Solving Linear Differential Equation

$$y' = ay + b \qquad y' = \frac{dy(t)}{dt}$$

first consider the case $b = 0$, so that

$y' = ay$ with $a \in \mathbb{R}$

then $y' = ay$

$$\Rightarrow \quad \frac{y'}{y} = a$$

using the derivative table: $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

$$\Rightarrow \quad (\ln y(t))' = a$$

$$\Rightarrow \quad \ln(y(t)) = at + C_0$$

$C_0 \in \mathbb{R}$ is an arbitrary constant of integration.

Now taking the exponentials on both the side

$$y(t) = \pm e^{at + C_0} = \pm e^{C_0} e^{at} = ce^{at}$$

$c \in \mathbb{R}$

Now, consider $b \neq 0$

So

$$y' = ay + b$$

$$\Rightarrow y' = a\left(y + \frac{b}{a}\right)$$

$$\Rightarrow \left(y + \frac{b}{a}\right)' = a\left(y + \frac{b}{a}\right)$$

as $\left(\frac{b}{a}\right)' = 0$ (constant)

let $\tilde{y} = y + \frac{b}{a}$

$$\boxed{\tilde{y}' = a\tilde{y}}$$

Proceeding as earlier,

$$\boxed{\tilde{y}(t) = Ce^{at}}$$

$$\Rightarrow y(t) + \frac{b}{a} = Ce^{at}$$

$$\Rightarrow y(t) = Ce^{at} - \frac{b}{a}$$