Homework 4 Solution

1) Expand

1. $x(t) = \cos(2\pi t) + \cos(3\pi t)$

$$\Omega_1 = 2\pi \Rightarrow T_1 = \frac{2\pi}{\Omega_1} = \frac{2\pi}{2\pi} = 1$$

$$\Omega_2 = 3\pi$$

$$\Rightarrow T_2 = \frac{2\pi}{3\pi} = \frac{2}{3}.$$

Fundamental time period of $x(t)$ is L.C.M of $\left(1, \frac{2}{3}\right)$

$$= \frac{LCM(1,2)}{GCD(1,3)} = \frac{2}{1}$$

Hence, the fundamental time period of $x(t)$ is

$T_0 = 2$ unit

$\Omega_0 = \frac{2\pi}{2} = \pi$ rad/unit

$$x(t) = \cos(2\pi t) + \cos(3\pi t)$$

For exponential Fourier series:

$$X_k = \frac{1}{2}\int_{t_0}^{t_0+T_0} \left(\cos(2\pi t) + \cos(3\pi t)\right) e^{-jk\Omega_0 t}\, dt$$

$$= \frac{1}{2}\int_0^2 \left(\frac{e^{j2\pi t} + e^{-j2\pi t} + e^{j3\pi t} + e^{-j3\pi t}}{2}\right) e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\int_0^2 \left(\frac{e^{j2\pi t - jk\pi t} + e^{-j2\pi t - jk\pi t} + e^{j3\pi t - jk\pi t} + e^{-j3\pi t - jk\pi t}}{2}\right) dt$$

$$= \frac{1}{4}\int_0^2 \left(e^{(j2\pi - jk\pi)t} + e^{(-j2\pi - jk\pi)t} + e^{(j3\pi - jk\pi)t} + e^{(-j3\pi - jk\pi)t}\right) dt$$

$$= \frac{1}{4}\left[\frac{e^{(j2\pi - jk\pi)t}}{j2\pi - jk\pi}\Bigg|_0^2 + \frac{e^{(-j2\pi - k\pi)t}}{(-j2\pi - k\pi)}\Bigg|_0^2 + \frac{e^{(j3\pi - jk\pi)t}}{j3\pi - jk\pi}\Bigg|_0^2 + \frac{e^{-j3\pi - jk\pi}}{-j3\pi - jk\pi}\Bigg|_0^2 \right]$$

$$= \frac{1}{4}\left[\frac{e^{(j2\pi - jk\pi)2} - 1}{j2\pi - jk\pi} + \frac{e^{(-j2\pi - k\pi)2} - 1}{(-j2\pi - k\pi)} + \frac{e^{(j3\pi - jk\pi)2} - 1}{j3\pi - jk\pi} + \frac{e^{-j3\pi - jk\pi}}{-j3\pi - jk\pi} \right]$$

Alternatively, we could also calculate trigonometric Fourier series as

$$x(t) = c_0 + 2\sum_{-\infty}^{\infty} \left[c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t) \right]$$

with
$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)\,dt$$

$k = 0, 1 \ldots$

$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)\,dt$$

$k = 0, 1 \ldots$

But it is easier to do integral for complex exponential than sinusoidals.

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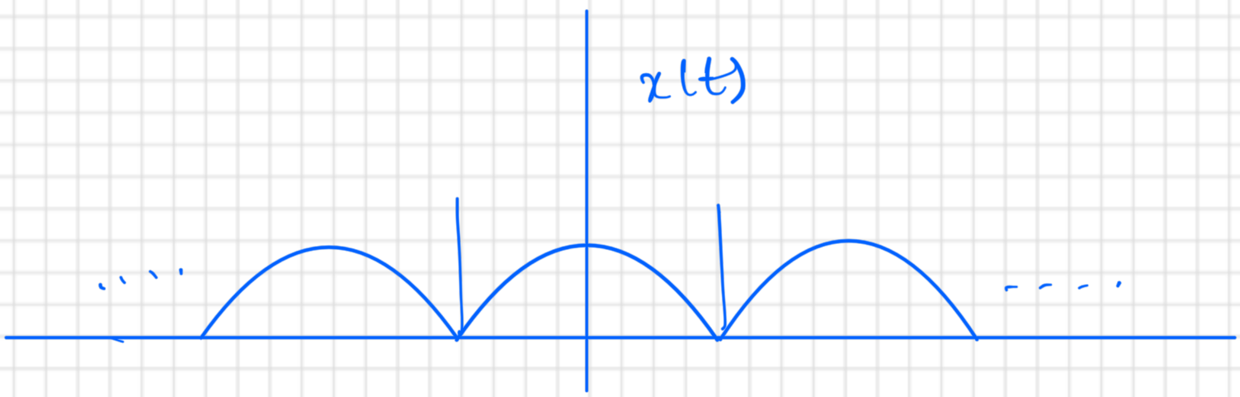
(2) $x(t) = |\cos(2\pi f_0 t)|$



For $\cos 2\pi f_0 t$ $\quad \Omega_0 = 2\pi f_0$

$$T_0 = \frac{2\pi}{\Omega_0} = \frac{2\pi}{2\pi f_0} = \frac{1}{f_0}$$

But as $|\cos 2\pi f_0 t|$ effectively makes frequency twice and time period half,

$$T_0 = \frac{1}{2f_0}$$

and $\Omega_0 = 4\pi f_0$

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{-jk\Omega_0 t}\, dt$$

Let $t_0 = -T_0/2$

So $$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\Omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-1/4f_0}^{1/4f_0} \cos(2\pi f_0 t)\, e^{-jk4\pi f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-1/4f_0}^{1/4f_0} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\, e^{-j4\pi f_0 t k}\, dt$$

$$= \frac{1}{2T_0}\int_{-1/4f_0}^{1/4f_0} \left[e^{j(2\pi f_0 - 4\pi f_0 k)t} + e^{j(-2\pi f_0 - 4\pi f_0 k)t}\right] dt$$

$$\frac{1}{2T_0} = f_0$$

$$= f_0 \int_{-1/4f_0}^{1/4f_0} \left(e^{-j2\pi f_0 (2k-1)t} + e^{-j2\pi f_0 (2k+1)t} \right) dt$$

$$= \cancel{f_0} \left[\frac{e^{-j2\pi f_0 (2k-1)t}}{-j\,2\pi \cancel{f_0}(2k-1)} \Bigg|_{-1/4f_0}^{1/4f_0} + \frac{e^{-j2\pi f_0 (2k+1)t}}{-j\,2\pi \cancel{f_0}(2k+1)} \Bigg|_{-\frac{1}{4f_0}}^{1/4f_0} \right]$$

$$= \frac{e^{-j2\pi \cancel{f_0}(2k-1)\frac{1}{4\cancel{f_0}}} - e^{+j2\pi \cancel{f_0}(2k-1)\frac{1}{4\cancel{f_0}}}}{-j\,2\pi (2k-1)}$$

$$+ \frac{e^{-j2\pi \cancel{f_0}(2k+1)\frac{1}{4\cancel{f_0}}} - e^{+j2\pi \cancel{f_0}(2k+1)\frac{1}{4\cancel{f_0}}}}{j\,2\pi\ (2k+1)}$$

$$= \frac{e^{-j\frac{\pi}{2}(2k-1)} - e^{j\frac{\pi}{2}(2k-1)}}{-j\,2\pi (2k-1)} + \frac{e^{-j\frac{\pi}{2}(2k+1)} - e^{j\frac{\pi}{2}(2k+1)}}{j\,2\pi (2k+1)}$$

$$= \frac{e^{-j\pi k + j\frac{\pi}{2}} + e^{j\pi k - j\pi/2}}{-j\,2\pi (2k-1)} \quad \frac{e^{-j\pi k - j\pi/2} - e^{j\pi k + j\pi/2}}{j\,2\pi (2k+1)}$$

$$= \frac{e^{-j\pi k} e^{j\pi/2} + e^{j\pi k} e^{-j\pi/2}}{-j\,2\pi (2k-1)} - \frac{e^{-j\pi k} e^{-j\pi/2} - e^{j\pi k} e^{j\pi/2}}{j\pi (2k+1)}$$

$$e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\pi/2 = j$$

$$e^{-j\pi k} = \cos\pi k - j\sin\pi k = (-1)^k$$

$$e^{j\pi k} = \cos\pi k + j\sin k = (-1)^k$$

$$e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\pi/2 = -j$$

$$= \frac{(-1)^k j - (-1)^k(-j)}{-j2\pi(2k-1)} - \frac{(-1)^k(-j) - (-1)^k j}{j2\pi(2k+1)}$$

$$= \frac{j(-1)^k + j(-1)^k}{-j2\pi(2k-1)} + \frac{(-1)^k j + (-1)^k j}{j2\pi(2k+1)}$$

$$= -\frac{\cancel{2}\cancel{j}(-1)^k}{\cancel{j}\cancel{2}\pi(2k-1)} + \frac{\cancel{2}\cancel{j}(-1)^k}{\cancel{j}\cancel{2}\pi(2k+1)}$$

$$= (-1)^k\left[\frac{-\pi(2k+1) + \pi(2k-1)}{\pi^2(4k^2-1)}\right]$$

$$= (-1)^k\left[\frac{\cancel{-2\pi k} - \pi \cancel{+2\pi k} - \pi}{\pi^2(4k^2-1)}\right]$$

$$x_k = \frac{(-1)^k \, -2\cancel{\pi}}{\pi^{\cancel{2}}(4k^2-1)} = \frac{-2(-1)^k}{\pi(4k^2-1)} = \frac{2(-1)^k}{\pi(1-4k^2)}$$

$$x_k = \frac{2(-1)^k}{\pi(1-4k^2)}$$