CPE 381: Fundamentals of Signals and Systems for Computer Engineers

Continuous-Time Systems

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Announcement

- Quiz 01, based on Chapter 01 Continuous-Time Signals from the Textbook. Available from September 05, 12:01 AM to September 07, 11:59 PM. 25 Questions, 45 Minutes.
- Homework 1 Solution is available.
- Homework 2 will be posted on Wednesday, due Sept 18, 2024.

Outline

1. Linear Time-Invariant (LTI) Continuous-Time Systems

2. Convolutional Integral

3. Causality and Stability

System Thinking

System is an abstraction.

Think of a device or a process as a system – something that transforms one signal into another.

$$x(t) \Longrightarrow \boxed{\mathcal{S}} \Longrightarrow y(t) = \mathcal{S}[x(t)]$$

Mathematical system vs Physical System.

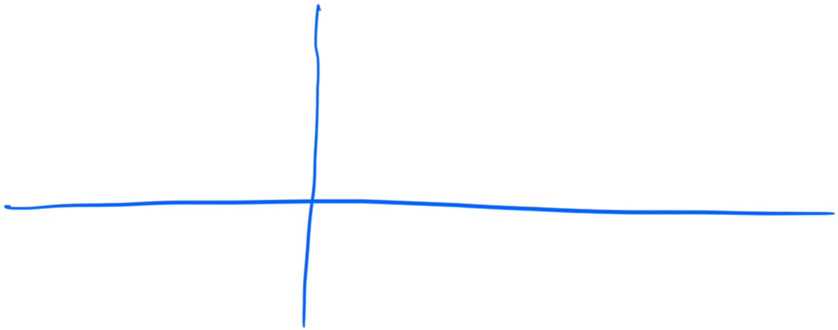
Continuous-time Systems

Multi-level System.

Discrete System

Digital System

When inputs and outputs are continuous, the system is continuous.



Linear Time-Invariant (LTI) Continuous-Time Systems

Linear Time-Invariant (LTI) Continuous-Time Systems

For most analyses at system-level, we care about the following properties:

- Linearity
- Time-invariance
- Causality
- Stability

Linearity

Scaled signal

$$\mathcal{S}[a \cdot x(t) + b \cdot y(t)] = \mathcal{S}[a \cdot x(t)] + \mathcal{S}[b \cdot y(t)] = a \cdot \mathcal{S}[x(t)] + b \cdot \mathcal{S}[y(t)]$$

Additivity

Superposition.

Time-invariance

If

$$x(t) \Rightarrow y(t) = \mathcal{S}[x(t)],$$

then, the time-invariance says:

$$x(t \mp \tau) \Rightarrow y(t \mp \tau) = \mathcal{S}[x(t \mp \tau)].$$

Linear-~~time~~ Time- Invariant (LTI) Systems

A system that satisfies both linearity and time-invariance is an LTI system.

Class Problem 1

$g(t-\tau) = a[f(t-\tau)]^2 - b[f(t-\tau)]$

$S[f(t-\tau)] = a[f(t-\tau)]^2 - b[f(t-\tau)]$

$S[Af_1(t) + Bf_2(t)] = a[Af_1(t) + Bf_2(t)]^2 - b[Af_1(t) + Bf_2(t)]$

$= aA^2 f_1^2(t) + aB^2 f_2^2(t) + 2aABf_1(t)f_2(t) - bAf_1(t) - bBf_2(t)$

$\neq$

Consider a system $g(t) = \mathcal{S}[f(t)] = a[f(t)]^2 - b[f(t)]$

- Is this system linear? Not a linear system.
- Is this system time-invariant? Time Invariant

$g_1(t)$ and $g_2(t)$

$g_1(t) = a[f_1(t)]^2 - b[f_1(t)]$

$g_2(t) = a[f_2(t)]^2 - b[f_2(t)]$

$A \cdot g_1(t) + B \cdot g_2(t) = Aa[f_1(t)]^2 - Abf_1(t) + Ba[f_2(t)]^2 - Bb[f_2(t)]$

Class Problem 2

Consider

$$g(y) = \mathcal{S}[f(t)] = \int_{-\infty}^{\infty} f(t)\text{rect}(y-t)dt$$

- Is this system linear? Linear!
- Is this system time-invariant? Time Varying.

$$S[f(t-\tau)] = \int_{-\infty}^{\infty} f(t-\tau)\,\text{rect}(y-t)dt$$

$$g(y-\tau) = \int_{-\infty}^{\infty} f(t)\,\text{rect}(y-t-\tau)dt$$

Linearity

$$a\,S[f_1(t)] + b\,S[f_2(t)] = a\int_{-\infty}^{\infty} f_1(t)\,\text{rect}(y-t)\,dt + b\int_{-\infty}^{\infty} f_2(t)\,\text{rect}(y-t)dt$$

$$S[a f_1(t) + b f_2(t)] = \int_{-\infty}^{\infty} (a f_1(t) + b f_2(t))\,\text{rect}(y-t)dt$$

$$= a\int_{-\infty}^{\infty} f_1(t)\,\text{rect}(y-t)dt + \int_{-\infty}^{\infty} b f_2(t)\,\text{rect}(y-t)dt$$

Static Systems vs Dynamic Systems

In static systems, output depends on the current input but not on the historical input or output

In dynamic systems, output depends on current input, past inputs, and past outputs.

Differential equations are used to represent dynamic systems (literature also refers to them as dynamical systems).

Representation of Dynamic Systems

Dynamic systems are represented by ordinary differential equations (ODE) with constant coefficients:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$
$$b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

with n initial conditions, $x(t) = 0$ for $t < 0$.

This equation represents a linear ordinary differential equation with constant coefficients, where $y(t)$ is the output, $x(t)$ is the input, and a_i and b_i are the constant coefficients.

Response of a Dynamic Systems

Assessing the output of the system:

- Consider output due to input only, considering initial conditions to be zero. **(zero-state response)**.
- Consider output due to initial conditions only, considering inputs to be zero. **(zero-input response)**.
- Complete response is the sum of the above two.

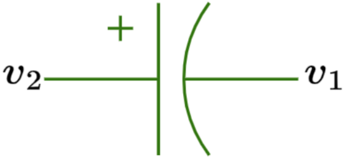
Differential Equations for Ideal Systems

Electrical Inductance

$$v_{21} = L\frac{di}{dt}$$

Electrical Capacitance

$$i = C\frac{dv_{21}}{dt}$$

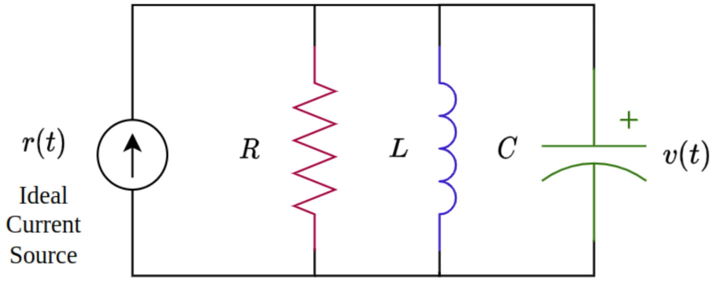


Electrical Resistance

$$i = \frac{v_{21}}{R}$$

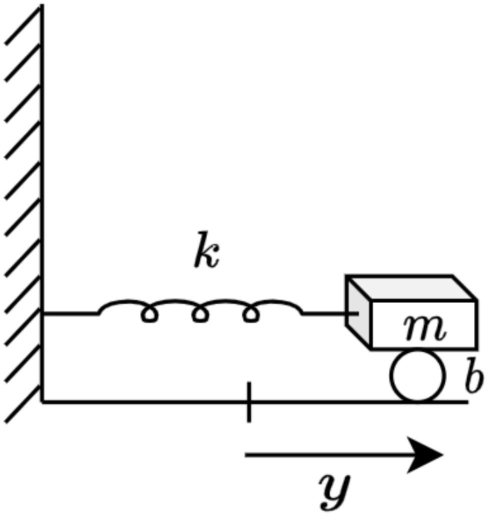
Parallel RLC Circuit

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)dt = r(t)$$



Damped Mass-spring Oscillator

$k =$ stiffness of the spring
$b =$ damping coefficient
$m =$ mass of the block
$y = y(t)$ time-dependent displacement from Equilibrium point.
$my'' = F - ky - by'$



Convolutional Integral

Response to an Impulse Signal

Output when a system is presented with a unit-impulse signal is called **response to an impulse signal**, or in short **impulse response**.

Why is Impulse Response Useful?

It allows us to predict what the system's output will look like in the **time-domain**.

Do you remember generalized signal representation using impulse signals?

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

What would we get if we pass $x(t)$ through an LTI system $\mathcal{S}$?

Impulse Response

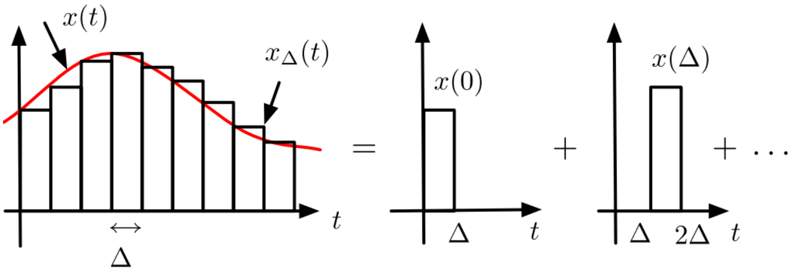
What would we get if we pass $x(t)$ through an LTI system $\mathcal{S}$?

Impulse respons

We get

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

where $h(t)$ is the impulse response. Out is the superposition of the responses of each term in the figure.



Definition of Convolution Integral

The response of an LTI system $\mathcal{S}$, represented by its impulse response $h(t)$, to any signal $x(t)$ is the **convolution integral**.

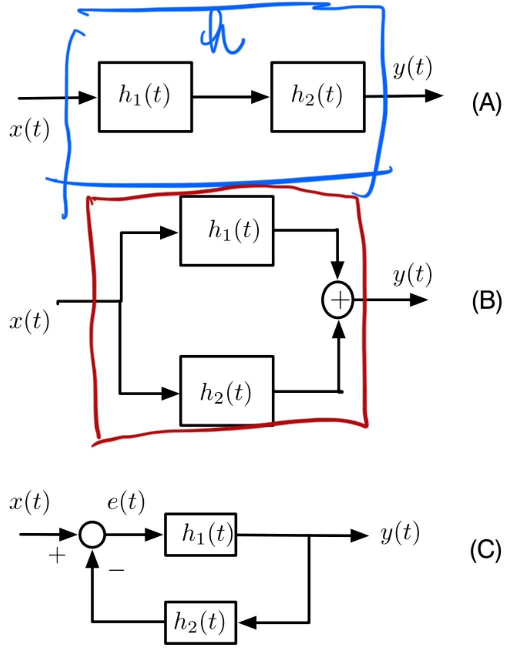
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = [x * h](t) = [h * x](t)$$

where $*$ stands for the convolution integral of the input signal $x(t)$ and the impulse response $h(t)$ of the system.

Some Points on Convolution Integral

- Impulse response is fundamental to characterizing an LTI system.
- A system represented by the convolution integral is LTI. It is a general representation of an LTI system.

Putting Multiple Systems Together



A: $h(t) = [h_1 * h_2](t) = [h_2 * h_1](t)$

B: $h(t) = h_1(t) + h_2(t)$

C: $h(t) = [h_1 - h * h_1 * h_2](t)$

Causality and Stability

Causality

- Cause-effect relationship,
- A system is causal if
  - whenever its input $x(t) = 0$, and there is no initial condition, $y(t) = 0$.
  - the output $y(t)$ doesn't depend on the future inputs.
- Causality is independent of linearity and time-invariance.

LTI system with impulse response $h(t)$ is causal if $h(t) = 0$ for $t < 0$, and

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

Bounded Input Bounded Output Stability

- If input is bounded, i.e. $|x(t)| \leq M, M < \infty$, then output is bounded under certain conditions.
- Condition for bounded output is

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|d\tau \leq M \int_{-\infty}^{\infty} |h(\tau)|d\tau \leq MK \cancel{<} \infty$$

$\leq MK < \infty$

where the integral $\int_{-\infty}^{\infty} |h(\tau)|d\tau \leq K < \infty$
i.e., the impulse is absolutely integrable.

To Calculate if an LTI system is BIBO Stable

- Compute its impulse response $h(t)$.
- Integrate it over $-\infty$ to ∞, effectively over the given support of the signal.
- If the integration is finite, the system is BIBO Stable.

Classwork

① A current impulse $5\delta(t)$ is forced through a capacitor C then the voltage =

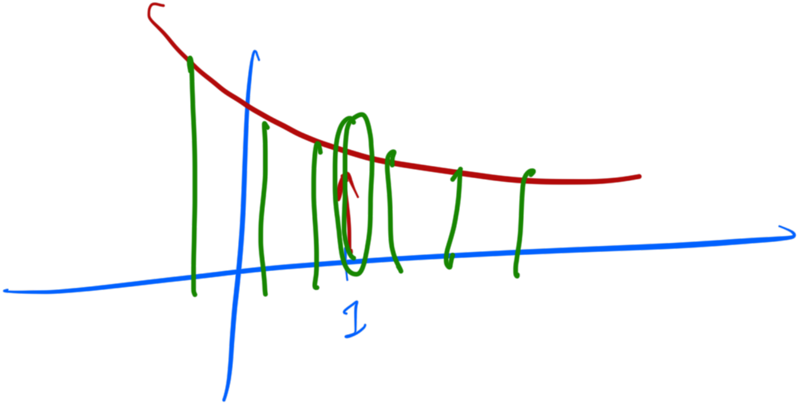
$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt = \frac{1}{C}\int_{-\infty}^{\infty} 5\delta(t)\,dt$$

$$= \frac{1}{C}5\int_{-\infty}^{\infty}\delta(t)\,dt$$

$$= \frac{1}{C}5\,u(t) \qquad \left[\delta(t) = \frac{d}{dt}u(t)\right]$$

Classwork

Q2. $\int_{-5}^{+6} e^{-2t}\delta(t-1)dt = e^{-2t}\Big|_{t=1} = e^{-2}$



$$\boxed{\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)}$$

Classwork

Q3. $\int_{-\infty}^{\infty} \delta(t) \cos\left(\frac{3t}{2}\right) dt = \cos\left(\frac{0 \cdot 3}{2}\right) = 1$

Q4. If a signal $f(t)$ has energy E, what is the energy of $f(2t)$?

$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$

Let $2t = u \Rightarrow 2dt = du$

$E_2 = \int_{-\infty}^{\infty} |f(2t)|^2 dt = \int_{-\infty}^{\infty} |f(u)|^2 \frac{du}{2} = \frac{1}{2}\int_{-\infty}^{\infty} |f(u)|^2 du$

$= \frac{1}{2} E$

Classwork

Properties of the unit impulse function

① $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$, $f(t)$ should be continuous at t_0

② $\int_{-\infty}^{\infty} f(t-t_0)\,\delta(t)\,dt = f(-t_0)$

③ $f(t)\,\delta(t-t_0) = f(t_0)\,\delta(t-t_0)$, "

④ $\delta(t-t_0) = \frac{d}{dt} u(t-t_0)$

⑤ $u(t-t_0) = \int_{-\infty}^{t} \delta(\tau-t_0)\,d\tau = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$

Classwork

6 $$\int_{-\infty}^{\infty} \delta(at - t_0)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta\left(t - \frac{t_0}{a}\right)dt$$

7 $$\delta(-t) = \delta(t)$$

Classwork

Classwork

Classwork

Classwork

Classwork

Classwork

Classwork

Up Next

- Laplace Transform, and its properties