$\boxed{Q1}$

1A

~~$m(y) = 1-2y$ or~~ $m : \{0,1\} \to \{-1,+1\}$

$m(y) = 1-2y$ or $2y-1$ } please check according to code

$$f(x) = \frac{1+\text{sign}(x)}{2}.$$

{note that m & f need not be inverse of each other)

$\boxed{\text{Ref Q 1}}$

2A ~~Proof~~

(2A) Proof :-

sub proof ~~sign(x_i x_j)~~ . $sig(x_i x_j) = sign(x_i)\,sig(x_j)$

$sign\,x = \dfrac{x}{|x|}$

case(i) for $x_i, x_j \neq 0 \;\forall x_i x_j$ $\qquad sign(x_i x_j) = sign\,x_i\; sign\,x_j$

$sig(x_i x_j) = \dfrac{|x_i x_j|}{x_i x_j}$

$= \cancel{\dfrac{x}{x_i x_j}}\; sign\left(\dfrac{1}{x_i x_j}\right)$ (∵ numerator is always positive)

$sign\,x_i\; sig\,x_j = \dfrac{|x_i|}{x_j} \cdot \dfrac{|x_j|}{x_j}$

$= \cancel{x}\; sign\left(\dfrac{1}{x_i x_j}\right)$ (∵ product term in numerator is always positive)

Case (ii) if $x_i = 0$ or if $x_j = 0$ or $x_i = x_j = 0$

$sign(x_i x_j) = sign(0) = 0$ $\qquad \boxed{\because sign(0) = 0}$

(i) $sign(|x_i|)(sign(x_j) = sign(0)\dfrac{|x_j|}{x} = 0$

(ii) $sign(x_i)\,sig(x_j) = \dfrac{|x_i|}{x}\,sign(0) = 0$

(iii) $sign(x_j)\,sig(x_j) = sign(0)\,sign(0) = 0$

Hence $\text{sign}(x_i x_j) = \text{sign}(x_i)\,\text{sign}(x_j)$

proof by induction $\text{sign}(x_1 x_2 \dots x_n) = \text{sign}(x_1)\,\text{sign}(x_2) \dots \text{sign}(x_n)$

for $i = 1$ $\quad \text{sign}(x_1) = \text{sign}(x_1)$

for $i = 2$ $\quad \text{sign}(x_1 x_2) = \text{sign}(x_1)(\text{sign}(x_2))$

from (sub proof)

~~for i = k~~

assum statement true of $i = k$

$$\text{sign}(x_1 x_2 \dots x_k) = \text{sign}(x_1)\,\text{sign}(x_2) \dots \text{sign}(x_k) \rightarrow ①$$

for $i = k+1$

$$\text{sign}(x_1 x_2 \dots x_{k+1}) = \text{sign}(x_1 x_2 x_3 \dots x_k) \cdot \text{sign}(x_{k+1})$$

(from subproof)

$$\text{sign}(x_1 \dots x_{k+1}) = \text{sign}(x_1)(\text{sign}(x_2) \dots \text{sign}(x_k)\,\text{sig}(x_{k+1})$$

Hence the statment is proved.

(3A) $b_1 = \dfrac{1 + \operatorname{sign}(\tilde{u}^T \tilde{x})}{2}$

$b_2 = \dfrac{1 + \operatorname{sign}(\tilde{v}^T \tilde{x})}{2}$

$m(b_2) = 2\left(\dfrac{1+\operatorname{sign}(\tilde{v}^T\tilde{x})}{2}\right) - 1$

$= \operatorname{sign}(\tilde{v}^T \tilde{x})$

$b_3 = \dfrac{1 + \operatorname{sign}(\tilde{w}^T \tilde{x})}{2}$

$XOR(b_1, b_2, b_3) = f\left(m(b_1) \cdot m(b_2) \cdot m(b_3)\right)$

$= f\left(\operatorname{sign}(\tilde{u}^T\tilde{x}) \cdot \operatorname{sign}(\tilde{v}^T\tilde{x}) \cdot \operatorname{sign}(\tilde{w}^T\tilde{x})\right)$

from question 2 proof

$= f\left(\operatorname{sign}\left[(\tilde{u}^T\tilde{x}) \cdot (\tilde{v}^T\tilde{x}) \cdot (\tilde{w}^T\tilde{x})\right]\right)$

$= \dfrac{1 + \operatorname{sign}\left\{\operatorname{sign}\left[(\tilde{u}^T\tilde{x}) \cdot (\tilde{v}^T\tilde{x}) \cdot (\tilde{w}^T\tilde{x})\right]\right\}}{2}$

$\boxed{\operatorname{sign}(\operatorname{sign}(x)) = \operatorname{sign} x}$

$= \dfrac{1 + \operatorname{sign}\left[(\tilde{u}^T\tilde{x}) \cdot (\tilde{v}^T\tilde{x}) \cdot (\tilde{w}^T\tilde{x})\right]}{2}$

Ref Q3 → The above calculation tells us that all we need to get hold of is the follow term

$(\tilde{u}^T\tilde{x}) \cdot (\tilde{v}^T\tilde{x}) \cdot (\tilde{w}^T\tilde{x})$

(3A) To express $(\tilde{u}^T\tilde{x})(\tilde{v}\cdot\tilde{x})(\tilde{w}^T\cdot\tilde{x})$.

$$\tilde{u}^T\tilde{x} = \sum_{i=1}^{n} u_i x_i$$

$$\tilde{v}^T\tilde{x} = \sum_{j=1}^{n} v_j x_j$$

$$\tilde{w}^T\tilde{x} = \sum_{k=1}^{n} w_k x_k$$

$$(\tilde{u}^T\tilde{x})(\tilde{v}\cdot\tilde{x})(\tilde{w}\cdot\tilde{x}) = \left(\sum_{i=1}^{n} u_i x_i\right)\cdot\left(\sum_{j=1}^{n} v_j x_j\right)\cdot\left(\sum_{k=1}^{n} w_k x_k\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} u_i v_j w_k (x_i x_j x_k)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_{ijk}\, \phi_{ijk}(x)$$

where $w_{ijk} = u_i v_j w_k$. $\phi_{ijk}(x) = x_i x_j x_k$.

$$= w^T \phi(\tilde{x})$$

$$w = [w_{ijk}] = [u_1 v_1 w_1, u_1 v_1 w_2 \; \cdots \; u_n v_n w_n]^T$$

$$\phi_{ijk}(\tilde{x}) = [x_1 x_1 x_1, x_1 x_1 x_2, \; \cdots \; x_n x_n x_n]$$