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Known results. (All results on this problem are unpublished. The results of Mark Thompson are part of his unpublished undergraduate thesis—Harvard University, 1970.)

Thompson considered a discrete version when the x_i are chosen uniformly from

$$\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\right\} \text{ and proved } w = 1/2 \text{ for } 1 \leqq n < 5.$$

It seems reasonable to generalize and allow x_1, x_2, x_3 to come from any (but the same) distribution of finite mean. Then the conjecture would be that the value w is that mean. This writer, for example, showed the conjecture true if the x_i are chosen uniformly from $\{0, 1, 8\}$.

Some work has been done on the generalization where n numbers $x_1, \cdots, x_n$ are chosen uniformly in $[0, 1]$ and k of them are changed by Player II. E. B. Keeler proved that if $n \leqq 2k$, $w = 1/2$. D. Kleitman and, independently, S. Zamir, proved that if $n = 4$, $k = 1$ then $w > 1/2$.

CLASSROOM NOTES

EDITED BY ROBERT GILMER

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TRAFFIC FLOW: LAPLACE TRANSFORMS

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Preface. We have found that students seem to develop a better appreciation of differential equations when presented with substantial applications. The following project is a modification of a handout we have used in our classes. Each question is preceded by "Q". Selected answers appear at the end. Section IV is written at an elementary, heuristic level, because our classes were studying differential equations as part of the basic calculus sequence. The handout referred to in VI applied stability theory to the Volterra-Lotka equations for predator-prey models and to a pendulum with a frictional force which is an arbitrary function of velocity.

1. Introduction. The mathematical study of traffic flow is relatively new. It requires little scientific background and uses primarily differential equations, probability and statistics. A book [**1**] is available; however, you may find it difficult if you have not had some probability and statistics. There are also brief surveys in [**2**, **3**].

We shall consider the following problem:

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> A single line of cars moves along a straight highway without passing. Under what conditions will acceleration and/or deceleration by the first driver cause a collision further back?

It is clear that *very rapid* action by the first driver can easily cause a pile up. We are interested in situations in which moderate action by the first diiver causes more and more violent responses as the effect travels back along the line of cars.

2. The model. This model is taken from **[4]**. The position of the lead car is given by $x_1(t)$, the position of the nth by $x_n(t)$. All drivers will be treated as identical. (This is not essential. It only simplifies calculations.) Time is measured in units of driver reaction time. Each driver's acceleration (deceleration) is proportional to the difference between the speed of his car and that of the car ahead. The first driver is free to do as he wishes. Thus we assume

$$x_n''(t) = C(x_{n-1}'(t-1) - x_n'(t-1)) \tag{1}$$

for some $C > 0$ and all $n > 1$. The $t - 1$ is due to reaction time lag. For $t \leqq 0$ we assume that $x_n'(t)$ is a constant independent of n; that is, the string of cars is moving, as a unit with constant velocity.

Q1: Comment on the reasonableness of (1). In particular, might C depend on car separation? In a qualitative way, how? Might it differ for acceleration and deceleration? How?

Q2: Introduce $z_n(t) = x_n(t) - x_n(0) - tx_n'(0)$. Interpret z_n and show that

$$\begin{cases} z_n''(t) = C(z_{n-1}'(t-1) - z_n'(t-1)) \text{ for } n > 1, \\ z_n(t) = z_n'(t) = 0 \text{ for } t \leqq 0 \text{ and } n \geqq 1. \end{cases} \tag{2}$$

(The fact that $z_n(0) = z_n'(0) = 0$ simplifies the next step.)

3. Some Laplace transforms.

Q3: Assume that the lead car varies its speed in some fashion when $t > 0$. Take Laplace transforms and show that

$$Z_{n+1}(s) = C^n(C + se^s)^{-n} Z_1(s), \tag{3}$$

where $Z_n(s) = \mathscr{L}(z_n(t))$.

Q4: Let $a_n(t) = z_n''(t) = x_n''(t)$, the acceleration of the nth car. Denote $\mathscr{L}(a_n(t))$ by $A_n(s)$. Using (3) deduce

$$A_{n+1}(s) = C^n(C + se^s)^{-n} A_1(s). \tag{4}$$

(You should recognize that this formula expresses the Laplace transformed description of the $n + 1$ car's behaviour in terms of the first car's behaviour!) When $A_1(s)$ is specified, these transforms can be inverted, (with work), but this

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approach leads to something messy and hard to use. Instead, one may rely on a bit of the theory of complez variables to obtain some simple approximate results.

4. Approximate inversion of Laplace transforms. We want to know roughly how the inverse transform of (4) behaves. In this section, we discuss without proof a well-known fact in the theory of Laplace transforms.

The transform $\mathscr{L}(e^{at}) = (s-a)^{-1}$ was derived in class for s a real number greater than a. However, $(s-a)^{-1}$ makes sense for all complex numbers $s \neq a$. All Laplace transforms can be extended to complex numbers in this fashion.

Let $g(s) = 1/\mathscr{L}(f)$. *Suppose* $g(s_0) = 0$ and no solution of $g(s) = 0$ has larger real part. Also suppose

$$g'(s_0) = g''(s_0) = \cdots = g^{(K)}(s_0) = 0$$

and $g^{(K+1)}(s_0) \neq 0$. One calls s_0 a zero of g of multiplicity $K+1$. Let $s_0 = a + bi$.

FACT: Under the above assumptions there is a constant E such that $f(t)$ grows like $Et^K e^{at}$. In addition, $f(t)$ may oscillate. If $b \neq 0$, then there is oscillation like $\cos(bt+d)$ for some d. Of course, there may be several roots of $g(s) = 0$ with real part a and different imaginary parts. Then there will be several components of the oscillation.

Examples. (The facts given here are without proof, you may wish to verify them.) Suppose

$$\mathscr{L}(f) = \left(\frac{1}{(s-a)^2 + b^2}\right)\left(\frac{1}{(s-a)^2 + b'^2}\right)$$

$b,\ b' \neq 0$. Then $g(s) = ((s-a)^2 + b^2)((s-a)^2 + b'^2)$ has roots $a \pm bi,\ a \pm b'i$. At each root $g'(s) \neq 0$. Hence $f(t)$ grows like

$$ce^{at}\cos(bt+d) + c'e^{at}\cos(b't+d')$$

for some constants $c,\ c',\ d,\ d'$. You can check this by finding $f(t)$. If $b' = 0$, then we get

$$ce^{at}\cos(bt+d) + \boxed{c'te^{at}}$$

and the term in the box is the important one. If $g(s) = ((s-a)^2 + b^2)(s-a)$, then $f(t)$ grows like

$$ce^{at}\cos(bt+d) + c'e^{at}.$$

Q5: Suppose $f = e^{-t} - e^{-2t}\cos t$. Then (complete this equation)

$$g(s) = \frac{1}{\mathscr{L}(f)} =$$

and $g(s) = 0$ has solutions $s =$

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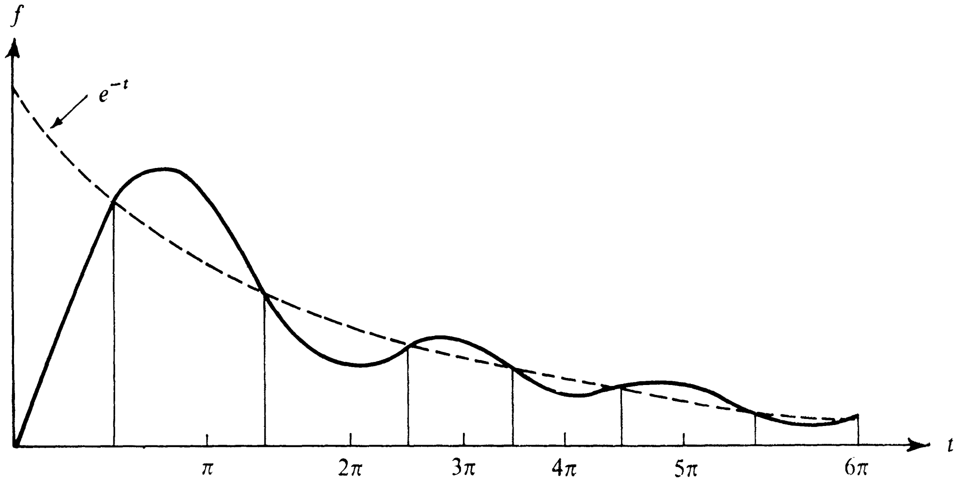
In this case, there is some oscillation due to the term $e^{-2t}\cos t$, but it dies out faster than e^{-t}. The graph of f is like that shown below. This relative unimportance of the oscillations is typical of the case in which there is only one root with largest real part and this root is a real number.



5. Analysis of our model using the previous sections. Now suppose the first driver accelerates and decelerates for a finite period of time; that is, $a_1(t) > 0$ for a bit, < 0 for a bit, and then $a_1(t) = 0$ for $t \geqq T$ for some T.

Q6: Use the physical behaviour of the first car to show that $1/(A_1(s)) = 0$ has no solutions.

The following can be shown concerning the zeros of $C + se^s$ which have largest real part equal to a.

Condition	Nature of $a + bi$	
$C > \pi/2$	$a > 0$	$b \neq 0$
$C = \pi/2$	$a = 0$	$b \neq 0$
$\pi/2 > C$	$a < 0$	$b \neq 0$
$1/e \geqq C$	$a < 0$	$b = 0$

Q7: Using the ideas developed in this handout, analyze each of the four cases for C. Describe the nature of $a_n(t)$, $(n > 1)$ for large t, where the first driver behaves as suggested at the beginning of this section. Show that a collision *must* take place if $C \geqq \pi/2$.

6. Some remarks. The above system is a "feedback mechanism" which can be characterized as follows:

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(i) a critical point (where $z_n(t) = 0$ for all n),

(ii) a "mechanism" which acts to restore the system when it deviates from the critical point (here the equations (2)),

(iii) often a time delay (here reaction time).

Similar ideas were studied in the stability theory handout where we studied the nature of critical points for non-linear models by linearization. A non-linear time delay situation can also be handled by a linearization approach, but it requires Laplace transforms as used here. Related ideas are presented in [6] and the references given there.

Feedback delay is important for stability, as the problem below shows. Suppose we measure things in units of ordinary time with Δ the reaction time. We have

$$x''_{n+1}(t) = k(x'_n(t-\Delta) - x'_n(t-\Delta)). \tag{5}$$

Q8: (a) Transform (5) to the form (1) and so express C in terms of k and Δ.

(b) Deduce that as reaction time increases the system tends to become unstable.

(c) Now assume $\Delta = 0$. Thus replace $t - 1$ by t in (1). Show that

$$Z_{n+1}(s) = C^n(C + s)^{-n}Z_1(s).$$

Deduce that the system is stable regardless of the value of $C > 0$. (In fact, large C gives greater stability in direct opposition to the case $\Delta \neq 0$!).

Another sort of instability can occur in our example. A plot of $z_n(t)$ versus time may level off as $t \to \infty$, but as n gets larger, the graphs may become wilder. This can be studied in various ways:

(i) Choose $x_1(t)$ so that (1) can be solved [5].

(ii) Study the inverse transform of (4) by approximate methods which are more accurate than those used above [4, 5].

(iii) Use other transforms to study other functions besides $a_n(t)$, for example $\int_0^\infty a_n(t)^2 dt$ [4].

We shall consider (i).

Let $x_n(t) = b_n e^{i\omega t}$ where b_n is to be found. In the end, we can let $x_n(t)$ be the real part of the above since $d/dt\,\mathrm{Re}(f(t)) = \mathrm{Re}(f'(t))$. (Re denotes "real part of.") This gives a sinusoidal motion.

Q9: (a) Let $f(t)$ be an arbitrary differentiable complex valued function of the real number t. Show that $d/dt\,\mathrm{Re} f(t) = \mathrm{Re}(f'(t))$.

(b) Using (1), show that

$$-\omega^2 b_n = iC\omega e^{-i\omega}(b_{n-1} - b_n).$$

(c) Deduce that

$$b_{n+1} = b_1 \Big/ \left(1 + \frac{i\omega}{C} e^{i\omega}\right)^n .$$

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(d) Show that we have instability if and only if

$$\left| 1 + \frac{i\omega}{C} e^{i\omega} \right| < 1$$

and hence when $C > \omega/2\sin\omega$.

(e) Conclude that if $C > \frac{1}{2}$, there is instability for ω near zero.

Note the difference between the result $C > \frac{1}{2}$ and our earlier result $C > \pi/2 = 1.57\cdots$.

Q10: Account for the difference just noted.

Experiments indieate that C is nearly $\frac{1}{2}$ for the actual drivers. See [**4**]. We have only considered one type of motion for the leader, but the ideas can be generalized by using Fourier series or approach (ii). Approach (iii) has another advantage. We can allow for the fact that (1) should be replaced by a less deterministic equation. This involves some elementary statistics and the Fourier transform.

SELECTED ANSWERS

A4: We have

$$\mathscr{L}(z_n''(t)) = C\{\mathscr{L}(z_{n-1}'(t-1)) - \mathscr{L}(z_n'(t-1))\}.$$

By tables of $\mathscr{L}$

$$s^2 Z_n(s) = C(e^{-s} s Z_{n-1}(s) - e^{-s} s Z_n(s)).$$

Hence $Z_n(s) = C(C + e^s)^{-1} Z_{n-1}(s)$.

A6: Let $\alpha = \max_{0 \leqq t \leqq T} |a_1(t)|$ and $\beta(s) = \max_{0 \leqq t \leqq T} |e^{st}|$. Then

$$|A_1(s)| \leqq \int_0^\infty |e^{st} a_1(t)|\, dt = \int_0^T |e^{st} a_1(t)|\, dt$$

$$\leqq \int_0^T \beta(s)\alpha dt = \alpha\beta(s)T.$$

Hence $(A_1(s))^{-1}$ has no roots.

A7: When $C > \pi/2$ acceleration is unstable: it oscillates with larger and larger amplitudes as time goes on. When $C = \pi/2$, there is stable oscillatory behavior for the second car and unstable oscillatory behavior for the rest of the string since

$$\frac{d}{ds} \frac{(C + se^s)^n}{A_1(s)} \Bigg|_{s = a + bi} = 0 \text{ for } n > 1.$$

A8: (a) Let the independent variable be $\tau = t/\Delta$. We obtain

$$x_{n+1}''(\tau) = \Delta k (x_n'(\tau) - x_n'(\tau - 1))$$

so the preceding analysis applies with $C = \Delta k$.

A10: We have considered *different* accelerations for the first car in the two cases. We have also given *different* answers.

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(a) In Section 5 we saw that the third car always develops wild acceleration if $C > \pi/2$.

(b) By Q9(c) we see that no car *ever* develops wild acceleration, but each car is wilder than its predecessor if $C > \omega/2\sin\omega$.

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IRRATIONAL NUMBERS

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For the past few years a clever proof has been making the rounds of the various mathematics departments.

THEOREM 1. *An irrational number raised to an irrational power may be rational.*

Proof: Consider the identity

$$[\sqrt{2}^{\sqrt{2}}]^{\sqrt{2}} = 2.$$

If $\sqrt{2}^{\sqrt{2}}$ is rational then we are finished. If not then $\sqrt{2}^{\sqrt{2}}$ is irrational so $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is the example.

This proof seems first to have been published by Dov Jarden as a curiosity in [3]. The proof was published again in [2]. Note that while the proof is elementary, it is non-constructive. The non-constructivity enters in the form of the logical principle of the excluded middle (*tertium non datur*) which the intuitionists reject.

Actually $\sqrt{2}^{\sqrt{2}}$ is irrational, being the square root of Hilbert's number $2^{\sqrt{2}}$, proved transcendental by Kuzmin [1] in 1930. But this result, which is not elementary, is not used above. Only the irrationality of $\sqrt{2}$ is used.

Consider next the related theorem.

THEOREM 2. *An irrational number raised to an irrational power may be irrational.*

Of course we can use set theoretical principles to prove that a^b is irrational for almost all real numbers b. Or we can use the result of Kuzmin [1] to prove Theorem 2 But does Theorem 2 have an elementary proof?

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