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ONLINE DEGREE

Mathematics for Data Science 2
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Affine subspace and Affine mappings

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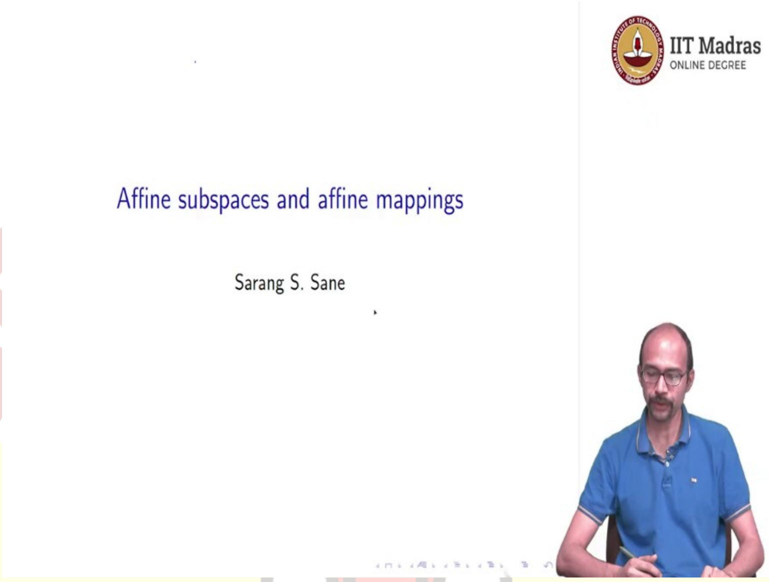

Hello, and welcome to the maths 2 component of the online BSc program on data science. In this video, we are going to talk about Affine sub-spaces and Affine mappings. So, before we start the video, I want to say that this video is not extremely essential to the general ideas in linear algebra that we have been looking at.

When it relates to data science, there is only one really important, I mean, important concept which we should address, and which is why we have to introduce this topic. So, I am going to be a bit brief in this video. And the idea is that if you want to really learn more, you should read a little bit of the literature or you can ask the tutors in the tutorial sessions. So, this is not really a very integral part of the course, but there is one thing that we have to understand in this video, and I will mention that when we come to it. So, let us look at Affine sub-spaces and Affine mappings.

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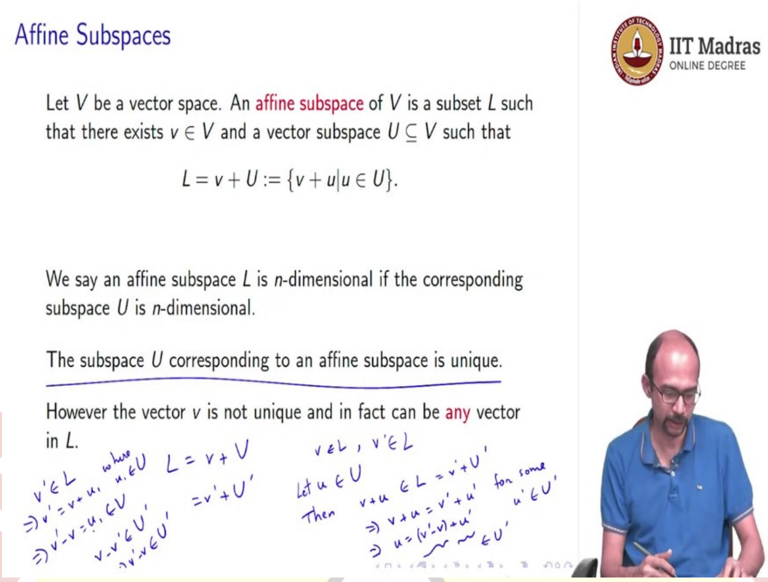

So let V be a vector space. So, an Affine subspace of V is a subset L, so it is a subset menu it is not a subspace, such that, there exists v in V and a vector subspace U in V such that L is literal v+u. So, what does that mean? That means, you will have to look at the set little v plus little u where little u varies over capital U.

So, essentially, it is taking this subspace U and then shifting it by V meaning you add the vector v to each vector in the subspace U. So, in particular, you added to 0 also. So, because U is a subspace, so, that means, 0 must be in U, which means the vector V must belong to this set L. So, any set L which can be obtained in this fashion is called as Affine subspace of V.

So, we say an Affine subspace is L n-dimensional if the corresponding subspace U is n-dimensional. So, if U is n-dimensional L is considered to be n-dimensional. So, the subspace U corresponding to an Affine sub-spaces is unique, so what do we mean by that. This definition for an Affine sub-spaces says that L is v+U, v is a vector U is an affine subspace. Maybe you could write it in a different way, as $v'+U'$

And what this statement is saying is that, maybe you can write it in a different way, but the U part must be the same. So, you can write it as v+ U and v'+U'. So, that is the only way to write it meaning there is no choice of subspace. So, the subspace that you are shifting must be the same.

And of course, there is this statement also implies that, I mean, the fact that I made it only for U and not for V tells us that the vector V can actually be chosen in a different way. In fact, it can be any vector in the set capital in the Affine subspace capital L. So, let us, maybe ask why that is the case. So, if I can write L as, so L = v + U = v'+U' how do I identify my set U'?

So, we just observed that both v and $v' \in L$ So, v belongs to L, v' also belongs to L by the same reasoning. So, then what we can try and do is we can try and look at how to express v' in terms of v + U and v in terms of v' + U' and that should sort of give the idea for why U = U'.

So, for example, if you take little u, so let $u \in U$ then $v+U \in L$ so that means it belongs to v'+U' that means, v+U = v'+u', for some $u' \in U'$ which implies that u=(v'-v)+u'.

So, now the claim is that v'-v is in U and U' both, and u' is of course, is in u', so that means, this u will be in U'. So, for every vector in U, it will be in U', conversely it will be in every vector in u' will be in U. So, why is v'-v in U or U'?

So, let us look at v'- v and let us look at how to write it in terms of v+U. So, well we know v'. Well, what I meant to say is, we know v' is in L that means v' is equal to V plus some, let us say u_1 where $u_1 \in U$ that means $v' - v = u_1 \in V$ And you can take the negative, so that will tell you also that $v - v' \in U$

And the same argument done for with v and v'. The rules changed; we will say that $v - v' \in U'$ So, that implies that $v' - v \in U'$ So, that means, both of these vectors belong to U', but U' is a vector space, it is a vector subspace. So, that tells us U is in U' and that completes the argument that I was giving here.

So, the main point here is if you can, if you write it as v+U and v'+U', then the U and U' must be the same. So, that is what this statement here is saying. And V is not unique. In fact, as we as, it can be any vector because you can add some vector of U to V and express that as your V.

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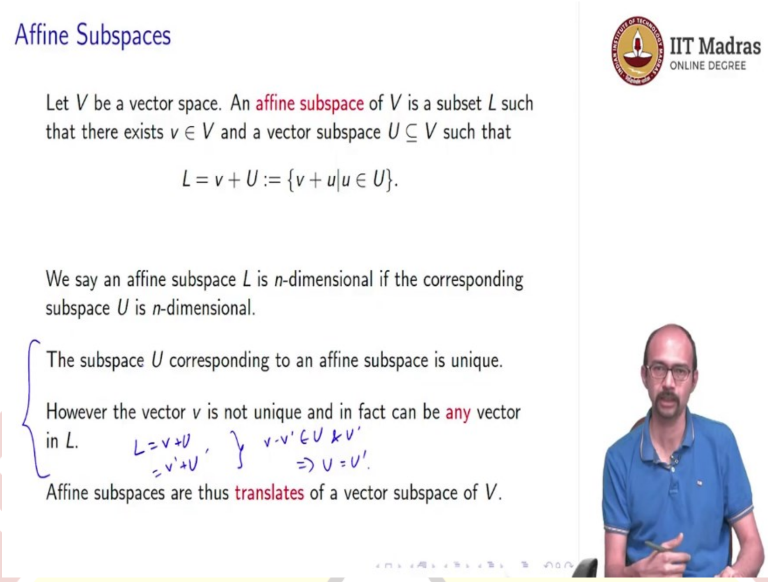

So, the point here is that Affine subspaces are translates of vectors of space of V. So, Affine subspaces are translates of a vector subspace of V. So, the main thing here is that, if you have L =v+U and v'+U', then $v-u' \in U, U'$ and using this you can claim that U=U', that is the thing here.

And for the other one, you can just take any vector in U and add it to v, and that will play the role of v'. So, v is not unique. So, Affine subspaces are the translates a vector subspace of V. So, you take this U, and then you translate it by v.

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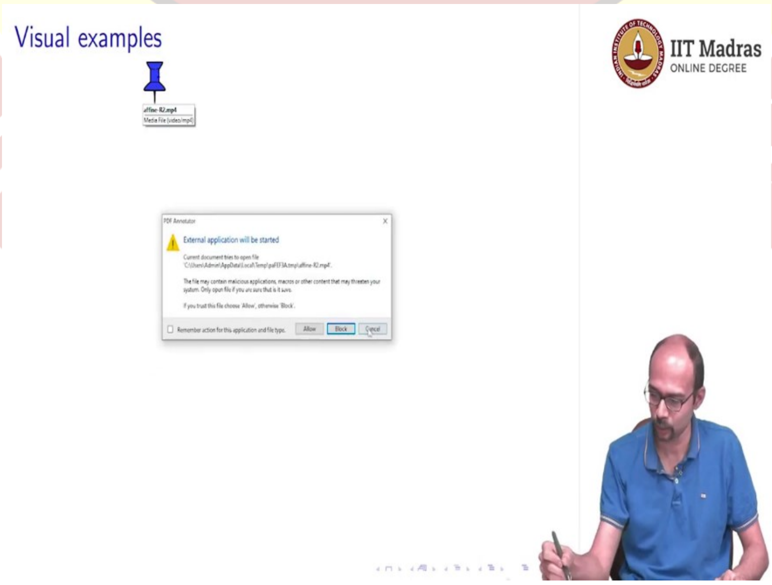

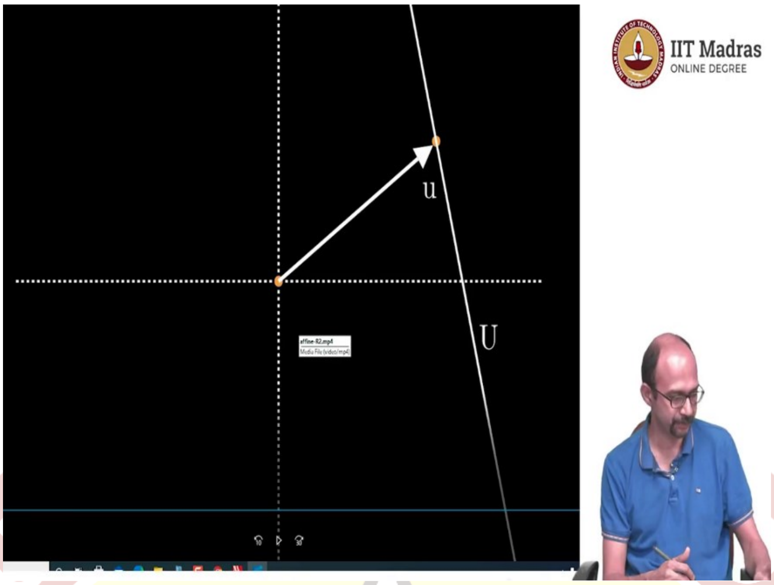

So, let us look at a few nice videos that our team has made to explain this idea clearly. So, this is a video in $\mathbb{R}^2$ so let us first draw the axis. So, once you have the axis, let us draw this vector u. This should have been labeled v and not u. This is literally v and not little u, and this is your subspace U, which is a line and then you translate it and that gives you your space L, so that is your new space L. So, this is an Affine space. So, this is an Affine subspace of $\mathbb{R}^2$

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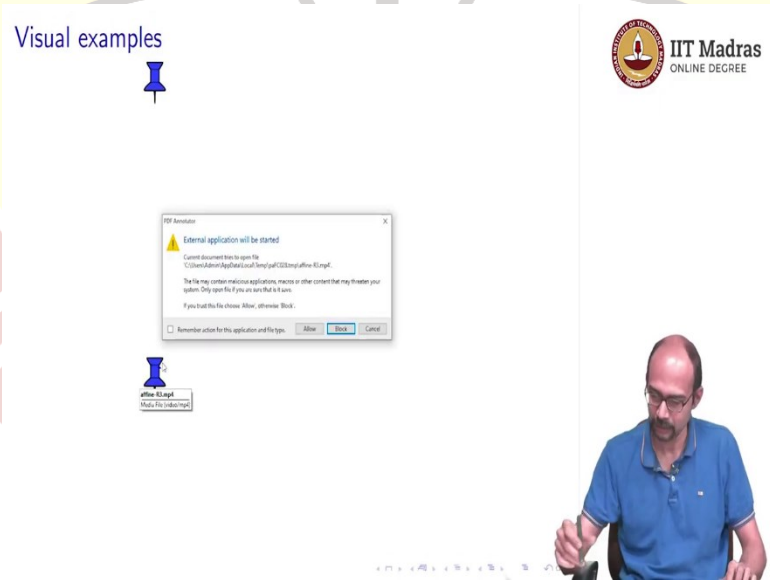

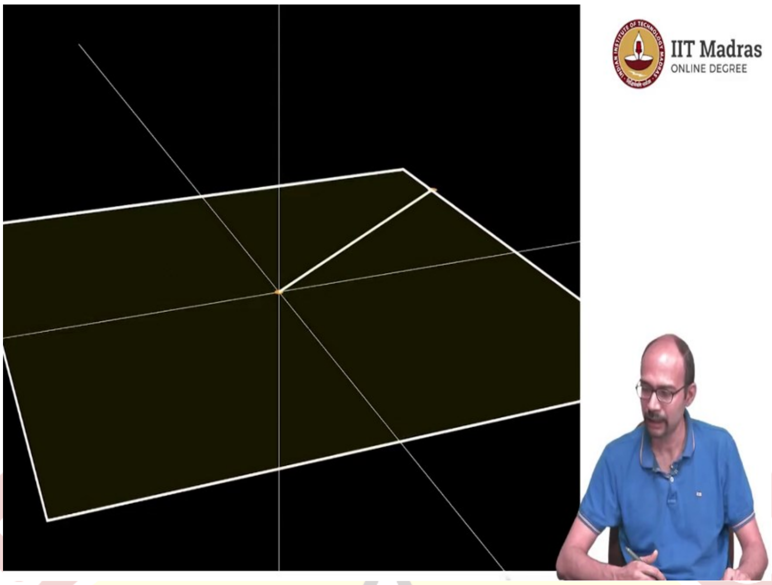
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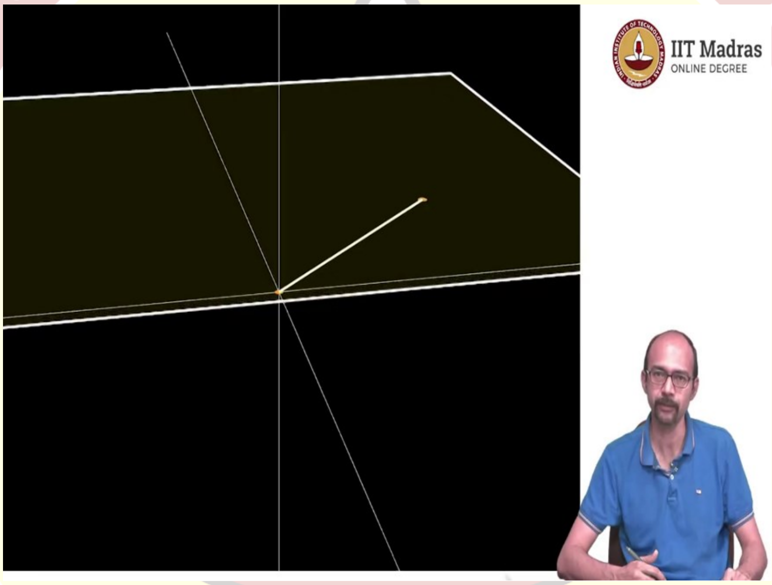
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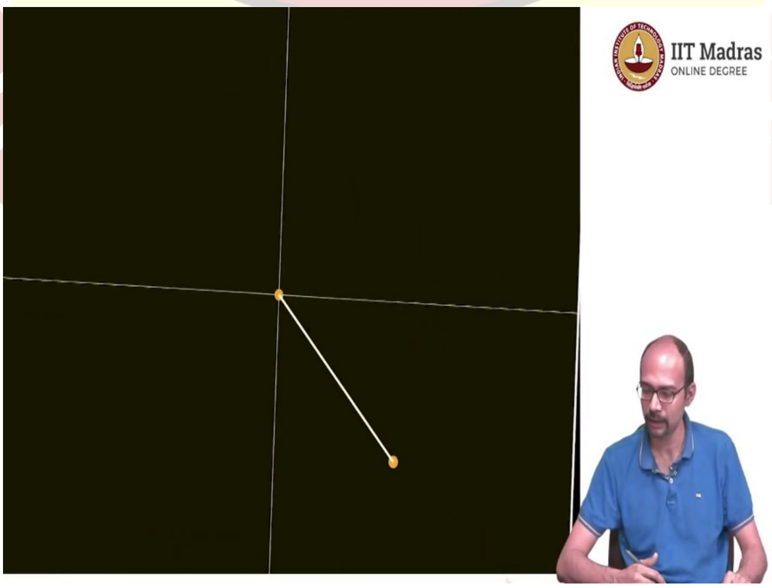
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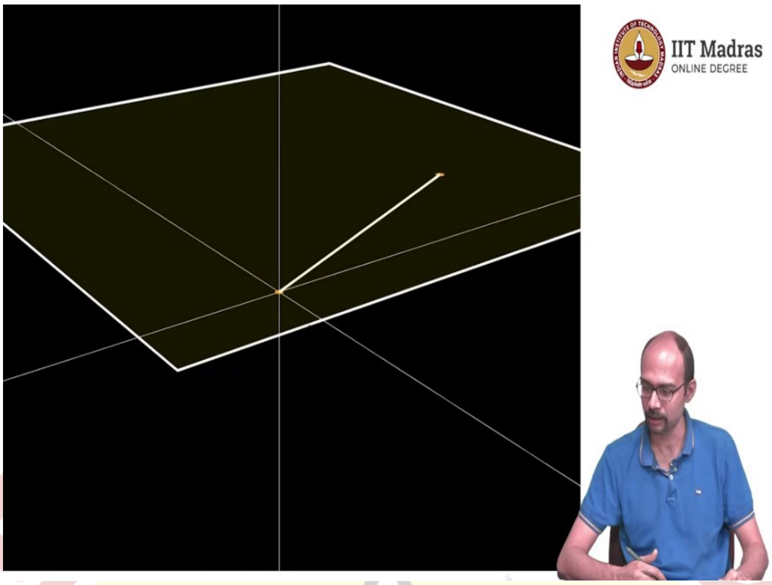

Let us maybe view another video in R^3 So, in time again, let us draw the axis. And that is your origin and then you draw point, you connect that so that is your vector, so this vector is being called V. And then you have a subspace this time it is the X, Y axis and then you shift it parallelly so that it contains this vector V, so it is a translate of the X, Y axis so that it contains the vector v that is this point v, that is what this video is showing. So, I hope you have understood the concept of a Affine subspace.

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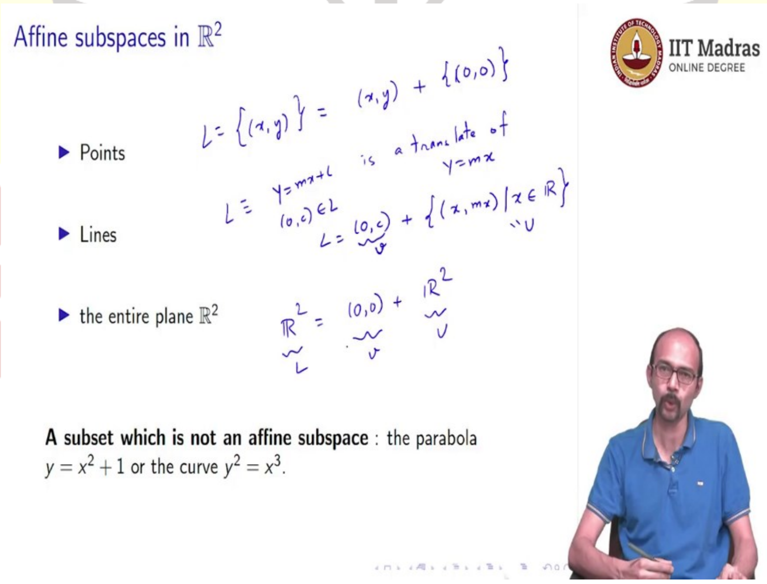

So, Affine subspaces are really not particularly hard they are just translates of subspaces that we have studied all along. So, you take a subspace and you move it. So let us put down what are the possible Affine subspaces in $\mathbb{R}^2$ So, in $\mathbb{R}^2$ you can look at points, because you take the origin, remember, the origin (0,0) is a subspace of $\mathbb{R}^2$ and then you can shift this by any vector V.

So, you take $\mathbb{R}^2$ the origin in $\mathbb{R}^2$ (0,0) and you shift it by V, so you do v + (0,0) and that gives you a point namely the point corresponding to v, and v can be anything, so it covers all possible points in $\mathbb{R}^2$ So, every point in $\mathbb{R}^2$ is an Affine subspace of $\mathbb{R}^2$ Similarly, we just saw the video about the first video about $\mathbb{R}^2$ where you had a line passing through the origin, so that is a subspace, and then you translated it and that gives you a line maybe not passing through the origin.

So, that is an Affine subspace, and then the entire plane $\mathbb{R}^2$ So, of course, there is nothing that rules out the entire plane from being an Affine subspace, because in particular, every vector subspace is an Affine subspace, you could take that vector v to be 0, so you, then you will get L to be 0 + U, which is just U. So, it could be in particular an actual vector subspace. So, $\mathbb{R}^2$ is a vector subspace of $\mathbb{R}^2$ so indeed it is a fine subspace as well.

So, let us also write down an example of something which is not an Affine subspace. So, Affine subspace is as we saw they are shifts of vectors of spaces linear subspaces. And linear subspaces or vector subspaces. So, those are always either the origin (0,0) or lines passing through the origin or the entire $\mathbb{R}^2$

So, if you have a curve, which is not a line. So, for example, if you take the parabola $y=x^2+1$ or you take a curve like $y^2=x^3$ if you have not seen these curves, I would recommend that you try to draw how these looks like, but they look curved, like this or like this, and they are not, so of course, they cannot be the translates of lines or planes or points. So, these are not Affine subspaces.

So, let me write down here. So, if you have a point, so if I take {(x,y)}. If I take this point, then how is it a subspace? So, this is my L. So, this is L, then this is the vector (x, y) plus the vector subspace (0, 0). And if you have a line, then in this case, well how do I write lines? Well, there is many ways of representing lines.

One is, maybe you have Y= mx + c and this line is a translate of Y = mx. So, how do I convert this into the form that I want? So, here you have to notice that the point $(0,c)\in L$ and that is exactly the point that you are shifting by. So, you write this $L=(0,c)+U$ where U is given by all points of the form $\{(x,mx)|x\in\mathbb{R}\}$

So, this is a line, this is a line passing through the origin, so this is your vectors subspace U, so you have written L as (0,c) plus this vector subspace. So, here this is your vector v. And the entire plane as I said, you can just write this as $\mathbb{R}^2=(0,0)+\mathbb{R}^2$ this is your U this is your little v, and this is your L. So, we have expressed each of these in the form that that we need to in order for the definition to work.

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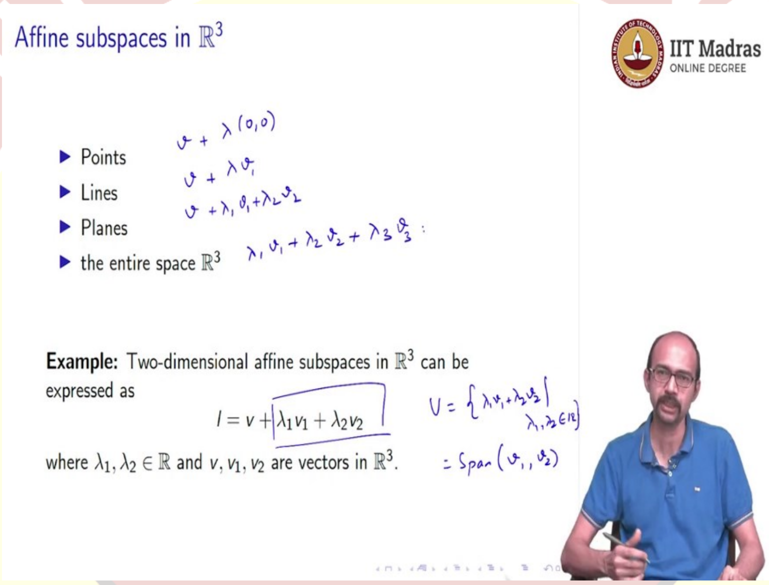

Let us ask the same question for $\mathbb{R}^3$ what are the Affine subspaces of in $\mathbb{R}^3$ So, I think by now you probably have a guess about what the answer is. You have points, you have lines, you have planes, yeah, this is something new. We did not have this in $\mathbb{R}^2$ because there is only one plane the entire space in $\mathbb{R}^2$ and then we have the entire space $\mathbb{R}^3$

So, what is let us also write down how any two-dimensional Affine subspace in $\mathbb{R}^3$ can be expressed. So, two-dimensional Affine subspace means a plane, because it is two-dimensional means you are translating a plane. So, then it can be written as $l=v+\lambda_1 v_1+\lambda_2 v_2$ where the $\lambda\in\mathbb{R}$ and $v, v_1, v_2\in\mathbb{R}$

So, v is the vector that was used in the definition of L = v +U. And what is U, U is exactly this part. So, this is, this part is U, so $U=\lambda_1 v_1+\lambda_2 v_2$ So, in other words, it is the span of v_1 and v_2 This is a $span(v_1, v_2)$ that is what is U.

So, I mean, if we use the same idea then points can be written as $v+\lambda(0,0)$ and then lines can be expressed as v plus λ times some vector on that line. So V_1 and planes as we have seen is $v+\lambda_1 v_1+\lambda_2 v_2$ and the entire space $\mathbb{R}^3$ is just $\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3$

If we want to do the same idea for the case of $\mathbb{R}^2$ we would have $v+\lambda(0,0)$ and then $v+\lambda v_1$

And then the third one would be just $\lambda_1 v_1+\lambda_2 v_2$ because that is the entire space it is only two-dimensional. So, I hope this gives you an idea of what are Affine subspaces and also the equations, the kinds of equations which govern them.

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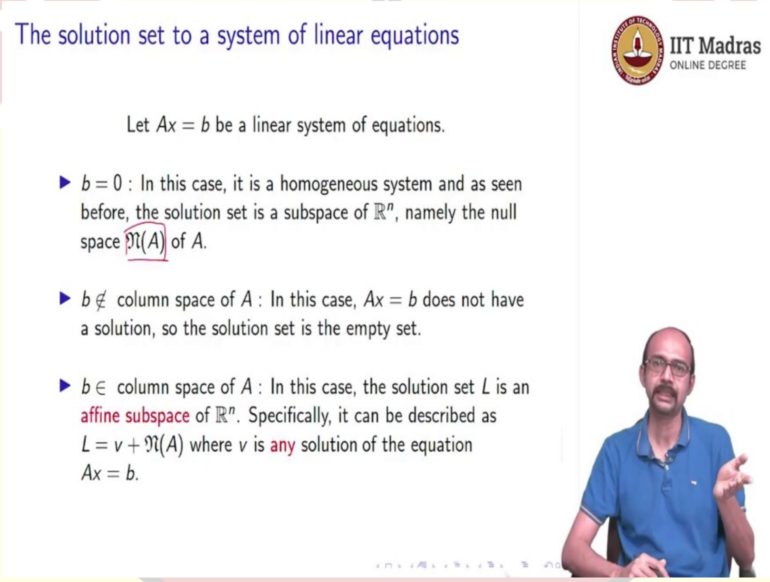

So, now this is the most important point about why we want to study Affine subspaces at all. That is because we, this particular example of the solution set to a system of linear equations, is an Affine subspace. So, let $Ax=b$ be a linear system of equations. And I want to pointedly add that I am not, I mean, this need not be homogeneous, and that is really the point.

If b is 0, then this is a homogeneous system. And we have seen before that the solution set is a subspace of $\mathbb{R}^n$ suppose this matrix is m by n, so this is a subspace of $\mathbb{R}^n$ So, we call that subspace the null space of A, the null space of A is exactly the solution set to this system of linear equations, Ax = 0 So, this is in the case where b is 0.

If b is not in the column space of A, so in this case, Ax=b does not have a solution. Because if b is not in the column space, that means no linear combination of the columns can equal b. If there was a linear combination which equal b, then it would be in the column space, so no linear combination can equal b, and that precisely means that there is no solution.

So, this solution set is the empty set. So, these two cases we have seen before and we know exactly how these two cases behave. And what we want to do in this video specifically is address the third case. Suppose b lies in the column space of A if b is 0, then we are back in case one, so nothing to do. So, if b is not 0, then how do I deal with it, then it need not be a vector subspace. In fact, it will not be a vector subspace, but it will be an Affine subspace.

In this case, the solution set L is an Affine subspace of $\mathbb{R}^n$ And how do we describe it, it can be described as $L=v+n(A)$ What is this n(A)? n(A) is exactly this n(A) that was here the null space. So, you have to translate the null space by a particular solution. So, you can take any particular solution and using that you translate the null space and the space that you obtain the Affine subspace is exactly their set of solutions.

So, the main mantra you have to take back here is the solution set of a non-inhomogeneous or a non-homogeneous system is an Affine subspace. And specifically, the way to write it down is get a particular solution, translate the null space and whatever you get is the entire set of solutions for the inhomogeneous system.

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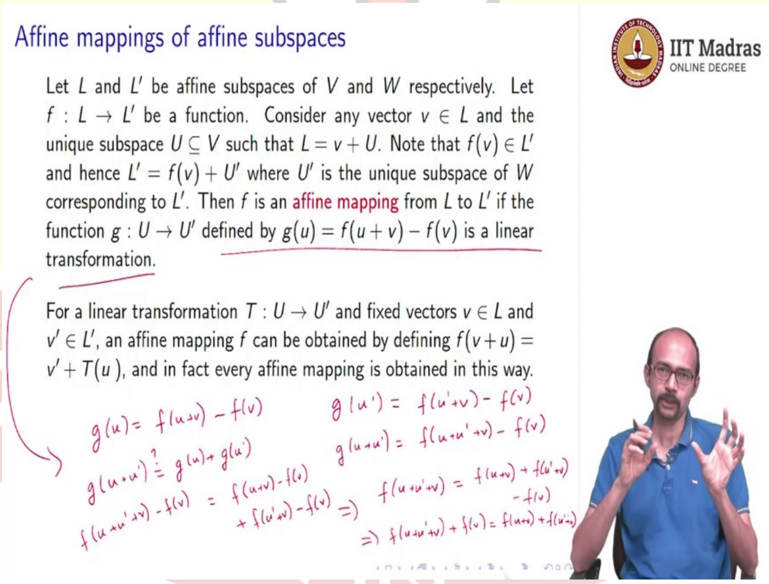

Now, let us come to the final thing that we want to deal with in this video. This is a slightly complicated notion, but we are we are just doing this for the sake of completeness. And one small idea. Let L and L’ be Affine subspaces of V and W respectively. So, we have seen how to define the notion of a linear transformation.

So, now suppose, so linear transformations work between vector spaces or vector subspaces. So now, suppose we have Affine subspaces, we would still like to know how to deal with functions on those, which in some sense, preserve the structures that we are interested in. So, how do we do that? So, that is the idea of an Affine mapping.

So, let L and L prime be Affine subspaces of V and W respectively. Let f: L → L' be a function. Consider any vector V in L and the unique subspace U contained in V such that L is v + U. So, we know that L= V + U. So, note that a f (v) in L' lies in L' and hence L' is f(v)+U' where U' is the unique subspace of W corresponding to L'.

Then f is an Affine mapping from L → L', if the function g that you obtain from U → U' defined by g(u)=f(u+v) - f(v) is a linear transformation. So, for a linear transformation T : U → U' and fixed vectors $v \in L$ and $v' \in L'$ an Affine mapping can be obtained by defining f (v+U) = v'+ T(u).

So, here basically what you are saying is put f (v)=v' and then for any other vector or any other point on this Affine subspace U, you look at v + U, there is some U such that you have that point or vector v + U, then you look at v + U and define f (v + u) to be v' + T(u), then this is kind of saying that it is linear. And in fact, every fine mapping is obtained in this way.

So, we have g(u) = f(u+v) - f(v). So, g(u)= f(u+v) - f(v). What does it mean for this g to be a linear transformation? So, I want that. So, let us also look at some other thing g(u'). So, the f(u'+v) - f(v). So, I am trying to explain what this statement means.

So, I want that g(u + u') = g(u) + g(u'). This is what I want. One of the things that I need for it to be a linear transformation. Let us unravel this definition. First of all, what is g(u + u')?

By definition, this is f(u +u'+v)- f(v).

So, let us put all these in here. This means that f(u +u'+v)- f(v) = f (u +v)- f(v) + f(u'+v)-f(v). I can cancel both sides - f(v). So, if I do that, what I get is $f(u+u'+v)=f(u+v)+f(u'+v)-f(v)$ and then I can take that f(u) on the other side. So, I get $f(u+u'+v)+f(v)=f(u+v)+f(u'+v)$ So, you can see here what is going on. You are taking u+u'+v. And then you look at it as u+v, u'+v, there is an extra v over there. So that extra v you get in here.

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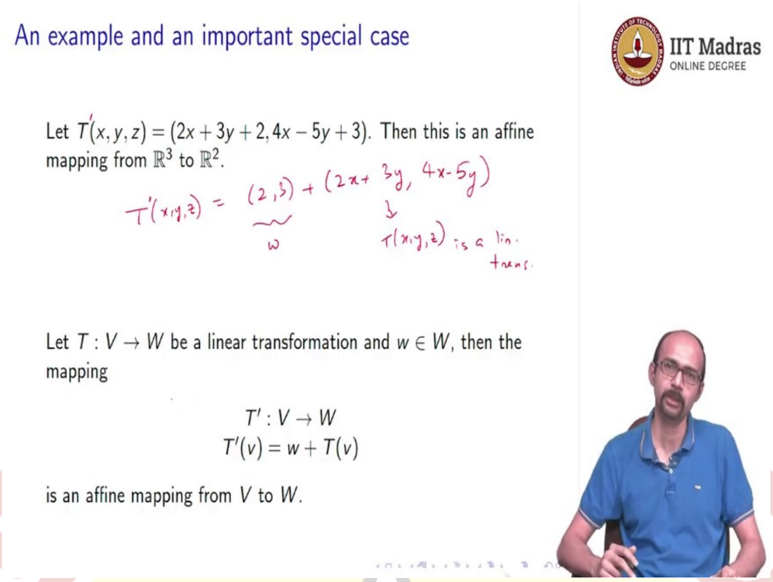

So, suppose you take T(x, y, z), to be maybe I should not have used T, let us call F, maybe, since I am using T in the previous slide for a linear transformation. So, let $T(x, y, z)=(2x+3y+2, 4x-5y+3)$ this is not a linear transformation. Why is that? Because if you look at T(0, 0, 0) then it becomes (2,3). And we know if it is a linear transformation, it would have been (0, 0) so this is not a linear transformation, but this is an Affine transformation.

And how do I know it is an Affine transformation. The reason I know it is an Affine transformation is because I can write this as $T(x, y, z)=(2,3)+(2x+3y, 4x-5y)$ and this is indeed a linear transformation. This is a linear transformation, and this is your vector of v'. This is what the second line in that slide, the previous slide said. So, this is why I know it is a linear transformation.

So, this is a general phenomenon. Namely, if you have a linear transformation, you can and some fixed vector W and W, then the mapping T'(v) = w + T(v) is and Affine mapping from V to W, this is exactly what I did here. Let us just see what I did here, what I did is, I took this T(x, y, z), which unfortunately this I could have called this T' or F.

So, T'(x, y, z) can be written as (2,3) + (2x+3y, 4x -5y), as I explained, you take this to be T(x, y, z) so this is indeed a linear transformation. So, this is a linear transformation. And this is your vector W, in here, so this is exactly what you are doing below. And this is an Affine mapping as a result.

So, essentially what we are saying is, this looks very much like a linear transformation, but you can shift it by a vector. And here, for example, you shifted it by (2,3), so, usually we will have to find out what that vector is and that vector can often be found or always be found by looking at T of the 0 vector. If it is in this form where you have it from V to W, then you can just look at what is T the 0 vector, and that will tell you what to shift by, what to translate by, that is the idea.

So, let us recall what we have done in this video. In this video, we have studied the notion of an Affine subspace, which is nothing but taking a usual vectors subspace and translating it by some vector V. The most important example that we saw is the case of the solution set of a homogeneous or in-homogeneous system of linear equations, which have a solution.

If you are, if you are looking at Ax=b and b and this does not have a solution at all, then of course it is the solution set is the empty set. So, there is nothing, there is no mathematics to do over there. But if it has a solution, then we know exactly what the solution set of solutions looks like.

You take a particular solution, and then you take the null space, and translate it by that, that is how we, that is how we look at. That is an example of an Affine subspace. And then we looked at Affine mappings, which are very similar to what we do for linear transformations, but you allow a translation again that is essentially what we have done in this video. Thank you.