IIT Madras
ONLINE DEGREE

Mathematics for Data Science 2
Professor Sarang S Sane
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 3
Systems of Linear Equations

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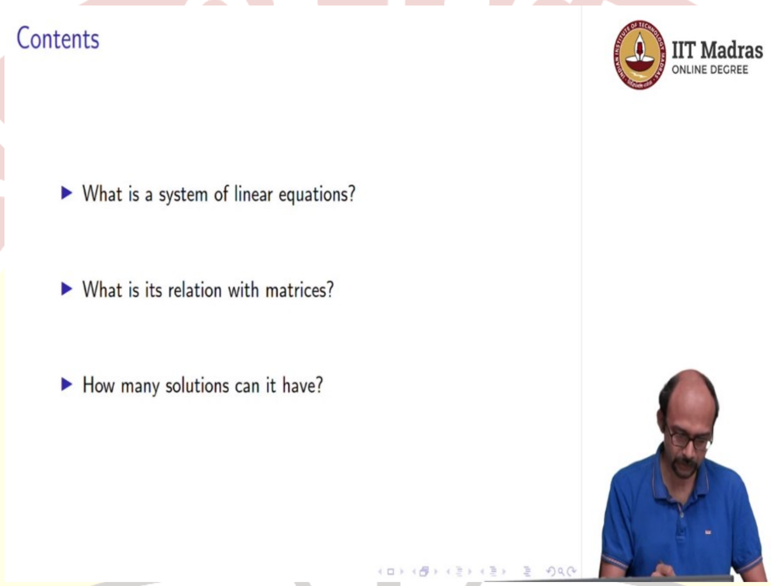

Welcome to the maths 2 component of the online B.Sc degree in data science. In today's video we are going to look at systems of linear equations. So, what are the contents of this video? So, let us see what is there, what is the system of linear equations? What is its relation with matrices? And finally, how many solutions can it have? So, these are the three things we are going to see today.

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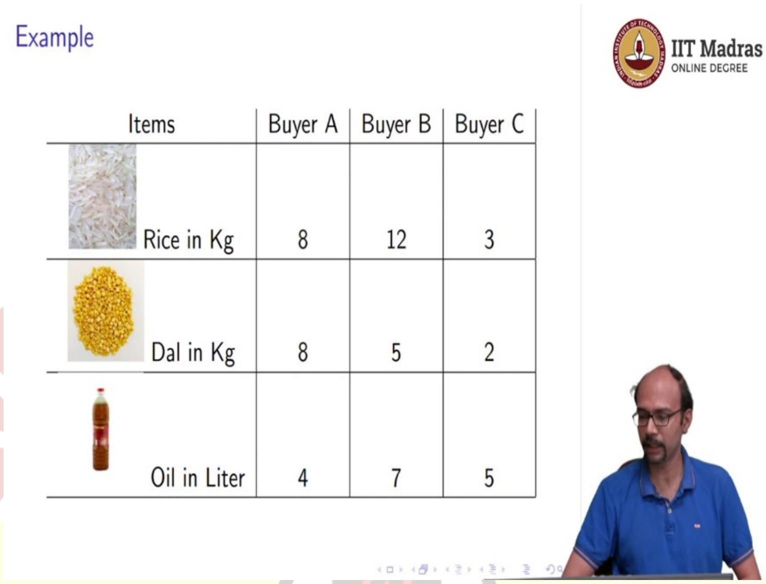

Let us begin with an example, so in this example we have 3 buyers and we have 3 items, so buyer A buys rice 8 kg's of rice, 8 kg's of dal and 4 litres of oil, buyer B buys 12 kg's of rice, 5 kg's of dal and 7 litres of oil and buyer C buys 3 kg's of rice, 2 kg's of dal and 5 litres of oil.

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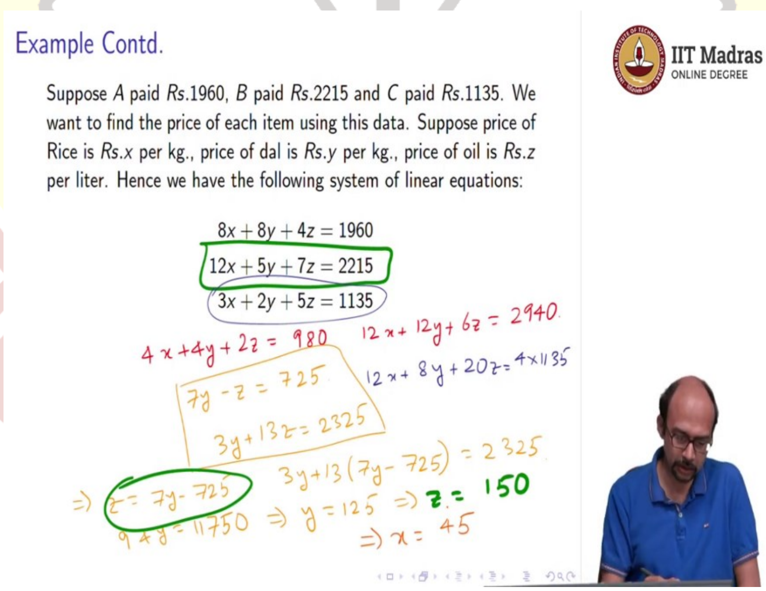

So, suppose these buyers paid some amount of money to the shopkeeper, namely A paid rupees 1960, B paid rupees 2215 and C paid rupees 1135, so I have this data. Now, we want to find the price of each item using this data. So, how do I write this as a system of linear equations? So, we will assume that the price of rice is rupees x per kg, we will assume that the price of dal is rupees y per kg and we will assume that the price of oil is rupees z per litre.

So, then what do we get for the first buyer? We get that 8 × x, because he bought he or she bought 8 kg of rice, so 8 × x + 8 × y 8 kg's of dal + 4 × z is 1960. Similarly, we get the next two equations. So, this is what a system of linear equations looks like and now from this data we would like to solve this system and get what is the price of rice, dal and oil.

So, I am going to quickly solve this. So, I think many of you may have solved such things in school, so how do we solve this? So, maybe we can take the first equation and divide it by 2, so if you do that, we get $4x + 4y + 2z$ is 980, this the first equation divided by 2 and then we will multiply this new equation by 3, so that I get a 12x, so $12x + 12y + 6z$ which is 2940.

Then I take this equation and subtract out the second equation over here, I will subtract out the second equation over here, so if I do that, what do I get? So, the 12x is subtracted out and what we get is $7y - z$ is 725. So, now we have an equation in terms of y and z, now similarly for the third equation what we can do is we can use the third equation and multiply the third equation by 4, so that we get a $12x$.

So, multiplying the third equation by 4, we will get $12x + 8y + 20z$ is 4 × 1135 and then we will use this equation and subtract out the second equation, so what we get is another equation in terms of y and z, so the equation we get in terms of y and z is $3y + 13z$ is 2325, so I will encourage it to do this. So, now we have two equations with two unknowns, namely y and z, so let us try and solve this.

So, well we can actually solve this by putting z = 7y - 725 and substitute in the second one, so if you substitute in the second one we get $3y + 13 \times 7y - 725$ is 2325. So, if you simplify this what you will get is 94 × y is 11750, so then y is 11750 divided by 94 which is 125. So, we have computed the price of dal, it is rupees 125 per kg, so we will substitute y is 125 in this equation over here and we will get z, so z is from here we will get that z is 150.

And then now we know y and z, so we will put them in some suitable equation and we can get x. So, x is going to be 45, so I will encourage you to check these. So, what was the point of this? The point of this was we had a real life problem in involving some costs and so on, so we have three equations.

And then we use some unknowns to denote the things that we wanted to compute and then we had three linear equations in those three unknowns and then we solve the system of linear equations. And we computed that x is 45, y is 125 and z is 150 which means the price of rice is rupees 45 per kg, the price of dal is rupees 125 per kg and the price of oil is rupees 150 per litre. So, this I hope this convinces you that solving a system of linear equations is something useful.

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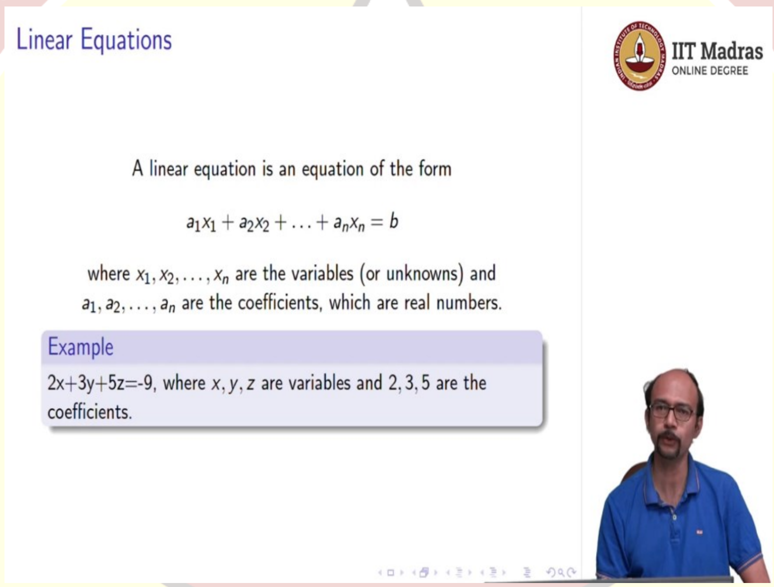

So, what is the system of linear equations? So, before that we have to ask, what is a linear equation? So, a linear equation is something of the form $a_1x_1 + a_2x_2 + \cdots . + a_nx_n = b$, this is what a linear equation looks like, this is one linear equation. So, what are the x_1,x_2, ... x_n these are variables or unknowns, these are the things that we want to compute, this was like the x y z in our previous example.

And then what are the a_1,a_2, ... a_ns? These are what are called coefficients, these are going to be real numbers, these are things we will assume, these are numbers will assume that we know. And then b is also a real number. So, here is an example, so $2x + 3y + 5z$ is - 9 and here x y z and are variables and the 2, 3, 5, are coefficients.

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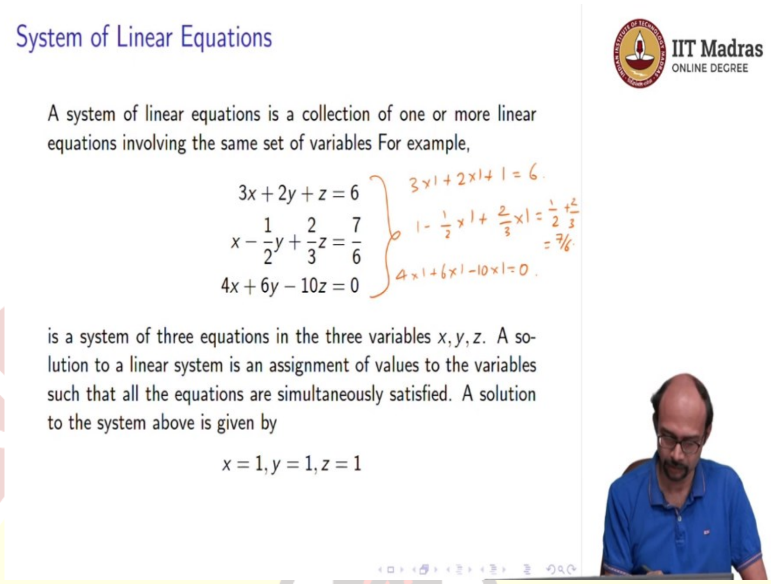

So, what is the system of linear equations? So, a system of linear equations, so that means you have several linear equations, we saw what is one linear equation, so it is something of the form $a_1x_1 + a_2x_2 + \cdots. + a_nx_n = b$, instead now we have many such equations but in the same set of unknowns, so $x_1, x_2, \ldots x_n$ remains the same but the coefficients and the constant at the end change.

So, here is an example, so we have $3x + 2y + z = 6$, $x - \frac{y}{2} + \frac{2}{3}z = \frac{7}{6}$ and $4x + 6y - 10z = 0$. So, I will encourage you to solve this equation similar to how we did it in the first example. So, if you do that you should get the answer x is 1, y is 1 and then 1 is 1. So, what are the unknowns or the variables here?

The x, y and z are the unknowns of the variable, so traditionally you represent if you have one unknown it x, if you have two unknowns x and y, if you have three unknown x, y and z and if you have four or more unknowns you take x_1, x_2, x_3, x_4 and up till as many unknowns as you have. So, what is the solution to this system of linear equations?

That means if you put x = 1, y = 1 and z = 1 in this system here, then these equations are solved. So, let us check that, so 3 × 1 + 2 × 1 + 1 is indeed 6, 1 - half × 1 + 2 3rd × 1, so let us 1 - so that is 1 - so that is half + 2 3rd which is indeed 7 by 6.

And then similarly you have 4 × 1 + 6 × 1 - 10 × 1 is 0. So, I hope it means you understand what it means, for a system of linear equations to have a set of solutions, so when you put x, y *and* z to be some particular numbers then these equations should be satisfied, that means these equality should be true, that means we say we have got solution.

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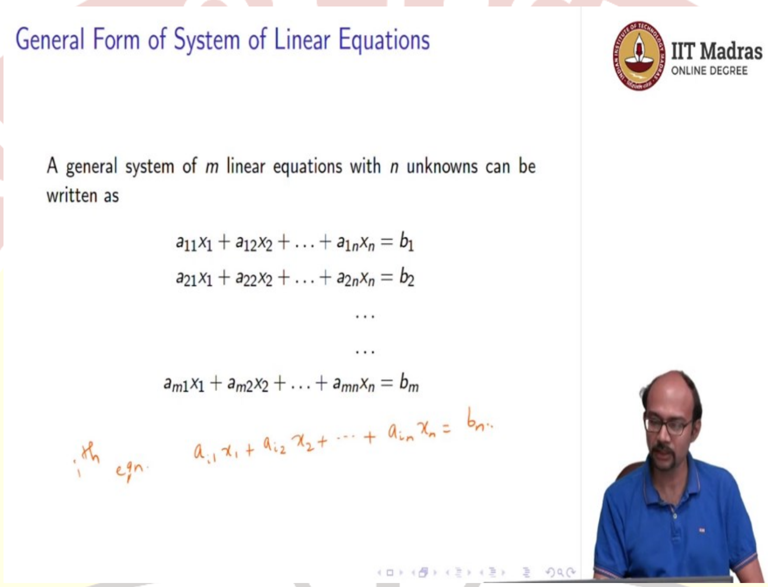

So, what is the general system of linear equations? So, you have m linear equations, that means you have m equations and you have n unknowns, so n unknowns mean you have $x_1, x_2, \ldots x_n$, then how do we write these? We write this as $a_{11}x_1 + a_{12}x_2 + \cdots. + a_{1n}x_n = b_1$, $a_{21}x_1 + a_{22}x_2 + \cdots. + a_{2n}x_n = b_2$ and all the way up to $a_{m1}x_1 + a_{m2}x_2 + \cdots. + a_{mn}x_n = b_m$. So, if you want the ith equation, then this is $a_{i1}x_1 + a_{i2}x_2 + \cdots. + a_{in}x_n = b_n$this is the ith equation in this system.

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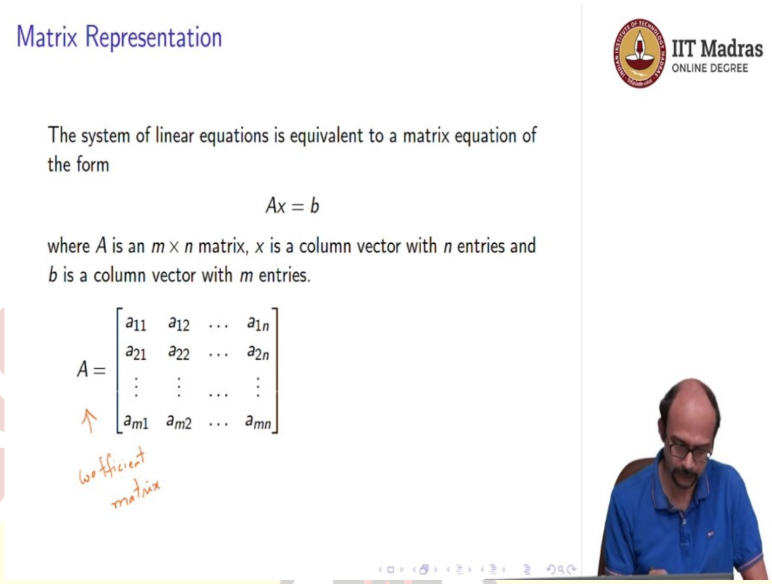

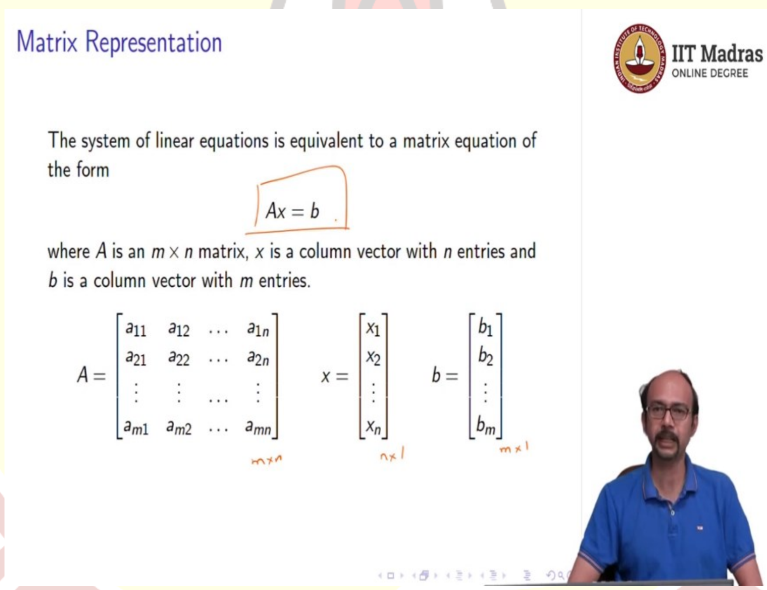

So, what is the connection with matrices? So, this is really important part of this video, so the system of linear equations is equivalent to a matrix equation of the form A × x =b, so in a previous video we have seen what it means to multiply matrices. So, here Ax and b are matrices, so A is an m by n matrix and what are the entries of this matrix? They are exactly the aij's that we saw in our previous slide.

So, this matrix A consists of the Aij's, x is a column vector with n entries, what are the entries? x_1,x_2, ... $x_n$exactly or unknowns and b is the column vector b_1,b_2, ... b_m, so this is A that from our previous slide these are the coefficient, so this A is called the coefficient matrix. What is x? x is a column matrix, so it that means it consist of one column, so it is the column x_1,x_2, ... x_n, it is a matrix with n rows and 1 column, it is a column matrix consisting of the unknowns.

And b is the set of constants, so it is a again a column matrix consisting on the constants. So, this is a m by 1 matrix, this is n by 1 and this is m by n. And now if you multiply these first of all let us ask does it make sense indeed this is m by n and this is n by 1, so it makes sense to multiply these and whatever the product is it will be of size m by 1.

So, we are saying that a product is exactly the left-hand side of the system of linear equations that we had on the previous slide and on the right hand side we will have these constant b_1,b_2, ... b_m. So, the previous system can be expressed in the form Ax = b, so the important part here is that any system of linear equations can be expressed in terms of matrices as a matrix by matrix multiplication. Let us, look at some examples.

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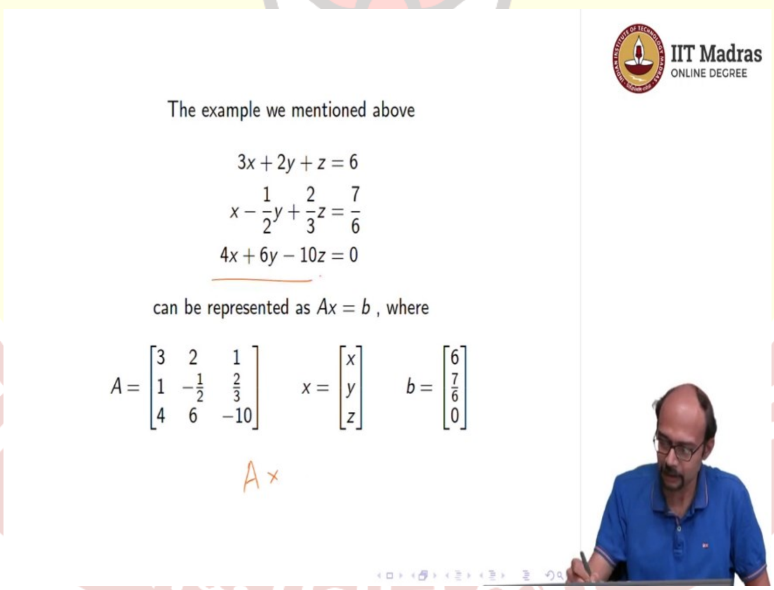

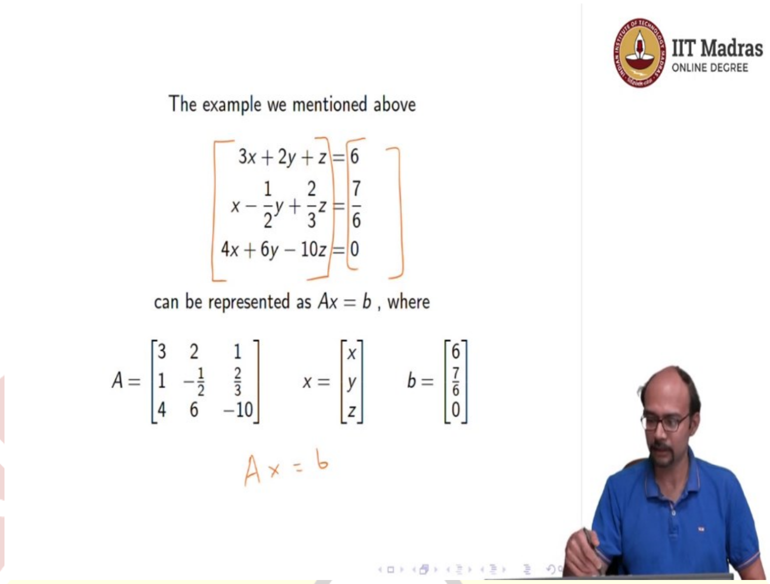

So, let us look at this example like we had before, $3x + 2y + z = 6$, $x - \frac{1}{2}y + \frac{2}{3}z = \frac{7}{6}$, $4x + 6y - 10z = 0$, so how do we represent this in terms of matrices? So, in terms of matrices this is the coefficient matrix, so we collect together all the coefficients and put them in the correct places. So, the first row is $[3 \quad 2 \quad 1]$, that is what occurs as a coefficient of in the first equation.

The second row is $\begin{bmatrix} 1 & \frac{-1}{2} & \frac{2}{3} \end{bmatrix}$ that is what occurs in the second equation and the third row is $[4 \quad 6 \quad -10]$, that is what occurs in the third equation. What is x? x is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the unknowns or the variable. And what is b? b is the set of constants on the right, so $\begin{bmatrix} 6 \\ \frac{7}{6} \\ 0 \end{bmatrix}$, and I will encourage you to multiply A and x.

So, multiply A by x and see that you get exactly this thing over here as a column matrix, so you get exactly this as your matrix. This is A × x and this thing on the right is b. And what are the system of linear equations saying? Saying Ax = b. So, we have rewritten this example as a in terms of matrix multiplication. Let us look at our first example.

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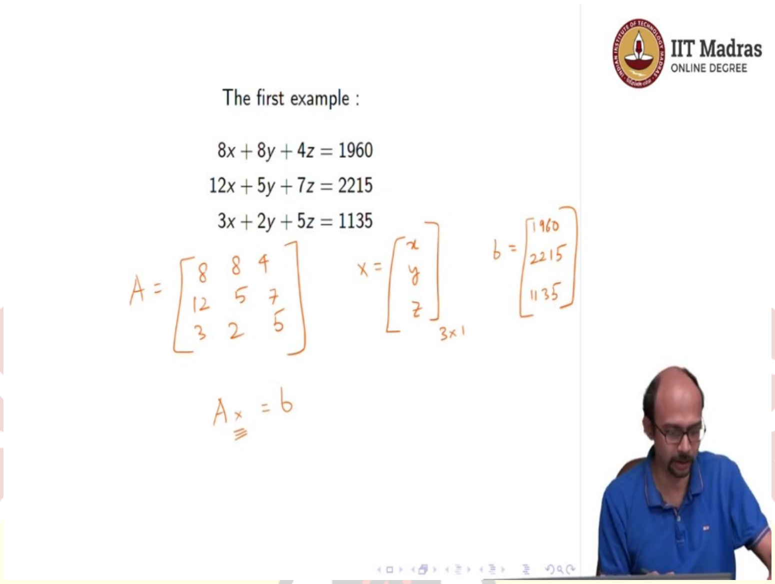

So, this was $8x + 8y + 4z = 1960, 12x + 5y + 7z = 2215, 3x + 2y + 5z = 1135$. How do we write this in terms of matrices? So, the coefficient matrix here is going to be $\begin{bmatrix} 8 & 8 & 4 \\ 12 & 5 & 7 \\ 3 & 2 & 5 \end{bmatrix}$, the matrix x which is a column matrix, that will be $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and the matrix of constants is going to be $\begin{bmatrix} 1960 \\ 2215 \\ 1135 \end{bmatrix}$. And once again, I will encourage you to check that $Ax = b$ exactly captures the systems of linear equations upstairs.

Now, I will point out one small strangeness in this entire notation you will see that the matrix here is called x, whereas you also have a variable called x, so be very careful to distinguish between these two, so which one is your variable and which one is your matrix, so when I when we are writing this matrix, this are 3 rows and 1 column, so is the matrix this is a column matrix of size 3, so 3 rows and 1 column.

And inside it is the one entry is x which is exactly our variable inside in our equations. So, this is an unfortunate fact that there is a clash of notations, but that is how it is and it with some practice you will get used to it. So, we have seen two examples of how the connection between system of linear equation and how to represent them in matrix terms.

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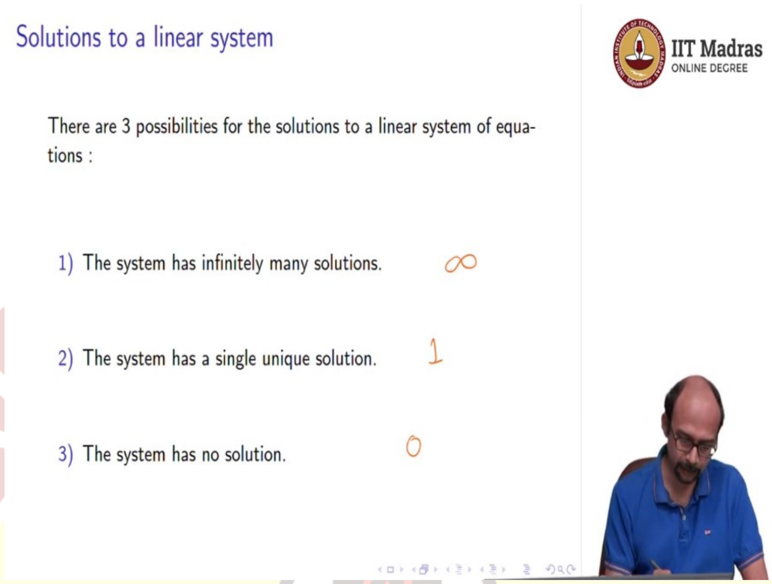

So, now let us talk about solutions, solutions to a linear system. So, we saw I explicitly found out the solutions for the first example and I asked you to check that x is 1, y is 1, z is 1 the solution for the second example, so in general what can we say about solutions of a linear system? So, it turns out that there are only three possibilities either a system has infinitely many solutions or a system has a single unique solution or a system has no solution.

So, these are the only three possibilities. So, it cannot happen that a system of linear equation over the real numbers has only three solutions, this is not possible. If it has three solutions it will have infinitely many solutions, it cannot have exactly three solutions, so it can have either 0 solutions, 1 solution or infinitely many solutions.

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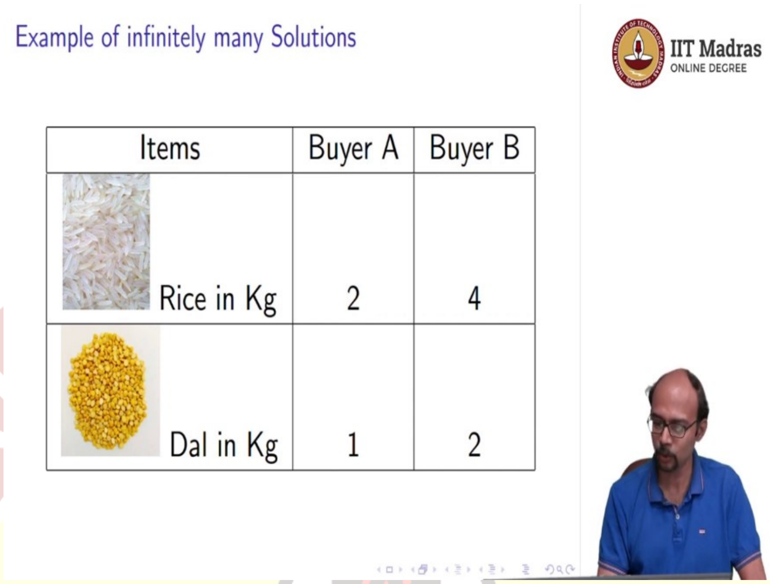

So, let us see examples of where this comes from. So, suppose we have to buyers A and B and buyer A buys 2 kg's of rice and 1 kg of dal, buyer B buys 4 kg's of rice and 2 kg's of dal.

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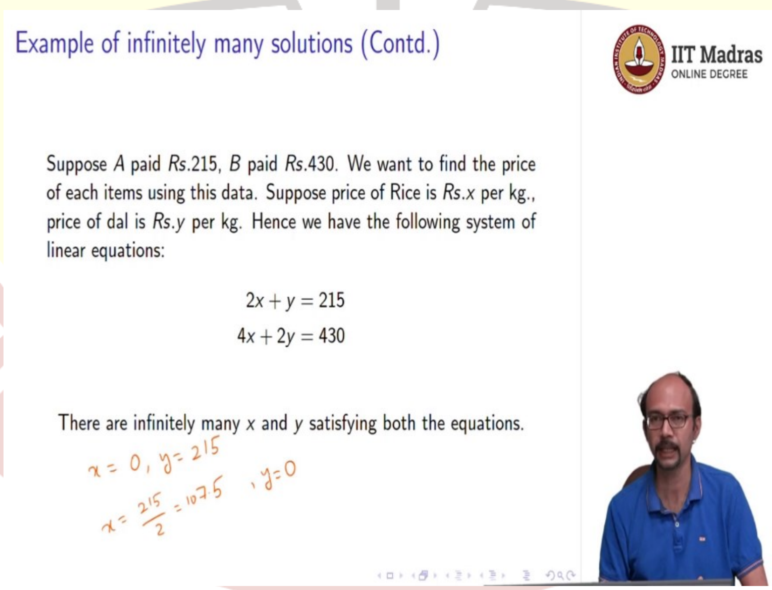

And they pay 215 rupees and 430 rupees respectively. So, again as in the first example, we want to find out the price of rice and the price of dal, so we put this into the form of linear equations. So, if you follow the first example you will quickly see that this is $2 \times x + y = 215, 4 \times x +$

$2y = 430$. And there are infinitely many x and y satisfying both the equations. So, I will encourage you to find out why that is the case.

Let me give you one example, I mean let me at least show you two examples, two solutions and you can find as many more as you want, one solution is $x = 0$ and $y = 215$, that is one solution. So, check that indeed this works. Another solution is $x = \frac{215}{2}$ so that is 107.5 and $y = 0$. So, this is another solution. And I will encourage you to find more solutions. So, what is happening here in terms of whether some geometry involved?

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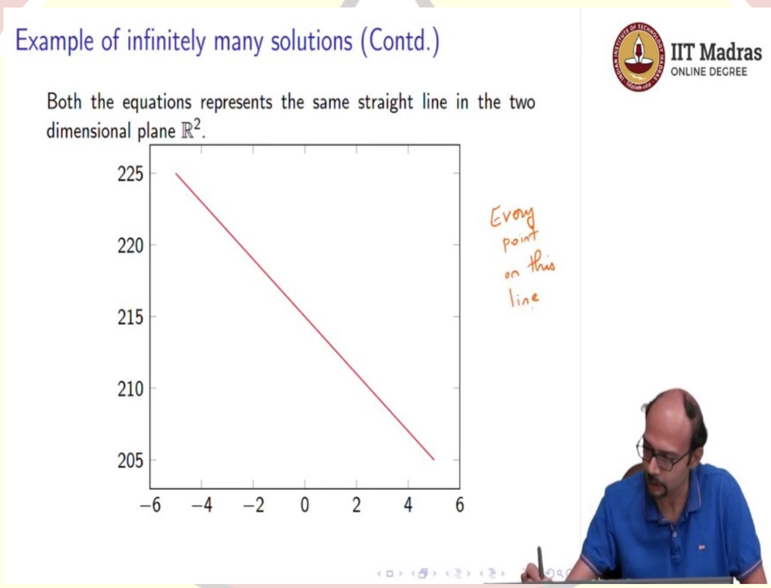

So, let us draw this picture. So, the idea is that if you draw remember that if you have the equations correspond to straight lines, yes, this is something you have seen in math 1, so you have $2x + y = 215$ and $4x + 2y = 430$ and both of them give you the same straight line, so any point on this line is going to satisfy is going to be a solution. So, every point on this line is a solution point. So, now you can find as many solutions as you want. So, this is the geometry behind why it has infinitely many solutions.

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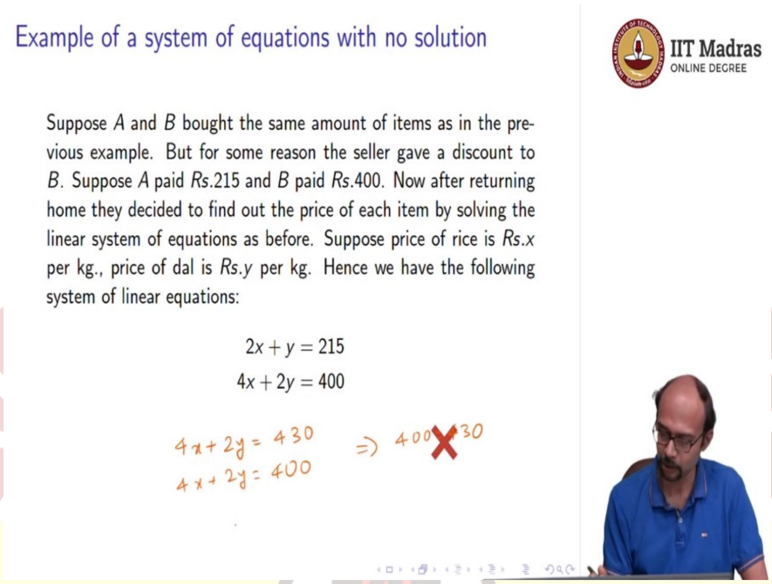

Let us, do an example where it has no solutions. So, again we have the same situation in the A and B go to the shop by same amount of rice and dal as the previous limit, so 4 kg's of rice and 2 kg's of dal and 2 kg's of rice and 1 kg of dal respectively, but this time the seller gives a discount to B, the seller is in a happy mood the shopkeeper and they give a discount to B.

So, A pays 215 rupees and B paid 400 rupees, in the previous example B had paid 430, this time they got a discount of 400 rupees. Now, after returning home, they tried to solve this system again put x and y to be the price of rice and dal respectively, so we get $2x + y = 215, 4x + 2y = 400$, let us try and solve this system of linear equations. So, if you try and solve this, so let us take the first equation and double it.

So if you double the first equation you get $4x + 2y = 430$, on the other hand the second equation is $4x + 2y = 400$, so what does that mean? That means 400 is equal to 430, so there is something really really wrong, this is this is something is going very wrong. So, of course, we know that 400 is not 430 that means there is no x and y for which this can be solved, so there is no solution here. What goes wrong?

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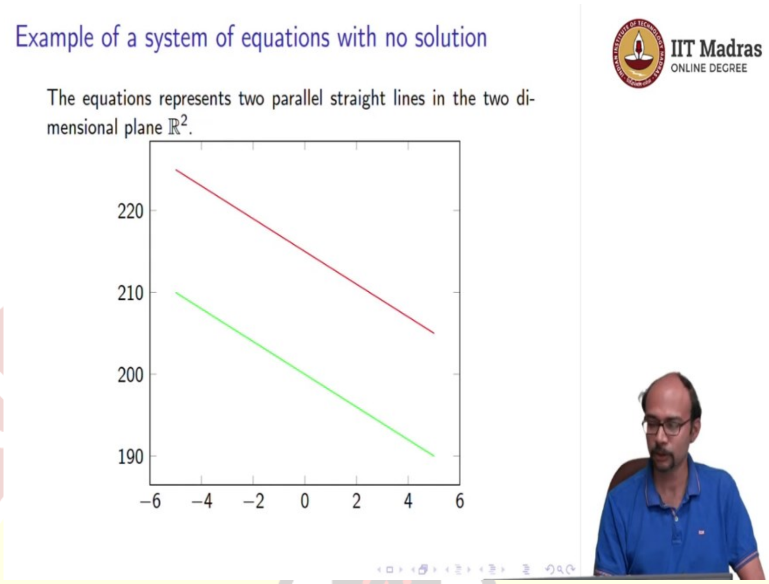

So, the reason there is no solution here is that if you draw the lines in this case, they are parallel lines, so they would not intersect at all. And the intersection is exactly what will give you the solution. So, in the previous case you had both the equations at the same line, so it was like you are two lines, which were the same, so the intersected in infinitely many points. So, here you have two lines which are parallel they do not intersect at all and so there is no solution.

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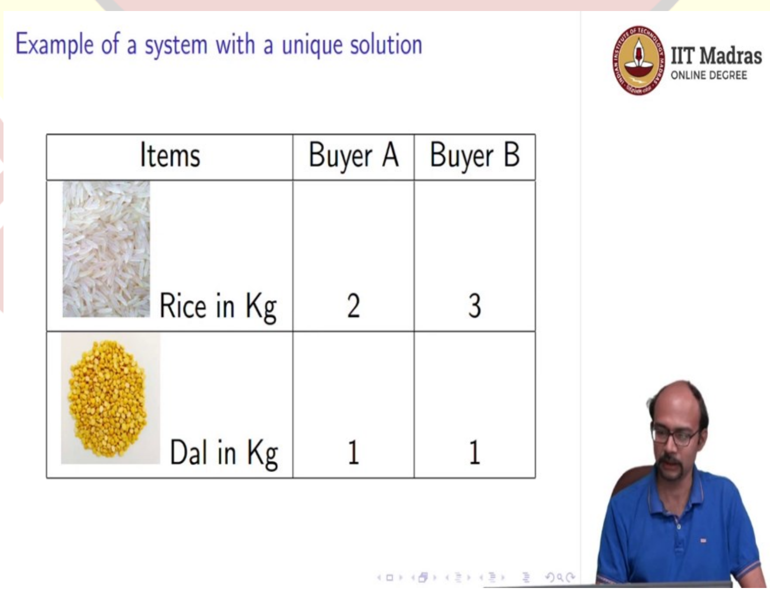

Finally, let us do the example the last example where you have a unique solution. So, in this case buyer A buys 2 kg's of rice and 1 kg of dal and buyer B buys rice 3 kg's of rice and 1 kg of dal.

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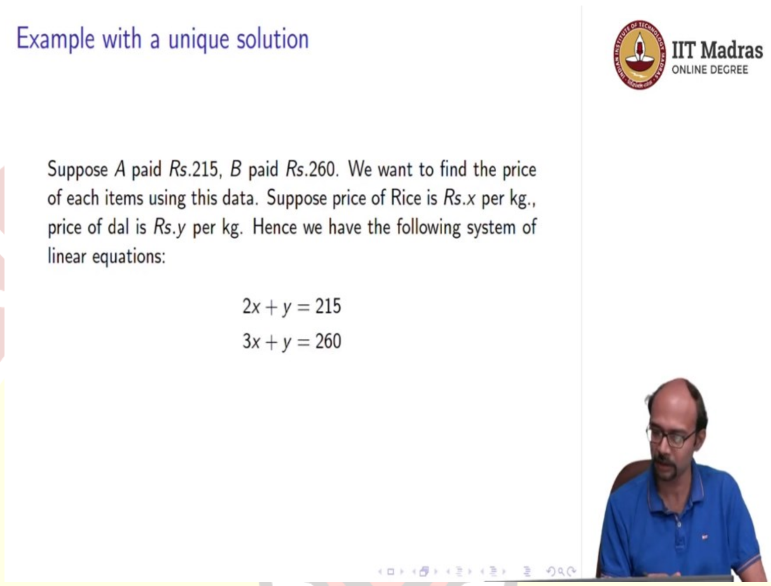

And buyer A pays 215 rupees buyer B pays 260 rupees, let us write down the equation so it is $2x + y = 253, 3x + y = 260$.

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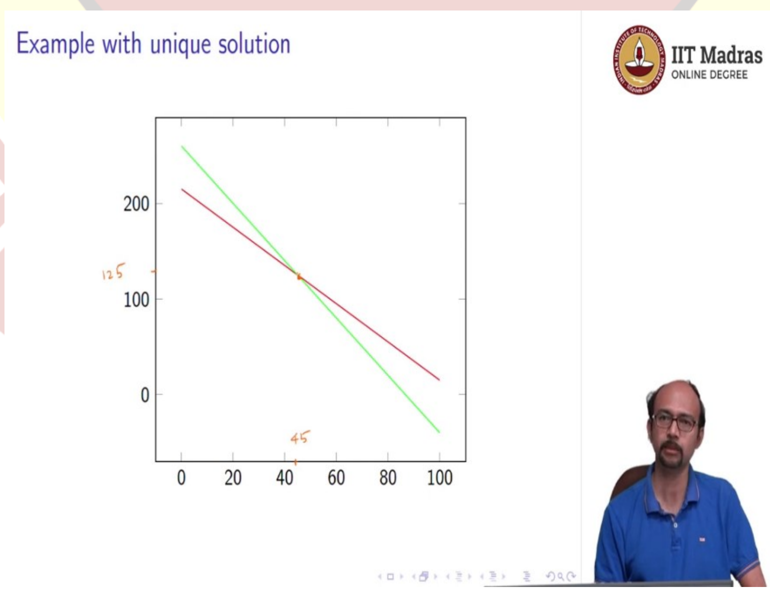

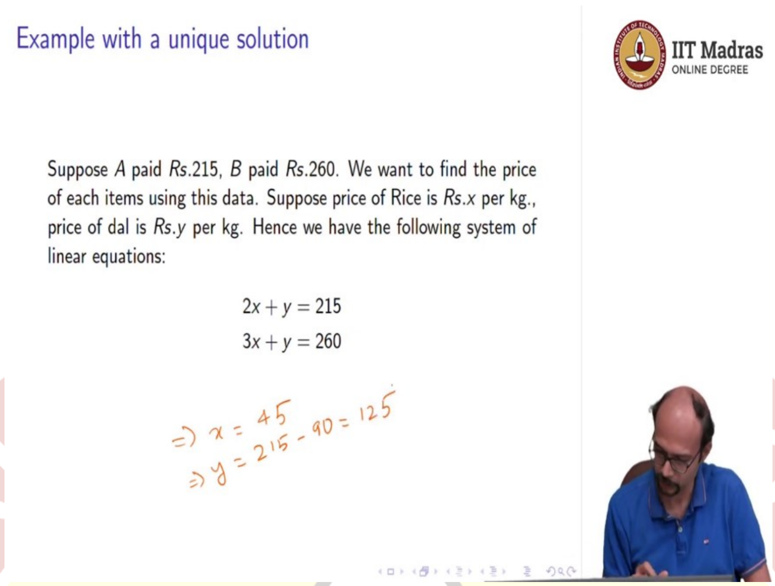

And I will encourage you to solve that equation. The point here is you have two lines, $2x + y$ is, maybe let us go back $2x + y\ is$ 215 and $3x + y\ is$ 260, so it is clear from here that x is 45 and once we know that x is 45 we can compute that y is, so 215 - 90, so that is 125. So, this is like in our first example, so the price of rice is 45 rupees per kg and the price of dal is 215 rupees per kg.

And you can check here that that is exactly where these intersect, so this is on the x axis will intersects at 45 and on the y axis, 125 and that is exactly the point of intersection. So, that is why there is a unique solution. So, you can draw these lines and wherever they intersect is where you have an equation, you have a solution. So, let us recall quickly what we study today.

We have studied what is the system of linear equations; we have study now to write it in terms of matrices, particularly using matrix multiplication with unknowns where x was a column matrix of variables. And then we have seen that what is the solution to system of linear equations and then we saw that there are three possible cases either a system has no solution or it has one unique solution or it has infinitely many solutions. That is all for today. Thank you.