EECS 16A Designing Information Devices and Systems I
Fall 2018 Midterm 2 Instructions

Read the following instructions before the exam.

There are 8 problems of varying numbers of points. You have 120 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can.

There are 28 pages on the exam, so there should be 14 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page before time is called. If a page is found without a student ID, we are not responsible for identifying the student who wrote that page.

You may consult ONE handwritten 8.5" × 11" note sheet (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate. **If you still run out of space, please use a boxed space for another part of the same problem and clearly tell us in the original problem space where to look.**

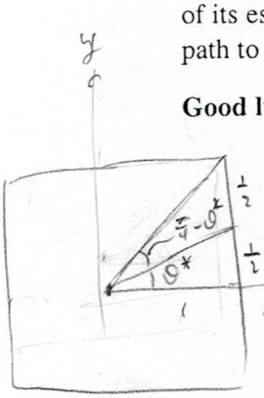
In general, show all of your work in order to receive full credit.

Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

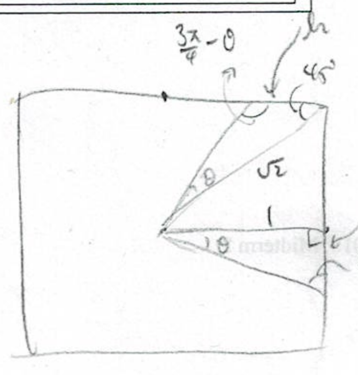
Our advice to you: if you can't solve the problem, state and solve a simpler one that captures at least some of its essence. You might get some partial credit, and more importantly, you will perhaps find yourself on a path to the solution.

Good luck!



$\theta^* = \tan^{-1}(\frac{1}{2})$
$= 26.56°$
$P(0 < \theta < 26.56°)$
$= P(26.56° < \theta < 45°)$

Do not turn this page until the proctor tells you to do so.



$\Rightarrow \begin{cases} l_1 = \tan\theta \\ \dfrac{l_2}{\sin\theta} = \dfrac{\sqrt{2}}{\sin(\frac{3\pi}{4}-\theta)} \end{cases}$

$\theta = \frac{\pi}{4}$:

$l_1 = \sqrt{2}$
$l_2 =$

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Doodle page!

Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

① $y_1 = \begin{bmatrix} |\vec{x}(0)\cdot\vec{d}(0)| \\ |A\vec{x}(0)\cdot\vec{d}(1)| \\ |A^2\vec{x}(0)\cdot\vec{d}(2)| \\ \vdots \end{bmatrix} \xrightarrow{\text{if } A\vec{x}=\lambda\vec{x}} \begin{bmatrix} |\vec{x}(0)\cdot\vec{d}(0)| \\ |\lambda\vec{x}(0)\cdot\vec{d}(1)| \\ |\lambda^2\vec{x}(0)\cdot\vec{d}(2)| \\ \vdots \end{bmatrix}$

② $y_2 = \log(y_1) = \begin{bmatrix} \log|\vec{x}(0)| + \log|\vec{d}(0)| \\ \log|\vec{x}(0)| + \log|\vec{d}(1)| + \log|\lambda| \\ \log|\vec{x}(0)| + \log|\vec{d}(2)| + 2\log|\lambda| \end{bmatrix}$

③ let $\begin{cases} \theta_i: \text{ angle between } A^i\vec{x}(0) \text{ and } \vec{d}(i), \quad \vec{d}(i): \text{unit norm} \\ X_i = |\cos\theta_i| \end{cases}$

$\Rightarrow |A^i\vec{x}(0)\cdot\vec{d}(i)| = |A^i\vec{x}(0)|\cdot X_i$

$E[x_{n} d_n][x_{n-1} \cdot d_{n-1}]$

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(f) (6 points) Using your design from Figure 8.5, find your V_{BB}, the signal going into the black box speaker system, as a function of x, the location of touch on the touch bar.

$$\begin{bmatrix} x_n d_n \\ x_{n-1} d_{n-1} \\ \vdots \\ x_1 d_1 \\ \cancel{x_0 d_0} \end{bmatrix} = \begin{bmatrix} x_{n-1} d_{n} \\ x_{n-2} d_{n-1} \\ \vdots \\ x_0 d_1 \end{bmatrix} \lambda + \begin{bmatrix} w_{n-1} d_n \\ w_{n-2} d_{n-1} \\ \vdots \end{bmatrix}$$

$$E\left(\frac{Y_1 \cdot \bar{X}_2}{Y_3 \cdot \bar{X}_4}\right) = E\left(\frac{N}{\cdots}\right)$$

(g) (1 point) Would you hit like, comment, and subscribe on Alan's YouTube channel? (*This is a fun question! Any answer will receive full credit.*)

$$\begin{bmatrix} x_{n-1}^T d_n \\ x_{n-2}^T d_{n-1} \\ \vdots \\ x_0^T d_1 \end{bmatrix} = \mathrm{diag}\left(\begin{bmatrix} x_{n-1}^T & x_{n-2}^T & \cdots & x_0^T \end{bmatrix} \begin{bmatrix} d_n \\ d_{n-1} \\ \vdots \\ d_1 \end{bmatrix} \right) = \quad \sim \frac{1}{}$$

Cost: $E[(\lambda-\hat{\lambda})^2]$ small

PRINT your student ID: $E[(\lambda/\hat{\lambda})]$ ~~small~~ 1 $E[(\hat{\lambda}/\lambda)]$ ~~small~~ 1.

∘ Stabilizing:

$E[X_n^2]$:

Extra page for scratchwork.

Work on this page will NOT be graded.

$x_{n+1} = \lambda x_n + w_n$.

$x_{n+1} = A x_n + w_n$.

$x_n \quad d_n$

$\boxed{x_{n+1} \cdot d_{n+1}} = \lambda \boxed{x_n \cdot d_{n+1}} + w_n \cdot d_{n+1}$

✓ X

$(x_n^T \cdot d_n)$

$E\left[\frac{\lambda}{X^2}\right] \approx$

$E[\log|\lambda - \hat{\lambda}|] \approx n^{-3}$

$E[\log(\boxed{\frac{\lambda}{\hat{\lambda}}})] \approx n^{-3}$.

$\log(\frac{\lambda}{\hat{\lambda}}) \approx n^{-3}$.

$\frac{1}{n^3} \sim 0$

$\frac{\lambda}{\hat{\lambda}} \approx e^{n^{-3}}$.

$\frac{\lambda}{\hat{\lambda}} \approx e^{n^{-3}}$

$\hat{\lambda} \approx \lambda \cdot \boxed{e^{-n^{-3}}}$

$\lambda \cdot e^{-n^{-3}} \approx \hat{\lambda}$

$E[(\lambda-\hat{\lambda})^2] \leq e^{-n}$

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Doodle page!

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$$\log \|\vec{x}[0]\| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \log|\lambda| \begin{bmatrix} 0 \\ 2 \\ \vdots \\ n-1 \end{bmatrix}$$

$$e^{n^{-3}} \approx 1 + n^{-3} = 1 + \varepsilon$$

$$\varepsilon = n^{-3}$$

$$\vec{x}_{n+1} = A\vec{x}_n + \vec{u}$$

$$\hat{v} = \vec{v}_1 + \vec{\varepsilon}$$

Does a mult. approx. of λ help control?

Can we go from $\log \lambda$ to λ?

(Phrase the estimation problem in that way)

How to estimate $x_n \cdot d_n$?

Can we estimate $x_{n-1}^T d_n$ from $x_{n-1}^T d_{n-1}$?

$$\vec{x}_{n+1} = A\vec{x}_n + \underline{\lambda_1 \hat{v}} - \hat{\lambda}_1, \hat{v}$$

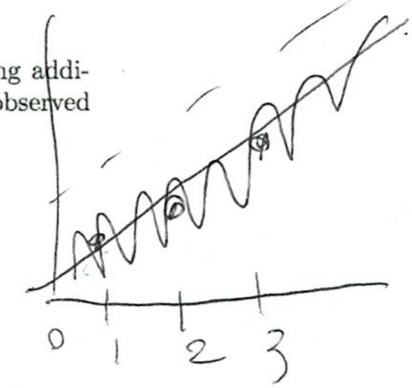
$$\vec{x}_{n+1} =$$

Probability of convergence of eigenvalues

$\Pr\left[e^{tX} \geq e^{ta}\right] \leq \frac{\mathrm{E}[e^{tX}]}{e^{ta}}$

$Y_2 \approx mx + c$

Variance



The approach is as follows: with some initial state $\vec{x}[0]$, keep observing additional states with no control applied, and consider the magnitude of the observed projections. We obtain the scalar observations

$\dfrac{\lambda^n \vec{x}[0] \cdot \vec{d}_n}{|\lambda|^n \|\vec{x}[0]\| \cos\theta_{\vec{d}_n, \vec{x}[0]}}$

$$\vec{y}_1 = \begin{bmatrix} \left\|\vec{x}[0] \cdot \vec{d}[0]\right\| \\ \left\|A\vec{x}[0] \cdot \vec{d}[1]\right\| \\ \left\|A^2\vec{x}[0] \cdot \vec{d}[2]\right\| \\ \vdots \end{bmatrix}$$

Let's consider a slightly simpler problem, Assume that $A\vec{x} = \lambda\vec{x}$. So then we have

$$\vec{y}_1 = \begin{bmatrix} \|\vec{x}[0]\| X_0 \\ \lambda\|\vec{x}[0]\| X_1 \\ \lambda^2\|\vec{x}[0]\| X_2 \\ \vdots \end{bmatrix},$$

where the X_i are independent nonnegative random variables, representing the absolute value of the x-coordinate of a random unit vector.

We take logs of all the entries of $\vec{y}_1$, to ob~~tain~~

$$\vec{y}_2 = \begin{bmatrix} \log\|\vec{x}[0]\| + \log X_0 \\ \log\|\vec{x}[0]\| + \log X_1 + \log\lambda \\ \log\|\vec{x}[0]\| + \log X_2 + 2\log\lambda \\ \vdots \end{bmatrix} = \begin{bmatrix} \log\|\vec{x}[0]\| \\ \log\|\vec{x}[0]\| \\ \log\|\vec{x}[0]\| \\ \vdots \end{bmatrix} + \begin{bmatrix} \log X_0 \\ \log X_1 \\ \log X_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \end{bmatrix} \log|\lambda|$$

$c = Y_2[0]$

$m + c = Y_2[1]$

$2m + c = Y_2[2]$

Using least squares, we aim to approximately solve the linear equation

$$A\vec{\beta} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \end{bmatrix} \boxed{\vec{\beta}} = \begin{bmatrix} \log\|\vec{x}[0]\| \\ \log\|\vec{x}[0]\| \\ \log\|\vec{x}[0]\| \\ \vdots \end{bmatrix} + \begin{bmatrix} \log X_0 \\ \log X_1 \\ \log X_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \end{bmatrix} \log\lambda$$

We wish to compute the variance of β. If the X_i were all zero, then we would obtain a $\hat{\beta}$ that recovers $\log\|\vec{x}[0]\|$ and $\log\lambda$ perfectly. Otherwise, we recover the approximation

$$\begin{bmatrix} w \\ l(\lambda) \end{bmatrix} \quad \beta = (A^T A)^{-1} A^T \vec{y} \neq \hat{\beta}.$$

$\beta = \begin{bmatrix} c \\ m \end{bmatrix}$

$\mathrm{Var}[X] \equiv E[XX^T]$

We have

$$\begin{aligned}
\mathrm{Var}(\beta) &= \mathrm{Var}((A^TA)^{-1}A^T\vec{y}) \\
&= (A^TA)^{-1}A^T\mathrm{Var}(\vec{y})A(A^TA)^{-1} \\
&= \mathrm{Var}(\vec{y})(A^TA)^{-1}A^TA(A^TA)^{-1} \\
\overset{?}{=}\quad &= \mathrm{Var}(X_i)(A^TA)^{-1},
\end{aligned}$$

since the X_i are independent with the same variance.

Now, we will make an aymptotic estimate on the variance of β (which we can refine later) to obtain a bound on the error of our estimate of λ. Observe that

$$A^TA = \begin{bmatrix} 1 & 1 & \dots & \\ 0 & 1 & 2 & \dots \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \Theta(N) & \Theta(N^2) \\ \Theta(N^2) & \Theta(N^3) \end{bmatrix},$$

so

$$(A^TA)^{-1} = \frac{1}{\Theta(N^4)} \begin{bmatrix} \Theta(N^3) & \Theta(N^2) \\ \Theta(N^2) & \Theta(N) \end{bmatrix} = \begin{bmatrix} \Theta(N^{-1}) & \Theta(N^{-2}) \\ \Theta(N^{-2}) & \Theta(N^{-3}) \end{bmatrix}.$$

Therefore, we find that

$$\begin{aligned}
\mathrm{Var}(\beta) &= \begin{bmatrix} \Theta(N^{-1}\mathrm{Var}(X_i)) & \Theta(N^{-2}\mathrm{Var}(X_i)) \\ \Theta(N^{-2}\mathrm{Var}(X_i)) & \Theta(N^{-3}\mathrm{Var}(X_i)) \end{bmatrix} \\
\implies \quad \mathrm{Var}(\log\lambda) &= \Theta(N^{-3}\mathrm{Var}(X_i)).
\end{aligned}$$

Thus, we can use Chebyshev's inequality to obtain bounds on the error for $\log\lambda$, since our estimate for it is unbiased.

$\lambda\left(1 \pm \frac{1}{n^3}\right)$

$E\left(\frac{x_n d_n}{x_{n+1} d}\right)$

E

$\lambda = x_{n-1} d_n$