Solution: Week 8 practice assignment.

1) Statement 1:

If T is a linear transformation, then

$$T(4,1) = T\big((2,3) + 2(1,-1)\big)$$
$$= T(2,3) + 2T(1,-1)$$
$$= (1,2) + 2(1,-1) = (3,0)$$

But it is given that $T(4,1) = (1,0) \neq (3,0)$

Hence, the Statement is wrong.

Statement 2:

$$T(1,0) = T\left(\tfrac{1}{5}(2,3) + \tfrac{3}{5}(1,-1)\right)$$
$$= \tfrac{1}{5}T(2,3) + \tfrac{3}{5}T(1,-1)$$
$$= \tfrac{1}{5}(1,2) + \tfrac{3}{5}(1,-1)$$
$$= \left(\tfrac{4}{5}, -\tfrac{1}{5}\right) = \tfrac{1}{5}(4,-1)$$

$$T(0,1) = T\left(\tfrac{1}{5}(2,3) - \tfrac{2}{5}(1,-1)\right)$$
$$= \tfrac{1}{5}T(2,3) - \tfrac{2}{5}T(1,-1)$$
$$= \tfrac{1}{5}(1,2) - \tfrac{2}{5}(1,-1) = \left(-\tfrac{1}{5}, \tfrac{4}{5}\right) = \tfrac{1}{5}(-1,4)$$

$$T(x,y) = T\left(x(1,0) + y(0,1)\right)$$
$$= x\,T(1,0) + y\,T(0,1) = \tfrac{1}{5}(4x,-x) + \tfrac{1}{5}(-y,4y)$$
$$= \tfrac{1}{5}(4x - y,\ -x + 4y)$$

Hence, the statement is true.

Statement 3:

Basis of $V = \{v_1, v_2, v_3\}$

Basis of $W = \{2v_1 + v_3,\ v_2 - v_3,\ v_3 - v_1\}$

$$T(v_1) = 2v_3 + v_1 = 1(2v_1 + v_3) + 0(v_2 - v_3) + 1(v_3 - v_1)$$
$$T(v_2) = 2v_1 = \tfrac{2}{3}(2v_1 + v_3) + 0(v_2 - v_3) - \tfrac{2}{3}(v_3 - v_1)$$
$$T(v_3) = 2v_2 = \tfrac{2}{3}(2v_1 + v_3) + 2(v_2 - v_3) + \tfrac{4}{3}(v_3 - v_1)$$

Hence, the matrix representation of T with respect to the given bases is,
$$\begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 0 & 2 \\ 1 & -2/3 & 4/3 \end{pmatrix}$$

Statement 4:

$$\begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 0 & 2 \\ 1 & -2/3 & 4/3 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 0 & 2 \\ 0 & -4/3 & 2/3 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & -4/3 & 2/3 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{-\frac{3}{4} R_2} \begin{pmatrix} 1 & 2/3 & 2/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{R_1 - \frac{2}{3} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{\frac{R_3}{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + \frac{1}{2} R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence rank of T is 3.

2) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$

$\gamma = \{(1,1,1), (0,1,1), (0,0,1)\}$

The matrix representation is (with respect to β and γ) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$T(1,0,0) = 1(1,1,1) + 0(0,1,1) + 0(0,0,1) = (1,1,1)$
$T(1,1,0) = 0(1,1,1) + 1(0,1,1) + 0(0,0,1) = (0,1,1)$
$T(1,1,1) = 0(1,1,1) + 0(0,1,1) + 1(0,0,1) = (0,0,1)$

Standard ordered basis: $\{(1,0,0), (0,1,0), (0,0,1)\}$

$(1,0,0) = 1(1,0,0) + 0(1,1,0) + 0(1,1,1)$
$(0,1,0) = -1(1,0,0) + 1(1,1,0) + 0(1,1,1)$
$(0,0,1) = 0(1,0,0) - 1(1,1,0) + 1(1,1,1)$

$T(1,0,0) = 1\,T(1,0,0) = (1,1,1)$
$T(0,1,0) = -1\,T(1,0,0) + 1\,T(1,1,0) = -(1,1,1) + (0,1,1)$
$= (-1,0,0)$

$T(0,0,1) = -1\,T(1,1,0) + 1\,T(1,1,1) = -1(0,1,1) + (0,0,1)$
$= (0,-1,0)$

Hence the matrix representation of T with respect to standard ordered basis is

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

3) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (x - y,\ y - z,\ z - x)$

$T(1,0,0) = (1, 0, -1)$

$T(0,1,0) = (-1, 1, 0)$

$T(0,0,1) = (0, -1, 1)$

Hence, $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$S : \mathbb{R}^3 \to \mathbb{R}^3$

$S(x, y, z) = (x + y,\ y + z,\ z + x)$

$S(1,0,0) = (1, 0, 1)$

$S(0,1,0) = (1, 1, 0)$

$S(0,0,1) = (0, 1, 1)$

Hence, $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$T \circ S\,(x, y, z) = T(x + y,\ y + z,\ z + x)$

$= (x - z,\ y - x,\ z - y)$

$S \circ T\,(x, y, z) = S(x - y,\ y - z,\ z - x)$

$= (x - z,\ y - x,\ z - y)$

Hence, $T \circ S = S \circ T$

Hence, $C = D$

$T \circ S(1,0,0) = (1,-1,0)$
$T \circ S(0,1,0) = (0,1,-1)$
$T \circ S(0,0,1) = (-1,0,1)$

Hence, $C = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = C = D.$$

Hence, only Statement 5 is not true.

4) i) $T : \mathbb{R}^2 \to \mathbb{R}^2$

Matrix representation of T with respect to standard ordered basis $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$T(1,0) = (1,-1)$, $T(0,1) = (-1,1)$

$T(x,y) = (x-y, -x+y)$

$S = \{(x, y) \mid x+y=1, \; x,y \in \mathbb{R}\}$

$= \{(x, 1-x) \mid x \in \mathbb{R}\}$

$T(x, 1-x) = (2x-1, \; -2x+1)$

$= (x', y')$ (say)

$x' = 2x-1, \quad y' = -2x+1$

$x' + y' = 2x-1-2x+1 = 0$

Hence, $T(S) = \{(x,y) \mid x+y=0, \; x,y \in \mathbb{R}\}$

$x+y=0$ is the st. line which passes through origin and has negative slope.

Hence, i) $\longrightarrow$ b) $\longrightarrow$ 3)

ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$

The matrix representation of T with respect to standard ordered basis is $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$T(1,0) = (0,1)$ and $T(0,1) = (-1,-1)$

$T(x,y) = (-y, x-y)$

$S = \{ (x, y) \mid x + y = 1 , x, y \in \mathbb{R} \}$

$= \{ (x, 1-x) \mid x \in \mathbb{R} \}$

$T(x, 1-x) = (-1+x, 2x-1)$

$= (x', y')$ (say)

$x' = -1 + x , \quad y' = 2x - 1$

$2x' = -2 + 2x , \quad -y' = -2x + 1$

$2x' - y' = -1$

Hence, $T(S) = \{ (x, y) \mid 2x - y = -1, x, y \in \mathbb{R} \}$

$2x - y = -1$ is a st. line which has positive slope and whose y-intercept is positive.

Hence, ii) $\rightarrow$ c $\rightarrow$ 2)

iii) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

The matrix representation of T with respect to standard ordered basis is

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$T(1,0) = (1,-1)$ and $T(0,1) = (1,0)$

$T(x, y) = (x + y, -x)$

$S = \{(x, y) \mid x + y = 1, \; x, y \in \mathbb{R}\}$

$= \{(x, 1-x) \mid x \in \mathbb{R}\}$

$T(x, 1-x) = (1, -x)$

$T(S) = \{(x, y) \mid x = 1, \; y \in \mathbb{R}\}$

$x = 1$ is the st. line parallel to Y-axis passing through $(1, 0)$.

Hence, iii) $\rightarrow$ a) $\rightarrow$ 1

5) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T(x, y) = \begin{cases} \left(x^2/y, \; y\right), & \text{if } y \neq 0 \\ (x, 0), & \text{otherwise} \end{cases}$$

Let $v_1 = (1, 1)$ and $v_2 = (1, 2)$

$T(1, 1) = (1, 1)$, $T(1, 2) = (1/2, 2)$

$T((1, 1) + (1, 2)) = T(2, 3) = (4/3, 3)$

Hence, $T(v_1 + v_2) \neq T(v_1) + T(v_2)$

If $y \neq 0$, then $cy \neq 0$ for all $c \in \mathbb{R}$.

$$T(cx, cy) = \left(\frac{c^2x^2}{cy}, cy\right)$$
$$= \left(cx^2/y, cy\right)$$
$$= c\left(x^2/y, y\right)$$

If $y = 0$, then,

$$T(cx, 0) = (cx, 0) = c(x, 0)$$

Hence, $T(cv) = cT(v)$ for all $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

Hence, P satisfies 2 but not 1.

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (x, xy)$

Let $v_1 = (1, 2)$, $v_2 = (1, 1)$

$$T(v_1 + v_2) = T(2, 3) = (2, 6)$$

$$T(v_1) + T(v_2) = T(1, 2) + T(1, 1)$$
$$= (1, 2) + (1, 1) = (2, 3)$$

Hence, 1 is not satisfied.

let $\vartheta = (1,2)$ and $c = 2$

then $T(c\vartheta) = T(2,4) = (2,8)$

$cT(\vartheta) = 2(1,2) = (2,4)$

Hence, $T(c\vartheta) \neq cT(\vartheta)$

Hence θ doesnot satisfy 1, 2 and 3.

$T : \mathbb{R}^3 \to \mathbb{R}$

$$T(x, y, z) = \frac{x + y + z}{5}$$

let $\vartheta_1 = (x_1, y_1, z_1)$ and $\vartheta_2 = (x_2, y_2, z_2)$

$$T(\vartheta_1 + \vartheta_2) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$= \frac{x_1 + x_2 + y_1 + y_2 + z_1 + z_2}{5}$$

$$= \frac{x_1 + y_1 + z_1}{5} + \frac{x_2 + y_2 + z_2}{5}$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

$$= T(\vartheta_1) + T(\vartheta_2)$$

Let $v = (x, y, z)$

Then $T(cv) = T(cx, cy, cz)$

$$= \frac{cx + cy + cz}{5}$$

$$= c\left(\frac{x+y+z}{5}\right) = cT(v)$$

Hence, R satisfies 1, 2 and 3.

6) $T(x,y,z) = (x+2y,\ x-y+cz,\ 2x+y+dz)$

$T(1,0,0) = (1,1,2)$
$T(0,1,0) = (2,-1,1)$
$T(0,0,1) = (0,c,d)$

Hence $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & c \\ 2 & 1 & d \end{pmatrix}$

$$\boxed{\begin{aligned} \det(A) &= 1(-d-c) \\ &\quad +1(-2d) + 2(2c) \\ &= 3c - 3d = 3(c-d) \end{aligned}}$$

$$B = \begin{pmatrix} 0 & 3 & 2 \\ -1 & -3 & -3 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & -3 & -3 \\ 0 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\Big\downarrow -R_1$$

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 3 & 2 \\ 0 & -3 & -2 \end{pmatrix} \xleftarrow{R_3 - 2R_1} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 3 & 2 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 3 & 2 \\ 0 & -3 & -2 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 2/3 \\ 0 & -3 & -2 \end{pmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank $B = 2$

If A is similar to B, then rank of A is also 2 (as, similar matrices have the same rank). Hence the rank of T is also 2.

As $\operatorname{rank}(B) = 2$, $\det(B) = 0$

We have already calculated $\det(A) = 3(c-d)$

If A and B are similar to each other then,

$B = P^{-1}AP$ for some matrix P.

In that case, $\det(B) = \det(P^{-1}AP)$

$$\begin{aligned} &= \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) \\ &= \det(A) \end{aligned}$$

Hence, det(A) should be 0.

$\Rightarrow 3(c-d) = 0 \Rightarrow \underline{\underline{c=d}}$.

If $c \neq d$, then matrix A cannot be similar to B.

If A and B are similar, and $c=1$, then $d=1$

Hence we have,

$$T(x,y,z) = (x+2y,\ x-y+z,\ 2x+y+z)$$

$$T(-2,1,3) = (0,0,0)$$

So, $(-2,1,3) \in \ker(T)$

As rank $T = 2$, then from rank nullity theorem we can conclude,

$$\text{rank } T + \text{nullity } T = \dim \mathbb{R}^3 = 3$$

$$\Rightarrow \text{nullity } T = 3-2 = 1.$$

Hence ker T is one dimensional subspace.

So, $\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}$ is a basis of ker T.

7) $(S+T): \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(S+T)(x,y,z) = S(x,y,z) + T(x,y,z)$

$= (x+y,\ y+z,\ z+x) + (x-y,\ y-z,\ z-x)$

$= (2x,\ 2y,\ 2z)$

$(S+T)(1,0,0) = (2,0,0)$
$(S+T)(0,1,0) = (0,2,0)$
$(S+T)(0,0,1) = (0,0,2)$

Hence, $C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= 2I$.

So, $n = 2$.

<u>Comprehension Type Question:</u>

$T(3,10,8) = 3$
$T(4,15,10) = 2$
$T(5,20,15) = 1$

8) $3T(5,20,15) - T(3,10,8) = 3 - 3 = 0$

$\Rightarrow T(15,60,45) - T(3,10,8) = 0$
$\Rightarrow T(12,50,37) = 0$

$$2T(5, 20, 15) - T(4,15,10) = 2-2=0$$

$$\Rightarrow T(10, 40, 30) - T(4, 15, 10) = 0$$

$$\Rightarrow T(6, 25, 20) = 0$$

$$\Rightarrow 2T(6,25,20) = 0$$

$$\Rightarrow T(12,50,40) = 0$$

Hence we have, $T(12,50,40) - T(12,50,37) = 0$

$$\Rightarrow T(0,0,3) = 0$$

$$\Rightarrow T(0,0,1) = 0.$$

Moreover we have, $20T(0,0,1) = 0$

$$\Rightarrow T(0,0,20) = 0$$

So, $T(6,25,20) - T(0,0,20) = 0$

$$\Rightarrow T(6,25,0) = 0$$

T is a linear transformation from a 3 dimensional vetor space to $\mathbb{R}$, which is a one dimensional vector space over $\mathbb{R}$.

As T is non-zero, rank $(T) = 1$

From rank nullity theorem, we get,

$$\text{nullity}(T) = 3-1 = 2$$

Now, $\{(6,25,0),(0,0,1)\}$ is a linearly independent set and we have already derived nullity$(T) = 2$.

So, $\{(6,25,0),(0,0,1)\}$ forms a basis of nullity(T).

In the given options the following will be bases of nullity(T):

option2: $\{(6,25,0), 25(0,0,1)\} = \{(6,25,0),(0,0,25)\}$

Option4: $\{2(6,25,0),(0,0,1)\} = \{(12,50,0),(0,0,1)\}$

option5: $\{4(6,25,0), 25(0,0,1)\} = \{(24,100,0),(0,0,25)\}$

9) $2T(3,10,8) - T(5,20,15) = (2\times 3) - 1 = 5$

$\Rightarrow T(6,20,16) - T(5,20,15) = 5$

$\Rightarrow T(1,0,1) = 5$

$\Rightarrow T(1,0,0) + T(0,0,1) = 5$

$\Rightarrow T(1,0,0) + 0 = 5$

$\Rightarrow T(1,0,0) = 5$

$T(5,20,15) = 1$

$\Rightarrow T(5,0,0) + T(0,20,0) + T(0,0,15) = 1$

$\Rightarrow \quad 5T(1,0,0) + 20\,T(0,1,0) + 15\,T(0,0,1) = 1$

$\Rightarrow \quad 25 + 20\,T(0,1,0) + 0 = 1$

$\Rightarrow \quad T(0,1,0) = -24/20 = -6/5$

$$T(x,y,z) = 5x - \frac{6y}{5}$$

$T(3,0,0) = 15$ and $T(0,0,1) = 0$

Hence the first three options are correct.

$T(9,0,0) + T(0,25,0) + T(0,0,1)$

$= 45 - 30 = 15$

Hence option 5 and 7 are also correct.

$T(25,9,0) = \left(125 - \frac{54}{5}\right)$ not a multiple of 15.

Hence option 6 and 8 are not correct.

$T(0,0,1) = 0$, which is a multiple of 15.

10) $$T(x,y,z) = 5x - \frac{6y}{5}$$

$y = 5$

$$T(x,5,z) = 5x - 6 = 4$$

$\Rightarrow \quad \underline{x = 2}$.