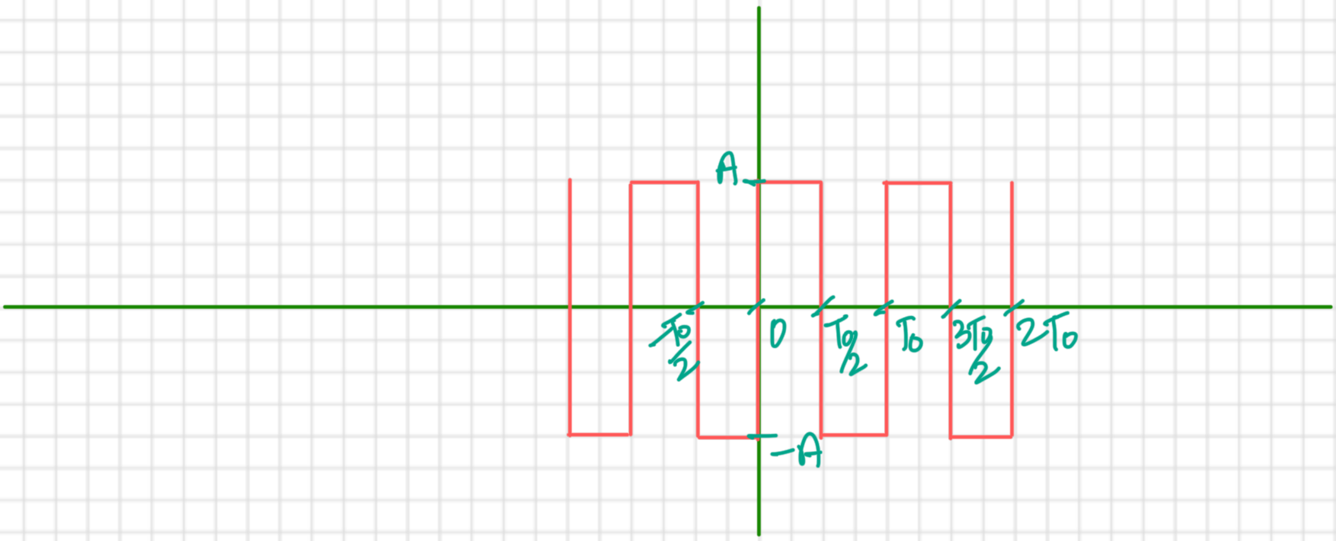
Worked Example 08. CPE 381 FA24 Instructor: Rahul Bhadani
10/07/2024

CQ1. What is Fourier Series?

CQ2. Is Fourier Series applicable only for a Period Signal?

1. Consider the periodic square wave $x(t)$ shown in Fig

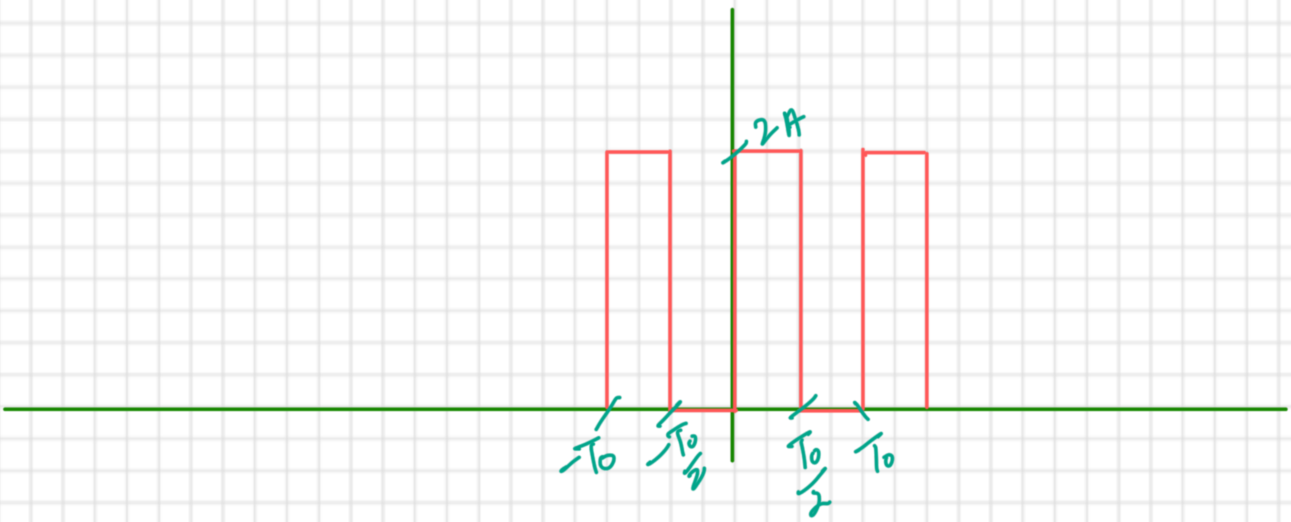


(1) Determine the complex exponential Fourier Series of $x(t)$.
(2) Determine the trigonometric Fourier series of $x(t)$.

Solution:

We can write $x(t)$ as $x(t) = x_1(t) - A$

$x_1(t)$ is given below.



In the Worked Example 7 we derived Fourier series for a similar waveform except amplitude was A instead of 2A.

We can use the same result for this waveform as:

$$x_1(t) = A + \frac{2A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\Omega_0 t}$$

$$\Omega_0 = \frac{2\pi}{T_0}$$

and $x(t) = x_1(t) - A$

$$= \frac{2A}{j\pi} \sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\Omega_0 t}$$

(ii) In terms of trigonometric series

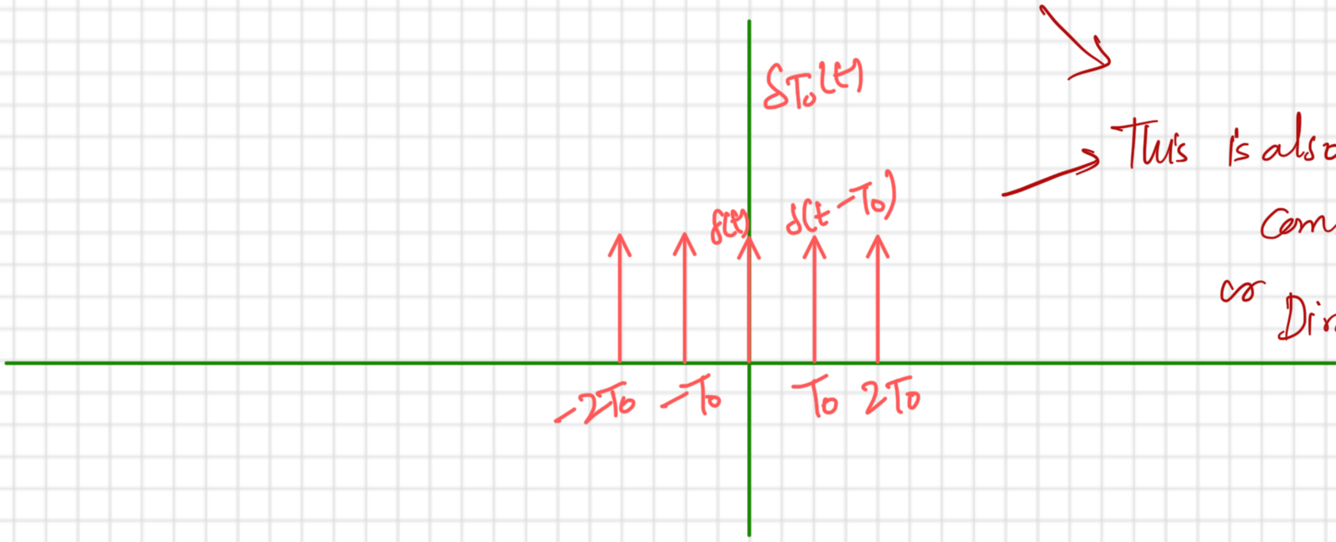
$$x_1(t) = A + \frac{4A}{\pi} \sum_{-\infty}^{\infty} \frac{1}{2m+1} \sin(2m+1)\Omega_0 t$$

and hence, $x(t) = \frac{4A}{\pi} \sum_{-\infty}^{\infty} \frac{1}{2m+1} \sin(2m+1)\Omega_0 t$

$$= \frac{4A}{\pi}\left(\sin\Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right)$$

Q2. Consider the periodic impulse train $\delta_{T_0}(t)$ shown

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta(t - kT_0)$$



This is also called comb function. or Dirac's Comb function.

(a) Determine the complex exponential Fourier signal of $\delta_{T_0}(t)$

(b) Determine the trigonometric Fourier series of $\delta_{T_0}(t)$

Solution

(a) From the definition of Fourier series:

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \qquad \Omega_0 = \frac{2\pi}{T_0}$$

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{-jk\Omega_0 t}\, dt \qquad (\text{set } t_0 = -T_0/2)$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-jk\Omega_0 t}\, dt = \frac{1}{T_0}$$

Hence $\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} e^{jk\Omega_0 t}$

$\Omega_0 = \frac{2\pi}{T_0}$

(b)

$$\delta_{T_0}(t) = X_0 + 2\sum_{k=1}^{\infty}\left[c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)\right]$$

$\Omega_0 = \frac{2\pi}{T_0}$

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cos(k\Omega_0 t)\, dt$$

$$= \frac{1}{T_0}$$

$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)\, dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\sin(k\Omega_0 t)\, dt = 0$$

$X_0 = \frac{1}{T}$ (From exponential Fourier Series expansion)

Thus

$$\delta_{T_0}(t) = \frac{1}{T_0} + \frac{2}{T_0}\sum_{k=1}^{\infty} \cos k\Omega_0 t \qquad \Omega_0 = \frac{2\pi}{T_0}$$

Q3: A period signal $x(t)$ has a fundamental frequency $\Omega_0 = 1$ and a period $x_1(t)$ is given by

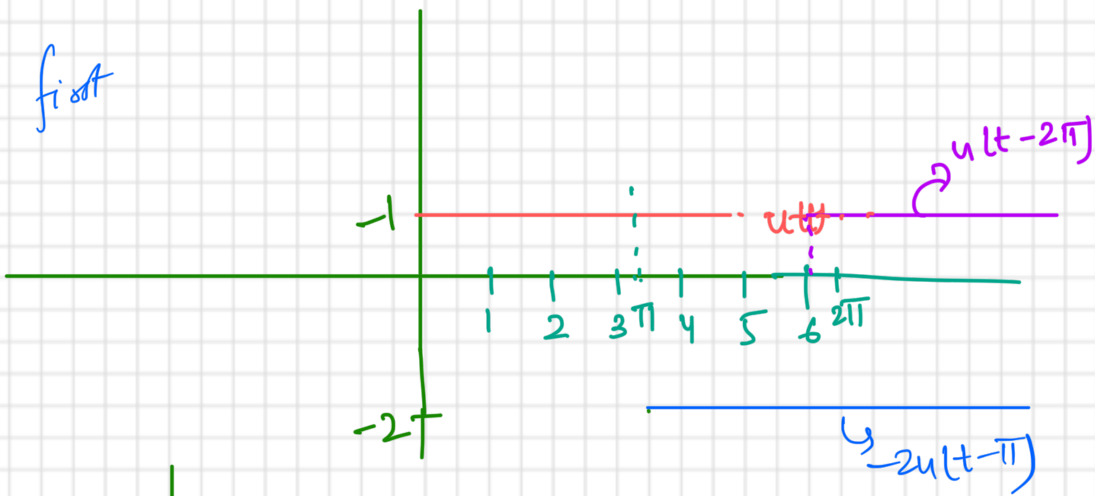
$$x_1(t) = u(t) - 2u(t-\pi) + u(t-2\pi)$$

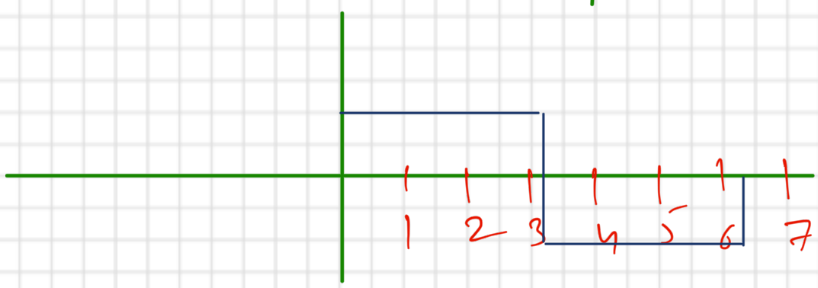
(a) Find the Fourier series coefficients X_k of $x(t)$ using their integral definition.

(b) Use the Laplace transform to find the Fourier series coefficients $\{X_k\}$ of $x(t)$.

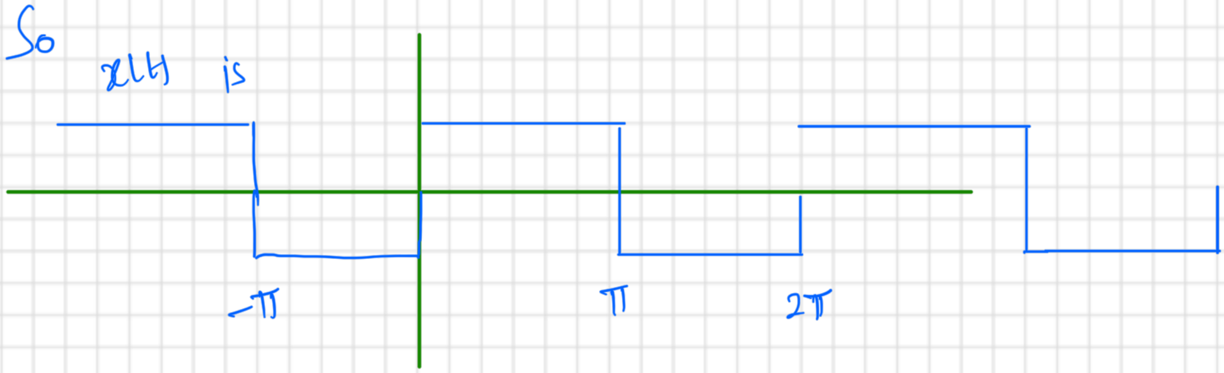
Solution

Let's plot first





So $x(t)$ is



$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, e^{-jk\Omega_0 t}\, dt \qquad T_0 = 2\pi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(t)\, e^{-jkt}\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} 1\cdot e^{-jkt}\, dt + \int_{\pi}^{2\pi} (-1)\, e^{-jkt}\, dt\right]$$

$$= \frac{1}{2\pi}\left[\left.\frac{e^{-jkt}}{-jk}\right|_0^{\pi} + (-1)\left.\frac{e^{-jkt}}{-jk}\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-jk\pi}-1}{-jk} - \left(\frac{e^{-jk\cdot 2\pi} - e^{-jk\pi}}{-jk}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-jk\pi}-1}{-jk} - \left(\frac{1-e^{-jk\pi}}{-jk}\right)\right] \qquad e^{-jk\cdot 2\pi} = 1$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{(e^{-jk\pi}-1)\cancel{2}}{-jk}\right] = \frac{1-e^{-jk\pi}}{jk\pi}$$

$$e^{-jk\pi} = \cos(\pi k) - j\sin(\pi k)$$
$$= \cos(\pi k) - 0$$
$$= \cos(\pi k)$$

Hence

$$X_k = \frac{1-\cos(\pi k)}{jk\pi} = \frac{1-(-1)^k}{jk\pi}$$

$$X_0 = \frac{1}{T_0}\int x(t)\,dt = \frac{1}{T_0}\left[\int_0^{\pi} 1\cdot dt + \int_{\pi}^{2\pi}(-1)\,dt\right]$$

$$= \frac{1}{T_0}\left[(\pi-0) + (-1)(2\pi-\pi)\right] = 0$$

Hence $X_k = \begin{cases} 0 \\ \dfrac{1-(-1)^k}{jk\pi} & k \neq 0 \end{cases}$

$$= \begin{cases} 0 & k \text{ is even} \\ \dfrac{-2j}{\pi k} & k \text{ is odd.} \end{cases}$$

(b) Using Laplace transform

the Laplace transform of one period is

$$X_1(s) = \frac{1}{s}\left(1 - 2e^{-\pi s} + e^{-2\pi s}\right)$$

$$X_k = \frac{1}{2\pi} X_1(s)\Big|_{s=jk} = \frac{1}{2\pi jk}\left(1 - 2e^{-\pi \cdot jk} + e^{-2\pi \cdot jk}\right)$$

$$= \frac{1}{2\pi jk}\left(1 - 2e^{-j\pi k} + 1\right)$$

$$= \frac{1}{\pi jk}\left(1 - e^{-j\pi k}\right)$$

which is what we got earlier too.

X_0 can be calculated using Lohiptal Rule

$$X_0 = \lim_{k \to \infty} \frac{1}{\pi jk}\left(1 - e^{-j\pi k}\right)$$