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Approx $x_{n-i} d_n$ as $x_{n-1} \cdot d_{n-1}$

Look @ total least sqs

Δx

Phase vs. Mag

$x_{n+1} = a \cdot x_n + u_n + w_n$

$y_n = z_n x_n$ $z_n \in \{-1, 1\}$

$y_1 = 2 = z_1 x_1$ $|x_1| = 2$.

$u_1 = \cancel{\$} -2a$

$x_2 = a \cdot x_1 + u_1 = a \cdot (\text{sgn}(x_1) \cdot 2 + \cancel{\$} -2a$

$y_2 = z_2 \cdot x_2$

What if we don't know a in the scalar case?

Higher dim case $C = \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \ddots \pm 1 \end{bmatrix}$

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Extra page for scratchwork.
Work on this page will NOT be graded.

$X_{n+1} = aX_n + u_n$

$Y_n = z_n X_n$ $\qquad$ $Y_n = d_n^T X_n$

$1\times1 \quad 1\times1$ $\qquad$ $1\times n \quad n\times1$

dots ⇒ $d[k]^T \cdot \vec{x}[0] = $

div ⇒ $[k+1]\,\frac{y[k]}{[k]}$

LHS ⇒ $y[k]$ $\quad 1:N$

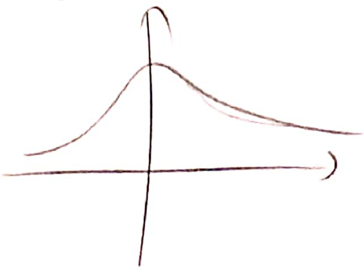
$$\begin{bmatrix} y[1] \\ \vdots \\ y[N] \end{bmatrix} = \begin{bmatrix} d[1]^T x[0] & \frac{y[0]}{d[2]^T x[0]} \\ \vdots & \\ d[N]^T x[0] & \frac{y[N-1]}{d[N-1]^T x[0]} \end{bmatrix} \lambda$$

LHS

div

$X \sim \text{Cauchy}(0,1)$

$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad \forall x \in \mathbb{R}$



$\Rightarrow Y = |X|$

$f_Y(y)$

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(b) (10 points) Your friend Vlad tells you that your transition matrix **M** was wrong, and gives you a new transition matrix **S**, which has a steady state. In order to find who wins the war, you need to find how many students end up in each section after everything has settled. **Find the vector $\vec{x}$ that represents the steady state of S.**

I don't feel like sufficiently touched on the concept of and using those as estimated states

$$\mathbf{S} = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 1 \end{bmatrix}$$

Threads ① bad estimate of λ from slope
estimate of eigenvector
good estimate of λ from method

feels like an observer except for no input but feels dynamical. How to make computation efficient?

② Do eigenvectors of unchanged eigenvalues remain the same if we change only one eigenvalue?

③ Can we control eigenvectors with FSFB?

④ Does choosing eigenvalues uniquely determine what eigenvector we get?

$$\lambda_1 \vec{v}_1 \vec{v}_1^T + \underbrace{\alpha \vec{v}_1 \vec{v}_2^T + \beta \vec{v}_2 \vec{v}_1^T + \gamma \vec{v}_2 \vec{v}_2^T}_{\text{BK components}} + \Delta \vec{v}_1 \vec{v}_1^T$$

$$\alpha \neq \beta = \gamma \propto \lambda_{des}, \quad \Delta = \lambda_{des} - \lambda_1$$

⑤ Can Schur form be leveraged? requires us knowing λ_1

⑥ How does estimation errors in $\hat{\lambda}$, & $\hat{v}$, effect things?

⑦ What does our C[k] do? Can we factor this in?

PRINT your student ID: Thread # 2

$$A = Q\Lambda Q^T = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix}$$

$\vec{b} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$ $\qquad$ $\vec{k} = \sum_{i=1}^{n} \beta_i \vec{v}_i^T \Rightarrow$ choose $\vec{u} = \vec{v}_1$

$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t]$
$u = \vec{k}^T\vec{x}[t]$ $\Rightarrow$ $A + \vec{b}\vec{k}^T = \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T + \sum_{i=1}^{n} \alpha_i \vec{v}_i \vec{k}^T$

with choice of $\vec{k} = \vec{v}_1$ $\bar{A} = A + \vec{b}\vec{v}_1^T = \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T + \sum_{i=1}^{n} \alpha_i \vec{v}_i \vec{v}_1^T$

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(c) (6 points) Your other friend Gireeja points out that the arguments are causing new people to join the sections and others to leave entirely. In other words, the system is not conservative! The new system can be modeled with a state transition matrix **A** that has the following eigenvalue/eigenvector pairings:

$$\lambda_1 = 1 : \vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} : \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2 : \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You want the number of students in sections to stabilize. Which of the vectors below represent **steady states** of the system, i.e. $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{x}$? **Fill in the circle(s) to the left of these vector(s).**

○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$

Space for scratchwork, **will NOT be graded**

$\lambda_1 \vec{v_1}\vec{v_1}^T + \alpha \vec{v_1}\vec{v_2}^T + \beta \vec{v_2}\vec{v_1}^T + \gamma \vec{v_2}\vec{v_2}^T + \Delta \vec{v_1}\vec{v_1}^T$

$(\lambda_1 + k)(\vec{v_1} + \vec{v_2})(\vec{v_1}^T + \vec{v_2}^T)$

$(\lambda_1 + k).(\vec{v_1}\vec{v_1}^T + \vec{v_2}\vec{v_2}^T + \vec{v_1}\vec{v_2}^T + \vec{v_2}\vec{v_1}^T)$

$\Delta = k, \ \alpha, \beta, \gamma = \lambda_1 + k = \lambda_{des}$

how to ~~choose~~ determine $\vec{v_2}$?

$\Delta = k = \lambda_1 + \lambda_{des}$

$\alpha = \beta = \gamma = \lambda_{des}$ choosing $\vec{v_2} = -\vec{v_1}$ is same as $\lambda_{des} = 0$

by choosing $\vec{v_2}$, we can choose $\vec{v_1}$?

Does controllability imply eigenvector placement?

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(a) (4 points) Express $\vec{y}[0]$ in terms of $\vec{x}[0]$.

$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1\ k_2]$

$\lambda^2-3\lambda+2$

$(\lambda-2)(\lambda-1)$

$\begin{bmatrix} 0 & 1 \\ -2-k_1 & 3-k_2 \end{bmatrix}$

$\lambda^2-\frac{1}{4}$

$\begin{bmatrix} 0 & 0 \\ 7/4 & -3 \end{bmatrix}$

$\begin{bmatrix} -\cancel{2} & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v}_1$

$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}[\frac{7}{4}\ -3]$ $\begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix}$ $\begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{4} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda_1\vec{v}_1\vec{v}_1^T + \cancel{\alpha}\vec{v}_1\vec{v}_2^T + \cancel{\alpha}\vec{v}_2\vec{v}_1^T + \cancel{\alpha}\vec{v}_2\vec{v}_2^T$ $\lambda_1=\frac{1}{2}$ $\lambda_2=-\frac{1}{2}$

$\lambda_1\vec{v}_1(\vec{v}_1^T+\vec{v}_2^T) + \cancel{\alpha}\vec{v}_2$

(b) (6 points) Express $\vec{x}[1]$ and $\vec{y}[1]$ in terms of $\vec{x}[0]$.

$\lambda_1\vec{v}_1\vec{v}_1^T+\lambda_2\vec{v}_2\vec{v}_2^T + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[\frac{7}{4}\ -3]$

$(\lambda-2)(\lambda-\frac{1}{2})$

$\lambda^2-\frac{5}{2}\lambda+1$

$(\lambda+\frac{1}{2})(\lambda-\frac{1}{2})$

$\cancel{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}[1\ 2] + \frac{1}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix}[2\ -1]$

$= \frac{1}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix}[2\ 1] + (-\frac{1}{2})\begin{bmatrix} 2 \\ -1 \end{bmatrix}[2\ -1]$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}[\frac{7}{4}\ -3]$

what happens if we keep an eigenvalue the same?

Eigenvector seems to be same.

$\begin{bmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{bmatrix}$ $\begin{bmatrix} -2 & 1 \\ -1 & +\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_1=\cancel{2}$ $\vec{v}_1$

$\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\lambda_2=\frac{1}{2}$ $\vec{v}_2$

$\begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix}$

$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix}$ $\begin{bmatrix} \alpha\vec{v}_1 \end{bmatrix}[k_1\ k_2]$

$\lambda_1\vec{v}_1\vec{v}_1^T + \lambda_2\vec{v}_2\vec{v}_2^T + \alpha\vec{v}_1 k_1^T$

PRINT your student ID: ~~something about this seems incorrect~~

$$2(\lambda_1+\alpha_1)\vec{v}_1\vec{v}_1^T + \sum_{i=1}^{n}\cancel{(\lambda_i+\alpha_i)\vec{v}_i(\vec{v}_i^T+\vec{v}_1^T)}$$

Do schur form, do calculation w/ orthogonality / how does it modify rest?

$$\bar{A} = \underbrace{\lambda_1\vec{v}_1\vec{v}_1^T + \cdots + \lambda_n\vec{v}_n\vec{v}_n^T}_{A} + \underbrace{\underbrace{\sum_{i=1}^{n}\alpha_i v_i}_{\vec{b}}\underbrace{\beta\vec{v}_1^T}_{\vec{k}}}_{\text{matrix}}$$

$\vec{k}\vec{x} \leftarrow$ scalar

$\vec{C}^T\vec{x} \leftarrow$ scalar

if I could do state feedback

$$\bar{A} = (\lambda_1+\alpha_i\beta)\vec{v}_1\vec{v}_1^T + (\lambda_2\vec{v}_2\vec{v}_2^T + \alpha_2\beta\vec{v}_2\vec{v}_1^T) + \cdots (\lambda_n\vec{v}_n\vec{v}_n^T + \alpha_n\beta\vec{v}_n\vec{v}_1^T)$$

$$\bar{A}\vec{v}_1 = (\lambda_1+\alpha_1\beta)\vec{v}_1 + \alpha_2\beta\vec{v}_2 + \cdots + \alpha_n\beta\vec{v}_n$$

if $\vec{x}$ is aligned w/ state

if $\boxed{-\lambda_1 = \alpha_1\beta}$ → The system is always pushed into the orthogonal subspace!

$\alpha_1 \Rightarrow$ we don't get to choose.

$$\vec{b}\cdot\vec{v}_1 = \sum_{i=1}^{n}\alpha_i\vec{v}_i^T\vec{v}_1 = \alpha_1$$

$\beta \Rightarrow$ we get to choose

$-\frac{\hat{\lambda}_1}{\hat{\alpha}_1} = \hat{\beta} \leftarrow$ estimated β factor

$$\vec{b}\cdot\hat{\vec{v}}_1 = \sum_{i=1}^{n}\alpha_i\vec{v}_i^T(\vec{v}_1+\vec{\varepsilon}) = \boxed{\alpha_1 + \underbrace{\sum_{i=2}\alpha_i\vec{v}_i^T\vec{\varepsilon}}_{\text{error term in } \hat{\alpha}}}$$

(g) (2 points) Does pineapple belong on pizza?

Observer structure? But for system A?

~~Expected bound on control capacity?~~

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$\vec{x}[t+1] = \Lambda \vec{x}[t] + \vec{B}u[t]$

$= \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix} \vec{x}[t] + \vec{B} \vec{K}^T \vec{v}_1 y$

$y = \vec{C}^T[t] \vec{x}[t]$

$\vec{x}[t+1] = \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T \vec{x}[t] + \vec{B} \vec{k}^T \vec{v}_1 \vec{y}$

$= \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T \vec{x}[t] + \vec{B} \vec{K}^T \vec{v}_1 \vec{C}^T[t] \vec{x}(t)$

$= \left(\sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T + \vec{B} \vec{K}^T \vec{v}_1 \vec{C}^T[t] \right) \vec{x}[t]$

If $\vec{B} \vec{K}^T \vec{v}_1 \vec{C}^T[k] = \lambda_1 \vec{v}_1 \vec{v}_1^T$

λ $\quad \underbrace{\vec{B} \vec{K}^T}_{\text{rank one matrix}} \vec{v}_1 C^T(k) = \lambda_1 \vec{v}_1 \vec{v}_1^T$

$\begin{bmatrix} \vec{v}_1 + \vec{v}_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T + \vec{v}_2^T \end{bmatrix}$

$\begin{bmatrix} \vec{v}_1 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T + \vec{v}_2^T \end{bmatrix}$

$+ \begin{bmatrix} \vec{v}_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T + \vec{v}_2^T \end{bmatrix}$

$\vec{v}_1 \vec{v}_1^T + \vec{v}_2 \vec{v}_1^T + \vec{v}_1 \vec{v}_2^T$

What effect does state feedback have?
Does it change eigenvectors too?

$\vec{b} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$

Schur form

CCF ←

Symmetric necessarily controllable?

\+ Adding to

$\vec{b} \vec{v}_1^T = \sum_{i=1}^{n} \alpha_i \vec{v}_i \vec{v}_1^T$

$\Lambda + \vec{b} \vec{v}_1^T = \sum \left(\lambda_i \vec{v}_i \vec{v}_i^T + \alpha_i \vec{v}_i \vec{v}_1^T \right)$

$\sum (\lambda_i + \alpha_i) (\vec{v}_i^T + \vec{v}_1^T)$