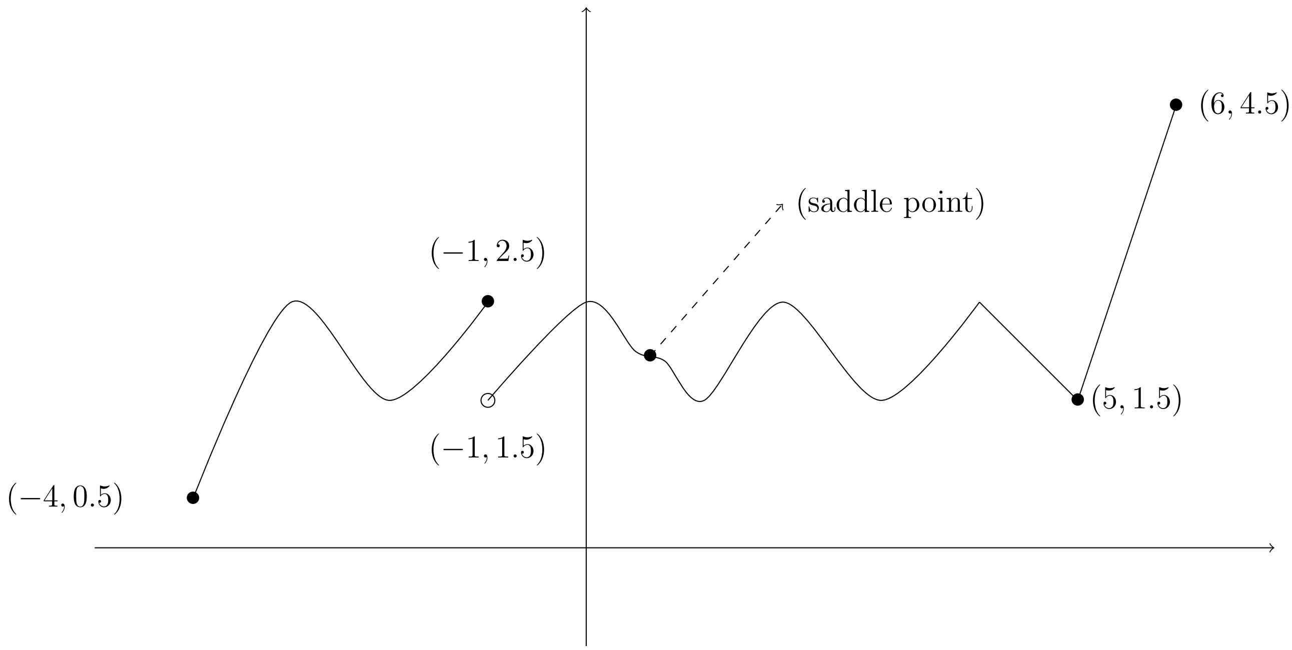


Figure M2W1PS1-3 1

9. Which of the following is(are) true?
   - ◯ **Option 1:** limit exists at $x = 5$.
   - ◯ Option 2: limit exists at $x = -1$.
   - ◯ **Option 3:** Limit exists at every point $x \in (-1, 6)$.
   - ◯ Option 4: The global minimum value of $f(x)$ is 1.5.
   - ◯ **Option 5:** $f(x)$ has the global maximum value at $x = 6$.
10. Find the number of critical points of $f(x)$ in the interval (-4, 6). [Ans: 10]
11. Find the number of discontinuities of $f(x)$ in the interval (-4, 6). [Ans: 1]
12. Find the number of points where $f(x)$ is not differentiable in the interval (-4, 6). [Ans: 3]
13. Find the number of points where $f(x)$ has local minimum in the interval [-4, 6] . [Ans: 5]
14. Find the number of points where $f(x)$ has local maximum in the interval [-4, 6] . [Ans: 6]

15. Let $f(x) = x(x-2)$ and $F(x)$ be the anti-derivative of $f(x)$ defined as $F(x) = \int_{-1}^{x} f(t)dt$. Choose the set of correct options about estimating the $\int_{-1}^{a} f(t)dt$ on the interval [-1, a] using Riemann sums, where a is a critical point of $F(x)$.

[**Hint:** Use $1^2+2^2+\ldots+(n-1)^2+n^2 = \dfrac{n(n+1)(2n+1)}{6}$ and $1+2+3+4+\ldots+n = \dfrac{n(n+1)}{2}$]

- ◯ Option 1: If $F(x)$ has a local maximum at $x = a$, then the value of the left Riemann sum using a partition of $[-1, a]$ into two sub-intervals of equal length is $\frac{5}{8}$.
- ◯ **Option 2:** If $F(x)$ has a local minimum at $x = a$, then the value of the right Riemann sum using a partition of $[-1, a]$ into three sub-intervals of equal length is -1.
- ◯ **Option 3:** Suppose $F(x)$ has a local maximum at $x = a$. Then the limit as n tends to ∞ of the values of the left Riemann sums using a partition of $[-1, a]$ into n sub-intervals of equal length is $\frac{4}{3}$.
- ◯ Option 4: Suppose $F(x)$ has a local minimum at $x = a$. Then the limit as n tends to ∞ of the values of the right Riemann sums using a partition of $[-1, a]$ into n sub-intervals of equal length is 4.

Week-1
Mathematics for Data Science - 2
Some topics of Maths 1, Functions of one variable, Graphs and tangents, Limits for sequence, Limits for functions of one variable
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. Let a_n and b_n be two sequences of real numbers. Consider the following statements.

   - **Statement 1:** If a_n and b_n both converge to some non-zero real number, then $a_n b_n$ also converges to some non-zero real number.
   - **Statement 2:** $a_n b_n$ may converge even if neither a_n nor b_n converge.
   - **Statement 3:** A constant sequence, i.e., $a_i = c$ for some real number c, for all $i \in \mathbb{N}$, is always convergent and it converges to c.

   Choose the correct option from the following.

   ◯ **Option 1:** All the three statements are true.
   ◯ Option 2: Statements 1 and 2 are true, but Statement 3 is false.
   ◯ Option 3: Statements 1 and 3 are true, but Statement 2 is false.
   ◯ Option 4: Only Statement 3 is true.
   ◯ Option 5: None of the statements is true.

   Solution:
   Statement 1:
   Assume the sequence a_n converges to the limit $\ell_1 \neq 0$ i.e., $\lim\limits_{n\to\infty} a_n = \ell_1$ and $\lim\limits_{n\to\infty} b_n = \ell_2 \neq 0$.
   Using multiplication rule of limits $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n . \lim\limits_{n\to\infty} b_n \implies \lim\limits_{n\to\infty} a_n b_n = \ell_1 \ell_2$.
   Hence, statement 1 is true.
   Statement 2: Assume $a_n = b_n = (-1)^n$. As we know that $(-1)^n$ is not convergent.
   Observe, $a_n b_n = (-1)^{2n} = (1)^n = 1$ which is a constant sequence and so converging to 1.
   Hence, statement 2 is also true.
   Statement 3:
   Since, any terms of the sequence $a_n = c$, if we take n larger and larger, still $a_n = c$ i.e., as $n \to \infty, a_n = c$.
   Hence, $\lim\limits_{n\to\infty} a_n = c$ and so statement 3 is also true.
   Hence, option 1 is true.

2. Match the given functions in Column A with their types in column B and their graphs in Column C, given in Table M2W1P1.

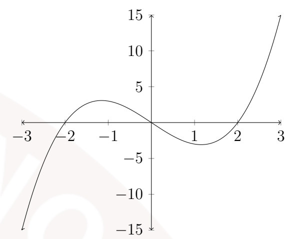
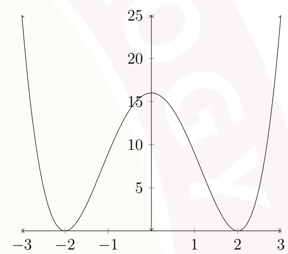
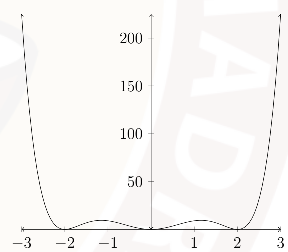
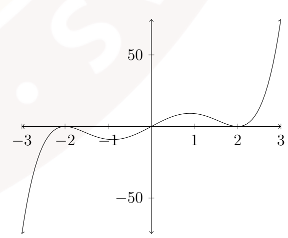
	Functions (Column A)		Types of functions (Column B)		Graphs (Column C)
i)	$f(x) = x^2(x-2)^2(x+2)^2$	a)	Polynomial of degree 3	1)	
ii)	$f(x) = x(x-2)(x+2)$	b)	Polynomial of degree 4	2)	
iii)	$f(x) = x(x-2)^2(x+2)^2$	c)	Polynomial of degree 5	3)	
iv)	$f(x) = (x-2)^2(x+2)^2$	d)	Polynomial of degree 6	4)	

Table: M2W1P1

◯ Option 1: i) → d) → 3), ii) → a) → 4), iii) → c) → 1), iv) → b) → 2).

◯ Option 2: i) $\rightarrow$ d) $\rightarrow$ 4), ii) $\rightarrow$ a) $\rightarrow$ 1), iii) $\rightarrow$ c) $\rightarrow$ 3), iv) $\rightarrow$ b) $\rightarrow$ 2).

◯ Option 3: i) $\rightarrow$ d) $\rightarrow$ 1), ii) $\rightarrow$ a) $\rightarrow$ 3), iii) $\rightarrow$ c) $\rightarrow$ 4), iv) $\rightarrow$ b) $\rightarrow$ 2).

◯ **Option 4:** i) $\rightarrow$ d) $\rightarrow$ 3), ii) $\rightarrow$ a) $\rightarrow$ 1), iii) $\rightarrow$ c) $\rightarrow$ 4), iv) $\rightarrow$ b) $\rightarrow$ 2).

Solution:
(i): $f(x) = x^2(x-2)^2(x+2)^2 = x^6 - 8x^4 + 16x^2$, which is of degree 6 polynomial and matches with (d) in column B. And having roots 0, 2 and -2 with all are even multiplicity. So the graph of the polynomial will touches the axis at 0, 2, and -2 and bounce back which follows only (3) in column C.
(ii): $f(x) = x(x-2)(x+2) = x^3 - 4x$, which is of degree 3 polynomial and matches with (a) in column B. And having roots 0, 2 and -2 with all are multiplicity 1. So the graph of the polynomial will cross the axis and look like straight line in small interval around 0, 2, and -2 which follows only (1) in column C.
(iii):$f(x) = x(x-2)^2(x+2)^2 = x^5 - 8x^3 + 16x^1$, which is of degree 5 polynomial and matches with (c) in column B. And having roots 0, 2 and -2 with 2 and -2 are even multiplicity and 0 is multiplicity 1. So the graph of the polynomial will cross the axis and look like straight line in small interval around 0 and will touches the axis at 2 and -2 and bounce back which follows only (4) in column C.
(iv): $f(x) = (x-2)^2(x+2)^2 = x^4 - 8x^2 + 16$, which is of degree 4 polynomial and matches with (b) in column B. And having roots 2 and -2 with all are even multiplicity. So the graph of the polynomial will touches the axis at 2 and -2 and bounce back which follows only (2) in column C.
Hence, option 4 is true.

2 Multiple Select Questions (MSQ)

3. Recall $n! = n.(n-1).(n-2)\ldots 3.2.1$. Define the sequence $\{a_n\}$ by $a_n = \frac{n}{(n!)^{\frac{1}{n}}}$. Note that $\lim a_n = e$. Which of the following option(s) is (are) true?

   ◯ Option 1: $\lim\limits_{n\to\infty} \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = 3e$

   ◯ Option 2: $\lim\limits_{n\to\infty} \dfrac{2n}{((2n)!)^{\frac{1}{2n}}} = 2e$

   ◯ **Option 3:** $\lim\limits_{n\to\infty} \dfrac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}} = e$

   ◯ **Option 4:** $\lim\limits_{n\to\infty} \ln \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = 1$

Solution:
Given $a_n = \frac{n}{(n!)^{\frac{1}{n}}}$ and $\lim a_n = e$.

Observe that, in option 1, in option 2 and in option 3, $\left\{\frac{3n}{((3n)!)^{\frac{1}{3n}}}\right\}, \left\{\frac{2n}{((2n)!)^{\frac{1}{2n}}}\right\}$, and $\left\{\frac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}}\right\}$ are the subsequences of the sequence $\{a_n\}$ and every subsequence of the sequence a_n converges to the same limit . Hence, $\lim\limits_{n\to\infty} \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = e$, $\lim\limits_{n\to\infty} \dfrac{2n}{((2n)!)^{\frac{1}{2n}}} = e$, and $\lim\limits_{n\to\infty} \dfrac{2n+1}{((2n+1)!)^{\frac{1}{2n+1}}} = e$

In option 4, as know we that if $a_n \to a$ and $a_n > 0$ for all $n \in \mathbb{N}$, and $a, c > 0$, then $log_c(a_n) \to log_c(a)$.

So, $\lim\limits_{n\to\infty} \ln \dfrac{3n}{((3n)!)^{\frac{1}{3n}}} = \lim\limits_{n\to\infty} \ln e = 1$.

4. The graph of some function is drawn below. Choose the set of correct statements about it.

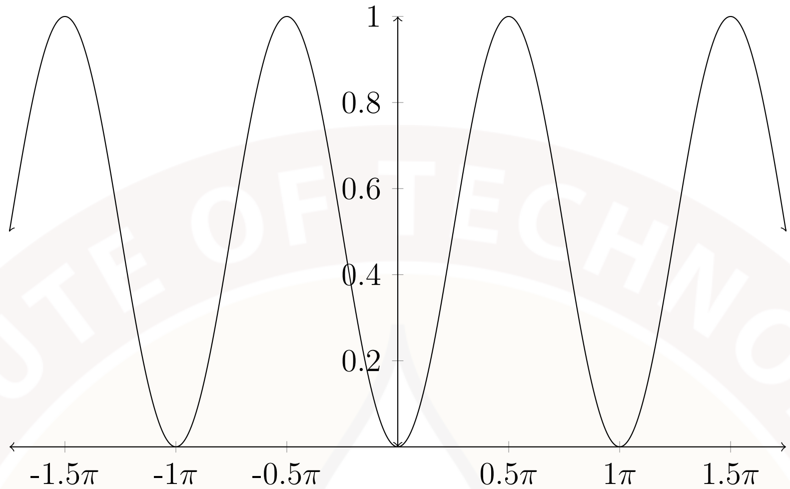


Figure: M2W1G1

- ◯ Option 1: Limit of the function as x tends to 0 is 1.
- ◯ **Option 2:** Limit of the function as x tends to 0 is 0.
- ◯ Option 3: Limit of the function as x tends to 0 is undefined.
- ◯ Option 4: There is a (unique) tangent at the point $x = \pi$, but not at $x = -\pi$.
- ◯ **Option 5:** There is a (unique) tangent at $x = \pi$, as well as at $x = -\pi$.
- ◯ Option 6: The given function is monotonically increasing in the interval $[-0.5\pi, 0]$.
- ◯ **Option 7:** The given function is monotonically decreasing in the interval $[-0.5\pi, 0]$.

Solution:
First, observe that as x approaches 0 from the left side or the right side, the value of the function in graph approaches 0. Hence limit at 0 exists and equal to 0. Hence, option 1 and option 3 are not true and option 2 is true.
Second, observer that x- axis touches the graph of the function at 0, 0.5π, -0.5π, π, $-\pi$, 1.5π, and -1.5π, so there is a unique tangent at the point $x = \pi$ as well as $x = -\pi$ which is x-axis. Hence, option 4 is not true and option 5 is true.
Third, observe that in the interval $[-0.5\pi, 0]$, the value of the function decreases from 1 to 0. Hence, option 6 is not true and option 7 is true.

5. Depending on the graphs given below, predict which do not have a (unique) tangent at the origin (i.e., $(0, 0)$) ?

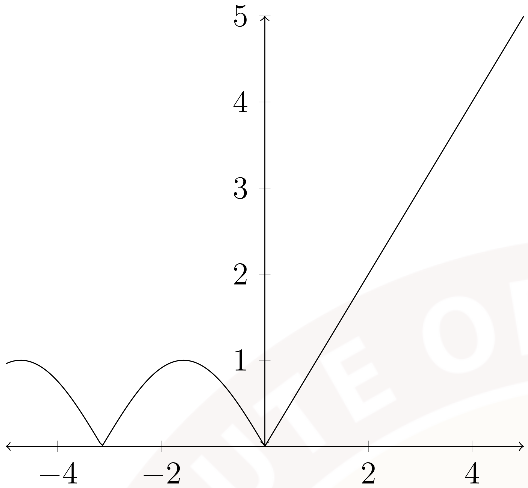


Figure: Curve 1

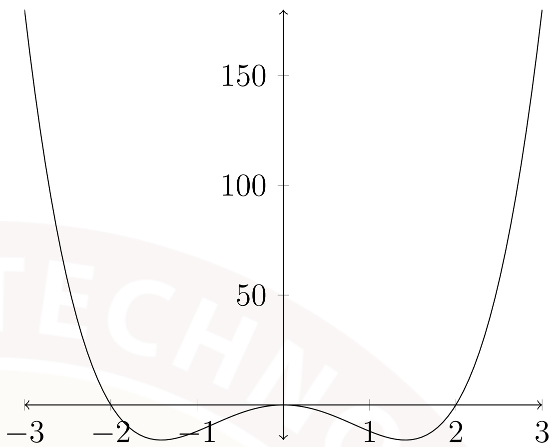


Figure: Curve 2

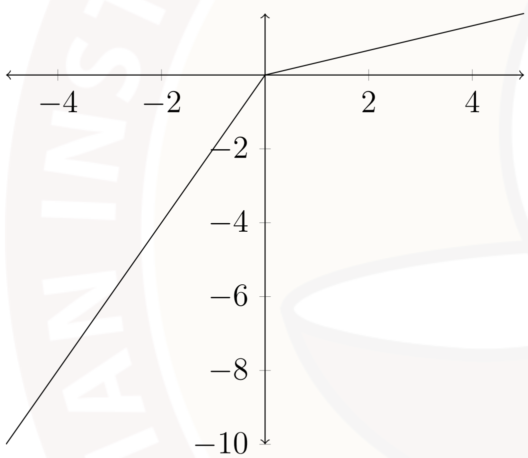


Figure: Curve 3

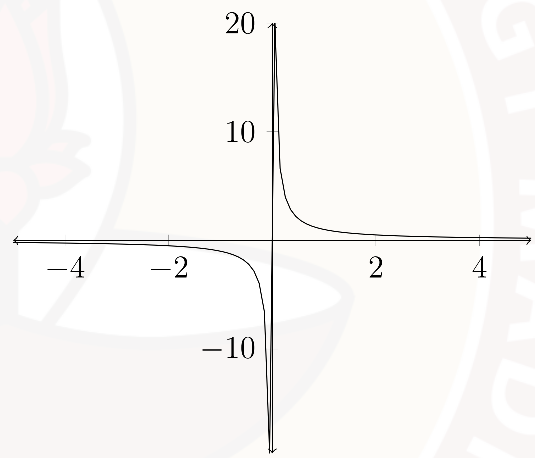


Figure: Curve 4

- ◯ **Option 1:** Curve 1
- ◯ Option 2: Curve 2
- ◯ **Option 3:** Curve 3
- ◯ **Option 4:** Curve 4

Solution:
Observe that in curve 1 and curve 3, at the origin if we approach from the left side, then there can be a tangent (the instantaneous direction of the curve) and if we approach from the right, then there can be another different tangent i.e., there is no unique tangent to the curve 1 and curve 3 at the origin. In curve 4, as x approaches the origin, the value of the function approaches to $-\infty$ from the left side and the value of the function

approaches to ∞ from the right side i.e., function is not defined at the origin, so there is no need to talk about the instantaneous direction of the curve 4. Hence, there is no tangent to curve 4 at the origin. But in curve 2, $x-$ axis touches the curve 2 at the origin, so there is a (unique) tangent to the curve 2 at the origin.

3 Numerical Answer Type (NAT)

6. Consider the following sequences $\{a_n\}, \{b_n\},$ and $\{c_n\}$:

$$a_n = \frac{1}{n^2}$$
$$b_n = 0$$
$$c_n = 1 - \frac{1}{n}$$

How many among the three sequences given above are subsequences of the sequence $\{\frac{1}{n}\}$? (Answer 1)

Solution:

A sequence which is obtained from the main sequence by removing finite or infinite terms from the main sequence, is called a subsequence of the sequence.

Observe that 0 and $\frac{2}{3}$ are the terms in sequence $\{b_n\}$ and $\{c_n\}$ for all value of n in b_n and for $n = 3$ in c_n respectively which are not in the sequence $\left\{\frac{1}{n}\right\}$. Hence the sequences $\{b_n\}$ and $\{c_n\}$ are not subsequences of the sequence $\left\{\frac{1}{n}\right\}$.

But, the sequence $\{a_n\}$, $a_n = \frac{1}{n^2}$, observe that if n is a natural number then n^2 is also a natural number. And all terms of the sequence $\left\{\frac{1}{n}\right\}$ are $\frac{1}{n}$ for some $n \in \mathbb{N}$. So the sequence $\{a_n\}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}$. Hence, answer is 1.

7. Consider the sequence $\{a_n\}$ given by $a_n = n^{\frac{1}{n}}$, which is known to converge to 1. What will be the limit of the sequence $\{b_n\}$ given by $b_n = n^{\frac{2}{n}} + 2n^{\frac{1}{n}} - 1$? (Answer 2)

Solution:

We know that if $a_n \to a$, then $f(a_n) \to f(a)$, where f is any polynomial function.

Observe that $b_n = f(a_n)$, where $f(x) = x^2 + 2x - 1$. Given that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$ so $\lim\limits_{n\to\infty} b_n = f(1) = 1 + 2 - 1 = 1$.

Hence, answer is 2.

4 Comprehension Type Question:

Suppose there are three schemes available for renting a studio room given in the table below.

Schemes	Rent (in ₹) for t hours
Scheme A	$p_1(t) = 100\lfloor t \rfloor + 200$
Scheme B	$p_2(t) = 100\lceil t \rceil + 200$
Scheme C	$p_3(t) = 100t + 200$

Table: M2W1P2

Where $\lfloor t \rfloor$ denotes the largest integer lesser than or equal to t, and $\lceil t \rceil$ denotes the smallest integer greater than or equal to t. Answer questions 7,8, and 9 using the given information.

8. If Rana wants to book the studio for 2.6 hours, which Scheme should he choose to avail the studio in minimum cost? (MCQ)

   ◯ Option 1: For 2.6 hours, costs for Scheme A and Scheme C are the same, and that is the minimum.

   ◯ **Option 2:** For 2.6 hours, cost for Scheme A is the minimum.

   ◯ Option 3: For 2.6 hours, cost for Scheme B is the minimum.

   ◯ Option 4: For 2.6 hours, costs for Scheme B and Scheme C are the same, and that is the minimum.

   ◯ Option 5: For 2.6 hours, cost for Scheme C is the minimum.

   Solution : Substitute $t = 2.6$ in $p_1(t), p_2(t)$ and $p_3(t)$, we get,
   $p_1(2.6) = 100\lfloor(2.6)\rfloor + 200 = 100 \times 2 + 200 = 400$
   $p_2(2.6) = 100\lceil(2.6)\rceil + 200 = 100 \times 3 + 200 = 500$
   $p_3(2.6) = 100 \times 2.6 + 200 = 460$
   Observe that for 2.6 hours, cost for Scheme A is the minimum and none of the cost for Scheme A, B and C is same. Hence, option 2 is true.

9. Consider the following graphs where X-axis denotes the time in hours and Y-axis denotes the cost for rent of the studio :

Figure: 1

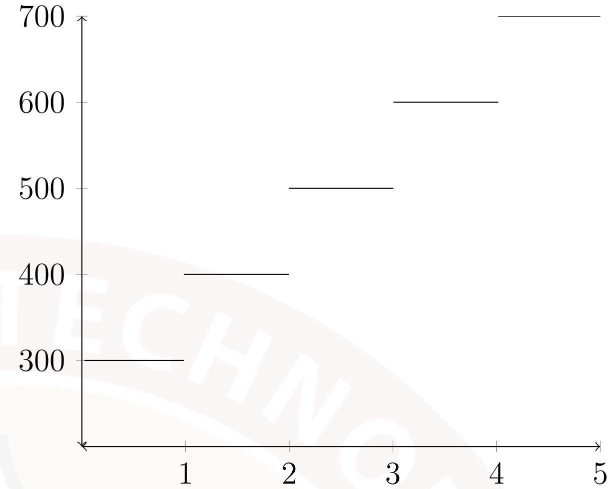


Figure: 2

Choose the correct option. (MCQ)

- ◯ **Option 1:** Figure 1 represents Scheme A, Figure 2 represents Scheme B.
- ◯ Option 2: Figure 2 represents Scheme A, Figure 1 represents Scheme B.
- ◯ Option 3: Figure 1 represents Scheme A, but Figure 2 does not represent Scheme B.
- ◯ Option 4: Figure 2 represents Scheme A, but Figure 1 does not represent Scheme B.

Solution:
As we know function $\lfloor t \rfloor$ denotes the largest integer lesser than or equal to t, and $\lceil t \rceil$ the smallest integer greater than or equal to t. From question 9, for Scheme A, $p_1(2.6) = 100\lfloor(2.6)\rfloor + 200 = 100 \times 2 + 200 = 400$, similarly we can check for other points. So the Figure 1 represents Scheme A.
For Scheme B, $p_2(2.6) = 100\lceil(2.6)\rceil + 200 = 100 \times 3 + 200 = 500$, similarly we can check for other points. So the Figure 2 represents Scheme B.
Hence, option 1 is true.

10. Choose the set of correct options. (MSQ)

- ◯ **Option 1:** The tangent to the function $p_1(t)$ which represents Scheme A does not exist at 1.
- ◯ **Option 2:** The tangent to the function $p_1(t)$ which represents Scheme A does not exist at a, for any natural number a.
- ◯ Option 3: The tangent to the function $p_3(t)$ which represents Scheme C does not exist at 1.
- ◯ **Option 4:** The tangent to the function $p_2(t)$ which represents Scheme B does not exist at a, for any natural number a.

Solution:
From question 10, the Figure M2W1PS1 represents the graph of p_1 and the Figure M2W1PS2 represents the graph of p_2. Observe that, p_3 represents a line with slope 100, so the Figure M2W1PS3 represents the graph of p_3.

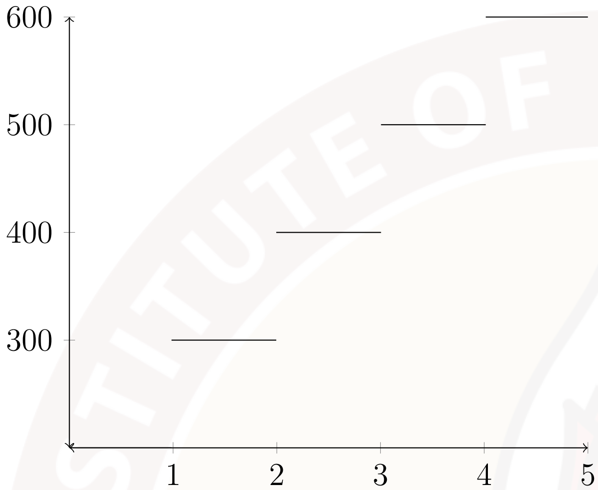


Figure: M2W1PS1

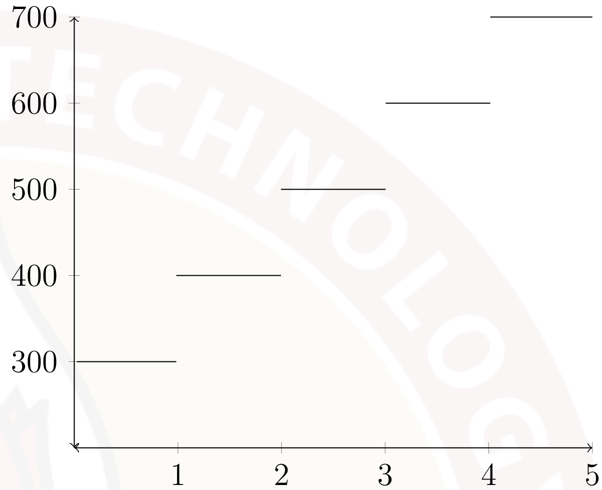


Figure: M2W1PS2

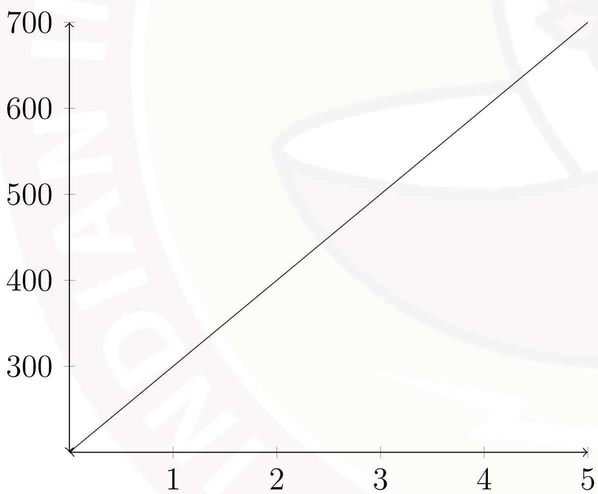


Figure: M2W1PS3

Now, for any natural number a, $p_1(a) = 100a + 200$ which is a natural number and the instantaneous direction of the curve (tangent) represented by the function $p_1(t)$) from left given by the line $y = 100a + 200$ and the instantaneous direction of the curve(tangent) from right given by the line $y = 100(a + 1) + 200$ i.e. there is no unique tangent line at a, i.e., there is no tangent to the function $p_1(t)$ at any natural number a.
for example, the instantaneous direction of the curve(tangent) at 1 from left given by

the line $y = 200$ and from right given by the line $y = 300$. So there is no tangent to the function $p_1(t)$ at 1.
Hence, option 1 and option 2 are true.
Similar things will happen with the function $p_2(t)$. So we can conclude that there is no tangent to the function $p_2(t)$ at any natural number a.
Now, the Figure M2W1PS3 represents the graph of the function $p_3(t)$. Observe that the instantaneous direction of the curve (tangent) represented by the function $p_3(t)$ exists at any point which is the line $y = 100t + 200$ itself. So the tangent to function $p_3(t)$ exists at 1 also.
Hence, option 3 is not true.

Week-1
Mathematics for Data Science - 2
Some topics of Maths 1, Functions of one variable, Graphs and tangents,
Limits for sequence, Limits for functions of one variable
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Consider the following statements.

   - **Statement 1:** If $\{a_n\}$ and $\{b_n\}$ both converge to some non-zero real number, then $\{a_n + b_n\}$ also converges to some non-zero real number.
   - **Statement 2:** If $\{a_n\}$ is an increasing sequence, i.e., $a_i \leq a_{i+1}$, for all $i \in \mathbb{N}$, then $\{(-1)^n a_n\}$ is a decreasing sequence.
   - **Statement 3:** If $\{a_n\}$ and $\{b_n\}$ both converge to the same real number, then $\{a_n - b_n\}$ must converge to 0.

   Choose the correction option from the following.

   - ○ Option 1: All the three statements are true.
   - ○ Option 2: Statements 1 and 2 are true, but Statement 3 is false.
   - ○ Option 3: Statements 1 and 3 are true, but Statement 2 is false.
   - ○ **Option 4:** Only Statement 3 is true.
   - ○ Option 5: None of the statements is true.

   Solution:

   - **Statement 1:** Suppose $\{a_n\}$ and $\{b_n\}$ both are constant sequences, such that $a_n = -1$ and $b_n = 1$, for all n. Both of them converges to some non-zero real number. As $\{a_n\}$ converges to -1 and $\{b_n\}$ converges to 1. But $a_n + b_n = 0$ for all n. Hence $\{a_n + b_n\}$ converges to 0. Hence the statement is false.
   - **Statement 2:** Suppose $a_n = n$. Hence the sequence $\{a_n\} = \{1, 2, 3, 4, \ldots\}$ is an increasing sequence. So the sequence $\{(-1)^n a_n\} = \{-1, 2, -3, 4, \ldots\}$, which is not a decreasing sequence. Hence the statement is false.
   - **Statement 3:** Suppose $\lim a_n = c = \lim b_n$ for some real number c. We know that, $\lim(a_n - b_n) = \lim a_n - \lim b_n = c - c = 0$. Hence the statement is true.

2. Match the given functions in Column A with their types in column B and their graphs in Column C, given in Table M2W1T1.

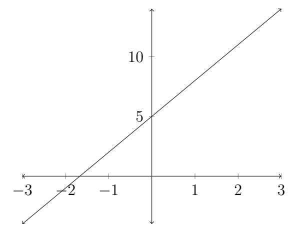
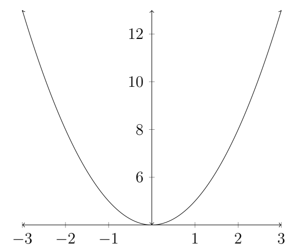
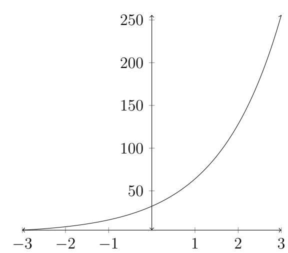
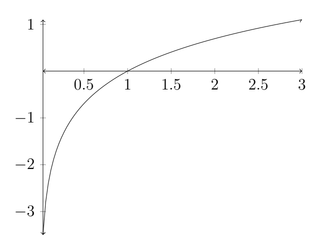
	Functions (Column A)		Types of functions (Column B)		Graphs (Column C)
i)	$f(x) = x^2 + 4$	a)	Logarithmic function	1)	
ii)	$f(x) = ln(x)$	b)	Exponential function	2)	
iii)	$f(x) = 2^{x+5}$	c)	Linear function	3)	
iv)	$f(x) = 3x + 5$	d)	Quadratic function	4)	

Table: M2W1G1

- ◯ Option 1: i) → d) → 2), ii) → a) → 4), iii) → c) → 3), iv) → b) → 1).
- ◯ **Option 2:** i) → d) → 2), ii) → a) → 4), iii) → b) → 3), iv) → c) → 1).

- ◯ Option 3: i) → d) → 2), ii) → a) → 3), iii) → b) → 4), iv) → c) → 1).
- ◯ Option 4: i) → d) → 3), ii) → a) → 4), iii) → b) → 2), iv) → c) → 1).

Solution:

- $f(x) = x^2 + 4$, is a quadratic function. The curve represented by the function f is a parabola. So, i) → d) → 2).
- $f(x) = ln(x)$ is a logarithmic function. So, ii) → a) → 4).
- $f(x) = 2^{x+5}$ is a exponential function. So, iii) → b) → 3).
- $f(x) = 3x + 5$ is a linear function. The curve represented by the function f is a straight line. So, iv) → c) → 1).

2 Multiple Select Questions (MSQ)

3. Limits of some standard functions are given below:

- $\lim_{x\to 0} \dfrac{sin(x)}{x} = 1$
- $\lim_{x\to 0} \dfrac{log(1+x)}{x} = 1$
- $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$

Using this given information choose the correct options.

○ **Option 1:** $\lim_{x\to 0} \dfrac{log(1+x)}{sin(x)} = 1.$

○ Option 2: $\lim_{x\to 0} \dfrac{log(1+x)}{sin(x)}$ is undefined.

○ Option 3: $\lim_{x\to 0} \dfrac{sin(5x)}{x} = 1.$

○ **Option 4:** $\lim_{x\to 0} \dfrac{sin(5x)}{x} = 5.$

○ Option 5: $\lim_{x\to 0} \dfrac{sin(5x)}{x} = \dfrac{1}{5}$

○ Option 6: $\lim_{x\to 0} \dfrac{e^{\frac{x}{2}} - 1}{sin2x} = 1.$

○ Option 7: $\lim_{x\to 0} \dfrac{e^{\frac{x}{2}} - 1}{sin2x} = 2.$

○ **Option 8:** $\lim_{x\to 0} \dfrac{e^{\frac{x}{2}} - 1}{sin2x} = \dfrac{1}{4}.$

○ Option 9: $\lim_{x\to 0} \dfrac{e^{\frac{x}{2}} - 1}{sin2x} = \dfrac{1}{2}.$

Solution:

- $\lim_{x\to 0} \dfrac{log(1+x)}{sin(x)} = \lim_{x\to 0} \dfrac{\frac{log(1+x)}{x}}{\frac{sin(x)}{x}} = \dfrac{\lim_{x\to 0} \frac{log(1+x)}{x}}{\lim_{x\to 0} \frac{sin(x)}{x}} = \dfrac{1}{1} = 1$
- $\lim_{x\to 0} \dfrac{sin(5x)}{x} = \lim_{5x\to 0} 5\dfrac{sin(5x)}{5x} = 5 \lim_{5x\to 0} \dfrac{sin(5x)}{5x} = 5$ (Since $x \to 0$, we have $5x \to 0$).
- $\lim_{x\to 0} \dfrac{e^{\frac{x}{2}} - 1}{sin2x} = \lim_{x\to 0} \dfrac{1}{4} \dfrac{\frac{e^{\frac{x}{2}}-1}{\frac{x}{2}}}{\frac{sin2x}{2x}} = \dfrac{1}{4} \dfrac{\lim_{\frac{x}{2}\to 0} \frac{e^{\frac{x}{2}}-1}{\frac{x}{2}}}{\lim_{2x\to 0} \frac{sin2x}{2x}} = \dfrac{1}{4}$ (Since $x \to 0$, we have $\frac{x}{2} \to 0$ and $2x \to 0$).

4. The graph of some function is drawn below in Figure M2W1G1. Choose the set of correct statements about it.

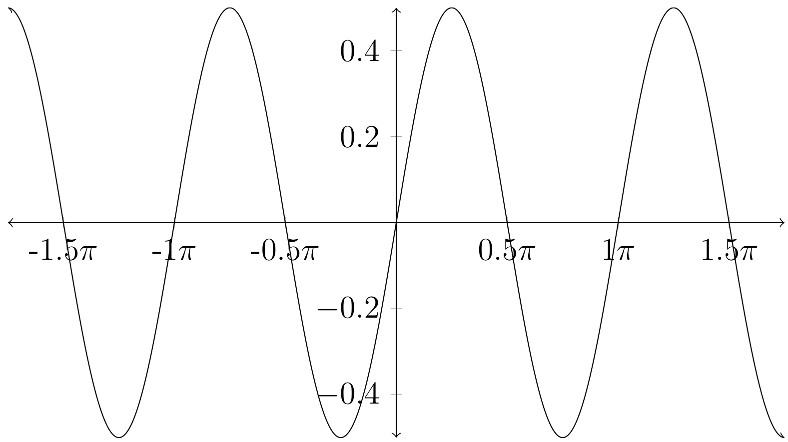
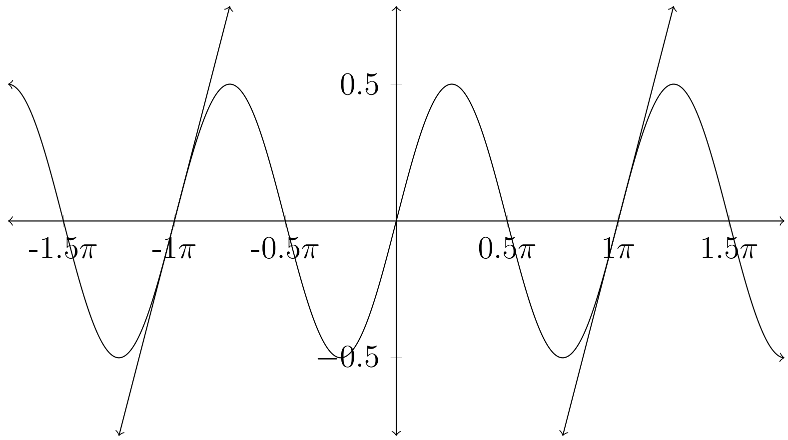


Figure: M2W1G1

- ◯ Option 1: Limit of the function as x tends to 0 is 1.
- ◯ **Option 2:** Limit of the function as x tends to 0 is 0.
- ◯ Option 3: Limit of the function as x tends to 0 is undefined.
- ◯ Option 4: There is a (unique) tangent at the point $x = \pi$, but not at $x = -\pi$.
- ◯ **Option 5:** There is a (unique) tangent at $x = \pi$, as well as at $x = -\pi$.
- ◯ Option 6: The given function is monotonically increasing in the interval $[-0.5\pi, 0.5\pi]$.
- ◯ Option 7: The given function is monotonically decreasing in the interval $[-0.5\pi, 0.5\pi]$.

Solution:

- As we are approaching from right of 0 towards 0, the value of the function is also approaching to 0. Similarly as we are approaching from the left of 0 towards 0, the value of the function is also approaching to 0. Hence limit of the function as x tends to 0 is 0.
- There is a (unique) tangent at $x = \pi$, as well as at $x = -\pi$ as shown in the figure below.



- The function is decreasing in the interval $[-0.5\pi, -0.25\pi]$ and increasing in the interval $[-0.25\pi, 0.25\pi]$. Again the function decreases in the interval $[0.25\pi, 0.5\pi]$.

5. Depending on the graphs given below, predict which have a (unique) tangent at the origin (i.e., $(0,0)$) ?

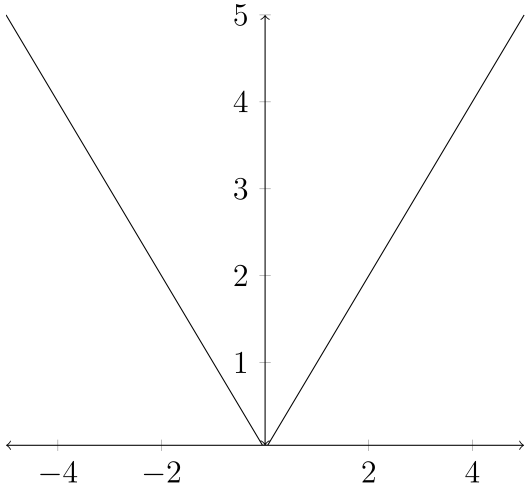


Figure: Curve 1

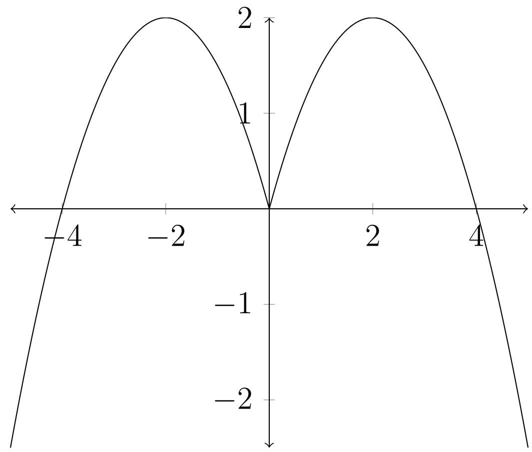


Figure: Curve 2

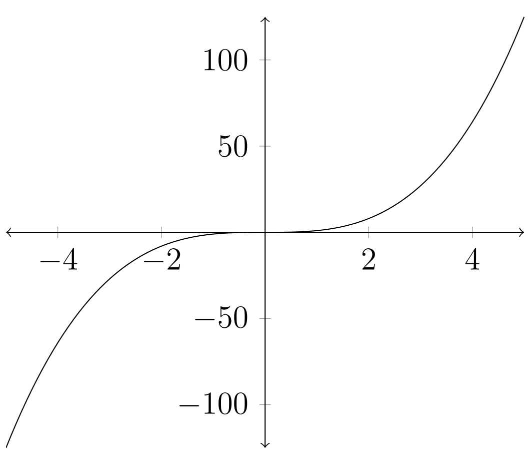


Figure: Curve 3

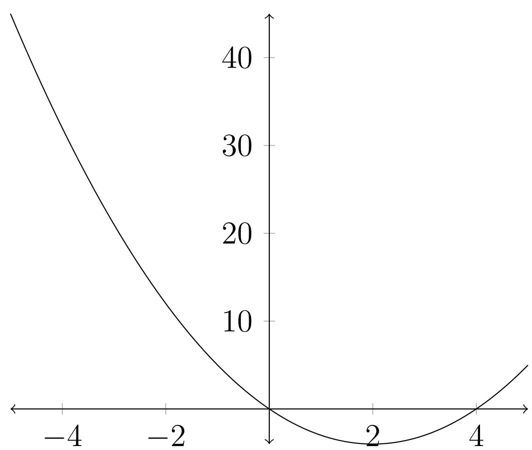


Figure: Curve 4

- ◯ Option 1: Curve 1
- ◯ Option 2: Curve 2
- ◯ **Option 3:** Curve 3
- ◯ **Option 4:** Curve 4

Solution:

- There are sudden changes in the slopes of Curve 1 and Curve 2, at the origin. So from the graph we can predict that the these two curves do not have a (unique) tangent at the origin $(0,0)$, where as Curve 3 and Curve 4 have.

3 Numerical Answer Type (NAT)

6. Find the limit of the sequence given by $a_n = \dfrac{2+4+6+\ldots+2n}{n^2}$, (where $n \in \mathbb{N} \setminus \{0\}$). (Answer: 1)

   Solution: $a_n = \dfrac{2(1+2+3+\ldots+n)}{n^2} = \dfrac{2\frac{n(n+1)}{2}}{n^2} = \dfrac{n(n+1)}{n^2} = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$

   As n increases, $\frac{1}{n} \to 0$. Hence $a_n \to 1$.

7. What will be the value of $\lim\limits_{x \to 2+} \lfloor x \rfloor - \lim\limits_{x \to 2-} \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x? (Answer: 1)

   Solution: $\lim\limits_{x \to 2^+} \lfloor x \rfloor = 2$ and $\lim\limits_{x \to 2^-} \lfloor x \rfloor = 1$. Hence $\lim\limits_{x \to 2+} \lfloor x \rfloor - \lim\limits_{x \to 2-} \lfloor x \rfloor = 1$

4 Comprehension Type Question:

Suppose a company runs three algorithms to predict its future growth. Suppose the error in the estimation depends on the available number (n) (where $n \in \mathbb{N} \setminus \{0\}$) of data as follows:

- Error in estimation by Algorithm 1: $a_n = \dfrac{n^2 + 5n}{3n^2 + 1}$.
- Error in estimation by Algorithm 2: $b_n = \dfrac{1}{2} + (-1)^n \dfrac{1}{n}$
- Error in estimation by Algorithm 3: $c_n = \dfrac{e^n + 4}{4e^n}$

Suppose the company has a large amount of data in their hand (we can assume n tends to ∞). Using the above set of information answer the questions 8, 9 and 10.

8. Which of the given algorithms should the company use to get the minimum error in the prediction of its growth? (MCQ)

   ◯ Option 1: Algorithm 1

   ◯ Option 2: Algorithm 2

   ◯ **Option 3:** Algorithm 3

   ◯ Option 4: Both Algorithm 1 and Algorithm 2 will give the same error and that will be the minimum.

   Solution: We can write $a_n = \dfrac{1 + \frac{5}{n}}{3 + \frac{1}{n^2}}$ and $c_n = \dfrac{1}{4} + \dfrac{1}{e^n}$.

   As $n \to \infty$, $a_n \to \frac{1}{3}$, $b_n \to \frac{1}{2}$, and $c_n \to \frac{1}{4}$, among which $\frac{1}{4}$ is the minimum.

   Hence, Algorithm 3 will give the minimum error in the prediction.

9. Which of the given algorithms gives the maximum error? (MCQ)

   ◯ Option 1: Algorithm 1

   ◯ **Option 2:** Algorithm 2

   ◯ Option 3: Algorithm 3

   ◯ Option 4: Both Algorithm 1 and Algorithm 2 will give the same error and that will be the maximum.

   Solution: From the solution of Question 8, it is clear that Algorithm 2 will give the maximum error in the prediction.

10. Suppose a new algorithm is designed to predict the growth of the company in future and the error in estimation by the new algorithm is given by $b_n - a_n$, where a_n and b_n are the same as defined earlier. Choose the set of correct options. (MSQ)

- ◯ **Option 1:** The error in estimation using the new algorithm is less than the error in estimation using Algorithm 1.
- ◯ Option 2: The error in estimation using the new algorithm is more than the error in estimation using Algorithm 2.
- ◯ **Option 3:** The error in estimation using the new algorithm is less than the error in estimation using Algorithm 3.
- ◯ Option 4: The error in estimation using the new algorithm cannot be compared with the error in estimation using Algorithm 3.

Solution: As $a_n \to \frac{1}{3}$ and $b_n \to \frac{1}{2}$, we have $b_n - a_n \to \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Hence, the error in estimation using the new algorithm is less than the error in estimation using Algorithm 1, Algorithm 2, or Algorithm 3.

Week-2
Mathematics for Data Science - 2
Limits, Continuity, Differentiability, and the derivative
Practice Assignment

1 Multiple Choice Questions (MCQ)

1. Match the given functions in Column A with the equation of their tangents at the origin $(0, 0)$ in column B and the plotted graphs and tangents in Column C, given in Table M2W2P1.

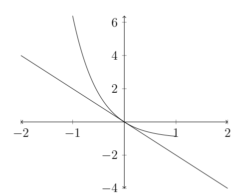
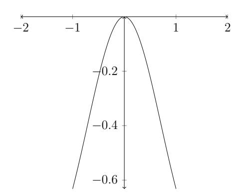
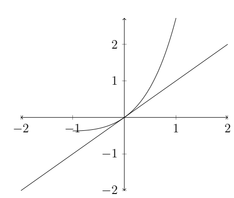
	Function (Column A)		It's tangent at (0,0) (Column B)		Graph (Column C)
i)	$f(x) = xe^x$	a)	$y = -2x$	1)	
ii)	$f(x) = e^{-2x} - 1$	b)	$y = x$	2)	
iii)	$f(x) = e^{-x^2} - 1$	c)	$y = 0$	3)	

Table: M2W2P1

○ Option 1: i) → b) → 3, ii) → c) → 2), iii) → a) → 1.

- ○ **Option 2:** i) $\rightarrow$ b) $\rightarrow$ 3, ii) $\rightarrow$ a) $\rightarrow$ 1), iii) $\rightarrow$ c) $\rightarrow$ 2.
- ○ Option 3: i) $\rightarrow$ c) $\rightarrow$ 3, ii) $\rightarrow$ a) $\rightarrow$ 2), iii) $\rightarrow$ b) $\rightarrow$ 1.
- ○ Option 4: i) $\rightarrow$ c) $\rightarrow$ 3, ii) $\rightarrow$ a) $\rightarrow$ 1), iii) $\rightarrow$ b) $\rightarrow$ 2.

Solution:

1. Given
$$f(x) = xe^x$$
$$f(1) = 1e^1 = e > 0$$
Only figure 3 has this property. Now differentiating the function,
$$f'(x) = 1e^x + xe^x$$
$$f'(0) = 1 + 0 = 1$$
Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = 1$, then the equation of tangent
$$y = x$$
Which is b) in column B. Therefore, $i) \rightarrow b) \rightarrow 3$.

2. Given
$$f(x) = e^{-2x} - 1$$
$$f(-1) = e^2 - 1 > 0$$
Only figure 1 has this property. Now differentiating the function,
$$f'(x) = e^{-2x}(-2) = -2e^{-2x}$$
$$f'(0) = -2$$
Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = -2$, then the equation of tangent
$$y = -2x$$
Which is a) in column B. Therefore, $ii) \rightarrow a) \rightarrow 1$.

3. Given
$$f(x) = e^{-x^2} - 1$$
$$f(-x) = e^{-x^2} - 1 = f(x)$$
The function is even and only figure 2 has this property. Now differentiating the function,
$$f'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$$

$$f'(0) = 0$$

Let the equation of tangent is $y = mx + c$. As the tangent passes through (0,0) therefore, c=0. And the slope of tangent is $m = f'(0) = 0$, then the equation of tangent

$$y = 0$$

Which is c) in column B. Therefore, $iii) \rightarrow c) \rightarrow 2$.

2 Multiple Select Questions (MSQ)

2. Consider the graphs given below:

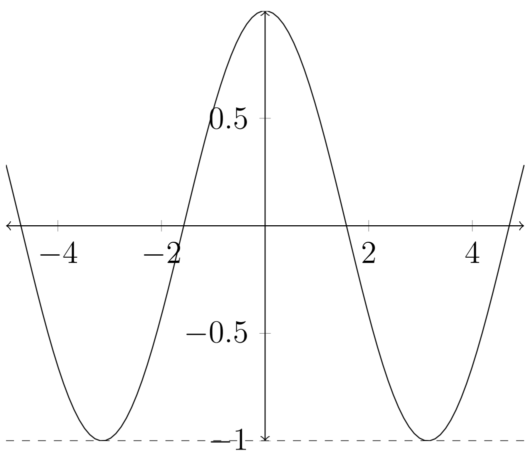


Figure: Curve 1

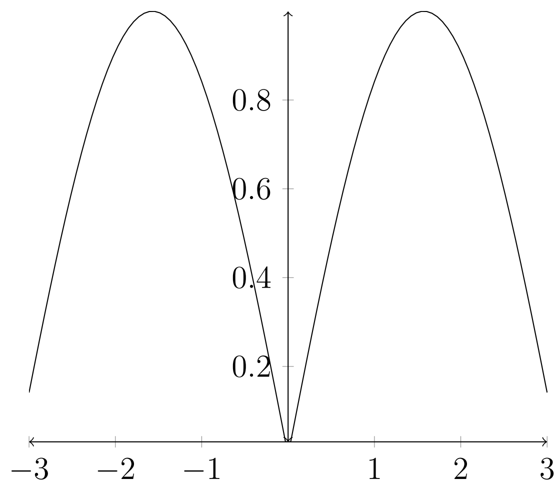


Figure: Curve 2

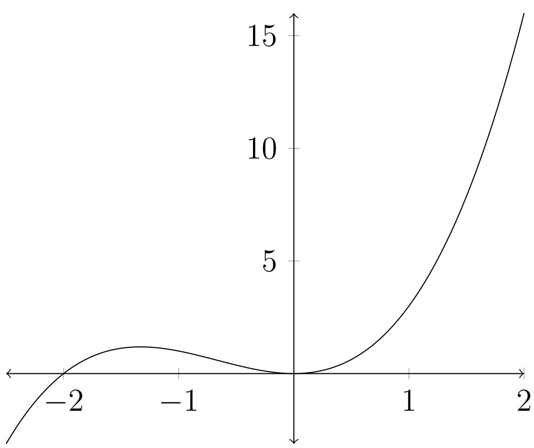


Figure: Curve 3

Figure: Curve 4

Choose the set of correct options:

- ◯ **Option 1:** There are at least two points on Curve 1, where the derivatives of the function corresponding to Curve 1, are equal.
- ◯ **Option 2:** At the origin the derivative of the function corresponding to Curve 2 does not exist.

- ◯ Option 3: The derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are the same.
- ◯ Option 4: The derivative of the function corresponding to Curve 4 does not exist at any point.

Solution:
Option 1: There are at least two points on Curve 1, where the derivatives of the function corresponding to Curve 1, are equal.

As it is shown in the figure, the straight line $y = -1$ is tangent at two point of the curve. So at those two points on Curve 1, the derivatives of the function corresponding to Curve 1, as slope of the tangents at those two points are the same.

Option 2: At the origin the derivative of the function corresponding to Curve 2 does not exist.

Curve 2 has a sharp corner at $x = 0$, which shows the derivative of the function corresponding to Curve 2 does not exist.That's why option 2 is correct.

Option 3: The derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are the same.

At origin the derivative of the function corresponding to Curve 3 is zero as the X-axis is the tangent of the curve at the origin. But at $x = -2$ the tangent is not parallel to the X-axis, hence the slope of the tangent at $x = -2$ must be different from 0. So the derivative of the function corresponding to Curve 3, at the origin and at the point $(-2, 0)$ are not the same.

Option 4: The derivative of the function corresponding to Curve 4 does not exist at any point.

The function corresponding to Curve 4 is a constant function, therefore, the derivative of the function corresponding to Curve 4 always exists and is 0.

3. Let f be a function and the Figure M2W2P1 represent the graph of function f. The solid points denote the value of the function at the points, and the values denoted by the hollow points are not taken by the functions.

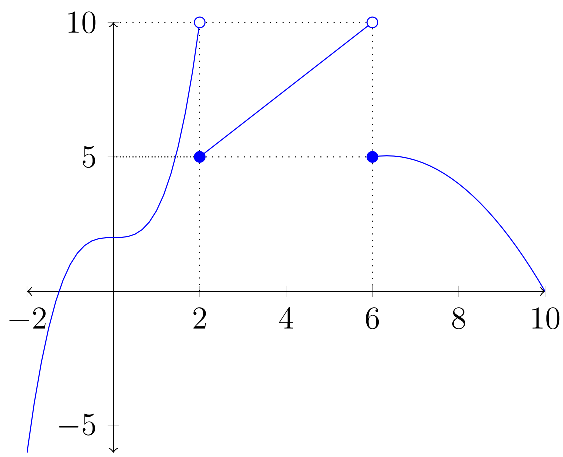


Figure: M2W2P1

Choose the set of correct options.

◯ **Option 1:** $\lim_{t \to 2-} f(t) = 10$

◯ **Option 2:** $\lim_{t \to 2+} f(t) = 5$

◯ **Option 3:** $\lim_{t \to 6-} f(t) = 10$

◯ Option 4: $\lim_{t \to 6+} f(t) = 10$

◯ Option 5: f is continuous at $x = 2$.

◯ **Option 6:** f is continuous at $x = 4$

Solution:

We can see that the curve is discontinuous at $t = 2$ and $t = 6$ only.

As t is approaching to 2 from left side, f is approaching to the value 10, which means the LHL (left hand limit) i.e., $t \to 2^-$ is 10.
As t is approaching to 2 from right side, f is approaching to the value 5, which means the RHL (right hand limit) i.e., $t \to 2^+$ is 5.
Hence limit at the function f does not exist at $t = 2$. So f is discontinuous at $t = 2$. Similar explanation can be given for $t = 6$.

4. Define a function f as follows:

$$f(x) = \begin{cases} x^3 & \text{if } x > 1, \\ x^2 & \text{if } 0 < x \leq 1 \\ x & \text{if } x < 0, \\ 0 & \text{if } x = 0 \end{cases}$$

Choose the set of correct options.

- ◯ **Option 1:** f is continuous, but not differentiable at $x = 1$.
- ◯ Option 2: f is both continuous and differentiable at $x = 1$.
- ◯ **Option 3:** f is continuous, but not differentiable at $x = 0$.
- ◯ Option 4: f is both continuous and differentiable at $x = 0$.
- ◯ Option 5: f is not continuous at $x = 0$.
- ◯ Option 6: f is not continuous at $x = 1$.

Solution:

For $x = 1$:

Left Hand Limit:

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x^2 = 1$$

Right Hand Limit:

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x^3 = 1$$

Moreover $f(1) = 1$. Hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1$$

Therefore, the function is continuous at $x = 1$.

For differentiability at $x = 1$:

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^3 - 1}{h} = \lim_{h \to 0^+} \frac{(1 + h^3 + 3h^2 + 3h) - 1}{h}$$

$$\lim_{h \to 0^+} \frac{h^3 + 3h^2 + 3h}{h} = \lim_{h \to 0^+} (h^2 + 3h + 3) = 3$$

Similarly,

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^+} \frac{(1 + h^2 + 2h) - 1}{h}$$

$$\lim_{h \to 0^+} \frac{h^2 + 2h}{h} = \lim_{h \to 0^+} (h + 2) = 2$$

Hence,

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \neq \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

So $\lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h}$ does not exist. Therefore the function f is not differentiable at $x = 1$. Similar argument can be given for $x = 0$.

5. Let f and g be two real valued functions defined as:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = e^x - 1$$

$$g : \mathbb{R} \to \mathbb{R}$$

$$g(x) = x$$

Choose the set of correct options.

- ○ **Option 1:** The linear function $ex - 1$ is the best linear approximation of the function $f(x)$ at the point $x = 1$.
- ○ Option 2: In this case, $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} g(x)}$.
- ○ **Option 3:** In this case, $(f+g)$ (where, $(f+g)(x)$ is defined by $f(x)+g(x)$) is continuous at $x = 0$.
- ○ **Option 4:** $\lim\limits_{x\to 0} f(x)g(x) = 0$.

Solution:

Given,

$$f(x) = e^x - 1$$

The linear approximation for this function at $x = 1$ would be

$$y = f'(1)(x-1) + f(1) = e^1(x-1) + e^1 - 1 = ex - 1$$

In option 2 given,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} g(x)}$$

LHS:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{e^x - 1}{x}$$

This is 0 divided by 0 case, therefore, we can use L'Hôpital's rule,

$$LHS = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{e^x}{1} = 1$$

Now

$$RHS = \frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} g(x)}$$

$RHS = \frac{\lim\limits_{x \to 0} e^x - 1}{\lim\limits_{x \to 0} x}$ is in indeterminate form, as both the numerator and denominator are 0.
Therefore,

$$LHS \neq RHS$$

In option 3,

$$(f+g)(x) = e^x - 1 + x$$

$$LHL = \lim_{x \to 0^-} (f+g)(x) = \lim_{x \to 0^-} (e^x - 1 + x) = 0$$

$$RHL = \lim_{x \to 0^+} (f+g)(x) = \lim_{x \to 0^+} (e^x - 1 + x) = 0$$

$$(f+g)(0) = 0$$

Therefore, $(f+g)(x)$ is continuous at $x = 0$.

In option 4,

$$\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} (e^x - 1)(x) = 0$$

3 Numerical Answer Type (NAT)

6. Let f be a differentiable function at $x = 0$. The tangent line to the curve represented by the function f at the point $(0, 5)$ passes through the point $(1, 5)$. What will be the value of $f'(0)$? [Answer: 0]

Solution:

Slope of the tangent line (as the line passes through (0,5) and (1,5)):

$$f'(0) = \frac{5-5}{1-0} = 0$$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x - 5$. Find the value of $(fg)'(0) - (f \circ g)'(0)$, where $f \circ g(x) = f(g(x))$ and $fg(x) = f(x)g(x)$. [Answer: 10]

   Solution:

   Take,

$$(fg)(x) = f(x)g(x) = x^2(x-5) = x^3 - 5x^2$$
$$(fg)'(x) = 3x^2 - 10x$$
$$(fg)'(0) = 0$$

   Now,

$$(f \circ g)(x) = (x-5)^2$$
$$(f \circ g)'(x) = 2(x-5)(1)$$
$$(f \circ g)'(0) = 2(0-5) = -10$$

   Therefore,

$$(fg)'(0) - (f \circ g)'(0) = 10$$

4 Comprehension Type Question:

The profit of Company A with respect to time (in months) is given by the function $f(t)$ (in lakhs) as follows:

$$f(t) = \begin{cases} \frac{(t-2)^n-1}{t-3} & \text{if } 0 \leq t < 3, \\ \lfloor t \rfloor & \text{if } t \geq 3 \end{cases}$$

for some integer n.
The profit of Company B with respect to time (in months) is given by the function $g(t)$ (in lakhs) as follows:

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^m & \text{if } t \geq 3 \end{cases}$$

for some integer m.
Use the information given above answer Questions 8,9 and 10.

8. If the functions $f(t)$ and $g(t)$ denoting the profits of Company A and Company B, respectively, are known to be continuous at $t = 3$, then what will be the values of n and m? (MCQ)

   - ○ Option 1: $n = 2$, and $m = 2$
   - ○ Option 2: $n = 2$, and $m = 3$
   - ○ **Option 3:** $n = 3$, and $m = 2$
   - ○ Option 4: $n = 3$, and $m = 3$

   Solution:

   Given,

   $$f(t) = \begin{cases} \frac{(t-2)^n-1}{t-3} & \text{if } 0 \leq t < 3, \\ \lfloor t \rfloor & \text{if } t \geq 3 \end{cases}$$

   For $f(t)$ to be continuous at $t = 3$,

   $$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^+} f(t) = f(3)$$

   $$\lim_{t \to 3^-} f(t) = \lim_{t \to 3^-} \frac{(t-2)^n - 1}{t-3}$$

   Using L'Hospital rule we get,

   $$\lim_{t \to 3^-} \frac{n(t-2)^{(n-1)}}{1} = n$$

   Further we have,

   $$\lim_{t \to 3^+} f(t) = \lim_{t \to 3^+} \lfloor t \rfloor = 3$$

Hence we have $n = 3 = f(3)$.

Now,

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^m & \text{if } t \geq 3 \end{cases}$$

For $g(t)$ to be continuous at $t = 3$,

$$\lim_{t \to 3^-} g(t) = \lim_{t \to 3^+} g(t) = g(3)$$

$$\lim_{t \to 3^-} g(t) = \lim_{t \to 3^-} \frac{t^3 - 3^3}{t - 3}$$

Using L'Hospital rule we get,

$$\lim_{t \to 3^-} \frac{3t^2}{1} = 27$$

Further we have,

$$\lim_{t \to 3^+} g(t) = \lim_{t \to 3^+} 3t^m = 3^{(m+1)}$$

Again we have $f(3) = 3^{(m+1)}$.
Therefore, $3^{(m+1)} = 27 \implies m + 1 = 3 \implies m = 2$.

9. Assuming g to be continuous at $t = 3$, choose the correct option from the following. (MCQ)

   - ◯ Option 1: $\lim_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 18$ and $\lim_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 18$, hence g is differentiable at $t = 3$.
   - ◯ Option 2: $\lim_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 9$ and $\lim_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 9$, hence g is differentiable at $t = 3$.
   - ◯ Option 3: $\lim_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 18$ and $\lim_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 9$, hence g is not differentiable at $t = 3$.
   - ◯ **Option 4:** $\lim_{t\to 3-} \frac{g(t)-g(3)}{t-3} = 9$ and $\lim_{t\to 3+} \frac{g(t)-g(3)}{t-3} = 18$, hence g is not differentiable at $t = 3$.

   Solution:

   As g is continuous at $t = 3$, we have

$$g(t) = \begin{cases} \frac{t^3-3^3}{t-3} & \text{if } 0 \leq t < 3, \\ 3t^2 & \text{if } t \geq 3 \end{cases}$$

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{\frac{t^3-3^3}{t-3} - (3 \times 3^2)}{t-3}$$

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{t^3 - 27 - 27t + 81}{(t-3)^2}$$

   Using L'Hospital rule two times consecutively,

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^-} \frac{3t^2-27}{2(t-3)} = \lim_{t\to 3^-} \frac{6t}{2} = 9$$

   Similarly we can calculate,

$$\lim_{t\to 3^+} \frac{g(t)-g(3)}{t-3} = \lim_{t\to 3^+} \frac{3t^2-27}{t-3} = \lim_{t\to 3^+} \frac{6t}{1} = 18$$

   As,

$$\lim_{t\to 3^-} \frac{g(t)-g(3)}{t-3} \neq \lim_{t\to 3^+} \frac{g(t)-g(3)}{t-3},$$

   g is not differentiable at $t = 3$.

10. Which of the following linear functions denotes the best linear approximation $L_f(t)$ of the function $f(t)$ at the point $t = 1$, assuming f to be continuous at $t = 3$? (MCQ)

   - ◯ **Option 1:** $L_f(t) = 2 - t$
   - ◯ Option 2: $L_f(t) = -t$
   - ◯ Option 3: $L_f(t) = 2 + t$
   - ◯ Option 4: $L_f(t) = -2 - t$

Solution:

For $t = 1 < 3$ and continuous at $t = 3$ means $n = 3$, then

$$f(t) = \frac{(t-2)^3 - 1}{t-3}$$

$$L_f(t) = f'(1)(t-1) + f(1)$$

$$f'(t) = \frac{3(t-2)^2(t-3) - ((t-2)^3 - 1)}{(t-3)^2}$$

$$f'(1) = \frac{-6 - (-2)}{4} = -1$$

And

$$f(1) = 1$$

Therefore,

$$L_f(t) = -1(t-1) + f(1)$$

$$L_f(t) = 2 - t$$

Week-2
Mathematics for Data Science - 2
Limits, Continuity, Differentiability, and the derivative
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Match the given functions in Column A with the equations of their tangents at the origin $(0, 0)$ in column B and the plotted graphs and the tangents in Column C, given in Table M2W2G1.

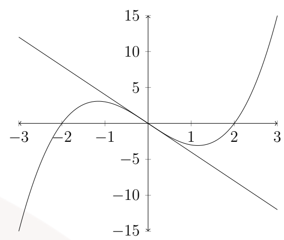
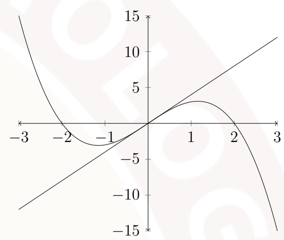
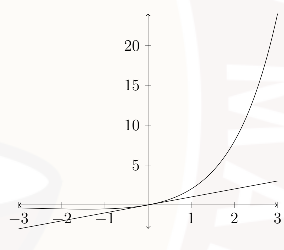
	Function (Column A)		It's tangent at (0,0) (Column B)		Graph (Column C)
i)	$f(x) = x2^x$	a)	$y = -4x$	1)	
ii)	$f(x) = x(x-2)(x+2)$	b)	$y = x$	2)	
iii)	$f(x) = -x(x-2)(x+2)$	c)	$y = 4x$	3)	

Table: M2W2G1

- ◯ Option 1: i) → b) → 3, ii) → c) → 1), iii) → a) → 2.
- ◯ **Option 2:** i) → b) → 3, ii) → a) → 1), iii) → c) → 2.
- ◯ Option 3: i) → b) → 3, ii) → a) → 2), iii) → c) → 1.
- ◯ Option 4: i) → c) → 3, ii) → a) → 1), iii) → b) → 2.

Solution:

i) Given $f(x) = x2^x \implies f'(x) = 2^x + x2^x \ln 2$.
So, $f(0) = 0$ and $f'(0) = 1$
Hence the equation of the tangent at the origin is

$$y - 0 = 1.(x - 0) \implies y = x.$$

In Column C, figure 3 has the line $y = x$ and exponential graph.
Hence i) $\rightarrow$ b) $\rightarrow$3).
ii) Given $f(x) = x(x-2)(x+2) = x^3 - 4x \implies f'(x) = 3x^2 - 4$.
So, $f(0) = 0$ and $f'(0) = -4$
Hence the equation of the tangent at the origin is

$$y - 0 = -4(x - 0) \implies y = -4x.$$

In Column C, figure 1 has the line $y = -4x$.
Hence ii) $\rightarrow$ a) $\rightarrow$1).
iii) Given $f(x) = -x(x-2)(x+2) = -x^3 + 4x \implies f'(x) = -3x^2 + 4$.
So, $f(0) = 0$ and $f'(0) = 4$
Hence the equation of the tangent at the origin is

$$y - 0 = 4(x - 0) \implies y = 4x$$

In Column C, figure 2 has the line $y = 4x$.
Hence iii) $\rightarrow$ c) $\rightarrow$2).

2 Multiple Select Questions (MSQ)

2. Consider the following two functions $f(x)$ and $g(x)$.

$$f(x) = \begin{cases} \frac{x^3-9x}{x(x-3)} & \text{if } x \neq 0, 3 \\ 3 & \text{if } x = 0 \\ 0 & \text{if } x = 3 \end{cases}$$

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \lfloor x \rfloor & \text{if } x > 2 \end{cases}$$

Choose the set of correct options.

- ◯ Option 1: $f(x)$ is discontinuous at both $x = 0$ and $x = 3$.
- ◯ Option 2: $f(x)$ is discontinuous only at $x = 0$.
- ◯ **Option 3:** $f(x)$ is discontinuous only at $x = 3$.
- ◯ Option 4: $g(x)$ is discontinuous at $x = 2$.
- ◯ **Option 5:** $g(x)$ is discontinuous at $x = 3$.

Solution:
(Options 1,2,3)
Given

$$f(x) = \begin{cases} \frac{x^3-9x}{x(x-3)} & \text{if } x \neq 0, 3 \\ 3 & \text{if } x = 0 \\ 0 & \text{if } x = 3 \end{cases}$$

Now, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{x^3-9x}{x(x-3)} = \lim\limits_{x \to 0} \frac{x(x-3)(x+3)}{x(x-3)} = \lim\limits_{x \to 0} x + 3 = 3 = f(0)$.
So $f(x)$ is continuous at $x = 0$.
Similarly, $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \frac{x^3-9x}{x(x-3)} = \lim\limits_{x \to 3} \frac{x(x-3)(x+3)}{x(x-3)} = \lim\limits_{x \to 3} x + 3 = 6 \neq f(3)$.
So $f(x)$ is not continuous at $x = 3$.
(Option 5)
Given

$$g(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ \lfloor x \rfloor & \text{if } x > 2 \end{cases}$$

Observe that, as $x > 2$, $g(x) = \lfloor x \rfloor$. And $\lim\limits_{x \to 3^+} g(x) = 3 \neq 2 = \lim\limits_{x \to 3^-} g(x)$. i.e, $\lim\limits_{x \to 3} g(x)$ does not exist.
Hence $g(x)$ is discontinuous at $x = 3$.
(Option 4)
Observe that $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} \lfloor x \rfloor = 2$
and $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} |x| = 2$.

Hence, $\lim_{x \to 2^+} g(x) = 2 = \lim_{x \to 2^-} g(x)$
i.e., $\lim_{x \to 2} g(x) = 2 = g(2)$.
So $g(x)$ is continuous at $x = 2$.

3. Consider the graphs given below:

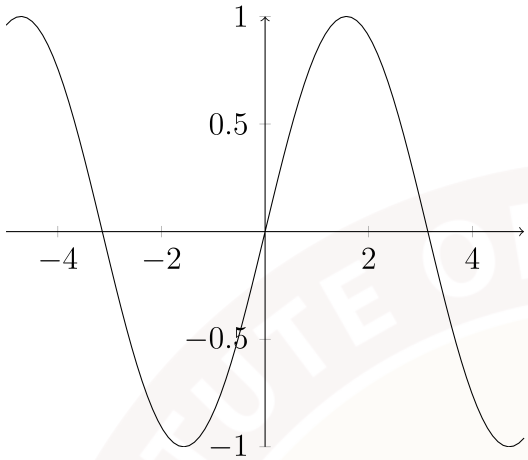


Figure: Curve 1

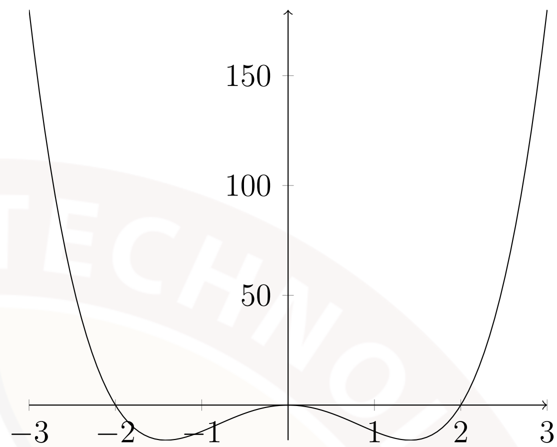


Figure: Curve 2

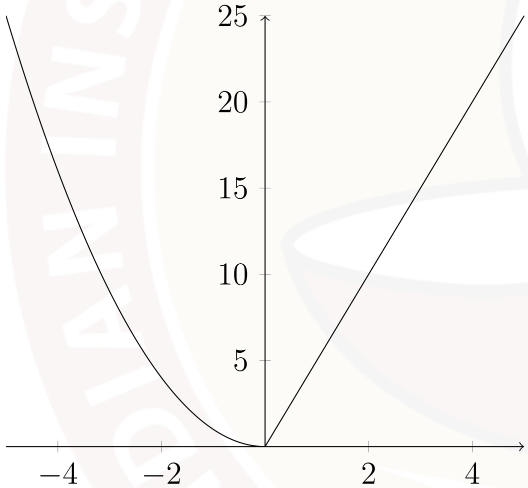


Figure: Curve 3

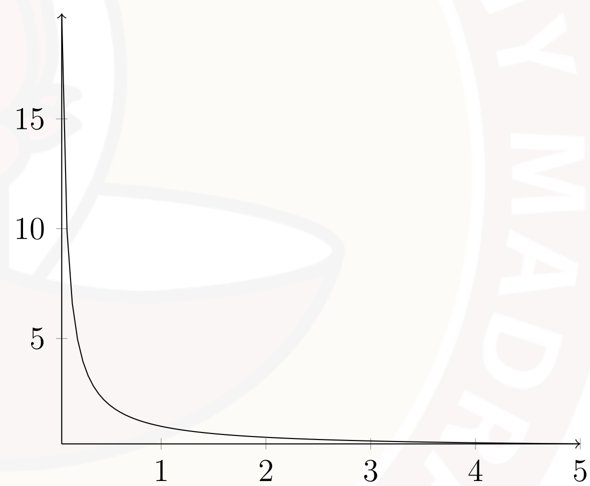


Figure: Curve 4

Choose the set of correct options.

- ◯ **Option 1:** Curve 1 is both continuous and differentiable at the origin.
- ◯ Option 2: Curve 2 is continuous but not differentiable at the origin.
- ◯ **Option 3:** Curve 2 has derivative 0 at $x = 0$.
- ◯ **Option 4:** Curve 3 is continuous but not differentiable at the origin.
- ◯ Option 5: Curve 4 is not differentiable anywhere.
- ◯ Option 6: Curve 4 has derivative 0 at $x = 0$.

Solution:
Option 1: Observe that if x approaches 0 from the left or from the right the value of the function represented by Curve 1 approaches 0. So, the limit of the function exists at $x = 0$ which is 0. And since the value of the function $f(x)$ is 0 at $x = 0$, the function represented by Curve 1 is continuous at $x = 0$.
And we can be draw a unique tangent to Curve 1 at the origin as shown in Figure M2W2GS (also observe that at $x = 0$, there does not exist any sharp corner).
Hence function is differentiable at the origin.

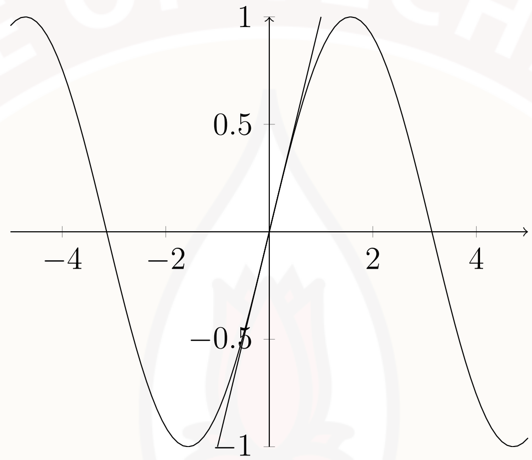


Figure M2W2GS

Options 2, 3: Observe that there is a unique tangent to the curve at the origin which is the $X-$axis itself and we know that slope of the $X-$axis is zero. Hence function represented by Curve 2 is differentiable with zero derivative at the origin.
And we know that a differentiable function is continuous.
Hence function represented by Curve 2 is continuous at the origin.

Option 4: Observe that there is sharp corner on Curve 3 at the origin. So function represented by Curve 3 is not differentiable at the origin.
But if x approaches 0 from the left or from the right the value of the function represented by Curve 3 approaches 0. So, the limit of the function exists at $x = 0$ which is 0. And since the value of the function $f(x)$ is 0 at $x = 0$, the function represented by Curve 3 is continuous at $x = 0$.
Option 6: If the derivative of the function represented by Curve 4 is 0 at the origin then at the origin the slope of of the tangent must be 0 i.e., the tangent must be parallel to the $X-$axis. For Curve 4, the tangent (if at all it exists) at the origin can never be parallel to the $X-$axis. Hence this statement is not true.

Option 5: Observe that at $x = 1$, there does not exist any sharp corner and at that

point, there exists a unique tangent (which is not vertical).
Hence function represented by Curve 4 is differentiable at $x = 1$.
Hence option 5 is not true.

4. Choose the set of correct options considering the function given below:

$$f(x) = \begin{cases} \frac{sin\ x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$$

- ◯ Option 1: $f(x)$ is not continuous at $x = 0$.
- ◯ **Option 2:** $f(x)$ is continuous at $x = 0$.
- ◯ Option 3: $f(x)$ is not differentiable at $x = 0$.
- ◯ **Option 4:** $f(x)$ is differentiable at $x = 0$.
- ◯ **Option 5:** The derivative of $f(x)$ at $x = 0$ (if exists) is 0.
- ◯ Option 6: The derivative of $f(x)$ at $x = 0$ (if exists) is 1.

Solution:
We know that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$. So $f(x)$ is continuous at $x = 0$.
Hence option 2 is true.
Now, $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{f(h)-1}{h} = \lim\limits_{h\to 0} \frac{\frac{\sin h}{h}-1}{h} = \lim\limits_{h\to 0} \frac{\sin h - h}{h^2} = \lim\limits_{h\to 0} \frac{\cos h - 1}{2h} = \lim\limits_{h\to 0} \frac{-\sin h}{2} = 0$
(using L'Hopital's rule twice).
Hence the derivative of $f(x)$ at $x = 0$ is 0.
So options 4 and 5 are true.

5. Let f be a polynomial of degree 5, which is given by

$$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Let $f'(b)$ denote the derivative of f at $x = b$. Choose the set of correct options.

- ◯ **Option 1:** $a_1 = f'(0)$
- ◯ **Option 2:** $5a_5 + 3a_3 = \frac{1}{2}(f'(1) + f'(-1) - 2f'(0))$
- ◯ **Option 3:** $4a_4 + 2a_2 = \frac{1}{2}(f'(1) - f'(-1))$
- ◯ Option 4: None of the above.

Solution:
Given $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \implies f'(x) = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$
So $f'(0) = a_1$, $f'(1) = 5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1$, and $f'(-1) = 5a_5 - 4a_4 + 3a_3 - 2a_2 + a_1$
Hence $5a_5 + 3a_3 = \frac{1}{2}(f'(1) + f'(-1) - 2f'(0))$ and $4a_4 + 2a_2 = \frac{1}{2}(f'(1) - f'(-1))$

3 Numerical Answer Type (NAT)

6. Let f be a differentiable function at $x = 3$. The tangent line to the graph of the function f at the point $(3, 0)$, passes through the point $(5, 4)$. What will be the value of $f'(3)$?
   [Answer: 2]
   Solution: slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.
   So Slope of the tangent at $x = 3$ is, $\frac{4-0}{5-3} = 2$.
   Since derivative of a function at a point equals the slope of the tangent at that point.
   Hence $f'(3) = 2$

7. Let f and g be two functions which are differentiable at each $x \in \mathbb{R}$. Suppose that, $f(x) = g(x^2 + 5x)$, and $f'(0) = 10$. Find the value of $g'(0)$. [Answer: 2]

Solution:

Given $f(x) = g(x^2 + 5x) \implies f'(x) = g'(x^2 + 5x)(2x + 5)$

So $f'(0) = 5g'(0) \implies g'(0) = \frac{10}{5} = 2$

4 Comprehension Type Questions:

The population of a bacteria culture of type A in laboratory conditions is known to be a function of time of the form

$$p : \mathbb{R} \to \mathbb{R}$$

$$p(t) = \begin{cases} \frac{t^3-27}{t-3} & \text{if } 0 \leq t < 3, \\ 27 & t = 3 \\ \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) & \text{if } t > 3 \end{cases}$$

where $p(t)$ represents the population (in lakhs) and t represents the time (in minutes).
The population of a bacteria culture of type B in laboratory conditions is known to be a function of time of the form

$$q : \mathbb{R} \to \mathbb{R}$$

$$q(t) = \begin{cases} (5t-9)^{\frac{1}{t-2}} & \text{if } 0 \leq t < 2, \\ e^4 & t = 2 \\ \frac{e^{t+2}-e^4}{t-2} & \text{if } t > 2 \end{cases}$$

where $q(t)$ represents the population (in lakhs) and t represents the time (in minutes).
Using the above information, answer the questions 8,9, and 10.

8. Choose the correct option from the following (a function is said to be continuous if it is continuous at all the points in the domain of the function). (MCQ)

   - ◯ Option 1: Both the functions $p(t)$ and $q(t)$ are continuous.
   - ◯ **Option 2:** $p(t)$ is continuous, but $q(t)$ is not.
   - ◯ Option 3: $q(t)$ is continuous, but $p(t)$ is not.
   - ◯ Option 4: Neither $p(t)$ nor $q(t)$ is continuous.

   Solution:
   Given

$$p(t) = \begin{cases} \frac{t^3-27}{t-3} & \text{if } 0 \leq t < 3, \\ 27 & t = 3 \\ \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) & \text{if } t > 3 \end{cases}$$

and

$$q(t) = \begin{cases} (5t-9)^{\frac{1}{t-2}} & \text{if } 0 \leq t < 2, \\ e^4 & t = 2 \\ \frac{e^{t+2}-e^4}{t-2} & \text{if } t > 2 \end{cases}$$

It is enough to check the continuty of $p(t)$ at $t = 3$ and of $q(t)$ at $t = 2$.
So right limit, $\lim\limits_{t\to 3^+} p(t) = \lim\limits_{t\to 3^+} \frac{1}{e^{81}(t-3)}(e^{27t} - e^{81}) = \lim\limits_{t\to 3^+} \frac{27e^{27t}}{e^{81}} = 27$ (Using L'Hopital's rule).

Left limit, $\lim\limits_{t\to3^-} p(t) = \lim\limits_{t\to3^-} \frac{t^3-27}{t-3} = \lim\limits_{t\to3^-} 3t^2 = 27$
Hence, $\lim\limits_{t\to3^-} p(t) = \lim\limits_{t\to3^+} p(t) = 27 = p(3)$.
So $p(t)$ is continuous at $x = 3$.
Now right limit, $\lim\limits_{t\to2^+} q(t) = \lim\limits_{t\to2^+} \frac{e^{t+2}-e^4}{t-2} = \lim\limits_{t\to2^+} e^{t+2} = e^4$ (using L'Hopital's rule).
Left limit, $\lim\limits_{t\to2^-} q(t) = \lim\limits_{t\to2^-} (5t-9)^{\frac{1}{t-2}}$, to get the left limit,
let $y = (5t-9)^{\frac{1}{t-2}}$.
Taking *log* with base e on both sides and $t > \frac{9}{5}$,
we get, $\ln y = \frac{\ln(5t-9)}{t-2} \implies \lim\limits_{t\to2^-} \ln y = \lim\limits_{t\to2^-} \frac{\ln(5t-9)}{t-2} = \lim\limits_{t\to2^-} \frac{5}{5t-9} = 5$ (using L'Hopital's rule)
Hence, $\lim\limits_{t\to2^-} \ln y = 5 \implies \lim\limits_{t\to2^-} y = e^5$.
So $\lim\limits_{t\to2^-} (5t-9)^{\frac{1}{t-2}} = e^5$.
Since $\lim\limits_{t\to2^+} q(t) \neq \lim\limits_{t\to2^-} q(t)$ i.e., $\lim\limits_{t\to2} q(t)$ does not exists, $q(t)$ is not continuous at $t = 2$.
Hence option 2 true.

9. Which of the following linear functions denotes the best linear approximation $L_p(t)$ of the function $p(t)$ at the point $t = 1$? (MCQ)

   - ◯ Option 1: $L_p(t) = 3t + 10$
   - ◯ Option 2: $L_p(t) = 3t + 8$
   - ◯ **Option 3:** $L_p(t) = 5t + 8$
   - ◯ Option 4: $L_p(t) = 5t + 10$

   Solution:
   $p(t) = \frac{t^3-27}{t-3}$ if $0 \leq t < 3 \implies p(1) = 13$
   $p'(t) = \frac{(t-3)(3t^2)-(t^3-27)}{(t-3)^2} \implies p'(1) = 5.$
   Therefore the best linear approximation $L_p(t)$ of the function $p(t)$ at the point $t = 1$ is $L_p(t) = p(1) + p'(1)(t-1) = 13 + 5(t-1) = 5t + 8$

10. Which of the following linear functions denotes the best linear approximation $L_q(t)$ of the function $q(t)$ at the point $t = 3$? (MCQ)

- ◯ Option 1: $L_q(t) = e^5 t - 2e^5 - e^4$
- ◯ Option 2: $L_q(t) = e^5 t + e^5 - 4e^4$
- ◯ Option 3: $L_q(t) = e^4 t - 2e^5 - e^4$
- ◯ **Option 4:** $L_q(t) = e^4 t + e^5 - 4e^4$

Solution:

$q(t) = \frac{e^{t+2} - e^4}{t-2}$ if $t > 2 \implies q(3) = e^5 - e^4$

$q'(t) = \frac{(t-2)e^{t+2} - (e^{t+2} - e^4)}{(t-2)^2} \implies q(3) = e^4$

Therefore the best linear approximation $L_q(t)$ of the function $q(t)$ at the point $t = 3$ is $L_q(t) = q(3) + q'(3)(t-3) = e^5 - e^4 + e^4(t-3) = e^4 t + e^5 - 4e^4$

Week-3
Mathematics for Data Science - 2
Critical points, Area under the curve, Integration
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. Suppose a wire of length m is cut into two pieces. One part is bent into a circle and other into a square. The minimum value of the combined area of the circle and the square is

   ◯ Option 1: $\dfrac{m^2}{\pi+4}$

   ◯ **Option 2:** $\dfrac{m^2}{4(\pi+4)}$

   ◯ Option 3: $\dfrac{m^2}{\pi+2}$

   ◯ Option 4: $\dfrac{m^2}{2(\pi+2}$)

   Solution: Let the piece that is bent into a circle have length x and the remaining piece of wire that bent into a square have length $m-x$. Then radius of the circle is $r = \dfrac{x}{2\pi}$ and side of the square is $a = \dfrac{m-x}{4}$.

$$\begin{aligned} \text{A} \;=\; \text{Total area} &= \text{Area of the circle} + \text{Area of the square} \\ &= \pi r^2 + a^2 \\ &= \pi \times \left(\frac{x}{2\pi}\right)^2 + \left(\frac{m-x}{4}\right)^2 \\ &= \frac{x^2}{4\pi} + \frac{(m-x)^2}{16} \end{aligned}$$

To get the minima we can equate $\dfrac{dA}{dx}$ to 0.

$$\frac{dA}{dx} = \frac{x}{2\pi} - \frac{(m-x)}{8} = 0 \implies \frac{x}{2\pi} = \frac{(m-x)}{8} \implies x = \frac{m\pi}{4+\pi}$$

$\dfrac{d^2A}{dx^2} = \dfrac{1}{2\pi} + \dfrac{1}{8}$ is always greater than 0. Hence, $x = \dfrac{m\pi}{4+\pi}$ is a point of minimum.

Then the minimum value of the combined area is

$$
\begin{aligned}
A_{min} &= \frac{(m\pi)^2}{4\pi(4+\pi)^2} + \frac{(4m+m\pi-m\pi)^2}{16(4+\pi)^2} \\
&= \frac{(m\pi)^2}{4\pi(4+\pi)^2} + \frac{(4m)^2}{16(4+\pi)^2} \\
&= \frac{(m\pi)^2}{4\pi(4+\pi)^2} + \frac{m^2}{(4+\pi)^2} \\
&= \frac{m^2\pi}{4(4+\pi)^2} + \frac{m^2}{(4+\pi)^2} \\
&= \frac{m^2\pi + 4m^2}{4(4+\pi)^2} \\
&= \frac{m^2(\pi+4)}{4(4+\pi)^2} \\
&= \frac{m^2}{4(4+\pi)}
\end{aligned}
$$

2. Match the given functions in Column A with the (signed) area between its graph and the interval $[-1, 1]$ on the X-axis in column B and the pictures of their graphs and the highlighted region corresponding to the area computation in Column C, given in Table M2W3P1.

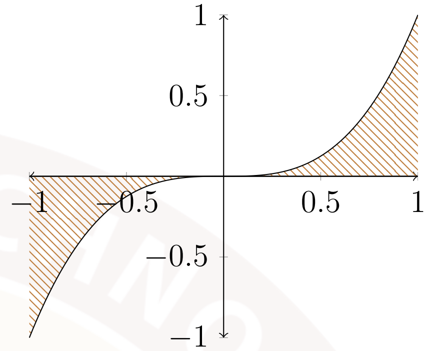
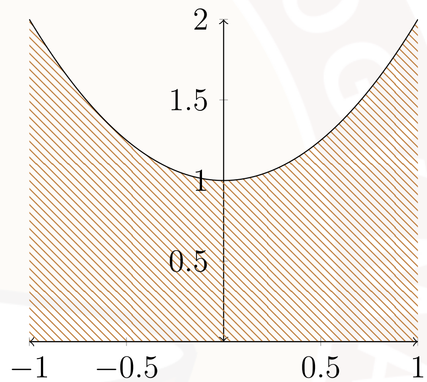
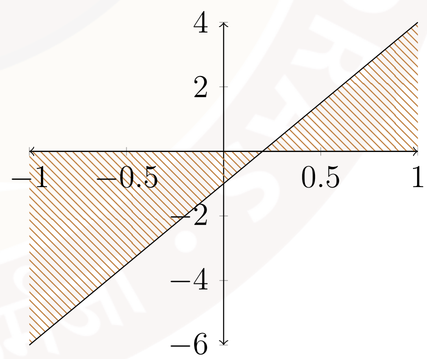
	Functions (Column A)		Area under the curve (Column B)		Graphs (Column C)
i)	$f(x) = 5x - 1$	a)	$\dfrac{\pi}{2}$	1)	
ii)	$f(x) = x^3$	b)	0	2)	
iii)	$f(x) = \dfrac{1}{x^2+1}$	c)	-2	3)	

Table: M2W3P1

◯ Option 1: i) → b) → 1), ii) → c) → 3), iii) → a) → 2).

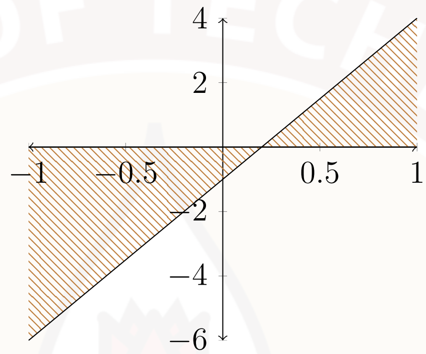
◯ Option 2: i) → b) → 3), ii) → a) → 1), iii) → c) → 2).

◯ **Option 3:** i) → c) → 3), ii) → b) → 1), iii) → a) → 2).

◯ Option 4: i) → b) → 3), ii) → c) → 1), iii) → a) → 2).

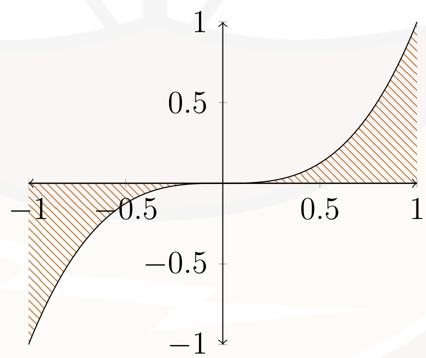
Solution:

- $f(x) = 5x - 1$



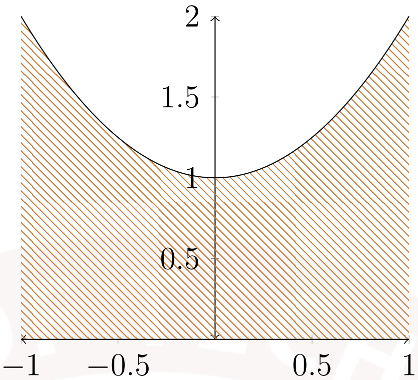
$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} (5x-1)\ dx = \int_{-1}^{1} 5x\ dx - \int_{-1}^{1} 1\ dx = (5 \times \frac{x^2}{2})\Big|_{-1}^{1} - x\Big|_{-1}^{1} = -2$$

- $f(x) = x^3$



$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} x^3\ dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

- $f(x) = \dfrac{1}{x^2+1}$



$$\int_{-1}^{1} f(x)\; dx = \int_{-1}^{1} \frac{1}{x^2+1}\; dx = \tan^{-1}(x)\Big|_{-1}^{1} = \tfrac{\pi}{2}$$

2 Multiple Select Questions (MSQ)

3. Suppose $\int x\ln(1+x)dx = f(x)\ln(x+1) - \frac{x^2}{4} + Ax + B$, where B is the constant of integration. Which of the following are correct?

   - ◯ **Option 1:** $f(x) = \frac{x^2-1}{2}$
   - ◯ Option 2: $f(x) = \frac{x^2-1}{4}$
   - ◯ Option 3: $A = \frac{1}{4}$
   - ◯ **Option 4:** $A = \frac{1}{2}$

   Solution: By using integration by parts:

$$
\begin{aligned}
\int x\ln(1+x)\ dx = \ln(1+x)\int x\ dx - \int\left\{\frac{d(\ln(1+x))}{dx}\int x\ dx\right\} dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2+x-x}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \int \frac{x^2+x}{2(1+x)}\ dx + \int \frac{x}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1-1}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \int \frac{x+1}{2(1+x)}\ dx - \int \frac{1}{2(1+x)}\ dx \\
&= \frac{x^2\ln(1+x)}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(1+x) + B \\
&= \frac{x^2-1}{2}\ln(1+x) - \frac{x^2}{4} + \frac{x}{2} + B
\end{aligned}
$$

   If we equate coefficients then $f(x) = \frac{x^2-1}{4}$, and $A = \frac{1}{2}$.

4. Consider the function $f(x) = x^3 - 6x$. Which of the following options are correct?
   - ◯ Option 1: f has neither local maxima nor local minima.
   - ◯ **Option 2**: $\sqrt{2}$ is a local minimum.
   - ◯ Option 3: $\sqrt{2}$ is a local maximum.
   - ◯ **Option 4:** $-\sqrt{2}$ is a local maximum.
   - ◯ Option 5: $-\sqrt{2}$ is a local minimum.
   - ◯ **Option 6:** f has two critical points.

   Solution: Number of critical points will be same as the number of solutions of the following equation,

$$f^{'}(x) = 3x^2 - 6 = 0 \implies x^2 - 2 = 0 \implies (x - \sqrt{2})(x + \sqrt{2}) = 0$$

Hence, the number of critical points is 2.

Now, $f^{''}(\sqrt{2}) > 0$, and $f^{''}(-\sqrt{2}) < 0$. Therefore, $\sqrt{2}$ is a local minimum and $-\sqrt{2}$ is a local maximum.

5. Choose the set of correct options.

 - ◯ **Option 1:** The left Riemann sum of the function $f(x) = x + 5$ on the interval $[1, 10]$ divided into three sub-intervals of equal length is 81.
 - ◯ **Option 2:** The middle Riemann sum of the function $f(x) = x^2$ on the interval $[0, 8]$ divided into four sub-intervals of equal length is 168.
 - ◯ **Option 3:** The left Riemann sum of the function $f(x) = x + 5$ on the interval $[3, 6]$ divided into n sub-intervals of equal length is $\frac{57}{2}$, as n tends to ∞.
 - ◯ Option 4: The right Riemann sum of the function $f(x) = \frac{1}{x}$ on the interval $[1, 9]$ divided into four sub-intervals of equal length is $\frac{16}{15}$.

Solution:

Option 1: If we divide $[1, 10]$ in three sub-intervals of equal length, we get the partition: $\{1, 4, 7, 10\}$. The left Riemann sum of the function $f(x) = x + 5$ is:

$$(4-1)f(1)+(7-4)f(4)+(10-7)f(7) = 3\times(f(1))+f(4)+f(7)) = 3\times(6+9+12) = 81$$

Option 2: If we divide $[0, 8]$ in four sub-intervals of equal length, we get the partition: $\{0, 2, 4, 6, 8\}$. The middle Riemann sum of the function $f(x) = x^2$ is:

$$(2-0)f(1)+(4-2)f(3)+(6-4)f(5)+(8-6)f(7) = 2\times(f(1))+f(3)+f(5)+f(7)) = 2 \times (1 + 9 + 25 + 49) = 168$$

Option 3: If we divide $[3, 6]$ in n sub-intervals of equal length, we get the partition: $\{3, 3 + \frac{3}{n}, 3 + \frac{6}{n} \ldots 6 - \frac{3}{n}, 6\}$. The left Riemann sum of the function $f(x) = x + 5$ is:

$$
\begin{aligned}
&\frac{3}{n}f(3)+\frac{3}{n}f\Big(3+\frac{3}{n}\Big)+\frac{3}{n}f\Big(3+\frac{6}{n}\Big)+\cdots+\frac{3}{n}f\Big(6-\frac{3}{n}\Big)\\
=&\frac{3}{n}\Bigg[(3+5)+(3+\frac{3}{n}+5)+(3+\frac{6}{n}+5)+\cdots+(6-\frac{3}{n}+5)\Bigg]\\
=&\Big(\frac{3}{n}\times 5n\Big)+\frac{3}{n}\Bigg[3+(3+\frac{3}{n})+(3+\frac{6}{n})+\cdots+(6-\frac{3}{n})\Bigg]\\
=&15+\frac{3}{n}\Bigg[3+(3+\frac{3}{n})+(3+\frac{6}{n})+\cdots+\Big(3+(3-\frac{3}{n})\Big)\Bigg]\\
=&15+\Big(\frac{3}{n}\times 3n\Big)+\frac{3}{n}\Bigg[\frac{3}{n}+\frac{6}{n}+\cdots+\frac{3n-3}{n}\Bigg]\\
=&15+9+\frac{9}{n^2}\Bigg[1+2+\cdots+(n-1)\Bigg]\\
=&24+\frac{9}{n^2}\frac{(n-1)n}{2}
\end{aligned}
$$

As n tends to ∞, the above sum converges to $24+\frac{9}{2}=\frac{57}{2}$.

Option 4: If we divide $[1,9]$ in four sub-intervals of equal length, we get the partition: $\{1,3,5,7,9\}$. The right Riemann sum of the function $f(x)=\frac{1}{x}$ is:

$(3-1)f(3)+(5-3)f(5)+(7-5)f(7)+(9-7)f(9) = 2\times(f(3))+f(5)+f(7)+f(9)) = 2\times(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}) = \frac{496}{315}$

3 Numerical Answer Type (NAT)

6. The value of $\int_0^{\frac{\pi^2}{4}} \sin\sqrt{x}\ dx$ is [Answer: 2]

Solution:

We make the substitution $t^2 = x \implies \frac{dx}{dt} = 2t \implies dx = 2tdt$, and the limits change to $t = \sqrt{0} = 0$ and $t = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$ The integral becomes

$$2\int_0^{\frac{\pi}{2}} t\sin t\ dt.$$

Now, $2\int_0^{\frac{\pi}{2}} t\sin t\ dt = \Big[-tcos(t) + sin(t)\Big]_0^{\frac{\pi}{2}} = 2$ [By using integration by parts].

7. Suppose $x + y = 16$. What is the value of xy when $x^3 + y^3$ is minimum? [Answer: 64]

Solution: It is given that $x + y = 16 \implies y = 16 - x$. so, $x^3 + y^3 = x^3 + (16 - x)^3$. Let $f(x) = x^3 + (16 - x)^3 = x^3 + 16^3 - x^3 - 768x + 48x^2 = 16^3 - 768x + 48x^2$. To get a minima we can equate $\dfrac{df}{dx}$ to 0.

$$\frac{df}{dx} = -768 + 96x = 0 \implies x = 8.$$

$\dfrac{d^2 f}{dx^2} = 96 > 0$. Hence, $x = 8$, and $y = 16 - x = 8$. The value of xy is 64.

4 Comprehension Type Question:

A car manufacturer determines that in order to sell x number of cars, the price per car(in lakh) must be $f(x) = 1000 - x$, if $x \leq 800$, and the manufacturer also determines that the total cost(in lakh) of producing x number of cars is

$$g(x) = \begin{cases} 30000 + 300x & \text{if } x \leq 400, \\ 100x + 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

Although in the above context, x can take only integer values, assume that x is a continuous variable in the interval $[0, 800]$ and that the functions $f(x)$ and $g(x)$ are defined as above on this entire interval.
Answer Questions 8,9, and 10 using the data given above.

8. Suppose the company can produce a maximum of 400 cars due to a production issue. The number of cars the company should produce and sell in order to maximize profit is

   ◯ **Option 1:** 350
   ◯ Option 2: 250
   ◯ Option 3: 300
   ◯ Option 4: 200

   Solution: If the company sells x number of cars then the total income is $I(x) = x(1000 - x)$. Total profit of the company is:

$$\text{Profit} = \text{Total income - Total cost}$$
$$P(x) = I(x) - g(x)$$

$$P(x) = \begin{cases} x(1000 - x) - (30000 + 300x) & \text{if } x \leq 400, \\ x(1000 - x) - (100x + 110000) & \text{if } 400 < x \leq 800, \end{cases}$$

   which is same as:

$$P(x) = \begin{cases} -x^2 + 700x - 30000 & \text{if } x \leq 400, \\ -x^2 + 900x - 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

   It is given that the company can produce a maximum of 400 cars due to a production issue, i.e, $x \leq 400$. So, the total profit is: $P(x) = -x^2 + 700x - 30000$.

   To get a maxima we can equate $\frac{dP}{dx}$ to 0.

$$\frac{dP}{dx} = -2x + 700 = 0 \implies x = 350.$$

$\frac{d^2P}{dx^2} = -2 < 0$. Hence $x = 350$ is a point of maximum. Therefore, company should produce and sell 350 numbers of cars in order to maximize its profit.

9. Suppose the company can produce a minimum of 401 cars and a maximum of 800 cars due to a production issue. The number of cars the company should produce and sell in order to maximize profit is

   - ◯ Option 1 750
   - ◯ Option 2: 650
   - ◯ Option 3: 550
   - ◯ **Option 4:** 450

   Solution: It is given that the company can produce a minimum of 401 cars and a maximum of 800 cars due to a production issue, i.e, $401 \leq x \leq 800$. So, the total profit is: $P(x) = -x^2 + 900x - 110000$.

   To get a maxima we can equate $\dfrac{dP}{dx}$ to 0.

   $$\frac{dP}{dx} = -2x + 900 = 0 \implies x = 450.$$

   $\dfrac{d^2P}{dx^2} = -2 < 0$. Hence $x = 450$ is a point of maximum. Therefore, company should produce and sell 450 numbers of cars in order to maximize its profit.

10. Let $P(x)$ denotes the function representing the profit of the company. Choose the set of correct statements.

- ◯ **Option 1:** $P(x)$ is continuous in the interval $[0, 800]$
- ◯ **Option 2:** The function $P(x)$ has two local maxima in the interval $[0, 800]$.
- ◯ Option 3: All the global maxima of $P(x)$ lie in the interval $[0, 400]$.
- ◯ **Option 4:** All the global maxima of $P(x)$ lie in the interval $[300, 500]$.

Solution:

$$P(x) = \begin{cases} -x^2 + 700x - 30000 & \text{if } x \leq 400, \\ -x^2 + 900x - 110000 & \text{if } 400 < x \leq 800 \end{cases}$$

Domain of P is $[0, 800]$. Clearly, P is continuous $[0, 400) \cup (400, 800]$. So, we need to check the continuity of the function only at $x = 400$.

$\mathbf{LHL\ of\ P(x)\ at\ x = 400}$:

$$\lim_{x \to 400^-} P(x) = \lim_{x \to 400^-} -x^2 + 700x - 30000 = -(400)^2 + 280000 - 30000 = 90000$$

$\mathbf{RHL\ of\ P(x)\ at\ x = 400}$:

$$\lim_{x \to 400^+} P(x) = \lim_{x \to 400^+} -x^2 + 900x - 110000 = -(400)^2 + 360000 - 110000 = 90000$$

Hence, $P(x)$ is continuous in the interval $[0, 800]$. From the solutions of Q8 and Q9, it is clear that the function $P(x)$ has two local maxima in the interval $[0, 800]$. Now,

$P(0) = -30000, P(350) = 92500, P(400) = 90000, P(450) = 92500,$ and $P(800) = -30000.$

So, both $x = 350$ and $x = 450$ are global maxima of $P(x)$.

Week-3
Mathematics for Data Science - 2
Critical points, Area under the curve, Integration
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Match the given functions in Column A with the (signed) area between its graph and the interval $[-1, 1]$ on the X-axis in column B and the pictures of their graphs and the highlighted region corresponding to the area computation in Column C, given in Table M2W3G1.

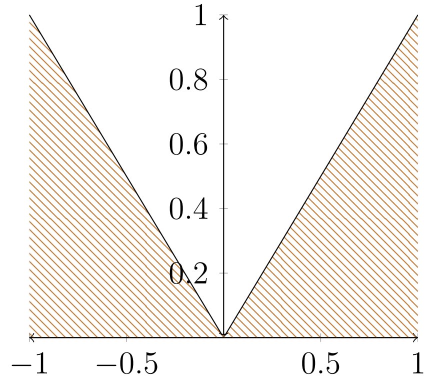
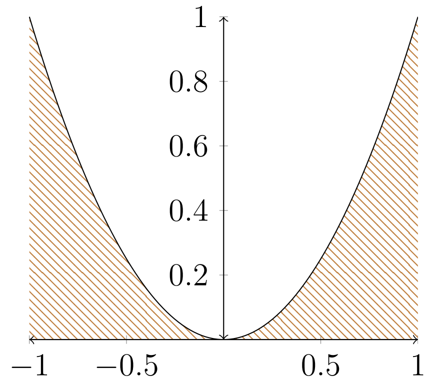
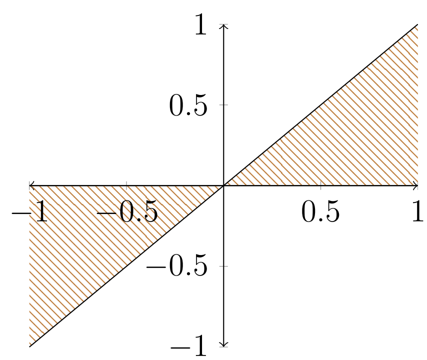
	Functions (Column A)		Area under the curve (Column B)		Graphs (Column C)
i)	$f(x) = x$	a)	$\frac{2}{3}$	1)	
ii)	$f(x) = \|x\|$	b)	0	2)	
iii)	$f(x) = x^2$	c)	1	3)	

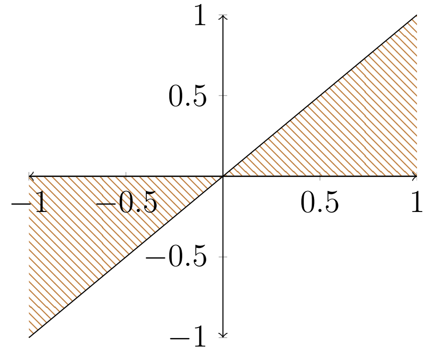
Table: M2W3G1

◯ Option 1: i) $\rightarrow$ b) $\rightarrow$ 1), ii) $\rightarrow$ c) $\rightarrow$ 3), iii) $\rightarrow$ a) $\rightarrow$ 2).

- ◯ Option 2: i) → b) → 3), ii) → a) → 1), iii) → c) → 2).
- ◯ Option 3: i) → c) → 3), ii) → b) → 1), iii) → a) → 2).
- ◯ **Option 4:** i) → b) → 3), ii) → c) → 1), iii) → a) → 2).

Solution:

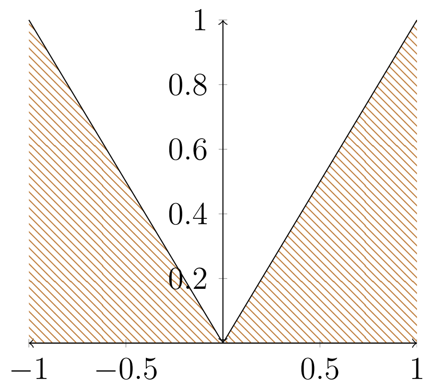
- $f(x) = x$



$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x\ dx = \frac{x^2}{2}\ \bigg|_{-1}^{1} = \left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

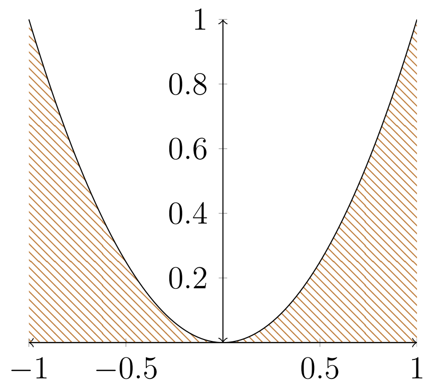
- $f(x) = |x|$

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$



$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} |x|\ dx = \int_{1}^{0} (-x)\ dx + \int_{0}^{1} x\ dx = -\frac{x^2}{2}\ \bigg|_{-1}^{0} + \frac{x^2}{2}\ \bigg|_{0}^{1}$$
$$= -\left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) = 1.$$

- $f(x) = x^2$



$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2 \, dx = \frac{x^3}{3} \Big|_{-1}^{1} = \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3}.$$

2 Multiple Select Questions (MSQ)

2. A cylinder of radius x and height $2h$ is to be inscribed in a sphere of radius R centered at O as shown in Figure M2W3G1

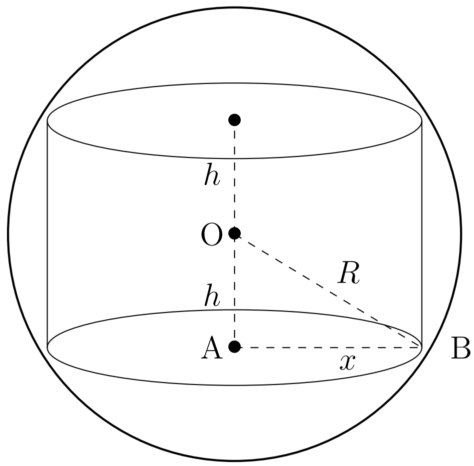


Figure M2W3G1

The volume of such a cylinder is given by $V = 2\pi x^2 h$ and the surface area of the outer curved surface is given by $S = 4\pi xh$. Choose the set of correct options.

- ○ Option 1: The cylinder has maximum volume amongst all cylinders which can be inscribed when $h = R$.
- ○ Option 2: The cylinder has maximum volume amongst all cylinders which can be inscribed when $h = \sqrt{3}R$.
- ○ **Option 3:** The cylinder has maximum volume amongst all cylinders which can be inscribed when $h = \frac{R}{\sqrt{3}}$.
- ○ Option 4: The cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed, when $h = 2R$.
- ○ **Option 5:** The cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed, when $h = \frac{R}{\sqrt{2}}$.
- ○ Option 6: The cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed, when $h = \sqrt{2}$.

Solution:
As the triangle ΔOAB is a right angle triangle, we have $h^2 + x^2 = R^2$, i.e., $x^2 = R^2 - h^2$.

For Volume:
The volume of the cylinder is $V = 2\pi(R^2 - h^2)h = 2\pi(R^2 h - h^3)$.
Hence, $\frac{dV}{dh} = 2\pi(R^2 - 3h^2)$.
To find the critical points, let us write the equation $\frac{dV}{dh} = 0$, i.e., $R^2 - 3h^2 = 0$, which

implies, $h = \frac{R}{\sqrt{3}}$ (as we can neglect the negative value of h).
Moreover we have, $\frac{d^2V}{dh^2} = 2\pi(-6h)$. For $h = \frac{R}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -12\pi\frac{R}{\sqrt{3}} < 0$.
So, $h = \frac{R}{\sqrt{3}}$ will give a maximum for V.
Therefore, the cylinder has maximum volume amongst all cylinders which can be inscribed when $h = \frac{R}{\sqrt{3}}$.

For Surface Area of the outer curved surface:
The surface area of the curved surface of the cylinder is $S = 4\pi xh = 4\pi h\sqrt{R^2 - h^2}$.
Hence, $\frac{dS}{dh} = 4\pi\left(\frac{1}{2}(-2h)(h)(R^2 - h^2)^{-\frac{1}{2}} + \sqrt{R^2 - h^2}\right)$
To find the critical points, let us write the equation $\frac{dS}{dh} = 0$, i.e.,

$$4\pi\left(-h^2(R^2 - h^2)^{-\frac{1}{2}} + \sqrt{R^2 - h^2}\right) = 0$$

So we have, $R^2 - h^2 = h^2$, i.e., $h = \frac{R}{\sqrt{2}}$.
Observe that, for $h = \frac{R}{\sqrt{2}}$, we have $\frac{d^2S}{dh^2} < 0$.
So, $h = \frac{R}{\sqrt{2}}$ will give a maximum for S.
Therefore, the cylinder has maximum surface area of its curved surface, amongst all cylinders which can be inscribed when $h = \frac{R}{\sqrt{2}}$.

3. Which of the following curves shown in the following figures enclose a negative area on the X axis in the interval $[0, 1]$?

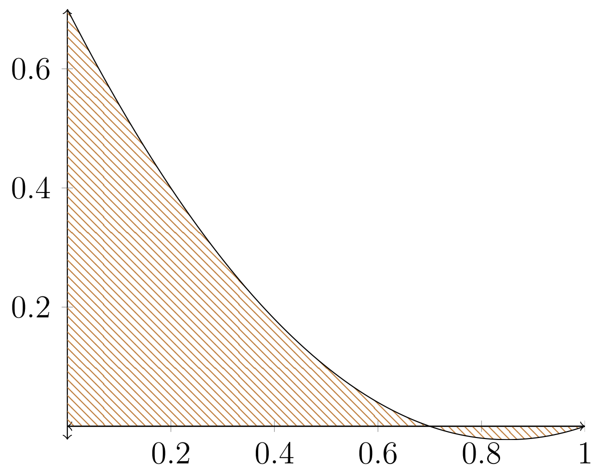


Figure: Curve 1

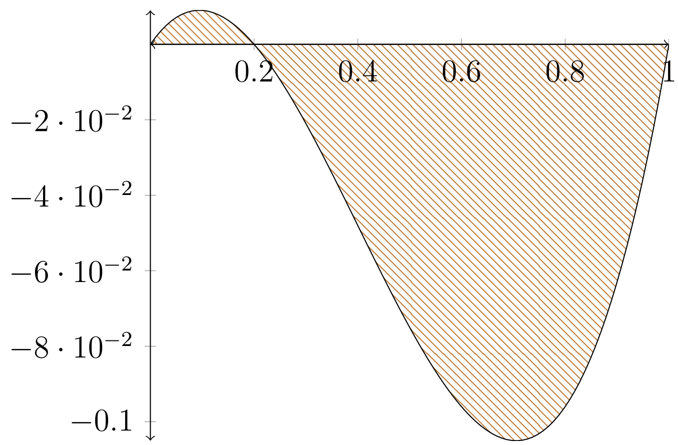


Figure: Curve 2

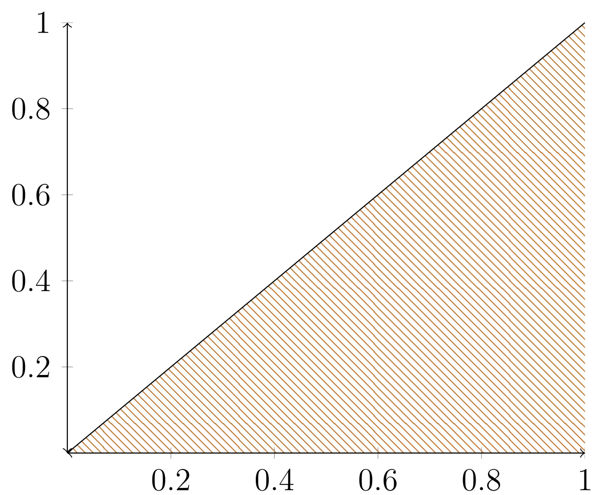


Figure: Curve 3

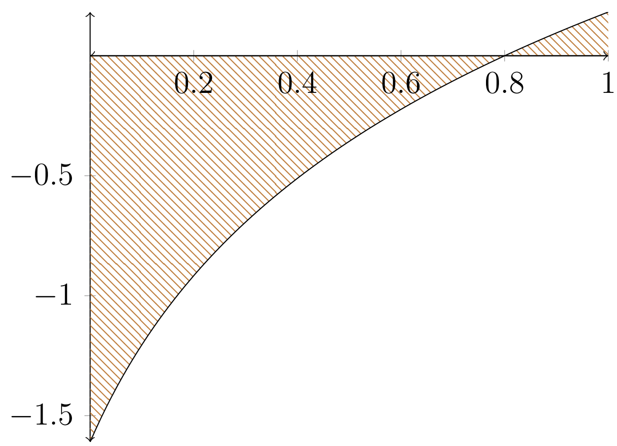


Figure: Curve 4

- ◯ Option 1: Curve 1
- ◯ **Option 2:** Curve 2
- ◯ Option 3: Curve 3
- ◯ **Option 4:** Curve 4

Solution:

The area enclosed above the X-axis, i.e., towards the positive direction of Y-axis, is positive, and the area enclosed below the X-axis, i.e., towards the negative direction of

Y-axis, is negative. So, if the portion of area enclosed by the curve above the X-axis is lesser than the portion of area enclosed by the curve below the X-axis, then the area enclosed by the curve as shown in the figure is negative. Hence, Curve 2 and Curve 4 enclose a negative area on the X-axis in the interval $[0, 1]$.

4. Suppose $\int x^2 sin\ 2x\ dx = Mx^2 cos\ 2x + Nx sin\ 2x + P cos\ 2x + C$, where C is the constant of integration. Which of the following are correct?

   - ◯ Option 1: $M = N = \frac{1}{2}$
   - ◯ **Option 2:** $M = -N = -\frac{1}{2}$
   - ◯ **Option 3:** $P = \frac{1}{4}$
   - ◯ Option 4: $P = 0$

Solution:

Using integration by parts:

$$\int x^2 sin\ 2x\ dx = x^2 \int sin\ 2x\ dx - \int \left(\frac{d}{dx}(x^2) \int sin\ 2x\ dx \right) dx$$

$$= x^2 \left(\frac{-cos\ 2x}{2} \right) - \int 2x \left(\frac{-cos\ 2x}{2} \right) dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + \int x\ cos\ 2x\ dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + x \int cos\ 2x\ dx - \int \left(\frac{d}{dx}(x) \int cos\ 2x\ dx \right) dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + x \left(\frac{sin\ 2x}{2} \right) - \int 1 \left(\frac{sin\ 2x}{2} \right) dx$$

$$= -\frac{1}{2} x^2 cos\ 2x + \frac{1}{2} x sin\ 2x + \frac{1}{4} cos\ 2x + C$$

where C is the constant of integration.

Hence, $M = -\frac{1}{2}$, $N = \frac{1}{2}$, and $P = \frac{1}{4}$.

5. Choose the set of correct options about estimating the area of the region bounded by the graph of function $f(x) = x^2 + 1$, above the interval [0,3] using Riemann sums.

   - ◯ **Option 1:** Estimated area will be 17 sq unit, by taking 3 subintervals of equal length and the right end points of the subintervals for the height of the rectangles.
   - ◯ Option 2: Estimated area will be 12 sq unit, by taking 3 subintervals of equal length and the left end points of the subintervals for the height of the rectangles.
   - ◯ **Option 3:** Estimated area will be $\frac{47}{4}$ sq unit, by taking 3 subintervals of equal length and the mid points points of the subintervals for the height of the rectangles.
   - ◯ **Option 4:** Estimated area will be 12 sq unit, by taking n subintervals of equal length and the right end points of the subintervals for the height of the rectangles, as n tends to ∞.

   Solution: If we divide $[0, 3]$ in 3 different sub-intervals of equal length, we get the partition: $\{0, 1, 2, 3\}$.

   - The estimated area by taking the right end points of the subintervals for the height of the rectangles is:
     $(1-0)f(1) + (2-1)f(2) + (3-2)f(3) = 1f(1) + 1(f(2) + 1f(3) = 2 + 5 + 10 = 17$ sq. units.
   - The estimated area by taking the left end points of the subintervals for the height of the rectangles is:
     $(1-0)f(0) + (2-1)f(1) + (3-2)f(2) = 1f(0) + 1(f(1) + 1f(2) = 1 + 2 + 5 = 8$ sq. units.
   - The estimated area by taking the mid points of the subintervals for the height of the rectangles is:
     $(1-0)f(\frac{1}{2})+(2-1)f(\frac{3}{2})+(3-2)f(\frac{5}{2}) = 1f(\frac{1}{2})+1(f(\frac{3}{2})+1f(\frac{5}{2}) = \frac{1}{4}+1+\frac{9}{4}+1+\frac{25}{4}+1 = \frac{47}{4}$ sq. units.

   If $[0, 3]$ is divided in n subintervals of equal length, then we get the partition:

   $$\left\{0, \frac{3}{n}, \frac{6}{n}, \frac{3(n-1)}{n}, \frac{3n}{n}\right\}$$

   The estimated area by taking the right end points of the subintervals for the height of the rectangles is:

   $$\frac{3}{n}f\left(\frac{3}{n}\right) + \frac{3}{n}f\left(\frac{6}{n}\right) + \ldots + \frac{3}{n}f\left(\frac{3(n-1)}{n}\right) + \frac{3}{n}f\left(\frac{3n}{n}\right)$$
   $$= \frac{3}{n}\left(\left(\frac{3}{n}\right)^2 1^2 + 1 + \left(\frac{3}{n}\right)^2 2^2 + 1 + \ldots + \left(\frac{3}{n}\right)^2 (n-1)^2 + 1 + \left(\frac{3}{n}\right)^2 n^2 + 1\right)$$

$$= \frac{3}{n}\left(\frac{3}{n}\right)^2\left(1^1 + 2^2 + \ldots + n^2\right) + \frac{3}{n}n$$

$$= \frac{3^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 3$$

$$= \frac{3^3}{6}1(1+\frac{1}{n})(2+\frac{1}{n}) + 3$$

As $n \to \infty$, this sum converges to $9 + 3 = 12$.

Hence the estimated area will be 12 sq unit, by taking n subintervals of equal length and the right end points of the subintervals for the height of the rectangles, as n tends to ∞.

Note: Observe that this value we obtained above is same as

$$\int_0^3 (x^2+1)\ dx$$

3 Numerical Answer Type (NAT)

6. Let $\int_0^{\frac{\pi}{2}} e^x sin\ x = \frac{1}{2}(e^{\frac{\pi}{2}} + a)$. What will be the value of a? [Answer: 1]

Solution: Let $I = \int_0^{\frac{\pi}{2}} e^x sin\ x\ dx$

By integrating by parts we get, $I = \int_0^{\frac{\pi}{2}} e^x sin\ x\ dx = -e^x cos\ x\ \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x cos\ x\ dx$

$$= -e^x cos\ x\ \Big|_0^{\frac{\pi}{2}} + e^x sin\ x\ \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x sin\ x\ dx$$

$$= -e^x cos\ x\ \Big|_0^{\frac{\pi}{2}} + e^x sin\ x\ \Big|_0^{\frac{\pi}{2}} - I = 1 + e^{\frac{\pi}{2}} - I$$

Hence, $2I = e^{\frac{\pi}{2}} + 1$, i.e., $I = \frac{1}{2}(e^{\frac{\pi}{2}} + 1)$.

Therefore, $a = 1$.

7. What will be the number of critical points of the function $f(x) = \frac{1}{6}(2x^3 + 3x^2 + 6)$? [Answer: 2]

Solution:
Number of critical points will be same as the number of solutions of the following equation,

$$f'(x) = x^2 + x = 0$$

The number of solutions of $x^2 + x$ is 2. Therefore, there are 2 critical points of the function $f(x) = \frac{1}{6}(2x^3 + 3x^2 + 6)$

4 Comprehension Type Question:

Suppose $f_1(x) = x^3$ and $f_2(x) = x$ denote the profits of Company A and Company B, respectively, throughout 1 year (the beginning of the year is denoted by $x = 0$ and the ending denoted by $x = 1$). The predicted profits of Company A and Company B of the same year are given by the functions $g_1(x) = \sqrt{x}$ and $g_2(x) = e^x$, respectively. The curves represented by the functions f_1 and g_1 are shown in Figure M2W3G2, and the curves represented by the functions f_2 and g_2 are shown in Figure M2W3G3.

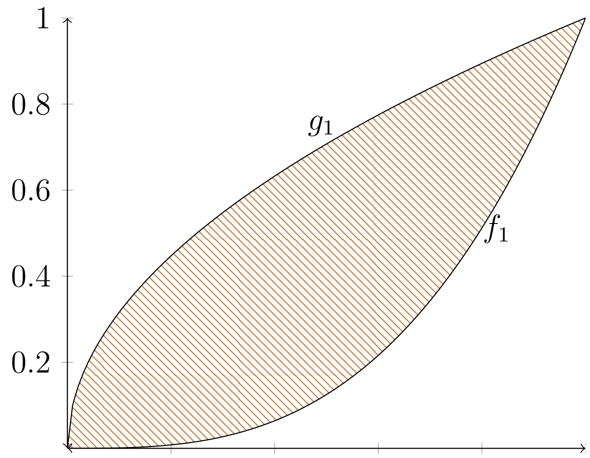


Figure: M2W3G2

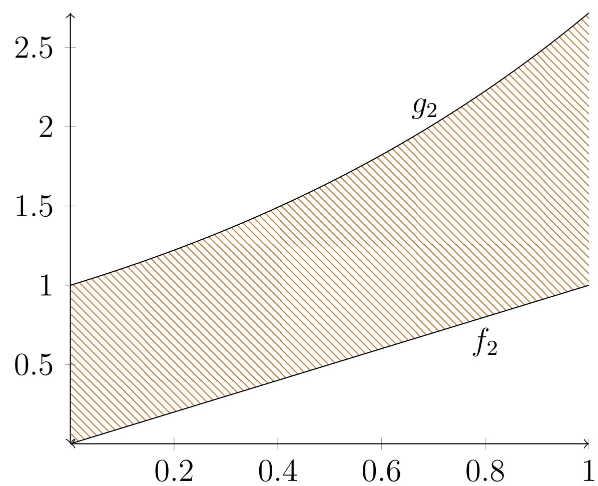


Figure: M2W3G3

Suppose the area of the region bounded by the two curves (the original curve and the predicted curve) in the interval $[0, 1]$ is defined to be the error in prediction.
Using the information above, answer the following questions.

8. What will be the difference between the minimum values of f_2 and g_2 in the interval $[0, 1]$?

   ○ Option 1: 0

   ○ **Option 2:** 1

   ○ Option 3: $e - 1$

   ○ Option 4: Cannot be determined from the given information.

   Solution: As both the functions f_2 and g_2 are increasing in the interval $[0, 1]$, the minimum values for both the functions will be at the origin, i.e. at $x = 0$. The minimum value of f_2 is 0 and the minimum value of g_2 is 1. Hence the difference between the minimum values of f_2 and g_2 will be 1, in the interval $[0, 1]$

9. What will be error in prediction for Company A?

- ◯ Option 1: $\frac{1}{4}$.
- ◯ Option 2: $\frac{2}{3}$
- ◯ **Option 3:** $\frac{5}{12}$
- ◯ Option 4: $\frac{11}{12}$

Solution: The error in prediction for Company A is the area enclosed by the functions f_1 and g_1 in the interval $[0, 1]$.
The area A_1 between the graph of f_1 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 f_1(x)\ dx = \int_0^1 x^3\ dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.$$

The area A_2 between the graph of g_1 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 g_1(x)\ dx = \int_0^1 \sqrt{x}\ dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

The area enclosed by the functions f_1 and g_1 in the interval $[0, 1]$ is$= A_2 - A_1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$.

10. Choose the correct option from the following.
    - ◯ Option 1: The error in prediction for company A is more than that for company B.
    - ◯ **Option 2:** The error in prediction for company B is more than that for company A.
    - ◯ Option 3: The errors in prediction for both companies are equal.
    - ◯ Option 4: The error in prediction for Company A and Company B, cannot be compared using the given information.

Solution: The error in prediction for Company B is the area enclosed by the functions f_2 and g_2 in the interval $[0, 1]$.
The area A_3 between the graph of f_2 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 f_2(x)\ dx = \int_0^1 x\ dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

The area A_4 between the graph of g_2 and the interval $[0, 1]$ in the X-axis is,

$$\int_0^1 g_2(x)\ dx = \int_0^1 e^x\ dx = e^x\Big|_0^1 = e - 1.$$

The area enclosed by the functions f_2 and g_2 in the interval $[0, 1]$ is$= A_4 - A_3 = e - 1 - \frac{1}{2} = e - \frac{3}{2}$.
Clearly, $e - \frac{3}{2} > \frac{5}{12}$.
Therefore, the error in prediction for company B is more than that for company A.

Week-4
Mathematics for Data Science - 2
Vectors and Matrices
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. If $A = \begin{bmatrix} 2x & 2z & y+x \\ -x+z & 2x & -y+z \\ x+y-2z & y-z-2x & 2y-2z \end{bmatrix}$, then $det(A)$ is

Hint:

- Replace column 1 with (column 1 - column 3)
  - ◯ Option 1: $(x-y)(y-z)(z-x)$
  - ◯ Option 2: $(x-y)(y+z)(z-x)$
  - ◯ **Option 3:** $(x-y)(y-z)(z+x)$
  - ◯ Option 4: $(x+y)(y-z)(z-x)$

Note:- Because of $det(A) = det(A^T)$, both the row operation and column operations have the same effects on the determinant of matrix A

Solution:

Given

$$A = \begin{bmatrix} 2x & 2z & y+x \\ -x+z & 2x & -y+z \\ x+y-2z & y-z-2x & 2y-2z \end{bmatrix}$$

Use the given hint, replacing column 1 with Column 1 - Column 3

$$det(A) = \begin{vmatrix} x-y & 2z & y+x \\ -x+y & 2x & -y+z \\ x-y & y-z-2x & 2y-2z \end{vmatrix}$$

$$= \begin{vmatrix} x-y & 2z & y+x \\ -(x-y) & 2x & -y+z \\ x-y & y-z-2x & 2y-2z \end{vmatrix}$$

Replacing, Row 3 + Row 2 in Row 3

$$\det(A) = \begin{vmatrix} x-y & 2z & y+x \\ -(x-y) & 2x & -y+z \\ 0 & y-z & y-z \end{vmatrix}$$

Replacing, Row 2 + Row1 in Row 2

$$\det(A) = \begin{vmatrix} x-y & 2z & y+x \\ 0 & 2(x+z) & x+z \\ 0 & y-z & y-z \end{vmatrix}$$

Observe, Row2 is multiple of $(x+z)$ & Row3 is multiple of $(y-z)$

$$\text{So, } \det(A) = (x+z)(y-z) \begin{vmatrix} x-y & 2z & y+x \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

To get determinant we can expand along the first column.

$$\text{So, } \det(A) = (x+z)(y-z)\Big[(x-y)(2-1) - 0\cdot(2z-(y+x)) + 0\,(2z-2(y+x))\Big]$$

$$\det(A) = (x+z)(y-z)(x-y)$$

Hence, third option is correct.

2. Match the systems of linear equations in Column A with their number of solutions in Column B and their geometric representation in Column C.

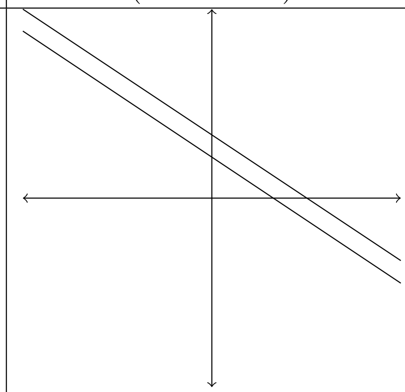
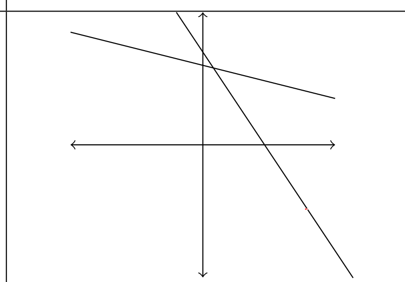
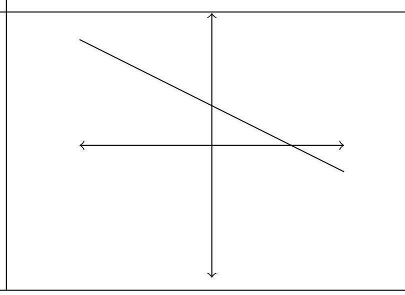
	System of linear equations (Column A)		Number of solutions (Column B)		Geometric representations (Column C)
i)	$3x + 2y = 7$, $x + 4y = 12$	a)	Infinite solutions	1)	
ii)	$2x + 3y = 5$, $8x + 12y = 13$	b)	No solution	2)	
iii)	$x + 2y = 3$, $4x + 8y = 12$	c)	Unique solution	3)	

Table: W4PT1

- ◯ **Option 1:** i) → c) → 2); ii) → b) → 1); iii) → a) → 3)
- ◯ Option 2: i) → c) → 1); ii) → b) → 3); iii) → a) → 2)
- ◯ Option 3: i) → b) → 3); ii) → c) → 2); iii) → a) → 1)
- ◯ Option 4: i) → a) → 3); ii) → b) → 1); iii) → c) → 2)

Solution:

System (i)

$$3x + 2y = 7$$
$$x + 4y = 12$$

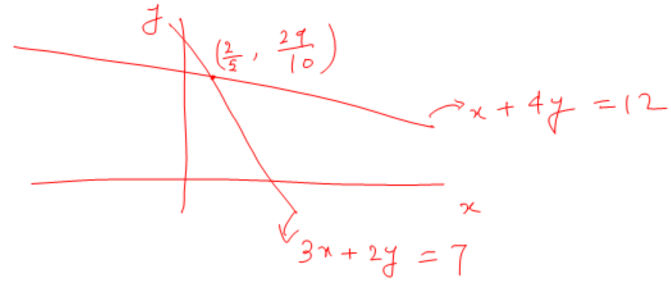
Multiplying 3 to the both sides of the second equation and substracting from first equation, will give $y = \frac{29}{10}$.

Similarly multiplying 2 to the (both sides of) first equation and substracting second equation from first equation, we'll give $x = \frac{2}{5}$

So, the system (i) has unique solution.

Geometrically, both equations in system (i) represent lines in X-Y co-ordinate plane and both lines intersect at a unique point and that is $\left(\frac{2}{5}, \frac{29}{10}\right)$

Below figure shows lines in X-Y co-ordinate plane.



So, (i) → (c) → (2)

System (ii)

$$2x + 3y = 5$$
$$8x + 12y = 13$$

Multiplying 4 to the both sides of the first equation, to get

$$8x + 12y = 20$$

& let the 2nd equation be as it is $8x + 12y = 13$

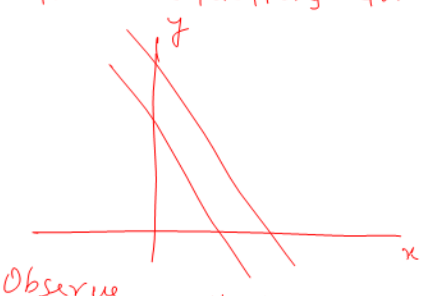
here, left sides of both the equations are the same.

So, $20 = 13$.

But this is not possible.

Hence, we can conclude that there is no solution for the system of linear equations (ii)

geometrically, these equations in the system (ii) represents parallel lines in X-Y plane. as shown in figure.



Observe that, there is no point of intersection of $8x + 12y = 13$ and $2x + 3y = 5$

So, there is no solution for system (ii)

So (ii) → (b) → (1)

System (iii)

$$x + 2y = 3$$
$$4x + 8y = 12$$

Multiply 4 to the both sides of the first equation to get, $4x + 8y = 12$

which is exactly the second equation.

Now $x + 2y = 3 \Rightarrow x = 3 - 2y$

let $y = t$, for some real number $t \in \mathbb{R}$,

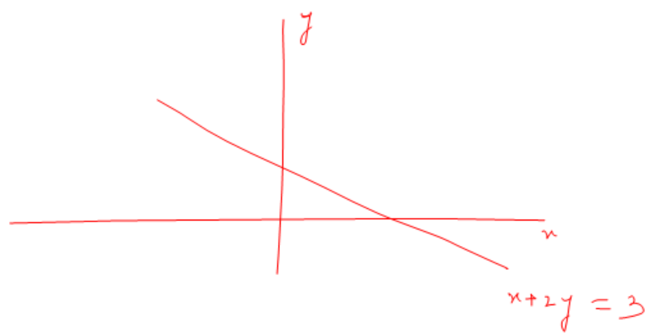
$\Rightarrow x = 3 - 2t$

Now, if we change value of 't', we will get different value of x

i.e there are infinitely many solutions.

Geometrically, Observe, second equation is multiple of first equation.

So, the figure below shows first line $x + 2y = 3$. This will be the same as $4x + 8y = 12$ i.e second line in $x-y$ plane.



So, (iii) → (a) → (3)

Hence, option 1 is the correct option.

2 Multiple Select Questions (MSQ):

3. Choose the set of correct options

   ◯ **Option 1:** If both A and B are 2×2 real matrices and $det(AB) = 0$, then $det(A) = 0$ or $det(B) = 0$.

   ◯ **Option 2:** If A is a 3×3 real matrix with non-zero determinant and k is some real number, then $det(k\ A) = k^3 \times det(A)$.

   ◯ **Option 3:** If $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$, then $A^{10} = 2^{10} \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$ (where A^n is the matrix $A \times A \times \ldots \times A$, n-times).

   ◯ **Option 4:** The number of scalar additions to be done to compute the matrix AB, where A is a 3×2 matrix and B is a 2×3 matrix, is 9.

Solution: Option 1:

Given $det(AB) = 0$

we know that, $det(AB) = det(A) \cdot det(B) = 0$

Observe that $det(A) = 0$ or $det(B) = 0$

So this option 1 is true.

Option 2:

Given $A_{3\times3}$ matrices

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

so, $kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{32} \end{bmatrix}$, Now $det(kA) = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$

Using property – If multiplying a real number 'k' with a row in a matrix then determinant of the new matrix is k times determinant of the earlier matrix.

Since, all three rows are multiple of k.

So, $\det(kA) = k \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k \cdot k \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$

$= k \cdot k \cdot k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k^3 \det(A)$

Hence, $\det(kA) = k^3 \det(A)$

So, Option 2 is true.

Option 3:

Given $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$B^3 = B^2 \cdot B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$B^4 = B^3 \cdot B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$\vdots$

$B^9 = B^8 \cdot B = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$

$$B^{10} = B^9 \cdot B = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

Now, $A^{10} = (2 \cdot B)^{10}$

$= 2^{10} \cdot B^{10}$

$A^{10} = 2^{10} \cdot \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$

So, option 3 is also true.

Option 4 :

Given A is a 3×2 matrix & B is 2×3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{bmatrix}$$

Observe that, AB is order of 3×3 matrix.

To obtain each row of AB, 3 scalar addition are performed.

So, total 9 scalar additions are performed in the matrix multiplication AB.

Hence, option 4 is also true.

4. Choose the set of correct options

- ◯ **Option 1:** A triangular 3×3 matrix has non-zero determinant if and only if all the diagonal entries are non-zero.
- ◯ Option 2: If A and B are 3×3 matrices then $det(A+B) = det(A) + det(B)$.
- ◯ Option 3: If A and B are 3×3 matrices then $adj(A+B) = adj(A) + adj(B)$.
- ◯ **Option 4:** If A is a 3×3 matrix and B is a matrix obtained from A by multiplying each column of A by its column number, then $det(B) = 6det(A)$.
- ◯ **Option 5:** If the sum of the first and the third row vectors of a 3×3 matrix A is equal to the second row vector of A, then $det(A) = 0$.

Solution: Option 1: let A be an upper triangular matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Observe det (A) = multiple of all diagonal entries.

i.e det (A) = $a_{11} a_{22} a_{33}$

clearly, det(A) = 0 if and only if one of the a_{11}, a_{22} or a_{33} will be zero which are the diagonal entries.

In other word, det(A) $\neq 0$ if and only if none of a_{11}, a_{22} or a_{33} will be zero i.e diagonal entries are non zero.

Similarly, the above statements are true for lower triangular matrices also.

Hence, option 1 is true.

Option 2: let $A = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

clearly, $det(A) = det(B) = 1 \Rightarrow det(A) + det(B) = 1+1 = 2$

$$A+B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Observe $\det(A+B) = 2^3 \cdot \det(I)$, where I is 3×3 identity matrix.

$\Rightarrow \det(A+B) = 8 \neq 2 = \det(A) + \det(B)$ (we know that $\det(I) = 1$)

i.e $\det(A+B) \neq \det(A) + \det(B)$

Hence option 2 is not true.

Option 3:

let $A = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Observe, $\operatorname{adj}(A) = \operatorname{adj}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \operatorname{adj}(A) + \operatorname{adj}(B) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Also, $A+B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

observe, $\operatorname{adj}(A+B) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

clearly, $\operatorname{adj}(A+B) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \operatorname{adj}(A) + \operatorname{adj}(B)$

Hence $\operatorname{adj}(A+B) \neq \operatorname{adj}(A) + \operatorname{adj}(B)$

Option 4:

let
$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{Column 1} & \text{Column 2} & \text{Column 3} \end{array} \end{array}$$

From the given condition, $B = \begin{bmatrix} a_{11} & 2\cdot a_{12} & 3\cdot a_{13} \\ a_{21} & 2\cdot a_{22} & 3\cdot a_{23} \\ a_{31} & 2\cdot a_{32} & 3\cdot a_{33} \end{bmatrix}$

Observe, $\det(B) = 1\cdot 2\cdot 3 \det(A) = 6\cdot \det(A)$

Hence option 4 is true.

Option 5 : Let A be a matrix of order 3×3 with given properties

i.e let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $a_{11} + a_{31} = a_{21}$

$a_{12} + a_{32} = a_{22}$

$a_{13} + a_{33} = a_{23}$

Use property - Adding a row with multiple of some real number to another row, which does not effect to determinant of the matrix.

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{31} & a_{12}+a_{32} & a_{13}+a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \left(\begin{array}{c}\text{Row} \\ \text{operation} \\ R_3+R_1\end{array}\right)$$

$$= \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$ (Because two row of the matrix are the same)

$\Rightarrow \det(A) = 0$

Hence, option 5 is true.

5. Mahesh bought 2 kg potato and c kg dal from a shop, and paid ₹200 to the shopkeeper. Gaurav bought 4 kg potato and 4 kg dal, and paid ₹d to the shopkeeper. If x_1 represents the price of 1 kg potato and x_2 represents the price of 1 kg dal, then choose the set of correct options.

- ○ Option 1: The matrix representation to find x_1 and x_2 is

$$\begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ d \end{bmatrix}$$

- ○ **Option 2:** The matrix representation to find x_1 and x_2 is

$$\begin{bmatrix} 2 & c \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ d \end{bmatrix}$$

- ○ **Option 3:** If Mahesh tries to find the price of 1 kg potato and 1 kg dal using appropriate matrix representation by taking $c = 2$ and $d = 400$, then the price of 1 kg potato that he thus arrives at, will not be unique.
- ○ Option 4: If Mahesh tries to find the price of 1 kg potato and 1 kg dal using appropriate matrix representation by taking $c = 2$ and $d = 400$, then the price of 1 kg potato that he thus arrives at, will be unique.
- ○ Option 5: If Mahesh tries to find the price of 1 kg potato and 1 kg dal using the appropriate matrix representation by taking $c = 2$ and $d \neq 400$, then he will be able to find the price (as a numerical value) of 1 kg potato.
- ○ **Option 6:** If Mahesh tries to find the price of 1 kg potato and 1 kg dal using the appropriate matrix representation by taking $c = 2$ and $d \neq 400$, then he will fail to find the price (as a numerical value) of 1 kg potato.

Solution: Given, x_1 represents the price of 1 kg potato.

x_2 reprsents the price of 1 kg dal

$\Rightarrow$ price of 2 kg potato $= 2x_1$

price of c kg dal $= cx_2$

price of 4 kg potato $= 4x_1$

price of 4 kg dal $= 4x_2$

Now, Gaurav bought 4 kg potato & 4 kg dal, & paid 'd' rupees.

i.e $4x_1 + 4x_2 = d$

again, Mahesh bought 2 kg potato & c kg dal from a shop & paid 200 rupees.

i.e $2x_1 + cx_2 = 200$

The system of linear equations will be

$$2x_1 + cx_2 = 200$$
$$4x_1 + 4x_2 = d$$

Now, Matrix form of the above system is $Ax = b$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & c \\ 4 & 4 \end{bmatrix}$ & $b = \begin{bmatrix} 200 \\ d \end{bmatrix}$

Hence, Option 2 is true, but Option 1 is not true.

Option 3:

If $c = 2$ & $d = 400$

then, the system becomes

$$2x_1 + 2x_2 = 200$$
$$4x_1 + 4x_2 = 400 \Rightarrow 2x_1 + 2x_2 = 200$$

Observe, both the equations are the same.

(Recall question 2)

this system of linear equations have infinitely many solution.

Hence, By solving the system, the price of 1 kg potato that Mahesh arrives at, will not be unique.

Hence, Option 3 is true.

Option 4: Observe from option 3, we conclude that the price of 1 kg potato that Mahesh thus arrives at, will not be unique.

Hence, option 4 is not true.

Option 5: Observe from option 3, $c = 2$ & $d = a$

where $a \in \mathbb{R} \setminus \{400\}$

The system is

$$2x_1 + 2x_2 = 200 \Rightarrow 4x_1 + 4x_2 = 400$$

$$4x_1 + 4x_2 = a$$

from the above equations $a = 400$

But it is given that $a \neq 400$

Hence, for $c = 2$ & $d \neq 400$ the the above system has no solution.

Hence, Mahesh will not be able to find the price of 1 kg potato

Hence, option 5 is not true & option 6 is true.

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6. The marks obtained by Safina, Ram and Pratiksha in Quiz 1, Quiz 2 and End sem (with the maximum marks for each exam being 100) are shown in Table W1PT2.

	Quiz 1	Quiz 2	End sem
Safina	89	95	88
Ram	92	81	98
Pratiksha	85	93	98

Table: W4PT2

The weightage of marks in final grade(in percent) of Quiz 1, Quiz 2, and End sem is shown in Table W1PT3.

	In percent (%)
Quiz 1	20
Quiz 2	20
End sem	60

Table: W4PT3

Choose the set of correct options.

- ◯ Option 1: Final grades (in 100) of Safina, Ram and Pratiksha can be represented by the matrix:

$$\begin{bmatrix} 93.4 \\ 94.4 \\ 89.6 \end{bmatrix}$$

- ◯ **Option 2:** Final grades (in 100) of Safina, Ram and Pratiksha can be represented by the matrix :

$$\begin{bmatrix} 89.6 \\ 93.4 \\ 94.4 \end{bmatrix}$$

- ◯ **Option 3:** The order of the matrix which represents final grades(in 100) of Safina, Ram and Pratiksha is 3×1

- ◯ **Option 4:** If bonus marks given to Safina, Ram and Pratiksha are represented by the following matrix $\begin{bmatrix} 1.4 \\ 2.6 \\ 0 \end{bmatrix}$ then the overall final grades (in 100) can be represented by the matrix:

$$\begin{bmatrix} 91 \\ 96 \\ 94.4 \end{bmatrix}$$

Solution:

Matrix form of marks obtained by Safina, Ram & Pratiksha in Quiz 1, Quiz 2 & End sem is $\begin{bmatrix} 89 & 95 & 88 \\ 92 & 81 & 98 \\ 85 & 93 & 98 \end{bmatrix}$

& matrix of the weightage of marks is $\begin{bmatrix} 20/100 \\ 20/100 \\ 60/100 \end{bmatrix}$

So, final grades of Safina, Ram & Pratiksha will be

$$\begin{bmatrix} 89 & 95 & 88 \\ 92 & 81 & 98 \\ 85 & 93 & 98 \end{bmatrix} \begin{bmatrix} 20/100 \\ 20/100 \\ 60/100 \end{bmatrix} = \begin{bmatrix} \frac{89\times 20}{100} + \frac{95\times 20}{100} + \frac{88\times 60}{100} \\ \frac{92\times 20}{100} + \frac{81\times 20}{100} + \frac{98\times 60}{100} \\ \frac{85\times 20}{100} + \frac{93\times 20}{100} + \frac{98\times 60}{100} \end{bmatrix}$$

$$= \begin{bmatrix} 89.6 \\ 93.4 \\ 94.4 \end{bmatrix}$$

Hence, the second option is true.

Observe that, the matrix which obtained in option 2

have 3 rows & 1 column

Hence the order of matrix is 3×1

So, option 3 is true.

Now, the matrix representing the bonous marks is $\begin{bmatrix} 1.4 \\ 2.6 \\ 0 \end{bmatrix}$

So overall final grades (in 100) can be obtained by

addition i.e, $\begin{bmatrix} 89.6 \\ 93.4 \\ 94.4 \end{bmatrix} + \begin{bmatrix} 1.4 \\ 2.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 91 \\ 96 \\ 94.4 \end{bmatrix}$

Hence, the fourth option is true.

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3 Numerical Answer Type (NAT):

7. Let A be a 3×3 matrix with non-zero determinant and B be a matrix obtained by adding 5 times of first row of A to the third row of A and adding 10 times of second row of A to the first row of A. What is the value of $det(3AB^{-1})$? [Ans: 27]

Solution:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Hence, $B = \begin{bmatrix} a_{11}+10a_{21} & a_{12}+10a_{22} & a_{13}+10a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+5a_{11} & a_{32}+5a_{12} & a_{33}+5a_{13} \end{bmatrix}$

Observe, $det(A) = det(B)$

We know $det(B^{-1}) = \dfrac{1}{det(B)} = \dfrac{1}{det(A)}$

Now, $3A = \begin{bmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$

So $\det(3A) = 3^3 \det(A)$

.

We know that

$$\det(3AB^{-1}) = \det(3A) \cdot \det(B^{-1})$$

$$= 3^3 \cdot \det(A) \cdot \frac{1}{\det(A)}$$

$$= 3^3$$

$$\det(3AB^{-1}) = 27$$

Hence $\det(3AB^{-1}) = 27$.

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4 Comprehension Type Question:

A shopkeeper sells three types of clothes- shirts, jeans, and T- shirts- in three different sizes: small, medium, and large. In a week, he sold 1 small, 1 medium and 2 large sized shirts; 2 small, c medium and 6 large sized jeans, and 1 small, 3 medium and $c-5$ large sized T-shirts (where c is an integer). The price of shirts, jeans, and T-shirts remain same for different sizes (i.e., small, medium, and large sized shirts have same price; similarly, small, medium, large sized jeans have same price; and small, medium, large sized T-shirts have same price). The shopkeeper earned ₹7 , ₹27 and ₹43 (in thousand) in that week, for small, medium, and large sized clothes respectively.
Answer the following questions using the given data.

8. If s, j, t represents the price of 1 shirt, 1 jeans and 1 T-shirt respectively and we want to find s, j, t by solving a system of linear equations represented by the matrix form $Ax = b$, where $x = (s, j, t)^T$, then which of the following options is correct? (MCQ)

   ○ Option 1: $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & c & 6 \\ 1 & 3 & c-5 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}$

   ○ **Option 2:** $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & c & 6 \\ 1 & 3 & c-5 \end{bmatrix}^T$ and $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}$

   ○ Option 3: $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & c & 6 \\ 1 & 3 & c-5 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}^T$

   ○ Option 4: $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & c & 6 \\ 1 & 3 & c-5 \end{bmatrix}^T$ and $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}^T$

Solution:

Given, the prices of one shirt, one jeans & one T-shirt is s, j, and t respectively. Also price of shirts in all different sized are the same, similarly for jeans and T-shirts

Since, the shopkeeper sold 1 small shirt, 2 small jeans & 1 small T-shirt and earned 7 rupees (in thousand) in a week, we get the equation:

$$s + 2j + t = 7$$

Similarly, the shopkeeper earned 27 rupees (in thousand) in that week for medium sized clothes after selling 1 medium shirt, c medium jeans & 3 medium T-shirts so we get the equation:

$$s + cj + 3t = 27$$

Similarly for large sized clothes,

$$2s + 6j + (c-5)t = 43$$

So, the system of linear equation is

$$s + 2j + t = 7$$
$$s + cj + 3t = 27$$
$$2s + 6j + (c-5)t = 43$$

Now, matrix representation of the system of linear equations is $Ax = b$.

where $x = \begin{bmatrix} s \\ j \\ t \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & c & 3 \\ 2 & 6 & c-5 \end{bmatrix}$ & $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}$

Observe that, in second option $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & c & 6 \\ 1 & 3 & c-5 \end{bmatrix}^T$

& $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}$

Hence, the second option is true.

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9. If $det(A) = 122$, then how many medium sized jeans were sold in the week? (NAT)
Ans : 16

Solution: Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & c & 3 \\ 2 & 6 & c-5 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 1 & c & 3 \\ 2 & 6 & c-5 \end{vmatrix}$$

$$= 1\left(c(c-5) - 18\right) - 2\left(c-5-6\right) + 1\left(6-2c\right)$$

$$= c^2 - 5c - 18 - 2c + 22 + 6 - 2c$$

$$\det(A) = c^2 - 9c + 10$$

also Given $\det(A) = 122$

$$\Rightarrow \quad c^2 - 9c + 10 = 122$$

$$\Rightarrow \quad c^2 - 9c - 112 = 0$$

$$\Rightarrow \quad (c-16)(c+7) = 0$$

$$\Rightarrow \quad c = 16 \quad \text{or} \quad c = -7$$

But c is the number of medium sized jeans so it cannot be negative.

Hence $c = 16$.

Hence, 16 medium sized peons sold in that week.

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10. If A is the matrix as above (in question 8) and $det(A) = 122$, then What is the price(in thousand) of a shirt, a jeans, and a T-shirt in the matrix form $x = (s, j, t)^T$? (MCQ)

○ Option 1: $x = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

○ Option 2: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

○ **Option 3:** $x = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

○ Option 4: $x = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

Solution:- Given $det(A) = 122$.

from Question 9 we got $c = 16$

Hence, $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 16 & 3 \\ 2 & 6 & 11 \end{bmatrix}$

Matrix representation of system of equations is $Ax = b$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 16 & 3 \\ 2 & 6 & 11 \end{bmatrix}$, $x = \begin{bmatrix} s \\ j \\ t \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix}$

So, to get the price (in thousand) of a shirt, a jeans, and a T-shirt, we have to choose such x which satisfies the equation $Ax = b$

Observe for $x = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 16 & 3 \\ 2 & 6 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+2+3 \\ 2+16+9 \\ 4+6+33 \end{bmatrix} = \begin{bmatrix} 7 \\ 27 \\ 43 \end{bmatrix} = b$$

Hence, the third option is true.

Week-4
Mathematics for Data Science - 2
Vectors and Matrices
Assignment

1 Multiple Choice Questions (MCQ)

1. Match the matrices in the column A with the properties of those in column B. (MCQ)

	Matrix (Column A)		Properties of matrix (Column B)
a)	$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$	i)	has determinant 0
b)	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	ii)	is a scalar matrix
c)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	iii)	is a lower triangular matrix but not a diagonal matrix
d)	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	iv)	is a diagonal matrix but not a scalar matrix

Table: W4AT1

○ Option 1: a) → i), b) → ii), c) → iii), d) → iv)
○ Option 2: a) → ii), b) → i), c) → iv), d) → iii)

○ Option 3: a) → iii), b) → iv), c) → i), d) → ii)

○ **Option 4:** a) → iii), b) → i), c) → iv), d) → ii)

Soln:

a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix}$ → all the elements above the diagonal are 0.
Hence it is a lower triangular matrix.
there are non-zero elements below the diagonal too. Hence it is not a diagonal matrix.

so, (a) → (iii)

b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Hence the determinant is 0.

so, (b) → (i)

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ → The non-zero elements are only on the diagonal, moreover, they are not the same.

so, it is a diagonal matrix, but not a scalar matrix. so, (c) → (iv)

d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ → It is a scalar matrix.

so, (d) → (ii)

Answer: option (4) (a) → (iii), (b) → (i), (c) → (iv), (d) → (ii).

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2. Match the systems of linear equations in Column A with their number of solutions in column B and their geometric representation in Column C.

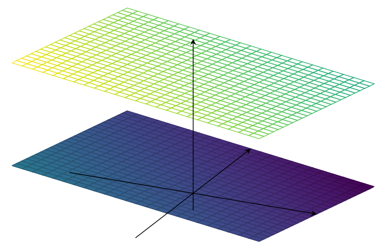
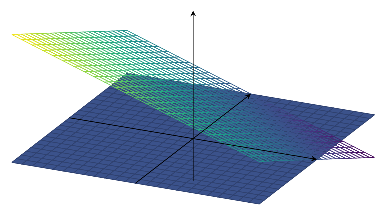
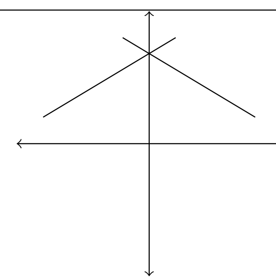
	System of linear equations (Column A)		Number of solutions (Column B)		Geometric representations (Column C)
i)	$x + y = 3$, $x - y = -3$	a)	Infinite solutions	1)	
ii)	$x + y + z = 1$, $x + y + z = 7$	b)	Unique solution	2)	
iii)	$z = 0$, $x + y + z = 1$	c)	No solution	3)	

Table: W4AT2

- ◯ Option 1: i) → a) → 3); ii) → c) → 1); iii) → b) → 2)
- ◯ **Option 2:** i) → b) → 3); ii) → c) → 1); iii) → a) → 2)
- ◯ Option 3: i) → b) → 3); ii) → c) → 2); iii) → a) → 1)
- ◯ Option 4: i) → a) → 3); ii) → c) → 1); iii) → b) → 2)

Soln: (i) $\left.\begin{array}{l} x+y=3 \\ x-y=-3 \end{array}\right\}$ This system of linear equations involves only two variable.

Add the 1st one with the 2nd, to get: $2x = 0$
$\Rightarrow x = 0$

Substituting $x=0$ in the 1st one, we get: $y = 3$

Hence, there is a unique solution.

In $\mathbb{R}^2$ both the equations represent straight lines, and they intersect at the point $(0, 3)$ which is on Y-axis.

So, i) $\longrightarrow$ b) $\longrightarrow$ 3)

(ii) $\left.\begin{array}{l} x+y+z=1 \\ x+y+z=7 \end{array}\right\}$ If there exists some solution (a,b,c) of this system of linear equations then the point (a, b, c) should lie on both the planes.

Which gives us $a+b+c=1$ and $a+b+c=7$
$\Rightarrow 1 = 7$ which is absurd.

Hence, there cannot exist any such point.
So, the system of linear equations has no soln.

Observe that: the above system of linear equations involve three variables. They represent planes on the co-ordinate system.

$ax+by+cz=d$ represents a plane on the co-ordinate system.

let $\left.\begin{array}{l} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{array}\right\}$ be the two linear equations.

Both of them represent planes.

Now if, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2}$

then the planes represented by them must be parallel (distinct) to each other.

ii) $\rightarrow$ c) $\rightarrow$ 1)

iii) $\left.\begin{array}{l} z=0 \\ x+y+z=1 \end{array}\right\}$ $z=0$ denotes the XY-plane.
$x+y+z=1$ denotes a plane on the co-ordinate system.

Substituting $z=0$ in the second equation we get,

$$x+y=1$$
$$\Rightarrow y=1-x.$$

Hence any point of the form $(a, 1-a, 0)$ will satisfy both the equations.

So, the system of linear equations has infinitely many solutions.

iii) $\longrightarrow$ a) $\longrightarrow$ 2)

<u>Answer:</u> Option 2 : i) $\rightarrow$ b) $\rightarrow$ 3), ii) $\rightarrow$ c) $\rightarrow$ 1), iii) $\rightarrow$ a) $\rightarrow$ 2)

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2 Multiple Select Questions (MSQ):

3. Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$. Which of the following options are true for a matrix A, such that $AB = C$? (MSQ)

- ○ Such a matrix does not exist.
- ○ There is a unique matrix A satisfying this property.
- ○ **There are infinitely many such matrices.**
- ○ ***A* should be a 2 × 3 matrix.**
- ○ A should be a 3×2 matrix.

Soln: Let A be a $m \times n$ matrix and B be a $n \times p$ matrix.

AB must be a $m \times p$ matrix.

Here, B is a 3×2 matrix, i.e., $n = 3$, $p = 2$

$C = AB$ is a 2×2 matrix, i.e., $m = 2$, $p = 2$

Hence, A must be a 2×3 matrix.

Let us take an arbitrary 2×3 matrix A and try to see the conditions on the elements of A for which $AB = C$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} - a_{13} & a_{12} \\ a_{21} - a_{23} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} = C$$

comparing the elements we get,

$a_{11} - a_{13} = 0$ $\Rightarrow a_{11} = a_{13}$	$a_{12} = 0$
$a_{21} - a_{23} = -1$ $\Rightarrow a_{21} = a_{23} - 1$	$a_{22} = 2$

Hence, $A = \begin{bmatrix} a_{11} & 0 & a_{11} \\ a_{23}-1 & 2 & a_{23} \end{bmatrix}$

where, a_{11} and a_{23} can take any arbitrary real number.

So, there are infinitely many such matrices A, such that $AB = C$

Answer: option 3: There are infinitely many such matrices.
option 4: A should be a 2×3 matrix.

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4. Let A be a 2×2 real matrix and let $trace(A)$ denote the sum of the elements in the diagonal of A. Which of the following is true? (MSQ)

   - ○ $det(A - cI)$ is a polynomial in c of degree 1.
   - ○ $\boldsymbol{det(A - cI)}$ **is a polynomial in** $\boldsymbol{c}$ **of degree 2.**
   - ○ $\boldsymbol{det(A - cI) = c^2 - trace(A)c + det(A)}$
   - ○ $det(A - cI) = c^2 + trace(A)c - det(A)$
   - ○ $det(A - cI) = trace(A)c - det(A)$
   - ○ $det(A - cI) = -trace(A)c + det(A)$

Soln. Let us choose an arbitrary 2×2 real matrix A as follows:

$$A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$trace(A) = p + s \quad , \quad det(A) = ps - qr$$

$$A - cI = \begin{bmatrix} p & q \\ r & s \end{bmatrix} - c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} - \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} p-c & q \\ r & s-c \end{bmatrix}$$

$$\begin{aligned} det(A - cI) &= (p-c)(s-c) - qr \\ &= ps - cs - cp + c^2 - qr \\ &= c^2 - c(p+s) + (ps - qr) \\ &= c^2 - trace(A)\cdot c + det A \end{aligned}$$

Answer: Option 2: $det(A - cI)$ is a polynomial in c of degree 2.

Option 3: $det(A - cI) = c^2 - trace(A)\cdot c + det(A)$

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5. Suppose there are two types of oranges and two types of bananas available in the market. Suppose 1 kg of each type of orange costs ₹50 and 1 kg of each type of banana costs ₹40. Gargi bought x kg of first type of each fruit, orange and banana, and y kg of second type of each fruit, orange and banana. She paid ₹250 for oranges and ₹200 for bananas. Which of the following options are correct with respect to the given information? (MSQ)

   ○ **Option 1:** The matrix representation to find x and y can be

   $$\begin{bmatrix} 50 & 50 \\ 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 250 \\ 200 \end{bmatrix}$$

   ○ Option 2: The matrix representation to find x and y can be

   $$\begin{bmatrix} 50 & 40 \\ 50 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 250 \\ 200 \end{bmatrix}$$

   ○ **Option 3:** The matrix representation to find x and y can be

   $$\begin{bmatrix} 40 & 40 \\ 50 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 250 \end{bmatrix}$$

   ○ **Option 4:** x can be 2 and y can be 3.

   ○ **Option 5:** There are infinitely many real values possible for x and y.

   ○ Option 6: There are only finitely many real values possible for x and y.

   ○ **Option 7:** There are only finitely many natural numbers possible for x and y.

<u>Soln.</u>

Orange type 1 cost	Orange type 2 cost.
50	50
Banana type 1 cost	Banana type 2 cost
40	40

Gargi bought

Orange type 1 : x kg

Banana type 1 : x kg

Orange type 2 : y kg

Banana type 2 : y kg

Gargi paid :

For oranges ₹ 250

For Bananas ₹ 200

Hence, $50x + 50y = 250 \longrightarrow (1)$

$40x + 40y = 200 \longrightarrow (2)$

It can be represented by

$$\begin{bmatrix} 50 & 50 \\ 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 250 \\ 200 \end{bmatrix}$$

If we interchange the order of the equations we get,

$$40x + 40y = 200$$
$$50x + 50y = 250.$$

It can be represented by

$$\begin{bmatrix} 40 & 40 \\ 50 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 250 \end{bmatrix}$$

From (1) we get, $50(x + y) = 250$

$\Rightarrow x + y = 5$

From (2) we get, $40(x + y) = 200$

$\Rightarrow x + y = 5.$

Hence, the solution of the system of linear equations is $(a, 5-a)$, where a can take any arbitrary real number.

But in this context, both of them have to be positive.

$a \geq 0, \quad 5 - a \geq 0$

$\Rightarrow 5 \geq a$

$\Rightarrow \underline{0 \leq a \leq 5}$

Hence, a can be any arbitrary real number in between 0 and 5.

Clearly, $x = 2$, $y = 3$ can be a solution.

Clearly, there are infinitely many real values possible for x and y.

As the solutions are of the form $(a, 5-a)$ and $0 \leq a \leq 5$,

there are finitely many natural numbers possible as solutions.

Those are,

$$\left.\begin{array}{l} x=0,\ y=5 \\ x=1,\ y=4 \\ x=2,\ y=3 \\ x=3,\ y=2 \\ x=4,\ y=1 \\ x=5,\ y=0 \end{array}\right\}$$

Answer: Option 1: The matrix representation to find x and y can be,

$$\begin{bmatrix} 50 & 50 \\ 40 & 40 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 250 \\ 200 \end{bmatrix}$$

Option 3: The matrix representation to find x and y can be

$$\begin{bmatrix} 40 & 40 \\ 50 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 250 \end{bmatrix}$$

Option 4: x can be 2, y can be 3

Option 5: there are infinitely many real values possible for x and y.

Option 7: There are only finitely many natural numbers possible for x and y.

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3 Numerical Answer Type (NAT):

6. Suppose $det(3A) = n \times det(A)$ for any 3×3 real matrix A. What is the value of n? (NAT) [Answer: 27]

Soln. If any real number c is multiplied with a row of a $p \times p$ matrix then the determinant of the new matrix will be c times the determinant of the earlier matrix.

Now, cA means c is multiplied with all the elements of matrix A.

If A is a $p \times p$ matrix, then there are p rows.

Hence, $\det(cA) = c^p \det(A)$

Here, $c = 3$ and $p = 3$, so, $\det(3A) = 3^3 \det(A) = 27 \det(A)$

Answer: $n = 27$

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7. Suppose $A = \begin{bmatrix} 2019 & 100 & 2119 \\ 2020 & 200 & 2220 \\ 2021 & 300 & 2321 \end{bmatrix}$. What will be the value of $det(A)$?

(NAT) [Answer: 0]

Soln:

$$\begin{vmatrix} 2019 & 100 & 2119 \\ 2020 & 200 & 2220 \\ 2021 & 300 & 2321 \end{vmatrix} \xrightarrow{R_3 - R_2} \begin{vmatrix} 2019 & 100 & 2119 \\ 2020 & 200 & 2220 \\ 1 & 100 & 101 \end{vmatrix}$$

$$\downarrow R_2 - R_1$$

$$\begin{vmatrix} 2019 & 100 & 2119 \\ 1 & 100 & 101 \\ 1 & 100 & 101 \end{vmatrix}$$

Recall: Row operation: adding scalar multiple of one row with other does not change the determinant of a matrix.

As the two rows of the matrix becomes identical, the determinant will be 0.

Answer: $det(A) = 0$

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4 Comprehension Type Question:

Suppose there are three families F_1, F_2, F_3 living in different cities and they pay ₹x_1, ₹x_2, ₹x_3 per unit respectively for electric consumption in each month. In January 2021, the electric consumption by F_1, F_2, and F_3 are 30 units, 20 units, and 25 units, respectively. In February 2021, it is 20 units, 35 units, and 25 units, respectively. In March 2021, it is 20 units, 10 units, and 15 units, respectively. The total amount paid by the three families together for the electricity consumption in January, February, and March are ₹670, ₹730, and ₹400 respectively.
Answer the following questions using this given data.

8. If we want to find x_1, x_2, x_3 by solving a system of linear equations represented by the matrix form $Ax = b$, where $x = (x_1, x_2, x_3)^T$, then which of the following options is correct? (MCQ)

   ○ Option A : $A = \begin{bmatrix} 30 & 20 & 20 \\ 20 & 35 & 10 \\ 25 & 25 & 15 \end{bmatrix}$ and $b = \begin{bmatrix} 670 \\ 730 \\ 400 \end{bmatrix}$

   ○ **Option B :** $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$ and $b = \begin{bmatrix} 670 \\ 730 \\ 400 \end{bmatrix}$

   ○ Option C : $A = \begin{bmatrix} 30 & 35 & 15 \\ 20 & 20 & 25 \\ 20 & 10 & 25 \end{bmatrix}$ and $b = \begin{bmatrix} 670 \\ 730 \\ 400 \end{bmatrix}$

   ○ Option D : $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$ and $b = \begin{bmatrix} 400 \\ 730 \\ 670 \end{bmatrix}$

Solution:

Month	electric consumption			Total payment by F_1, F_2, F_3 together
	F_1	F_2	F_3	
In Jan 2021	30	20	25	670
In Feb 2021	20	35	25	730
In March 2021	20	10	15	400.

F_1 pays ₹ x_1 per unit

F_2 pays ₹ x_2 per unit

F_3 pays ₹ x_3 per unit.

Hence we have,

$$\left.\begin{aligned} 30x_1 + 20x_2 + 25x_3 &= 670 \\ 20x_1 + 35x_2 + 25x_3 &= 730 \\ 20x_1 + 10x_2 + 15x_3 &= 400 \end{aligned}\right\}$$

Matrix representation:

$$\begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 670 \\ 730 \\ 400 \end{bmatrix}$$

$$Ax = b.$$

Answer: option B: $A = \begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix}$ and $b = \begin{bmatrix} 670 \\ 730 \\ 400 \end{bmatrix}$

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9. Which of the following is the possible solution of $Ax = b$, where $x = (x_1, x_2, x_3)^T$? (MCQ)

○ Option A: $x = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$

○ Option B: $x = \begin{bmatrix} 9 \\ 8 \\ 10 \end{bmatrix}$

○ **Option C:** $x = \begin{bmatrix} 9 \\ 10 \\ 8 \end{bmatrix}$

○ Option D: $x = \begin{bmatrix} 8 \\ 10 \\ 9 \end{bmatrix}$

Soln We have the matrix representation of the System of linear equations.

$$\begin{bmatrix} 30 & 20 & 25 \\ 20 & 35 & 25 \\ 20 & 10 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 670 \\ 730 \\ 400 \end{bmatrix}$$

The System of linear equations is as follows:

$$30x_1 + 20x_2 + 25x_3 = 670 \quad \text{—— (1)}$$
$$20x_1 + 35x_2 + 25x_3 = 730 \quad \text{—— (2)}$$
$$20x_1 + 10x_2 + 15x_3 = 400 \quad \text{—— (3)}$$

(2) − (1) gives: $-10x_1 + 15x_2 = 60$

$$\Rightarrow -2x_1 + 3x_2 = 12 \quad \text{—— (4)}$$

$6 \times (2) - 10 \times (3)$ gives:

$$120x_1 + 210x_2 + 150x_3 = 4380$$
$$\underset{(-)}{200x_1} \underset{(-)}{+} 100x_2 \underset{(-)}{+} 150x_3 \underset{(-)}{=} 4000$$

$$-80x_1 + 110x_2 = 380$$
$$\Rightarrow -8x_1 + 11x_2 = 38 \quad \text{—— (5)}$$

$4 \times (4) - (5)$ gives:

$$\begin{array}{r} -8x_1 + 12x_2 = 48 \\ -8x_1 + 11x_2 = 38 \\ (+) \quad (-) \quad\quad (-) \\ \hline x_2 = 10 \end{array}$$

Substituting the value of x_2 in (4) we get,

$$-2x_1 + 3(10) = 12$$

$$\Rightarrow -2x_1 + 30 = 12$$

$$\Rightarrow -2x_1 = -18$$

$$\Rightarrow x_1 = 9$$

Substituting the values of x_1 ad x_2 in (1) we get,

$$30(9) + 20(10) + 25x_3 = 670$$

$$\Rightarrow 270 + 200 + 25x_3 = 670$$

$$\Rightarrow 470 + 25x_3 = 670$$

$$\Rightarrow 25x_3 = 670 - 470 = 200$$

$$\Rightarrow x_3 = 8 .$$

Hence, $x_1 = 9$, $x_2 = 10$, $x_3 = 8$.

Answer: $x = \begin{bmatrix} 9 \\ 10 \\ 8 \end{bmatrix}$

10. Which of the following is(are) correct? (MSQ)

- ○ **Option A:**

$$det(A) = 30 \times det\left(\begin{bmatrix}35 & 25\\10 & 15\end{bmatrix}\right) - 20 \times det\left(\begin{bmatrix}20 & 25\\20 & 15\end{bmatrix}\right) + 25 \times det\left(\begin{bmatrix}20 & 35\\20 & 10\end{bmatrix}\right)$$

- ○ **Option B:**

$$det(A) = 30 \times det\left(\begin{bmatrix}35 & 25\\10 & 15\end{bmatrix}\right) + 20 \times det\left(\begin{bmatrix}25 & 20\\15 & 20\end{bmatrix}\right) + 25 \times det\left(\begin{bmatrix}20 & 35\\20 & 10\end{bmatrix}\right)$$

- ○ **Option C:**

$$det(A) = -20 \times det\left(\begin{bmatrix}20 & 25\\10 & 15\end{bmatrix}\right) + 35 \times det\left(\begin{bmatrix}30 & 25\\20 & 15\end{bmatrix}\right) - 25 \times det\left(\begin{bmatrix}30 & 20\\20 & 10\end{bmatrix}\right)$$

- ○ Option D:

$$det(A) = 20 \times det\left(\begin{bmatrix}20 & 25\\10 & 15\end{bmatrix}\right) - 35 \times det\left(\begin{bmatrix}30 & 25\\20 & 15\end{bmatrix}\right) + 25 \times det\left(\begin{bmatrix}30 & 20\\20 & 10\end{bmatrix}\right)$$

Soln:

$$A = \begin{bmatrix}\boxed{30 \quad 20 \quad 25}\\ 20 \quad 35 \quad 25\\ 20 \quad 10 \quad 15\end{bmatrix}$$

$$det(A) = 30 \times det\left(\begin{bmatrix}35 & 25\\10 & 15\end{bmatrix}\right) - 20 \times det\left(\begin{bmatrix}20 & 25\\20 & 15\end{bmatrix}\right) + 25 \times det\left(\begin{bmatrix}20 & 35\\20 & 10\end{bmatrix}\right)$$

$$= 30\,(35 \times 15 - 25 \times 10) - 20\,(20 \times 15 - 25 \times 20) + 25\,(20 \times 10 - 35 \times 20)$$

$$= 30\,(525 - 250) - 20\,(300 - 500) + 25\,(200 - 700)$$

$$= 30\,(275) - 20\,(-200) + 25\,(-500)$$

$$= 8250 + 4000 - 12500$$

$$= 12250 - 12500 = \underline{\underline{-250}}.$$

$$det(A) = 30 \times det\left(\begin{bmatrix}35 & 25\\10 & 15\end{bmatrix}\right) \underbrace{- 20 \times det\left(\begin{bmatrix}20 & 25\\20 & 15\end{bmatrix}\right)}_{} + 25 \times det\left(\begin{bmatrix}20 & 35\\20 & 10\end{bmatrix}\right)$$

Interchanging the column will change the sign of the determinant.

Hence, $det\left(\begin{bmatrix}20 & 25\\20 & 15\end{bmatrix}\right) = -det\left(\begin{bmatrix}25 & 20\\15 & 20\end{bmatrix}\right)$

which gives us the expression in option B.

Option C:

$$-20 \times \det\left(\begin{bmatrix} 20 & 25 \\ 10 & 15 \end{bmatrix}\right) + 35 \times \det\left(\begin{bmatrix} 30 & 25 \\ 20 & 15 \end{bmatrix}\right) - 25 \times \det\left(\begin{bmatrix} 30 & 20 \\ 20 & 10 \end{bmatrix}\right)$$

$$= -20\,(20\times 15 - 25\times 10) + 35\,(30\times 15 - 25\times 20) - 25\,(30\times 10 - 20\times 20)$$

$$= -20\,(300 - 250) + 35\,(450 - 500) - 25\,(300 - 400)$$

$$= -20\,(50) + 35\,(-50) - 25\,(-100)$$

$$= -1000 - 1750 + 2500$$

$$= -2750 + 2500 = -250 = \det(A)$$

Option D:

$$20 \times \det\left(\begin{bmatrix} 20 & 25 \\ 10 & 15 \end{bmatrix}\right) - 35 \times \det\left(\begin{bmatrix} 30 & 25 \\ 20 & 15 \end{bmatrix}\right) + 25 \times \det\left(\begin{bmatrix} 30 & 20 \\ 20 & 10 \end{bmatrix}\right)$$

$$= 20\,(300 - 250) - 35\,(450 - 500) + 25\,(300 - 400)$$

$$= 20\,(50) - 35\,(-50) + 25\,(-100)$$

$$= 1000 + 1750 - 2500$$

$$= 2750 - 2500 = 250 \neq \det(A)$$

Answer: Option A, Option B, and Option C are correct.

Week - 5
Mathematics for Data Science - 2
Solutions of system of linear equations
Practice Assignment Solution.

1 Multiple Choice Questions (MCQ)

1. Consider the following systems of equations and choose the correct option.

System I:
$$\begin{aligned} -x + 2y - 2z &= 2 \\ 2x + z &= -1 \\ x - 3y + z &= 3 \end{aligned}$$

System II:
$$\begin{aligned} -2x + y + z &= 0 \\ \frac{3}{2}x + 2y - z &= -2 \\ 3x + 4y - 2z &= 5 \end{aligned}$$

System III:
$$\begin{aligned} x + 3z &= -5 \\ -\frac{2}{5}x - \frac{1}{5}y - 2z &= 3 \\ 2x + y + 10z &= -15 \end{aligned}$$

- ○ Option 1: All the three systems have a unique solution.
- ○ Option 2: System I has a unique solution, whereas, System II and System III have no solution.
- ○ Option 3: System I has a unique solution, whereas, System II and System III have infinitely many solutions.
- ○ **Option 4:** System I has a unique solution, System II has no solution, and System III has infinitely many solutions.
- ○ Option 5: System I has no solution, System II and System III have infinitely many solutions.

Solution :- Matrix representation of System I is

$$\begin{bmatrix} -1 & 2 & -2 \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & -2 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & -3 & 1 & 3 \end{array}\right] \xrightarrow{(-1)R_1} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 2 & 0 & 1 & -1 \\ 1 & -3 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 4 & -3 & 3 \\ 0 & -1 & -1 & 5 \end{array}\right] \xrightarrow{\substack{\frac{1}{4}R_2 \\ (-1)R_3}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 1 & 1 & -5 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 7/4 & \frac{-23}{4} \end{array}\right]$$

$$\xrightarrow{\frac{4}{7}R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 1 & -23/7 \end{array}\right]$$

Observe that the row echelon form of Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -2 \\ 0 & 1 & -3/4 & 3/4 \\ 0 & 0 & 1 & -23/7 \end{array}\right]$$

, So in system form we can write

$$x - 2y + 2z = -2$$
$$y - \frac{3}{4}z = \frac{3}{4}$$
$$z = -23/7$$

So solution is $z = -23/7$

$$y - \frac{3}{4}z = 3/4$$
$$\Rightarrow \quad y - \frac{3}{4} \times \left(-\frac{23}{7}\right) = \frac{3}{4} \quad \Rightarrow \quad y = -12/7$$

&

$$x - 2y + 2z = -2$$
$$\Rightarrow \quad x = 8/7$$

Hence the system has a unique solution.

System II: Matrix representation of the system II is

$$\begin{bmatrix} -2 & 1 & 1 \\ 3/2 & 2 & -1 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix of the system II is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 3/2 & 2 & -1 & -2 \\ 3 & 4 & -2 & 5 \end{array}\right]$$

Row echelon form of Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1/2 & -1/2 & 0 \\ 0 & 1 & -1/11 & -8/11 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

In system form we can write

$$x - \frac{1}{2}y - \frac{1}{2}z = 0,$$

$$y - \frac{z}{11} = -8/11$$

$$\& \quad 0 = 1$$

which is absured

So, the system II has no solution.

System III   The matrix representation of system III is

$$\begin{bmatrix} 1 & 0 & 3 \\ -2/5 & -1/5 & -2 \\ 2 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -15 \end{bmatrix}$$

Augmented matrix of the system III is

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ -2/5 & -1/5 & -2 & 3 \\ 2 & 1 & 10 & -15 \end{array}\right]$$

Row echelon form of Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

So, in system of linear equations form

$$x + 3z = -5$$
$$y + 4z = -5.$$

So $y = -5 - 4z$

& $x = -5 - 3z$

Let $z = t$ any real number then

$$x = -5 - 3t$$
$$y = -5 - 4t$$
$$z = t$$

So system III has infinitely many solutions

Hence, option 4 is true.

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2. Match the matrices in Column A with their row operation steps (in the exact sequence given) in Column B, and their corresponding reduced row Echelon forms in Column C of Table M2W2PT1.

	Matrices (Column A)		Steps for row operation (Column B)		Reduced row Echelon form (Column C)
i)	$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	a)	$[\ ] \overset{\frac{1}{2}R_1}{\rightsquigarrow} [\ ] \overset{R_2+(-1)R_1}{\rightsquigarrow} [\ ] \overset{R_3+R_1}{\rightsquigarrow} [\ ] \overset{R_3+(-1)R_2}{\rightsquigarrow} [\ ]$	1)	$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
ii)	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	b)	$[\ ] \overset{R_2+2R_1}{\rightsquigarrow} [\ ] \overset{R_2 \leftrightarrow R_3}{\rightsquigarrow} [\ ] \overset{\frac{1}{3}R_3}{\rightsquigarrow} [\ ] \overset{R_1+(-1)R_3}{\rightsquigarrow} [\ ]$	2)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
iii)	$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$	c)	$[\ ] \overset{R_1 \leftrightarrow R_2}{\rightsquigarrow} [\ ] \overset{\frac{1}{2}R_1}{\rightsquigarrow} [\ ] \overset{R_3+(-1)R_2}{\rightsquigarrow} [\ ] \overset{R_1+(-\frac{1}{2})R_2}{\rightsquigarrow} [\ ]$	3)	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Table: M2W5PT1

Find the correct option.

- ◯ Option 1: i) → b) → 3); ii) → c) → 2); iii) → a) → 1)
- ◯ Option 2: i) → a) → 3); ii) → c) → 1); iii) → b) → 2)
- ◯ **Option 3**: i) → b) → 3); ii) → c) → 1); iii) → a) → 2)
- ◯ Option 4: i) → c) → 1); ii) → b) → 3); iii) → a) → 2)

Solution :

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3/3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now collect the row operation steps,

first $R_2 + 2R_1$ then $R_2 \leftrightarrow R_3$ then $R_3/3$ then $R_1 - R_3$

& Reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $(i) \to (b) \to 3$

Now, let $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now, collect the row operation steps,

first $R_2 \leftrightarrow R_1$ then $R_1/2$ then $R_3 - R_2$ then $R_1 - \frac{1}{2}R_2$

& Reduced row echolon form of B is $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $(ii) \to (c) \to 1$

Now, let $C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

collection of row operation steps,

first $\frac{1}{2}R_1$ then $R_2 - R_1$ then $R_3 + R_1$ then $R_3 - R_2$

& Reduced row echolon form of C is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $(iii) \to (a) \to 2$

Hence, option 3 is true.

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3. There are two laptop manufacturers, one is at Adyar and the other is at Tambaram. Suppose the production costs (in crore of ₹) at Adyar and Tambaram are represented by the equations $A(x) = a_1x^2 + b_1x + c_1$ and $T(x) = a_2x + c_2$, respectively, where x represents the number (in hundreds) of laptops produced. At Adyar, the initial investment is known to be ₹3 crore, and the production costs for manufacturing 100 (i.e., $x = 1$) and 300 laptops (i.e., $x = 3$) are ₹4 crore and ₹12 crore, respectively. At Tambaram, the production costs for manufacturing 100 and 200 laptops are ₹6 crore and ₹7 crore, respectively. Suppose, Parveena and Amenla need new laptops for their start up companies. Parveena needs 500 laptops and Amenla needs 150 laptops. Both of them want their laptops with minimum production cost. Choose the correct option from the given set of options below.

   - ○ Option 1: Parveena should place her order at Adyar and Amenla should place her order at Tambaram to avail the minimum production cost.
   - ○ **Option 2:** Parveena should place her order at Tambaram and Amenla should place her order at Adyar to avail the minimum production cost.
   - ○ Option 3: Both of them should place their order at Tambaram to avail the minimum production cost.
   - ○ Option 4: Both of them should place their order at Adyar to avail the minimum production cost.

Solution :- Given production cost at Adyar is $A(x) = a_1x^2 + b_1x + c_1$

& production cost at Tambaram is $T(x) = a_2x + c_2$

where x represents the number (in hundred) of laptops produced.

Given intial investment is ₹ 3 crore at Adyar

i.e if we substitute $x = 0$ in $A(x)$ then

$c_1 = 3$.

So $A(x) = a_1x^2 + b_1x + 3$

Given, at Adyar, production cost for manufacturing 100 (i.e $x = 1$) is ₹ 4 crore i.e $4 = a_1 + b_1 + 3 \Rightarrow a_1 + b_1 = 1$ —(1)

Production cost for manufacturing 300 laptops is ₹ 12 crore.

i.e $$9a_1 + 3b_1 + 3 = 12$$
$$\Rightarrow 3a_1 + b_1 = 3 \quad —(2)$$

equation (1) & (2) form system of linear equations which having a unique solution which is, $a_1 = 1$ & $b_1 = 0$

So $A(x) = x^2 + 3$

Again, at Tambaram the production cost for manufacturing 100 laptops is ₹ 6 crore

i.e $$a_2 + c_2 = 6 \quad —— (3)$$

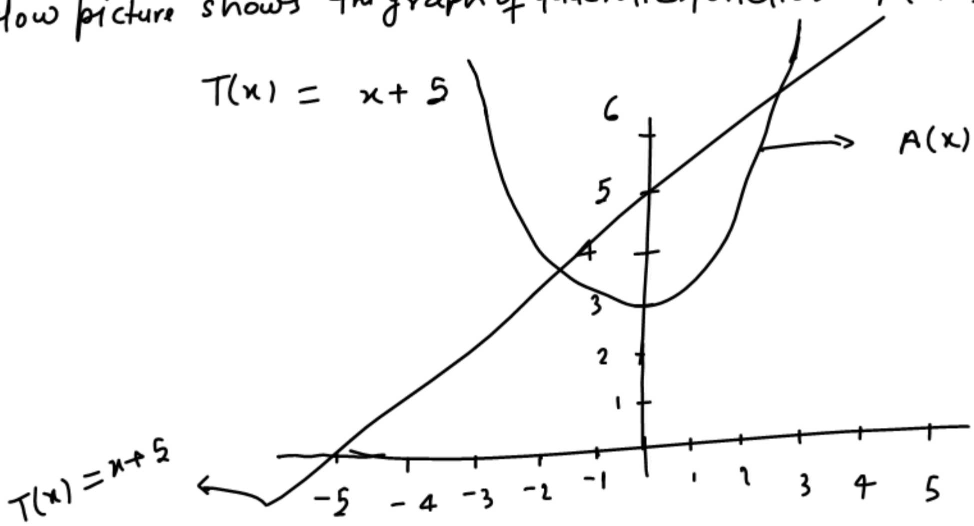
The production cost for manufacturing 200 laptops is ₹ 7 crore

i.e $$2a_2 + c_2 = 7 \quad —— (4)$$

After solving equations (3) & (4) we get $a_2 = 1$, $c_2 = 5$

So $T(x) = x + 5$

Below picture shows the graph of quadratic function $A(x) = x^2 + 3$ and line



Now, production cost for manufacturing 500 laptops at Adyar is $A(5) = 5^2+3 = 28$ crore rupees,

and at Tambaram is $T(5) = 5+5 = 10$ crore rupees..

The production cost for manufacturing 150 laptops at Adyar is $A(1\cdot5) = (1\cdot5)^2+3 = 2\cdot25+3$

$= 5\cdot25$ crore rupees,

and at Tamabaram is $T(1\cdot5) = 1\cdot5+5 = 6\cdot5$ crore rupees

So, the production cost for manufacturing 500 laptops at Tambaram makes minimum cost, and the production cost for manufacturing 150 laptops at Adyar makes minimum cost.

So, Praveena should place her order Tambaram and Amenla should place her order at Adyar to avail the minimum production cost.

Hence, option 2 is true.

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2 Multiple Select Questions (MSQ)

4. Choose the set of correct options.

   ◯ **Option 1:** If A is an upper triangular 3×3 matrix, then the adjoint matrix of A is also an upper triangular matrix.

   ◯ **Option 2:** If A is an invertible upper triangular 3×3 matrix, then the inverse matrix of A is also an upper triangular matrix.

   ◯ Option 3: Let A is an arbitrary real 3×3 matrix. If C is the adjoint matrix of A, then C is also the adjoint matrix of A^T.

   ◯ Option4: C_{jk} denotes the cofactor with respect to the j-th row and the k-th column of a 3×3 matrix A. If another matrix B is obtained from A by replacing the j-th row of A with $\begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$, then $det(B) = 3C_{jk}$

   ◯ Option 5: If A is an invertible 3×3 matrix and $C = adj(adj(A))$, then $det(C) = det(A)^9$

Solution :-

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} a_{22}a_{33} & -a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ 0 & a_{11}a_{33} & -a_{11}a_{23} \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$$

which is an upper triangular matrix

Hence, option 1 is true.

We know that $A^{-1} = \dfrac{adj(A)}{|A|}$ where $|A| = det(A)$

$$\text{So} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22}a_{33} & -a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ 0 & a_{11}a_{33} & -a_{11}a_{23} \\ 0 & 0 & a_{11}a_{22} \end{bmatrix}$$

which is also an upper triangular matrix.

Hence, option 2 is also true.

Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $adj(A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Now $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. then $adj(A^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Here $adj(A) \neq adj(A^T)$

Hence, option 3 is not true.

Now option 4: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

let $j = 2$ & $k = 2$, then

$$C_{22} = 1$$

As given, B is obtained from A by replacing the j^{th} row of A with $[3 \ 0 \ 0]$

So, for case $j = 2$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so, $det(B) = 0 \neq 3\,C_{22}$

So, option-4 is not true.

Option-5 : first Method :

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

then $adj(A) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

and $adj(adj(A)) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$

We know that determinant of a diagonal matrix is just multiplication of diagonal entries.

So, $det(A) = 6$

and $det(adj(adj(A))) = 6 \times 12 \times 18$

$\Rightarrow \quad det(adj(adj(A))) = 6^4 \neq det(A)^9 = 6^9$

Second Method:

Given $C = adj(adj(A))$, let $m = det(A) \Rightarrow \frac{1}{m} = det(A)$

$$det(C) = det(adj(adj(A)))$$

$$= det(adj(det(A) \cdot A^{-1})) \quad , \left(\text{use, } A^{-1} = \frac{adj(A)}{det(A)} \right)$$

$$= det(adj(m \cdot A^{-1}))$$

$$= det(det(mA^{-1}) \cdot (mA^{-1})^{-1}) \quad , \left(\text{use, } A^{-1} = \frac{adj(A)}{det(A)} \right)$$

$$= det\left(m^3 det(A^{-1}) \cdot \frac{1}{m}(A^{-1})^{-1}\right) \quad \left(\begin{array}{c} \text{Since } A \text{ is} \\ 3\times 3 \text{ matrix} \\ \text{so } det(KA) = K^3 det(A) \end{array} \right)$$

$$= det\left(m^3 (det(A))^{-1} \cdot \frac{1}{m} \cdot A\right) \quad , \left(\because (A^{-1})^{-1} = A \right)$$

$$= det\left(m^3 \cdot \frac{1}{m} \cdot \frac{1}{m} \cdot A\right)$$

$$= det(m \cdot A)$$

$$= m^3 \cdot det(A)$$

$$= m^3 \cdot m \quad \left(\because \text{we have assume } det(A) = m \right)$$

$$= m^4$$

$$\Rightarrow \quad det(C) = det(A)^4$$

Hence, Option 5 is not true.

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5. Choose the correct set of options based on the matrices given in Table M2W2PT2.

Column A	Column B
$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$B_2 = \begin{bmatrix} -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \\ -\frac{1}{3} & 0 & 0 \end{bmatrix}$
$A_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}$	$B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$A_4 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$	$B_4 = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

Table: M2W5PT2

- ◯ **Option 1:** A_1 and B_3 are inverses to each other.
- ◯ Option 2 : A_1 and B_1 are inverses to each other.
- ◯ **Option 3 :**A_2 and B_1 are inverses to each other.
- ◯ **Option 4:** A_3 and B_4 are inverses to each other.
- ◯ Option 5: A_2 and B_3 are inverses to each other.
- ◯ Option 6: A_3 and B_2 are inverses to each other.
- ◯ Option 7: A_4 and B_4 are inverses to each other.
- ◯ **Option 8:** A_4 and B_2 are inverses to each other.
- ◯ Option 9: A_2 and A_3 have different reduced row echelon form.
- ◯ Option 10: A_1 and A_2 have different reduced row echelon form.
- ◯ **Option 11:** All the matrices in column A have the same reduced row echelon form and that is the identity matrix of order 3.

○ Option 12: All the matrices in column A have the same reduced row echelon form but that is not the identity matrix of order 3.

Solution: Given $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$|A_1| = 1(-1) = -1$$

$$\operatorname{adj}(A_1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\text{so} \quad A_1^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = B_3$$

similarly, we can calculate A_2^{-1} which is B_1, $A_3^{-1} = B_4$

& $A_4^{-1} = B_2$

Now $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Reduced row echelon form

$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Reduced row echelon form

$A_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix}$

$$\xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow[R_3+3R_2]{R_1+2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Reduced row echelon form}$$

$$A_4 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\downarrow \left(-\tfrac{1}{3}\right) R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xleftarrow[R_3+R_2]{R_1+2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Reduced row echelon form}$$

Clearly, all matrices in column A have same reduced row echelon form which is Identity matrix of order 3.

Hence option 1, option 3, option 4, option 8 and option 11 are true.

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3 Numerical Answer Type (NAT):

6. A gym trainer suggested Pranjal to include banana, mozzarella cheese, and avocado in his daily diet, for his fitness. In 1 banana, there are 1 unit of protein, 20 units of carbohydrate, and 1 unit of fat. In $\frac{1}{2}$ cup mozzarella cheese, there are 10 units of protein, 50 units of carbohydrate and 0 unit of fat. In 1 avocado there are 3 units of protein, 10 units of carbohydrate, and 10 units of fat. Suppose the calories intake from 1 banana, $\frac{1}{2}$ cup mozzarella cheese, and 1 avocado are 105, 90 and 115, respectively. If the gym trainer suggested Pranjal to take 18 units of protein, 110 units of carbohydrate, and 22 units of fat by taking only these three items, then find out the calories intake by Pranjal each day from these three items only. [Answer: 530]

Solution: Given

	Protein	Carbohydrate	Fat	Calories
1 Banana	1	20	1	105
½ cup mozzarella Cheese	10	50	0	90
1 Avocado	3	10	10	115

Let, to take 18 units of protein, 110 units of carbohydrate & 22 units of fat, Pranjal takes x banana, y number of ½ cup mozzarella cheese & z avocado, then system of linear equations is

$$x + 10y + 3z = 18$$

$$20x + 50y + 10z = 110 \Rightarrow 2x + 5y + z = 11$$

$$x + 10z = 22$$

Matrix representation of the above system of linear equations is

$$\begin{bmatrix} 1 & 10 & 3 \\ 2 & 5 & 1 \\ 1 & 0 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \\ 22 \end{bmatrix}$$

Augmented matrix of the above system is

$$\left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 2 & 5 & 1 & 11 \\ 1 & 0 & 10 & 22 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & -15 & -5 & -25 \\ 0 & -10 & 7 & 4 \end{array}\right]$$

$$\downarrow \left(\frac{-1}{15}\right) R_2$$

$$\left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 31/3 & 62/3 \end{array}\right] \xleftarrow{R_3 + 10R_2} \left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & -10 & 7 & 4 \end{array}\right]$$

$$\downarrow \frac{3}{31} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 10 & 3 & 18 \\ 0 & 1 & 1/3 & 5/3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Hence, the above system has a unique solution &

$$z = 2$$

$$y + \frac{z}{3} = 5/3 \Rightarrow y = \frac{5}{3} - \frac{2}{3} = 1$$

$$x + 10y + 3z = 18 \Rightarrow x + 10 + 6 = 18$$

$$\Rightarrow x = 2$$

So, Pronjal will take 2 banana, Only one $\frac{1}{2}$ cup mozzarella cheese & 2 avocado.

Calories intake from 1 banana, $\frac{1}{2}$ cup mozzarella & 1 avocado are 105, 90 & 115 respectively.

Hence, total calories taken by Pronjal is

$$2 \times 105 + 1 \times 90 + 115 \times 2 = 210 + 90 + 230$$
$$= 530$$

So Answer is 530.

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7. Consider the system of linear equations $Ax = b$, where $A = \begin{bmatrix} 2 & a & 3 \\ a & -2 & -1 \\ -1 & a & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ 5/4 \\ x_3 \end{bmatrix}$,

   and $b = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$, and the solution for x is partially known.

   What is the value of a, if $a > 1$ is given? [Answer: 2]

Solution: Given $A = \begin{bmatrix} 2 & a & 3 \\ a & -2 & -1 \\ -1 & a & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ 5/4 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$

Observe, we have given value of x_2 which is $5/4$

$$|A| = \begin{vmatrix} 2 & a & 3 \\ a & -2 & -1 \\ -1 & a & 0 \end{vmatrix}, \quad \text{where } |A| = \det(A)$$

Expand along R_3

$$|A| = -1(-a+6) - a(-2-3a) + 0$$

$$= a - 6 + 3a^2 + 2a$$

$$|A| = 3a^2 + 3a - 6$$

replacing second column with b in A

we get

$$A_{x_2} = \begin{bmatrix} 2 & 1 & 3 \\ a & a & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Expand along R_3

$$\Rightarrow |A_{x_2}| = -1(-1-3a) - 1(-2-3a)$$

$$= 1 + 3a + 2 + 3a = 6a + 3$$

Now, using Cramer rule

$$x_2 = 5/4 = \frac{|A_{x_2}|}{|A|} = \frac{6a+3}{3a^2+3a-6}$$ this is well

defined because $|A| \neq 0$ for $a > 1$

$$\Rightarrow \quad 5(3a^2+3a-6) = 4(6a+3)$$

$$\Rightarrow \quad 5a^2+5a-10 = 8a+4$$

$$\Rightarrow \quad 5a^2-3a-14 = 0$$

$$\Rightarrow \quad 5a^2-10a+7a-14 = 0 \qquad -14 \times 5$$

$$\Rightarrow \quad (a-2)(5a+7) = 0$$

$$\Rightarrow \quad a = 2 \quad \text{or} \quad a = -7/5$$

but $a > 1$

$$\Rightarrow \quad a = 2$$

Hence, Answer is 2.

.

4 Comprehension Type Question:

In genetics, a classic example of dominance is the inheritance of seed shape (pea shape) in peas. Peas may be round (associated with genotype R) or wrinkled (associated with genotype r). In this case, three combinations of genotypes are possible: RR, rr, and Rr. The RR individuals have round peas and the rr individuals have wrinkled peas. In Rr individuals the R genotype masks the presence of the r genotype, so these individuals also have round peas. Thus, the genotype R is completely dominant to genotype r, and genotype r is recessive to genotype R. First, assume the crossing of RR with RR. This always gives the genotype RR, therefore the probabilities of an offspring to be RR, Rr, and rr respectively are equal to 1, 0, and 0. Second, assume crossing of Rr with RR. The offspring will have equal chances to be of genotype RR and genotype Rr, therefore the probabilities of RR, Rr, and rr repectively are 1/2, 1/2, and 0. Third, consider crossing of rr with RR. This always results in genotype Rr. Therefore, the probabilities of genotypes RR, Rr, and rr repectively are 0, 1, and 0, respectively.
This can be veiwed as the following table:

Parents' genotypes			Genotypes of offspring
RR-RR	RR-Rr	RR-rr	
1	1/2	0	RR
0	1/2	1	Rr
0	0	0	rr

Table: M2W5 PT 3

The matrix representing this observation is given by $P = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Let the probabilities of RR, Rr, and rr in the initial (i.e., at $t = 0$) sample space be $X_0^1, X_0^2,$ and X_0^3, respectively. This is represented by the initial distribution vector (3×1 matrix) is denoted by $X_0 = \begin{bmatrix} X_0^1 \\ X_0^2 \\ X_0^3 \end{bmatrix}$.

For any positive integer n, the distribution vector after n generations (i.e., at $t = n$) is denoted by X_n and given by the equation $PX_{n-1} = X_n$.
Using the above information answer the following questions.

8. Find out the correct set of options from the following. (MSQ)
   - ◯ Option 1: The row reduced echelon form of P and P^2 are different in this case.
   - ◯ **Option 2:** The row reduced echelon form of P and P^2 are same in this case.

◯ **Option 3:** The row reduced echelon form of P is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

◯ Option 4: The row reduced echelon form of P is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Given $P = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Reduced row echelon form of P

Now $P^2 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

So $P^2 = \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{4R_2} \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow{R_1 - \frac{3}{4}R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Reduced row echelon form of P^2

Observe, Reduced row echelon form of P & P^2 are the same.

Hence, Option 2 & Option 3 are true.

.

9. Suppose after 2 years the distribution vector i.e., X_2 is calculated to be $\begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}$, and the initial distribution of RR is $\frac{1}{3}$. Find out the initial distibution of Rr and rr. (MCQ)
   - ◯ Option 1: The initial distibution of Rr and rr : $\frac{2}{3}$, 0, respectively.
   - ◯ Option 2: The initial distibution of Rr and rr : 0, $\frac{2}{3}$, respectively.
   - ◯ **Option 3:** The initial distibution of Rr and rr : $\frac{1}{3}$, $\frac{1}{3}$, respectively.
   - ◯ Option 4: Cannot be determined from the given information.

Solution :- Given $PX_{n-1} = X_n$ & initial distribution of RR is $1/3$

Let initial distribution is $X_0 = \begin{bmatrix} 1/3 \\ y \\ z \end{bmatrix}$,

For simplicity of notations we are writing

$X_0^1 = x, \quad X_0^2 = y, \quad \& \quad X_0^3 = z.$

Now, $PX_0 = X_1 \quad \& \quad PX_1 = X_2$

$\Rightarrow \quad P \cdot P X_0 = PX_1 = X_2$

$\Rightarrow \quad P^2 X_0 = X_2$

$$\Rightarrow \begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

Now, this is a system of linear equations, where $x = 1/3$

Observe, we can not apply cramer's rule because determinant of coefficient matrix is zero.

Now, Augmented matrix of the above system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 3/4 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{4R_2} \left[\begin{array}{ccc|c} 1 & 3/4 & 1/2 & 3/4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 - \tfrac{3}{4}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, from reduced row echelon form of Augmented matrix

$$x - z = 0 \quad , \text{ but } x = 1/3$$

$$\Rightarrow \quad z = 1/3$$

again from above reduced row echelon form of Augmented matrix,

$$y + 2z = 1$$

$$\Rightarrow \quad y + \frac{2}{3} = 1$$

$$\Rightarrow \quad y = 1 - 2/3 = 1/3$$

But y & z denote the initial distribution of Rr & rr respectively

Hence initial distribution of Rr & rr are $1/3$, $1/3$ respectively.

Hence, the third option is correct.

.

10. Suppose after 3 generations the distribution vector i.e., X_3 is calculated to be $\begin{bmatrix}1\\0\\0\end{bmatrix}$, and recall that $0 \leq X_0^1, X_0^2, X_0^3 \leq 1$. Find out the correct set of options. (MSQ)

   ◯ **Option 1:** $X_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

   ◯ Option 2: $X_0 = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$

   ◯ Option 3: $X_0 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

   ◯ Option 4: X_0 cannot be determined from the given information.

   ◯ **Option 5:** $X_0 = X_n$ for all positive integer n.

   ◯ Option 6: There can be some positive integer n for which $X_0 \neq X_n$.

Solution:

Given $PX_{n-1} = X_n$ & $X_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$\Rightarrow$ $PX_0 = X_1$ & $PX_1 = X_2$ & $PX_2 = X_3$

$\Rightarrow$ $P^2X_0 = PX_1 = X_2$

$\Rightarrow$ $P^3X_0 = PX_2 = X_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Now $P^3 = P^2 \cdot P = \begin{bmatrix}1 & 3/4 & 1/2\\0 & 1/4 & 1/2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1/2 & 0\\0 & 1/2 & 1\\0 & 0 & 0\end{bmatrix}$

$= \begin{bmatrix}1 & 7/8 & 3/4\\0 & 1/8 & 1/4\\0 & 0 & 0\end{bmatrix}$

For simplicity of notation we are writing $x_0^1 = x$, $x_0^2 = y$, $x_0^3 = z$

So, let initial distribution $X_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

then $P^3 X_0 = X_3$ form a system of linear equations

$$\begin{bmatrix} 1 & 7/8 & 3/4 \\ 0 & 1/8 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, Augmented matrix of the above system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 7/8 & 3/4 & 1 \\ 0 & 1/8 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{8R_2} \left[\begin{array}{ccc|c} 1 & 7/8 & 3/4 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 - \tfrac{7}{8} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is reduced row echelon from of Augmented matrix.

So,
$$x - z = 1$$
$$y + 2z = 0$$

$$\Rightarrow \quad x = 1 + z \quad \text{——— ①}$$
$$y = -2z \quad \text{—— ②}$$

Observe that, it is given that $0 \le x, y\ z \le 1$

From equation (1). we get $x \geq 1$ as $z \geq 0$

But we also have, $x \leq 1$

Hence, $x = 1$

Hence, from equation (1)

$$z = 0$$

and from equation (2) $z = 0 \Rightarrow y = 0$

So $$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now again, $P x_{n-1} = x_n$

$$\Rightarrow \quad P x_0 = x_1 \quad \Rightarrow x_1 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, $$x_2 = P x_1 = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vdots \qquad \vdots$$

$$x_n = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x_0$$

Hence, option 1 & option 5 are true.

Week-5
Mathematics for Data Science - 2
Solutions of System of Linear Equations
Graded Assignment Solutions

1 Multiple Choice Questions (MCQ)

1. Consider the system of equations given below:

$$\frac{x^{2021}}{2021} + \frac{y^{2021}}{2021} - \frac{z^{2021}}{2021} = \pi$$

$$\frac{x^{2021}}{2021} + \frac{y^{2021}}{2021} - \frac{z^{2021}}{2021} = e$$

$$\frac{x^{2021}}{2021} - \frac{y^{2021}}{2021} + \frac{z^{2021}}{2021} = 1729.$$

The system has

- ◯ **Option 1:** no solution.
- ◯ Option 2: a unique solution.
- ◯ Option 3: infinitely many solutions.
- ◯ Option 4: finitely many solutions.

Solution:
Observe that left side of first two equations in the system of linear equations are the same.
This implies that $\pi = e$ which is not true.
Hence the system of linear equations has no solution.

2. Let the reduced row echelon form of a matrix A be

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix}.$$

If the first, second and fourth columns of A are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ then the third column of matrix A is,

- ○ Option 1: $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$
- ○ Option 2: $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$
- ○ **Option 3:** $\begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$
- ○ Option 4: $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Solution:
Let us assume the elements of the third column of matrix A be x, y, z such that

$$A = \begin{bmatrix} 1 & 3 & x & -1 \\ 0 & 2 & y & 0 \\ -1 & 1 & z & 0 \end{bmatrix}$$

To transform this matrix into reduced row echelon form, we do the following row operations in the same sequence:
$R_3 + R_1, R_2/2, R_3 - 4R_2, R_1 - 3R_2, R_3/(-6), R_2 - R_3, R_1 + 3R_3$.

Now, the transformed matrix is $\begin{bmatrix} 1 & 0 & \frac{x-y-z}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{y+z+x}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{-z-x+2y}{6} & \frac{1}{6} \end{bmatrix}$.

Compare this matrix with the reduced row echelon form of the matrix A.
We get,

$$\frac{x-y-z}{2} = 0, \frac{y+z+x}{6} = 0, \frac{-z-x+2y}{6} = 1.$$

Solving these three equations, we get, $x = 0, y = 2, z = -2$ which are the elements of the third column of the matrix A.

2 Multiple Select Questions (MSQ)

3. In a particular year, the profit (in lakhs of ₹) of Star Fish company is given by the polynomial $P(x) = ax^2 + bx + c$ where x denotes the number of months since the beginning of the year (i.e., $x = 1$ denotes January, $x = 2$ denotes February, and so on). In January and February the company made a loss of ₹45(in lakhs), and ₹19(in lakhs) respectively, and in March the company made profit of ₹3(in lakhs). Let the loss be represented by negative of profit.
Choose the correct set of options based on the given information.

- ○ Option 1: The maximum profit will be in the month of May.
- ○ **Option 2:** The maximum profit will be in the month of August.
- ○ **Option 3:** The maximum monthly profit amount is ₹53 lakh.
- ○ Option 4: The maximum monthly profit amount is ₹35 lakh.

Solution: Based on the given information of losses in January and February, and profit in March, we formulate the following set of equations:

$$\begin{aligned} a + b + c &= -45 \\ 4a + 2b + c &= -19 \\ 9a + 3b + c &= 3 \end{aligned}$$

This is a system of linear equations where a, b, c being the unknowns. The augmented matrix is written as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -45 \\ 4 & 2 & 1 & -19 \\ 9 & 3 & 1 & 3 \end{array}\right]$$

The following sequence of row operations are performed to transform this augmented matrix into reduced row echelon form:

$$R_2 - 4R_1, R_3 - 9R_1, \frac{-1}{2}R_2, R_1 - R_2, R_3 + 6R_2, R_1 + \frac{1}{2}R_3, R_2 - \frac{3}{2}R_3.$$

The resultant augmented matrix is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 32 \\ 0 & 0 & 1 & -75 \end{array}\right]$$

Thus, we find that $a = -2, b = 32, c = -75$. Now, we can represent the polynomial as $P(x) = -2x^2 + 32x - 75$. We have to find the point of global maximum and the corresponding maximum value of this function. Differentiating $P(x)$, we get $P'(x) = -4x + 32$. We get $x = 8$ as a critical point. Note that $P''(x) = -4$. Thus, the function has a maximum at the critical point. So, $P(8) = 53$ which is the maximum monthly profit and it happens in the month of August ($x = 8$).

4. If A be a 3×4 matrix and b be a 4×1 matrix, then choose the set of correct options.

   - ◯ **Option 1:** If $(A|b)$ be the a ugmented matrix and $(A'|b')$ be the matrix obtained from $(A|b)$ after a finite number of elementary row operations then the system $Ax = b$ and the system $A'x = b'$ have the same set of solutions.
   - ◯ Option 2: If $(A'|b')$ is the reduced row echelon form of $(A|b)$ then the system $A'x = b'$ has at least one solution.
   - ◯ **Option 3:** If $(A'|b')$ is the reduced row echelon form of $(A|b)$, then A' is also in reduced row Echelon form.
   - ◯ **Option 4:** If $(A'|b')$ is the reduced row echelon form of $(A|b)$ and there is no row such that the only non zero entry lies in the last column of $(A'|b')$ then the system $Ax = b$ has at least one solution.

Solution:

Option 1: We know from the Gauss elimination method that any number of elementary row operations on an augmented matrix $(A|b)$ does not alter the solutions of $Ax = b$. Hence this option is true.

Option 2: Note that if there is a row with all zeros in the row echelon form such that the corresponding row in the b vector is non-zero, then we have no solution for the system $Ax = b$.

For example, Let

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

be the reduced row echelon form of a augmented matrix of a system of linear equations. Observe that the system of linear equation has no solution.

Option 3: Observe that when we transform the augmented matrix into reduced row echelon form, the coefficient matrix A also gets transformed into reduced row echelon form.

Option 4: Observe that if $(A'|b')$ is the reduced row echelon form of $(A|b)$ and there is no row such that the only non zero entry lies in the last column of $(A'|b')$, then all variable can be dependent or there can be at least one variable which will be independent. In both the cases the system of linear equations has at least one solution.

To better understand this statement, consider the following example

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & b_1 \\ 0 & 1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & 0 & b_3 \end{array}\right]$$

Since there is no row such that the only non-zero entry lies in the last column, x_4, the unknown variable, becomes an independent variable. Thus, there can be infinitely many solutions for the system $Ax = b$.

5. Choose the set of correct options

   ◯ **Option 1:** If the sum of all the elements of each row of a matrix A is 0, then A is not invertible.

   ◯ **Option 2:** If E is a matrix of order 3×3 obtained from the identity matrix by a finite number of elementary row operations then E is invertible.

   ◯ Option 3: Any system of linear equations has at least one solution.

   ◯ Option 4: If A is a matrix of order 3×3 and $det(A) = 3$ then $det(Adj(A)) = 3$.

   ◯ **Option 5:** If A is a matrix of order 3×3 and $det(A) = 3$ then $det(Adj(A)) = 9$.

Solution:

Note: we have shown a 3×3 matrix here only as an example. We cannot consider A to be a $m \times n$ matrix $(m \neq n)$ because inverse exists only for square matrices (where determinant is defined).

Option 1: Consider a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Given

$$a_{11} + a_{12} + a_{13} = 0$$
$$a_{21} + a_{22} + a_{23} = 0$$
$$a_{31} + a_{32} + a_{33} = 0$$

$$\text{So, } det(A) = \text{ determinant of } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$= \text{ determinant of } \begin{bmatrix} a_{11} & a_{12} & a_{11} + a_{12} + a_{13} \\ a_{21} & a_{22} & a_{21} + a_{22} + a_{23} \\ a_{31} & a_{32} & a_{31} + a_{32} + a_{33} \end{bmatrix}$$
$$= \text{ determinant of } \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix}$$

$\implies det(A) = 0$

Hence matrix A is not invertible. Similarly, we can show for a square matrix of any order.

Option 2: Observe that if E is a 3×3 matrix obtained from $I_{3\times3}$ using finite number of elementary row operations, then $det(E) = c.det(I_{3\times3})$, where c is any non zero real number. so $det(E) = c \neq 0$ (Since det($I_{3\times3} = 1$).

Hence, E is invertible.

Option 3: A system of linear equations can have a unique solution, infinitely many solutions or no solution.

For example, consider a system linear equation with only one equation $x + y + z = 1$.

Observe that this system of linear equation has infinite many solutions.

Options 4 and 5: Let A be a square matrix of order 3.
we know that $A^{-1} = \frac{adj(A)}{det(A)}$
$\implies (det(A))A^{-1} = adj(A)$
$\implies det((det(A))A^{-1}) = det(adj(A))$
$\implies det(adj(A)) = det(A)^n det(A^{-1})$ Since $det(k.A) = k^n det(A)$
$\implies det(adj(A)) = det(A)^{n-1}$ Since $det(A^{-1}) = \frac{1}{det(A)}$

So, $det(Adj(A)) = 3^{3-1} = 9$.

6. Ramya bought 1 comic book, 2 horror books, and 1 novel from a bookshop which cost her ₹1000. Romy bought 2 comic books, 5 horror books, and 1 novel which cost him ₹2000. Farjana bought 4 comic books, 5 horror books, and c novel from a shop which cost her ₹d. If $x_1, x_2,$ and x_3 represent the price of each comic book, horror book, and novel, respectively, then choose the set of correct options.

- ◯ **Option 1:** The matrix representation to find x_1, x_2 and x_3 is
$$\begin{bmatrix}1 & 2 & 1\\ 2 & 5 & 1\\ 4 & 5 & c\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1000\\ 2000\\ d\end{bmatrix}$$
- ◯ **Option 2:** The matrix representation to find x_1, x_2 and x_3 is
$$\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}1 & 2 & 4\\ 2 & 5 & 5\\ 1 & 1 & c\end{bmatrix} = \begin{bmatrix}1000 & 2000 & d\end{bmatrix}$$
- ◯ Option 3: The matrix representation to find x_1, x_2 and x_3 is
$$\begin{bmatrix}1 & 2 & 4\\ 2 & 5 & 5\\ 1 & 1 & c\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix} = \begin{bmatrix}1000 & 2000 & d\end{bmatrix}$$
- ◯ Option 4: If Farjana tries to find $x_1, x_2,$ and x_3 using appropriate matrix representation by taking $c = 2$ and $d = 4000$, then the price of each comic book that she thus arrives at, will not be unique.
- ◯ **Option 5:** If $c = 7$ and $d = 4000$, then the price of each comic book cannot be determined from this data.
- ◯ **Option 6:** If $c = 7$ and $d = 3000$, then the shopkeeper has made a mistake.
- ◯ **Option 7:** If $c = 2$ and $d = 3000$, then the price of each comic book can be determined from the data.

Solution:
Given $x_1, x_2,$ and x_3 represent the price of each comic book, horror book, and novel, respectively.
So, prices of 1 comic book, 2 horror books, and 1 novel are $x_1, 2x_2$ and x_3 respectively. Similarly, we can get for others.
So, we write three equations for the total price of all the books purchased by Ramya, Romy and Farjana respectively as the following:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1000\\ 2x_1 + 5x_2 + x_2 &= 2000\\ 4x_1 + 5x_2 + cx_3 &= d \end{aligned}$$

Option 1: The matrix representation of the system of linear equations $Ax = b$ is given by

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 4 & 5 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ d \end{bmatrix}$$

Option 2 and 3: Observe that $(Ax)^T = b^T \implies x^T A^T = b^T$. Thus,

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 5 \\ 1 & 1 & c \end{bmatrix} = \begin{bmatrix} 1000 & 2000 & d \end{bmatrix}$$

Now, the augmented matrix of the above system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1000 \\ 2 & 5 & 1 & 2000 \\ 4 & 5 & c & d \end{array}\right]$$

In order to find whether the system of equations has no solution, unique solution or infinitely many solutions, we transform the obtained augmented matrix to a row echelon form using the following sequence of row operations:

$$R_2 - 2R_1, R_3 - 4R_1, R_1 - 2R_2, R_3 + 3R_2.$$

The resultant augmented matrix is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & c-7 & d-4000 \end{array}\right]$$

Option 4: Substitute $c = 2, d = 4000$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

Observe that in this case the system has a unique solution.
Option 5: Now, substitute $c = 7, d = 4000$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Observe that in this case the system has infinitely many solutions.

Hence the price of each comic book cannot be determined from this data.
Option 6: Now, substitute $c = 7, d = 3000$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1000 \end{array}\right]$$

Observe that in this case the system has no solutions.
Hence the shopkeeper has made a mistake.
Option 7: Now, substitute $c = 2, d = 3000$, we get,

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1000 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & -1000 \end{array}\right]$$

Thus, we get $x_3 = 200, x_1 + 3x_3 = 1000 \implies x_1 = 400$.
Hence the price of each comic book can be determined from the data.

3 Numerical Answer Type (NAT):

7. Three mobile shops- shop A, shop B and shop C, sell three brands of mobile phones: brand R, brand S and brand T. In a week, shop A sold 1 mobile phone of brand R, $3k$ mobile phones of brand S, and $3k+4$ mobile phones of brand T. Shop B sold 1 mobile phone of brand R, $k+4$ mobile phones of brand S, and $4k+2$ mobile phones of brand T. Shop C sold 1 mobile phone of brand R, $2k+2$ mobile phones of brand S, and $3k+4$ mobile phones of brand T (assume, $k \neq 2$). Assume that the price of a given model of a given brand is the same in all the shops. Shop A, shop B, and shop C earned ₹61, ₹65 and ₹66 (in thousands), respectively by selling these three brands of mobile phones. If the price of each mobile phone of brand S is ₹5 (in thousands), then what is the price of each mobile phone of brand T (in thousands)? [Note: Suppose the price comes out to be 20,000, then the answer should be 20]? [Answer: 6]

Solution:
Let x_R, x_S, x_T be the price of each mobile of brand R, S, T respectively.
So, In a week, shop A earned ₹$(x_R + 3kx_S + (3k+4)x_T)$ by selling 1 mobile phone of brand R, $3k$ mobile phones of brand S, and $3k+4$ mobile phones of brand T which is equal to ₹61 (in thousands). Similarly we can calculate amount in a week for the shop B and shop C.
Hence, The system of equations representing the total earnings of the shops A, B, C can be represented by the following:

$$\begin{aligned} x_R + 3kx_S + (3k+4)x_T &= 61 \\ x_R + (k+4)x_S + (4k+2)x_T &= 65 \\ x_R + (2k+2)x_S + (3k+4)x_T &= 66. \end{aligned}$$

So augmented matrix of the above system of linear equations is

$$\left[\begin{array}{ccc|c} 1 & 3k & 3k+4 & 61 \\ 1 & k+4 & 4k+2 & 65 \\ 1 & 2k+2 & 3k+4 & 66 \end{array}\right].$$

Now, using the following sequence of elementary row operations:

$$R_2 - R_1, R_3 - R_1, R_2/(4-2k), R_3/(2-k), R_3 - R_2, R_2 + R_3, 2R_3$$

Transformed the above augmented matrix into a row echelon form given below:

$$\left[\begin{array}{ccc|c} 1 & 3k & 3k+4 & 61 \\ 0 & 1 & 0 & \frac{5}{2-k} \\ 0 & 0 & 1 & \frac{6}{2-k} \end{array}\right]$$

Note that $k \neq 2$.
Based on the above, we can write $x_S = \frac{5}{2-k}, x_T = \frac{6}{2-k}$.
Since, the price of each mobile phone of brand S is ₹5 (in thousands) i.e., $x_S = 5$.
After substituting the value $x_S = 5$ in the equation $x_S = \frac{5}{2-k}$, we got $k = 1$ and so $x_T = 6$. (We can use cramer's rule also to solve this system of linear equations.)
Hence, the price of each mobile phone of brand T is ₹6 (in thousands).

4 Comprehension Type Question:

The network in Figure: M2W5GA1 shows a proposed plan for flow of traffic around a park. All the streets are assumed to be one-way and the arrows denote the direction of flow of traffic. The plan calls for a computerized traffic light at the South Street. Let $2x_1, 3x_2, 2x_3,$ and x_4 denote the average number (per hour) of vehicles expected to pass through the connecting streets (e.g., $2x_1$ denote the average number (per hour) of vehicles expected to pass through the street connecting the North Street and West Street as shown in Figure: M2W5GA1). 400, 1000, 900, and c denote the average number (per hour) of vehicles expected to pass through West, North, East, and South Streets respectively.

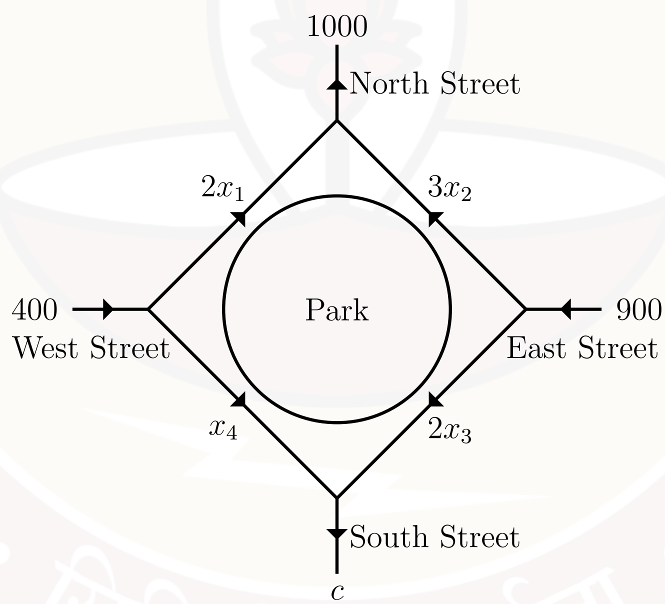


Figure: M2W5GA1

8. The system of equations corresponding to the flow of expected traffic according to the given data above, will be

◯ **Option 1:**

$$\begin{aligned} 2x_1 + 3x_2 &= 1000 \\ 3x_2 + 2x_3 &= 900 \\ 2x_3 + x_4 &= c \\ 2x_1 + x_4 &= 400 \end{aligned}$$

◯ Option 2:

$$\begin{aligned} 2x_1 + 3x_2 &= 900 \\ 3x_2 + 2x_3 &= 1000 \\ 2x_3 + x_4 &= 400 \\ 2x_1 + x_4 &= c \end{aligned}$$

◯ Option 3:

$$\begin{aligned} 2x_1 + 3x_2 &= 1000 \\ 3x_2 + 2x_3 &= c \\ 2x_3 + x_4 &= 900 \\ 2x_1 + x_4 &= 400 \end{aligned}$$

◯ Option 4:

$$\begin{aligned} 2x_1 + 3x_2 &= 400 \\ 3x_2 + 2x_3 &= 900 \\ 2x_3 + x_4 &= c \\ 2x_1 + x_4 &= 1000 \end{aligned}$$

Solution:
The average number of vehicles expected to pass through West street is 400. $2x_1$ and x_4 are the average number of vehicles expected to pass through the street connecting West and North and West and South streets respectively. Hence, $2x_1 + x_4 = 400$. Similarly, we arrive at the other three equations:
$2x_3 + x_4 = c$
$3x_2 + 2x_3 = 900$
$2x_1 + 3x_2 = 1000$.
Hence, option 1 is true.

9. How many vehicles per hour in average are expected to pass through the South Street? (NAT) [Answer: 300]
Solution:
According to the question we need to find the value of c.
The system of linear equation are:

$$\begin{aligned} 2x_1 + 3x_2 &= 1000 \quad ...(1) \\ 3x_2 + 2x_3 &= 900 \quad ...(2) \\ 2x_3 + x_4 &= c \quad ...(3) \\ 2x_1 + x_4 &= 400 \quad ...(4) \end{aligned}$$

After subtracting of the both side of the equation from equation (4) to equation (3), and from equation (1) to equation (2) we get,
$2x_1 - 2x_3 = 400 - c$ and
$2x_1 - 2x_3 = 100$
Since left side of the both equations are the same.
Hence, $400 - c = 100 \implies c = 300$.

10. Match the names of the street in Column A with the maximum and minimum number of vehicles expected to pass through the street in average (per hour) in Column B and Column C, respectively; in Table M1W5GA1.

	Name of the connecting street		The maximum number of vehicles expected to pass through the street (per hour)		The minimum number of vehicles expected to pass through the street (per hour)
	Column A		Column B		Column C
a)	Connecting West and North street	i)	300	1)	0
b)	Connecting East and North street	ii)	300	2)	100
c)	Connecting East and South street	iii)	400	3)	600
d)	Connecting West and South street	iv)	900	4)	0

Table : M1W5GA1

(MCQ)

- ◯ Option 1: a $\rightarrow$ iv $\rightarrow$ 1; b $\rightarrow$ ii $\rightarrow$ 3; c $\rightarrow$ iii $\rightarrow$ 2; d $\rightarrow$ i $\rightarrow$ 4
- ◯ Option 2: a $\rightarrow$ ii $\rightarrow$ 1; b $\rightarrow$ iii $\rightarrow$ 3; c $\rightarrow$ iv $\rightarrow$ 2; d $\rightarrow$ i $\rightarrow$ 4
- ◯ Option 3: a $\rightarrow$ ii $\rightarrow$ 2; b $\rightarrow$ iv $\rightarrow$ 3; c $\rightarrow$ iii $\rightarrow$ 4; d $\rightarrow$ i $\rightarrow$ 1
- ◯ **Option 4:** a $\rightarrow$ iii $\rightarrow$ 2; b $\rightarrow$ iv $\rightarrow$ 3; c $\rightarrow$ i $\rightarrow$ 4; d $\rightarrow$ ii $\rightarrow$ 1

Solution:
As we get the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 1000 \\ 3x_2 + 2x_3 &= 900 \\ 2x_3 + x_4 &= 300 \\ 2x_1 + x_4 &= 400 \end{aligned}$$

So we can write last two equations $2x_1 = 400 - x_4$ and $2x_3 = 300 - x_4$
Observe that the average number of vehicles expected to pass can not be negative, that means x_4 can not be grater than 300. i.e., $x_4 \leq 300$.
Therefore, the maximum number of vehicles expected to pass through West street to south street is 300. But the minimum number of vehicles expected to pass through West street to south street can be 0. So, $0 \leq x_4 \leq 300$.

Again, since the third equation in the system of linear equations is $2x_3 = 300 - x_4$ and the second equation in the system of linear equations is $3x_2 + 2x_3 = 900 \implies 3x_2 = 900 - 2x_3 = 900 - 300 + x_4 = 600 + x_4$

Therefore, $600 \leq 3x_2 \leq 900$, (as $0 \leq x_4 \leq 300$).

Now, since the average number of vehicles expected to pass through West street is 400 and the maximum number of vehicles expected to pass through West street to south street is 300 (as we obtained) so remaining 100 vehicles pass through West street to North street that means the minimum number of vehicles expected to pass through West street to North street is 100. i.e., $100 < 2x_1$

From the last equation of the system of linear equation $2x_1 = 400 - x_4 \implies 2x_1 \leq 400$ (as $0 \leq x_4 \leq 300$).

So, $100 \leq 2x_1 \leq 400$.

Now, since the third equation in the system of linear equations is $2x_3 = 300 - x_4$. So, $0 \leq 2x_3 \leq 300$ (as $0 \leq x_4 \leq 300$).

Week-6
Mathematics for Data Science - 2
Introduction to Vector Space
Practice Assignment

1 Multiple Choice Questions (MCQ)

1. Consider the set $V = \{(-1, x, -y) \mid x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with the usual addition and scalar multiplication as in $\mathbb{R}^3$ and associated statements given below.

- **P:** V is not closed under addition.
- **Q:** V is not closed under scalar multiplication.
- **R:** V has zero element with respect to addition. i.e., there exists some element 0 such that $v + 0 = v$, for all $v \in V$.
- **S:** V is a vector space.

Which of the following statements is true?

◯ Option 1: Only P is true.
◯ Option 2: Only Q is true.
◯ Option 3: Both P and R are true.
◯ Option 4: Both R and S are true.
◯ **Option 5:** Both P and Q are true.

Solution:

- Let $v_1 = (-1, x_1, -y_1)$ and $v_2 = (-1, x_2, -y_2)$ be in V.
$v_1 + v_2 = (-2, x_1 + x_2, -y_1 - y_2)$, which is clearly not in V as the first coordinate is -2. Therefore, V is not closed under addition.
- Now for any $c(\neq 1) \in \mathbb{R}$ and any $v = (-1, x, -y) \in V$, $cv = (-c, x, -y) \in V$ as the first coordinate is not -1. Hence V is not closed under scalar multiplication.
- Let assume that there exists a zero element $(-1, a, -b)$ in V with respect to addition. $v + 0 = (-1, x, -y) + (-1, a, -b) = (-2, x + a, -y - b)$, which can never be same as v for any $v = (-1, x, -y) \in V$, as the first coordinate is -2. So V does not have any zero element with respect to addition.
- As we have already proved that V is not closed under addition and scalar multiplication, we need not have to check the other conditions of vector space. The given set V is not a vector space.

2. Match the vector spaces (with the usual scalar multiplication and vector addition as in $M_{3\times3}(\mathbb{R})$) in column A with their bases in column B in Table : M2W6P1.

	Vector space (Column A)		Basis (Column B)
a)	$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \mid x+y+z=0, \text{ and } x,y,z \in \mathbb{R} \right\}$	i)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$
b)	$V = \{A \mid A \in M_{3\times3}(\mathbb{R}),$ A is a diagonal matrix $\}$	ii)	$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$
c)	$V = \{A \mid A \in M_{3\times3}(\mathbb{R}),$ A is a scalar matrix $\}$	iii)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Table : M2W6P1

Choose the correct option.

- ◯ Option 1: a → ii, b → i, c → iii.
- ◯ **Option 2:** a → ii, b → iii, c → i.
- ◯ Option 3: a → i, b → ii, c → iii.
- ◯ Option 4: a → iii, b → ii, c → i.

Solution:

- $$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \;\middle|\; x+y+z=0,\ \text{and } x,y,z \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} x & y & z \\ 0 & z & x \\ y & 0 & 0 \end{bmatrix} \;\middle|\; z=-x-y,\ \text{and } x,y,z \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} x & y & -x-y \\ 0 & -x-y & x \\ y & 0 & 0 \end{bmatrix} \;\middle|\; x,y \in \mathbb{R} \right\}$$
Putting $x=1$ and $y=0$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and putting $x=0$ and $y=1$, we get $\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Therefore the following set,
$$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$
is linearly independent set and it also spans the given vector space V. Hence it forms a basis of V.

- $$V = \{A \mid A \in M_{3\times 3}(\mathbb{R}), A \text{ is a diagonal matrix } \}$$
$$= \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \;\middle|\; x,y,z \in \mathbb{R} \right\}$$
Putting $x=1$ and $y=z=0$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Putting $y=1$ and $x=z=0$, we get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Putting $z=1$ and $x=y=0$, we get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Therefore the following set,

$$\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

is linearly independent set and it also spans the given vector space V. Hence it forms a basis of V.

- $$V = \{A \mid A \in M_{3\times 3}(\mathbb{R}), A \text{ is a scalar matrix }\}$$

$$= \left\{\begin{bmatrix}x&0&0\\0&x&0\\0&0&x\end{bmatrix} \middle| x \in \mathbb{R}\right\}$$

Therefore the following set,

$$\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$$

is linearly independent set and it also spans the given vector space V. Hence it forms a basis of V.

2 Multiple Select Questions (MSQ)

3. The marks obtained by Karthika, Romy and Farzana in Quiz 1, Quiz 2 and End sem (with the maximum marks for each exam being 100) are shown in Table M2W6P2.

	Quiz 1	Quiz 2	End sem
Karthika	20	10	50
Romy	30	20	70
Farzana	40	30	c

Table: M2W6P2

Let Quiz 1, Quiz 2 and End sem marks obatined by Karthika, Romy and Farzana be represented by vectors. Use the above information, to choose the correct option(s).

- ○ Option 1: If Farzana obtained 90 marks in End sem (i.e $c = 90$), then the marks in Quiz 1, Quiz 2 and End sem represent linearly independent vectors.
- ○ **Option 2:** If Farzana obtained 80 marks in End sem (i.e $c = 80$), then the marks in Quiz 1, Quiz 2 and End sem represents linearly independent vectors.

○ **Option 3:** If 20% of Quiz 1, 30% of Quiz 2, and 50% of End sem are used to obtain total marks, then the vector representing the total marks is linear combination of vectors representing the marks of Quiz 1, Quiz 2 and End sem.

○ Option 4: If 0% of Quiz 1, 50% of Quiz 2 and 50% of End sem are used to obtain total marks, then the vector representing the total marks is not a linear combination of vectors representing the marks of Quiz 1, Quiz 2 and End sem.

Solution: The marks obtained by Karthika, Romy, and Farzana in Quiz 1 can be expressed as the vector $Q_1 = (20, 30, 40)$, the marks obtained by Karthika, Romy, and Farzana in Quiz 2 can be expressed as the vector $Q_2 = (10, 20, 30)$, and the marks obtained by Karthika, Romy, and Farzana in End sem can be expressed as the vector $S = (50, 70, c)$.

- **For Option 1:** If $c = 90$, then $S = (50, 70, 90)$. We have,

$$3Q_1 - Q_2 - S = 0$$

  Hence, the set $\{Q_1, Q_2, S\}$ is linearly dependent.

- **For Option 2:** If $c = 90$, then $S = (50, 70, 80)$. We can write the vectors Q_1, Q_2 and S as columns of a matrix as follows:

$$A = \begin{bmatrix} 20 & 10 & 50 \\ 30 & 20 & 70 \\ 40 & 30 & 80 \end{bmatrix}$$

  $det(A) \neq 0$. Hence, the vectors are linearly independent.

- **For Option 3:** Let the vector representing the total marks be v.

$$v = \frac{20}{100}Q_1 + \frac{30}{100}Q_2 + \frac{50}{100}S = \frac{1}{5}Q_1 + \frac{3}{10}Q_2 + \frac{1}{2}S$$

  Hence, v is a linear combination of the vectors Q_1, Q_2, and S.

- **For Option 4:** Let the vector representing the total marks be v.

$$v = \frac{0}{100}Q_1 + \frac{50}{100}Q_2 + \frac{50}{100}S = 0\ Q_1 + \frac{1}{2}Q_2 + \frac{1}{2}S$$

  Hence, v is a linear combination of the vectors Q_1, Q_2, and S.

4. Which of the following sets with the given addition and scalar multiplication (scalars are real numbers in every case) form vector spaces?

$V_1 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_1$
Scalar multiplication:

$$c(x, y, z) = \begin{cases} (0, 0, 0) & c = 0 \\ (cx, cy, \frac{z}{c}) & c \neq 0 \end{cases} \qquad (x, y, z) \in V_1,\ c \in \mathbb{R}$$

$V_2 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_2$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_2,\ c \in \mathbb{R}$

$V_3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_3$
Scalar multiplication: $c(x, y, z) = (cx, cy, z);\ (x, y, z) \in V_3,\ c \in \mathbb{R}$

$V_4 = \{(x, y, z) \mid x, y, z \in \mathbb{R}, x + y + z = 1\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_4$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_4,\ c \in \mathbb{R}$

$V_5 = \{(x, y) \mid x, y \in \mathbb{R}, x + 2y = 0\}$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_5$
Scalar multiplication: $c(x, y, z) = (cx, cy);\ (x, y) \in V_5,\ c \in \mathbb{R}$

[**Hint:** Check the axioms related to scalar multiplication in V_1.]

- ○ Option 1: V_1 is a vector space.
- ○ Option 2: V_2 is a vector space.
- ○ **Option 3:** V_3 is not a vector space.
- ○ Option 4: V_4 is a vector space.
- ○ **Option 5:** V_5 is a vector space.

Solution:

- **For V_1:**
$$5(x,y,z) = \left(5x, 5y, \frac{z}{5}\right)$$
and
$$(2+3)(x,y,z) = 2(x,y,z)+3(x+y,z) = \left(2x, 2y, \frac{z}{2}\right)+\left(3x, 3y, \frac{z}{3}\right) = \left(5x, 5y, \frac{z}{2} + \frac{z}{3}\right)$$
Observe that, $\frac{z}{5} \neq \frac{z}{2} + \frac{z}{3}$ is not in general.
Hence,
$$(2+3)(x,y,z) \neq 2(x,y,z) + 3(x,y,z)$$
Therefore V_1 is not a vector space.

- **For V_2:**
$$2(x,y,z) = (2x, 2y, 2z)$$
and
$$3(x,y,z) = (3x, 3y, 3z)$$
.
Again we have, $(2+3)(x,y,z) = 5(x,y,z) = (5x, 5y, 5z)$
$$2(x,y,z)+3(x,y,z) = (2x,2y,2z)+(3x,3y,3z) = (2x+3x, 2y+3y, 2z\times 3z) = (5x, 5y, 6z^2)$$
$5z$ and $6z^2$ are not equal in general, as for example if we choose $z = 1$, then $5z = 5$ and $6z^2 = 6$. So we have,
$$(2+3)(x,y,1) \neq 2(x,y,1) + 3(x,y,1)$$
Therefore V_2 is not a vector space.

- **For V_3:**
$$2(x,y,z) = (2x, 2y, z)$$
and
$$3(x,y,z) = (3x, 3y, z)$$
Again we have $(2+3)(x,y,z) = 5(x,y,z) = (5x, 5y, z)$
$$2(x,y,z)+3(x,y,z) = (2x,2y,z)+(3x,3y,z) = (2x+3x, 2y+3y, z+z) = (5x, 5y, 2z)$$
$2z = z$ is true only for $z = 0$. Hence for all $z \in \mathbb{R} \setminus \{0\}$,
$$(2+3)(x,y,z) \neq 2(x,y,z) + 3(x,y,z)$$
Therefore V_3 is not a vector space.

- **For V_4:** Observe that both the addition and scalar multiplication are the usual addition and scalar multiplication. Hence the zero respect to the addition should be $(0,0,0)$ as usual. But, $(0,0,0) \notin V_4$, as $0+0+0=0 \neq 1$.
  Therefore V_4 is not a vector space.
- **For V_5:** Observe that both the addition and scalar multiplication are the usual addition and scalar multiplication. Hence the zero respect to the addition should be $(0,0,0)$ as usual and $(0,0) \in V_5$. Any vector $v \in V_5$ is of the form $v = (-2y, y)$ where $y \in \mathbb{R}$. Now it is easy to check that V_5 is closed under vector addition and scalar multiplication. All the other axioms of vector space are satisfied as the addition and scalar multiplication are the usual ones.
  Therefore V_5 is a vector space.

5. Choose the set of correct options.

   ◯ **Option 1:** Any subset of any linearly independent set of vectors is linearly independent.

   ◯ Option 2: Any subset of any linearly dependent set of vectors is linearly dependent.

   ◯ Option 3: Any subset of any linearly dependent set of vectors is linearly independent.

   ◯ **Option 4:** {(1,1,0), (0,1,0), (0,-1,1)} is a basis of vector space $\mathbb{R}^3$ with usual addition and scalar multiplication.

   ◯ **Option 5:** {(1,4), (7,2)} is a basis of vector space $\mathbb{R}^2$ with usual addition and scalar multiplication.

   Solution:

   - **For Option 1:** Let S_1 be a subset of linearly independent set of vectors (say, S). Now if we take any finite collection of vectors $v_1, v_2, \ldots, v_n$ from S_1, then they also belongs to S. So $v_1, v_2, \ldots, v_n$ are linearly independent vectors. It implies that any finite collection of set of vectors from S is linearly independent. So S is linearly independent set. As we have chosen S_1 and S arbitrarily, we can conclude that, Any subset of any linearly independent set of vectors is linearly independent.
   - **For Option 2:** Observe that, the set $\{(1,0),(0,1),(1,1)\}$ is a linearly dependent set in $\mathbb{R}^2$. But it's subset $\{(1,0),(0,1)\}$ is linearly independent.
     Moreover, any singleton subset of a set is linearly independent. So if we choose any singleton subset of a linearly dependent set, then it must be linearly independent.
   - **For Option 3:** Observe that, the set $S = \{(1,0),(0,1),(1,1),(2,2)\}$ is a linearly dependent set in $\mathbb{R}^2$. $\{(1,0),(0,1),(1,1)\}$ is a subset of S which is again a linearly dependent set.
   - **For Option 4:** Consider the following equality:

     $$a(1,1,0) + b(0,1,0) + c(0,-1,1) = (a, a+b-c, c) = (0,0,0)$$

     Hence we have $a = 0, a+b-c = 0$, and $c = 0$, which gives us $a = b = c = 0$. Hence the given set is linearly independent. Moreover, let (x,y,z) be any arbitrary vector in $\mathbb{R}^3$, then we can express (x,y,z) in linear combination of the vectors given in the set as follows:

     $$(x,y,z) = x(1,1,0) + (y+z-x)(0,1,0) + z(0,-1,1)$$

     Therefore the given set spans $\mathbb{R}^3$. So {(1,1,0), (0,1,0), (0,-1,1)} is a basis of vector space $\mathbb{R}^3$ with usual addition and scalar multiplication.

- **For Option 5:** Consider the following equality:

$$a(1,4) + b(7,2) = (a + 7b, 4a + 2b) = (0,0)$$

Hence we have $a + 7b = 0, 4a + 2b = 0$, which gives us $a = b = 0$. Hence the given set is linearly independent. Moreover, let (x, y) be any arbitrary vector in $\mathbb{R}^2$, then we can express (x, y) in linear combination of the vectors given in the set as follows:

$$(x, y) = \frac{7y - 2x}{26}(1,4) + \frac{4x - y}{26}(7,2)$$

Therefore the given set spans $\mathbb{R}^2$. So {(1,4), (7,2)} is a basis of vector space $\mathbb{R}^2$ with usual addition and scalar multiplication.

3 Numerical Answer Type (NAT):

6. Consider the set of three vectors $S = \{(1, c, -1), (-1, 0, c), (c, c, 2c)\}$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. Find the number of real values of c, so that the above set S will be linearly dependent? [Answer: 1]

   Solution: We can write these vectors as the columns of a matrix as follows:

$$A = \begin{bmatrix} 1 & -1 & c \\ c & 0 & c \\ -1 & c & 2c \end{bmatrix}$$

   The given set S is linearly dependent if $det(A) = 0$
   $det(A) = 1(-c^2) - (-1)(2c^2 + c) + c(c^2) = -c^2 + 2c^2 + c + c^3 = c^3 + c^2 + c = c(c^2 + c + 1) = 0$
   The only real solution of this equation is 1. Hence, the number of real values of c so that the set S will be linearly dependent is 1.

7. Find out the value of a for which the matrix $\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix}$ will be in the spanning set of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$ with usual matrix addition and scalar multiplication. [Answer: 5]

Solution: If the matrix $\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix}$ will be in the spanning set of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $M_{2\times 2}(\mathbb{R})$ with usual matrix addition and scalar multiplication, then the matrix $\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix}$ is linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} a & 3 \\ 0 & -5 \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}$$

Hence $x = a$, $y = 3$, and $-x = -5$. Therefore $a = x = 5$.

4 Comprehension Type Question:

Suppose in a village there are four farmers A, B, C and D, each owning 1 acre of land. They cultivate paddy, pulses and/or sugarcane in their lands as follows: Farmer A uses 50% of his land for paddy, 30% for pulses and the remaining for sugarcane. Farmer B uses 40% of her land for paddy and she divides her remaining land equally for pulses and sugarcane. Farmer C uses the whole land for paddy only, and Farmer D uses 30% for paddy, 30% for pulses and the remaining for sugarcane. Using the above information to answer the following questions.

8. Suppose the area used by a farmer for different crops is denoted by a row vector. Let S be the span of the resulting four row vectors . Choose the correct set of options. (MSQ)

   ◯ **Option 1:** The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmers A, B, and C.

   ◯ **Option 2:** The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

   ◯ Option 3: The row vectors corresponding to the area used for different crops by Farmer C can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

   ◯ Option 4: The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer C.

Solution:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{2}{10}$
B	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{3}{10}$
C	$\frac{10}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
D	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Table: M2W6PS1

For Option 1: The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmers A, B, and C, as follows:

$$\left(\frac{3}{10}, \frac{3}{10}, \frac{4}{10}\right) = -1\left(\frac{5}{10}, \frac{3}{10}, \frac{2}{10}\right) + 2\left(\frac{4}{10}, \frac{3}{10}, \frac{3}{10}\right) + 0\left(\frac{10}{10}, \frac{0}{10}, \frac{0}{10}\right)$$

For Option 2: The row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B, as we have already found.

$$\left(\frac{3}{10}, \frac{3}{10}, \frac{4}{10}\right) = -1\left(\frac{5}{10}, \frac{3}{10}, \frac{2}{10}\right) + 2\left(\frac{4}{10}, \frac{3}{10}, \frac{3}{10}\right)$$

For Option 3: Let the row vectors corresponding to the area used for different crops by Farmer A, Farmer B, and Farmer C be written as the columns of a matrix (say A).

$$A = \begin{bmatrix} \frac{5}{10} & \frac{4}{10} & \frac{10}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{0}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{0}{10} \end{bmatrix}$$

$$det(A) \neq 0$$

Hence, the row vectors corresponding to the area used for different crops by Farmer A, Farmer B, and Farmer C are linearly independent. Therefore, the row vectors corresponding to the area used for different crops by Farmer C cannot be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

For Option 4: Let the row vectors corresponding to the area used for different crops by Farmer A, Farmer C, and Farmer D be written as the columns of a matrix (say A).

$$A = \begin{bmatrix} \frac{5}{10} & \frac{10}{10} & \frac{3}{10} \\ \\ \frac{3}{10} & \frac{0}{10} & \frac{3}{10} \\ \\ \frac{2}{10} & \frac{0}{10} & \frac{4}{10} \end{bmatrix}$$

$$det(A) \neq 0$$

Hence, the row vectors corresponding to the area used for different crops by Farmer A, Farmer C, and Farmer D are linearly independent. Therefore, the row vectors corresponding to the area used for different crops by Farmer D cannot be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer C.

9. Let S be the vectors space defined in the previous question, with the usual addition and scalar multiplication on $\mathbb{R}^3$. Which of the following sets will be a basis of S? (MCQ)

- ○ Option 1: $\{(5,3,2)\}$
- ○ Option 2: $\{(5,3,2),(4,3,3)\}$
- ○ **Option 3:** $\{(5,3,2),(4,3,3),(10,0,0)\}$
- ○ Option 4: $\{(5,3,2),(4,3,3),(10,0,0),(3,3,4)\}$

Solution: In the previous question we have already proved that, the row vectors corresponding to the area used for different crops by Farmer A, Farmer B, and Farmer C are linearly independent and the row vectors corresponding to the area used for different crops by Farmer D can be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmers A, B, and C. But the row vectors corresponding to the area used for different crops by Farmer C cannot be obtained as a linear combination of the row vectors corresponding to the area used for different crops by Farmer A and Farmer B.

Hence the set $\{\frac{1}{10}(5,3,2), \frac{1}{10}(4,3,3)\}$ does not span S, which also implies that the set $\{\frac{1}{10}(5,3,2)\}$ does not span S. Hence the set given in Option 1 and 2 does not span S.
The set $\{\frac{1}{10}(5,3,2), \frac{1}{10}(4,3,3), \frac{1}{10}(10,0,0), \frac{1}{10}(3,3,4)\}$ is linearly dependent. Hence the set given in Option 4 does not span S.
The set $\{\frac{1}{10}(5,3,2), \frac{1}{10}(4,3,3), \frac{1}{10}(10,0,0)\}$ is linearly independent and it also spans S. Hence the set given in Option 3 is a basis of S.

10. Suppose Farmer B buys the same amount of land (1 acre) and uses it in the same ratio for different crops as she was using for her land earlier. Choose the correct set of options. (MCQ)

- ○ Option 1: Farmer A sells his whole land to Farmer D, and Farmer D uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B is using more amount of area of land for pulses than Farmer D.
- ○ **Option 2:** Farmer A sells his whole land to Farmer D, and Farmer D uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B and Farmer D are using the same amount of area of land for paddy.
- ○ Option 3: Farmer A sells his whole land to Farmer C, and Farmer C uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B and Farmer C are using the same amount of area of land for paddy.
- ○ Option 4: Farmer A sells his whole land to Farmer C, and Farmer C uses the land she bought from Farmer A, in the same ratio for different crops as Farmer A was using earlier. In this new scenario, Farmer B is using more amount of area of land for paddy than Farmer C.

Solution:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{2}{10}$
B	$\frac{4}{10}+\frac{4}{10}=\frac{8}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$
C	$\frac{10}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
D	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Table: M2W6PS2

For Option 1 and 2:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	0	0	0
B	$\frac{4}{10}+\frac{4}{10}=\frac{8}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$
C	$\frac{10}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
D	$\frac{3}{10}+\frac{5}{10}=\frac{8}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$	$\frac{4}{10}+\frac{2}{10}=\frac{6}{10}$

Table: M2W6PS2

Hence Option 1 is not true, but Option 2 is true.
For Option 3 and 4:

	Paddy (in acre)	Pulses (in acre)	Sugarcane (in acre)
A	0	0	0
B	$\frac{4}{10}+\frac{4}{10}=\frac{8}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$	$\frac{3}{10}+\frac{3}{10}=\frac{6}{10}$
C	$\frac{10}{10}+\frac{5}{10}=\frac{15}{10}$	$\frac{0}{10}+\frac{3}{10}=\frac{3}{10}$	$\frac{0}{10}+\frac{2}{10}=\frac{2}{10}$
D	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Table: M2W6PS3

Hence both Option 3 and Option 4 are false.

Week-3
Mathematics for Data Science - 2
Introduction to Vector Space
Graded Assignment

1 Multiple Choice Questions (MCQ)

1. Which of the following sets with the given addition and scalar multiplication operations (scalars are real numbers in every case) form vector spaces?

$V_1 = \{(x, y) | x, y \in \mathbb{R}\}$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1);\ (x_1, y_1), (x_2, y_2) \in V_1$
Scalar multiplication: $c(x, y) = (cx, 1);\ (x, y) \in V_1,\ c \in \mathbb{R}$

$V_2 = \{(x, y) | x, y \in \mathbb{R}\}$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);\ (x_1, y_1), (x_2, y_2) \in V_2$
Scalar multiplication: $c(x, y) = (cx, 0);\ (x, y) \in V_2,\ c \in \mathbb{R}$

$V_3 = \{(x, y) | x, y \in \mathbb{R}\}$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2);$
$(x_1, y_1), (x_2, y_2) \in V_3$
Scalar multiplication: $c(x, y) = (cx, cy);\ (x, y) \in V_3,\ c \in \mathbb{R}$

$V_4 = \{(x, y, z) | x, y, z \in \mathbb{R}, x + y = z\}$
Addition: $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2);$
$(x_1, y_1, z_1), (x_2, y_2, z_2) \in V_4$
Scalar multiplication: $c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_4,\ c \in \mathbb{R}$

○ Option 1: V_1 is a vector space, but others are not.
○ Option 2: V_2 is a vector space but others are not.
○ Option 3: V_3 is a vector space but others are not.

○ **Option 4:** V_4 is a vector space but others are not.

○ Option 5: V_2 and V_4 are vector spaces, V_1 and V_3 are not.

○ Option 6: V_2 and V_3 are vector spaces, V_1 and V_4 are not.

Solution:

1. Given V_1,

$$V_1 = \{(x, y)|x, y \in \mathbb{R}\}$$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1);\ (x_1, y_1), (x_2, y_2) \in V_1$
Scalar multiplication: $c(x, y) = (cx, 1);\ (x, y) \in V_1,\ c \in \mathbb{R}$

Given addition,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 1)$$

Let $(x_1, y_1) = v_1$ and (a, b) be a zero vector in the vector space V_1, then

$$0 + v_1 = (a, b) + (x_1, y_1) = (a + x_1, 1) \neq v_1$$

Which shows V_1 does not have any zero vector.
Therefore, V_1 is not a vector space.

2. Given V_2,

$$V_2 = \{(x, y)|x, y \in \mathbb{R}\}$$
Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2);\ (x_1, y_1), (x_2, y_2) \in V_2$
Scalar multiplication: $c(x, y) = (cx, 0);\ (x, y) \in V_2,\ c \in \mathbb{R}$

Here we can see that addition follows all the four laws of addition for a vector space.
But given multiplication,

$$c(x, y) = (cx, 0)$$

Let $c = 1$, then

$$1(x, y) = (1x, 0) = (x, 0)$$

If we write $(x, y) = v$, where $v \in V_2$, then

$$1v \neq v$$

Which fails the unitary law of multiplication in a vector space i.e.,

$$1.v = v$$

Therefore, V_2 is not a vector space.

3. Given V_3,

$$\begin{aligned}
&V_3 = \{(x, y) | x, y \in \mathbb{R}\} \\
&\textit{Addition: } (x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2); \\
&\qquad\qquad\qquad\qquad\qquad (x_1, y_1), (x_2, y_2) \in V_3 \\
&\textit{Scalar multiplication: } c(x, y) = (cx, cy);\ (x, y) \in V_3,\ c \in \mathbb{R}
\end{aligned}$$

Given addition,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 + y_1 + y_2)$$

Let $(x_1, y_1) = v_1$ and (a, b) be a zero vector in the vector space V_3, then

$$0 + v_1 = (a, b) + (x_1, y_1) = (a + x_1 + b + y_1, a + x_1 + b + y_1) \neq v_1$$

Which shows V_1 does not have any zero vector.
Therefore, V_3 is not a vector space.

4. Given V_4,

$$\begin{aligned}
&V_4 = \{(x, y, z) | x, y, z \in \mathbb{R}, x + y = z\} \\
&\textit{Addition: } (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2); \\
&\qquad\qquad\qquad\qquad\qquad (x_1, y_1, z_1), (x_2, y_2, z_2) \in V_4 \\
&\textit{Scalar multiplication: } c(x, y, z) = (cx, cy, cz);\ (x, y, z) \in V_4,\ c \in \mathbb{R}
\end{aligned}$$

Clearly, V_4 follows all the laws of Addition and Multiplication for a vector space.
Therefore, V_4 is a vector space.

2. Choose the set of correct options

   ◯ Option 1: If V is a real vector space, then $(\alpha+\beta)(x+y) = \alpha x + \beta y$, for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$.

   ◯ Option 2: A vector space can have more than one zero vector.

   ◯ **Option 3:** (-1, 0 ,0), (-1, 1, -1) and (0, 2, 3) are linearly independent vectors in $\mathbb{R}^3$.

   ◯ Option 4: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are linearly dependent vectors in $M_{2\times 2}(\mathbb{R})$.

Solution:

Option 1: Let us assume $\alpha = 2, \beta = 5$, then according to option 1

$$(\alpha+\beta)(x+y) = \alpha x + \beta y$$

$$\implies (2+3)(x+y) = 2x + 3y$$

But we know that if V is a real vector space then,

$$a(x+y) = ax + by, a \in \mathbb{R}$$

Therefore,

$$(\alpha+\beta)(x+y) = (\alpha+\beta)x + (\alpha+\beta)y$$

$$\implies (2+3)(x+y) = (2+3)x + (2+3)y = 5x + 5y \neq 2x + 3y$$

Therefore, V is not a vector space.

Option 2: A vector space can have more than one zero vector.

Incorrect, as the zero vector is unique in a vector space. Let v and v' are two zero vectors in a vector space then,

$$v = v + v' = v' + v = v'$$

both are the same i.e, the zero vector is unique in a vector space.

Option 3: (-1, 0 ,0), (-1, 1, -1) and (0, 2, 3) are linearly independent vectors in $\mathbb{R}^3$.

Let $a, b, c \in R$ and

$$a \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -a & -b & 0 \\ 0 & b & 2c \\ 0 & b & 3c \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies Ax = B$$

Where,

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}, x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$det(A) = -1(3-2) \neq 0$$

It means the system has unique solution (trivial solution only) i.e., $a = b = c = 0$.
Therefore, the vectors are linearly independent.

Option 4: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ are linearly dependent vectors in $M_{2\times 2}(\mathbb{R})$.

Take $a, b, c \in \mathbb{R}$ and

$$a \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} -a-c & -2b+c \\ b+c & a-c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Which means all the element in the matrix should be zero. i.e.,

$$-a - c = 0 \implies a = -c$$

$$a - c = 0 \implies a = c$$

$$\implies a = c = 0$$

and

$$-2b + c = 0 \implies b = 0$$

As **we can see** $a = b = c = 0$**, therefore, the vectors are linearly independent.**

3. A healthy juice consist of 30 units of protein, 11 units of carbohydrate (in tens), 53 units of fat, and 213 units of calcium (in average). A juice maker makes two types of juice, Type A and Type B. Type A consists of banana, milk, and almond, where as Type B consists of apple, milk, and almond. Table M2W3G1 shows the amount of protein, carbohydrate (in tens), fat, and calcium present in each banana, apple, and almond, and in 100 ml of milk.

Items	Protein	Carbohydrate (in tens)	Fat	Calcium
Banana (1 piece)	2	3	1	5
Apple (1 piece)	1	2	1	6
Almond (1 piece)	6	1	15	1
Milk (100 ml)	4	1	3	100

Table: M2W3G1

Use the above information to choose the correct option.

- ○ **Option 1:** With the right quantities of ingredients of Type A, it can be a healthy juice, and those amounts are unique.
- ○ Option 2: With the right quantities of ingredients of Type B, it can be a healthy juice, and those amounts are unique.
- ○ Option 3: With the right quantities of ingredients of Type A, it can be a healthy juice, and those amounts are not unique.
- ○ Option 4: With the right quantities of ingredients of Type B, it can be a healthy juice, and those amounts are not unique.

Solution:

Let Type A juice is made by mixing a unit of banana, b unit of milk, and c unit of almond and it is a healthy juice, then

$$a(2,3,1,5) + b(4,1,3,100) + c(6,1,15,1) = (30,11,53,213)$$

$$\implies \begin{bmatrix} 2 & 4 & 6 \\ 3 & 1 & 1 \\ 1 & 3 & 15 \\ 5 & 100 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 30 \\ 11 \\ 53 \\ 213 \end{bmatrix}$$

For solving the linear system take $A|b$,

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 30 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{array}\right]$$

$$\downarrow \frac{R_1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 3 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 5 & 100 & 1 & 213 \end{array}\right]$$

$$\downarrow R_4 - 5R_1$$

$$\downarrow R_3 - R_1$$

$$\downarrow R_2 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & -5 & -8 & -34 \\ 0 & 1 & 12 & 38 \\ 0 & 90 & -14 & 138 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & -5 & -8 & -34 \\ 0 & 90 & -14 & 138 \end{array}\right]$$

$$\downarrow R_3 + 5R_2$$

$$\downarrow R_4 - 90R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & 0 & 52 & 156 \\ 0 & 0 & -1094 & -3282 \end{array}\right]$$

$$\downarrow R_3/52$$

$$\downarrow R_4/2094$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$\downarrow R_4 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 15 \\ 0 & 1 & 12 & 38 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We can see the rank of matrix is 3, which means unique solution.
Therefore, option 1 is correct.

Let Type B juice is made by mixing a unit of apple, b unit of milk, and c unit of almond and it is a healthy juice, then

$$a(1, 2, 1, 6) + b(4, 1, 3, 100) + c(6, 1, 15, 1) = (30, 11, 53, 213)$$

$$\implies \begin{bmatrix} 1 & 4 & 6 \\ 2 & 1 & 1 \\ 1 & 3 & 15 \\ 6 & 100 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 30 \\ 11 \\ 53 \\ 213 \end{bmatrix}$$

For solving the linear system take $A|b$,

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 2 & 1 & 1 & 11 \\ 1 & 3 & 15 & 53 \\ 6 & 100 & 1 & 213 \end{array}\right]$$

$$\downarrow R_4 - 6R_1$$

$$\downarrow R_3 - R_1$$

$$\downarrow R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & -7 & -11 & -59 \\ 0 & -1 & 9 & 23 \\ 0 & 76 & -35 & 33 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\downarrow -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & -7 & -11 & -59 \\ 0 & 76 & -35 & 33 \end{array}\right]$$

$$\downarrow R_3 + 7R_2$$

$$\downarrow R_4 - 76R_2$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & 0 & -74 & -172 \\ 0 & 0 & 649 & 1781 \end{array}\right]$$

$$\downarrow -R_3/74$$

$$\downarrow R_3/649$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & 0 & 1 & \frac{86}{37} \\ 0 & 0 & 1 & \frac{1781}{649} \end{array}\right]$$

$$\downarrow R_4 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 30 \\ 0 & 1 & -9 & -23 \\ 0 & 0 & 1 & \frac{86}{37} \\ 0 & 0 & 0 & k \end{array}\right]$$

here $k = \frac{1781}{649} - \frac{86}{37} \neq 0$ Which means no solution therefore, options 2,3, and 4 are incorrect.

4. Match the vector spaces (with the usual scalar multiplication and vector addition as in $M_{3\times3}(\mathbb{R})$) in column A with their bases in column B in Table : M2W3G2.

	Vector space (Column A)		Basis (Column B)
a)	$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid x+y+z=0, \text{ and } x,y,z \in \mathbb{R} \right\}$	i)	$\left\{ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$
b)	$V = \{A \mid A \in M_{3\times3}(\mathbb{R})$, A is a diagonal matrix, with sum of the elements in the diagonal is zero $\}$	ii)	$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$
c)	$V = \left\{ \begin{bmatrix} 0 & z & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \mid x+y+z=0, \text{ and } x,y,z \in \mathbb{R} \right\}$	iii)	$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$

Table : M2W3G2

Choose the correct option.

- ◯ Option 1: a → ii, b → i, c → iii.
- ◯ **Option 2:** a → ii, b → iii, c → i.
- ◯ Option 3: a → i, b → ii, c → iii.
- ◯ Option 4: a → iii, b → ii, c → i.

Solution:

a) Given,

$$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid x+y+z=0, x,y,z \in \mathbb{R} \right\}$$

$$V = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid z=-x-y, x,y,z \in \mathbb{R} \right\}$$

$$V = \left\{ \begin{bmatrix} x & y & -x-y \\ 0 & 0 & y \\ 0 & 0 & x \end{bmatrix} \mid z=-x-y, x,y,z \in \mathbb{R} \right\}$$

Putting $x=0, y=1$, we get $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and putting $x=1, y=0$, we get $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
Therefore, the following set,

$$\left\{ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V . Hence it forms a basis of V.

b) Given, $V = \{A \mid A \in M_{3\times 3}(\mathbb{R})$, A is a diagonal matrix with sum of the elements in the diagonal is zero.$\}$
Let $x, y, z \in \mathbb{R}$ and

$$V = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}, x+y+z=0 \text{ or } z=-x-y \right\}$$

$$\implies V = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -x-y \end{bmatrix} \right\}$$

Putting $x=0, y=1$, we get $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and Putting $x=1, y=0$, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Therefore, the following set,

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V . Hence it forms a basis of V.

Given in iii) in B column.

c) If $x, y, z \in R$, Given,

$$V = \left\{ \begin{bmatrix} 0 & z & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}, x + y + z = 0 \; or \; x = -z - y \right\}$$

$$\implies V = \left\{ \begin{bmatrix} 0 & z & -z-y \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

Putting $z = 1, y = 0$, we get $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and Putting $0 = 1, y = 1$, we get $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore, the following set,

$$\left\{ \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is linearly independent set and it also spans the given vector space V . Hence it forms a basis of V.

2 Multiple Select Questions (MSQ)

5. Consider the set of vectors $V = \{(-1,1,5),(2,1,3),(2,1,2),(1,-1,7),(-1,3,-5)\}$ from $\mathbb{R}^3$, choose the set of correct options

   - ◯ Option 1: The singleton set of vector $\{(-1,1,5)\}$ is linearly dependent.
   - ◯ **Option 2:** If $\alpha, \beta \in V$ and α, β are distinct then $\{\alpha, \beta\}$ is a linearly independent set of vectors.
   - ◯ Option 3: The set {(-1, 1, 5), (2, 1, 3),(-2, 2, 10)} is a linearly dependent set of vectors.
   - ◯ Option 4: The set V is a linearly independent set of vectors.
   - ◯ Option 5: The set $\{\alpha, \beta, \gamma\}$ is a linearly dependent set of vectors for any $\alpha, \beta, \gamma \in V$, where all the three are distinct vectors.
   - ◯ Option 6: The set $\{\alpha, \beta, \gamma, \delta\}$ is a linearly independent set of vectors for any $\alpha, \beta, \gamma, \delta \in V$, where all the four are distinct vectors.
   - ◯ **Option 7:** The system $AX = b$, where $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 7 \\ -1 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have infinitely many solutions.
   - ◯ **Option 8:** The system $AX = b$, where $A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have unique solution.
   - ◯ Option 9: The system $AX = b$ where $A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -5 \\ -1 & 1 & 5 \end{bmatrix}^T$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ have infinitely many solutions.

Solution:

Option 1 is incorrect as singleton set is always independent.

Given,

$$V = \{(-1,1,5),(2,1,3),(2,1,2),(1,-1,7),(-1,3,-5)\}$$

Here no vector is multiple of any other vector which means any set of vectors $V' \subset V$ two vectors is a linearly independent set of vectors. Therefore, option 2 is correct.

In option 3 given set is $\{(-1,1,5),(2,1,3),(-2,2,10)\}$. Visualize,

$$(-2,2,10) = 2(-1,1,5) + 0(2,1,3)$$

Hence linearly dependent.

We know that m vectors in $\mathbb{R}^n$ are always linearly dependent for $m > n$. Therefore, maximum three vectors can only be linearly independent. That's why option 4 and 6 are incorrect.

Now to find the maximum number of linearly independent vectors,

$$a(-1,1,5) + b(2,1,3) + c(2,1,2) + d(1,-1,7) + e(-1,3,-5) = 0$$

$$\implies \begin{bmatrix} -1 & 2 & 2 & 1 & -1 \\ 1 & 1 & 1 & -1 & 3 \\ 5 & 3 & 2 & 7 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A|b$

$$\left[\begin{array}{ccccc|c} -1 & 2 & 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 3 & 0 \\ 5 & 3 & 2 & 7 & -5 & 0 \end{array}\right]$$

$$\downarrow -R_1$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 3 & 0 \\ 5 & 3 & 2 & 7 & -5 & 0 \end{array}\right]$$

$$\downarrow R_2 - R_1$$

$$\downarrow R_3 - 5R_1$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 0 & 3 & 3 & 0 & 2 & 0 \\ 0 & 13 & 12 & 12 & -10 & 0 \end{array}\right]$$

$$\downarrow R_2/3$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2/3 & 0 \\ 0 & 13 & 12 & 12 & -10 & 0 \end{array}\right]$$

$$\downarrow R_3 - 13R_2$$

$$\implies \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2/3 & 0 \\ 0 & 0 & -1 & 12 & -32/3 & 0 \end{array}\right]$$

$$\downarrow R1 + 2R_2$$

$$\downarrow R_2 + R_3$$

$$\Longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 7/4 & 0 \\ 0 & 1 & 0 & 12 & -10 & 0 \\ 0 & 0 & -1 & +12 & -32/3 & 0 \end{array}\right]$$

$$\downarrow -R_3$$

$$\Longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 7/4 & 0 \\ 0 & 1 & 0 & 12 & -10 & 0 \\ 0 & 0 & 1 & -12 & 32/3 & 0 \end{array}\right]$$

Clearly we can see that the rank is 3, which means there are atleast three linearly independent vectors. Therefore, option 5 are incorrect.

In option 7, four vectors of V are used. We saw in the above explanation that the rank is 3, which means the system will have one independent variable with the three dependent variable, depending on the value of fourth variable, therefore infinite solutions.

In option 8,

$$A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \end{bmatrix}^t = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ -5 & 3 \end{bmatrix}$$

Which means

$$\begin{bmatrix} -1 & 2 \\ 3 & 1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix,

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & 1 & 0 \\ -5 & 3 & 0 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 7 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore unique solution.

In option 9 given,

$$A = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & -5 \\ -1 & 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 3 & 1 \\ 2 & -5 & 5 \end{bmatrix}$$

$$det(A) = 2(15 + 5) - 1(-5 - 5) + 2(-1 + 3) = 40 + 10 + 4 \neq 0$$

Therefore, unique solution. So, the option is incorrect.

3 Numerical Answer Type (NAT):

6. Find the value of c for which the vector $(3, 2, c)$ will be in the spanning set of the vectors $(1, 0, 1)$ and $(0, 1, -1)$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. [Answer: 1]

Solution:

If vector $(3, 2, c)$ is in the spanning set of the vectors $(1, 0, 1)$ and $(0, 1, -1)$, then we get $(3, 2, c)$ as linear combination of the other two vectors.

$$x(1, 0, 1) + y(0, 1, -1) = (3, 2, c)$$

$$\implies x = 3, y = 2$$

$$\implies x - y = c \implies c = 1$$

7. Consider the set of three vectors $S = \{(c,-1,-2),(1,0,-1),(-1,-3,c)\}$ in $\mathbb{R}^3$ with usual addition and scalar multiplication. For which value of c, the above set S will be linearly dependent? [Answer: 2.5]

Solution:

If set $S = \{(c,-1,-2),(1,0,-1),(-1,-3,c)\}$ is linearly dependent, then the determinant of the matrix created by S will be 0.

$$S = \begin{bmatrix} c & 1 & -1 \\ -1 & 0 & -3 \\ -2 & -1 & c \end{bmatrix}$$

$$det(S) = c(0-3) - 1(-c-6) - 1(1-0) = 0$$

$$-3c + c + 6 - 1 = 0 \implies c = 2.5$$

4 Comprehension Type Question:

In genetics, a classic example of dominance is the inheritance of shape of seeds in peas. Peas may be round (associated with genotype R) or wrinkled (associated with genotype r). In this case, three combinations of genotypes are possible: RR, rr, and Rr. The RR individuals have round peas and the rr individuals have wrinkled peas. In Rr individuals the R genotype masks the presence of the r genotype, so these individuals also have round peas. Thus, the genotype R is completely dominant to genotype r, and genotype r is recessive to genotype R. First, assume the crossing of RR with RR. This always gives the genotype RR, therefore the probabilities of an offspring to be RR, Rr, and rr respectively are equal to 1, 0, and 0. Second, assume crossing of Rr with RR. The offspring will have equal chances to be of genotype RR and genotype Rr, therefore the probabilities of RR, Rr, and rr repectively are 1/2, 1/2, and 0. Third, consider crossing of rr with RR. This always results in genotype Rr. Therefore, the probabilities of genotypes RR, Rr, and rr are 0, 1, and 0, respectively.
This can be veiwed as the following table:

Parents' genotypes			Genotypes of offspring
RR-RR	RR-Rr	RR-rr	
1	$\frac{1}{2}$	0	RR
0	$\frac{1}{2}$	1	Rr
0	0	0	rr

Table: M2W3G3

The matrix representing this observation is given by $P = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and the initial distribution vector (3×1 matrix) is denoted by $X_0 = \begin{bmatrix} X_0^1 \\ X_0^2 \\ X_0^3 \end{bmatrix}$, where X_0^1 denotes the distribution of RR, X_0^2 denotes the distribution of Rr, and X_0^3 denotes the distribution of rr. For any positive integer n, the distribution vector after n generations (i.e., at $t = n$) is denoted by X_n and given by the equation $PX_{n-1} = X_n$.
Using the above information, answer the following questions.

8. Suppose, in an experiment, 100 pairs of parents with genotype combinations RR-RR, 100 pairs of parents with genotype combinations RR-Rr, and 200 pairs of parents with genotype combination RR-rr are taken to observe the genotypes of their offspring. Suppose from crossing of each pair of parents a single offspring is produced. Find the set of correct options from the following. (MSQ)

   ◯ Option 1: There will be at least 200 offspring with wrinkled peas.

   ◯ **Option 2:** There will be no offspring with wrinkled peas.

- ◯ Option 3: There will be no offspring with round peas.
- ◯ **Option 4:** All the offspring will have round peas.
- ◯ Option 5: All the offspring will have wrinkled peas.
- ◯ **Option 6:** There will be at least 100 offspring with combination of genotypes RR.
- ◯ **Option 7:** There will be at least 200 offspring with combination of genotypes Rr.

Solution:

The offspring result would be

$$100(1,0,0) + 100(1/2,1/2,0) + 200(0,1,0) = (150,250,0)$$

Which means $RR = 150(round - shaped), Rr = 250(round - shaped), rr = 0(wrinkled)$ Using this information, therefore, option 4 is correct.

As it is given that the probability of RR-RR Parents' genotypes is 1 if Genotypes of offspring is RR. In the experiment 100 pairs of parents with genotype combinations RR-RR are used therefore, there will be at least $100 \times 1 = 100$ offspring with combination of genotypes RR.

Similarly, probabilty of RR-rr with Rr is 1. In the experiment 200 pairs of parents with genotype combinations RR-Rr are used therefore, there will be at least $200 \times 1 = 200$ offspring with combination of genotypes Rr.

9. Suppose the vector space X spanned by the column vectors of P (i.e., the probability vectors) with the usual addition and scalar multiplication, is known as the probability space. Which of the following options are correct? (MSQ)

   - ○ Option 1: $\{(\frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0), (0, 0, \frac{1}{2})$ is a linearly independent set in X.
   - ○ **Option 2:** $\{(\frac{1}{2}, 0, 0), (0, \frac{1}{2}, 0)$ is a linearly independent set in X.
   - ○ **Option 3:** The vector space X can be expressed as the set $\{(a, b, 0) | a, b \in \mathbb{R}\}$ with the usual addition and scalar multiplication.
   - ○ Option 4: The vector space X can be expressed as the set $\{(a, 0, b) | a, b \in \mathbb{R}\}$ with the usual addition and scalar multiplication.
   - ○ Option 5: The vector space X can be expressed as the set $\{(a, b, c) | a, b, c \in \mathbb{R}\}$ with the usual addition and scalar multiplication.

   Solution:

   Given vector space X is spanned by the column vectors P. Therefore, a vector X_s in X for $x, y, z \in \mathbb{R}$ would be

$$X_s = x(1, 0, 0) + y(1/2, 1/2, 0) + z(0, 1, 0)$$

$$X_s = (x + \frac{y}{2}, \frac{y}{2} + z, 0)$$

   Here the third element of vector should always be zero.Therefore, (0, 0, 1/2) is not in X as X. So, option 1 is incorrect.

   Let us take $x + \frac{y}{2} = a$, $\frac{y}{2} + z = b$, then $X_s = (a, b, 0)$. So, option 3 is correct.

   Take $a = 1/2$ and $b = 1/2$ then $X_s = (1/2, 1/2, 0)$. Therefore, option 2 is correct.

10. Supoose $X_0 = \begin{bmatrix} \frac{1}{3} \\ \\ \frac{1}{3} \\ \\ \frac{1}{3} \end{bmatrix}$. Find out the correct set of correct options. (MSQ)

◯ Option 1: X_0 and X_1 are linearly dependent.

◯ **Option 2:** X_0 and X_1 are linearly independent.

◯ Option 3: The set $\{X_0, X_1, X_2\}$ is a linearly dependent set.

◯ **Option 4:** The set $\{X_0, X_1, X_2\}$ is a linearly independent set.

Solution:

Given,

$$PX_{n-1} = X_n$$

$$\implies X_1 = PX_0$$

$$X_1 = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\implies X_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

Clearly X_0 ans X_1 are linearly independent.

Now,

$$\implies X_2 = PX_1$$

$$X_1 = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$\implies X_2 = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \end{bmatrix}$$

Now the set $X = \{X_0, X_1, X_2\} = \{(1/3, 1/3, 1/3), (1/2, 1/2, 0), (3/4, 1/4, 0)\}$. Let matrix A be the matrix made by the elements of X as column vectors then,

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

$$det(A) = \frac{1}{3}(0) + \frac{1}{3}(0) + \frac{1}{3}(\frac{1}{8} - \frac{3}{8}) \neq 0$$

Which means columns of A are linearly independent. Therefore, set X is a linearly independent set.

Week-7
Mathematics for Data Science - 2
Basis of a vector space, Rank and dimension of a matrix
Practice Assignment Solution

1 Multiple Choice Questions (MCQ)

1. A molecule is composed of atoms. A molecule of Ethylene, with the chemical formula C_2H_4, consists of two Carbon atoms and four Hydrogen atoms. A molecule of Oxygen, with the formula O_2, consists of two Oxygen atoms. Note that Carbon, Hydrogen and Oxygen are denoted by the letters C, H, and O respectively in the formula. When Ethylene comes in contact with Oxygen (O_2); Carbon dioxide (CO_2) and water (H_2O) are produced as the products of the chemical reaction . The equation corresponding to the chemical reaction (R) is given below

$$x_1C_2H_4 + x_2O_2 \longrightarrow x_3CO_2 + x_4H_2O. \qquad \ldots \text{(R)}$$

To balance the chemical equation we have to choose $x_1, x_2, x_3, \text{and} x_4$ such that both sides have the same number of carbon atoms on each side, the same number of hydrogen atoms on each side, and the same number of oxygen atoms on each side.
Note: An example to write the system of linear equations for balancing the chemical equation is the following :

$$x_1C + x_2O \longrightarrow x_3CO_2 \qquad \begin{aligned} x_1 &= x_3 \\ x_2 &= 2x_3 \end{aligned}$$

Consider the system of linear equations obtained for balancing the chemical equation (R) to answer the question.
Consider the following statements:

- **Statement 1:** The nullity of the matrix corresponding to this system is 1.
- **Statement 2:** $\{(1, 3, 1, 1)\}$ is a basis of the null space of the matrix corresponding to this system.
- **Statement 3:** There are an infinite number of ways to balance the chemical equation (R).

Which of the following statements is true?

- ◯ Option 1: Only Statement 1 is true.
- ◯ Option 2: Only Statement 2 is true.
- ◯ Option 3: Only Statement 3 is true.
- ◯ **Option 4:** Both, Statement 1 and Statement 3 are true.
- ◯ Option 5: Both Statement 2 and Statement 3 are true.
- ◯ Option 6: Both Statement 1 and Statement 2 are true.

Solution: The system of linear equations for balancing the chemical equation (R) is

$$\begin{aligned} 2x_1 &= x_3 \\ 4x_1 &= 2x_4 \\ 2x_2 &= 2x_3 + x_4 \end{aligned} \Rightarrow \begin{aligned} 2x_1 - x_3 &= 0 \\ 2x_1 - x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Matrix corresponding to the above system is

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Number of Independent variable is "1". Hence, Nullity of A is 1. [Statement 1 is true]

$\Rightarrow$ There are infinitely many solutions for the above system of linear equations.

Therefore, there are an infinite number of ways to balance the chemical equation "R".
(Statement-3 is correct)

Now, $A\begin{bmatrix}1\\3\\1\\1\end{bmatrix} = \begin{bmatrix}2 & 0 & -1 & 0\\2 & 0 & 0 & -1\\0 & 2 & -2 & -1\end{bmatrix}\begin{bmatrix}1\\3\\1\\1\end{bmatrix} \neq \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\Rightarrow$ $\{(1,3,1,1)\}$ can not be a basis of the null space of A. [Statement-2 is not correct].

2. Match the sets of vectors in column A with their properties of linear dependence or independence in column B and the dimension of the vector spaces in column C spanned by the sets.

	Set of vectors (Column A)		Linear dependence or independence (Column B)		Dimension of the vector space spanned by the set (Column C)
a)	$\{(1,0,0),(1,1,0),(1,1,1),(1,0,1)\}$	i)	Linearly independent	1)	1
b)	$\{(1,0,-1),(-1,2,0),(-2,0,0)\}$	ii)	Linearly dependent	2)	2
c)	$\{(1,-1,2),(-1,1,-2),(2,-2,4)\}$	iii)	Linearly dependent	3)	3
d)	$\{(1,0,1),(1,1,0),(0-1,1)\}$	iv)	Linearly dependent	4)	3

Table : M2W7G1

Choose the correct option.

- ◯ Option 1: a → ii → 3, b → iii → 2, c → i → 4, d → iv → 1
- ◯ Option 2: a → ii → 3, b → i → 4, c → iii → 2, d → iv → 1
- ◯ **Option 3:** a → ii → 4, b → i → 3 , c → iv → 1, d → iii → 2
- ◯ Option 4: a → ii → 4, b → i → 3 , c → iv → 2, d → iii → 1

Solution: a) $S_1 := \{(1,0,0),(1,1,0),(1,1,1),(1,0,1)\}$

Four vectors can't be linearly independent in $\mathbb{R}^3$.

$\Rightarrow$ S_1 is Linearly dependent.

$(1,0,1)-(1,0,0)+(1,1,0)=(1,1,1)$

Let $S=\{(1,0,1),(1,0,0),(1,1,0)\} \Rightarrow \text{Span}(S) = \text{Span}(S_1)$

Now consider the matrix using the vectors of S

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \det(A) \neq 0 \Rightarrow S_1 \text{ is Linearly independent.}$$

$$\Rightarrow \dim(\text{span}(S)) = \dim(\text{span}(S_1)) = 3.$$

b) $S_2 = \{(1,0,-1), (-1,2,0), (-2,0,0)\}$

Consider the matrix using the vectors of S_2: $B = \begin{bmatrix} 1 & 0 & -1 \\ -1, & 2, & 0 \\ -2, & 0, & 0 \end{bmatrix}$

$\det(B) \neq 0 \Rightarrow S_2$ is linearly independent $\Rightarrow \dim(\text{Span}(S_2)) = 3.$

c) $S_3 = \{(1,-1,2), (-1,1,-2), (2,-2,4)\}$

$$-1\,(1,-1,2) = (-1,1,-2) \text{ and } 2(1,-1,2) = (2,-2,4)$$

$$\Rightarrow \text{Span}(S_3) = \text{Span}\{(1,-1,2)\}$$

$$\Rightarrow \dim(\text{span}(S_3)) = \dim(\text{span}\{(1,-1,2)\}) = 1.$$

d) $S_4 = \{(1,0,1), (1,1,0), (0,-1,1)\}$

$$(1,0,1) - (1,1,0) = (0,-1,1)$$

$$\Rightarrow \text{Span}(S_4) = \text{Span}\{(1,0,1), (1,1,0)\}$$

Clearly, $(1,0,1) \neq \alpha(1,1,0)$ for any $\alpha \in \mathbb{R}$.

$\Rightarrow \{(1,0,1), (1,1,0)\}$ is Linearly independent.

Hence, $\dim(\text{Span}(S_4)) = \dim(\text{span}\{(1,0,1), (1,1,0)\}) = 2.$

3. Which of the following option(s) is (are) true?

- ◯ Option 1: The number of linearly independent vectors in a vector space can be more than the dimension of the vector space.
- ◯ Option 2: Dimension of the vector space spanned by the vectors $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is 2.
- ◯ **Option 3:** Dimension of the vector space spanned by the vectors $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, and $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ is 1.
- ◯ Option 4: Nullity of two different matrices cannot be equal.

Solution: Option 1:

No of linearly independent vectors in a vector space $V \leq \dim(V)$. [By the definition of the dimension of a vector space]

Option 2:

$$S := \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Suppose
$$\alpha \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \beta \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \gamma \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha-\beta-\gamma & -\alpha+\beta+\gamma \\ \alpha-\beta+\gamma & \alpha+\beta+\gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} \alpha-\beta-\gamma &= 0 \quad \text{— (1)} \\ -\alpha+\beta+\gamma &= 0 \quad \text{— (2)} \\ -\alpha-\beta+\gamma &= 0 \quad \text{— (3)} \\ \alpha+\beta+\gamma &= 0 \quad \text{— (4)} \end{aligned}$$

eq (1) + eq (4) $\Rightarrow \alpha = 0$

Substitue the value of α in eq (2) and eq (3)

$$\Rightarrow \quad \beta + \gamma = 0 \text{ and } -\beta + \gamma = 0$$

add the above two equations

$$\Rightarrow 2\gamma = 0 \Rightarrow \gamma = 0 \Rightarrow \beta = 0$$

Hence, S is a linearlyindependent set.

Therefore $\dim(\operatorname{Span}(S)) = 3$.

Option 3:

$$S_1 := \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \right\}$$

$$-1 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } -2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$\Rightarrow \operatorname{Span}(S_1) = \operatorname{Span}\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(\operatorname{Span}(S_1)) = \dim\left(\operatorname{Span}\left\{ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right\}\right) = 1.$$

Option 4: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A \neq B$, but nullity $(A) =$ nullity $(B) = 0$.

[Because both A and B are Invertible matrix].

2 Multiple Select Questions (MSQ)

4. Which of the following option(s) is(are) true?

- ◯ **Option 1:** If $A_{2\times 3}$ is a non zero matrix, then nullity of the matrix ≤ 2.
- ◯ **Option 2:** If $A_{3\times 2}$ is a non zero matrix, then nullity of the matrix ≤ 1.
- ◯ **Option 3:** Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix A is also zero.
- ◯ **Option 4:** Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix B is also zero.

Solution: Option: 1

For an $m\times n$ matrix, $\text{rank}(A)+\text{nullity}(A)=n$

In particular, If A is a 2×3 matrix

then $\text{rank}(A)+\text{nullity}(A)=3$

Since A is a non-zero matrix $\Rightarrow \text{rank}(A)\geq 1$

$\Rightarrow \text{nullity}(A)\leq 2.$

Option 2: If A is a non-zero 3×2 matrix

then $\text{rank}(A)+\text{nullity}(A)=2$

Since A is non-zero $\Rightarrow \text{rank}(A)\geq 1$

$\Rightarrow \text{nullity}(A)\leq 1$

Option 3 & 4: Nullity of AB is zero

$\Rightarrow$ AB is an invertible matrix

$\Rightarrow \det(AB)\neq 0 \Rightarrow \det(A)\det(B)\neq 0$

$\Rightarrow \det(A) \neq 0$ and $\det(B) \neq 0$.

$\Rightarrow$ Both A and B are invertible.

$\Rightarrow \text{nullity}(A) = \text{nullity}(B) = 0$.

5. Let A be a nonzero 4×4 matrix. Which of the following options are true?

- ◯ **Option 1:** The rank of A must be at least 1.
- ◯ Option 2: The rank of A may be 0.
- ◯ **Option 3:** The rank of A must be less than 4.
- ◯ **Option 4:** If the rank of the matrix is 2, then the dimension of the vector space spanned by the vectors corresponding to each column of A, must be 2.
- ◯ **Option 5:** If the rank of the matrix is 3, then there exists one vector corresponding to one column of A, which can be expressed as a linear combination of the vectors corresponding to each of the remaining columns of A.

Option 1:- Since A is a non-zero matrix then at least one of the rows of A must be non-zero.

$\Rightarrow$ RowSpace of $A \neq \{(0,0,0,0)\}$

$\Rightarrow \text{Rank}(A) \geq 1$.

$\Rightarrow$ option 2 can not be correct.

Option 3: A is a 4×4 matrix

By Rank-Nullity theorem

$$\text{Rank}(A) + \text{Nullity}(A) = 4$$

$$\Rightarrow \text{Rank}(A) \leq 4$$

Option 4: For an $m \times n$ matrix A

Row rank(A) = Column rank(A)

Hence, option 4 is correct.

<u>Option 5</u> :- If the rank of the matrix is 3, then the column rank is also 3.

A is a 4×4 matrix $\Rightarrow$ There are four columns in A and they can not be linearly independent (Column rank = 3)

Hence, one column can be expressed as a linear combination of the remaining three columns.

3 Numerical Answer Type (NAT)

6. Find the dimension of the vector space $V = M_{2\times 3}(\mathbb{R})$. [Answer: 6]

Solution:

General form of 2x3 matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$ where a_{ij}'s are real numbers.

Consider the set

$$S := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Clearly, the set S is linearly independent and $\text{span}(S) = V = M_{2\times 3}(\mathbb{R})$

Hence, $\dim(V) = |S| = 6$.

7. If rank of the matrix $\begin{bmatrix} 0 & -1 & a \\ 2 & 0 & -4 \\ 3 & -9 & -6 \end{bmatrix}$ is 2 then find the value of a. [Answer: 0]

Solution: $A := \begin{bmatrix} 0 & -1 & a \\ 2 & 0 & -4 \\ 3 & -9 & -6 \end{bmatrix}$

$(2,0,-4) \neq \alpha(3,-9,6)$ for any $\alpha \in \mathbb{R}$

$\Rightarrow$ The set $\{(2,0,-4),(3,-9,6)\}$ is linearly independent.

$\Rightarrow \quad \text{Rank}(A) \geq 2$

Now, $\text{Rank}(A) = 2$ Iff $\det(A) = 0$.

$\det(A) = 1(-12+12) + a(-18)$

$\det(A) = 0 \Rightarrow -18a = 0 \Rightarrow \boxed{a = 0}$

4 Comprehension Type Question:

Ayesha, Pritha, Sabya, and Wang went on a trip to Manali and Kasol. Accommodation costs ₹1500 per day in Manali and ₹800 per day in Kasol. The total food cost is ₹2000 per day in Manali and ₹1200 per day in Kasol. They plan to spend 2 days in Manali and 2 days in Kasol. The first and second rows of the Table M2W7G2 shows the percentage of contribution by each of them for the accomodation at Manali and Kasol, respectively. Similarly, the first and the second row of the Table M2W7G3 shows the percentage of contribution by each of them for the food at Manali and Kasol, respectively.

Table for Accommodation cost:

	Ayesha	Pritha	Sabya	Wang
Manali	$x_1\%$	$x_2\%$	$x_3\%$	$x_4\%$
Kasol	$y_1\%$	$y_2\%$	$y_3\%$	$y_4\%$

Table: M2W7G2

Table for cost of food:

	Ayesha	Pritha	Sabya	Wang
Manali	$v_1\%$	$v_2\%$	$v_3\%$	$v_4\%$
Kasol	$w_1\%$	$w_2\%$	$w_3\%$	$w_4\%$

Table: M2W7G3

Suppose $T(x, y)$ denotes the contribution of a person for accommodation per day, where the first variable x denotes the percentage of contribution by that person for accommodation in Manali and the second variable y denotes the percentage of contribution by that person for accommodation in Kasol. (i.e., if a and b denote the costs for accommodation per day at Manali and Kasol, respectively, then $T(x, y) = \frac{1}{100}(ax + by)$). Similarly, $T'(v, w)$ denotes the contribution by a person for food per day, where the first variable v denotes the percentage of contribution for food in Manali and the second variable w denotes the percentage of contribution for food in Kasol. Answer the following questions based on the given information.

8. Choose the set of correct options. (MSQ)

   ◯ Option 1: $T(x, y) = 30x + 16y$

   ◯ **Option 2:** $T(x, y) = 15x + 8y$

   ◯ **Option 3:** $T'(v, w) = 20v + 12w$

○ Option 4: $T'(v, w) = 40v + 24w$

Solution: It is given that if a and b denote the costs for accommodation per day at Manali and Kasol, respectively, then $T(x,y) = \frac{1}{100}(ax+by)$.

- Accommodation costs 1500 per day in Manali and 800 per day in Kasol.

$$\Rightarrow T(x,y) = \frac{1}{100}(1500x + 800y) = 15x + 8y.$$

- The total food cost is 2000 per day in Manali and 1200 per day in Kasol.

$$\Rightarrow T'(v,w) = \frac{1}{100}(2000v + 1200w)$$

$$= 20v + 12w$$

Clearly, Both T and T' are linear transformations.

9. Choose the set of correct options. (MSQ)

- ◯ **Option 1:** Suppose Ayesha contributes for herslef and also on behalf of Sabya. Then the contribution per day by Ayesha is given by $T(x_1 + x_3, y_1 + y_3)$ and $T'(v_1 + v_3, w_1 + w_3)$, for accommodation and food, respectively.
- ◯ **Option 2:** The total contribution (for the whole trip) for the accommodation by Pritha is given by $2T(x_2, y_2)$, which is equal to $T(2x_2, 2y_2)$.
- ◯ Option 3: The total contribution (for the whole trip) for the accommodation by Pritha is given by $2T(x_2, y_2)$, which is not equal to $T(2x_2, 2y_2)$.
- ◯ Option 4: Suppose Pritha contributes for herself and also on behalf of Wang for food. Then the contribution per day by Pritha for food is given by $T'(v_2, w_2) + T'(v_4, w_4)$, which is not equal to $T'(v_2 + v_4, w_2 + w_4)$.
- ◯ **Option 5:** Suppose Pritha contributes for herself and also on behalf of Wang for food. Then the contribution per day by Pritha for food is given by $T'(v_2, w_2) + T'(v_4, w_4)$, which is equal to $T'(v_2 + v_4, w_2 + w_4)$.

Solution:

Option 1: T(x, y) denotes the contribution of a person for accommodation per day, where the first variable "x" denotes the percentage of contribution by the person for accommodation at Manali and "y" denotes the percentage of contribution by that person for accommodation in Kasol.

From Table: M2W7G2.

Accommodation contribution of Ayesha at Manali = x_1%
at Kasol = y_1%

" " " Sabya at Manali = x_3%
at Kasol = y_3%

If Ayesha contributes for herself and Sabya,

Then the total accom: contribution by Ayesha
at Manali $= (x_1+x_3)\%$
at Kasol $= (y_1+y_3)\%$

Total contribution by Ayesha $= T(x_1+x_3, y_1+y_3)$

Similarly, total contribution for food by Ayesha is $T'(v_1+v_3, w_1+w_3)$.

Option 2 & 3:

Total contribution for accomodation per day by Pritha is $T(x_2, y_2)$.

Contribution for whole trip (two days) is

$$2T(x_2, y_2) = T(2x_2, 2y_2)$$

(because T is a linear transformation).

Option 4 & 5.

Similarly we can check, If Pritha contributes for herself and Wang for food then the total contribution by Pritha is $T'(v_2+v_4, w_2+w_4)$

$$= T'(v_2, w_2) + T'(v_4, w_4)$$

[Because T' is a linear transformation].

10. The total contribution (accommodation and food) per day can be denoted by: (MCQ)

- ◯ Option 1: $f(x, y, v, w) = T(x, y) + T'(v, w)$ which is not a linear mapping.
- ◯ **Option 2:** $f(x, y, v, w) = T(x, y) + T'(v, w)$ which is a linear mapping.
- ◯ Option 3: $f(x, y, v, w) = T(x, y)T'(v, w)$ which is not a linear mapping.
- ◯ Option 4: $f(x, y, v, w) = T(x, y)T'(v, w)$ which is a linear mapping.

Solution: Option 1 & 2:

$$f(x, y, v, w) = T(x, y) + T'(v, w)$$

☆ $f(x_1+x_2, y_1+y_2, v_1+v_2, w_1+w_2)$

$$= T(x_1+x_2, y_1+y_2) + T'(v_1+v_2, w_1+w_2)$$

$$= T(x_1, y_1) + T(x_2, y_2) + T'(v_1, w_1) + T'(v_2, w_2)$$

$$= T(x_1, y_1) + T'(v_1, w_1) + T(x_2, y_2) + T'(v_2, w_2)$$

$$= f(x_1, y_1, v_1, w_1) + f(x_2, y_2, v_2, w_2).$$

☆ $f(cx, cy, cv, cw)$

$$= T(cx, cy) + T'(cv, cw)$$

$$= cT(x, y) + cT'(v, w)$$

$$= c\,\big(T(x,y) + T'(v,w)\big)$$

$$= c\,\big(F(x,y,v,w)\big)$$

<u>Option 3 & 4.</u>

$$F(x,y,v,w) = T(x,y)\,T'(v,w)$$
$$= (15x+8y)(20v+12w).$$

$$F(1,0,0,0) = 15$$
$$F(0,0,1,0) = 20$$

$$F(1,0,1,0) = 300 \neq F(1,0,0,0) + F(0,0,1,0) = 35$$

So, F is not a linear transformation.

Week-7
Mathematics for Data Science - 2
Basis of a vector space, Rank and dimension of a matrix
Practice Assignment

1 Multiple Choice Questions (MCQ)

1. A molecule is composed of atoms. A molecule of Ethylene, with the chemical formula C_2H_4, consists of two Carbon atoms and four Hydrogen atoms. A molecule of Oxygen, with the formula O_2, consists of two Oxygen atoms. Note that Carbon, Hydrogen and Oxygen are denoted by the letters C, H, and O respectively in the formula. When Ethylene comes in contact with Oxygen (O_2); Carbon dioxide (CO_2) and water (H_2O) are produced as the products of the chemical reaction . The equation corresponding to the chemical reaction (R) is given below

$$x_1C_2H_4 + x_2O_2 \longrightarrow x_3CO_2 + x_4H_2O. \qquad \ldots \text{(R)}$$

To balance the chemical equation we have to choose $x_1, x_2, x_3, \text{and} x_4$ such that both sides have the same number of carbon atoms on each side, the same number of hydrogen atoms on each side, and the same number of oxygen atoms on each side.
Note: An example to write the system of linear equations for balancing the chemical equation is the following :

$$x_1C + x_2O \longrightarrow x_3CO_2$$

corresponding system of linear equations is:

$$x_1 = x_3$$
$$x_2 = 2x_3$$

Consider the system of linear equations obtained for balancing the chemical equation (R) to answer the question.
Consider the following statements:

- **Statement 1:** The nullity of the matrix corresponding to this system is 1.
- **Statement 2:** $\{(1, 3, 1, 1)\}$ is a basis of the null space of the matrix corresponding to this system.
- **Statement 3:** There are an infinite number of ways to balance the chemical equation (R).

Which of the following statements is true?

- ◯ Option 1: Only Statement 1 is true.
- ◯ Option 2: Only Statement 2 is true.
- ◯ Option 3: Only Statement 3 is true.
- ◯ **Option 4:** Both, Statement 1 and Statement 3 are true.
- ◯ Option 5: Both Statement 2 and Statement 3 are true.
- ◯ Option 6: Both Statement 1 and Statement 2 are true.

2. Match the sets of vectors in column A with their properties of linear dependence or independence in column B and the dimension of the vector spaces in column C spanned by the sets.

	Set of vectors (Column A)		Linear dependence or independence (Column B)		Dimension of the vector space spanned by the set (Column C)
a)	$\{(1,0,0),(1,1,0),(1,1,1),(1,0,1)\}$	i)	Linearly independent	1)	1
b)	$\{(1,0,-1),(-1,2,0),(-2,0,0)\}$	ii)	Linearly dependent	2)	2
c)	$\{(1,-1,2),(-1,1,-2),(2,-2,4)\}$	iii)	Linearly dependent	3)	3
d)	$\{(1,0,1),(1,1,0),(0-1,1)\}$	iv)	Linearly dependent	4)	3

Table : M2W7G1

Choose the correct option.

- ◯ Option 1: a → ii → 3, b → iii → 2, c → i → 4, d → iv → 1
- ◯ Option 2: a → ii → 3, b → i → 4, c → iii → 2, d → iv → 1
- ◯ **Option 3:** a → ii → 4, b → i → 3 , c → iv → 1, d → iii → 2
- ◯ Option 4: a → ii → 4, b → i → 3 , c → iv → 2, d → iii → 1

3. Which of the following option(s) is (are) true?

   ◯ Option 1: The number of linearly independent vectors in a vector space can be more than the dimension of the vector space.

   ◯ Option 2: Dimension of the vector space spanned by the vectors $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, and $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is 2.

   ◯ **Option 3:** Dimension of the vector space spanned by the vectors $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, and $\begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$ is 1.

   ◯ Option 4: Nullity of two different matrices cannot be equal.

2 Multiple Select Questions (MSQ)

4. Which of the following option(s) is(are) true?

   ◯ **Option 1:** If $A_{2\times 3}$ is a non zero matrix, then nullity of the matrix ≤ 2.

   ◯ **Option 2:** If $A_{3\times 2}$ is a non zero matrix, then nullity of the matrix ≤ 1.

   ◯ **Option 3:** Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix A is also zero.

   ◯ **Option 4:** Let A and B be two square matrices of order 3, if nullity of matrix AB is 0, then nullity of matrix B is also zero.

5. Let A be a nonzero 4×4 matrix. Which of the following options are true?

   ◯ **Option 1:** The rank of A must be at least 1.

   ◯ Option 2: The rank of A may be 0.

   ◯ **Option 3:** The rank of A must be less than 4.

   ◯ **Option 4:** If the rank of the matrix is 2, then the dimension of the vector space spanned by the vectors corresponding to each column of A, must be 2.

   ◯ **Option 5:** If the rank of the matrix is 3, then there exists one vector corresponding to one column of A, which can be expressed as a linear combination of the vectors corresponding to each of the remaining columns of A.

3 Numerical Answer Type (NAT)

6. Find the dimension of the vector space $V = M_{2\times 3}(\mathbb{R})$. [Answer: 6]

7. If rank of the matrix $\begin{bmatrix} 0 & -1 & a \\ 2 & 0 & -4 \\ 3 & -9 & -6 \end{bmatrix}$ is 2 then find the value of a. [Answer: 0]

4 Comprehension Type Question:

Ayesha, Pritha, Sabya, and Wang went on a trip to Manali and Kasol. Accommodation costs ₹1500 per day in Manali and ₹800 per day in Kasol. The total food cost is ₹2000 per day in Manali and ₹1200 per day in Kasol. They plan to spend 2 days in Manali and 2 days in Kasol. The first and second rows of the Table M2W7G2 shows the percentage of contribution by each of them for the accomodation at Manali and Kasol, respectively. Similarly, the first and the second row of the Table M2W7G3 shows the percentage of contribution by each of them for the food at Manali and Kasol, respectively.

Table for Accommodation cost:

	Ayesha	Pritha	Sabya	Wang
Manali	$x_1\%$	$x_2\%$	$x_3\%$	$x_4\%$
Kasol	$y_1\%$	$y_2\%$	$y_3\%$	$y_4\%$

Table: M2W7G2

Table for cost of food:

	Ayesha	Pritha	Sabya	Wang
Manali	$v_1\%$	$v_2\%$	$v_3\%$	$v_4\%$
Kasol	$w_1\%$	$w_2\%$	$w_3\%$	$w_4\%$

Table: M2W7G3

Suppose $T(x, y)$ denotes the contribution of a person for accommodation per day, where the first variable x denotes the percentage of contribution by that person for accommodation in Manali and the second variable y denotes the percentage of contribution by that person for accommodation in Kasol. (i.e., if a and b denote the costs for accommodation per day at Manali and Kasol, respectively, then $T(x, y) = \frac{1}{100}(ax + by)$). Similarly, $T'(v, w)$ denotes the contribution by a person for food per day, where the first variable v denotes the percentage of contribution for food in Manali and the second variable w denotes the percentage of contribution for food in Kasol. Answer the following questions based on the given information.

8. Choose the set of correct options. (MSQ)

- ○ Option 1: $T(x, y) = 30x + 16y$
- ○ **Option 2:** $T(x, y) = 15x + 8y$
- ○ **Option 3:** $T'(v, w) = 20v + 12w$

- ◯ Option 4: $T'(v, w) = 40v + 24w$

9. Choose the set of correct options. (MSQ)
   - ◯ **Option 1:** Suppose Ayesha contributes for herslef and also on behalf of Sabya. Then the contribution per day by Ayesha is given by $T(x_1 + x_3, y_1 + y_3)$ and $T'(v_1 + v_3, w_1 + w_3)$, for accommodation and food, respectively.
   - ◯ **Option 2:** The total contribution (for the whole trip) for the accommodation by Pritha is given by $2T(x_2, y_2)$, which is equal to $T(2x_2, 2y_2)$.
   - ◯ Option 3: The total contribution (for the whole trip) for the accommodation by Pritha is given by $2T(x_2, y_2)$, which is not equal to $T(2x_2, 2y_2)$.
   - ◯ Option 4: Suppose Pritha contributes for herself and also on behalf of Wang for food. Then the contribution per day by Pritha for food is given by $T'(v_2, w_2) + T'(v_4, w_4)$, which is not equal to $T'(v_2 + v_4, w_2 + w_4)$.
   - ◯ **Option 5:** Suppose Pritha contributes for herself and also on behalf of Wang for food. Then the contribution per day by Pritha for food is given by $T'(v_2, w_2) + T'(v_4, w_4)$, which is equal to $T'(v_2 + v_4, w_2 + w_4)$.

10. The total contribution (accommodation and food) per day can be denoted by: (MCQ)
    - ◯ Option 1: $f(x, y, v, w) = T(x, y) + T'(v, w)$ which is not a linear mapping.
    - ◯ **Option 2:** $f(x, y, v, w) = T(x, y) + T'(v, w)$ which is a linear mapping.
    - ◯ Option 3: $f(x, y, v, w) = T(x, y)T'(v, w)$ which is not a linear mapping.
    - ◯ Option 4: $f(x, y, v, w) = T(x, y)T'(v, w)$ which is a linear mapping.

Week-8
Mathematics for Data Science - 2
Rank of a matrix and Linear Transformation
Practice Assignment

1 Multiple Choice Questions (MCQ)

1. Consider the following statements

   - **Statement 1:** There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(2,3) = (1,2)$, $T(1,-1) = (1,-1)$, and $T(4,1) = (1,0)$.
   - **Statement 2:** If there is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(2,3) = (1,2)$, and $T(1,-1) = (1,-1)$, then $T(x,y) = \frac{1}{5}(4x - y, -x + 4y)$.
   - **Statement 3:** Let $\beta = \{v_1, v_2, v_3\}$ and $\gamma = \{2v_1 + v_3, v_2 - v_3, v_3 - v_1\}$ be bases of a vector space V. Consider a linear transformation $T : V \to V$ such that $T(v_1) = 2v_3 + v_1, T(v_2) = 2v_1,$ and $T(v_3) = 2v_2$. The matrix representation of T, with respect to β and γ for the domain and codomain respectively, is $\begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 2 \\ 1 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix}$
   - **Statement 4:** Let $\{v_1, v_2, v_3\}$ be a basis of a vector space V and $\{2v_1 + v_3, v_2 - v_3, v_3 - v_1\}$ be a basis of vector space W. Consider a linear transformation $T : V \to W$ such that $T(v_1) = 2v_3 + v_1, T(v_2) = 2v_1,$ and $T(v_3) = 2v_2$. Then, the rank of T is 3.

   Which of the following options is correct?

   - ○ Option 1: Only Statement 1 is correct.
   - ○ Option 2: All the statements are correct.
   - ○ Option 3: Statement 2 and Statement 4 are not correct.
   - ○ **Option 4:** All the statements except Statement 1 are correct.
   - ○ Option 5: All the statements except Statement 3 are correct.

2. Suppose the matrix representation of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered bases $\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$ for the domain and $\gamma = \{(1,1,1), (0,1,1), (0,0,1)\}$ for the range, is $I_{3\times3}$, i.e., the identity matrix of order 3. Which one is the correct matrix representation of the linear transformation T with respect to standard ordered basis of $\mathbb{R}^3$ for both domain and range?

○ Option 1: $I_{3\times 3}$ i.e., identity matrix of order 3.

○ Option 2: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

○ **Option 3:** $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$

○ Option 4: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

○ Option 5: $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

3. Let T and S be two linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, which are defined as follows:

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad\qquad S : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x, y, z) = (x - y, y - z, z - x) \qquad S(x, y, z) = (x + y, y + z, z + x)$$

Let A and B be the matrices corresponding to T and S respectively, with respect to the standard ordered bases of $\mathbb{R}^3$. Let C and D be the matrices corresponding to $T \circ S$ and $S \circ T$ respectively, with respect to the standard ordered basis.

Consider the following set of statements:

- **Statement 1:** $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$
- **Statement 2:** $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
- **Statement 3:** $C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
- **Statement 4:** $T \circ S = S \circ T$
- **Statement 5:** $C \neq D$
- **Statement 6:** $D = BA$

Which of the following options is correct?

○ Option 1: Only Statement 4 is not true.

- ◯ **Option 2:** Only Statement 5 is not true.
- ◯ Option 3: All the statements are true.
- ◯ Option 4: Only Statement 6 is not true.
- ◯ Option 5: Only Statement 3 is not true.

4. Consider the following set $S = \{(x, y) \mid x + y = 1, x, y \in \mathbb{R}\}$. In column A the matrix representation of some linear transformations are given with respect to the standard ordered bases. Match the entries in column A with the set $T(S)$ given in column B and the geometric representations of S and $T(S)$ in column C.

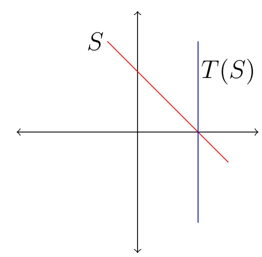
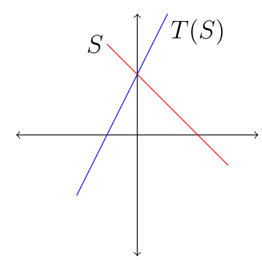
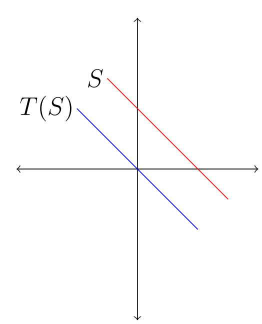
	Matrix form of linear transformation (Column A)		Image of the given set (Column B)		Geometric representations (Column C)
i)	$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	a)	$T(S) = \{(x, y) \mid x = 1, y \in \mathbb{R}\}$	1)	
ii)	$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$	b)	$T(S) = \{(x, y) \mid x + y = 0, x, y \in \mathbb{R}\}$	2)	
iii)	$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$	c)	$T(S) = \{(x, y) \mid 2x - y = -1, x, y \in \mathbb{R}\}$	3)	

Table: M2W8P1

○ Option 1: i → b → 2, ii → c → 1, iii → a → 3.

- ◯ **Option 2:** i → b → 3, ii → c → 2, iii → a → 1.
- ◯ Option 3: i → c → 2, ii → b → 3, iii → a → 1.
- ◯ Option 4: i → c → 2, ii → a → 3, iii → b → 1

2 Multiple Select Questions (MSQ)

5. Consider the following two groups. In Group 1, there are 3 functions from one vector space to another, and in Group 2, there are three properties of functions. All the vector spaces have usual addition and scalar multiplication.

Group-1

- **P:** $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(x, y) = \begin{cases} \left(\frac{x^2}{y}, y\right) & \text{if } y \neq 0 \\ \\ (x, 0) & \text{otherwise} \end{cases}$$

- **Q:** $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x, y) = (x, xy)$
- **R:** $T : \mathbb{R}^3 \to \mathbb{R}$ such that $T(x, y, z) = \frac{x+y+z}{5}$

Group-2

- **1:** $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$
- **2:** $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{R}$.
- **3:** T is a linear transformation

Choose the set of correct options.

- ◯ Option 1: P satisfies 1 but not 2.
- ◯ **Option 2:** P satisfies 2 but not 1
- ◯ Option 3: Q satisfies 1.
- ◯ Option 4: Q satisfies 3.
- ◯ Option 5: P satisfies 3.
- ◯ **Option 6:** R satisfies 2.
- ◯ **Option 7:** R satisfies 3.
- ◯ Option 8: R satisfies 1 but not 2.

6. Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x, y, z) = (x + 2y, x - y + cz, 2x + y + dz)$. Let A be the matrix representation of T with respect to the standard ordered bases of $\mathbb{R}^3$ and $B = \begin{bmatrix} 0 & 3 & 2 \\ -1 & -3 & -3 \\ 2 & 3 & 4 \end{bmatrix}$.

   Based on the above information which of the followings option(s) is(are) true?

   - ◯ **Option 1:** If A is similar to B, then rank of the linear transformation T is 2.
   - ◯ Option 2: If A is similar to B, then nullity of the linear transformation T is 2.
   - ◯ **Option 3:** If $c \neq d$, then A cannot be similar to B.
   - ◯ **Option 4:** If A is similar to B and $c = 1$, then a basis of the kernel is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}$.

3 Numerical Answer Type (NAT):

7. Let T and S be two linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, which are defined as follows:

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \qquad\qquad S : \mathbb{R}^3 \to \mathbb{R}^3$$
$$T(x, y, z) = (x - y, y - z, z - x) \qquad S(x, y, z) = (x + y, y + z, z + x)$$

   Let $S + T$ be defined to be the linear transformation as follows:

$$(S + T) : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(S + T)(x, y, z) = S(x, y, z) + T(x, y, z)$$

   Let C be the matrix representation of $S + T$ with respect to the standard ordered bases of $\mathbb{R}^3$. If $C = nI$, where I denotes the identity matrix of order 3, then what will be the value of n? [Answer: 2]

4 Comprehension Type Question:

For each video on YouTube, consider the vector (Number of comments in thousands, Number of shares in thousands, Number of views in thousands) and consider the subset S of $\mathbb{R}^3$ consisting of these vectors. It is known that the ranking of a YouTube video can be thought of as the restriction of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ to S where $\mathbb{R}^3$ comes with usual addition and scalar multiplication. (assume ranking 0 (zero) is possible and in practice negative rankings can be ignored).

The following data in Table M2W8P2 shows the correlation between comments, shares, and views - with YouTube rankings of a video.

Number of comments (in thousands)	Number of shares (in thousands)	Number of views (in thousands)	Ranking of a video
3	10	8	3
4	15	10	2
5	20	15	1

Table: M2W8P2

Use the above information to answer questions 8, 9 and 10.

8. Which of the following sets of vectors in S have the property that the YouTube video corresponding to every linear combination of that set which is in S has ranking 0? (MSQ)

   ◯ Option 1: $\{(1, 1, 0), (0, 0, 1)\}$
   ◯ **Option 2:** $\{(6, 25, 0), (0, 0, 25)\}$
   ◯ Option 3: $\{(25, 6, 0), (0, 0, 25)\}$
   ◯ **Option 4:** $\{(12, 50, 0), (0, 0, 1)\}$
   ◯ **Option 5:** $\{(24, 100, 0), (0, 0, 25)\}$
   ◯ Option 6: $\{(6, 0, 0), (0, 25, 0), (0, 0, 1)\}$
   ◯ Option 7: $\{(25, 0, 0), (0, 6, 0), (0, 0, 1)\}$

9. Which of the following sets of vectors in S have the property that the YouTube video corresponding to every linear combination of that set which is in S has ranking a multiple of 15? (MSQ)

   ◯ **Option 1:** $\{(3, 0, 0), (0, 0, 1)\}$
   ◯ **Option 2:** $\{(3, 0, 1)\}$
   ◯ **Option 3:** $\{(3, 0, 1), (0, 0, 1)\}$
   ◯ **Option 4:** $\{(0, 0, 1)\}$
   ◯ **Option 5:** $\{(9, 25, 0), (0, 0, 1)\}$
   ◯ Option 6: $\{(25, 9, 0), (0, 0, 1)\}$
   ◯ **Option 7:** $\{(9, 0, 0), (0, 25, 0), (0, 0, 1)\}$
   ◯ Option 8: $\{(25, 0, 0), (0, 9, 0), (0, 0, 1)\}$

10. Suppose that for a YouTube video there are 5 (in thousands) shares. What must be the number of comments (in thousands) so that the ranking of the video will be 4? [Note: Suppose your answer is 6000, then you have to enter 6 as the answer.]
    (NAT) [Answer: 2]

Solution: Week 8 practice assignment.

1) Statement 1:

If T is a linear transformation, then

$$T(4,1) = T((2,3) + 2(1,-1))$$

$$= T(2,3) + 2T(1,-1)$$

$$= (1,2) + 2(1,-1) = (3,0)$$

But it is given that $T(4,1) = (1,0) \neq (3,0)$

Hence, the Statement is wrong.

Statement 2:

$$T(1,0) = T\left(\tfrac{1}{5}(2,3) + \tfrac{3}{5}(1,-1)\right)$$

$$= \tfrac{1}{5}\, T(2,3) + \tfrac{3}{5}\, T(1,-1)$$

$$= \tfrac{1}{5}(1,2) + \tfrac{3}{5}(1,-1)$$

$$= \left(\tfrac{4}{5}, -\tfrac{1}{5}\right) = \tfrac{1}{5}(4,-1)$$

$$T(0,1) = T\left(\tfrac{1}{5}(2,3) - \tfrac{2}{5}(1,-1)\right)$$

$$= \tfrac{1}{5}\, T(2,3) - \tfrac{2}{5}\, T(1,-1)$$

$$= \tfrac{1}{5}(1,2) - \tfrac{2}{5}(1,-1) = \left(-\tfrac{1}{5}, \tfrac{4}{5}\right) = \tfrac{1}{5}(-1,4)$$

$$T(x,y) = T\left(x(1,0) + y(0,1)\right)$$
$$= x\,T(1,0) + y\,T(0,1) = \tfrac{1}{5}(4x, -x) + \tfrac{1}{5}(-y, 4y)$$
$$= \tfrac{1}{5}(4x - y, -x + 4y)$$

Hence, the statement is true.

Statement 3:

Basis of $V = \{v_1, v_2, v_3\}$

Basis of $W = \{2v_1 + v_3, v_2 - v_3, v_3 - v_1\}$

$$T(v_1) = 2v_3 + v_1 = 1(2v_1 + v_3) + 0(v_2 - v_3) + 1(v_3 - v_1)$$
$$T(v_2) = 2v_1 = \tfrac{2}{3}(2v_1 + v_3) + 0(v_2 - v_3) - \tfrac{2}{3}(v_3 - v_1)$$
$$T(v_3) = 2v_2 = \tfrac{2}{3}(2v_1 + v_3) + 2(v_2 - v_3) + \tfrac{4}{3}(v_3 - v_1)$$

Hence, the matrix representation of T with respect to the given bases is,
$$\begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 0 & 2 \\ 1 & -2/3 & 4/3 \end{pmatrix}$$

Statement 4 :

$$\begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 0 & 2 \\ 1 & -2/3 & 4/3 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 0 & 2 \\ 0 & -4/3 & 2/3 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & -4/3 & 2/3 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{-3/4 R_2} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_1 - 2/3 R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{\frac{R_3}{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + 1/2 R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence rank of T is 3 .

2) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$

$\gamma = \{(1,1,1), (0,1,1), (0,0,1)\}$

The matrix representation is (with respect to β and γ) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$T(1,0,0) = 1(1,1,1) + 0(0,1,1) + 0(0,0,1) = (1,1,1)$
$T(1,1,0) = 0(1,1,1) + 1(0,1,1) + 0(0,0,1) = (0,1,1)$
$T(1,1,1) = 0(1,1,1) + 0(0,1,1) + 1(0,0,1) = (0,0,1)$

Standard ordered basis: $\{(1,0,0), (0,1,0), (0,0,1)\}$

$(1,0,0) = 1(1,0,0) + 0(1,1,0) + 0(1,1,1)$
$(0,1,0) = -1(1,0,0) + 1(1,1,0) + 0(1,1,1)$
$(0,0,1) = 0(1,0,0) - 1(1,1,0) + 1(1,1,1)$

$T(1,0,0) = 1\,T(1,0,0) = (1,1,1)$
$T(0,1,0) = -1\,T(1,0,0) + 1\,T(1,1,0) = -(1,1,1) + (0,1,1)$
$= (-1,0,0)$

$T(0,0,1) = -1\,T(1,1,0) + 1\,T(1,1,1) = -1(0,1,1) + (0,0,1)$
$= (0,-1,0)$

Hence the matrix representation of T with respect to standard ordered basis is

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

3) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (x - y,\ y - z,\ z - x)$

$T(1,0,0) = (1, 0, -1)$
$T(0,1,0) = (-1, 1, 0)$
$T(0,0,1) = (0, -1, 1)$

Hence, $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$S : \mathbb{R}^3 \to \mathbb{R}^3$

$S(x, y, z) = (x + y,\ y + z,\ z + x)$

$S(1,0,0) = (1, 0, 1)$
$S(0,1,0) = (1, 1, 0)$
$S(0,0,1) = (0, 1, 1)$

Hence, $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$$T \circ S\,(x, y, z) = T(x + y,\ y + z,\ z + x)$$
$$= (x - z,\ y - x,\ z - y)$$

$$S \circ T\,(x, y, z) = S(x - y,\ y - z,\ z - x)$$
$$= (x - z,\ y - x,\ z - y)$$

Hence, $T \circ S = S \circ T$

Hence, $C = D$

$T \circ S(1,0,0) = (1,-1,0)$
$T \circ S(0,1,0) = (0,1,-1)$
$T \circ S(0,0,1) = (-1,0,1)$

Hence, $C = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = C = D.$$

Hence, only Statement 5 is not true.

4) i) $T : \mathbb{R}^2 \to \mathbb{R}^2$

Matrix representation of T with respect to standard ordered basis $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$T(1,0) = (1,-1)$, $T(0,1) = (-1,1)$

$T(x,y) = (x-y, -x+y)$

$$S = \{(x, y) \mid x+y=1,\ x, y \in \mathbb{R}\}$$

$$= \{(x, 1-x) \mid x \in \mathbb{R}\}$$

$$T(x, 1-x) = (2x-1,\ -2x+1)$$

$$= (x', y') \quad (\text{say})$$

$$x' = 2x-1, \quad y' = -2x+1$$

$$x'+y' = 2x-1-2x+1 = 0$$

Hence, $T(S) = \{(x, y) \mid x+y=0,\ x, y \in \mathbb{R}\}$

$x+y=0$ is the st. line which passes through origin and has negative slope.

Hence, i) $\longrightarrow$ b) $\longrightarrow$ 3)

ii) $T: \mathbb{R}^2 \to \mathbb{R}^2$

The matrix representation of T with respect to standard ordered basis is

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$T(1,0) = (0,1)$ and $T(0,1) = (-1,-1)$

$T(x, y) = (-y,\ x-y)$

$S = \{(x,y) \mid x+y=1, x,y \in \mathbb{R}\}$

$= \{(x, 1-x) \mid x \in \mathbb{R}\}$

$T(x, 1-x) = (-1+x, 2x-1)$

$= (x', y')$ (say)

$x' = -1+x, \quad y' = 2x-1$

$2x' = -2+2x, \quad -y' = -2x+1$

$2x' - y' = -1$

Hence, $T(S) = \{(x,y) \mid 2x-y=-1, x,y \in \mathbb{R}\}$

$2x-y=-1$ is a st. line which has positive slope and whose y-intercept is positive.

Hence, ii) → c → 2)

iii) $T: \mathbb{R}^2 \to \mathbb{R}^2$

The matrix representation of T with respect to standard ordered basis is

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$T(1,0) = (1,-1)$ and $T(0,1) = (1,0)$

$T(x, y) = (x+y, -x)$

$S = \{(x, y) \mid x+y=1,\ x, y \in \mathbb{R}\}$

$= \{(x, 1-x) \mid x \in \mathbb{R}\}$

$T(x, 1-x) = (1, -x)$

$T(S) = \{(x, y) \mid x=1,\ y \in \mathbb{R}\}$

$x=1$ is the st. line parallel to Y-axis passing through $(1,0)$.

Hence, iii) $\longrightarrow$ a) $\longrightarrow$ 1

5) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(x, y) = \begin{cases} (x^2/y,\ y), & \text{if } y \neq 0 \\ (x, 0), & \text{otherwise} \end{cases}$$

Let $v_1 = (1, 1)$ and $v_2 = (1, 2)$

$T(1,1) = (1, 1)$, $T(1,2) = (1/2, 2)$

$T((1,1)+(1,2)) = T(2,3) = (4/3, 3)$

Hence, $T(v_1 + v_2) \neq T(v_1) + T(v_2)$

If $y \neq 0$, then $cy \neq 0$ for all $c \in \mathbb{R}$.

$$T(cx, cy) = \left(\frac{c^2x^2}{cy}, cy\right)$$

$$= \left(cx^2/y, cy\right)$$

$$= c\left(x^2/y, y\right)$$

If $y = 0$, then,

$$T(cx, 0) = (cx, 0) = c(x, 0)$$

Hence, $T(cv) = cT(v)$ for all $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

Hence, P satisfies 2 but not 1.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (x, xy)$

Let $v_1 = (1, 2)$, $v_2 = (1, 1)$

$$T(v_1 + v_2) = T(2, 3) = (2, 6)$$

$$T(v_1) + T(v_2) = T(1, 2) + T(1, 1)$$

$$= (1, 2) + (1, 1) = (2, 3)$$

Hence, 1 is not satisfied.

let $v = (1,2)$ and $c = 2$

then $T(cv) = T(2,4) = (2,8)$

$cT(v) = 2(1,2) = (2,4)$

Hence, $T(cv) \neq cT(v)$

Hence θ doesnot satisfy 1,2 and 3.

$T: \mathbb{R}^3 \to \mathbb{R}$

$$T(x,y,z) = \frac{x+y+z}{5}$$

let $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$

$T(v_1 + v_2) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$$= \frac{x_1 + x_2 + y_1 + y_2 + z_1 + z_2}{5}$$

$$= \frac{x_1 + y_1 + z_1}{5} + \frac{x_2 + y_2 + z_2}{5}$$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= T(v_1) + T(v_2)$

Let $v = (x, y, z)$

Then $T(cv) = T(cx, cy, cz)$

$$= \frac{cx + cy + cz}{5}$$

$$= c\left(\frac{x + y + z}{5}\right) = cT(v)$$

Hence, R satisfies 1, 2 and 3.

6) $T(x, y, z) = (x + 2y, x - y + cz, 2x + y + dz)$

$T(1,0,0) = (1, 1, 2)$
$T(0,1,0) = (2, -1, 1)$
$T(0,0,1) = (0, c, d)$

Hence $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & c \\ 2 & 1 & d \end{pmatrix}$

$$\boxed{\begin{aligned} \det(A) &= 1(-d - c) + 1(-2d) + 2(2c) \\ &= 3c - 3d = 3(c - d) \end{aligned}}$$

$$B = \begin{pmatrix} 0 & 3 & 2 \\ -1 & -3 & -3 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -3 & -3 \\ 0 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\xrightarrow{-R_1} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_3 - 2R_1} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 3 & 2 \\ 0 & -3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 3 & 2 \\ 0 & -3 & -2 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 2/3 \\ 0 & -3 & -2 \end{pmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank $B = 2$

If A is similar to B, then rank of A is also 2 (as, similar matrices have the same rank). Hence the rank of T is also 2.

As rank$(B) = 2$, $\det(B) = 0$

We have already calculated $\det(A) = 3(c-d)$

If A and B are similar to each other then,

$B = P^{-1}AP$ for some matrix P.

In that case, $\det(B) = \det(P^{-1}AP)$

$$= \det(P^{-1})\det(A)\det(P)$$

$$= \frac{1}{\det(P)}\det(A)\det(P)$$

$$= \det(A)$$

Hence, det(A) should be 0.

$\Rightarrow \; 3(c-d) = 0 \;\Rightarrow\; \underline{\underline{c=d}}$.

If $c \neq d$, then matrix A cannot be similar to B.

If A and B are similar, and $c=1$, then $d=1$

Hence we have,

$$T(x,y,z) = (x+2y,\; x-y+z,\; 2x+y+z)$$

$$T(-2,1,3) = (0,0,0)$$

So, $(-2,1,3) \in \ker(T)$

As rank $T = 2$, then from rank nullity theorem we can conclude,

$$\text{rank } T + \text{nullity } T = \dim \mathbb{R}^3 = 3$$

$$\Rightarrow \text{nullity } T = 3-2 = 1.$$

Hence ker T is one dimensional subspace.

So, $\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}$ is a basis of ker T.

7) $(S+T): \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(S+T)(x,y,z) = S(x,y,z) + T(x,y,z)$

$= (x+y,\ y+z,\ z+x) + (x-y,\ y-z,\ z-x)$

$= (2x,\ 2y,\ 2z)$

$(S+T)(1,0,0) = (2,0,0)$
$(S+T)(0,1,0) = (0,2,0)$
$(S+T)(0,0,1) = (0,0,2)$

Hence, $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= 2I$.

So, $n = 2$.

<u>Comprehension Type Question:</u>

$T(3,10,8) = 3$
$T(4,15,10) = 2$
$T(5,20,15) = 1$

8) $3T(5,20,15) - T(3,10,8) = 3 - 3 = 0$

$\Rightarrow T(15,60,45) - T(3,10,8) = 0$
$\Rightarrow T(12,50,37) = 0$

$$2T(5,20,15) - T(4,15,10) = 2-2=0$$

$$\Rightarrow T(10,40,30) - T(4,15,10) = 0$$

$$\Rightarrow T(6,25,20) = 0$$

$$\Rightarrow 2T(6,25,20) = 0$$

$$\Rightarrow T(12,50,40) = 0$$

Hence we have, $T(12,50,40) - T(12,50,37) = 0$

$$\Rightarrow T(0,0,3) = 0$$

$$\Rightarrow T(0,0,1) = 0.$$

Moreover we have, $20T(0,0,1) = 0$

$$\Rightarrow T(0,0,20) = 0$$

So, $T(6,25,20) - T(0,0,20) = 0$

$$\Rightarrow T(6,25,0) = 0$$

T is a linear transformation from a 3 dimensional vector space to $\mathbb{R}$, which is a one dimensional vector space over $\mathbb{R}$.

As T is non-zero, $\operatorname{rank}(T) = 1$

From rank nullity theorem, we get,

$$\operatorname{nullity}(T) = 3-1 = 2$$

Now, $\{(6,25,0), (0,0,1)\}$ is a linearly independent set and we have already derived nullity $(T) = 2$.

So, $\{(6,25,0), (0,0,1)\}$ forms a basis of nullity(T).

In the given options the following will be bases of nullity (T):

option 2: $\{(6,25,0), 25(0,0,1)\} = \{(6,25,0), (0,0,25)\}$

option 4: $\{2(6,25,0), (0,0,1)\} = \{(12,50,0), (0,0,1)\}$

option 5: $\{4(6,25,0), 25(0,0,1)\} = \{(24,100,0), (0,0,25)\}$

9) $2T(3,10,8) - T(5,20,15) = (2\times 3) - 1 = 5$

$\Rightarrow T(6,20,16) - T(5,20,15) = 5$

$\Rightarrow T(1,0,1) = 5$

$\Rightarrow T(1,0,0) + T(0,0,1) = 5$

$\Rightarrow T(1,0,0) + 0 = 5$

$\Rightarrow T(1,0,0) = 5$

$T(5,20,15) = 1$

$\Rightarrow T(5,0,0) + T(0,20,0) + T(0,0,15) = 1$

$\Rightarrow 5T(1,0,0) + 20\,T(0,1,0) + 15\,T(0,0,1) = 1$

$\Rightarrow 25 + 20\,T(0,1,0) + 0 = 1$

$\Rightarrow T(0,1,0) = -24/20 = -6/5$

$$T(x,y,z) = 5x - \frac{6y}{5}$$

$T(3,0,0) = 15$ and $T(0,0,1) = 0$

Hence the first three options are correct.

$T(9,0,0) + T(0,25,0) + T(0,0,1)$

$= 45 - 30 = 15$

Hence option 5 and 7 are also correct.

$T(25,9,0) = \left(125 - \frac{54}{5}\right)$ not a multiple of 15.

Hence option 6 and 8 are not correct.

$T(0,0,1) = 0$, which is a multiple of 15.

10) $T(x,y,z) = 5x - \frac{6y}{5}$

$y = 5$

$T(x,5,z) = 5x - 6 = 4$

$\Rightarrow \underline{x = 2}$.

Week-8
Mathematics for Data Science - 1
Exponential and Logarithm
Assignment

1 Multiple Choice Questions(MCQ)

1. If $18^x - 12^x - (2 \times 8^x) = 0$, then the value of x is.

    1. $\frac{\ln 2}{\ln 3 - \ln 2}$
    2. $\frac{\ln 18}{\ln 12 - \ln 8}$
    3. $\ln 2$
    4. $\ln 18$

    Answer: Option 1

Solution:

$$18^x - 12^x - (2 \times 8^x) = 0$$

Domain = R as all the terms are exponential functions.

we can write: $2^x . 9^x - 2^x . 6^x - 2^x (2 \times 4^x) = 0$

2^x is a positive number. Dividing by $2^x \Rightarrow$

$$9^x - 6^x - (2 \times 4^x) = 0 \quad \text{—— (1)}$$

$\boxed{\text{Let } a = 3^x \text{ and } b = 2^x}$

then $9^x = (3^2)^x = 3^{2x} = (3^x)^2 = a^2$

$6^x = 2^x 3^x = ba$

$4^x = (2^x)^2 = a^2$

Therefore equation (1) would be

$$a^2 - ab - 2b^2 = 0$$

$$\Rightarrow \quad a^2 - 2ab + ab - 2b^2 = 0$$

$$a(a-2b) + b(a-2b) = 0$$

$$(a-2b)(a+b) = 0$$

If $a - 2b = 0 \Rightarrow a = 2b \Rightarrow 3^x = 2 \times 2^x$

taking log $\Rightarrow$ $x \log 3 = \log 2 + x \log 2$

$$\boxed{x = \frac{\log 2}{\log 3 - \log 2}} \text{ — Answer.}$$

If $a + b = 0$

$3^x + 2^x = 0 \Rightarrow$ Not possible

2. Suppose three distinct persons A, B and C are standing on the X- axis of the XY- plane (as shown in the figure M1W9G-1) and the distance between B and A is same as the distance between C and B. The coordinates of A, B and C are $(\log_5 3, 0)$, $(\log_5(3^x - \frac{9}{2}), 0)$ and $(\log_5(3^x - \frac{9}{4}), 0)$ respectively. What is the distance between C and B?
(MCQ), (Ans:(a))

1. $\log_5(2)$ units.
2. $\log_5(\frac{5}{4})$ units.
3. $\log_5(\frac{3}{2})$ units
4. $\log_5(\frac{7}{3})$ units.

$AB = \sqrt{\left\{\log_5(3^x - 9/2) - \log_5 3\right\}^2 + 0} = \log_5 \frac{(3^x - 9/2)}{3} \Rightarrow 3^x > 9/2$

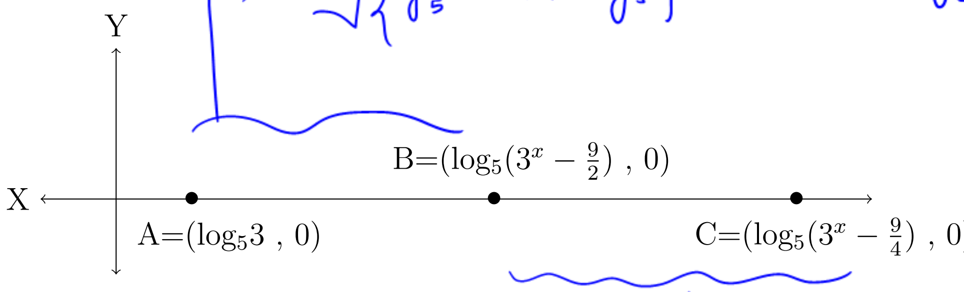


Figure M1W9G-1

$BC = \sqrt{\left\{\log_5(3^x - 9/4) - \log_5(3^x - 9/2)\right\}^2 + 0}$

$= \log_5 \left\{\frac{3^x - 9/4}{3^x - 9/2}\right\} \Rightarrow 3^x - 9/4 > 0$

Answer: Option 3

Given: $AB = BC$

$\log_5 \left\{\frac{3^x - 9/2}{3}\right\} = \log_5 \left\{\frac{3^x - 9/4}{3^x - 9/2}\right\}$

$\frac{3^x - 9/2}{3} = \frac{3^x - 9/4}{3^x - 9/2} \Rightarrow (3^x - 9/2)^2 = 3(3^x - 9/4)$

$\Rightarrow (3^x)^2 + (9/2)^2 - 2(3^x)(9/2) = 3(3^x) - 3(9/4)$

take $3^x = a \Rightarrow$ $a^2 + (\frac{9}{2})^2 - 9a = 3a - 3(9/4) \Rightarrow a^2 - 12a + \frac{81}{4} + 27/4 = 0$

$\Rightarrow a^2 - 12a + \frac{108}{4} = 0 \Rightarrow a^2 - 12a + 27 = 0 \Rightarrow (a-3)(a-9) = 0$

If $a = 3 \Rightarrow 3^x = 3 \Rightarrow x = 1$ but $3^1 \not> 9/2$. Therefore, $\boxed{x = 2}$

If $a = 9 \Rightarrow 3^x = 9 \Rightarrow x = 2$ Now $3^2 = 9 > 9/2$

$$BC = \log_5 \left\{ \frac{3^x - 9/4}{3^x - 9/2} \right\} = \log_5 \left\{ \frac{3^2 - 9/4}{3^2 - 9/2} \right\} = \log_5 \left\{ \frac{3/4}{1/2} \right\}$$

$$\boxed{BC = \log_5 \{ 3/2 \}} \Longrightarrow \text{Answer.}$$

3. In a city, a rumour is spreading about the safety of corona vaccination. Suppose N number of people live in the city and $f(t)$ is the number of people who **have not** yet heard about the rumour after t days. Suppose $f(t)$ is given by $f(t) = Ne^{-kt}$, where k is a constant. If the population of the city is 1000, and suppose 40 have heard the rumor after the first day. After how many days (approximately) half of the population would have heard the rumor?

   1. 20
   2. 17
   3. 13
   4. 12

   Answer: Option 2

After first day $\Rightarrow$ $t=1$

40 have heard therefore, $1000 - 40 = 960$ people have not heard i.e.,

$$960 = Ne^{-kt}$$

$$t=1 \Rightarrow \quad 960 = 1000e^{-k} \Rightarrow \boxed{e^{-k} = \frac{960}{1000}}$$

Half of population will heard then $f(t) = \frac{1000}{2} = 500$

therefore,

$$500 = 1000e^{-kt} \Rightarrow \frac{500}{1000} = \left(e^{-k}\right)^t$$

$$\frac{500}{1000} = \left(\frac{960}{1000}\right)^t$$

taking log:

$$\ln\left\{\frac{500}{1000}\right\} = t \ln\left\{\frac{960}{1000}\right\}$$

$$\ln(1/2) = t \ln(.96) \Rightarrow t = \frac{\ln(0.5)}{\ln(.96)}$$

$$t \simeq 16.97$$

t can be condisered as 17 days.

4. Consider the function $f(x) = \log_2(12 + 4x - x^2)$. The range of f is

1. $(-\infty, 4]$
2. $(-\infty, \infty)$
3. $(0, \infty)$
4. $(0, \log 12]$

Answer: Option 1

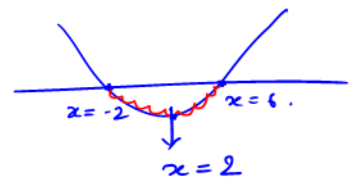
Domain:

$$12 + 4x - x^2 > 0.$$

$$x^2 - 4x - 12 < 0$$

$$(x-6)(x+2) < 0$$

$$\boxed{x \in (-2, 6)}$$



at $x = 2$

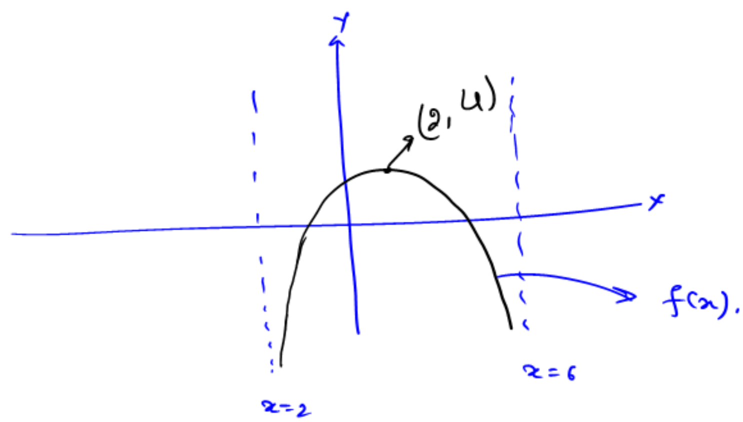
$$12 + 4x - x^2 = 12 + 8 - 4 = 16$$

$$f(x) = \log_2(16) = 4$$

→ $x = -2$ and $x = 6$ will work as assymptotes:

→ As at $x = 0$, $\log x$ tends towards $-\infty$, similarly at $x = -2$ and $x = 6$, $f(x)$ will tends towords $-\infty$.

→ $f(x)$ will be symmetric Arround $x = 2$.



$\boxed{\text{Range: } (-\infty, 4]}$ — Ans.

Use the following information for the questions 5 and 6.

Consider the function $f(x) = \frac{2e^x}{3e^x+1}$ from $\mathbb{R}$ to $\mathbb{R}$.

5. Which of the following is true about f?
   1. f is not a one to one function.
   2. f is a one to one function.
   3. Range of f is $\mathbb{R}$.
   4. f is a bijective function.

   Answer: Option 2

6. The inverse of f would be
   1. $\ln(\frac{2x}{2-3x})$
   2. $\ln(\frac{2x}{2x-x})$
   3. $\ln(\frac{x}{2-3x})$
   4. $\ln(\frac{x}{2x-x})$

   Answer: Option 3

$$f(x) = \frac{2e^x}{3e^x+1}.$$

To find one to one nature:

take $x_1 > x_2$.

$$f(x_1) = \frac{2e^{x_1}}{3e^{x_1}+1}$$

$$f(x_2) = \frac{2e^{x_2}}{3e^{x_2}+1}$$

Let $f(x_1) > f(x_2)$

then $\frac{\cancel{2}e^{x_1}}{3e^{x_1}+1} > \frac{\cancel{2}e^{x_2}}{3e^{x_2}+1}$

$$\cancel{3e^{(x_1+x_2)}} + e^{x_1} > \cancel{3e^{(x_1+x_2)}} + e^{x_2}$$

$$e^{x_1} > e^{x_2}$$

we know that: e^x is an exponetial and increasing function.

therefore if $e^{x_1} > e^{x_2} \Rightarrow x_1 > x_2$

which is true with our assumption.

$\boxed{\text{Therefore, } f(x) \text{ is an increasing function and thats why one to one function.}}$

Now for Range: $f(x) = \frac{2e^x}{\underbrace{3e^x+1}_{\text{always positive.}}}$ $2e^x \rightarrow$ always positive $\rightarrow$ $f(x)$ is always positive

which means, codain $\neq$ Range.

$\boxed{f(x) \text{ is not Onto function.}}$

As $f(x)$ is one to one function, inverse of $f(x)$ is possible:

$$f(x) = \frac{2e^x}{3e^x+1}$$

Replace x by $f^{-1}(x)$ and $f(x)$ by x:

$$x = \frac{2e^{f^{-1}(x)}}{3e^{f^{-1}(x)}+1}$$

$$3x\,e^{f^{-1}(x)} + x = 2e^{f^{-1}(x)}$$

$$3x\,e^{f^{-1}(x)} - 2e^{f^{-1}(x)} = -x$$

$$e^{f^{-1}(x)}\left\{3x-2\right\} = -x$$

$$e^{f^{-1}(x)} = \frac{x}{2-3x}$$

$$\boxed{f^{-1}(x) = \ln\left\{\frac{x}{2-3x}\right\}}$$ — Answer.

2 Multiple Select Questions (MSQ)

Use the following information for the questions 7 and 8.

The amount of gold (in kilograms) sold by a jeweler on the mth day of 2019 is given by the function $f(m) = \log_{10}(m+1) - \frac{1}{2}\log_{m+1}(0.01)$ (where $m = 1$ corresponds to the 1st January, 2019, and $m = 365$ corresponds to the 31st December, 2019). Find the correct set of options.

7. If $m > n > 9$, then choose the correct option(s).

   1. $f(m) > f(n)$
   2. $f(m) < f(n)$
   3. $f(m) = f(n)$
   4. $f(m) \leq f(n)$

   Answer: Option 1

8. Choose the correct option(s).

   1. The jeweler sold at least 540 kg gold in 2019.
   2. The jeweler sold at least 730 kg gold in 2019.
   3. The jeweler sold at least 2 kg gold daily throughout the year 2019.
   4. The jeweler sold at least 10 kg gold daily throughout the year 2019.

   Answer: Options 2 and 3

Solution

Given $f(m) = \log_{10}(m+1) - \frac{1}{2}\log_{m+1}(0.01)$

$= \log_{10}(m+1) - \frac{1}{2}\log_{m+1} 10^{-2}$

$= \log_{10}(m+1) - (-2) \times \frac{1}{2}\log_{m+1} 10$

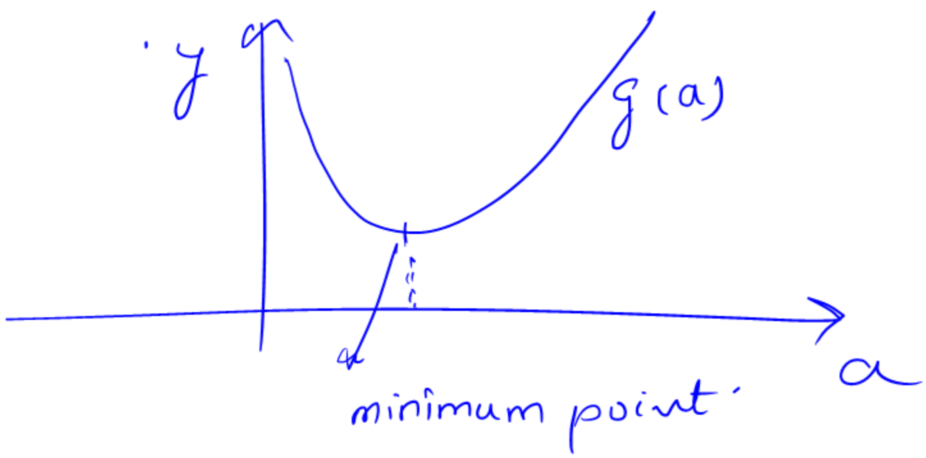
$= \log_{10}(m+1) + \frac{1}{\log_{10}(m+1)}$

Let $\log_{10}(m+1) = a$ then $f(m) = a + \frac{1}{a} = g(a)$ (let)

If $a \to \infty$ the $g(a) \to \infty$

if $a \to 0$ then $g(a) \to \infty$ as $\frac{1}{a} \to \infty$.

therefore the curve will look like

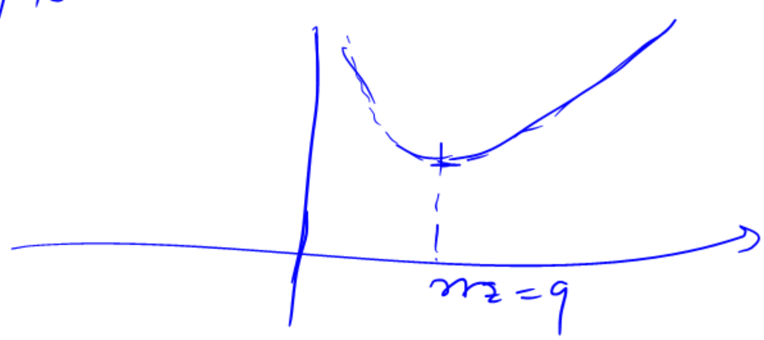


The minimum point will occure at $a = 1$.

We can use Desmos to see the behaviour.

$a = 1 \Rightarrow \log_{10} m+1 = 1 \Rightarrow m = 9$

Therefore:



Therefore After $m=9$ $f(m)$ is an increasing function that's why $f(m) > f(n)$ for $m > n > 9$. → Answer Question 7.

The minimum value of $f(m)$ would be at $m=9 \Rightarrow$

$$f(m)\Big|_{min} = \log_{10}(9+1) + \frac{1}{\log_{10}(9+1)}$$

$$= 2$$

$\Rightarrow$ Therefore jeweller sells atleast 2 kg gold per day. → Answer.

$\Rightarrow$ And a year contains 365 days, therefore atleast $365 \times 2 = 730$ kg gold will be sold in a year. → Answer.

9. The stock market chart of a tourism company (A) is shown roughly in the Figure M1W9G-2. This company was listed in February ($x = 2$) and experiences a logarithmic fall after the COVID−19 outbreak which is given by $y = -a\ log(x-h)+a$. x represents the number of months since the beginning of the year and y represents the stock price in ₹(1000). During the 10^{th} month the pharmacy company announced that the vaccine is made for the COVID−19. Thereafter, the stock price of the company (A) is raised exponentially $y = 10^{\frac{x}{b}} - b$. Choose the correct set of options. (Note: a is any positive real number, b is a positive integer and h is a constant.)

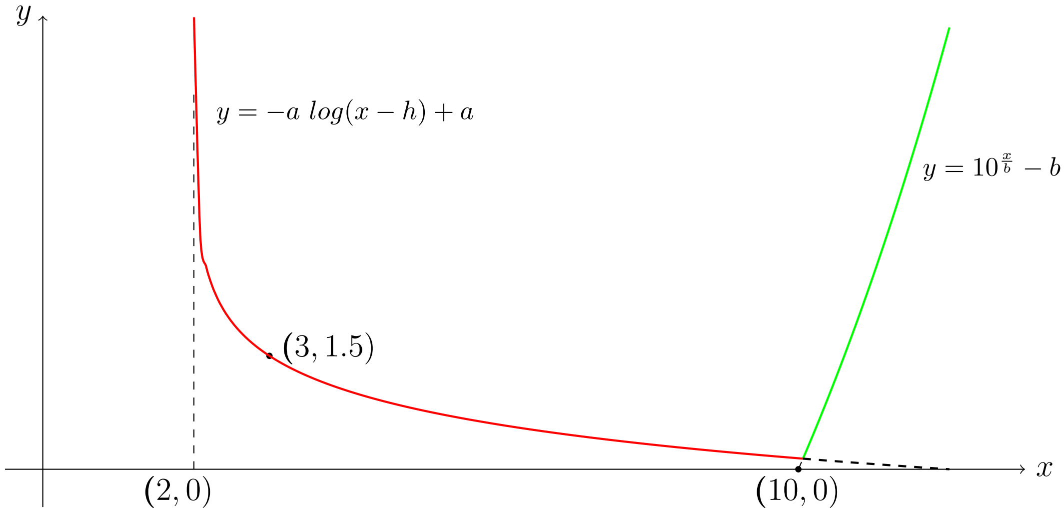


Figure M1W9G-2

1. For logarithmic fall the value of $a = 1.5$ and $h = 2$.
2. For exponential rise passing through (10, 0) the value of $b = 10$.
3. The stock price in 12^{th} month is ₹4000.
4. If the vaccine was not made and the stock price just followed the same logarithmic function through out, then the investor would have lost his/her entire investment on the 12^{th} month.

Answer: Options 1, 2, and 4

The assymptote will occure when $x - h = 0$

$\Rightarrow x = h = 2.$

At $x = 3 \Rightarrow y = -a\log(3-2) + a = 1.5$

$= -a \log 1 + a = 1.5$

$= 0 + a = 1.5 \Rightarrow \boxed{a = 1.5}$

Given $y = 10^{x/b} - b = 0$ at $x = 10$

$\Rightarrow 10^{10/b} - b = 0 \Rightarrow 10^{10/b} = b$

Taking log at base 10 $\Rightarrow \frac{10}{b} \log_{10} 10 = \log_{10} b$

$\Rightarrow \frac{10}{b} = \log_{10} b \Rightarrow 10 = b \log_{10} b$

Taking Antilog :

$\Rightarrow 10^{10} = b^b \Rightarrow \boxed{b = 10}$

for 12th month :

$y = -a \log_{10}(12-2) + a = -1.5 \log_{10} 10 + 1.5$

$y = 0$

option (4) is correct.

10. If $m^{\log_3 2} + 2^{\log_3 m} = 16$. Then, what is the value of m ?
(NAT)
Answer: 27

$$m^{\log_3 2} + 2^{\log_3 m} = 16$$

$$2^{\log_3 m} + 2^{\log_3 m} = 16$$

$$2\left\{2^{\log_3 m}\right\} = 16$$

$$2^{\log_3 m} = 8 = 2^3$$

$$\log_3 m = 3$$

$$m = 3^3 = 27 \text{ Ans.}$$

Week - 8
Prcactice Assignment
Mathematics for Data Science - 1

1. If $b > 0$ and $\boxed{4\log_x b + 9\log_{b^5x} b = 1,}$ then the possible value(s) of x is(are)
   (a) $\boldsymbol{b^{10}}$
   (b) b^9
   (c) $\boldsymbol{b^{-2}}$
   (d) b^5
   (e) b^4

Soln:

$$4\log_x b + 9\log_{b^5 x} b = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{\log_b(b^5 x)} = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{\log_b b^5 + \log_b x} = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{5\cancel{\log_b b}^{\,1} + \log_b x} = 1$$

Let $p = \log_b x$

$$\Rightarrow \frac{4}{p} + \frac{9}{5+p} = 1$$

$$\Rightarrow \frac{4(5+p)+9p}{p(5+p)} = 1 \implies 20 + 4p + 9p = 5p + p^2$$

$$\Rightarrow \boxed{p^2 - 8p - 20 = 0}$$

$\Rightarrow p^2 - 8p - 20 = 0$

$\Rightarrow p^2 - 10p + 2p - 20 = 0$

$\Rightarrow p(p-10) + 2(p-10) = 0$

$\Rightarrow (p+2)(p-10) = 0$

$\boxed{p = -2, 10}$

We know that $\boxed{p = \log_b x}$

If $p = -2$

$-2 = \log_b x$

$\boxed{x = b^{-2}}$

If $p = 10$

$10 = \log_b x$

$\boxed{x = b^{10}}$

Note:-

$$\boxed{\log_{a^b} c = \frac{1}{b} \log_a c}$$ ✓ ∗ Can be used later

Proof:- LHS $= \log_{a^b} c$

$$= \frac{1}{\log_c a^b} = \frac{1}{b \log_c a} = \frac{1}{b} \times \frac{1}{\log_c a} = \frac{1}{b} \log_a c$$

$= RHS$

$\boxed{LHS = RHS}$

2. George deposits ₹5L in a bank that compounded quarterly at the rate of 20% per year. How long will it take to increase his money to 16 times the principal amount(in year)?

(a) $\frac{\ln 16}{4}$

(b) $\frac{\ln 16}{4 \ln \frac{21}{20}}$

(c) $\frac{\ln 2}{\ln \frac{21}{20}}$

(d) $\log_{\frac{21}{20}} 2$

(e) $\frac{\ln 2^{\frac{1}{4}}}{\ln \frac{21}{20}}$

Soln Formula for compound interest

$$\boxed{A = P\left(1 + \frac{R}{100}\right)^{t}}$$

where

t = time period (years)

R = Interest rate per year

P = Principal or initial deposits

A = Amount after t years

n = No. of times it compounded in a year.

$$\Rightarrow \boxed{A = P\left(1 + \frac{R}{n \times 100}\right)^{nt}}$$

$n = 4$

$$A = P\left(1 + \frac{20}{400}\right)^{4t}$$

$$\boxed{A = 16P}$$

$$\Rightarrow 16\cancel{P} = \cancel{P}\left(1 + \frac{1}{20}\right)^{4t}$$

$$\Rightarrow 16 = \left(\frac{21}{20}\right)^{4t}$$

$$\boxed{\text{formula: } \ln a^b = b \ln a}$$

$$\Rightarrow \ln 16 = 4t \ln\left(\frac{21}{20}\right)$$

$$\boxed{t = \frac{1}{4}\left(\frac{\ln 16}{\ln\left(\frac{21}{20}\right)}\right)}$$

$$t = \frac{\ln(16)^{1/4}}{\ln\left(\frac{21}{20}\right)} = \frac{\ln(2^4)^{1/4}}{\ln\left(\frac{21}{20}\right)} = \boxed{\frac{\ln 2}{\ln\left(\frac{21}{20}\right)}}$$

We have

$$\boxed{t = \frac{\ln 2}{\ln \frac{21}{20}}}$$

$$\boxed{\text{Formula : } \log_b a = \frac{\log a}{\log b}}$$

Using change of base formula, we get,

$$\boxed{t = \log_{\frac{21}{20}} 2}$$

3. Choose the set of correct options.

(a) ~~$\log_5 2$ is a rational number~~

(b) ~~If $0 < b < 1$ and $0 < x < 1$ then $\log_b x < 0$~~

(c) **If $\log_3(\log_5 x) = 1$ then $x = 125$**

(d) ~~If $0 < b < 1, 0 < x < 1$ and $x > b$ then $\log_b x > 1$~~

(e) **If $0 < b < 1$ and $0 < x < y$ then $\log_b x > \log_b y$**

Soln

(a) Let $\log_5 2$ be rational number, thus it can be written in $\frac{p}{q}$ form,

$\log_5 2 = p/q$

$\Rightarrow 2 = 5^{\frac{p}{q}}$

$\boxed{2^q = 5^p}$

2 & 5 are co-primes & 2 cannot divide 5

thus assumption is wrong.

So it should irrational

(b) Given:- $0 < b < 1$ & $0 < x < 1$ then $\boxed{\log_b x < 0}$

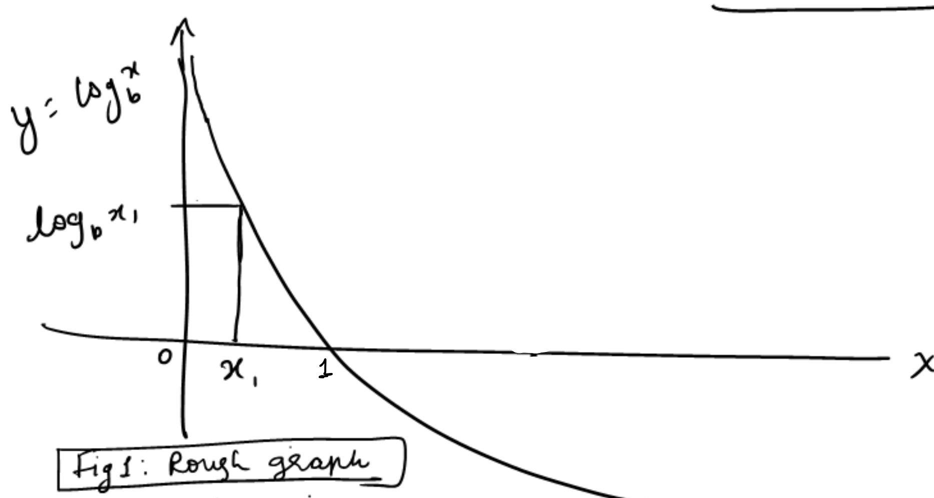


Fig 1: Rough graph

Let x_1 be in $\boxed{0 < x_1 < 1}$; then from above graph (Fig 1)

$\boxed{\log_b x_1 > 0}$ $\therefore$ statement of option(b) is wrong.

✓ option (c): Given that,

$$\boxed{\log_3 \log_5 (x) = 1}$$ then $x = ?$

$\log_5 x = 3^1 = 3$

$x = 5^3 = 125$

$$\boxed{x = 125}$$

Formula:

$\log_a x = b$

$$\boxed{x = a^b}$$

option (d) Given that:-

$0 < b < 1$, $0 < x < 1$, & $\boxed{x > b}$ then $\boxed{\log_b x > 1}$

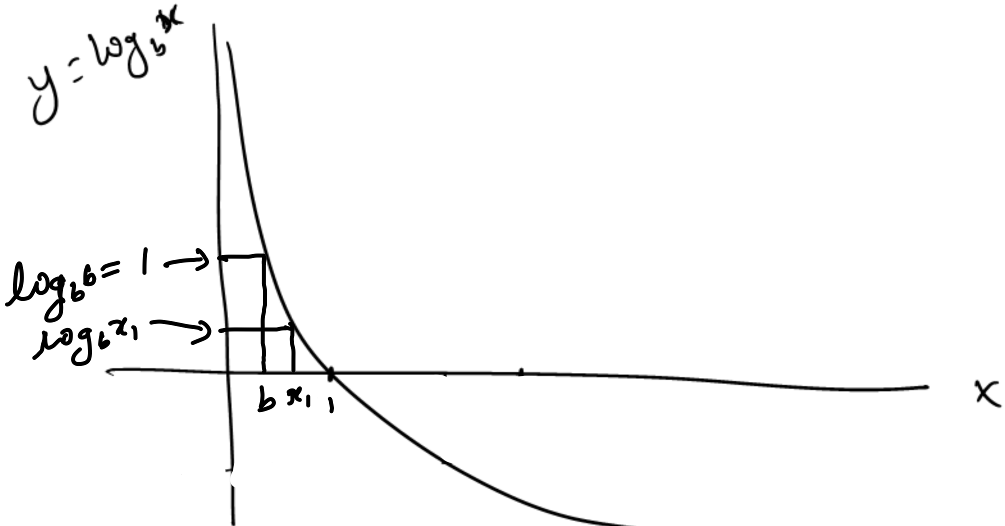


Fig 2: Rough graph

Let $b < x_1 < 1$

then, $\log_b x_1 < 1$ (see Fig 2; notice: $\log_b b = 1$)

Thus the statement of option (d) is wrong.

option (e) Given that:

$\underline{0 < b < 1}$ & $0 < x < y$ then $\boxed{\log_b x > \log_b y}$

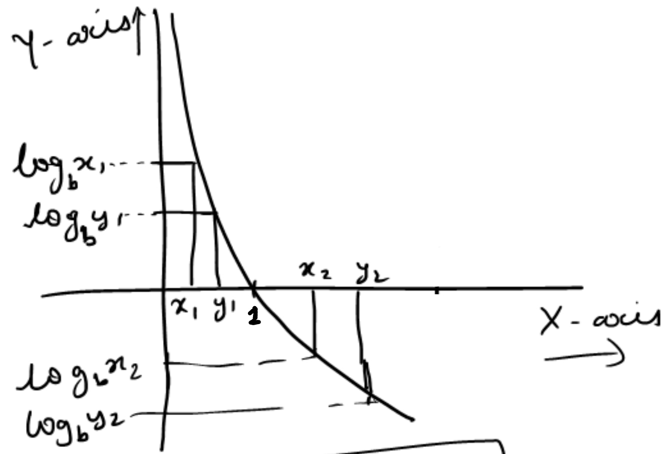


Fig 3: Rough graph

Case I: Let

$x_1 < y_1 < 1$

From graph (Fig 3)

$\log_b x_1 > \log_b y_1$

Thus in case I, the given statement is right

Case II: $b < x_2 < 1$ & $b < y_2 < 1$

also $1 < x_2 < y_2$

From graph (Fig 3)

$\log_b x_2 > \log_b y_2$

Thus in case II, the given statement is right again

Thus overall the given statement is right, thus the option (e) is correct.

4. Suppose that two types of insects are found in a pond. Their growth in number after t seconds is given by the equations $f(t) = 5^{3t-2}$ and $h(t) = 3^{2t+1} (t \neq 0)$. For what value of t will both insects be of same number in the pond?

(a) $\mathbf{\frac{\ln 3+2\ln 5}{3\ln 5-2\ln 3}}$ ✓

(b) $\mathbf{\frac{\ln 75}{\ln \frac{125}{9}}}$ ✓

(c) $\mathbf{\log_{\frac{125}{9}} 75}$ ✓

(d) ~~$\frac{\ln 5+2\ln 3}{3\ln 3-2\ln 5}$~~

Soln:-

Insects number will be same when

$$f(t) = h(t)$$

$$5^{3t-2} = 3^{2t+1}$$

$$\Rightarrow (3t-2)\ln 5 = (2t+1)\ln 3$$

$$\Rightarrow (3\ln 5)t - 2\ln 5 = (2\ln 3)t + \ln 3$$

$$(3\ln 5)t - (2\ln 3)t = \ln 3 + 2\ln 5$$

$$\Rightarrow t(3\ln 5 - 2\ln 3) = \ln 3 + 2\ln 5$$

$$\Rightarrow \boxed{t = \frac{\ln 3 + 2\ln 5}{3\ln 5 - 2\ln 3}}$$ ✓

formula:

① $\ln a^b = b\ln a$

② $\ln(ab) = \ln a + \ln b$

③ $\ln \frac{a}{b} = \ln a - \ln b$

$$\Rightarrow t = \frac{\ln 3 + \ln 5^2}{\ln 5^3 - \ln 3^2}$$

$$\Rightarrow = \frac{\ln 3 \times 25}{\ln\left(\frac{125}{9}\right)}$$

$$\Rightarrow \boxed{t = \frac{\ln 75}{\ln\left(\frac{125}{9}\right)}}$$

Using change of base formula

$$\boxed{t = \log_{\frac{125}{9}} 75}$$

Formula:

$$\log_b a = \frac{\log a}{\log b}$$

7. If $\log_{\sqrt{2}}(x+4) - \log_2(\frac{1}{2}x+2) = 1$ then x is

(a) $\mathbf{-3}$

(b) 1

(c) -4

(d) 5

Soln:-

Given that:

$$\log_{2^{1/2}}(x+4) - \log_2\left(\frac{1}{2}x+2\right) = 1$$

Using derived formula

$$\log_{a^b} c = \frac{1}{b}\log_a c$$

$$\Rightarrow \frac{1}{2}\log_2(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$$

$$\Rightarrow \log_2(x+4)^{1/2} - \log_2\left(\frac{x+4}{2}\right) = 1$$

$$\Rightarrow \log_2\left(\frac{(x+4)^{1/2}}{\left(\frac{x+4}{2}\right)}\right) = 1$$

formula:

$$\log_b x = a$$
$$x = b^a$$

$$\Rightarrow \frac{(x+4)^{1/2}}{\left(\frac{x+4}{2}\right)} = 2$$

$$\Rightarrow (x+4)^{1/2} = \cancel{2}\left(\frac{x+4}{\cancel{2}}\right)$$

Squaring on b.s

$$(x+4) = (x+4)^2$$

$$(x+4)^2 - (x+4) = 0$$

$$(x+4)\left((x+4)-1\right) = 0$$

we get

$x+4=0$ or $x+4-1=0$

$\boxed{x=-4}$ $\boxed{x=-3}$

From Question we have

$\log_{\sqrt{2}}(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$

$\boxed{\text{when } x=-4}$

$\log_{\sqrt{2}}(-4+4) - \log_2\left(\frac{-4}{2}+2\right) = 1$

Notice: $-4+4=0$

Thus -4 is out of domain of log function.

$\boxed{\text{Now when } x=-3}$

$\cancel{\log(-3+4)}^{0} - \log_2\left(\frac{-3}{2}+2\right) = 1$

$-\log_2\left(\frac{1}{2}\right) = 1$

$-\left[\cancel{\log_2 1}^{0} - \cancel{\log_2 2}^{1}\right] = 1$

$1 = 1$

Thus $x=-3$ is the right option.

8. Seismologists use the Richter scale to measure and report the magnitude of earthquake as given by the equation $\boxed{R = \ln I - \ln I_0,}$ where I is the intensity of an earthquake with respect to a minimal or reference intensity I_0(i.e $I = cI_0$, where c is a constant). The reference intensity is the smallest earth movement that can be recorded on a seismograph. If an earthquake in city A recorded of magnitude 8.0 in Richter scale and intensity of the earthquake in city B is the reference intensity, then what is the ratio of intensity of earthquake in city A with respect to city B?

(a) $e^0 : 1$

(b) $e^1 : 2$

(c) $\mathbf{e^8 : 1}$

(d) $e^5 : 1$

(e) $e^8 : 2$

Soln Using the given equation

To find: $\frac{I}{I_0} = ?$

$$\boxed{R = \ln I - \ln I_0}$$

$$\Rightarrow R = \ln \frac{I}{I_0}$$

$$\Rightarrow 8 = \ln \frac{I}{I_0}$$

$$\Rightarrow e^8 = \frac{I}{I_0}$$

$$\Rightarrow \frac{I}{I_0} = \frac{e^8}{1} \Rightarrow e^8 : 1$$

$\therefore$ The ratio of intensity of earthquake in city A w.r.t city B is $e^8 : 1$

9. Suppose that the number of bacteria present in a loaf of rotten bread after t minutes is given by the equation $G(t) = G_0 3^{kt}$, where G_0 represents the number of bacteria at $t = 0$, k is a constant (Given $\ln 730 = 6.59$ and $\ln 3 = 1.09$). If the initial number of bacteria is 1000 and it takes 1 min to increase to 9000 then how long(in minutes) would it take for the bacteria count to grow to 730000(integer value of t)?

(a) 2

(b) 1

(c) **3**

(d) 6

Soln :-

Given :- $G_0 = 1000$

At, $t = 1$ min $\quad G(t) = 9000$

To find :-

At what time (t), $G(t) = 7,30,000$

Solve :-

$$G(t) = G_0 3^{kt} \quad \text{——} \ (1)$$

$$\Rightarrow \frac{G(t)}{G_0} = 3^{kt}$$

At $t = 1$ min

$$\Rightarrow \frac{9\cancel{000}}{1\cancel{000}} = 3^{kt}$$

$$\Rightarrow \ln 3^2 = kt \ln 3$$

$$\Rightarrow 2 \cancel{\ln 3} = kt \cancel{\ln 3}$$

$$\Rightarrow \boxed{k = 2}$$

On substituting the values of K & G_0, equation ① becomes

$G(t) = 1000\ 3^{2t}$; when $G(t) = 7,30,000$, then

$$\Rightarrow \frac{730\cancel{000}}{1\cancel{000}} = 3^{2t}$$

$$\Rightarrow 730 = 3^{2t}$$

$$\Rightarrow \ln 730 = 2t \ln 3$$

$$\Rightarrow t = \frac{\ln 730}{2 \ln 3} = \frac{6.59}{2 \times 1.09}$$

$$\boxed{t = 3 \text{ min}}$$

Thus at $t = 3$ min (integer value) bacteria count would be 7,30,000.

Let c_A and c_B be the luminosity(luminous efficacy) of the bulbs A and B respectively. The bulb A is $f(x)$ times brighter than the B, if $f(x) = 3^{x^2+1}$(i.e $c_A = f(x) \times c_B$), where x is the difference of the magnitude of supply voltage between the bulb A and the bulb B. Answer the questions ~~8 and 9~~ based on above information.

10. If the bulb A is 10 times brighter than the bulb B, then the difference of the magnitude of supply voltage between the two bulbs is

(a) $\sqrt{\log_3 5 - 1}$

(b) $\sqrt{\log_3 10}$

(c) $\sqrt{\frac{\ln 10}{\ln 3}}$

(d) $\sqrt{\mathbf{\log_3 \frac{10}{3}}}$

Soln:-

given: $C_A = 10\, C_B$ — (1)

Luminosity is the measure of brightness

$C_A = C_B \times f(x)$

$C_A = C_B \times 3^{x^2+1}$ — (2)

solve

$10 \cancel{C_B} = \cancel{C_B}\, 3^{x^2+1}$

$\Rightarrow \log_3 10 = (x^2+1) \cancel{\log_3 3}$

$\Rightarrow \log_3 10 = x^2+1$

$\Rightarrow x^2 = \log_3 10 - 1$

$\Rightarrow x = \sqrt{\log_3 10 - 1}$

$\Rightarrow x = \sqrt{\frac{\log 10}{\log 3} - 1}$

$$\Rightarrow x = \sqrt{\frac{\log 10 - \log 3}{\log 3}}$$

$$\Rightarrow x = \sqrt{\frac{\log \frac{10}{3}}{\log 3}}$$

formula: $\log_b a = \frac{\log a}{\log b}$

$$\boxed{x = \sqrt{\log_3 \frac{10}{3}}}$$

The difference b/w the magnitude of 2 bulbs is $\sqrt{\log_3 \frac{10}{3}}$

11. If 4 voltage and 3 voltage are the supply voltages for the bulbs A and B respectively then how many times the bulb A is brighter than the bulb B?
Ans : 9

Soln:- Since x is the difference b/w the supply voltage of A & B, thus

$x = 4 - 3 = 1$

We know that

$$C_A = C_B \times f(x)$$

We have to find $f(x)$

$$f(x) = 3^{x^2+1}$$

$$= 3^{1+1} = 3^2 = 9$$

$$f(x) = 9$$

$$\boxed{C_A = 9\, C_B}$$

$\therefore$ 9 times brighter

12. Find the number of values of x satisfying the equation $(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1$.
Ans: 3

Soln:- Given that:

$$(5x)^{\log_{(5x)^{1/5}}(6x^3-36x^2+66x-35)^{1/5}} = 1$$

Using derived formula

$$\log_{a^b} c = \frac{1}{b}\log_a c$$

$$(5x)^{5\log_{5x}(6x^3-36x^2+66x-35)^{1/5}} = 1$$

Formula:

$$a^{\log_a x} = x$$

$$\Rightarrow (5x)^{\log_{5x}(6x^3-36x^2+66x-35)^{\cancel{5}/\cancel{5}}} = 1$$

$$\Rightarrow 6x^3 - 36x^2 + 66x - 35 = 1$$

$$\Rightarrow 6x^3 - 36x^2 + 66x - 36 = 0$$

$$\Rightarrow 6(x^3 - 6x^2 + 11x - 6) = 0$$

$$\Rightarrow x^3 - 6x^2 + 11x - 6 = 0$$

Hit & trial method

when $x = 1$

$$1 - 6 + 11 - 6 = 0$$
$$0 = 0$$

Synthetic division to find other roots

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
x-1 & x^3 - 6x^2 + 11x - 6 \\
 & \cancel{x^3} - x^2 \\
 & (-) \quad (+) \\
\hline
 & -5x^2 + 11x - 6 \\
 & -5x^2 + 5x \\
 & (+) \quad (-) \\
\hline
 & 6x - 6 \\
 & 6x - 6 \\
 & (-) \quad (+) \\
\hline
 & 0
\end{array}
$$

Factorizing to get other roots

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 2x - 3x + 6 = 0$

$\Rightarrow x(x-2) - 3(x-2)$

$\Rightarrow (x-3)(x-2) = 0$

$x - 3 = 0$ or $x - 2 = 0$

$\underline{\underline{x = 2, 3, 1}}$

We got 3 values all together.

13. Which of the following is/are true?
(MSQ), (Ans:(a), (c))

- ◯ If m and n are positive real numbers, then $m^{\log(n)} = n^{\log(m)}$.
- ◯ $\log_5 12345678999999999999999$ is an rational number.
- ◯ The function $f(x) = \log_{10}(x^2 + x + 1)$ is one-one on the interval $(-0.5, \infty)$.
- ◯ None of the above.

Answer

(a) Given m & n are positive real number then $m^{\log(n)} = n^{\log(m)}$

Taking log on both side

$$\cancel{\log n \cdot \log} m = \cancel{\log m} \cdot \log n$$

$$1 = 1$$

$$LHS = RHS$$

(b) Consider $a = 123456789\ldots9$

If $\log_5 a$ is a rational number then it can be represented in p/q form

$$\log_5 a = \frac{p}{q}$$

$$a = 5^{p/q}$$

$$a^q = 5^p$$

But 5 cannot divide a, thus $a^q = 5^p$ is never going to be satisfied. Thus it must be irrational.

(c) Given: $f(x) = \log_{10}(x^2+x+1)$ is one-one in interval $(-0.5, \infty)$.

Soln:-

$f(x) = g(q(x))$; $q(x) = x^2+x+1$; $g(x) = \log_{10} x$

Plotting $q(x)$

- y-intercept $q(0) = 1$
- No x-intercept as discriminant < 0
  $\left[\begin{array}{c} b^2-4ac < 0 \\ -3 < 0 \end{array}\right]$
- coefficient of $x^2 > 0$ thus parabola opens upward.
- Vertex $(-b/2a, q(-b/2a))$
  $(-0.5, 0.75)$
- A rough diagram is shown in Figure 3.

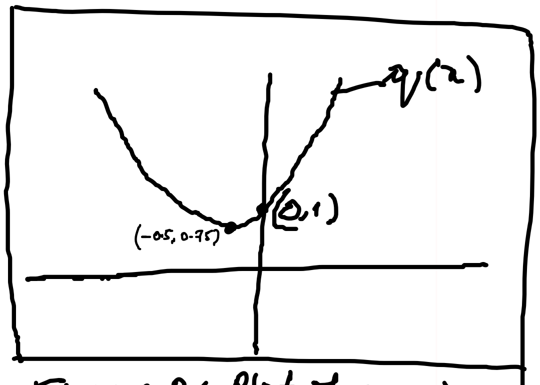


Figure 3: Plot of $q(x)$

Plotting $f(x)$

$f(x) = \log_{10}(x^2+x+1)$

- Minimum value $f(x)$ will be at minimum value of $q(x)$
  $\to f(-0.5) = \log_{10}(.75) = -0.125$
- x-intercept ($f(x)=0$)
  This will happen when
  $x^2+x+1 = 1 \Rightarrow x(x+1) = 0$
  $x = 0, \quad x = -1$

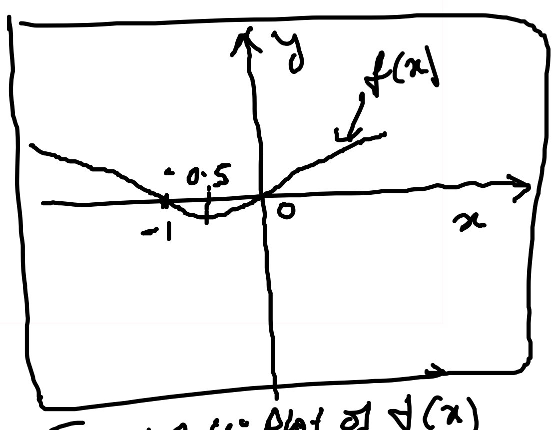


Figure 4: Plot of $f(x)$

- Notice $f(x) > 0$ for all values of x except values between 0 and 1
- Rough plot of $f(x)$ is shown in figure 4
- Clearly, $f(x)$ is strictly increasing function in the domain $(-0.5, \infty)$ and thus it is one-one fun on in domain $(-0.5\ \infty)$.

14. Which of the following is/are true?
(MSQ), (Ans:(a), (b), (c), (d))

- ◯ Suppose D is an arbitrary subset of $\mathbb{R}$ and f is one-one function on D. $\log f(x)$ whenever defined is also an one-one function on D.
- ◯ $(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}}$, where ! denotes the factorial function, and for a non negative integer n, the value of $n!$ is $n \times (n-1) \times ... \times 2 \times 1$.
- ◯ The function $f(x) = 2^x + 3^x + ... + 100^x$ is one-one function on $\mathbb{R}$.
- ◯ There exists a function $f(x)$ on $\mathbb{R}$, such that $\log(f(x)) \geq 100$ for all $x \in \mathbb{R}$.

(a) Given: f is one-one function on D.
Also, $\log f(x)$ is defined on D (given)
We know that log function is one-one function and therefore log of an one-one function (strictly increasing or decreasing) will give one-one function.

(b) Given: $(14!)^{\frac{1}{14}} < (15!)^{\frac{1}{15}}$
Taking $\log_{14!}$ on both side, we have

only if

$$\frac{1}{14}\log_{14!} 14! < \frac{1}{15}\log_{14!} 15!$$

$$\frac{1}{14} < \frac{1}{15}\log_{14!}(15 \times 14!)$$

$$\frac{1}{14} < \frac{1}{15}\left[\log_{14!} 15 + \log_{14!} 14!\right]$$

$$\frac{15}{14} < \log_{14!} 15 + 1 \Rightarrow \frac{15}{14} - 1 < \log_{14!} 15$$

$$\frac{1}{14} < \log_{14!} 15 \Rightarrow 1 < 14\log_{14!} 15$$

$$\Rightarrow 1 < \log_{14!}(15)^{14} \Rightarrow \boxed{14! < 15^{14}}$$

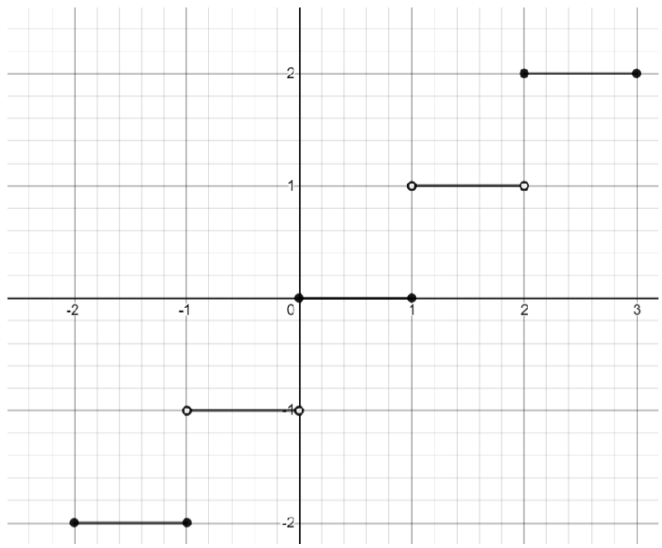
which is true.

(c) We know that, exponential function of a natural number (except 0 & 1) is strictly increasing function & thus one-one function.
Algbraic sum (linear combination) of such exponential function is also one-one function.

d) by properl of log fn.
over any range of a defined f
logf exists

Week 7 Graded Assignment Question
Mathematics for Data Science - 1

1. A graph is shown in Figure M1W8A-8.1, ∘ symbol signifies that the point is not included in the graph of $f(x)$ and the • symbol signifies that the point is included in the graph of $f(x)$.
 Choose the correct option.



M1W8A-8.1

- ◯ The graph cannot be a function, because it fails the vertical line test.
- ◯ The graph cannot be a function, because it passes the horizontal line test but fails the vertical line test.
- ◯ **The graph can be a function, because it passes the vertical line test.**
- ◯ The graph cannot be a function, because it passes the vertical line test but fails the horizontal line test

Solution
First, to check the given graph represents a function, we have to use vertical line test. Now, $x = c$ where c is a constant, shown already(in grid vertical lines) in the figure M1W8A-8.1, crosses the graph once (including • and ∘ per definition). Therefore, the given graph represents a function.

2. For $y = x^n$,where n is a positive integer and $x \in \mathbb{R}$, which of the following statement is true?

   - ◯ For all values of n, y is not a injective function.
   - ◯ For all values of n, y is an injective function.
   - ◯ y is not a function.
   - ◯ **If n is an even number, then y is not an injective function. If n is an odd number, then y is an injective function.**

Solution:
To check y is a function for all positive integer, let there exist two elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$x_1 = x_2$$

taking the power n on both sides, we get,

$$\implies x_1^n = x_2^n$$

$$\implies y(x_1) = y(x_2).$$

Observe that for single input we get a unique output. So, $y = x^n$ is a function for all $n \in \mathbb{Z}^+$.(For vertical line test, see Figure M1W8AS-8.1 and M1W8AS-8.2.
If $n > 1$ is odd positive integer then graph of the function is similar to Figure M1W8AS-8.1, where vertical and horizontal lines are for vertical and horizontal line test respectively.

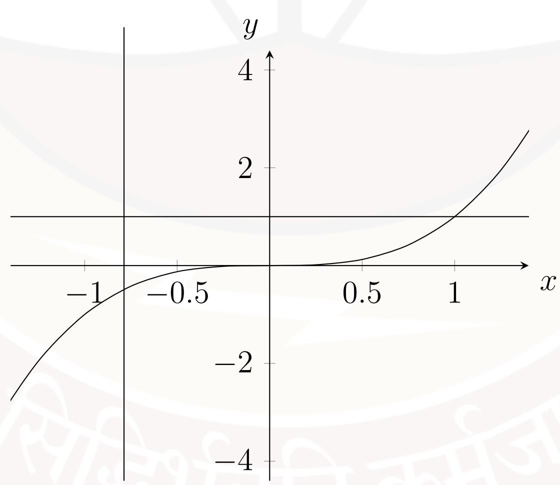


Figure M1W8AS-8.1

If n is even positive integer then graph of the function is similar to Figure M1W8AS-8.2, where vertical and horizontal lines are for vertical and horizontal line test respectively).

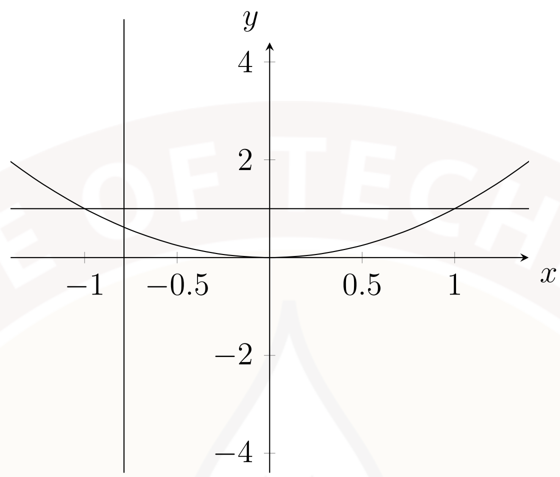


Figure M1W8AS-8.2

So, third option is not correct.
If $n = 1$ which is a positive integer then given function becomes $y = x$.
Using horizontal line test, this function is one-to-one.
So, first option is not correct.
If $n = 2$ which is an even positive integer then given function becomes $y = x^2$.
Using the horizontal line test, this function is not one-to-one(For horizontal line test, see the Figure M1W8AS-8.2).
So, second option is not true.
Now, to check for odd positive integer n, $y = x^n$ is injective. Let $n = 2m + 1$ be an odd integer and $m \in \mathbb{Z}$ (For horizontal line test, see the Figure M1W8AS-8.1). Let there exist two distinct elements in the domain $x_1, x_2 \in \mathbb{R}$ such that

$$f(x_1) = f(x_2)$$

$$\implies x_1^{2m+1} = x_2^{2m+1}$$

Taking the power $\frac{1}{2m+1}$ on both sides we get,

$$x_1 = x_2.$$

From above we see that as no two distinct elements in the domain give the same image. Hence $x_1, x_2 \in \mathbb{R}$ can't be distinct. So, this shows that this function is injective.
Now, to check for even positive integer $n, y = x^n$ is injective or not. Let $n = 2m$ be an even integer and $m \in \mathbb{Z}$, $y = x^{2m} = (x^2)^m$ (See Figure M1W8AS-8.2).
Now, for $x = a, -a$, we get same output $y = a^m$.
Therefore, for even positive integer $n, y = x^n$ not one to one function.
Hence, fourth option is correct.

3. If $4m - n = 0$, then the value of

$$(\frac{16^m}{2^n} + \frac{27^n}{9^{6m}})$$

is

Answer: **2**

Solution:

Given $4m - n = 0$.

Now

$$\begin{aligned} & \frac{16^m}{2^n} + \frac{27^n}{9^{6m}} \\ = & \frac{(2^4)^m}{2^n} + \frac{(3^3)^n}{(3^2)^{6m}} \\ = & \frac{2^{4m}}{2^n} + \frac{3^{3n}}{3^{2\times 6m}} \\ = & 2^{4m-n} + 3^{3n-12m} \\ = & 2^{4m-n} + 3^{-3(4m-n)} \\ = & 2^0 + 3^0 \\ = & 1 + 1 = 2 \end{aligned}$$

4. Half-life of an element is the time required for half of a given sample of radioactive element to change to another element.The rate of change of concentration is calculated by the formula $A(t) = A_o(\frac{1}{2})^{(\frac{t}{\gamma})}$ where γ is the half-life of the material, A_o is the initial concentration of the radioactive element in the given sample, $A(t)$ is the concentration of the radioactive element in the sample after time t.
If Radium has a half-life of 1600 years and the initial concentration of Radium in a sample was 100%, then calculate the percentage of Radium in that sample after 2000 years.

- ◯ 35%
- ◯ **42%**
- ◯ 19%
- ◯ 21%

Solution
Given $A(t) = A_o(\frac{1}{2})^{(\frac{t}{\gamma})}$ and half life of radium is 1600 i.e $\gamma = 1600$

$$\implies A(2000) = A_o(\frac{1}{2})^{(\frac{2000}{1600})}$$

$$= A_o(\frac{1}{2})^{(\frac{5}{4})}$$

At initial time($t = 0$) the concentration of Radium is 100% i.e at initial time the concentration of Radium is A_0
So, the percentage of Radium in that sample after 2000 years is

$$(\frac{A_o(\frac{1}{2})^{(\frac{5}{4})}}{A_0} \times 100)\% = ((\frac{1}{2})^{(\frac{5}{4})} \times 100)\% \approx 42\%$$

5. If $f(x) = (1-x)^{\frac{1}{2}}$ and $g(x) = (1-x^2)$, then find the domain of the composite function $g \circ f$.

- ○ $\mathbb{R}$
- ○ $\boldsymbol{\big((-\infty, 1] \cap [-2, \infty)\big) \cup (-\infty, -2)}$
- ○ $[1, \infty)$
- ○ $\boldsymbol{\mathbb{R} \setminus (1, \infty)}$

Answer: Option 2, Option 4

Solution:

Given $f(x) = (1-x)^{\frac{1}{2}}$.

To define $f(x)$, $1 - x \geq 0 \implies x \leq 1$.

So, the domain of $f(x)$ is $(-\infty, 1] = \mathbb{R} \setminus (1, \infty)$.

Now, the domain of $g(x) = (1-x^2)$ is $\mathbb{R}$ and the range of $f(x)$ is $[0, \infty)$.

Hence, when we use the two rules as in the video lecture to determine the domain of $g \circ f$.

Here, the domain of $g \circ f$ = The domain of $f = (-\infty, 1] = \mathbb{R} \setminus (1, \infty)$.

Now, $\big((-\infty, 1] \cap [-2, \infty)\big) \cup (-\infty, -2) = [2, 1] \cup (-\infty, -2) = (-\infty, 1]$.

Hence, second and fourth option is true.

6. Find the domain of the inverse function of $y = x^3 + 1$.

   ◯ $\mathbb{R}$
   ◯ $\mathbb{R} \setminus \{1\}$
   ◯ $[1, \infty)$
   ◯ $\mathbb{R} \setminus [1, \infty)$

   Answer: Option 1
   Solution:
   We know that the domain of the inverse of a given function is the range of the given function.
   Since the range of the function $y = x^3 + 1$ is $\mathbb{R}$, domain of the inverse function of $y = x^3 + 1$ is $\mathbb{R}$.
   Hence, first option is correct.

7. If $f(x) = x^2$ and $h(x) = x - 1$, then which of the following options is(are) correct?

   - ◯ $f \circ h$ is not an injective function.
   - ◯ $f \circ h$ is an injective function
   - ◯ $f(f(h(x))) \times h(x) = (x-1)^4$
   - ◯ $f(f(h(x)) \times h(x) = (x-1)^5$

Solution:
Given $f(x) = x^2$ and $h(x) = x - 1$.
Using horizontal line test, $f \circ h = (x-1)^2$ is not a injective function.
Again, $f(f(h(x))) \times h(x) = ((x-1)^2)^2 \times (x-1) = (x-1)^{2\times 2+1} = (x-1)^5$.
Hence, first and fourth options are correct.

8. If $f(x) = x^3$, then which of the following options is the set of points where the graphs of the functions $f(x)$ and $f^{-1}(x)$ intersect each other?

   - ◯ {(-1,1),(0,0),(1,-1)}
   - ◯ {(-2,-8),(1,1),(2,8)}
   - ◯ **{(-1,-1),(0,0),(1,1)}**
   - ◯ {(-2,-8),(0,0),(2,8)}

   Solution:
   Let $g(x) = x^{\frac{1}{3}}$ be a function such that $f \circ g = (x^{\frac{1}{3}})^3 = x$ and $g \circ f = (x^3)^{\frac{1}{3}} = x$
   Hence, g is the inverse function of f.
   To get intersection point

$$
\begin{aligned}
& f = g \\
\implies & x^3 = x^{\frac{1}{3}} \\
\implies & x^9 = x \\
\implies & x^9 - x = 0 \\
\implies & x(x^8 - 1) = 0 \\
\implies & x((x^4)^2 - 1) = 0 \\
\implies & x(x^4 + 1)(x^4 - 1) = 0 \\
\implies & x(x^4 - 1)(x + 1)(x - 1)(x^2 + 1) = 0
\end{aligned}
$$

   Observe that for real value 0, -1, 1, $f = g$ and $g(0) = 0, g(-1) = -1, g(1) = 1$.
   It follows that the set of points where the graphs of the functions $f(x)$ and $f^{-1}(x)$ intersect each other is {(-1,-1),(0,0),(1,1)}

9. In a survey, the population growth in an area can be predicted according to the equation $\alpha(T) = \alpha_o(1 + \frac{d}{100})^T$ where d is the percentage growth rate of population per year and T is the time since the initial population count α_o was taken. If in 2015, the population of Adyar was 30,000 and the population growth rate is 4% per year, then what will be the approximate population of Adyar in 2020?($T = 0$ corresponds to the year 2015, $T = 1$ corresponds to the year 2016 and so on..)

   - ◯ 60251
   - ◯ 71255
   - ◯ 91000
   - ◯ **36500**

   Solution:
   Given $\alpha(T) = \alpha_o(1 + \frac{d}{100})^T$ and initial population count $\alpha_o = 30000$.
   The population growth rate $d = 4\%$ per year and $T = 5$.
   Hence, the approximate population of Adyar in 2020 is $\alpha(5) = 30000 \times (1 + \frac{4}{100})^5 \approx 36500$

10. An ant moves along the curve whose equation is $f(x) = x^2 + 1$ in the restricted domain $[0, \infty)$. Let a mirror be placed along the line $y = x$. If the reflection of the ant with respect to the mirror moves along the curve $g(x)$, then which of the following options is(are) correct?

- ◯ $\boldsymbol{g(x) = f^{-1}(x)}$
- ◯ $g(x) = f(x)$
- ◯ $\boldsymbol{g(x) = \sqrt[2]{(x-1)}}$
- ◯ $g(x) = \sqrt[2]{(x+1)}$

Solution:
Given $f(x) = x^2 + 1$ in the restricted domain $[0, \infty)$.
Using horizontal test, given function f is one to one function(See Figure M1W8AS-8.3).

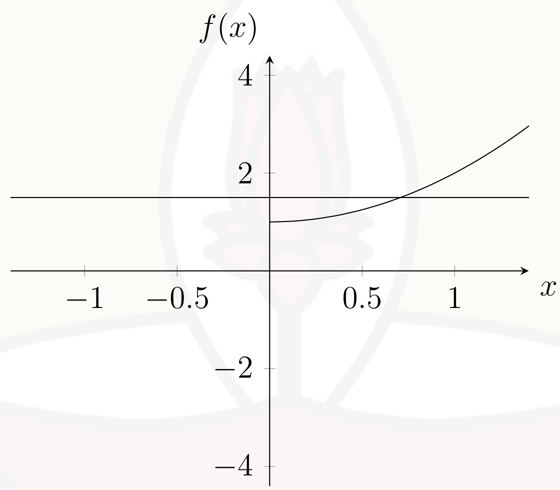


Figure M1W8AS-8.3

Hence, f is reversible function.
We know that the graph of the inverse of a function is symmetric along the line $y = x$ and the mirror placed along the the line $y = x$.
It follows that $g(x)$ is inverse of the function f i.e $g(x) = f^{-1}(x)$.
Now, consider a function $f^{-1}(x) = \sqrt{x-1}$.
Since, $f \circ f^{-1} = (\sqrt{x-1})^2 + 1 = x = I$, similarly, $f^{-1} \circ f = x = I$, where I is the identity function.
Therefore, inverse function of $f(x) = x^2 + 1$ is $f^{-1}(x) = \sqrt{x-1}$,
where f^{-1} is just a notation of inverse function.
Hence, first and third options are correct.

11. Suppose a textile shop provides two different types of offers during a festival season. The first offer(D_1) is "shop for more than ₹14,999 and pay only ₹9,999". The second offer(D_2) is "avail 30% discount on the total payable amount". If Shalini wants to buy two dresses each of which costs more than ₹8,000 and she is allowed to avail both offers simultaneously, then which of the following options is(are) correct?

    - ◯ The minimum amount she should pay after applying two offers cannot be determined because the exact values of the dresses she wanted to buy are unknown.
    - ◯ **The minimum amount she should pay after applying the two offers simultaneously is approximately ₹6, 999.3.**
    - ◯ The amount she is supposed to pay after applying D_2 only is ₹11,199.
    - ◯ **The amount she is supposed to pay after applying $\boldsymbol{D_1}$ only is ₹9, 999.**
    - ◯ **Suppose the total payable amount is ₹17, 999 for the two dresses. In order to pay minimum amount Shalini should avail offer $\boldsymbol{D_1}$ first and offer $\boldsymbol{D_2}$ next.**
    - ◯ **Suppose the total payable amount is ₹17, 999 for the two dresses. If Shalini avails offer $\boldsymbol{D_2}$ first, then she cannot avail offer $\boldsymbol{D_1}$.**
    - ◯ Suppose the total payable amount is ₹17,999 for the two dresses. In order to pay minimum amount Shalini should avail offer D_2 first and offer D_1 next.

    Solution:
    Given each dress cost more than ₹8000.
    If Shalini wants to buy two dresses then total payable amount is greater than ₹16000 which is greater than ₹14999.
    So she can avail offer D_1.
    If she avails offer D_1 first and offer D_2 next, then the total payable amount = ₹$9999(1 - \frac{30}{100}) \approx$ ₹6999
    If she avails offer D_2 first, then the total payable amount can be less than ₹14999 or greater than or equal to ₹14999. After that she may or not avail the offer D_1. If she avails offer D_1 after D_2, then also the total payable amount can not be less than ₹6999.
    In any case the minimum amount she should pay after applying the two offers(without any order) simultaneously is at least ₹6, 999.
    Hence, first option is not correct and second option is correct.
    Since the total payable amount is unknown, therefore we can not say how much she needs to pay after applying the offer D_2.
    Hence, third option is not correct.
    As we see above if Shalini avails offer D_1 only, then payable amount = ₹9999.
    Hence, fourth option is correct.
    Suppose the total payable amount is ₹17, 999 for the two dresses (which is greater than ₹14999).
    If Shalini avails offer D_2 first then payable amount = ₹$17999(1 - \frac{30}{100}) \approx$ ₹12599 which

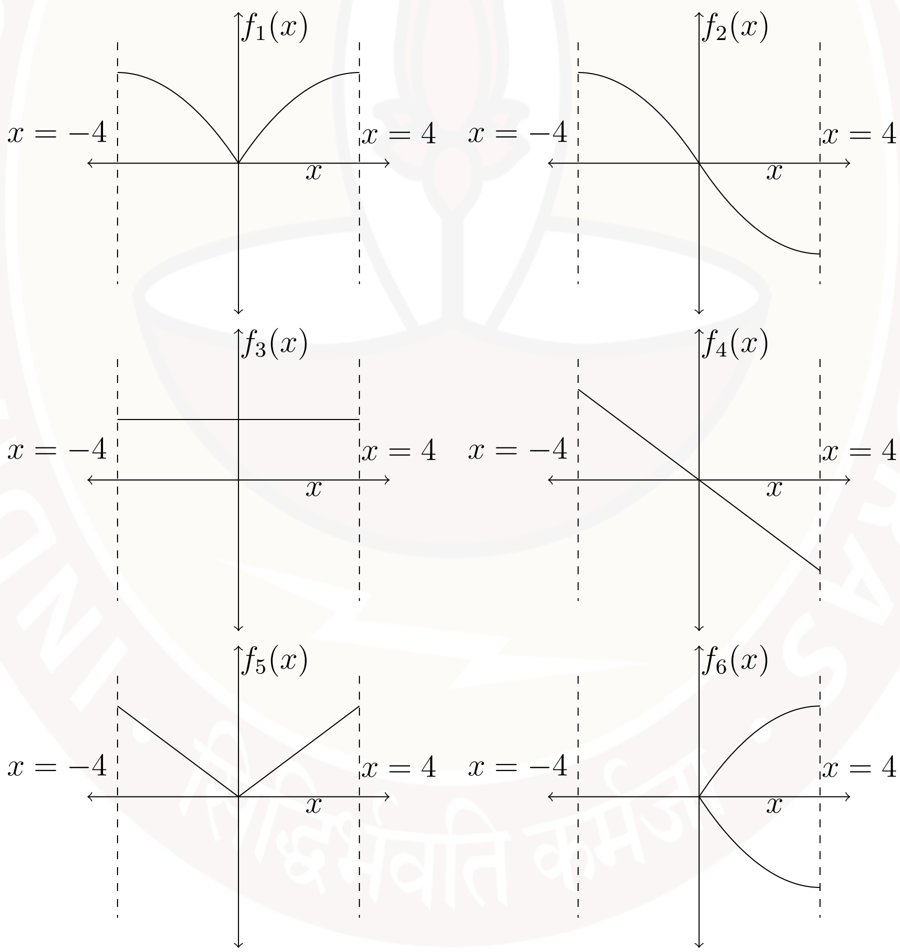
is less than ₹14999.
So, she can not avail offer D_1 next and she has to pay $\approx$ ₹12599.
Hence, sixth option is correct and seventh option is not correct.
And if Shalini avails offer D_1 first then payable amount = ₹9999
and then offer D_2 then total payable amount = ₹$9999(1 - \frac{30}{100}) \approx$ ₹6999
Hence, from above in order to pay minimum amount Shalini should avail offer D_1 first and offer D_2 next.
So, fifth option is correct.

Week - 7
Practice assignment Solution
Exponential Functions
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

Answer the questions 1, 2, and 3 based on the given graphs.



Domain for each one is $[-4, 4]$.

1. Choose the correct option.

   - ◯ f_3 is not a function.
   - ◯ **f_6 is not a function.**
   - ◯ f_5 is not a function.
   - ◯ All of the above are functions.

   Solution:

   Vertical line test fails only for f_6 and therefore $f_6(x)$ is not a function.

2. Choose the correct option.

   - ◯ f_1 and f_3 are one-one functions in the given domain.
   - ◯ **f_2 and f_4 are one-one functions in the given domain.**
   - ◯ f_3 and f_5 are one-one functions in the given domain.
   - ◯ f_5 is one-one function in the given domain.

   Solution:

   The function f_2 and f_4 are strictly decreasing function in the domain [-4, 4], therefore these are one to one functions.

   Or

   The functions f_2 and f_4 are the only functions which satisfy the conditions of horizontal and vertical line tests in the domain [-4, 4], therefore these are one to one functions.

3. Choose the correct option.

   - ◯ f_1 and f_5 are strictly increasing functions in the given domain.
   - ◯ **f_2 and f_4 are strictly decreasing functions in the given domain.**
   - ◯ f_4 and f_5 are strictly decreasing functions in the given domain.
   - ◯ f_5 is strictly increasing function in the given domain.

   Solution:

   A function $f(x)$ is said to be strictly decreasing on a given interval if $f(b) < f(a)$ for all $b > a$, where $a,\ b$ belong to the domain. On the other hand, if $f(b) <= f(a)$ for all $b > a$, then the function is said to be simply decreasing function.
   Clearly from the given graph, f_2 and f_4 are strictly decreasing functions in the domain [-4, 4].

Use the following information for the questions 4 and 5.
Let N_0 be the number of atoms of a radioactive material at the initial stage i.e., at time $t = 0$, and $N(t)$ be the number of atoms of the same radioactive material at a given time t, which is given by the equation $N(t) = N_0 e^{-\lambda t}$, where λ is the decay constant.

4. If at time t_1, the number of atoms reduces to the half of N_0 and at the time t_2 the number of atoms reduces to the one fourth of N_0, then which one of the following equations is correct?
   - ◯ $e^{\frac{t_1}{t_2}} = 2$
   - ◯ $e^{\frac{t_2}{t_1}} = 2$
   - ◯ $\boldsymbol{e^{\lambda(t_2 - t_1)} = 2}$
   - ◯ $e^{\lambda(t_1 - t_2)} = 2$

   Solution:

   According to the question, at t_1,
   $$N(t) = \frac{N_0}{2}$$
   According to the equation,
   $$N(t) = N_0\ e^{-\lambda t}$$
   Therefore for $t = t_1$,
   $$\frac{1}{2} \times N_0 = N_0\ e^{-\lambda t_1}$$
   $$\frac{1}{2} = e^{-\lambda t_1} \tag{1}$$
   It is also given that at t_2, $N = \frac{N_0}{4}$
   $$\frac{1}{4} \times N_0 = N_0\ e^{-\lambda t_2}$$
   $$\frac{1}{4} = e^{-\lambda t_2} \tag{2}$$
   On dividing (1) by (2) we get,
   $$e^{\lambda(t_2 - t_1)} = 2$$

5. If $N_{\frac{1}{\lambda}}$ is the number of atoms at time $t = \frac{1}{\lambda}$, then what is the ratio of N_0 to $N_{\frac{1}{\lambda}}$?
   - ◯ $1 : e$
   - ◯ $\boldsymbol{e : 1}$
   - ◯ $1 : e^{-\lambda}$

◯ $1 : e^{\lambda}$

Solution:

It is given that at $t = \frac{1}{\lambda}$, $N = N'$

$$N' = N_0 \ e^{-\frac{\lambda}{\lambda}}$$

$$N' = \frac{N_0}{e}$$

$$\frac{N_0}{N'} = \frac{e}{1}$$

Therefore,

$N_0 : N' = e : 1$

2 Multiple Select Questions (MSQ):

6. Selvi deposits ₹P in a bank A which provides an interest rate of 10% per year. After 10 years, she withdraws the whole amount from bank A and deposits it in another bank B for n years which provides an interest rate of 12.5% per year. $M_A(x)$ represents the amount in Selvi's account after x years of depositing in bank A. $M_B(y)$ represents the amount in Selvi's account after y years of depositing in bank B. If the interests are compounded yearly, then choose the set of correct options.

   - ○ $\boldsymbol{M_A(x)}$ **is an one-one function of** $\boldsymbol{x}$**, for** $\boldsymbol{x \in (0, 10)}$**.**
   - ○ $\boldsymbol{M_B(y)}$ **is an one-one function of** $\boldsymbol{y}$.
   - ○ $M_A(12) = P \times 1.1^{12}$
   - ○ $\boldsymbol{M_A(12) = 0}$
   - ○ $\boldsymbol{M_A(x)}$ **is a strictly increasing function of** $\boldsymbol{x}$**, for** $\boldsymbol{x \in (0, 10)}$**.**
   - ○ $M_B(y)$ is a decreasing function of y.
   - ○ $\boldsymbol{M_B(n) = (P \times 1.1^{10}) \times (1.125)^n}$
   - ○ $M_B(n) = (P \times 1.1^n) \times (1.125)^{10}$

Solution:

When the principal amount P is compounded annually, the amount M after q years is given by

$$M = P \times \left(1 + \frac{\text{Interest rate}}{100}\right)^q$$

Amount $M_A(x)$ after x years in bank A will be

$$M_A(x) = P \times \left(1 + \frac{10}{100}\right)^x$$

So after 10 years the amount $M_A(10)$ will be

$$M_A(10) = P \times (1.1)^{10}$$

As Selvi has withdrawn all the amounts from bank A after 10 years so amount left in bank A after 12 years will be $M_A(12) = 0$.

After 10 years the new principal amount $P \times (1.1)^{10}$ is deposited in another bank B, so for any years y the amount will be $M_B(y)$ which is given by

$$M_B(y) = P \times (1.1)^{10} \times \left(1 + \frac{12.5}{100}\right)^y$$

So for n years

$$M_B(n) = P \times (1.1)^{10} \times (1.125)^n$$

Clearly $M_A(x)$ and $M_B(y)$ are strictly increasing functions therefore both are one-to-one functions of x and y respectively.

Use the following information for questions 7 and 8.

There are two offers in a shop. In the first offer, the discount in total payable amount is $M(n)\%$ if the number of products bought at a time is n. The second offer involves a discount of ₹1000 on the total payable amount. If Geeta shops of ₹15,000, then answer the following questions.

7. If the total payable amounts after applying the first and second offers (one at a time) are represented by the functions $f(n)$ and $g(n)$ respectively and the total payable amount after applying both the offers together is represented by $T(n)$, then choose the set of correct options.

   - ◯ $f(n) = (100 - M(n)) \times 15000$ and $g(n) = 14000$
   - ◯ $f(n) = (100 - M(n)) \times 1500$ and $g(n) = (100 - M(n)) \times 15000 - 1000$
   - ◯ $\boldsymbol{f(n) = (100 - M(n)) \times 150}$ and $\boldsymbol{g(n) = 14000}$
   - ◯ $T(n) = (100 - M(n)) \times 15000$ is the total payable amount when the first offer is applied after the second.
   - ◯ $\boldsymbol{T(n) = (100 - M(n)) \times 140}$ **is the total payable amount when the first offer is applied after the second.**
   - ◯ $\boldsymbol{T(n) = (100 - M(n)) \times 150 - 1000}$ **is the total payable amount when the second offer is applied after the first.**

   Solution:

   It is given that total payable amount without any offer is ₹15,000. Then, total payable amount after first offer is

$$f(n) = \frac{(100 - M(n))}{100} \times 15,000 = (100 - M(n)) \times 150$$

   And total payable amount if second offer is applied will be

$$g(n) = 15,000 - 1000 = ₹14,000.$$

   Now, total payable amount when the first offer is applied after the second will be

$$T(n) = \frac{100 - M(n)}{100} \times g(n)$$

$$T(n) = \frac{(100 - M(n))}{100} \times 14000 = (100 - M(n)) \times 140$$

And total payable amount when the second offer is applied after the first will be

$$T(n) = f(n) - 1000$$

$$T(n) = \frac{(100 - M(n))}{100} \times 15000 - 1000 = (100 - M(n)) \times 150 - 1000$$

8. If Geeta is allowed to use the offer in any sequence and $M(n) = -n^2 + 18n - 72$, where $n \in \{6, 7, 8, 9\}$, then choose the set of correct options which minimizes the total payable amount.

   ◯ Total payable amount is same irrespective of the order in which the offers are applied.

   ◯ **She should choose offer one and then offer two i.e., $\boldsymbol{gof(M(n))}$.**

   ◯ She should choose offer two and then offer one i.e. $gof(M(n))$.

   ◯ **If she chooses offer one and then offer two, the minimum payable amount will be ₹12650.**

Solution:

Total payable amount when she choose offer one and then offer two is

$$T_1(n) = (100 - M(n)) \times 150 - 1000$$

It is given that $M(n) = -n^2 + 18n - 72$, so

$$T_1(n) = (100 - (-n^2 + 18n - 72)) \times 150 - 1000$$

On solving we get,

$$T_1(n) = 150n^2 - 2700n + 24800$$

And total payable amount when she chooses offer two and then offer one is

$$T_2(n) = (100 - M(n)) \times 140$$

On substituting $M(n)$ and solving we get,

$$T_2(n) = 140n^2 - 2520n + 24080$$

Since the coefficient of n^2 is positive for both $T_1(n)$ and $T_2(n)$ therefore minimum value i.e., minimum payable amount of these function can be calculated as follows
For $T_1(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2700)}{2 \times 150} = 9$$

The minimum payable amount will be

$$T_1(9) = 150(9)^2 - 2700(9) + 24800 = ₹12{,}650$$

For $T_2(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2520)}{2 \times 140} = 9$$

The minimum payable amount will be

$$T_2(9) = 140(9)^2 - 2520(9) + 24080 = ₹12{,}740$$

Thus if she chooses offer one and then offer two, the minimum payable amount will be ₹12,650.

n	$T_1(n)$ ₹	$T_2(n)$ ₹
6	14000	14000
7	13250	13300
8	12800	12880
9	12650	12740

Table: M1W8PAS-1

From Table: M1W8PAS-1, it is clear that for all the values of n the total payable amount is lower for $T_1(n)$ as compared to $T_2(n)$ therefore she should choose offer one and then offer two.
Note: This can be also identified by plotting the graph for $T_1(n)$ and $T_2(n)$.

3 Numerical Answer Type (NAT):

Use the following information for questions 9-15.
Given two real valued functions $f(x) = \frac{5x+9}{2x}$, $g(y) = \sqrt{y^2 - 9}$. If $h(x) = f(g(x))$, then answer the following questions.

9. If domain of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $\mathbb{R} \setminus (-n, n)$ respectively, then find the value of $m + n$. [Ans: 3]
Solution:

At $x = 0$ the function $f(x) \to \infty$ or the function is undefined at $x = 0$ thus the domain of $f(x)$ is $\mathbb{R} \setminus 0$.
We can also write the domain as $(-\infty, 0) \cup (0, \infty)$ therefore, $m = 0$.

It is given that $g(y) = \sqrt{y^2 - 9}$ on changing the variable in terms of x we get $g(x) = \sqrt{x^2 - 9}$.

$g(x)$ will be defined when $x^2 - 9 \geq 0$. On solving

$$x^2 \geq 9$$

$$x \geq 3$$

or

$$x \leq -3$$

Thus the domain will be $\mathbb{R} \setminus (-3, 3)$, hence $n = 3$. So, $m + n = 0 + 3 = 3$

10. If range of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $[n, \infty)$ respectively, then find the value of $2(m + n)$. [Ans: 5]

Solution:
As $f(x)$ is defined everywhere except 0, therefore there will be an asymptote at $x = 0$.
If we draw a graph of $f(x)$:
End behaviour:
As $x \to \infty$, $f(x) \to \frac{5}{2}$.
As $x \to -\infty$, $f(x) \to \frac{5}{2}$.
The end behaviours show that the function has another asymptote at $f(x) = y = \frac{5}{2}$.

Intercept:

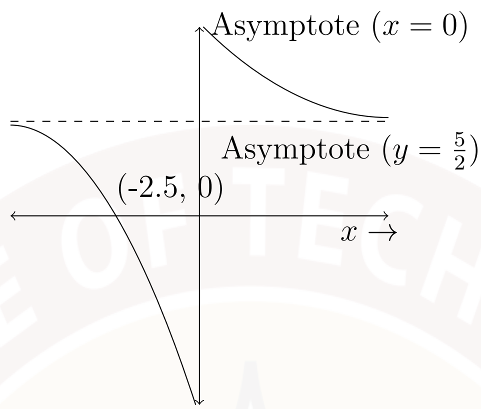
$$f(x) = 0 \implies \frac{5x + 9}{2x} = 0$$

$$x = -\frac{9}{5}$$

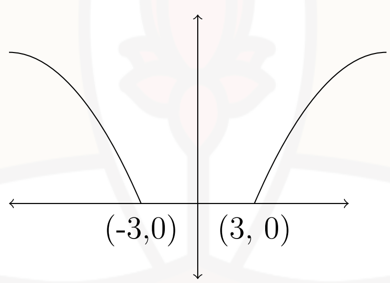
It means $f(x)$ might change the sign at $x = -\frac{9}{5}$.
For $-\infty < x < 0$, $f(x)$ will have value from $-\infty$ to $\frac{5}{2}$.

Similarly for $0 < x < \infty$, $f(x)$ will have value from $\frac{5}{2}$ to ∞.

Therefore the range of $f(x)$ is $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$. A rough diagram of $f(x)$ is shown below.



As $g(x) = \sqrt{x^2 - 9}$ is a positive square root function so it will have only the positive values including zero at $x = 3$ and $x = -3$.
A rough diagram is created using the facts that the $g(x)$ is not defined from (-3, 3) and at $x = 3$ the function gives the value zero. At ∞ the function provides the value ∞. As the quadratic function involved and the $b = 0$ the function will be symmetric about $y-$axis.



Therefore the range will be $[0, \infty)$. Thus $m = 2.5$ and $n = 0$, so,

$$2(m + n) = 2(2.5 + 0) = 5$$

11. If domain of $h(x)$ is $(-\infty, -3) \cup (m, \infty)$, then find the value of m. [Ans: 3]

Solution:
Given,

$$h(x) = f(g(x))$$
$$h(x) = f(\sqrt{x^2 - 9})$$
$$= 2.5 + \frac{4.5}{\sqrt{x^2 - 9}}$$

There are two possibilities when the function is undefined. Firstly when the denominator is zero and secondly when the function in square root provides negative value. It means

$\sqrt{x^2-9} \neq 0$ and $x^2 - 9 \geq 0$.
Combining both the conditions we can say the function is defined only when

$$x^2 - 9 > 0$$

$$x^2 > 9 \implies -3 < x < 3$$

Thus the domain will be $(-\infty, -3) \cup (3, \infty)$, hence $m = 3$.

12. If domain of $f^{-1}(x)$ is $(-\infty, m) \cup (m, \infty)$, then find the value of $2m$. [Ans: 5]

Solution:
Given that $f(x) = \frac{5x+9}{2x}$ let us say $f(x) = y$ so $y = \frac{5x+9}{2x}$ on rearranging,

$$y = \frac{5}{2} + \frac{9}{2x}$$

$$\frac{2y-5}{2} = \frac{9}{2x}$$

$$x = \frac{9}{2y-5}$$

Therefore $f^{-1}(x) = \frac{9}{2x-5}$. This function will be defined when

$$2x - 5 \neq 0$$

$$x \neq \frac{5}{2}$$

The domain of this function is $(-\infty, 2.5) \cup (2.5, \infty)$ thus $m = 2.5$ therefore $2m = 5$

13. If $f^{-1}(5) = 9/m$, then find the value of m. [Ans: 5]

Solution:
$f^{-1}(5) = \frac{9}{2\times5-5} = \frac{9}{5}$, thus $m = 5$.

Week - 6
Assignment
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Select Questions (MSQ):

Consider the following graphs:

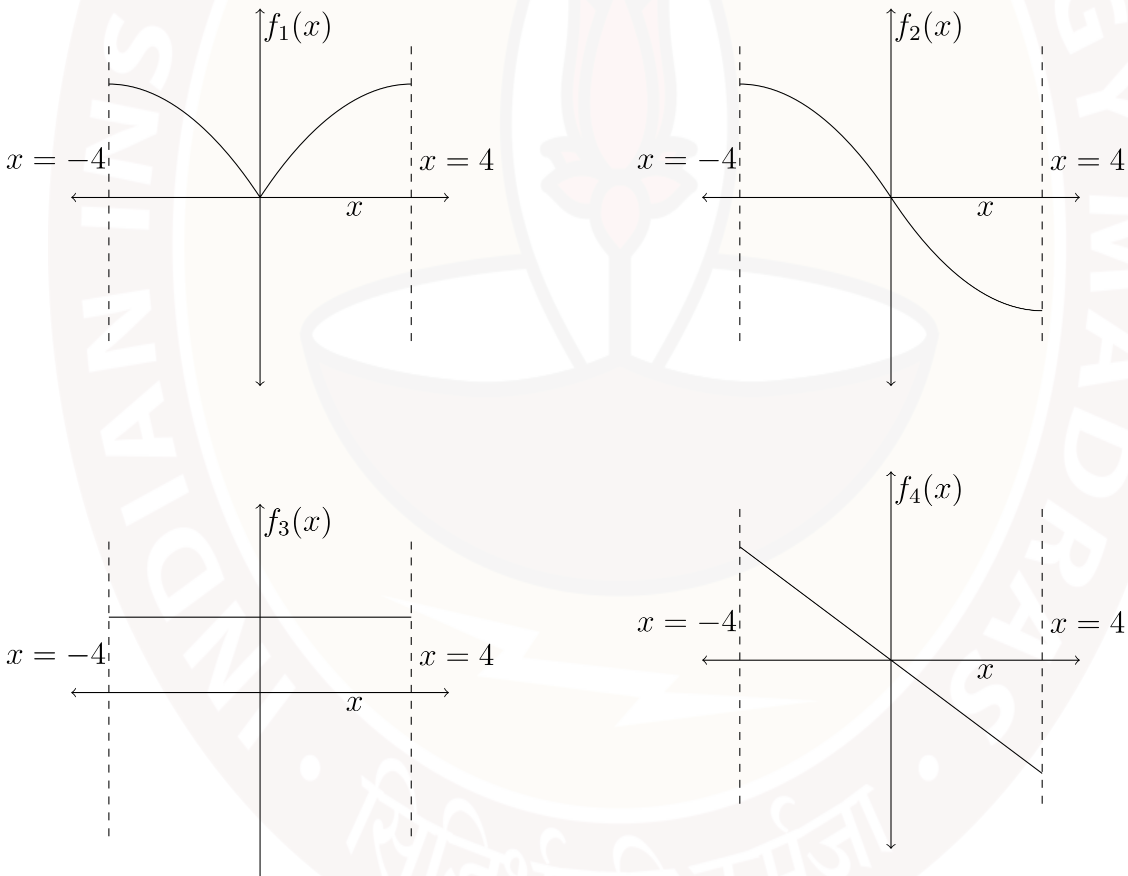
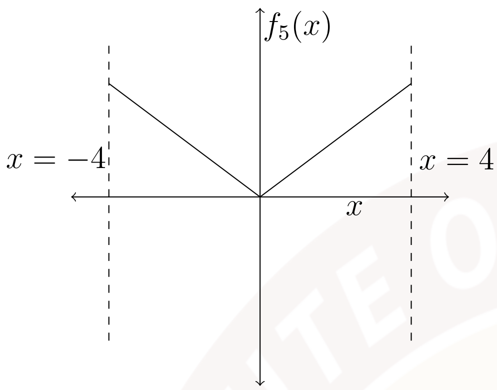
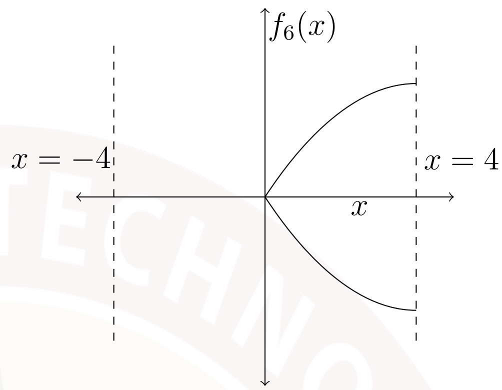


Figure: A-7.4





Domain for each one is $[-4, 4]$.

1. Find out the correct answers.
   - ◯ **f_1 is an even function in the given domain.**
   - ◯ f_2 is an even function in the given domain.
   - ◯ **f_3 is an even function in the given domain.**
   - ◯ f_4 is an even function in the given domain.
   - ◯ **f_5 is an even function in the given domain.**
   - ◯ f_6 is an even function in the given domain.

2. Find out the correct answers.
   - ◯ f_1 is an odd function in the given domain.
   - ◯ **f_2 is an odd function in the given domain.**
   - ◯ f_3 is an odd function in the given domain.
   - ◯ **f_4 is an odd function in the given domain.**
   - ◯ f_5 is an odd function in the given domain.
   - ◯ f_6 is an odd function in the given domain.

Solution:
A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ and is said to be an odd function if $f(-x) = -f(x)$. Now, $f(-x) = f(x)$ if and only if the graph of the function $f(x)$ is symmetric about Y- axis and $f(-x) = -f(x)$ if and only if the graph of the function $f(x)$ is symmetric about the origin.
Since the graphs of $f_1(x), f_3(x)$ and $f_5(x)$ are symmetric about Y- axis, $f_1(x), f_3(x)$ and $f_5(x)$ are even functions in the given domain.
Similarly, the graphs of $f_2(x)$ and $f_4(x)$ are symmetric about origin, $f_2(x)$ and $f_4(x)$ are odd functions in the given domain.
In the Figure: A-7.6, since we get two outputs corresponding to one input, it follows that $f_6(x)$ is not a function.

2 Multiple Choice Questions (MCQ):

A box has width which is 3 cm more than double the length, and has height which is 2 cm more than thrice the length of the box. The box is filled with small bricks each of whose lengths is one fourth of the length of the box. The width of each brick is 1 cm more than six times the length of the brick, and the height of each brick is 1 cm more than $\frac{8}{3}$ times the length of the brick.

3. Assuming x (in cm) to be the length of the box, what is the volume (in cubic cm) of the box?

   ◯ $6x^3 + 6$

   ◯ $6x^3 + 5x^2 + 6x$

   ◯ $\boldsymbol{6x^3 + 13x^2 + 6x}$

   ◯ $6x^2 + 13x^3 + 6$

4. The maximum number of bricks can be kept in the box is

   ◯ 4

   ◯ 6

   ◯ 12

   ◯ **24**

Solution:

Let x be the length of the box (in cm). So width of the box will be $2x + 3$ and height will be $3x + 2$.
Also, length of a brick $= \frac{x}{4}$,
width of a brick $= 6 \times \frac{x}{4} + 1 = \frac{3x}{2} + 1 = \frac{3x+2}{2}$,
height of a brick $= \frac{8}{3} \times \frac{x}{4} + 1 = \frac{2x}{3} + 1 = \frac{2x+3}{3}$.
So, volume of a brick = length $\times$ width $\times$ height
i.e volume of a brick$(V_{br}) = \frac{x}{4} \times (\frac{3x}{2} + 1) \times (\frac{2x}{3} + 1) = \frac{x(3x+2)(2x+3)}{24}$.
Also, volume of the box $(V_b) = x \times (2x + 3) \times (3x + 2) = 6x^3 + 13x^2 + 6x$.
Now, the maximum number of bricks $= \frac{V_b}{V_{br}} = \frac{x(2x+3)(3x+2)}{\frac{x(3x+2)(2x+3))}{24}} = 24$.

5. The consumption of new plastics in year x after opening a company is given as a polynomial $N(x)$ (in tonnes). The company also recycles the used plastics and regenerates them for use. The regenerated amount of plastic in year x after opening the company is given as the polynomial $R(x)$ (in tonnes). These polynomials are known to be applicable for the first 15 years of the company's functioning.
Use the following notes to solve the question:

- $N(x) = -0.005x^4 + 0.2x^3 - 3x^2 + 14x + 70$
- $R(x) = 0.005x^4 - 0.1x^3 + x^2 + x$
- $P(x) = 0.01x^4 - 0.3x^3 + 4x^2 - 13x - 70$ has exactly two real roots.
- $Q(x) = 0.01x^3 - 0.2x^2 + 2x + 7$ has exactly one real root and it is negative.
- $Q(x)$ is a factor of $P(x)$.

When will the company regenerate more plastic than it would have consumed? (Years from opening the company.)

- ◯ After 4 years.
- ◯ After 6 years.
- ◯ After 8 years.
- ◯ **After 10 years.**
- ◯ Never
- ◯ None of the above.

Solution:
We have to find the value of x for which the polynomial $R(x) - N(x) > 0$.
Now, from the given data $R(x) - N(x) = P(x)$ and $Q(x)$ divides $P(x)$.
So,

$$\begin{aligned}
P(x) &= 0.01x^4 - 0.3x^3 + 4x^2 - 13x^2 - 70 \\
&= (0.01x^4 - 0.2x^3 + 2x^2 + 7x) + (-0.1x^3 + 2x^2 - 20x - 70) \\
&= x(0.01x^3 - 0.2x^2 + 2x + 7) - 10(0.01x^3 - 0.2x^2 + 2x + 7) \\
&= (x - 10)(0.01x^3 - 0.2x^2 + 2x + 7)
\end{aligned}$$

Now, as given $P(x) = 0.01x^4 - 0.3x^3 + 4x^2 - 13x - 70$ has exactly two real roots and $Q(x)$ has exactly one real root which is negative.
Hence, $P(x)$ has only one positive real root which is $x = 10$.
That means after 10 years, the company will regenerate more plastic than its consumption of new plastic.

6. Given that $p(x) = (2x^2 + mx + 8)(4x^2 + nx + 1)$, M is the set of values of m, N is the set of values of n, and $C = M \cap N$. If $p(x)$ always has four real distinct root, then choose the correct option.

   - ○ $M = \{z \mid z \in (-\infty, -4) \cup (4, \infty)\}$
   - ○ $\boldsymbol{C = \{z \mid z \in (-\infty, -8) \cup (8, \infty)\}}$
   - ○ $M = \{8, -8\}$
   - ○ $C = \{4, -4\}$
   - ○ $M = \{z \mid z \in \mathbb{R}\}$

   Solution:
   Given $p(x) = (2x^2 + mx + 8)(4x^2 + nx + 1)$.
   So, $p(x)$ has degree 4.
   Now, $p(x) = (2x^2 + mx + 8)(4x^2 + nx + 1) = 0$
   $\implies (2x^2 + mx + 8) = 0$ or $(4x^2 + nx + 1) = 0$.
   Now, for four distinct real root of the $p(x)$, the discriminant $D_1, D_2 > 0$ for both quadratic equations $(2x^2 + mx + 8) = 0$ and $(4x^2 + nx + 1) = 0$ respectively.
   Now, discriminant of the quadratic equation $(2x^2 + mx + 8) = 0$ is

$$D_1 = m^2 - 4 \times 2 \times 8 > 0$$
$$\implies m^2 > 64$$
$$\implies |m| > 8.$$

   In interval form $m \in (-\infty, -8) \cup (8, \infty)$.
   Since M is the set of values of m so $M = \{z \mid z \in (-\infty, -8) \cup (8, \infty)\}$.
   Similarly, discriminant of the quadratic equation $(4x^2 + nx + 1) = 0$ is

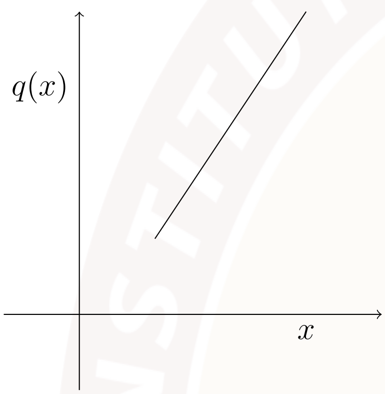
$$D_2 = n^2 - 4 \times 4 \times 1 > 0$$
$$\implies n^2 > 16$$
$$\implies |n| > 4.$$

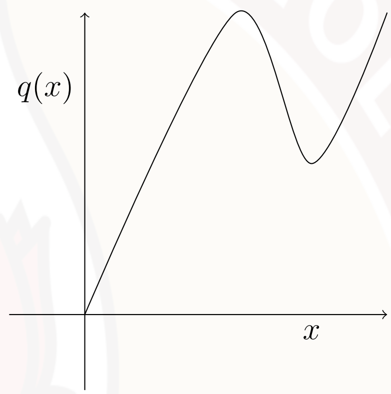
   In interval form $n \in (-\infty, -4) \cup (4, \infty)$.
   Since N is the set of values of n so $N = \{z \mid z \in (-\infty, -4) \cup (4, \infty)\}$.
   Now, $C = M \cap N = ((-\infty, -8) \cup (8, \infty)) \cap ((-\infty, -4) \cup (4, \infty)) = (-\infty, -8) \cup (8, \infty)$.
   In set notation form $C = \{z \mid z \in (-\infty, -8) \cup (8, \infty)\}$.

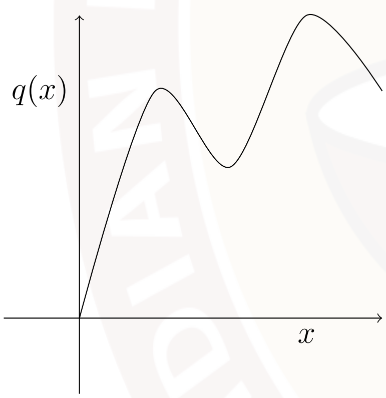
Suppose there are three polynomial functions with the following properties:

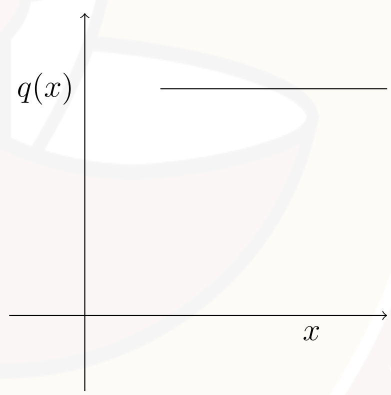
- $deg\ f(x) > deg\ g(x)$
- $deg\ f(x) = deg\ h(x) = 5$
- $h(x)$ divides $f(x)$.

7. Which of the following graphs is most suitable to represent $q(x) = \frac{f(x)}{h(x)}$?
   [Answer: 4th Option]

◯ 

◯ 

◯ 

◯ 

8. What will be the maximum possible degree of the remainder when $f(x)+g(x)$ is divided by $h(x)$?
   - ◯ 5
   - ◯ 4
   - ◯ $\boldsymbol{deg\ g(x)}$
   - ◯ $(deg\ g(x)) - 1$

Solution (a):
Since $h(x)$ divides $f(x)$, the remainder is zero. Further $q(x)$ is the quotient. It follows that

$$f(x) = q(x)h(x) + 0.$$

Now,

$$deg\ q(x)h(x) = deg\ q(x) + deg\ h(x) = deg\ f(x) = deg\ h(x) \implies deg\ q(x) = 0.$$

So, $q(x)$ is a constant function i.e $q(x) = c$ where c is a constant.
Now, in options (a), (b) and (c), the value of $q(x)$ varies with x. So, these options are false.
In option (d), the value of $q(x)$ remains constant as x is varying. So, this option is true.
Solution (b):
Since $h(x)$ divides $f(x)$ and $deg\ g(x) < deg\ f(x) = deg\ h(x)$.
So, while dividing $f(x) + g(x)$ by $h(x)$, $g(x)$ will be the remainder.
Hence, the maximum possible degree of the remainder when $f(x) + g(x)$ is divided by $h(x)$ is $deg\ g(x)$.

3 Numerical Answer Type (NAT):

9. A company tracks their profits with respect to number of years (t) from the year of establishment. At the end of the second year (i.e. $t = 2$) the company registers neither profit nor loss. The same situation arises at the end of fifth and seventh year. If the equation relating the profit (in thousands) and the number of years t, is a cubic polynomial in t, with leading coefficient being 20. What will be the profit (in thousands) at the end of 10 years?

   Answer: 2400

   Solution:

   Let $p(t)$ be the cubic polynomial denoting the profit of the company with respect to the number of years(t) from the year of establishment.
   Since $p(t)$ is cubic polynomial, the maximum number of possible roots is 3.
   At the end of second year (i.e $t = 2$) the company register neither profit nor loss. So $p(2) = 0$.
   Similarly $p(5) = p(7) = 0$.
   Hence, 2, 5 and 7 are zeroes of the given polynomial and

$$p(t) = a(t-2)(t-5)(t-7) = a(t^3 - 14t^2 + 59t - 70)$$

   where a is stretch factor.
   Since 20 is leading coefficient of the given polynomial $p(t)$, we have $a = 20$.
   Hence, $p(t) = 20(t-2)(t-5)(t-7)$.
   Now, $p(10) = 20 \times 8 \times 5 \times 3 = 2400$.
   Hence, the profit of the company after 10 years is 2400.

A roller coaster ride follows the curve defined by the polynomial function $f(x) = 9x^3 + 9x^2 - x - 1$ (considering some fixed horizontal reference plane) in the domain $[-1, 0.5]$.

10. How many bends are there in the roller coaster?
Answer: 2

11. In the given domain, how many times does the roller coaster reach the same height as that of the starting point (i.e. for $x = -1$)(taking $x = -1$ in to the account)?
Answer: 3

Solution:
Given

$$\begin{aligned} f(x) &= 9x^3 + 9x^2 - x - 1 \\ &= 9x^2(x+1) - (x+1) \\ &= (x+1)(9x^2 - 1) \\ \implies f(x) &= (x+1)(3x+1)(3x-1). \end{aligned}$$

So, $f(x)$ has three zeros -1, $\frac{1}{3}, -\frac{1}{3}$ which are in the given domain.
The rough sketch of the graph of $f(x)$ is shown in the Figure M1W7AS- 7.1

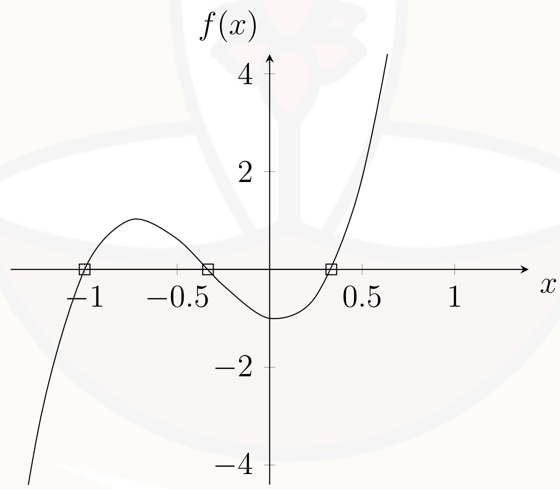


Figure M1W7AS-7.1

From Figure M1W7AS- 7.1 it is clear that that there are two turning points and the number of X- intercepts are 3.
So, for (a): there are two bends in the roller coaster and
for (b): in the given domain, the roller coaster reaches three times the same height as that of the starting point.

A chemical substance A is the reactant in a chemical reaction which gets converted into a product B. The concentrations (in mol/L) of A and B depend on the reaction time t as $C_A(t) = t^3 - t^2 - 21t + 45$ and $C_B(t) = t^3 + t^2 + 22t$ respectively as shown in Figure.

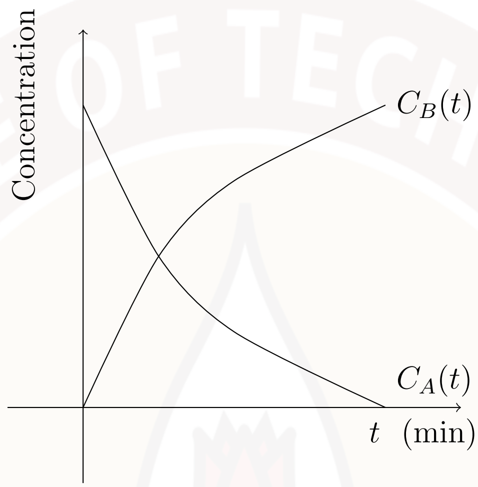


Fig A-7.11

12. At what time t, since the reaction started, the concentrations of A and B would be same?
Answer: 1
Solution
If concentrations of A and B are equal
then

$$
\begin{aligned}
& C_A(t) = C_B(t) \\
\implies & t^3 - t^2 - 21t + 45 = t^3 + t^2 + 22t \\
\implies & 2t^2 + 43t - 45 = 0 \\
\implies & 2t^2 - 2t + 45t - 45 = 0 \\
\implies & 2t(t-1) + 45(t-1) = 0 \\
\implies & (t-1)(2t+45) = 0
\end{aligned}
$$

So, $t = 1$ or $t = -\frac{45}{2}$ but t represents time so t can not be negative. Hence, after 1 min of starting the reaction, the concentrations of A and B become equal.

13. Find the concentration (in mol/L) of either substance when their concentrations become equal.
Answer: 24
Solution
Since at $t = 1$ concentrations of A and B are equal, value of C_A at $t = 1$ is $1-1-21+45 = 24$.

14. Find the concentration (in mol/L) of the product when the concentration of the reactant becomes zero.
Answer: 102
Solution
The concentration of the reactant becomes zero when

$$\begin{aligned}
&C_A(t) = 0 \\
\implies &t^3 - t^2 - 21t + 45 = 0 \\
\implies &t^3 - 3t^2 + 2t^2 - 6t - 15t + 45 = 0 \\
\implies &t^2(t-3) + 2t(t-3) - 15(t-3) = 0 \\
\implies &(t-3)(t^2 + 2t - 15) = 0 \\
\implies &(t-3)(t^2 + 5t - 3t - 15) = 0 \\
\implies &(t-3)^2(t+5) = 0 \\
\implies &t = 3 \ or \ t = -5
\end{aligned}$$

But, t represents time and time can not be negative. So, at $t = 3$ concentration of the reactant becomes zero.
Now, concentration of the product at $t = 3$ is $C_B(3) = 3^3 + 3^2 + 22 \times 3 = 102$.

Week - 6
Practice Assignment
Graphs of polynomials
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

Syllabus Covered:

- Graphs of Polynomials: Identification and Characterization
- Zeroes of Polynomial Functions
- Graphs of Polynomials: Multiplicities
- Graphs of Polynomials: Behavior at X-intercepts
- Graphs of Polynomials: End behavior
- Graphs of Polynomials: Turning points
- Graphs of Polynomials: Graphing & Polynomial creation

1 Multiple Select Questions (MSQ):

1. Figure: M1W7PA-7.1 shows the graph of polynomial $p(x)$. Choose the set of correct options.

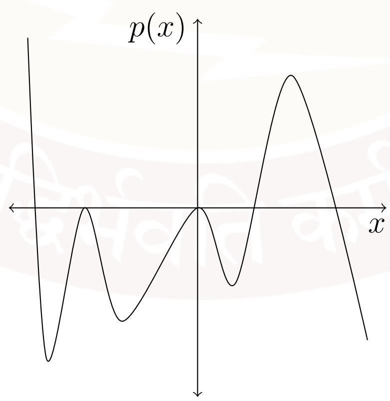


Figure: M1W7PA-7.1

- ◯ The degree of $p(x)$ is minimum 5.
- ◯ **The degree of $p(x)$ is minimum 7.**
- ◯ **x^4 could be a factor of $p(x)$.**
- ◯ $p(x)$ is an odd function.
- ◯ Multiplicity of a positive root of $p(x)$ can be even.
- ◯ **Multiplicities of zero and at least one negative root could be the same.**

Solution:

Option (b): Correct
Let a_1, a_2, a_3, a_4, and a_5 be the points at which the value of $p(x) = 0$ are as shown in Figure: M1W7PAS-7.1. At points a_1, a_4, a_5, the curve crosses in a linear fashion hence the degree should be 1, which accounts for total 3 degrees. At points a_2 and a_3, the curve bounces back, therefore it can have at least 2 degrees each, which accounts for 4 degrees together.
Therefore all together the degree of $p(x)$ is minimum 7.

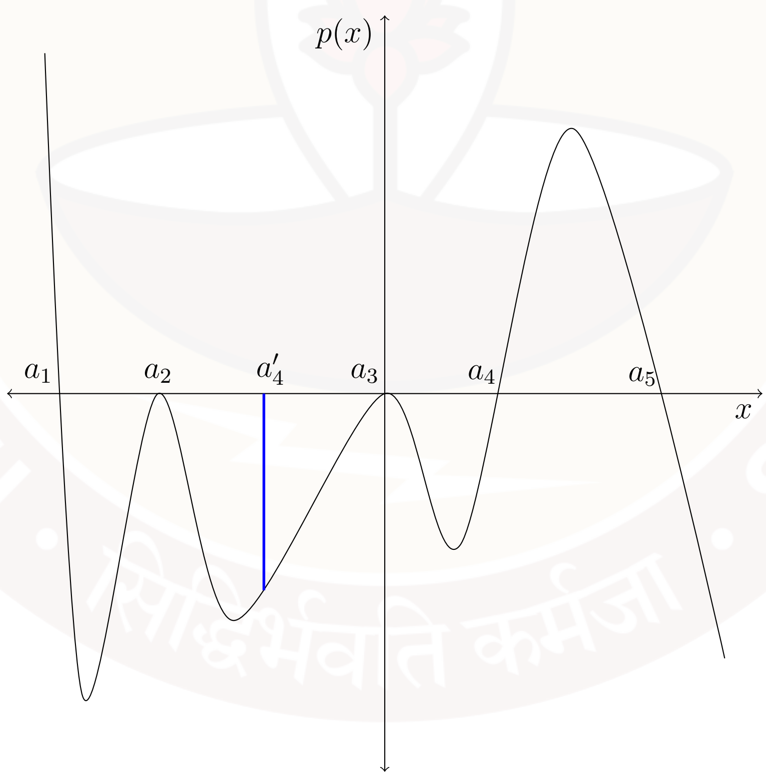


Figure: M1W7PAS-7.1

Option (c): Correct
Since at $x = 0$, the curve bounces back therefore, at point $(0, 0)$ the factor will be of the form x^n where n is an even number. Hence when $n = 4$, x^4 could be a factor of $p(x)$.

Option (d): Incorrect
A function is odd when $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ which means the graph is symmetric with respect to the origin. But it is not the case as in Figure: M1W7PAS-7.1. Therefore, $p(x)$ is not an odd function.

Option (e): Incorrect
Multiplicities of a positive root of $p(x)$ cannot be even because at points a_4 and a_5 the curve crosses in a linear fashion hence the multiplicity should be 1.

Option (f): Correct
At a_2 the root is negative and at a_3 it has zero root and the curve bounces back at both point. Therefore at those points the factor will be of the form x^n where n is an even natural number and they can be same.

2. Choose the correct options.

   ◯ Every function must be either an odd function or an even function.

   ◯ **A function is an even function if $f(x) = f(|x|)$.**

   ◯ **$f(x) = 0$, for all $x \in \mathbb{R}$, is an even function.**

   ◯ Every even degree polynomial is an even function.

Solution:

Option (a): Incorrect
Some functions could be neither odd nor even. For example, $f(x) = x^3 + x^2$ then $f(-x) = -x^3 + x^2$.
It is not an even function because $f(-x) \neq f(x)$ and not an odd function because $f(-x) \neq -f(x)$.

Option (b): Correct
Given, $f(x) = f(|x|) \implies f(x) = f(x)$ or $f(x) = f(-x)$.
Function is an even function when $f(x) = f(-x)$.

Option (c): Correct
As $f(x) = 0$, $f(-x) = 0$ and $f(x) = f(-x)$ therefore it is an even function.

Option (d): Incorrect
Because it may have other terms which will influence the nature of the function.
Example:$f(x) = x^4 + x^3$
It is an example of even degree polynomial but it is not even function because $f(-x) \neq f(x)$ where $f(-x) = x^4 - x^3$

3. The polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ has the following properties:

- $p(x)$ is an even degree polynomial.
- $p(x)$ has at least one positive real root and at least one negative real root.
- $(x-2)^n,\ max(n) = 2$ is a factor of $p(x)$.
- $p(0) \neq 0$

From the options given, choose the the possible representations of $p(x)$.
[Ans: Options C, E]

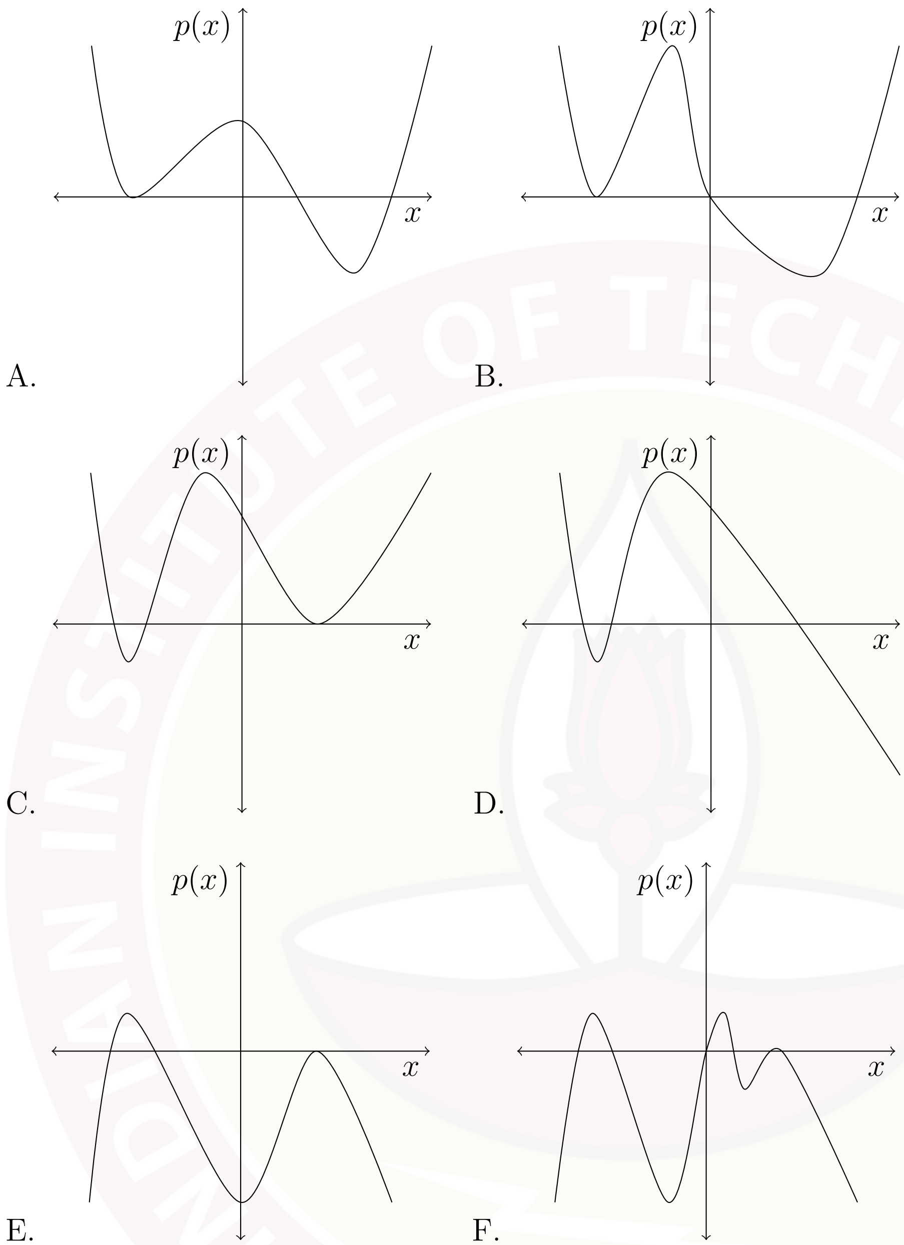


Figure: M1W7PA-7.2

Solution:

According to the condition $(x-2)^n$, $max(n) = 2$, options C, E and F qualify to be right options. But option F does not fulfil the condition $p(0) \neq 0$ condition. Hence options C and E are correct.

2 Multiple Choice Questions (MCQ):

4. Suppose a cubic polynomial f intersects the X-axis at $x = 1$ and $x = -2$. Moreover, $f(x) < 0$ when $x \in (0, 1)$, and $f(x) > 0$ when $x \in (-2, 0)$. Find out the equation of the polynomial.

   - ◯ Inadequate information.
   - ◯ $a(x^3 - x^2 - 2x),\ a > 0$
   - ◯ $\boldsymbol{a(x^3 + x^2 - 2x),\ a > 0}$
   - ◯ $a(x^3 + 3x^2 - x - 3),\ a < 0$

   Solution:

   A cubic polynomial can have at most three roots. It is given that f intersects the X-axis at $x = 1$ and $x = -2$ which accounts for the two factors $(x - 1)$ and $(x - (-2))$. Therefore, the equation is of the form $f(x) = a(x - b)(x - 1)(x + 2)$ where a and b are constants.
   Based on the end behavior of $f(x)$ two possible rough diagrams are shown in Figure: M1W7PAS-7.2.

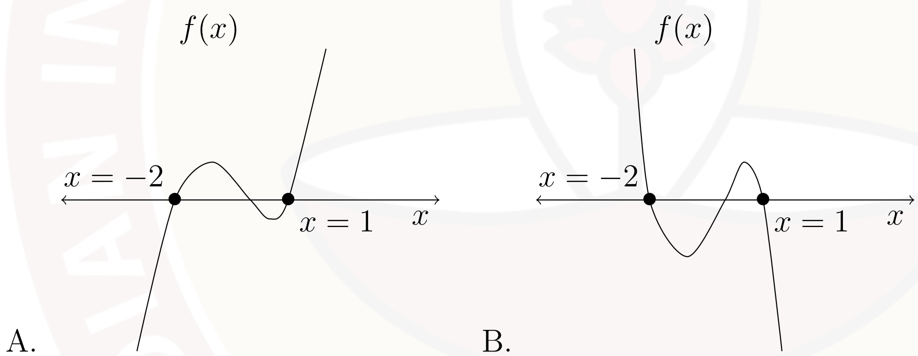


Figure: M1W7PAS-7.2

Also $f(x) < 0$ when $x \in (0, 1)$, $f(x) > 0$ when $x \in (-2, 0)$ so only A in Figure: M1W7PAS-7.2 can represent the function. When $x \to \infty$, $f(x) = \infty$, and $x \to -\infty$, $f(x) = -\infty$, shows that $a > 0$.

Clearly, the function is changing the sign at $x = 0$ which means $(x - 0)$ is a factor of $f(x)$ which means $b = 0$. Therefore,
$f(x) = a(x)(x - 1)(x + 2) \implies \boldsymbol{f(x) = a(x^3 + x^2 - 2x).}$

5. The volume of a box V, varies with some variable x as $V(x) = x^3 + 12x^2 + 39x + 28$ cubic metres. If $(x + a)$ metre is the measurement of one side of the box, then choose the correct option for a.

 - ◯ $a = -1, 5, 3$
 - ◯ $a = 1, 5, 3$
 - ◯ $a = -7, -4, 1$
 - ◯ $a = 7, 2, 2$
 - ◯ $\boldsymbol{a = 7, 1, 4}$
 - ◯ $a = 28, 1, 1$

Solution:

We know that the volume of a box is determined by multiplying the lengths of sides of the box. If $(x + a)$ is the measurement of one side, then it will be a factor of the volume polynomial.
By hit and trial method, one of the roots of $V(x) = x^3 + 12x^2 + 39x + 28$ is -1. Hence $(x + 1)$ is one of the factors of the cubic polynomial $V(x)$. On dividing $V(x)$ by $(x + 1)$, we get $x^2 + 11x + 28$ which on factorization gives the other factors $(x + 7)$ and $(x + 4)$. Thus the possible values of a are 1, 4, and 7.

Use the following information for questions 6 and 7.
Ankita has to travel to various locations for advertising her company's products. The company reimburses her expenses such as accommodation, food etc. The company also blacklists an employee whenever the employee's expenditure in a given month exceeds ₹ 9000. The accounts department fits the data of Ankita's monthly expenditure to a polynomial $E(x)$ (in ₹) where x is the number of months since her joining the company. The polynomial fit is known to be applicable for a period of two years.

6. If $E(x) - 9000 = a(x-4.5)(x-12)(x-20),\ a > 0$, then how many times has Ankita been black listed in two years?

   - ◯ 7
   - ◯ 4
   - ◯ **11**
   - ◯ 3
   - ◯ 15
   - ◯ 9

   Solution:

   Company blacklists an employee whenever employee's expenditure in given month is more than 9000 which means if $E(x) > 9000 \implies E(x) - 9000 > 0$.
   On solving,

$$\begin{aligned} E(x) - 9000 &> 0 \\ a(x-4.5)(x-12)(x-20) &> 0 \end{aligned}$$

   Clearly the zeros of the polynomial are 4.5, 12, and 20. The above polynomial is a cubic polynomial (odd degree polynomial) and $a > 0$, then using end behavior a rough plot is shown in Figure: M1W7PAS-7.3.

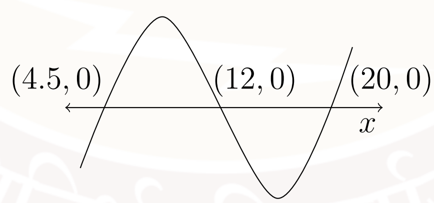


Figure: M1W7PAS-7.3

   From the Figure: M1W7PAS-7.3 clearly $E(x) - 9000 > 0$ when $x \in (4.5, 12)$ and $x \in (20, 24]$ i.e, when the number of months are $x = 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 21,\ 22,\ 23,$ and 24. Therefore, Ankita will be black listed 11 times in two years.

7. Choose the information that is not required to solve question number 6.

   - ◯ End behavior of $E(x)$.
   - ◯ Zeroes of the function.
   - ◯ Degree of the polynomial $E(x)$.
   - ◯ **Exact value of $\boldsymbol{a}$.**

   Solution:

   The end behavior of a polynomial function is the behavior of the graph of $E(x)$ as x approaches $+\infty$ or $-\infty$. So, the knowledge of sign of the leading coefficient is used to predict the end behavior of the function. In question number 6, the fact that $a > 0$ was crucial to solve the problem.

   Degree of the polynomial $E(x)$, is used to determine the maximum number of solutions it can have and also the number of times it will cross the $X-$axis when graphed. In question number 6 the degree is 3.

   Zeroes of the function are critical to determine where the function touches or crosses $X-$axis. In question number 6, three zeroes were given.

   Exact value of a in the question number 6 will simply increase the $y-$coordinate of the vertex value, which is not of concern in the above problem when $a > 0$. Thus this information is not required to solve question number 6.

8. An equipment shows the reading $y(x)$ upon applying load x (in tonnes). Starting from $x = 0$ tonne, the load is steadily increased and thus the reading $y(x)$ is also observed to increase. The first stage of failure is observed at a certain load x_1 after which increasing the load results in a decrease in the reading. The load is continually increased after the first stage of failure and the second stage of failure occurs at load x_2 where the reading reaches 80 and the equipment stops working.

   Use the information provided below and find the maximum load x_2 (in tonnes) that can be applied to this equipment so that it does not stop working.

   Useful information:

   (a) $y(x) - 80 = ax(bx^2 + cx + d)(x + 1)(x - 4)$

   (b) $c^2 - 4bd < 0$

   (c) $ab < 0$

   Choose the correct option.

   - ◯ 0
   - ◯ -1
   - ◯ **4**.
   - ◯ None of the above.

Solution:

As the reading y is a dependent function of x, it can not be less than zero as load x cannot be negative. Initially the equipment reading $y(x)$ increases as the load x increases but it starts to decrease as load increases after the 1^{st} failure i.e, when load $x = x_1$ and stops working when $y(x) = 80$. (see Figure: M1W7PAS-7.4).

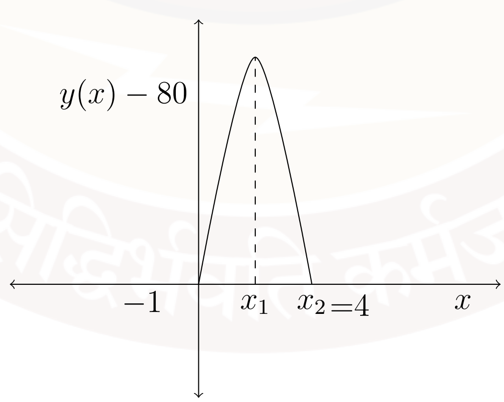


Figure: M1W7PAS-7.4

So the equipment works when $y(x) \geq 80 \implies y(x) - 80 \geq 0$.
On solving,

$$\begin{aligned} y(x) - 80 &\geq 0 \\ ax(bx^2 + cx + d)(x+1)(x-4) &\geq 0 \end{aligned}$$

Since the polynomial is of degree 5, it could have at most 5 roots.
Finding the roots,

$$\begin{aligned} x &= 0 \\ x &= -1 \\ x &= 4 \end{aligned}$$

For other two roots,

$$bx^2 + cx + d = 0$$

Given that $c^2 - 4bd < 0$ which indicates $bx^2 + cx + d$ has no real root. The curve represented by $y(x) - 80$ is shown in the Figure: M1W7PAS-7.4. Clearly, $y(x) - 80 \geq 0$ when $x \leq 4$. Thus maximum load will be $x_2 = 4$ (in tonnes) that can be applied to this equipment so that it does not stop working.

3 Numerical Type Questions (NAT):

9. Let A be the interval $[\alpha, \beta]$, where α and β are the smallest and the largest roots respectively of $f_1(x) = x^4 - 3x^3 - 9x^2 - 3x - 10$. If B is the largest proper subset of A such that elements of B are integers, then what is the cardinality of B? [Ans: 8]

Solution:

By hit and trial method, one of the roots is -2, thus the factor $(x + 2)$ when divides $f_1(x)$ gives $x^3 - 5x^2 + x - 5$. Now we can write it as $x^2(x - 5) + 1(x - 5)$, thus we get $(x - 5)(x^2 + 1)$.
$(x^2 + 1)$ has no real root, therefore the interval of A is all real values in $[-2,\ 5]$. Since B is the largest proper subset of A and it contains only integers, thus the elements of B will be -2, -1, 0, 1, 2, 3, 4, 5 and its cardinality is 8.

10. A train follows a path along the curve $y = x^3 + 12x^2 + 3x$ and Riya is travelling on a path $y = 0$. How many places can Riya catch the train? [Ans: 3]

Solution:

Riya takes the path $y = 0$ which means she is travelling along the $X-$axis. So, Riya can catch the train at $x-$intercepts of the curve $y = x^3 + 12x^2 + 3x$. On factorizing we get $x(x^2 + 12x + 3)$. So x is one factor and other factors can be obtained from $x^2 + 12x + 3$. The discriminant of $x^2 + 12x + 3$ is given by $b^2 - 4ac = 12 \times 12 - 4 \times 1 \times 3 = 132 > 0$, thus it will have 2 real and distinct roots different from 0. So altogether Riya can catch the train at 3 places.

Mathematics for Data Science - 1

Graded assignment solutions

Week - 6

1 Multiple Choice Questions (MCQ):

1. What should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$?

    1. $4x$
    2. $4x - 3$
    3. $6x - 3$
    4. $2x - 3$

    Answer: Option 2.

    Solution:

    Using 4 step division algorithm, we find the remainder when $P(x)$ is divided by $2x^2+x-1$. If we subtract the obtained remainder from $P(x)$ then the resultant polynomial will be divisible by $2x^2 + x - 1$.
    Now,

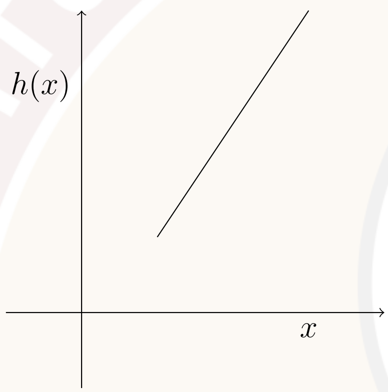
$$
\begin{array}{r|l}
 & \qquad\qquad\quad 3x^2 \quad + x + 1 \\
\cline{2-2}
2x^2 + x - 1 & \; 6x^4 + 5x^3 \qquad\quad + 4x - 4 \\
 & -6x^4 - 3x^3 + 3x^2 \\
\cline{2-2}
 & \qquad\quad 2x^3 + 3x^2 + 4x \\
 & \qquad -2x^3 \; - x^2 \; + x \\
\cline{2-2}
 & \qquad\qquad\quad 2x^2 + 5x - 4 \\
 & \qquad\qquad -2x^2 \; - x + 1 \\
\cline{2-2}
 & \qquad\qquad\qquad\quad 4x - 3
\end{array}
$$

Therefore, when $P(x)$ is divided by $2x^2 + x - 1$, we get $4x - 3$ as the remainder.
Hence, $\boldsymbol{4x - 3}$ should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$.

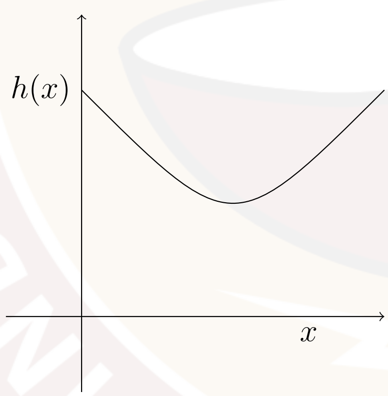
2. Table A-5.1 provides the information regarding some polynomials. Which is the most suitable (not exact) representation of $h(x)$ where $h(x)$ is known to be a polynomial in x, and if $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$?

Polynomial	Degree	Condition
$P(x)$	m	$m > 0$
$Q(x)$	n	$m > 2n > 0$
$R(x)$	k	$k = m - n$
$S(x)$	t	$t = 2n$

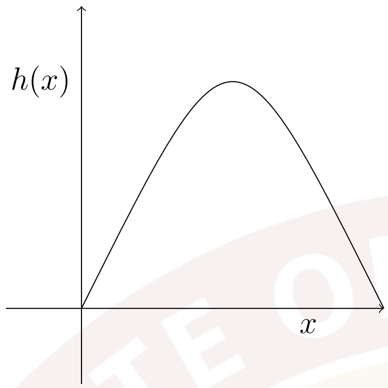
Table A-5.1



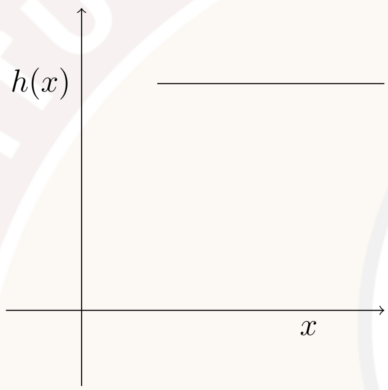
1.



2.



3.



4.

Answer: option 4

Solution:

Given, the degree of $P(x)$ is 'm' where $m > 0$, the degree of $Q(x)$ is 'n' where $m > 2n > 0$, the degree of $R(x)$ is 'k' where $k = m - n$, and the degree of $S(x)$ is 't' where $t = 2n$.

Also, $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$ and $h(x)$ is known to be a polynomial. The degree of $h(x)$ will be the difference between the degree of the numerator and the degree of the denominator. The degree of the numerator will be the degree of the term which has the highest degree in the numerator. Similarly, the degree of the denominator will be the degree of the term which has the highest degree in the denominator.

Now, the degree of the polynomial $P(x)Q(x)$ will be '$m + n$', the degree of $R(x)S(x)$ will be '$k+t = m-n+2n = m+n$', and the degree of $S(x)P(x)$ will be '$t+m = 2n+m$'.

Therefore, the degree of the numerator (polynomial $P(x)Q(x) - R(x)S(x) + S(x)P(x)$) will be '$m + 2n$'.
Similarly, the degree of the denominator (polynomial $P(x) + P(x)Q^2(x)$) will be $m+2n$.

As $h(x)$ is given to be a polynomial and also the degrees of the polynomials in the numerator and the denominator are same, we can conclude that the degree of $h(x)$ is zero i.e. $h(x)$ should be a constant.
So, option 4 is the most suitable representation of $h(x)$.

Use the following information to solve questions 3-5.

A manufacturing company produces three types of products A, B, and C from one raw material in a single continuous process. This process generates total solid wastes (W) (in kg) as $W(r) = -0.0001r^3 + 0.1r^2 + r$, where r is the amount of raw material used in kg. If instead, the company uses three different batch-processes (one batch process for one product) to produce the above products, then the amount of waste generated because of products A, B, and C are given as $W_A = -0.00001r^4 + 0.015r^3$, $W_B = -0.005r^3 + 0.05r^2$ and $W_C = 0.05r^2$ respectively. (See the Figure A-5.1 for the reference.)

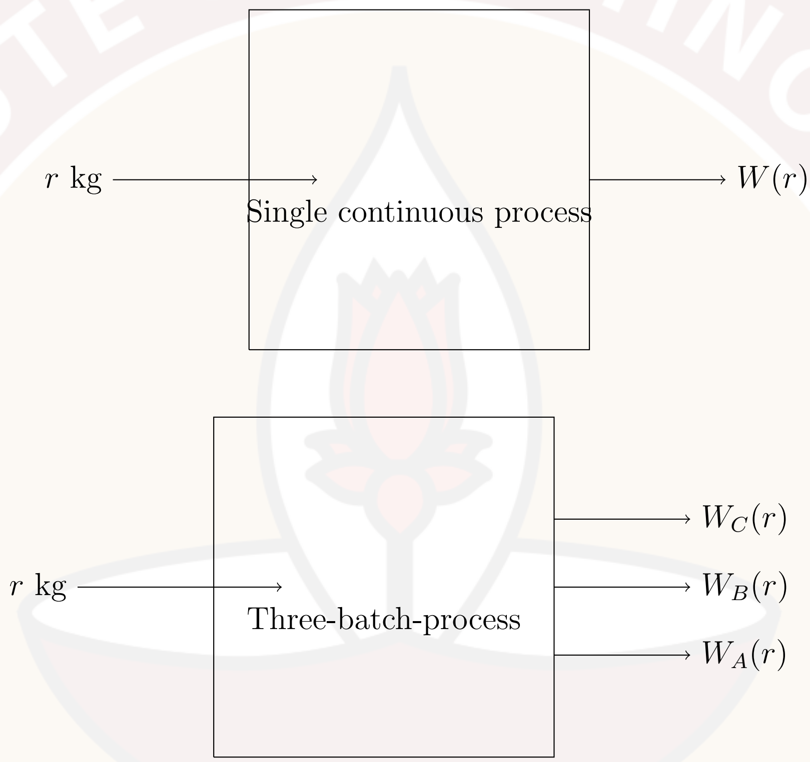


Figure A-5.1

3. What is the total amount of waste generated because of the three different batch-processes?

   1. $-0.00001r^4 + 0.01r^3 + 1.5r^2$
   2. $-0.00001r^4 + 0.015r^3 + 1.5r^2$
   3. $-0.00001r^4 + 0.01r^3 + 0.1r^2$
   4. $-0.00001r^4 + 0.01r^3 + 0.5r^2$

   Answer: Option 3

Solution:

The total amount of waste generated because of the three different batch-processes is

$$\begin{aligned} W_A + W_B + W_C &= -0.00001r^4 + 0.015r^3 - 0.005r^3 + 0.05r^2 + 0.05r^2 \\ &= \boldsymbol{-0.00001r^4 + 0.01r^3 + 0.1r^2} \end{aligned}$$

4. What is the ratio of the total waste generated by the three-batch-processes with respect to the single continuous process?

   1. $-0.01r$
   2. $-0.1r$
   3. $0.1r$
   4. $0.01r$

   Answer = Option 3.

   Solution:

   The total waste generated by the three-batch-processes is
   $W_A + W_B + W_C = -0.00001r^4 + 0.01r^3 + 0.1r^2$.
   The waste generated in the single continuous process is $W(r) = -0.0001r^3 + 0.1r^2 + r$.
   The ratio of the total waste generated by the three-batch-processes with respect to the single continuous process is

$$\begin{aligned}\frac{W_A + W_B + W_C}{W(r)} &= \frac{-0.00001r^4 + 0.01r^3 + 0.1r^2}{-0.0001r^3 + 0.1r^2 + r} \\ &= \frac{(0.1r)(-0.0001r^3 + 0.1r^2 + r)}{-0.0001r^3 + 0.1r^2 + r} \\ &= \boldsymbol{0.1r}\end{aligned}$$

5. Let the company wastes Rs. 5,000 in waste treatment when it uses the single continuous process by consuming 100 kg of raw material. If instead of continuous process the company uses the three-batch-processes, then how much extra amount (in Rs.) will the company have to pay for waste treatment with respect to the continuous process?

   1. 50,000
   2. 500
   3. 45,000
   4. 5,000
   5. 4,000

   Answer: Option 3

   Solution:

   As the ratio for waste generation (continuous to batch) is 10 we can calculate cost for waste management from batch process will be ten times of the continuous process.
   Therefore the cost for waste management from the batch process will be $5,000 \times 10 = 50,000$.
   So the the extra amount required is $50,000 - 5,000 = 45,000$.

2 Multiple Select Questions (MSQ):

6. Let $P(x)$ and $Q(x)$ be two non zero polynomials of degrees m and n respectively. If $f(x) = P(x) + Q(x)$, $g(x) = P(x)Q(x)$, and $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\}$, where $h(x)$ is known to be a polynomial in x, then choose the set of correct options.

   1. The degree of $f(x)$ is $m + n$.
   2. The degree of $g(x)$ is $m + n$.
   3. The degree of $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
   4. The degree of $h(x)$ is m^3.
   5. The degree of $h(x)$ is n^3.
   6. The degree of $h(x)$ is $2m + n$.

   Answer: Options 2, 3, and 6.

   Solution:

   Given, $P(x)$ and $Q(x)$ are two non zero polynomials of degree m and n respectively.
   Also, $f(x) = P(x) + Q(x)$.
   If $m > n$, then the degree of the polynomial $f(x)$ will be m, else if $m < n$, then the degree of the polynomial $f(x)$ will be n, else if $m = n$, then the degree of the polynomial will be less than or equal to m(or n).
   Therefore, we can conclude that the degree of the polynomial $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
   Hence, option 1 is incorrect, and option 3 is correct.

   Now, $g(x) = P(x)Q(x)$, the degree of the polynomial $g(x)$ will be the sum of the degrees of the polynomials $P(x)$ and $Q(x)$.
   Therefore, the degree of $g(x)$ is $m + n$. Hence, option 2 is correct.

   Finally, $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\} = (P(x))^2Q(x) + \frac{(P(x))^2}{Q(x)}$.
   The degree of the polynomial $(P(x))^2Q(x)$ will be $2m + n$ and as given that $h(x)$ is a polynomial implies $Q(x)$ divides $(P(x))^2$, so the degree of the polynomial $\frac{(P(x))^2}{Q(x)}$ will be $2m - n$.
   Since $2m + n > 2m - n$, the degree of the polynomial $h(x)$ is $2m + n$. Hence, options 4 and 5 are incorrect, and option 6 is correct.

7. Given a polynomial $P(x) = (2x+5)(1-3x)(x^2+3x+1)$, then choose the set of correct options.

   1. Coefficient of x^5 is 0.
   2. Coefficient of x^3 is -18.
   3. Degree of P is 4.
   4. Coefficient of x^3 is -13.
   5. Degree of P is 7.
   6. All of the above.

   Answer: Options 1 and 3.

   Solution:

$$\begin{aligned}
\text{Given}, P(x) &= (2x+5)(1-3x)(x^2+3x+1) \\
&= (2x+5-6x^2-15x)(x^2+3x+1) \\
&= (5-6x^2-13x)(x^2+3x+1) \\
&= 5x^2-6x^4-13x^3+15x-18x^3-39x^2+5-6x^2-13x \\
&= -6x^4-31x^3-40x^2+2x+5
\end{aligned}$$

Option 1 is correct, because there is no x^5 term in the polynomial $P(x)$. So, the coefficient of x^5 is 0.
The degree of the polynomial $P(x)$ is 4. Hence, option 3 is correct and option 5 is incorrect.
The coefficient of x^3 is -31. Hence, options 2 and 4 are incorrect.

8. A sheet $ABCD$ of dimensions 10 ft x 3 ft is shown in Figure A-5.2. A box is made by removing two squares of equal dimensions $AEFG$ and $DHIJ$ and two rectangles of equal dimensions $BKLM$ and $CNOP$ respectively.

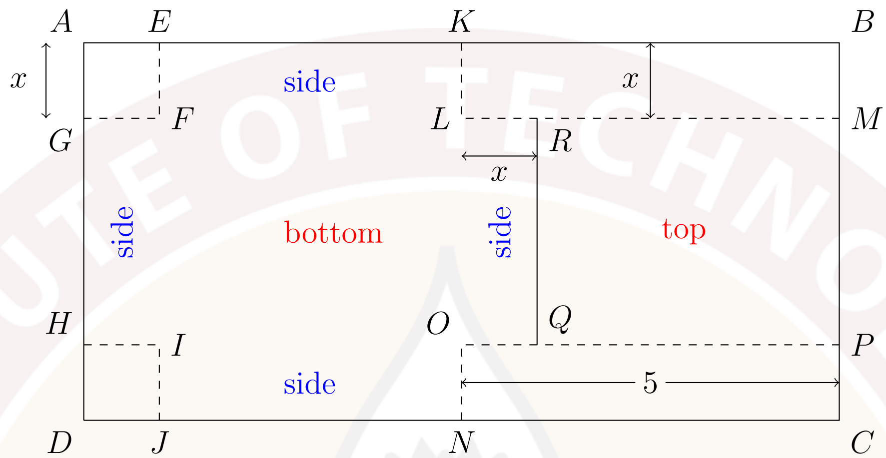


Figure A-5.2

Choose the set of correct options.

1. The volume of the box is $2x^2 - 23x + 30$.
2. The volume of the box is $2x^3 - 13x^2 + 15x$.
3. If $x = 0.5$, then the volume of the box is 5.625 cubic ft.
4. Value of x should always be greater than 0 but less than 1.5.

Answer: Options 2 and 4.

Solution:

From Figure A-5.2, the length of the box will be $EK = AB - KB - AE = 10 - 5 - x = 5 - x$, the breadth of the box will be $GH = AD - AG - HD = 3 - x - x = 3 - 2x$, and the height of the box will be $AE = x$.
Therefore, the volume of the box V given by length $\times$ breadth $\times$ height will be

$$V = (5 - x)(3 - 2x)(x)$$
$$V = (15 - 3x - 10x + 2x^2)(x)$$
$$V = 2x^3 - 13x^2 + 15x$$

Hence, options 1 is incorrect, and option 2 is correct.

If $x = 0.5$, then the volume of the box

$$\begin{aligned}
V &= 2x^3 - 13x^2 + 15x \\
V &= 2(0.5)^3 - 13(0.5)^2 + 15(0.5) \\
V &= 2(0.625) - 13(0.25) + 7.5 \\
V &= 1.25 - 3.25 + 7.5 = 5.5
\end{aligned}$$

Hence, option 3 is incorrect.

Now, to create a box, length of every side of box should always have a positive value. Therefore, $x > 0$ and $5 - x > 0 \implies x < 5$ and $3 - 2x > 0 \implies x < 1.5$. Combining all the conditions we get $x \in (0, 1.5)$.

3 Numerical Answer Type (NAT):

9. A curious student created a performance profile of his favourite cricketer as $R = -x^5 + 6x^4 - 30x^3 + 80x^2 + 70x + c$, where R is the total cumulative runs scored by the cricketer in x matches. He picked three starting values shown in Table A-5.2 and tried to find the value of c. If he uses Sum Squared Error method, then what will be the value of c?

No. of matches	Total score
1	120
2	285
3	361

Table A-5.2

Answer: -2

Solution:

Let us calculate the predicted cumulative runs scored by the player in the first three matches.
Substituting $x = 1, 2, 3$ in the given function, we get

$$\begin{aligned}
R(1) &= -(1)^5 + 6(1)^4 - 30(1)^3 + 80(1)^2 + 70(1) + c \\
&= -1 + 6 - 30 + 80 + 70 + c \\
&= 125 + c \\
R(2) &= -(2)^5 + 6(2)^4 - 30(2)^3 + 80(2)^2 + 70(2) + c \\
&= -32 + 96 - 240 + 320 + 140 + c \\
&= 284 + c \\
R(3) &= -(3)^5 + 6(3)^4 - 30(3)^3 + 80(3)^2 + 70(3) + c \\
&= -243 + 486 - 810 + 720 + 210 + c \\
&= 363 + c
\end{aligned}$$

Now, let us find the sum squared error of cumulative score for these three matches.

$$
\begin{aligned}
\text{SSE} &= \sum_{n=1}^{3}(R(n) - y_n)^2, \text{where } y_n \text{ is the total cumulative score in } n \text{ matches.}\\
&= (R(1) - y_1)^2 + (R(2) - y_2)^2 + (R(3) - y_3)^2\\
&= (125 + c - 120)^2 + (284 + c - 285)^2 + (363 + c - 361)^2\\
&= (5 + c)^2 + (c - 1)^2 + (2 + c)^2\\
&= 25 + 10c + c^2 + c^2 - 2c + 1 + 4 + 4c + c^2\\
&= 3c^2 + 12c + 30
\end{aligned}
$$

We have to find the value of c such that SSE becomes minimum, this is equal to the minimum value of the quadratic equation $3c^2 + 12c + 30$.
We know that the minimum value of any quadratic function of form $f(x) = Ax^2+Bx+D$, occurs at $x = \frac{-B}{2A}$. Here, $A = 3, B = 12$
So, the minimum value of the quadratic equation $3c^2 + 12c + 30$, occurs at
$c = \frac{-B}{2A} = \frac{-12}{2(3)} = -2$
Therefore, the minimum SSE is obtained when the value of c is $\mathbf{-2}$.

10. What is the minimum value of $x-$coordinate for the points of intersection of functions $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$?

Answer: 0

Solution:

At the points of intersection, observe that $f(x) = g(x)$.
Here, $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$.
Equating the functions we get,

$$\begin{aligned}
-x^5 + 5x^4 - 7x - 2 &= -x^5 + 5x^4 - x^2 - 2 \\
-7x &= -x^2 \\
x^2 - 7x &= 0 \\
x(x-7) &= 0 \\
\implies x &= 0 \text{ (or) } x = 7
\end{aligned}$$

Therefore, the minimum value of x- coordinate for the points of intersection of functions $f(x)$ and $g(x)$ is $\mathbf{0}$.

Mathematics for Data Science - 1

Graded assignment solutions

Week - 6

1 Multiple Choice Questions (MCQ):

1. What should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$?

   1. $4x$
   2. $4x - 3$
   3. $6x - 3$
   4. $2x - 3$

   Answer: Option 2.

   Solution:

   Using 4 step division algorithm, we find the remainder when $P(x)$ is divided by $2x^2+x-1$. If we subtract the obtained remainder from $P(x)$ then the resultant polynomial will be divisible by $2x^2 + x - 1$.
   Now,

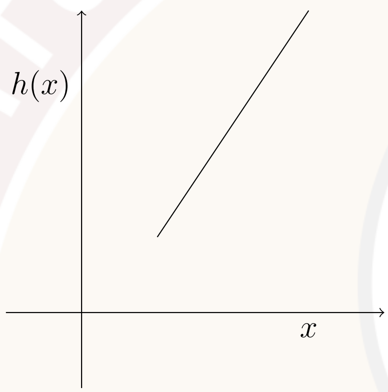
$$
\begin{array}{r|l}
 & \quad\quad\quad\quad\quad 3x^2 \quad + x + 1 \\
\hline
2x^2 + x - 1 & 6x^4 + 5x^3 \quad\quad\quad + 4x - 4 \\
 & -6x^4 - 3x^3 + 3x^2 \\
\cline{2-2}
 & \quad\quad 2x^3 + 3x^2 + 4x \\
 & \quad\quad -2x^3 - x^2 + x \\
\cline{2-2}
 & \quad\quad\quad\quad 2x^2 + 5x - 4 \\
 & \quad\quad\quad\quad -2x^2 - x + 1 \\
\cline{2-2}
 & \quad\quad\quad\quad\quad\quad 4x - 3
\end{array}
$$

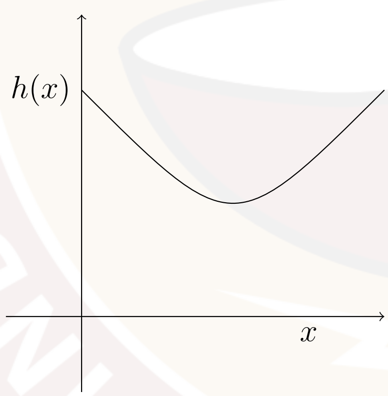
   Therefore, when $P(x)$ is divided by $2x^2 + x - 1$, we get $4x - 3$ as the remainder.
   Hence, $\boldsymbol{4x - 3}$ should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$.

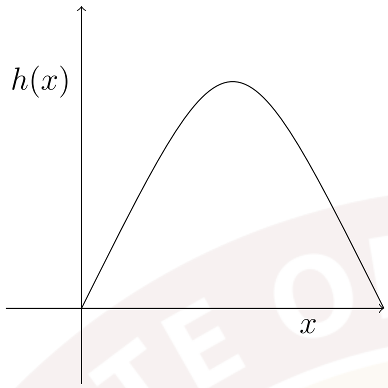
2. Table A-5.1 provides the information regarding some polynomials. Which is the most suitable (not exact) representation of $h(x)$ where $h(x)$ is known to be a polynomial in x, and if $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$?

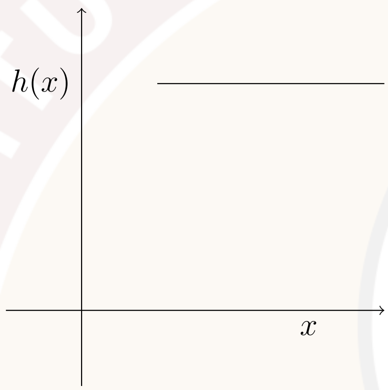
Polynomial	Degree	Condition
$P(x)$	m	$m > 0$
$Q(x)$	n	$m > 2n > 0$
$R(x)$	k	$k = m - n$
$S(x)$	t	$t = 2n$

Table A-5.1

1. 

2. 

3. 

4. 

Answer: option 4

Solution:

Given, the degree of $P(x)$ is 'm' where $m > 0$, the degree of $Q(x)$ is 'n' where $m > 2n > 0$, the degree of $R(x)$ is 'k' where $k = m - n$, and the degree of $S(x)$ is 't' where $t = 2n$.

Also, $h(x) = \frac{P(x)Q(x)-R(x)S(x)+S(x)P(x)}{P(x)+P(x)Q^2(x)}$ and $h(x)$ is known to be a polynomial. The degree of $h(x)$ will be the difference between the degree of the numerator and the degree of the denominator. The degree of the numerator will be the degree of the term which has the highest degree in the numerator. Similarly, the degree of the denominator will be the degree of the term which has the highest degree in the denominator.

Now, the degree of the polynomial $P(x)Q(x)$ will be '$m + n$', the degree of $R(x)S(x)$ will be '$k+t = m-n+2n = m+n$', and the degree of $S(x)P(x)$ will be '$t+m = 2n+m$'.

Therefore, the degree of the numerator (polynomial $P(x)Q(x) - R(x)S(x) + S(x)P(x)$) will be '$m + 2n$'.
Similarly, the degree of the denominator (polynomial $P(x) + P(x)Q^2(x)$) will be $m+2n$.

As $h(x)$ is given to be a polynomial and also the degrees of the polynomials in the numerator and the denominator are same, we can conclude that the degree of $h(x)$ is zero i.e. $h(x)$ should be a constant.
So, option 4 is the most suitable representation of $h(x)$.

Use the following information to solve questions 3-5.

A manufacturing company produces three types of products A, B, and C from one raw material in a single continuous process. This process generates total solid wastes (W) (in kg) as $W(r) = -0.0001r^3 + 0.1r^2 + r$, where r is the amount of raw material used in kg. If instead, the company uses three different batch-processes (one batch process for one product) to produce the above products, then the amount of waste generated because of products A, B, and C are given as $W_A = -0.00001r^4 + 0.015r^3$, $W_B = -0.005r^3 + 0.05r^2$ and $W_C = 0.05r^2$ respectively. (See the Figure A-5.1 for the reference.)

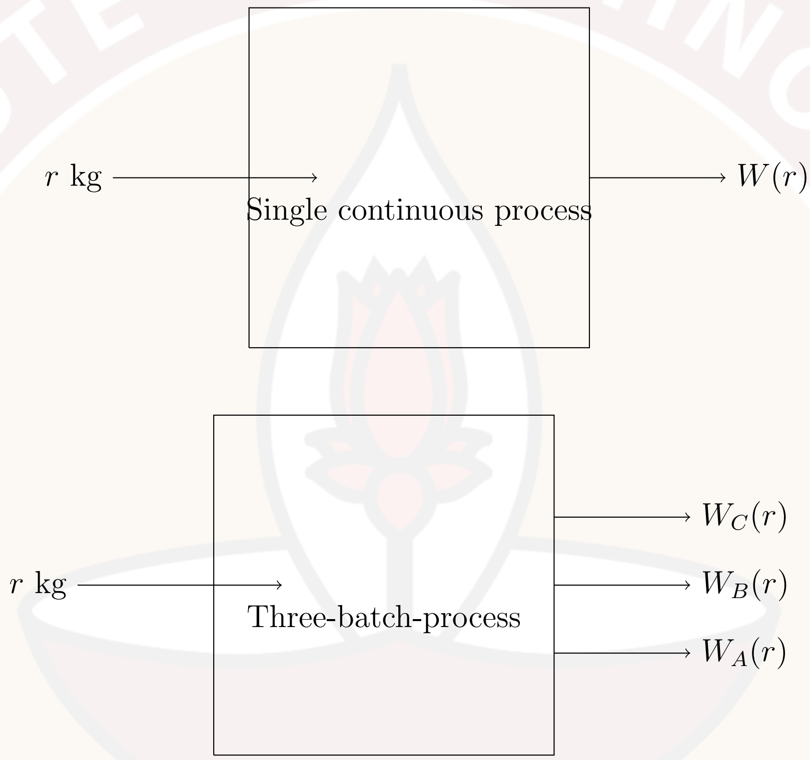


Figure A-5.1

3. What is the total amount of waste generated because of the three different batch-processes?

    1. $-0.00001r^4 + 0.01r^3 + 1.5r^2$
    2. $-0.00001r^4 + 0.015r^3 + 1.5r^2$
    3. $-0.00001r^4 + 0.01r^3 + 0.1r^2$
    4. $-0.00001r^4 + 0.01r^3 + 0.5r^2$

    Answer: Option 3

Solution:

The total amount of waste generated because of the three different batch-processes is

$$\begin{aligned} W_A + W_B + W_C &= -0.00001r^4 + 0.015r^3 - 0.005r^3 + 0.05r^2 + 0.05r^2 \\ &= \boldsymbol{-0.00001r^4 + 0.01r^3 + 0.1r^2} \end{aligned}$$

4. What is the ratio of the total waste generated by the three-batch-processes with respect to the single continuous process?

    1. $-0.01r$
    2. $-0.1r$
    3. $0.1r$
    4. $0.01r$

    Answer = Option 3.

    Solution:

    The total waste generated by the three-batch-processes is
    $W_A + W_B + W_C = -0.00001r^4 + 0.01r^3 + 0.1r^2$.
    The waste generated in the single continuous process is $W(r) = -0.0001r^3 + 0.1r^2 + r$.
    The ratio of the total waste generated by the three-batch-processes with respect to the single continuous process is

$$\begin{aligned}\frac{W_A + W_B + W_C}{W(r)} &= \frac{-0.00001r^4 + 0.01r^3 + 0.1r^2}{-0.0001r^3 + 0.1r^2 + r} \\ &= \frac{(0.1r)(-0.0001r^3 + 0.1r^2 + r)}{-0.0001r^3 + 0.1r^2 + r} \\ &= \boldsymbol{0.1r}\end{aligned}$$

5. Let the company wastes Rs. 5,000 in waste treatment when it uses the single continuous process by consuming 100 kg of raw material. If instead of continuous process the company uses the three-batch-processes, then how much extra amount (in Rs.) will the company have to pay for waste treatment with respect to the continuous process?

    1. 50,000
    2. 500
    3. 45,000
    4. 5,000
    5. 4,000

    Answer: Option 3

    Solution:

    As the ratio for waste generation (continuous to batch) is 10 we can calculate cost for waste management from batch process will be ten times of the continuous process.
    Therefore the cost for waste management from the batch process will be $5,000 \times 10 = 50,000$.
    So the the extra amount required is $50,000 - 5,000 = 45,000$.

2 Multiple Select Questions (MSQ):

6. Let $P(x)$ and $Q(x)$ be two non zero polynomials of degrees m and n respectively. If $f(x) = P(x) + Q(x)$, $g(x) = P(x)Q(x)$, and $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\}$, where $h(x)$ is known to be a polynomial in x, then choose the set of correct options.

    1. The degree of $f(x)$ is $m + n$.
    2. The degree of $g(x)$ is $m + n$.
    3. The degree of $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
    4. The degree of $h(x)$ is m^3.
    5. The degree of $h(x)$ is n^3.
    6. The degree of $h(x)$ is $2m + n$.

    Answer: Options 2, 3, and 6.

    Solution:

    Given, $P(x)$ and $Q(x)$ are two non zero polynomials of degree m and n respectively.
    Also, $f(x) = P(x) + Q(x)$.
    If $m > n$, then the degree of the polynomial $f(x)$ will be m, else if $m < n$, then the degree of the polynomial $f(x)$ will be n, else if $m = n$, then the degree of the polynomial will be less than or equal to m(or n).
    Therefore, we can conclude that the degree of the polynomial $f(x)$ is $max\{m, n\}$ if $m \neq n$, where $max\{m, n\}$ represents the maximum value from m and n.
    Hence, option 1 is incorrect, and option 3 is correct.

    Now, $g(x) = P(x)Q(x)$, the degree of the polynomial $g(x)$ will be the sum of the degrees of the polynomials $P(x)$ and $Q(x)$.
    Therefore, the degree of $g(x)$ is $m + n$. Hence, option 2 is correct.

    Finally, $h(x) = P(x)\{P(x)Q(x) + \frac{P(x)}{Q(x)}\} = (P(x))^2Q(x) + \frac{(P(x))^2}{Q(x)}$.
    The degree of the polynomial $(P(x))^2Q(x)$ will be $2m + n$ and as given that $h(x)$ is a polynomial implies $Q(x)$ divides $(P(x))^2$, so the degree of the polynomial $\frac{(P(x))^2}{Q(x)}$ will be $2m - n$.
    Since $2m + n > 2m - n$, the degree of the polynomial $h(x)$ is $2m + n$. Hence, options 4 and 5 are incorrect, and option 6 is correct.

7. Given a polynomial $P(x) = (2x+5)(1-3x)(x^2+3x+1)$, then choose the set of correct options.

    1. Coefficient of x^5 is 0.
    2. Coefficient of x^3 is -18.
    3. Degree of P is 4.
    4. Coefficient of x^3 is -13.
    5. Degree of P is 7.
    6. All of the above.

    Answer: Options 1 and 3.

    Solution:

$$\begin{aligned}
\text{Given, } P(x) &= (2x+5)(1-3x)(x^2+3x+1) \\
&= (2x+5-6x^2-15x)(x^2+3x+1) \\
&= (5-6x^2-13x)(x^2+3x+1) \\
&= 5x^2-6x^4-13x^3+15x-18x^3-39x^2+5-6x^2-13x \\
&= -6x^4-31x^3-40x^2+2x+5
\end{aligned}$$

Option 1 is correct, because there is no x^5 term in the polynomial $P(x)$. So, the coefficient of x^5 is 0.
The degree of the polynomial $P(x)$ is 4. Hence, option 3 is correct and option 5 is incorrect.
The coefficient of x^3 is -31. Hence, options 2 and 4 are incorrect.

8. A sheet $ABCD$ of dimensions 10 ft x 3 ft is shown in Figure A-5.2. A box is made by removing two squares of equal dimensions $AEFG$ and $DHIJ$ and two rectangles of equal dimensions $BKLM$ and $CNOP$ respectively.

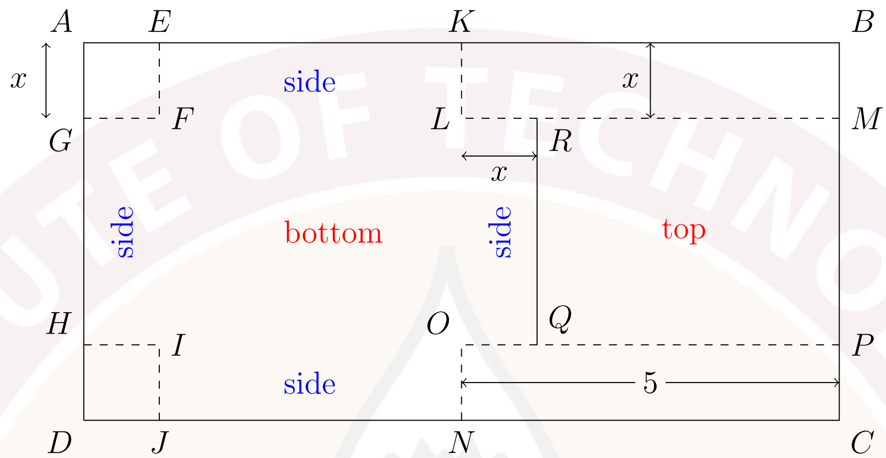


Figure A-5.2

Choose the set of correct options.

1. The volume of the box is $2x^2 - 23x + 30$.
2. The volume of the box is $2x^3 - 13x^2 + 15x$.
3. If $x = 0.5$, then the volume of the box is 5.625 cubic ft.
4. Value of x should always be greater than 0 but less than 1.5.

Answer: Options 2 and 4.

Solution:

From Figure A-5.2, the length of the box will be $EK = AB - KB - AE = 10 - 5 - x = 5 - x$, the breadth of the box will be $GH = AD - AG - HD = 3 - x - x = 3 - 2x$, and the height of the box will be $AE = x$.
Therefore, the volume of the box V given by length $\times$ breadth $\times$ height will be

$$V = (5 - x)(3 - 2x)(x)$$
$$V = (15 - 3x - 10x + 2x^2)(x)$$
$$V = 2x^3 - 13x^2 + 15x$$

Hence, options 1 is incorrect, and option 2 is correct.

If $x = 0.5$, then the volume of the box

$$V = 2x^3 - 13x^2 + 15x$$
$$V = 2(0.5)^3 - 13(0.5)^2 + 15(0.5)$$
$$V = 2(0.625) - 13(0.25) + 7.5$$
$$V = 1.25 - 3.25 + 7.5 = 5.5$$

Hence, option 3 is incorrect.

Now, to create a box, length of every side of box should always have a positive value. Therefore, $x > 0$ and $5 - x > 0 \implies x < 5$ and $3 - 2x > 0 \implies x < 1.5$. Combining all the conditions we get $x \in (0, 1.5)$.

3 Numerical Answer Type (NAT):

9. A curious student created a performance profile of his favourite cricketer as $R = -x^5 + 6x^4 - 30x^3 + 80x^2 + 70x + c$, where R is the total cumulative runs scored by the cricketer in x matches. He picked three starting values shown in Table A-5.2 and tried to find the value of c. If he uses Sum Squared Error method, then what will be the value of c?

No. of matches	Total score
1	120
2	285
3	361

Table A-5.2

Answer: -2

Solution:

Let us calculate the predicted cumulative runs scored by the player in the first three matches.
Substituting $x = 1, 2, 3$ in the given function, we get

$$\begin{aligned}
R(1) &= -(1)^5 + 6(1)^4 - 30(1)^3 + 80(1)^2 + 70(1) + c \\
&= -1 + 6 - 30 + 80 + 70 + c \\
&= 125 + c \\
R(2) &= -(2)^5 + 6(2)^4 - 30(2)^3 + 80(2)^2 + 70(2) + c \\
&= -32 + 96 - 240 + 320 + 140 + c \\
&= 284 + c \\
R(3) &= -(3)^5 + 6(3)^4 - 30(3)^3 + 80(3)^2 + 70(3) + c \\
&= -243 + 486 - 810 + 720 + 210 + c \\
&= 363 + c
\end{aligned}$$

Now, let us find the sum squared error of cumulative score for these three matches.

$$\begin{aligned}
\text{SSE} &= \sum_{n=1}^{3}(R(n)-y_n)^2, \text{where } y_n \text{ is the total cumulative score in } n \text{ matches.}\\
&= (R(1)-y_1)^2+(R(2)-y_2)^2+(R(3)-y_3)^2\\
&= (125+c-120)^2+(284+c-285)^2+(363+c-361)^2\\
&= (5+c)^2+(c-1)^2+(2+c)^2\\
&= 25+10c+c^2+c^2-2c+1+4+4c+c^2\\
&= 3c^2+12c+30
\end{aligned}$$

We have to find the value of c such that SSE becomes minimum, this is equal to the minimum value of the quadratic equation $3c^2+12c+30$.
We know that the minimum value of any quadratic function of form $f(x) = Ax^2+Bx+D$, occurs at $x = \frac{-B}{2A}$. Here, $A = 3, B = 12$
So, the minimum value of the quadratic equation $3c^2+12c+30$, occurs at
$c = \frac{-B}{2A} = \frac{-12}{2(3)} = -2$
Therefore, the minimum SSE is obtained when the value of c is $\mathbf{-2}$.

10. What is the minimum value of $x-$coordinate for the points of intersection of functions $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$?

Answer: 0

Solution:

At the points of intersection, observe that $f(x) = g(x)$.
Here, $f(x) = -x^5 + 5x^4 - 7x - 2$ and $g(x) = -x^5 + 5x^4 - x^2 - 2$.
Equating the functions we get,

$$\begin{aligned}-x^5 + 5x^4 - 7x - 2 &= -x^5 + 5x^4 - x^2 - 2 \\ -7x &= -x^2 \\ x^2 - 7x &= 0 \\ x(x-7) &= 0 \\ &\implies x = 0 \text{ (or) } x = 7\end{aligned}$$

Therefore, the minimum value of x- coordinate for the points of intersection of functions $f(x)$ and $g(x)$ is $\mathbf{0}$.

Week - 5
Practice Assignment
Algebra of polynomials
Mathematics for Data Science - 1

1 Multiple Choice Questions (MCQ):

1. Let x be the number of years since the year 2000 (i.e., $x = 0$ denotes the year 2000). The total amount of profit (in ₹) on books in a shop is given by the function $T(x) = 5x^3 + 3x + 1$. The shop sells books of four languages English, Bengali, Hindi, and Tamil. The profits from selling English and Bengali books are given by $E(x) = 3x^3 - 5x^2 + x$ and $B(x) = x^2 + 4x + 5$ respectively. The profit from selling Hindi and Tamil books are found to be the same.

   (a) Which of the following polynomial functions represents the profit from selling Tamil books?

   - ◯ $2x^3 + 4x^2 - 2x - 4$
   - ◯ $x^3 - 2x^2 - x + 2$
   - ◯ $\boldsymbol{x^3 + 2x^2 - x - 2}$
   - ◯ $2x^3 - 4x^2 - 2x + 4$

   (b) In which year was the profit from Hindi books zero?

   - ◯ **2001**
   - ◯ 2002
   - ◯ 2004
   - ◯ 2010

   Solution:

   (a) The total profit from selling English and Bengali books is= $E(x) + B(x) = (3x^3 - 5x^2 + x) + (x^2 + 4x + 5) = 3x^3 - 4x^2 + 5x + 5$. Hence the total profit from selling Hindi and Tamil books is= $T(x) - (3x^3 - 4x^2 + 5x + 5) = 5x^3 + 3x + 1 - 3x^3 + 4x^2 - 5x - 5 = 2x^3 + 4x^2 - 2x - 4$.
   As the profit from selling Hindi and Tamil books are found to be the same, the profit from selling Tamil books is= $\frac{1}{2}(2x^3 + 4x^2 - 2x - 4) = x^3 + 2x^2 - x - 2$

   (b) Profit from selling Hindi books (which is same as the profit from selling Tamil books) is $x^3 + 2x^2 - x - 2$.

   $$x^3 + 2x^2 - x - 2 = x^2(x + 2) - 1(x + 2) = (x + 2)(x^2 - 1) = (x + 2)(x + 1)(x - 1)$$

   So the profit will be zero if $(x + 2)(x + 1)(x - 1) = 0$, i.e., at $x = -2, -1, 1$ the profit can be 0. But in this context, x cannot be negative. So $x = 1$ is the only possibility. Hence in the year 2001 the profit from Hindi books was zero.

2. Find the quadratic polynomial which when divided by x, $x-1$, and $x+1$ gives the remainders 7, 14, and 8 respectively.

   - ◯ $4x^2 - 3x + 7$
   - ◯ $x^2 + 7x + 7$
   - ◯ $7x^2 + x + 7$
   - ◯ $\mathbf{4x^2 + 3x + 7}$

   Solution: Let the quadratic polynomial which is satisfying the given condition be $p(x) = ax^2 + bx + c$.
   When it is divided by x the remainder is 7. It implies that if we substitute $x = 0$ in $p(x)$ we will get 7, i.e., $p(0) = 7$. Similarly we have $p(1) = 14$ and $p(-1) = 8$.
   Hence we have the following equations:

$$\begin{aligned}
p(0) &= a(0)^2 + b(0) + c \\
&= c \\
&= 7 \\
p(1) &= a.(1)^2 + b.1 + c \\
&= a + b + c \\
&= 14 \\
p(-1) &= a(-1)^2 + b(-1) + c \\
&= a - b + c \\
&= 8
\end{aligned}$$

   So, we have $c = 7$, and substituting c in the second and third equation we get, $a+b = 7$, and $a - b = 1$. By solving these two equations we get $a = 4$ and $b = 3$.
   Hence the quadratic polynomial is $4x^2 + 3x + 7$.

3. Box A has length x unit, breadth $(x+1)$ unit, and height $(x+2)$ unit. Box B has length $(x+1)$ unit, breadth $(x+1)$ unit, and height $(x+2)$ unit. There are two more boxes C and D of cubic shape with side x unit. The total volume of A and B is y cubic unit more than the total volume of C and D. Find y in terms of x.

   - ◯ $x^3 + 7x^2 + 7x + 2$
   - ◯ $\mathbf{7x^2 + 7x + 2}$
   - ◯ $7x^2 - 7x - 2$
   - ◯ $x^3 + 7x^2 - 7x - 2$

   Solution: The volume of box A is $x(x+1)(x+2) = x^3 + 3x^2 + 2x$ cubic unit.
   The volume of box B is $(x+1)(x+1)(x+2) = (x^2+2x+1)(x+2) = x^3+4x^2+5x+2$ cubic unit.
   The volume of box C and D is x^3 cubic unit each. So the total volume of A and B is $2x^3 + 7x^2 + 7x + 2$ and the total volume of C and D is $2x^3$.
   Hence $y = (2x^3 + 7x^2 + 7x + 2) - 2x^3 = 7x^2 + 7x + 2$.

4. The population of a bacteria culture in laboratory conditions is known to be a function of time of the form $p(t) = at^5 + bt^2 + c$, where p represents the population (in lakhs) and t represents the time (in minutes). Suppose a student conducts an experiment to determine the coefficients a, b, and c in the formula and obtains the following data:

   - $p(0) = 3$
   - $p(1) = 5$
   - $p(2) = 39$

   Which of the following options is correct?

   - ○ $p(t) = 3t^5 - t^2 + 3$
   - ○ $p(t) = 4t^5 - 2t^2 + 3$
   - ○ $\boldsymbol{p(t) = t^5 + t^2 + 3}$
   - ○ $p(t) = 39t^5 + 5t^2 + 3$

   Solution: Given that, $p(t) = at^5 + bt^2 + c$.
   $p(0) = c = 3$
   $p(1) = a + b + c = 5$, putting $c = 3$, we get $a + b = 2$.
   $p(2) = a(2)^5 + b(2)^2 + c = 32a + 4b + c = 39$, substituting $c = 3$, we get $32a + 4b = 36$, implies, $8a + b = 9$ (cancelling 4 from both sides)
   By solving these two equations we get $a = 1$, and $b = 1$.
   Hence, $p(t) = t^5 + t^2 + 3$.

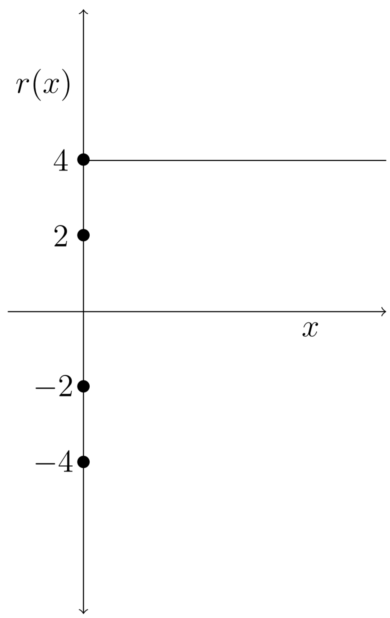
5. If the polynomials $x^3 + ax^2 + 5x + 7$ and $x^3 + 2x^2 + 3x + 2a$ leave the same remainder when divided by $(x - 2)$, then the value of a is:

   - ◯ $\frac{3}{2}$
   - ◯ $\boldsymbol{-\frac{3}{2}}$
   - ◯ $\frac{5}{2}$
   - ◯ $-\frac{5}{2}$

   Solution: Given that both the polynomials leave same remainder when divided by $(x - 2)$. By substituting $x = 2$ both the polynomial should have same value.
   By substituting $x = 2$ in $x^3 + ax^2 + 5x + 7$, we get $8 + 4a + 10 + 7 = 4a + 25$.
   By substituting $x = 2$ in $x^3 + 2x^2 + 3x + 2a$, we get $8 + 8 + 6 + 2a = 2a + 22$.
   So we have,

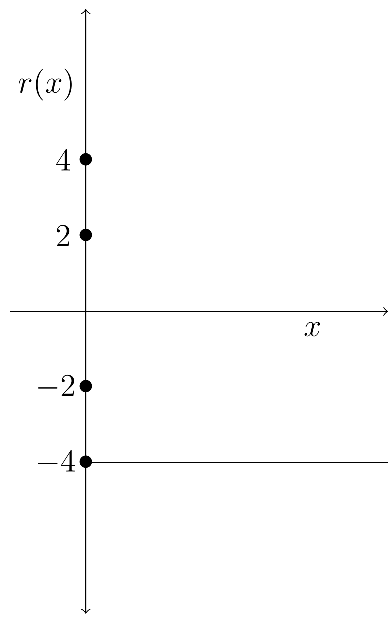
$$4a + 25 = 2a + 22$$
$$2a = -3$$
$$a = -\frac{3}{2}$$

6. Let $r(x)$ be a polynomial function which is obtained as the remainder after dividing the polynomial $2x^3 + x^2 - 5$ by the polynomial $2x - 3$. Choose the correct option which represents the polynomial $r(x)$ most appropriately.



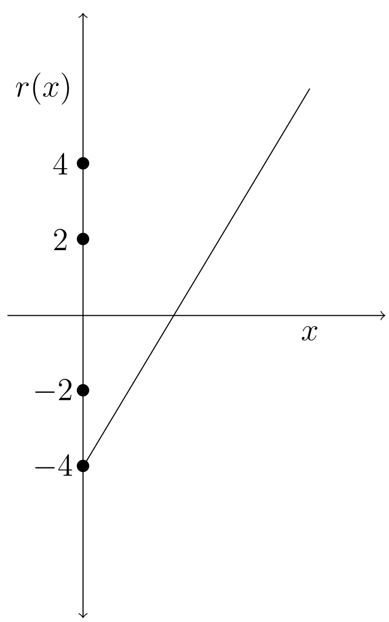
○ **Option A**

Fig P-6.2



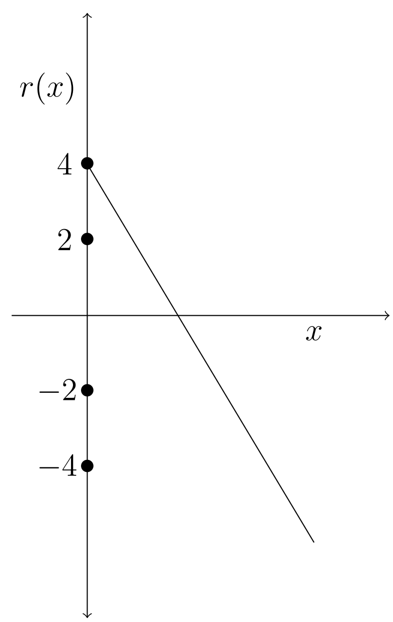
○ Option B

Fig P-6.3



○ Option C

Fig P-6.4



○ Option D

Fig P-6.5

Solution We get 4 as the remainder if $2x^3 + x^2 - 5$ is divided by the polynomial $2x - 3$.

$$2x^3 + x^2 - 5 = (2x - 3)(x^2 + 2x + 3) + 4$$

Hence $r(x) = 4$, which is a constant polynomial. Hence, the first option is the correct.

2 Multiple Select Questions (MSQ):

7. By dividing a polynomial $p(x)$ with another polynomial $q(x)$ we get $h(x)$ as the quotient and $r(x)$ as the remainder.

   (a) The maximum degree of $r(x)$ can be,

   - ○ $deg\ p(x)$
   - ○ $deg\ (p(x)) - 1$
   - ○ $deg\ q(x)$
   - ○ $\boldsymbol{deg\ (q(x)) - 1}$

   (b) If $deg\ p(x) < deg\ q(x)$, then choose the set of correct answers:

   - ○ $\boldsymbol{h(x) = 0}$
   - ○ $deg\ h(x) = deg\ q(x)$
   - ○ $deg\ r(x) = deg\ q(x)$
   - ○ $\boldsymbol{deg\ r(x) = deg\ p(x)}$

Solution:

(a) The degree of the remainder $r(x)$ should be strictly less than the degree of the polynomial $q(x)$. So the maximum degree of $r(x)$ is $deg\ (q(x)) - 1$.

(b) If $deg\ p(x) < deg\ q(x)$, then quotient will be zero polynomial, hence $deg\ h(x) = 0$. The remainder will be $p(x)$ itself. So $deg\ r(x) = deg\ p(x)$.

3 Numerical Answer Type (NAT):

8. An open box can be made from a piece of cardboard of length $7x$ unit and breadth $5x$ unit, by cutting squares of side x unit out of the corners of the rectangular cardboard, then folding up the sides as shown in the Figure P-6.1.

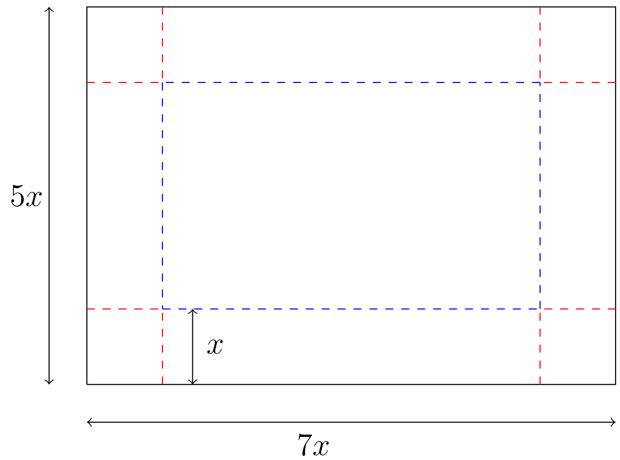


Figure P-6.1

(a) What will be the coefficient of x^3 in the polynomial representing the volume of the box? [Answer:15]

(b) What will be the coefficient of x^2 in the polynomial representing the volume of the box? [Answer:0]

Solution: As the sides of the piece of the cardboard has been cut out, the length of the box made will be $7x - (x + x) = 5x$ unit and the breadth of the box made will be $5x - (x + x) = 3x$ unit, and the height will be x unit.
Hence the volume of the box will be $5x \times 3x \times x = 15x^3$ cubic unit.

(a) The coefficient of x^3 in the polynomial representing the volume of the box is 15.

(b) The coefficient of x^2 in the polynomial representing the volume of the box is 0.

Week - 4
Assignment
Quadratic Equations
Mathematics for Data Science - 1
Full marks: 25

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

1. Find out the points where the curve $y = 4x^2 + x$ and the straight line $y = 2x - 3$ intersect with each other. **1 mark**

   - ◯ $(\frac{3}{2}, 0)$ and $(\frac{3}{2}, \frac{21}{2})$.
   - ◯ Only at the origin.
   - ◯ **The curve and the straight line do not intersect.**
   - ◯ $(1, -1)$ and $(1, 5)$.

   Solution: The following Figure M1W5AS-1 shows that the curves $y = 4x^2 + x$ and $y = 2x - 3$ are not intersecting with each other at any point.

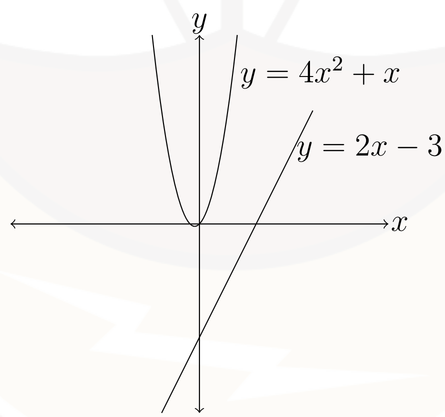


Figure: M1W5AS-1

Suppose that the curves $y = 4x^2 + x$ and $y = 2x - 3$ are intersecting at the point (a, b). Let us try to find the point. As both curves are passing through the point (a, b), it should satisfy both the equations. So we have $b = 4a^2 + a$ and $b = 2a - 3$. Which implies,

$$4a^2 + a = 2a - 3$$

i.e.

$$4a^2 - a + 3 = 0$$

The discriminant of the obtained quadratic is

$$(-1)^2 - 4(4)(3) = 1 - 48 = -47 < 0$$

As the discriminant is negative, the quadratic equation has no real root. So there cannot be any point (a, b) on the real plane at which these curves intersect each other.

2. Three friends A, B and C started their journey at 08:00 am from the points $(2,31)$, $(3,51)$, and $(6,0)$ respectively. A followed the path along the curve $3x^2+2x+15$ and B followed the path along the curve $2x^2+10x+3$. They all meet at 11:00 am at a point P whose x coordinate is greater than 4. If C followed a straight-line path, and one unit is equal to 1 km then what was the speed of C? **3 marks**

   - ◯ 31.26 km/hr
   - ◯ 32 km/hr
   - ◯ **45 km/hr**
   - ◯ 45.5 km/hr
   - ◯ $\frac{4}{3}$ km/hr
   - ◯ $\frac{3}{4}$ km/hr

Solution: A and B followed the the path along the curve $3x^2+2x+15$ and $2x^2+10x+3$ respectively and met at the point P. Hence P is one of the points of intersections of these two curves. Let the coordinate of the point P be (a,b). Hence we have,

$$3a^2+2a+15 = 2a^2+10a+3$$
$$a^2-8a+12 = 0$$
$$(a-6)(a-2) = 0$$

Hence there are two possible values for a: 2 and 6. It is given in the problem that the x coordinate of the point P is greater than 4. Hence a must be 6. Now substituting the value of a in any one of the equations we get the value of b as 135.

Hence coordinate of the point P is $(6,135)$.

C starts its journey from the point $(6,0)$ and reaches the point P, whose coordinate is $(6,135)$, along a straight-line path. So the distance covered by C is 135 units. Now in the problem it is given that one unit is equal to 1 km. So in 3 hours (i.e., 8:00 am to 11:00 am) C has covered 135 km.

So the speed of C is $\frac{135}{3} = 45$ km/hr.

3. Consider the curve of the quadratic function $y = (x - \frac{1}{p})(x + \frac{1}{q})$, with $p, q \neq 0$. Suppose the distance between its x intercepts is 2, and the y intercept is at a distance 1 from the origin. Then which of the following equations is correct for this given curve? **1 mark**

- ◯ $y = (x - \frac{1}{3})(x - \frac{5}{3})$
- ◯ $y = (x - 1)^2$
- ◯ $\boldsymbol{y = (x - 1)(x + 1)}$
- ◯ None of the above.

Solution: Given the quadratic function $y = (x - \frac{1}{p})(x + \frac{1}{q})$.

$$x \text{ intercepts}: (\tfrac{1}{p}, 0) \text{ and } (-\tfrac{1}{q}, 0).$$
$$y \text{ intercept}: (0, -\tfrac{1}{pq})$$

From the given information in the problem we have,

$$\frac{1}{p} + \frac{1}{q} = 2 \tag{1}$$
$$\frac{1}{pq} = 1 \tag{2}$$

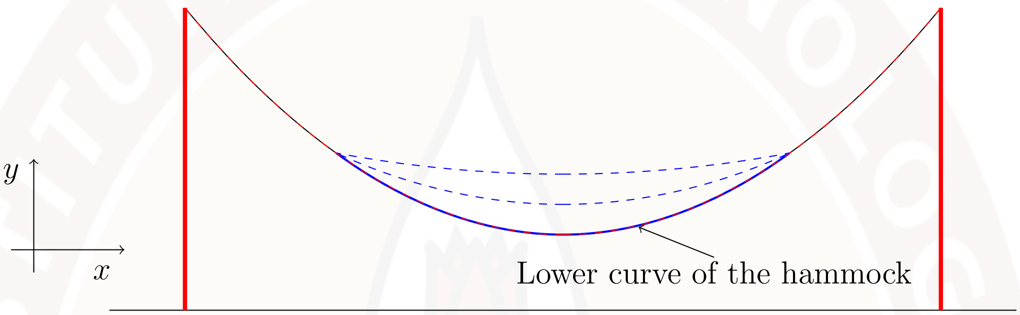
From equation (2) we have, $q = \frac{1}{p}$. By substituting this in equation (1) we get,

$$\begin{aligned} \frac{1}{p} + p &= 2 \\ \frac{1+p^2}{p} &= 2 \\ 1 + p^2 &= 2p \\ p^2 - 2p + 1 &= 0 \\ (p-1)^2 &= 0 \\ p &= 1 \end{aligned}$$

Hence, $q = \frac{1}{p} = 1$. So the equation of the given curve is $y = (x - 1)(x + 1)$.

4. A hammock is a cloth swing as shown in the picture below. The height (y) from ground of any point of the lower curve of the hammock varies with respect to the horizontal distance (x) from some fixed origin. The origin is in the plane of y and x. Consider the following equations and choose the correct option. (It is expected that the hammock does not touch the ground.)

A: $y = x^2 + 6x + 8$
B: $y = x^2 + 4x + 8$
C: $y = x^2 + 4x + 2$ **6 marks**



- ◯ Only A can represent the hammock.
- ◯ **Only B can represent the hammock.**
- ◯ Only C can represent the hammock.
- ◯ Both A and B can represent the hammock.
- ◯ Both A and C can represent the hammock.
- ◯ None of these equations is appropriate to represent the hammock.

Solution: As it is expected that the hammock does not touch the ground, the y-coordinate of the vertex of the parabola should be strictly greater than 0.

$$\text{Vertex of A} : (-3, -1)$$
$$\text{Vertex of B} : (-2, 4)$$
$$\text{Vertex of C} : (-2, -2)$$

So only for B, the y-coordinate of the vertex is strictly greater than 0. Hence, only B can represent the hammock.

5. A bird is hunting for fish in a pond. She swoops down from a height and picks up a fish from the water surface and flies back up, all along a path $y = 4x^2 - (2k+2)x + 1$, where $k \geq 0$, y is its height from the water surface, and x is the horizontal distance from a fixed origin which is in the plane of the path. Then, the value of k can be: **3 marks**

   - ◯ 3
   - ◯ 2
   - ◯ **1**
   - ◯ 5

   Solution: As the bird swoops down from a height and picks up a fish from the water surface and flies back up, all along a path $y = 4x^2 - (2k+2)x + 1$, there should be only one real root of the equation $4x^2 - (2k+2)x + 1 = 0$. Hence the discriminant should be 0. So we have,

$$\begin{aligned}
(-(2k+2))^2 - 4(4)(1) &= 0 \\
4k^2 + 4 + 8k - 16 &= 0 \\
4k^2 + 8k - 12 = 0 & \\
k^2 + 2k - 3 &= 0 \\
(k+3)(k-1) &= 0
\end{aligned}$$

   The value of k can be 1 or -3. As k is given to be non-negative, we have $\boldsymbol{k = 1}$.

6. If the height of a right-angled triangle is 2 cm more than its base, and the hypotenuse is 4 cm more than its base, then what is the height (cm) of the triangle? **1 mark**

   - ◯ 10
   - ◯ **8**
   - ◯ 6
   - ◯ 4
   - ◯ 3
   - ◯ None of the above.

   Solution: Let the length of the base of the given triangle be x cm. The height will be $x+2$ cm and the length of the hypotenuse will be $x+4$ cm. As it is a right angled triangle, using the Pythagorean theorem we have,

$$x^2 + (x+2)^2 = (x+4)^2$$
$$x^2 + x^2 + 4 + 4x = x^2 + 16 + 8x$$
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$

   As the length of base cannot be negative, x must be positive. Hence $x = 6$.
   So the height of the triangle is 8 cm.

2 Multiple Select Questions (MSQ):

7. A company opens two new branches A and B in 2010. A and B make yearly profits in lakhs as $P_1(y) = 10y - y^2$ and $P_2(y) = y^2 - 6y$ respectively, where y is the number of years after opening the branch. Let loss be represented as $-ve$ of profit. Then choose the correct set of options from the following. **3 marks**

   ◯ **The range of profit made by branch B for the first 9 years is $[-9, 27]$.**

   ◯ **The range of profit made by branch A for the first 10 years is $[0, 25]$.**

   ◯ Till 2016 branch A was making a loss.

   ◯ **In 2018, both companies made the same profit.**

   ◯ **Going by the trajectory of branch A, 2020 is the suitable time to shut down the branch for avoiding a loss.**

   ◯ B never goes into a loss.

Solution: Firstly observe that we are given two quadratic functions of y. So y will be plotted along the horizontal axis and $p(y)$ will be plotted along the vertical axis. Also observe in Figure: M1W5AS-7 that the parabola represented by the function $p_2(y)$ opens towards the positive direction of y-axis, i.e. open upwards, as the coefficient of y^2 is positive and the parabola represented by the function $p_1(y)$ opens towards the negative direction of y-axis, i.e. open downwards, as the coefficient of y^2 is negative in this case.

- $p_2(y)$ will be minimum at the vertex of the parabola represented by the corresponding curve. The coordinate of the vertex is $(3, -9)$. Moreover, $p_2(y)$ will be increasing for $y > 3$. So in the first 9 years the maximum value of $p_2(y)$ will be at $y = 9$, and $p_2(9) = 27$. Hence, the range of profit made by branch B for the first 9 years is $[-9, 27]$. So the first option is correct.
- $p_1(y)$ will be minimum at the vertex of the parabola represented by the corresponding curve. The coordinate of the vertex is $(5, 25)$. Moreover, $p_1(y)$ increases till $y = 5$ starting from $y = 0$, and then again decreases again for $y > 5$. $p_1(10) = p_1(0) = 0$. Hence, the range of profit made by the branch A for the first 10 years is $[0, 25]$. So the second option is also correct.
- A is never making a loss. So the third option is not correct.
- If both the branches have to make same profit at some point then $p_1(y)$ must be equal with $p_2(y)$ for some y.

$$\begin{aligned} 10y - y^2 &= y^2 - 6y \\ 2y^2 - 16y &= 0 \\ y(y-8) &= 0 \end{aligned}$$

  Hence at $y = 0$ and $y = 8$ the profit will be same. At $y = 0$ both the branches are opening so there is no profit for both of them, so $p_1(0) = 0 = p_2(y)$. At $y = 8$, i.e in 2018, both the companies made the same profit. Hence the fourth option is correct.

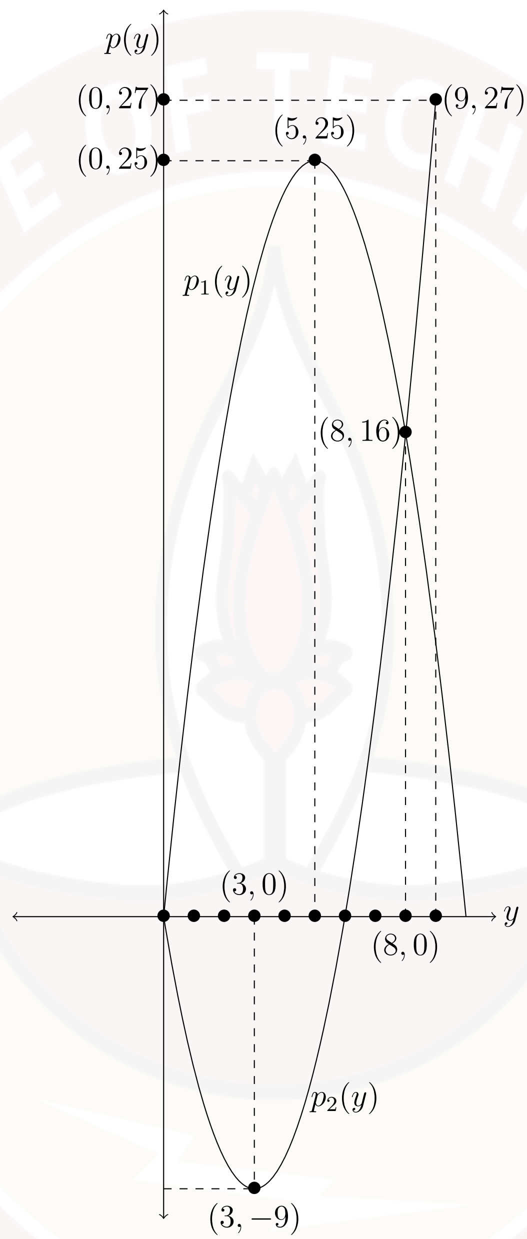


Figure: M1W5AS-7

- We have already calculated that $p_1(10) = 0$ and for $y > 10$, $p_1(y) < 0$. So after 2020 the branch A will be making a loss. Hence going by the trajectory of branch A, 2020 is the suitable time to shut down the branch for avoiding a loss. Hence the fifth option is also correct.

- For $0 \leq y \leq 6$, $p_2(y) \leq 0$. Hence till 2016 the branch B is making a loss. So the last option is not the correct one.

8. Choose the correct set of options regarding the quadratic equation $(5m+9)^2x^2-(3n+4)x+1=0$ where $m, n \in \mathbb{Z}$. **3 marks**

   - ◯ Both roots are equal if and only if $10m-3n+14=0$.
   - ◯ **Both roots are equal if $\boldsymbol{10m-3n+14=0}$.**
   - ◯ **Both roots are equal if $\boldsymbol{m=n=-2}$.**
   - ◯ Discriminant is 91 if $m=n=-1$.
   - ◯ Both roots will always be distinct and real if $m=n$.
   - ◯ Discriminant is 91 if $m=n=4$.

Solution: In order to talk about the roots of the quadratic equation $(5m+9)^2x^2-(3n+4)x+1=0$ where $m, n \in \mathbb{Z}$, we have to calculate the discriminant of $(5m+9)^2x^2-(3n+4)x+1$.

Discriminant of $(5m+9)^2x^2-(3n+4)x+1$ is

$$\begin{aligned}
&=(-3n-4)^2-4(5m+9)^2(1)\\
&=(3n+4)^2-(2(5m+9))^2\\
&=(3n+4+10m+18)(3n+4-10m-18)\\
&=(3n+10m+22)(3n-10m-14)
\end{aligned}$$

Both the roots of the equation will be equal if the discriminant is 0. Then either $(3n+10m+22)=0$ or $(3n-10m-14)=0$.
So both the roots are equal if $3n-10m-14=0$, i.e. $10m-3n+14=0$. But from this we can not say that if both the roots are equal then $10m-3n+14=0$, because both the roots can be equal even if $3n+10m+22$ is 0 and $10m-3n+14\neq 0$, as even in this case the discriminant will be 0.
So the first option is not correct, whereas the second one is correct.

If $m=n=-2$, then $(10m-3n+14)=(-20+6+14)=0$, So the discriminant will be 0. Hence both roots are equal if $m=n=-2$, i.e **the third option is also correct.**

If $m=n=-1$, then the discriminant is $(-3-10+22)(-3+10-14)=-63$. So **the fourth option is not correct.**
We have already seen that if $m=n=-2$ then the roots are equal. So **the fifth option is also not correct.**
If $m=n=4$, then the discriminant is $(12+40+22)(12-40-14)=-3108\neq 91$. Hence **the sixth option is also not correct.**

3 Numerical Answer Type (NAT):

9. In order to cover a fixed distance of 48 km, two vehicles start from the same place. The faster one takes 2 hrs less and has a speed 4 km/hr more than the slower one. Using the given information, answer the following questions. **1 mark+ 1 mark**

   (a) What is the speed (in km/hr) of the slower vehicle? [Ans: 8]

   (b) What is the time (in hrs) taken by the faster one? [Ans: 4]

   Solution: Let the speed of the slower vehicle be x km/hr. The time taken by the slower one to cover 48 km is $\frac{48}{x}$ hr. The speed of the faster one is $x + 4$ km/hr. So the time taken by the faster one to cover 48 km is $\frac{48}{x+4}$. It is given than the faster one takes 2 hrs less than the slower one to cover the distance. So we have,

$$\begin{aligned}\frac{48}{x} - \frac{48}{x+4} &= 2\\ \frac{x+4-x}{x(x+4)} &= \frac{2}{48}\\ \frac{x(x+4)}{4} &= 24\\ x^2 + 4x - 96 &= 0\\ (x+12)(x-8) &= 0\end{aligned}$$

   As the speed has to be positive, x must be 8.
   Hence the speed of the slower vehicle is 8 km/hr.
   The speed of the faster one is 12 km/hr. So the time taken by the faster one is $\frac{48}{12} = 4$ hrs.

10. Let the ratio of the length to the breadth of a flag be 3:2. Let the cost of the cloth required to make the flag be Rs. 4 per square metre and the cost of stitching along its perimeter be Rs. 2 per metre. If the cost of making (the cost of cloth and the cost of stitching) 6 such flags is Rs. 24, then answer the following questions.

1 mark+ 1 mark

(a) How much length (in metre) has to be stitched along the perimeter to make 6 flags in total ? [Ans: 10]

(b) What is the total area (in square metres) of 6 flags? [Ans: 1]

Solution: Let the length and the breadth of the flag be $3x$ and $2x$ metre. So the cloth required to make a flag is $6x^2$ square metre. The length of the perimeter is $10x$ metre. Hence the cost of the cloth required is $4 \times 6x^2 = 24x^2$ Rs. and the cost of stitching is $2 \times 10x = 20x$ Rs.

The cost of making of 6 flag is 24 Rs. So the cost of making 1 flag is 4 Rs.

Hence we have,

$$
\begin{aligned}
24x^2 + 20x &= 4 \\
6x^2 + 5x - 1 &= 0 \\
(x+1)(6x-1) &= 0
\end{aligned}
$$

x cannot be negative as the length and breadth cannot be negative. So $x = \frac{1}{6}$. Therefore the length of a flag is $\frac{1}{2}$ metre and the breadth of a flag is $\frac{1}{3}$. Hence, the perimeter is $5/3$ and the area of a flag is $\frac{1}{6}$ square metre.

Hence 10 metre has to be stitched along the perimeter to make 6 flags in total and the total area of 6 flags is 1 square metres.

Week - 5
Practice Assignment Solution
Quadratic Equations
Mathematics for Data Science - 1

NOTE:

- There are some questions which have functions with discrete-valued domains (such as month or year). For simplicity, we treat them as continuous functions.
- For a given quadratic equation $ax^2 + bx + c = 0, \ \ a \neq 0$:
  - ◯ Sum of roots $= -\frac{b}{a}$.
  - ◯ Product of roots $= \frac{c}{a}$.

1 Multiple Choice Questions (MCQ):

1. What will be the value of parameter k, if the discriminant of equation $4x^2+9x+10k = 0$ is 1?

   ◯ $\frac{82}{80}$
   ◯ $\frac{41}{80}$
   ◯ $\mathbf{\frac{1}{2}}$
   ◯ $\frac{41}{160}$
   ◯ 1
   ◯ None of the above.

   Solutions:
   Comparing the given equation $4x^2 + 9x + 10k = 0$ with the standard quadratic equation $ax^2 + bx + c = 0$:

$$a = 4, \ b = 9, \text{ and } c = 10k$$
$$\text{Discriminant } (d) = b^2 - 4ac$$
$$d = 9^2 - 4 \times 4 \times 10k$$
$$1 = 81 - 160k$$
$$\boldsymbol{k = \frac{1}{2}}$$

2. A boat has a speed of 30 km/hr in still water. In flowing water, it covers a distance of 50 km in the direction of flow and comes back in the opposite direction. If it covers this total of 100 km in 10 hours, then what is the speed of flow of the water (in km/hr)?

   - ◯ $5 - 5\sqrt{37}$
   - ◯ $-10\sqrt{6}$
   - ◯ $\mathbf{10\sqrt{6}}$
   - ◯ $20\sqrt{3}$
   - ◯ $-20\sqrt{3}$
   - ◯ 2

Solutions:
Total time taken by the boat = time taken by the boat in the direction of flow + time taken by the boat in the opposite direction of flow.
We know that:

$$time(t) = \frac{distance}{net\ speed}$$

Considering the direction of flow of water to be positive:
The net speed in the direction of flow (v_f) = speed of the boat in still water + speed of flow.
The net speed in the opposite direction of flow (v_b) = speed of the boat in still water - speed of flow.
Let the speed of flow be x then,

$$10 = \frac{50}{v_f} + \frac{50}{v_b}$$

$$10 = \frac{50}{30+x} + \frac{50}{30-x}$$

$$1 = \frac{5}{30+x} + \frac{5}{30-x}$$

$$1 = \frac{5(30-x+30+x)}{(30+x)(30-x)}$$

$$(30+x)(30-x) = 300$$

$$30^2 - x^2 = 300$$

$$x^2 = 600x = \pm 10\sqrt{6}$$

Speed of flow can not be negative therefore, the correct answer is $\mathbf{10\sqrt{6}}$.

3. A stunt man performs a bike stunt between two houses of the same height as shown in Figure 1. His bike (lowest part of the bike) makes an angle of θ at house A with the horizontal at the beginning of the stunt, follows a parabolic path and lands at house B with an angle of $(180 - \theta)$ with the horizontal.

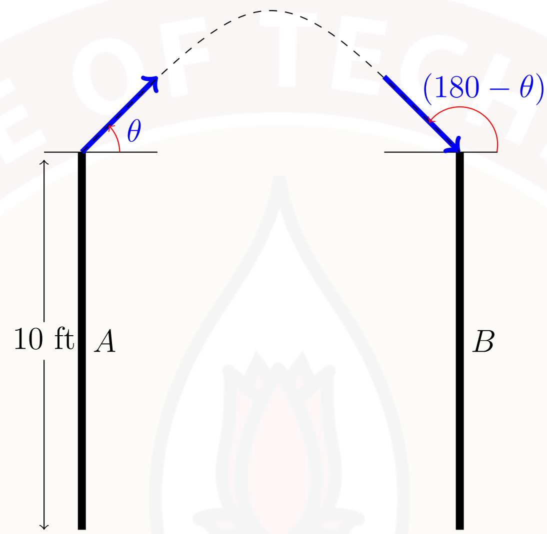


Figure PA-5.1

If the maximum height achieved by the bike is 12.5 ft from the ground and $tan\ \theta = 1$, then find the distance between the two houses.

- ○ 1 ft
- ○ 2.5 ft
- ○ 5 ft
- ○ **10 ft**
- ○ 15 ft
- ○ 20 ft

Solution:
Assuming the top of the house A to be origin, the horizontal direction as $X-$ axis, and the vertical direction as $Y-$ axis.

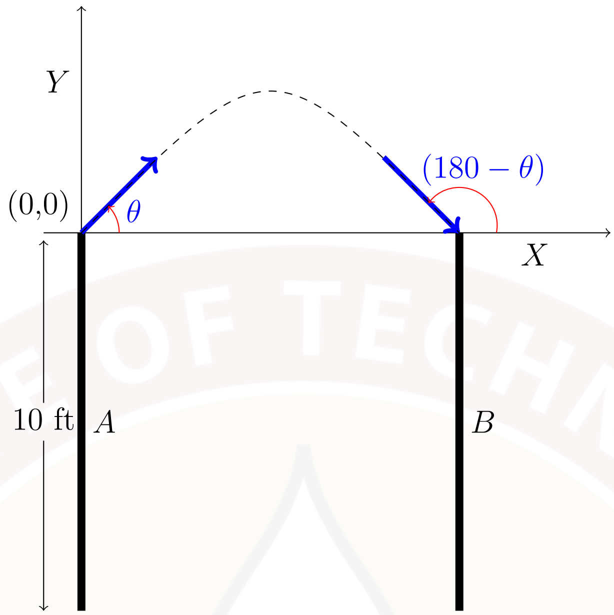


Figure M1W5PAS-3.1

Let the quadratic function representing the above curve be $f(x) = ax^2 + bx + c$.
Since the curve passes through the origin, we have $c = 0$.
The curve is making an angle θ with respect to positive $X-$ axis which means the slope of the tangent at the curve is $\tan\theta$.
We also know that the slope of the curve represented by quadratic function at $x = x$ is $2ax + b$. Therefore,

$$\begin{aligned} 2ax + b &= \tan\theta \\ 2a \times 0 + b &= 1 \\ b &= 1 \end{aligned}$$

The maximum height achieved by the bike is 12.5 ft which means the $y-$ coordinate of the vertex is $12.5 - 10 = 2.5$.
The $x-$ coordinate of the vertex for a curve represented by function $ax^2 + bx + c$ is

$$-\frac{b}{2a} = -\frac{1}{2a}$$

Therefore,

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
f(-\frac{1}{2a}) &= 2.5 \\
a \times (-\frac{1}{2a})^2 + 1 \times (-\frac{1}{2a}) + 0 &= 2.5 \\
\frac{1}{4a} - \frac{1}{2a} &= 2.5 \\
-\frac{1}{4a} &= 2.5 \\
a &= -\frac{1}{10}
\end{aligned}$$

Axis of symmetry,

$$x = -\frac{b}{2a} = -\frac{1}{2 \times (-1/10)} = 5$$

Because of symmetricity, the coordinate of landing point will be (10, 0).
Therefore two houses A and B are 10 ft apart.

2 Multiple Select Question (MSQ):

4. Given that $f_1(x) = -x^2 - 6x$ and $f_2(x) = x^2 + 6x + 10$. Let $f(x)$ be a function such that the domain of $f(x)$ is $[\alpha,\ \beta]$, where $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, then choose the set of correct options.

   - ○ Range of $f(x)$ is $[-1, 3]$.
   - ○ Range of $f(x)$ is $[0, 5]$.
   - ○ Domain of $f(x)$ is $[-5, 5]$.
   - ○ **Domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[-5, -1]}$.**
   - ○ **Inadequate information provided for finding the range of $\boldsymbol{f(x)}$.**
   - ○ Inadequate information provided for finding the domain of $f(x)$.

Solution:
Since $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, we have α and β are the abscissa of intersection points of both the curves.
To find the intersection points of the curves represented by $f_1(x)$ and $f_2(x)$:

$$f_1(x) = f_2(x)$$
$$-x^2 - 6x = x^2 + 6x + 10$$
$$2x^2 + 12x + 10 = 0$$

Here,

$$a = 2, b = 12, \text{ and } c = 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 2 \times 10}}{2 \times 2}$$

$$x = \frac{-12 \pm 8}{4} = -1, -5$$

Therefore,

$$\alpha = -5 \text{ and } \beta = -1.$$

Since the Domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[\alpha, \beta]}$ domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[-5, -1]}$.

The figure below gives a rough pictorial representation of $f_1(x)$ and $f_2(x)$ (drawn with smooth lines).
$f(x)$ can have any shape. An example is shown in the figure (drawn with dashed lines) for $f(x)$.

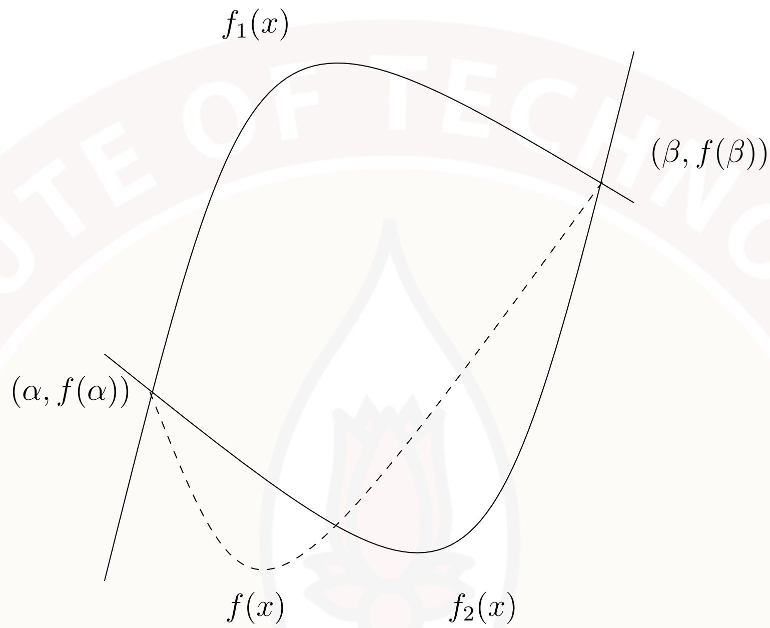


Figure M1W5PAS-4.1

As it is clear from figure that we do not know the minimum and maximum value of $f(x)$, we do not have proper data to comment on the range.

5. If $f(x) = 2x^2 + (5+k)x + 7$, $g(x) = 5x^2 + (3+k)x + 1$, $h_1(x) = f(x) - g(x)$, and $h_2(x) = g(x) - f(x)$, then choose the set of correct options.

- ◯ **Roots for $\boldsymbol{h_1(x) = 0}$ and roots for $\boldsymbol{h_2(x) = 0}$ are real, distinct, and the roots are the same for $\boldsymbol{h_1(x) = 0}$ and $\boldsymbol{h_2(x) = 0}$.**
- ◯ Roots for $h_1(x) = 0$ and roots for $h_2(x) = 0$ are real and distinct but the roots are not the same for $h_1(x) = 0$ and $h_2(x) = 0$.
- ◯ **Sum of roots of quadratic equation $\boldsymbol{h_1(x) = 0}$ will be $\boldsymbol{\frac{2}{3}}$.**
- ◯ **Product of roots of quadratic equation $\boldsymbol{h_2(x) = 0}$ will be -2.**
- ◯ **Axis of symmetry for both the functions $\boldsymbol{h_1(x)}$ and $\boldsymbol{h_2(x)}$ will be the same.**
- ◯ Vertex for both the functions $h_1(x)$ and $h_2(x)$ will be the same.

Solution:
Given that,

$$h_1(x) = f(x) - g(x)$$
$$h_1(x) = -(g(x) - f(x))$$
$$h_1(x) = -h_2(x)$$

Negative sign before any function does not make any changes on zeros of the function. Therefore, roots of $h_1(x) = 0$ and roots of $h_2(x) = 0$ will be same.

Now, for the properties of $h_1(x)$:

$$h_1(x) = f(x) - g(x) = 2x^2 + (5+k)x + 7 - (5x^2 + (3+k)x + 1)$$
$$h_1(x) = -3x^2 + 2x + 6$$
$$d = 2^2 - 4(-3) \times 6 > 0$$

It means the roots of $h_1(x)$ are real and distinct.
The roots of $h_1(x) = 0$ has the same as the roots of $h_2(x) = 0$, which means the roots for $h_2(x) = 0$ will also be real and distinct.

Sum of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $-\frac{b}{a} = -\frac{2}{(-3)} = \frac{2}{3}$.

Product of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $\frac{c}{a} = \frac{6}{(-3)} = -2$.

Multiplying a quadratic function by the minus sign does not make any changes in the

axis of symmetry.
The answer to all the above questions can be seen in the given figure.

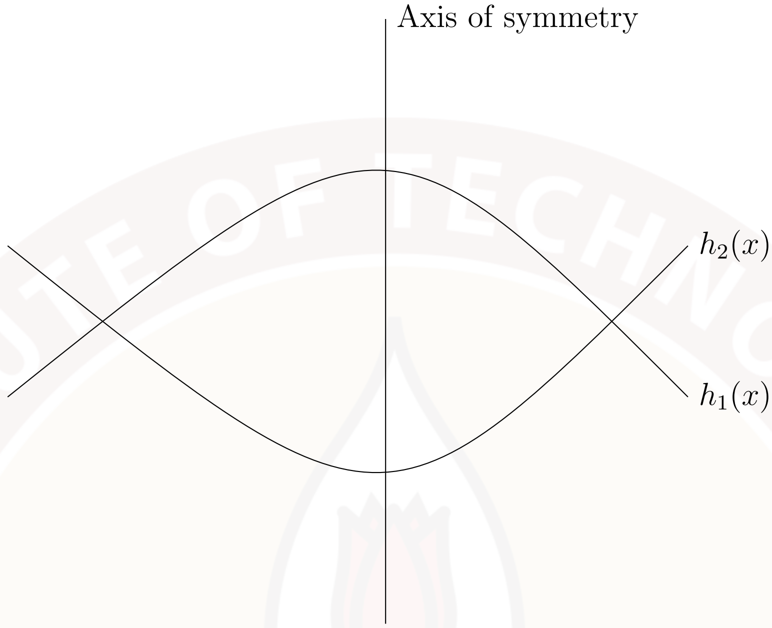


Figure M1W5PAS-5.1

Use following information for questions 6-8.
Vaishali wants to set up a small plate making machine in her village. Table P-5.1 shows the different costs involved in making the plates. Figure 5 shows her survey regarding the demand (number of packets of the plate) versus selling price of plate per packet (in ₹) per day.

Cost type	Cost
Electricity	₹1.5 per packet
Miscellaneous	₹6.5 per packet
Raw material	₹10 per packet

Table P-5.1

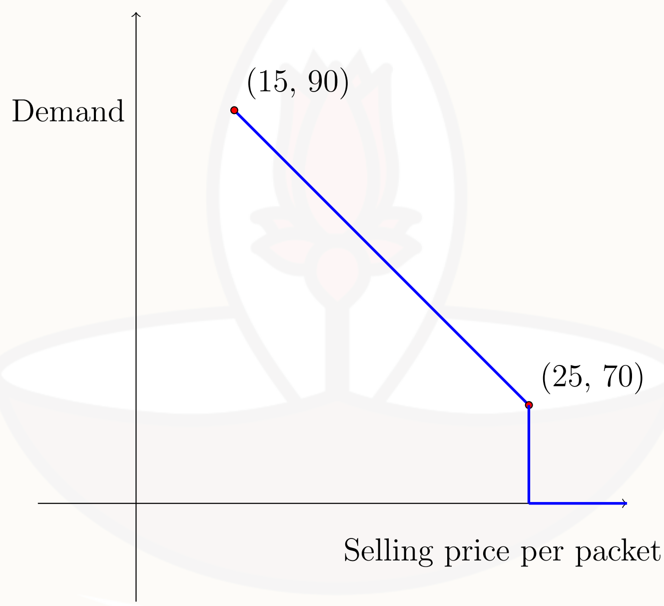


Figure PA-5.2

6. Choose the correct option which shows the profit obtained by Vaishali per day. Here, x is the selling price per packet.
   - ◯ $2(60 - x)$
   - ◯ $x(x - 18)$
   - ◯ $\mathbf{2(x - 18)(60 - x)}$
   - ◯ $2(x + 18)(60 - x)$
   - ◯ Inadequate information.

Solution:
From the figure, it is clear that the demand is dependent on the selling price of plates. Let y be the demand of the numbers of packets, then from two-points form of a line,

$$y - 90 = \frac{70 - 90}{25 - 15}(x - 15)$$
$$y - 90 = -2(x - 15)$$
$$y = -2x + 120$$

From the table, total cost per packet (in ₹)$= 1.5 + 6.5 + 10 = 18$

Per day profit = Demand per day$\times$(Selling price per packet - Cost per packet)

$$Profit = y(x - 18)$$
$$Profit = (-2x + 120)(x - 18)$$
$$Profit = 2(x - 18)(60 - x)$$

7. Choose the set of correct options.
   - ◯ **Vaishali should sell a packet with a minimum price of ₹18 so as not to incur any loss.**
   - ◯ Vaishali should sell a packet with a minimum price of ₹12 so as not to incur any loss.
   - ◯ To make maximum profit per day, the selling price per packet should be ₹39.
   - ◯ **To make maximum profit per day, the selling price per packet should be ₹25.**
   - ◯ Vaishali should sell a packet with maximum price of ₹60 so as not to incur any loss.
   - ◯ **Vaishali should sell a packet with a maximum price of ₹25 so as not to incur any loss.**

Solution:
From question 6,

$$Profit = 2(x - 18)(60 - x)$$

$$Profit = -2x^2 + 156x - 2160 \tag{1}$$

To get minimum selling price with no loss, profit should be zero. Therefore,

$$2(x - 18)(60 - x) = 0$$
$$x = 18 \text{ or } 60$$

From the graph given in question, it is clear that we can not sell a packet at ₹60, because the demand will be zero.
Therefore, the minimum selling price will be ₹18 per packet.

Since the profit is a quadratic function of the selling price (x) in equation (1) with negative coefficient of x^2.
Therefore, the maximum profit will occur at

$$x = -\frac{b}{2a} = -\frac{156}{2 \times (-2)} = 39$$

A rough pictorial representation is shown in Figure below,

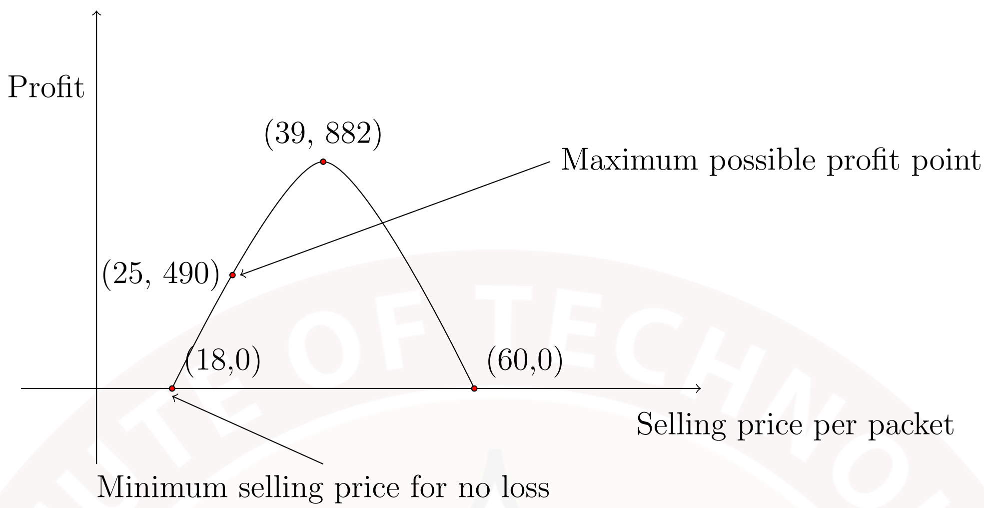


Figure M1W5PAS-7.1

The increase in selling price will result in profit increment till 39. But the maximum acceptable selling price is ₹25, therefore the maximum profit will occur at a selling price of ₹25.

So, from the figure it is clear that the maximum selling price for no loss is ₹60 but we can not increase the price beyond ₹25. Therefore, the maximum profit to incur any loss will be ₹25.

3 Numerical Answer type (NAT):

8. What should be the price of plate per packet (₹) to make a profit of ₹490 per day?
[Hint: $(x-53)$ a factor of $2(-x^2+78x-1325)$.] [Ans: 25]

Solution:
From equation (1) $Profit = -2x^2 + 156x - 2160$

$$-2x^2 + 156x - 2160 = 490$$
$$-2x^2 + 156x - 2650 = 0$$
$$2(-x^2 + 78x - 1325) = 0$$

It is given that $(x-53)$ a factor of $2(-x^2+78x-1325)$. So dividing $2(-x^2+78x-1325)$ by $(x-53)$ we will get $-2x+50$.
Therefore,

$$2(-x^2 + 78x - 1325) = 0$$

$$(x-53)(2x-50) = 0$$

If

$$x - 53 = 0$$

$$x = 53$$

But selling price can not go beyond 25.
Now if,

$$2x - 50 = 0$$
$$x = \frac{50}{2}$$
$$x = 25$$

Therefore, the selling price of plate should be ₹25 .

9. What will be the value of $m+n$ if the sum of the roots and the product of the roots of equation $(5m+5)x^2-(4n+3)x+10=0$ are 3 and 2 respectively?

Solution:

We know that the sum of the roots of an equation $ax^2+bx+c=0$ is $\frac{-b}{a}$ and the product of its roots is $\frac{c}{a}$.

Here, $a=5m+5, b=-(4n+3), c=10$. Substituting these values we get,
The product of the roots of the given equation

$$\frac{c}{a}=\frac{10}{5m+5}=2$$

$$5m+5=5$$

$$m+1=1$$

$$\boldsymbol{m=0}$$

The sum of the roots as

$$\frac{-b}{a}=\frac{-(-(4n+3))}{5m+5}=3$$

$$4n+3=3(5m+5)$$

For $m=0$

$$4n+3=3\times 5$$

$$4n=12$$

$$\boldsymbol{n=3}$$

Therefore,

$$m+n=0+3=3.$$

10. What will the sum of two positive integers be if the sum of their squares is 369 and the difference between them is 3?. **Solution:**

Let a and b be the two positive integers. Given that

$$a^2 + b^2 = 369 \tag{2}$$
$$a - b = 3 \tag{3}$$

Squaring equation (3) on both sides, we get

$$\begin{aligned} (a-b)^2 &= 3^2 \\ a^2 - 2ab + b^2 &= 9 \\ 369 - 2ab &= 9 \\ 2ab &= 369 - 9 \end{aligned}$$

$$2ab = 360$$

Now, to find the sum of the integers

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^2 &= 369 + 360 \\ (a+b)^2 &= 729 \\ a+b &= \pm\sqrt{729} = \pm 27 \end{aligned}$$

As a and b are positive integers, their sum should also be a positive integer.
Therefore, $a + b = 27$.

Mathematics for Data Science - 1

Practice Assignment Solutions

Week-4

1. Multiple Choice Questions (MCQ):

1. What will be the equation of the tangent to the curve $f(x) = 2x^2 + 9x + 20$ at point (-3,11)?

- ◯ $y = 3x$
- ◯ $\boldsymbol{y = -3x + 2}$
- ◯ $y = -3x + 20$
- ◯ $y = -\frac{x}{3} + 2$
- ◯ $y = \frac{x}{3} + 20$
- ◯ $y = -\frac{x}{3}$

Solution:

A rough diagram is given in the Figure PS-4.1 .

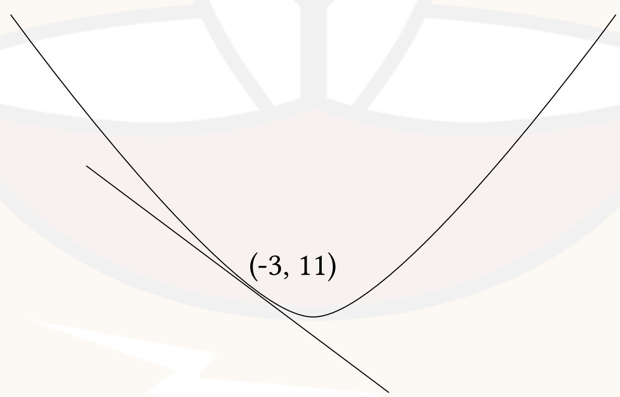


Figure PS-4.1

Let the equation of the tangent be $y = mx + c$, where m is the slope of the tangent line.
Note that m is also the slope of f at (-3, 11).
The slope of any quadratic function $g(x) = ax^2 + bx + c$, where $a \neq 0$ at x will be $2ax + b$.

Therefore, at $x = -3$,

$$m = 2ax + b \implies m = 2 \times 2 \times (-3) + 9 \implies m = -3$$

Since the tangent passes through the point (-3, 11), it should satisfy the equation of the tangent.

$$y = mx + c \implies 11 = -3 \times (-3) + c \implies c = 2.$$

So, the equation of the tangent will be $\boldsymbol{y = -3x + 2}$.

2. Find the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$.

- ◯ $\frac{1}{\sqrt{2}}$
- ◯ $\boldsymbol{\sqrt{2}}$
- ◯ $1 + \sqrt{2}$
- ◯ $1 + \frac{1}{\sqrt{2}}$

Solution:

Given $y = 4x^2$. Observe that, on comparing the above with the the general form of a quadratic function $f(x) = ax^2 + bx + c$, we have $b = 0$ which means Y-axis is the axis of symmetry. Also $c = 0 \implies$ the curve represented by this function will pass through the origin.
$-b/2a = 0$ and at $x = 0 \implies y = 0$ which means the vertex is at the origin and $a > 0 \implies$ the parabola is upward opened.
$y = 2$ is a constant function and it will pass through the point (0, 2). A rough diagram is given in the Figure PS-4.2

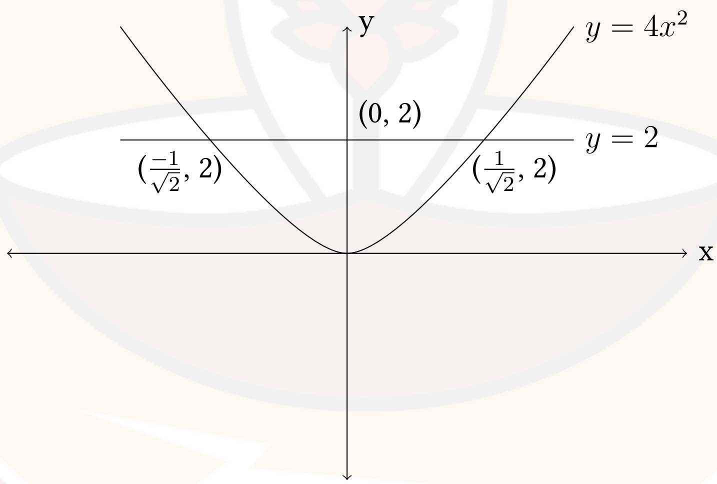
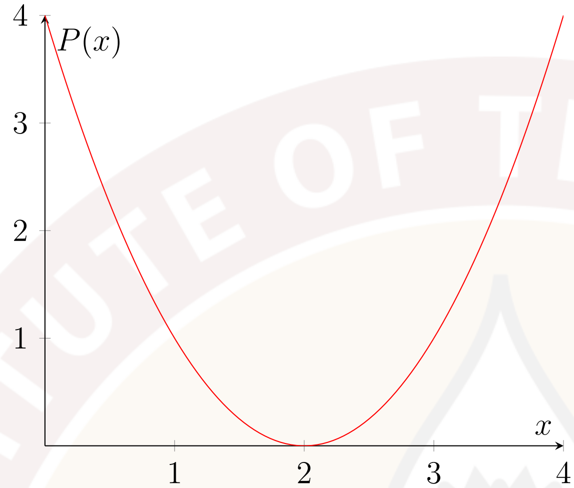
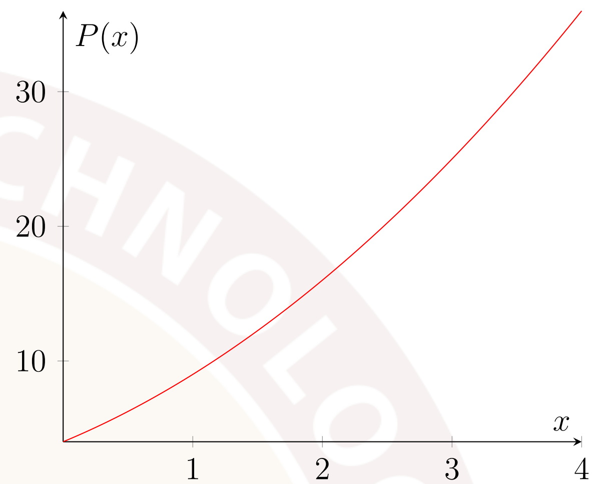


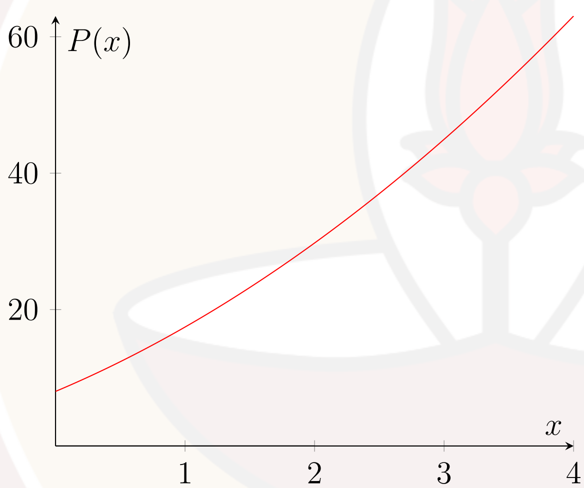
Figure PS-4.2

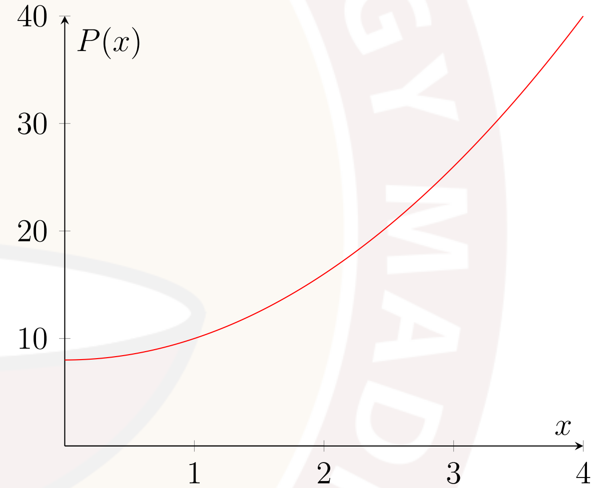
At the intersection points, $4x^2 = 2 \implies x = \pm\frac{1}{\sqrt{2}}$ which means the intersection points will be $(\frac{-1}{\sqrt{2}}, 2)$ and $(\frac{1}{\sqrt{2}}, 2)$.
Observe that these intersecting points will be the end points of the required line segment on the straight line $y = 2$.
Therefore, the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$ will be $\sqrt{(2-2)^2 + (\frac{1}{\sqrt{2}} - (\frac{-1}{\sqrt{2}})^2} = \sqrt{0 + (\frac{2}{\sqrt{2}})^2} = \sqrt{0 + (\sqrt{2})^2} = \boldsymbol{\sqrt{2}}$.

3. Mr. Mehta has two sons. Both sons send money to their father each month separately as $M_1(x) = (x-2)^2$ and $M_2(x) = (x+2)^2$ respectively. If x denotes the month, then choose the curve which best represents the total amount (P(x)) received by Mr. Mehta every month.

◯ 

◯ 

◯ 

◯ 

Solution:

Given,

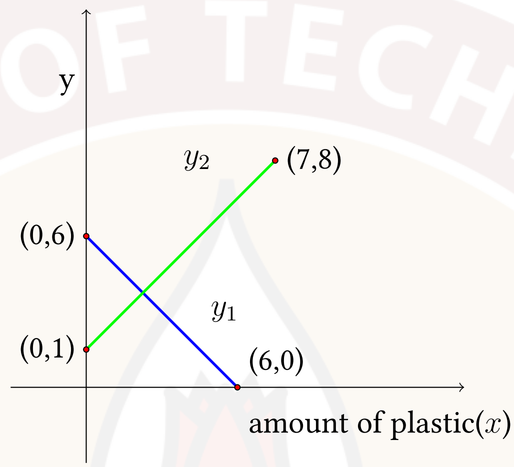
$$M_1(x) = (x-2)^2$$
$$M_2(x) = (x+2)^2.$$

So, the total amount received by Mr. Mehta is:

$$P(x) = M_1(x) + M_2(x) = (x-2)^2 + (x+2)^2 = x^2 - 4x + 4 + x^2 + 4x + 4$$
$$\Rightarrow P(x) = 2x^2 + 8.$$

In $P(x),\ b = 0$ which means Y-axis will be the axis of symmetry of the curve $p(x)$.
Now, the curve shown in the first option is not symmetric about the line $x = 0$. So, option 1 is incorrect.
The curve in the second option, passes through the origin but that is not the case for $P(x)$ as $x = 0 \implies P(x) = 8$. So, option 2 is incorrect.
The curve in the third option, does not pass through (4, 40). So, option 3 is also incorrect.
Now, the curve in the last option will pass through the points (0, 8), (1, 10), and (4,40). So, the curve in the fourth option will be the best curve that represents the total amount received by Mr.Mehta every month.

4. A civil engineer found that the durability d of the road she is laying depends on two functions y_1 and y_2 as follows: $d = ay_1y_2$ where $a > 0$. Functions y_1 and y_2 depend on the amount of plastic (x) mixed in bitumen, and their variations are shown in the graph given below. Find the values of functions y_1 and y_2 such that the durability of the road is maximum.



Solution:

Given, the durability of the road $d = ay_1y_2$.
From the given graph, the equations of the lines:

$$y_1 = 6 - x$$
$$y_2 = x + 1$$
$$\implies d = ay_1y_2 = a(6-x)(x+1) = -ax^2 + 5ax + 6a$$

Here $a > 0 \implies -a < 0$ which means the curve represented by d is open downward and the durability d of the road is the maximum at $x = \frac{-b}{2a} = \frac{-5a}{2(-a)} = \frac{5}{2}$.
Therefore, the value of $y_1 = 6 - x = 6 - \frac{5}{2} = \frac{7}{2}$ and the value of $y_2 = x + 1 = \frac{5}{2} + 1 = \frac{7}{2}$.

5. Let A be the set of all points on the curve defined by the function $f_1(x) = x^2 - x - 42$ and let B be the set of all points on the curve f_2 defined by the reflection of the curve f_1 with respect to X axis. If C is the set of all points on the axes then choose the correct option regarding the cardinality of set D where $D = (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

   - ◯ infinite.
   - ◯ 8
   - ◯ **4**
   - ◯ 6
   - ◯ 2
   - ◯ zero.

   Solution:

   For the function $f_1(x) = x^2 - x - 42$, $a > 0 \implies$ opening upward, $-\frac{b}{2a} = \frac{1}{2} \implies x = \frac{1}{2}$ is the axis of symmetry.
   $x = 0 \implies f_1(0) = -42$ so, it will pass through the point (0, -42).
   The reflection of $f_1(x)$ with respect to $X-$ axis i.e. $f_2(x)$ will pass through the point (0, 42).
   For intersection points of both curves:
   Both the curves will be intersecting on same place on $X-$ axis as they are mirror image of each other around $X-$ axis. A rough diagram is given in the Figure PS-4.3

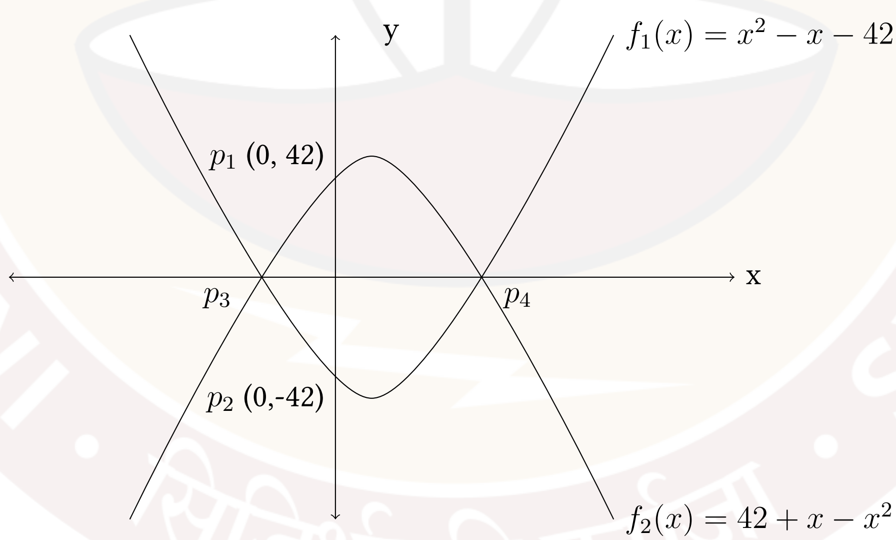


Figure PS-4.3

   Since A is the set of all points on the curve f_1, B will be the set of all points on the curve f_2 and C will be the set of all points on the X-axis or Y-axis.

From Figure PS-4.3,
$A\cap\ B$ is the set of all points which are on f_1 and f_2. Therefore, $A\cap\ B$ = $\{p_3, p_4\}$.
$A\cap C$ is the set of all points which are on the curve f_1 and on the X-axis or Y-axis. Therefore, $A\cap\ C$ = $\{p_3, p_4, p_2\}$.
$B\cap\ C$ is the set of all points which are on the curve f_2 and on the $X-$axis or $Y-$axis. Therefore, $B\cap\ C$ = $\{p_3, p_1, p_4\}$.
Now, $D = (A\cap\ B)\cup\ (A\cap\ C)\cup\ (B\cap\ C) = \{p_1, p_1, p_1, p_4\}$ and therefore, the cardinality of D is 4.

6. Let $f_1(x) = x^2 - 25$. Let A be the set of all points inside the region by the curves representing $f_1(x)$ and its reflection $f_2(x)$ with respect to $X-$ axis (excluding the points on curve). Choose the correct option.

- ◯ The cardinality of A is 2.
- ◯ The cardinality of A is 4.
- ◯ **$\boldsymbol{Y-}$ coordinates of the points in set A belong to the interval $\mathbf{(-25, 25)}$.**
- ◯ $Y-$ coordinates of the points in set A belong to the interval $[-25, 25]$.
- ◯ $X-$ coordinates of the points in set A belong to the interval $[-5, 5]$.
- ◯ $X-$ coordinates of the points in set A will be all real numbers because f_1 is a quadratic function.

Solution:

A rough diagram is shown in the Figure PS-4.4

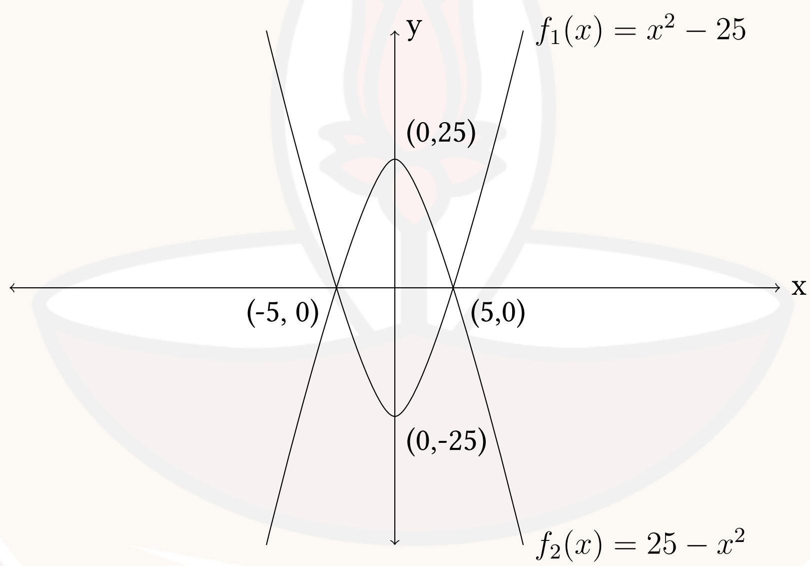


Figure PS-4.4

From the Figure PS-4.4, observe that the set A is infinite, because the region between the two curves f_1 and f_2 has infinitely many points. Therefore, the cardinality of A is not finite. So, options 1 and 2 are wrong.

Also, the region is in between the lines y = +25 and y = -25. Therefore, Y-coordinates of all the points in set A lie between -25 and +25 (-25 and +25 are excluded because they are points on the curves). So, option 3 is correct and option 4 is incorrect because -25 and +25 are included.

Also, the points in A are in between the lines x = -5 and x = +5 (-5 and +5 are excluded because they are points on the curves). Therefore, the X-coordinates of the points in set A belong to the interval (-5, 5). So, options 5 and 6 are incorrect.

2. Multiple Select Questions (MSQ):

1. **Choose the correct set of options regarding the function $\mathbf{f(x) = x^2 + 6x + 8}$**

   ◯ $y = -3$ is the axis of symmetry.

   ◯ **-2 and -4 are the zeroes of the above function**.

   ◯ The maximum value of the above function is -1.

   ◯ **Slope of the function at $(-3, -1)$ is zero**.

   ◯ **$2x + 6$ is the slope of this curve at any given x**.

   ◯ The function is symmetric around $x = 3$.

Solution:

Given, $f(x) = x^2 + 6x + 8$.
The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-6}{2} = -3$.
Therefore, $x = -3$ is the axis of symmetry of curve $f(x)$. So, options 1 and 6 are incorrect.

For zeros:

$$f(-2) = (-2)^2 + 6(-2) + 8 = 4 - 12 + 8 = 0$$
$$f(-4) = (-4)^2 + 6(-4) + 8 = 16 - 24 + 8 = 0$$

Hence, -2 and -4 are the zeros of the given function. So, option 2 is correct.

As $f(x)$ is an upward parabola, the maximum value of the function is + ∞ at $x = +\infty$. So, option 3 is incorrect.

Now, at $x = -3$, $f(x) = f(-3) = (-3)^2 + 6(-3) + 8 = 9 - 18 + 8 = -1$.
Therefore, the point (-3, -1) is the vertex of the given function. Also, the slope of the function at vertex is always 0. So, option 4 is correct.

We know that the slope of any given quadratic function $g(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}$ at point $(x, g(x))$ is $2ax + b$. Here, $a = 1, b = 6$ and $c = 8$
Therefore, the slope of $f(x)$ is $2x + 6$ at any given x. So, option 5 is correct.

2. A quadratic function f is such that its value decreases over the interval $(-\infty, -2)$ and increases over the interval $(-2, \infty)$, and $f(0) = f(-4) = 23$. Then, f can be

- ◯ $-3x^2 - 12x + 23$
- ◯ $\mathbf{3x^2 + 12x + 23}$
- ◯ $5(x-2)^2 + 3$
- ◯ $\mathbf{5(x+2)^2 + 3}$
- ◯ $\boldsymbol{ax^2 + 4ax + 23,\ a > 0}$
- ◯ $ax^2 + 4ax + 23,\ a < 0$

Solution:

Given, the values of f decreases over $(-\infty, -2)$ and increases over interval $(-2, \infty)$. Also, $f(0) = f(-4) = 23$.
The curve f is roughly shown in the Figure PS-4.5.

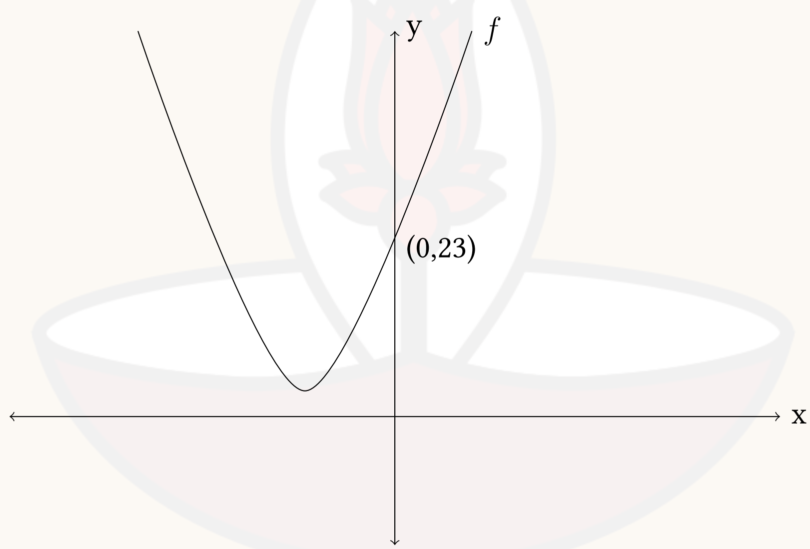


Figure PS-4.5

Suppose $f(x) = ax^2 + bx + c$, for any $a, b, c \in \mathbb{R}$.
We have $f(0) = 23 = a(0)^2 + b(0) + c = c \Rightarrow c = 23$.
Now, $f(-4) = 23 = a(-4)^2 + b(-4) + 23 \Rightarrow 16a - 4b = 0 \Rightarrow b = 4a$.
As the curve f which is shown in the Figure PS-4.5 is an upward parabola, the value of a should be positive.

Therefore, the quadratic function that satisfies the given conditions will be of the form $f(x) = ax^2 + 4ax + 23$, for all $a > 0$. So, option 5 is correct and option 6 is incorrect.

If $a = 3$, then f can be $3x^2 + 12x + 23$. So, option 2 is correct.

If $a = 5$, then f can be $5x^2 + 20x + 23 = 5(x + 2)^2 + 3$. So, option 4 is correct.

In option 1, the leading coefficient of the given function is $-3 = a < 0$. So, it is incorrect.

3. Suppose one root of a quadratic equation of the form $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$, is $2 + \sqrt{3}$. Then choose the correct set of options.

   - ◯ **There can be infinitely many such quadratic equations.**
   - ◯ There is no such quadratic equation.
   - ◯ There is a unique quadratic equation satisfying the properties.
   - ◯ $\boldsymbol{x^2 - 4x + 1 = 0}$ **is one such quadratic equation**.
   - ◯ $x^2 - 2x - 3 = 0$ is one such quadratic equation.

Solution:

Given, $2 + \sqrt{3}$ is a root of $ax^2 + bx + c = 0$. One root of the quadratic equation is known. The other root can be any real number k.
For each value of k we will have a different quadratic equation. Therefore, there can be infinitely many quadratic equations that have $2 + \sqrt{3}$ as a root. So, option 1 is correct and options 2,3 are incorrect.

Now, option 4 is correct because the function value (at $x = 2 + \sqrt{3}$) is

$$(2+\sqrt{3})^2 - 4(2+\sqrt{3}) + 1 = 4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1 = 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is a root of } x^2 - 4x + 1 = 0.$$

Option 5 is incorrect because the function value (at $x = 2 + \sqrt{3}$) is

$$(2+\sqrt{3})^2 - 2(2+\sqrt{3}) - 3 = 4 + 4\sqrt{3} + 3 - 4 - 2\sqrt{3} - 3 = 2\sqrt{3} \neq 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is not a root of } x^2 - 2x - 3 = 0.$$

4. A company's profits are known to be dependent on the months of a year. The profit pattern (in lakhs of Rupees) from January to December is $P(x) = -2x^2 + 25x$. Here, x represents the month number, starting from 1 (for January) and ending at 12 (for December). On this basis, choose the correct option.

   - ◯ **The maximum profit in a month is Rs.78 lakhs.**
   - ◯ The maximum profit in a month is Rs.78.125 lakhs.
   - ◯ The maximum profit in a month is Rs.77 lakhs.
   - ◯ **The maximum profit is recorded in June.**
   - ◯ The profit in December is 144 lakhs.
   - ◯ None of the above.

Solution:

The profit of the company is given as $P(x) = -2x^2 + 25x$. Observe $P(x)$ is downward open. So, the maximum profit will be recorded at vertex.
The X-Coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-25}{2(-2)} = 6.25$
So, the vertex lies between the lines $x = 6$ and $x = 7$
Therefore, the maximum profit will be recorded in the month of June($x = 6$) or July($x = 7$).
The profit(in lakhs of Rupees) in June is

$$P(6) = -2(6)^2 + 25(6) = -72 + 150 = 78$$

and profit(in lakhs of Rupees) in July is

$$P(7) = -2(7)^2 + 25(7) = -98 + 175 = 77$$

Therefore, the maximum profit of Rs.78 lakhs is recorded in the month of June. So, options 1 and 4 are correct.

The profit (in lakhs of Rupees) in December is

$$P(12) = -2(12)^2 + 25(12) = -288 + 300 = 12$$

So, option 5 is incorrect.

5. Raghav sells 2000 packets of bread for Rs. 20000 each day, and makes a profit of Rs. 4,000 per day. He finds that if the cost price increases by Rs. x per packet, he can increase the selling price by Rs. $2x$ per packet. However, when this price increase happens, he loses $200x$ of his customers. Choose the correct options.

   - ◯ **For the maximum profit per day, cost price is Rs. 12 per packet**.
   - ◯ For the maximum profit per day, cost price is Rs. 4 per packet.
   - ◯ For the maximum profit per day, the sale price increases by Rs. 4 per packet.
   - ◯ For the maximum profit per day, Raghav will lose 400 customers.
   - ◯ **The maximum difference in profit per day could be Rs. 3200**.
   - ◯ The maximum difference in profit per day could be Rs. 7200.

Solution:

The selling price of bread $\frac{20000}{2000}$ = 10 Rupees per packet.
We know that, selling price - cost price = profit ⇒ 20000 - cost price = 4000 ⇒ cost price per day= 16000.
Therefore, the cost price is = $\frac{16000}{2000}$ = 8 Rupees per packet.

Now, if the cost price of each packet increases to $8 + x$ and the selling price of each packet is increased to $10 + 2x$, then the customers left will be $2000 - 200x$.

So, the total profit (say P) in terms of x:

profit = (selling price of each packet- cost price of each packet) x (number of customers)

$$\implies P(x) = \{(10 + 2x) - (8 + x)\} \times (2000 - 200x)$$
$$\implies P(x) = (2 + x)(2000 - 200x)$$
$$\implies P(x) = 4000 + 1600x - 200x^2.$$

The maximum profit occurs at $x = -\frac{b}{2a}$ = $-\frac{1600}{2(-200)}$ = 4.

Hence, for the maximum profit per day:

cost price per packet = $8 + x$ = 8 + 4 = **12**.
sale price per packet = $10 + 2x$ = 10 + 8 = **18**
The customers he loses = $200x$ = 200(4) = **800**.
Maximum profit = $4000 + 1600x - 200x^2$ = $4000 + 1600(4) - 200(4)^2$ = **7200**

Therefore, maximum difference in profit = 7200 - 4000 = **3200 Rupees**.
So, the options 1 and 5 are correct.

3. Numerical answer type(NAT):

1. A farmer has a wire of length 576 metres. He uses it to fence his rectangular field to protect it from animals. If he fences his field with four rounds of wire, and the field has the maximum area possible to accommodate such a fencing, what is the area (in square metres) of the field?

   Solution:

   Suppose, the length of the rectangular field is 'l' metres and breadth of the rectangular field is 'm' metres. So, the perimeter of the rectangular field will be $2(l+m)$.
   Now, as he fences his field with four rounds of wire, we have four times the perimeter of the field which, in turn, is equal to the length of the wire. i.e,

$$4(2(l+m)) = 576$$
$$\Rightarrow l+m = \frac{576}{8} = 72$$
$$\Rightarrow m = 72 - l$$

Area of field (A) = lm

$$\Rightarrow A = l(72-l) = 72l - l^2$$

The maximum area of the field (A_{max})= $-\frac{b^2}{4a} + c$

$$A_{max} = -\frac{72^2}{4\times(-1)} + 0$$

$$\boldsymbol{\Rightarrow A_{max} = 1296} \textbf{ square metres}$$

2. Consider the quadratic function $f(x) = x^2 - 2x - 8$. Two points P and Q are chosen on this curve such that they are 2 units away from the axis of symmetry. R is the point of intersection of axis of symmetry and the $X - axis$. And S is the vertex of the curve. Based on this information, answer the following:
   (a) What is the height of $\triangle PQR$ taking PQ as the base?
   (b) What is the height of $\triangle PQS$ taking PQ as the base?

Solution:

The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$ and two units away points will be $x = 1 + 2 = 3$ and $x = 1 - 2 = -1$.
At $x = 3 \Rightarrow f(x) = -5$ and at $x = -1 \Rightarrow f(x) = -5$. Also, the vertex of the curve is (1, -9).
A rough diagram can be drawn with this information as shown in Figure PS-4.6.

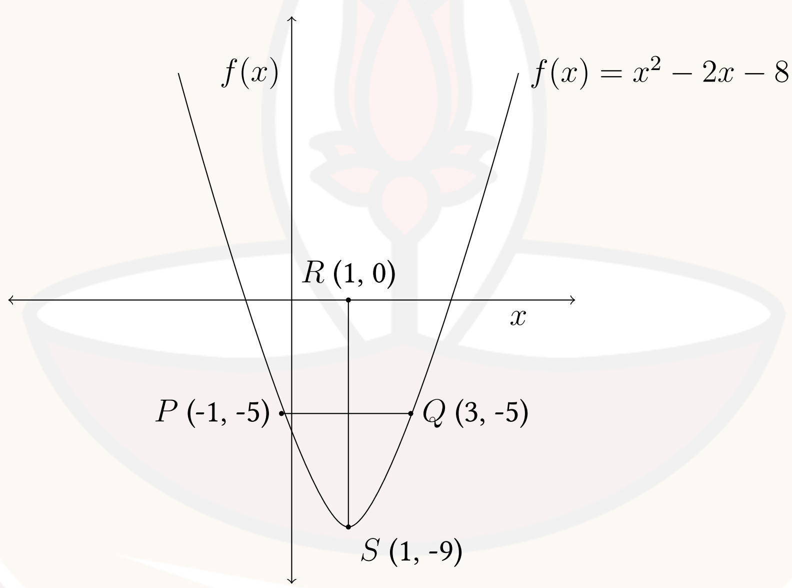


Figure PS-4.6

(a) From the above figure, the height of $\triangle PQR$ taking PQ as the base will be the distance between lines $y = 0$ and $y = -5$ and that is equal to 0 - (-5) = 5 units.

(b) From the above figure, the height of $\triangle PQS$ taking PQ as the base will be the distance between lines $y = -5$ and $y = -9$ and that is equal to (-5) - (-9) = 4 units.

Week - 3
Practice problems
Straight line
Mathematics for Data Science - 1

1 Multiple Choice Questions (MCQ):

1. A vehicle is travelling on a straight line path and it passes through the points $A(-4, 2)$, $B(-1, 3)$, and $C(2, \mu)$. The value of μ is:

   - ◯ 2
   - ◯ **4**
   - ◯ -2
   - ◯ 10

   Solution:

   Since the vehicle is travelling on a straight line path and passes through the points A, B, and C, it follows that A, B, and C are collinear. Hence the slope of the straight line path joining A and B will be equal to the slope of the straight line path joining B and C. Using the slope formula for two points, we have

$$\frac{3-2}{-1+4} = \frac{\mu - 3}{2+1}$$

$$\implies \boldsymbol{\mu = 4}.$$

2. Suppose two boats are starting their journey from the ferry ghat A (considered as the origin), one towards ferry ghat B along the straight line $y = -2x$ and the other towards the ferry ghat C along a straight line perpendicular to the path followed by B. The river is 1 km wide uniformly and parallel to the X-axis. Suppose Rahul wants to go to the exact opposite point of A along the river.

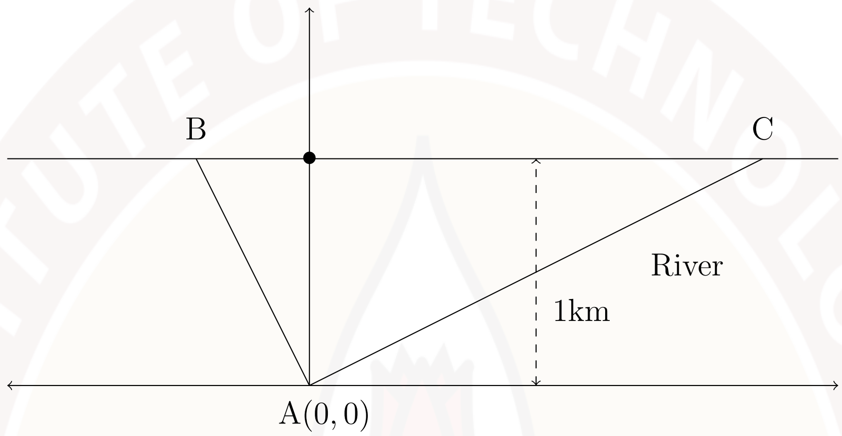


Figure PS-3.1

Then, answer the following questions.

(a) How much total distance does Rahul have to travel to reach his destination if he takes the boat towards ferry ghat B?

- ○ $\sqrt{5}$
- ○ $\sqrt{5} + 2$
- ○ $\frac{\sqrt{5}}{2}$
- ○ $\boldsymbol{\frac{\sqrt{5}+1}{2}}$

Solution:

See the Figure PS-3.2 for reference:

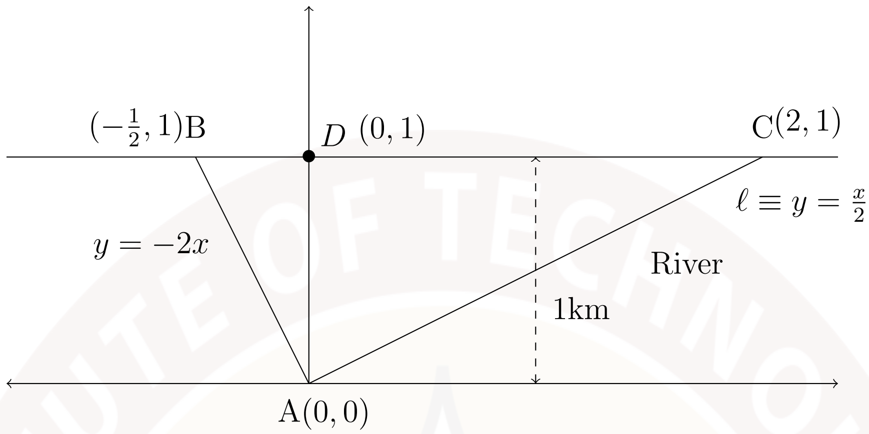


Figure PS-3.2

Since the point A is assumed to be the origin, the side of the river from which Rahul is starting his journey is considered to be the X-axis. The path towards Rahul's destination, which is perpendicular to the X-axis, is hence the Y-axis. Let D be Rahul's destination, which is 1 km away from the point A and is on the opposite side of the river. It follows that the point D is $(0, 1)$.

Hence, the equation of the line representing the opposite side of the river is $y = 1$. Solution of the equations $y = 1$ and $y = -2x$ gives the location of ferry ghat B which is the point $(-\frac{1}{2}, 1)$.

Using the distance formula between two points, the distance between ferry ghat A and ferry ghat B is given by

$$\sqrt{(-\frac{1}{2} - 0)^2 + (1 - 0)^2} = \frac{\sqrt{5}}{2}\text{units}$$

Similarly, the distance between ferry ghat B and the point D is $\frac{1}{2}$ units.

Hence, the total distance that Rahul has to travel to reach his destination D if he takes the boat toward ferry ghat B is given by

$$\frac{\sqrt{5}}{2} + \frac{1}{2} = \frac{\sqrt{5}+1}{2} \text{ units}$$

(b) How much total distance does Rahul have to travel to reach his destination if he takes the boat towards ferry ghat C?

- ◯ $\sqrt{5}$
- ◯ $\boldsymbol{\sqrt{5}+2}$
- ◯ $\frac{\sqrt{5}}{2}$
- ◯ $\frac{\sqrt{5}+1}{2}$

Solution: Let ℓ denote the path towards ferry ghat C from A. The equation of path ℓ will be $y = mx$ since it passes through the origin. Since ℓ is perpendicular to the line $y = -2x$, which has a slope $m_1 = -2$, it follows that

$$m = -\frac{1}{m_1} = \frac{1}{2}$$

$\implies$ the equation of ℓ is $y = \frac{x}{2}$.

Solution of the equations $y = \frac{x}{2}$ and $y = 1$ gives the location of ferry ghat C which is (2,1).

Using the distance formula between two points, the distance between ferry ghat A and ferry ghat C is

$$\sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5} \text{ units}$$

Similarly, the distance between ferry ghat C and the destination point D is 2 units.

Hence, the total distance that Rahul has to travel to reach his destination D if he takes the boat towards ferry ghat C is $\sqrt{5} + 2$ units.

3. Suppose a bird is flying along the straight line $4x - 5y = 20$ on the plane formed by the path of the flying bird and the line of eye point view of a person who shoots an arrow which passes through the origin and the point $(10, 8)$. What is the point on the co-ordinate plane where the arrow hits the bird?

   - ◯ $(20, 12)$
   - ◯ $(25, 16)$
   - ◯ **The arrow will miss the bird.**
   - ◯ Inadequate information.

   Solution:
   Using the two point form of a line, the equation of the path of arrow passing through the origin and the point (10,8) is

$$(y - 0) = \frac{8 - 0}{10 - 0}(x - 0) \implies 8x - 10y = 0$$

The slope intercept form of the above line is given by

$$y = \frac{8}{10}x$$

From the above line, we obtain the slope as

$$m_1 = \frac{8}{10} = \frac{4}{5}$$

Similarly, for the path of the bird along the straight line $4x - 5y = 20$, we get the slope

$$m_2 = \frac{4}{5}$$

Here, $m_1 = m_2$,
That is, the lines $8x - 10y = 0$ and $4x - 5y = 20$ have the same slope. Therefore, the path of flying bird and the path of the arrow are parallel to each other. Hence, the arrow will miss the bird.

4. We plot the displacement (S) versus time (t) for different velocities as it follows the equation $S = vt$, where v is the velocity. Identify the best possible straight lines in the Figure P-3.2 for the given set of velocities.

Table PS-3.1

v_1	v_2	v_3	v_4
1	-2	0.5	-1

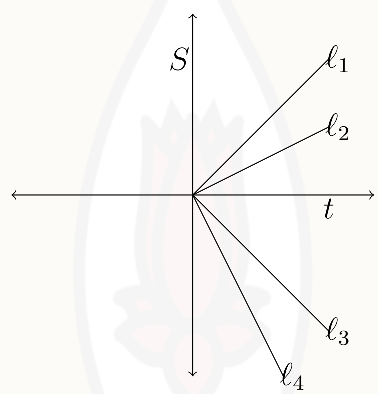


Figure PS-3.3

- ◯ $v_1 \to \ell_1$, $v_2 \to \ell_2$, $v_3 \to \ell_3$, and $v_4 \to \ell_4$.
- ◯ $v_1 \to \ell_1$, $v_2 \to \ell_4$, $v_3 \to \ell_3$, and $v_4 \to \ell_2$.
- ◯ $\boldsymbol{v_1 \to \ell_1, v_2 \to \ell_4}$**,** $\boldsymbol{v_3 \to \ell_2}$**, and** $\boldsymbol{v_4 \to \ell_3}$**.**
- ◯ $v_1 \to \ell_2$, $v_2 \to \ell_4$, $v_3 \to \ell_1$, and $v_4 \to \ell_3$.

Solution:
From Figure PS-3.3, ℓ_1 and ℓ_2 have positive slope and the slope of ℓ_1 is greater than the slope of ℓ_2. Similarly the slopes of ℓ_3 and ℓ_4 are negative and the slope of line ℓ_3 is greater than the slope of line ℓ_4.
Substituting the value of v in equation $s = vt$, we get the equations

$$s = t, s = -2t, s = 0.5t, s = -t$$

By comparing the above equations of lines and the lines in Figure PS-3.3, we conclude that v_1 corresponds to the line ℓ_1, v_2 corresponds to the line ℓ_4, v_3 corresponds to the line ℓ_2, and v_4 corresponds to the line ℓ_3.

2 Multiple Select Questions (MSQ):

5. A constructor is asked to construct a road which is at a distance of $\sqrt{2}$ km from the municipality office and perpendicular to a road which can be defined by the equation of the straight line $x - y = 8$ (considering the municipality office to be the origin). Find out the possible equations of the straight lines to represent the new road to be constructed.

   ○ $x - y - 2 = 0$

   ○ $\boldsymbol{x + y + 2 = 0}$

   ○ $x - y + 2 = 0$

   ○ $\boldsymbol{x + y - 2 = 0}$

   Solution:

   See the Figure PS-3.4 for reference:

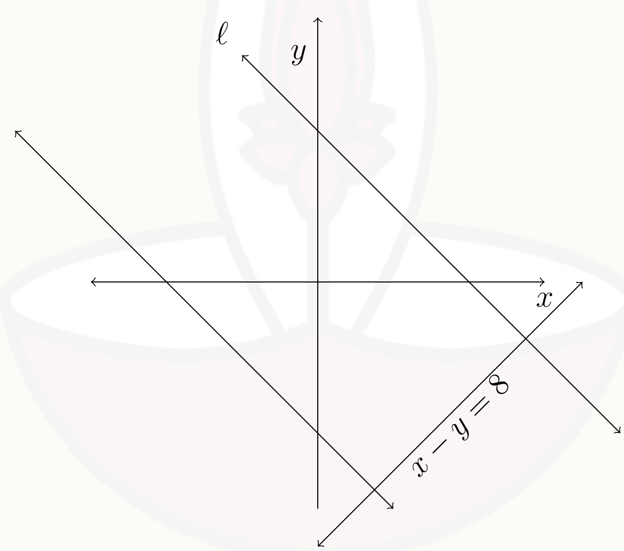


Figure PS-3.4

Let the new road constructed be denoted by ℓ. Given, ℓ is perpendicular to the straight line $x - y = 8$. That is, ℓ is perpendicular to the line $y = x - 8$ whose slope is $m_1 = 1$. Therefore, the slope of ℓ is

$$m_2 = -\frac{1}{m_1} = -1$$

By the slope intercept form, the equation of ℓ is

$$y = m_2\ x\ +\ c$$

$$\implies y = -x + c \text{ , where } c \text{ is a constant}$$

That is, ℓ is the line given by

$$x + y - c = 0$$

It is given that the distance of ℓ from the municipality office is $\sqrt{2}$.
The distance formula of a point (x_1, y_1) from a line $(Ax + By + C = 0)$ is given by $\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$. Substituting $x_1 = 0, y_1 = 0, A = 1, B = 1, C = -c$ in formula, we will get the distance of the point(0,0) from the line ℓ,

$$\frac{|1 \times 0 + 1 \times 0 - c|}{\sqrt{1^2 + 1^2}}$$

which is equal to $\sqrt{2}$. So,

$$\frac{|0 + 0 - c|}{\sqrt{1+1}} = \sqrt{2}$$

$$\implies \frac{|c|}{\sqrt{2}} = \sqrt{2}$$

$$\implies |c| = 2$$

$\implies c = +2$ or $c = -2$.
Hence, the equation of the new road ℓ is

$$x + y + 2 = 0$$

or

$$x + y - 2 = 0.$$

6. Suppose there are two roads perpendicular to each other which are both at the same distance from Priya's house (considered as the origin). The meeting point of the two roads is on the x-axis and at a distance of 5 units from Priya's house.
Choose the correct possible equations representing the roads.

- ◯ Inadequate information.
- ◯ $y = \frac{1}{2}x + 5, y = -2x - 5$
- ◯ $\boldsymbol{y = -x - 5, y = x + 5}$
- ◯ $y = 2x - 10, y = -2x - 10$
- ◯ $y = 2x - 5, y = -\frac{1}{2}x - 5$
- ◯ $\boldsymbol{y = -x + 5, y = x - 5}$
- ◯ $x = 5, x = -5$

Solution:
Denote the two roads by ℓ_1 and ℓ_2. The meeting point of ℓ_1 and ℓ_2 are on the X-axis and at a distance of 5 units from Priya's house (origin) i.e x-intercepts of the roads are 5 or -5 and passing through the points (5,0) or (-5,0) respectively.

Case 1: when x-intercept is 5 and passes through (5,0)
Using intercept form of a line on the axes, the equation of line ℓ_1 is

$$\frac{x}{5} + \frac{y}{b} = 1$$

where b is a constant.

That is, ℓ_1 is

$$bx + 5y - 5b = 0 \tag{1}$$

See Figure PS-3.5 for reference.

The slope of the road ℓ_1 is $m_1 = -\frac{b}{5}$.
Since the road ℓ_2 is perpendicular to ℓ_1, the slope of road ℓ_2 is

$$m_2 = -\frac{1}{m_1} = \frac{5}{b}$$

Using the slope intercept form, the equation of the road ℓ_2 is

$$y = \frac{5}{b}x + c \Longrightarrow by - 5x - bc = 0 \text{ where } b \text{ and } c \text{ are constant}$$

The roads ℓ_1 and ℓ_2 are at the same distance from Priya's house (origin).
Using distance formula of a line from a point, we get

$$\frac{|-5b|}{\sqrt{b^2+25}} = \frac{|-bc|}{\sqrt{b^2+25}} \Longrightarrow |c| = |5| \Longrightarrow c = 5 \quad or - 5$$

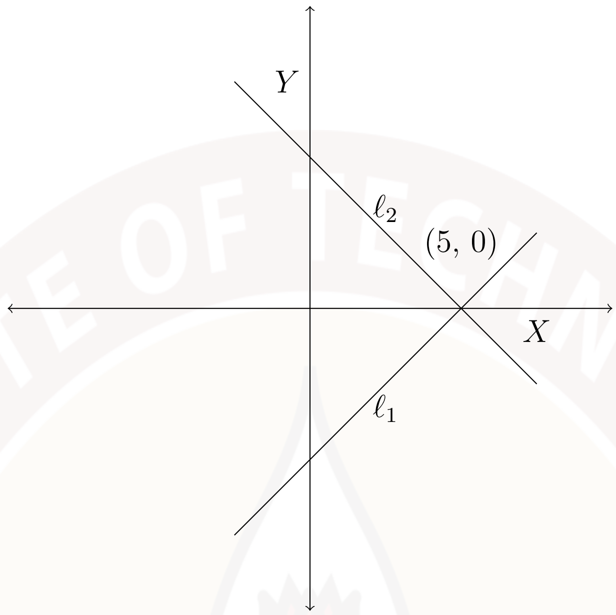


Figure PS-3.5

When $c = 5$, the equation of road ℓ_2 becomes $by - 5x - 5b = 0$. Since ℓ_2 passes through $(5, 0)$, we get $b = -5$.
Therefore, the equation of the road ℓ_2 is $y = -x + 5$.
Substituting $b = -5$ in Equation (1), we will get the equation of the road ℓ_1 as $y = x - 5$.
When $c = -5$, we will get the same equation alternatively.

Case 2: when x-intercept is -5 and passing through (-5,0)
We follow the same process as in Case 1 and we get the equation of the road ℓ_2 as $y = x + 5$ and the equation of the road ℓ_1 as $y = -x - 5$.
See Figure PS-3.6 for reference.

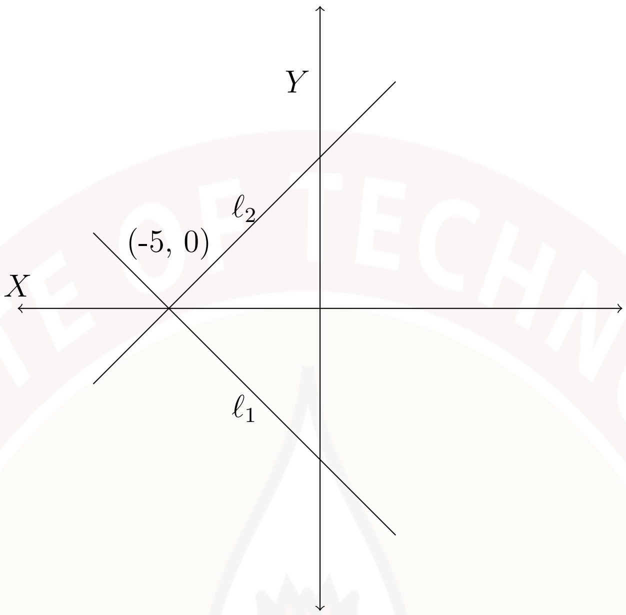


Figure PS-3.6

7. Consider the following two diagrams.

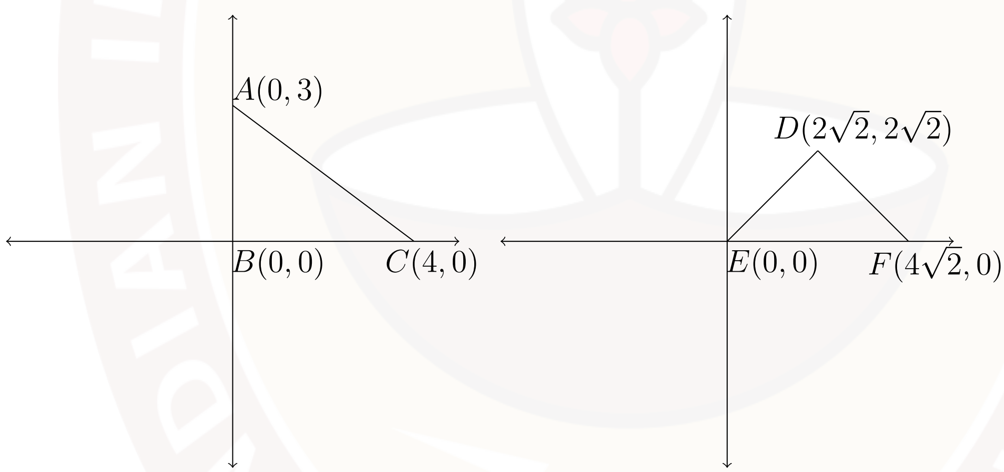


Figure PS-3.7

Which of the following option(s) is(are) true about the triangles ΔABC and ΔDEF given in Figure PS-3.7?

- ◯ Only ΔABC is a right angled triangle while ΔDEF is not.
- ◯ **Both ΔABC and ΔDEF are right angled triangles.**
- ◯ The area of ΔABC is greater than the area of ΔDEF.
- ◯ Both the triangles have the same area.

◯ **The area of ΔDEF is 8 sq.unit.**

Solution:

In Figure PS-3.7, vertices A and C are on $Y-$axis and $X-$ axis respectively and the vertex B is at the origin itself.
Therefore, $\triangle ABC$ is a right angle triangle.
The distance formula between two points $(x_1, y_1), (x_2, y_2)$ is given by

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Using the above formula, in $\triangle DEF$, the length of side DE is

$$\sqrt{(2\sqrt{2} - 0)^2 + (2\sqrt{2} - 0)^2} = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$$

Similarly, the length of side DF is

$$\sqrt{(4\sqrt{2} - 2\sqrt{2})^2 + (0 - 2\sqrt{2})^2} = 4$$

The length of side EF is $4\sqrt{2}$. We have

$$DE^2 + DF^2 = 16 + 16 = 32 = (4\sqrt{2})^2 = EF^2$$

Hence, by the Pythagoras theorem, $\triangle DEF$ is also a right angled triangle.
Area of the right angled $\triangle ABC = \frac{1}{2} \times 4 \times 3 = 6$ sq. unit.
Area of the right angled $\triangle DEF = \frac{1}{2} \times 4 \times 4 = 8$ sq. unit.

8. Let the diagonals of a quadrilateral with one vertex at $(0,0)$ bisect each other perpendicularly at the point $(1,2)$. Further, let one of the diagonals be on the straight line $y = 2x$. Then, which of the following is (are) correct statements?

 - ◯ **The diagonally opposite vertex of $(0,0)$ is $(2,4)$.**
 - ◯ The other diagonal is on the straight line $y = -\frac{1}{2}x$.
 - ◯ **The other diagonal is on the straight line $y = -\frac{1}{2}x + \frac{5}{2}$.**
 - ◯ The diagonally opposite vertex of $(0,0)$ is $(\frac{3}{2}, 3)$.

Solution:

Figure PS-3.8 shows a sketch of the quadrilateral.

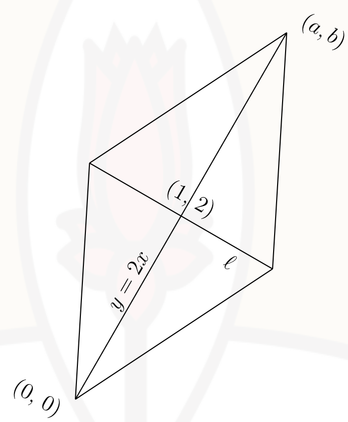


Figure PS-3.8

The diagonal $y = 2x$ has slope $m_1 = 2$.
Let the other diagonal, perpendicular to the line $y = 2x$, be on the line ℓ.
So, the slope of the line ℓ is

$$m_2 = -\frac{1}{m_1} = -\frac{1}{2}$$

From the slope intercept form, the equation of the line ℓ is $y = -\frac{x}{2} + c$, where c is a constant.
Since both the diagonals intersect at the point (1,2) and one diagonal is on line ℓ, the point (1,2) belongs to ℓ and hence $c = \frac{5}{2}$.
Hence, the equation of the line ℓ is $\boldsymbol{y = -\frac{1}{2}x + \frac{5}{2}}$.

Let the opposite vertex of $(0,0)$ be (a,b).
Since the point (1, 2) is the bisection point of the both diagonals, it follows that the point $(1,2)$ is mid-point of the line segment joining the points $(0,0)$ and (a,b).
Using the section formula of a line segment,

$$\frac{a}{2} = 1 \implies a = 2$$

$$\frac{b}{2} = 2 \implies b = 4$$

Hence the diagonally opposite vertex of $(0, 0)$ is $(2, 4)$.

9. A woman is reported missing in a locality. The police department finds a human femur bone during their investigation. They estimate the height H of a female adult (in cm) using the relationship $H = 1.8f + 70$, where f is the length (in cm) of the femur bone. The length of the femur found is 35 cm, and the missing woman is known to be 130 cm tall. In the particular locality, maximum height of a female is 195 cm and the minimum length of a female femur bone is 15 cm. Based on the given data answer the following questions.

   (a) Choose the set of correct options.

   - ◯ If an error of 1 cm is allowed, bone could belong to missing female.
   - ◯ **If an error of 3 cm is allowed, bone could belong to missing female.**
   - ◯ If the height as a function of femur length is known to be accurate, the range of the function is [70, 195].
   - ◯ **If the height as a function of femur length is known to be accurate, the range of the function is [97, 195].**
   - ◯ **If the height as a function of femur length is known to be accurate, the domain of the function is [15, $\frac{625}{9}$].**

   Solution:

   The relationship between height of a woman H and the length of her femur bone f is given by

$$H = 1.8f + 70. \tag{2}$$

   Since the length of the femur bone found during the investigation is 35 cm, we have

$$H = 1.8 \times 35 + 70 = 133 \text{ cm}$$

   The height of missing woman is known to be 130cm. Since $133 - 130 = 3 \leq 3$ and by our assumption, an error of 3 cm is allowed, it is possible that the femur bone found during the investigation belongs to the missing woman.

   Given that the maximum height of a female in that location is 195 cm.
   Substituting $H = 195$ in Equation (2) , we get the maximum length of female femur bone in that location i.e maximum $f = \frac{625}{9}$ cm.
   Since the minimum length of of femur bone known in that location is 15 cm and if height as a function of femur length is known to be accurate then the domain of the function is $[15, \frac{625}{9}]$.

   Given that the minimum length of the female femur bone in that location is 15 cm.
   The minimum height of a female in that location is $H = 1.8 \times 15 + 70 = 97$ cm.
   Since the maximum height of a female in that location is 195 cm, the range of the height function is $[97, 195]$

(b) A new detective agency came up with a relationship $H = mf + 70$, where H is the height of a male adult (in cm) and f is the length (in cm) of the femur bone. They have used the following sample set given below in the Table P-3.2 , such that the sum squared error is minimum.

height(H) (in cm)	150	160	170	180
length of femur bone(f) (in cm)	40	42	48	56

Table PS-3.2

Choose the correct option (only one option is correct).

- ○ $m = 1$
- ○ $m = 1.5$
- ○ $\boldsymbol{m = 2}$
- ○ $m = 2.5$

Solution:

From Table PS-3.3, we can see that the minimum SSE is for $m = 2$.

H (in cm)	f (in cm)	$(H - mf - 70)^2$			
		$m = 1$	$m = 2$	$m = 1.5$	$m = 2.5$
150	40	1600	0	400	400
160	42	2304	36	729	225
170	48	2704	16	784	400
180	56	2916	4	676	900
SSE		$\sum = 9524$	$\sum = 56$	$\sum = 2589$	$\sum = 1925$

Table PS-3.3

3 Numerical Answer Type (NAT):

10. What will be the slopes of the straight lines perpendicular to the following lines?
a) $2x + 5y - 9 = 0$

Answer:2.5
Solution:
Using the slope intercept form, the slope of the line $2x + 5y - 9 = 0$ is $m_1 = -\frac{2}{5}$.
Let the slope of the perpendicular line be m_2. Then

$$m_1.m_2 = -1$$

$$\implies m_2 = \frac{5}{2} = 2.5$$

b) $-5x + 25y + 28 = 0$

Answer:-5
Solution:
Using the slope intercept form, the slope of the line $-5x + 25y + 28 = 0$ is $m_1 = \frac{1}{5}$.
Let the slope of the perpendicular line be m_2, then

$$m_1.m_2 = -1$$

$$\implies m_2 = -5.$$

Week - 3
Practice Assignment-2 Solution
Straight line - 2
Mathematics for Data Science - 1

1 Multiple Choice Questions (MCQ)

1. In a triangle made by the points of the intersection of the two axes and the line $bx+ay = ab$, d is the height of the perpendicular from the origin to the opposite side. Then $\frac{1}{a^2}+\frac{1}{b^2}$ is equal to

 ○ $\sqrt{2}d^2$
 ○ $\frac{\sqrt{2}}{d^2}$
 ○ $\boldsymbol{\frac{1}{d^2}}$
 ○ d^2

Solution:
The intercept form of the equation of the line $bx + ay = ab$, where $a > 0, b > 0$ (without loss of generality) is given by $\frac{x}{a} + \frac{y}{b} = 1$.
Clearly x-intercept $= a$ and y-intercept $= b$.

See Figure AS-3.1 for reference:

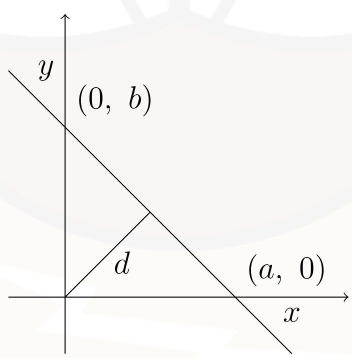


Figure AS-3.1

Using perpendicular distance formula, we have

$$d = \frac{|\ Ax_1 + By_1 + C\ |}{\sqrt{A^2 + B^2}}$$

Here, $x_1 = 0$, $y_1 = 0$, $A = b$, $B = a$, and $C = \ -ab$
Therefore,

$$d = \frac{|0\ +\ 0\ -ab|}{\sqrt{a^2 + b^2}}$$

Squaring both sides:

$$d^2 = \frac{a^2b^2}{a^2+b^2}$$

$$\implies \frac{1}{d^2} = \frac{a^2+b^2}{a^2b^2}$$

$$\implies \boldsymbol{\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2}}$$

2. An elephant standing at the junction of two straight roads represented by the equations $x-y+2=0$ and $y-1=0$ wants to reach another road whose equation is $x-y-3=0$. If the elephant can move in any direction and wants to cover the shortest distance to its destination road, then the equation of the path that the elephant should follow is:

- ◯ $\boldsymbol{x+y=0}$
- ◯ $-x+y=0$
- ◯ $x-y+1=0$
- ◯ $2x-y=0$

Solution:

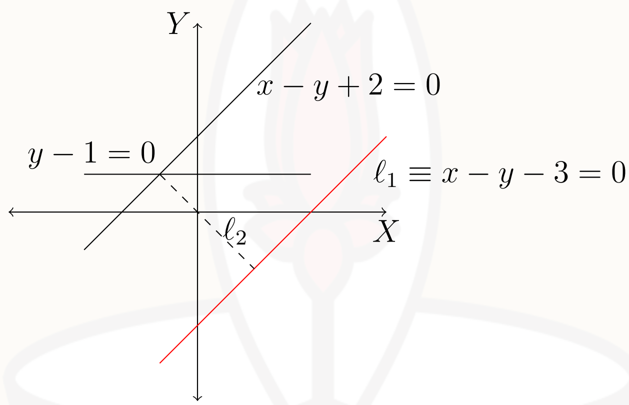


Figure AS-3.2

The junction of the two roads is the intersection point of the lines $x-y+2=0$ and $y=1$. That is, the solution of the equations $x-y+2=0$ and $y=1$ is the point $(-1,1)$.
Let ℓ_1 denote the road that the elephant wants to reach. Then $\ell_1 \equiv x-y-3=0$.
Let the shortest route from the junction to the road ℓ_1 be denoted by ℓ_2. Then, ℓ_2 will be a road that is perpendicular to ℓ_1.
The slope of ℓ_1 is $m_1=1$.
If the slope of ℓ_2 is m_2, then $m_1m_2=-1 \implies m_2=-1$.
Using the slope intercept form, the equation of ℓ_2 is $y=-x+c$. Since ℓ_2 passes through the point (-1,1), it will satisfy ℓ_2. Therefore,

$$1=-(-1)+c \implies c=0$$

Hence, the elephant should follow the path

$$y=-x+0 \implies \boldsymbol{\ell_2 \equiv x+y=0}$$

3. Slope of a line which cuts intercepts of equal lengths on the positive sides of the axes is

   - ◯ 0
   - ◯ **-1**
   - ◯ 1
   - ◯ $\sqrt{3}$

   Solution:

   Let the line cut an intercept of a $(a > 0)$ units on both axes.
   So, the intercept form of the line will be

   $$\frac{x}{a} + \frac{y}{a} = 1$$

   After rearranging the equation, the slope intercept form of the equation is $y = -x + a$.
   Clearly, the slope is -1.

4. Distance between the lines $15x + 9y + 14 = 0$ and $10x + 6y - 14 = 0$ is

   ◯ $\mathbf{\frac{35}{3\sqrt{34}}}$

   ◯ $\frac{35}{2\sqrt{34}}$

   ◯ $\frac{5}{3\sqrt{34}}$

   ◯ $\frac{35}{\sqrt{34}}$

Solution:

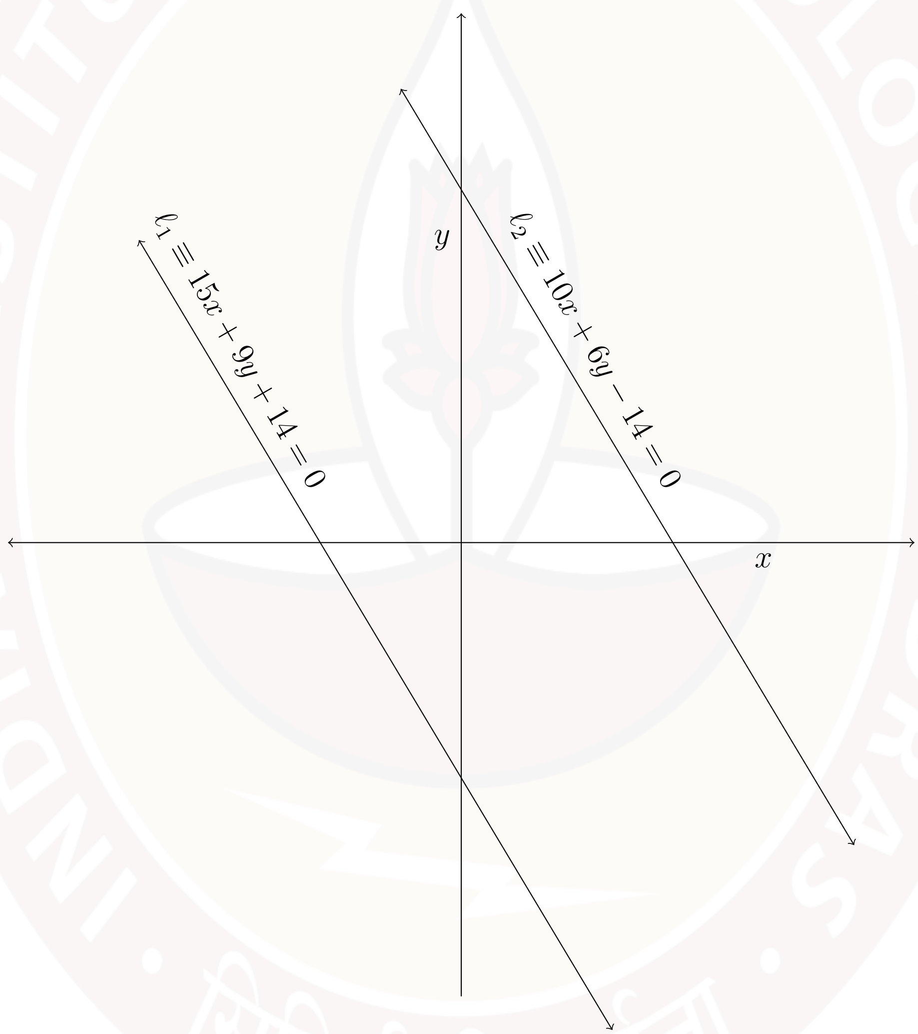


Figure AS-3.3

Let ℓ_1 be the line $15x + 9y + 14 = 0$. Let m_1 be the slope of ℓ_1. Then $m_1 = -\frac{5}{3}$. Let ℓ_2 be the line $10x + 6y - 14 = 0$. Let m_2 be the slope of ℓ_2. Then $m_2 = -\frac{5}{3}$.
This shows that the lines ℓ_1 and ℓ_2 are parallel.

ℓ_1 and ℓ_2 represent straight lines.
Divide both sides of the equations of ℓ_1 and ℓ_2 by 3 and 2 respectively.
Then, ℓ_1 becomes $5x + 3y + \frac{14}{3} = 0$ and ℓ_2 becomes $5x + 3y - 7 = 0$.
Using the distance formula between two parallel lines, the distance d between the lines is

$$\begin{aligned} d &= \frac{|\frac{14}{3} - (-7)|}{\sqrt{5^2 + 3^2}} \\ &= \frac{35}{3\sqrt{34}} units \end{aligned}$$

5. The equation of the straight road which is equidistant from two parallel straight roads represented by $6x + 4y - 5 = 0$ and $3x + 2y + 4 = 0$ is

   ◯ $12x - 8y + 3 = 0$

   ◯ $8x - 12y - 3 = 0$

   ◯ $\mathbf{12x + 8y + 3 = 0}$

   ◯ $8x + 12y + 3 = 0$

Solution:

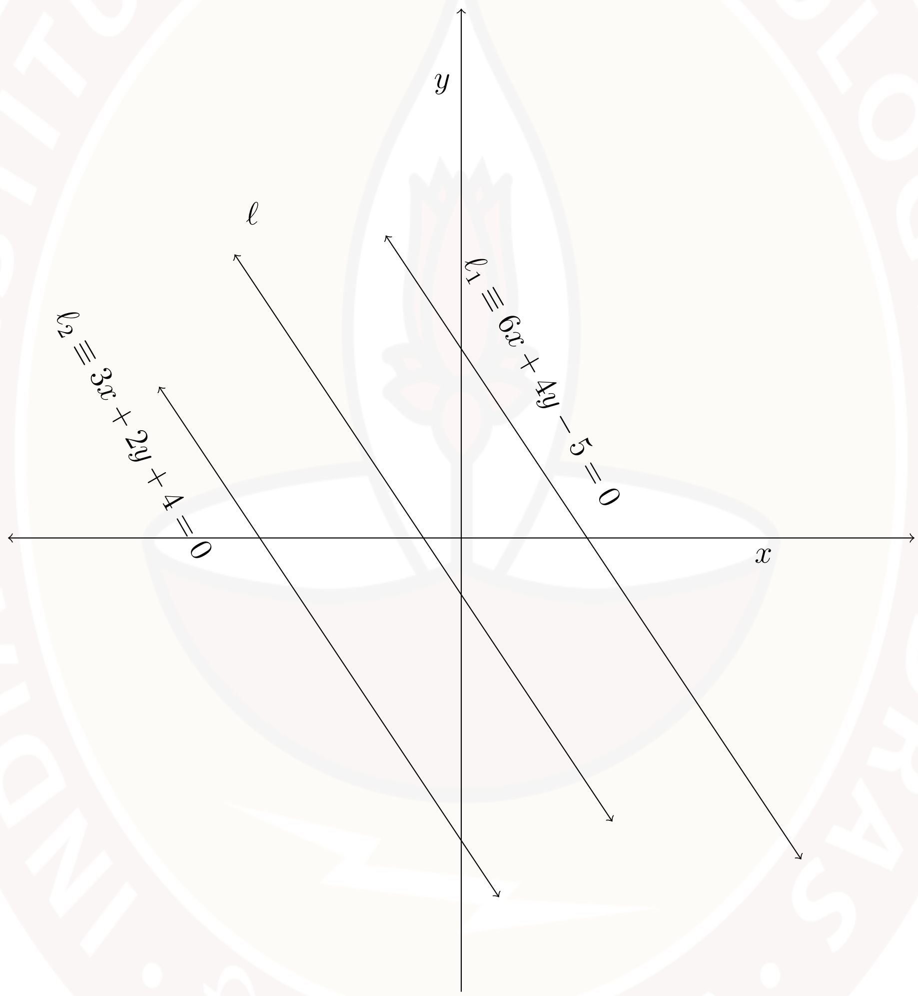


Figure AS-3.4

Let ℓ_1 and ℓ_2 represent the roads

$$\ell_1 \equiv 6x + 4y - 5 = 0, \text{ with slope } m_1 = -\tfrac{3}{2}$$
$$\text{and } \ell_2 \equiv 3x + 2y + 4 = 0, \text{ with slope } m_2 = -\tfrac{3}{2}.$$

Multiplying the both sides of equation of ℓ_2 by 2, we get $\quad \ell_2 \equiv 6x + 4y + 8 = 0.$

Let ℓ be the equation of the required road. Using the slope intercept form, the equation ℓ (with same slope) will be:

$$\ell \equiv 6x + 4y + c = 0$$

If ℓ is equidistant from ℓ_1 and ℓ_2, then

$$\frac{\mid c - (-5) \mid}{\sqrt{6^2 + 4^2}} = \frac{\mid c - 8 \mid}{\sqrt{6^2 + 4^2}} \Longrightarrow |c + 5| = |c - 8|$$

Case 1:

$$c + 5 = c - 8 \Longrightarrow 5 = -8 \text{ which is not possible.}$$

Case 2:

$$c + 5 = -(c - 8) \Longrightarrow c = \tfrac{3}{2}.$$

Therefore,

$$\ell \equiv 6x + 4y + \frac{3}{2} = 0$$

or

$$\ell \equiv \boldsymbol{12x + 8y + 3 = 0}$$

Hence the equation of the required road is $\ell \equiv \boldsymbol{12x + 8y + 3 = 0}$.

6. A line perpendicular to the line segment joining the points $A\,(1,0)$ and $B\,(2,3)$, divides it at C in the ratio of $1:3$. Then the equation of the line is

   - ◯ $2x + 6y - 9 = 0$
   - ◯ $\mathbf{2x + 6y - 7 = 0}$
   - ◯ $2x - 6y - 9 = 0$
   - ◯ $2x - 6y + 7 = 0$

Solution:

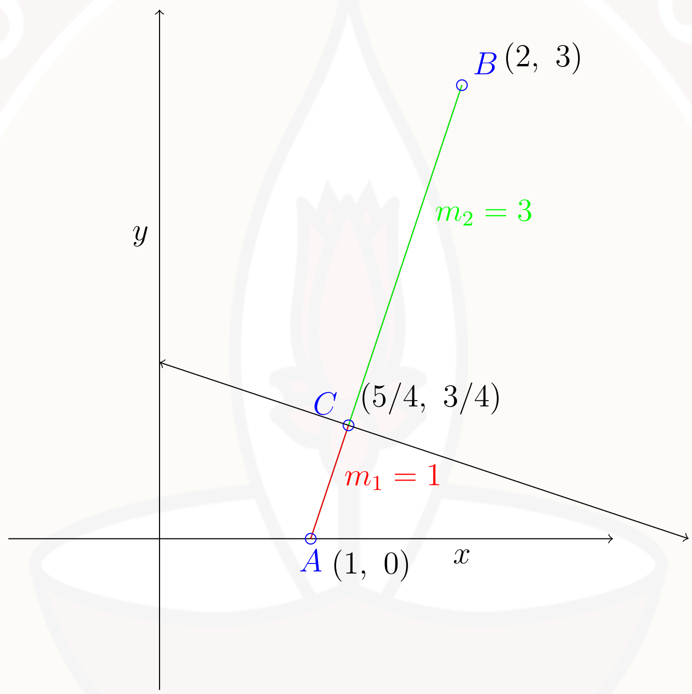


Figure AS-3.5

Since the perpendicular line divides the line segment joining the points A and B at C in the ratio of $1:3$, (see Figure AS-3.5) the coordinate of C is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

where $x_1 = 1, y_1 = 0, x_2 = 2, y_2 = 3, m_1 = 1$ and $m_2 = 3$ (using the section formula).
Hence coordinate of C is $(\frac{5}{4}, \frac{3}{4})$.
Using the slope formula, the slope of the line joining the points A and B is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{2 - 1} = 3$$

Therefore, the slope of the perpendicular line which is perpendicular to the line segment

joining the points A and B and passing through the point C (say ℓ) is $m' = -\frac{1}{m} = -\frac{1}{3}$.
From the slope intercept form, the equation of the line ℓ is $x + 3y = d$, where d is a constant.
Since the line ℓ passes through point C, C satisfies the equation of ℓ. We get $d = \frac{7}{2}$.
Hence the equation of the required line ℓ is $\boldsymbol{2x + 6y - 7 = 0}$.

7. An experiment is performed in an ideal condition to find out the velocity of a particle obeying the equation $S = vt$ where the distance travelled (S) inside a labeled glass tube by a particle with an uniform velocity (v) is measured with respect to time taken(t). The data collected by repeating the experiment five times is shown in the table below. If we use the method to minimize the squared sum error, what is the most likely velocity

S (in cm)	3	9	13	14	17
t (in sec)	2	4	6	8	10

Table AS-3.1

of the particle among the given options?

- ◯ 2.5
- ◯ 1.5
- ◯ **2**
- ◯ 1

Solution:

Consider Table AS-3.2.

S (in cm)	t (in cm)	$(S - vt)^2$			
		$v = 2.5$	$v = 1.5$	$v = 2$	$v = 1$
3	2	4	0	1	1
9	4	1	9	1	25
13	6	4	16	1	49
14	8	36	4	4	36
17	10	64	4	9	49
SSE		$\sum = 109$	$\sum = 33$	$\sum = 16$	$\sum = 160$

Table AS-3.2

From the table, it is clear that $v = 2$ gives the minimum Sum Squared Error. Hence **most likely velocity of the particle is 2.**

Second Method:

Observe Figure AS-3.6, which shows the lines represented by different slopes (v values) and the points mentioned in Table AS-3.1.

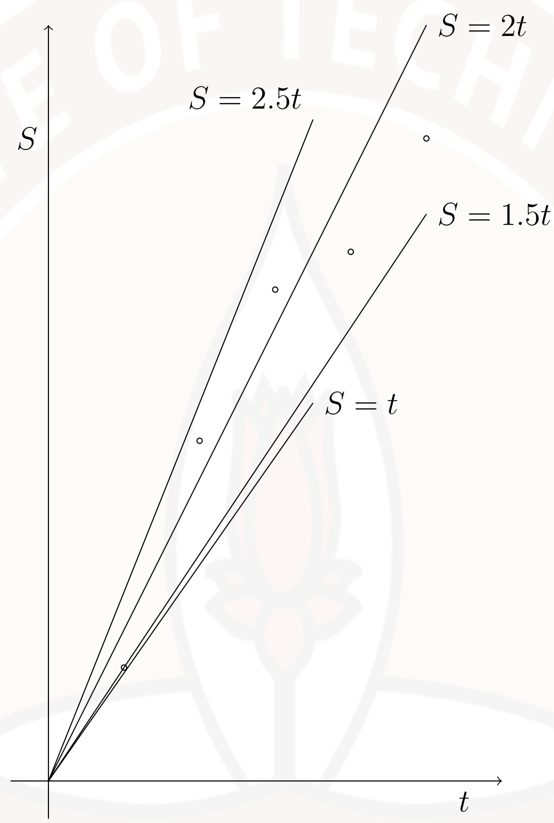


Figure AS-3.6

From Figure AS-3.6 also, it is clear that the most likely velocity of the particle is 2.

2 Multiple Select Questions (MSQ)

1. The equations of lines with slope 3 and the length of the perpendicular from the origin equal to $\sqrt{10}$ are

   ◯ $x = y + 10$

   ◯ $y = 2x - 5$

   ◯ $\boldsymbol{y = 3x + 10}$

   ◯ $x = 3y - 10$

   ◯ $\boldsymbol{y = 3x - 10}$

   ◯ $x = 3y + 10$

   Solution:

   Given, the slope of the line is 3 and the distance from the origin is $\sqrt{10}$.
   From the slope intercept form, the equation of the line is $y = 3x + c$, where c is a constant.
   The general form of the line is $-3x + y - c = 0$.

   Using perpendicular distance formula, the distance of the line from the origin is

   $$d = \frac{|\ 0\ +\ 0\ -\ c\ |}{\sqrt{(-3)^2 + 1}} = \frac{|-c|}{\sqrt{10}}$$

   Therefore, we get

   $$\frac{|c|}{\sqrt{10}} = \sqrt{10}$$

   After solving the above equation, we get $|c| = 10 \Rightarrow \quad c = +10 \ \ or \ c = -10$.
   Hence, the equation of the line is $\boldsymbol{-3x + y + 10 = 0 \ or \ -3x + y - 10 = 0}$.

2. Ankit is located at $(3, 3)$. He called Ajay to ask his location. Ajay describes the path he had taken from home (located at the origin) as: " I walked three units towards East and then nine units towards North. And I repeated the same pattern thrice." Now Ankit wants a direct path to reach Ajay, then choose the correct options.

   - ◯ He should follow $3x - y = 0$.
   - ◯ **He should follow $4x - y - 9 = 0$.**
   - ◯ **Ankit will have to walk a distance of $6\sqrt{17}$ units**.
   - ◯ Ankit will have to walk a distance of $3\sqrt{17}$ units.
   - ◯ Ajay has walked a distance of $9\sqrt{10}$ units from his home.
   - ◯ **Ajay has walked a distance of 36 units from his home.**

Solution:

Figure AS-3.7 shows the location of Ankit and the path travelled by Ajay.

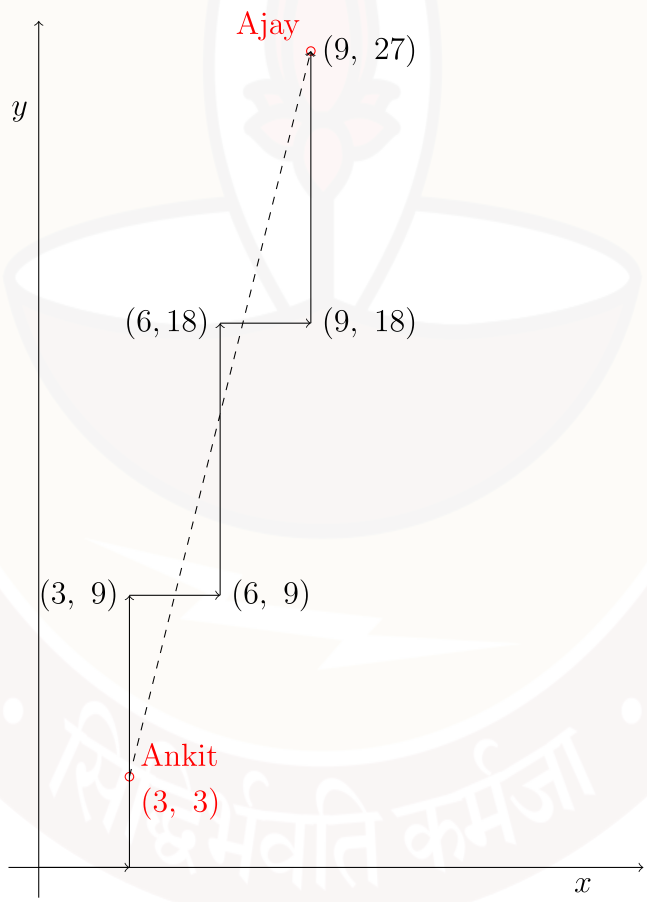


Figure AS-3.7

From Figure AS-3.7, the location of Ajay is (9,27).
Ankit will have to follow the path which connects their locations directly.
Using two point form of a line, the equation of the line passing through the points (3,3) and (9,27) is

$$4x - y - 9 = 0.$$

The distance which Ankit will have to walk will be the distance between their locations (3,3) and (9,27).
Using distance formula, the distance between the points (3,3) and (9,27) is

$$d = \sqrt{(9-3)^2 \ + \ (27-3)^2} = 6\sqrt{17} \ units$$

Also, the travelled distance by Ajay $= 3+9+3+9+3+9 = 36$ units.

3. To determine the gas constant R, two students A and B perform an experiment based on the ideal gas equation given as $Pv = RT$. Both use the same gaseous sample having $v = 16.6\ m^3/mol$ and reported the approximate value of R as $8.3\ J/(K \triangle mol)$ using the minimisation of sum squared error. The data collected by both the students are reported below. Choose the correct option:

$P(T)(Pa)$	137	139	142	142	141
$T(K)$	274	276	278	280	282

Table AS-3.3: Data collected by student A.

$P(T)(Pa)$	137	141	142	148	145
$T(K)$	276	280	284	288	290

Table AS-3.4: Data collected by student B.

- ○ **$\boldsymbol{A}$ has better fit than $\boldsymbol{B}$.**
- ○ B has better fit than A.
- ○ A and B both have same fit.
- ○ SSE calculated by B is 14.
- ○ **SSE calculated by $\boldsymbol{A}$ is 14.**
- ○ SSE calculated by both A and B is 14.

Solution:
Given, the ideal gas equation

$$Pv = RT$$
$$\implies P = \frac{R}{v}T$$
$$\implies P = \frac{8.3}{16.6}T$$
$$\implies P = \frac{1}{2}T$$

Now, from Table AS-3.5 and Table AS-3.6, we can say that $\boldsymbol{A}$ is a better fit than $\boldsymbol{B}$ and SSE calculated by $\boldsymbol{A}$ is 14.

$T(K)$	$P(T)(Pa)$	$(P-\frac{1}{2}T)^2$
274	137	0
276	139	1
278	142	9
280	142	4
282	141	0
		$\sum = 14$

Table AS-3.5: SSE calculated by ***A***

$T(K)$	$P(T)(Pa)$	$(P-\frac{1}{2}T)^2$
276	137	1
280	141	1
284	142	0
288	148	16
290	145	0
		$\sum = 18$

Table AS-3.6: SSE calculated by ***B***

3 Numerical Answer Type (NAT)

1. Find the values of r for which the line $(r+5)x-(r-1)y+r^2+4r-5=0$ is
   (a) parallel to the x-axis [Ans: -5]
   (b) parallel to the line $2x-y+4=0$, [Ans: 7]
   (c) parallel to the y-axis. [Ans: 1]
   (d) perpendicular to the line $x-y+32=0$ [Ans: -2]

   Solution:

   Using the slope intercept form of the line $\ell \equiv (r+5)x-(r-1)y+r^2+4r-5=0$, the slope of the line is
   $$m_1 = \frac{r+5}{r-1}$$
   a.
   Since line ℓ is parallel to the X-axis and the slope of the X-axis is $m_2 = 0$,
   $$m_1 = m_2 \implies \frac{r+5}{r-1} = 0 \implies r = -5.$$
   Hence $r = -5$.

   b.
   Since line ℓ is parallel to the line $2x-y+4=0$ and the slope of the line $2x-y+4=0$

is $m_2 = 2$, we have

$$m_1 = m_2 \implies \frac{r+5}{r-1} = 2 \implies r = 7.$$

Hence $r = 7$.

c.

Since line ℓ is parallel to the Y-axis ($x = 0$), using parallel line condition, we have

$$a_1 b_2 = a_2 b_1$$

where $a_1 = 1, b_1 = 0, a_2 = r+5, b_2 = -(r-1) = 1-r$,

$$\implies 1 - r = 0$$

Hence $r = 1$.

d.

Since line ℓ is perpendicular to the line $x - y + 32 = 0$ which has slope $m_2 = 1$, we have

$$m_1 \times m_2 = -1 \implies \frac{r+5}{r-1} = -1 \implies r = -2.$$

Hence $r = -2$.

2. A, B and C have coordinates $(2, 9), (10, -7)$ and $(6, p)$ respectively. Line AB is perpendicular to line BC. Then the value of p is
[Ans: -9]
Solution:
From the two point form of a line, the equation of the line passing through the points A and B is

$$2x + y = 13$$

The slope of line $2x + y = 13$ is $m_1 = -2$.
Again from the two point form of a line, the equation of the line passing through points the B and C is

$$y = -\frac{(p+7)}{4}x + \frac{10(p+7)}{4} - 7$$

The slope of this line is,

$$m_2 = \frac{-(p+7)}{4}$$

Since lines AB and BC are perpendicular to each other, we have

$$m_1 \times m_2 = -1$$

$$\implies \frac{2(p+7)}{4} = -1 \implies p = -9$$

Hence $p = -9$.

Mathematics for Data Science - 1
Graded Assignment Solutions
Week 3 - March Qualifier, '21

1 MULTIPLE CHOICE QUESTIONS:

1. Which of the following equations represents the general form of a straight line? (Ans: a)

General form of a straight line ; $Ax + By + C$

✓ ○ $5x + 3y + 2 = 0$
○ $5x^2 + 3y + 3 = 0$ – not a straight line
○ $y = 3x + 2$ – slope form : $y = mx + c$
○ $x/2 + y/3 = 1$ – intercept form : $\frac{x}{a} + \frac{y}{b} = 1$

2. Distance between the lines $3y - 2x - 4 = 0$ and $4x - 6y + 7 = 0$ is (Ans: b)

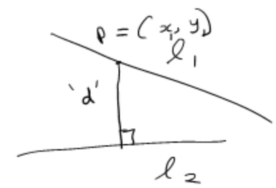
○ $\frac{15}{2\sqrt{13}}$
✓ ○ $\frac{1}{2\sqrt{13}}$
○ $\frac{15}{\sqrt{13}}$
○ $\frac{1}{\sqrt{13}}$

Soln. : To find the distance between two straight lines, consider a point on one of the lines and then find perpendicular distance of that point from the second line.

Consider $3y - 2x - 4 = 0$. To find a point on this line, put $y = 0$ $\Rightarrow$ $0 - 2x - 4 = 0$ $\Rightarrow$ $x = -2$

i.e. $(-2, 0)$ is a point on this line

Now, perpendicular distance 'd' of the point $(-2, 0)$ from the line $4x - 6y + 7 = 0$ is to be found.



distance 'd' of the point $p = (x_1, y_1)$ from the line $Ax + By + C = 0$ is : $d = \frac{|Ax_1 + By_1 + c|}{\sqrt{A^2 + B^2}}$

for the line $4x - 6y + 7 = 0$, $A = 4$, $B = -6$, $C = 7$

$\therefore d = \frac{|4 \times (-2) + -6 \times 0 + 7|}{\sqrt{4^2 + (-6)^2}} = \frac{|-8+7|}{\sqrt{16+36}} = \frac{1}{2\sqrt{13}}$

distance between two lines can also be found using the formula $d = \frac{|c_2 - c_1|}{\sqrt{A^2 + B^2}}$ if the equations are in the form $Ax + By + C_1 = 0$
$Ax + By + C_2 = 0$

3. In a fluid flow domain, stream lines and equipotential lines are perpendicular to each other. If the equation of a stream line is given by: $7x + 8y - 2 = 0$ and the point of intersection with the corresponding equipotential line is marked as $(3, 6)$ then the equation of the corresponding equipotential line is: (Ans: d)

   - ○ $7x - 8y - 5 = 0$
   - ○ $8x - 7y + 36 = 0$
   - ○ $8x + 7y + 18 = 0$
   - ✓ ○ $24x - 21y + 54 = 0$

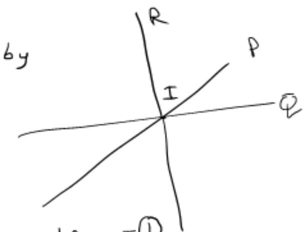
   ADDITIONAL INFO: Stream lines represent the directions of flow and the equipotential lines join the points with equal velocity potential.

THIS QUESTION WILL BE REMOVED FROM WEEK 3 ASSIGNMENT.

4. Consider the two lines $P := 3x - 4y + 5 = 0$ and $Q := 4x + 5y - 45 = 0$. There is another line $R := Q + \lambda P = 0$ passing through the intersection of these two lines. Value of the constant λ is ten times the length of the perpendicular distance of the line P from the origin $(0, 0)$. The exact equation of the line R is given by: (Ans: b)

- ○ $7x - 8y + 5 = 0$
- ✓○ $34x - 35y + 5 = 0$
- ○ $19x - 20y - 20 = 0$
- ○ $5x + 6y - 55 = 0$

ADDITIONAL INFO: In general, the equation of a line passing through the intersection of two lines $P := a_p x + b_p y + c_p = 0$ and $Q := a_q x + b_q y + c_q = 0$ is $Q + \lambda P = 0$, where the constant λ is real and can be determined based on additional conditions.



ADDITIONAL INFO: Equation of a line R passing through the intersection of two line P and Q is given by

$Q + \lambda P = 0$ where λ is real

Here, $Q + \lambda P = (a_q + \lambda a_p)x + (b_q + \lambda b_p)y + c_q + \lambda c_p = 0$

$= a_R x + b_R y + c_R$ —(1)

$a_R = a_q + \lambda a_p$, $b_R = b_q + \lambda b_p$, $c_R = c_q + \lambda c_p$

Let I be the intersection point with the coordinates (x_i, y_i) i.e. they are

Let us assume lines P, Q intersect exactly at one point I

not parallel.

Solving for x, y from $P := 0$, $Q := 0$ we would obtain coordinates of I

$a_p x + b_p y + c_p = 0$

$a_p x = -c_p - b_p y$

$x = \frac{-c_p - b_p y}{a_p}$

$a_q x + b_q y + c_q = 0$

substituting $x = \frac{-c_p - b_p y}{a_p}$

$\frac{a_q}{a_p}(-c_p - b_p y) + b_q y + c_q = 0$

$\Rightarrow -\frac{a_q c_p}{a_p} + y\left[b_q - \frac{b_p a_q}{a_p}\right] + c_q = 0$

$\therefore y = \frac{c_p \frac{a_q}{a_p} - c_q}{b_q - b_p \frac{a_q}{a_p}}$

$= \frac{c_p a_q - c_q a_p}{b_q a_p - b_p a_q}$

and $x = \frac{-c_p - b_p y}{a_p}$

$= \frac{-c_p - b_p\left(\frac{c_p a_q - c_q a_p}{b_q a_p - b_p a_q}\right)}{a_p} = \frac{-c_p b_q a_p + \cancel{c_p b_p a_q} - \cancel{b_p c_p a_q} + b_p c_q a_p}{a_p(b_q a_p - b_p a_q)}$

$= \frac{a_p}{a_p}\left(\frac{-c_p b_q + b_p c_q}{b_q a_p - b_p a_q}\right)$

$\therefore I = \left(\frac{-c_p b_q + b_p c_q}{b_q a_p - b_p a_q}, \frac{c_p a_q - c_q a_p}{b_q a_p - b_p a_q}\right)$

where $b_q a_p - b_p a_q \neq 0$

Now, line R must also pass through I

$a_R x + b_R y + c_R = 0$

substituting coordinates of I in the above equation,

$a_R [-c_p b_q + b_p c_q] + b_R [c_p a_q - c_q a_p] + c_R (b_q a_p - b_p a_q) = 0$

substituting the values of a_R, b_R and c_R from ① we can show that a linear relation holds between the coefficients a's, b's, c's of the 3 lines for any real λ.

Q4 Soln. : $P : = 3x - 4y + 5 = 0$, $Q : 4x + 5y - 45 = 0$

$R : = Q + \lambda P = 0 = (4 + 3\lambda)x + (5 - 4\lambda)y - 45 + 5\lambda$

To find λ,

Let distance of the line P from the origin $(0, 0)$ be 'd'

distance of a line $Ax + By + C = 0$ from a point (x_0, y_0) is given by

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

so,
$$d = \frac{|3 \times 0 - 4 \times 0 + 5|}{\sqrt{3^2 + 4^2}} = \frac{5}{\sqrt{25}} = \frac{5}{5} = 1$$

given that $\lambda = 10d = 10$

substituting $\lambda = 10$ in $R : = 0$ we get,

$R : = (4 + 30)x + (5 - 40)y - 45 + 50 = 0$

$\therefore R : = 34x - 35y + 5 = 0$

Answer the questions Q5, Q6 based on the following passage: In general, if p is the perpendicular distance of a line ℓ from the origin, then the equation of ℓ can be written in the form: $x \cos\alpha + y \sin\alpha = p$ where α is the angle (measured in the anticlockwise direction) made by the perpendicular, from the origin to ℓ, with the x-axis. Hints: $\sin 30 = 1/2, \sin 120 = \frac{\sqrt{3}}{2}, \cos 30 = \frac{\sqrt{3}}{2}, \cos 120 = -1/2$ (all angles are measured in degrees).

5. The equation of a line is given by $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$. Value of the angle (measured in the anticlockwise direction) made by the line with the x- axis (in degrees) is? (Ans: a)

   - ✓ ○ 120
   - ○ 60
   - ○ 150
   - ○ 30

$\frac{\sqrt{3}}{2} x + \frac{1}{2} y = 5$

$\sqrt{3} x + y = 10$

slope $m = -\sqrt{3} = \tan\theta \Rightarrow \theta = 120$ degrees

6. The perpendicular from the origin intersects the line $\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$ at the coordinates: (Ans: b)

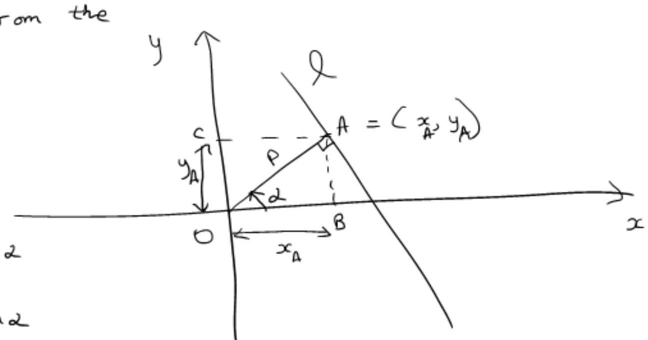
   - ○ $(\frac{5}{2}, \frac{5\sqrt{3}}{2})$
   - ✓ ○ $(\frac{5\sqrt{3}}{2}, \frac{5}{2})$
   - ○ $(\frac{-5\sqrt{3}}{2}, \frac{5}{2})$
   - ○ $(\frac{5\sqrt{3}}{2}, \frac{-5}{2})$

ADDITIONAL INFO:

Consider a line $\ell := ax + by + c = 0$

p = distance of the perpendicular from the origin O to the line ℓ

Let A be a point on ℓ with coordinates (x_A, y_A).



$OB = x_A$, $OC = y_A$

$\cos\alpha = \frac{OB}{OA} = \frac{x_A}{p} \Rightarrow x_A = p\cos\alpha$

$\sin\alpha = \frac{AB}{OA} = \frac{y}{p} \Rightarrow y_A = p\sin\alpha$

We know that perpendicular distance of O from a line ℓ is,

$$\frac{|a \times 0 + b \times 0 + c|}{\sqrt{a^2+b^2}} = \frac{|c|}{\sqrt{a^2+b^2}} = OA = p$$

Now, A is a point on ℓ.

$x_A \cos\alpha + y_A \sin\alpha = p\cos^2\alpha + p\sin^2\alpha = p(\cos^2\alpha + \sin^2\alpha) = p$

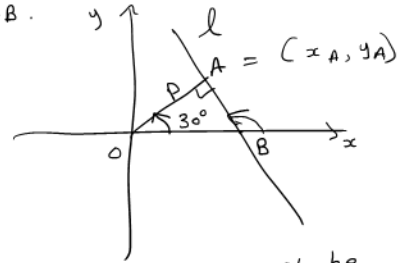
$(\because \cos^2\alpha + \sin^2\alpha = 1$ is a trigonometric identity$)$

Q5 soln. : $l: \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5$ is of the form $x\cos\alpha + y\sin\alpha = p$

based on Passage

$\therefore$ $p = 5$, $\cos\alpha = \frac{\sqrt{3}}{2}$ $\Rightarrow$ $\alpha = 30$ degrees

Consider the right-angled triangle ΔOAB.



$180 = \angle ABO + 30 + 90$

($\because$ sum of interior angles of a triangle must be 180 degrees)

$\Rightarrow$ $\angle ABO = 180 - 30 - 90$

$= 60$

$\therefore$ angle made by the line l with x-axis must be $180 - 60 = 120$ degrees

($\because$ angle in a line must be 180)

Q6 soln. : perpendicular from origin O to the line l intersects it at the point A.

we know that, $x_A = p\cos\alpha = 5 \times \cos 30 = \frac{5\sqrt{3}}{2}$

$y_A = p\sin\alpha = 5 \times \sin 30 = 5 \times \frac{1}{2}$

$\therefore$ $A = (x_A, y_A) = \left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$

2 MULTIPLE SELECT QUESTIONS:

7. Let X be the set of all straight lines in the coordinate plane. Let us define a relation R on X as follows, $R := \{(l_1, l_2) \in X \times X \mid \text{The lines } l_1 \text{and } l_2 \text{ intersect at least at one point}\}$. Which of the following statements is/are true? (Ans: a, c)

   ✓ ○ R is symmetric.

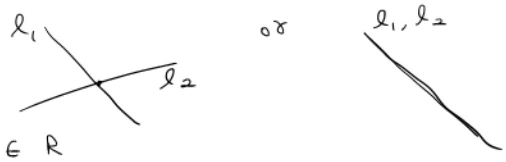
   ○ R is transitive.

   ✓ ○ R is reflexive.

   ○ R is an equivalence relation.

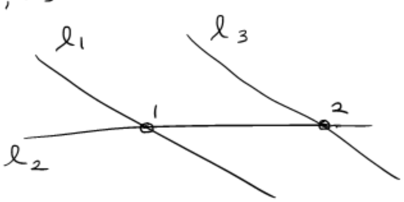
Q 7 Soln. : Given that $R := \{(l_1, l_2) \in X \times X \mid l_1 \text{ and } l_2 \text{ intersect at least once}\}$

(a) if two lines l_1, l_2 intersect at least at one point, then l_2, l_1 also intersect i.e. order of the pair (l_1, l_2) is not significant. Hence, $(l_1, l_2) \in R$ and $(l_2, l_1) \in R$. $\therefore$ R is symmetric.



(b) Consider 3 lines l_1, l_2, l_3. Assume that l_1, l_2 intersect at a point.
Assume that l_2, l_3 also intersect each other.

Is it always true that l_1, l_3 must intersect at least once?
one example is $\Longrightarrow$



l_1, l_2 intersect at 1,
l_2, l_3 intersect at 2,
but l_1, l_3 never intersect.
So, at least for one combination of l_1, l_2, l_3, l_1 and l_3 do not intersect.
$\therefore$ R is not transitive

(d) A relation is an equivalence relation if and only if it satisfies symmetric, reflexive and transitive properties. Here, R is not transitive.
Hence, R is not an equivalence relation.

(c) Consider one element l in the set X of straight lines.
l intersect with itself at all the points.
Thus, R is reflexive. $\{$ reflexive property : $(l, l) \in R$

8. Consider three points on the xy- coordinate plane A = $(3, -7)$, B = $(6, -14)$ and C = $(-9, 21)$. Which of the following statements is/are true? (Ans: a, c, d)

✓ ◯ If we consider these three points to be the vertices of a triangle, then the area is 0.

◯ If we consider these three points to be the vertices of a triangle, then the area is 168 square units.

✓ ◯ The points A, B and C are collinear.

✓ ◯ In general, if the area of a triangle considering any three points be zero then the three points are collinear.

Soln. : $A = (3, -7) = (x_A, y_A)$, $B = (6, -14) = (x_B, y_B)$, $C = (-9, 21) = (x_C, y_C)$

Area of the triangle ΔABC

$$= \frac{1}{2} \left| x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B) \right|$$

$$= \frac{1}{2} \left| 3(-14-21) + 6(21-(-7)) + (-9)(-7-(-14)) \right|$$

$$= \frac{1}{2} \left| -105 + 168 - 63 \right| = 0 \text{ units}$$

Hence, option (a) is correct and option (b) is wrong.

option (c) : To check if A, B, C are collinear, compare the slopes of the line segments AB, BC and AC. If they are all of equal value, then the 3 points must be on the same straight line.

$$\text{slope of } AB = \frac{-14-(-7)}{6-3} = \frac{-7}{3}$$

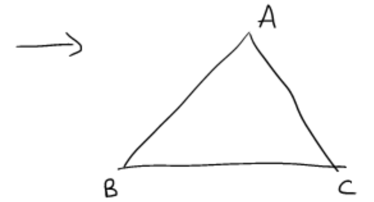
$$\text{slope of } BC = \frac{21-(-14)}{-9-6} = \frac{35}{-15} = \frac{-7}{3}$$

$$\text{slope of } AC = \frac{21-(-7)}{-9-3} = \frac{28}{-12} = -\frac{7}{3}$$

All the slopes are equal to $-7/3$. Hence, A, B, C are collinear. option (c) is also correct.

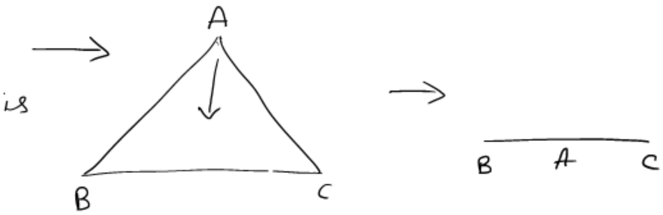
option (d) : If the area of a triangle is zero, then the 3 points are collinear.

Consider 3 points A, B, C →



If the area of the $\Delta ABC = 0$, then the third vertex must lie on the opposite side.

Thus, the points A, B, C must be collinear. option (d) is also correct.



__Additional Info__ : The statement in option (d) can be proved with mathematical rigour using the concept of linear Algebra.

$$\text{area of } \Delta ABC = \frac{1}{2}\left| x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)\right|$$

The expression inside the modulus $|\quad|$ can be written in a matrix determinant form shown below

$$\begin{array}{c|ccc|}
 & \text{Col. 1} & \text{Col. 2} & \text{Col. 3} \\
\text{Row 1} & x_A & x_B & x_C \\
\text{Row 2} & y_A & y_B & y_C \\
\text{Row 3} & 1 & 1 & 1
\end{array} = D$$

without changing the value of D we can perform some Row and Column operations such that,

$$D = \begin{vmatrix} x_A & 1 & 1 \\ y_A & \dfrac{y_B - y_A}{x_B - x_A} & \dfrac{y_C - y_B}{x_C - x_B} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= x_A(0 - 0) + 1\left(\frac{y_C - y_B}{x_C - x_B} - 0\right) + 1\left(0 - \frac{y_B - y_A}{x_B - x_A}\right)$$

If area of $\Delta ABC = 0$, then $D = 0 \Rightarrow \dfrac{y_C - y_B}{x_C - x_B} = \dfrac{y_B - y_A}{x_B - x_A}$.

$\Rightarrow$ slope of line segment BC = slope of line segment BA

Thus, the points A, B, C must be collinear.

__NOTE__ : This additional info is not part of this qualifier course. It will be discussed in Maths Term 2.

9. You have been closely monitoring your bike's mileage recently. Here is a table showing two columns representing the amount paid for fuel and the corresponding mileage in Km. You have noted down the distance travelled each time when the fuel meter falls back to a fixed reference mark. By computing the SSE for each, identify which one of the given options is the best fit (with the least error)? Consider y to be the amount paid and x to be the corresponding distance in Km. (Ans: c)

(y_i) (x_i) $i = 1$ to 7

Amount paid (₹)	Distance (Km)
70	20
50	15
40	14
20	8
80	28
100	40
90	35

Table 1

- ◯ $y = 2x + 35$
- ◯ $y = x + 32$
- ✓ ◯ $y = 1.5x + 36$ is the best fit.
- ◯ $y = 3.5x + 5$

ADDITIONAL INFO: Do you see how laborious the calculations can get for such an elementary data set? Can you think of better ways to minimize the error function?

Solution : For a straight line $y = mx + c$, SSE is given by

$$SSE = \sum_{i=1}^{n} (y_i - y)^2 = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

where, n = total number of recordings of data. Here, $n = 7$

For each of the options compute SSE and then find the best straight line fit by identifying the option with the least SSE.

option 1 : $SSE = (70 - 2\times 20 - 35)^2 + (50 - 2\times 15 - 35)^2 + (40 - 2\times 14 - 35)^2 + (20 - 2\times 8 - 35)^2 + (80 - 2\times 28 - 35)^2 + (100 - 2\times 40 - 35)^2 + (90 - 2\times 35 - 35)^2$

Similarly find SSE for all the 4 options and observe that option (c) : $y = 1.5x + 36$ has the least SSE.

Additional Info :

$$SSE = \sum_{i=1}^{n} (y_i - y)^2$$

$$= \sum_{i=1}^{n} (y_i - m x_i - c)^2 \quad - (1)$$

It is laborious to compute SSE for every option and compare the values. There must be a better way to find the values of m and c such that SSE is minimum.

If SSE is reduced in this form,

$$SSE = k_1^2 + (k_m m - d_m)^2 + (k_c c - d_c)^2 \quad - (2)$$

where, k_1, k_m, d_m, k_c, d_c are real constants, then SSE is minimum when $m = \frac{d_m}{k_m}$ and $c = \frac{d_c}{k_c}$.

In many practical cases, it may not be possible to reduce SSE given by (1) to the form given by (2) as there would be additional terms containing mc.

Thus, to minimize SSE in (1), multivariate differential calculus can be used. This topic is out of scope of the Qualifier syllabus. It will be discussed in Maths Term 2.

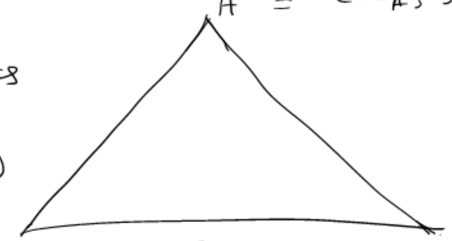
3 NUMERICAL ANSWER TYPE:

10. The area of a triangle is 5 square units. Two of its vertices are $(2, 1)$ and $(3, -2)$. The third vertex with coordinates (x, y) lies on the line $y = x + 3$. Given the condition that $7 < x + y < 11$, what is the value of $x + y$? $= 10$ units (Ans: 10)

Soln. :

Let $A = (2, 1)$, $B = (3, -2)$

and $C = (x, y)$



Given, area $\Delta ABC = 5$ sq. units

$$= \frac{1}{2} \left| x_A (y_B - y_C) + x_B (y_C - y_A) + x_C (y_A - y_B) \right|$$

$$= \frac{1}{2} \left| 2(-2 - y) + 3(y - 1) + x(1 - (-2)) \right|$$

$$= \frac{1}{2} \left| -4 - 2y + 3y - 3 + 3x \right| = \frac{1}{2} \left| y + 3x + 3 \right| = 5$$

$$\Rightarrow \left| y + 3x - 7 \right| = 10$$

$y + 3x - 7 = \pm 10$ since the expression within $| \ |$ can take both positive and negative value.

Case 1

$y + 3x - 7 = 10$

$y + 3x = 17$ — (1.1)

Point $C = (x, y)$ also lies on the line $y = x + 3$

solving $y + 3x = 17$ &

$y = x + 3$ — (2)

multiplying (2) by 3,

$3y = 3x + 9$ — (3)

(1.1) + (3),

$y + \cancel{3x} + 3y = 17 + \cancel{3x} + 9$

$4y = 26 \Rightarrow y = 26/4$

$x = y - 3 = \frac{26}{4} - 3 = \frac{14}{4}$

$\therefore \quad x + y = \frac{14 + 26}{4} = 10$

Case 2

$y + 3x - 7 = -10$

$y + 3x = -3$ — (1.2)

(1.2) + (3),

$y + \cancel{3x} + 3y = -3 + \cancel{3x} + 9$

$4y = 6$

$y = \frac{6}{4}$

$x = y - 3$

$= \frac{6}{4} - 3 = \frac{-6}{4}$

$x + y = \frac{-6 + 6}{4} = 0$

Since $7 < x + y < 11$,

$\therefore \quad x + y = 10$ is the answer.

Week - 2
Assignment Solutions
Mathematics for Data Science - 1, March '21

1 Multiple Choice Questions (MCQ):

1. You are running a car rental company. The company charges ₹500 per day and ₹8 per kilo-metre. A customer returns a car with the odometer readings (shows the distance travelled since the time of renting the car) of 300 kilo-metre and the bill amounts to ₹5900. How many days was this car rented? (Ans: c)
   - ◯ 5
   - ◯ 6
   - ◯ 7
   - ◯ 8

Q1. Soln: Company charges ₹500 per day and ₹8 per km.
if no. of days a car was rented be 'x' and total km travelled be 'y', then the total cost can be represented as

$$\text{cost} = ₹500x + ₹8y \quad -① \Rightarrow \text{eqn. of a straight line}$$

Here, total cost = ₹5900
total distance covered = 300 km

$\therefore$ eqn. ① becomes $5900 = 500x + 8 \times 300$

$\Rightarrow 500x = 5900 - 2400 = 3500$

$\therefore x = \frac{3500}{500} = 7$ days

2. You are climbing a ladder which is slanted at an angle of 45 degrees (measured in the anticlockwise direction) with respect to the ground. The ladder, leaning against a wall, is at a vertical distance of 1 metre from the ground. If you are at a location which cuts the ladder in the ratio 2 : 1 from the top to bottom, what are the coordinates of your location? Assume origin $(0, 0)$ to be at the intersection of the ladder and the ground. (Ans: b)

   - ◯ $(1/2, 1/2)$
   - ◯ $(1/3, 1/3)$
   - ◯ $(2/3, 2/3)$
   - ◯ $(1/3, 2/3)$

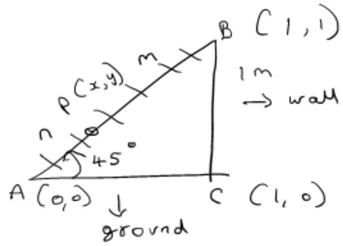
Solution:

Consider a ladder AB leaning on a wall BC.

AB makes 45° (in the anticlockwise direction) with the ground AC

$\tan 45 = 1 = \frac{BC}{AC}$

$\Rightarrow AC = BC = 1$ m



Coordinates of A are $(0, 0)$

Coordinates of C are $(1, 0)$

Coordinates of B are $(1, 1)$

Your location is at P with the coordinates (x, y)

Let (x_A, y_A), (x_B, y_B) represent the coordinates of the points A and B respectively. Then,

$$\frac{m}{n} = \frac{2}{1} = \frac{x - x_B}{x_A - x} = \frac{y - y_B}{y_A - y}$$

Since $x_A = 0$, $x_B = 1$, $y_A = 0$, $y_B = 1$

$$\frac{x-1}{0-x} = 2 \Rightarrow x - 1 = -2x, \quad 3x = 1 \text{ or } x = \frac{1}{3}$$

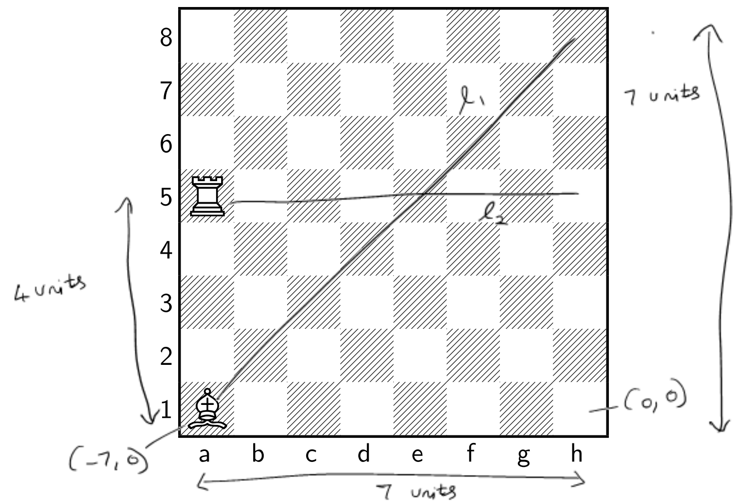
and, $$\frac{y-1}{0-y} = 2 \Rightarrow y - 1 = -2y \text{ or } y = \frac{1}{3}$$

∴ Coordinates of the point P (your location on the ladder) are $\left(\frac{1}{3}, \frac{1}{3}\right)$.

Read the following passage for answering Q3, Q4, Q5

In a chessboard, the Bishop (shown in the cell '$a1$') can move diagonally along the same coloured squares and a Rook (shown in the cell '$a5$') can move only in the horizontal or in vertical directions. There is no restriction on the maximum number of cells the Bishop and the Rook can move in the chessboard. Both of them can never move out of the board.

Assume each cell in the board is a point on the coordinate plane such that the bottom right corner cell '$h1$' is the origin (0,0). Consider the right direction to be +ve X-axis and upward direction to be +ve Y-axis. Each cell consists of unit distance in both axes, i.e., co-ordinate of '$g1$' to be $(-1, 0)$, co-ordinate of '$h2$' to be $(0, 1)$ etc.



3. If B and R represent the initial coordinates of Bishop and Rook respectively, then which of the following is correct? (Ans: b)

   B = a1 = $(-7, 0)$
   R = a5 = $(-7, 4)$

   - ○ $B = (-8, 1), R = (-8, 5)$
   - ○ $B = (-7, 0), R = (-7, 4)$
   - ○ $B = (-5, 1), R = (-5, 5)$
   - ○ $B = (-6, 1), R = (-6, 4)$

4. The Bishop is initially located in the cell '$a1$' and moves to the cell '$h8$' following a straight line 'l_1'. l_1 can be represented as: (Ans: a)
   - ○ $y = x + 7$
   - ○ $y = 2x - 5$

- $y = 3x + 1$
- $y = x + 8$

5. The Rook moves from the cell '$a5$' to the cell '$h5$' along a straight line 'l_2'. l_2 can be represented as: (Ans: a)
   - $y = 4$
   - $y = 5$
   - $y = x + 4$
   - $y + x = -3$

Q4. Solution: Bishop B moves from a1 to h8

h8 = (0, 7), a1 = (−7, 0)

slope of the line l_1 joining a1 and h8

$$= \frac{0-7}{-7-0} = 1$$

equation of l_1 : $y = mx + c$

$m = 1$,

substituting coordinates of h8 in the equation,

$7 = 0 + c \Rightarrow c = 7$

$\therefore$ equation of l_1 is $y = x + 7$. So, option (a) is correct

Q5 Solution: R moves from a5 to h5

Coordinates of h5 are (0, 4) and a5 are (−7, 4)

slope of the line l_2 = $\frac{4-4}{-7-0} = 0$

equation of the line l_2 : $y = mx + c$

$m = 0$

$y = c$

substituting coordinates of h5 in the equation,

$y = 4 = c$

$\therefore c = 4$

Thus, l_2 equation is $y = 4$. So, option (a) is correct

6. Three points $P(-4,4), Q(3,-3)$ and $R(g,h)$ are collinear. Identify the coordinates of the point R from the following: (Ans: c)

   - ◯ $(-1,-1)$
   - ◯ $(2,2)$
   - ◯ $(1,-1)$
   - ◯ $(1,1)$

Let $P = (x_P, y_P) = (-4, 4)$, $Q = (x_Q, y_Q) = (3,-3)$

and $R = (x_R, y_R) = (g, h)$

P, Q, R all lie on the same line.

slope of the line $= \dfrac{y_Q - y_P}{x_Q - x_P} = \dfrac{y_R - y_Q}{x_R - x_Q}$

i.e. $\dfrac{-3-4}{3-(-4)} = \dfrac{h-(-3)}{g-3}$

$\dfrac{-7}{7} = \dfrac{h+3}{g-3} = -1 \Rightarrow h \cancel{+3} = -g \cancel{+3}$

$\underline{h = -g}$

Also, slope $= \dfrac{y_Q - y_P}{x_Q - x_P} = \dfrac{y_R - y_P}{x_R - x_P}$

i.e. $-1 = \dfrac{h-4}{g-(-4)} \Rightarrow h \cancel{-4} = -g \cancel{-4}$

$\underline{h = -g}$

From the given options, only the point $(1,-1)$ satisfies the condition $h = -g$. Hence, option (c) is correct.

2 Multiple Select Questions (MSQ):

7. A carpenter has a call out fee (basic charges) of ₹200 and also charges ₹80 per hour. Which of the following are true? (Ans: a,c,d)

   - ◯ If y is the total cost in (₹) and x is the total number of working hours, then the equation of the total cost is represented by $y = 80x + 200$.
   - ◯ Following the same notations of y, x, equation of the total cost is represented by $y = 200x + 80$.
   - ◯ The total charges, if the carpenter has worked for 4 hours, would be ₹520.
   - ◯ If the carpenter charged ₹350 for fixing a L-stand and changing door locks, then the number of working hours would be approximately one hour and 53 minutes.

Basic charges = ₹200 , x = no. of working hours, y = total cost

y = ₹200 + ₹80x

∴ option (a) is correct and (b) is wrong

option (c) : no. of working hours = 4 = x

total cost $y = 200 + 80x = 200 + 80 \times 4$

$= 200 + 320 = ₹520$

so, option (c) is correct.

option (d) : Total charges = ₹350

$= y = 200 + 80x$

$\Rightarrow 80x = 350 - 200 = 150$

$x = \frac{150}{80} = \frac{80}{80} + \frac{70}{80}$ hours

$= 1$ hour $+ \frac{70}{80} \times 60$ minutes (with 70/80 × 60 reduced to 7 × 15 / 2)

$= 1$ hour $+ 52.5$ minutes

Hence, option (d) is correct.

8. A swimming pool leaks at a slow rate of 0.5 gallons per hour. If the pool holds 325 gallons of water when it is full, then which of the following statements are false? (Ans: b,d)

- ◯ 301 gallons of water would remain in the pool after two full days.
- ◯ 324 gallons of water would remain in the pool after two full days.
- ◯ Quantity of water remaining in the pool (y in gallons) after x number of hours can be computed by the equation: $y = 325 - 0.5x$.
- ◯ The pool would be empty on the 25^{th} day since the beginning of the leak.

leak rate = 0.5 gallons per hour

Full capacity of the pool = 325 gallons

so, amount of water after 'x' no. of hours since the leak

$= y = 325 - 0.5x$

$\therefore$ option (c) is correct

After 2 full days, $x = 2 \times 24 = 48$ hours

$y = 325 - 0.5 \times 48 = 325 - 24 = 301$ gallons

$\therefore$ option (a) is correct and option (b) is false

Pool would be empty when $y = 0 = 325 - 0.5x$

$\Rightarrow x = \frac{325}{0.5} = 325 \times 2 = 650$ hours

$= \frac{650}{24}$ days

$= \frac{648}{24} + \frac{2}{24}$

$= 27 + \frac{2}{24}$ days

i.e. on the 28^{th} day pool would get empty

$\therefore$ option (d) is false

9. The skeleton of an aircraft wing comprises of the ribs and spars. Note that ribs and spars are arranged perpendicular to each other during the manufacturing of a wing. We have eight structural parts with the two end-points of each of them known. Identify which of the following pairs of structures (indicated with their end-points) can potentially be chosen for the ribs and spars combination. R and S represent structures designed for the ribs and spars respectively (Ans b,c)

   ◯ R: $(1,2),(2,3)$, S: $(1,3),(1,4)$

   ◯ R: $(0.5,1),(1.5,3)$, S: $(2.5,1),(0.5,2)$

   ◯ R: $(2.5,6),(3.5,7)$, S: $(4.5,8),(3.5,9)$

   ◯ R: $(3,2),(5,5)$, S: $(2,3),(4,4)$

3 Numerical Answer Type (NAT):

10. You are keen to know the area of the plot, which you want to buy, in the outskirts of Chennai city. Interestingly, the plan of the plot indicates that it is of a triangular boundary. The corners of the plot are marked by the points A $(1,3)$, B $(-5,7)$ and C $(8,12)$ with respect to some fixed coordinate system. Compute the area of the plot in square units. (Ans: 41)

Q9 Soln.: Condition for R and S to be perpendicular

slope of R $(m_R) = \dfrac{-1}{\text{slope of S } (m_S)}$

i.e. $m_R \cdot m_S = -1$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

(a) R: $(1,2)$, $(2,3)$ are the end-points

$m_R = \dfrac{3-2}{2-1} = \dfrac{1}{1}$

S: $(1,3)$, $(1,4)$ $\Rightarrow m_S = \dfrac{4-3}{1-1} \to \infty$

R and S are not perpendicular since $m_R \cdot m_S \to \infty$

∴ option (a) is not correct

1

(b) R : (0.5, 1), (1.5, 3)

$$m_R = \frac{3-1}{1.5-0.5} = \frac{2}{1} = 2$$

S : (2.5, 1), (0.5, 2)

$$m_S = \frac{2-1}{0.5-2.5} = \frac{1}{-2} = -\frac{1}{2}$$

$$m_R \times m_S = 2 \times \left(\frac{-1}{2}\right) = -1$$

∴ these 2 structures can be used as a rib-spar pair ∴ option (b) is correct

(c) ✓ R : (2.5, 6), (3.5, 7)

$$m_R = \frac{7-6}{3.5-2.5} = \frac{1}{1} = 1$$

S : (4.5, 8), (3.5, 9)

$$m_S = \frac{9-8}{3.5-4.5} = \frac{1}{-1} = -1$$

$$m_R \times m_S = 1 \times (-1) = -1$$

∴ these 2 structures can be used as a rib-spar pair ∴ option (c) is correct

(d) R : (3, 2), (5, 5)

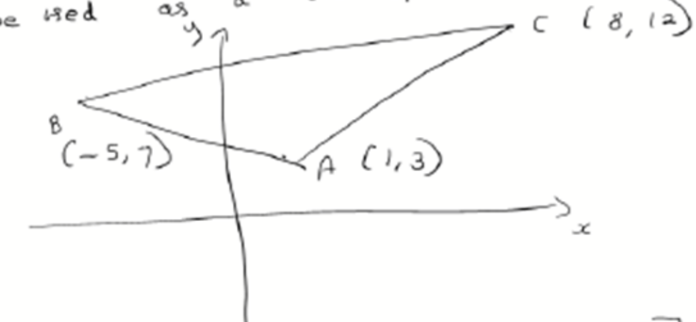
$$m_R = \frac{5-2}{5-3} = \frac{3}{2}$$

S : (2, 3), (4, 4)

$$m_S = \frac{4-3}{4-2} = \frac{1}{2}$$

$$m_R \times m_S = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4} \neq -1$$

∴ these 2 structures are not perpendicular and cannot be used as a rib-spar combination



Q10. Soln.:

Let (x_A, y_A) indicate coordinates of point A and similarly for points B and C respectively

Area of the plot = Area of Δ ABC

$$= \frac{1}{2}\left|\left[x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)\right]\right|$$

Note that $x_A = 1$, $y_A = 3$, $x_B = -5$, $y_B = 7$, $x_C = 8$, $y_C = 12$

$$\Rightarrow \text{Area} = \frac{1}{2}\left|\left[1(7-12) + (-5)(12-3) + 8(3-7)\right]\right|$$

$$= \frac{1}{2}\left|\left[-5 + (-5)\times 9 + 8 \times (-4)\right]\right|$$

$$= \frac{1}{2} \cdot |-5 - 45 - 32| = \frac{1}{2}|-82|$$

$$= 41 \text{ square units}$$

Ans : 41

Week - 2
Solutions for Practice Assignment-2
Straight line - 1
Mathematics for Data Science - 1

1 Multiple Choice Questions (MCQ):

1. A publishing house purchases a printing machine for ₹50,000. At the end of 5 years the value of the machine is supposed to be ₹10,000 only. If the loss in value is assumed to be linear then what is the yearly loss in the value of machine?

   ○ ₹ 5,000 only.
   ○ ₹ 6,000 only.
   ○ ₹ 10,000 only.
   ○ ₹ 12,000 only.
   ○ **₹ 8,000 only.**
   ○ None of the above.

   Solution:
   Let y represents the value of machine and x represents the number of years after purchasing the machine.

   $$\text{When, } x = 0 \implies y = 50,000.$$
   $$\text{And when, } x = 5 \implies y = 10,000.$$

   Applying equation of the line in two point form:

   $$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
   $$\implies y - 50,000 = \frac{10{,}000 - 50{,}000}{5 - 0}(x - 0)$$

   On rearranging the above equation, we have:

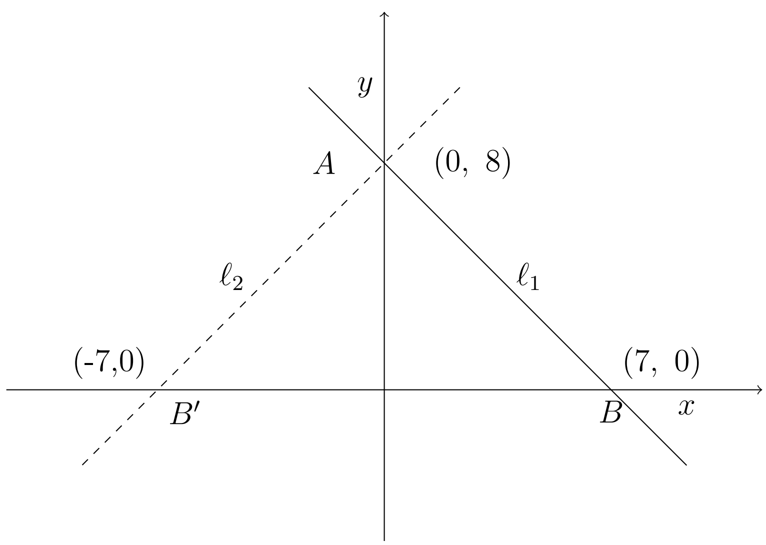
   $$y = -8000x + 50,000$$

   Then the slope of the line will be the yearly loss in the value of machine i.e. ₹ 8,000.

2. A line is represented by $7y = 56 - 8x$. If mirror image is taken with respect to the $Y - axis$, a new line is formed. What will be the equation of the new line?

   - ○ $7x + 8y = 56$
   - ○ $7x + 8y = -56$
   - ○ $\frac{x}{7} + \frac{y}{8} = 1$
   - ○ $\frac{x}{8} + \frac{y}{7} = 1$
   - ○ $\boldsymbol{y = \frac{8}{7}x + 8}$
   - ○ $y = \frac{7}{8}x - 7$

Solution:
Rearranging the equation $7y = 56 - 8x$ in intercept form, we have $\ell_1 \equiv \frac{y}{8} + \frac{x}{7} = 1$.
Then, the points of intercept are (0,8) and (7,0) as shown in (Figure 1)

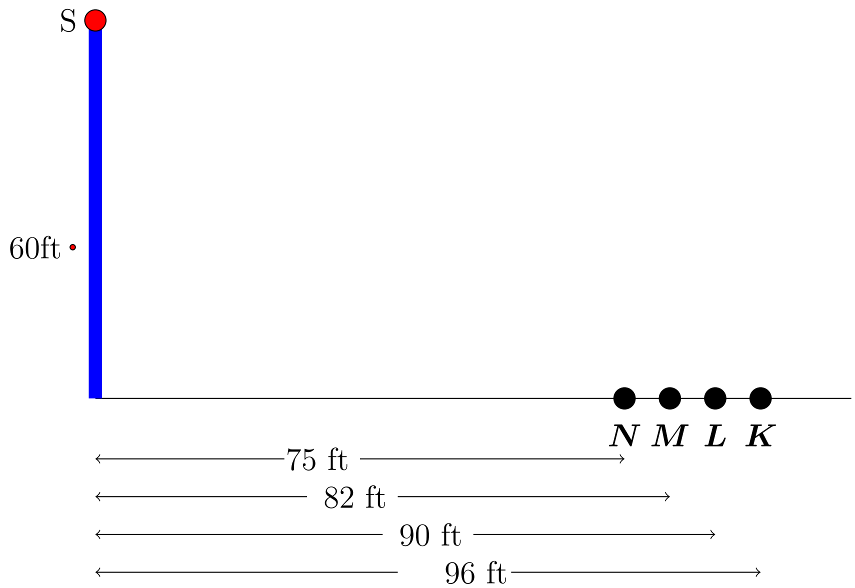


If we treat $Y-$ axis as mirror then the mirror image line ℓ_1 will be line ℓ_2 and the mirror image of points A $(0,8)$ and B $(7,0)$ will be A $(0,8)$ and B $(-7,0)$ respectively shown in (Figure 1).

Then the equation of new line passing through A and B' using intercept form $\frac{x}{a} + \frac{y}{b} = 1$ will be $\ell_2 \equiv \frac{y}{8} - \frac{x}{7} = 1$.
On rearranging:

$$\boldsymbol{y = \frac{8}{7}x + 8}$$

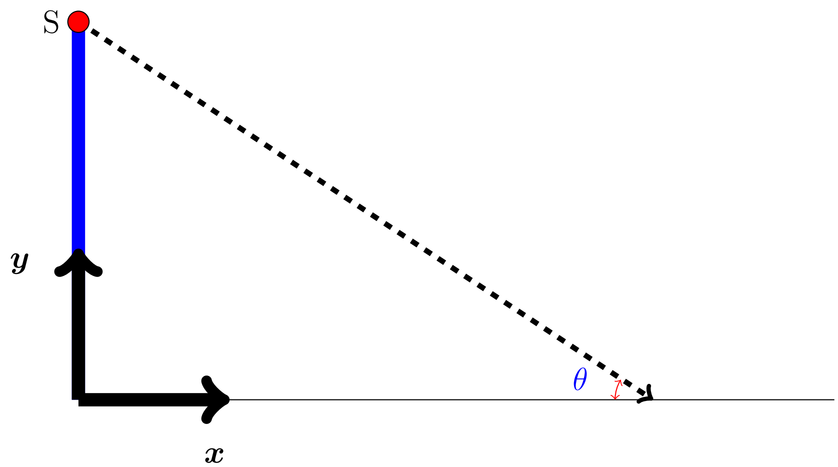
3. A sniper is sitting on top of a tower at a height of 60 ft. There are four workers K, L, M and N standing at a distance of 96 ft, 90 ft, 82 ft, and 75 ft respectively from the base of the tower. The heights of K, L, M, and N are 6 ft, 5.5 ft, 5.7 ft, and 5.2 ft respectively. The sniper misfires a bullet at an angle θ with the horizontal. Since the range covered by the bullet is short, the path of the bullet is assumed to be a straight-line path. If $\tan\theta = \frac{2}{3}$, choose the correct option.



- ○ All workers are safe.
- ○ All the workers are safe except K.
- ○ Only K and N are safe.
- ○ No one is safe.
- ○ Only K is safe.
- ○ **All the workers are safe except M.**

Solution:

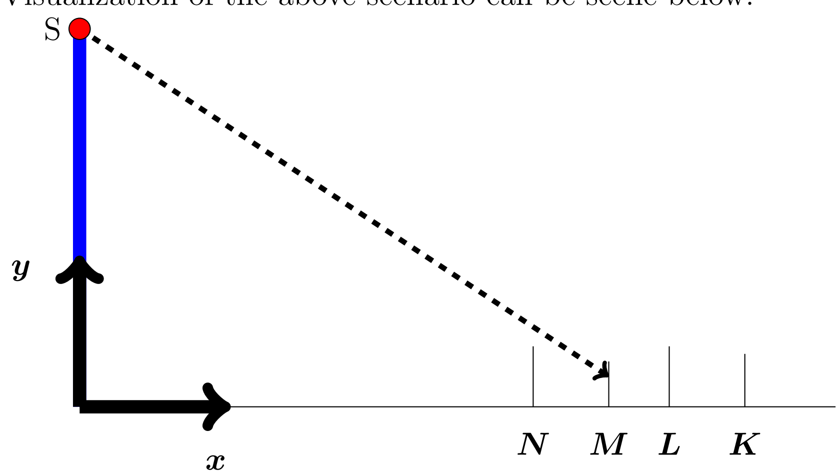
Let us treat height from ground as $Y-$ axis and horizontal distance on ground from tower base as $X-$ axis as shown in Figure.



From figure it is clear that if $x = 0 \longrightarrow y = 60$.
The path of bullet can be written using slope intercept form as $y = mx + c$.
Here $c = 60$ then the path of bullet will be $y = mx + 60$.
Now m will be the slope of line and it is known as $m = tan\ \theta$. According to figure, θ is from -ve $x-$ axis therefore $m = -tan\theta = -\frac{2}{3}$.
There the path of bullet will be $y = -\frac{2}{3}x + 60$.

Worker N is standing at 75 ft away from the tower base which means $x = 75$. Putting this value in the path of bullet $y = -\frac{2}{3} \times 75 + 60 = 10$, where the height of worker is 6 ft. Therefore the bullet will even not touch the worker N.

Similarly for M, $y = 5.333$ which is lower than the height of worker M. It means the bullet will hit worker M and then it will not cross $x = 82$, therefore we do not need to check for others.
Visualization of the above scenario can be scene below:



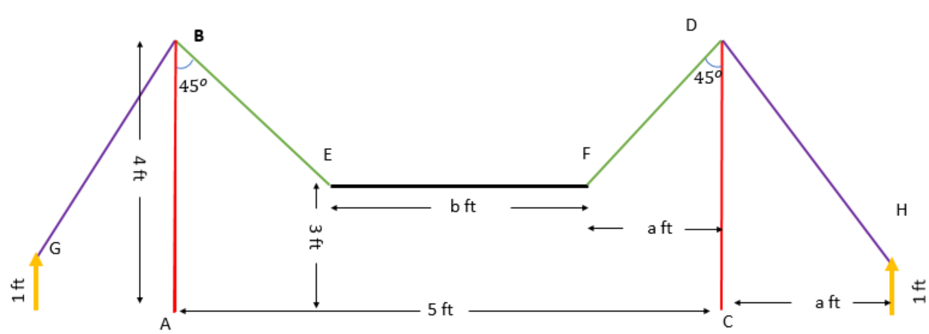
2 Multiple Select Questions (MSQ):

1. Choose the correct options regarding mirror image of a given point.

   - ◯ Point (4,6) is the mirror image of (-4,6) when X-axis is treated as mirror.
   - ◯ **Point (-2,8) is the mirror image of (-2,-8) when X-axis is treated as mirror.**
   - ◯ Point (-2,8) is the mirror image of (2,8) when X-axis is treated as mirror.
   - ◯ **Point (-2,8) is the mirror image of (2,8) when Y-axis is treated as mirror.**
   - ◯ Point (4,6) is the mirror image of (-4,-6) when X-axis is treated as mirror.
   - ◯ Point (-4,6) is the mirror image of (4,-6) when Y-axis is treated as mirror.

   Solution:
   Only the sign of a ordinate ($Y-$ coordinate) changes when $X-$axis is treated as mirror. Similarly, only the sign of abscissa ($X-$ coordinate) changes when $Y-$axis is treated as mirror. Therefore the options (a), (c), (e), (f) are incorrect.

2. A long horizontal piece of wood EF is suspended above the ground, as shown in the diagram, by two ropes EB and FD which are tied to two bamboo poles AB and CD of equal lengths. The bamboo poles are at equal distances from the edge of wooden piece. The bamboo poles are supported by two different ropes GB and HD respectively as shown in the figure. Choose the correct options.



   - ◯ $b = 4$
   - ◯ If A is treated as origin then the coordinate of point E will be $(3, 4)$.
   - ◯ **If A is treated as origin then the coordinate of point E will be $(1, 3)$.**
   - ◯ **If A is treated as origin then the coordinate of point F will be $(4, 3)$.**
   - ◯ **The slope of thread DH is -3.**

○ The slope of thread DH is -$\frac{1}{3}$.

Solution:

Let us consider point A as origin and distance from A on ground as $X-$axis. Therefore, its coordinate is A $(0,0)$ and $y-$ coordinate of E will be 3 in rectangular coordinate system.

Then, the coordinate of B is (0,4). The slope (m) of the line segment BE is $-tan\,45 = -1$;
Let coordinate of E be $(x, 3)$, then the slope of BE:

$$m = -1 = \frac{4\ -\ 3}{0-x}$$
$$\longrightarrow x = 1$$

Similarly, coordinates of F is (4,3) and $a = 1$ (Using slope of FD).
Therefore, $b = 4 - 1 = 3$.

Coordinates of D and H are $(5, 4)$ and $(6, 1)$ respectively, using D and H, slope of thread DH can be found:

$$\text{Slope of thread } DH \ = \ \frac{y_2\ -\ y_1}{x_2\ -\ x_1} = \frac{1\ -\ 4}{6\ -\ 5}$$
$$\textbf{Slope of thread} \ = \textbf{-3}$$

3. A line represented by $\frac{y}{3} - \frac{x}{7} = 1$ intersects the $X - axis$ and the $Y - axis$ at points A and B respectively. The mirror image of this line (taken with $X - axis$ as the mirror) intersects the $Y - axis$ at point C. The mirror image of the point A (with $Y - axis$ as the mirror) is denoted by D. Choose the correct options.

   - ○ **Equation of line segment $\boldsymbol{CD}$ is $\boldsymbol{\frac{x}{7} - \frac{y}{3} = 1}$.**
   - ○ Equation of line segment CD is $\frac{x}{7} + \frac{y}{3} = 1$.
   - ○ **$\boldsymbol{DB}$ is parallel to $\boldsymbol{AC}$.**
   - ○ DC is perpendicular to AC.
   - ○ **The area of the geometry enclosed by above four lines is 42 square units.**
   - ○ The area of the geometry enclosed by above four lines is 10.5 square units.

Solution:
Option (a): Correct
Let, l_1: $\frac{y}{3} - \frac{x}{7} = 1$ as shown in (Figure 2) represents the line in intercept form of a given.
Then, the Coordinates of $X-$intercept is A(-7, 0) and $Y-$intercept is B(0,3)
The mirror image of this line l_1 will be line l_2 as shown in (Figure 2).
Since, when $X-$axis is treated as mirror only the sign of $Y-$coordinate changes. Therefore, coordinate of C is (0,-3).
Similarly, coordinate of D is (7,0) as $Y-$axis is mirror, hence, only the sign of $X-$coordinate of point A changes.
Equation of line segment CD can be obtained using the points C and D in the intercept form i.e. l_2: $\frac{x}{7} - \frac{y}{-3} = 1$

Option (b): Incorrect
Because, as seen above the equation of line segment CD is,
$\frac{x}{7} - \frac{y}{-3} = 1$
Option (c): Correct
Slope of BD can be found using B(0,3) and D(7,0)

$$\text{Slope of } BD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{7 - 0}$$

$$\text{Slope of } BD = -\frac{3}{7}$$

Slope of AC can be found using A(-7,0) and C(0,-3)

$$\text{Slope of } AC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 0}{0 - (-7)}$$

$$\text{Slope of } AC = -\frac{3}{7}$$

Since, slpoe of $BD = AC$, therefore are form parallel to each other.
Option (d): Incorrect

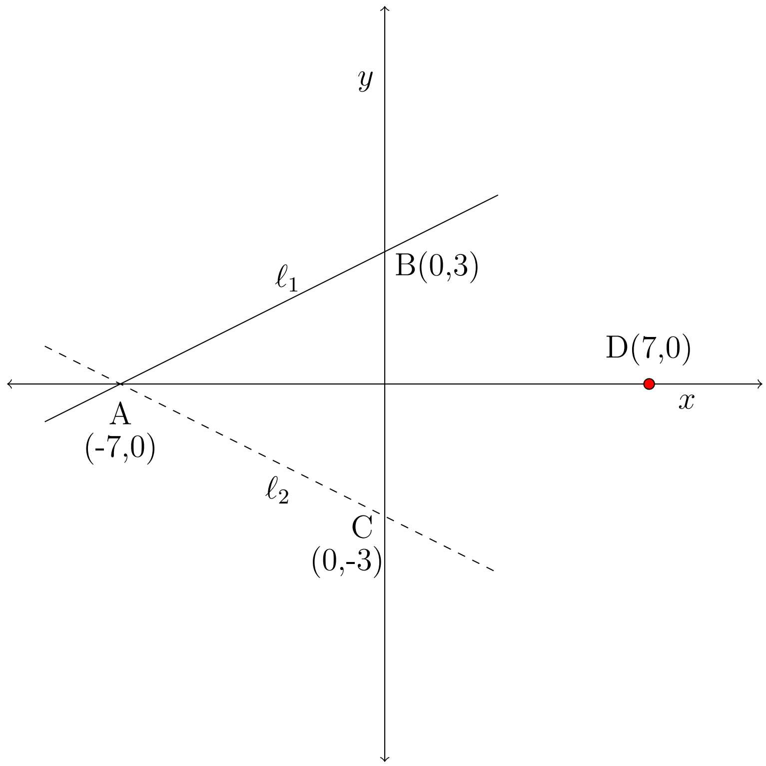


Figure 1: representation of ℓ_1 and ℓ_2 on coordinate plane

Slope of AC (m_1) = -1
Slope of DC (m_2) can be found using the points D(7,0) and C(0,-3)

$$\text{Slope of } DC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3)}{7 - 0}$$

$$\text{Slope of } DC \ (m_2) = \text{-}\frac{3}{7}$$

For a line to be perpendicular to each other,

$$m_1 \times m_2 = \text{-1}$$
$$\text{-1} \times (-\tfrac{3}{7}) \neq \text{-1}$$

Therefore, line segment DC and AC are not perpendicular to each other.
Option (e): Correct

The area of ABC = area of $DBC = a$ (geometric similarity)
The area formed by geometry $ABDC$ = Area of triangle ABC + area of triangle DBC $= a + a = 2a$
Area of a triangle ABC can be obtained using the vertices A(-7,0), B(0,3) and C(0,-3) is:

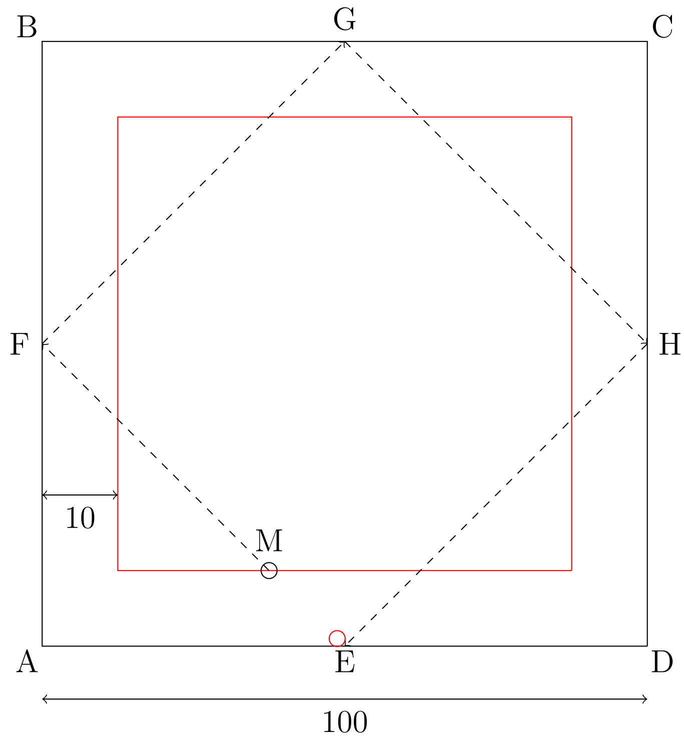
$$a = \frac{1}{2} \mid x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \mid$$

$$a = \frac{1}{2} \mid -7(3 - (-3)) + 0(-3 - (-7)) + 0(0 - 3) \mid$$

$$a = \mid -21 \mid = 21$$

Therefore, area of geometry $ABDC = 2 \times 21 = 42$ square units

4. A carom board is a square board with a symmetrical design as shown in the diagram below. E, F, G, and H are the midpoints of AD, BA, CB, and DC respectively. Anand wants to pocket his last coin which is at E. Taking the laws of reflection to be applicable, he strikes from the point M, so that the striker reflects at F, then at G, then at H and finally hits the coin at E. Choosing D to be the origin (right direction is $+ve\, X - axis$ and upward direction is +ve Y-axis), choose the correct options.



- ○ M does not lie on the line segment EF.
- ○ **The coordinate of point M is $(-60, 10)$.**
- ○ The coordinate of point M is $(-40, 10)$.
- ○ **The slope of MF is -1.**
- ○ **FG and GH are perpendicular line segments.**
- ○ Equation of GH will be $y + x = 150$.
- ○ **Segment HE intersects the inner square at $(-10, 40)$.**

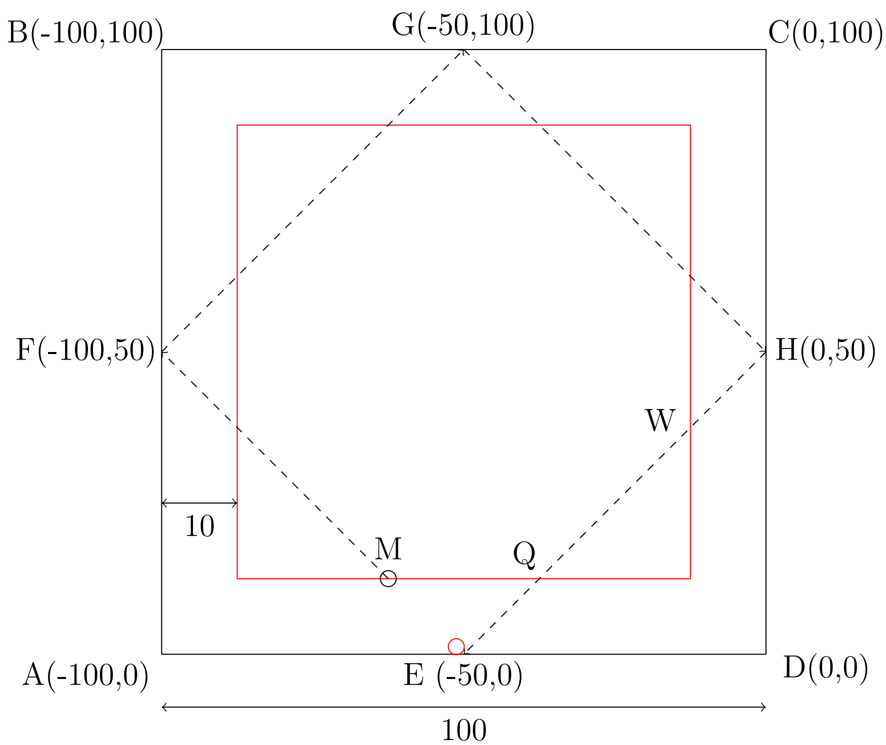
Solution:

Considering corner D of the carom board as origin in coordinate system.

Then coordinates of the corners will be:$D(0, 0), C(0, 100), B(-100, 100), A(-100, 0)$.

Therfore coordinates of mid points on carom will be:

$H(0, 50), G(-50, 100), F(-100, 50), E(-50, 0)$ as shown in below Figure.



Option (a) is incorrect because point M comes in the path of FE as per the law of reflection.
Option (b): Correct
Because using the point E(-50,0), F(-100,50), we can form a equation of line in two-point form.

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - (0) = \frac{50 - 0}{-100 - (-50)}(x - (-50)) \implies y = -x - 50$$

Since, $Y-$coordinate of M is 10 (given: Carom symmetry). On solving the above equation with $y = 10$:

$$10 = -x - 50$$
$$x = -60$$

Therefore the coordinate of M is (-60,10)

Option (c): Incorrect, because coordinate of M is (-60,10) as explained above.

Option (d): Correct, because using the point M(-60,10), F(-100,50)

$$\text{Slope of MF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{50 - (10))}{-100 - (-60)}$$

$$\text{Slope of MF} = \text{-1}$$

Option (e): Correct
Because the slopes of FG and GH can be found using the points F(-100,50), G(-50,100), H(0,50).

$$\text{Slope of FG } (m_1) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{100 - 50}{-50 - (-100)}$$

$$\text{Slope of FG } (m_1) = 1$$

$$\text{Slope of GH } (m_2) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{50 - 100}{0 - (-50)}$$

$$\text{Slope of GH } (m_2) = \text{-1}$$

Since, $(m_1)\times(\mathrm{m}_2)$ = -1, therefore they are perpendicular to each other.

Option (f): Incorrect
Equation of line segment GH can be found using the slope-intercept form.
Where slope (m) is -1 (found in above option) and Y-intercept (c) is 50 at point H.

$$\text{Equation of line in slope-intercept form is: } y = mx + c$$
$$y = -x + 50 \implies \boldsymbol{y + x = 50}$$

Option (g): Correct
Clearly from the above figure there are two points lets say Q and W where the line segment HE intersects the inner square.
Equation of a line segment HE can be obtained in two-points form, using the coordinates H(0,50), E(-50,0).

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - (50) = \frac{0 - 50}{-50 - 0}(x - 0) \implies y = x + 50$$

$Y-$coordinate of Q is 10. Therefore, the other coordinate can be found by substituting in equation of line segment HE.

$$10 = x + 50$$
$$x = -40$$
$$\text{Q(-40,10)}$$

Similarly $X-$coordinate of W is -10 and the other coordinate can be obtained in same way.

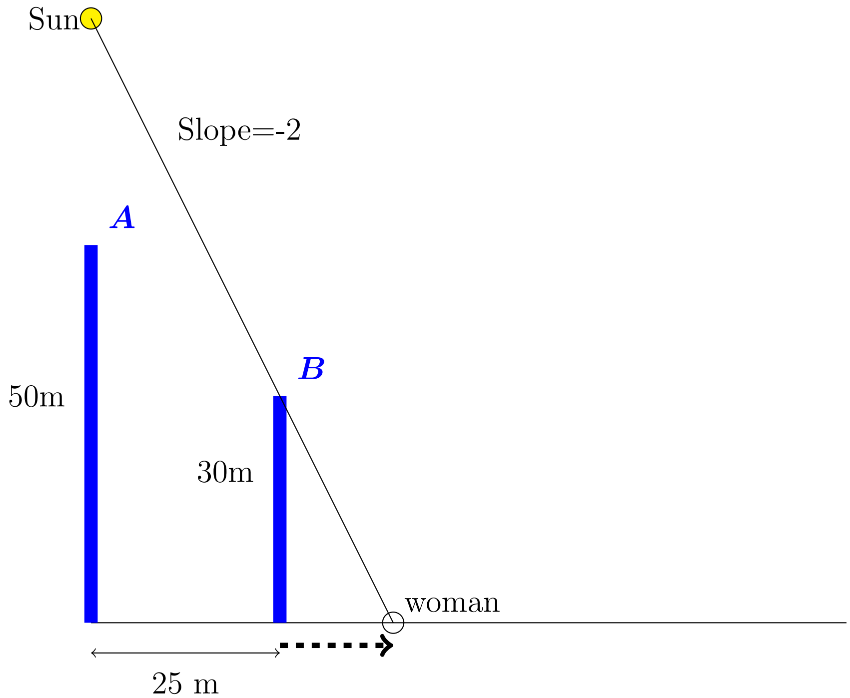
$$y = -10 + 50$$
$$x = 40$$
W(-10,40)

Therefore, W(-10,40) is one of the point in the inner square where the line segment HE intersects.

3 Numerical Answer Type (NAT):

1. Two buildings A and B are as shown in the diagram. A woman walks out of building B to get some sunshine. How much minimum distance in metres will she have to walk in the right direction of building B, according to the situations given below? Assume the right direction of buildings as positive and ground as $X - axis$.

   (a) The first sunray not blocked by B has a slope of -2. [Ans: 15]



Solution:

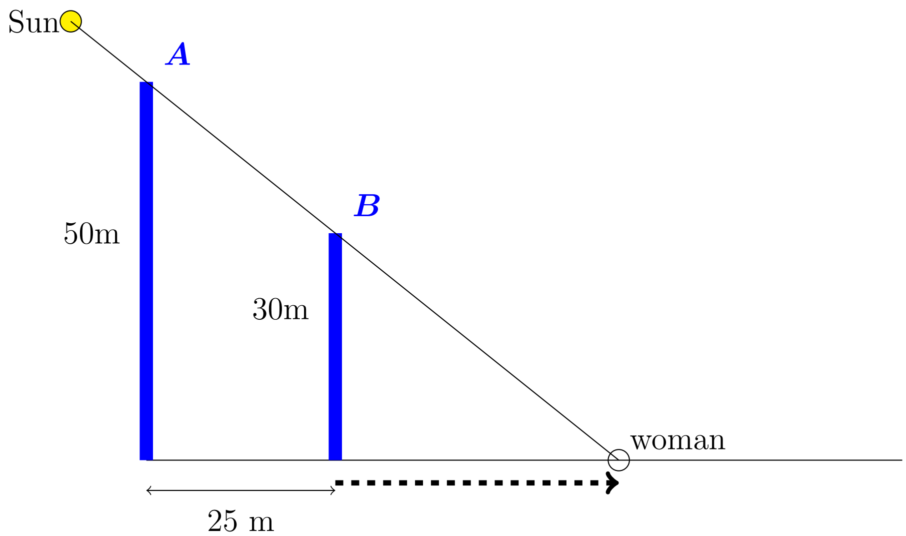
Let us consider the sunray as a line and the foot of the building B as origin in the coordinate plane.
Then, the coordinate B(0,30) forms the $Y-$intercept and the slope of the sunray is given as -2.
Therefore, a equation of line in a slope intercept form can be established:
y represents the height of building and x representing the minimum distance travelled by women to get sunshine from the foot of building B.

$$y = mx + c, \quad (\text{ c = 30m, } Y\text{−intercept})$$
$$y = -2x + 30$$

On solving the above equation with $y = 0$:

$$2x = 30$$
$$x = 15m$$

(b) The first sunray not blocked by A is also the first sunray not blocked by B.
[Ans: 37.5]



Solution:
Let us again consider the sunray as a line and this time foot of the building A as origin in the coordinate plane.
Then, the coordinate A(0,50), and the coordinate of B is (25,30).
Therefore, a equation of line in two point form can be established:
y represents the height of building and x representing the minimum distance travelled by women to get sunshine from the foot of building A.

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

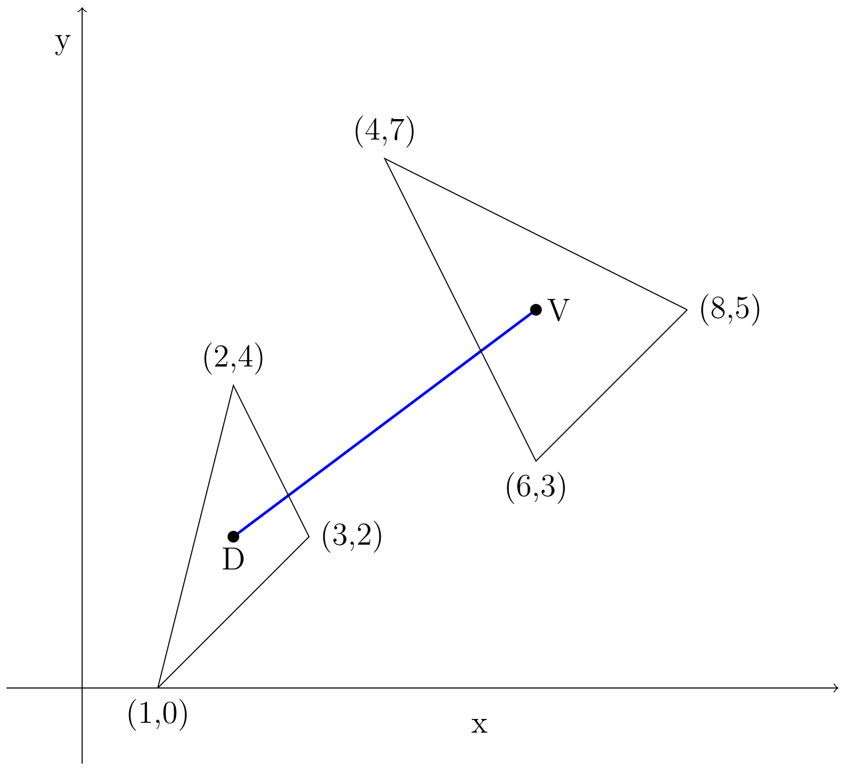
$$\implies y - 50 = \frac{30 - 50}{25 - 0}(x - 0) \implies y = -0.8x + 50$$

On solving the above equation with $y = 0$:

$$0.8x = 50$$
$$x = 62.5m$$

Therefore, women has to travel 62.5 - 25 = **37.5m** from the foot of building B

2. Veeru (V) and his horse Dhanno (D) are running towards each other from their locations which are the centroids of two different triangles shown in figure below. The x - coordinate x_c of the centroid of a triangle can be found using the formula $x_c = \frac{x_1+x_2+x_3}{3}$, where x_1, x_2, x_3 are the x-coordinates of the vertices of the triangle. Corresponding formula can be applied for the y-coordinate, too. Then answer the following questions.



(a) If Veeru and Dhanno follow the path DV then what will $tan\,\theta$ be, if θ is the angle which DV makes with the horizontal? [Ans = $\frac{3}{4}$]

Solution:

Using the formula for the centroid of a triangle (i.e. $x_c = \frac{x_1+x_2+x_3}{3}, y_c = \frac{y_1+y_2+y_3}{3}$), the coordinates of D and V can be given as:

$$D \equiv (x_c = \tfrac{x_1+x_2+x_3}{3}, y_c = \tfrac{y_1+y_2+y_3}{3}) \implies D(\tfrac{1+2+3}{3}, \tfrac{0+4+2}{3})$$

$$\implies D(2,2)$$

$$V \equiv (x_c = \tfrac{x_1+x_2+x_3}{3}, y_c = \tfrac{y_1+y_2+y_3}{3}) \implies V(\tfrac{6+4+8}{3}, \tfrac{3+7+5}{3})$$

$$\implies V(6,5)$$

Now, the slope of line segment DV:

$tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{5-2}{6-2}$

$= \mathbf{\frac{3}{4}}$

(b) If Veeru can run two units per minute and Dhanno can run three units per minute and they meet at a point M on line segment DV, what will be the ordinate of M? [Ans = $\frac{19}{5}$]

Solution:
The distance between the points D and V can be obtained by distance formula:

$$DV = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$= \sqrt{(5-2)^2 + (6-2)^2} \implies \sqrt{3^2 + 5^2}$$

$$DV = 5 \text{ units}$$

Distance travelled by Dhanno and Veeru in 1 min are 3 units and 2 units respectively which accounts for 5 units.
Since they meet at M so we can consider the travelled distance as the ratio in which the line segment DV is divided.
Therefore,

$$\text{Ordinate } (y) = \frac{m \times y_2 + n \times y_1}{m + n}$$
$$y = \frac{3 \times 5 + 2 \times 2}{3+2}$$

$\boldsymbol{y = \frac{19}{5}}$

Week - 2
Practice Assignment Solutions
Straight line - 1
Mathematics for Data Science - 1

NOTE: There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

1. If R is the set of all points which are 5 units away from the origin and are on the axes then R is:

   - ○ $R = \{(5,5), (-5,5), (-5,-5), (5,-5)\}$
   - ○ $R = \{(5,0),\ (5,-5),\ (5,5),\ (-5,0)\}$
   - ○ $R = \{(5,0),\ (0,5),\ (5,5),\ (0,-5)\}$
   - ○ $\boldsymbol{R = \{(5,0),\ (0,5),\ (-5,0),\ (0,-5)\}}$
   - ○ $R = \{(5,0),\ (0,5),\ (-5,0),\ (-5,5)\}$
   - ○ There is no such set.

   Solution:

   The points on the $x-$axis are represented by $(\pm a,\ 0)$, and on the $y-$axis are represented by $(0,\ \pm b)$, where a and b are the distances of the points $(\pm a,\ 0)$ and $(0,\ \pm b)$, respectively, from the origin. Therefore, the points (5, 0), (0, 5), (-5, 0), (0, -5) lie on the axes and are 5 units away from the origin. See Figure PS-2.1 for reference.

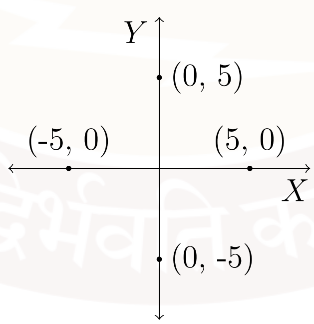


Figure PS-2.1

2. A point P divides the line segment MN such that $MP : PN = 2 : 1$. The coordinates of M and N are $(-2, 2)$ and $(1, -1)$ respectively. What will be the slope of the line passing through P and the point $(1, 1)$?

   - ◯ $\frac{4}{3}$
   - ◯ **1**
   - ◯ Inadequate information.
   - ◯ $-\frac{4}{3}$
   - ◯ $\tan(\frac{4}{3})$
   - ◯ None of the above.

Solution:

By the sectional formula, the coordinates of a point (x, y) that divides a line segment defined by two points $(x_1, y_1), (x_2, y_2)$ in the ratio $m : n$ is given by

$$x = \frac{m \times x_2 + n \times x_1}{m + n}$$

$$y = \frac{m \times y_2 + n \times y_1}{m + n}$$

Since point P divides the line segment formed by the points $M(-2, 2)$ and $N(1, -1)$ in the ratio 2:1, we obtain the coordinates of point P denoted by, say (x_p, y_p), using the sectional formula as follows.

$$x_p = \frac{2 \times 1 + 1 \times (-2)}{2 + 1} = 0$$

$$y_p = \frac{2 \times (-1) + 1 \times 2}{2 + 1} = 0$$

Hence point $P = (0, 0)$ denotes the origin as shown in Figure PS-2.2

Now, we compute the slope of the line passing through P and (1,1) as,

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{1 - 0} = 1$$

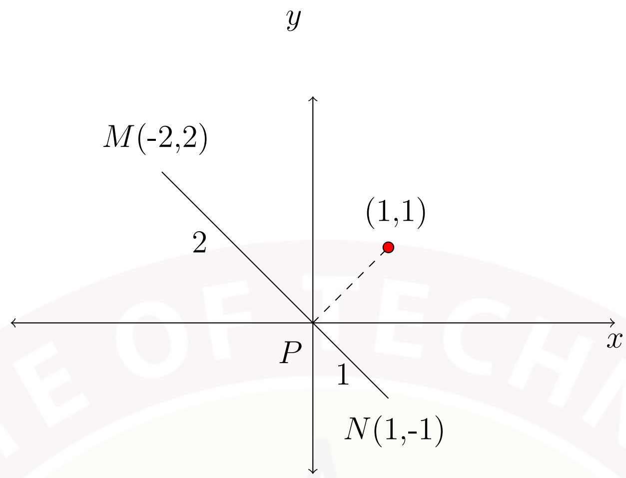


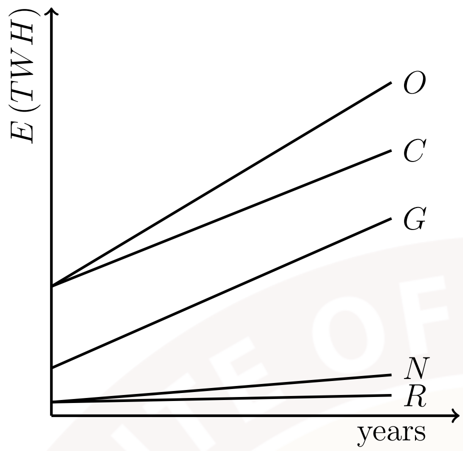
Figure PS-2.2

Use the following information to solve questions 3 and 4.
Table PS-2.1 shows the different types of energies consumed (approximate values) in years 1965 and 2015 across the world

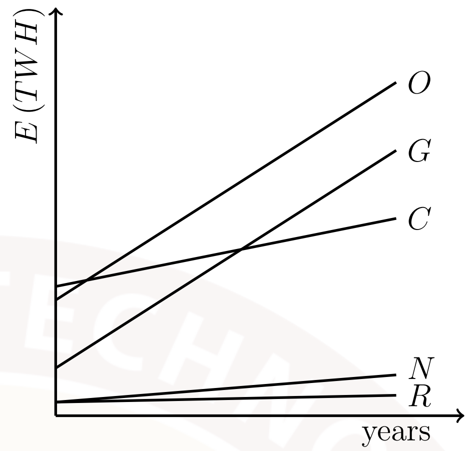
Energy type	Approximate energy used (TWH)	
	1965	2015
Oil (O)	19000	49000
Coal (C)	17000	38000
Gas (G)	7000	29000
Nuclear (N)	2000	6000
Renewable (R)	2000	3000

Table PS-2.1

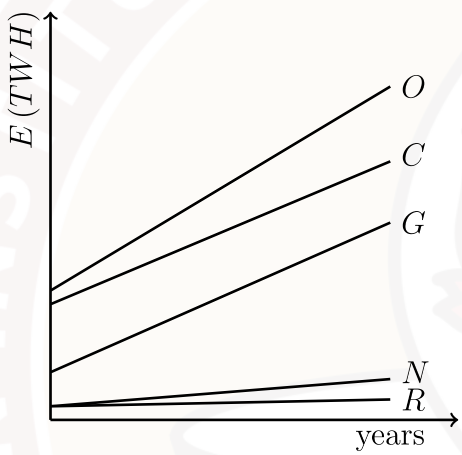
3. A student assumes a linear relationship between energy consumed (E) and the number of years after 1965. Choose the option which best represents the linear relationships assumed by the student (from 1965 to 2015). [Ans: Option C]



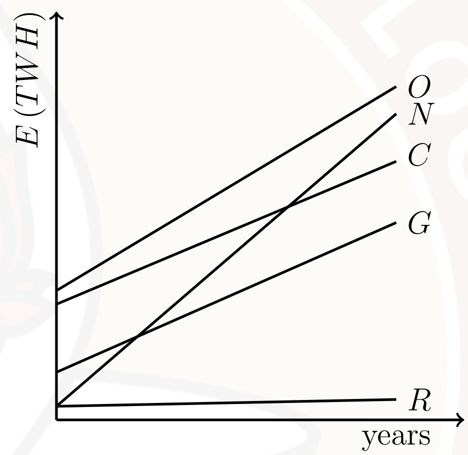
A.



B.



C.



D.

Solution:

Let x-axis and y-axis represent the years and the energy consumption respectively. The energy consumption in 2015 is in the order $O > C > G > N > R$, which is represented correctly in options (A) and (C). However, option (A) shows the energy consumption of O and C being same in the year 1965, which is not true. Hence, option (A) is not correct. Therefore, the correct answer is option (C).

4. The student estimated the energy consumption in 2025 and created Table PS-2.2. Choose the correct option.

Energy type	Approximate energy used (TWH)		
	1965	2015	2025
Oil (O)	19000	49000	o
Coal (C)	17000	38000	c
Gas (G)	7000	29000	g
Nuclear (N)	2000	6000	n
Renewable (R)	2000	3000	r

Table PS-2.2

- ◯ $o = 64000$
- ◯ $c = 48500$
- ◯ $g = 38500$
- ◯ $n = 8000$
- ◯ $r = 3500$
- ◯ **None of the above**.

Solution:

As earlier, let x-axis and y-axis represent the years and the energy consumption respectively. Using the data provided for two years, we can find the equation of the line in two-point form. Equation for the energy type *oil* (O) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 19000 = \frac{49000 - 19000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 19000 = \frac{49000 - 19000}{2015 - 1965}(2025 - 1965)$$

$$y = 55000$$

Equation for the energy type *coal* (C) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 17000 = \frac{38000 - 17000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$:

$$\implies y - 17000 = \frac{38000 - 17000}{2015 - 1965}(2025 - 1965)$$

$$y = 42200$$

Equation for the energy type *gas* (G) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 7000 = \frac{29000 - 7000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 7000 = \frac{29000 - 7000}{2015 - 1965}(2025 - 1965)$$

$$y = 33400$$

Equation for the energy type *nuclear* (N) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 2000 = \frac{6000 - 2000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 2000 = \frac{6000 - 2000}{2015 - 1965}(2025 - 1965)$$

$$y = 6800$$

Equation for the energy type *renewable* (R) will be:

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 2000 = \frac{3000 - 2000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$:

$$\implies y - 2000 = \frac{3000 - 2000}{2015 - 1965}(2025 - 1965)$$

$$y = 3200$$

Thus, none of the options given is correct.

2 Multiple Select Questions (MSQ):

1. The elements of a relation R are shown as points in the graph in Figure P-2.3. Choose the set of correct options:

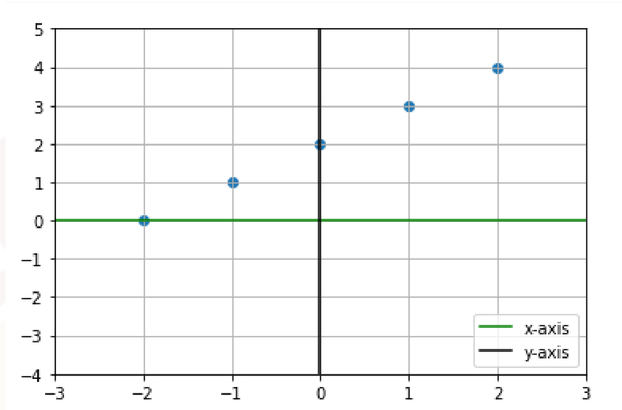


Figure PS-2.3

- ○ **$\boldsymbol{R}$ can be represented as $\boldsymbol{R = \{(-2,0),\ (-1,1),\ (0,2),\ (1,3),\ (2,4)\}}$.**
- ○ We can write R as $R = \{(a,b)|(a,b) \in X \times Y, b = a+2\}$, where X is the set of all values on the $x-axis$, and Y is the set of all values on the $y-axis$.
- ○ R cannot be a function because it is a finite set.
- ○ R is a symmetric relation.
- ○ **$\boldsymbol{R}$ is a function because it has only one output for one input.**
- ○ If R is a function then it is a bijective function on $X \times Y$, where X is the set of all values on the $x-axis$, and Y is the set of all values on the $y-axis$.
- ○ **We can write $\boldsymbol{R}$ as $\boldsymbol{R = \{(a,b)|(a,b) \in X \times Y, b = a+2\}}$, where $\boldsymbol{X = \{-2,-1,0,1,2\}}$ and $\boldsymbol{Y = \{0,1,2,3,4\}}$.**

Solution:

- Option (a) is correct since the coordinates of the points in the Figure P-2.3 are as is defined by the function.
- Option (b) is incorrect. We can write R as $\{R = (a,b) \mid (a,b)\ in\ X \times Y,\ b = a+2\}$, where $X = \{-2,-1,0,1,2\}$ and $Y = \{0,1,2,3,4\}$. Here R is a finite set so we can not write for all values of x-axis or y-axis.
- Option (c) is incorrect since R can be a function of a finite set.
- Option (d) is incorrect since R is not a symmetric relation. For example, corresponding to the element (-2, 0), there is no element (0, -2) in R.

- Option (e) is correct since for every value of X there is single corresponding value in Y.
- Option (f) is incorrect since R as a function is not defined for all values on the x-axis, and Y is not the set of all values on the y-axis, whereas $X = \{-2, -1, 0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$.
- Option (g) is correct, and explained in accordance with definition of function.

2. Find the values of a for which the triangle ΔABC is an isosceles triangle, where A, B, and C have the coordinates $(-1,1)$, $(1,3)$, and $(3,a)$ respectively.

- ◯ **If $\boldsymbol{AB = BC}$, then $\boldsymbol{a = 1}$.**
- ◯ If $AB = BC$, then $a = -1$ or -5.
- ◯ **If $\boldsymbol{BC = CA}$, then $\boldsymbol{a = -1}$.**
- ◯ If $BC = CA$, then $a = 1$.

Solution:

As we know, for an isosceles triangle two of its sides are equal. According to the question the vertices of C is $(3,a)$ therefore, depending on the value of a we can have length of $AB = BC$ or $BC = CA$

Since the vertices of triangle are given, we can find the length of each side using distance formula.

Value of a when length of $AB = BC$:

Length of any side of triangle is given by

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$\implies \sqrt{(3-1)^2 + (1-(-1))^2} \;=\; \sqrt{(a-3)^2 + (3-1)^2}$$

$$\implies \sqrt{8} \;=\; \sqrt{4 + (a-3)^2}$$

Squaring them on both sides, we have

$$\implies (a-3)^2 \;=\; 4 \implies a - 3 \;=\; \pm 2$$

Therefore,

$$a \;=\; 5,\; 1$$

But, if $a = 5$ then the three points will be co-linear therefore,

$$\boldsymbol{a = 1}$$

Value of a when length of $BC = CA$:

$$\sqrt{(a-3)^2 + (3-1)^2} \;=\; \sqrt{(a-1)^2 + (3-(-1))^2}$$

$$\implies \sqrt{4 + (a-3)^2} \;=\; \sqrt{16 + (a-1)^2}$$

Squaring on both sides of the equation, we get

$$\implies 4 + (a-3)^2 \;=\; 16 + (a-1)^2 \implies (a-3)^2 - (a-1)^2 \;=\; 12$$

$$\implies (2a-4)(-2) \;=\; \implies 12 \implies a = -1$$

Therefore,

$$\boldsymbol{a \;=\; -1}$$

3. A plane begins to land when it is at a height of 1500 metre above the ground. It follows a straight line path and lands at a point which is at a horizontal distance of 2700 metre away. There are two towers which are at horizontal distances of 900 metre and 1800 metre away in the same direction as the landing point. Choose the correct option(s) regarding the plane's trajectory and safe landing.

   - ◯ The trajectory of the path could be $\frac{y}{27} + \frac{x}{15} = 100$ if $x - axis$ and $y - axis$ are horizontal and vertical respectively.
   - ◯ The maximum safe height of the towers are 1000 metre and 1500 metre respectively.
   - ◯ **The trajectory of the path could be $\frac{y}{15} + \frac{x}{27} = 100$ if $x - axis$ and $y - axis$ are horizontal and vertical respectively.**
   - ◯ The maximum safe height of the towers are 1500 metre and 500 metre respectively.
   - ◯ **The maximum safe height of the towers are 1000 metre and 500 metre respectively.**
   - ◯ None of the above.

Solution:

Let us consider the height of plane from ground as $y-$axis and horizontal distance on ground as $x-$axis as shown in Figure PS-2.4

Then, the point P(0,1500) represents the position of the airplane when it began its descent and point Q(2700,0) represents the point where the plane landed.

The two towers which are 900m and 1800m away from the $y-$axis are represented by A and B respectively.

The equation of a straight line path traced by plane from $P(0, 1500)$ to $Q(2700, 0)$ can be obtained using the intercept-form.

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\implies \frac{x}{2700} + \frac{y}{1500} \;=\; 1$$

On rearranging:

$$\frac{y}{15} + \frac{x}{27} \;=\; 100$$

Now, to check the maximum safe height of towers:

For tower A at $X-$coordinate $= 900m$, the maximum safe height will be:

$$\frac{y}{15} + \frac{900}{27} \;=\; 100$$

$$\implies y = 1000m$$

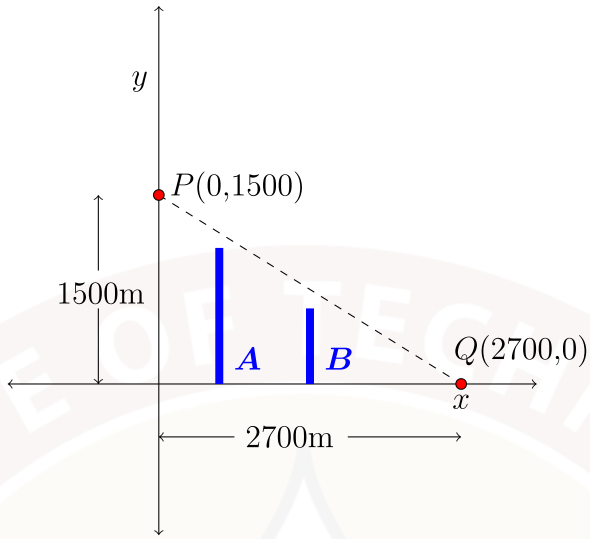


Figure PS-2.4

For tower B at $X-$coordinate $= 1800m$, the maximum safe height will be:

$$\frac{y}{15} + \frac{1800}{27} = 100$$
$$\implies y = 500m$$

3 Numerical Answer Type (NAT):

Use the following information to solve the question 1-2.
The coordinates of points A, B, C and E are shown in the figure PS-2.5 below. Points D and F are the midpoints of lines BC and AD respectively. Using the data given and Figure PS-2.5, answer the questions below.

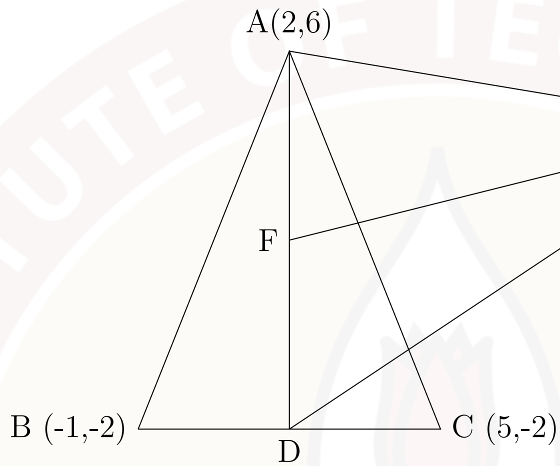


Figure PS-2.5

1. Find the area of triangle ADE. [Ans: 32]

   Solution:

   By the sectional formula, the coordinates of a point (x, y) that divides a line segment defined by two points $(x_1, y_1), (x_2, y_2)$ in the ratio $m : n$ is given by

$$x = \frac{m \times x_2 + n \times x_1}{m + n}$$

$$y = \frac{m \times y_2 + n \times y_1}{m + n}$$

   Since point D is the midpoint of the line segment BC formed by the points $B(-1, -2)$ and $C(5, -2)$ so they are in the ratio 1:1. Thus, we can obtain the coordinates of the point D denoted by, say (x_d, y_d), using the sectional formula as follows.

$$x_d = \frac{1 \times 5 + 1 \times (-1)}{1 + 1} = 2$$

$$y_d = \frac{1 \times (-2) + 1 \times (-2)}{1 + 1} = -2$$

Therefore,

$$\implies D(2, -2)$$

Now, area of triangle ADE with vertices A(2, 6), D(2, -2) and E(10, 5) can be obtained as:

$$= \frac{1}{2}| \ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \ |$$
$$= \frac{1}{2}| \ 2(-2 - 5) + 2(5 - 6) + 10(6 - (-2)) \ |$$
$$= 32$$

2. Let the slope of a line FG be 2 and the coordinate of the point G be (a, 9). Then, what is the value of a? [Ans: 5.5]

Solution:

As seen earlier, the point F is the midpoint of the line segment AD formed by the points $A(2, 6)$ and $D(2, -2)$ so they are in the ratio 1:1. Thus we can obtain the coordinates of the point F denoted by, say (x_f, y_f), using the sectional formula as follows.

$$x_f = \frac{1 \times 2 + 1 \times 2}{1 + 1} = 2$$

$$y_f = \frac{1 \times (-2) + 1 \times 6}{1 + 1} = 2$$

Therefore,

$$\implies F(2, 2)$$

Now, the slope of FG will be $= \frac{9 \ - \ 2}{a \ - \ 2} = 2$
On solving the above equation, we get $\boldsymbol{a = 5.5}$

3. Leo rents a motorcycle for 2 days. Hence, the rental company provides the motorcycle at Rs. 500 per day with 100 km free per day. The additional charges after 100 km are Rs. 2 per km Leo drives the motorcycle for a total of 500 km. How much (Rs.) will he have to pay to the rental company? [Ans: 1600]

Solution:

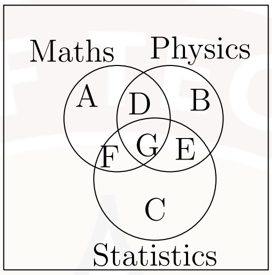
Leo has rented a motorcycle for 2 days, thus he has to pay Rs. 1,000 for free 200 km ride. Thereafter, he has to pay Rs. 2 per km. for rest of 300km, which accounts for Rs. 600. Thus, in total he has to pay Rs. 1,600.

Week - 1

Solutions for Practice Assignment

Mathematics for Data Science - 1

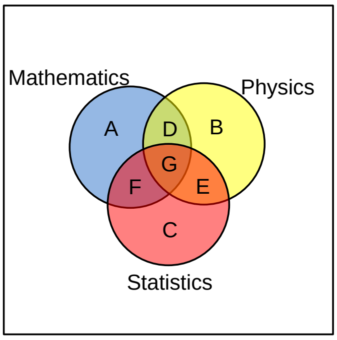
1. Given below is a Venn diagram for sets of students who take $Maths$, $Physics$, and $Statistics$. Which of the option(s) is(are) correct? [Notation: For sets P and Q, $P \setminus Q$ denotes the set of elements in P which are not in Q.]



- ◯ D is the set of students who take both $Maths$ and $Statistics$.
- ◯ $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects.
- ◯ E is a subset of the set of the students who have not taken $Maths$.
- ◯ $Maths \setminus D$ is the set of all students who have taken only $Maths$.
- ◯ $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$.

Solution: According to Figure 1, D is the set of students who take both $Maths$ and $Physics$. Hence the first statement is not valid.

The second option - $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects - is correct. This is because D is the set of students who take both $Maths$ and $Physics$, E is the set of students who take both $Physics$ and $Statistics$, F is the set of students who take both $Maths$ and $Statistics$ and G is the set of students who take all three subjects.



PS-1.1: Figure for Question 1

Third option - E is a subset of the set of the students who have not taken $Maths$ - is also correct. E is the set of students who take both $Physics$ and $Statistics$ and G is the set of students who take $Maths$ in addition to $Physics$ and $Statistics$. $(B \cup E \cup C)$ is the set of students who have not taken $Maths$. Clearly, E is a subset of this set. As E and G are two different sets, this option is correct.

Fourth option - $Maths \setminus D$ is the set of all students who have taken only $Maths$ - is not correct. $Maths \setminus D$ represents the students of $Maths$ who have not taken $Physics$ and may or may not have taken $Statistics$. This implies that students who take only $Maths$ (set A), or the students who take both $Maths$ and $Statistics$ (set F) or the students who take all three subjects (set G) are also included in $Maths \setminus D$ set. Hence this option is not correct.

Fifth option - $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$ - is correct. $(D \cup G \cup E)$ represents the students who take only $Maths$ and $Physics$ or all three subjects or $Physics$ and $Statistics$. $Physics \setminus (D \cup G \cup E)$ represents B, which is the set of students who only take $Physics$. Hence this option is correct.

2. Let A be the set of natural numbers less than 6 and whose greatest common divisor with 6 is 1. Let B be the set of divisors of 6. What are the cardinalities of $A, B, A \cup B,$ and $A \cap B$?

   ◯ (1,5,6,0)

   ◯ (1,4,5,0)

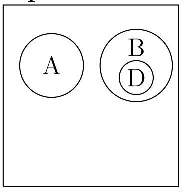
   ◯ (2,4,5,1)

   ◯ (2,4,6,1)

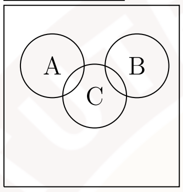
   Solution: We have set A=$\{1, 5\}$, B=$\{1, 2, 3, 6\}$, $A \cup B$ =$\{1, 2, 3, 5, 6\}$ and $A \cap B$=$\{1\}$. It follows that the cardinalities (i.e. number of elements) of A, B, $A \cup B$ and $A \cap B$ are respectively 2, 4, 5 and 1. Hence, the third option - $\{2, 4, 5, 1\}$ - is correct.

3. Let A be the set of all even natural numbers (including zero), B be the set of all odd natural numbers, C be the set of all natural numbers which divide 100, and D be the set of all prime numbers less than 100. Which of the following is(are) correct representation of these sets? [Note: A region represented in a Venn diagram could be empty. Take the set of real numbers to be the universal set.]

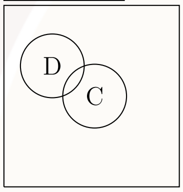
○ Option 1



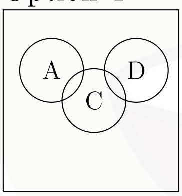
○ Option 2



○ Option 3



○ Option 4



Solution: By definition, $A = \{0, 2, 4, 6, 8, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$, $C = \{2, 4, 5, 10, \ldots, 50\}$ and $D = \{2, 3, 5, 7, 11, .., 97\}$.
Option 1 shows D as a subset of all odd natural numbers. But D contains element 2, whereas B does not. Hence, this option is wrong.
Option 2 has overlap between A and C and overlap between B and C, but no overlap between A and B. A and B are sets of even and odd natural numbers which have no overlap. C is the set of natural numbers which divide 100. $A \cap C = \{2, 4, 10, 20, 50\}$ and $B \cap C = \{1, 5, 25\}$. Hence, this option is correct.
Option 3 represents C and D sets with an overlap between them. The overlapping area includes the set of all prime numbers which can divide 100. This is the set $\{2, 5\}$. Hence, option 3 is also correct.
$A \cap D = \{2\}$, but there is no overlap between A and D in Option 4. Hence, this option is wrong.

4. Let A be the set of natural numbers which are multiples of 5 strictly less than 100, and B be the set of natural numbers which divide 100. What are the cardinalities of the following sets?
$B \setminus A$ (the set of elements in B but not in A), $A \cap B$, and B

- ◯ (2, 5, 7)
- ◯ (4, 5, 9)
- ◯ (3, 4 , 7)
- ◯ (3, 5, 8)

Solution: By definition, $A = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$, $B = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$, $B \setminus A = \{1, 2, 4, 100\}$ and $A \cap B = \{5, 10, 20, 25, 50\}$. It follows that the cardinalities of sets $B \setminus A$, $A \cap B$ and B are, respectively, 4, 5 and 9. Hence, option 2 is correct.

5. Suppose the cardinality of set A is 2 and the cardinality of set B is 3, what are the cardinalities of the cartesian product $A \times B$ and the power set of $A \times B$?

- ◯ 6 and 36
- ◯ 5 and 32
- ◯ <u>6 and 64</u>
- ◯ 5 and 25

Solution: Let the cardinality of set A be $\text{n}(A)$ and the cardinality of set B be $\text{n}(B)$. Then, the cardinality of the cartesian product $(A \times B)$, $\text{n}(A \times B) = \text{n}(A) \times \text{n}(B) = 3 \times 2 = 6$. If a set A has cardinality n, then the cardinality of power set of A is 2^n. It follows that the cardinality of the power set of $(A \times B)$ is $2^6 = 64$. Hence, the third option is correct.

6. In a survey, it is found that in a particular locality 64 houses buy English newspapers, 94 houses buy Tamil newspapers, and 26 houses buy both English and Tamil newspapers. How many houses buy newspapers of only one language?
   Answer: 106
   Solution: Number of houses which buy only English newspapers is (64 - 26) = 38.
   Number of houses which buy only Tamil newspapers is (94 - 26) = 68.
   Therefore, number of houses which buy either English or Tamil newspaper is
   (68 + 38) = 106.

7. Which of the following numbers is(are) irrational?

- ◯ $\underline{\sqrt{2+\sqrt{3}}}$
- ◯ $(2+\sqrt{3})(2-\sqrt{3})$
- ◯ $(2+\sqrt{3})+(2-\sqrt{3})$
- ◯ $\underline{2\sqrt{3}+3\sqrt{2}}$

Solution: Since $\sqrt{3}$ is an irrational number, it follows that $(2+\sqrt{3})$ and hence $\sqrt{(2+\sqrt{3})}$ are also irrational.
In the second option, $(2+\sqrt{3})(2-\sqrt{3})$ = 4 - 3 = 1, which is a rational number.
In the third option, $(2+\sqrt{3})+(2-\sqrt{3})$ = 4, which is also a rational number.
Since both $\sqrt{3}$ and $\sqrt{2}$ are irrational numbers, we have $(2\sqrt{3}+3\sqrt{2})$ is an irrational number.

8. Which of the following is(are) true for the relation R given below?
$R = \{(a, b)|$ both a and b are even non-zero integers and $\frac{a}{b}$ is an integer $\}$

- ◯ R is a reflexive relation.
- ◯ R is a symmetric relation.
- ◯ R is a transitive relation.
- ◯ R is an equivalence relation.

Solution: A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$. R is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$, and R is called transitive if (a, b) and (b, c) is in R implies $(a, c) \in R$. If a relation R is reflexive, symmetric and transitive, then it is called equivalence relation.
For any non-zero even integer a, $\frac{a}{a} = 1$ is an integer. Hence, $(a, a) \in R$, which implies that R is reflexive.
Now, let a = 4, and b = 2. Then, $\frac{a}{b} = \frac{4}{2} = 2$ is an integer. Hence, $(a, b) \in R$. But $\frac{b}{a} = \frac{2}{4} = \frac{1}{2}$ is not an integer. Therefore, $(b, a) \notin R$. It follows that R is not symmetric.
Let $(a, b) \in R$ and $(b, c) \in R$. That is, both $\frac{a}{b}$ and $\frac{b}{c}$ are integers. Hence, their product $\frac{a}{b} . \frac{b}{c} = \frac{a}{c}$ is also an integer. It follows that $(a, c) \in R$. Therefore, R is transitive.
Although R is reflexive and transitive but not symmetric, it is not an equivalence relation.

9. Find the domain and range of the following real valued function.
$f(x) = \sqrt{3-x}$ (Note: $\sqrt{}$ denotes the positive square root)

- ◯ domain=$\{x \in \mathbb{R} \mid x \neq 3\}$
  range=$\{x \in \mathbb{R} | x \geq 3\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \geq 3\}$
  range=$\{x \in \mathbb{R} \mid x \geq 0\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
  range=$\{x \in \mathbb{R} \mid x \geq 0\}$
- ◯ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
  range=$\{x \in \mathbb{R} \mid x \leq 0\}$

Solution: The set of real numbers $\mathbb{R}$ includes all rational and irrational numbers.
$\sqrt{a}$ is real valued if $a \geq 0$. If f has to be real valued, then

$$3 - x \geq 0$$

$$\Rightarrow 3 \geq x$$

Hence, domain of the function f is $\{x \in \mathbb{R} \mid x \leq 3\}$.
Since $\sqrt{}$ denotes the positive square root (as given in the question statement), the range of function f is nothing but all the positive real numbers, i.e. $\{x \in \mathbb{R} \mid x \geq 0\}$.

10. Which of the following is(are) true for the given function?
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2 + 2$

- ◯ f is not injective.
- ◯ f is surjective.
- ◯ f is not surjective.
- ◯ f is bijective.

Solution: A function f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e. no two elements in the domain will have the same image. f is called surjective if for any element in the co-domain there is a pre-image in the domain, i.e. for any y in the co-domain, there exists an x in the domain such that $f(x) = y$. A function f is said to be bijective if it is both injective and surjective.
Since $f(x) = x^2 + 2$, we have $f(-1) = 3 = f(1)$. Hence, f is not injective. Now, the co-domain of the function is given as $\mathbb{R}$.
Now if f is surjective then codomain and the range should be same, that means every element in the codomain should have a preimage. Now let us try to find a preimage for 1 (observe that $1 \in \mathbb{R}$, as codomain of the function is given as $\mathbb{R}$). To find the preimage of 1, we have to find an element a from the domain for which $f(a) = 1$, i.e. $a^2 + 2 = 1$, i.e. $a^2 = -1$. Now we know that the square of any real number cannot be negative. Hence there cannot exist any real number a (in the domain) for which $f(a) = 1$. Hence 1 has no preimage. So codomain and range is not same. Hence f is not surjective. Also, $1 \in \mathbb{R}$. Let x be such that $x \in \mathbb{R}$, and $f(x) = 1$.
As the function is neither injective, nor surjective, therefore it is not bijective.

11. Find the domain of the following real valued function.
$f(x) = \frac{\sqrt{x+2}}{x^2-9}$

- ◯ $\underline{\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}}$
- ◯ $\{x \in \mathbb{R} \mid x \leq -2, x \geq 3\}$
- ◯ $\{x \in \mathbb{R} \mid x \neq -2, x \leq 3\}$
- ◯ $\{x \in \mathbb{R} \mid x \neq -2, x \neq 3\}$

Solution: $f(x) = \frac{\sqrt{x+2}}{x^2-9}$. For f to be a well-defined function, the denominator must be non-zero. That is,

$$x^2 - 9 \neq 0$$
$$\Rightarrow x \neq \pm 3$$

Further, if f has to be real valued, then $\sqrt{x+2}$ has to be real valued. Hence $x+2$ must be non-negative. That is,

$$x + 2 \geq 0$$
$$\Rightarrow x \geq -2$$

It follows that the domain of the function $f(x)$ is $\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}$.

12. Let S be the set {January, February, March, April, May, June, July, August, September, October, November, December} of months in a year. Define the following three relations:

- $R_1 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same last four letters. } \}$
- $R_2 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same number of days. } \}$
- $R_3 := \{(a, c) \mid a, c \in S,\ \text{for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$

For example,(December, June)$\in R_3$ since (December, September)$\in R_1$, (September, June)$\in R_2$.

(a) Choose the correct option(s).

◯ R_3 is symmetric.

◯ R_3 is reflexive.

◯ R_3 is transitive.

◯ None of the above.

(b) What is the cardinality of R_3?

Answer: 85

Solution: For definitions of types of relations, please refer to solution of Question 8. Every month has the same last four letters as itself (except *May* which has only three letters). In Table 1, the months whose name has been shown in red color have the same last four letters as each other. Similarly, the months whose name has been shown in blue color also have the same last four letters as each other.

Name of the months (Elements of S)
January
February
March
April
May
June
July
August
September
October
November
December

Table 1: Question 12 : R_1 relation

Hence $R_1 = \{$(Jan, Jan), (Jan, Feb), (Feb, Jan), (Feb, Feb), (Mar, Mar), (April, April), (June, June), (July, July), (Aug, Aug), (Oct, Oct), (Sept, Sept), (Sept, Nov), (Sept, Dec), (Nov, Sept), (Nov, Nov), (Nov, Dec), (Dec, Sept), (Dec, Nov), (Dec, Dec)$\}$

The relation R_2 consists of the pairs of months with the same number of days. In Table 2, the months whose name has been shown in red color have 31 days each. The months whose name has been shown in black color have 30 days each.

Name of the months
January
February
March
April
May
June
July
August
September
October
November
December

Table 2: Question 12: R_2 relation

Observe that it is a equivalence relation. The partition formed by this equivalence relation is as follows:

Class 1: Jan, Mar, May, July, Aug, Oct, Dec [Months with 31 days each]

Class 2: April, June, Sept, Nov [Months with 30 days each]

Class 3: Feb [Month with 28 or 29 days]

Now, R_3 is defined as follows:

$$R_3 = \{(a, c) \mid a, c \in S, \text{ for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$$

If $(a, c) \in R_3$, then there must exist some pair $(a, b) \in R_1$.
Let us list out the number of elements of R_3 by listing out pairs starting with as shown below :

January: (Jan, Jan) $\in R_1$, Now we assume three partitions in the set S, formed by the relation R_2. These partitions are class 1, class 2, class 3. Hence from these classes, 7 pairs will be there in R_3 starting with January. These are {(Jan, Jan), (Jan, Mar), (Jan, May), (Jan, July), (Jan, Aug), (Jan, Oct), (Jan, Dec)}. Moreover, (Jan, Feb) is in R_1, and Feb is in another partition in S due to R_2. So there are total 8 pairs (adding (Jan, Feb) with previous 7 elements) in R_3 starting with Jan.

February: Since (Feb, Jan) is in R_1, then due to class 1 there will be 7 pairs : {(Feb, Jan), (Feb, Mar), (Feb, May), (Feb, July), (Feb, Aug), (Feb, Oct), (Feb, Dec)}. The element (Feb, Feb) will be in R_3 due to class 3. Hence 8 pairs are there in R_3 starting with Feb.

March: Due to class 1, seven pairs {(Mar, Jan), (Mar, Mar), (Mar, May), (Mar, July), (Mar, Aug), (Mar, Oct), (Mar, Dec)}.

April: Due to class 2, four pairs {(April, April), (April, June), (April, Sept), (April, Nov)}.

May: No pair will start with May as there is no pair in R_1 starting with May.

June: Due to class 2, 4 pairs: {(June, April), (June, June), (June, Sept), (June, Nov)}

July: Due to class 1, 7 pairs. {(July, Jan), (July, March), (July, July), (July, Aug), (July, Oct), (July, Dec)}

August: Due to class 1, 7 pairs. {(Aug, Jan), (Aug, Mar), (Aug, May), (Aug, July), (Aug, Aug), (Aug, Oct), (Aug, Dec)}

September: As (Sept, Dec) is a pair in R_1, it will pair up will all months in class 1, and as (Sept, Sept) is in R_1, it will pair up with all months with class 2. Hence there are total 11 pairs in R_3 starting with Sept : {(Sept, Jan), (Sept, Mar), (Sept, May), (Sept, July), (Sept, Aug), (Sept, Oct), (Sept, Dec), (Sept, April), (Sept, June), (Sept, Sept), (Sept, Nov)}

October: Due to class 1, 7 pairs are there : {(Oct, Jan), (Oct, Mar), (Oct, May), (Oct, July), (Oct, Aug), (Oct, Oct), (Oct, Dec)}

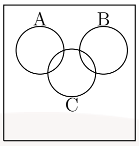
November: Due to both class 1 and class 2, 11 pairs : {(Nov, Jan), (Nov, Mar), (Nov, May), (Nov, July), (Nov, Aug), (Nov, Oct), (Nov, Dec), (Nov, April), (Nov, June), (Nov, Sept), (Nov, Nov)}

December: Due to both class 1 and class 2, 11 pairs: {(Dec, Jan), (Dec, Mar), (Dec, May), (Dec, July), (Dec, Aug), (Dec, Oct), (Dec, Dec), (Dec, April), (Dec, June), (Dec, Sept), (Dec, Nov)}

12. (b)
Hence cardinality of R_3 is $8+8+7+4+4+7+7+11+7+11+11 = 85$.
12. (a)

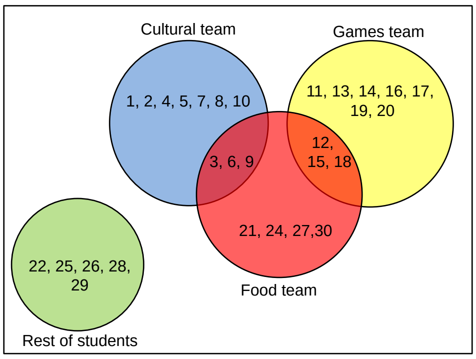
- (May, May) is not in R_3, hence R_3 is not reflexive.
- (Jan, May) is in R_3, but (May, Jan) is not in R_3, hence R_3 is not symmetric.
- (Mar, Dec) is in R_3, (Dec, Sept) is in R_3, but (Mar, Sept) is not in R_3. Hence R_3 is not transitive.

13. For a college event, thirty student volunteers were given id numbers from 1 to 30 such that each student gets a unique number. The students with id numbers from 1 to 10 are in Team 1 which coordinates the cultural program. The students with id numbers from 11 to 20 are in Team 2 which coordinates the games. The students whose roll numbers are multiples of 3 are in Team 3 which takes care of food. Now consider the following Venn diagram and choose the correct option(s).



- ◯ C, B , and A can represent Team 1, Team 2, and Team 3 respectively.
- ◯ A, B, and C can represent Team 1, Team 2, and Team 3 respectively.
- ◯ Roll number 15 has been assigned two jobs and is in both B and C.
- ◯ Roll number 25 is not in $A \cup B \cup C$.
- ◯ Roll number 10 is in both A and C.
- ◯ Cardinality of C is 20.

Solution:



PS-1.2: Venn diagram for Question 13

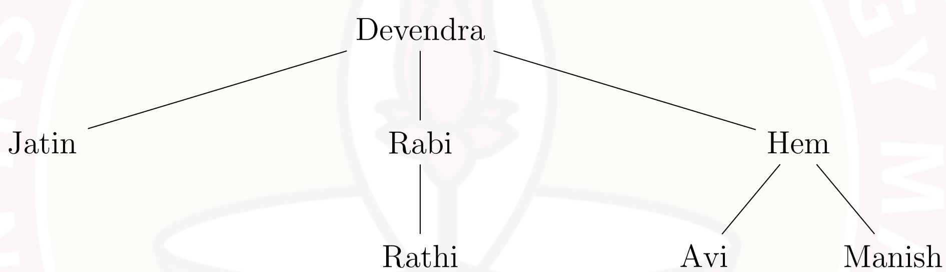
Figure PS-1.2 shows the Venn diagram corresponding to Question 13. Team 1, responsible for coordination of cultural programs, is represented by the blue circle. Team 2, responsible for game events, is represented by the yellow circle. Team 3, that takes care of food, is represented by the red circle. Rest of the students are represented using the green circle. Clearly, set A can correspond to the blue circle, B can denote the yellow circle and C can denote the red circle. That is, A, B, and C can represent Team 1, Team 2, and Team 3 respectively. Hence, option 2 is correct and option 1 is wrong. Roll number 15 is a common element between games team and food team, hence, option 3 is correct. Roll number 25 is located in the range of students with Roll number 21 to 30 but 25 is not divisible by 3. Hence, 25 does not belong to the set $A \cup B \cup C$ and so option 4 is correct. The number 10 is not divisible by 3, hence Roll number 10 is not in the set C. Therefore, option 5 is wrong. Further, since cardinality of C is 10, option 6 is also wrong.

Week - 1
Graded Assignment
Mathematics for Data Science - 1

1. (2 points) Devendra has three sons (Jatin, Rabi, and Hem). Rabi has one son named Rathi. Hem has two sons (Avi and Manish). This family tree has been shown in the figure below.
Let us define a relation R as follows,

- $R := \{(A, B) | A$ and B are first cousins, i.e. their parents are siblings$\}$.

- $S := \{(A, B) | A$ is son of $B\}$.



Which of the following is (are) true?

○ R is an equivalence relation.

√ **(Rathi, Rabi) $\in S$ but (Rabi, Rathi) $\notin S$.**

○ (Rathi, Hem) $\in R$.

√ **(Jatin, Devendra) $\in S$ but (Rathi, Devendra) $\notin S$.**

Solution:
An equivalence relation is a binary relation that is reflexive, symmetric and transitive. Now a binary relation is reflexive if it relates every element to itself. Here (Rathi, Rathi) is not a member of the relation R, so it will not be a reflexive relation. Hence R is not an equivalence relation. So option 1 is not correct.

Rathi is the son of Rabi, so (Rathi, Rabi)$\in S$ but Rabi is not the son of Rathi, hence (Rabi, Rathi) $\notin S$.So option 2 is correct.

Rabi is Rathi's father , while Devendra is the father of Hem. Rabi is the son of Devendra and not a sibling. Hence, (Rathi, Hem)$\notin R$. So, the third option, (Rathi, Hem)$\in R$ is not correct.

Jatin is the son of Devendra, so (Jatin, Devendra)$\in S$ but Rathi is the son of Rabi (and not the son of Devendra), so (Rathi, Devendra)$\notin S$.So option 4 is correct.

2. (2 points) If A and B are two sets, U is the universal set, ϕ is an empty set and $((A \cap U) \cup (B \cup \phi)) = ((A \cap \phi) \cup A) \cap (B \cap B)$ then which of the following is(are) true?

   ◯ Either A or B is an empty set.

   √ $A = B$.

   √ **A is a subset of B and B is a subset of A.**

   ◯ B is proper subset of A.

   Solution:
   From left hand side, we get
   $((A \cap U) \cup (B \cup \phi))$
   $= ((A \cup (B \cup \phi))$ [as the interesection of the universal set U and A, denoted by $A \cap U = A$]
   Again, union of B and the empty set ϕ will be the set B only, i.e., $(B \cup \phi) = B$
   Therefore, $(A \cup (B \cup \phi)) = (A \cup B)$.
   Now, from right hand side, we get,
   $((A \cap \phi) \cup A) \cap (B \cap B) = (\phi \cup A) \cap (B \cap B)$ [as the intersection of the set A and the empty set ϕ is, the empty set ϕ only, i.e. $\phi \cap A$=ϕ]
   Now, the union of these two sets will be $\phi \cup A = A$ and the intersection of two sets $B \cap B = B$
   Therefore,
   $(\phi \cup A) \cap (B \cap B) = (A \cap B)$.
   Hence, from LHS and RHS we get,
   $A \cup B = A \cap B$
   This means, either $A = B$ or A is a subset of B and B is a subset of A.
   Hence, second and third options will be correct. As we have obtained $A = B$ so in some cases, A and B can not be empty sets, hence option 1 is incorrect.
   As $A = B$, therefore $A \subseteq B$ (A is a subset of B) or $B \subseteq A$. For B to be a proper subset of A, i.e. $B \subset A$, the set B is to be properly contained in the set A, hence option 4 is not correct.

Read the text below for answering Q3 and Q4
Let a relation R be defined as $R = \{(A, B) \mid A \text{ is an ancestor of } B\}$.

3. (2 points) State whether true or false: R is symmetric.

   √ **False.**

   ◯ True.

   Solution:
   A relation R over a set X is symmetric if for all elements $a, b \in X$, if $(a, b) \in R$ then $(b, a) \in R$.
   An example of the symmetric relation is the relation "is equal to(=)", because if $a = b$ is true then $b = a$ is also true.
   In the given question, if A is an ancestor of B, then B can not be an ancestor of A. Therefore this relation is not symmetric.

4. (2 points) State whether true or false: R is transitive.

   √ **True**

   ◯ False

   Solution:
   A relation R over a set X is transitive if for all elements a, b, c in X, whenever R relates a to b and b to c, then R also relates a to c. In mathematical terms, if for three elements $\{a, b, c\}$, if $(a, b) \in R$ and $(b, c) \in R$ then if $(a, c) \in R$ then R is a transitive relation.
   In the given question, if a person A is an ancestor of a person B, and B is the ancestor of a person C, then A is also an ancestor of C.
   Hence this relation R is transitive.

5. (2 points) Let X be the set of all natural numbers divisible by 5, Y be the set of all natural numbers divisible by 10, and Z be the set of all natural numbers that are not perfect squares. Let A, B, C, D, E, F, G be the regions marked in the following Venn diagram. For example, $A = X \setminus (Y \cup Z)$, $B = Y \setminus (X \cup Z)$, and $C = Z \setminus (X \cup Y)$. (Note : A region in the Venn-diagram can be empty)

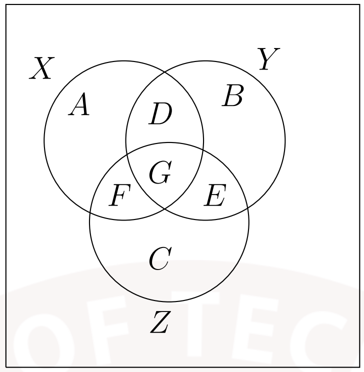


Figure 1: Fig A-1.1

Choose the correct option(s) from below.

- √ **250 is in G.**
- ◯ A is empty.
- √ **E is empty.**
- √ **The region C consists of the numbers which are not perfect squares, and are neither divisible by 5 nor divisible by 10.**

Solution:
X: set of all natural numbers divisible by 5
Y: set of all natural numbers divisible by 10
Z: set of all natural numbers that are not perfect squares.
Now, 250 is divisible by 5 (as $\frac{250}{5}$=50) and 250 is divisible by 10, (as $\frac{250}{10} = 25$) and 250 is not a perfect square.
Hence, 250 lies in the overlapping area of all 3 sets, i.e. in the G region.
The set A consists of the numbers which are divisible by 5, but not by 10 and are perfect squares.The numbers like $\{25, 625, ..\}$ are elements of this set, hence A is not empty, so option 2 is incorrect.
$E = (Y \cap Z) \backslash G = (X^c \cap Y \cap Z)$.
E is an empty set, because, a number which is divisible by 10 (element of the set Y), will always be divisible by 5 (element of the set X). So, third option is correct.
According to the diagram, the region $C = Z \backslash ((X \cap Z) \cup (Y \cap Z)) = (Z \cap X^c \cap Y^c)$.
Therefore, the region C consists of the numbers which are not perfect squares, and are neither divisible by 5 nor divisible by 10. So, option 4 is also correct.

6. (2 points) Let L denote the following list of words: Marvel, Daredevil, Natasha, Stephen, Peter.
Let R and S be relations defined on pairs of words formed from L as given below.

- A pair of words are in R if they have at least two distinct letters in common.
- A pair of words are in S if they have at least three distinct letters in common.

For example, (Marvel, Daredevil) is both in R and S, as the letters A,R,V,E and L are in common.
Which of the following statements are correct?

√ **(Daredevil, Marvel) is in both R and S.**

○ (Stephen, Peter) is in S but not in R.

○ (Daredevil, Natasha) is in S.

√ **There is a pair which is neither in R nor in S.**

Solution:
Let us write the words given in first option:
D A R E D E V I L
M A R V E L
Both words have 5 common letters - A, R, E, V, L
So both of these words will be in both the sets R and S as a pair of words are in R if they have at least two distinct letters in common and a pair of words are in S if they have at least three distinct letters in common.
For the second option, the words are
S T E P H E N
P E T E R
These two words have 3 common letters- T, E and P. Hence, these two words are included in both the sets R and S. So, option 2 is incorrect.
For option 3,
D A R E D E V I L
N A T A S H A
only the letter 'A' is common. Hence this pair is neither in the set S nor in the set R.
So, option 3 is incorrect but option 4 is correct.

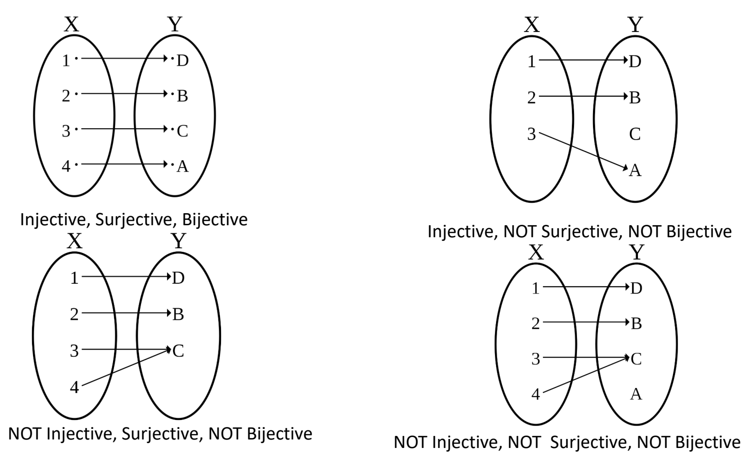
7. (2 points) Consider the following table of few materials and their dielectric constant values.

Material	Dielectric constant
Air	1
Vacuum	1
Paper	3
Porcelain	6
Nerve membrane	7
Silicon	12

Table 1: Table A-1.1

We can think of this as a function f from the set of materials to the set of dielectric constant values consisting of the elements $\{1, 3, 6, 7, 12\}$. Now pick out the correct statement from the following.

- ◯ f is neither one to one nor onto.
- ◯ f is one to one but not onto.
- √ **f is onto but not one to one.**
- ◯ f is bijective.



https://en.wikipedia.org/wiki/Injective_function

Solution:
An injective function (or one-to-one function) is a function that maps distinct elements of its domain to distinct elements of its codomain. In other words,

every element of the function's codomain is the image of at most one element of its domain. If a function f is mapped from the domain X to the co-domain Y in a manner, such that, different inputs from X to f produce different inputs in Y, then f will be an injective function. Here both the elements "Air" and "Vacuum" are mapped to the same element "1" in the set of dielectric constants. Hence this function is not injective.
A function f from a set X to a set Y is surjective (also known as onto), if for every element y in the codomain Y of f, there is at least one element x in the domain X of f such that $f(x) = y$. For a surjective function, the range is same as the co-domain.
In the given table, dielectric constant values of every materials is from the set $\{1, 3, 6, 7, 12\}$ and for all the elements of the set $\{1, 3, 6, 7, 12\}$ there is at least one material such that f(Material)=dielectric constant. Hence this is an onto function.
So option 3 is correct.
A function is bijective when it is both injective and surjective function. As this function is not injective, hence it is not a bijective function. Hence option 4 is wrong.

8. (2 points) Suppose $A = \{a, b, c, d, e\}$. How many subsets of 2 elements are possible?

   ◯ 6

   ◯ 9

   √ **10**

   ◯ 8

   Solution:
   The subsets containing 2 elements can be written as:
   $\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}$
   $\{b, c\}, \{b, d\}, \{b, e\}$
   $\{c, d\}, \{c, e\}$
   $\{d, e\}$
   Hence 10 such subsets are possible.

9. (2 points) Let us consider the following sets,

   - $A = \{x \in \mathbb{N} \mid x \bmod 3 = 0 \text{ and } 1 \leq x \leq 11\}$
   - $B = \{x \in \mathbb{N} \mid x \bmod 4 = 0 \text{ and } 1 \leq x \leq 11\}$

- $C = \{x \in \mathbb{N} \mid x \bmod 5 = 0 \text{ and } 1 \leq x \leq 11\}$

Which of the following Venn diagrams accurately describes these sets.

◯ Diagram A

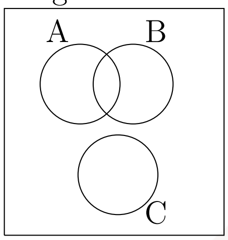


Figure 2: Fig A-1.2

◯ Diagram B

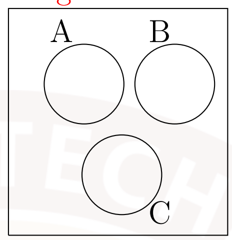


Figure 3: Fig A-1.3

◯ Diagram C

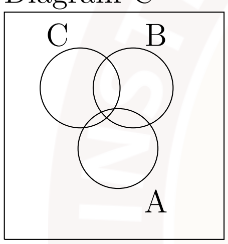


Figure 4: Fig A-1.4

◯ Diagram D

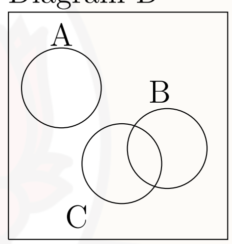


Figure 5: Fig A-1.5

Solution:
For two given integers a and b, $b \bmod a = 0$ means, if we divide b by a, the remainder will be 0.
The elements of the first set A, given in the question as $A = \{x \in \mathbb{N} \mid x \bmod 3 = 0 \text{ and } 1 \leq x \leq 11\}$ will be $\{3, 6, 9\}$.
The elements of the set B, given in the question as $B = \{x \in \mathbb{N} \mid x \bmod 4 = 0 \text{ and } 1 \leq x \leq 11\}$ will be $\{4, 8\}$.
The elements of the set C, given in the question as $C = \{x \in \mathbb{N} \mid x \bmod 5 = 0 \text{ and } 1 \leq x \leq 11\}$ will be $\{5, 10\}$.
So, none of the elements are common among the three sets A, B, C. Hence, there is no overlapping or intersection among the regions A, B, C in the Venn diagram. Among the given options, only option B satisfies this criteria. Therefore, option B is the correct answer.

10. (3 points) If X, Y and Z are 3 sets, complement of any set A is denoted as A^c and U is the universal set then find $[(X \cap Y)^c \cup (Z^c \cap U)]^c$ from the given Venn diagram

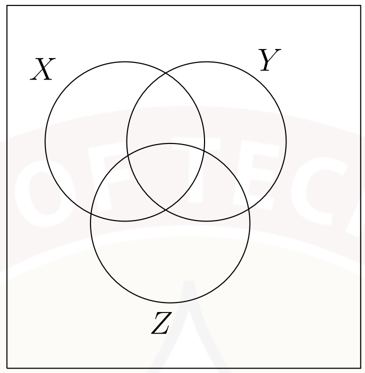


Figure 6: Fig A-1.6

- ◯ $(X^c \cup Y^c \cup Z)$
- ◯ ϕ
- √ $(X \cap Y \cap Z)$
- ◯ $(X \cup Y \cup Z)$

Solution:
$(X \cap Y)$ is the blue colored region in the figure below and $(X \cap Y)^c$ is the uncolored region in the universal set in the Figure 7.

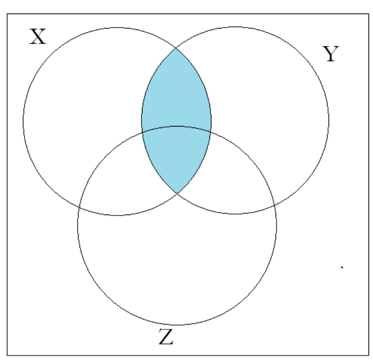


Figure 7: $X \cap Y$ region marked in blue color

Now $(Z^c \cap U) = Z^c$. Now, $(X \cap Y))^c \cup Z^c$ is shown as the yellow colored portion in Figure 8. Therefore, the white colored portion in Figure 8 is $((X \cap Y)^c \cup Z^c)^c$. From Figure 8 we can see that this area is $(X \cap Y \cap Z)$. Hence option 3 is correct.

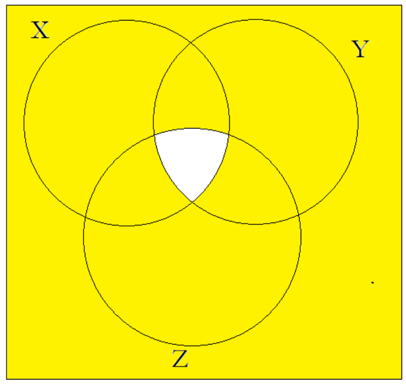


Figure 8: $(X \cap Y)^c \cup Z^c$ region marked in yellow color

11. (3 points) In a class of 50 students numbered 1 to 50, all even numbered students opt for Physics, those whose numbers are divisible by 5 opt for Chemistry and those whose numbers are divisible by 7 opt for Mathematics. How many opt for none of the three subjects?

   ◯ 19

   √ **17**

   ◯ 21

   ◯ 11

   Solution:
   Students who opt for Physics will be -$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50\}$
   Students who opt for Chemistry will be -$\{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$
   Students who opt for Mathematics will be -$\{7, 14, 21, 28, 35, 42, 49\}$
   Other students have not opted for any of the three subjects. These students are- $\{1, 3, 9, 11, 13, 17, 19, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47\}$.
   Number of these students or the cardinality of this set is 17. Hence option 2 is correct.

Read the following passage to answer Q12, Q13, Q14

In the 'Rock-Paper- Scissors' game, there are 3 shapes- 'rock', 'paper' and 'scissors'. The rule of the game is as follows :

- Rock defeats scissors
- Paper defeats rock
- Scissors defeats paper.

If a relation R is defined as $R = \{(A, B) \mid A \text{ defeats } B\}$ where A, B are any of the three shapes {rock, paper, scissors} then answer the following questions.

12. (2 points) State 'True' or 'False' : R is a symmetric relation.

   ◯ True

   √ **False**

   Solution:
   In the question, the given relation is $R = \{(A, B) \mid A \text{ defeats } B\}$. So, if R is symmetric then, for example, if rock defeats scissors, then scissors also should defeat rock, which is not true. This is observed for the other element-paper as well. Hence it is a false statement that R is a symmetric relation.

13. (2 points) State 'True' or 'False' : R is a transitive relation.

   ◯ True

   √ **False**

   Solution:
   By the question, paper defeats rock and rock defeats scissors. Now, if the relation is transitive, then paper should defeat scissors, but according to the question, that is not true as scissors defeats paper. Therefore it is false that R is a transitive relation.

14. (2 points) State 'True' or 'False' : R is an equivalence relation.

   ◯ True

   √ **False**

   Solution:
   A given relation will be an equivalence relation if it is reflexive, symmetric and transitive. The given relation is neither symmetric, nor transitive, hence it is not an equivalence relation.

Week - 1
Solutions for Practice Assignment-2
Mathematics for Data Science - 1

1 Multiple Select Questions (MSQ):

1. (a) $(3+3\sqrt{5})(2-2\sqrt{5}) = (6+6\sqrt{5}-6\sqrt{5}-30) = -24$. Hence this is a rational number.

   (b) $\frac{\sqrt{64}}{\sqrt{25}} = \frac{8}{5}$. This is a rational number.

   (c) $\frac{(3+\sqrt{5})}{(3-\sqrt{5})} = \frac{(3+\sqrt{5})(3-\sqrt{5})}{3^2-5} = \frac{(9+6\sqrt{5}+5)}{4} = \frac{7}{2} + \frac{3\sqrt{5}}{2}$. It is an irrational number as $\sqrt{5}$ is irrational.

   (d) $\sqrt[3]{2} = 2^{\frac{1}{3}}$ is also an irrational number.

2. According to given question,
   $R = \{$(Satyajit, Kalyani), (Satyajit, Nalini), (Kalyani, Satyajit), (Nalini, Satyajit)$\}$
   $S = \{$(Satyajit, Sukumar), (Sukumar, Upendra)$\}$ A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$. R is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$, and R is called transitive if (a, b) and (b, c) is in R also implies $(a, c) \in R$.

   (a) Hence (Satyajit, Sukumar) $\in$ S but (Sukumar, Satyajit) $\notin$ S.

   (b) (Kalyani, Satyajit) and (Satyajit, Nalini) are in R, but (Kalyani, Nalini) is not in R. Hence, R is not a transitive relation.

   (c) Also, by observing the elements of S, (Sukumar,Upendra) $\in$ S and (Satyajit, Sukumar) $\in$ S but (Satyajit, Upendra) $\notin$ S.

   (d) (Satyajit, Sukhalata) is neither in R nor in S.

3. Suppose $x \in A$, then $x \in A \cup B$. Now it is given that $A \cup B = A \cap B$. Hence $x \in A \cap B$. Hence $x \in B$. Hence every element of A must be in B. Hence, $A \subseteq B$.
   Now, let $x \in B$. Hence, $x \in A \cup B$. As $A \cup B = A \cap B$, $x \in A \cap B$. Hence $x \in A$. Which implies that every element of B must be in A. Hence $B \subseteq A$.
   Hence we have $A = B$.

4. $R = \{(A, B) | A \subseteq B\}$

   (a) For any set A, we have $A \subseteq A$. Hence $(A, A) \in R$. Hence R is reflexive. Moreover, If $(A, B) \in R$ and $(B, C) \in R$, then we have $A \subseteq B \subseteq C$. Hence $A \subseteq C$. Which implies that $(A, C) \in R$. Hence R is transitive.

   (b) Let $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$. Hence $A \subseteq B$, but the converse is not true. Hence $(A, B) \in R$, does not imply $(B, A) \in R$. Hence R is not symmetric.

   (c) If $A \subseteq B$, and $B \subseteq A$, then $A = B$. Hence if $(A, B) \in R$ and $A \neq B$, then $(B, A) \notin R$. Hence R is anti-symmetric.

   (d) As R is transitive, the last option is not correct.

5. $R = \{(A, B) \mid$ Both A and B live in the same city $\}$.
   (a) A and B live in the same city is same as saying B and A live in the same city. Hence $(A, B) \in R$, implies $(B, A) \in R$.
   (b) As it is symmetric, clearly it is not anti-symmetric.
   (c) Suppose $(A, B) \in R$ and $(B, C) \in R$. Hence A and B live in same city, and B and C also live in the same city. Hence A and C live in the same city. Hence $(A, C) \in R$. Hence R is transitive.
   (d) Clearly $(A, A) \in R$, for all A. Hence R is reflexive.

6. Elements in set $X = \{$ 100, 200, 300, 400, 500, 600,..$\}$
   Elements in set $Y = \{$ 25, 50, 75, 100, 125, 150, 175, 200, ...$\}$
   Elements in set $Z = \{$ 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...$\}$

   - A contains the elements of X which are not included in $(Y \cup Z)$
   - B contains the elements of X which are not included in $(X \cup Z)$
   - C contains the elements of Z which are not included in $(X \cup Y)$
   - D represents the set of natural numbers which are divisible by both 25 and 100, but are not perfect squares
   - E represents the set of natural numbers which are divisible by 25 and are perfect squares but are not divisible by 100
   - F represents the set of natural numbers which are divisible by 100 and are perfect squares, but are not divisible by 25
   - G represents the set of natural numbers which are divisible by 25, 100 and are perfect squares

   (a) 625 is a perfect square and divisible by 25, but not divisible by 100. Hence it is in E.
   (b) 2500 is a perfect square which is divisible by both 100 and 25. Hence it is an element of G. Hence G is not an empty set.
   (c) 200 is divisible by 100 and 25, but not a perfect square. Hence it is not in G.
   (d) As we have seen earlier 2500 is in G, not in F.
   (e) As we have seen earlier 2500 is in G.
   (f) 25 is a perfect square, but it is not divisible by 100. Hence it is not in F.
   (g) Any number which is divisible by 100 must be divisible by 25. Hence there cannot be any element which is in X but not included in Y and as a result in $Y \cup Z$. Hence A is empty. For the similar reason F is also empty.

7. (a) (University, Institute) has the letters 'i', 'e', 'u' in common. Therefore they are in both R and S.
   (b) (Science, Literature) has only 'e' and 'i' in common, hence they are in R but not in S.

(c) (Movies, Technology) has only 'o' and 'e' in common, hence they are not in S

(d) The pair (Company, Literature) is neither in R, nor in S, as they have only one letter in common.

2 Multiple Choice Questions (MCQ):

8. The codomain for the function is nothing but the set of integers between 50 and 70.

   (a) Hasan and Rahul has same weight. Hence the function is not one to one.

   (b) There is no student in the list whose weight is 60, but 60 is in the codomain. Hence the function is not onto. The range of the function is $\{51, 54, 55, 62\}$, which is not same as the codoamin of the function.

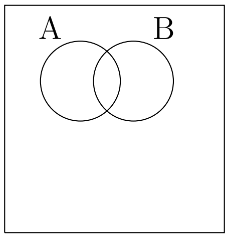
   (c) Clearly the function is neither one to one nor onto.

   (d) A function is called bijective if it is both one to one and onto, which is not the case here. Hence the function is not bijective.

9. Possible subsets of two elements $= \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$
   Number of subsets $= 6$.

10. Elements of $A = \{3, 6, 9, 12\}$
    Elements of $B = \{4, 8, 12\}$
    Elements of $C = \{5, 10\}$
    The element 12 is common in both sets A and B. There is no common element between B and C.
    Hence diagram A is accurate representation.

11. $(B \cup A)^c$ is the region out side both the circles. Hence it does not have any element common with A. Hence $A \cap (B \cup A)^c$ is empty set.



12. As even numbered students have taken Physics, odd numbered students have chances of taking none of the subjects. Apart from this, in this set there should not be any student whose number is divisible by either 5 or 7.
    Hence the elements are $\{1, 3, 9, 11, 13, 17, 19, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 51, 53, 57, 59, 61, 67, 69, 71, 73, 79, 81, 83, 87, 89, 93, 97, 99, 101, 103, 107, 109, 111, 113, 117\}$
    Hence the cardinality of this set is 41.
    Another procedure:
    Number of students who have taken physics is 60.
    Number students who have taken chemistry is 24.
    Number of students who have taken maths is 17.
    The students who have taken both physics and chemistry are the students who have been numbered as multiples of both 2 and 5, i.e. multiples of 10. Hence the number of students who have taken both physics and chemistry is 12.
    The students who have taken both physics and maths are the students who have been numbered as multiples of both 2 and 7, i.e. multiples of 14. Hence the number of students who have taken both physics and chemistry is 8.
    The students who have taken both maths and chemistry are the students who have been numbered as multiples of both 7 and 5, i.e. multiples of 35. Hence the number of students who have taken both physics and chemistry is 3.
    The students who have taken all the subjects are the students who have been numbered as the multiples of all the three numbers 2, 5 and 7, i.e. multiples of 70. Hence the number of students who have taken all the three subjects is 1.
    Hence the number of students who have not taken at least one subject is$= 60+24+17-12-8-3+1 = 79$. Hence the number of students who have not taken any subject is$= 120 - 79 = 41$.

13. It is clear that $f(1) = f(3) = f(-1) = 0$ as $f(x) = 0$, for all odd integer x. So f is not one to one. Moreover, for any integer y, $f(2y) = y$ as $2y$ is always an even integer. Hence f is onto.
    Hence f is onto but not one to one.