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Signals and Systems for Computer Engineers

WORKSHEET

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Content

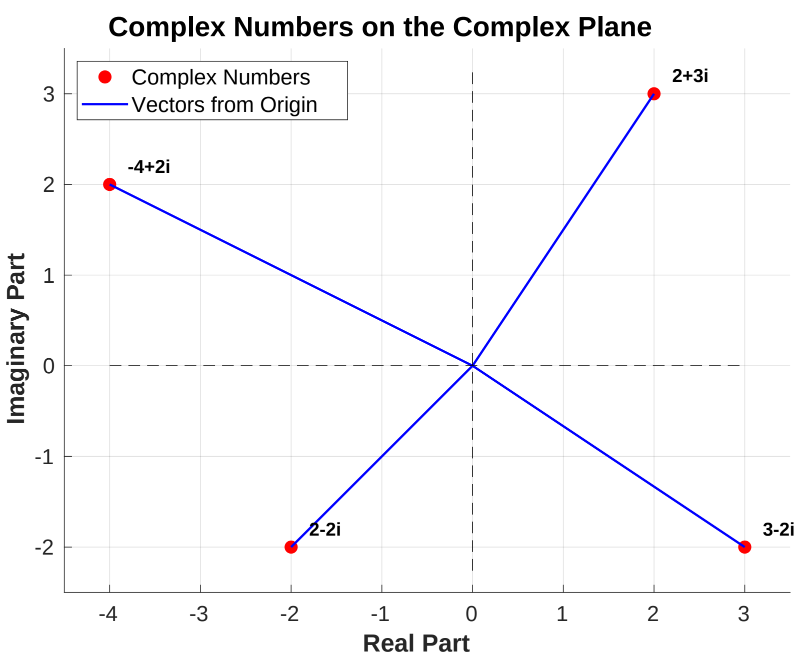
1. Mathematical Preliminaries

1.1 Complex Numbers

1. **4pts** Plot the following complex numbers on the complex plane:

   - $2 + i3$
   - $3 - i2$
   - $-2 - i2$
   - $-4 + j2$

   Solution:



2. **4pts** Express $\frac{-1+3i}{2+5i}$ in the form $a + ib$.

Solution:

$$\begin{aligned}
\frac{-1+3i}{2+5i} &= \frac{-1+3i}{2+5i} \times \frac{2-5i}{2-5i} \\
&= \frac{-2+5i+6i-15i^2}{2^2+5^2} \\
&= \frac{-2+11i+15}{4+25} \\
&= \frac{13+11i}{29} \\
&= \frac{13}{29} + \frac{11}{29}i
\end{aligned} \tag{1.1}$$

3. **2pts** If $Z = 3 + i5$ is a complex number, what is the value of the modulus $|Z|$?

   Solution:

   $$|Z| = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$$

4. **4pts** Find the roots of the equation $x^2 + x + 1 = 0$.

   Solution:

   For a quadratic equation $ax^2 + bx + c = 0$, the roots are given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. In this case, $a = 1$, $b = 1$, $c = 1$.

   $$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

5. **2pts** Write the following complex numbers in the polar form:

   (a) $z = 1 + i$

   (b) $w = \sqrt{3} - i$

   Solution for (a): $|z| = \sqrt{1^2+1^2} = \sqrt{2}$ $\tan\theta = \frac{1}{1} = 1$, so $\theta = \frac{\pi}{4}$. In polar form: $z = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$.

   Solution for (b): $|w| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$ $\tan\theta = \frac{-1}{\sqrt{3}}$, so $\theta = -\frac{\pi}{6}$ (since the number is in the 4th quadrant). In polar form: $w = 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$.

6. **4pts** Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in the polar form.

   Solution:

   From the previous problem, we have: $z_1 = 1 + i = \sqrt{2}e^{i\pi/4}$

   $z_2 = \sqrt{3} - i = 2e^{-i\pi/6}$

   $z_1z_2 = (\sqrt{2}e^{i\pi/4})(2e^{-i\pi/6}) = 2\sqrt{2}e^{i(\pi/4-\pi/6)} = 2\sqrt{2}e^{i(\pi/12)}$

   In standard polar form: $2\sqrt{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$.

7. **2pts** Find $(\frac{1}{2} + \frac{1}{2}i)^{10}$.

   Solution:

   First, convert to polar form:

   $\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

   Using De Moivre's Theorem:

   $(\frac{1}{2} + \frac{1}{2}i)^{10} = (\frac{\sqrt{2}}{2})^{10}(\cos(10 \cdot \frac{\pi}{4}) + i\sin(10 \cdot \frac{\pi}{4}))$

   $= (\frac{2^{1/2}}{2})^{10}(\cos(\frac{5\pi}{2}) + i\sin(\frac{5\pi}{2}))$

   $= (\frac{1}{2^{1/2}})^{10}(\cos(\frac{\pi}{2} + 2\pi) + i\sin(\frac{\pi}{2} + 2\pi))$

   $= (\frac{1}{2^5})(0 + i \cdot 1) = \frac{i}{32}$

8. **2pts** Evaluate or Simplify:

   (a) $e^{i\pi}$

   (b) $e^{-1+i\pi/2}$

   Solution for (a): $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$.

   Solution for (b): $e^{-1+i\pi/2} = e^{-1}e^{i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{e}\left(0 + i(1)\right) = \frac{i}{e}$

9. **2pts** Evaluate the expression below and write your answer in the form $a + ib$.

   (a) $(5 - i6) + (3 + i2)$

   (b) $\frac{3}{4-i3}$

   Solution for (a): $(5 - i6) + (3 + i2) = (5 + 3) + (-6 + 2)i = 8 - 4i$.

   Solution for (b): $\frac{3}{4-i3} = \frac{3}{4-i3} \times \frac{4+i3}{4+i3} = \frac{12+i9}{4^2+3^2} = \frac{12+i9}{16+9} = \frac{12+i9}{25} = \frac{12}{25} + \frac{9}{25}i$.

10. **2pts** Find the complex conjugate and modulus of the number:

    (a) $12 + i5$

    (b) $-1 + 2\sqrt{2}i$

    Solution for (a):

    Complex conjugate: $12 - i5$.

    Modulus: $|12 + i5| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$.

    Solution for (b):

    Complex conjugate: $-1 - 2\sqrt{2}i$.

    Modulus: $|-1 + 2\sqrt{2}i| = \sqrt{(-1)^2 + (2\sqrt{2})^2} = \sqrt{1+8} = \sqrt{9} = 3$.

11. **2pts** Apply De Moivre's Theorem to simplify:

(a) $(1+i)^{20}$

(b) $(1-\sqrt{3}i)^5$

(c) $(1-i)^8$

Solution for (a):

Convert $1+i$ to polar form: $r = \sqrt{1^2+1^2} = \sqrt{2}$,
$\theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$.
$1+i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$.
$(1+i)^{20} = (\sqrt{2})^{20}(\cos(20\cdot\frac{\pi}{4}) + i\sin(20\cdot\frac{\pi}{4}))$
$= 2^{10}(\cos(5\pi) + i\sin(5\pi)) = 1024(-1+0i) = -1024$.

Solution for (b):

Convert $1-\sqrt{3}i$ to polar form: $r = \sqrt{1^2+(-\sqrt{3})^2} = 2$, $\theta = \tan^{-1}(\frac{-\sqrt{3}}{1}) = -\frac{\pi}{3}$.
$1-\sqrt{3}i = 2(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$.
$(1-\sqrt{3}i)^5 = 2^5(\cos(5\cdot(-\frac{\pi}{3})) + i\sin(5\cdot(-\frac{\pi}{3})))$
$= 32(\cos(-\frac{5\pi}{3}) + i\sin(-\frac{5\pi}{3})) = 32(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}))$
$= 32(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 16(1+i\sqrt{3})$.

Solution for (c):

Convert $1-i$ to polar form: $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$, $\theta = \tan^{-1}(\frac{-1}{1}) = -\frac{\pi}{4}$.
$1-i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$.
$(1-i)^8 = (\sqrt{2})^8(\cos(8\cdot(-\frac{\pi}{4})) + i\sin(8\cdot(-\frac{\pi}{4})))$
$= 2^4(\cos(-2\pi) + i\sin(-2\pi)) = 16(1+0i) = 16$.

12. **3pts** Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

(a) $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

(b) $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

Solution: Euler's formula states: $e^{ix} = \cos x + i\sin x$ $e^{-ix} = \cos x - i\sin x$

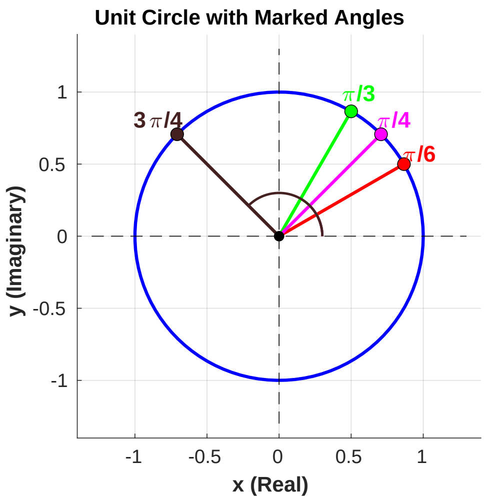
To prove (a), add the two equations: $e^{ix} + e^{-ix} = (\cos x + i\sin x) + (\cos x - i\sin x) = 2\cos x$. Therefore, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$.

To prove (b), subtract the second equation from the first: $e^{ix} - e^{-ix} = (\cos x + i\sin x) - (\cos x - i\sin x) = 2i\sin x$. Therefore, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.

1.2 *Trigonometry*

1. **4 pts** Draw a circle and mark $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{3\pi}{4}$.

   Solution:

2. **1 pts** Convert $\frac{3\pi}{2}$ radians into degrees.

   Solution:

$$\frac{3\pi}{2} \times \frac{180^\circ}{\pi} = 270^\circ \tag{1.2}$$

3. **3 pts** Find the value of $\theta : 4\sin^2\theta = 3$.

   Solution:

$$\begin{aligned} 4\sin^2\theta &= 3 \\ \sin^2\theta &= \frac{3}{4} \\ \sin\theta &= \pm\frac{\sqrt{3}}{2} \end{aligned} \tag{1.3}$$

   If $\sin\theta = \frac{\sqrt{3}}{2}$, then $\theta = 60^\circ$ or $\theta = 120^\circ$ (Quadrant I and II).

   If $\sin\theta = -\frac{\sqrt{3}}{2}$, then $\theta = 240^\circ$ or $\theta = 300^\circ$ (Quadrant III and IV).

4. **3 pts** Find the value of $x : 2\sin^2 x - 3\sin x + 1 = 0$.

Solution:

$$\begin{aligned} 2\sin^2 x - 3\sin x + 1 &= 0 \\ 2\sin^2 x - 2\sin x - \sin x + 1 &= 0 \\ 2\sin x(\sin x - 1) - 1(\sin x - 1) &= 0 \\ (2\sin x - 1)(\sin x - 1) &= 0 \end{aligned} \tag{1.4}$$

This gives two possible solutions:

$2\sin x - 1 = 0 \implies \sin x = \frac{1}{2}$.

$\sin x - 1 = 0 \implies \sin x = 1$.

For $\sin x = \frac{1}{2}$, $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

For $\sin x = 1$, $x = \frac{\pi}{2}$.

The general solutions are $x = \frac{\pi}{6} + 2\pi K$, $x = \frac{5\pi}{6} + 2\pi K$, and $x = \frac{\pi}{2} + 2\pi K$.

5. **5 pts** Prove: $(1 - \sin^2(t))(1 + \tan^2(t)) = 1$.

   Solution:

$$\begin{aligned} (1 - \sin^2(t))(1 + \tan^2(t)) &= (\cos^2(t))(1 + \tan^2(t)) \\ &= \cos^2(t) + \cos^2(t)\tan^2(t) \\ &= \cos^2(t) + \cos^2(t)\frac{\sin^2(t)}{\cos^2(t)} \\ &= \cos^2(t) + \sin^2(t) \\ &= 1 \end{aligned} \tag{1.5}$$

6. **5 pts** Prove: $\frac{\sin^3(t)+\cos^3(t)}{\sin(t)+\cos(t)} = 1 - \sin(t)\cos(t)$.

   Solution:

   Using the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

   Let $a = \sin(t)$ and $b = \cos(t)$.

$$\begin{aligned} \frac{\sin^3(t) + \cos^3(t)}{\sin(t) + \cos(t)} &= \frac{(\sin(t) + \cos(t))(\sin^2(t) - \sin(t)\cos(t) + \cos^2(t))}{\sin(t) + \cos(t)} \\ &= \sin^2(t) + \cos^2(t) - \sin(t)\cos(t) \\ &= 1 - \sin(t)\cos(t) \end{aligned} \tag{1.6}$$

7. **2 pts** What is the value of $\sin\theta$ and $\cos\theta$ given $\tan\theta = \frac{4}{3}$?

   Solution:

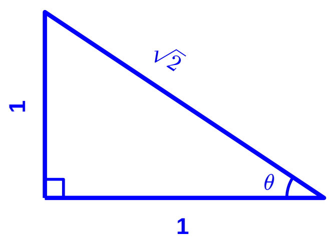
$\tan\theta = \frac{4}{3} = \frac{\text{Perpendicular}}{\text{Base}}$.

The hypotenuse is $h = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}$.

$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{5}$.

8. **2 pts** Find the value of $\sin\theta$ and $\cos\theta$ from the triangle.



Solution:

From the right-angled triangle with sides 1, 1 and hypotenuse $\sqrt{2}$:
$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{1}{\sqrt{2}}$.

$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{1}{\sqrt{2}}$.

9. **3 pts** Prove $\sin^{-1}(\frac{\sqrt{2}}{2}) - \sin^{-1}(\frac{1}{2}) = \frac{\pi}{12}$.

Solution:

$\sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.

$\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

$\sin^{-1}(\frac{\sqrt{2}}{2}) - \sin^{-1}(\frac{1}{2}) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi - 2\pi}{12} = \frac{\pi}{12}$.

10. **2 pts** Find the value of x given $2\sin x = 1$.

Solution:

$2\sin x = 1 \implies \sin x = \frac{1}{2}$.

For $\sin x = \frac{1}{2}$, $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

1.3 Calculus

1. Using the first principle, differentiate the function $f(x) = e^{2x}$ with respect to x.

Solution: We are given $f(x) = e^{2x}$. Thus, $f(x+h) = e^{2(x+h)}$.
The definition of the derivative is $\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

Substituting the function:

$$\begin{aligned}
\frac{d}{dx}(e^{2x}) &= \lim_{h\to 0}\frac{e^{2(x+h)}-e^{2x}}{h} \\
&= \lim_{h\to 0}\frac{e^{2x}e^{2h}-e^{2x}}{h} \\
&= \lim_{h\to 0}\frac{e^{2x}(e^{2h}-1)}{h} \\
&= e^{2x}\lim_{h\to 0}\frac{e^{2h}-1}{h}
\end{aligned} \tag{1.7}$$

We use the known limit $\lim_{x\to 0}\frac{e^x-1}{x} = 1$. To apply this, we multiply the limit by $\frac{2}{2}$:

$\lim_{x\to 0}\frac{e^x-1}{x} = 1$

$$\begin{aligned}
&= e^{2x}\left(\lim_{h\to 0}\frac{e^{2h}-1}{2h}\right)\times 2 \\
&= e^{2x}\cdot 1\cdot 2 \\
&= 2e^{2x}
\end{aligned} \tag{1.8}$$

So, $\frac{d}{dx}(e^{2x}) = 2e^{2x}$.

2. If $y = \sin x + e^x$, find $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(\sin x + e^x) \\
&= \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x) \\
&= \cos x + e^x
\end{aligned} \tag{1.9}$$

3. If $y = x^2 + \sin^{-1} x + \log_e x$, find $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\log_e x) \\
&= 2x^{2-1} + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \\
&= 2x + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}
\end{aligned} \tag{1.10}$$

4. If $y = e^x \sin x$, find $\frac{dy}{dx}$.

Hint: Use the product rule if $y = u(x)v(x)$, then $\frac{dy}{dx} = \{\frac{d}{dx}u(x)\}v(x) + u(x)\{\frac{d}{dx}v(x)\}$

Solution: Let $u(x) = e^x$ and $v(x) = \sin x$.

$$\begin{aligned}
\frac{dy}{dx} &= \left(\frac{d}{dx}(e^x)\right)\sin x + e^x\left(\frac{d}{dx}(\sin x)\right) \\
&= e^x \sin x + e^x \cos x \\
&= e^x(\sin x + \cos x)
\end{aligned} \tag{1.11}$$

5. If $y = \frac{x}{x^2+1}$, find $\frac{dy}{dx}$.

Hint: Use the quotient rule: $\frac{dy}{dx} = \frac{\{\frac{d}{dx}u(x)\}v(x) - \{\frac{d}{dx}v(x)\}u(x)}{\{v(x)\}^2}$

Solution: Let $u(x) = x$ and $v(x) = x^2 + 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{(x^2+1)\cdot 1 - x \cdot 2x}{(x^2+1)^2} \\ &= \frac{x^2+1-2x^2}{(x^2+1)^2} \\ &= \frac{1-x^2}{(x^2+1)^2}\end{aligned} \tag{1.12}$$

6. Evaluate $\int \frac{x+1}{x^3+x^2-6x} dx$.

Solution: First, use partial fraction decomposition:

$$\begin{aligned}\frac{x+1}{x^3+x^2-6x} &= \frac{x+1}{x(x^2+x-6)} \\ &= \frac{x+1}{x(x+3)(x-2)} \\ &= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}\end{aligned} \tag{1.13}$$

Setting the numerators equal: $A(x-2)(x+3) + Bx(x+3) + Cx(x-2) = x+1$. The coefficients are found to be $A = -1/6$, $B = 3/10$, and $C = -2/15$.

$$\begin{aligned}\int \frac{x+1}{x^3+x^2-6x} dx &= \int \left(\frac{-1/6}{x} + \frac{3/10}{x-2} + \frac{-2/15}{x+3} \right) dx \\ &= -\frac{1}{6}\int \frac{1}{x} dx + \frac{3}{10}\int \frac{1}{x-2} dx - \frac{2}{15}\int \frac{1}{x+3} dx \\ &= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C\end{aligned} \tag{1.14}$$

7. Find the indefinite integral of $f(x) = 3x^2 + 4x - 2$.

Solution:

$$\begin{aligned}\int f(x)dx &= \int (3x^2 + 4x - 2)dx \\ &= \int 3x^2 dx + \int 4x dx - \int 2dx \\ &= 3\frac{x^3}{3} + 4\frac{x^2}{2} - 2x + C \\ &= x^3 + 2x^2 - 2x + C\end{aligned} \tag{1.15}$$

8. Find $\int x \sin x dx$.

Hint: Use integration by parts: $\int u dv = uv - \int v du$

Solution: Let $u = x$ and $dv = \sin x dx$. Then $du = dx$ and $v =$

$\int \sin x dx = -\cos x$.

$$\begin{aligned}\int x \sin x dx &= x(-\cos x) - \int (-\cos x) dx \\ &= -x\cos x + \int \cos x dx \\ &= -x\cos x + \sin x + C\end{aligned} \tag{1.16}$$

9. Solve the differential equation $\frac{dy}{dx} = 3x^2$.

   Solution:

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 \\ dy &= 3x^2 dx \\ \int dy &= \int 3x^2 dx \\ y &= \frac{3x^3}{3} + C \\ y &= x^3 + C\end{aligned} \tag{1.17}$$

   Thus, the correct answer is $y = x^3 + C$.

10. Solve the equation: $(1+y^2)y' = \frac{3}{x}$.

    Solution: Separate the variables:

$$\begin{aligned}(1+y^2)\frac{dy}{dx} &= \frac{3}{x} \\ (1+y^2)dy &= \frac{3}{x}dx \\ \int (1+y^2)dy &= \int \frac{3}{x}dx \\ \int 1dy + \int y^2 dy &= 3\int \frac{1}{x}dx \\ y + \frac{y^3}{3} &= 3\ln|x| + \ln C \\ y + \frac{y^3}{3} &= \ln(C|x|^3) \\ e^{y+y^3/3} &= C|x|^3\end{aligned} \tag{1.18}$$