CPE 381 Exam 1

100 points

September 30, 2024 11:20 AM

Your only source of references for this exam is your one-sided cheatsheet on letter size paper or smaller, handwritten by you, calculator, and pen/pencil, eraser. No other booklets, additional paper, textbooks, or other materials may be referenced during this examination. You may ask for additional sheets from the examiner. **However, please use all the extra space provided in this exam paper.**

Read every question on this examination carefully. Although portions may seem familiar—do not assume that the information presented in this examination is duplicated from any examples (written or otherwise) that you may have seen before.

Total number of pages (including this page and three additional pages at the end for writing solution, excluding supplementary tables, if any): 14

Name: ______________________________

SCORE EARNED:

Q1	Q2	Q3	Q4	Q5	Q6	Total
10	10	30	10	10	30	100

Note:: **Assume i and j are complex units, i.e.** $\sqrt{-1}$ for the entire exam.

1 Toss a Coin (10 points)

Indicate true (T) or false (F) for each of the below statements.

T **F** **(a)** A system can be represented by a differential equation.

T ***F*** **(b)** Anticausal signal is defined only on positive time-axis.

T **F** **(c)** A finite-energy signal has zero power.

T ***F*** **(d)** Inverse Laplace transform of the function

$$Y(s) = \frac{1}{s^2 + 4}$$

is

$$y(t) = \sin(t^2 + 4)$$

T **F** **(e)** $x(t) \rightarrow x(-t)$ denotes reflection of a signal along the y-axis, i.e. taking a mirror image by assuming y-axis as a mirror.

T ***F*** **(f)** A system is said to be stable when all poles of its transfer function lay on the right half of the s-plane.

T **F** **(g)** An LTI System represented by its impulse response $h(t)$ is causal if $h(t) = 0$ for $t < 0$.

T ***F*** **(h)** $x(-t) = -x(t)$ denotes an even signal.

T **F** **(i)** Area under the impulse is unity.

T **F** **(j)** Linearity and time invariance are independent of each other.

2 Four-sided Dice (10 points)

A **B** ***C*** **D** **(a)** A control system is defined by the following mathematical relationship:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 5x = 12(1 - e^{-2t}). \tag{1}$$

The response of the system (i.e. what does the system output) at $t \to \infty$ is:

(A) $x = 6$.

(B) $x = 2$.

(C) $x = 2.4$.

(D) $x = -2$.

$$\begin{aligned} X(s) &= \frac{12\left(\dfrac{1}{s} - \dfrac{1}{s+2}\right)}{s^2 + 6s + 5} \\ &= \frac{24}{s(s+2)(s+1)(s+5)} \\ \Rightarrow \lim_{t\to\infty} x(t) &= \lim_{s\to 0}\Big[sX(s)\Big] = \frac{24}{10} = 2.4 \end{aligned} \tag{2}$$

(We used Final Value Theorem from Laplace transform)

A **B** ***C*** **D** **(b)** $e^{j\pi/4}$ equals

(A) 1

(B) $\sqrt{2} + j\sqrt{2}$

(C) $\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

(D) $\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

A ***B*** **C** **D** **(c)** Consider the graph in Figure 1 where a complex number z is indicated on the complex plane. Circle the correct answer to the complex number z.

(A) $\angle z = \cos^{-1}\frac{8}{6}$.

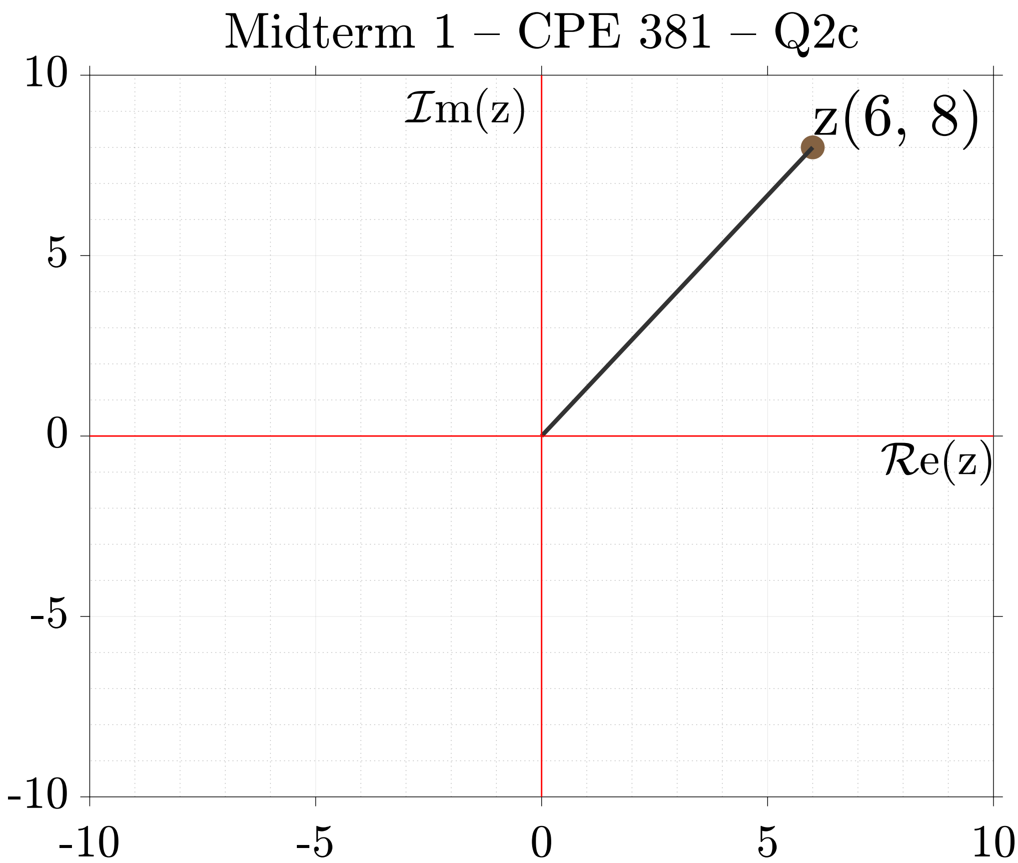

Figure 1: Q2d

(B) The complex conjugate , z^* of z is $6 - 8i$.

(C) $|z| = 9$.

(D) $z = 8 + 6i$.

A B *C* D (d) Which of the following is equal to $\sin(2x)\cos(2x)$?

(A) $\dfrac{e^{j4x} - e^{-j4x}}{4}$

(B) $\dfrac{e^{j4x} + e^{-j4x}}{4}$

(C) $\dfrac{e^{j4x} - e^{-j4x}}{4j}$

(D) $\dfrac{e^{j2x} - e^{-j2x}}{4j}$

A ***B*** **C** **D** **(e)** Consider the graph in Figure 2. Circle the correct signal that represents the graph.

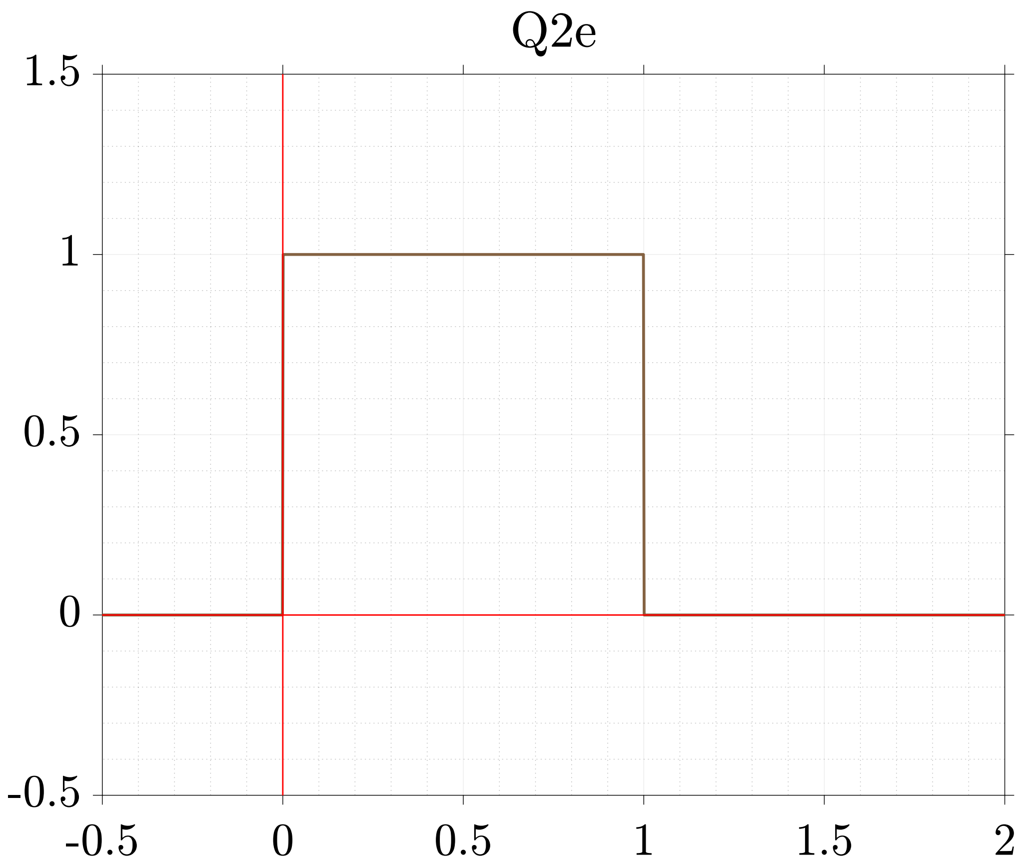

Figure 2: Q2e

(A) $u(t) + u(t-1)$

(B) $u(t) - u(t-1)$

(C) $\delta(t) - \delta(t-1)$

(D) $\delta(t) - u(t-1)$

3 Journey to the s-verse (30 points)

Consider a ramp signal $r(t)$ as shown in Figure 3 by the dashed line. The x-axis denotes the time-axis and the y-axis denotes the signal value.

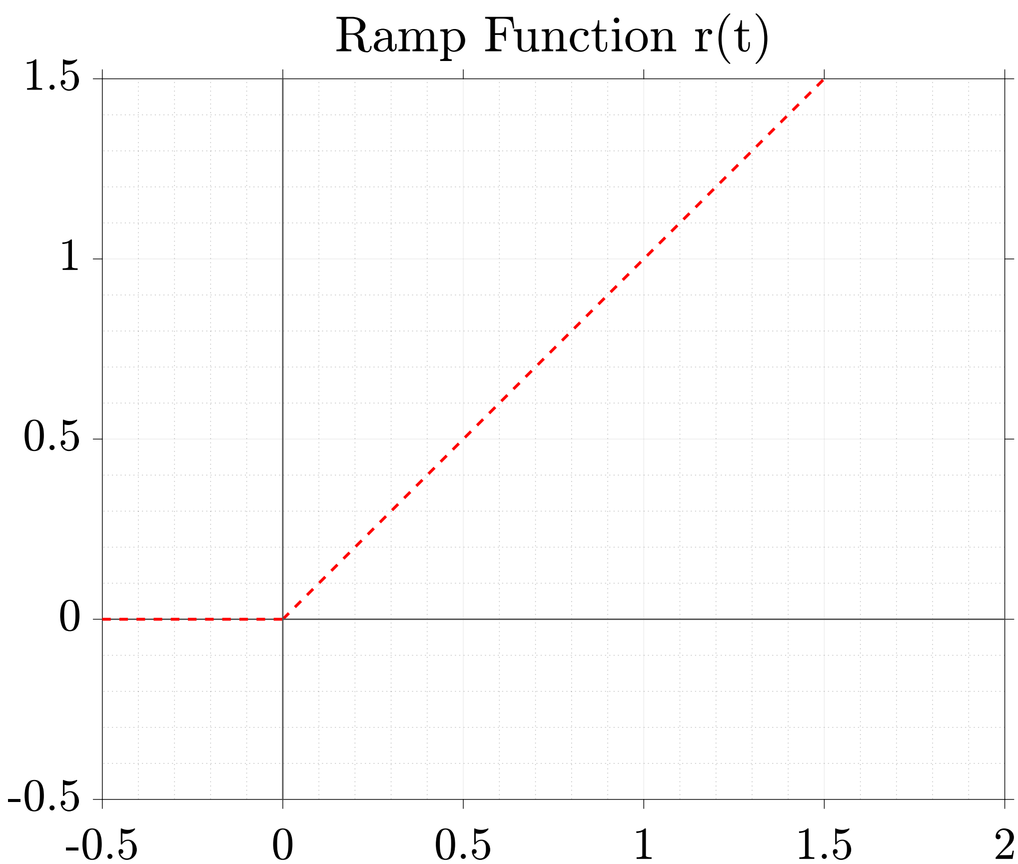

Figure 3: Q3: Ramp Signal $r(t)$

1. Write the equation of the ramp signal in terms of time t, and the unit-step signal $u(t)$. **(3 points)**

2. Derive the Laplace transform $R(s)$ of the ramp signal from the definition of the Laplace transform integral, (i.e. do not use Laplace Transform Table). **(9 points)** *Hint: Use integration by parts* $\int f(t)g(t)'dt = f(t)g(t) - \int f'(t)g(t)dt$.

3. What is the region of convergence (ROC) for the Laplace transform $R(s)$ to exist? **(3 points)**

4. Consider a frequency shift e^{-2s} in s-domain applied to the ramp signal $r(t)$. Using the properties of the Laplace transform, write down the resulting signal in terms of time t and the unit-step signal $u(t)$. You may use the provided tables to facilitate your answer. **(6 points)**

5. The ramp signal is time-differentiated and used as an input to a system represented by the transfer function $H(s) = \dfrac{1}{s+2}$. What's the output signal $y(t)$ in the time domain? **(6 points)**

6. Find $\lim_{t\to\infty} y(t)$. **(3 points)**

Answer:

1. From the graph in Figure 3, we see that $x = y$ for every point on the red line on the positive time-axis. Further, the line is zero on the negative time axis. Hence, we can write the ramp signal $r(t)$ as $tu(t)$ or

$$r(t) = \begin{cases} t, & t > 0 \\ 0, & t \le 0 \end{cases} \tag{3}$$

2. The Laplace transform $R(s)$ of the ramp signal $r(t)$ can be derived as follows:

$$\begin{aligned}
R(s) &= \int_{-\infty}^{\infty} tu(t)e^{-st}dt = \int_{0}^{\infty} te^{-st}dt \\
&\text{Use integration by parts: } \int f(t)g(t)'dt = f(t)g(t) - \int f'(t)g(t)dt \\
&f(t) = t, g'(t) = e^{-st} \\
&f'(t) = 1, g(t) = -\frac{e^{-st}}{s} \\
\Rightarrow R(s) &= \frac{-\,te^{-st}}{s}\bigg|_{t=0}^{\infty} - \int_0^{\infty} -\frac{e^{-st}}{s}dt \\
&\text{Let } u = -st \Rightarrow \frac{du}{dt} = -s \Rightarrow \int_0^{\infty} -\frac{e^{-st}}{s}dt = \frac{1}{s^2}\int e^u du = \frac{e^u}{s^2} = \frac{e^{-st}}{s^2} \\
\Rightarrow R(s) &= \frac{-\,te^{-st}}{s}\bigg|_{t=0}^{\infty} - \frac{e^{-st}}{s^2}\bigg|_{t=0}^{\infty} = [(0-0) - (0 - \tfrac{1}{s^2})] = \frac{1}{s^2}
\end{aligned} \tag{4}$$

3. The above integral is only finite when $\mathcal{R}e(s) > 0$. Hence ROC is $\mathcal{R}e(s) > 0$.

4. Frequency-shift in s-domain causes time-shift in the time-domain. Hence, based Laplace transform property, $f(t-\alpha)u(t-\alpha) \longleftrightarrow e^{-\alpha s}F(s)$, the resulting signal is $(t-2)u(t-2)$.

5. When the ramp signal is time-differentiated, we get the unit-step signal $u(t)$ whose Laplace transfer is $\dfrac{1}{s}$. The output $Y(s) = H(s)X(s) = \dfrac{1}{(s+2)s}$. Using partial fraction expansion, $Y(s) = -\dfrac{1}{2(s+2)} + \dfrac{1}{2s}$. From the Laplace transfer table, the time-domain signal for the output is $y(t) = \left[-\dfrac{1}{2}e^{-2t} + \dfrac{1}{2}\right]u(t)$.

6. Using final value theorem $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \dfrac{s}{(s+2)s} = \dfrac{1}{2}$.

4 Artist and Equations (10 points)

4.1 Sketching

Sketch the graph for the following signals. Please label the x-axis and the y-axis appropriately. Unlabeled sketches of graphs will result in a penalty.

1. $y(t) = u(t) - u(t-1) + u(t-2) + u(t-3) + u(t-4)$
2. $x(t) = 2u(-t)$

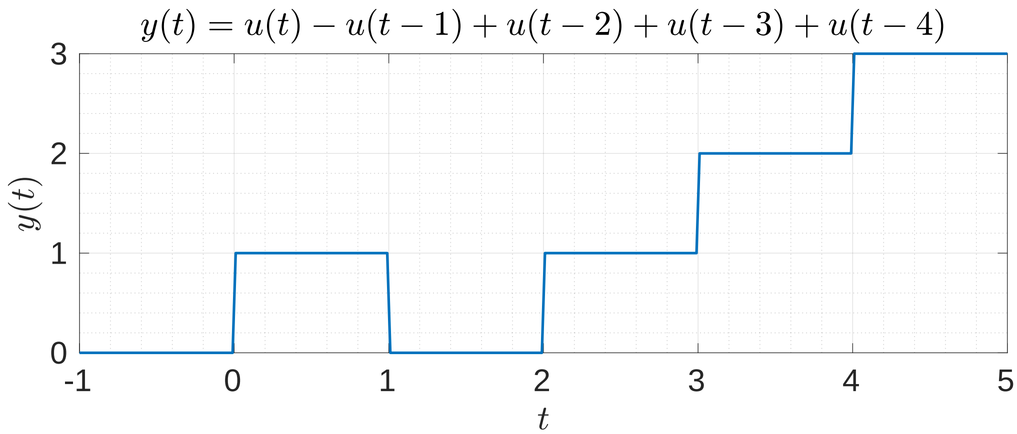

Figure 4: Question 4.1 (1)

1.

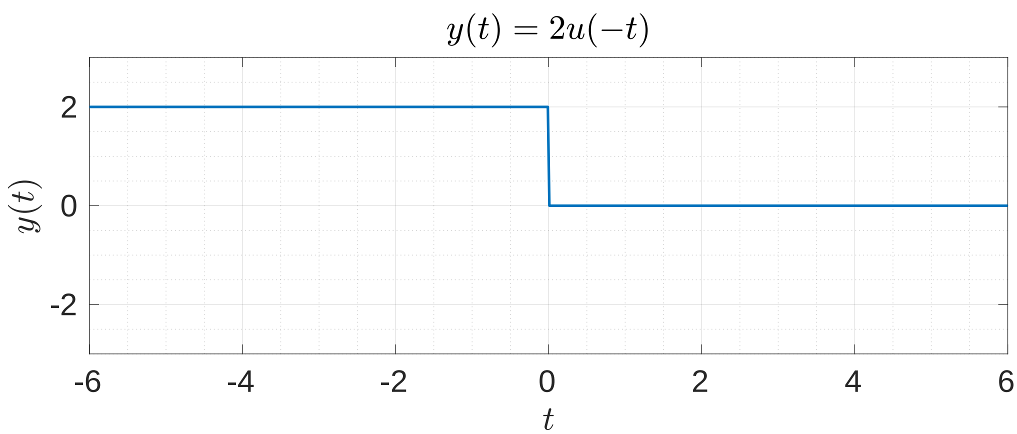

Figure 5: Question 4.1 (2)

2.

4.2 Writing Equations

Represent each of the following graphs as mathematical equations using basic signals (their transformations) such as unit-step functions $u(t)$, unit-impulse function $\delta(t)$, ramp function $r(t)$, etc. Assume signals stretching to $t = \pm\infty$ on either side of the time axes.

1. $p(t)$

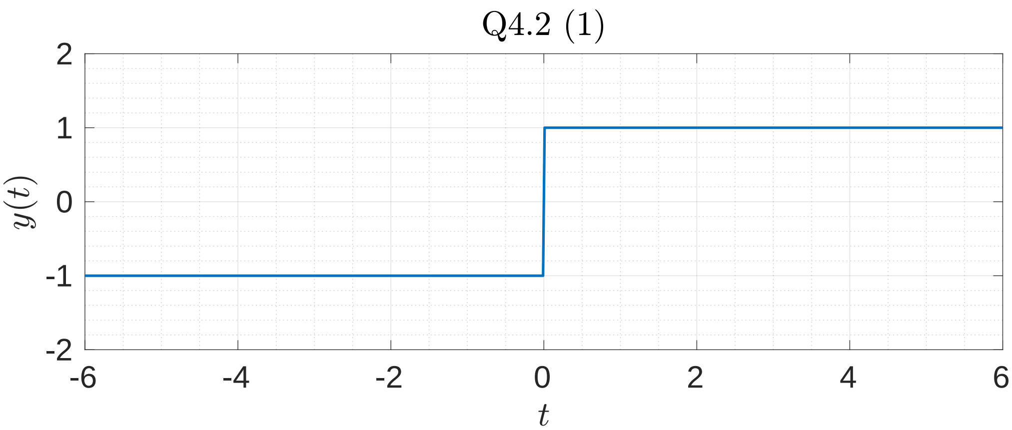

Figure 6: Question 4.2 (1)

2. $q(t)$

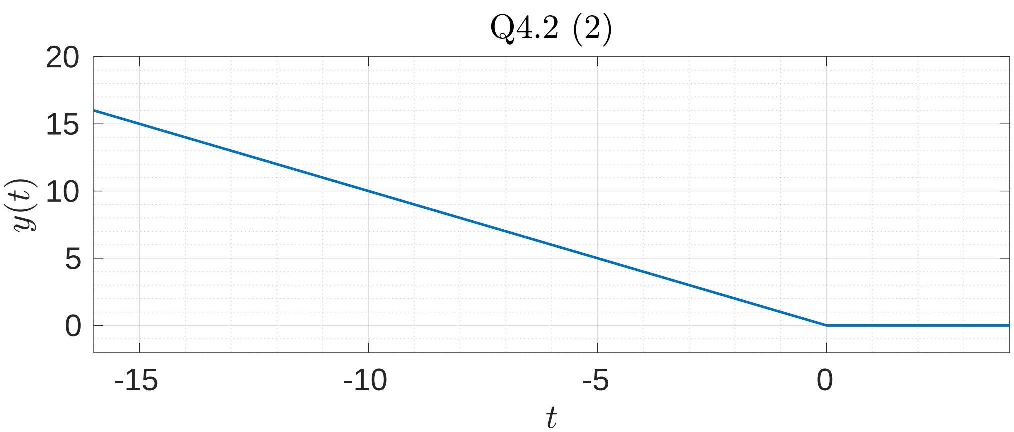

Figure 7: Question 4.2 (2)

1. $p(t) = u(t) - u(-t)$
2. $q(t) = r(-t)$ or $q(t) = -tu(-t)$

5 Transformers Duo (10 points)

For each of the following, provide a specific transformation as being asked.

1. Determine the Laplace transform of

$$f(t) = \begin{cases} 1 & 0 \le t \le 4 \\ 3 & 4 \le t \le 5 \\ 0 & 5 \le t \le \infty \end{cases} \tag{5}$$

   Hint: Write down $f(t)$ using commonly known signals such as $u(t)$, $r(t)$, etc. or their transformations, then take its Laplace transform.

2. Determine the Inverse Laplace Transform of

$$X(s) = \frac{5s + 13}{s(s^2 + 4s + 13)} \tag{6}$$

   Hint: $s^2 + 4s + 13 = (s + 2 - j3)(s + 2 + j3)$

1. $f(t)$ can be written as

$$\begin{aligned} f(t) &= (u(t) - u(t-4)) + 3(u(t-4) - u(t-5)) \\ &= u(t) + 2u(t-4) - 3u(t-5) \end{aligned} \tag{7}$$

   Using the Laplace-transform table, its Laplace transform is

$$F(s) = \frac{1}{s} + \frac{2}{s}e^{-4s} - \frac{3}{s}e^{-5s} \tag{8}$$

2. $s^2 + 4s + 13$ can be factorized as $(s+2)^2 + 9 = (s + 2 - j3)(s + 2 + j3)$.
   Hence writing partial fraction expansion for $X(s)$:

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s - (-2 + j3)} + \frac{C_3}{s - (-2 - j3)} \tag{9}$$

   Matching likewise terms give $C_1 = 1$, $C_2 = -0.5(1 + j)$, $C_3 = -0.5(1 - j)$
   Hence,

$$X(s) = \frac{1}{s} - \frac{0.5(1 + j)}{s - (-2 + j3)} - \frac{0.5(1 - j)}{s - (-2 - j3)} \tag{10}$$

   Using the Laplace transform table, we find that

$$x(t) = u(t) - 0.5(1 + j)e^{(-2+j3)t}u(t) - 0.5(1 - j)e^{(-2-j3)t}u(t) \tag{11}$$

6 Systems (30 points)

Consider a causal LTI continuous system described by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t) \tag{12}$$

where $y(t)$ is the system output and $x(t)$ is the input.

1. Consider zero initial condition. Find the transfer function $H(s)$ of the system. **(10 points)**
2. Find its poles and zeros. From its poles and zeros, determine whether the system is BIBO stable. Hint: first, write down the conditions for stability. **(10 points)**
3. If $x(t) = u(t)$ and initial conditions are zero, determine the steady-state response $y_{\text{ss}}(t)$ **(10 points)**

6.1 Solution:

1. Consider zero initial condition. The Laplace transform of the ordinary differential equation is

$$(s^2 + 3s + 2)Y(s) = X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

2. Poles are at $s = -1, -2$. Both are in the left-hand s-plane so the system is BIBO-stable. Equivalently, the impulse response $h(t) = \mathcal{L}^{-1}[H(s)]$ can be shown to be absolutely integrable.

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{-1}{s+2}\right] = [e^{-t} - e^{-2t}]u(t)$$

   And then

$$\int_{-\infty}^{\infty} |h(t)|dt = -e^{-t} + 0.5e^{-2t}\Big|_0^{\infty} = 1 - 0.5 = 0.5$$

3.

$$X(s) = \frac{1}{s}$$

   then,

   Consider, the initial condition as zero,

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

where $A = Y(s)s\Big|_{s=0} = 05.$

Hence, the stead state is $y_{ss}(t) = 0.5$ since other terms will disappear.

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Table 3.1 Basic properties of one-sided Laplace transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t)+\beta g(t)$	$\alpha F(s)+\beta G(s)$
Time-shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\ f(t)$	$-\dfrac{dF(s)}{ds}$
Derivative	$\dfrac{df(t)}{dt}$	$sF(s)-f(0-)$
Second derivative	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0-)-f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt'$	$\dfrac{F(s)}{s}$
Expansion/contraction	$f(\alpha t),\ \alpha\neq 0$	$\dfrac{1}{\lvert\alpha\rvert}F\left(\dfrac{s}{\alpha}\right)$
Initial value	$f(0-)=\lim_{s\to\infty} sF(s)$	
Derivative Duality	$\dfrac{df(t)}{dt}$	$sF(s)$
	$tf(t)$	$-\dfrac{dF(s)}{ds}$
Integration Duality	$\int_{0-}^{t} f(\tau)d\tau$	$F(s)/s$
	$f(t)/t$	$\int_{-\infty}^{-s} F(-\rho)d\rho$
Time and Frequency Duality	$f(t-\alpha)u(t-\alpha)$	$F(s)e^{-\alpha s}$
	$f(t)e^{-\alpha t}u(t)$	$F(s+\alpha)$
Time Scaling Duality	$f(\alpha t)u(t)$	$(1/\lvert\alpha\rvert)F(s/\alpha)$
	$(1/\lvert\alpha\rvert)f(t/\alpha)u(t)$	$F(\alpha s)$
Convolution	$[f*g](t)$	$F(s)G(s)$
Initial value	$f(0-)=\lim_{s\to\infty} sF(s)$	

Table 3.2 One-sided Laplace transforms

$\delta(t)$	1, whole s-plane
$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
$2A\ e^{-at}\cos(\Omega_0 t+\theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
$\frac{1}{(N-1)!}\ t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$

B.8 APPENDIX: USEFUL MATHEMATICAL FORMULAS

We conclude this chapter with a selection of useful mathematical facts.

B.8-1 Some Useful Constants

$\pi \approx 3.1415926535$

$e \approx 2.7182818284$

$\frac{1}{e} \approx 0.3678794411$

$\log_{10} 2 \approx 0.30103$

$\log_{10} 3 \approx 0.47712$

B.8-2 Complex Numbers

$e^{\pm j\pi/2} = \pm j$

$e^{\pm jn\pi} = \begin{cases} 1 & n \,\text{even} \\ -1 & n \,\text{odd} \end{cases}$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$a + jb = re^{j\theta} \qquad r = \sqrt{a^2+b^2},\, \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$(re^{j\theta})^k = r^k e^{jk\theta}$

$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$

B.8-3 Sums

$$\sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r-1} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} k\, r^k = \frac{r + [n\,(r-1) - 1]\, r^{n+1}}{(r-1)^2} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k^2\, r^k = \frac{r[(1+r)(1-r^n) - 2n(1-r)r^n - n^2(1-r)^2 r^n]}{(1-r)^3} \qquad r \neq 1$$

B.8-4 Taylor and Maclaurin Series

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

$$f(x) = f(0) + \frac{x}{1!}\dot{f}(0) + \frac{x^2}{2!}\ddot{f}(0) + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0)$$

B.8-5 Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$(1+x)^n \approx 1 + nx \qquad |x| \ll 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

B.8-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j \sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp \sin x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm \cos x$$

$$2 \sin x \cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$a\cos x + b\sin x = C\cos(x+\theta) \qquad C = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\left(\frac{-b}{a}\right)$

B.8-7 Common Derivative Formulas

$\dfrac{d}{dx}f(u) = \dfrac{d}{du}f(u)\dfrac{du}{dx}$

$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dx^n}{dx} = nx^{n-1}$

$\dfrac{d}{dx}\ln(ax) = \dfrac{1}{x}$

$\dfrac{d}{dx}\log(ax) = \dfrac{\log e}{x}$

$\dfrac{d}{dx}e^{bx} = be^{bx}$

$\dfrac{d}{dx}a^{bx} = b(\ln a)a^{bx}$

$\dfrac{d}{dx}\sin ax = a\cos ax$

$\dfrac{d}{dx}\cos ax = -a\sin ax$

$\dfrac{d}{dx}\tan ax = \dfrac{a}{\cos^2 ax}$

$\dfrac{d}{dx}(\sin^{-1} ax) = \dfrac{a}{\sqrt{1-a^2x^2}}$

$\dfrac{d}{dx}(\cos^{-1} ax) = \dfrac{-a}{\sqrt{1-a^2x^2}}$

$\dfrac{d}{dx}(\tan^{-1} ax) = \dfrac{a}{1+a^2x^2}$

B.8-8 Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$

$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$

$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$

$$\int x^2\cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int \sin ax\cos bx\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)$$

B.8-9 L'Hôpital's Rule

If $\lim f(x)/g(x)$ results in the indeterministic form $0/0$ or ∞/∞, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{\dot{f}(x)}{\dot{g}(x)}$$

B.8-10 Solution of Quadratic and Cubic Equations

Any *quadratic* equation can be reduced to the form

$$ax^2 + bx + c = 0$$

The solution of this equation is provided by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A general *cubic* equation

$$y^3 + py^2 + qy + r = 0$$

may be reduced to the *depressed cubic* form

$$x^3 + ax + b = 0$$

by substituting

$$y = x - \frac{p}{3}$$

This yields

$$a = \tfrac{1}{3}(3q - p^2) \qquad b = \tfrac{1}{27}(2p^3 - 9pq + 27r)$$

Now let

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The solution of the depressed cubic is

$$x = A + B, \qquad x = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \qquad x = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}$$

and

$$y = x - \frac{p}{3}$$

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3. Hogben, Lancelot. *Mathematics in the Making.* Doubleday, New York, 1960.

Complex Number Formulas

Basic Definitions

- Complex Number: $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
- Real Part: $\Re(z) = a$
- Imaginary Part: $\Im(z) = b$
- Conjugate: $\overline{z} = a - bi$
- Modulus: $|z| = \sqrt{a^2 + b^2}$
- Argument: $\arg(z) = \theta$ where $z = |z|(\cos\theta + i\sin\theta)$

Operations

- Addition: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
- Subtraction: $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$
- Multiplication: $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$
- Division: $\frac{z_1}{z_2} = \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + \frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}i$

Polar Form

- $z = r(\cos\theta + i\sin\theta)$
- Euler's Formula: $z = re^{i\theta}$
- Multiplication: $z_1 \cdot z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
- Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

Exponential Form

- $e^{i\theta} = \cos\theta + i\sin\theta$
- De Moivre's Theorem: $(re^{i\theta})^n = r^ne^{in\theta}$

Roots of Complex Numbers

- $z^{1/n} = r^{1/n}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$ for $k = 0, 1, \ldots, n-1$