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# Signals and Systems for Computer Engineers

WORKSHEET

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# Content

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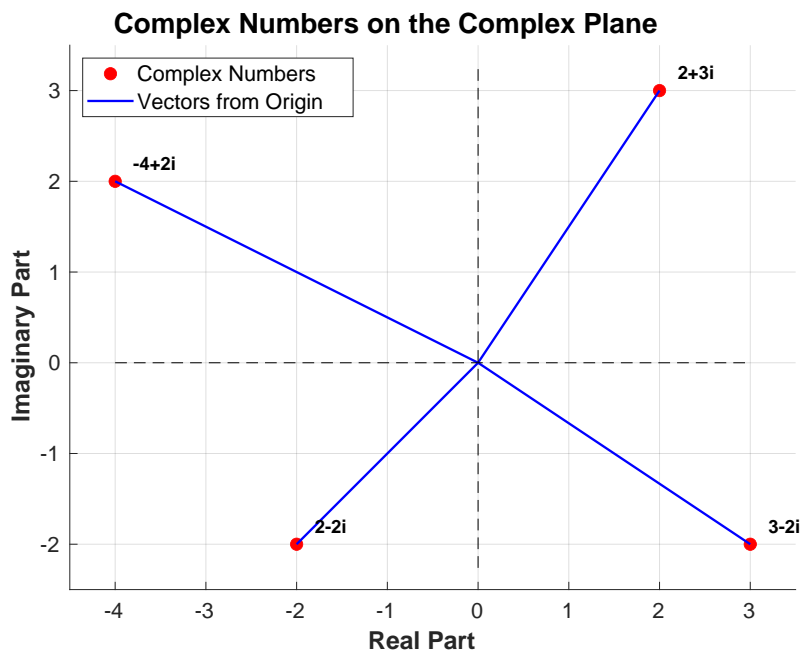
# 1. Mathematical Preliminaries

## 1.1 Complex Numbers

1. **4pts** Plot the following complex numbers on the complex plane:

- $2 + i3$
- $3 - i2$
- $-2 - i2$
- $-4 + j2$

Solution:



2. **4pts** Express  $\frac{-1+3i}{2+5i}$  in the form  $a + ib$ .

**Solution:**

$$\begin{aligned}
\frac{-1+3i}{2+5i} &= \frac{-1+3i}{2+5i} \times \frac{2-5i}{2-5i} \\
&= \frac{-2+5i+6i-15i^2}{2^2+5^2} \\
&= \frac{-2+11i+15}{4+25} \quad (1.1) \\
&= \frac{13+11i}{29} \\
&= \frac{13}{29} + \frac{11}{29}i
\end{aligned}$$

3. **2pts** If  $Z = 3 + i5$  is a complex number, what is the value of the modulus  $|Z|$ ?

**Solution:**

$$|Z| = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34}$$

4. **4pts** Find the roots of the equation  $x^2 + x + 1 = 0$ .

**Solution:**

For a quadratic equation  $ax^2 + bx + c = 0$ , the roots are given by  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . In this case,  $a = 1$ ,  $b = 1$ ,  $c = 1$ .

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

5. **2pts** Write the following complex numbers in the polar form:

(a)  $z = 1 + i$

(b)  $w = \sqrt{3} - i$

**Solution for (a):**  $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$   $\tan \theta = \frac{1}{1} = 1$ , so  $\theta = \frac{\pi}{4}$ . In polar form:  $z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$ .

**Solution for (b):**  $|w| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$   
 $\tan \theta = \frac{-1}{\sqrt{3}}$ , so  $\theta = -\frac{\pi}{6}$  (since the number is in the 4th quadrant).  
 In polar form:  $w = 2(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))$ .

6. **4pts** Find the product of the complex numbers  $1 + i$  and  $\sqrt{3} - i$  in the polar form.

**Solution:**

From the previous problem, we have:  $z_1 = 1 + i = \sqrt{2}e^{i\pi/4}$

$$z_2 = \sqrt{3} - i = 2e^{-i\pi/6}$$

$$z_1 z_2 = (\sqrt{2}e^{i\pi/4})(2e^{-i\pi/6}) = 2\sqrt{2}e^{i(\pi/4 - \pi/6)} = 2\sqrt{2}e^{i(\pi/12)}$$

In standard polar form:  $2\sqrt{2}(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12})$ .

7. **2pts** Find  $(\frac{1}{2} + \frac{1}{2}i)^{10}$ .

**Solution:**

First, convert to polar form:

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

Using De Moivre's Theorem:

$$\begin{aligned} (\frac{1}{2} + \frac{1}{2}i)^{10} &= (\frac{\sqrt{2}}{2})^{10}(\cos(10 \cdot \frac{\pi}{4}) + i \sin(10 \cdot \frac{\pi}{4})) \\ &= (\frac{2^{1/2}}{2})^{10}(\cos(\frac{5\pi}{2}) + i \sin(\frac{5\pi}{2})) \\ &= (\frac{1}{2^{1/2}})^{10}(\cos(\frac{\pi}{2} + 2\pi) + i \sin(\frac{\pi}{2} + 2\pi)) \\ &= (\frac{1}{2^5})(0 + i \cdot 1) = \frac{i}{32} \end{aligned}$$

8. **2pts** Evaluate or Simplify:

(a)  $e^{i\pi}$

(b)  $e^{-1+i\pi/2}$

**Solution for (a):**  $e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$ .

**Solution for (b):**  $e^{-1+i\pi/2} = e^{-1}e^{i\pi/2} = e^{-1}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = \frac{1}{e}\left(0 + i(1)\right) = \frac{i}{e}$

9. **2pts** Evaluate the expression below and write your answer in the form  $a + ib$ .

(a)  $(5 - i6) + (3 + i2)$

(b)  $\frac{3}{4-i3}$

**Solution for (a):**  $(5 - i6) + (3 + i2) = (5 + 3) + (-6 + 2)i = 8 - 4i$ .

**Solution for (b):**  $\frac{3}{4-i3} = \frac{3}{4-i3} \times \frac{4+i3}{4+i3} = \frac{12+i9}{4^2+3^2} = \frac{12+i9}{16+9} = \frac{12+i9}{25} = \frac{12}{25} + \frac{9}{25}i$ .

10. **2pts** Find the complex conjugate and modulus of the number:

(a)  $12 + i5$

(b)  $-1 + 2\sqrt{2}i$

**Solution for (a):**

Complex conjugate:  $12 - i5$ .

Modulus:  $|12 + i5| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$ .

**Solution for (b):**

Complex conjugate:  $-1 - 2\sqrt{2}i$ .

Modulus:  $|-1 + 2\sqrt{2}i| = \sqrt{(-1)^2 + (2\sqrt{2})^2} = \sqrt{1 + 8} = \sqrt{9} = 3$ .

11. **2pts** Apply De Moivre's Theorem to simplify:

(a)  $(1 + i)^{20}$

(b)  $(1 - \sqrt{3}i)^5$

(c)  $(1 - i)^8$

**Solution for (a):**

Convert  $1 + i$  to polar form:  $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ ,

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

$$1 + i = \sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right).$$

$$\begin{aligned}(1 + i)^{20} &= (\sqrt{2})^{20}(\cos(20 \cdot \frac{\pi}{4}) + i \sin(20 \cdot \frac{\pi}{4})) \\ &= 2^{10}(\cos(5\pi) + i \sin(5\pi)) = 1024(-1 + 0i) = -1024.\end{aligned}$$

**Solution for (b):**

Convert  $1 - \sqrt{3}i$  to polar form:  $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ ,  $\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$ .

$$1 - \sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right).$$

$$\begin{aligned}(1 - \sqrt{3}i)^5 &= 2^5(\cos(5 \cdot (-\frac{\pi}{3})) + i \sin(5 \cdot (-\frac{\pi}{3}))) \\ &= 32(\cos(-\frac{5\pi}{3}) + i \sin(-\frac{5\pi}{3})) = 32(\cos(\frac{\pi}{3}) + i \sin(\frac{\pi}{3})) \\ &= 32(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 16(1 + i\sqrt{3}).\end{aligned}$$

**Solution for (c):**

Convert  $1 - i$  to polar form:  $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ ,  $\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$ .

$$1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right).$$

$$\begin{aligned}(1 - i)^8 &= (\sqrt{2})^8(\cos(8 \cdot (-\frac{\pi}{4})) + i \sin(8 \cdot (-\frac{\pi}{4}))) \\ &= 2^4(\cos(-2\pi) + i \sin(-2\pi)) = 16(1 + 0i) = 16.\end{aligned}$$

12. **3pts** Use Euler's formula to prove the following formulas for  $\cos x$  and  $\sin x$ :

(a)  $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

(b)  $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

**Solution:** Euler's formula states:  $e^{ix} = \cos x + i \sin x$   $e^{-ix} = \cos x - i \sin x$

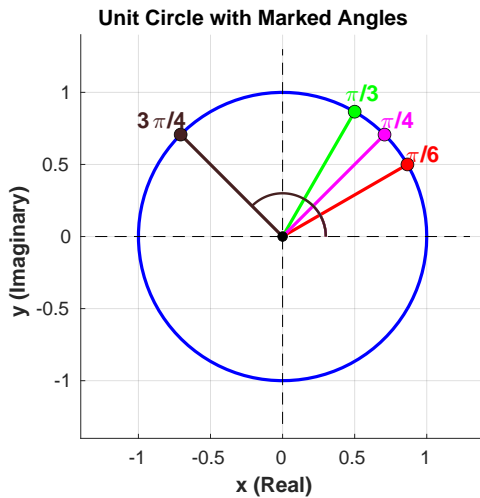
To prove (a), add the two equations:  $e^{ix} + e^{-ix} = (\cos x + i \sin x) + (\cos x - i \sin x) = 2 \cos x$ . Therefore,  $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ .

To prove (b), subtract the second equation from the first:  $e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x) = 2i \sin x$ . Therefore,  $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ .

## 1.2 Trigonometry

1. **4 pts** Draw a circle and mark  $\frac{\pi}{6}$ ,  $\frac{\pi}{3}$ ,  $\frac{\pi}{4}$ , and  $\frac{3\pi}{4}$ .

**Solution:**



2. **1 pts** Convert  $\frac{3\pi}{2}$  radians into degrees.

**Solution:**

$$\frac{3\pi}{2} \times \frac{180^\circ}{\pi} = 270^\circ \quad (1.2)$$

3. **3 pts** Find the value of  $\theta$  :  $4 \sin^2 \theta = 3$ .

**Solution:**

$$\begin{aligned} 4 \sin^2 \theta &= 3 \\ \sin^2 \theta &= \frac{3}{4} \\ \sin \theta &= \pm \frac{\sqrt{3}}{2} \end{aligned} \quad (1.3)$$

If  $\sin \theta = \frac{\sqrt{3}}{2}$ , then  $\theta = 60^\circ$  or  $\theta = 120^\circ$  (Quadrant I and II).

If  $\sin \theta = -\frac{\sqrt{3}}{2}$ , then  $\theta = 240^\circ$  or  $\theta = 300^\circ$  (Quadrant III and IV).

4. **3 pts** Find the value of  $x$  :  $2 \sin^2 x - 3 \sin x + 1 = 0$ .

**Solution:**

$$\begin{aligned} 2 \sin^2 x - 3 \sin x + 1 &= 0 \\ 2 \sin^2 x - 2 \sin x - \sin x + 1 &= 0 \\ 2 \sin x (\sin x - 1) - 1 (\sin x - 1) &= 0 \\ (2 \sin x - 1)(\sin x - 1) &= 0 \end{aligned} \quad (1.4)$$

This gives two possible solutions:

$$2 \sin x - 1 = 0 \implies \sin x = \frac{1}{2}.$$

$$\sin x - 1 = 0 \implies \sin x = 1.$$

$$\text{For } \sin x = \frac{1}{2}, x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}.$$

$$\text{For } \sin x = 1, x = \frac{\pi}{2}.$$

The general solutions are  $x = \frac{\pi}{6} + 2\pi K$ ,  $x = \frac{5\pi}{6} + 2\pi K$ , and  $x = \frac{\pi}{2} + 2\pi K$ .

5. **5 pts** Prove:  $(1 - \sin^2(t))(1 + \tan^2(t)) = 1$ .

**Solution:**

$$\begin{aligned} (1 - \sin^2(t))(1 + \tan^2(t)) &= (\cos^2(t))(1 + \tan^2(t)) \\ &= \cos^2(t) + \cos^2(t) \tan^2(t) \\ &= \cos^2(t) + \cos^2(t) \frac{\sin^2(t)}{\cos^2(t)} \quad (1.5) \\ &= \cos^2(t) + \sin^2(t) \\ &= 1 \end{aligned}$$

6. **5 pts** Prove:  $\frac{\sin^3(t) + \cos^3(t)}{\sin(t) + \cos(t)} = 1 - \sin(t) \cos(t)$ .

**Solution:**

Using the identity  $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ .

Let  $a = \sin(t)$  and  $b = \cos(t)$ .

$$\begin{aligned} \frac{\sin^3(t) + \cos^3(t)}{\sin(t) + \cos(t)} &= \frac{(\sin(t) + \cos(t))(\sin^2(t) - \sin(t) \cos(t) + \cos^2(t))}{\sin(t) + \cos(t)} \\ &= \sin^2(t) + \cos^2(t) - \sin(t) \cos(t) \\ &= 1 - \sin(t) \cos(t) \end{aligned} \quad (1.6)$$

7. **2 pts** What is the value of  $\sin \theta$  and  $\cos \theta$  given  $\tan \theta = \frac{4}{3}$ ?

**Solution:**

$$\tan \theta = \frac{4}{3} = \frac{\text{Perpendicular}}{\text{Base}}.$$

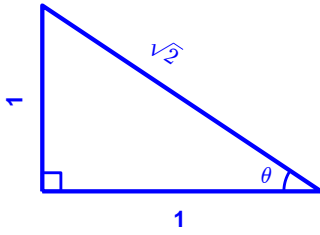
The hypotenuse is  $h = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ .

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}.$$

$$\cos \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{5}.$$



8. **2 pts** Find the value of  $\sin \theta$  and  $\cos \theta$  from the triangle.



**Solution:**

From the right-angled triangle with sides 1, 1 and hypotenuse  $\sqrt{2}$ :

$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{1}{\sqrt{2}}.$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{1}{\sqrt{2}}.$$

9. **3 pts** Prove  $\sin^{-1}(\frac{\sqrt{2}}{2}) - \sin^{-1}(\frac{1}{2}) = \frac{\pi}{12}$ .

**Solution:**

$$\sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}.$$

$$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}.$$

$$\sin^{-1}(\frac{\sqrt{2}}{2}) - \sin^{-1}(\frac{1}{2}) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi - 2\pi}{12} = \frac{\pi}{12}.$$

10. **2 pts** Find the value of  $x$  given  $2 \sin x = 1$ .

**Solution:**

$$2 \sin x = 1 \implies \sin x = \frac{1}{2}.$$

$$\text{For } \sin x = \frac{1}{2}, x = \frac{\pi}{6} \text{ or } x = \frac{5\pi}{6}.$$

### 1.3 Calculus

1. Using the first principle, differentiate the function  $f(x) = e^{2x}$  with respect to  $x$ .

**Solution:** We are given  $f(x) = e^{2x}$ . Thus,  $f(x+h) = e^{2(x+h)}$ .

The definition of the derivative is  $\frac{d}{dx}(f(x)) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ .

Substituting the function:

$$\begin{aligned} \frac{d}{dx}(e^{2x}) &= \lim_{h \rightarrow 0} \frac{e^{2(x+h)} - e^{2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{2x}e^{2h} - e^{2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{2x}(e^{2h} - 1)}{h} \\ &= e^{2x} \lim_{h \rightarrow 0} \frac{e^{2h} - 1}{h} \end{aligned} \quad (1.7)$$

We use the known limit  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ . To apply this, we multi-

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

ply the limit by  $\frac{2}{2}$ :

$$\begin{aligned}
 &= e^{2x} \left( \lim_{h \rightarrow 0} \frac{e^{2h} - 1}{2h} \right) \times 2 \\
 &= e^{2x} \cdot 1 \cdot 2 \\
 &= 2e^{2x}
 \end{aligned} \tag{1.8}$$

So,  $\frac{d}{dx}(e^{2x}) = 2e^{2x}$ .

2. If  $y = \sin x + e^x$ , find  $\frac{dy}{dx}$ .

**Solution:**

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(\sin x + e^x) \\
 &= \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \\
 &= \cos x + e^x
 \end{aligned} \tag{1.9}$$

3. If  $y = x^2 + \sin^{-1} x + \log_e x$ , find  $\frac{dy}{dx}$ .

**Solution:**

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\log_e x) \\
 &= 2x^{2-1} + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \\
 &= 2x + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}
 \end{aligned} \tag{1.10}$$

4. If  $y = e^x \sin x$ , find  $\frac{dy}{dx}$ .

**Solution:** Let  $u(x) = e^x$  and  $v(x) = \sin x$ .

$$\begin{aligned}
 \frac{dy}{dx} &= \left( \frac{d}{dx}(e^x) \right) \sin x + e^x \left( \frac{d}{dx}(\sin x) \right) \\
 &= e^x \sin x + e^x \cos x \\
 &= e^x (\sin x + \cos x)
 \end{aligned} \tag{1.11}$$

Hint: Use the product rule if  $y = u(x)v(x)$ , then  $\frac{dy}{dx} = \left\{ \frac{d}{dx}u(x) \right\}v(x) + u(x)\left\{ \frac{d}{dx}v(x) \right\}$

5. If  $y = \frac{x}{x^2+1}$ , find  $\frac{dy}{dx}$ .

**Solution:** Let  $u(x) = x$  and  $v(x) = x^2 + 1$ .

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
 &= \frac{(x^2+1) \cdot 1 - x \cdot 2x}{(x^2+1)^2} \\
 &= \frac{x^2+1-2x^2}{(x^2+1)^2} \\
 &= \frac{1-x^2}{(x^2+1)^2}
 \end{aligned} \tag{1.12}$$

Hint: Use the quotient rule:  $\frac{dy}{dx} = \frac{\left\{ \frac{d}{dx}u(x) \right\}v(x) - \left\{ \frac{d}{dx}v(x) \right\}u(x)}{\{v(x)\}^2}$

6. Evaluate  $\int \frac{x+1}{x^3+x^2-6x} dx$ .

**Solution:** First, use partial fraction decomposition:

$$\begin{aligned}\frac{x+1}{x^3+x^2-6x} &= \frac{x+1}{x(x^2+x-6)} \\ &= \frac{x+1}{x(x+3)(x-2)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}\end{aligned}\quad (1.13)$$

Setting the numerators equal:  $A(x-2)(x+3) + Bx(x+3) + Cx(x-2) = x+1$ . The coefficients are found to be  $A = -1/6$ ,  $B = 3/10$ , and  $C = -2/15$ .

$$\begin{aligned}\int \frac{x+1}{x^3+x^2-6x} dx &= \int \left( \frac{-1/6}{x} + \frac{3/10}{x-2} + \frac{-2/15}{x+3} \right) dx \\ &= -\frac{1}{6} \int \frac{1}{x} dx + \frac{3}{10} \int \frac{1}{x-2} dx - \frac{2}{15} \int \frac{1}{x+3} dx \\ &= -\frac{1}{6} \ln|x| + \frac{3}{10} \ln|x-2| - \frac{2}{15} \ln|x+3| + C\end{aligned}\quad (1.14)$$

7. Find the indefinite integral of  $f(x) = 3x^2 + 4x - 2$ .

**Solution:**

$$\begin{aligned}\int f(x) dx &= \int (3x^2 + 4x - 2) dx \\ &= \int 3x^2 dx + \int 4x dx - \int 2 dx \\ &= 3 \frac{x^3}{3} + 4 \frac{x^2}{2} - 2x + C \\ &= x^3 + 2x^2 - 2x + C\end{aligned}\quad (1.15)$$

8. Find  $\int x \sin x dx$ .

Hint: Use integration by parts:  
 $\int u dv = uv - \int v du$

**Solution:** Let  $u = x$  and  $dv = \sin x dx$ . Then  $du = dx$  and  $v = \int \sin x dx = -\cos x$ .

$$\begin{aligned}\int x \sin x dx &= x(-\cos x) - \int (-\cos x) dx \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C\end{aligned}\quad (1.16)$$

9. Solve the differential equation  $\frac{dy}{dx} = 3x^2$ .

**Solution:**

$$\begin{aligned}
 \frac{dy}{dx} &= 3x^2 \\
 dy &= 3x^2 dx \\
 \int dy &= \int 3x^2 dx \\
 y &= \frac{3x^3}{3} + C \\
 y &= x^3 + C
 \end{aligned} \tag{1.17}$$

Thus, the correct answer is  $y = x^3 + C$ .

10. Solve the equation:  $(1 + y^2)y' = \frac{3}{x}$ .

**Solution:** Separate the variables:

$$\begin{aligned}
 (1 + y^2) \frac{dy}{dx} &= \frac{3}{x} \\
 (1 + y^2) dy &= \frac{3}{x} dx \\
 \int (1 + y^2) dy &= \int \frac{3}{x} dx \\
 \int 1 dy + \int y^2 dy &= 3 \int \frac{1}{x} dx \\
 y + \frac{y^3}{3} &= 3 \ln |x| + \ln C \\
 y + \frac{y^3}{3} &= \ln(C|x|^3) \\
 e^{y+y^3/3} &= C|x|^3
 \end{aligned} \tag{1.18}$$

## 1.4 Sequences and Series

1. Show that a sequence given by the term  $u_n = \frac{2n}{3n+7}$  is strictly increasing.

**Solution:** We can prove this by computing and simplifying the

expression  $u_{n+1} - u_n$ :

$$\begin{aligned}
 u_{n+1} - u_n &= \frac{2(n+1)}{3(n+1)+7} - \frac{2n}{3n+7} \\
 &= \frac{2(n+1)(3n+7) - 2n(3(n+1)+7)}{(3(n+1)+7)(3n+7)} \\
 &= \frac{(6n+14)(3n+10) - (6n^2+20n)}{(3n+10)(3n+7)} \\
 &= \frac{(6n^2+40n+14) - (6n^2+20n)}{(3n+10)(3n+7)} \\
 &= \frac{14}{(3n+10)(3n+7)} > 0, \quad \forall n \in \mathbb{N}.
 \end{aligned} \tag{1.19}$$

2. Show that the sequence  $u_n = \frac{5n}{n^2+1}$  is strictly decreasing.

**Solution:**

$$\begin{aligned}
 u_{n+1} - u_n &= \frac{5(n+1)}{(n+1)^2+1} - \frac{5n}{n^2+1} \\
 &= \frac{5(n+1)(n^2+1) - 5n((n+1)^2+1)}{((n+1)^2+1)(n^2+1)} \\
 &= \frac{5(n+1)(n^2+1) - 5n(n^2+2n+1+1)}{(n^2+2n+2)(n^2+1)} \\
 &= \frac{5(n^3+n+n^2+1) - 5n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)} \\
 &= \frac{5n^3+5n+5n^2+5 - (5n^3+10n^2+10n)}{(n^2+2n+2)(n^2+1)} \\
 &= \frac{-5n^2-5n+5}{(n^2+2n+2)(n^2+1)} \\
 &= \frac{-5(n^2+n-1)}{(n^2+2n+2)(n^2+1)} < 0, \quad \forall n \in \mathbb{N}.
 \end{aligned} \tag{1.20}$$

3. For  $f(x) = \sin(x)$ , write down its Maclaurin series.

**Solution:**

We know that  $f \in C^\infty(\mathbb{R})$  and  $f^{(n)}(x) = \sin(x + \frac{n\pi}{2})$ ,  $\forall n \in \mathbb{N}$ .

Then  $f^{(n)}(0) = \sin(\frac{n\pi}{2})$ , and therefore, the Maclaurin series of  $f$  is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

4. Find the Taylor series expansion for  $e^x + \cos(x)$ .

**Solution:**

Using Taylor series with  $a = 0$ ,

Taylor series expansion gives  $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Hence,

$$\begin{aligned} e^x + \cos(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= 1 + x + \frac{x^3}{3!} + 2\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \end{aligned} \tag{1.21}$$