CPE 381: Fundamentals of Signals and Systems for Computer Engineers

05 Laplace Transform

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Announcement

- Homework 2 has been posted. Due Date: Sept 18, 2024
- Quiz 2 will open on Friday the 13th - and will be available until the 15th.
- Homework 3 will be posted later with a due date of Sept 25th.
- Quiz 3 will be open Friday 20th.
- Reading Assignment: Chapter 2, and Section 3.1-3.3 of the textbook;
- Chapters 2, 3, and 7 of of Signals, systems, and transforms (Parr, John M. Phillips, Charles L. Riskin etc.) available at https://github.com/rahulbhadani/CPE381_FA24/blob/master/Books/

Outline

1. **Motivating Laplace Transform**

2. **Laplace Transform**

Motivating Laplace Transform

From Calculus to Algebra

Solving the differentiation equation is hard, but solving a linear algebraic equation is easier.

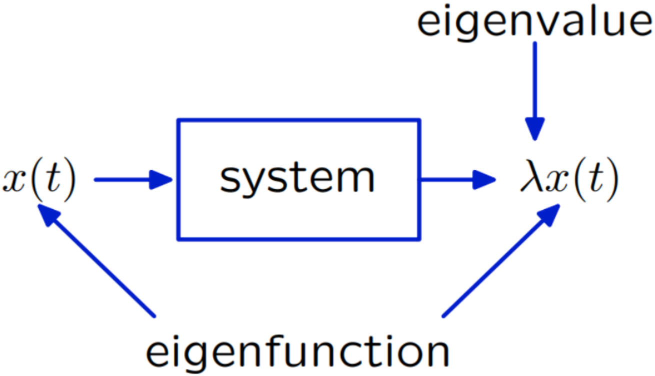
Euler found that when using an appropriate **Kernel function**, multiplied to a function, and then integrated transform it into another function of different variable that can be operated using algebraic operators.

$$F(p) = \int_a^b K(t,p) f(t) dt$$

where $K(t,p)$ is a Kernel function. This goes back to the integral transform invented by Leonhard Euler https://www.jstor.org/stable/41133757.

Eigenfunctions

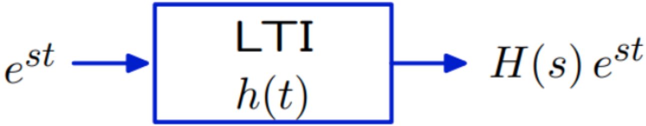
If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.



Complex Exponentials as Eigenfunction of LTI Systems

Complex exponentials are eigenfunctions of LTI systems.
If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s) e^{st}$$



$H(s)$ is the Laplace transform of $h(t)$. This definition is only applicable to LTI systems.
That's why we need to know first if a system is an LTI system.

Complex Exponentials

Consider $s = \sigma + j\Omega$, then

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-\tau s}d\tau$$

From the above equations, it seems obvious that $H(s)$ is an infinite combination of complex exponentials, weighted by the impulse response.

Laplace Transform

Two-sided or Bilateral Laplace Transform

Definition

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt, \quad s \in \mathrm{ROC} \tag{1}$$

ROC $=$ Region of Convergence, the region in the complex plane (henceforth referred to as s plane, as $s = \sigma + j\Omega$ is a complex number).

- σ : damping factor;
- Ω : frequency in `rad/s`.

Inverse Laplace Transform

Definition

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds, \quad \sigma \in \mathrm{ROC} \tag{2}$$

The relationship between a function and its Laplace transform can be written as:

$$f(t) \iff F(s) \tag{3}$$

Transfer Function

For a given system $\mathcal{S}$, if $f(t) = h(t)$, then $H(s)$ is called the Transfer Function and characterizes the system in s-domain (or frequency domain).

Why Damping Factor and Frequency in s? I

Going back to Euler's transform:

$$F(p) = \int_a^b K(t,p) f(t) dt$$

e^{-st} is a special case Kernel function.

When the exponential is raised to a negative value, then the exponential is damped. Hence, even with limits from $-\infty$ to ∞, we may expect the integral to converge (or within other specified limits).

Why Damping Factor and Frequency in s? II

If s is a complex number $\sigma + j\Omega$, then $e^{-\sigma t}$ will damp the input signal, but $e^{-j\Omega t}$ will have an effect or rotating phasor with Ω as the angular frequency. Laplace built his idea on the top of Euler's work.

Read more: https://www.cfm.brown.edu/people/dobrush/am33/MuPad/part6.html or https://web.archive.org/web/20230829205701/https://www.cfm.brown.edu/people/dobrush/am33/MuPad/part6.html

Region of Convergence (ROC)

Definition

The region in the s plane for which the integral converges is called the region of convergence.

We denote the ROC set as $\mathcal{R}$.

Example

Laplace Transform

For a signal $x(t) = e^{-at}u(t)$, find the Laplace transform $X(s)$ and its ROC.