Lecture 16 - Notes

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1 Overview

Last time, we saw how to linearize scalar systems of a particular form - specifically, systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t)) + bu(t).$$

We will now complete our discussion of linearization, by discovering how to linearize arbitrry vector systems of the most general form

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x}(t), \vec{u}(t)).$$

2 Linearizing Multivariate Scalar Functions

Last time, we saw how to use the linearization of a scalar function $f(x)$ in order to linearize its associated scalar system. It would make sense to do something similar, linearly approximating $\vec{f}(\vec{x}, \vec{u})$ in order to linearize its associated vector system.

A key difference between $\vec{f}(\vec{x}, \vec{u})$ and the functions $f(x)$ that we have seen before is that $\vec{f}$ takes in multiple scalar parameters (contained within the two vector inputs), rather than just a single one. So a good first step would be to figure out how to linearly approximate a scalar function of two variables, before generalizing to functions of vectors of arbitrary dimension.

Consider some $g(a, b)$, and imagine approximating g near $a = a^*$ and $b = b^*$. Intuitively, we can imagine starting at $g(a^*, b^*)$, moving by some δg_a to $g(a, b^*)$, and then by some further δg_b to $g(a, b)$.

Expressed algebraically, let

$$\begin{aligned}
\delta g_a &= g(a, b^*) - g(a^*, b^*) = g(a^* + \delta a, b^*) - g(a^*, b^*) \\
\delta g_b &= g(a, b) - g(a, b^*) = g(a, b + \delta b) - g(a, b^*).
\end{aligned}$$

Notice that the sum of δg_a and δg_b represents the total change in g from (a^*, b^*) to (a, b).

But since both δg_a and δg_b express the change in g as we vary just one parameter, we can construct their linear approximations using the same techniques we saw previously, for functions taking in a single scalar input! Specifically, consider the scalar functions

$$\begin{aligned} g_a(x) &= g(x, b^*) \\ g_b(x) &= g(a, x). \end{aligned}$$

By definition, we have that

$$\begin{aligned} \delta g_a &= g_a(a) - g_a(a^*) \\ \delta g_b &= g_b(b) - g_b(b^*). \end{aligned}$$

But since $g_a(x)$ and $g_b(x)$ are both scalar functions, we have the linear approximations

$$\begin{aligned} \delta g_a &\approx (a - a^*) \frac{\mathrm{d}g_a}{\mathrm{d}t}\bigg|_{x=a^*} \\ \delta g_b &\approx (b - b^*) \frac{\mathrm{d}g_b}{\mathrm{d}t}\bigg|_{x=b^*}. \end{aligned}$$

Therefore, we can write

$$\delta g = \delta g_a + \delta g_b \approx \frac{\mathrm{d}g_a}{\mathrm{d}t}(x = a^*) + \frac{\mathrm{d}g_b}{\mathrm{d}t}(x = b^*).$$

So, are we done? We've certainly expressed our δg as an approximation of some kind. But what are these derivatives of g_a and g_b doing here? How do they relate to g itself, as a function of two scalar parameters?

First, let's look at $g_a(x)$. Intuitively, $g_a(x)$ describes the behavior of $g(a, b)$ as we vary a and hold b constant at b^*. So its derivative at $x = a$ should express the rate of change of g at (a^*, b^*) as we vary a, treating $b = b^*$ as a constant. This quantity is more conventionally referred to as the *partial derivative* of $g(a, b)$ at (a^*, b^*) with respect to a, and is written as

$$\frac{\partial g}{\partial a}\bigg|_{\substack{a=a^* \\ b=b^*}}.$$

In a similar manner, $g_b(x)$ describes the behavior of $g(a, b)$ as we vary b and hold a constant. But whereas $g_a(x)$ held b constant at the starting point b^*, $g_b(x)$ holds a constant at our final value of a! So taking the derivative of $g_b(x)$ at $x = b^*$ can be written as the partial derivative

$$\frac{\partial g}{\partial b}\bigg|_{\substack{a=a \\ b=b^*}}$$

evaluated at $a = a$, not $a = a^*$. Putting these partial derivatives into our approximations, we find that

$$\delta g \approx \frac{\partial g}{\partial a}\bigg|_{\substack{a=a^* \\ b=b^*}} \delta a + \frac{\partial g}{\partial b}\bigg|_{\substack{a=a \\ b=b^*}} \delta b.$$

This approximation seems pretty good! There's only one lingering issue - it isn't actually linear! Notice that the partial derivative acting as the coefficient of δb is not a constant, but actually varies with a. This is undesirable, since it introduces some nasty nonlinearities. To address this, notice that $\frac{\partial g}{\partial b}$ is typically nonzero at (a^*, b^*). So when we move to (a, b^*), its small variation will be insignificant, relative to its original value, if δa is small (since this is a second-order effect). Thus, we can approximate its value as

$$\frac{\partial g}{\partial b}\bigg|_{\substack{a=a\\b=b^*}} \approx \frac{\partial g}{\partial b}\bigg|_{\substack{a=a^*\\b=b^*}},$$

to finally obtain the purely linear approximation

$$\delta g \approx \frac{\partial g}{\partial a}\bigg|_{\substack{a=a^*\\b=b^*}} \delta a + \frac{\partial g}{\partial b}\bigg|_{\substack{a=a^*\\b=b^*}} \delta b.$$

The above approximation is known as the *total derivative* of g, and generalizes in the natural manner to functions g taking in arbitrarily many arguments.

3 Linearizing Vector-Valued Functions

Now we know how to linearize functions producing scalar outputs, we can go further and linearize functions with vector inputs and outputs. Consider some function $\vec{f}(\vec{x})$. Let $\vec{x}$ have k components, and let the output of $\vec{f}$ have n components. Thus, we can express any input as

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix}^T,$$

where all these x_i are scalars. Saying that $\vec{x}$ is provided as a vector input to $\vec{f}(\vec{x})$ is the same thing as stating that $\vec{f}$ takes k scalar inputs x_1 through x_k. Thus, we will treat the two representations below of $\vec{f}$ as equivalent:

$$\vec{f}(\vec{x}) \text{ and } \vec{f}(x_1, x_2, \ldots, x_k).$$

Similarly, the output of $\vec{f}$ can be thought of as a set of n scalars f_1 through f_n. Thus, we can represent $\vec{f}(\vec{x})$ in terms of multivariate scalar functions as follows:

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_k) \\ f_2(x_1, x_2, \ldots, x_k) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_k) \end{bmatrix}.$$

Now, we can apply our known linear approximations for scalar functions near a DC operating point $\vec{x}^* = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_k^* \end{bmatrix}^T$, to obtain

$$\delta \vec{f} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots \frac{\partial f_1}{\partial x_k}\delta x_k \\ \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots \frac{\partial f_2}{\partial x_k}\delta x_k \\ \vdots \\ \frac{\partial f_n}{\partial x_1}\delta x_1 + \frac{\partial f_n}{\partial x_2}\delta x_2 + \ldots \frac{\partial f_n}{\partial x_k}\delta x_k \end{bmatrix}.$$

This complicated matrix can be simplified somewhat as the product of a matrix and a vector, so we obtain

$$\delta \vec{f} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_k} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_k \end{bmatrix}.$$

We refer to the column vector on the right as $\delta\vec{x}$, and the matrix on the right as the *Jacobian matrix* of f and $\vec{x}$, expressed using the notation $J_{\vec{x}}$ or $\nabla_{\vec{x}}\vec{f}$. Thus, we may express

$$\delta \vec{f} = J_{\vec{x}} \delta \vec{x}.$$

4 Linearizing a General System

Finally, we can put all this machinery together, and linearize a general system, of the form

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x}, \vec{u}).$$

As before, given a DC input $\vec{u}^*$, we can solve for the DC operating point $\vec{x}^*$, by solving for the equilibrium point where $\vec{f}(\vec{x}^*, \vec{u}^*) = \vec{0}$. Then, treating $\vec{f}$ as a function of the parameters $x_1, x_2, \ldots, x_n$ and $u_1, u_2, \ldots, u_k$, we may express

$$\delta \vec{f} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots \frac{\partial f_1}{\partial x_n}\delta x_n + \frac{\partial f_1}{\partial u_1}\delta u_1 + \frac{\partial f_1}{\partial u_2}\delta u_2 + \ldots \frac{\partial f_1}{\partial u_k}\delta u_k \\ \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots \frac{\partial f_2}{\partial x_n}\delta x_n + \frac{\partial f_2}{\partial u_1}\delta u_1 + \frac{\partial f_2}{\partial u_2}\delta u_2 + \ldots \frac{\partial f_2}{\partial u_k}\delta u_k \\ \vdots \\ \frac{\partial f_n}{\partial x_1}\delta x_1 + \frac{\partial f_n}{\partial x_2}\delta x_2 + \ldots \frac{\partial f_n}{\partial x_n}\delta x_n + \frac{\partial f_2}{\partial u_1}\delta u_1 + \frac{\partial f_2}{\partial u_2}\delta u_2 + \ldots \frac{\partial f_2}{\partial u_k}\delta u_k \end{bmatrix}.$$

Rearranging this using matrix-vector form and the Jacobian, we obtain

$$\begin{aligned} \delta \vec{f} &\approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_k} \\ \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_k} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \vdots \\ \delta u_k \end{bmatrix} \\ &= J_{\vec{x}} \delta \vec{x} + J_{\vec{u}} \delta \vec{u}. \end{aligned}$$

Thus, we can approximate our state transition equation as

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \approx \vec{f}(\vec{x}^*, \vec{u}^*) + \delta \vec{f} = \vec{f}(\vec{x}^*, \vec{u}^*) + J_{\vec{x}} \delta \vec{x} + J_{\vec{u}} \delta \vec{u}.$$

But since we chose our DC operating point to be such that $\vec{f}(\vec{x}^*, \vec{u}^*) = \vec{0}$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\vec{x}) \approx J_{\vec{x}}\delta\vec{x} + J_{\vec{u}}\delta\vec{u},$$

so we can now obtain a fully linearized approximation of any general nonlinear system near an equilibrium $(\vec{x}^*, \vec{u}^*)$, which was our original goal.