# CPE 381: Fundamentals of Signals and Systems for Computer Engineers

## 08 Sampling Theorem

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# Outline

# Motivation

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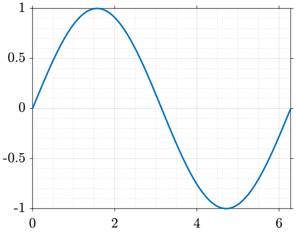
- Nature has continuous signals.
- In order to process them using computers, we need them to sample, quantize, and code to obtain digital signals – both in time and amplitude.

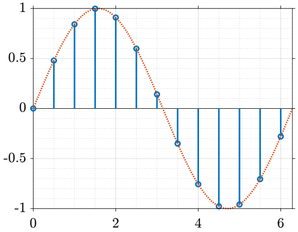
# Uniform Sampling

Discretize the time-variable

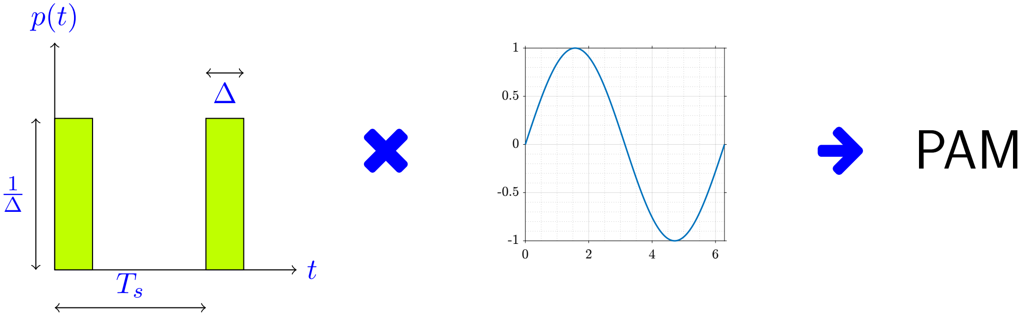
$$x(nT_s) = x(t)\Big|_{t=nT_s}$$

where $n$ is an integer.



# Pulse Amplitude Modulation (PAM)




PAM can be thought of as a switch that closes every $T_s$ seconds $\Delta$ seconds and remains open otherwise. PAM $x_{\text{PAM}}$ is a multiplication of a continuous-time signal $x(t)$ bu a periodic pulse $p(t)$.

# PAM

For a small pulse width $\Delta$, PAM signal is

$$x_{\text{PAM}}(t) = x(t)p(t) = \frac{1}{\Delta}\sum_{m} x(mT_s)[u(t - mT_s) - u(t - mT_s - \Delta)]$$

As periodic signal can be written as a Fourier series:

$$p(t) = \sum_{k=-\infty}^{\infty} P_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_s}$$

where $P_k$ are the Fourier series coefficients. Hence, the PAM signal can be written as:

$$x_{\text{PAM}}(t) = \sum_{k=-\infty}^{\infty} P_k x(t) e^{jk\Omega_0 t}$$

Its Fourier transform is $X_{\text{PAM}}(\Omega) = \sum_{k=-\infty}^{\infty} P_k X(\Omega - k\Omega_0)$ (using frequency-shift property).

# Sampling Theorem

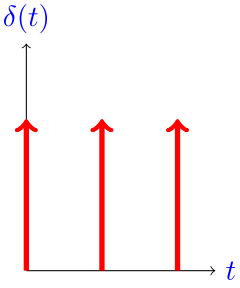
# Modeling Sampled Signal using Impulses

If $\Delta \to 0$, we have a train of impulses. In the limiting conditions, we define the impulse sampling function as

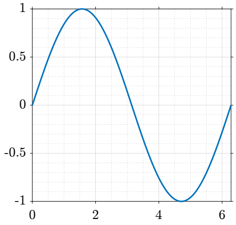
$$\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$$

Thus, the sampled signal is:

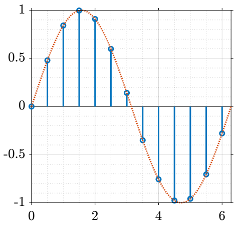
$$x_s(t) = x(t)\delta_{T_s}(t)$$



$\delta(t)$
$t$

1
0.5
0
-0.5
-1
0
2
4
6

1
0.5
0
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6

# Fourier Series of Impulse Train

$$\delta_{T_s}(t) = \sum_{=-\infty}^{\infty} kD_k e^{jk\Omega_s T}$$

where $\Omega_s = \dfrac{2\pi}{T_s}$ is the sampling frequency.

and,

$$D_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta_{T_s}(t)e^{-jk\Omega_s t}dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jk\Omega_s t}dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-j0}dt = \frac{1}{T_s}$$

# Sampled Signal in Continuous-Time and Discrete-Time

## Continuous-Time Version

$$x_s(t) = x(t)\delta_{T_s}(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\Omega_s t}$$

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

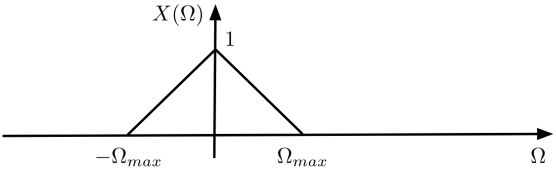
## Discrete-time Version

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\Omega T_s n}$$

# Nyquist Sampling Rate Condition I

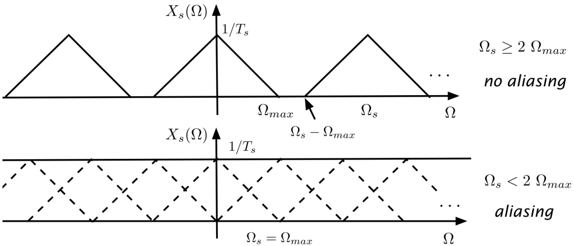
Depending on the maximum frequency present in the spectrum of $x(t)$ and on the chosen sampling frequency $\Omega_s$ (or the sampling period $T_s$) it is possible to have overlaps when the spectrum of $x(t)$ is shifted and added to obtain the spectrum of the sampled signal $x_s(t)$.




Band-limited

Band-limited signal $x(t)$ with a low-pass spectrum, finite support, i.e. $X(\Omega) = 0$ for $\Omega > \Omega_{\max}$.

# Nyquist Sampling Rate Condition II




Sampling frequency $\Omega_s$ such that the spectrum of the sampled signal consists of shifted non-overlapping versions of $\frac{1}{T_s}X(\Omega)$. This is only possible when $\underline{\Omega_s - \Omega_{\max} \geq \Omega_{\max}}$ or $\underline{\Omega_s \geq 2\Omega_{\max}}$ which is called $\boxed{\textbf{Nyquist sampling rate condition}}$.

If $\Omega_s < 2\Omega_{\max}$, then when creating $X_s(\Omega)$ the shifted spectra of $x(t)$ overlap. . In this case, due to the overlap, it will not be possible to recover the original continuous-time signal from the sampled signal, and thus the sampled signal does not share the same information with the original continuous-time signal. This overlapping phenomenon is called **frequency aliasing**.

# Sampling $x(t)$ with Infinite Support

- The signal is not band-limited.
- Aliasing will always be present.

The only way to sample a non-band-limited signal $x(t)$ without aliasing –at the cost of losing information provided by the high-frequency components of $x(t)$ – is by obtaining an approximate signal $x_a(t)$ lacking the high-frequency components of $x(t)$ and thus permitting us to determine a maximum frequency for it. This is accomplished by anti-aliasing filtering, commonly used in samplers.

# Example 1

Consider the signal $x(t) = 2\cos(2\pi t + \pi/4)$, $0 - \infty < t < \infty$, determine if it is bandlimited. Use $T_s = 0.4, 0.5$, and $1$ seconds/sample as sampling periods, and for each of these find out whether the Nyquist sampling rate condition is satisfied and if the sampled signal looks like the original signal or not.

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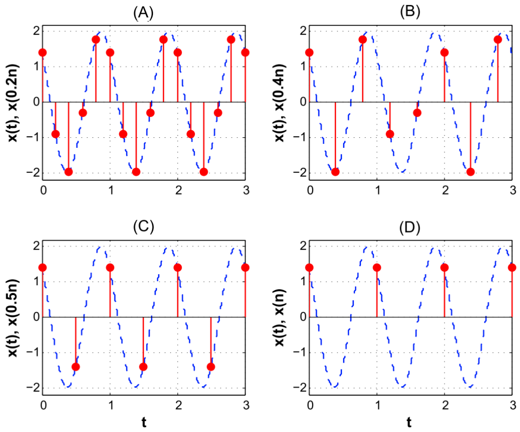
# Example 1: Assessing Nyquist Criteria



# Example 2

Consider the following signals: (i) $x_1(t) = u(t + 0.5) - u(t - 0.5)$; (ii) $x_2(t) = e^{-t}u(t)$. Determine if they are band-limited or not. If not, determine the frequency for which the energy of the non-band-limited signal corresponds to 99% of its total energy and use this result to approximate its maximum frequency.

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# Reconstruction of Original Continuous Signal

If $\Omega_s > 2\Omega_{\max}$, The spectrum of the sampled signal $x_s(t)$ displays a superposition of shifted versions of the spectrum $X(\Omega)$ multiplied by $\frac{1}{T_s}$ with no overlaps.

In such a case, we can get continuous signals by passing the sampled signal through an ideal low-pass filter with a frequency response

$$H_{\ell p} = \begin{cases} T_s, & -\Omega_s/2 \leq \Omega \leq \Omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the reconstructed signal in the form of a Fourier transform can be written as

$$X_r(\Omega) = H_{\ell p} X_s(\Omega) = \begin{cases} X(\Omega), -\Omega_s/2 \leq \Omega \leq \Omega_s/2 \\ 0, \quad \text{otherwise} \end{cases}$$

# The Exact Recovery of the Original Signal May Not be Possible

- Because the continuous-time signal is not exactly band-limited.
- Sampling is not done exactly at the uniform-rate, some variations always occur.
- The filter required for the exact recovery is an ideal low-pass filter which in practice cannot be realized.
- For non-bandlimited signals, we can use an anti-aliasing filter which is a low-pass filter that enforces to generate an approximate signal with maximum frequency.

# Signal Reconstruction from Sinc Interpolation

Ideal low-pass filter has impulse response:

$$h_{\ell p} = \frac{T_s}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega t} d\Omega = \frac{\sin(\pi t/T_s)}{\pi t/T_s}, \quad \Omega_s = \frac{2\pi}{T_s}$$

Reconstruction is the convolution of sampled signal and the low-pass filter's impulse response.

$$\begin{aligned} x_r(t) &= [x_s * h_{\ell p}](t) = \int_{-\infty}^{\infty} x_s(\tau) h_{\ell p}(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(\tau - nT_s)\right] h_{\ell p}(t-\tau) d\tau = \sum_n x(nT_s)\frac{\sin(\pi(t-nT_s)/T_s)}{\pi(t-nT_s)/T_s} \\ &= \sum_n x(nT_s)\mathrm{sinc}\Big(\pi(t-nT_s)/T_s\Big) \end{aligned}$$

# Signal Reconstruction from Sinc Interpolation

Let $t = kT_s$,

$$x_r(kT_s) = \sum_n x(nT_s)\frac{\sin(\pi(k-n))}{\pi(k-n)} = x(kT_s),$$

since

$$\frac{\sin(\pi(k-n))}{\pi(k-n)} = x(kT_s) = \begin{cases} 1, & k-n=0, \quad \text{or} \quad n=k \\ 0, & n \neq k \end{cases}$$

# Sampling of Modulated Signals

For modulated signals, the sampling rate depends on the bandwidth of the message or modulating signal, rather than on the maximum frequency of the modulated signal.

A modulated signal: $x(t) = m(t)\cos(\Omega_c t)$.
$m(t)$: Message
$\cos(\Omega_c t)$: Carrier
$\Omega_c$: Carrier Frequency
$\Omega_{\max}$: Maximum frequency present in the message

# Sampling of Modulated Signals

The sampling of $x(t)$ with a sampling period Ts generates in the frequency domain a superposition of the spectrum of $x(t)$ shifted in frequency by $\Omega_s$ and multiplied by $\frac{1}{T_s}$.

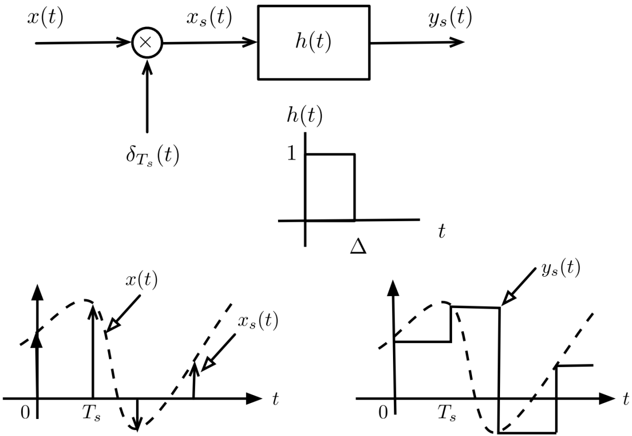
To avoid aliasing in the frequency domain, i.e. no overlapping of the shifted spectrum,

$$(\Omega_c + \Omega_{\max}) - \Omega_s < (\Omega_c - \Omega_{\max}) \Rightarrow \Omega_s > 2\Omega_{\max}$$

**Hence, the sampling period depends on the bandwidth $\boldsymbol{\Omega_{\max}}$** of the actual message $m(t)$.

# Sample-and-Hold Sampling

A sample-and-hold sampling system acquires a sample and holds it long enough for quantization and coding to be done before the next sample is acquired.



# Zero-order Hold (ZOH)

Zero-order hold filter is an LTI system that facilitates sample-and-hold.

Its impulse response $h(t)$ is a pulse of desired width $\Delta \leq T_s$.

The output of the sample-and-hold system is a weighted sequence of shifted versions of the impulse response.

Ideal Sampler: $x_s(t) = x(t)\delta_{T_s}(t)$
ZOH Output: $y_s(t) = (x_s * h)(t)$.

# Spectrum of ZOH sampled signal

$$y_s(t) = (x_s * h)(t)$$

Frequency response of $h(t)$ is:

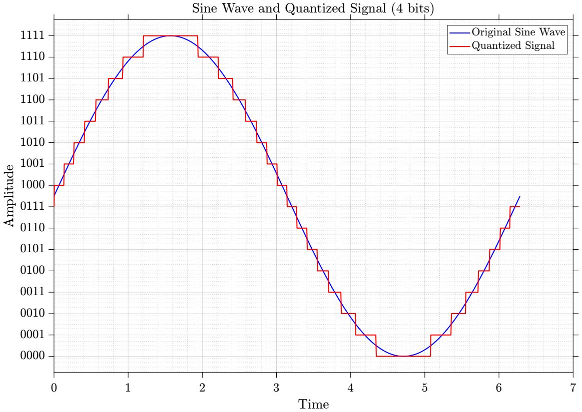
$$H(\Omega) = \frac{e^{-\Delta s/2}}{s}(e^{\Delta s/2} - e^{-\Delta s/2})\Big|_{s=j\Omega} = \frac{\sin(\Delta\Omega/2)}{\Omega/2}e^{-j\Delta\Omega/2}$$

**Thus,** $$Y_s(\Omega) = X_s(\Omega)H(\Omega) = \left[\frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)\right]H(\Omega)$$

$$= X_s(\Omega)H(\Omega) = \left[\frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)\right]\frac{\sin(\Delta\Omega/2)}{\Omega/2}e^{-j\Delta\Omega/2}$$

# Quantization and Coding

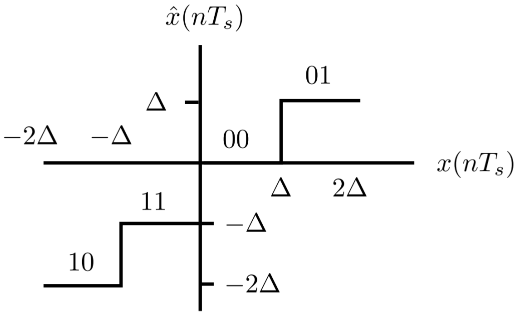
- Amplitudes are quantized – they have certain levels.
- To facilitate the quantization, we have a quantizer.
- A quantizer is a nonlinear system.
- Number of quantization levels is specified by how many bits we use to encode.
- Of course, quantization introduces error as quantization is merely an approximation. The error is called **quantization error**.




**Code:** https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/quantizing_signal.m

# Quantization Step

$$\Delta = \frac{\text{dynamic range of signal}}{2^b}$$



# Quantization Error

Sampled Signal: $x(nT_s) = x(t)|_{t=nT_s}$

**Example: 4-level quantizer**

$k\Delta \le x(nT_s) < (k+1)\Delta \Rightarrow \hat{x}(nT_s)) = k\Delta, \quad k = -2, -1, 0, 1.$

$$\begin{array}{rcrcccc}
 & & & & \hat{x}(nT_s)) & \Rightarrow & \text{Binary Code} \\
-2\Delta \le x(nT_s) < -\Delta & \Rightarrow & \hat{x}(nT_s)) = -2\Delta & | & -2\Delta & & 10 \\
-\Delta \le x(nT_s) < 0 & \Rightarrow & \hat{x}(nT_s)) = -\Delta & | & -\Delta & & 11 \\
0 \le x(nT_s) < \Delta & \Rightarrow & \hat{x}(nT_s)) = 0 & | & 0\Delta & & 00 \\
\Delta \le x(nT_s) < 2\Delta & \Rightarrow & \hat{x}(nT_s)) = \Delta & | & \Delta & & 01
\end{array}$$

# Quantization Error

We define quantization error as $\varepsilon(nT_s) = x(nT_s) - \hat{x}(nT_s)$

$$\begin{aligned} \hat{x}(nT_s) \leq x(nT_s) &\leq \hat{x}(nT_s) + \Delta \\ 0 \leq \varepsilon(nT_s) &\leq \Delta \end{aligned}$$

# Example

Suppose we are trying to decide between an 8 and a 9-bit A/D converter for a certain application where the signals in this application are known to have frequencies that do not exceed 5 kHz. The dynamic range of the signals is $10V$, so that the signal is bounded as $-5 \leq x(t) \leq 5$. Determine an appropriate sampling period and compare the percentage of error for the two A/Ds of interest.

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