CPE 381: Fundamentals of Signals and Systems for Computer Engineers

11 Discrete Fourier Analysis

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Outline

1. Motivation
2. Discrete-Time Fourier Transform
3. Fourier Series of Discrete-Time Periodic Signals
4. Discrete Fourier Transform
5. Fast Fourier Transform

Motivation

Discrete-Time Representation of Fourier Transform

Just as the discrete analogous of Laplace transform is Z-transform, we have a discrete analogous of (continuous-time) Fourier transform called **Discrete-time Fourier Transform (DTFT)**.

Discrete-Time Fourier Transform

The Discrete-Time Fourier Transform (DTFT)

For a discrete-time signal $x[n]$, the DTFT is

$$X(e^{j\omega}) = \sum_{n} x[n]e^{-j\omega n}, \quad -\pi \leq \omega \leq \pi$$

and the inverse DTFT is

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

Points about DTFT

1. DTFT measures the frequency content of a discrete-time signal.
2. DTFT is periodic in frequency with a period of 2π

$$X(e^{j\omega+2\pi k}) = \sum_n x[n]e^{-j(\omega 2\pi k)n} = X(e^{j\omega}), \quad k \quad \text{is an integer.}$$

3. For DTFT $X(e^{j\omega})$ to converge, being an infinite sum, it is necessary that

$$|X(e^{j\omega})| \leq \sum_n |x[n]||e^{j\omega n}| = \sum_n |x[n]| < \infty$$

Sampling and DTFT

Sampled signal: $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$

Its FT is $X_s(\Omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-jn\Omega T_s}$.

If we let $\omega = \Omega T_s$, the discrete frequency in radians, and $x[n] = x(nT_s)$, then FT can be

$X_s(e^{j\omega}) = \sum_n x[n]e^{-jn\omega}$ where $x[n] = x(nT_s) = x(t)\Big|_{t=nT_s}$.

At the same time, the spectrum of the sampled signal can be written as

$$X_s(e^{j\Omega T_s}) = X_s(e^{j\omega}) = \sum_k \frac{1}{T_s} X\left(\frac{\omega}{T_s} - \frac{2\pi k}{T_s}\right), \quad \omega = \Omega T_s$$

which is a periodic repetition with a fundamental period of $2\pi/T_s$ rad/s in the continuous frequency or 2π radian in the discrete frequency.
The sampling converts the continuous-time signal into a discrete-time signal with a periodic spectrum in a continuous frequency.

Z-transform and DTFT

We can write DTFT using Z-transform computed on the unit circle as:

$$X_s(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

For this relation to be valid, $X(z)$ must have ROC that includes the unit circle $z = 1e^{j\omega}$. In other cases, **we can use duality property**.

Example

Consider the noncausal signal $x[n] = \alpha^{|n|}$. Determine its DTFT. Use the obtained DTFT to find

$$\sum_{n=-\infty}^{\infty} \alpha^{|n|}$$

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Duality in Time and Frequency

First, we consider the DTFT of $\delta[n-k]$.
Z-transform $\mathcal{Z}[\delta[n-k]] = z^{-k}$ with ROC the whole z-plane except for the origin.
Then, DTFT of $\delta[n-k]$ is $e^{-j\omega k}$ after setting $z = e^{j\omega}$.
From the duality we would expect the signal $e^{-j\omega_0 n}, -\pi \leq \omega_0 \leq \pi$, would have $2\pi\delta(\omega+\omega_0)$.
We can see below, how:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega+\omega_0)e^{j\omega n}d\omega = e^{-j\omega_0 n}\int_{-\pi}^{\pi}\delta(\omega+\omega_0)d\omega = e^{-j\omega_0 n}$$

Here, we used the property of the δ function.
This gives the following dual pairs:

1. $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \Longleftrightarrow \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$
2. $\sum_{k=-\infty}^{\infty} X[k]e^{-\omega_k n} \Longleftrightarrow \sum_{k=-\infty}^{\infty} 2\pi X[k]\delta(\omega+\omega_k)$

DTFT of a Constant Signal

$$y[n] = A, -\infty < n < \infty - \infty < n < \infty$$

We use duality to calculate DTFT of $y[n]$ as calculating DTFT using integral is not possible because $y[n]$ is not absolutely summable.

$$x[n] = A\delta[n] \Longleftrightarrow X(e^{j\omega}) = A$$

Then, by duality

$$y[n] = A, -\infty < n < \infty \Longleftrightarrow Y(e^{j\omega}) = 2\pi A\delta(\omega), \quad -\pi \leq \omega \leq \pi$$

The signal $y[n]$ doesn't change from $-\infty$ to ∞ so that its frequency is $\omega = 0$, and thus its DTFT $Y(e^{j\omega})$ is concentrated in that frequency.

Example

The DTFT of a signal $x[n]$ is

$$X(e^{j\omega}) = 1 + \delta(\omega - 4) + \delta(\omega + 4) + 0.5\delta(\omega - 2) + 0.5\delta(\omega + 2)$$

The signal $x[n] = A + B\cos(\omega_0 n)\cos(\omega_1 n)$ is given as a possible signal that has $X(e^{j\omega})$ as its DTFT. Determine whether you can find A, B ω_0, and ω_1 to obtain the desired DTFT. If not, provide a better $x[n]$.

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Time and Frequency Supports

Just like with continuous-time signals, the frequency support of the DTFT of a discrete-time signal is inversely proportional to the time support of the signal.

Example

Consider a discrete-pulse

$$p[n] = u[n] - u[n-N]$$

Find its DTFT $P(e^{j\omega})$ and discuss the relation between its frequency support and the time support of $p[n]$ when $N = 1$ and when $N \to \infty$.

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Downsampling

Definition

Down-sampling a signal $x[n]$ means getting rid of samples, i.e., contracting the signal. The signal down-sampled by an integer factor $M > 1$ is given by $x_d[n] = x[Mn]$.

Inverse DTFT $x[Mn]$ is

$$x[Mn] = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} X(e^{j\omega})e^{jMn\omega}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\rho/M})e^{jn\rho}d\rho$$

with $\rho = M\omega$. Thus, the DTFT of $x_d[n] = \frac{1}{M}X(e^{j\omega/M})$, i.e. an expansion of the DTFT of $x[n]$ by a factor of M.

Upsampling

Definition

Up-sampling a signal $x[n]$ on the other hand, consists in adding $L-1$ zeros (for some integer $L>1$) in between the samples of $x[n]$, i.e., the up-sampled signals is

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

.

thus expanding the original signal.

Inverse DTFT $x_u[n]$ is $X_u(e^{j\omega}) = X(e^{jL\omega}), \quad -\pi \le \omega < \pi$ We can see that $X_u(e^{j\omega}) = \sum_{n=0,\pm L,\pm 2L,\cdots} x[n/L]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega Lm} = X(e^{jL\omega})$ indicating a contraction of the DTFT of $x[n]$.

Example

Consider an ideal low-pass filter with frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & -\pi/2 \leq \omega \leq \pi/2 \\ 0, & -\pi \leq \omega < -\pi/2, \quad \text{and} \quad \pi/2 < \omega \leq \pi \end{cases}$$

which is the DTFT of an impulse response $h[n]$. Determine $h[n]$. Suppose that we down-sample $h[n]$ with a factor $M = 2$. Find the down-sampled impulse response $h_d[n] = h[2n]$ and its corresponding frequency response $H_d(e^{j\omega})$.

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Example

A discrete pulse is given by $x[n] = u[n] - u[n-4]$. Suppose we down-sample $x[n]$ by a factor of $M = 2$, so that the length 4 of the original signal is reduced to 2, giving $x_d[n] = x[2n] = u[2n] - u[2n-4] = u[n] - u[n-2]$. Find the corresponding DTFTs for $x[n]$ and $x_d[n]$, and determine how are they related.

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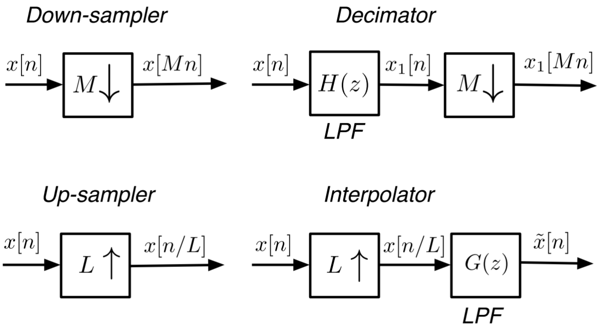
Decimator and Interpolator

Cascading a low-pass filter to a down-sampler, to avoid aliasing, gives a **decimator**. Cascading a low-pass filter to an up-sampler, to smooth the signal, gives an **interpolator**



Anti-aliasing Low-Pass Filter for the Downsampler

When down-sampling by a factor of M, to avoid aliasing (caused by the signal not being band-limited to $[-\pi/M, \pi/M]$) an antialiasing discrete low-pass filter is used before the down-sampler. The frequency response of the filter is

$$H(e^{j\omega}) = \begin{cases} 1, & -\pi/M \leq \omega \leq \pi/M \\ 0, & \text{otherwise in}[-\pi, \pi) \end{cases}$$

Low-pass Filter for Interpolation

When up-sampling by a factor of L, to change the zero samples in the up-sampler into actual samples, we smooth out the up-sampled signal using an ideal low-pass with frequency response

$$G(e^{j\omega}) = \begin{cases} L, & -\pi/M \leq \omega \leq \pi/M \\ 0, & \text{otherwise in}[-\pi, \pi) \end{cases}$$

Energy and Power of Aperiodic Discrete-Time Signals

- **Parseval's Energy Equivalence:** $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$
- **Parseval's Power Equivalence:** $P_x = \lim_{N\to\infty} \dfrac{1}{2N+1}\sum_{n=-N}^{N} |y[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S^x(e^{j\omega}) d\omega$

where

$$S^x(e^{j\omega}) = \lim_{N\to\infty} \frac{|X_N(e^{j\omega})|^2}{2N+1}$$

$X_N(e^{j\omega}) = \mathcal{F}(y[n]W_{2N+1}[n])$ which is DTFT of $x_N[n]$.
$W_{2N+1} = u[n+N] - u[n-(N+1)]$ is a rectangular window.
Remark: The significance of the above results is that for any signal, whether of finite energy or of finite power, we obtain a way to determine how the energy or power of the signal is distributed over frequency. Suppose the signal is known to be infinite energy and finite power. In that case, the windowed computation of the power allows us to approximate the power and the power spectrum for a finite number of samples.

Time and Frequency Shifts

- Shifting a signal in time does not change its frequency content. Thus the magnitude of the DTFT of the signal is not affected, only the phase is.

$$\mathcal{F}x(n-N) = \sum_n x[n-N]e^{-j\omega n} = \sum_m x[m]e^{-j\omega(m+N)} = e^{-j\omega N}X(e^{j\omega})$$

- In its dual, multiplying a signal by a complex exponential $e^{j\omega_0 n}$ for some frequency ω_0, then the spectrum of the signal is shifted in frequency.
  Assuming $x_1[n] = x[n]e^{j\omega_0 n}$,

$$X_1(e^{j\omega}) = \sum_n x_1[n]e^{-j\omega n} = \sum_n x[n]e^{-j(\omega-\omega_0)n} = X(e^{j(\omega-\omega_0)}).$$

Symmetry

$$\mathrm{Re}[X(e^{j\omega})] = \mathrm{Re}[X(e^{-j\omega})]$$

and

$$\mathrm{Im}[X(e^{j\omega})] = -\,\mathrm{Im}[X(e^{-j\omega})]$$

Example

. For the signal $x[n] = \alpha^n u[n]$, $0 < \alpha < 1$, find the magnitude and the phase of its DTFT $X(e^{j\omega})$.

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Convolution Sum in Frequency Domain

If $h[n]$ is the impulse response of a stable LTI system, its output $y[n]$ can be computed by means of the convolution sum

$$y[n] = \sum_k x[k]h[n-k]$$

where $x[n]$ is the input. The Z-transform of the $y[n]$ is the product

$$Y(z) = H(z)X(z), \quad ROC : \mathcal{R}_Y = \mathcal{R}_H \cap \mathcal{R}_X.$$

If the unit circle is included in $\mathcal{R}_Y$, then

- $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- $|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$
- $\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$

Discrete-Time Fourier Transform (DTFT) properties

Table 11.1 Discrete-Time Fourier Transform (DTFT) properties

Z-transform:	$x[n],\ X(z),\ \lvert z\rvert = 1 \in \text{ROC}$	$X(e^{j\omega}) = X(z)\vert_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)})$, k integer
Linearity:	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N} X(e^{j\omega})$
Frequency shift:	$x[n]e^{j\omega_o n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\lvert X(e^{j\omega})\rvert$ even function of ω
		$\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\displaystyle\sum_{n=\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

Fourier Series of Discrete-Time Periodic Signals

Discrete-Time Periodic Signals

A discrete-time signal $x[n]$ is periodic if there is a positive integer N such that $x[n + kN] = x[n]$, for any integer k.

Discrete-Time LTI System

If the an input signal $x[n]$ to an LTI system of frequency response $H(e^{j\omega_0})$ has a form

$$x[n] = \sum_k A[k]e^{k\omega_k n}$$

then the output is

$$y[n] = \sum_k A[k]e^{k\omega_k n} H(e^{j\omega_k})$$

This property is valid whether the frequency components of the input signal are harmonically related (when $x[n]$ is periodic), or not.

Fourier Series Recap for a Periodic Signal

Recall that a signal $x(t)$, periodic of fundamental period T_0, can be represented by its Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{X}[k] e^{\frac{2\pi kt}{T_0}}$$

Fourier Series for Periodic Sampled Signal

For a sampled signal $x_s(t) = x(nT_s) = x[n]$ of $x(t)$ with a sampling period of $T_s = T_0/N$ (i..e $\Omega_s = N\Omega_0$, satisfying Nyquist sampling condition) where N is a positive integer, then

$$x[n] = x(nT_s) = \sum_{k=-\infty}^{\infty} \hat{X}[k]e^{\frac{2\pi knT_s}{T_0}} = \sum_{k=-\infty}^{\infty} \hat{X}[k]e^{\frac{2\pi knT_0}{T_0 N}} = \sum_{k=-\infty}^{\infty} \hat{X}[k]e^{\frac{2\pi kn}{N}}$$

Fourier Series for Periodic Sampled Signal

$$x[n] = \sum_{k=-\infty}^{\infty} \hat{X}[k] e^{\frac{2\pi kn}{N}}$$

The summation repeats the frequencies between 0 to 2π. To avoid these repetitions, we let $k = m + rN$ where $0 \leq m \leq N-1$ and $r = 0, \pm 1, \pm 2, \cdots$, i.e., we divide the infinite support of k.

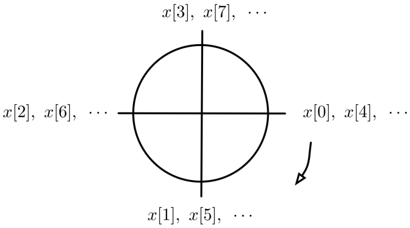
$$x[n] = \sum_{m=0}^{N-1} \sum_{r=-\infty}^{\infty} \hat{X}[m+rN] e^{j\frac{2\pi(m+rN)n}{N}} = \sum_{m=0}^{N-1} \left[\sum_{r=-\infty}^{\infty} \hat{X}[m+rN] \right] e^{j\frac{2\pi mn}{N}}$$

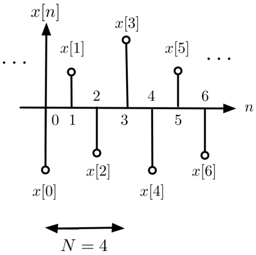
$$= \sum_{m=0}^{N-1} X[m] e^{j\frac{2\pi mn}{N}}$$

where we write $X[m] = \left[\sum_{r=-\infty}^{\infty} \hat{X}[m+rN] \right]$ and the exponential repeats after a full circular period, so we remove **m+rN**. This representation is in terms of complex exponentials with frequencies $2\pi m/N$, $m = 0, 1, \cdots, N-1$, from $0 to 2\pi(N-1)/N$. We will see this Fourier series representation next.

Circular Representation of Discrete-Time Periodic Signals

- Since both the fundamental period N of a periodic signal $x[n]$, and the samples in a first period $x_1[n]$ completely characterize a periodic signal $x[n]$, a circular rather than a linear representation would more efficiently represent the signal.
- The circular representation is obtained by locating uniformly around a circle, the values of the first period starting with $x[0]$ and putting in a clockwise direction the remaining terms $x[1]$,···, $x[N-1]$.
- For $m = kN + r$, $0 \leq r < N$, $x[m] = x[kN + r] = x[r]$. This is a circular representation.





Response of LTI Systems to Periodic Signals

If $x[n]$ is a periodic signal of fundamental period N, and it is an input to an LTI system with the transfer function $H(z)$, then

$$y[n] = \sum_{k=0}^{N-1} X[k]H(e^{jk\omega_0})e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}, \quad \text{fundamental frequency}$$

with $x[n]$ having the Fourier series

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\omega_0 n}$$

and $Y[k] = X[k]H(jk\omega_0 n)$

Discrete Fourier Transform

DTFT of Periodic Signals

Recall:

- Direct DTFT: $X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n}, \quad -\pi \leq \omega < \pi$.
- Inverse DTFT: $\frac{1}{2\pi}\int_{-\pi}^{\pi}$

Computer implementation of DTFT has two problems:

1. **ω varies continuously from $-\pi$ to π , hence computing an exact $X(e^{j\omega})$ needs just uncountable number of continuously varying frequencies.**
2. **The inverse DTFT requires integration which is an approximate on the computer.**

Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT) which is **not** DTFT is computed at discrete frequencies and its inverse also doesn't require integration. It is also suitable for computer implementation using an algorithm called the Fast-Fourier Transform (FFT).

Note that the Fast-Fourier Transform (FFT) is not a new transform, it is an efficient algorithm for computer implementation.

This representation of the discrete-time periodic signal is discrete and periodic in both time and frequency.

Fourier Series in Discrete-Time and Discrete-Frequency

For a periodic $\tilde{x}[n]$ of fundamental period N and represented by N values in a period, its Fourier series is

$$\tilde{x}[n] = \sum_{0}^{N-1} \tilde{X}[k] e^{j\omega_0 nk}, \quad 0 \leq k, n \leq N-1$$

$\omega_0 = 2\pi/N$ is the fundamental discrete frequency.

- **The coefficients $\tilde{X}[k]$ correspond to harmonic frequencies $k\omega_0$, for $0 \leq k \leq N-1$, so that $\tilde{x}[n]$ has no frequency components at any other frequencies.**
- **Thus $\tilde{x}[n]$ and $\tilde{X}[k]$ are both discrete and periodic of the same fundamental period N.**

Calculating Fourier Series Coefficients

The Fourier series coefficients can be calculated using the Z-transform

$$\tilde{X}[k] = \frac{1}{N}\mathcal{Z}[\tilde{x}_1[n]]|_{z=e^{jk\omega_0}}, \quad 0 \leq k \leq N-1, \quad \omega_0 = 2\pi/N$$

where $\tilde{x}_1[n] = \tilde{x}[n]W[n]$ is a period of $\tilde{x}[n]$. $W[n]$ is a rectangular window

$$W[n] = u[n] - u[n-N] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

And, we already know $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \tilde{x}_1[n+rN]$

DFT Formula

Then we can write the DFT of a periodic signal $\tilde{x}[n]$ as

$$X[k] = N\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\omega_0 nk} = \tilde{x}[n]e^{-j2\pi nk/N}, \quad 0 \le k \le N-1, \omega_0 = 2\pi/N$$

and the inverse DFT is

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}, \quad 0 \le n \le N-1$$

Aperiodic Discrete-Time Signal

- We obtain the DFT of an aperiodic signal $y[n]$ by sampling in frequency its DTFT $Y(e^{j\omega})$.
- Sampling frequencies: $\{\omega_k = 2\pi k/L, \quad k = 0, 1, \cdots, L-1\}$, L to be determined.
- From sampling discussion in Chapter 8: sampling in frequency generates a periodic signal in time:

$$\tilde{y}[n] = \sum_{r=-\infty}^{\infty} y[n + rL]$$

If $y[n]$ is of finite length, N, and $L \geq N$, then the periodic extension $\tilde{y}[n]$ displays a period equal to $y[n]$ (may be some padded zeros if $L > N$)

Time-Aliasing

If $L < N$, the first period of $\tilde{y}[n]$ does not coincide with $y[n]$ because of the superposition of shifted versions of it—this corresponds to **time aliasing**, the dual of frequency-aliasing which occurs in time-sampling.

DFT of Aperiodic Signal

For $L \geq N$, we have

Discrete Fourier Transform (DFT)

$$Y[k] = Y(e^{j2\pi k/L}) = \sum_{n=0}^{N-1} y[n] e^{-j2\pi nk/L}, \quad 0 \leq k \leq L-1$$

Inverse DFT of Aperiodic Signal

For $L \geq N$, we have

InverseDiscrete Fourier Transform (DFT)

$$y[n] = \frac{1}{L}\sum_{k=0} Y[k]e^{j2\pi nk/L}, \quad 0 \leq n \leq L-1$$

Some Practical Consideration for DFT Calculation

- The generation of the periodic extension $\tilde{y}[n]$ is not needed, we just need to generate a period that either coincides with $y[n]$ when $L = N$, or that is $y[n]$ with a sequence of $L - N$ zeros attached to it (i.e., $y[n]$ is padded with zeros) when $L > N$.
- To avoid time aliasing we do not consider choosing $L < N$.
- For a very long sequence $N \to \infty$, it does not make sense to compute its DFT, even if we could. Such a DFT would give the frequency content of the whole signal and since a large support signal could have all types of frequencies its DFT would just give no valuable information.

DFT of a Large Sequence

- To obtain the frequency content of a signal with a large time-support is to window it and compute the DFT of each of these segments.
- Thus when $y[n]$ is of infinite length, or its length is much larger than the desired or feasible length L, we use a window $W_L[n]$ of length L, and represent $y[n]$ as the superposition

$$y[n] = \sum_m y_m[n]$$

where $y_m[n] = y[n]W_L[n - mL]$

- By linearity:

$$Y[K] = \sum_m DFT(y_m[n]) = \sum_m Y_m[k]$$

Fast Fourier Transform

Computing DFT via FFT

Recall, for a sequence of length $L = N$,

DFT:
$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}$, $k = 0, 1, \cdots, N-1$
Inverse DFT:
$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N}$, $n = 0, 1, \cdots, N-1$

Employing Symmetry for FFT

Let $W_N = e^{-j2\pi/N}$, then

- $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$
- $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$

Radix-2 FFT Decimation-in-Time

Algorithm: Assuming $N = 2^\gamma$, $\gamma > 1$, then

1. W_N^{nk} is periodic.

$$W_N^{nk} = \begin{cases} W_N^{(n+N)k} \\ W_N^{n(k+N)} \end{cases}$$

2. W_N^{nk} exhibits symmetry.
3. $W_N^{k(2n)} = e^{-j2\pi(2kn)/N}$

$$W_N^{k(2n)} = e^{-j2\pi kn/(N/2)} = W_{N/2}^{kn}$$

4. $W_N^{k(2n+1)} = W_N^k W_{N/2}^{kn}$
5. W is called **Twiddle Factor**.

Divide and Conquer Approach for FFT Algorithm

Using the divide and conquer principle:

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} = \sum_{n=0}^{N/2-1} \left[\underbrace{x[2n]W_N^{k(2n)}}_{\textit{Even Terms}} + \underbrace{x[2n+1]W_N^{k(2n+1)}}_{\textit{Odd Terms}} \right], \quad k = 0, \cdots, N-1$$

Using properties W, we can write

$$X[k] = \underbrace{\sum_{n=0}^{N/2-1} x[2n]W_{N/2}^{kn}}_{\textit{N/2 point DFT}} + W_N^k \underbrace{\sum_{n=0}^{N/2-1} x[2n+1]W_{N/2}^{kn}}_{\textit{N/2 point DFT}} = Y[k] + W_N^k Z[k]$$

Divide and Conquer Approach for FFT Algorithm

$Y[k]$ and $Z[k]$ are DFTs of the length $N/2$ of even-number sequence an odd-numbered sequence respectively.

For $N/2$ points, we would employ a similar computation:

$$X[k] = Y[k] + W_N^k Z[k], \quad k = 0, \cdots, (N/2) - 1$$

For $k \leq N/2$, we use :

$$X[k + N/2] = Y[k + N/2] + W_N^{k+N/2} Z[k + N/2] = Y[k] - W_N^k Z[k], \quad k = 0, \cdots, N/2 - 1$$

We used, $W_N k + N/2 = -W_N^k$.

Decimation in Time FFT Algorithm

We can write in matrix form:

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{\Omega}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{\Omega}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_{N/2} \\ \mathbf{Z}_{N/2} \end{bmatrix} = \mathbf{A}_1 \begin{bmatrix} \mathbf{Y}_{N/2} \\ \mathbf{Z}_{N/2} \end{bmatrix}$$

where $\mathbf{I}_{N/2}$ is the unit matrix (or identity matrix), and $\mathbf{\Omega}_{N/2}$ is a diagonal matrix with entries $W_N^k, \quad k = 0, 1, \cdots, N/2 - 1$.
The vectors $\mathbf{X}_N$, $\mathbf{Y}_{N/2}$, and $\mathbf{Z}_{N/2}$ are vector form of $X[k]$, $Y[k]$, and $Z[k]$.

Example

Consider the decimation-in-time FFT algorithm for $N = 4$. Give the equations to compute the four DFT values $X[k]$, $k = 0, \cdots, 3$ in the matrix form.

Blank space for calculation

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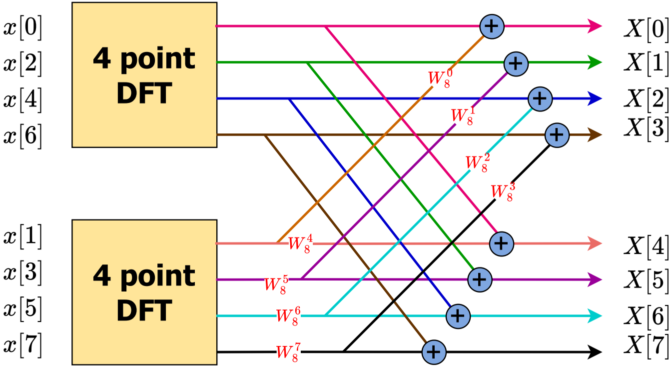
Graphical Representation of FFT

$$X[k] = \underbrace{\sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{kn}}_{N/2 \text{ point DFT}} + W_N^k \underbrace{\sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{kn}}_{N/2 \text{ point DFT}}$$

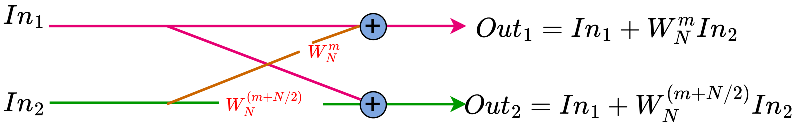
Each $N/2$ point DFT is only valued for $k = 0, 1, 2, \cdots N/2-1$ because $W_{N/2}^{k}$ is periodic with the period $N/2$. For $N = 8$,



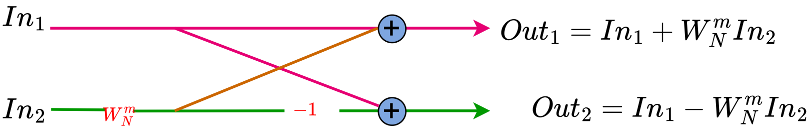
W factor gets multiplied and then gets added with another input. The diagram is called butterfly structure.

Decomposition of Butterfly Structure

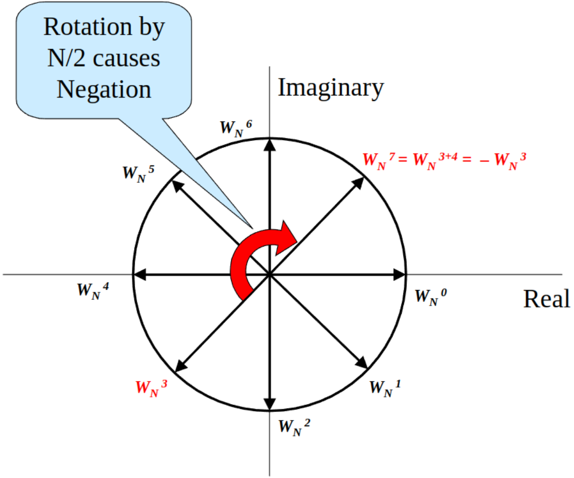
This decomposition into half-length DFTs can be done again to each of the two $N/2$-pt. DFTs, and then again, and then again ... until reaching 2-pt DFTs.



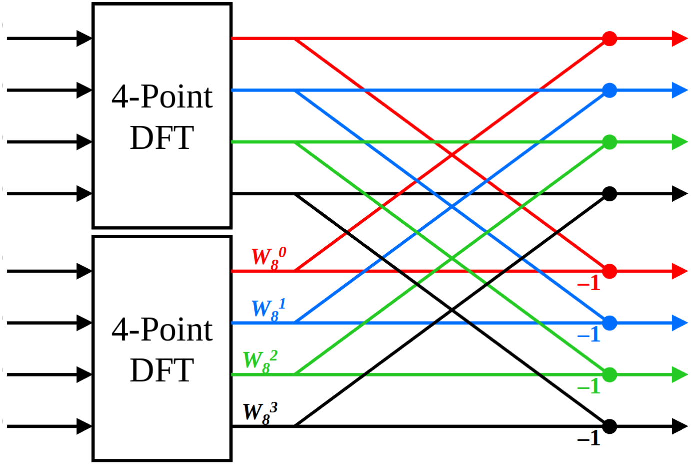
We can also exploit some properties of W to make the computation more efficient ($W_N^{(m+N/2)} = -W_N^m$):



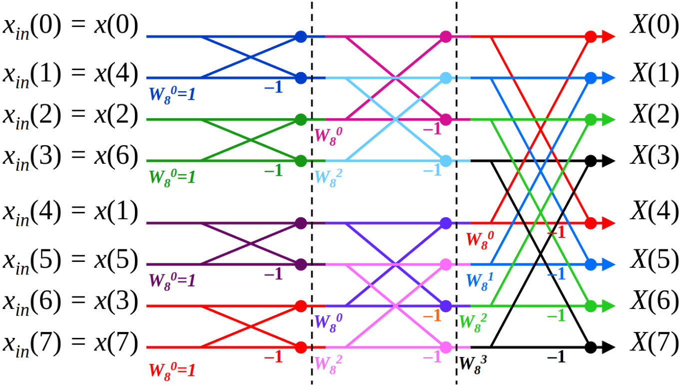
Exploitable Structure of Twiddle Factors



Improved 8-Point DFT



Complete Divide and Conquer Strategy for FFT



Inverse DFT using FFT

The FFT algorithm can be used to compute the inverse DFT without any changes in the algorithm. Assuming the input $x[n]$ is complex ($x[n]$ being real is a special case), the complex conjugate of the inverse DFT equation, multiplied by N, is

$$Nx^*[n] = \sum_{k=0}^{N-1} X^*[k]W^{nk}$$