Week-3
Mathematics for Data Science - 2
Activity Slides

Second derivative test: Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then,

- $x = c$ is a local maximum if $f'(c) = 0$ and $f''(c) < 0$.
- $x = c$ is a local minimum if $f'(c) = 0$ and $f''(c) > 0$.
- The test fails if $f'(c) = 0$ and $f''(c) = 0$.
- A saddle point or inflection point is a critical point which is not a local maximum or a local minimum.

1. Let $f(x)$ be defined as $f(x) = \frac{x^4}{2} - \frac{13x^3}{3} + 11x^2 - 8x$. Which of the following options is true?

   - **Statement 1:** The number of critical points are 3.
   - **Statement 2:** $x = 1$ is a critical point.
   - **Statement 3:** $x = 2$ and $x = 4$ both are critical points.
   - **Statement 4:** $x = \frac{1}{2}$ is a critical points.

     ◯ Option 1: All the statements are true.

     ◯ Option 2: Statements 1 and 2 are true.

     ◯ Option 3: Statements 1,3 and 4 are true.

     ◯ Option 4: Only Statement 3 is true.

Solution:

- **Step 1:** $f'(x) = 2x^3 - 13x^2 + 22x - 8$
- **Step 2:** Try to solve the equation $f'(x) = 0$, i.e., $2x^3 - 13x^2 + 22x - 8 = 0$.
- **Step 3:** $2x^3 - 13x^2 + 22x - 8 = (x - 2)(x - 4)(2x - 1)$

Option 3 is correct.

2. Let $f(x)$ be defined as $f(x) = \frac{x^4}{2} - \frac{13x^3}{3} + 11x^2 - 8x$. Which of the following options is true?

   - **Statement 1:** $x = 2$ is a local maxima.
   - **Statement 2:** $x = 4$ is a local maxima.
   - **Statement 3:** $x = \frac{1}{2}$ is a local minima.
   - **Statement 4:** $x = 4$ and $x = \frac{1}{2}$ are local minima.

     ○ Option 1: Only Statement 1 is true.

     ○ Option 2: Statements 2 and 3 are true.

     ○ Option 3: Only Statement 4 is true.

     ○ Option 4: Statements 1,3 and 4 are true.

Solution:

- **Step 1:** $f''(x) = 6x^2 - 26x + 22$
- **Step 2:** $f''(2) = -6 < 0$, $f''(4) = 14 > 0$, and $f''(\frac{1}{2}) = \frac{21}{2} > 0$.

Option 4 is correct.

3. Consider the function defined as follows:

$$f(x) = \begin{cases} -x^2 + 2x + 3 & \text{if } 0 \leq x \leq 50 \\ x^3 + 3 & \text{if } -50 \leq x < 0. \end{cases}$$

Which of the following options are correct?

- **Statement 1:** 1 is a local maximum.
- **Statement 2:** -50 is the global minimum.
- **Statement 3:** 0 is the global maximum.
- **Statement 4:** 50 is the global minimum.

  ◯ Option 1: Only Statements 1 and 2 are true.

  ◯ Option 2: Only Statements 3 and 4 are true.

  ◯ Option 3: Only Statement 1 is true.

  ◯ Option 4: None of the statements is true.

Solution:

- **Step 1:** If $0 < x \leq 50$, then $f'(x) = -2x + 2$ and $f''(x) = -2$. It is clear that $x = 1$ will be the solution of $f'(x) = -2x + 2 = 0$ and $f''(1) < 0$. Hence $x = 1$ is a local maximum.
- **Step 2:** If $-50 \leq x < 0$, then $f'(x) = 3x^2$. Hence $f'(x)$ is nonzero in the given interval. So there is no critical points excepts the boundary points.
- **Step 3:** As in the interval $-50 \leq x < 0$, $f'(x) > 0$, f is an increasing function in this interval. Hence it attains a minimum at $x = -50$, and $f(-50) = (-50)^3 + 3$
- **Step 4:** In the interval $0 \leq x \leq 1$, f is an increasing function, as in that interval $f'(x) = -2x + 2 \geq 0$, and in the interval $1 \leq x \leq 50$, the function f is a decreasing function, as in that interval $f'(x) = -2x + 2 \leq 0$.
- **Step 5:** In the boundary points $x = 0$ and $x = 50$, the function $-x^2 + 2x + 3$ attains the minimum in the interval $0 \leq x \leq 50$, and $f(0) = 3$ and $f(50) = -(50)^2 + 2(50) + 3$.
- **Step 6:** Moreover, $f(-50) < f(50)$. Hence -50 is the global minimum.

Option 1 is correct.

4. Let $f(x) = \sin x + \cos x + 5$. Which of the following is the maximum value of the function in the interval $[0, \pi]$?

- ○ Option 1: $\sqrt{2}$.
- ○ Option 2: 5.
- ○ Option 3: $\sqrt{2} + 5$.
- ○ Option 4: 2.

Solution:

- **Step 1:** The critical points (interior) of $f(x)$ are the solutions of the equation $f'(x) = \cos x - \sin x = 0$.
- **Step 2:** $x = \frac{\pi}{4}$ is the only solution of the above equation in the interval $[0, \pi]$. Moreover, $f''(x) = -\sin x - \cos x$, and $f''(\frac{\pi}{4}) < 0$. Hence $x = \frac{\pi}{4}$ is the maximum.
- **Step 3:** $f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 5 = \sqrt{2} + 5$.
- **Step 4:** We also have to check for the boundary points. $f(0) = 6$, and $f(\pi) = 4$.
- **Step 5:** $4 < 6 < \sqrt{2} + 5$. Hence, $\sqrt{2} + 5$ is the maximum value of the function in the interval $[0, \pi]$.

Option 3 is correct.

Estimation of the area of the region bounded by the graph of function $\boldsymbol{f(x)}$, above the interval $[\boldsymbol{a}, \boldsymbol{b}]$ using Riemann Sum by taking $\boldsymbol{n}$ subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$ of equal length.:

- **Step 1:** So the partition of $[a, b]$ is

$$\left\{a, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{(n-1)(b-a)}{n}, b\right\}$$

- **Step 2:**
  - Estimated area by taking left end points of the subintervals for the height of rectangles is,

$$\lim_{n\to\infty}\left[f(a)\frac{1}{n} + f\left(a + \frac{(b-a)}{n}\right)\frac{1}{n} + \ldots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\frac{1}{n}\right]$$

  - Estimated area by taking right end points of the subintervals for the height of rectangles is,

$$\lim_{n\to\infty}\left[f\left(a + \frac{(b-a)}{n}\right)\frac{1}{n} + \ldots + f\left(a + \frac{(n-1)(b-a)}{n}\right)\frac{1}{n} + f(b)\frac{1}{n}\right]$$

5. Choose the set of correct options about estimating the area of the region bounded by the graph of function $f(x) = 2x + 3$, above the interval [0,3] using Riemann sums.

   - **Statement 1:** Estimated area will be 21 sq unit, by taking 3 subintervals of equal length and the right end points of the subintervals for the height of the rectangles.
   - **Statement 2:** Estimated area will be 12 sq unit, by taking 3 subintervals of equal length and the left end points of the subintervals for the height of the rectangles.
   - **Statement 3:** Estimated area will be 18 sq unit, by taking 3 subintervals of equal length and the mid points points of the subintervals for the height of the rectangles.
   - **Statement 4:** Estimated area will be 18 sq unit, by taking n subintervals of equal length and the right end points of the subintervals for the height of the rectangles, as n tends to ∞.

     ◯ Option 1: All the Statements are correct.

     ◯ Option 2: Only Statement 4 is correct.

     ◯ Option 3: Only Statements 1, 3, and 4 are correct.

     ◯ Option 4: Only Statements 1 and 3 are correct.

Solution: If we divide $[0,3]$ in 3 different sub-intervals of equal length, we get the partition: $\{0,1,2,3\}$.

- The estimated area by taking the right end points of the subintervals for the height of the rectangles is:
  $(1-0)f(1)+(2-1)f(2)+(3-2)f(3) = 1f(1)+1f(2)+1f(3) = 5+7+9 = 21$ sq. units.
- The estimated area by taking the left end points of the subintervals for the height of the rectangles is:
  $(1-0)f(0)+(2-1)f(1)+(3-2)f(2) = 1f(0)+1f(1)+1f(2) = 3+5+7 = 15$ sq. units.
- The estimated area by taking the mid points of the subintervals for the height of the rectangles is:
  $(1-0)f(\frac{1}{2})+(2-1)f(\frac{3}{2})+(3-2)f(\frac{5}{2}) = 1f(\frac{1}{2})+1f(\frac{3}{2})+1f(\frac{5}{2}) = 4+6+8 = 18$ sq. units.

If $[0, 3]$ is divided in n subintervals of equal length, then we get the partition:

$$\left\{0, \frac{3}{n}, \frac{6}{n}, \frac{3(n-1)}{n}, \frac{3n}{n}\right\}$$

The estimated area by taking the right end points of the subintervals for the height of the rectangles is:

$$\frac{3}{n}f\left(\frac{3}{n}\right) + \frac{3}{n}f\left(\frac{6}{n}\right) + \ldots + \frac{3}{n}f\left(\frac{3(n-1)}{n}\right) + \frac{3}{n}f\left(\frac{3n}{n}\right)$$

$$= \frac{3}{n}\left(2\left(\frac{3}{n}\right)1 + 3 + 2\left(\frac{3}{n}\right)2 + 3 + \ldots + 2\left(\frac{3}{n}\right)(n-1) + 3 + 2\left(\frac{3}{n}\right)n + 3\right)$$

$$= \frac{3}{n} \times 2\left(\frac{3}{n}\right)(1 + 2 + \ldots + n) + \frac{3}{n} \times 3n$$

$$= \frac{3^2}{n^2} \times 2\left(\frac{n(n+1)}{2}\right) + 9$$

$$= 9 \times 1\left(1 + \frac{1}{n}\right) + 9$$

As $n \to \infty$, this sum converges to $9 + 9 = 18$.

Hence the estimated area will be 18 sq unit, by taking n subintervals of equal length and the right end points of the subintervals for the height of the rectangles, as n tends to ∞.

Option 3 is correct.

Note: Observe that this value we obtained above is same as

$$\int_0^3 (2x + 3)\, dx$$

Anti-derivative and Integration: If $f(x)$ is a continuous function and $F(x)$ is its anti-derivative, i.e., $F'(x) = f(x)$, then,

$$\int f(x)\ dx = F(x) + c$$

.

6. Let $f(x) = x^3 - x^2 + x$. Let $F(x)$ be the anti-derivative of $f(x)$ such that $F(2) = 6$. Then $F(x)$ equals

   - ◯ Option 1: $\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + 8$
   - ◯ Option 2: $\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} + \frac{8}{3}$
   - ◯ Option 3: $\frac{x^3}{3} - \frac{x^2}{2} + x + \frac{8}{3}$
   - ◯ Option 4: $\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + \frac{8}{3}$

Solution:

$$\int f(x)\ dx = \int (x^3 - x^2 + x) = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$$

where C is the constant of integration.

Moreover it is given that $F(2) = 6$. As $F(x)$ is anti-derivative of $f(x)$, we must have $F(x) = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + c$ for some constant c.

Hence, $c = \frac{8}{3}$.

Option 4 is correct.

7. If $f(x) = x^2 + 1$ then find the area under the curve represented by $f(x)$, which lie above to X- axis in the interval [-1,2].

   - ◯ Option 1: 3
   - ◯ Option 2: 6
   - ◯ Option 3: 9
   - ◯ Option 4: 1

Solution: The area (A) under the curve represented by $f(x)$, which lie above to X-axis in the interval [-1,2]

$$\int_{-1}^{2} f(x)\ dx = \int_{-1}^{2} (x^2+1)\ dx = \left(\frac{x^3}{3}+x\right)\Big|_{-1}^{2}$$

Hence, $A = 6$

Option 2 is correct.

8. Find the value of given definite integral $\int_{-2021}^{2021}(x^{2021}.\cos 2021x+\sin 2021x)\,dx$.

- ◯ Option 1: 4042
- ◯ Option 2: 2021
- ◯ Option 3: 0
- ◯ Option 4: 1

Solution: If $f(x)$ is an odd function, i.e., if $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\ dx = 0$.
Consider $f(x) = x^{2021}. \cos 2021x + \sin 2021x$.
Hence

$$\begin{aligned} f(-x) &= (-x)^{2021} \cos(-2021x) + \sin(-2021x) \\ &= -x^{2021} \cos 2021x - \sin 2021x = -f(x) \end{aligned}$$

Option 3 is correct.

9. Let $f_1(x) = 3x^2$ and $f_2(x) = 4 - x^2$ represent two curves (see Figure M2W3AQ1 for reference). If A is the area which is enclosed by the curves $f_1(x)$ and $f_2(x)$, then find the value of $3A$.

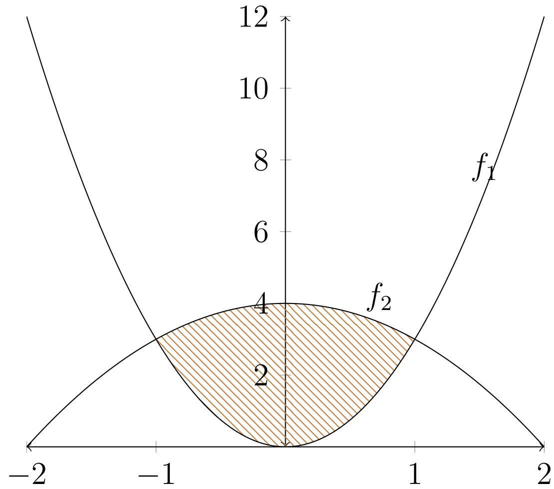
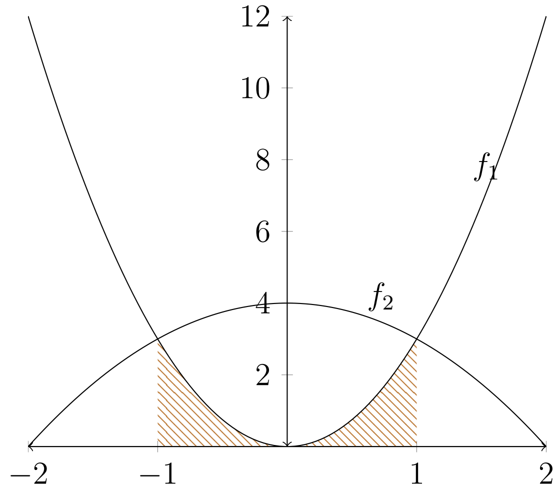


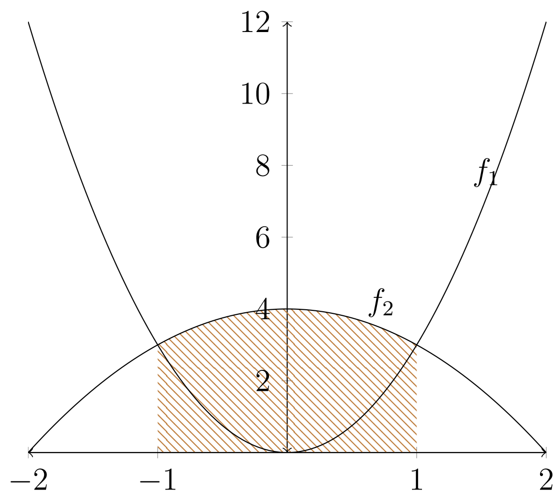
Figure: M2W3AQ1

- ◯ Option 1: 2π
- ◯ Option 2: π
- ◯ Option 3: 9
- ◯ Option 4: 16

Solution:

- **Step 1:** Find out the intersection of f_1 and f_2. These two functions are intersecting at $x = -1$ and $x = 1$.
- **Step 2:** Find the area under the curve f_1 and f_2 separately in the interval $[-1, 1]$ on the X-axis. The area between the curve f_1 and the interval $[-1, 1]$ on the X-axis is 2 sq. units. The area between the curve f_2 and the interval $[-1, 1]$ on the X-axis is $\frac{22}{3}$ sq. units.





- **Step 3:** If A is the area which is enclosed by the curves $f_1(x)$ and $f_2(x)$, then find the value of $A = \frac{22}{3} - 2 = \frac{16}{3}$ sq. units.
- Hence, $3A = 16$

Option 4 is correct.