Worked Example 07 CPE 381, FA24, 10/02/2024
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Q1. Find the fourier coefficients for $x(t) = \sin^2(t)$.

We can write $\sin^2(t) = \frac{1-\cos 2x}{2}$.

$\cos 2x$ has period $T_0 = \frac{2\pi}{2} = \pi$

Hence $\Omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$

Hence,

$$x(t) = \sin^2(t) = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^2$$

$$= -\frac{1}{4}\left(e^{j2t\cdot 1} + e^{j2t\cdot(-1)} - 2\right)$$

$$= -\frac{1}{4}e^{-j2t\cdot 1} - \frac{1}{4}e^{j2t(-1)} + \frac{1}{2}$$

Comparing it with $x(t) = \sum X_k e^{jk\Omega_0 t}$

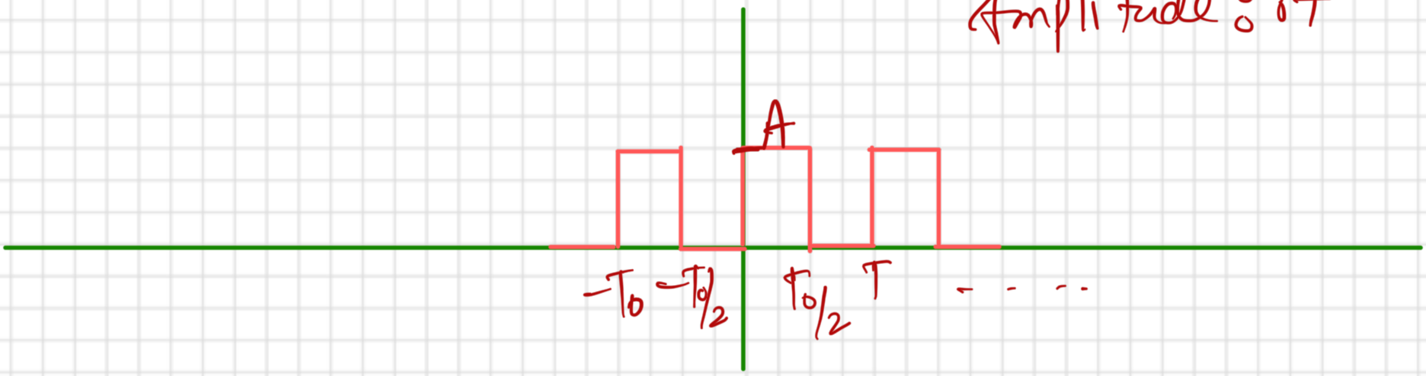
$$X_0 = \frac{1}{2}$$

$$X_{-1} = -\frac{1}{4}$$

$$X_1 = -\frac{1}{4}$$

Q2. Represent the following signal in terms of complex exponential fourier series

Amplitude : A



$$x(t) = \sum_{k=-\infty} X_k e^{jk\Omega_0 t}$$

$$\Omega_0 = \frac{2\pi}{T_0}$$

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\Omega_0 t}$$

$$= \frac{1}{T_0}\int_0^{T_0/2} A e^{-jk\Omega_0 t} dt$$

$$= -\frac{A}{jk\Omega_0 T_0} e^{-jk\Omega_0 t}\Big|_0^{T_0/2}$$

$$= \frac{-A}{jk\Omega_0 T_0}\left(e^{-jk\Omega_0 T_0/2} - 1\right)$$

$$= \frac{A}{jk2\pi}\left(1 - e^{-jk\pi}\right) = \frac{A}{jk2\pi}\left[1 - (-1)^k\right]$$

(As $\Omega_0 T_0 = 2\pi$, and $e^{-jk\pi} = (-1)^k$)

Hence

$$X_k = 0 \quad \text{for} \quad k = 2m \neq 0 \quad (\text{or when } k \text{ is even})$$

$$X_k = \frac{A}{jk\pi}, \quad \text{for } k = 2m+1 \quad (\text{or when } k \text{ is odd})$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} A\,dt = \frac{A}{2}$$

Hence $X_0 = \frac{A}{2}$

$$X_{2m} = 0$$

$$X_{2m+1} = \frac{A}{j(2m+1)\pi}$$

Thus,

$$x(t) = \frac{A}{2} + \frac{A}{j\pi}\sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\Omega_0 t}$$

In terms of trigonometric Fourier series:

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} \left(c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t) \right)$$

$$c_{2m} = d_{2m} = 0, \quad m \neq 0$$

$$c_{2m+1} = 2\,\mathrm{Re}\left[X_{2m+1}\right] = 0$$

$$d_{2m+1} = -2\,\mathrm{Im}\left[X_{2m+1}\right] = \frac{2A}{(2m+1)\pi}$$

Hence

$$x(t) = \frac{A}{2} + \frac{2A}{j\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin\left((2m+1)\Omega_0 t\right)$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left(\sin \Omega_0 t + \frac{1}{3}\sin \Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right)$$