IIT Madras

ONLINE DEGREE

Mathematics of Data Science 2
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What is a linear mapping – Part 1

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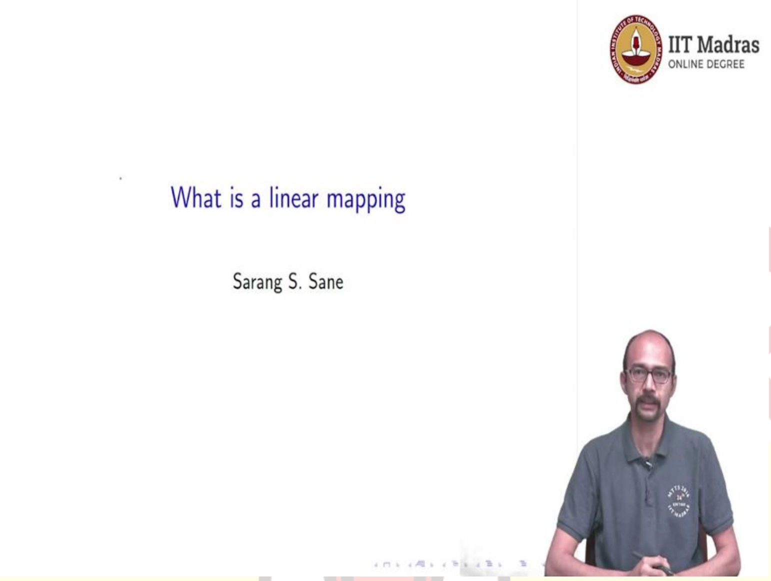

Hello, and welcome to the Maths 2 component of the online BSc Program on Data Science. In this video, we are going to talk about what is a linear mapping.

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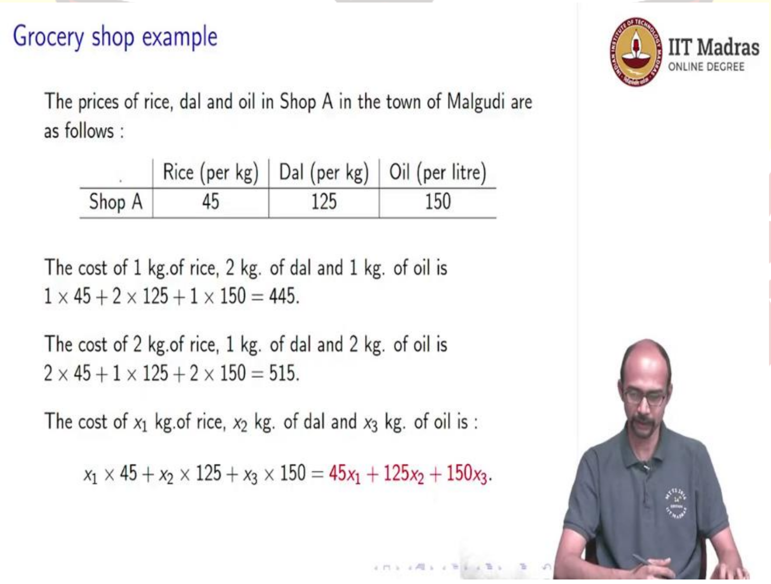

So, let us start with an example. So, we have a grocery shop, a usual kirana shop that you go to, and let us say it's in the town of Malgudi. The shop is Shop A, and we have three items that we are interested in rice, dal and oil. And the prices in these, in Shop A of these items are 45 rupees per kg for rice, 125 rupees per kg for dal and 150 rupees per liter for oil.

So, if you want to compute, for example, the cost of 1 kg of rice, 2 kg of dal and 1 kg of oil, then what do we do, we multiply $1 * 45 + 2 * 125 + 1 * 150 = 445$, if I have done this computation correctly.

Similarly, if you want to calculate the cost of 2 kgs of rice, 1 kgs of dal and 2 kgs of, 2 liters of oil both of these will be liters, then that is given by multiplying $45 * 2 + 125 * 1 + 150 * 2$, which, if I have done it correctly, should give us 515 rupees.

So, in general, if you have x_1 kgs of rice, x_2 kgs of dal and x_3 liters of oil, not kgs then how do we do that, we have to multiply $45 * x_1 + 125 * x_2 + 150 * x_3$. So usually, we write it in the opposite way. Where we write it as $45x_1 + 125x_2 + 150x_3$. So, this is notice a linear combination of x_1, x_2, x_3 with coefficients 45, 125 and 150.

So, the point is if you want to get the cost of certain number of items, and you know the cost of each individual item, then the total cost is a linear combination of the amounts of the items that you want to buy. So, we have seen this kind of idea before also. This is not the first time we are seeing it in this course. But I wanted to remind you of this.

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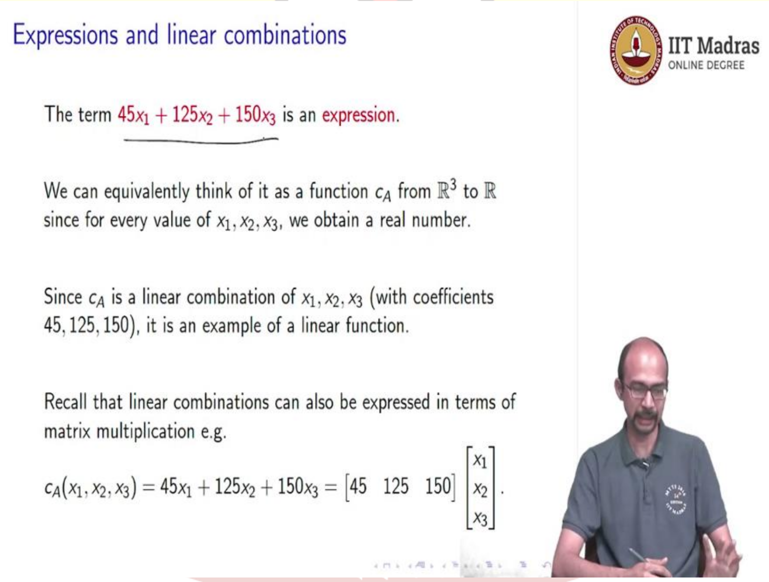

So let us quickly summarize this in terms of what is an expression and what is a linear combination. So, the term $45x_1 + 125x_2 + 150x_3$ is an expression. I want to explicitly say this because sometimes there is a confusion about what is an expression and what is an equation. So, this is not an equation, this is an expression.

So, we can equivalently think of this as a function, and that is what this video is going to be about. We can think of this as a function c_A, c for cost, A for the Shop A from $\mathbb{R}^3$ to $\mathbb{R}$. Since for every value of x_1, x_2, x_3 we get a real number. How do we do that? Because if you plug-in some values for x_1, x_2, x_3 you can evaluate, that is exactly what we did in the previous slide, where we evaluated this with specific values of x_1, x_2, x_3. So that means this is a function. Given a particular triple, meaning given values of x_1, x_2, x_3 it will give us a real number. So, we can take this as a function from $\mathbb{R}^3$ to $\mathbb{R}$.

So, since c_A is a linear combination of these three variables, x_1, x_2, x_3 and with coefficients as we noted 45, 125, and 150 this is an example of a linear function. Why linear? Because it has something to do with a line, which we may study all, which you may have seen already in maths 1, and which we will anyway refresh after something. But in our context, it is linear because there are no powers involved here or no exponents. It is just $45x_1 + 125x_2 + 150x_3$.

There are no other terms, where you have other powers of x_i is coming in or you have some other kinds of functions, logarithm and so on. That is why it is a linear, we call it a linear function. And we will explicitly say what we mean by a linear function in a little while and that is what this video is about, about linear mappings.

So, recall, we have done this before that, linear combinations can also be expressed in terms of matrix multiplication. And this is what we have done all of maybe a week back or two weeks ago, where we studied in great detail matrices and how they operate and various kinds of manipulations. So, it is important to remember this because we are going to use this soon for various other purposes.

So, we can express the function C_A, which was defined as a linear combination, $45x_1 + 125x_2 + 150x_3$ as the matrix multiplication, where we have the coefficients as a row matrix $\begin{bmatrix} 45 & 125 & 150 \end{bmatrix}$, and the variables, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. We have seen this before I just want to want you to remember this for what is coming next.

So, I again want to reiterate remember that this is, you can think of this either as a linear combination. When I say this, I mean, this term $45x_1 + 125x_2 + 150x_3$ as a linear combination or an expression. It is not an equation. An equation is when you equate it to something, and then we will try to solve. So an equation is when you have an equal sign somewhere and you say, this expression is equal to some other expression. That is when we will say it is an equation.

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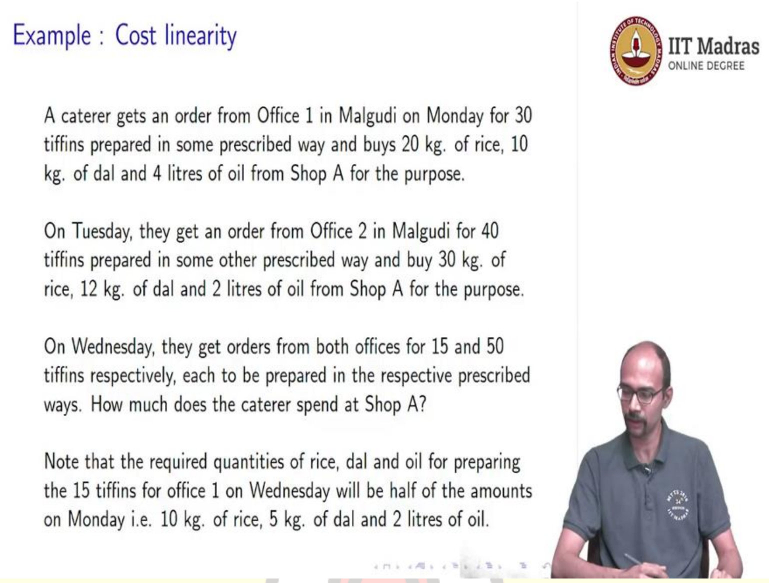

So let us go on. So, let us talk about cost linearity. So, this is what was hidden in that statement about it being a linear function. So, a caterer gets an order from Office 1 in Malgudi on Monday, for 30 tiffins prepared in some prescribed way. Maybe they want just rice and dal prepared separately and with more oil.

So, for Office 1, they want 30 tiffins so the caterer estimates that they will need 20 kgs of rice, 10 kgs of dal and 4 litres of oil, and they go to Shop A buy these. This is for Office 1 in Malgudi. So, on Tuesday, they get an order from Office 2, so which is a different office from Office 1 and they want a different way of preparation. They want less oil, but maybe they want the rice to be more fluffy or less fluffy or something and for whatever purpose, whatever way in which they want it to be made.

The caterer estimates that for 40 tiffins they will need 40 kgs of rice, 12 kgs of dal and 2 liters of oil, less oil. So, the Office 2 is health conscious. And then they go to Shop A for buying this material. On Wednesday, they get offers, they get orders from both offices, Office 1 and Office 2. Office 1 wants 15 tiffins, Office 2 wants 50 tiffins, maybe Office 1 did not like so much.

Some people in Office 1 did not like how they repaired it, so only half of them ordered, but they still wanted to be prepared in their way, whatever that was. And Office 2 wants it to be repaired in their way, whatever that was. So, the question is, how much does the caterer spend at Shop A. What is the cost to the caterer or how much the Shop A earn from the caterer?

Now, the natural way of doing this, I mean, one way of doing this maybe is to ask, well, for 50 tiffins for Office 1, I know how much was required for 30 tiffins for Office 1, so 15 tiffins will require half of that amount, so 10 kgs of rice, 5 kgs of dal and 2 litres of oil. And 50 tiffins for Office 2 will require five by four times whatever was used up on Tuesday, because that time they had offered 40 tiffins. So, you would have five by four times 30, five by four times 12 and five by four times 2 respectively kgs, kgs and liters of rice, dal and oil. So, this is what we are noting here. On Wednesday, for Office 1, we will require 10 kgs of rice 5 kgs of dal and 2 liters of oil.

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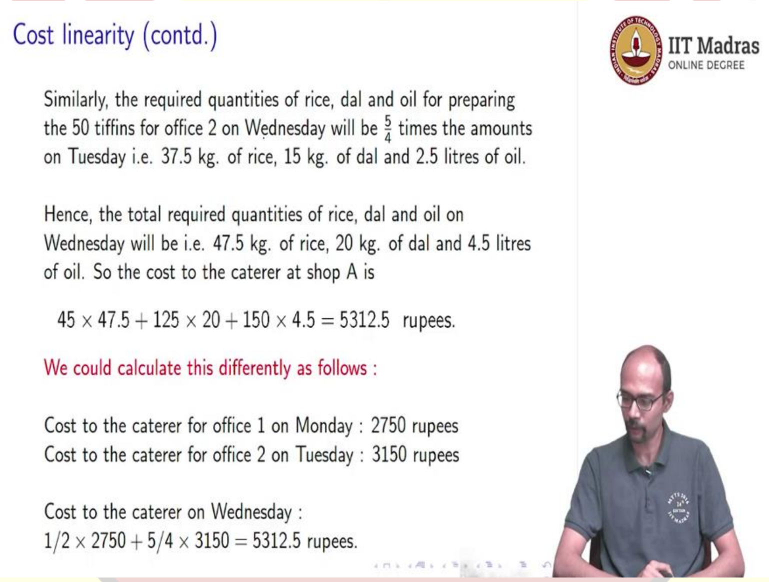

Wednesday for Office 2, which remember has 50 tiffins, we will do require five by four times the amount on Tuesday, which means you need 37.5 kgs of rice, earlier it was 30 kgs of rice on Tuesday for 40 tiffins, 15 kgs of dal. So, I think we had 12 kgs of dal on Tuesday, and 2.5 liters of oil, we had 2 liters of oil on Tuesday. So we know for each office, how much rice, dal and oil is going to be required.

So, the caterer will go to Shop A and will buy the sum of these two. So, the total required quantities of rice, dal and oil on Wednesday will be 47.5 kgs of rice, 20 kgs of dal and 4.5 liters of oil. So, I am just adding up whatever we need for Office 1 on Wednesday to whatever we need for Office 2 on Wednesday.

So, the cost to the caterer at Shop A, they always buy from Shop A, this is what we know is 45 times 47.5, so you substitute plus 125 times, 15 plus 150 times 2.5, so you substitute these values 47.5, 20 and 4.5 in our linear combination or in the cost function c_A. You evaluate the cost function at 47.5, 20, 4.5. So that gives us $45 * 47.5 + 125 * 20 + 150 * 4.5 = 5312.5$. This is something you have to calculate. And if I have done my calculation, right, hopefully this is what you will get.

So now, we could calculate this a bit differently. How do I do that? So, we can calculate what was the cost to the caterer on Monday for Office 1. So how do I do that? Well, on Monday how much was bought? On Monday, we had, I think 20 kgs of rice, 15 kgs of dal and sorry, 10 kgs of dal and 4 litres of oil. So, you put in this vector in $\mathbb{R}^3$, 20,10, 4, you put it into the cost function C_A or you evaluate the linear combination with $x_1 = 20$, $x_2 = 10$ and $x_3 = 4$, and then you get 2,750 rupees, that is for Office 1 on Monday.

On Tuesday, we had 40 tiffins for Office 2, and for that we had 30 kgs of rice, 12 kgs of dal and 2 liters of oil, so you take this vector, 30, 12, 2 and evaluate the cost function on this vector or this element of $\mathbb{R}^3$ or equivalently you put it into the linear combination, and what we get is 3,150 rupees. So, what I have done, I have computed the costs on Tuesday and Monday.

So, now, I am claiming that the cost to the caterer on Wednesday is $\frac{1}{2} * 2750 + \frac{5}{4} * 3150 = 5312.5$. So, the claim is that both of these are the same, and that is basically what I want to say. Now why is that happening? We will see that.

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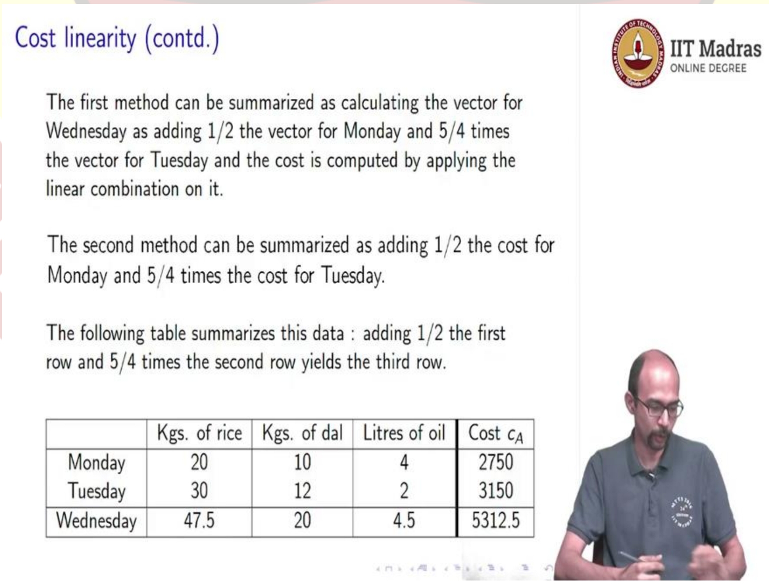

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So, the first method can be summarized, as calculating the vector for Wednesday, by adding half times a vector for Monday, by the vector I mean the demand vector. The amounts of quantity that are required. So, the demands on Monday, you took $½\, of\, that + 5 * 4$ the demands on Tuesday. That is exactly how we got this, these numbers.

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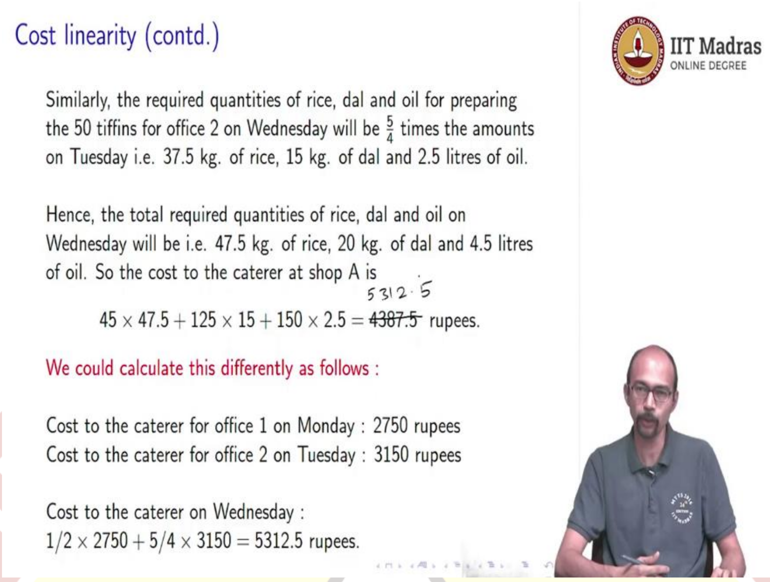

So, these numbers 47.5, 15 and 2.5. That is exactly how we got those numbers.

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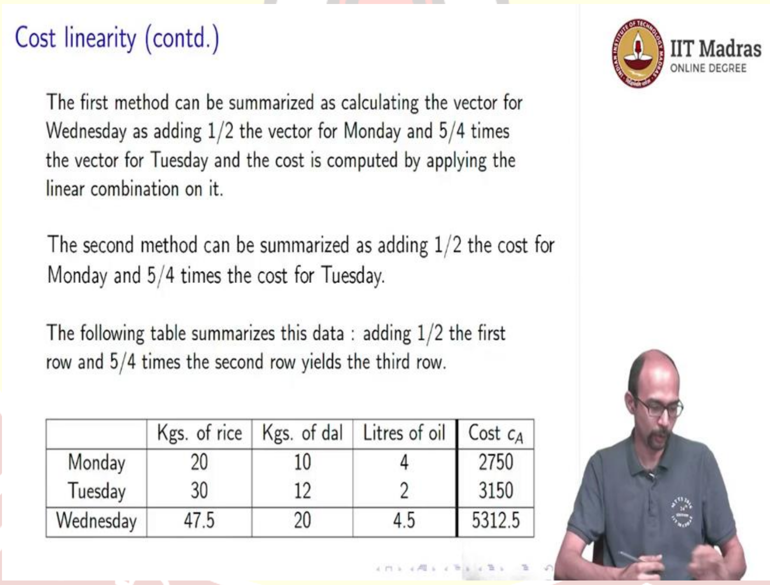

And then we compute the cost function on those numbers and that is how we did the first method. The second method can be summarized, as adding ½ times a cost for Monday, and 5/4 times a cost on Tuesday.

So, I will summarize this in this table. So, these are the quantities of rice, dal and oil that we require on Monday, Tuesday and Wednesday, respectively. And we see that the third column, sorry, the third row is obtained as half times the first row plus 5 by 4 times the second row. And this is where linearity is coming in. We are so now I have converted whatever I have done before to a sort of vector problem.

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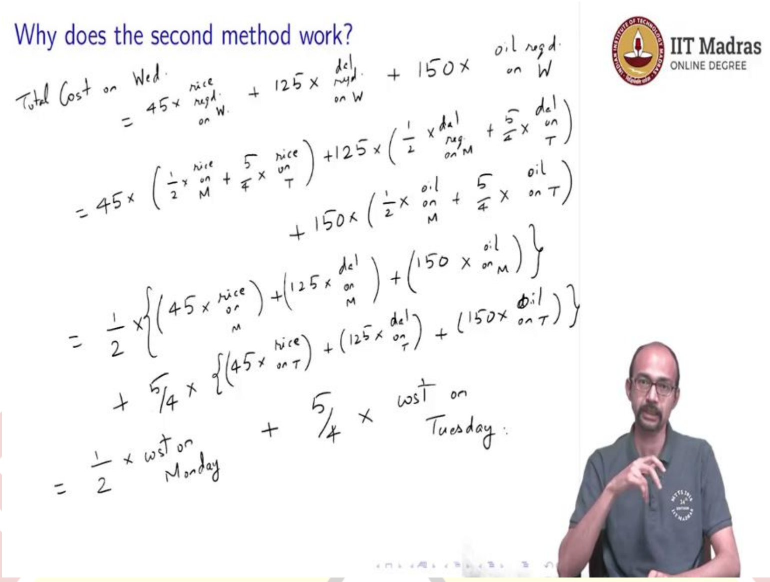

So, let us now understand why does the second method work? So, for the second method what did we do, so I want the cost on Wednesday. Total cost on Wednesday, which we know is certainly given by the evaluating. So, 45 times rice required on Wednesday plus 125 times dal required on Wednesday plus 150 times oil required on Wednesday. But how much rice was required on Wednesday?

Well, we just saw that, that is half times the rice on Monday, plus 5 by 4 times the rice on Tuesday. And we can do the same thing for dal and oil, so this is half times, excuse me, half times the dal required on Monday, plus 5 by 4 times the dal required on Tuesday, plus 150 times, half times oil on Monday, plus 5 by 4 times the oil on Tuesday.

Now, we can separate out the quantities for Monday and Tuesday. So, if we do that, we get, so I will do it in one step. Ideally, I should have written one more step, but I think you can follow what is happening. So, this is half times 45 times rice on Monday plus, excuse me, it is a bracket here plus 125 times. I do not need to put the brackets by BODMAS, but I am anyway just be careful.

Dal on Monday plus 150 times oil on Monday. These are all the terms for Monday. And then plus 5 by 4 times 45 times rice on Tuesday, the same expression, but now for the Tuesday quantities, 125 times dal on Tuesday plus 150 times oil on Tuesday. But now, whatever we have inside the curly brackets is exactly evaluating the linear combination that I have on the quantities for Monday and Tuesday respectively.

So, this is exactly whatever we have in the curly brackets is exactly the cost on Monday, that is the first curly bracket, and whatever we have in the second curly brackets is the cost on Tuesday. And this explains why the second method works. The first method is clearly the sort of long way of doing it, and the neat way of doing it. But the second method is clearly much shorter. And yields the same answer I mean, which otherwise, we would not have discussed it.

And this is the reason why. So, what is the, what really worked here? Why did this work? What worked here was the fact that we had a linear combination. So that is why this function, I said this function is linear, because these terms can split up, you can break them into two. And I will, we will make this more explicit as we go on. So, I, but I hope I have convinced you that both the methods give you the same answer.