CPE 381: Fundamentals of Signals and Systems for Computer Engineers

10 Z-transform

Rahul Bhadani
Electrical & Computer Engineering, The University of Alabama in Huntsville

Outline

1. Motivation
2. Two sided Z-transform
3. Discrete Transfer Function
4. One-Sided Z-transform Inverse
5. Solution of Difference Equations

Motivation

Z-transform and Laplace Transform

The discrete equivalent of Laplace Transform that follows:

- If the sampled signal is: $x(t) = \sum x(nT_s)\delta(t - nT_s)$
  its Laplace transform is : $X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n x(nT_s) = e^{-nsT_s}$
- By letting $z = e^{sT_s}$,

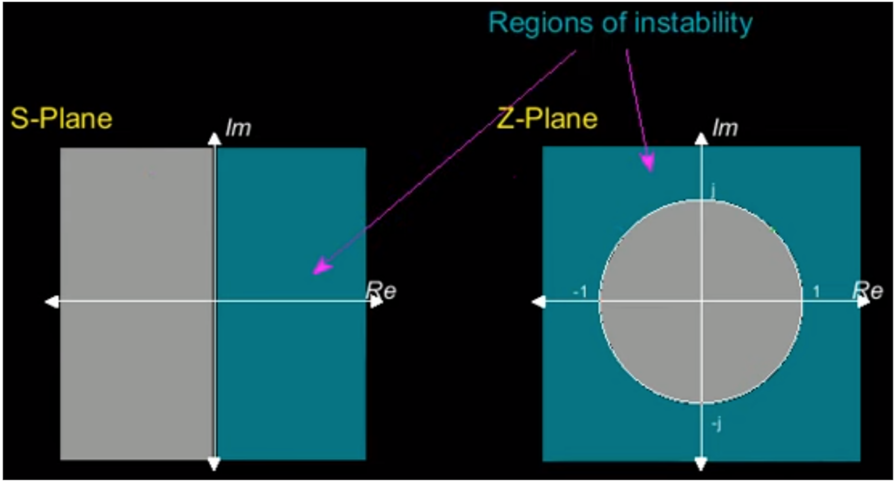
$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$$

which is called the Z-transform of the sampled signal.

Z-transform in Practice

Although Z-transform seems to be a discrete version of Laplace transform, the practical use doesn't necessarily bother to find the relationship between Laplace and Z-transform.

- Complex Z-plane is a polar form rather than a 2D cartesian plane.
- Radius corresponds to the damping factor, and angle corresponds to the discrete frequency ω in radians.
- Thus, the unit circle in the Z-plane is analogous to the $j\Omega$-axis in the Laplace plane.
- Inside of the unit circle is analogous to the left-hand s-plane.

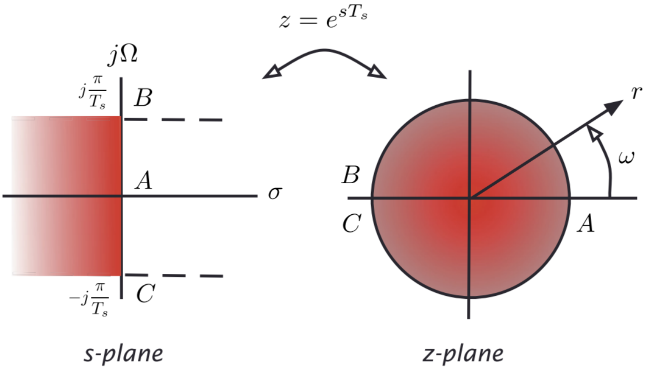


See: Source:https://www.youtube.com/watch?v=acQecd6dmxw

Relationship between s-plane and z-plane

By letting $z = e^{sT_s}$,

- $z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} = e^{\sigma T_s} e^{j\Omega T_s}$
- $r = e^{\sigma T_s}$, $\omega = \Omega T_s$, and hence, $z = re^{j\omega}$.



Example

To see the possibility of infinite zeros and poles in the Laplace transform of a sampled signal, consider a pulse $x(t) = u(t) - u(t - T_0)$ sampled with a sampling period $T_s = \frac{T_0}{N}$, N being a positive integer.

Two-sided Z-transform

Two sided Z-transform

Given a discrete-time signal $x[n]$, $-\infty < n < \infty$, its two-sided Z-transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

defined in a region of convergence (ROC) on the Z-plane.

Remark on Z-transform

The two-sided Z-transform is not useful in solving difference equations with nonzero initial conditions, just as the two-sided Laplace transform was not useful either in solving ordinary differential equations with nonzero initial conditions.

Hence, we need to look at one-sided Z-transform.

One-sided Z-transform

Defined for causal signals, $x[n] = 0$ for $n < 0$

$$X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n}^{\infty} x[n]u[n]z^{-n}$$

The ROC for this z-transform is denoted by $\mathcal{R}_1$.

Two-sided Z-transform as One-sided Z-transform

$X(z) = \mathcal{Z}(x[n]u[n]) + \mathcal{Z}(x[-n]u[n])|_z - x[0]$.
The first z-transform has ROC $\mathcal{R}_1$, and the second z-transform has ROC $\mathcal{R}_2$.
Proof:

$$\begin{aligned}
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} &= \sum_{n=0}^{\infty} x[n]u[n]z^{-n} + \sum_{-\infty}^{0} x[n]u[-n]z^{-n} - x[0] \\
&= \mathcal{Z}(x[n]u[n]) + \sum_{m=0}^{\infty} x[-m]u[m]z^{m} - x[0] \\
&= \mathcal{Z}(x[n]u[n]) + \mathcal{Z}(x[-n]u[n])|_z - x[0]
\end{aligned}$$

Region of Convergence for Z-transform

Infinite summation we see in the Z-transform has to converge under some conditions for it to be meaningful. It means

$$|X(z)| = \left| \sum_n x[n] z^{-n} \right| \leq \sum_n |x[n]| |r^{-n} e^{j\omega n}| = \sum_n |x[n]| r^{-n} < \infty$$

We see that for a given discrete signal or sequence, the convergence depends on the value of r.

Poles and Zeros of Z-transform

The poles of a Z-transform $X(z)$ are complex values $\{p_k\}$ such that

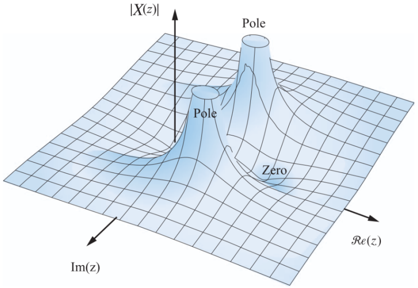
$$X(p_k) \to \infty$$

The zeros of $X(z)$ are the complex values $\{z_k\}$ that make

$$X(z_k) = 0$$

Visualization of Z-transform

The magnitude $|X(z)|$ of the z-transform represents a surface in the z-plane. There are two zeros at $z_1 = 0$, $z_2 = 1$ and two poles at $p_{1,2} = 0.9e^{\pm j\pi/4}$.



Example

Find the poles and zeros of the following Z-transforms

- (i) $X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$
- (ii) $X_2(z) = \dfrac{(z^{-1} - 1)(z^{-1} + 2)^2}{z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1)}$

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

ROC of Finite-Support Signals

The region of convergence (ROC) of the Z-transform of a signal $x[n]$ of a finite support $[N_0, N_1]$, where $-\infty < N_0 \leq n \leq N_1 < \infty$,

$$X(z) = \sum_{n=N_0}^{N_1} x[n]z^{-n}$$

is the whole Z-plane, excluding the origin $z = 0$, and/or $z = \pm\infty$, depending on N_0 and N_1.

Example

Find the Z-transform of a discrete-time pulse $x[n] = u[n] - u[n-10]$. Determine the region of convergence of $X(z)$.

Blank space for calculation

Blank space for calculation

Blank space for calculation

ROC of Infinite-Support Signals

Signals of infinite support are either causal, anticausal or a combination of these-noncausal. For the Z-transform of a causal signal $x_c[n]$, i.e., $x_c[n] = 0, n < 0$.

$$X_c(z) = \sum_{n=0}^{\infty} x_c[n] z^{-n} = \sum_{n=0}^{\infty} x_c[n] r^{-n} e^{-jn\omega}$$

to converge we need to determine appropriate values of r, the damping factor, as the frequency ω has no effect on the convergence.

ROC of Infinite-Support Signals for a Causal Signal

If R1 is the radius of the farthest out pole of $X_c(z)$, then there is an exponential $R_1^n u[n]$ such that $|x_c[n]| < MR_1^n$ for $n \geq 0$. Then, for$X(z)$ to converge we need that

$$|X_c(z)| \leq \sum_{n=0}^{\infty} |x_c[n]| r^{-n} < M \sum_{n=0}^{\infty} \left(\frac{R_1}{n}\right)^n < \infty$$

or that $\frac{R_1}{r} < 1$, which is equivalent to $|z| = r|R_1$. ROC is thus the outside of a circle containing all the poles of $X_c(z)$, i.e., it does not include any poles of $X_c(z)$.

ROC of Infinite-Support Signals for an Anti-causal Signal

Likewise, for an anti-causal signal $x_a[n]$ (i.e., $x_a[n] = 0$, for $n > 0$) if we choose a radius R_2 which is smaller than the radius of all the poles of $X_a(z)$, the region of convergence is $|z| = r < R_2$. This ROC is the inside of a circle that does not include any poles of $X_a(z)$.

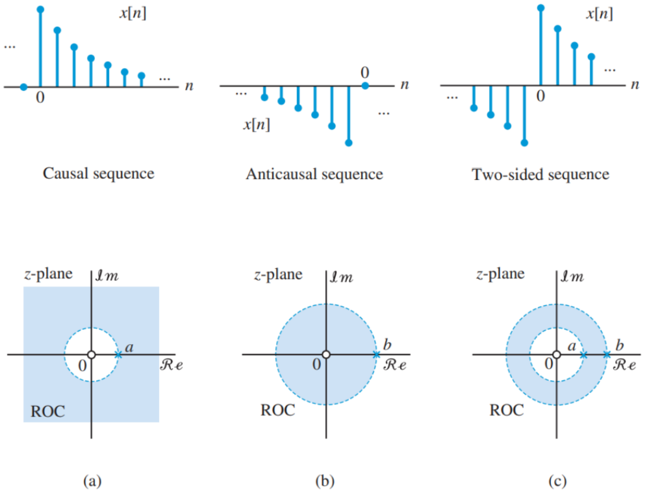
ROC of Infinite-Support Signals for a Non-causal (Two-sided) Signal

If the signal$x[n]$ is noncausal, it can be expressed as

$$x[n] = x_c[n] + x_a[n]$$

$x_c[n]$ is causal whereas $x_a[n]$ is anti-causal.
The corresponding ROC of $X(z)$ would be $0 < R_1 < |z| < R_2 < \infty$.

ROC Visually



Example 1: ROC

Consider the sequence $x[n] = a^n u[n]$, a is real, find the z-transform and sketch its ROC.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Example 2: ROC

Consider the sequence $x[n] = -a^n u[-n-1]$, a is real, find the z-transform, and sketch its ROC.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Properties of ROC

- The ROC does not contain any poles.
- If $x[n]$ is a finite sequence (that is, $x[n] = 0$ except in a finite interval $N_1 \leq n \leq N_2$, where N_1, and N_2, are finite) and $X(z)$ converges for some value of z, then the ROC is the entire z-plane except possibly $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence (that is, $x[n] = 0$ for $n < N_1 < \infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| > r_{max} \quad \text{or} \quad \infty > |z| > r_{max}$$

where r_{max} equals the largest magnitude of any of the poles of $X(z)$. Thus, the ROC is the exterior of the circle $|z| = r_{max}$, in the z-plane with the possible exception of $z = \infty$.

Properties of ROC

- If $x[n]$ is a left-sided sequence (that is, $x[n] = 0$ for $n > N_2 > -\infty$) and $X(z)$ converges for some value of z, then the ROC is of the form

$$|z| < r_{min} \qquad \text{or} \quad 0 < |z| < r_{min}$$

where r_{min} equals the smallest magnitude of any of the poles of $X(z)$. Thus, the ROC is the interior of the circle $|z| = r_{min}$, in the z-plane with the possible exception of $z = 0$.

- If $x[n]$ is a two-sided sequence (that is, $x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of z, then the ROC is of the form

$$r_1 < |z| < r_2$$

where r_1 and r_2 are the magnitudes of the two poles of$X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z| = r_1$ and $|z| = r_2$ not containing any poles.

Example

The poles of $X(z)$ are $z = 0.5$ and $z = 2$; find all the possible signals that can be associated with $X(z)$ according to different regions of convergence.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Example

Find the regions of convergence of the Z-transforms of the following signals

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

Find then the Z-transform of $x_1[n] + x_2[n]$.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Derivative Property

If $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$
its derivative with respective to z is

$$\frac{dX(z)}{dz} = \sum_{n}^{\infty} x[n]\frac{dz^{-n}}{dz} = -z^{-1}\sum_{n}^{\infty} nx[n]z^{-n}$$

or

$$nx[u]u[n] \longleftrightarrow -z\frac{dX(z)}{dz}$$

Discrete Transfer Function

Transfer Function in Z-domain

The output $y[n]$ of a causal LTI system is calculated using the convolution sum

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$$

where $x[n]$ is a causal input and $h[n]$ is the impulse response of the system. Z-transform of the $y[n]$ is the product

$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)$$

and thus the transfer function of a discrete system is defined as

$$H(z) = \frac{Y(z)}{X(z)}$$

Convolution Sum

Fun video: https://www.youtube.com/watch?v=KuXjwB4LzSA

What is convolution sum?

The convolution sum property can be seen as a way to obtain the coefficients of the product of two polynomials. s. Whenever we multiply two polynomials $X_1(z)$ and $X_2(z)$, of finite or infinite order, the coefficients of the resulting polynomial can be obtained using the convolution sum.

Example

Consider computing the output of an FIR filter

$$y[n] = \frac{1}{2}(x[n] + x[n-1] + x[n-2])$$

for an input $x[n] = u[n] - u[n-4]$ using the convolution sum, analytically and graphically, and the Z-transform.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Example

Consider a discrete-time IIR system represented by the difference equation

$$y[n] = 0.5y[n-1] + x[n]$$

with $x[n]$ as input and $y[n]asoutput.Determinethetransferfunctionofthesystemandfr$om it find the impulse and the unit-step responses. Determine under what conditions the system is BIBO stable. If stable, determine the transient and steady-state responses of the system.

Blank space for calculation

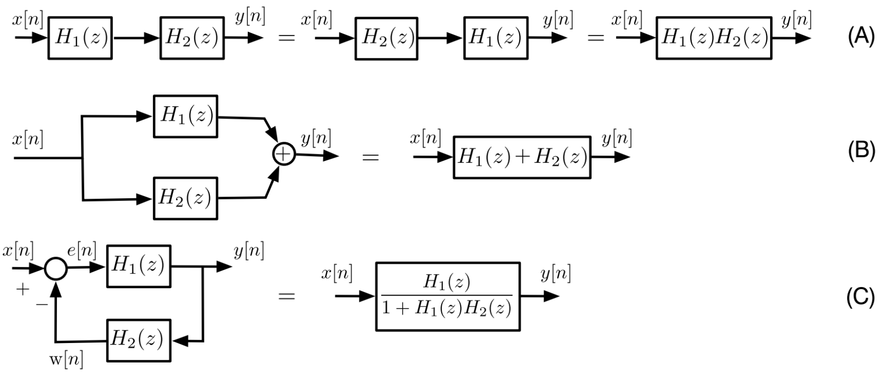
Blank space for calculation

Blank space for calculation

Blank space for calculation

Interconnection of Discrete Systems

Connections of LTI systems: (A) cascade, (B) parallel, and (C) negative feedback.

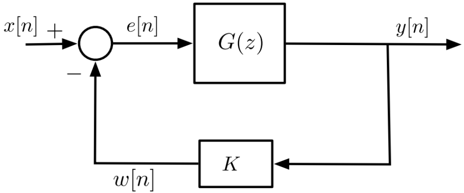


Initial Value and Final Value Theorem for Z-transform

If $X(z)$ is the Z-transform of a causal signal $x[n]$ with an initial value $x[0]$ and a final value $lim_{n\to\infty} x[n]$ are obtained from $X(z)$ according to

- Initial value: $x[0] : \lim_{z\to\infty} X(z)$
- Final value: $lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

Example



Consider a negative feedback connection of a plant given in the figure with a transfer function

$$G(z) = \frac{1}{1 - 0.5z^{-1}}$$

and a constant feedback gain of K.
If the reference signal is a unit-step, $x[n] = u[n]$, determine the behavior of the error signal $e[n]$. What is the effect of the feedback, from the error point of view, on an unstable plant $G(z) = \frac{1}{1-z^{-1}}$

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

One-Sided Z-transform Inverse

Long Division Method

When a rational function $X(z) = B(z)/A(z)$, having as ROC the outside of a circle of radius R (i.e., $x[n]$ is causal), is expressed as

$$X(z) = x[0] + x[1]z^{-1} + x[2]^{-2} + \cdots$$

by dividing $B(z)$ by $A(z)$, then the inverse is the sequence $\{\cdots 0, 0, x[0], x[1], x[2], \cdots\}$ or

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Example

Find the inverse Z-transform of

$$X(z) = \frac{1}{1 + 2z^{-2}}, \quad |z| > \sqrt{2}$$

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Partial Fraction Method with Example

Consider the non-proper rational function

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}$$

(numerator and denominator of the same degree in powers of z^{-1}).
Determine how to obtain an $X(z)$ expansion containing a proper rational term to find $x[n]$.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Find the inverse Z-transform of

$$X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{z(z+1)}{(z+0.5)(z-0.5)}, \quad |z| > 0.5$$

by using the negative and the positive powers of z representations.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

One-Sided Z-transforms Pair Table

Table 10.1 One-sided Z-transforms

$\delta[n]$	1, whole z-plane
$u[n]$	$\dfrac{1}{1-z^{-1}}, \quad \lvert z\rvert > 1$
$nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}, \quad \lvert z\rvert > 1$
$n^2u[n]$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}, \quad \lvert z\rvert > 1$
$\alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$\dfrac{1}{1-\alpha z^{-1}}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$
$\alpha^n\cos(\omega_0 n)u[n], \quad \lvert\alpha\rvert < 1$	$\dfrac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$\alpha^n\sin(\omega_0 n)u[n], \quad \lvert\alpha\rvert < 1$	$\dfrac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$

Basic Properties of One-sided Z-transform

Table 10.2 Basic properties of one-sided Z-transform

Causal signals	$\alpha x[n],\ \beta y[n]$	$\alpha X(z),\ \beta Y(z)$
Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(z) + \beta Y(z)$
Convolution sum	$\sum_k x[n]y[n-k]$	$X(z)Y(z)$
Time-shifting	$x[n-N],\ N > 0$	$z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication	$n\ x[n]$	$-z\dfrac{dX(z)}{dz}$
	$n^2\ x[n]$	$z^2\dfrac{d^2X(z)}{dz^2} + z\dfrac{dX(z)}{dz}$
Finite difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\dfrac{X(z)}{1-z^{-1}}$
Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Solution of Difference Equations

Example

The following first-order difference equation represents a discrete IIR system

$$y[n] = ay[n-1] + x[n], \quad n \geq 0$$

where $x[n]$ is the system's input and $y[n]$ its output. Discuss how to solve it using recursive methods and the Z-transform. Obtain a general form for the complete solution $y[n]$ in terms of the impulse response $h[n]$ of the system.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Example

Solve the difference equation

$$y[n] = y[n-1] - 0.25y[n-2] + x[n], \quad n \geq 0$$

with zero initial conditions and $x[n] = u[n]$.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Approximate Solution of Ordinary Differential Equations

The solution of ordinary differential equations requires converting them into difference equations, which can then be solved in closed form by means of the Z-transform.

Example

Consider an RLC circuit represented by the second-order ordinary differential equation

$$\frac{d^2 v_c(t)}{dt^2} + \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

where the voltage across the capacitor $v_c(t)$ is the output and the source $v_s(t) = u(t)$ is the input. Let the initial conditions be zero. Approximate the derivatives by their definition, find and solve the resulting difference equation.

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation

Blank space for calculation