

# CPE 381: Fundamentals of Signals and Systems for Computer Engineers

## 07 Frequency Analysis: Fourier Transform

**Rahul Bhadani**

**Electrical & Computer Engineering, The University of Alabama in Huntsville**

# Outline

---

1. Fourier Transform of an Aperiodic Signal
2. Properties of Fourier Transform
3. Spectral Representation
4. Filtering



# Fourier Transform of an Aperiodic Signal

# Aperiodic Signal

---

- ⚡ Aperiodic signal can be thought of as a periodic signal with infinite fundamental frequency.
- ⚡ In practice, there are no periodic signals, but they are nice mathematical tools.

$$x(t) = \lim_{T_0 \rightarrow \infty} \tilde{x}(t)$$

# From Fourier Series to Fourier Transform

Consider

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

and its Fourier coefficient is

$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jn\Omega_0 t} dt$$

In the limiting case  $T_0 \rightarrow \infty$ ,  $X_n \rightarrow 0$ , so to avoid it,  $X(\Omega_n) = T_0 X_n$ , and  $\Omega_n = n\Omega_0$  are the harmonic frequencies.

# From Fourier Series to Fourier Transform

Rewriting  $\Omega_0 = \Delta\Omega = \frac{2\pi}{T_0}$

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X(\Omega_n) e^{j\Omega_n t} = \sum_{n=-\infty}^{\infty} X(\Omega_n) e^{j\Omega_n t} \frac{\Delta\Omega}{2\pi}$$

$$X(\Omega_n) = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-j\Omega_n t} dt$$

As  $T_0 \rightarrow \infty$ ,  $\Delta\Omega \rightarrow d\Omega$  and sum becomes integral. Thus  $\Omega_n = n\Omega_0 = n\Delta\Omega \rightarrow \Omega$ .

# Fourier Transform

Hence, in the limiting case,  $T_0 \rightarrow \infty$ ,

Fourier Transform

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt$$

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t} d\Omega$$

# Obtaining Fourier Transform

- ⚡ If a signal's Laplace transform includes  $j\Omega$  axis in its ROC, then we can simply compute the Fourier Transform by first computing its Laplace transform and set  $s = j\Omega$ .
- ⚡ If  $x(t)$  has a finite time support and in that support  $x(t)$  is bounded, its Fourier transform exists. To find it use the integral definition or the Laplace transform of  $x(t)$ .
- ⚡ If  $x(t)$  is periodic, its Fourier transform is obtained using the signal's Fourier series.
- ⚡ For some functions, e.g. the integral is not well-defined, or has discontinuities then the integration formula cannot be used for calculating the Fourier transform. We will use properties of Fourier transform in such a case.



**It appears that almost nothing has a Fourier transform—nothing except practical communication signals. No signal amplitude goes to infinity and no signal lasts forever; therefore, no practical signal can have an infinite area under it, and hence all have Fourier transforms.**

**- E. Craig**

# Frequency Content with Fourier Transform

---

- ⚡ The Fourier transform measures the frequency content of a signal.
- ⚡ Time-domain vs Frequency-domain: The characterization in one domain provides information that is not clearly available in the other.

# Fourier Transform using Laplace Transform

For  $X(\Omega)$  to exist,

- ⚡  $x(t)$  must be absolutely integrable.
- ⚡  $x(t)$  has only a finite number of discontinuities and a finite number of minima and maxima in any finite interval.
- ⚡ ROC of its Laplace transform  $X(s)$  must have  $j\Omega$  axis.

$$\mathcal{F}[x(t)] = \mathcal{L}[x(t)] \Big|_{s=j\Omega} = X(s) \Big|_{s=j\Omega}$$

The duality between time and frequency allows the computation of Fourier transforms.

# Fourier Series from Laplace Transform

1.  $x(t) = u(t)$ ,  $X(s) = \frac{1}{s}$ , ROC:  $\sigma > 0$ ,  $j\Omega$ -axis is not included, FT cannot be obtained using Laplace transform.

2.  $x(t) = e^{-2t}u(t)$ ,  $X(s) = \frac{1}{s+2}$ , ROC:  $\sigma > -2$ , then

$$X(\Omega) = \left. \frac{1}{s+2} \right|_{s=j\Omega} = \frac{1}{j\Omega+2}$$

3.  $x(t) = e^{-|t|}$ ,  $X(s) = \frac{1}{s+1} + \frac{1}{1-s}$ , ROC:  $-1 < \sigma < 1$ , then

$$X(\Omega) = \left. X(s) \right|_{s=j\Omega} = \frac{2}{1-(j\Omega)^2} = \frac{2}{1+\Omega^2}$$



# Properties of Fourier Transform

# Linearity

---

$$\mathcal{F}[\alpha x(t) + \beta y(t)] = \alpha \mathcal{F}[x(t)] + \beta \mathcal{F}[y(t)] = \alpha X(\Omega) + \beta Y(\Omega)$$

# Fourier Transform an Impulse Function

$$x(t) = \delta(t)$$
$$X(\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = e^{-j0} = \int_{-\infty}^{\infty} \delta(t) dt = 1, \quad -\infty < \Omega < \infty$$

Hence, if the delta function has finite support as seen, then its Fourier transform has infinite support in frequency.

It means  $\delta(t)$  changes so much in a very short time that its FT has all possible frequency components.

# Duality Property

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\rho) e^{j\rho t} d\rho$$

Replacing  $t$  with  $-\Omega$ ,

$$2\pi x(-\Omega) = \int_{-\infty}^{\infty} e^{-j\rho\Omega} d\rho = \int_{-\infty}^{\infty} X(t) e^{-j\Omega t} dt = \mathcal{F}[X(t)]$$

after replacing  $\rho$  with  $t$  as they are dummy variables. Hence

$$X(t) \Leftrightarrow 2\pi X(-\Omega)$$



# Fourier Transform of a Constant Signal

$$x(t) = A, \quad -\infty < t < \infty$$

As, it is a constant, natural  $\Omega = 0$  is the frequency. We cannot find its frequency in the usual way it is not absolutely integrable. Then we can use the **duality property**.

$$\delta(t) \Leftrightarrow 1$$

$$A\delta(t) \Leftrightarrow A$$

$$A \Leftrightarrow 2\pi A\delta(-\Omega) = 2\pi A\delta(\Omega)$$

Notice the last step because  $\delta$  is an even function.

# Fourier Transform of Shifted Delta and Cosines

## Shifted Delta

$$\delta(t - b) + \delta(t + b) \Leftrightarrow e^{-jb\Omega} + e^{jb\Omega} = 2 \cos(b\Omega)$$

Apply Laplace Transform to  $\delta(t \pm b)$ , and set  $s = j\Omega$ , and  $\delta(t) \leftrightarrow 1$ .

## Cosine

We use duality property:

$$2 \cos(bt) \Leftrightarrow 2\pi[\delta(\Omega - b) + \delta(\Omega + b)]$$

$$\cos(bt) \Leftrightarrow \pi[\delta(\Omega - b) + \delta(\Omega + b)]$$

# Example

---

Use duality to find the Fourier transform of  $x(t) = 10\text{sinc}(0.5t)$ .

Blank space for calculation

Blank space for calculation

Blank space for calculation

# Time Frequency Inverse Proportionality

Support of  $X(\Omega)$  is inversely proportional to the support of  $x(t)$ . If  $x(t)$  has Fourier transform  $X(\Omega)$ , and  $\alpha \neq 0$ , a real number, then

$x(\alpha t)$

⚡ is a contracted signal when  $\alpha > 1$

⚡, is a contracted and reflected signal when  $\alpha < -1$

⚡ is an expanded signal when  $0 < \alpha < 1$

⚡ is an expanded and reflected signal when  $-1 < \alpha < 0$

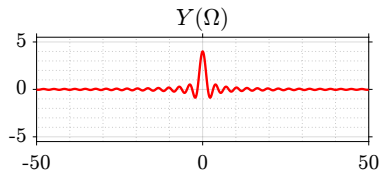
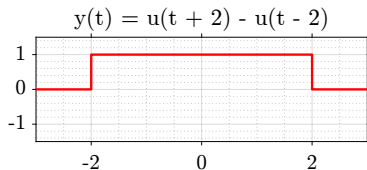
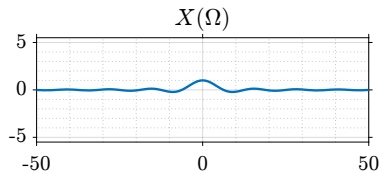
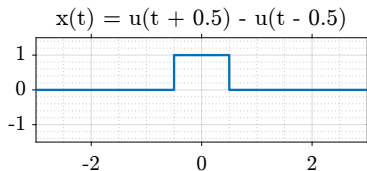
⚡ reflect signal when  $\alpha = -1$

and,

$$x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{\Omega}{\alpha}\right)$$

# Inverse Proportionality of Time and Frequency:

## Example 1



Notice that the wider the pulse the more concentrated in frequency its Fourier transform.

Code: [https://github.com/rahulbhadani/CPE381\\_FA24/blob/master/Code/fourier\\_transform\\_rect.m](https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/fourier_transform_rect.m)



# Inverse proportionality of time and frequency:

## Example 2

---

Consider a pulse  $x(t) = u(t) - u(t - 1)$ . Find its Fourier transform, and the Fourier Transform of  $x(2t)$ .

# Inverse proportionality of time and frequency:

## Example 2

---

Blank space for calculation

# Inverse proportionality of time and frequency:

## Example 2

---

Blank space for calculation

# Inverse proportionality of time and frequency:

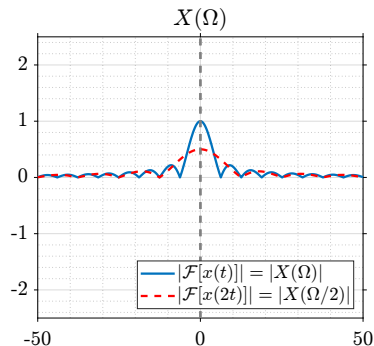
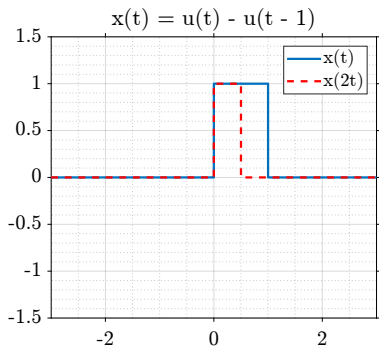
## Example 2

---

Blank space for calculation

# Inverse proportionality of time and frequency:

## Example 2



$x(t)$  has finite support, while  $X(\Omega)$  has infinite support.

Code: [https://github.com/rahulbhadani/CPE381\\_FA24/blob/master/Code/FourierTransform\\_Rect\\_Expanded.m](https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/FourierTransform_Rect_Expanded.m)

## Example 3

---

Apply the reflection property to find the Fourier transform of  $x(t) = e^{-a|t|}$ ,  $a > 0$ .

# Example 3

---

Blank space for calculation

# Example 3

---

Blank space for calculation

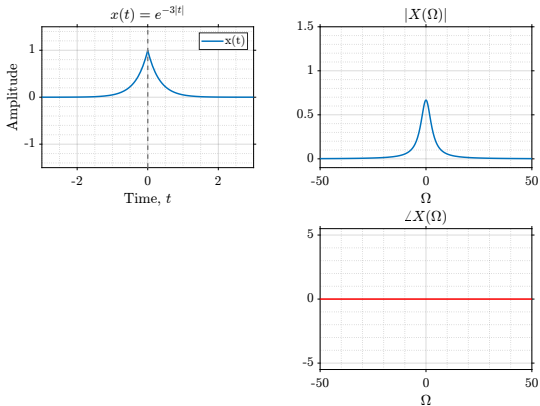


# Example 3

---

Blank space for calculation

## Example 3



From this plot, we see that its energy is concentrated mostly in low frequencies, hence the signal is called **low-pass**.

Code: [https://github.com/rahulbhadani/CPE381\\_FA24/blob/master/Code/FourierTransform\\_Exp\\_Abs.m](https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/FourierTransform_Exp_Abs.m)



# Spectral Representation

# Modulation

$$\mathcal{F}[x(t)e^{j\Omega_0 t}] = \int_{-\infty}^{\infty} [x(t)e^{j\Omega_0 t}]e^{-j\Omega t} dt = \int x(t)e^{j(\Omega - \Omega_0)t} dt = X(\Omega - \Omega_0)$$

Then applying frequency shifting property to  $x(t) \cos(\Omega_0 t)$ , we have

$$x(t) \cos(\Omega_0 t) \Leftrightarrow 0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

Here,  $x(t)$  is the main signal, called the **message**,  $\cos(\Omega_0 t)$  is called the **carrier**, and  $x(t) \cos(\Omega_0 t)$  is the **modulated signal**.

The scheme discussed is called Amplitude Modulation

# Why Amplitude Modulation?

- ⚡ Music signals are audible at up to 22 KHz, while speech signals are at 100 Hz - 5 KHz. They are relatively low frequency.
- ⚡ The length of an antenna that is used to radiate the signal is a quarter of a wavelength, given by

$$\lambda = \frac{c}{f} \text{ meter}$$

$c = 3 \times 10^8 \text{ m/s}$  is the speed of light, and  $f$  is the frequency in Hz of the signal to be radiated.

- ⚡ Assuming  $f = 30$  KHz, normal for music and speech with some noise, the  $\lambda \approx 10$  km, hence antenna size would be 2.5 km. A very giant antenna.
- ⚡ Choosing an appropriate modulation frequency helps in reducing the size of the antenna.

# Fourier Transform of Periodic Signals

If the function is periodic, we know that it can be represented as a Fourier series, i.e.

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_k t}$$

where

$$X_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-j\Omega_k t} dt$$

From the linearity and frequency-shifting property, we can write

$$X(\Omega) = \sum_k \mathcal{F}[X_k e^{j\Omega_k t}] = \sum_k 2\pi X_k \delta(\Omega - \Omega_k)$$

$$\Omega_k = k\Omega_0.$$

## Example 4

---

Consider a periodic signal  $x(t)$  with a period  $x_1(t) = r(t) - 2r(t - 0.5) + r(t - 1)$ . Its fundamental frequency is  $\Omega_0 = 2\pi$ . Determine its Fourier transform analytically and using MATLAB.

# Example 4

---

Blank space for calculation



# Example 4

---

Blank space for calculation

# Example 4

---

Blank space for calculation

# Parseval's Energy Relationship

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t) dt = \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\Omega) e^{j\Omega t} d\Omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\Omega) \left[ \int_{-\infty}^{\infty} x(t) e^{j\Omega t} dt \right] d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega \end{aligned}$$

Thus  $|X(\Omega)|^2$  is an energy density—indicating the amount of energy at each of the frequencies. The plot  $|X(\Omega)|^2$  vs. is called the energy spectrum of  $x(t)$ , and displays how the energy of the signal is distributed over frequency.

## Example 5: Pulse Signal and its Energy

Consider  $p(t) = u(t + 1) - u(t - 1)$ , use its Fourier transform  $P(\Omega)$  and Parseval's energy relation to show that

$$\int_{-\infty}^{\infty} \left( \frac{\sin(\Omega)}{\Omega} \right)^2 = \pi$$

# Example 5: Pulse Signal and its Energy

---

Blank space for calculation

# Example 5: Pulse Signal and its Energy

---

Blank space for calculation

# Example 5: Pulse Signal and its Energy

---

Blank space for calculation

# Spectral Representation

---

- ⚡ Magnitude spectrum:  $|X(\Omega)|$  vs  $\Omega$ : even function
- ⚡ Phase spectrum:  $\angle X(\Omega)$  vs  $\Omega$ : odd function
- ⚡ Energy/Power spectrum:  $|X(\Omega)|^2$  vs  $\Omega$ : even function





# Filtering

# Frequency Response

$H(j\Omega)$  (or  $H(\Omega)$ ) is the frequency response of the system. Thus,

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [X(\Omega)H(\Omega)]e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} [Y(\Omega)]e^{j\Omega t} d\Omega$$

Thus  $Y(\Omega) = X(\Omega)H(\Omega)$  For a periodic signal  $X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - \Omega_k)$ ,

$$Y(\Omega) = X(\Omega)H(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(\Omega_k) \delta(\Omega - \Omega_k)$$

Thus Fourier coefficient of the output  $y(t)$  is  $Y_k = X_k H(\Omega_k)$ .

# Filtering

---

In practice  $s(t) = x(t) + \eta(t)$ , where  $\eta$  is the noise. We want to get rid of noise. So we design a special system called **Filters** that will by assuming that noise has certain frequencies, and filters will remove those frequencies.

The problem of filtering is finding the transfer function  $H(s)$  of the filter (or the coefficients of the denominator and numerator of the transfer function).

## Example 6: From Full-wave Rectifier to DC Source

---

Let  $x(t) = |\cos(\pi t)|$  is a full-wave rectifier be an input to a filter  $H(\Omega)$ , and the output is a DC signal. What should  $H(\Omega)$  look like?

# Example 6: From Full-wave Rectifier to DC Source

---

Blank space for calculation

# Example 6: From Full-wave Rectifier to DC Source

---

Blank space for calculation

# Example 6: From Full-wave Rectifier to DC Source

---

Blank space for calculation

# Low Pass Filters

## Ideal Low Pass Filter

A filter that keeps low frequencies is called a low-pass filter. The magnitude response of an ideal low-pass filter is

$$|H_{lp}(\Omega)| = \begin{cases} 1, & -\Omega_1 \leq \Omega \leq \Omega_1 \\ 0, & \text{otherwise} \end{cases}$$

and the frequency response is  $\angle H_{lp}(\Omega) = -\alpha\Omega$  which is a linear phase.

⚡  $\Omega_1$  is called cut-off frequency.



# Band-pass Filter

## Ideal Band-pass Filter

*Keeps a band of frequencies*

Magnitude Response:

$$|H_{bp}(\Omega)| = \begin{cases} 1, & \Omega_1 \leq \Omega \leq \Omega_2 \text{ and } -\Omega_2 \leq \Omega \leq -\Omega_1 \\ 0, & \text{otherwise} \end{cases}$$

Its phase is also linear in the pass band.

# High-pass Filter

## Ideal High-pass Filter

*Keeps only high frequencies*

Magnitude Response:

$$|H_{hp}(\Omega)| = \begin{cases} 1, & \Omega \geq \Omega_2 \text{ and } -\Omega \leq -\Omega_2 \\ 0, & \text{otherwise} \end{cases}$$

Its phase is also linear in the pass band.

An ideal multi-band filter can be obtained by a combination of low, high, and band-pass filter.

# Band-stop Filter (Band-eliminating Filter)

## Ideal High-pass Filter

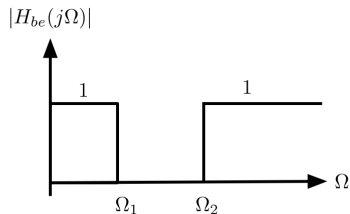
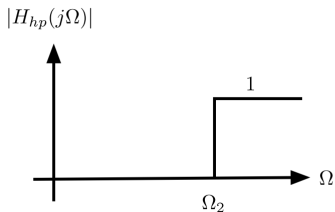
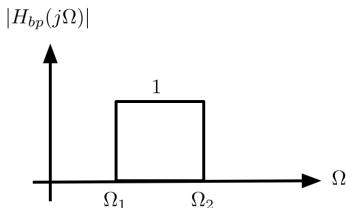
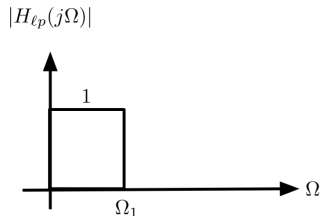
*Removes a range of frequencies*

Magnitude Response:

$$|H_{be}(\Omega)| = 1 - |H_{bp}(\Omega)|$$

Its phase is also linear in the pass band.

# Magnitude Response of Filter



# Frequency Response from Poles and Zeros

Consider a transfer function of the form  $G(s) = K \frac{s - z}{s - p}$  where  $K \neq 0$ , and  $p$  is a pole,  $z$  is a zero. Frequency response at some frequency  $\Omega_0$  is obtained by setting  $s = j\Omega_0$ . Hence,

$$G(\Omega) = K \frac{j\Omega_0 - z}{j\Omega_0 - p} = K \frac{\vec{Z}(\Omega_0)}{\vec{P}(\Omega_0)} = |K| e^{j\angle K} \frac{|\vec{Z}(\Omega_0)|}{|\vec{P}(\Omega_0)|} e^{j(\angle \vec{Z}(\Omega_0) - \angle \vec{P}(\Omega_0))}$$

Magnitude response is then

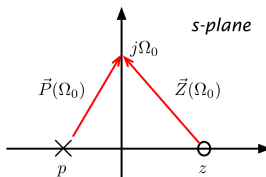
$$|G(\Omega)| = |K| \frac{|\vec{Z}(\Omega_0)|}{|\vec{P}(\Omega_0)|}$$

and the phase response as

$$\angle G(\Omega) = \angle K + \angle \vec{Z}(\Omega_0) - \angle \vec{P}(\Omega_0)$$

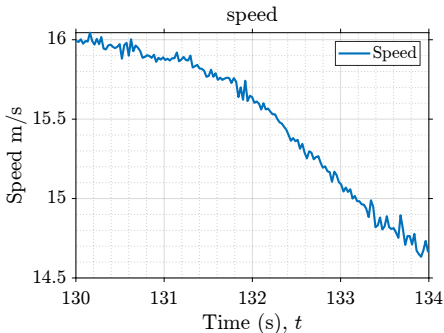
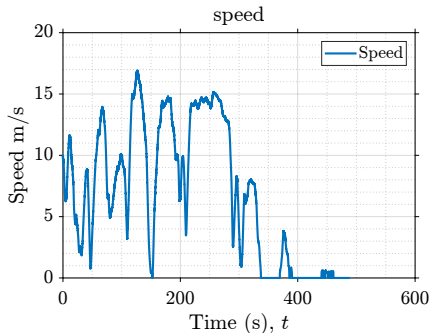
# Frequency Response from Poles and Zeros

- ⚡ Poles create **hills** at frequencies in the  $j\Omega$ -axis in front of the imaginary parts of the poles. The closer the pole is to the  $j\Omega$ -axis, the narrower and higher the hill. If, for instance, the poles are on the  $j\Omega$ -axis (this would correspond to an unstable and useless filter) the magnitude response at the frequencies of the poles will be infinity.
- ⚡ Zeros create **valleys** at the frequencies in the  $j\Omega$ -axis in front of the imaginary parts of the zeros. The closer a zero is to the  $j\Omega$ -axis approaching it from the left or the right (as the zeros are not restricted by stability to be in the open left-hand s-plane) the closer the magnitude response is to zero. If the zeros are on the  $j\Omega$ -axis, the magnitude response at the frequencies of the zeros is zero.



# Filtering a Real Signal I

Consider a data file that contains a timestamp and speed in  $km/h$ . We will convert it to  $m/s$  first.



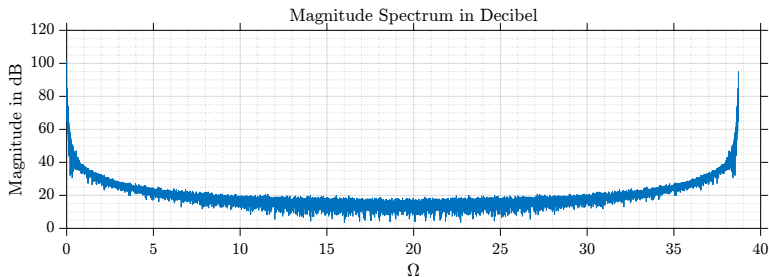
# Filtering a Real Signal II

```
csv_file = '../Data/vel.csv';  
data = readtable(csv_file);  
data.Time = data.Time - data.Time(1);  
% Convert the velocity data to m/s  
velocity_ms = data.Message * 1000 / 3600;
```

Compute Fourier Transform using FFT and Plot Magnitude Spectrum, we plot the magnitude spectrum in Decibel ( $20 \cdot \log_{10}$ ) for clarity.



# Filtering a Real Signal III

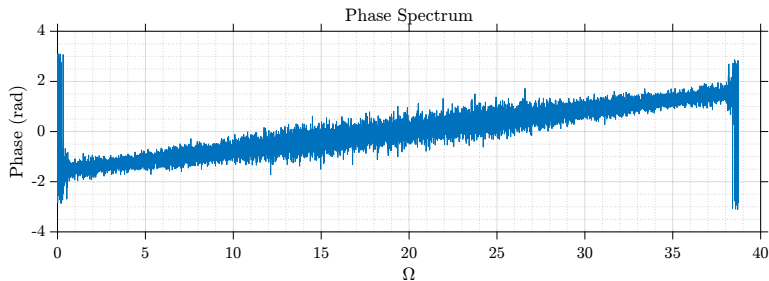


# Filtering a Real Signal IV

```
% Compute the Fourier transform of the velocity data
fs = 1 / mean(diff(data.Time)); % sampling frequency
velocity_fft = fft(velocity_ms);

% Plot the magnitude spectrum
f = figure;
f.Position = [ 810      819      1320      419];
plot((0:length(velocity_fft)-1) * fs / length(velocity_fft), 20*log10(abs(velocity_fft)));
```

# Filtering a Real Signal V



# Filtering a Real Signal VI

```
% Compute the Fourier transform of the velocity data
fs = 1 / mean(diff(data.Time)); % sampling frequency
velocity_fft = fft(velocity_ms);

% Plot the phase spectrum
f = figure;
f.Position = [ 810      819      1320      419];
plot((0:length(velocity_fft)-1) * fs / length(velocity_fft), angle(velocity_fft));
```

Now, we design a filter

# Filtering a Real Signal VII

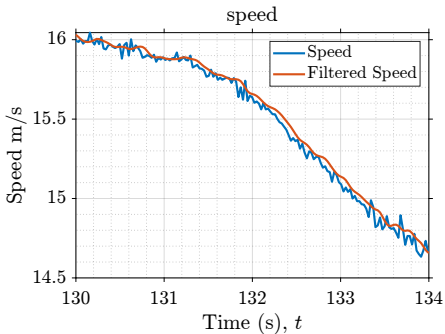
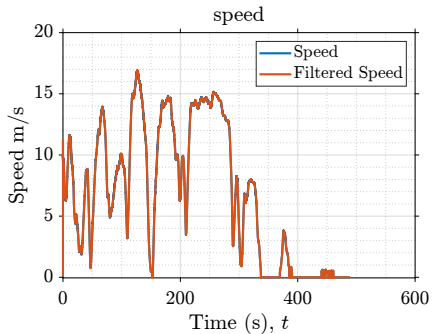
```
cutoff_freq = 5; % Hz
order = 4;

% Normalize the cutoff frequency to the Nyquist frequency
% Design the low-pass filter
[b, a] = butter(order, cutoff_freq / (fs/2), 'low')

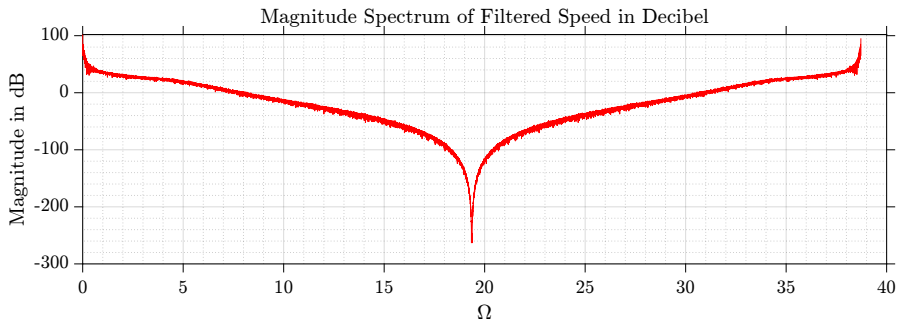
% Apply the filter to the velocity data
velocity_filtered = filter(b, a, velocity_ms);
```

```
b = [0.0003, 0.0013, 0.0019, 0.0013, 0.0003]
a = [1.0000, -3.2380, 3.9911, -2.2127, 0.4647]
```

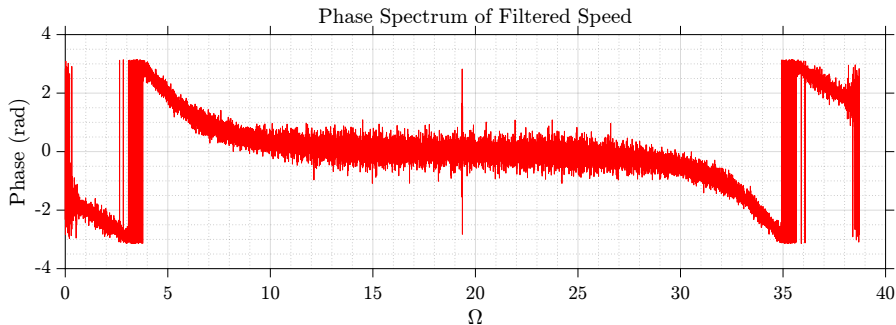
# Filtering a Real Signal VIII



# Filtering a Real Signal IX



# Filtering a Real Signal X



Full code at

[https://github.com/rahulbhadani/CPE381\\_FA24/blob/master/Code/speed\\_fourier.m](https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/speed_fourier.m)



# Time-shift Properties

---

$$x(t \pm t_0) \Leftrightarrow X(\Omega)e^{\pm j\Omega t_0}$$

# Differentiation Property

---

$$\frac{d^N x(t)}{dt^N} \Leftrightarrow (j\Omega)^N X(\Omega)$$

# Integration Property

---

$$\int_{-\infty}^t x(\sigma) d\sigma \Leftrightarrow \frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$$

where  $X(0) = \int_{-\infty}^{\infty} x(t) dt$

# Integration Property

---

Blank space for calculation

# Integration Property

---

Blank space for calculation

# Integration Property

---

Blank space for calculation

## Example 7: Fourier Transform of Triangular Pulse

---

$$x(t) = r(t) - 2r(t - 1) + r(t - 2)$$

Find the Fourier transform of  $x(t)$  using the derivative property.

# Example 7: Fourier Transform of Triangular Pulse

---

Blank space for calculation



# Example 7: Fourier Transform of Triangular Pulse

---

Blank space for calculation

# Example 7: Fourier Transform of Triangular Pulse

---

Blank space for calculation

# Fourier Optics

---

<https://www.youtube.com/watch?v=EGYVquku8r4>