Tutorial 1
Proofs and Logic

Tutorials are meant to reinforce the material taught in lecture. Therefore, please try these exercises before going to class. In doing so, you may discover gaps in your understanding. You may be asked to present your solutions in class, and this can help you see alternative solutions your classmates may have. Keep in mind that the goal of tutorials is *not* to answer every exercise, but to clarify doubts and reinforce concepts. Solutions to all tutorial exercises will be given in the following week.

> Define $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ to be the set of natural numbers.
>
> Also, an integer n is said to be *odd* if it can be written as $n = 2k + 1$; and *even* if it can be written as $n = 2k$, for some $k \in \mathbb{Z}$.
>
> Every integer is either odd or even, but not both.

Q1. You are given these two statement forms $(\sim p \vee q) \wedge q$ and $(\sim p \wedge q) \vee q$.

a. Use a truth table to make your conclusion about these two statement forms.

b. Aiken concluded from part (a) that if a statement form contains both $\vee$ and $\wedge$, then you can always interchange $\vee$ and $\wedge$ to obtain an equivalent statement form. Dueet does not agree with him. If you were Dueet, what would you do to prove your point by using these two statement forms $(\sim p \vee q) \wedge p$ and $(\sim p \wedge q) \vee p$?

Q2. Use Theorem 2.1.1 (Epp) to verify the logical equivalences for the following parts. Supply a reason for each step (eg: "by the commutative law").

a. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$.

b. $(p \wedge \sim(\sim p \vee q)) \vee (p \wedge q) \equiv p$.

Q3. Some of the arguments below are valid, whereas others exhibit the converse or the inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity. Otherwise, state whether the converse or the inverse error is made.

a. Sandra knows Java and Sandra knows C++.
$\therefore$ Sandra knows C++.

b. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
$\therefore$ The product of these two number is not divisible by 6.

c. If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.
The set of all irrational numbers is infinite.
$\therefore$ There are as many rational numbers as there are irrational numbers.

d. If I get a Christmas bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.
$\therefore$ If I get a Christmas bonus or I sell my motorcycle, then I'll buy a stereo.

Q4. *Knights and Knaves*
The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and have the following encounters with natives.

(a) Two natives A and B address you as follows:
A says: Both of us are knights.
B says: A is a knave.
What are A and B?

(b) Another two natives C and D approach you but only C speaks:
C says: Both of us are knaves.
What are C and D?

(c) You then encounter natives E and F.
E says: F is a knave.
F says: E is a knave.
How many knights and knaves are there?

Q5. Prove that the product of two odd natural numbers is an odd natural number.

Q6. Prove that $\exists! x \in \mathbb{R}, \exists! y \in \mathbb{R}$ such that $(x-a)^2 + (y-b)^2 = 0$, where a, b are real constants.

Q7. I.M. Smart, your CS1231 classmate, came across this question:

Prove that if a, b, c are integers such that $a^2 + b^2 = c^2$, then a, b cannot both be odd.

(a) Smart attempts to prove the above:

Proof (by Contradiction).

1. Suppose a, b are both odd.
2. Then $\exists m, n \in \mathbb{Z}$ such that $a = 2m+1$, and $b = 2n+1$.
3. Then $a^2 + b^2 = (2m+1)^2 + (2n+1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2 = c^2$.
4. Then $c = \sqrt{4m^2 + 4m + 4n^2 + 4n + 2}$.
5. But the right hand side is not an integer.
6. This contradicts the fact that c is an integer.
7. Hence a, b cannot both be odd. ∎

Explain why Smart's proof is incomplete.

(b) Not to be left out, Aiken also tries to prove the statement:

Proof (by Contraposition).

1. (Want to prove: if a, b are both odd, then $a^2 + b^2 \neq c^2$)
2. Suppose a, b are both odd.
3. Then $\exists m, n \in \mathbb{Z}$ such that $a = 2m+1$, and $b = 2n+1$.
4. Then $a^2 + b^2 = (2m+1)^2 + (2n+1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2$.

5. This number is even, and has remainder 2 when divided by 4.
6. Now, c is either odd or even.
7. <u>Case 1: c is odd</u>
   7.1. Then c^2 is odd.
   7.2. Then $c^2 \neq a^2 + b^2$, because the right hand side is even.
8. <u>Case 2: c is even</u>
   8.1. Then c^2 is even.
   8.2. Then $\exists k \in \mathbb{Z}$ such that $c = 2k$.
   8.3. Then $c^2 = 4k^2$.
   8.4. But this has remainder 0 when divided by 4.
   8.5. Hence $c^2 \neq a^2 + b^2$, since the right hand side has remainder 2 when divided by 4 (from Line 5.).
9. In all cases, $c^2 \neq a^2 + b^2$.
10. Therefore, by Contraposition, the original statement is true. ■

Is Aiken's proof correct? If so, how can it be improved? If not, where is it wrong?