Lecture 25 - Notes

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1 Overview

Having established some mathematical preliminaries in the previous lecture, we will now begin to consider the DFT itself. Specifically, we will look at a linear transformation that, when applied on a discrete-time signal, allows us to write it as a sum of discrete-time sinusoidal signals. In this lecture, we will focus on the properties of this linear transformation in its matrix representation. We will explain why it is useful in converting between the time and frequency domains (which, of course, is the DFT matrix's most important property!) in future lectures.

2 The DFT Matrix

Recall that we define a particular kth root of unity

$$\omega_k = e^{2\pi j/k}.$$

We define the DFT matrix for a signal of length N to be

$$F_N = \begin{bmatrix} \omega_N^{-0 \cdot 0} & \omega_N^{-0 \cdot 1} & \cdots & \omega_N^{-0 \cdot (N-1)} \\ \omega_N^{-1 \cdot 0} & \omega_N^{-1 \cdot 1} & \cdots & \omega_N^{-1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{-(N-1) \cdot 0} & \omega_N^{-(N-1) \cdot 1} & \cdots & \omega_N^{-(N-1) \cdot (N-1)} \end{bmatrix},$$

where the entry at the ath row and bth column is

$$(F_N)_{a,b} = \omega_N^{-a \cdot b},$$

with our row and column numbering both starting at 0[1].

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[1] From this lecture on, we will continue to use the convention of 0-indexing our enumerations, unless we explicitly state otherwise or it is extremely obvious from context. Note also that we will not use i or j as indices in most contexts, to avoid confusion when working with complex numbers.

What does this matrix look like? Let's consider this matrix for a particular value of N - say, $N = 3$. Substituting directly into the definition, we find that

$$F_3 = \begin{bmatrix} \omega_3^{-0\cdot0} & \omega_3^{-0\cdot1} & \omega_3^{-0\cdot2} \\ \omega_3^{-1\cdot0} & \omega_3^{-1\cdot1} & \omega_3^{-1\cdot2} \\ \omega_3^{-2\cdot0} & \omega_3^{-2\cdot1} & \omega_3^{-2\cdot2} \end{bmatrix} = \begin{bmatrix} \omega_3^0 & \omega_3^0 & \omega_3^0 \\ \omega_3^0 & \omega_3^{-1} & \omega_3^{-2} \\ \omega_3^0 & \omega_3^{-2} & \omega_3^{-4} \end{bmatrix}.$$

To simplify this matrix representation, recall that ω_3, being a cube root of unity, has the property that $\omega_3^3 = 1$. Thus, we can multiply any number by this quantity and not change its value. In particular, observe that

$$\omega^{-1} = \omega^{-1} \cdot \omega^3 = \omega^2$$
$$\omega^{-2} = \omega^{-2} \cdot \omega^3 = \omega^1 = \omega$$
$$\omega^{-4} = \omega^{-4} \cdot \omega^3 \cdot \omega^3 = \omega^2.$$

Notice further that clearly $\omega_3^0 = 1$. Thus, we can simplify our representation of F_3 to become

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega_3^2 & \omega_3 \\ 1 & \omega_3 & \omega_3^2 \end{bmatrix}.$$

Thus, we see that there are three distinct values in the matrix: 1, ω_3, and ω_3^2. Notice that these values are exactly the three cube roots of unity, and so will be evenly spaced on a unit circle if plotted on the complex plane, as shown:
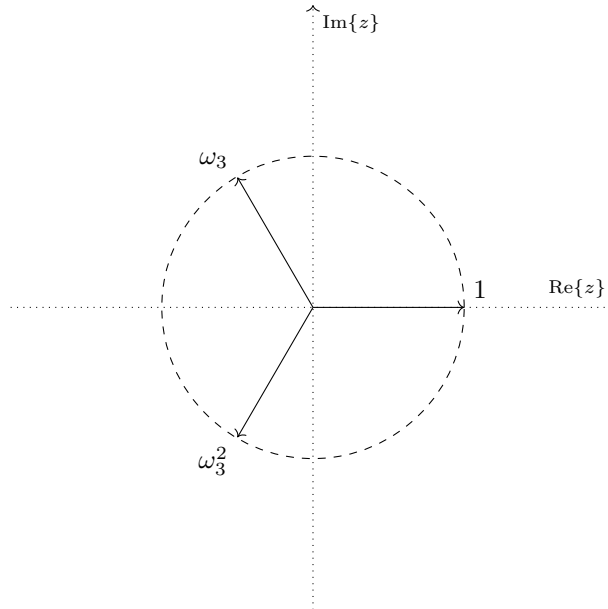
Looking at each of the rows of F_3, we can view them as representing motion around the circle with different angular frequencies - the first row represents

the angular frequency $\omega = 0$, the second represents $\omega = -\pi/3$ (treating each entry as representing one unit of time), and the third represents $\omega = \pi/3$. This interpretation of the rows will be important in future lectures, when we rigorously consider writing signals in the frequency domain.

3   Simple Properties of the DFT Matrix

We will now establish some properties of DFT matrices, that are fairly straightforward consequences. First, notice that for all valid a, b, and N,

$$(F_N)_{a,b} = \omega_N^{-ab} = \omega_N^{-ba} = (F_N)_{b,a},$$

so F_N is symmetric. Written algebraically, $F_N = F_N^T$.

Moreover, observe that each entry of F_N is a root of unity raised to some real power, so each entry has unit magnitude. Consider the rows of F_N, written as $\vec{u}_k$ such that

$$F_N = \begin{bmatrix} - & \vec{u}_0^T & - \\ - & \vec{u}_1^T & - \\ & \vdots & \\ - & \vec{u}_{N-1}^T & - \end{bmatrix}.$$

Observe that, by definition, for any valid k,

$$\|\vec{u}_k\| = \sqrt{\sum_{a=0}^{N-1} |(F_N)_{k,a}|}$$

$$= \sqrt{\sum_{a=0}^{N-1} 1}$$

$$= \sqrt{N},$$

using our observation about the magnitudes of the entries of F_N. Since F_N is symmetric, it is clear that the magnitude of each of its columns is also $\sqrt{N}$. Thus, if we divide F_N by $\sqrt{N}$, we will obtain the matrix $(1/\sqrt{N})F_N$ of a similar form but with rows and columns of unit length. Some other texts call this matrix (rather than F_N) the DFT matrix, but we will not do so here.

Finally, consider the entries of $\vec{u}_k$:

$$\vec{u}_k^T = \begin{bmatrix} \omega_N^{-0 \cdot k} & \omega_N^{-1 \cdot k} & \omega_N^{-2 \cdot k} & \cdots & \omega_N^{-(N-1) \cdot k} \end{bmatrix}.$$

Observe that we can rewrite these entries as

$$\vec{u}_k^T = \begin{bmatrix} (\omega_N^{-k})^0 & (\omega_N^{-k})^1 & (\omega_N^{-k})^2 & \cdots & (\omega_N^{-k})^{N-1} \end{bmatrix},$$

so they form the entries of a geometric sequence with first element 1. By definition, we have thus shown that F_N is a Vandermonde matrix, since the common ratios of each of its rows is clearly distinct.

4   Orthogonality of the DFT Matrix

Above, we showed that the norms of the rows and columns of F_N are both $\sqrt{N}$, using a straightforward algebraic argument. However, we claim that these rows are all also orthogonal. To prove this, it is natural to take the inner product of two rows. We need to be careful, though, to use the inner product generalized to the complex field, so we have for valid $a \neq b$

$$\langle \vec{u}_a, \vec{u}_b \rangle = \overline{\vec{u}_b}^T \vec{u}_a$$

$$= \sum_{k=0}^{N-1} \overline{(F_N)_{b,k}} (F_N)_{a,k}$$

$$= \sum_{k=0}^{N-1} \overline{\omega_N^{-bk}} \omega_N^{-ak}$$

$$= \sum_{k=0}^{N-1} \omega_N^{bk} \omega_N^{-ak}$$

$$= \sum_{k=0}^{N-1} (\omega_N^{b-a})^k,$$

since, from basic properties of the polar form of complex numbers, we have that

$$\overline{\omega_N^{-bk}} = \overline{(e^{2\pi/N})^{-bk}}$$

$$= \overline{e^{-2\pi bk/N}}$$

$$= e^{2\pi bk/N}$$

$$= (e^{2\pi/N})^{bk}$$

$$= \omega_N^{bk}.$$

Now, observe that since $a \neq b$, our summation is actually just the summation of a geometric series with initial term 1 and common ratio ω_N^{b-a}. Thus, we may apply the well-known formula for summing a geometric series, to obtain

$$\langle \vec{u}_a, \vec{u}_b \rangle = \sum_{k=0}^{N-1} (\omega_N^{b-a})^k$$

$$= \frac{1 - (\omega_N^{b-a})^N}{1 - \omega_N^{b-a}}.$$

Observe that for $a \neq b$, the denominator is nonzero, so this expression is defined. However, observe that since ω_N^{b-a} is an Nth root of unity raised to an integer power, it is itself a root of unity. By definition, therefore, $(\omega_N^{b-a})^N = 1.$[2]

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[2]If you're not comfortable with the properties of roots of unity, you can of course show this directly by swapping the order of the powers. But going about it this way helps build intuition for complex numbers and is marginally simpler.

Substituting into the above expression, we obtain

$$\langle \vec{u}_a, \vec{u}_b \rangle = \frac{1 - (\omega_N^{b-a})^N}{1 - \omega_N^{b-a}}$$
$$= \frac{0}{1 - \omega_N^{b-a}}$$
$$= 0,$$

so $\vec{u}_a$ and $\vec{u}_b$ are indeed orthogonal.

We thus have that all the rows and columns of F_N have the same magnitude and are mutually orthogonal. This sounds very similar, but not the same, as orthogonality - the only difference being that the magnitudes of these vectors are all $\sqrt{N}$, not 1. However, by normalizing these magnitudes, we thus have that $(1/\sqrt{N})F_N$ is an orthogonal matrix!

One property of orthogonal matrices, that we have seen many times before, is that their Hermitian transpose is their own inverse. Thus,

$$\left(\frac{1}{\sqrt{N}} F_N \right)^{-1} = \left(\frac{1}{\sqrt{N}} F_N \right)^*$$
$$\implies \qquad \sqrt{N} F_N^{-1} = \frac{1}{\sqrt{N}} F_N^*$$
$$\implies \qquad F_N F_N^* = NI,$$

where I is the $N \times N$ identity matrix.

5   Complex Conjugates within the DFT Matrix

Above, we saw that the F_N's conjugate transpose was essentially its inverse (up to a factor of N). Indeed, since F_N is symmetric, we have that

$$F_N \overline{F_N} = NI,$$

so the conjugate of F_N is also essentially its inverse. Now, we will look at this conjugate from a different perspective. Writing it out elementwise, we have

that

$$\overline{F_N} = \begin{bmatrix} \overline{\omega_N^{-0 \cdot 0}} & \overline{\omega_N^{-0 \cdot 1}} & \cdots & \overline{\omega_N^{-0 \cdot (N-1)}} \\ \overline{\omega_N^{-1 \cdot 0}} & \overline{\omega_N^{-1 \cdot 1}} & \cdots & \overline{\omega_N^{-1 \cdot (N-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\omega_N^{-(N-1) \cdot 0}} & \overline{\omega_N^{-(N-1) \cdot 1}} & \cdots & \overline{\omega_N^{-(N-1) \cdot (N-1)}} \end{bmatrix}$$

$$= \begin{bmatrix} \omega_N^{0 \cdot 0} & \omega_N^{0 \cdot 1} & \cdots & \omega_N^{0 \cdot (N-1)} \\ \omega_N^{1 \cdot 0} & \omega_N^{1 \cdot 1} & \cdots & \omega_N^{1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_N^{(N-1) \cdot 0} & \omega_N^{(N-1) \cdot 1} & \cdots & \omega_N^{(N-1) \cdot (N-1)} \end{bmatrix},$$

since taking the conjugate is the same thing as flipping the sign of the complex term in the exponent.

We claim that taking the conjugate is the same thing as rearranging the order of the columns (or rows) of F_N. For $a = 0$, $\vec{u}_a = \vec{u}_0$ is a real vector consisting of all 1s, so it is not affected by taking the conjugate. However, we claim that the order of all the other columns is reversed after taking the conjugate. Specifically, we assert that, for all valid $a \geq 1$,

$$\vec{u}_a = \overline{\vec{u}_{N-a}}.$$

This can be proved via a straightforward algebraic argument. Consider the bth element of $\vec{u}_a$. By definition, since $\omega_N^N = 1$, we have that

$$\begin{aligned} (\vec{u}_a)_b &= F_{a,b} \\ &= \omega_N^{-ab} \\ &= \omega_N^{-ab}(\omega_N^N)^b \\ &= \omega_N^{-ab}\omega_N^{Nb} \\ &= \omega_N^{(N-a)b} \\ &= \overline{\omega_N^{-(N-a)b}} \\ &= \overline{F_{N-a,b}} \\ &= \overline{(\vec{u}_{N-a})_b}. \end{aligned}$$

Since every element of $\vec{u}_{N-a}$ is the conjugate of its corresponding term from $\vec{u}_a$, we have that

$$\vec{u}_a = \overline{\vec{u}_{N-a}},$$

as desired.

Applying this result on all the columns of $\overline{F_N}$, it is clear that we may write

$$\overline{F_N} = \begin{bmatrix} | & | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_{N-1} \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ \vec{u}_0 & \vec{u}_{N-1} & \vec{u}_{N-2} & \cdots & \vec{u}_1 \\ | & | & | & & | \end{bmatrix}.$$

An analogous result can immediately be obtained for the rows by taking the transpose of the above equation.