ECE368: Probabilistic Reasoning
Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \ldots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

   Forward

   Initialization, $\alpha(z_0) = p(z_0)\, p((\tilde{x}_0, \tilde{y}_0) | z_0)$

   Recursion, $\alpha(z_i) = p((\tilde{x}_i, \tilde{y}_i) | z_i) \cdot \sum_{z_{i-1}} (\alpha(z_{i-1}) \cdot p(z_i | z_{i-1}))$

   Backward

   Initialization, $\beta(z_{n-1}) = 1$

   Recursion, $\beta(z_i) = \sum_{z_{i+1}} (\beta(z_{i+1}) \cdot p((\tilde{x}_{i+1}, \tilde{y}_{i+1}) | z_{i+1}) \cdot p(z_{i+1} | z_i))$

   Marginal distribution, $p(z_n | (\tilde{x}_0, \tilde{y}_0), \ldots (\tilde{x}_{n-1}, \tilde{y}_{n-1})) = \dfrac{\alpha(z_n) \cdot \beta(z_n)}{\sum_{z_n} \alpha(z_n) \cdot \beta(z_n)}$ Normalized.

   (b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

   $$p(z_{99} | (\hat{x}_0, \hat{y}_0) \ldots (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.81026 & (11, 0, \text{'stay'}) \\ 0.179608 & (11, 0, \text{'right'}) \\ 0.010123 & (10, 1, \text{'down'}) \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0, \hat{y}_0), \ldots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

   $$p(z_{30} | (\hat{x}_0, \hat{y}_0) \ldots (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.91304 & (6, 7, \text{'right'}) \\ 0.043478 & (5, 7, \text{'right'}) \\ 0.04347 & (5, 7, \text{'stay'}) \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

Initialization

$$w_0(z_0) = \ln\left(p(z_0) * P((\hat{x}_0, \hat{y}_0) | z_0) \right) \triangleq w_1(z_1)$$

Recursion

$$w_i(z_i) = \ln p((\hat{x}_i, \hat{y}_i) | z_i) + \max_{z_{i-1}} \left\{ \ln p(z_i | z_{i-1}) + w_{i-1}(z_{i-1}) \right\}$$

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

i	MAP Estimate
90	11,5, 'down'
91	11,6, 'down'
92	11,7, 'down'
93	11,7, 'stay'
94	11,7, 'stay'
95	10,7, 'left'
96	9,7, 'left'

97) 8,7, 'left'

98) 7,7, 'left'

99) 6,7, 'left'

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

$z_{64} = (3, 7, \text{'stay'})$ $z_{65} = (2, 7, \text{'stay'})$

At z_{64}, action is 'stay' so position should not change in z_{65} but in z_{65}, the position has changed $\therefore$ it is incorrect.