# CPE 381: Fundamentals of Signals and Systems for Computer Engineers

## 10 Z-transform


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# Outline

# Motivation

# Z-transform and Laplace Transform

The discrete equivalent of Laplace Transform that follows:

- If the sampled signal is: $x(t) = \sum x(nT_s)\delta(t - nT_s)$
  its Laplace transform is : $X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n x(nT_s) = e^{-nsT_s}$
- By letting $z = e^{sT_s}$,

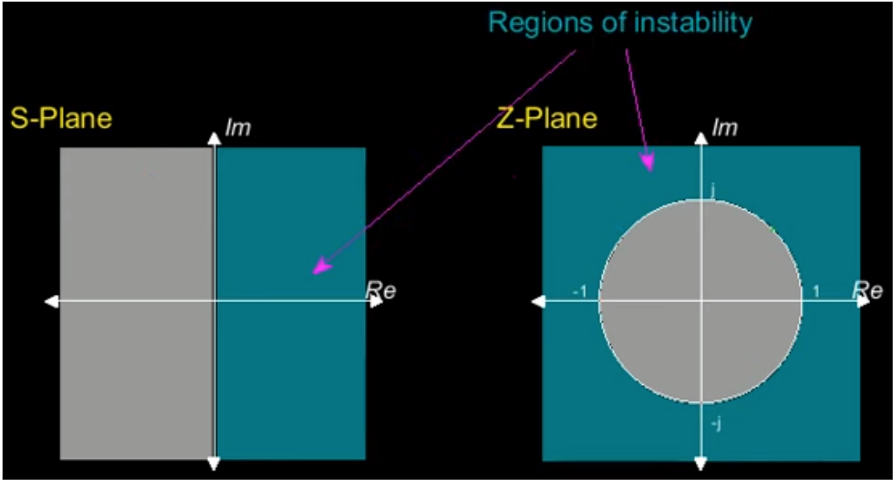
$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n}$$

**which is called the Z-transform of the sampled signal.**

# Z-transform in Practice

Although Z-transform seems to be a discrete version of Laplace transform, the practical use doesn't necessarily bother to find the relationship between Laplace and Z-transform.

- Complex Z-plane is a polar form rather than a 2D cartesian plane.
- Radius corresponds to the damping factor, and angle corresponds to the discrete frequency $\omega$ in radians.
- Thus, the unit circle in the Z-plane is analogous to the $j\Omega$-axis in the Laplace plane.
- Inside of the unit circle is analogous to the left-hand $s$-plane.

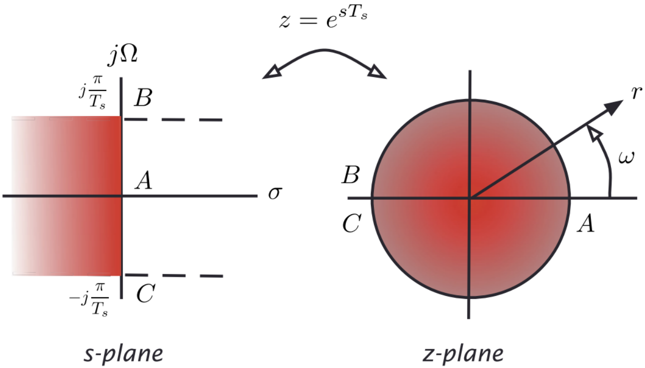



See: Source:https://www.youtube.com/watch?v=acQecd6dmxw

# Relationship between s-plane and z-plane

By letting $z = e^{sT_s}$,

- $z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} = e^{\sigma T_s} e^{j\Omega T_s}$
- $r = e^{\sigma T_s}$, $\omega = \Omega T_s$, and hence, $z = re^{j\omega}$.



# Example

To see the possibility of infinite zeros and poles in the Laplace transform of a sampled signal, consider a pulse $x(t) = u(t) - u(t - T_0)$ sampled with a sampling period $T_s = \frac{T_0}{N}$, $N$ being a positive integer.

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# Two-sided Z-transform

# Two sided Z-transform

Given a discrete-time signal $x[n]$, $-\infty < n < \infty$, its two-sided Z-transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

defined in a region of convergence (ROC) on the Z-plane.

# Remark on Z-transform

The two-sided Z-transform is not useful in solving difference equations with nonzero initial conditions, just as the two-sided Laplace transform was not useful either in solving ordinary differential equations with nonzero initial conditions.

Hence, we need to look at one-sided Z-transform.

# One-sided Z-transform

Defined for causal signals, $x[n] = 0$ for $n < 0$

$$X_1(z) = \mathcal{Z}(x[n]u[n]) = \sum_{n}^{\infty} x[n]u[n]z^{-n}$$

The ROC for this z-transform is denoted by $\mathcal{R}_1$.

# Two-sided Z-transform as One-sided Z-transform

$X(z) = \mathcal{Z}(x[n]u[n]) + \mathcal{Z}(x[-n]u[n])|_z - x[0]$.
The first z-transform has ROC $\mathcal{R}_1$, and the second z-transform has ROC $\mathcal{R}_2$.
Proof:

$$
\begin{aligned}
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} &= \sum_{n=0}^{\infty} x[n]u[n]z^{-n} + \sum_{-\infty}^{0} x[n]u[-n]z^{-n} - x[0] \\
&= \mathcal{Z}(x[n]u[n]) + \sum_{m=0}^{\infty} x[-m]u[m]z^{m} - x[0] \\
&= \mathcal{Z}(x[n]u[n]) + \mathcal{Z}(x[-n]u[n])|_z - x[0]
\end{aligned}
$$

# Region of Convergence for Z-transform

Infinite summation we see in the Z-transform has to converge under some conditions for it to be meaningful. It means

$$|X(z)| = \left| \sum_{n} x[n] z^{-n} \right| \leq \sum_{n} |x[n]| |r^{-n} e^{j\omega n}| = \sum_{n} |x[n]| r^{-n} < \infty$$

We see that for a given discrete signal or sequence, the convergence depends on the value of $r$.

# Poles and Zeros of Z-transform

The poles of a Z-transform $X(z)$ are complex values $\{p_k\}$ such that

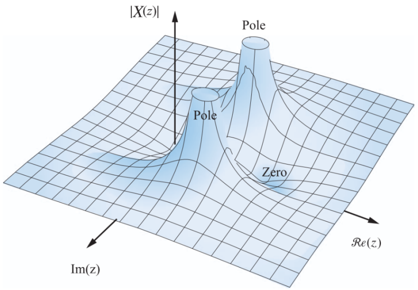
$$X(p_k) \to \infty$$

The zeros of $X(z)$ are the complex values $\{z_k\}$ that make

$$X(z_k) = 0$$

# Visualization of Z-transform

The magnitude $|X(z)|$ of the z-transform represents a surface in the z-plane. There are two zeros at $z_1 = 0$, $z_2 = 1$ and two poles at $p_{1,2} = 0.9e^{\pm j\pi/4}$.



# Example

Find the poles and zeros of the following Z-transforms

- (i) $X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$
- (ii) $X_2(z) = \dfrac{(z^{-1} - 1)(z^{-1} + 2)^2}{z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1)}$

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# ROC of Finite-Support Signals

The region of convergence (ROC) of the Z-transform of a signal $x[n]$ of a finite support $[N_0, N_1]$, where $-\infty < N_0 \leq n \leq N_1 < \infty$,

$$X(z) = \sum_{n=N_0}^{N_1} x[n] z^{-n}$$

is the whole Z-plane, excluding the origin $z = 0$, and/or $z = \pm\infty$, depending on $N_0$ and $N_1$.

# Example

Find the Z-transform of a discrete-time pulse $x[n] = u[n] - u[n-10]$. Determine the region of convergence of $X(z)$.

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# ROC of Infinite-Support Signals

Signals of infinite support are either causal, anticausal or a combination of these-noncausal. For the Z-transform of a causal signal $x_c[n]$, i.e., $x_c[n] = 0, n < 0$.

$$X_c(z) = \sum_{n=0}^{\infty} x_c[n] z^{-n} = \sum_{n=0}^{\infty} x_c[n] r^{-n} e^{-jn\omega}$$

to converge we need to determine appropriate values of $r$, the damping factor, as the frequency $\omega$ has no effect on the convergence.

# ROC of Infinite-Support Signals for a Causal Signal

If R1 is the radius of the farthest out pole of $X_c(z)$, then there is an exponential $R_1^n u[n]$ such that $|x_c[n]| < MR_1^n$ for $n \geq 0$. Then, for$X(z)$ to converge we need that

$$|X_c(z)| \leq \sum_{n=0}^{\infty} |x_c[n]| r^{-n} < M \sum_{n=0}^{\infty} \left(\frac{R_1}{n}\right)^n < \infty$$

or that $\dfrac{R_1}{r} < 1$, which is equivalent to $|z| = r|R_1$. ROC is thus the outside of a circle containing all the poles of $X_c(z)$, i.e., it does not include any poles of $X_c(z)$.

# ROC of Infinite-Support Signals for an Anti-causal Signal

Likewise, for an anti-causal signal $x_a[n]$ (i.e., $x_a[n] = 0$, for $n > 0$) if we choose a radius $R_2$ which is smaller than the radius of all the poles of $X_a(z)$, the region of convergence is $|z| = r < R_2$. This ROC is the inside of a circle that does not include any poles of $X_a(z)$.

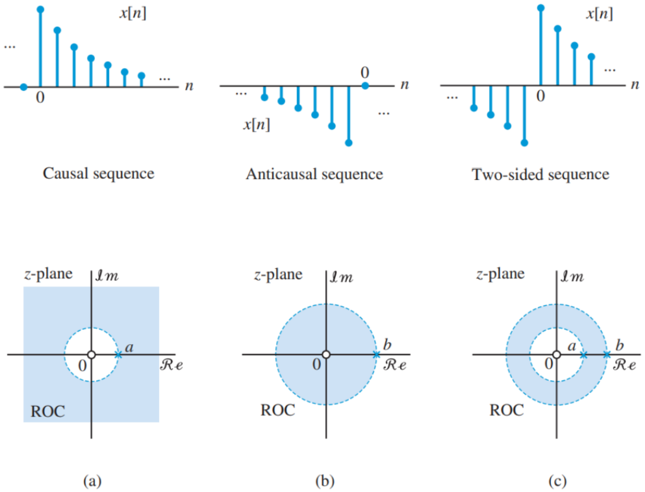
# ROC of Infinite-Support Signals for a Non-causal (Two-sided) Signal

If the signal$x[n]$ is noncausal, it can be expressed as

$$x[n] = x_c[n] + x_a[n]$$

$x_c[n]$ is causal whereas $x_a[n]$ is anti-causal.
The corresponding ROC of $X(z)$ would be $0 < R_1 < |z| < R_2 < \infty$.

# ROC Visually



# Example 1: ROC

Consider the sequence $x[n] = a^n u[n]$, $a$ is real, find the z-transform and sketch its ROC.

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# Example 2: ROC

Consider the sequence $x[n] = -a^n u[-n-1]$, $a$ is real, find the z-transform, and sketch its ROC.

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# Properties of ROC

- The ROC does not contain any poles.
- If $x[n]$ is a finite sequence (that is, $x[n] = 0$ except in a finite interval $N_1 \leq n \leq N_2$, where $N_1$, and $N_2$, are finite) and $X(z)$ converges for some value of $z$, then the ROC is the entire z-plane except possibly $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence (that is, $x[n] = 0$ for $n < N_1 < \infty$) and $X(z)$ converges for some value of $z$, then the ROC is of the form

$$|z| > r_{max} \quad \text{or} \quad \infty > |z| > r_{max}$$

where $r_{max}$ equals the largest magnitude of any of the poles of $X(z)$. Thus, the ROC is the exterior of the circle $|z| = r_{max}$, in the z-plane with the possible exception of $z = \infty$.

# Properties of ROC

- If $x[n]$ is a left-sided sequence (that is, $x[n] = 0$ for $n > N_2 > -\infty$) and $X(z)$ converges for some value of $z$, then the ROC is of the form

$$|z| < r_{min} \quad \text{or} \quad 0 < |z| < r_{min}$$

where $r_{min}$ equals the smallest magnitude of any of the poles of $X(z)$. Thus, the ROC is the interior of the circle $|z| = r_{min}$, in the z-plane with the possible exception of $z = 0$.

- If $x[n]$ is a two-sided sequence (that is, $x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of $z$, then the ROC is of the form

$$r_1 < |z| < r_2$$

where $r_1$ and $r_2$ are the magnitudes of the two poles of$X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z| = r_1$ and $|z| = r_2$ not containing any poles.

# Example

The poles of $X(z)$ are $z = 0.5$ and $z = 2$; find all the possible signals that can be associated with $X(z)$ according to different regions of convergence.

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# Example

Find the regions of convergence of the Z-transforms of the following signals

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

Find then the Z-transform of $x_1[n] + x_2[n]$.

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# Derivative Property

If $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$
its derivative with respective to z is

$$\frac{dX(z)}{dz} = \sum_{n}^{\infty} x[n]\frac{dz^{-n}}{dz} = -z^{-1}\sum_{n}^{\infty} nx[n]z^{-n}$$

or

$$nx[n]u[n] \longleftrightarrow -z\frac{dX(z)}{dz}$$

# Discrete Transfer Function

# Transfer Function in Z-domain

The output $y[n]$ of a causal LTI system is calculated using the convolution sum

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$$

where $x[n]$ is a causal input and $h[n]$ is the impulse response of the system. Z-transform of the $y[n]$ is the product

$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)$$

and thus the transfer function of a discrete system is defined as

$$H(z) = \frac{Y(z)}{X(z)}$$

# Convolution Sum

Fun video: https://www.youtube.com/watch?v=KuXjwB4LzSA

**What is convolution sum?**

The convolution sum property can be seen as a way to obtain the coefficients of the product of two polynomials. s. Whenever we multiply two polynomials $X_1(z)$ and $X_2(z)$, of finite or infinite order, the coefficients of the resulting polynomial can be obtained using the convolution sum.

# Example

Consider computing the output of an FIR filter

$$y[n] = \frac{1}{2}(x[n] + x[n-1] + x[n-2])$$

for an input $x[n] = u[n] - u[n-4]$ using the convolution sum, analytically and graphically, and the Z-transform.

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# Time-shift Property

If

$$x[n] \Leftrightarrow X(z)$$

then

$$x[n-N] \Leftrightarrow z^{-N}X(z) + x[-1]z^{-N+1}$$

For a causal system $x[-1] = 0$, then

$$x[n-N] \Leftrightarrow z^{-N}X(z)$$

# Example

Consider a discrete-time IIR system represented by the difference equation

$$y[n] = 0.5y[n-1] + x[n]$$

with $x[n]$ as input and $y[n]$ as output. Determine the system's transfer function and, from it, find the impulse and unit-step responses. Determine under what conditions the system is BIBO stable. If stable, determine the system's transient and steady-state responses.

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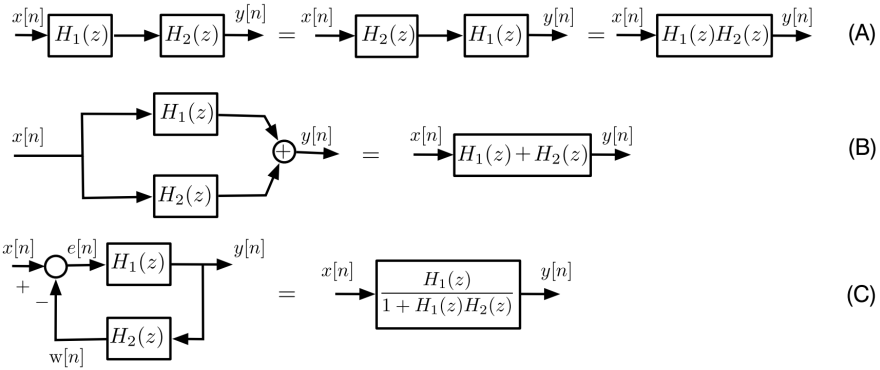
# Interconnection of Discrete Systems

Connections of LTI systems: (A) cascade, (B) parallel, and (C) negative feedback.

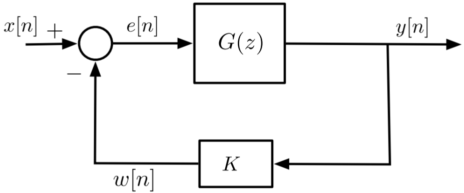


# Initial Value and Final Value Theorem for Z-transform

If $X(z)$ is the Z-transform of a causal signal $x[n]$ with an initial value $x[0]$ and a final value $lim_{n\to\infty} x[n]$ are obtained from $X(z)$ according to

- Initial value: $x[0] : \lim_{z\to\infty} X(z)$
- Final value: $lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

# Example




Consider a negative feedback connection of a plant given in the figure with a transfer function

$$G(z) = \frac{1}{1 - 0.5z^{-1}}$$

and a constant feedback gain of $K$.
If the reference signal is a unit-step, $x[n] = u[n]$, determine the behavior of the error signal $e[n]$. What is the effect of the feedback, from the error point of view, on an unstable plant $G(z) = \frac{1}{1-z^{-1}}$

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# One-Sided Z-transform Inverse

# Long Division Method

When a rational function $X(z) = B(z)/A(z)$, having as ROC the outside of a circle of radius R (i.e., $x[n]$ is causal), is expressed as

$$X(z) = x[0] + x[1]z^{-1} + x[2]^{-2} + \cdots$$

by dividing $B(z)$ by $A(z)$, then the inverse is the sequence $\{\cdots 0, 0, x[0], x[1], x[2], \cdots\}$ or

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

# Example

Find the inverse Z-transform of

$$X(z) = \frac{1}{1 + 2z^{-2}}, \quad |z| > \sqrt{2}$$

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# Partial Fraction Method with Example

Consider the non-proper rational function

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}$$

(numerator and denominator of the same degree in powers of $z^{-1}$).
Determine how to obtain an $X(z)$ expansion containing a proper rational term to find $x[n]$.

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# Example

Find the inverse Z-transform of

$$X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{z(z+1)}{(z+0.5)(z-0.5)}, \quad |z| > 0.5$$

by using the negative and the positive powers of z representations.

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# One-Sided Z-transforms Pair Table

| **Table 10.1 One-sided Z-transforms** | |
|---|---|
| $\delta[n]$ | 1, whole z-plane |
| $u[n]$ | $\dfrac{1}{1-z^{-1}}, \quad \lvert z\rvert > 1$ |
| $nu[n]$ | $\dfrac{z^{-1}}{(1-z^{-1})^2}, \quad \lvert z\rvert > 1$ |
| $n^2u[n]$ | $\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}, \quad \lvert z\rvert > 1$ |
| $\alpha^n u[n], \quad \lvert\alpha\rvert < 1$ | $\dfrac{1}{1-\alpha z^{-1}}, \quad \lvert z\rvert > \lvert\alpha\rvert$ |
| $n\alpha^n u[n], \quad \lvert\alpha\rvert < 1$ | $\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad \lvert z\rvert > \lvert\alpha\rvert$ |
| $\cos(\omega_0 n)u[n]$ | $\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$ |
| $\sin(\omega_0 n)u[n]$ | $\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$ |
| $\alpha^n \cos(\omega_0 n)u[n], \quad \lvert\alpha\rvert < 1$ | $\dfrac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$ |
| $\alpha^n \sin(\omega_0 n)u[n], \quad \lvert\alpha\rvert < 1$ | $\dfrac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$ |

# Basic Properties of One-sided Z-transform

**Table 10.2 Basic properties of one-sided Z-transform**

| | | |
|---|---|---|
| Causal signals | $\alpha x[n],\ \beta y[n]$ | $\alpha X(z),\ \beta Y(z)$ |
| Linearity | $\alpha x[n] + \beta y[n]$ | $\alpha X(z) + \beta Y(z)$ |
| Convolution sum | $\sum_k x[n]y[n-k]$ | $X(z)Y(z)$ |
| Time-shifting | $x[n-N],\ N > 0$ | $z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N]$ |
| Time reversal | $x[-n]$ | $X(z^{-1})$ |
| Multiplication | $n\ x[n]$ | $-z\dfrac{dX(z)}{dz}$ |
| | $n^2\ x[n]$ | $z^2\dfrac{d^2X(z)}{dz^2} + z\dfrac{dX(z)}{dz}$ |
| Finite difference | $x[n] - x[n-1]$ | $(1-z^{-1})X(z) - x[-1]$ |
| Accumulation | $\sum_{k=0}^{n} x[k]$ | $\dfrac{X(z)}{1-z^{-1}}$ |
| Initial value | $x[0]$ | $\lim_{z\to\infty} X(z)$ |
| Final value | $\lim_{n\to\infty} x[n]$ | $\lim_{z\to 1}(z-1)X(z)$ |

# Solution of Difference Equations

# Example

The following first-order difference equation represents a discrete IIR system

$$y[n] = ay[n-1] + x[n], \quad n \geq 0$$

where $x[n]$ is the system's input and $y[n]$ its output. Discuss how to solve it using recursive methods and the Z-transform. Obtain a general form for the complete solution $y[n]$ in terms of the impulse response $h[n]$ of the system.

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# Example

Solve the difference equation

$$y[n] = y[n-1] - 0.25y[n-2] + x[n], \quad n \geq 0$$

with zero initial conditions and $x[n] = u[n]$.

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# Approximate Solution of Ordinary Differential Equations

The solution of ordinary differential equations requires converting them into difference equations, which can then be solved in closed form by means of the Z-transform.

# Example

Consider an RLC circuit represented by the second-order ordinary differential equation

$$\frac{d^2 v_c(t)}{dt^2} + \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

where the voltage across the capacitor $v_c(t)$ is the output and the source $v_s(t) = u(t)$ is the input. Let the initial conditions be zero. Approximate the derivatives by their definition, find and solve the resulting difference equation.

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