# Homework 3
# CPE381

Canvas: `hw03`

Due: 27 September 2024, 11:59 PM

You are allowed to use a generative model-based AI tool for your assignment. However, you must submit an accompanied reflection report on how you use the AI tool, what was the query to the tool, and how it improved your understanding of the subject. You must also add your thoughts on how you would tackle the assignment if there was no such tool available. Failure to provide a reflection report for every single assignment where an AI tool was used may result in a penalty and subsequent actions will be taken in line with plagiarism policy.

**Submission instruction:**

Upload a .pdf on Canvas with the format {firstname.lastname}_cpe381_hw03.pdf. If there is a programming assignment, then you should include your source code along with your PDF files in a zip file firstname.lastname}_cpe381_hw03.zip. If a plot is being asked, your PDF file must also contain plots generated by your MATLAB code. Your submission must contain your name, and UAH Charger ID or the UAH email address. Please number your pages as well.

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## 1 Autobots, transform and roll out! (10 points)

Find the Laplace transform of the following and specify their region of convergence (ROC):

1. $x(t) = -e^{-at}u(-t)$. Use the first principle. where $u(t)$ is the unit-step signal.
2. $x(t) = e^{at}u(-t)$. Use the first principle.
3. $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$. Use the Laplace transform table.
4. $x(t) = e^{-3t}u(t) + e^{2t}u(-t)$. Use the Laplace transform table.
5. $x(t) = 5e^{-0.3t}u(t-2)$ Use the Laplace transform table.

### 1.1 Solution:

1.

$$\begin{aligned} X(s) &= -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0^-} e^{-(s+a)t}dt \\ &= \frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0^-} = \frac{1}{s+a}, \quad \text{Re}(s) < -a \end{aligned} \tag{1}$$

2.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{at}u(-t)e^{-st}dt = \int_{-\infty}^{0^-} e^{-(s-a)t}dt \\ &= -\frac{1}{s-a}e^{-(s-a)t}\Big|_{-\infty}^{0^-} = -\frac{1}{s-a}, \quad \text{Re}(s) < a \end{aligned} \tag{2}$$

3. Using the table of Laplace Transform:

$$\begin{aligned} e^{-2t}u(t) &\leftrightarrow \frac{1}{s+2}, \quad \text{Re}(s) > -2 \\ e^{-3t}u(t) &\leftrightarrow \frac{1}{s+3}, \quad \text{Re}(s) > -3 \end{aligned} \tag{3}$$

We see that poles are at $\sigma = -2$, $\sigma = -3$. Hence, ROC is at the intersection, as the ROC of these poles overlap. Thus ROC is $\text{Re}(s) > -2 \cap \text{Re}(s) > -3 = \text{Re}(s) > -2$.

4. Using the table of Laplace Transform:

$$\begin{aligned} e^{-3t}u(t) &\leftrightarrow \frac{1}{s+3}, \quad \text{Re}(s) > -3 \\ e^{2t}u(-t) &\leftrightarrow \frac{1}{s-2}, \quad \text{Re}(s) < 2 \end{aligned} \tag{4}$$

We see that poles are at $\sigma = 2$, $\sigma = -3$. There is no zero. so ROC of the overall is $\text{Re}(s) > -3 \cap \text{Re}(s) < 2 = -3 < \text{Re}(s) < 2$.

5. Using the table of Laplace Transform:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$$

$$f(t-\alpha)u(t-\alpha) \leftrightarrow e^{-\alpha s}F(s)$$

Hence , $5e^{-0.3t}u(t-2) \leftrightarrow \dfrac{5e^{-2s}}{s+0.3}$

# 2 Time is mysterious. (10 points)

1. The unilateral Laplace transform of $f(t)$ is $\dfrac{1}{s^2+s+1}$. Calculate the Laplace transform of $tf(t)$.

2. The transfer function of a system is given by $H(s) = \dfrac{1}{s^2(s-2)}$. Find out the impulse response of the system if $u(t)$ is used to denote the unit-step signal.

## 2.1 Solution:

1.

$$\mathcal{L}[f(t)] = \frac{1}{s^2+s+1} = F(s) \tag{5}$$

From the property of Laplace transform,

$$\mathcal{L}[tf(t)] = (-1)^1 \frac{d}{ds}F(s) = -\frac{d}{ds}\frac{1}{s^2+s+1} = -\frac{-(2s+1)}{(s^2+s+1)^2} = \frac{2s+1}{(s^2+s+1)^2} \tag{6}$$

2. **Method 1:**

Using, partial fraction expansion,

$$\frac{1}{s^2(s-2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-2} = -\frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s-2)} \tag{7}$$

From the Laplace transform table (Table 3.2 of the textbook), we have

$$h(t) = -\frac{1}{2}r(t) - \frac{1}{4}u(t) + \frac{1}{4}e^{2t}u(t) \tag{8}$$

where $r(t)$ is the ramp function, $u(t)$ is the unit-step signal.
**Method 2:**

$$h(t) = \mathcal{L}[H(s)] = \mathcal{L}\left[\frac{1}{s^2(s-2)}\right] = \mathcal{L}\left[\frac{1}{s^2} \times \frac{1}{(s-2)}\right] = r(t) * e^{2t}u(t) = tu(t) * e^{2t}u(t) = (t * e^{2t})u(t) \tag{9}$$

Method 2 uses the duality property of the Laplace transform that the product of two functions in the s-domain is equal to convolution in the time-domain. (See the textbook Table 3.1).

# 3 Are you the Riddler? (20 points)

1. A system is described by the following differential equation, where $u(t)$ is the input to the system and $y(t)$ is the output of the system

$$y'(t) + 5y(t) = u(t) \tag{10}$$

When $y(0^-) = 1$ and $u(t)$ is a unit step function, find $y(t)$ using the Laplace transform method.

2. A continuous-time LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t) \tag{11}$$

   Assuming zero initial condition, find out the response $y(t)$ of the above system for the input $x(t) = e^{-2t}u(t)$.

## 3.1 Solution:

1. Taking the Laplace transform on both sides,

$$\begin{aligned} sY(s) - y(0^-) + 5Y(s) &= \frac{1}{s} \\ (s+5)Y(s) - 1 &= \frac{1}{s} \\ (s+5)Y(s) &= 1 + \frac{1}{s} \\ (s+5)Y(s) &= \frac{1+s}{s} \\ Y(s) &= \frac{1+s}{s(s+5)} \end{aligned} \tag{12}$$

   Taking partial fractions,

$$\begin{aligned} \frac{1+s}{s(s+5)} &= \frac{A}{s} + \frac{B}{s+5} \\ \Rightarrow A = \frac{1}{5}, B = \frac{4}{5} \end{aligned} \tag{13}$$

   Thus,

$$Y(s) = \frac{1}{5s} + \frac{4}{5(s+5)} \tag{14}$$

   Using Table 3.2 from the textbook,

$$y(t) = \frac{1}{5}u(t) + \frac{4}{5}e^{-5t}u(t) \tag{15}$$

2. Taking the Laplace transform both sides

$$\begin{aligned} s^2Y(s) + 4sY(s) + 3Y(s) &= 2sX(s) + 4X(s) \\ (s^2+4s+3)Y(s) &= (2s+4)X(s) \\ (s^2+4s+3)Y(s) &= (2s+4)\frac{1}{(s+2)}, \quad \text{As } x(t) = e^{-2t}u(t) \\ Y(s) &= \frac{2(s+2)}{(s^2+4s+3)(s+2)} = \frac{2(s+2)}{(s+1)(s+3)(s+2)} \\ Y(s) &= \frac{2}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{s+3} \end{aligned} \tag{16}$$

   Using Laplace transform table 3.2, the solution is

$$y(t) = (e^{-t} - e^{-3t})u(t) \tag{17}$$

# 4 Ready steady, go! (10 points)

1. For the system shown below, $x(t) = \sin(t)u(t)$, find out the response $y(t)$ in the steady-

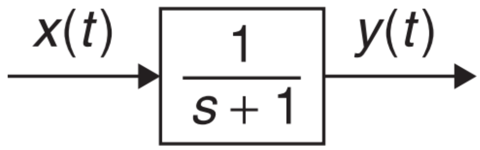


Figure 1: Question 4.1

state.

Note that here, the impulse response $h(t)$ is shown in $s-$domain in the diagram. The impulse response in $s-$domain is called transfer function.

## 4.1 Solution:

The transfer function $H(s)$ based on the system representation given in Figure 1 is given by

$$H(s) = \frac{1}{s+1} \tag{18}$$

We define the transfer function as the ration of Laplace transform of output and the Laplace transform of input, assuming the zero initial condition:

$$H(s) = \frac{Y(s)}{X(s)} \tag{19}$$

Given $x(t) = \sin(t)u(t)$, its Laplace transform $X(s) = \dfrac{1}{s^2+1}$, $Re(s) > 0$. Thus,

$$Y(s) = H(s)X(s) = \frac{1}{s+1} \times \frac{1}{s^2+1} \tag{20}$$

We can write $(s^2+1) = (s+j)(s-j)$ where $j$ is the imaginary number $\sqrt{-1}$. Thus,

$$\begin{aligned} Y(s) &= \frac{1}{s+1} \times \frac{1}{s^2+1} \\ &= \frac{1}{(s+1)(s^2+1)} \\ &= \frac{A}{s+1} + \frac{Bs+C}{s^2+1} \\ &= \frac{As^2 + A + Bs^2 + sC + Bs + C}{(s+1)(s^2+1)} \end{aligned} \tag{21}$$

Equation coefficients on both sides

$$\begin{aligned}
&A + B = 0 \Rightarrow B = -A \\
&A + C = 1 \\
&B + C = 0 \Rightarrow C = -B \\
&\qquad \Rightarrow A - B = 1 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}, B = -\frac{1}{2}, C = \frac{1}{2}
\end{aligned} \tag{22}$$

Thus, $Y(s) = \frac{1}{2(s+1)} + \frac{1-s}{2(s^2+1)} = \frac{1}{2(s+1)} + \frac{1}{2(s^2+1)} - \frac{s}{2(s^2+1)}$.

Taking the Laplace transform, $y(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}\sin(t)u(t) - \frac{1}{2}\cos(t)u(t)$.

## 5 Journey to the *s*-verse (30 points)

Consider a ramp signal $r(t)$ as shown in Figure 2 by the dashed line. The x-axis denotes the time-axis and the y-axis denotes the signal value.

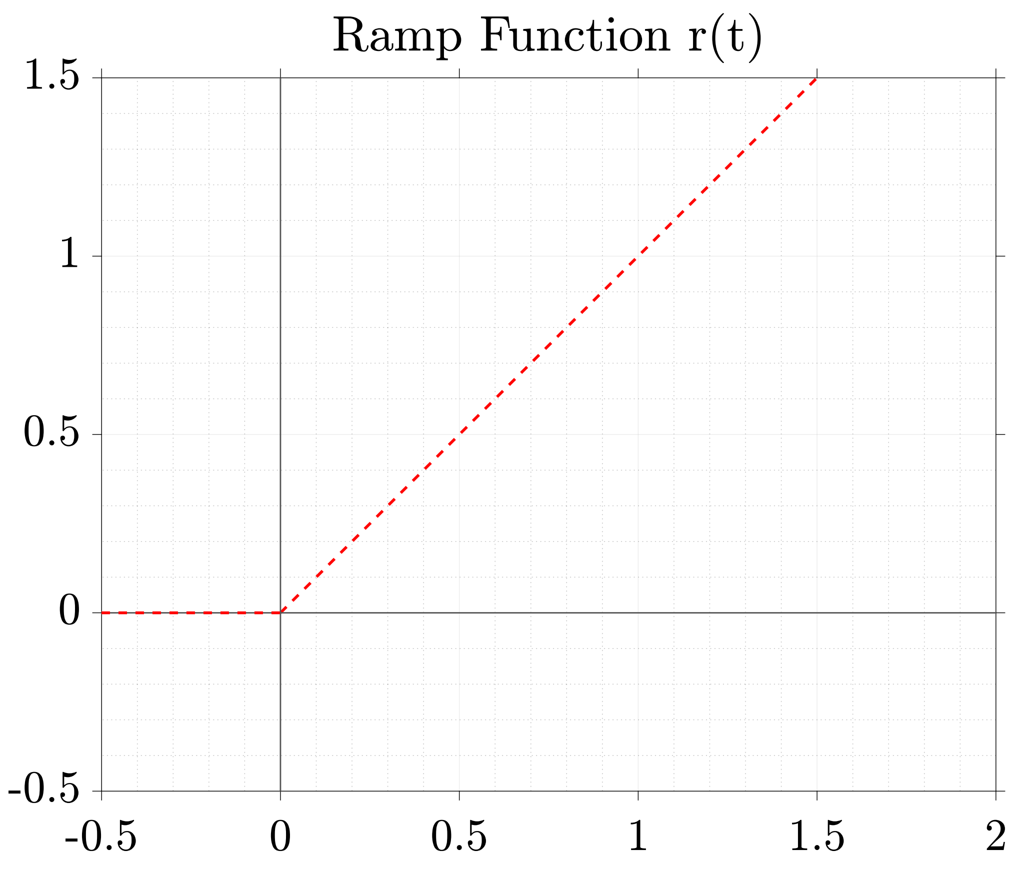


Figure 2: Ramp Signal $r(t)$

1. Write the equation of the ramp signal in terms of time $t$, and the unit-step signal $u(t)$. **(3 points)**

2. Derive the Laplace transform $R(s)$ of the ramp signal from the definition of the Laplace transform integral, (i.e. do not use Laplace Transform Table). **(9 points)** *Hint: Use integration by parts* $\int f(t)g(t)'dt = f(t)g(t) - \int f'(t)g(t)dt$.

3. What is the region of convergence (ROC) for the Laplace transform $R(s)$ to exist? **(3 points)**

4. Consider a frequency shift $e^{-2s}$ in s-domain applied to the ramp signal $r(t)$. Using the properties of the Laplace transform, write down the resulting signal in terms of time $t$ and the unit-step signal $u(t)$. You may use the provided tables to facilitate your answer. **(6 points)**

5. The ramp signal is time-differentiated and used as an input to a system represented by the transfer function $H(s) = \dfrac{1}{s+2}$. What's the output signal $y(t)$ in the time domain? **(6 points)**

6. Find $\lim_{t\to\infty} y(t)$. **(3 points)**

## 5.1 Solution:

1. From the graph in Figure 2, we see that $x = y$ for every point on the red line on the positive time-axis. Further, the line is zero on the negative time axis. Hence, we can write the ramp signal $r(t)$ as $tu(t)$ or

$$r(t) = \begin{cases} t, & t > 0 \\ 0, & t \leq 0 \end{cases} \tag{23}$$

2. The Laplace transform $R(s)$ of the ramp signal $r(t)$ can be derived as follows:

$$\begin{aligned}
R(s) &= \int_{-\infty}^{\infty} tu(t)e^{-st}dt = \int_{0}^{\infty} te^{-st}dt \\
&\quad \text{Use integration by parts: } \int f(t)g(t)'dt = f(t)g(t) - \int f'(t)g(t)dt \\
&\quad f(t) = t, g'(t) = e^{-st} \\
&\quad f'(t) = 1, g(t) = -\frac{e^{-st}}{s} \\
\Rightarrow R(s) &= \frac{-te^{-st}}{s}\bigg|_{t=0}^{\infty} - \int_0^{\infty} -\frac{e^{-st}}{s}dt \\
&\quad \text{Let } u = -st \Rightarrow \frac{du}{dt} = -s \Rightarrow \int_0^{\infty} -\frac{e^{-st}}{s}dt = \frac{1}{s^2}\int e^u du = \frac{e^u}{s^2} = \frac{e^{-st}}{s^2} \\
\Rightarrow R(s) &= \frac{-te^{-st}}{s}\bigg|_{t=0}^{\infty} - \frac{e^{-st}}{s^2}\bigg|_{t=0}^{\infty} = [(0-0) - (0 - \tfrac{1}{s^2})] = \frac{1}{s^2}
\end{aligned} \tag{24}$$

3. The above integral is only finite when $\mathcal{R}e(s) > 0$. Hence ROC is $\mathcal{R}e(s) > 0$.

4. Frequency-shift in s-domain causes time-shift in the time-domain. Hence, based Laplace transform property, $f(t-\alpha)u(t-\alpha) \longleftrightarrow e^{-\alpha s}F(s)$, the resulting signal is $(t-2)u(t-2)$.

5. When the ramp signal is time-differentiated, we get the unit-step signal $u(t)$ whose Laplace transfer is $\frac{1}{s}$. The output $Y(s) = H(s)X(s) = \frac{1}{(s+2)s}$. Using partial fraction expansion, $Y(s) = -\frac{1}{2(s+2)} + \frac{1}{2s}$. From the Laplace transfer table, the time-domain signal for the output is $y(t) = \left[-\frac{1}{2}e^{-2t} + \frac{1}{2}\right]u(t)$.

6. Using final value theorem $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s}{(s+2)s} = \frac{1}{2}$.

# 6 Systems (10 points)

Consider a causal LTI continuous system described by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = 2x'(t) - x(t) \tag{25}$$

1. Using Laplace transform technique, determine the zero-input response $y_{zi}(t)$ if $y'(0^-) = 2$ and $y(0^-) = -3$. **(3 points)**
2. Using Laplace transform technique, determine the zero-state response $y_{zs}$ if the input is $x(t) = u(t)$. **(2 points)**
3. Find the complete response of the system for $y'(0^-) = 2$ and $y(0^-) = -3$ if the input is $x(t) = u(t)$. **(3 points)**
4. Is the system stable with the $x(t) = u(t)$ ? Provide your argument. **(2 points)**

## 6.1 Solution:

From the property of Laplace transform, we have

$$y''(t) = s^2Y(s) - sy(0^-) - y'(0^-)$$

and

$$y'(t) = sY(s) - y(0^-)$$

For this problem, $y(0^-)$ and $y(0)$ are the same. Similarly for $y'(0^-)$ and $y'(0)$.

1. For zero-input, we can write the Laplace transform of Equation (25) as follows:

$$\begin{aligned}
&s^2Y_{zi}(s) - sy(0^-) - y'(0^-) + 3sY_{zi}(s) - 3y(0^-) + 2Y_{zi}(s) = 0 \\
&\Rightarrow \because y'(0^-) = 2, y(0^-) = -3 \\
&s^2Y_{zi}(s) - (s \times -3) - 2 + 3(sY_{zi}(s) - (s \times -3)) + 2Y_{zi}(s) = 0 \\
&Y_{zi}(s)(s^2 + 3s + 2) + 3s + 7 = 0 \\
&Y_{zi}(s) = \frac{-(3s+7)}{s^2+3s+2} = \frac{-4}{s+1} + \frac{1}{s+2} \quad \text{(Using partial fraction expansion)} \\
&y_{zi}(t) = -4e^{-t}u(t) + e^{-2t}u(t)
\end{aligned} \tag{26}$$

2. $y'(0^-) = 0, y(0^-) = 0$ for the case of zero-state response.

   For the input $x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}$, taking the Laplace transform of the differential equation, we can write:

$$
\begin{aligned}
&sY_{zs}(s) + 3sY_{zs}(s) + 2Y(s) = 2sX(s) - X(s) \\
&\Rightarrow Y_{zs}(s)(s^2 + 3s + 2) = (2s - 1)\frac{1}{s} \\
&Y_{zs}(s) = \frac{(2s-1)}{s(s+1)(s+2)} = \frac{-1/2}{s} + \frac{3}{s+1} + \frac{-5/2}{s+2} \quad \text{(Use partial fraction expansion)} \\
&y_{zs}(t) = \left[-\frac{1}{2} + 3e^{-t} - \frac{5}{2}e^{-2t}\right]u(t)
\end{aligned}
\tag{27}
$$

3. The complete response of the system of the system is $y(t) = y_{zs}(t) + y_{zi}(t)$.

4. System is stable as the poles are $s = -1, -2$ which is in the left half plane on the $\sigma-$axis which correspond to decaying exponentials.

# 7 Laplace Transform in MATLAB (10 points)

Use Matlab symbolic computation to find the Laplace transform of a real exponential

$$x(t) = 5e^{-2t}\cos(8t)u(t)$$

Plot the signal and the poles and zeros of their Laplace transform.
Repeat the analysis and plot the results for

$$x(t) = 5e^{-4t}\cos(8t)u(t)$$

Discuss the changes in the $s$ plane and describe their effect on function in the time domain.

## 7.1 Solution:

```
%% (C) Rahul Bhadani
t = -0.5:0.01:5;

%% Signal plotting without symbolic function

x1 = 5.*exp(-2.*t).*cos(8.*t).*heaviside(t);
fig = figure(1);
fig.Position =[995         779        1219         447];
plot(t, x1, "Color", "red", "LineStyle","-", "LineWidth", 3);
grid on;
grid minor;
xlabel('$t$', 'Interpreter', 'latex');
ylabel('$x_1(t)$', 'Interpreter', 'latex');
```

```
title('HW03 Q7, Part 1: x_1(t) = 5e^{-2t}cos(8t)u(t)', 'Interpreter', 'latex');
% Set the font size of the axes
ylim([-7, 7])
set(gca, 'FontSize', 24);

%% We can also make a plot using symbolic function
syms t s
x1 = 5*exp(-2*t)*cos(8*t)*heaviside(t);
x2 = 5*exp(-4*t)*cos(8*t)*heaviside(t);

% Compute the Laplace transform
X1 = laplace(x1, t, s);

% Extract numerator and denominator polynomial
[num, den] = numden(X1);
num_coeffs = fliplr(coeffs(num, s)); % coeffs returns lowest order coefficient
denom_coeff = fliplr(coeffs(den, s));
z1=roots(num_coeffs); p1=roots(denom_coeff);

% Compute the Laplace transform
X2 = laplace(x2, t, s);
[num, den] = numden(X2);
num_coeffs = fliplr(coeffs(num, s)); % coeffs returns lowest order coefficient
denom_coeff = fliplr(coeffs(den, s));
z2=roots(num_coeffs); p2=roots(denom_coeff);

% Plot the signals
fig = figure(1);
fig.Position =[995         347        1219         879];

subplot(2,1,1);
ezplot(x1, [-0.5, 5]);
grid on;
grid minor;
xlabel('$t$', 'Interpreter', 'latex');
ylabel('$x_1(t)$', 'Interpreter', 'latex');
title('HW03 Q7: $x_1(t) = 5e^{-2t}cos(8t)u(t)$', 'Interpreter', 'latex');
ylim([-7, 7]);
set(gca, 'FontSize', 24);

subplot(2,1,2);
ezplot(x2, [-0.5, 5]);
grid on;
grid minor;
xlabel('$t$', 'Interpreter', 'latex');
ylabel('$x_2(t)$', 'Interpreter', 'latex');
title('HW03 Q7: $x_2(t) = 5e^{-4t}cos(8t)u(t)$', 'Interpreter', 'latex');
ylim([-7, 7]);
```

```
set(gca, 'FontSize', 24);
exportgraphics(fig, '../figures/HW03_Q7_Signal.pdf');


%%

% Plot the poles and zeros of the Laplace transform
f = figure(2);
plot(real(z1), imag(z1), 'b.', 'MarkerSize',90);
hold on;
plot(real(p1), imag(p1), 'bx', 'MarkerSize',15);
plot(real(z2), imag(z2), 'r.', 'MarkerSize',90);
plot(real(p2), imag(p2), 'rx', 'MarkerSize',15);
grid on;
grid minor;
set(gca, 'FontSize', 16);
set(gca, 'XColor', [0, 0, 0], 'YColor', [0, 0, 0], 'TickDir', 'out');
xaxis = get(gca, 'XAxis');
xaxis.TickLabelInterpreter = 'latex';
yaxis = get(gca, 'YAxis');
ylim([-10, 10]);
xlim([-6, 2]);
yaxis.TickLabelInterpreter = 'latex';
title('Poles and Zeros: blue for $x_1$, red for $x_2$', 'Interpreter', 'latex');
exportgraphics(f, '../figures/HW03_Q7_PZ.pdf');
```

Comparing the two plots, we can see that the poles of $X_2(s)$ are shifted to the left by 2 units compared to the poles of $X_1(s)$. This corresponds to a faster exponential decay in the time domain, as seen in the plot of $x_2(t)$. The zeros of $X_2(s)$ are also shifted to the left by 2 units compared to the zeros of $X_1(s)$. This corresponds to a change in the phase response of the system but does not affect the system's stability. You should be able to realize that these observations are in line with the duality properties of Laplace transform.

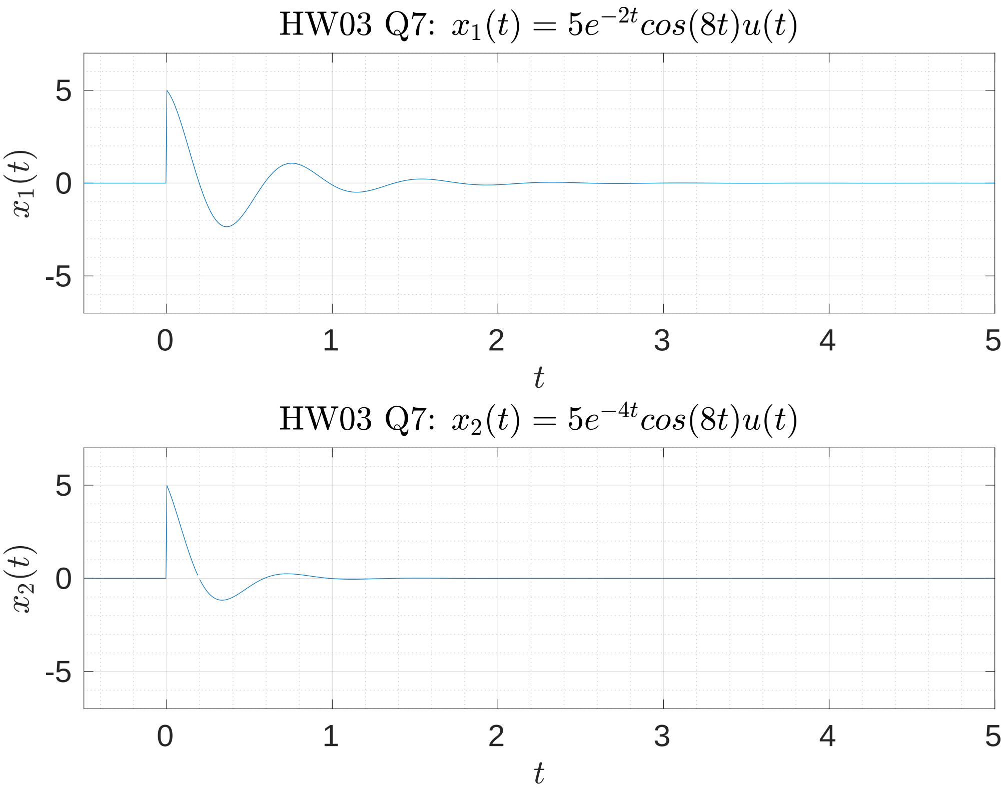


Figure 3: Ramp Signal $r(t)$

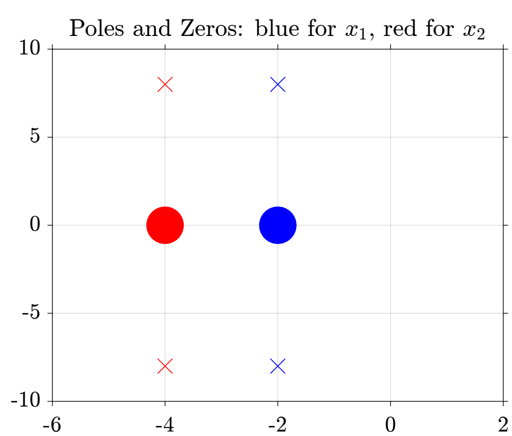


Figure 4: Ramp Signal $r(t)$