CPE 381 Exam 2

100 points

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Your only source of references for this exam is your one-sided cheatsheet on letter size paper or smaller, handwritten by you, calculator, and pen/pencil, eraser. No other booklets, additional paper, textbooks, or other materials may be referenced during this examination. You may ask for additional sheets from the examiner. **However, please use all the extra space provided in this exam paper.**

Read every question on this examination carefully. Although portions may seem familiar—do not assume that the information presented in this examination is duplicated from any examples (written or otherwise) that you may have seen before.

Total number of pages (including this page and three additional pages at the end for writing solution, excluding supplementary tables, if any): 13

Name: ______________________________________

SCORE EARNED:

Q1	Q2	Q3	Q4	Q5	Q6	Total
5	10	10	20	30	25	100

***Note*: Assume i and j are complex units, i.e.** $\sqrt{-1}$ for the entire exam.

1 Toss a Coin (5 points)

Indicate true (T) or false (F) for each of the below statements.

T **F** **(a)** Fourier Series exists only for periodic signal.

T **F** **(b)** Fourier Transform for a period can be calculated using the Fourier series and duality principle.

T **F** **(c)** Fourier transform is a special case of Laplace transform where $\sigma = 0$, given $s = \sigma + j\Omega$.

T ***F*** **(d)** Assuming $\delta[n]$ as the discrete-time unit-impulse function (or unit-sample function), $\delta[-3] = 1$.

T **F** **(e)** In order to use Laplace transform to calculate the Fourier transform of a function $x(t)$, the region of convergence must include j-Ω axis.

2 Four-sided Dice (10 points)

A **B** ***C*** **D** **(a)** The period of signal $\cos(4t) + \sin(6t)$

(A) 4π.

(B) 2π.

(C) π.

(D) 2.

A **B** **C** **D** **(b)** The Fourier series expression of a real periodic signal with fundamental frequency (in Hz) f_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{2\pi f_0 t}$$

If the Fourier coefficient $X_2 = 3 + j4$, then what is the value of $|X_{-2}|$

(A) 5.

(B) -5.

(C) 3

(D) 4.

A ***B*** **C** **D** **(c)** Consider a signal $x(t)$ in its Fourier series form:

$$x(t) = (2 + j2)e^{-3jt} + j2e^{-jt} + 3 - 2e^{jt} + (2 - j2)e^{j3t}$$

Considering the fundamental frequency $\Omega_0 = 1$ rad/s.

What's the value of $|X_{-3}|$?

(A) $2 - j2$.

(B) $2\sqrt{2}$.

(C) 3

(D) $j2$.

A **B** **C** **D** **(d)** Select the correct Fourier transform expression for $x(t) = A[u(t) - u(t-b)]$ where A and b are positive constant numbers.

Consider Ω is the frequency variable.

(A) $Ae^{-j\Omega b/2}\frac{b}{2}\text{sinc}\left(\frac{\Omega b}{2}\right)$

(B) $Ae^{-j\Omega b/2}\frac{b}{2}\cos\left(\frac{\Omega b}{2}\right)$

(C) $Ae^{-j\Omega b/2}\delta\left(\frac{\Omega b}{2}\right)$

(D) $Ae^{-j\Omega b/2}\frac{b}{2}$

A B C *D* **(e)** The frequency range of the C-band signal is 4.0 GHz to 8.0 GHz (1 Giga = 10^9). The suitable choice of sampling rate for such a signal is

(A) 8×10^9 samples/s or higher

(B) 4×10^9 samples/s or lower

(C) 8×10^9 samples/s or lower

(D) 16×10^9 samples/s or higher

3 Numerophile (10 points)

Write down the numerical values for the following

1. $\delta[2] + u[4]$

   Answer

   1

2. $\delta[-3]$

   Answer

   0

3. $e^{j\pi k}$

   Answer

   $(-1)^k$

4. $x[4]$ where

$$x[n] = \begin{cases} 1, & n = -1, 0, 1 \\ 0.5, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

   Answer

   0

5. Given a frequency of the signal as $5Hz$, its sampling period T_s.

   Answer

   $\frac{1}{5}$ seconds = 0.2 seconds

6. The DC term (i.e. constant term X_0 in the Fourier series expansion of $x(t) = \sin^2(t)$. Use $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$

Answer

$\frac{1}{2}$

7. The value $|X(2)|$ where $X(\Omega)$ is the Fourier transform of $x(t) = 0.5e^{-|t|}$.

Fourier Transform of $x(t)$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

Answer

$$\begin{aligned}
X(\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\
&= \int_{-\infty}^{\infty} 0.5e^{-|t|}e^{-j\Omega t}dt = 0.5\left[\int_{-\infty}^{0} e^{t}e^{-j\Omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\Omega t}dt\right] \\
&= \frac{0.5}{1-j\Omega} + \frac{0.5}{1+j\Omega} = \frac{1}{1+\Omega^2} \\
X(2) &= \frac{1}{1+4} = \frac{1}{5} = 0.2 \\
|X(2)| &= 0.2
\end{aligned}$$

8. Period of the signal in Figure 1.

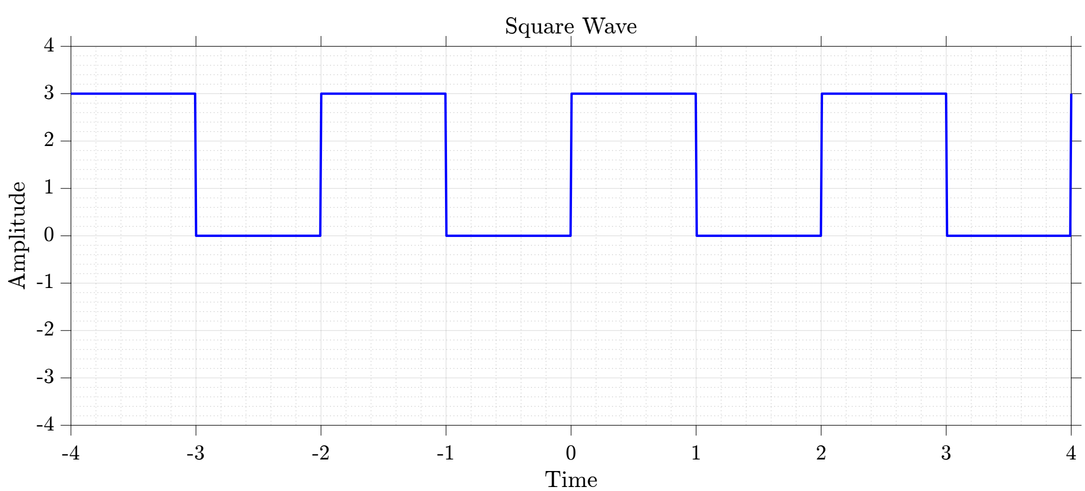


Figure 1: Square Wave $x(t)$

Answer

2 units

9. The maximum value of $u(t+1) - u(t-1)$ where $u(t)$ is the continuous-time unit-step signal.

Answer

1

10. The number of terms in the complex exponential Fourier series expansion of $x(t) = 0.5 + 4\cos(2\pi t) - 8\cos(4\pi t)$ with non-zero Fourier series coefficients.

Answer

5

4 The Interview (20 points)

Explain in a few sentences

1. What does the spectrum of a signal tell us about and why it is important to study the spectrum through various mathematical tools such as the Fourier series and Fourier transform?

   Answer

   The spectrum of a signal tells its frequency content. It is important to study the spectrum to understand what frequencies dominate the signal and it can help us in choosing sampling frequencies that may satisfy Nyquist sampling criteria and to design filters to remove unwanted frequency components.

2. If a signal is not band-limited what is one important issue we might encounter when sampling such a signal and how to avoid the issue?

   Answer

   If a signal is not band-limited then we will see that the sampled signal will consist of overlapped spectra causing frequency aliasing. To avoid that we can select a maximum frequency that can capture the majority of the energy of the signal (say 90% of the energy of the original signal), making the signal band limited. In this way, we can sample the signal without aliasing.

3. What is a low-pass filter and what concept do we employ to remove high-frequency noise?

   Answer

   A low-pass filter is a filter that filters out high-frequency content from a signal. In order to remove high-frequency content, we select a transfer function of the filter in such a way that the magnitude is zero for high-frequency components. We are required to choose a cutoff frequency above which the filter rejects any high-frequency component.

4. Consider a signal $x(t)$ whose Fourier transform is $X(\Omega)$, what changes you would expect to see in the Functional form of its Fourier transform when the signal $x(t)$ is delayed by a time-units.

Answer

We will see phase change in its Fourier transform compared to the original Fourier transform when a signal is delayed by a, and the magnitude will remain the same. Mathematically, the Fourier transform will be multiplied by a complex exponential.

5. Any signal $x(t)$ can be represented by a special infinite sum called Fourier series. However, there is a problem when writing computer programs to implement such a Fourier series. Explain the problem you might encounter when attempting to represent a signal $x(t)$ as its Fourier series through a computer program.

Answer

As infinite terms exist in the Fourier series expansion, it is infeasible to implement the exact Fourier series in computer programming. However, a few Fourier series terms can be chosen to approximate the original signal. As we increase the terms of terms, the sum becomes closer to the actual signal, although it suffers from the ripple effect or Gibbs phenomenon.

5 Duality (30 points)

Consider a signal

$$x(t) = \frac{20\sin(0.5t)}{0.5t}$$

1. Use the duality property to find its Fourier transform. **(10 points)**
2. Is $x(t))$ bandlimited? If so, write down its maximum frequency Ω_{max}. **(10 points)**
3. Sketch the Fourier transform $X(\Omega)$ of $x(t)$ **(5 points)**
4. Suppose the sampling period to sample $x(t)$ is chosen as $T_s = \pi$. Does the satisfy the Nyquist sampling criteria? **(5 points)**

Solution

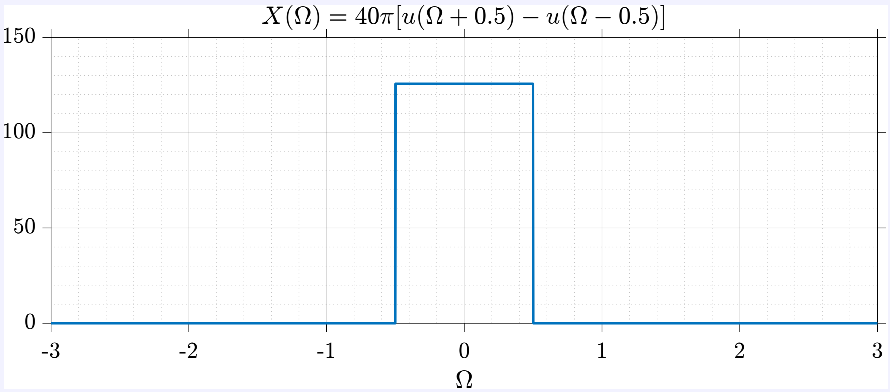
1. Consider a signal $p = 20[u(t+0.5) - u(t-0.5)]$.
   Its Fourier transform is calculated to be

$$\begin{aligned}
P(\Omega) &= 20\int_{-0.5}^{0.5} e^{-j\Omega t} dt \\
&= 20\frac{e^{-j\Omega t}}{-j\Omega}\Big|_{-0.5}^{0.5} \\
&= 20\frac{e^{-j\Omega 0.5} - e^{j\Omega 0.5}}{-j\Omega} \\
&= 20\frac{e^{j\Omega 0.5} - e^{-j\Omega 0.5}}{j\Omega}\cdot\frac{2}{2} \\
&= 40\frac{e^{j\Omega 0.5} - e^{-j\Omega 0.5}}{2j\Omega} \\
&= 40\frac{\sin(0.5\Omega)}{\Omega} \\
P(\Omega) &= \frac{40\sin(0.5\Omega)}{\Omega}\cdot\frac{0.5}{0.5} = 20\text{sinc}(0.5\Omega)
\end{aligned}$$

Then, by duality,

$$\begin{aligned}
x(t) = P(t) = \frac{20\sin(0.5t)}{0.5t} &= 20\text{sinc}(0.5\Omega) \\
\Rightarrow X(\Omega) = 2\pi p(-\Omega) = 2\pi p(\Omega) &= 2\pi \times p = 40\pi[u(\Omega+0.5) - u(\Omega-0.5)]
\end{aligned}$$

Because $p(\ )$ is an even function.

2. Yes, the signal is bandlimited, as we can see from its Fourier transform that its Ω_{max} is 0.5 rad/unit.
3. 

4. $T_s = \pi$ equals to $\Omega_s = 2\pi/\pi = 2$ rad/unit.

   Nyquist sampling criteria says $\Omega_s \geq 2\Omega_{max}$. As $2 \geq 0.5$, Nyquist sampling criteria is satisfied for the sampling time $T_s = \pi$.

6 Its Discrete World (25 points)

Consider a system represented by the difference-equation

$$y[n] = 0.5y[n-1] + x[n] + x[n-1], \quad n \geq 0$$

1. Write down the expression for the impulse response $h[n]$.

   Hint: Determine $h[0], h[1], \cdots$, and discover the pattern to write the closed form expression for $h[n]$.

   (10 points)

2. Let the initial condition be $y[-1] = -2$, and the input $x[n] = u[n]$ first and then $x[n] = 2u[n]$. Find the corresponding outputs. **(5 points)**

3. Let the initial condition be $y[-1] = 0$, and the input $x[n] = u[n]$ first and then $x[n] = 2u[n]$. Find the corresponding outputs. **(5 points)**

4. Use the above result to determine when the system can be linear. **(5 points)**

Solution

1. Impulse response is obtained by setting $x[n] = \delta[n]$, then $y[n] = h[n]$. Hence,

$$
\begin{aligned}
h[n] &= 0.5h[n-1] + \delta[n] + \delta[n-1] \\
h[0] &= 0.5h[-1] + \delta[0] + \delta[-1], \quad \delta[-1] = 0, \delta[0] = 1 \\
&\quad \text{noting casually from } n \geq 0, \quad \text{we can write } h[-1] = 0 \\
h[0] &= 1 = u[0] \\
h[1] &= 0.5h[0] + \delta[1] + \delta[0] \\
&= 0.5 + 0 + 1 = 1.5 = 1.5u[1] \\
h[2] &= 0.5h[1] + \delta[2] + \delta[1] \\
&= 0.5 \times 1.5 = 0.5 \times 1.5u[2] \qquad \text{note all } \delta[n] \text{ for } n > 0 \text{ are } 0. \\
h[3] &= 0.5h[2] = 0.5 \times 0.5 \times 1.5 = 0.5 \times 0.5 \times 1.5u[3] \\
h[4] &= 0.5h[3] = 0.5 \times 0.5 \times 0.5 \times 1.5u[4] \\
&\vdots \\
h[n] &= 0.5^{n-1} \times 1.5 = 0.5^n \times 3u[n] = 3 \times 0.5^n u[n]
\end{aligned}
$$

2. $y[-1] = -2$, $x[n] = u[n]$, then

$$
\begin{aligned}
y[0] &= 0.5y[-1] + x[0] + x[-1] = 0 \\
y[1] &= 0.5y[0] + x[1] + x[0] = 2 \\
y[2] &= 0.5y[1] + x[2] + x[1] = 3
\end{aligned}
$$

Now, lets use $x[n] = 2u[n]$, and call the response $y_1[n]$, then

$$
\begin{aligned}
y_1[0] &= 0.5y[-1] + x[0] + x[-1] = 1 \\
y_1[1] &= 0.5y[0] + x[1] + x[0] = 4.5 \\
y_1[2] &= 0.5y[1] + x[2] + x[1] = 6.25
\end{aligned}
$$

Clearly $y_1[n] \neq 2y[n]$. Due the initial condition being non-zero, the system is not linear.

3. If $y[-1] = 0$, for $x[n] = u[n]$,

$$
\begin{aligned}
y[0] &= 0.5y[-1] + x[0] + x[-1] = 1 \\
y[1] &= 0.5y[0] + x[1] + x[0] = 2.5 \\
y[2] &= 0.5y[1] + x[2] + x[1] = 3.25
\end{aligned}
$$

Now, let's double our input $x[n] = 2u[n]$, and call our response $y_1[n]$, then

$$y_1[0] = 0.5y[-1] + x[0] + x[-1] = 2$$
$$y_1[1] = 0.5y[0] + x[1] + x[0] = 5.0$$
$$y_1[2] = 0.5y[1] + x[2] + x[1] = 6.5$$

Clearly, with the zero initial condition, $y_1[n] = 2y[n]$.

4. From above computation, we see that the system will be linear when the initial condition is zero.

– §§§ –

B.8 APPENDIX: USEFUL MATHEMATICAL FORMULAS

We conclude this chapter with a selection of useful mathematical facts.

B.8-1 Some Useful Constants

$\pi \approx 3.1415926535$

$e \approx 2.7182818284$

$\frac{1}{e} \approx 0.3678794411$

$\log_{10} 2 \approx 0.30103$

$\log_{10} 3 \approx 0.47712$

B.8-2 Complex Numbers

$e^{\pm j\pi/2} = \pm j$

$e^{\pm jn\pi} = \begin{cases} 1 & n\,\text{even} \\ -1 & n\,\text{odd} \end{cases}$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$a + jb = re^{j\theta} \qquad r = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$(re^{j\theta})^k = r^k e^{jk\theta}$

$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$

B.8-3 Sums

$$\sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r-1} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} k\,r^k = \frac{r + [n\,(r-1) - 1]\,r^{n+1}}{(r-1)^2} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k^2\,r^k = \frac{r[(1+r)(1-r^n) - 2n(1-r)r^n - n^2(1-r)^2 r^n]}{(1-r)^3} \qquad r \neq 1$$

B.8-4 Taylor and Maclaurin Series

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

$$f(x) = f(0) + \frac{x}{1!}\dot{f}(0) + \frac{x^2}{2!}\ddot{f}(0) + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0)$$

B.8-5 Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$(1+x)^n \approx 1 + nx \qquad |x| \ll 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

B.8-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j \sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp \sin x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm \cos x$$

$$2 \sin x \cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$a\cos x + b\sin x = C\cos(x+\theta) \qquad C = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\left(\frac{-b}{a}\right)$

B.8-7 Common Derivative Formulas

$\dfrac{d}{dx}f(u) = \dfrac{d}{du}f(u)\dfrac{du}{dx}$

$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dx^n}{dx} = nx^{n-1}$

$\dfrac{d}{dx}\ln(ax) = \dfrac{1}{x}$

$\dfrac{d}{dx}\log(ax) = \dfrac{\log e}{x}$

$\dfrac{d}{dx}e^{bx} = be^{bx}$

$\dfrac{d}{dx}a^{bx} = b(\ln a)a^{bx}$

$\dfrac{d}{dx}\sin ax = a\cos ax$

$\dfrac{d}{dx}\cos ax = -a\sin ax$

$\dfrac{d}{dx}\tan ax = \dfrac{a}{\cos^2 ax}$

$\dfrac{d}{dx}(\sin^{-1} ax) = \dfrac{a}{\sqrt{1-a^2x^2}}$

$\dfrac{d}{dx}(\cos^{-1} ax) = \dfrac{-a}{\sqrt{1-a^2x^2}}$

$\dfrac{d}{dx}(\tan^{-1} ax) = \dfrac{a}{1+a^2x^2}$

B.8-8 Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$

$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$

$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$

$$\int x^2\cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int \sin ax\cos bx\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)$$

B.8-9 L'Hôpital's Rule

If $\lim f(x)/g(x)$ results in the indeterministic form $0/0$ or ∞/∞, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{\dot{f}(x)}{\dot{g}(x)}$$

B.8-10 Solution of Quadratic and Cubic Equations

Any *quadratic* equation can be reduced to the form

$$ax^2 + bx + c = 0$$

The solution of this equation is provided by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A general *cubic* equation

$$y^3 + py^2 + qy + r = 0$$

may be reduced to the *depressed cubic* form

$$x^3 + ax + b = 0$$

by substituting

$$y = x - \frac{p}{3}$$

This yields

$$a = \tfrac{1}{3}(3q - p^2) \qquad b = \tfrac{1}{27}(2p^3 - 9pq + 27r)$$

Now let

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The solution of the depressed cubic is

$$x = A + B, \qquad x = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \qquad x = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}$$

and

$$y = x - \frac{p}{3}$$

REFERENCES

1. Asimov, Isaac. *Asimov on Numbers.* Bell Publishing, New York, 1982.
2. Calinger, R., ed. *Classics of Mathematics.* Moore Publishing, Oak Park, IL, 1982.
3. Hogben, Lancelot. *Mathematics in the Making.* Doubleday, New York, 1960.