CPE 381: Fundamentals of Signals and Systems for Computer Engineers

Continuous-Time Signals

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Announcement

- Homework 01 Due September 01 11:59 PM
- Quiz 01, based on Chapter 01 Continuous-Time Signals from the Textbook. Available from September 05, 12:01 AM to September 07, 11:59 PM. 30 Questions, 45 Minutes.
- Office hour: 08/28 Aug Wednesday, 1 PM - 3:30 PM.

Outline

Motivation

Signals and Systems is ‘Grandfather’ of Data Science for Electrical and Computer Engineers

Classification of Signals

We care about the following properties when dealing with signals:

- Predictability: Random or Deterministic
- Variations of time and amplitude: continuous, discrete (time or x-axis) / quantized (amplitude or y-axis)
- Periodic/Aperiodic
- Finite energy/finite power; Infinite energy/Infinite power

Operation on Signals

Basic Mathematical Operations

- Addition: $x(t) + y(t)$
- Subtraction: $x(t) - y(t)$
- Constant multiplication: $kx(t)$ where k is a constant

Time-shift

- $x(t - \tau) \rightarrow$ Signal is delayed
- $x(t + \tau) \rightarrow$ Signal is advanced

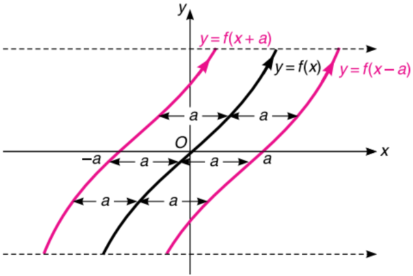
$f(x)$ transforms to $f(x - a)$

i.e., $f(x) \longrightarrow f(x - a)$; a is positive. Shift the graph of $f(x)$ through 'a' unit towards right

$f(x)$ transforms to $f(x + a)$.

i.e., $f(x) \longrightarrow f(x + a)$; a is positive. Shift the graph of $f(x)$ through 'a' units towards left.

Graphically it could be stated as



Time Reflection

- $x(t) \rightarrow x(-t)$: take mirror image **along the y-axis**

Note: The book doesn't specify whether to take the mirror image along the y-axis or not and it is confusing because the signal used in example 1.3.1 is symmetric with respect to both the x and y axes.

$f(x)$ transforms to $f(-x)$

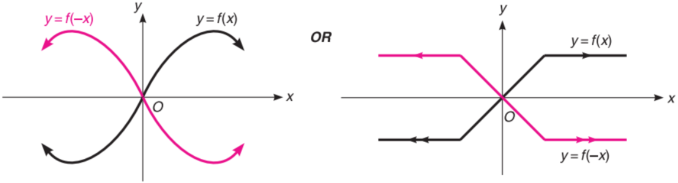
i.e., $$f(x) \longrightarrow f(-x)$$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in y-axis as plane mirror.

OR

"Turn the graph of $f(x)$ by 180° about y-axis."

Graphically it is stated as;



Signal Stretching along y-axis

- $f(x) \to af(x); \quad a > 1$: Stretch the graph of $f(x)$ 'a' times along y-axis.
- $f(x) \to \frac{1}{a}f(x); \quad a > 1$: Shrink the graph of $f(x)$ 'a' times along y-axis.

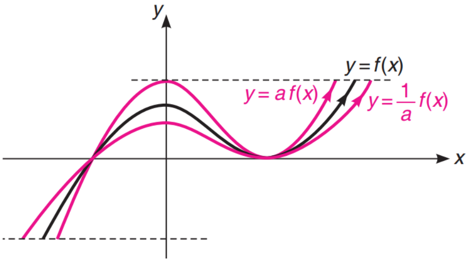
$f(x)$ transforms to $a\, f(x)$

i.e., $$f(x) \longrightarrow af(x); \quad a > 1$$

Stretch the graph of f(x) 'a' times along y-axis.

$$f(x) \longrightarrow \frac{1}{a} f(x); \; a > 1.$$

Shrink the graph of f(x) 'a' times along y-axis.



Signal Stretching along x-axis

- $f(x) \to af(ax);\quad a > 1$: Stretch the graph of $f(x)$ 'a' times along x-axis.
- $f(x) \to f\left(\frac{1}{a}x\right);\quad a > 1$: Shrink the graph of $f(x)$ 'a' times along x-axis.

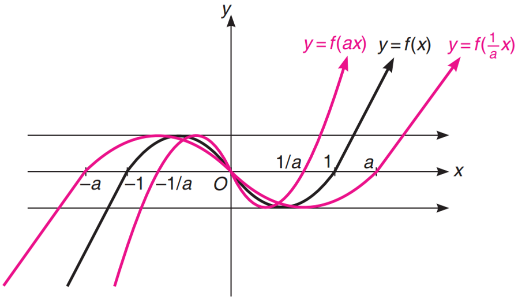
***f*(x) transforms to *f*(*ax*)**

i.e., $\quad f(x) \longrightarrow f(ax);\quad a > 1$

Shrink (or contract) the graph of f(x) '*a*' times along *x*-axis.

again $\quad f(x) \longrightarrow f\left(\frac{1}{a}x\right);\quad a > 1$

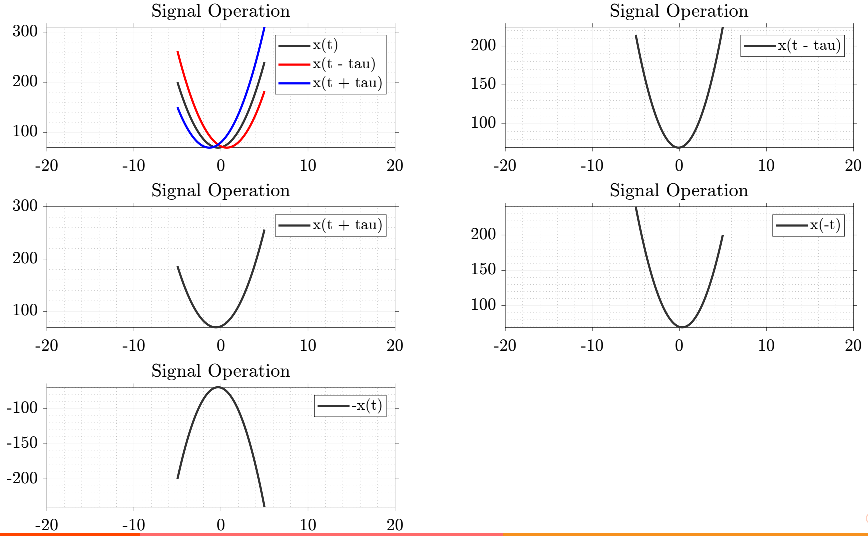
Stretch (or expand) the graph of f(x) '*a*' times along *x*-axis.



Example and MATLAB Code

$$x(t) = 6t^2 + 4t + 70$$

Code: https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/signal_operation.m



Even and Odd Signals

- Even Signal: $x(t) = x(-t)$
- Odd Signal: $x(t) = -x(-t)$
- Any signal can be represented by the sum of even and odd signals
  $y(t) = y_e(t) + y_o(t)$

$$y_e(t) = 0.5[y(t) + y(-t)]$$

$$y_o(t) = 0.5[y(t) - y(-t)]$$

Periodic Signals

- Defined for all possible values of $t, -\infty < t < \infty$.
- There is the real value $T_0 \in \mathbb{R}^+$, called the fundamental frequency such that $x(t + kT_0) = x(t), k \in \mathbb{I}$.
- A constant signal is periodic of a non-definable fundamental period.
- A $\cos(\omega t + \theta)$, $\omega = 2\pi/T_0$, $\omega = 2, \theta = -\pi/2, A = 2$.

  What's the fundamental frequency, $1/T_0$?

Energy and Power of Signals

What's the instantaneous power of a resistor?

- **Energy:**

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- **Power:**

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

A signal is called finite power if the signal power is finite.

Basic Signals as Building Blocks

Complex Exponentials

Consider $A = |A|e^{j\theta}$, $a = r + k\Omega_0$

- $x(t) = Ae^{at} = ...$
- Real part $f(t) = \text{Re}\{x(t)\}, = ...$

  $-|A|e^{rt} \leq f(t)) \geq |A|e^{rt}$. $r < 0$, $f(t)$ is damped, $r > 0$, $f(t)$ grows.
- Imaginary part $g(t) = \text{Im}\{x(t)\}, = ...$

Sinusoids

A sinusoid of the general form:

$$A\cos(\Omega_0 t + \theta) = A\sin(\Omega_0 t + \theta + \pi/2), \quad -\infty < t < \infty$$

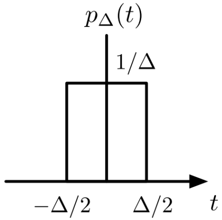
- A is the amplitude
- $\Omega_0 = 2\pi f_0$ is angular frequency in `rad/s`.
- θ is phase shift
- Fundamental period T_0 is

$$T_0 = \frac{2\pi}{\Omega_0} = \frac{1}{f_0}$$

Rectangular pulse and Unit impulse

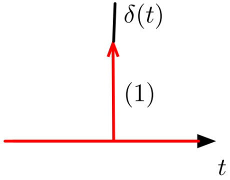
- A rectangular pulse of duration Δ and unit area:

$$p_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & -\Delta/2 \le t \le \Delta/2 \\ 0, & \text{otherwise} \end{cases}$$



- Unit Impulse:

$$\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$$



- **Calculate**

$$\int_{-\infty}^{t} p_\Delta(t)$$

Unit Step

- Integration of rectangular pulse:

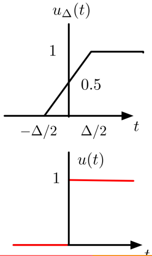
$$u_\Delta(t) = \int_{-\infty}^{t} p_\Delta(t) = \begin{cases} 1, & t \geq \dfrac{\Delta}{2} \\ \frac{1}{\Delta}(t + \frac{\Delta}{2}), & \dfrac{\Delta}{2} \leq t \leq \dfrac{\Delta}{2} \\ 0, & t < -\dfrac{\Delta}{2} \end{cases}$$

- Limit case:

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 1, & t \geq 0 \\ \frac{1}{\Delta}(t + \frac{\Delta}{2}), & t = 0 \\ 0, & t < 0 \end{cases}$$

- A common case is to ignore $t = 0$ case, which gives us unit step function as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

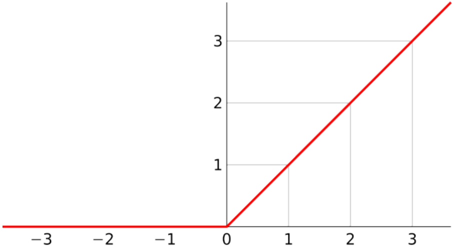


Ramp Signal

The ramp signal is $r(t) = tu(t)$

- The relation between the ramp, the unit step, and the unit impulse:

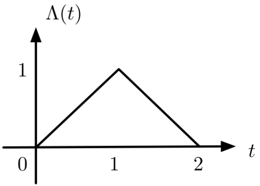
$$\frac{dr(t)}{dt} = u(t)$$
$$\frac{d^2t}{dt^2} = \delta(t)$$

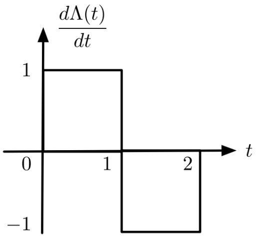


Triangular Pulse

The triangular pulse is

$$\Lambda(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ -t+2, & 1 < t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$





- $\Lambda(t)$ can also be written as:

$$\Lambda(t) = r(t) - 2r(t-1) + r(t-2)$$

Sifting Property

The product of $f(t)$ and $\delta(t)$ gives zero everywhere except at the origin where we get an impulse of area $f(0)$, that is, $f(t)\delta(t) = f(0)\delta(t)$.
Hence,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \int_{-\infty}^{\infty} f(0)\delta dt = f(0)\int_{-\infty}^{\infty} \delta(t)dt = f(0)$$

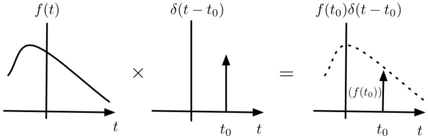
This is called **Sifting Property**.
If we delay or advance the $\delta(t)$ function in the integrated, the result is that all values of $f(t)$ are sifted out except for the value corresponding to the location of the delta function, that is,

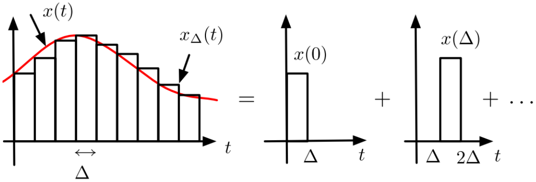
$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{\infty} \delta(t-\tau)dt = f(\tau) \quad \text{for any } \tau$$

Generic Representation of Signals

Hence, if we do integration in terms of variable τ, we get a generic representation of signals in terms of impulse and shifted impulse.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$





Approximation of $x(t)$:

$$x_\Delta(t) = \sum_{-\infty}^{\infty} x_\Delta(t-k\Delta) = \sum_{-\infty}^{\infty} x(k\Delta)p_\Delta(t-k\Delta)\Delta$$

In the limit as $\Delta \to 0$ these pulses become impulses, separated by an infinitesimal value:
$\lim_{\Delta\to 0} x_\Delta(t) \to x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

Modulation and Windowing

Modulation

Multiplication by a complex exponential shifts the frequency of the original signal.

Definition

Superimposing a low-frequency signal on a high-frequency carrier signal is called **Modulation**.

Example:

Consider an exponential signal $x(t) = e^{j\Omega_0 t}$ of frequency Ω_0. If we multiply an exponential $e^{j\phi t}$ with $x(t)$, then:

$$x(t)e^{j\phi t} = e^{j(\Omega_0+\phi)t} = \cos((\Omega_0 + \phi)t) + j\sin((\Omega_0 + \phi)t)$$

$\phi > 0$: the frequency of new exponential is greater than Ω_0, otherwise lower.

Various Types of Modulation

$A(t)cos(\Omega(t)t + \theta(t))$

- $A(t)$ changes: Amplitude Modulation
- $\Omega(t)$ changes: Frequency Modulation
- $\theta(t)$ changes: Phase Modulation

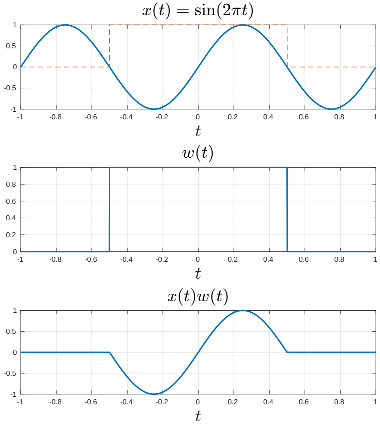
Windowing

For a window signal $w(t)$, the time-windowed signal
$x(t)w(t)$ displays $x(t)$ within the support of $w(t)$.

Example:

$x(t) = \sin(2\pi t)$

$$w(t) = \begin{cases} 1 & \text{if } -0.5 \leq t \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$



Code for the graph:
https://github.com/rahulbhadani/CPE381_FA24/blob/master/Code/windowing.m

Up Next

- Continuous-time Systems
  - Linear-Time Invariance
  - Static vs Dynamic Systems
  - Convolutional Integral
  - BIBO Stability