Q1 (a) Suppose a fair coin is tossed repeatedly till a head is obtained. What is the probability that the number of tosses is equal to 5?

A_5: no. of tosses $= 5$

$A_5 = (\text{tails in } 1^{st})$ AND $(\text{tails in } 2^{nd})$ AND $(\text{tails in } 3^{rd})$ AND $(\text{tails in } 4^{th})$ AND $(\text{tails in } 5^{th})$

$$P(A) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$$

(b) What is the probability that the number of tosses is greater than or equal to 5?

Method 1: $A_{\geq 5} = A_5$ OR A_6 OR A_7 OR ....

$$P(A_{\geq 5}) = \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots = \frac{1/2^5}{1 - 1/2} = \frac{1}{16}$$

Method 2: $A_{\geq 5} = (\text{tails in } 1^{st})$ AND $(\text{tails in } 2^{nd})$ AND $(\text{tails in } 3^{rd})$ AND $(\text{tails in } 4^{th})$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

Q2 (a) Suppose a fair die is thrown repeatedly till a 1 is obtained. What is the probability that the number of throws is equal to 5?

A_5: no. of throws $= 5$

$A_5 = $ (not 1 in 1^{st}) AND (not 1 in 2^{nd}) AND (not 1 in 3^{rd}) AND (not 1 in 4^{th}) AND (1 in 5^{th})

$$P(A_5) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} = \frac{5^4}{6^5}$$

(b) What is the probability that the number of throws is greater than or equal to 5?

$A_{\geq 5} = A_5$ OR A_6 OR A_7 OR .... $=$ (not 1 in 1^{st}) AND (not 1 in 2^{nd}) AND.... AND (not 1 in 4^{th})

$$P(A_{\geq 5}) = \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^5 \cdot \frac{1}{6} + \cdots\cdots = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$$

$$= \frac{(5/6)^4 \cdot 1/6}{1 - 5/6} = \left(5/6\right)^4$$

Q3 (a) Suppose a volcano erupts in a particular month with probability 0.01, independent of all prior non-eruptions. What is the probability that the volcano will erupt within the next one year?

$A =$ erupt in one year $= (\text{Erupt in M1})$ OR (NOT in M1, Erupt in M2) OR .... OR (NOT in M1–M11, Erupt in M12)

$A^c = (\text{Not in M1})$ AND (Not in M2) AND ... AND (NOT in M12)

$P(A^c) = 0.99 \times 0.99 \times \dots \times 0.99 = 0.99^{12} \Rightarrow P(A) = 1 - 0.99^{12}$

(b) Suppose the volcano does not erupt for two years. What is the conditional probability that the volcano will erupt during the third year?

Independence

$1 - 0.99^{12}$

Q4 (a) Suppose a fair coin is tossed 10 times. What is the probability that the number of heads is equal to 8?

heads : Binomial(10, 1/2)

$$P(\#\text{heads} = 8) = \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 = \frac{45}{1024}$$

(b) What is the probability that the number of heads is greater than or equal to 8?

$$\#\text{heads} \geq 8 = (\#\text{heads} = 8) \text{ or } (\#\text{heads} = 9) \text{ or } (\#\text{heads} = 10)$$

$$P(\#\text{heads} \geq 8) = P(\#\text{heads} = 8) + P(\#\text{heads} = 9) + P(\#\text{heads} = 10)$$

$$= \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + \binom{10}{9}\cdot\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + \binom{10}{10}\left(\frac{1}{2}\right)^{10}$$

$$= \frac{56}{1024}$$

Q5 (a) Suppose a fair die is thrown 10 times. What is the probability that 1 never appears?

$$\#1\text{'s} = \text{Binomial}(10, 1/6)$$

$$P(\#1\text{'s} = 0) = \binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10}$$

(b) What is the probability that either 2 or 3 appear at least once?

$$\#\text{times } 2 \text{ (or) } 3 \text{ appear} = \text{Binomial}\left(10, \frac{2}{6}\right)$$

$$P(\#\text{times } 2 \text{ (or) } 3 \text{ appear} \geq 1) = 1 - P(\#\text{times } 2 \text{ (or) } 3 \text{ appear} < 1)$$

$$= 1 - P(\#\text{times } 2 \text{ or } 3 \text{ appear} = 0)$$

$$= 1 - \binom{10}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{10} = 1 - \left(\frac{2}{3}\right)^{10}$$

Q6 (a) In a 12-week monsoon season, the probability of rain in a particular week is 0.8, independent of all other weeks. What is the probability that it rains for 10 or more weeks?

$$W = \#\text{ weeks of rain} = \text{Binomial}(12, 0.8)$$

$$P(\#\text{ weeks of rain} \geq 10) = P(W=10) + P(W=11) + P(W=12)$$

$$= \binom{12}{10}(0.8)^{10}(0.2)^{2} + \binom{12}{11}(0.8)^{11}(0.2) + \binom{12}{12}(0.8)^{12}$$

(b) What is the probability that it rains for fewer than 4 weeks?

$$P(W<4) = P(W=3) + P(W=2) + P(W=1) + P(W=0)$$

$$= \binom{12}{3}(0.8)^{3}(0.2)^{9} + \binom{12}{2}(0.8)^{2}(0.2)^{10} + \binom{12}{1}(0.8)(0.2)^{11} + \binom{12}{0}(0.2)^{12}$$

Q7 Urn 1 contains 5 red and 10 blue marbles. Urn 2 contains 10 red and 5 blue marbles. One of the urns is chosen at random and a marble is chosen from that urn at random. Given that the marble is blue, what is the probability that it came from Urn 2?

$$P(\text{Urn } 2 \mid \text{blue}) = \frac{P(\text{blue} \mid \text{Urn } 2)\, P(\text{Urn } 2)}{\underbrace{\left(P(\text{blue} \mid \text{Urn } 2)\, P(\text{Urn } 2) + P(\text{blue} \mid \text{Urn } 1)\, P(\text{Urn } 1)\right)}_{P(\text{blue})}}$$

$$= \frac{5/15 \cdot 1/2}{5/15 \cdot 1/2 + \frac{10}{15} \cdot 1/2} = 1/3$$

Q8 In an exam, female students score marks in the set {71, 72, …, 100} with uniform probability. Male students score marks in the set {61, 62, …, 100} with uniform probability. Ratio of number of male to female students is 60:40. Suppose a student's mark is known to be 80, what is the probability that the student is male?

$$P(m \mid 80) = \frac{P(80 \mid m) \cdot P(M)}{\underbrace{\left(P(80 \mid m)\, P(m) + P(80 \mid F) \cdot P(F)\right)}_{P(80)}}$$

$$= \frac{1/40 \cdot 3/5}{\frac{1}{40} \cdot 3/5 + \frac{1}{30} \cdot 2/5} = \frac{9}{17}$$

Q9 (a) The number of passengers arriving at a bus terminus over a 10-minute interval is approximately Poisson with a mean of 5. What is the probability that no passenger arrives in a 10-minute interval?

$$P(\#\text{passengers} = k) = e^{-5} \cdot \frac{5^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$P(\#\text{passengers} = 0) = e^{-5}$$

(b) What is the probability that at least 3 passengers arrive in a 10-minute interval?

$$A = \#\text{passengers} \geq 3 = (\# = 3) \text{ OR } (\# = 4) \text{ OR } \ldots$$

$$A^c = \#\text{passengers} \leq 2 = (\# = 0) \text{ OR } (\# = 1) \text{ OR } (\# = 2)$$

$$= e^{-5} + e^{-5} \cdot 5 + e^{-5} \cdot \frac{5^2}{2}$$

Q10 (a) The number of customers visiting an online shopping portal to buy Item A in any interval of 1 hour is approximately Poisson with a mean of 10. If the available stock for Item A is 15 at 9am, what is the probability that Item A will go out of stock at 10am?

item A goes out of stock at 10am $= \left(\#\text{customers}_{9am-10am} \geq 15\right)$

$$P\left(\#\text{customers}_{9am-10am} \geq 15\right) = \sum_{k=15}^{\infty} e^{-10} \cdot \frac{10^k}{k!}$$

(b) The number of customers visiting an online shopping portal to buy Item A in any interval of 1 hour is approximately Poisson with a mean of 10, independent of the number arriving in any other 1 hour interval. If the available stock for Item A is 15 at 9am, what is the probability that Item A will go out of stock at 11am?

$\#\text{customers}_{9am-11am} \sim \text{Poisson}(20)$

$$P\left(\#\text{customers}_{9am-11am} \geq 15\right) = \sum_{k=15}^{\infty} e^{-20} \cdot \frac{20^k}{k!}$$

Q11 (a) The number of meteorites hitting earth's atmosphere in a month is approximately Poisson with a mean of 3. Given that at most 3 meteorites hit earth's atmosphere in a month, what is the conditional probability that no meteorite hits?

$$P(\#\text{ meteorites} = 0 \mid \#\text{ meteorites} \le 3) = \frac{P((\#\text{ meteorites} = 0) \cap (\#\text{ meteorites} \le 3))}{P(\#\text{ meteorites} \le 3)}$$

$$= \frac{P(\# = 0)}{P(\# = 0) + P(\# = 1) + P(\# = 2) + P(\# = 3)} = \frac{e^{-3}}{e^{-3} + e^{-3}\cdot 3 + e^{-3}\cdot 3^2/2 + e^{-3}\cdot 3^3/3!} = \,?$$

(b) Given that at least 3 meteorites hit earth's atmosphere in a month, what is the conditional probability that 5 meteorites hit?

$$P(\#\text{ meteorites} = 5 \mid \#\text{ meteorites} \ge 3) = \frac{P(\# = 5 \cap \# \ge 3)}{P(\# \ge 3)}$$

$$= \frac{P(\# = 5)}{P(\# = 3) + P(\# = 4) + \cdots} = \frac{P(\# = 5)}{1 - P(\# = 0) - P(\# = 1) - P(\# = 2)} = \frac{e^{-3}\cdot 3^5/5!}{1 - e^{-3} - e^{-3}\cdot 3 - e^{-3}\cdot \frac{3^2}{2}}$$

Q12 Let X be a random variable uniformly distributed in {-5,-4,...,5}. Find the PMF of $X^2 - 4$.

$P(x)$	x	$y = x^2 - 4$
$\vdots$	-5	21
	-4	12
	-3	5
	-2	0
$1/11$	-1	-3
	0	-4
	1	-3
	2	0
	3	5
	4	12
$\vdots$	5	21

$\longrightarrow$

y	$A_y = \{x : x^2 - 4 = y\}$	$P(y)$
-4	0	$1/11$
-3	$\{1, -1\}$	$2/11$
0	$\{2, -2\}$	$2/11$
5	$\{3, -3\}$	$2/11$
12	$\{4, -4\}$	$2/11$
21	$\{5, -5\}$	$2/11$

Q13 The annual salary of an employee is the sum of a base pay of Rs. 5 lakhs and a variable pay that is a random variable V. Assume that V is uniform in {10000k: k=1,2,3,4,5}. What is the distribution of the annual salary?

$P(V=x)$	x	$y = x + 500000$
1/5	10000	510000
1/5	20000	520000
1/5	30000	530000
1/5	40000	540000
1/5	50000	550000

y	A_y	$P(y)$
510000	10000	1/5
520000	20000	1/5
530000	30000	1/5
540000	40000	1/5
550000	50000	1/5

Q14 A company manufactures 10 appliances a day. An appliance fails quality check with probability 0.1 independent of all others. Every appliance that passes the quality check results in a profit of Rs. 1000, while every appliance that fails the quality check results in a loss of Rs. 5000. What is the distribution of the overall daily profit of the company?

QC fail

$$\text{Profit} = (10-x)\cdot 1000 - x\cdot 5000$$

$$= 10000 - 6000x$$

$P(\# \text{QC fail} = x)$	x	$y = 10000 - 6000x$
0.9^{10}	0	10000
$10(0.1)(0.9)^9$	1	4000
$\binom{10}{2}(0.1)^2(0.9)^8$	2	-2000
	3	-8000
⋮	⋮	
$(0.1)^{10}$	10	-50000

y	$P(y)$
10000	0.9^{10}
4000	$10(0.1)(0.9)^9$
-2000	$45(0.1)^2(0.9)^8$
-8000	$120(0.1)^3(0.9)^7$
⋮	⋮
-50000	$(0.1)^{10}$

Q15 Let a random variable X be Poisson distributed with mean 1. Let

$$g(x) = \begin{cases} |x-5|, & x = 0, 1, \ldots, 10 \\ 5, & x > 10 \end{cases}$$

What is the distribution of g(X)?

x	$y=g(x)$
0	5
1	4
2	3
3	2
4	1
5	0
6	1
7	2
8	3
9	4
10	5
11	5
$\vdots$	$\vdots$

y	$A_y = \{x : y = g(x)\}$	$P(y)$
0	5	$e^{-1}/5!$
1	$\{4, 6\}$	$e^{-1}\left(1/4! + 1/6!\right)$
2	$\{3, 7\}$	$e^{-1}\left(1/3! + 1/7!\right)$
3	$\{2, 8\}$	$e^{-1}\left(1/2! + 1/8!\right)$
4	$\{1, 9\}$	$e^{-1}\left(1 + 1/9!\right)$
5	$\{0, 10, 11, 12, \ldots\}$	$e^{-1}\left(1 + \frac{1}{10!} + \frac{1}{11!} + \cdots\right)$

Q16, Q17

x	y	P(X=x,Y=y)
1	1	0.1
1	2	0.2
1	3	0.2
2	1	0.3
2	2	0.1
2	3	0.1

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x	$P(X=x)$
1	0.5
2	0.5

y	$P(Y=y)$
1	0.4
2	0.3
3	0.3

X & Y: dependent

x	$P(X=x \mid Y=2)$
1	$2/3$
2	$1/3$

y	$P(Y=y \mid X=1)$
1	$1/5$
2	$2/5$
3	$2/5$

17

x	y	$x+y$	min	max
1	1	2	1	1
1	2	3	1	2
1	3	4	1	3
2	1	3	1	2
2	2	4	2	2
2	3	5	2	3

s	$P(X+Y=s)$
2	0.1
3	0.5
4	0.3
5	0.1

z	$P(\min=z)$
1	0.8
2	0.2

z	$P(\max=z)$
1	0.1
2	0.6
3	0.3

Q18, Q19, Q20 Suppose random variables X and Y are independent and uniformly distributed in {1,2,...,100}. Find the distribution of X + Y, max(X,Y), min(X,Y).

$X+Y \in \{2, 3, \ldots, 200\}$

$P(\underbrace{X+Y=2}_{(X=1,Y=1)}) = P(X=1, Y=1) = P(X=1)P(Y=1)$

$= 1/100 \cdot 1/100$

$P(\underbrace{X+Y=50}_{}) = 49 \cdot \frac{1}{100} \cdot \frac{1}{100}$

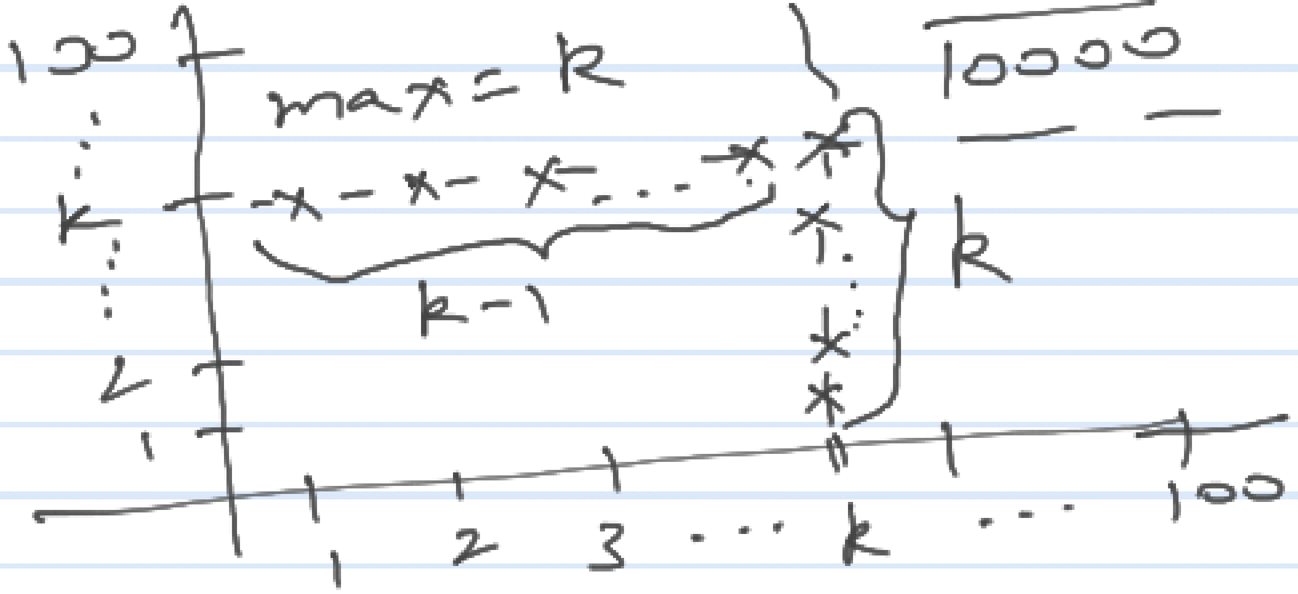
$(X=1, Y=49)$ or $(X=2, Y=48)$ (or) ... (or) $(X=49, Y=1)$

$P(\underbrace{X+Y=150}_{}) = 51 \cdot \frac{1}{100} \cdot \frac{1}{100}$

$(X=50, Y=100)$ (or) $(X=51, Y=99)$ (or) ... $(X=100, Y=50)$

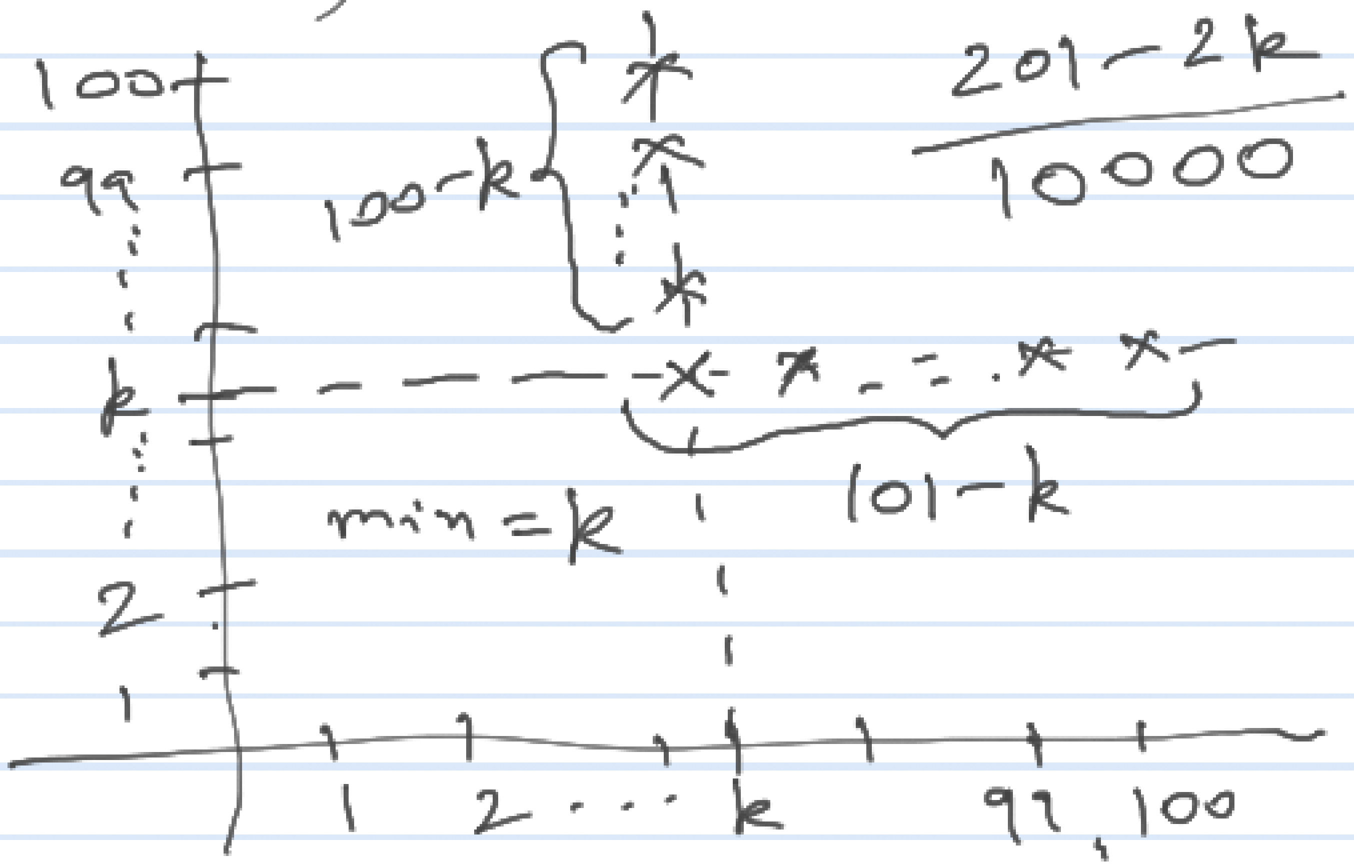
$$P(X+Y=k) = \begin{cases} \frac{k-1}{10000} & k=2, \ldots, 101 \\ \frac{201-k}{10000} & k=102, \ldots, 200 \end{cases}$$

$\max(X,Y) \in \{1, 2, \ldots, 100\}$



$\frac{2k-1}{10000}$

$\min(X,Y) \in \{1, 2, \ldots, 100\}$



$\frac{201-2k}{10000}$

Q21, Q22, Q23, Q24 Suppose random variables X and Y are independent and Geometric(1/3). Find the distribution of X + Y, max(X,Y), min(X,Y).

$X+Y \in \{2, 3, 4, \ldots\}$

$x+y$	(x,y)
2	$(1,1)$
3	$(1,2),(2,1)$
$\vdots$	
k	$\underbrace{(1,k-1),(2,k-2),\ldots,(k-1,1)}_{k-1}$
$\vdots$	

$$P[X+Y=k] = p(1-p)^{k-2}p + (1-p)p(1-p)^{k-3}p + \cdots + (1-p)^{k-2}p \cdot p$$

$$= p^2(1-p)^{k-2}(k-1)$$

$\max(X,Y) \in \{1, 2, 3, \ldots\}$

$\max(x,y)$	(x,y)
k	$(1,k),(2,k),\ldots,(k-1,k),(k,k),$
	$(k,k-1),(k,k-2),\ldots,(k,1)$

$$P(\max(X,Y)=k) = \underbrace{p(1-p)^{k-1}p}_{(1,k)} + \underbrace{(1-p)p(1-p)^{k-1}p}_{(2,k)} + \ldots$$

Anything smart for max??

$(\max(X,Y)=k) \Leftrightarrow$

$(\max(X,Y)\le k) \cap (\max(X,Y)\le k-1)^c$

$P(\max=k) = P(\max\le k) - P(\max\le k-1)$

Q25, Q26, Q27 Suppose X ~ Binomial(10,1/3) and Y ~ Binomial(15,1/3) are independent. Find the distribution of X + Y, max(X,Y).

$x+y$	(x,y)	P
0	$(0,0)$	$\left(\frac{2}{3}\right)^{10}\left(\frac{2}{3}\right)^{15}=\left(\frac{2}{3}\right)^{25}$
1	$(0,1),(1,0)$	$\underbrace{\left(\frac{2}{3}\right)^{10}\cdot 15\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{14}}_{(0,1)}+\underbrace{10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{9}\left(\frac{2}{3}\right)^{15}}_{(1,0)}=25\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{24}$
2	$(0,2),(1,1),(2,0)$	$\binom{25}{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{23}$
$\vdots$		
k	$\cdots$	$\binom{25}{k}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{25-k}$

why?

X: # of successes in 10 trials
Y: # of successes in 15 trials
$\}$ same success probability

X+Y: # of successes in 25 trials

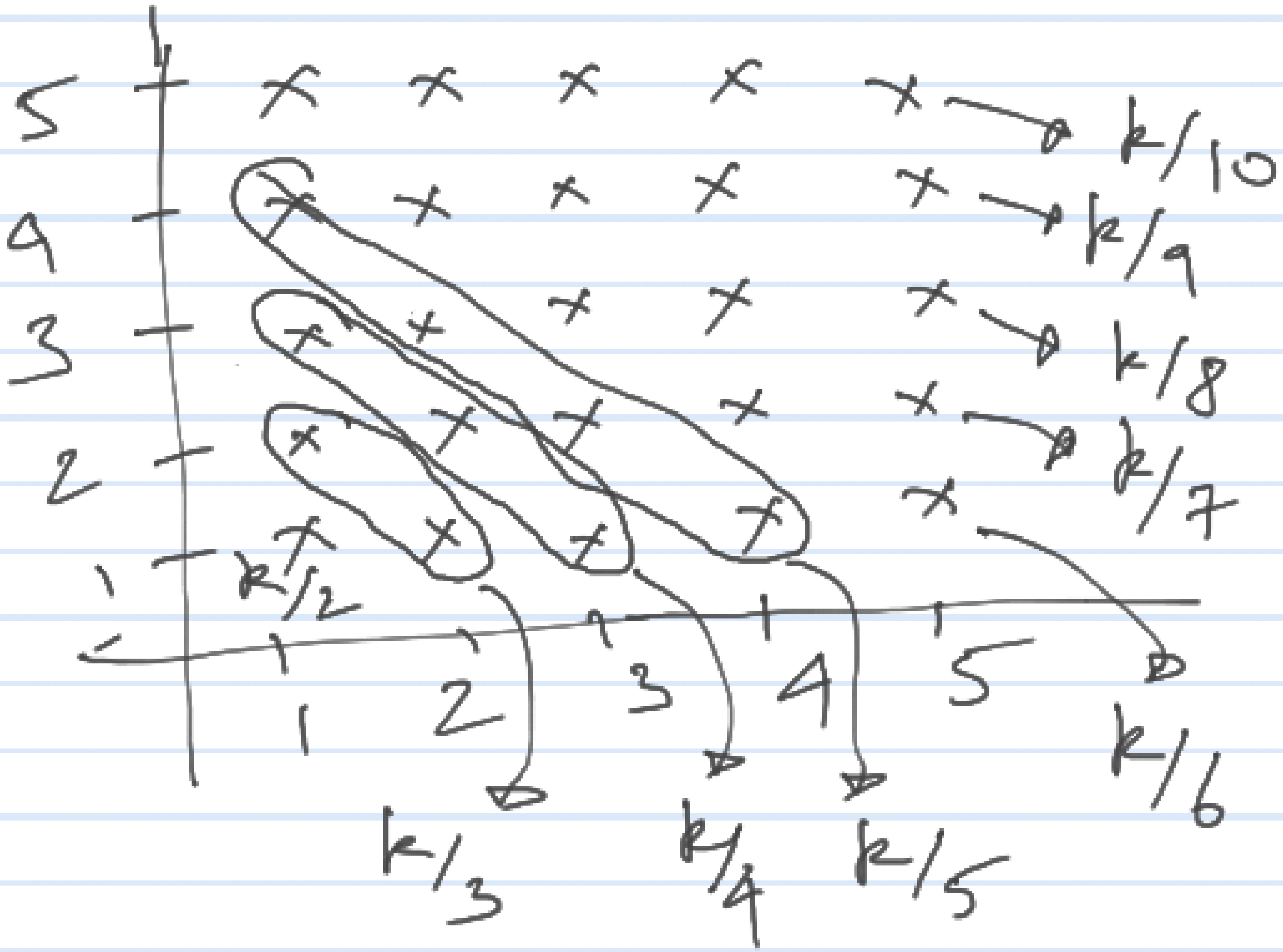
max(X,Y)

→ Complicated expression

Q28 Suppose X and Y have a joint PMF

$$P(X=x, Y=y) = \frac{k}{x+y}, x, y \in \{1, 2, \ldots, 5\}$$

What is the distribution of X+Y?



$$\frac{k}{2} + 2\cdot\frac{k}{3} + 3\cdot\frac{k}{4} + 4\cdot\frac{k}{5} + 5\cdot\frac{k}{6}$$
$$+ 4\cdot\frac{k}{7} + 3\cdot\frac{k}{8} + 2\cdot\frac{k}{9} + \frac{k}{10} = 1$$
$$k = ?$$

$x+y$	$P(x+y)$
2	$k/2$
3	$2k/3$
4	$3k/4$
5	$4k/5$
6	$5k/6$
7	$4k/7$
8	$3k/8$
9	$2k/9$
10	$k/10$

Q29 Suppose X and Y are independent and Geometric(1/3). Find P(max(X,Y) <= k), P(min(X,Y) <= k).

$$(\max(X,Y) \le k) = (X \le k \text{ AND } Y \le k) \Rightarrow P(\max \le k) = P(X \le k)\, P(Y \le k)$$
$$= (1-(1-p)^k)(1-(1-p)^k)$$

$$(\min(X,Y) > k) = (X > k \text{ AND } Y > k) \Rightarrow P(\min > k) = P(X > k) \cdot P(Y > k)$$
$$= (1-p)^k \cdot (1-p)^k$$
$$P(\min \le k) = 1 - (1-p)^{2k}$$

Q30 Suppose random variables X and Y are independent and uniformly distributed in {1,2,...,100}. What is the distribution of (X | X+Y = 50)?
What is the distribution of (X | max(X,Y) = 50)?

$$P(X=x \mid X+Y=50) = \frac{P(X=x \cap X+Y=50)}{P(X+Y=50)} = \frac{P(X=x, Y=50-x)}{P(X+Y=50)} = \frac{1/10000}{49/10000}$$
$(x = 1, 2, \dots, 49)$

$$P(X=x \mid \max(X,Y)=50) = \frac{P(X=x \cap \max(X,Y)=50)}{P(\max(X,Y)=50)} = \begin{cases} \dfrac{P(X=x, Y=50)}{P(\max(X,Y)=50)} = 1/99 & x=1,\dots,49 \\ \dfrac{P(X=50, Y\in\{1,2,\dots,50\})}{P(\max(X,Y)=50)} = \dfrac{50}{99} & x=50 \end{cases}$$
$(x = 1, 2, \dots, 50)$