IIT Madras

ONLINE DEGREE

Mathematics for Data Science - 2

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What is the rank/dimension for a vector space?

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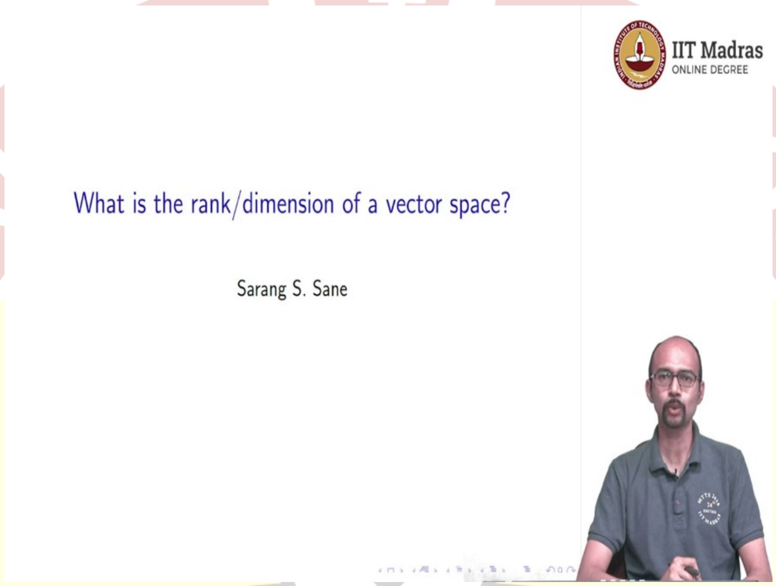

Hello, and welcome to the Maths 2 component of the online B.Sc. degree on data science. In this video we are going to talk about what is the rank or the dimension of a vector space.

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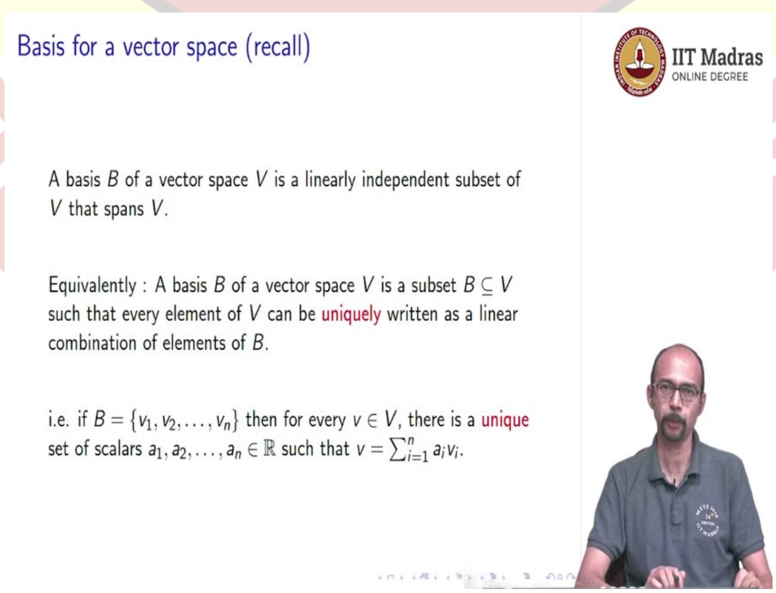

So, let us recall before, we have defined what is called the basis for a vector space. What is the basis? A basis for vector space is a linearly independent subset which spans that vector space. What does that mean? That means, we can write every vector uniquely as a linear combination of vectors from this set B. So, just to make that last remark clear, if V is v_1, v_2, ... v_n then for every v in V there is a unique set of scalars, again, the emphasis on unique, a_1, a_2, a_n, so that V is summation $\sum_{i=1}^{n} a_i v_i$.

So, spanning says that there is a set of vectors and linear independence says that that set of vectors is, sorry, there is a set of scalars, so that V is summation aivi and linear independence says that that set of scalars must be unique. No other scalars will give you this vector V, only a1, a2, an, this is the only possible choice which gives you V. That is, if set B satisfies this property, then it is called a basis. And if it is a basis, it satisfies this property.

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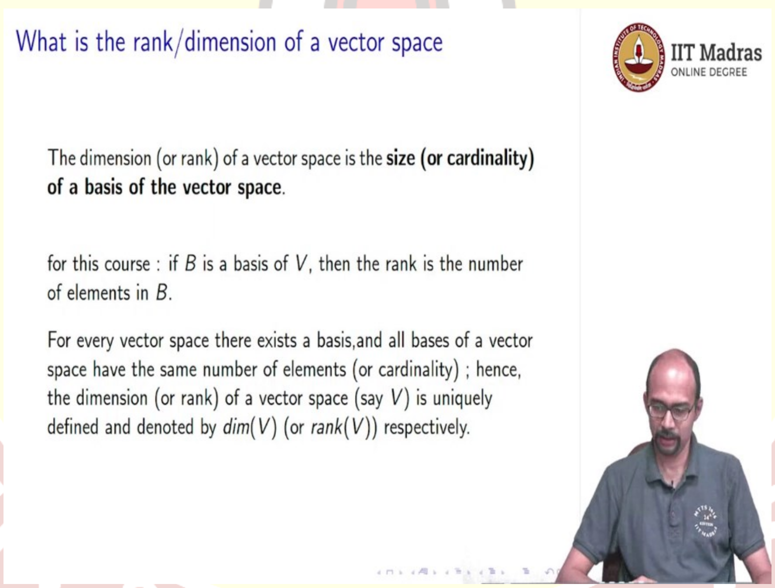

So, what is the rank or the dimension of a vector space? It is the size of a basis or the cardinality of a basis. So, for this course, what that means is, it is the number of elements in a basis B. What is the difference in these two statements; even in the previous slide I had a slight difference in how I put it down. So, the point is in general your vectors could have an infinite number of, your basis size could be in finite and that is why we have to talk about cardinality and size and so on.

So, for the, for those who want to do more serious or rigorous mathematics, we have to bother about such things. But for this course, we want to use all these ideas to analyze data. And of course, we, there is no notion of an infinite amount of data. It can be very large, still finite. So, we will be content with sizes which are finite. So, what this means is, if B is the basis of vector space V, you just count the number of elements in B.

For example, in the standard basis, if you have $\mathbb{R}^n$, we have the standard basis e1, e2, en, the number of elements in that basis is n and that is exactly the dimension or the rank of the vector space $\mathbb{R}^n$. So, first of all, we have to ask various questions for, because we have made this definition. Does every vector space have a basis? Indeed, that is true. Again, as I said, we will restrict ourselves to easy examples where they are subspaces of $\mathbb{R}^n$.

And we, for such things, we can go about doing it in the way we did in the previous video. Namely, we can start with the empty set and keep appending vectors so that the set that you have at every stage remains linearly independent. And at some stage, we will get a maximal linearly independent set and that will be your basis. So, that happens in a finite number of steps. So, the number of steps tells you what is the number of basis vectors. So, indeed, every vector space has a basis. And indeed, we can count it using that method that we had in the previous video.

Now, of course, you might say that you can use that method, but we had various choices of applying that method. We saw various different basis for a single vector space. How do we know that the different basis have the same size, that is a fact. This is a fact that we are not going to prove. We may indicate a proof at some point, but we will not give a formal proof.

So, the point we are trying to make here is every vector space has a basis, every basis has the same number of elements in it and that number is called the dimension or the rank of that vector space. So, the dimension of vector space is uniquely defined, it is, meaning it is well defined, because every basis has the same number of elements and basis do exist. So, it is denoted by dim of V or rank of V.

So, the reason for the slight haziness in nomenclature, meaning we have two different names, is because when you do higher algebra dimension starts having a different meaning. But for this course, we are going to stick to dimension of V. However, the notion of rank will come and that will come for a matrix at the end of this video. And to reconcile these, it is a good idea to remember the dimension of is the same as rank of V. They are the same, they have the same meaning.

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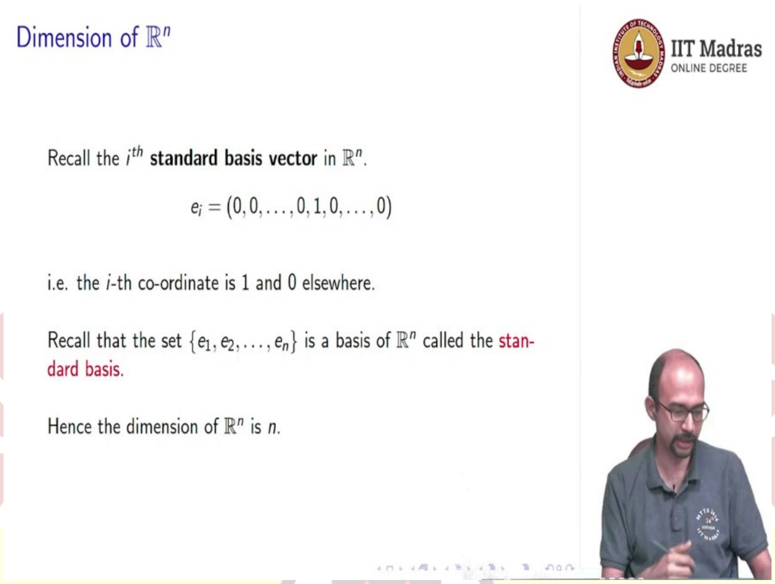

With that caveat, let us go ahead. So, let us recall the standard basis vector in $\mathbb{R}^n$. E_i is 0, 0, 0 up till the ith vector, ith coordinate which is 1 and then again a bunch of 0s in the other coordinates. So, the set e_1, e_2, ... e_n was called the standard basis and that means that the dimension of $\mathbb{R}^n$ is n. So, we noted this idea in some previous videos. We said the largest, if you have $n + 1$ vectors in $\mathbb{R}^n$, they cannot be linearly independent.

They have to be linearly dependent. So, n seemed like the optimal size. This was what we talked about in our previous video that having linear independence and spanning means that in some sense your, the size has to meet in some place that is exactly the dimension. So, here the dimension is n, for $\mathbb{R}^n$ the dimension is n.

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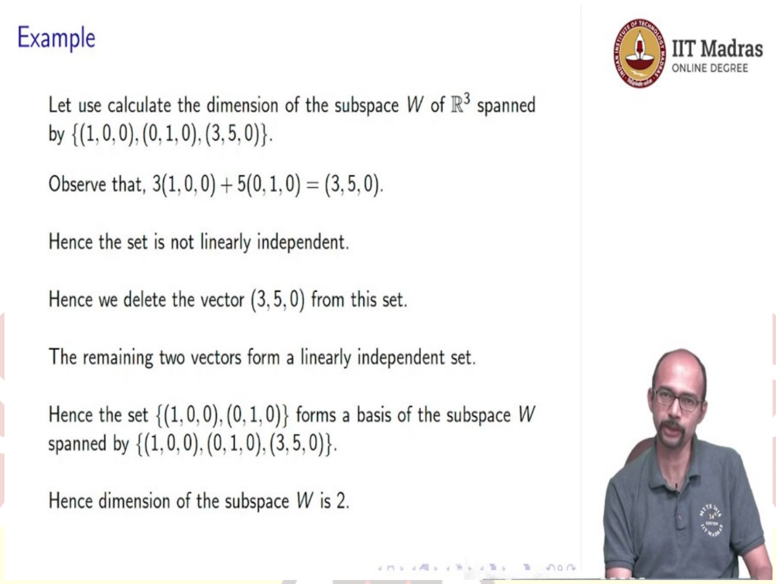

Let us calculate the dimension of the subspace W of $\mathbb{R}^3$ spanned by (1, 0, 0), (0, 1, 0) and (3, 5, 0). So, observe that the third vector 3, 5, 0 is a linear combination of the first two vectors. So, this set is not linearly independent, but we can do the following. So, this was a method that we discussed in the previous video. If you have a spanning set, how to get a basis from that spanning set, namely do remove or delete vectors which are linear combinations of the other vectors.

So, we delete the vector (3, 5, 0) from the set, the remaining two vectors (1, 0, 0) and (0, 1, 0) do form a linearly independent set. You can check this. It is a subset of the standard basis. So, remember that if you have a set of vectors which are linearly independent, then every subset is also linearly independent. Something you have either already done in your tutorials or we will do soon.

So, the set 1, 0, 0, and 0, 1, 0 we know it is a spanning set, because we obtained it by deleting vector 3, 5, 0. I will warn you it is a spanning set for the subspace W not for $\mathbb{R}^3$. We are not saying it is a spanning set for $\mathbb{R}^3$. We are saying it is a spanning set for the subspace W. So, it is a, so and we know it is linearly independent, the set (1, 0, 0) and (0, 1, 0). That means it is linearly independent + spanning for W. That means it forms a basis for W.

And what is W? W is spanned by these three vectors. That is what we started with. So, what we are really saying is that, well, we have a spanning set for W. We started with these three vectors, but we actually found that a basis for W which is (1, 0, 0) and 0, 1, 0 and now we know exactly what W is that means. W is exactly the XY plane. W is points on the XY plane. So, in particular, this also tells us that the dimension of W is 2. Why is it 2, because the basis of W has size 2. What is the basis, (1, 0, 0) and (0, 1, 0).

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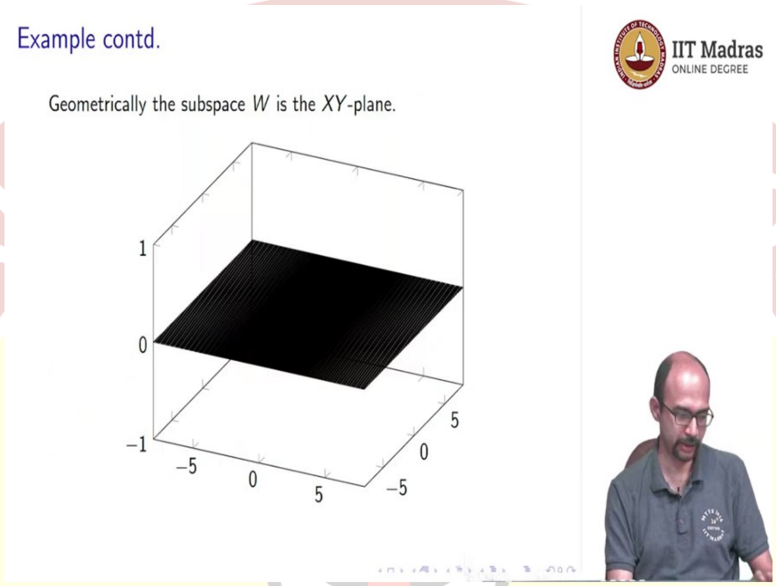

So, as I noted, this is the XY plane.

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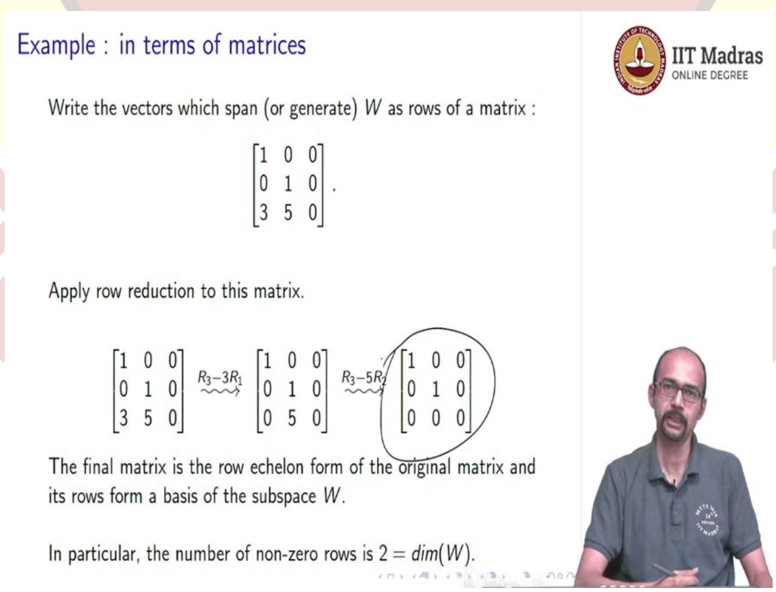

Let us do this same example in terms of matrices. And this is important because we are going to use these ideas in the next video and we are going to demonstrate explicitly how to find basis and how to find the dimension. So, write the vectors which span W as rows of a matrix. So, in this case that matrix is going to be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}$. Apply row reduction to this matrix. This is something we have studied in the last week or maybe two weeks ago.

So, if we do that, this is a very easy row reduction. So, the, it is written here, but I will suggest you quickly check this. So, you subtract 3 times the first row from the third row and you subtract 5 times the second row from the third row. And what do you get is the last matrix which is in a reduced row echelon form. So, we get the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So, this is in row echelon form, actually reduced row echelon form. And its rows form a basis of the subspace W.

So, what are the rows? The rows are 1, 0, 0 and 0, 1, 0, which is exactly what we saw was a basis for W. So, in particular, what that means is, if you want to know only what is the dimension, you do not care about what is the basis, but just the dimension, we could have read it off as the number of non-zero rows in this last matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. The number of non-zero rows in this matrix is 2 and that is a dimension of W. So, this is just an example. But this is a fact that we will use later that this always happens.

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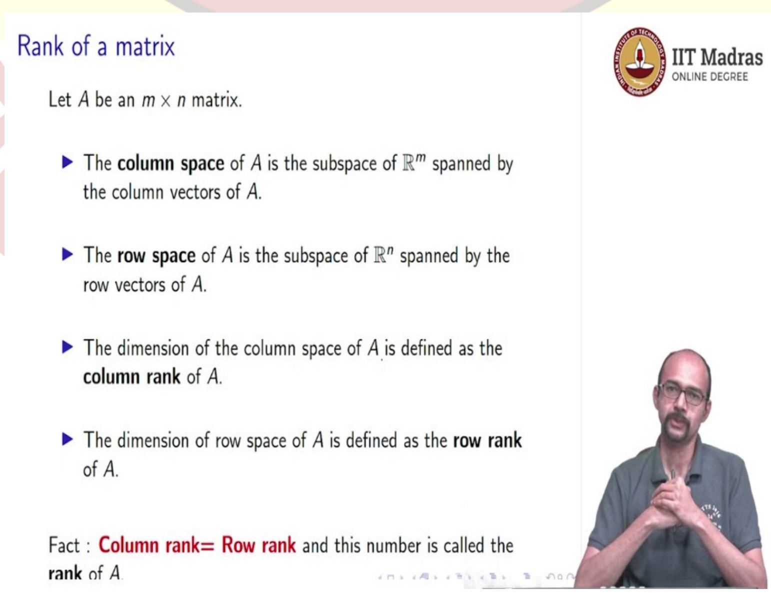

Let us use this idea. I mean, so somehow this is related to matrices. So, let us define rank of a matrix. This is one of the reasons as I mentioned earlier that there are these two, I specifically talked about rank of V or dimension of V, which are both the same things, but remember that both are the same because we are only going to use that to define what is the rank of a matrix.

Let A be an m by n matrix. The column space of A is the subspace of $\mathbb{R}^n$. So, remember that the columns have m components. So, they are subspaces of, sorry, sub, they are elements of $\mathbb{R}^m$, they are vectors in $\mathbb{R}^m$. So, the column space of A is a subspace of $\mathbb{R}^m$ generated by the column vectors of A. Generated means spanned by the column vectors of A. So, you take all the column vectors and take span of that set that is the column space of A.

Similarly, we can talk about the row space. This is a subspace of $\mathbb{R}^n$. So, you take the rows of A and you take the span of that that is the row space. So, the dimension of the column space of A is defined as the column rank of A. The dimension of the row space of A is defined as the row rank of A. And here is a fact which we will not prove very important fact.

And this is at the heart of that previous fact also that I stated, which I did not prove that two different basis have the same size. So, the fact is that the column rank is the same as the row rank. That previous fact we can prove directly, but this is another way of proving it. So, column rank is the same as row rank. And we call this number as the rank of A.

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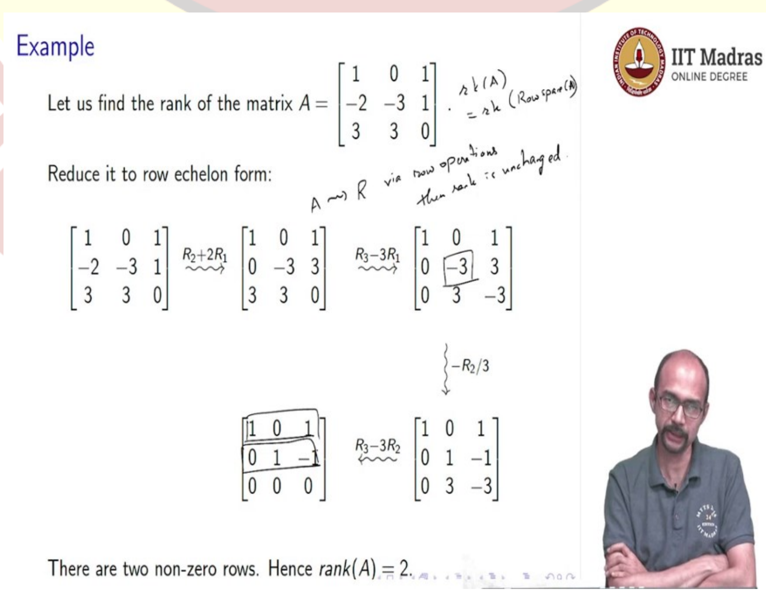

So, let us find the rank of A. This is an example, not only justify why this is used, that will come later. So, what we do is we take this matrix and we do again the same thing that we did before. You reduce it to row echelon form. So, I am, this is rather, these are the kinds of computations we have done before. So, I will run through this quickly.

So, this is $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{bmatrix}$. So, we want to make the first column, there is a 1 already in the one 1th position, so make everything else in the first column as 0, that means we add 2 times the first row to the second row and we add 3 times, sorry, subtract 3 times the first row from the third row, that gives us the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$.

So, now let us get the two 2th element, make it what we call a pivot. So, this - 3 we want to make 1. So, we divide R2 by - 3. So, if you do that, you get the row 0, 1, - 1 in place of the, as the second row in place of what we have as the second row. And then use the 1 in the two 2th position to sweep out the second column below that 1.

And if we do that, actually the third column becomes 0. So, you do subtract 3 times the second row from the third row and that gives you the final matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ 1, 0, 1, 0, 1, - 1, 0, 0, 0. And this matrix has two non-zero rows, and therefore, rank of this matrix is 2. So, the point here is, I mean, what did we use in order to prove this? Remember that our definition was that the rank of the matrix A, so rank of A is the rank of the row space of A or column space of A.

But what we are saying is that if you reduce it to, reduce row echelon form or row echelon form, so if you change A to R via row operations, then the rank remains unaffected. So, the rank does not change under elementary operations. This is really the main point. And hence, what I can do is I can put the matrix into, a matrix in reduced row echelon form or row echelon form and then I can just look at how many rows are non-zero, because here what I really have is that these two rows 1, 0, 1 and 0, 1, - 1, they span the row space of A.

And because of the special nature of these roles that they form a matrix which is in reduced row echelon form, they are also linearly independent, which means it is a basis. So, that is why rank of A is 2.

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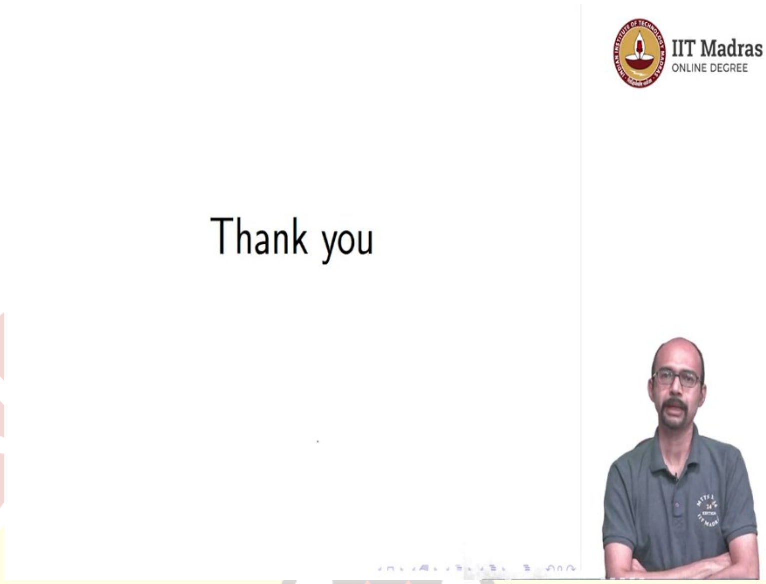

So, let us recall quickly what we have seen in this video. So, in this video, we have talked about the dimension or the rank of a vector space. So, what is the dimension of the rank of vector space it is the number of elements in the basis. Why does this statement even make sense? It makes sense because every vector space has a basis and all possible basis have the same size. These are statements we have to prove.

They are not entirely trivial, not hard either, but they need some proof. And maybe this is not, we would not be studying that in this course. But these are important statements. So, now that we know what is the dimension we can go ahead and ask how do we compute it. So, one way of computing it is to find a basis.

How do we find a basis? Well, in the previous video we have seen one method is to keep choosing linearly independent vectors, appending linearly independent vectors to your set and, well, we have to be careful, when I say linearly independent vectors, that means you have a set which is linearly independent and you append a vector which maintains the linear independence of that set. And you keep doing this until this is no longer possible.

That was one way of finding a basis. And then you ask how many, what is the size of such a basis. The other way is to have a spanning set and go the other way, keep throwing out elements, deleting. But right now we saw a slightly simpler method which was that if you have a spanning set, you put that into a matrix form, you take the reduced row echelon form of that matrix or even just the row echelon form.

And then what is, whatever is the number of non-zero rows, that is the dimension of your vector space V. Now, the idea here is sort of dependent on matrices. So, we can talk about the rank of a matrix, which we defined as the column, the dimension of the column space or equally as a dimension of the row space because both of them match. That is again a theorem which we have not proved.

But which I suggest you can prove by yourself, maybe a bit hard, but you can at least check on some examples, check the row rank and the column rank and so see that they match. And how do I get row rank or column rank, again it is by reducing to the echelon form. In the next video, we are going to take this up more systematically and do many more examples of computing dimensions and basis. Thank you.