Week-4

Mathematics for Data Science - 2

Solve with us-2

1. Choose the set of correct options.

   ◯ Option 1: If A is a square matrix of order 2 and $A^2 = I$, then $A = I$ or $A = -I$ where I is the identity matrix of order 2.

   ◯ Option 2: If A is a square matrix of order 2 and $A^2 = 0$, then $A = 0$.

   ◯ Option 3: If A and B are square matrices of order 2 and $AB = 0$, then $A = B = 0$.

   ◯ Option 4: If A is a scalar matrix of order 2, B is a non-zero square matrix of order 2 and $AB = 0$, then $A = 0$.

Solution:

Option 1:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \implies A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Option 2:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Option 3:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Option 4:

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, B = \begin{bmatrix} x & y \\ w & z \end{bmatrix} \implies AB = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x & y \\ w & z \end{bmatrix} = \begin{bmatrix} ax & ay \\ aw & az \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\implies ax = ay = aw = az = 0$

B is non zero $\implies a = 0 \implies A = 0$

2. Choose the set of correct options.

   ◯ Option 1: There exist some real matrices A and B, such that $AB = BA$.

   ◯ Option 2: There do not exist any real matrices A and B, such that $AB = BA$.

   ◯ Option 3: There does not exist any real matrix A, such that $A^2 = A$.

   ◯ Option 4: There exists some real 3×3 matrix A such that $A^2 + I = 0$

   Solution:

   Option 1:

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = BA$$

   Option 3:

   $$A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \implies A^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} (3 \times 3) + (-6 \times 1) & (3 \times -6) + (-6 \times -2) \\ (1 \times 3) + (-2 \times 1) & (1 \times -6) + (-2 \times -2) \end{bmatrix}$$
   $$= \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}.$$

   $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$ is an example of an idempotent matrix.

   Option 4:

   Suppose there is a 3×3 matrix A such that $A^2 + I = 0 \implies A^2 = -I \implies \text{Det}(A^2) = \text{Det}(-I) \implies \text{Det}(A)^2 = -1$, which is not possible.

Consider a system of linear equations:

$$\begin{aligned} -2x_1 + 3x_2 + x_3 &= 1 \\ -x_1 + x_3 &= 0 \\ 2x_2 &= 5 \end{aligned} \tag{1}$$

3. The above System has

   [Hint: Solve for x_1, x_2, and x_3.]

   - ◯ Option 1: a unique solution.
   - ◯ Option 2: no solution.
   - ◯ Option 3: infinitely many solutions.
   - ◯ Option 4: None of the above.

   Solution:

   Step 1:

   From the above system, we have:

   $$2x_2 = 5 \implies x_2 = \frac{5}{2}$$

   Substitute the value of x_2 in $-2x_1 + 3x_2 + x_3 = 1$:

   $$-2x_1 + 3 \times \frac{5}{2} + x_3 = 1 \implies -2x_1 + x_3 = 1 - \frac{15}{2} \implies -2x_1 + x_3 = \frac{-13}{2}.$$

   Step 2:

   Now, we have

   $$-x_1 + x_3 = 0 \tag{2}$$

   $$-2x_1 + x_3 = \frac{-13}{2} \tag{3}$$

   From Eq (2), $x_3 = x_1$. Now, replace x_3 with x_1 in Eq (3)

   $$-2x_1 + x_1 = \frac{-13}{2} \implies -x_1 = \frac{-13}{2} \implies x_1 = \frac{13}{2}$$

   Hence, $x_1 = x_3 = \frac{13}{2}$, and $x_2 = \frac{5}{2}$

4. Consider a system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -2x_1 + kx_2 &= d \\ 4x_1 + 6x_2 &= 12 \end{aligned}$$

Which of the following statement is wrong?

- ◯ Option 1: $Ax = b$ represents the above system, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 3 \\ -2 & k \\ 4 & 6 \end{bmatrix}$, and $b = \begin{bmatrix} 6 \\ d \\ 12 \end{bmatrix}$
- ◯ Option 2: The system has no solution if $k = -3$, $d = 0$.
- ◯ Option 3: The system has a unique solution if $k = 3$, $d = 0$.
- ◯ Option 4: The system has infinitely many solutions if $k = -3$, $d = 6$.

Solution:

Option : 2

If $k = -3$, $d = 0$, then

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -2x_1 - 3x_2 &= 0 \\ 4x_1 + 6x_2 &= 12 \end{aligned}$$

$\implies -2x_1 - 3x_2 = 0 \implies 2x_1 + 3x_2 = 0 = 6$, which is not possible.

Option 3:

If $k = 3$, $d = 0$, then

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \qquad (4) \\ -2x_1 + 3x_2 &= 0 \qquad (5) \\ 4x_1 + 6x_2 &= 12 \qquad (6) \end{aligned}$$

From Eq (5), $-2x_1 + 3x_2 = 0 \implies 3x_2 = 2x_1$.
Now, replace $2x_1$ with $3x_2$ in Eq (4):

$$3x_2 + 3x_2 = 6 \implies 6x_2 = 6 \implies x_2 = 1.$$

Hence, $x_1 = \frac{3}{2}, x_2 = 1$.

Option 4:

If $k = -3$, $d = 6$, then

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -2x_1 - 3x_2 &= 6 \\ 4x_1 + 6x_2 &= 12 \end{aligned}$$

$\implies -2x_1 - 3x_2 = 6 \implies 2x_1 + 3x_2 = -6 = 6$, which is not possible.

5. Let v be a solution of the systems of linear equations $A_1x = b$ and $A_2x = b$. Which of the following options are correct ?

   - ◯ Option 1: v is a solution of the system of linear equations $(A_1 + A_2)x = b$.
   - ◯ Option 2: v is a solution of the system of linear equations $(A_1 + A_2)x = -b$.
   - ◯ Option 3: v is a solution of the system of linear equations $(A_1 - A_2)x = 0$.
   - ◯ Option 4: v is a solution of the system of linear equations $(A_1 - A_2)x = b$.

   Solution:

   v is a solution of the systems of linear equations $A_1x = b$ and $A_2x = b$

   $$\implies A_1v = b \text{ and } A_2v = b$$

   - $(A_1 + A_2)v = A_1v + A_2v = b + b = 2b$.
   - $(A_1 - A_2)v = A_1v - A_2v = b - b = 0$.

6. If all the elements of a 3×3 real matrix A are the same, then which of the following is (are) correct?

   - ◯ Option 1: Determinant of matrix A cannot be determined from the given information.
   - ◯ Option 2: Determinant of matrix A will be the sum of the elements of a row.
   - ◯ Option 3: Determinant of matrix $A + A^T$ is 0, where A^T denotes the transpose of A.
   - ◯ Option 4: Determinant of matrix $A + A^T$ cannot be determined from the given information, where A^T denotes the transpose of A.

Solution:

- $A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$

  $det(A) = a \times det\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}\right) - a \times det\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}\right) + a \times det\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}\right) = 0$

- If $A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$, then $A^T = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$.

  $$A + A^T = \begin{bmatrix} 2a & 2a & 2a \\ 2a & 2a & 2a \\ 2a & 2a & 2a \end{bmatrix} \implies \det(A + A^T) = 0.$$

7. Which of the following option is wrong?

   - ◯ Option 1: If two rows are the same in a 3×3 real matrix, then the determinant of that matrix is zero.
   - ◯ Option 2: If two columns are the same in a 3×3 real matrix, then the determinant of that matrix is zero.
   - ◯ Option 3: If one row is a non-zero multiple of another row in a 3×3 real matrix, then the determinant of that matrix is not zero.
   - ◯ Option 4: If one row is a non-zero multiple of another row in a 3×3 real matrix, then the determinant of that matrix is zero.

   solution:

   Option 1::

$$A = \begin{bmatrix} x & y & z \\ a & a & a \\ a & a & a \end{bmatrix}$$

$$det(A) = x \times det\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}\right) - y \times det\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}\right) + z \times det\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}\right)$$

$$= x \times 0 + y \times 0 + z \times 0 = 0.$$

   Option 2::

$$A = \begin{bmatrix} x & a & a \\ y & a & a \\ z & a & a \end{bmatrix} \implies A^T = \begin{bmatrix} x & y & z \\ a & a & a \\ a & a & a \end{bmatrix}$$

   From the previous argument $\det(A^T) = 0 = \det(A)$.

   Option 3::

$$A = \begin{bmatrix} x & y & z \\ a & b & c \\ ka & kb & kc \end{bmatrix}$$

$$det(A) = x \times det\left(\begin{bmatrix} b & c \\ kb & kc \end{bmatrix}\right) - y \times det\left(\begin{bmatrix} a & c \\ ka & kc \end{bmatrix}\right) + z \times det\left(\begin{bmatrix} a & b \\ ka & kb \end{bmatrix}\right)$$

$$= x \times 0 + y \times 0 + z \times 0 = 0.$$

8. Which of the following option is wrong?

   - ◯ Option 1: If both A and B are 2×2 real matrices and $det(AB) = 0$, then $det(A) = 0$ or $det(B) = 0$.
   - ◯ Option 2: If A is a 2×2 real matrix with non-zero determinant and k is some real number, then $det(k\ A) = k^2 \times det(A)$.
   - ◯ Option 3: A triangular 3×3 matrix has non-zero determinant if and only if all the diagonal entries are non-zero.
   - ◯ Option 4: If A and B are 3×3 matrices then $det(A + B) = det(A) + det(B)$.

   Solution:

   Option 1::

   $det(AB) = det(A)det(B) = 0 \implies det(A) = 0$ or $det(B) = 0$.

   Option 2::

   If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$.

   $\det\ (kA) = k^2ad - k^2bc = k^2\ \det(A)$

   Option 3::

   $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ or $A = \begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}$

   $\det(A) = adf$.

   Option 4::

   $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$