Directional derivatives in terms of the gradient

Sarang S. Sane

Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a

scalar-valued multivariable function.

Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a

scalar-valued multivariable function.

If $\underset{\sim}{a}$ is a point in $\mathbb{R}^n$, then an open ball of radius r around $\underset{\sim}{a}$ is the set defined as

$$\{\underset{\sim}{x} \in \mathbb{R}^n \mid ||\underset{\sim}{x} - \underset{\sim}{a}|| < r\}.$$

Very important notation

Unless specifically mentioned otherwise, further ahead in this video a function will mean a

scalar-valued multivariable function.

If $\underset{\sim}{a}$ is a point in $\mathbb{R}^n$, then an open ball of radius r around $\underset{\sim}{a}$ is the set defined as

$$\{\underset{\sim}{x} \in \mathbb{R}^n \mid ||\underset{\sim}{x} - \underset{\sim}{a}|| < r\}.$$

$e_1, e_2, \ldots, e_n$ is the standard ordered basis of $\mathbb{R}^n$.

Recall : partial and directional derivatives

Recall : partial and directional derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$.

Recall : partial and directional derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$.

The directional derivative of f in the direction of the unit vector u is the function denoted by $f_u\left(\underset{\sim}{x}\right)$ and defined as

$$f_u\left(\underset{\sim}{x}\right) = \lim_{h \to 0} \frac{f\left(\underset{\sim}{x} + hu\right) - f\left(\underset{\sim}{x}\right)}{h} \quad .$$

Recall : partial and directional derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$.

The directional derivative of f in the direction of the unit vector u is the function denoted by $f_u\left(\underset{\sim}{x}\right)$ and defined as

$$f_u\left(\underset{\sim}{x}\right) = \lim_{h \to 0} \frac{f\left(\underset{\sim}{x} + hu\right) - f\left(\underset{\sim}{x}\right)}{h} \quad .$$

Its domain consists of those points of D at which the limit exists.

Recall : partial and directional derivatives

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$.

The directional derivative of f in the direction of the unit vector u is the function denoted by $f_u\left(\underset{\sim}{x}\right)$ and defined as

$$f_u\left(\underset{\sim}{x}\right) = \lim_{h\to 0} \frac{f\left(\underset{\sim}{x} + hu\right) - f\left(\underset{\sim}{x}\right)}{h} \quad .$$

Its domain consists of those points of D at which the limit exists.

When $u = e_i$, the directional derivative is called the partial derivative of f w.r.t. x_i and is denoted by $f_{x_i}\left(\underset{\sim}{x}\right)$ or $\dfrac{\partial f}{\partial x_i}\left(\underset{\sim}{x}\right)$.

Computation : directional derivatives vs. partial derivatives

Example : $f(x,y,z) = xy + yz + zx$

$$u = (u_1, u_2, u_3)$$

$$f_u(\underset{\sim}{x}) = \lim_{h\to 0} \frac{f(x+hu_1,\, y+hu_2,\, z+hu_3) - f(x,y,z)}{h}$$

$$= \lim_{h\to 0} \frac{(x+hu_1)(y+hu_2) + (y+hu_2)(z+hu_3) + (z+hu_3)(x+hu_1) - (xy+yz+zx)}{h}$$

$$= \lim_{h\to 0} \frac{h^2(u_1u_2 + u_2u_3 + u_3u_1) + h(xu_2 + yu_1 + yu_3 + zu_2 + zu_1 + xu_3)}{h}$$

$$= \lim_{h\to 0} \boxed{h(\quad)} + u_1(y+z) + u_2(x+z) + u_3(x+y)$$

$$= u_1(y+z) + u_2(x+z) + u_3(x+y).$$

$$f_x = y+z, \quad f_y = x+z, \quad f_z = y+x.$$

$$\nabla f(\underset{\sim}{x}) \cdot u$$

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Suppose all the partial derivatives of f at $\underset{\sim}{a}$ exist.

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Suppose all the partial derivatives of f at $\underset{\sim}{a}$ exist. Then the gradient vector of f at $\underset{\sim}{a}$ is the vector $\left(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \ldots, f_{x_n}(\underset{\sim}{a})\right)$ in $\mathbb{R}^n$.

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Suppose all the partial derivatives of f at $\underset{\sim}{a}$ exist. Then the gradient vector of f at $\underset{\sim}{a}$ is the vector $\left(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \ldots, f_{x_n}(\underset{\sim}{a})\right)$ in $\mathbb{R}^n$. It is denoted by $\nabla f(\underset{\sim}{a})$.

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Suppose all the partial derivatives of f at $\underset{\sim}{a}$ exist. Then the gradient vector of f at $\underset{\sim}{a}$ is the vector $\left(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \ldots, f_{x_n}(\underset{\sim}{a})\right)$ in $\mathbb{R}^n$. It is denoted by $\nabla f(\underset{\sim}{a})$.

The gradient function of f is the function taking values in $\mathbb{R}^n$ obtained by associating to every point $\underset{\sim}{a}$ its gradient vector $\nabla f(\underset{\sim}{a})$.

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Suppose all the partial derivatives of f at $\underset{\sim}{a}$ exist. Then the gradient vector of f at $\underset{\sim}{a}$ is the vector $\left(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \ldots, f_{x_n}(\underset{\sim}{a})\right)$ in $\mathbb{R}^n$. It is denoted by $\nabla f(\underset{\sim}{a})$.

The gradient function of f is the function taking values in $\mathbb{R}^n$ obtained by associating to every point $\underset{\sim}{a}$ its gradient vector $\nabla f(\underset{\sim}{a})$. It is denoted by $\nabla f(\underset{\sim}{x})$.

The gradient vector/function

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Suppose all the partial derivatives of f at $\underset{\sim}{a}$ exist. Then the gradient vector of f at $\underset{\sim}{a}$ is the vector $\left(f_{x_1}(\underset{\sim}{a}), f_{x_2}(\underset{\sim}{a}), \ldots, f_{x_n}(\underset{\sim}{a})\right)$ in $\mathbb{R}^n$. It is denoted by $\nabla f(\underset{\sim}{a})$.

The gradient function of f is the function taking values in $\mathbb{R}^n$ obtained by associating to every point $\underset{\sim}{a}$ its gradient vector $\nabla f(\underset{\sim}{a})$. It is denoted by $\nabla f(\underset{\sim}{x})$.

The domain of $\nabla f(\underset{\sim}{x})$ is the set of points in D where all partial derivatives exist.

Examples

1. $f(x, y) = sin(xy)$

$\frac{\partial f}{\partial x} = y\cos(xy)$, $\frac{\partial f}{\partial y} = x\cos(xy)$

$\nabla f(x,y) = (y\cos(xy), x\cos(xy))$.

$\nabla f(0,0) = (0,0)$.

$\nabla f(\pi,1) = (1\cos(\pi), \pi\cos(\pi))$
$= (-1,-\pi)$.

2. $f(x, y, z) = x^2 + y^2 + z^2$

$\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$\partial f/\partial z = 2z$

$\nabla f(x,y,z) = (2x, 2y, 2z)$.

$\nabla f(1,2,3) = (2,4,6)$.

3. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

$$\nabla f(x,y) = \begin{cases} \frac{1}{(x^2+y^2)^2}(x^3-y^2x, y^3-x^2y) & (x,y) \neq (0,0) \\ (0,0) & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^3-y^2x}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{y^3-x^2y}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Properties of gradients

Properties of gradients

Linearity :

Properties of gradients

Linearity : Let $c \in \mathbb{R}$.

$$\nabla(cf + g)\left(\underset{\sim}{x}\right) = c\nabla f\left(\underset{\sim}{x}\right) + \nabla g\left(\underset{\sim}{x}\right) \quad .$$

Properties of gradients

Linearity : Let $c \in \mathbb{R}$.

$$\nabla(cf + g)\left(\underset{\sim}{x}\right) = c\nabla f\left(\underset{\sim}{x}\right) + \nabla g\left(\underset{\sim}{x}\right) \quad .$$

The product rule

Properties of gradients

Linearity : Let $c \in \mathbb{R}$.

$$\nabla(cf+g)\left(\underset{\sim}{x}\right) = c\nabla f\left(\underset{\sim}{x}\right) + \nabla g\left(\underset{\sim}{x}\right) \quad .$$

The product rule

$$\nabla(fg)\left(\underset{\sim}{x}\right) = g\left(\underset{\sim}{x}\right)\nabla f\left(\underset{\sim}{x}\right) + f\left(\underset{\sim}{x}\right)\nabla g\left(\underset{\sim}{x}\right) \quad .$$

Properties of gradients

Linearity : Let $c \in \mathbb{R}$.

$$\nabla(cf+g)\left(\underset{\sim}{x}\right) = c\nabla f\left(\underset{\sim}{x}\right) + \nabla g\left(\underset{\sim}{x}\right) \quad .$$

The product rule

$$\nabla(fg)\left(\underset{\sim}{x}\right) = g\left(\underset{\sim}{x}\right)\nabla f\left(\underset{\sim}{x}\right) + f\left(\underset{\sim}{x}\right)\nabla g\left(\underset{\sim}{x}\right) \quad .$$

The quotient rule

Properties of gradients

Linearity : Let $c \in \mathbb{R}$.

$$\nabla(cf+g)\left(\underset{\sim}{x}\right) = c\nabla f\left(\underset{\sim}{x}\right) + \nabla g\left(\underset{\sim}{x}\right) \quad .$$

The product rule

$$\nabla(fg)\left(\underset{\sim}{x}\right) = g\left(\underset{\sim}{x}\right)\nabla f\left(\underset{\sim}{x}\right) + f\left(\underset{\sim}{x}\right)\nabla g\left(\underset{\sim}{x}\right) \quad .$$

The quotient rule

$$\nabla(f/g)\left(\underset{\sim}{x}\right) = \frac{1}{g\left(\underset{\sim}{x}\right)^2}\left(g\left(\underset{\sim}{x}\right)\nabla f\left(\underset{\sim}{x}\right) - f\left(\underset{\sim}{x}\right)\nabla g\left(\underset{\sim}{x}\right)\right) \quad .$$

Directional derivatives and gradients

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Directional derivatives and gradients

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Theorem

Suppose ∇f exists and is continuous on some open ball around the point $\underset{\sim}{a}$.

Directional derivatives and gradients

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined on a domain D in $\mathbb{R}^n$ containing some open ball around the point $\underset{\sim}{a}$.

Theorem

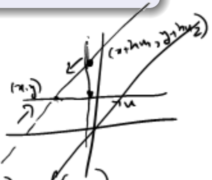
Suppose ∇f exists and is continuous on some open ball around the point $\underset{\sim}{a}$. Then for every unit vector u, the directional derivative $f_u(\underset{\sim}{a})$ exists and equals $\nabla f(\underset{\sim}{a}) \cdot u$.

$$\lim_{h \to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+hu_1, y+hu_2) - f(x+hu_1, y) + f(x+hu_1, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+hu_1, y+hu_2) - f(x+hu_1, y)}{h} + \lim_{h \to 0} \frac{f(x+hu_1, y) - f(x,y)}{hu_1} \cdot u_1$$

$$\approx \lim_{h \to 0} \frac{\partial f}{\partial y}(x+hu_1, y) \times u_2 + \frac{\partial f}{\partial x}(x,y) \cdot u_1$$



Examples

- $f(x, y) = x + y$

$$f_u(x,y) = (1,1)\cdot u$$
$$= u_1 + u_2.$$

$$\frac{\partial f}{\partial x} = 1$$
$$\frac{\partial f}{\partial y} = 1.$$
$$\nabla f = (1,1).$$

- $f(x, y, z) = xy + yz + zx$

$$f_u(x,y,z) = \nabla f \cdot u$$
$$= (y+z,\ z+x,\ x+y)\cdot(u_1, u_2, u_3)$$
$$= u_1(y+z) + u_2(z+x) + u_3(x+y).$$

- $f(x, y, z) = sin(xy)$

$$\nabla f = (y\cos(xy),\ x\cos(xy))$$

$$f_u(x,y) = \nabla f \cdot u$$
$$= (y\cos(xy),\ x\cos(xy))\cdot u$$
$$= u_1 y\cos(xy) + u_2 x\cos(xy).$$

A cautionary tale

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

$$\boxed{\nabla f(x,y)=}\begin{cases}\left(\frac{x^3-y^2x}{(x^2+y^2)^2},\ \frac{y^3-x^2y}{(x^2+y^2)^2}\right) & \text{if } (x,y)\neq(0,0)\\ \boxed{(0,0)} & \text{if } (x,y)=(0,0).\end{cases}$$

at $(0,0)$ is not continuous

$$f_u(x,y)=\nabla f(x,y)\cdot u \quad \text{if } (x,y)\neq(0,0)$$

$$=\frac{1}{(x^2+y^2)^2}\left(u_1(x^3-y^2x)+u_2(y^3-x^2y)\right).$$

$$f_u(0,0)=\lim_{h\to 0}\frac{u_1u_2}{h}=\begin{cases}0 & \text{if } u_1 \text{ or } u_2 \text{ is } 0\\ \text{DNE}\end{cases}$$

Thank you