CPE 381: Fundamentals of Signals and Systems for Computer Engineers

07 Frequency Analysis: Fourier Transform

Rahul Bhadani

Electrical & Computer Engineering, The University of Alabama in Huntsville

Outline

Fourier Transform of an Aperiodic Signal

Aperiodic Signal

- Aperiodic signal can be thought of as a periodic signal with infinite fundamental frequency.
- In practice, there are no periodic signals, but they are nice mathematical tools.

$$x(t) = \lim_{T_0 \to \infty} \tilde{x}(t)$$

From Fourier Series to Fourier Transform

Consider

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

and its Fourier coefficient is

$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jn\Omega_0 t} dt$$

In the limiting case $T_0 \to \infty$, $X_n \to 0$, so to avoid it, $X(\Omega_n) = T_0 X_n$, and $\Omega_n = n\Omega_0$ are the harmonic frequencies.

From Fourier Series to Fourier Transform

Rewriting $\Omega_0 = \Delta\Omega = \dfrac{2\pi}{T_0}$

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X(\Omega_n) e^{j\Omega_n t} = \sum_{n=-\infty}^{\infty} X(\Omega_n) e^{j\Omega_n t} \frac{\Delta\Omega}{2\pi}$$

$$X(\Omega_n) = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-j\Omega_n t} dt$$

As $T_0 \to \infty$, $\Delta\Omega \to d\Omega$ and sum becomes integral. Thus $\Omega_n = n\Omega_0 = n\Delta\Omega \to \Omega$.

Fourier Transform

Hence, in the limiting case, $T_0 \to \infty$,

Fourier Transform

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega$$

Obtaining Fourier Transform

- If a signal's Laplace transform includes $j\Omega$ axis in its ROC, then we can simply compute the Fourier Transform by first computing its Laplace transform and set $s = j\Omega$.
- If $x(t)$ has a finite time support and in that support $x(t)$ is bounded, its Fourier transform exists. To find it use the integral definition or the Laplace transform of $x(t)$.
- If $x(t)$ is periodic, its Fourier transform is obtained using the signal's Fourier series.
- For some functions, e.g. the integral is not well-defined, or has discontinuities then the integration formula cannot be used for calculating the Fourier transform. We will use properties of Fourier transform in such a case.

It appears that almost nothing has a Fourier transform—nothing except practical communication signals. No signal amplitude goes to infinity and no signal lasts forever; therefore, no practical signal can have an infinite area under it, and hence all have Fourier transforms.

- E. Craig

Frequency Content with Fourier Transform

- The Fourier transform measures the frequency content of a signal.
- Time-domain vs Frequency-domain: The characterization in one domain provides information that is not clearly available in the other.

Fourier Transform using Laplace Transform

For $X(\Omega)$ to exist,

- $x(t)$ must be absolutely integrable.
- $x(t)$ has only a finite number of discontinuities and a finite number of minima and maxima in any finite interval.
- ROC of its Laplace transform $X(s)$ must have $j\Omega$ axis.

$$\mathcal{F}[x(t)] = \mathcal{L}[x(t)]\Big|_{s=j\Omega} = X(s)\Big|_{s=j\Omega}$$

The duality between time and frequency allows the computation of Fourier transforms.

Fourier Series from Laplace Transform

1. $x(t) = u(t)$, $X(s) = \dfrac{1}{c}$, ROC: $\sigma > 0$, $j\Omega$-axis is not included, FT cannot be obtained using Laplace transform.
2. $x(t) = e^{-2t}u(t)$, $X(s) = \dfrac{1}{s+2}$, ROC: $\sigma > -2$, then
$$X(\Omega) = \left.\frac{1}{s+2}\right|_{s=j\Omega} = \frac{1}{j\Omega + 2}$$
3. $x(t) = e^{-|t|}$, $X(s) = \dfrac{1}{s+1} + \dfrac{1}{1-s}$, ROC: $-1 < \sigma < 1$, then
$$X(\Omega) = \left.X(s)\right|_{s=j\Omega} = \frac{2}{1-(j\Omega)^2} = \frac{2}{1+\Omega^2}$$

Properties of Fourier Transform

Linearity

$$\mathcal{F}[\alpha x(t) + \beta y(t)] = \alpha\mathcal{F}[x(t)] + \beta\mathcal{F}[y(t)] = \alpha X(\Omega) + \beta Y(\Omega)$$

Fourier Transform an Impulse Function

$$x(t) = \delta(t)$$
$$X(\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = e^{-j0} = \int_{-\infty}^{\infty} \delta(t) dt = 1, \quad -\infty < \Omega < \infty$$

Hence, if the delta function has finite support as seen, then its Fourier transform has infinite support in frequency.

It means $\delta(t)$ changes so much in a very short time that its FT has all possible frequency components.

Duality Property

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\rho)e^{j\rho t}d\rho$$

Replacing t with $-\Omega$,

$$2\pi x(-\Omega) = \int_{-\infty}^{\infty} e^{-j\rho\Omega}d\rho = \int_{-\infty}^{\infty} X(t)e^{-j\Omega t}dt = \mathcal{F}[X(t)]$$

after replacing ρ with t as they are dummy variables. Hence

$$X(t) \Leftrightarrow 2\pi X(-\Omega)$$

Fourier Transform of a Constant Signal

$$x(t) = A, \quad -\infty < t < \infty$$

As, it is a constant, natural $\Omega = 0$ is the frequency. We cannot find its frequency in the usual way it is not absolutely integrable. Then we can the **duality property**.

$$\begin{aligned} \delta(t) &\Leftrightarrow 1 \\ A\delta(t) &\Leftrightarrow A \\ A &\Leftrightarrow 2\pi A\delta(-\Omega) = 2\pi A\delta(\Omega) \end{aligned}$$

Notice the last step because δ is an even function.

Fourier Transform of Shifted Delta and Cosines

Shifted Delta

$$\delta(t-b)+\delta(t+b) \Leftrightarrow e^{-jb\Omega}+e^{jb\Omega} = 2\cos(b\Omega)$$

Apply Laplace Transform to $\delta(t \pm b)$, and set $s = j\Omega$, and $\delta(t) \leftrightarrow 1$.

Cosine

We use duality property:

$$2\cos(bt) \Leftrightarrow 2\pi[\delta(\Omega-b)+\delta(\Omega+b)]$$
$$\cos(bt) \Leftrightarrow \pi[\delta(\Omega-b)+\delta(\Omega+b)]$$

Example

Use duality to find the Fourier transform of $x(t) = 10\text{sinc}(0.5t)$.

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Time Frequency Inverse Proportionality

Support of $X(\Omega)$ is inversely proportional to the support of $x(t)$. If $x(t)$ has Fourier transform$X(\Omega)$, and $\alpha \neq 0$, a real number, then

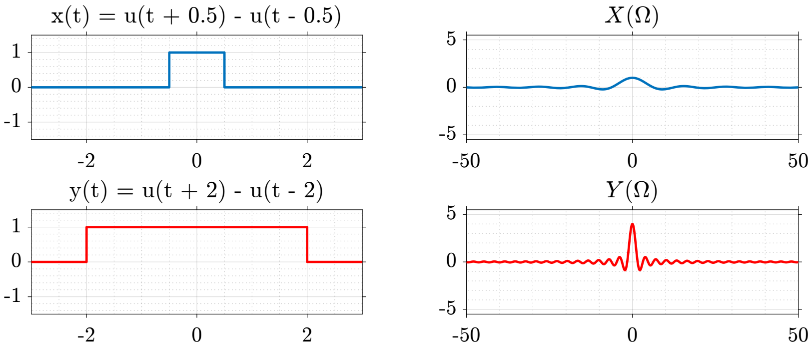
$x(\alpha t)$

- is a contracted signal when $\alpha > 1$
- , is a contracted and reflected signal when $\alpha < -1$
- is an expanded signal when $0 < \alpha < 1$
- is an expanded and reflected signal when $-1 < \alpha < 0$
- reflect signal when $\alpha = -1$

and,

$$x(\alpha)t \leftrightarrow \frac{1}{|\alpha|}X(\frac{\Omega}{\alpha})$$

Inverse Proportionality of Time and Frequency: Example 1



Notice that the wider the pulse the more concentrated in frequency its Fourier transform.

Code: https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/fourier_transform_rect.m

Inverse proportionality of time and frequency: Example 2

Consider a pulse $x(t) = u(t) - u(t-1)$. Find its Fourier transform, and the Fourier Transform of $x(2t)$.

Inverse proportionality of time and frequency: Example 2

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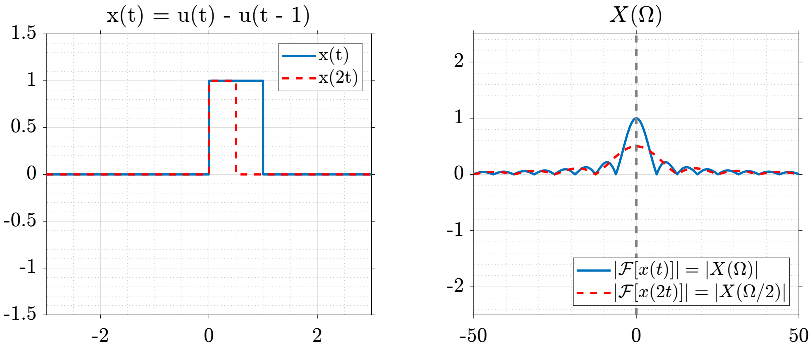

$x(t)$ has finite support, while $X(\Omega)$ has infinite support.

Code: https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/FourierTransform_Rect_Expanded.m

Example 3

Apply the reflection property to find the Fourier transform of $x(t) = e^{-a|t|}$, $a > 0$.

Example 3

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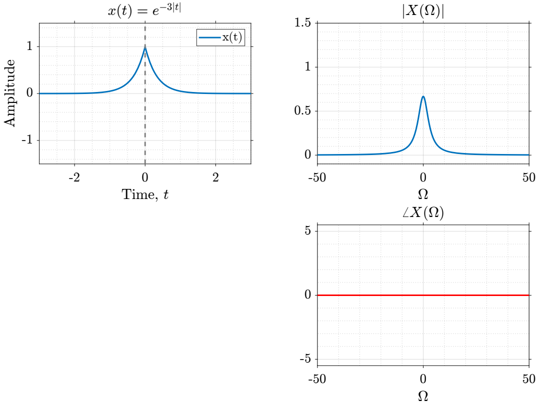
Example 3



From this plot, we see that its energy is concentrated mostly in low frequencies, hence the signal is called **low-pass**.

Code: https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/FourierTransform_Exp_Abs.m

Spectral Representation

Modulation

$$\mathcal{F}[x(t)e^{j\Omega_0 t}] = \int_{-\infty}^{\infty}[x(t)e^{j\Omega_0 t}]e^{-j\Omega t}dt = \int x(t)e^{j(\Omega-\Omega_0)t}]dt = X(\Omega - \Omega_0)$$

Then applying frequency shifting property to $x(t)\cos(\Omega_0 t)$, we have

$$x(t)\cos(\Omega_0 t) \Leftrightarrow 0.5[X(\Omega - \Omega_0) + X(\Omega - \Omega_0)]$$

Here, $x(t)$ is the main signal, called the **message**, $\cos(\Omega_0 t)$ is called the **carrier**, and $x(t)\cos(\Omega_0 t)$ is the **modulated signal**.
The scheme discussed is called Amplitude Modulation

Why Amplitude Modulation?

- Music signals are audible at up to 22 KHz, while speech signals are at 100 Hz - 5 KHz. They are relatively low frequency.
- The length of an antenna that is used to radiate the signal is a quarter of a wavelength, given by

$$\lambda = \frac{c}{f} \ \ meter$$

  $c = 3 \times 10^8 m/s$ is the speed of light, and f is the frequency in Hz of the signal to be radiated.
- Assuming $f = 30$ KHz, normal for music and speech with some noise, the $\lambda \approx 10$ km, hence antenna size would be 2.5 km. A very giant antenna.
- Choosing an appropriate modulation frequency helps in reducing the size of the antenna.

Fourier Transform of Periodic Signals

If the function is periodic, we know that it can be represented as a Fourier series, i.e.

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_k t}$$

where

$$X_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-j\Omega_k t} dt$$

From the linearity and frequency-shifting property, we can write

$$X(\Omega) = \sum_k \mathcal{F}[X_k e^{j\Omega_k t}] = \sum_k 2\pi X_k \delta(\Omega - \Omega_k)$$

$\Omega_k = k\Omega_0$.

Example 4

Consider a periodic signal $x(t)$ with a period $x_1(t) = r(t) - 2r(t - 0.5) + r(t - 1)$. Its fundamental frequency is $\Omega_0 = 2\pi$. Determine its Fourier transform analytically and using MATLAB.

Example 4

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Parseval's Energy Relationship

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\Omega)e^{-j\Omega t}d\Omega\right]dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\Omega)\left[x(t)e^{-j\Omega t}dt\right]d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|^2 d\Omega$$

Thus $|X(\Omega)|^2$ is an energy density—indicating the amount of energy at each of the frequencies. The plot $|X(\Omega)|^2$ vs. is called the energy spectrum of $x(t)$, and displays how the energy of the signal is distributed over frequency.

Example 5: Pulse Signal and its Energy

Consider $p(t) = u(t+1) - u(t-1)$, use its Fourier transform $P(\Omega)$ and Parseval's energy relation to show that

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\Omega)}{\Omega} \right)^2 d\Omega = \pi$$

Example 5: Pulse Signal and its Energy

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Spectral Representation

- Magnitude spectrum: $|X(\Omega)|$ vs Ω: even function
- Phase spectrum: $\angle X(\Omega)$ vs Ω: odd function
- Energy/Power spectrum: $|X(\Omega)|^2$ vs Ω: even function

Filtering

Frequency Response

$H(j\Omega)$ (or $H(\Omega)$) is the frequency response of the system. Thus,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[X(\Omega)H(\Omega)]e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}[Y(\Omega)]e^{j\Omega t}d\Omega$$

Thus $Y(\Omega) = X(\Omega)H(\Omega)$ For a periodic signal $X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - \Omega_k)$,

$$Y(\Omega) = X(\Omega)H(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(\Omega_k)\delta(\Omega - \Omega_k)$$

Thus Fourier coefficient of the output $y(t)$ is $Y_k = X_k H(\Omega_k)$.

Filtering

In practice $s(t) = x(t) + \eta(t)$, where η is the noise. We want to get rid of noise. So we design a special system called **Filters** that will by assuming that noise has certain frequencies, and filters will remove those frequencies.
The problem of filtering is finding the transfer function $H(s)$ of the filter (or the coefficients of the denominator and numerator of the transfer function).

Example 6: From Full-wave Rectifier to DC Source

Let $x(t) = |\cos(\pi t)|$ is a full-wave rectifier be an input to a filter $H(\Omega)$, and the output is a DC signal. What should $H(\Omega)$ look like?

Example 6: From Full-wave Rectifier to DC Source

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Example 6: From Full-wave Rectifier to DC Source

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Example 6: From Full-wave Rectifier to DC Source

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Low Pass Filters

Ideal Low Pass Filter

A filter that keeps low frequencies is called a low-pass filter. The magnitude response of an ideal low-pass filter is

$$|H_{\ell p}(\Omega)| = \begin{cases} 1, & -\Omega_1 \leq \Omega \leq \Omega_1 \\ 0, & \text{otherwise} \end{cases}$$

and the frequency response is $\angle H_{\ell p}(\Omega) = -\alpha\Omega$ which is a linear phase.

- Ω_1 is called cut-off frequency.

Band-pass Filter

Ideal Band-pass Filter

Keeps a band of frequencies

Magnitude Response:

$$|H_{bp}(\Omega)| = \begin{cases} 1, & \Omega_1 \leq \Omega \leq \Omega_2 \ \text{ and } \ -\Omega_2 \leq \Omega \leq -\Omega_1 \\ 0, & \text{otherwise} \end{cases}$$

Its phase is also linear in the pass band.

High-pass Filter

Ideal High-pass Filter

Keeps only high frequencies

Magnitude Response:

$$|H_{hp}(\Omega)| = \begin{cases} 1, & \Omega \geq \Omega_2 \text{ and } -\Omega \leq -\Omega_2 \\ 0, & \text{otherwise} \end{cases}$$

Its phase is also linear in the pass band.

An ideal multi-band filter can be obtained by a combination of low, high, and band-pass filter.

Band-stop Filter (Band-eliminating Filter)

Ideal High-pass Filter

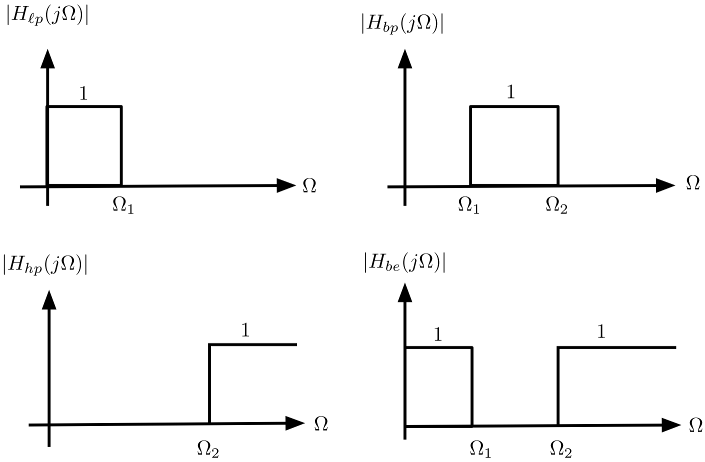
Removes a range of frequencies

Magnitude Response:

$$|H_{be}(\Omega)| = 1 - |H_{bp}(\Omega)|$$

Its phase is also linear in the pass band.

Magnitude Response of Filter



Frequency Response from Poles and Zeros

Consider a transfer function of the form $G(s) = K\frac{s-z}{s-p}$ where $K \neq 0$, and p is a pole, z is a zero. Frequency response at some frequency Ω_0 is obtained by setting $s = j\Omega_0$. Hence,

$$G(\Omega) = K\frac{j\Omega_0 - z}{j\Omega_0 - p} = K\frac{\vec{Z}(\Omega_0)}{\vec{P}(\Omega_0)} = |K|e^{j\angle K}\frac{|\vec{Z}(\Omega_0)|}{|\vec{P}(\Omega_0)|}e^{j(\angle\vec{Z}(\Omega_0) - \angle\vec{P}(\Omega_0))}$$

Magnitude response is then

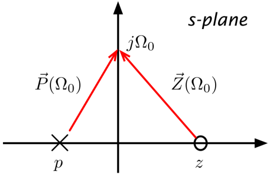
$$|G(\Omega)| = |K|\frac{|\vec{Z}(\Omega_0)|}{|\vec{P}(\Omega_0)|}$$

and the phase response as

$$\angle G(\Omega) = \angle K + \angle\vec{Z}(\Omega_0) - \angle\vec{P}(\Omega_0)$$

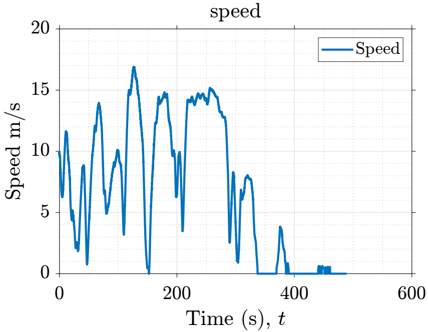
Frequency Response from Poles and Zeros

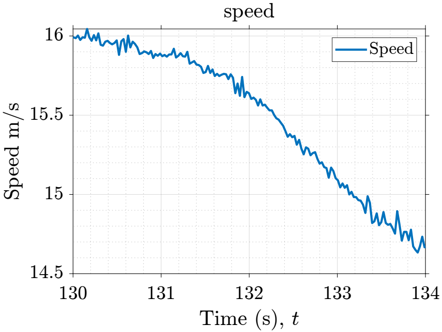
- Poles create $j\Omega$ at frequencies in the $j\Omega$-axis in front of the imaginary parts of the poles. The closer the pole is to the $j\Omega$-axis, the narrower and higher the hill. If, for instance, the poles are on the $j\Omega$-axis (this would correspond to an unstable and useless filter) the magnitude response at the frequencies of the poles will be infinity.
- Zeros create **valleys** at the frequencies in the $j\Omega$-axis in front of the imaginary parts of the zeros. The closer a zero is to the $j\Omega$-axis approaching it from the left or the right (as the zeros are not restricted by stability to be in the open left-hand s-plane) the closer the magnitude response is to zero. If the zeros are on the $j\Omega$-axis, the magnitude response at the frequencies of the zeros is zero.



Filtering a Real Signal I

Consider a data file that contains a timestamp and speed in km/h. We will convert it to **m/s** first.



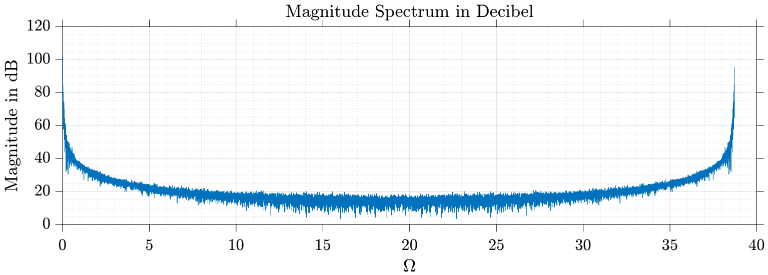


Filtering a Real Signal II

```
csv_file = '../Data/vel.csv';
data = readtable(csv_file);
data.Time = data.Time - data.Time(1);
% Convert the velocity data to m/s
velocity_ms = data.Message * 1000 / 3600;
```

Compute Fourier Transform using FFT and Plot Magnitude Spectrum, we plot the magnitude spectrum in Decibel (20*log10) for clarity.

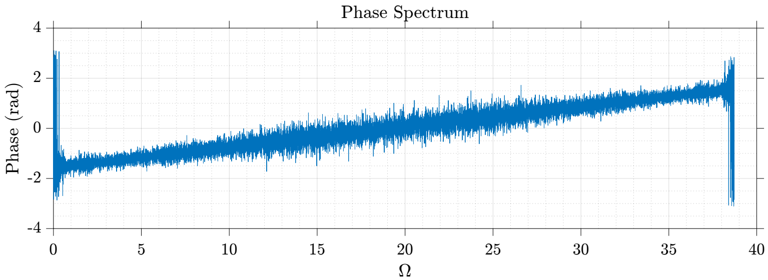
Filtering a Real Signal III



Filtering a Real Signal IV

```
% Compute the Fourier transform of the velocity data
fs = 1 / mean(diff(data.Time));  % sampling frequency
velocity_fft = fft(velocity_ms);

% Plot the magnitude spectrum
f = figure;
f.Position = [ 810          819          1320          419];
plot((0:length(velocity_fft)-1) * fs / length(velocity_fft), 20*log10(abs(velocity_fft)));
```

Filtering a Real Signal V



Filtering a Real Signal VI

```
% Compute the Fourier transform of the velocity data
fs = 1 / mean(diff(data.Time));  % sampling frequency
velocity_fft = fft(velocity_ms);

% Plot the phase spectrum
f = figure;
f.Position = [ 810          819          1320          419];
plot((0:length(velocity_fft)-1) * fs / length(velocity_fft), angle(velocity_fft));
```

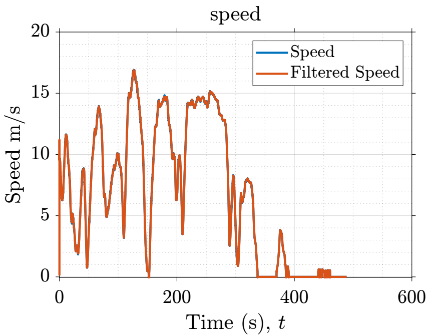
Now, we design a filter

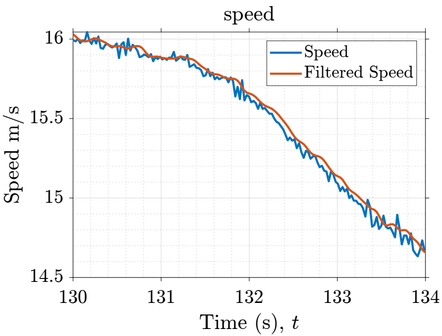
Filtering a Real Signal VII

```
cutoff_freq = 5;   % Hz
order = 4;

% Normalize the cutoff frequency to the Nyquist frequency
% Design the low-pass filter
[b, a] = butter(order, cutoff_freq / (fs/2), 'low')

% Apply the filter to the velocity data
velocity_filtered = filter(b, a, velocity_ms);
```

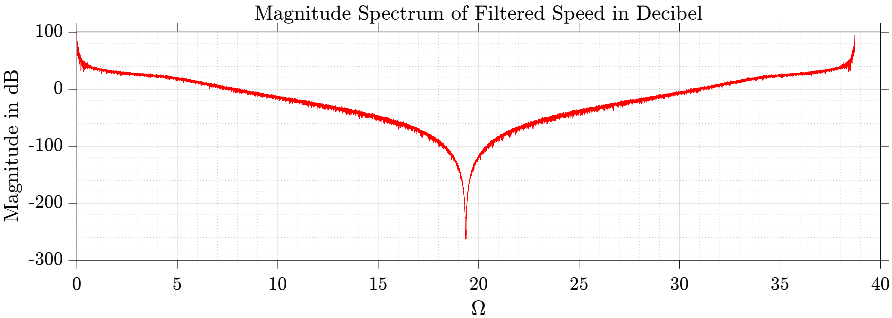
$$b = [0.0003, 0.0013, 0.0019, 0.0013, 0.0003]$$
$$a = [1.0000, -3.2380, 3.9911, -2.2127, 0.4647]$$

Filtering a Real Signal VIII

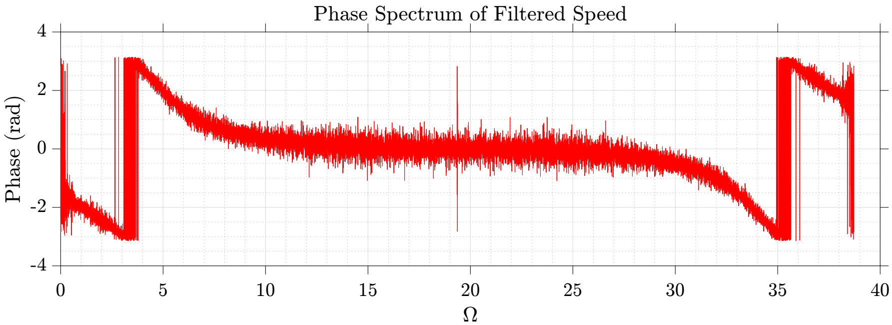




Filtering a Real Signal IX



Filtering a Real Signal X



Full code at
https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/speed_fourier.m

Time-shift Properties

$$x(t \pm t_0) \Leftrightarrow X(\Omega)e^{\pm j\Omega t_0}$$

Differentiation Property

$$\frac{d^N x(t)}{dt^N} \Leftrightarrow (j\Omega)^N X(\Omega)$$

Integration Property

$$\int_{-\infty}^{t} x(\sigma)d\sigma \Leftrightarrow \frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$$

where $X(0) = \int_{-\infty}^{\infty} x(t)dt$

Integration Property

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Integration Property

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Integration Property

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Example 7: Fourier Transform of Triangular Pulse

$$x(t) = r(t) - 2r(t-1) + r(t-2)$$

Find the Fourier transform of $x(t)$ using the derivative property.

Example 7: Fourier Transform of Triangular Pulse

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Example 7: Fourier Transform of Triangular Pulse

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Fourier Optics

https://www.youtube.com/watch?v=EGYVquku8r4