CPE 381: Fundamentals of Signals and Systems for Computer Engineers

09 Discrete Signals and Systems

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Outline

$\downarrow_9^1$ Discrete-time Signals

Discrete-time Signals

A discrete-time signal $x[n]$ can be thought of as a real- or complex-valued function of the integer sample index n:

$$
\begin{aligned}
x[.] &: \mathcal{I} \to \mathcal{R}(\mathcal{C}) \\
n &\quad x[n].
\end{aligned}
$$

Example

$$x(t) = 3\cos\left(2\pi t + \frac{\pi}{4}\right), \quad -\infty < t < \infty$$

$$T_s \leq \frac{\pi}{\Omega_{\max}} = \frac{\pi}{2\pi} = 0.5s/sample$$

Then its discrete version is

$$x[n] = 3\cos\left(2\pi t + \frac{\pi}{4}\right)|_{t=0.5n} = 3\cos\left(\pi n + \frac{\pi}{4}\right), \quad -\infty < t < \infty$$

Periodic Discrete-time Signal

$$x[n + kN] = x[n]$$

k: any integer
N: Fundamental period of $x[n]$

Aperiodic discrete-time signals don't satisfy the above property.

Sampling Analog Sinusoid

When sampling an analog sinusoid,

$$x(t) = A\cos(\Omega_0 t + \theta), \quad -\infty < t < \infty$$

of fundamental period $T_0 = 2\pi/\Omega_9$, $\Omega_0 > 0$, we obtain a **periodic discrete sinusoid**.

$$x[n] = A\cos(\Omega_0 T_s n + \theta) = A\cos\left(\frac{2\pi T_s}{T_0} n + \theta\right)$$

provided that

$$\frac{T_s}{T_0} = \frac{m}{N}$$

for the positive integers N and m which are not divisible by each other. To avoid frequency aliasing, the sampling period should also satisfy the Nyquist sampling condition,

$$T_s \leq \frac{\pi}{\Omega_0} = \frac{T_0}{2}$$

Finite-energy, Finite-power Discrete-time Signal

- **Energy:** $\varepsilon_x = \sum_{n=-\infty}^{\infty} = |x[n]|^2$
- **Power:** $P_x = \lim_{N\to\infty} \dfrac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$
- $x[n]$ is said to have **finite energy** or to be **square summable** if $\varepsilon_x < \infty$.
- $x[n]$ is called **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} = |x[n]| < \infty$$

- $x[n]$ is said to have finite power if $P_x < \infty$.

Example

$$x(t) = \begin{cases} 2\cos(\Omega_0 t - \pi/4) & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- Determine its discrete-time signal. Choose $T_s = 0.1$.
- Determine if this discrete-time signal has finite energy, finite power and compare these characteristics with those of the continuous-time signal for $\Omega_0 = \pi$ and $\Omega_0 = 3.2$ rad/s.

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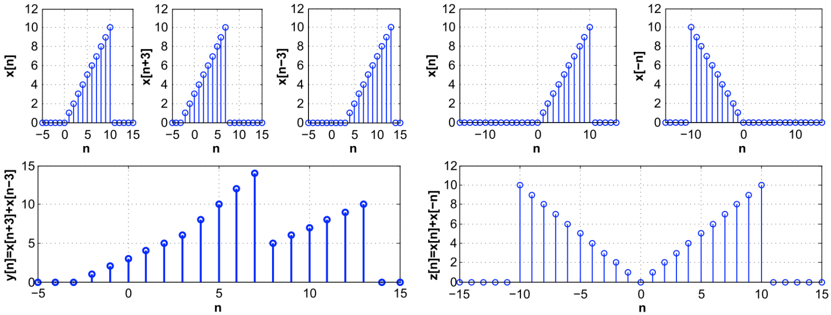
Operations on Discrete-time Signals

Discrete-time Signal Manipulation

A discrete-time signal $x[n]$ is said to be

- delayed by L (an integer) samples if $x[n-L]$ is $x[n]$ shifted to the right L samples,
- advanced by M (an integer) samples if $x[n+M]$ is $x[n]$ shifted to the left M samples,
- reflected if the variable n in $x[n]$ is negated, i.e., $x[-n]$.



Even and Odd Discrete-Time Signal

- $x[n]$ is an **even** signal if $x[n] = x[-n]$.
- $x[n]$ is an **odd** signal if $x[n] = -x[-n]$.

Decomposing a signal into add and even signal:

$$x[n] = \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n])$$

where $x_e[n] = \frac{1}{2}(x[n] + x[-n])$ is the even signal component and $x_o[n] = \frac{1}{2}(x[n] - x[-n])$ is the odd signal component.

Basic Discrete-Time Signals

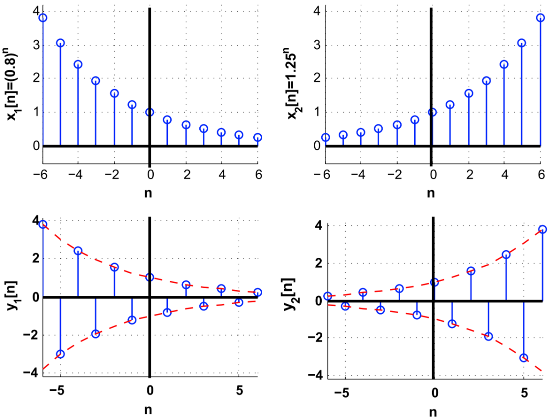
Discrete-Time Complex Exponential

Given a complex number $A = |A|e^{j\theta}$ and $\alpha = |\alpha|e^{j\omega_0}$, a discrete-time complex exponential is a signal

$$\begin{aligned} x[n] = A\alpha^n &= |A||\alpha|^n e^{j(\omega_0 n+\theta)} \\ &= |A||\alpha|^n[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)] \end{aligned}$$

where ω_0 is a discrete frequency in radians.

Notice that ω will represent discrete frequencies in our discussion, while Ω is used for the continuous frequencies.



Real exponential $x_1[n] = 0.8^n$, $x_2[n] = 1.25^n$ **(top)**

and

Modulated exponential $y_1[n] = x_1[n]\cos(\pi n)$ and $y_2[n] = x_2[n]\cos(\pi n)$ **(bottom)**.

Discrete-time Sinusoids

A special case of discrete-complex exponential:

- $\alpha = e^{j\omega_0}$
- $x[n] = A\alpha^n = |A|e^{j(\omega_0 n+\theta)} = |A|\cos(\omega_0 n + \theta) + j|A|\sin(\omega_0 n + \theta)$
- The real part of $x[n]$ is a cosine, while the imaginary part is a sine.
- Discrete sinusoids of amplitude A and phase shift θ are periodic if

$$A\cos(\omega_0 n + \theta) = A\sin(\omega_0 n + \theta + \pi/2), \quad -\infty < n\infty$$

- $\omega_0 = 2\pi m/N$ (rad) is the discrete frequency, for integers m and $N > 0$ which are not divisible. Otherwise, discrete-time sinusoids are not periodic.
- $\omega_0 + 2\pi k = \omega_0$, where k is an integer.

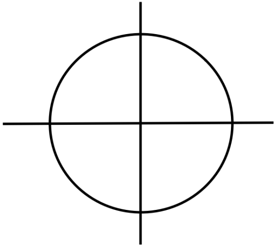
Limiting range for Discrete Frequencies

To avoid ambiguity in discrete-frequency values, we limit its range

$$-\pi < \omega \leq \pi$$

$\omega = \pi/2, 5\pi/2, 9\pi/2 \cdots = \pi/2$

$\omega = \pi, 3\pi, 5\pi \cdots = \pi = -\pi$



$\omega = 0, 2\pi, 4\pi, \cdots = 0$

$\omega = 3\pi/2, 7\pi/2, 11\pi/2, \cdots = -\pi/2$

Discrete-Time Unit-Step and Unit-Sample Signals

- The unit-step $u[n]$ and the unit-sample $\delta[n]$ discrete-time signals are defined as

$$u[n] = \begin{cases} 1, & n \geq 0, 0, & n < 0 \end{cases}$$

$$\delta[n] = \begin{cases} 1, & n = 0, 0, & \text{otherwise} \end{cases}$$

- $\delta[n] = u[n] - u[n-1]$
- $u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m]$

Note: $u[n]$ and $\delta[n]$ are NOT sampled versions of the continuous signals $u(t)$ and $\delta(t)$. $u[n]$ and $\delta[n]$ are entirely different signals.

Discrete-Ramp Functions

- $r[n] = nu[n]$
- $r[n] = \sum_{k=0}^{\infty} k\delta[n-k] = \sum_{k=0}^{\infty} u[n-k]$

Generic Representation of Discrete-Time Signals

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Sifting property of the unit-sample signal

- $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
- $\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & \text{otherwise} \end{cases}$
- $x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Discrete-time Systems and Their Properties

Discrete-time Systems Representation

A discrete-time system is a transformation of a discrete-time input signal $x[n]$ into a discrete-time output signal $y[n]$

$$y[n] = \mathcal{S}\{x[n]\}$$

Linearity

- **Scaling:** $\mathcal{S}\{ax[n]\} = a\mathcal{S}\{x[n]\}$
- **Additivity:** $\mathcal{S}\{x[n] + v[n]\} = \mathcal{S}\{x[n]\} + \mathcal{S}\{v[n]\}$
- Equivalently, superposition applies:

$$\mathcal{S}\{ax[n] + bv[n]\} = a\mathcal{S}\{x[n]\} + b\mathcal{S}\{v[n]\}$$

Time-invariance

- If for an input$x[n]$ the corresponding output is $y[n] = \mathcal{S}\{x[n]\}$, the output corresponding to an advanced or a delayed version of $x[n]$, $x[n \pm M]$, for an integer M, is $y[nM] = \mathcal{S}\{x[n \pm M]\}$, or the same as before but shifted as the input.
- In other words, the system is not changing with time.

Recursive And Non-Recursive Discrete-Time Systems

- Input: $x[n]$
- Output: $y[n]$
- **Recursive System (infinite impulse response (IIR) system)**:

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m], \quad n \geq 0$$

- **Non-Recursive System (Finite impulse response (FIR) system):**

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m]$$

These two equations shown above are called difference equations.

Autoregressive Discrete System

$$y[n] = ay[n-1] + bx[n, \quad n \geq 0$$

with initial condition of $y[-1]$ given.

Example: Autoregressive Moving-Average System

Consider a system represented by the difference-equation:

$$y[n] = 0.5y[n-1] + x[n] + x[n-1] n \geq 0, y[-1]$$

Consider two cases:

- Let the initial condition be $y[-1] = -2$, and the input $x[n] = u[n]$ first and then $x[n] = 2u[n]$. Find the corresponding outputs.
- Let the initial condition be $y[-1] = 0$, and the input $x[n] = u[n]$ first and then $x[n] = 2u[n]$. Find the corresponding outputs.

Use the above results to determine in each case if the system is linear. Find the steady-state response, i.e., $\lim_{n\to\infty} y[n]$.

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Dynamic Discrete-Time Systems Represented By Difference Equations

A recursive discrete-time system is represented by a difference equation

$$y[n] = -\sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m], \quad n \geq 0$$

with the initial condition given as $y[-k]$ for $k = 1, \cdots, N-1$.

This difference equation could be the approximation of an ordinary differential equation representing a continuous-time system being processed discretely.

Zero-input and Zero-state Responses

Just as in the continuous-time case, the system being represented by the difference equation is not LTI unless the initial conditions are zero and the input is causal.

The complete response of a system represented by the difference equation can be shown to be composed of zero-input and zero-state responses,

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

The component $y_{zi}[n]$ is the response when the input $x[n]$ is set to zero, thus it is completely due to the initial conditions.
The response $y_{zs}[n]$ is due to the input only, as we set the initial conditions to zero.

Convolution Sum

- Remember the generic representation of the signal: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- The Convolution Sum gives the output of the LTI system:
  $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} x[n-m]h[m]$

Example

Consider an autoregressive system represented by a first-order difference equation
$y[n] = 0.5y[n-1] + x[n], \quad n \geq 0.$
Find the impulse response $h[n]$ of the system and then compute the response of the system to $x[n] = u[n] - u[n-3]$ using the convolution sum.

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Linear And Non-Linear Filtering of Discrete Signals

Linear Filtering

In general, the filters under consideration are linear and shift-invariant. Thus, the output such as signals or images are characterized by the convolution sum between the input signal and the filter impulse response.

Example: Averaging Filter

Let $y[n] = x[n] + \eta[n]$

- $x[n]$: The averaging filter.
- $\eta[n]$: Gaussian noise.

Averaging filter of M-th order:

$$z[n] = \frac{1}{M}\sum_{k=0}^{M-1} y[n-k]$$

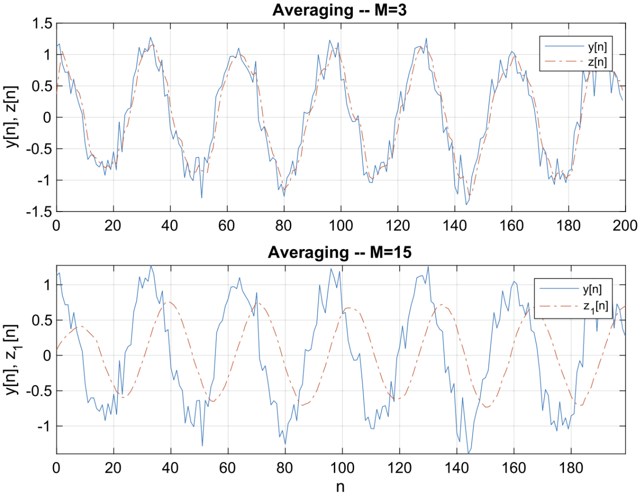
Note that higher-order filters will have a larger delay.

Averaging Filter in MATLAB

```
N=200;n=0:N-1;
x=cos(pi*n/16); % input signal
noise=0.2*randn(1,N); % noise
y=x+noise; % noisy signal
% averaging linear filter with M=3
z=averager(3,y);
% averaging linear filter with M=15
z1=averager(15,y);
```

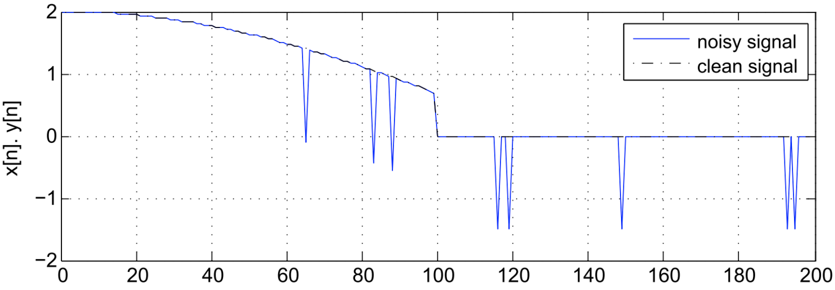
```
function y=averager(M,x)
% Moving average of signal x
% M: order of averager
% x: input signal
%
b=(1/M)*ones(1,M);
y=filter(b,1,x);
```

Averaging Filter in MATLAB



Nonlinear Filtering

Not all linear filters are capable of removing noise, such as impulsive noise. In such a case, we can use non-linear filtering methods, such as median filters.



Median Filtering

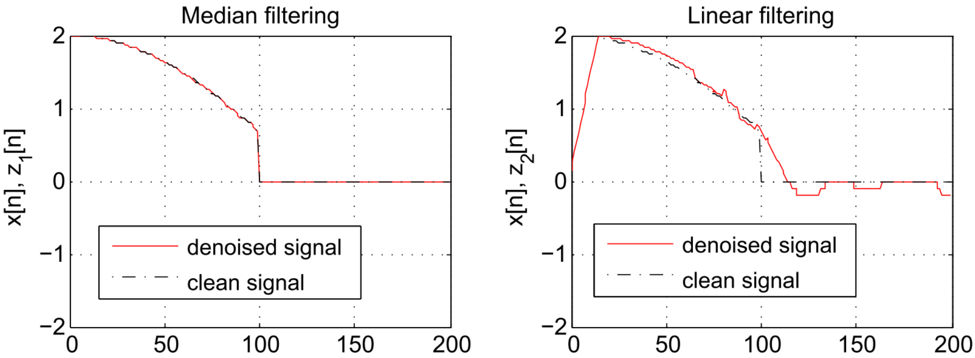
A median filter considers a certain number of samples (the example shows a 5th-order median filter), orders them according to their values, and chooses the one in the middle (i.e., the median) as the filter's output. Such a filter is non-linear as it does not satisfy superposition.

Median Filter in MATLAB

```
clear all; clf
N=200;n=0:N-1;
% impulsive noise
for m=1:N,
    d=rand(1,1);
    if d>=0.95,
        noise(m)=-1.5;
    else
        noise(m)=0 ;
    end
end
```

```
x=[2*cos(pi*n(1:100)/256) zeros(1,100)];
y1=x+noise;
% linear filtering
z2=averager(15,y1);
% non-linear filtering -- median filtering
z1(1)=median([0 0 y1(1) y1(2) y1(3)]);
z1(2)=median([0 y1(1) y1(2) y1(3) y1(4)]);
z1(N-1)=median([y1(N-3) y1(N-2) y1(N-1) y1(N) 0]);
z1(N)=median([y1(N-2) y1(N-1) y1(N) 0 0]);
for k=3:N-2,
    z1(k)=median([y1(k-2) y1(k-1) y1(k) y1(k+1) y1(k+2)]);
end
```

Median Filter vs Averaging Filter



Causality of a Discrete-Time LTI System

A discrete-time system $\mathcal{S}$ is causal if:

- whenever the input $x[n] = 0$, and there are no initial conditions, the output is $y[n] = 0$,
- the present output $y[n]$ does not depend on future inputs.

An LTI system can be noncausal, such is the case of the following LTI system that computes the moving average of the input:

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

Causality of a Discrete-Time LTI System

- An LTI discrete-time system is causal if the impulse response of the system is such that $h[n] = 0$ for $n < 0$.
- A signal $x[n]$ is said to be causal if $x[n] = 0$ for $n < 0$.
- For a causal LTI discrete-time system with a causal input $x[n]$ its output $y[n]$ is given by

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k], \quad n \geq 0$$

where the lower limit of the sum depends on the input causality, x$[k] = 0$ for $k < 0$, and the upper limit on the causality of the system, $h[n-k] = 0$ for $n - k < 0$ or $k > n$.

Bounded Input–Bounded Output (BIBO) Stability

- Stability characterizes useful systems.
- A stable system provides well-behaved outputs for well-behaved inputs.
- Bounded input–bounded output (BIBO) stability establishes that for a bounded (which is what is meant by 'well-behaved') input $x[n]$ the output of a BIBO stable system $y[n]$ is also bounded.
- Hence, if there is a finite bound $M < \infty$ such that $|x[n]| < M$ for all n (you can think of it as an envelope $[-M, M]$ inside which the input $x[n]$ is) the output is also bounded, i.e., $|y[n]| < L$ for $L < \infty$ and all n.

Bounded Input–Bounded Output (BIBO) Stability

- An LTI discrete-time system is said to be BIBO stable if its impulse response $h[n]$ is absolutely summable

$$\sum_k |h[k]| < \infty$$

- Or,

$$|y[n]| \leq \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq |x[n-k]||h[k]| \leq M \sum_{k=-\infty}^{\infty} |h[k]| \leq MN < \infty$$

provided that $\sum_{k=-\infty}^{\infty} |h[k]| < N < \infty$, or that the impulse response be absolutely summable.
Consider $L = MN$.

Example

Consider an autoregressive system $y[n] = 0.5y[n-1] + x[n]$.
Determine if the system is BIBO stable

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Two-Dimensional Discrete-Time Signals

Two-dimensional discrete signals

A discrete two-dimensional signal $x[m, n]$ is a mapping of integers $[m, n]$ into real values that is not defined for non-integer values.

Two-dimensional Impulse Signal

$$\delta[m,n] = \begin{cases} 1, & [m,n] = [0,0] \\ 0, & [m,n] \neq [0,0] \end{cases}$$

A signal $x[m,n]$ defined in a support $[M_1, N_1] \times [M_2, N_2]$, $M_1 < M_2$, $N_1 < N_2$ can be written as

$$x[m,n] = \sum_{k=M_1}^{M_2} \sum_{\ell=N_1}^{N_2} x[k,\ell]\delta[m-k, n-\ell]$$

Two-dimensional Unit-step Signal

Two-dimensional unit-step signal $u_1[m,n]$, with support in the first quadrant

$$u_1[m,n] = \begin{cases} 1, & m \geq 0, n \geq 0, \\ 0, & \text{otherwise} \end{cases} \qquad = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \delta[m-k, n-\ell]$$

Two-dimensional Unit-ramp Signal

A two-dimensional unit-ramp signal $r_1[m,n]$, with support in the first quadrant

$$r_1[m,n] = \begin{cases} mn, & m \geq 0, n \geq 0, \\ 0, & \text{otherwise} \end{cases} \quad = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} k\ell\delta[m-k, n-\ell]$$

Separable Signals

A class of two-dimensional signals of interest are separable signals $y[m,n]$ that are the product of two one-dimensional signals, one being a function of m and the other of n

$$y[m,n] = y_1[m]y_2[n]$$

$\delta[m,n] = \delta[m]\delta[n]$ is separable.

Two-Dimensional Discrete-Time Systems

Two-dimensional System

A two-dimensional system is an operator $\mathcal{S}$ that maps an input $x[m,n]$ into a unique output $y[m,n] = \mathcal{S}(x[m,n])$.
We only consider the LTI 2-D system:

- Linearity: $\mathcal{S}\left(\sum_{i=1}^{I} a_i x_i[m,n]\right) = \sum_{i=1}^{I} a_i \mathcal{S}(x_i[m,n]) = \sum_{i=1}^{I} a_i y_i[m,n]$
- Shift-Invariance: $\mathcal{S}(x_i[m-M, n-N]) = y_i[m-M, n-N]$
  A system satisfying these two conditions is called **linear shift-invariant** or **LSI**.

Impulse Response in LSI Two-dimensional System

Suppose then the input $x[m, n]$ of an LTI system and that the response of the system to $\delta[m, n]$ is $h[m, n]$ or the impulse response of the system. Then,

$$
\begin{aligned}
y[m, n] &= \sum_{k} \sum_{\ell} x[k, \ell] \mathcal{S}(\delta[m-k, n-\ell]) \\
&= \sum_{k} \sum_{\ell} x[k, \ell] h[m-k, n-\ell] = (x * h)[m, n]
\end{aligned}
$$

This is also called a 2-d Convolution Sum.

Separable LSI Systems

An LSI system is **separable** if its impulse response $h[m,n]$ is a separable sequence, i.e., $h[m,n] = h_1[m]h_2[n]$. Then we can write the convolution sum as

$$
\begin{aligned}
y[m,n] &= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} x[k,\ell] h_1[m-k] h_2[n-\ell] \\
&= \sum_{k=-\infty}^{\infty} h_1[m-k] \left[\sum_{\ell=-\infty}^{\infty} x[k,\ell] h_2[n-\ell] \right]
\end{aligned}
$$

Notice that the term in the bracket is the convolution of the sum of $h_2[n]$ and the input for fixed values of k.

Separable LSI Systems

Let

$$y_1[k,n] = \sum_{\ell=-\infty}^{\infty} x[k,\ell]h_2[n-\ell] = \sum_{\ell=0}^{\infty} x[k,\ell]h_2[n-\ell]$$

then,

$$y[m,n] = \sum_{k=-\infty}^{\infty} y_1[k,n]h_1[m-k] = \sum_{k=0}^{\infty} y_1[k,n]h_1[m-k]$$

which is one-d convolution sum of $h_1[m]$ and $y_1[k,n]$ for fixed values of n.

The final forms above are obtained using that both input and impulse response are supported in the first quadrant.

Images as 2-dimensional Discrete Signals

Represented as a matrix of integer values

Pixel Value

62	93	43	94	32	49	43	32
34	56	56	64	29	67	10	0
52	200	45	58	29	4	2	84
84	61	76	95	62	85	95	74
63	193	95	85	42	43	45	13

Filtering on Images

Form a new image whose pixels are modified version of original pixel values.

Goals of filtering:

- Extract useful information from the images
  - Features (edges, corners, blobs. . .)
- Modify or enhance image properties
  - super-resolution; in-painting; de-noising

2D discrete-space systems (filters)

$$f[n,m] \to \boxed{System \quad \mathcal{S}} \to g[n,m]$$

$$g = \mathcal{S}[f], \quad g[n,m] = \mathcal{S}\{f[n,m]\}$$

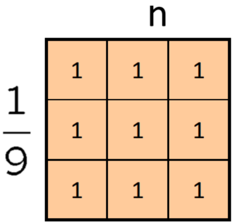
$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$$

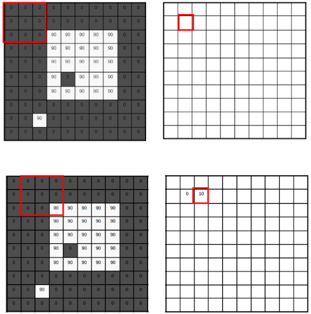
2D Filter Example

2D discrete-space moving average over a 3×3 window of a neighborhood

$$g[n,m] = \frac{1}{9}\sum_{k=n-1}^{n+1}\sum_{\ell=m-1}^{m+1} f[k,\ell] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{\ell=-1}^{1} f[n-k,m-\ell]$$

Or, $(f * h)[m,n] = \frac{1}{9}\sum_{k,\ell} f[k,\ell]h[m-k,n-\ell]$





Example

Consider a separable impulse response

$$h[m,n] = \begin{cases} 1, & 0 \le m \le 1, 0 \le n \le 1 \\ 0, & \text{, otherwise} \end{cases}$$
$$= (u[m] - u[m-2])(u[n] - u[n-2]) = h_1[m]h_2[n]$$

For an input

$$x[m,n] = \begin{cases} 1, & 0 \le m \le 1, 0 \le n \le 1 \\ 0, & \text{, otherwise} \end{cases}$$

find the output of the system $y[m,n]$.

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