CPE 381: Fundamentals of Signals and Systems for Computer Engineers

06 Frequency Analysis: Fourier Series

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Outline

From Laplace to Fourier

Recall:

$$F(p) = \int_a^b K(t,p) f(t) dt$$

where $K(t,p)$ is a Kernel function.
For Laplace transform, e^{-st} is the kernel function where $s = \sigma + j\Omega$.

If we set $\sigma = 0$, we get **Fourier Transform**.
We will come back to it later.

Complex Exponentials and Frequency Representation

Frequency Representation of a Signal

Normally we think signals as a function of time.

In the 19th century, Joseph Fourier, a French mathematician showed in his work about heat flow that represents a signal as a sum of sinusoids.

This idea gave rise to what is now known as the frequency domain, where we think of signals as a function of frequency.

Spectrum of a Signal

How the power or energy of a signal is distributed over different frequency components is called the **Spectrum** of the Signal.

A periodic signal's spectrum is discrete.

For an aperiodic signal, its spectrum is continuous.

Frequency representation of an LTI system

- Frequency response (related to the transfer function) determines how an LTI system responds to sinusoids of different frequencies.
- Permits computation of steady-state response.

$$x(t) \;=\; A\sin(2\pi f t + \phi)$$

Sinusoids

Fourier Analysis Vs Laplace Analysis

- Fourier Analysis: Steady State (Communication Systems, Filter Design)
- Laplace Analysis: Steady State + Transient State (e.g. Control Theory)

Recall Impulse Response and Transfer Function

Transfer Function – Impulse Response Relationship

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-\tau s} d\tau$$

If we set $s = j\Omega_0$ (i.e. $\sigma = 0$), we get the frequency response of the system at Ω_0.

Now think of Inverse Laplace Transfer

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

If we discretize the above and put $s = j\Omega_k$

$$x(t) = \sum_k X_k e^{j\Omega_k t}$$

where X_k is a complex value.

If we have a system with frequency response $H(j\Omega_k)$, then the output of the system is

$$y(t) = \sum_k X_k e^{j\Omega_k t} H(j\Omega_k)$$

Hence, we can write any signal as linear combination of complex exponentials.

Some Remarks

- Stability of an LTI system is necessary to ensure that $H(j\Omega)$ exists for all frequencies.
- For sinusoid input,
  $x(t) = A\cos(\Omega_0 t + \theta)$, the steady-state output is given by

$$y_{ss}(t) = \frac{Ae^{j\theta}}{2} e^{j\Omega_0 t} H(j\Omega_0) + \frac{Ae^{-j\theta}}{2} e^{-j\Omega_0 t} H(-j\Omega_0) =$$

$$= A|H(j\Omega_0)|\cos(\Omega_0 t + \theta + \angle H(j\Omega_0))$$

Complex Exponential Fourier Series

Fourier Series as a Representation of a Periodic Signal using Complex Exponentials

- Helps in spectral characterization
- Mathematically, the Fourier series is an expansion of periodic signals in terms of normalized orthogonal complex exponentials.

Orthonormal Functions

Consider a set of complex functions $\psi_k(t)$defined in an interval $[a, b]$, and such that for any pair of these functions, let us say $\psi_\ell(t)$ and $\psi_m(t)$, then the inner product of $\psi_\ell(t)$ and $\psi_m(t)$ is

$$\int_a^b \psi_\ell(t)\psi_m^*(t)dt = \begin{cases} 0, & \ell \neq m \\ 1, & \ell = m \end{cases}$$

Such functions are called orthonormal (orthogonal + normalized).

A function x(t) can be approximated as sum of orthonormals

$$\hat{x} = \sum_k \alpha_k \psi_k(t)$$

We minimize the total error as $\varepsilon(t) = x(t) - \hat{x}(t)$

$$\int_a^b |\varepsilon(t)|^2 dt = \int_a^b \left| x(t) - \sum_k \alpha_k \psi_k(t) \right|^2 dt$$

Periodic Function's Fourier Series

We can see that one such orthonormal functions are exponentials.
If we consider periodic functions with period T_0, then

$$x(t) = \int_{-\infty}^{\infty} X_k e^{j\Omega_0 t}, \quad \Omega_0 = 2\pi/T_0$$

Fourier coefficients:

$$X(k) = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Fourier Functions are Orthonormal over a Period

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{-jk\Omega_0 t} \times (e^{-j\ell\Omega_0 t})^* dt = \begin{cases} 0, & \ell \neq k \\ 1, & \ell = k \end{cases}$$

An Interesting Video on Fourier Series

https://www.youtube.com/watch?v=ds0cmAV-Yek

Fourier Series to Represent Periodic Signal

$$x(t) = \sum_k X_k e^{jk\Omega_0 t}, \quad \Omega_0 = 2\pi T_0$$

Example

Find the exponential Fourier series of a raised cosine signal ($B \geq A$),

$$x(t) = B + A\cos(\Omega_0 t + \theta)$$

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Power Distribution over Frequency

The power spectrum provides information as to how the power of the signal is distributed over the different frequencies present in the signal.

Periodic signals are infinite energy signals, they have finite power.

Parseval's Theorem for Power

The power of a periodic signal $x(t)$ of fundamental period T_0 is given by

$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt$$

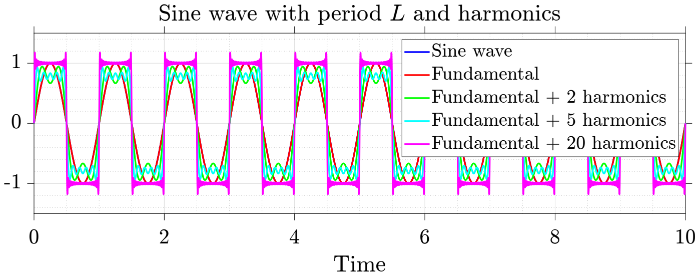
Replacing the Fourier series of x(t) in the power equation we have:

$$\begin{aligned}
\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} X_k X_m^* e^{j\Omega_0(k-m)t} dt \\
&= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} X_k X_m^* \frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{j\Omega_0(k-m)t} dt = \sum_{k=-\infty}^{\infty} |X_k|^2
\end{aligned}$$

Harmonics

Definition

Harmonics with respect to Fourier series and analysis means the sine and cosine components that constitute a function, or to put more simply, the simplest functions that a given function can be broken down into.



Signals as Sum of Harmonics

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

where $x_k(t) = X_k e^{jk\Omega_0 t}$
The power of each of these components $x_k(t)$ is given by

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x_k(t)|^2 dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |X_k e^{jk\Omega_0 t}|^2 dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |X_k|^2 = |X_k|^2$$

The plot of $|X_k|^2$ versus the harmonics displays how the power of the signal is distributed over the harmonics.

Line Spectra

Line spectra refer to a graphical representation of the frequency content of a signal, where the frequency axis is discrete and the amplitude axis represents the magnitude of the signal at each frequency.

Line Spectra are Symmetrical

- $|X_k| = |X_{-k}|$: magnitude $|X_k|$ is an even function of $k\Omega_0$.
- $\angle X_k = -\angle X_{-k}$: phase $\angle X_k$ is an odd function of $k\Omega_0$.

Trigonometric Fourier Series: Fourier Series using Sinusoids

For orthogonality in terms of sinusoids:

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{-jk\Omega_0 t} \times (e^{-j\ell\Omega_0 t})^* dt = 0$$

$$\Rightarrow \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \cos((k-\ell)\Omega_0 t)dt + j\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sin((k-\ell)\Omega_0 t)dt = 0$$

We can use Trigonometric Identities to expand the above:

$$\sin(\alpha)\sin(\beta) = 0.5[\cos(\alpha-\beta) - \cos(\alpha+\beta)], \quad \cos(\alpha)\cos(\beta) = 0.5[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

It shows that cosine and sine functions are orthogonal when k and ℓ are not equal to each other.

Back to Exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

- $|X_k| = |X_{-k}|$: magnitude $|X_k|$ is an even function of $k\Omega_0$.
- $\angle X_k = -\angle X_{-k}$: phase $\angle X_k$ is an odd function of $k\Omega_0$.

Then, we separate them out:

$$\begin{aligned} x(t) &= X_0 + \sum_{k=1}^{\infty}[X_k e^{jk\Omega_0 t} + X_{-k} e^{-jk\Omega_0 t}] \\ &= X_0 + \sum_{k=1}^{\infty}\left[|X_k| e^{jk\Omega_0 t + \theta_k} + X_{-k} e^{-jk\Omega_0 t - \theta_k}\right] = X_0 + 2\sum_{k=1}^{\infty}|X_k|\cos(k\Omega_0 t + \theta_k) \end{aligned}$$

Alternative Formula

$$X_k = X^*_{-k}$$
$$z = a + jb$$
$$z + z^* = (a + jb) + (a - jb) = 2\,\mathrm{Re}(z)$$

$$\begin{aligned}
x(t) &= X_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}[X_k e^{jk\Omega_0 t + \theta_k}] \\
&= X_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}[X_k]\cos(k\Omega_0 t) - 2\,\mathrm{Im}[X_k]\sin(k\Omega_0 t) \\
&= X_0 + 2\sum_{k=1}^{\infty} (c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t))
\end{aligned}$$

$$|X_k| = \sqrt{c_k^2 + d_k^2}$$
$$\theta_k = -\tan^{-1}\frac{d_k}{c_k}$$

Finding Fourier Coefficients

For a periodic signal with period T_0:

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}dt$$

for any time t_0.

Fourier Coefficients using Laplace Transform

If we can calculate the Laplace transform for one period of a $x(t)$, we can easily calculate the Fourier coefficients.
The equation for one period:

$$x_1(t) = x(t)[u(t-t_0) - u(t-t_0-T_0)], \quad \text{for any} \quad t_0$$

$s = jk\Omega_0$

$$X_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x_1(t)e^{-st}dt \Rightarrow X_k = \frac{1}{T_0}\mathcal{L}[x_1(t)]\Big|_{s=jk\Omega_0}$$

$\Omega_0 = \dfrac{2\pi}{T_0}$ is **the fundamental frequency.**

Fourier Series of Reflected Signal

$$x(-t) = \sum_{m} X_m e^{-jm\Omega_0 t} = \sum_{k} X_{-k} e^{jk\Omega_0 t}$$

Even Signals

For even signal, we have $x(t) = x(-t)$ such that $X_k = X_{-k}$.
Also, we saw in a previous slide, in general: $X_k = X^*_{-k} = X_{-k}$ which basically means that coefficients are real-valued for even signals.

Hence, writing down the DC term separately, and the negative and positive indices terms separately:

$$
\begin{aligned}
x(t) &= X_0 + \sum_{-\infty}^{-1} X_k e^{jk\Omega_0 t} + \sum_{-\infty}^{-1} X_k e^{jk\Omega_0 t} + \sum_{k=1}^{\infty} X_k e^{jk\Omega_0 t} \\
&= X_0 + \sum_{1}^{\infty} X_k [e^{jk\Omega_0 t} + e^{-jk\Omega_0 t}] = X_0 + 2\sum_{1}^{\infty} X_k \cos(k\Omega_0 t)
\end{aligned}
$$

Odd Signals

Similarly, for odd signals, $x(t) = -x(-t)$, and $X_k = -X_{-k}$ which results in

$$x(t) = 2\sum_{1}^{\infty} jX_k \sin(k\Omega_0 t)$$

Even and Odd Signals

From our previous discussion, and knowing that any signal can be broken down into even and odd signals, $x(t) = x_e(t) + x_o(t)$,
following holds, and we derived that differently earlier:

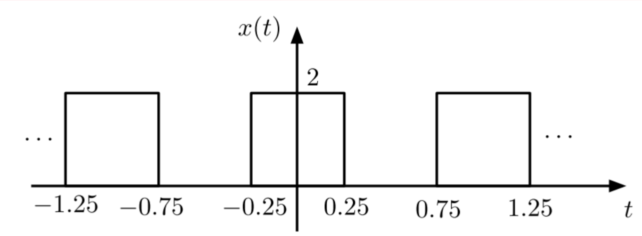
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} = X_0 + 2\sum_{k=1}^{\infty}(c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t))$$

As, cosine is even and sine is odd, effectively we can write the signal as the sum of odd and even signals.

$$X_k = X_{ek} + X_{ok}$$
$$X_{ek} = 0.5[X_k + X_{-k}]$$
$$X_{ok} = 0.5[X_k - X_{-k}]$$

Example

Find the Fourier Series of Period Pulse Train with $T_0 = 1$.



Start with calculating the fundamental frequency.

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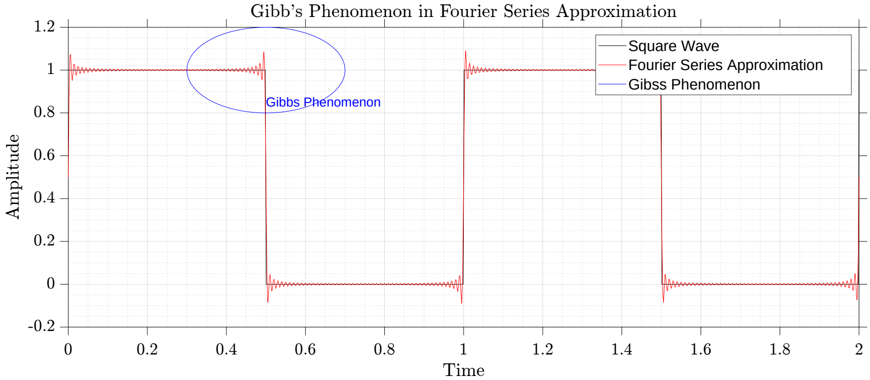
Convergence of Fourier Series

For the Fourier series to converge to the periodic signal $x(t)$, the signal should satisfy the following sufficient (not necessary) conditions over a period:

- be absolutely integrable
- has a finite number of maxima, minima and discontinuities

Gibb's Phenomenon.

Although the Fourier series converges to the arithmetic average at discontinuities, it can be observed that there is some ringing before and after the discontinuity points. This is called the Gibb's phenomenon.



Time and Frequency Shifting

If X_k are the Fourier coefficients of $x(t)$, then for $x(t-t_0)$, $x(t)$ delayed t_0 seconds, its Fourier series coefficients can be determined as follows:

- Fundamental frequency is Ω_0.

$$x(t) = \sum_k X_k e^{jk\Omega_0 t}$$

$$x(t-t_0) = \sum_k X_k e^{jk\Omega_0(t-t_0)} = \sum_k [X_k e^{-jk\Omega_0 t_0}] e^{jk\Omega_0 t}$$

$$x(t+t_0) = \sum_k X_k e^{jk\Omega_0(t-t_0)} = \sum_k [X_k e^{jk\Omega_0 t_0}] e^{jk\Omega_0 t}$$

We see that only a change in phase is caused by the time shift; the magnitude spectrum remains the same.

Centering around $\pm\Omega_1$

We multiply the original signal by a cosine signal to make it real-valued and centered around $\pm\Omega_1$.

$$y_1(t) = x(t)\cos(\Omega_1 t) = \sum_k 0.5X_k[e^{j(k\Omega_0+\Omega_1)t} + e^{j(k\Omega_0-\Omega_1)t}]$$

Response of LTI Systems to Periodic Signal

$$x(t) = \sum_k X_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

The output in the steady state, if the impulse response is $h(t)$:

$$y(t) = \sum_{k=-\infty}^{\infty} [X_k H(jk\Omega_0)] e^{jk\Omega_0 t}$$

Fourier Coefficients of $y(t)$ is $Y_k = H_k(jk\Omega_0)$. As we write $x(t) = \sum_k X_k e^{jk\Omega_0 t} = X_0 + \sum_{k=1}^{\infty} 2|X_k| \cos(k\Omega_0 t + \angle X_k)$, we can write the steady-state output $y(t)$ as $y(t) = X_0 |H(j0)| + 2 \sum_{k=1}^{\infty} 2|X_k||H(jk\Omega_0)| \cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$

Filtering of Periodic Signals

What is Filter?

A filter is an LTI system that allows us to retain, get rid of, or attenuate frequency components of the input, i.e., to “filter” the input.
$y(t) = X_0|H(j0) + 2\sum_{k=1}^{\infty}|X_k||H(jk\Omega_0)|\cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$

- Keeping a certain frequency: $|H(j\ell\Omega_0)| = 1$
- Removing a certain frequency: $|H(j\ell\Omega_0)| = 0$

Operations using Fourier Series

Addition

If

$$z(t) = \alpha x(t) + \beta y(t)$$

for constants α and β, then,

$$Z_k = \alpha X_k + \beta Y_k$$

Case of Different Fundamental Frequencies

If $x(t)$ is periodic with fundamental period of T_1, and $y(t)$ has fundamental period of T_2 such that $T_2/T_1 = N/M$ for non-divisible integer N, and M, then $z(t) = \alpha x(t) + \beta y(t)$ is periodic with fundamental period $T_0 = MT_2 = NT_1$, and its Fourier coefficients are

$$Z_k = \alpha X_{k/N} + \beta Y_{k/M}$$

for $k = 0, \pm 1, \pm 2, \ldots$ such that k/N, and k/M are integers,
where X_k, and Y_k are the Fourier coefficients of $x(t)$, and $y(t)$.

Multiplication

$$z(t) = x(t)y(t)$$

Fourier coefficients are the convolution sum of the Fourier coefficients of $x(t)$ and $y(t)$:

$$Z_k = \sum_m X_m Y_{k-m}$$

$$x(t)y(t) = z(t) = \sum_m X_m e^{jm\Omega_0 t} \sum_\ell Y_\ell e^{j\ell\Omega_0 t} = \sum_m \sum_\ell X_m Y_\ell e^{(m+\ell)\Omega_0 t}$$
$$= \sum_k \left[\sum_m X_m Y_{k-m} \right] e^{jk\Omega_0 t}$$

Derivatives

Fourier Coefficients of $\frac{dx(t)}{dt}$ is $jk\Omega_0 X_k$.

$$x(t) = \sum_k X_k e^{jk\Omega_0 t},$$

then

$$\frac{dx(t)}{dt} = \sum_k X_k \frac{de^{jk\Omega_0 t}}{dt} = \sum_k [jk\Omega_0 X_k] e^{jk\Omega_0 t}$$

Integral

For a zero-mean, periodic signal $y(t)$, let $z(t) = \int_{-\infty}^{t} y(\tau)d\tau$, we have Fourier coefficients as $Z_k = \dfrac{Y_k}{jk\Omega_0}$, $k \neq 0$, $Z_0 = -\sum_{m\neq 0} Y_m \dfrac{1}{jm\Omega_0}$. Derivation:

$$z(t) = \int_{-\infty}^{t} y(\tau)d\tau = \int_{-\infty}^{MT_0} y(\tau)d\tau + \int_{MT_0}^{t} y(\tau)d\tau = 0 + \int_{MT_0}^{t} y(\tau)d\tau$$

Replacing $y(t)$ by its Fourier series gives

$$\begin{aligned} z(t) &= \int_{MT_0}^{t} \sum_{k\neq 0} Y_k e^{jk\Omega_0\tau} d\tau = \sum_{k\neq 0} \int_{MT_0}^{t} e^{jk\Omega_0\tau} d\tau \\ &= \sum_{k\neq 0} Y_k \frac{1}{jk\Omega_0}[e^{jk\Omega_0 t} - 1] = -\sum_{k\neq 0} Y_k \frac{1}{jk\Omega_0} + \sum_{k\neq 0} Y_k \frac{1}{jk\Omega_0} e^{jk\Omega_0 t} \end{aligned}$$

Implications of Derivatives and Integrations

1. The derivative of a periodic signal $x(t)$ enhances its higher harmonics.
2. The integration of a zero-mean periodic signal $x(t)$ smooths out the signal.

Amplitude and Time Scaling

$$x(t) = \sum_k X_k e^{jk\Omega_0 t},$$

has period T_0.

$$y(t) = Ax(\alpha t) = \sum_{k=-\infty}^{\infty} (AX_k) e^{jk\alpha\Omega_0 t}, \quad T_1 = 2\pi/\Omega_1 = T_0/\alpha$$

for amplitude A and scaling factor $\alpha \neq 0$.

1. If $\alpha > 1$, then $x(\alpha t)$ is compressed, and the fundamental frequencies are expanded.
2. If only time scaling is done, i.e., $A = 1$, for $\alpha > 0$ the Fourier coefficients of the original signal and the time-scaled signal are identical, it is the fundamental frequencies that are changed by the scaling factor.

Negative Scaling

$$y(t) = Ax(-|\alpha|t) = \sum_{k=-\infty}^{\infty} (AX_k)e^{jk-|\alpha|\Omega_0 t} = \sum_{=-\infty}^{\infty} \ell(AX_{-\ell})e^{j|\ell|\alpha\Omega_0 t} = \sum_{=-\infty}^{\infty} \ell(AX_\ell^*)e^{j\ell\alpha\Omega_0 t}$$

where $Y_\ell = (AX_{-\ell}) = (AX_\ell^*)$
Time period is $T_1 = 2\pi/\Omega_1 = T_0/\alpha$
and we have new set of harmonics as $\ell\Omega_1 = \ell|\alpha|\Omega_0$.

Applications of Fourier Series

Signal Processing

- Filtering: Fourier series are used to design filters that remove unwanted frequencies from signals.
- Modulation Analysis: Fourier series help analyze and separate modulated signals in communication systems.

Example: Audio filtering - Fourier series are used in audio processing to remove noise and improve sound quality.

Image Processing

- Image Compression: Fourier series are used in image compression algorithms like JPEG to reduce image size.
- Image Reconstruction: Fourier series help reconstruct images from compressed data.

Example: Medical Imaging - Fourier series are used in medical imaging techniques like MRI and CT scans to reconstruct images of the body.

Power Systems

- Power Quality Analysis: Fourier series are used to analyze and monitor power quality in electrical power systems.
- Harmonic Analysis: Fourier series help identify and mitigate harmonic distortions in power systems.

Example: Power Grid Management - Fourier series are used in power grid management to monitor and control power flow.

Vibration Analysis

- Vibration Analysis: Fourier series are used to analyze and predict vibrations in mechanical systems.
- Condition Monitoring: Fourier series help monitor the condition of mechanical systems and predict maintenance needs.

Example: Predictive Maintenance - Fourier series are used in predictive maintenance to detect anomalies in mechanical systems.

Video on Fourier Series

https://youtu.be/r6sGWTCMz2k

Additional Resources

- Drawing everything with Fourier Transform:
- https://www.youtube.com/watch?v=MY4luNgGfms
- https://olgaritme.com/posts/drawing-with-the-fourier-series/index.html
- https://dsp.stackexchange.com/questions/59068/how-to-get-fourier-coefficients-to-draw-any-shape-using-dft
- https://www.reddit.com/r/3Blue1Brown/comments/cvpdn7/make_your_own_fourier_circle_drawings/
- https://contra.medium.com/drawing-anything-with-fourier-series-using-blender-and-python-c0881e1b738c
- https://www.fourierart.com/

Upnext

Fourier Transform