CPE 381: Fundamentals of Signals and Systems for Computer Engineers

08 Sampling Theorem

Rahul Bhadani

Electrical & Computer Engineering, The University of Alabama in Huntsville

Outline

1. **Motivation**
2. **Sampling Theorem**

Motivation

Motivation

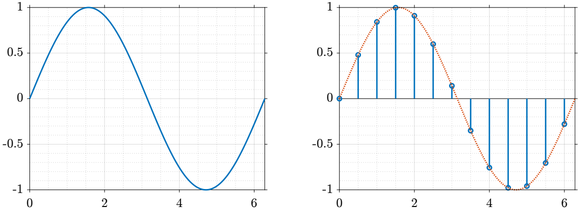
- Nature has continuous signals.
- In order to process them using computers, we need them to sample, quantize, and code to obtain digital signals – both in time and amplitude.

Uniform Sampling

Discretize the time-variable

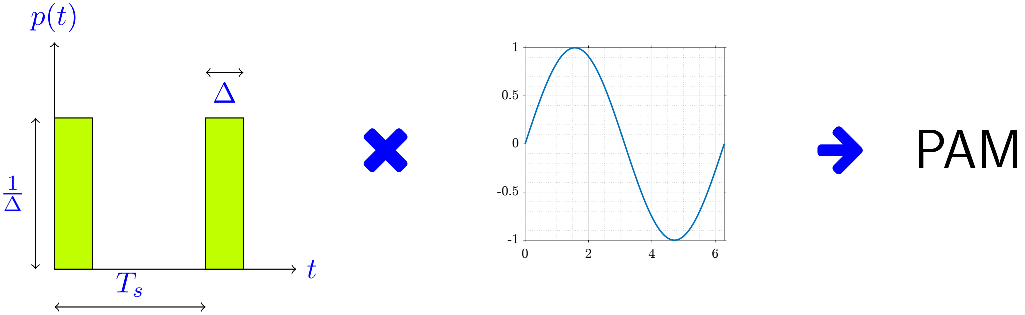
$$x(nT_s) = x(t)\Big|_{t=nT_s}$$

where n is an integer.



Sampling processing can be thought of as a modulation processing coming from Pulse Amplitude Modulation, with $\Delta << T_s$; Δ: width of the pulse, T_s: sampling period.

Pulse Amplitude Modulation (PAM)



PAM can be thought of as a switch that closes every T_s seconds Δ seconds and remains open otherwise. PAM x_{PAM} is a multiplication of a continuous-time signal $x(t)$ by a periodic pulse $p(t)$.

PAM

For a small pulse width Δ, PAM signal is

$$x_{\mathrm{PAM}}(t) = x(t)p(t) = \frac{1}{\Delta}\sum_{m} x(mT_s)[u(t - mT_s) - u(t - mT_s - \Delta)]$$

As periodic signal can be written as a Fourier series:

$$p(t) = \sum_{k=-\infty}^{\infty} P_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_s}$$

where P_k are the Fourier series coefficients. Hence, the PAM signal can be written as:

$$x_{\mathrm{PAM}}(t) = \sum_{k=-\infty}^{\infty} P_k x(t) e^{jk\Omega_0 t}$$

Its Fourier transform is $X_{\mathrm{PAM}}(\Omega) = \sum_{k=-\infty}^{\infty} P_k X(\Omega - k\Omega_0)$ (using frequency-shift property).

Sampling Theorem

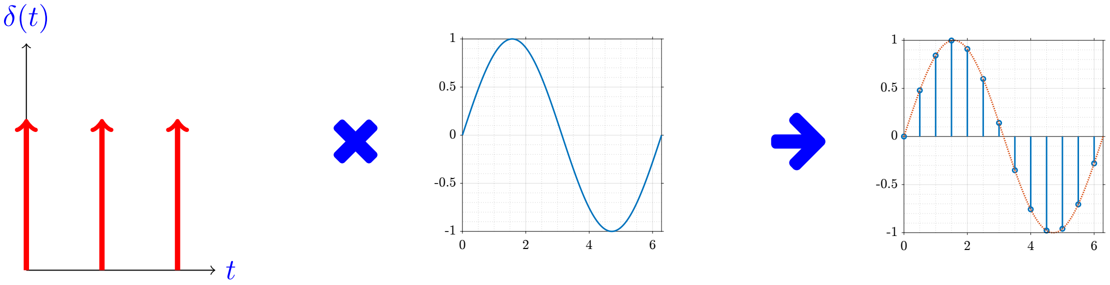
Modeling Sampled Signal using Impulses

If $\Delta \to 0$, we have a train of impulses. In the limiting conditions, we define the impulse **sampling function** as

$$\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$$

Thus, the sampled signal is:

$$x_s(t) = x(t)\delta_{T_s}(t)$$



δ(t)
t
1
0.5
0
-0.5
-1
0
2
4
6

Fourier Series of Impulse Train

$$\delta_{T_s}(t) = \sum_{=-\infty}^{\infty} kD_k e^{jk\Omega_s T}$$

where $\Omega_s = \dfrac{2\pi}{T_s}$ is the sampling frequency.

and,

$$D_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta_{T_s}(t)e^{-jk\Omega_s t}dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jk\Omega_s t}dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-j0}dt = \frac{1}{T_s}$$

Sampled Signal in Continuous-Time and Discrete-Time

Two ways to imagine the sampled signal $x_s(t)$:

Continuous-Time Version

$$x_s(t) = x(t)\delta_{T_s}(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\Omega_s t}$$

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$

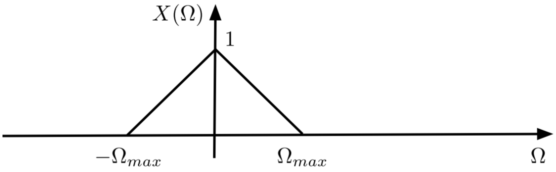
Discrete-time Version

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(\Omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\Omega T_s n}$$

Nyquist Sampling Rate Condition I

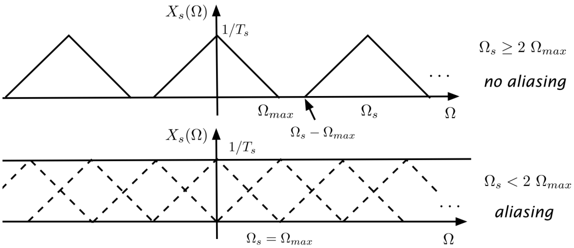
Depending on the maximum frequency present in the spectrum of $x(t)$ and on the chosen sampling frequency Ω_s (or the sampling period T_s) it is possible to have overlaps when the spectrum of $x(t)$ is shifted and added to obtain the spectrum of the sampled signal $x_s(t)$.



Band-limited

Band-limited signal $x(t)$ with a low-pass spectrum, finite support, i.e. $X(\Omega) = 0$ for $\Omega > \Omega_{\max}$.

Nyquist Sampling Rate Condition II

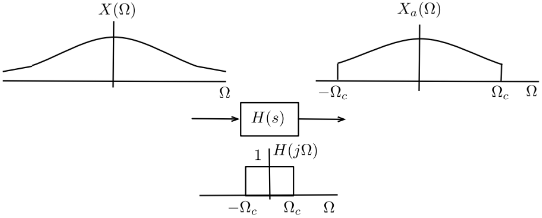


Sampling frequency Ω_s such that the spectrum of the sampled signal consists of shifted non-overlapping versions of $\frac{1}{T_s}X(\Omega)$. This is only possible when $\Omega_s - \Omega_{\max} \geq \Omega_{\max}$ or $\Omega_s \geq 2\Omega_{\max}$ which is called **Nyquist sampling rate condition**.

If $\Omega_s < 2\Omega_{\max}$, then when creating $X_s(\Omega)$ the shifted spectra of $x(t)$ overlap. . In this case, due to the overlap, it will not be possible to recover the original continuous-time signal from the sampled signal, and thus the sampled signal does not share the same information with the original continuous-time signal. This overlapping phenomenon is called **frequency aliasing**.

Sampling $x(t)$ with Infinite Support

- The signal is not band-limited.
- Aliasing will always be present.
- See the figure on the right for the concept of an anti-aliasing filter.



The only way to sample a non-band-limited signal $x(t)$ without aliasing — at the cost of losing information provided by the high-frequency components of $x(t)$ — is by obtaining an approximate signal $x_a(t)$ lacking the high-frequency components of $x(t)$ and thus permitting us to determine a maximum frequency for it. This is accomplished by **anti-aliasing filtering**, commonly used in samplers.

Example 1

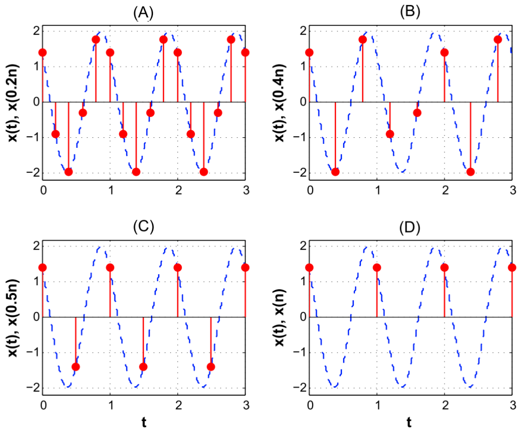
Consider the signal $x(t) = 2\cos(2\pi t + \pi/4)$, $0 - \infty < t < \infty$, determine if it is bandlimited. Use $T_s = 0.4, 0.5$, and 1 seconds/sample as sampling periods, and for each of these find out whether the Nyquist sampling rate condition is satisfied and if the sampled signal looks like the original signal or not.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Example 1: Assessing Nyquist Criteria



Example 2

Consider the following signals: (i) $x_1(t) = u(t+0.5) - u(t-0.5)$; (ii) $x_2(t) = e^{-t}u(t)$. Determine if they are band-limited or not. If not, determine the frequency for which the energy of the non-band-limited signal corresponds to 99% of its total energy and use this result to approximate its maximum frequency.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Reconstruction of Original Continuous Signal

If $\Omega_s > 2\Omega_{\text{max}}$, The spectrum of the sampled signal $x_s(t)$ displays a superposition of shifted versions of the spectrum $X(\Omega)$ multiplied by $\dfrac{1}{T_s}$ with no overlaps.

In such a case, we can get continuous signals by passing the sampled signal through an ideal low-pass filter with a frequency response

$$H_{\ell p} = \begin{cases} T_s, & -\Omega_s/2 \le \Omega \le \Omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the reconstructed signal in the form of a Fourier transform can be written as

$$X_r(\Omega) = H_{\ell p} X_s(\Omega) = \begin{cases} X(\Omega), & -\Omega_s/2 \le \Omega \le \Omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

The Exact Recovery of the Original Signal May Not be Possible

- Because the continuous-time signal is not exactly band-limited.
- Sampling is not done exactly at the uniform-rate, some variations always occur.
- The filter required for the exact recovery is an ideal low-pass filter which in practice cannot be realized.
- For non-bandlimited signals, we can use an anti-aliasing filter which is a low-pass filter that enforces to generate an approximate signal with maximum frequency.

Choosing Cutoff Frequency

- The selection of the cutoff frequency depends on the application and the prior knowledge.
- Speech signal has frequencies ranging from 100 Hz - 5 kHz. When sampling speech signal, a cutoff frequency of 5 kHz may be chosen, and the sampling rate is set to 10,000 samples/s.
- Music signals may range from 0 to 22 kHz. Thus, when sampling music signals, the antialiasing filter cutoff frequency is set to 22 kHz, and the sampling rate may be 44,000 samples/s or higher.

Signal Reconstruction from Sinc Interpolation

Ideal low-pass filter has impulse response:

$$h_{\ell p} = \frac{T_s}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega t} d\Omega = \frac{\sin(\pi t/T_s)}{\pi t/T_s}, \quad \Omega_s = \frac{2\pi}{T_s}$$

Reconstruction is the convolution of sampled signal and the low-pass filter's impulse response.

$$\begin{aligned} x_r(t) &= [x_s * h_{\ell p}](t) = \int_{-\infty}^{\infty} x_s(\tau) h_{\ell p}(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(\tau - nT_s)\right] h_{\ell p}(t-\tau) d\tau = \sum_n x(nT_s) \frac{\sin(\pi(t-nT_s)/T_s)}{\pi(t-nT_s)/T_s} \\ &= \sum_n x(nT_s) \operatorname{sinc}\Big(\pi(t-nT_s)/T_s\Big) \end{aligned}$$

Signal Reconstruction from Sinc Interpolation

Let $t = kT_s$,

$$x_r(kT_s) = \sum_n x(nT_s) \frac{\sin(\pi(k-n))}{\pi(k-n)} = x(kT_s),$$

since

$$\frac{\sin(\pi(k-n))}{\pi(k-n)} = x(kT_s) = \begin{cases} 1, & k-n=0, \quad \text{or} \quad n=k \\ 0, & n \neq k \end{cases}$$

The Nyquist-Shannon Sampling Theorem

Consider, a low-pass continuous signal $x(t)$ is band-limited. $X(\Omega) = 0$ for $|\Omega| > \Omega_{\max}$.

- The information in $x(t)$ is preserved when the sampled signal $x_s(t)$ with samples $x(nT_s) = x(t)|_{t=nT_s}$ is created with the sampling frequency $\Omega_s = 2\pi/T_s$ rad/s such that $\Omega\_s \geq 2\Omega_{\max}$ or $f_s = 1/T_s \geq \dfrac{\Omega_{\max}}{\pi}$.

  This is called **Nyquist sampling rate condition**.
- $x(t)$ can be reconstructed by using passing the $x_s(t)$ through an ideal low-pass filter with the frequency response $H(\Omega) = T_s$ for $-\Omega_s/2 < \Omega < \Omega_s/2$, and $H(\Omega) = 0$ otherwise.
- $2\Omega_{\max}$ is called Nyquist sampling frequency.
- $\Omega_s/2$ is called the folding frequency.

Sampling of Modulated Signals

Sampling of band-pass signals is used in simulations of communication systems and in implementing modulation systems in software-defined radio. For modulated signals, the sampling rate depends on the bandwidth of the message or modulating signal, rather than on the maximum frequency of the modulated signal.

A modulated signal: $x(t) = m(t)\cos(\Omega_c t)$.
$m(t)$: Message
$\cos(\Omega_c t)$: Carrier
Ω_c: Carrier Frequency
$\Omega_{\max}$: Maximum frequency present in the message
$\Omega_c >> \Omega_{\max}$

Sampling of Modulated Signals

The sampling of $x(t)$ with a sampling period Ts generates in the frequency domain a superposition of the spectrum of $x(t)$ shifted in frequency by Ω_s and multiplied by $\frac{1}{T_s}$.

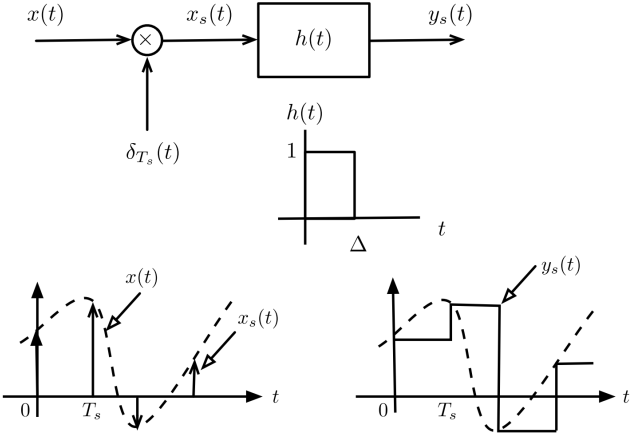
To avoid aliasing in the frequency domain, i.e. no overlapping of the shifted spectrum,

$$(\Omega_c + \Omega_{\max}) - \Omega_s < (\Omega_c - \Omega_{\max}) \Rightarrow \Omega_s > 2\Omega_{\max} \quad \text{or} \quad T_s < \frac{\pi}{\Omega_{\max}}$$

Hence, the sampling period depends on the bandwidth $\boldsymbol{\Omega_{\max}}$ of the actual message $m(t)$.

Sample-and-Hold Sampling

A sample-and-hold sampling system acquires a sample and holds it long enough for quantization and coding to be done before the next sample is acquired.



Zero-order Hold (ZOH)

Zero-order hold filter is an LTI system that facilitates sample-and-hold.

Its impulse response $h(t)$ is a pulse of desired width $\Delta \leq T_s$.

The output of the sample-and-hold system is a weighted sequence of shifted versions of the impulse response.

Ideal Sampler: $x_s(t) = x(t)\delta_{T_s}(t)$
ZOH Output: $y_s(t) = (x_s * h)(t)$.

Spectrum of ZOH sampled signal

$$y_s(t) = (x_s * h)(t) = \sum_n x(nT_s)h(t - nT_s)$$

Frequency response of $h(t)$ is:

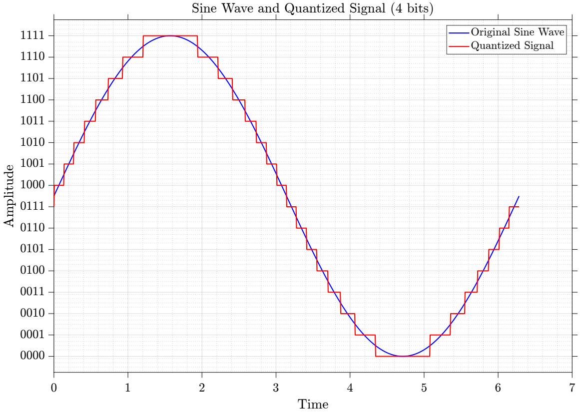
$$H(\Omega) = \frac{e^{-\Delta s/2}}{s}(e^{\Delta s/2} - e^{-\Delta s/2})\Big|_{s=j\Omega} = \frac{\sin(\Delta\Omega/2)}{\Omega/2}e^{-j\Delta\Omega/2}$$

Thus, $$Y_s(\Omega) = X_s(\Omega)H(\Omega) = \left[\frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)\right]H(\Omega)$$

$$= X_s(\Omega)H(\Omega) = \left[\frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)\right]\frac{\sin(\Delta\Omega/2)}{\Omega/2}e^{-j\Delta\Omega/2}$$

Quantization and Coding

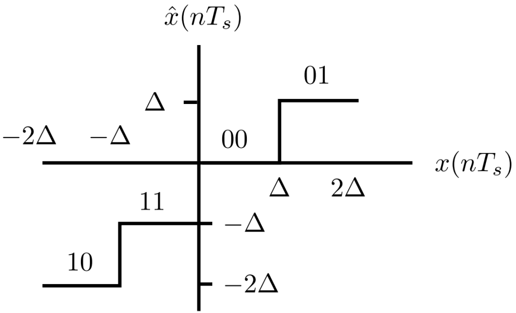
- Amplitudes are quantized – they have certain levels.
- To facilitate the quantization, we have a quantizer.
- A quantizer is a nonlinear system.
- Number of quantization levels is specified by how many bits we use to encode.
- Of course, quantization introduces error as quantization is merely an approximation. The error is called **quantization error**.



Code: https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/quantizing_signal.m

Quantization Step

$$\Delta = \frac{\text{dynamic range of signal}}{2^b}$$



Quantization Error

Sampled Signal: $x(nT_s) = x(t)|_{t=nT_s}$

Example: 4-level quantizer

$k\Delta \leq x(nT_s) < (k+1)\Delta \Rightarrow \hat{x}(nT_s)) = k\Delta, \quad k = -2, -1, 0, 1.$

$$
\begin{array}{rcrcc}
 & & & \hat{x}(nT_s)) & \Rightarrow \quad \text{Binary Code} \\
-2\Delta \leq x(nT_s) < -\Delta & \Rightarrow & \hat{x}(nT_s)) = -2\Delta & | & -2\Delta \qquad 10 \\
-\Delta \leq x(nT_s) < 0 & \Rightarrow & \hat{x}(nT_s)) = -\Delta & | & -\Delta \qquad 11 \\
0 \leq x(nT_s) < \Delta & \Rightarrow & \hat{x}(nT_s)) = 0 & | & 0\Delta \qquad 00 \\
\Delta \leq x(nT_s) < 2\Delta & \Rightarrow & \hat{x}(nT_s)) = \Delta & | & \Delta \qquad 01
\end{array}
$$

Quantization Error

We define quantization error as $\varepsilon(nT_s) = x(nT_s) - \hat{x}(nT_s)$

$$
\begin{aligned}
\hat{x}(nT_s) \leq x(nT_s) &\leq \hat{x}(nT_s) + \Delta \\
0 \leq \varepsilon(nT_s) &\leq \Delta
\end{aligned}
$$

Example

Suppose we are trying to decide between an 8 and a 9-bit A/D converter for a certain application where the signals in this application are known to have frequencies that do not exceed 5 kHz. The dynamic range of the signals is $10V$, so that the signal is bounded as $-5 \leq x(t) \leq 5$. Determine an appropriate sampling period and compare the percentage of error for the two A/Ds of interest.

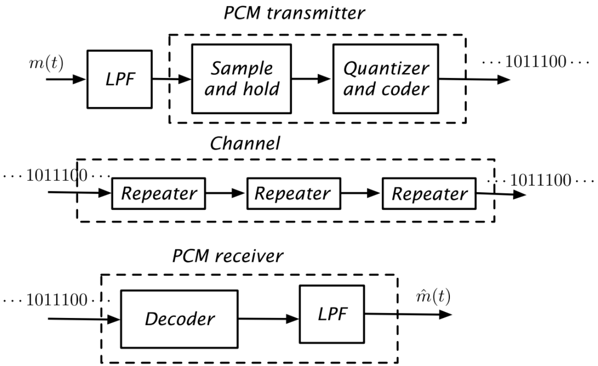
Blank space for calculation

Blank space for calculation

Blank space for calculation

Application to Digital Communication: PCM

- Sampling theorem by Nyquist and Shannon led to the era of digital communication.
- Modern digital communication using Pulse Code Modulation (PCM).



Application to Digital Communication: TDM

- Pulse-modulated signals have large bandwidths, causing frequency overlap and interference when transmitted together.
- However, these signals provide information only at the sampling times.
- Between these sampling times, samples from other signals can be inserted, generating a combined signal with a smaller sampling time. At the receiver, the samples corresponding to each of the multiplexed signals are separated.
- This forms the principle of **time-division multiplexing (TDM)**, where pulses from different signals are interspersed into one signal, converted into a PCM signal, and transmitted.