CHAPTER

01

Trigonometric Functions and Identities

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Session 1

Measurement of Angles

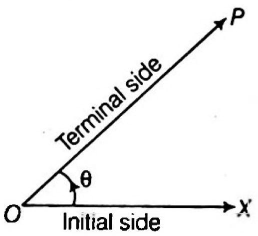
The word 'Trigonometry' is derived from two Greek words.

(i) trigonon (ii) metron

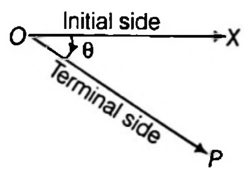
The word trigonon means a triangle and the word metron mean a measure. Hence, trigonometry means measuring the sides and angle of triangle. The subject was originally develop to solve geometric problems involving triangle.

Angle

In trigonometry, as in case of geometry. Angle is measure of rotation from the direction of one ray about its initial point. The original ray called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anti-clockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative



(i) Positive angle



(ii) Negative angle

Measurement of Angles

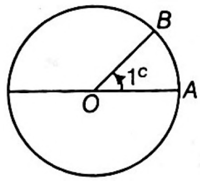
There are three systems used for the measurement of angles.

1. Sexagesimal system or English system (degree)
2. Circular measurement (radian)
3. Centesimal system or French system (grade)

We shall describe the units of measurement of angle which are most commonly used, i.e sexagesimal system (degree measure) and circular measurement (radian measure)

1. **Sexagesimal or Degree measure** If a rotation from the initial side to the terminal side is (1/360)th of a revolution, the angle is said to have a measure of one degree, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as $1'$; one sixtieth of minute is called a second, written as $1''$. Thus, $1^\circ = 60'$ and $1' = 60''$.

2. **Circular measurement or Radian measure** The angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle is called a radian and denoted by 1^c.



3. **Centesimal or French system** In this system of measurement a right angle is divided into 100 equal parts called **Grades**. Each grade is then divided into 100 equal parts called **minutes** and each minute is further divided into 100 equal parts called **Seconds**.

Thus, right angle $= 100^g$

$$1^\circ = 100'$$

$$1' = 100''$$

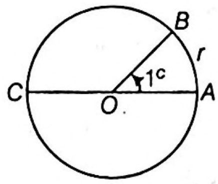
Note

Angle of 90° is called a right angle $1'$ of centesimal system $\neq 1'$ of sexagesimal system $1''$ of centesimal system $\neq 1''$ of sexagesimal system.

This system of measurement of angles is not commonly used and so here we will not study this system of measurement of angles.

Radian is a Constant Angle

Let ABC be a circle whose centre is O and radius is r. Let the length of arc AB of the circle by equal to r. Then by the definition of radian.



$$\angle AOB = 1 \text{ radian}$$

Produce AO and let it cut the circle at C. Then AC is a diameter of the circle and arc ABC is equal to half the circumference of the circle.

Also $\angle AOC = 2$ right angle $= 180^\circ$

By geometry, we know that angles subtended at the centre of a circle are proportional to the lengths of the arcs which subtend them

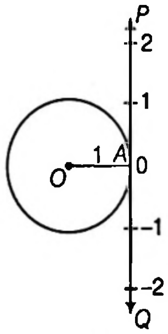
$$\therefore \qquad \frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } ABC} \text{ or } \frac{1^c}{180^\circ} = \frac{r}{\frac{2\pi r}{2}}$$

[$\because$ circumference of the circle $= 2\pi r$]

$$\therefore \qquad 1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{2 \text{ right angle}}{\pi} = \text{constant}$$

[since a right angle and π are constants]

Relation between Radians and Real Numbers



Consider a unit circle with center O. Let A be any point on the circle. Consider OA as the initial side of an angle. Then the length of an arc of the circle gives the radian measure of the angle which the arc subtends at the center of the circle. Consider line PAQ which is tangent to the circle at A. Let point A represents the real number zero, AP represents a positive real number, and AQ represents a negative real number. If we rope line AP in the counter-clockwise direction along the circle, and AQ in the clockwise direction, then every real number corresponds to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

Relation between Degree and Radian

It follows that the magnitude in radian of one complete revolution (360 degree) is the length of the entire circumference divided by the radius, or $\frac{2\pi r}{r}$ or 2π.

Therefore, 2π radian $= 360^\circ$

or $\qquad \pi$ radian $= 180^\circ$

or $\quad 1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ 16'$ (approximately)

Again, $180^\circ = \pi$ radian

$\therefore 1^\circ = \frac{\pi}{180}$ radian $= 0.01746$ radian (approximately)

Thus radian measure of an angle $= \frac{\pi}{180} \times$ degree measure of the angle and degree measure of an angle $= \frac{180}{\pi} \times$ radian measure of the angle.

Thus if the measure of an angle in degrees, and radians be D and C respectively, then

$$\frac{D}{180} = \frac{C}{\pi}$$

The Relation between Degree Measures and Radian Measures of Some Common Angles

Degree	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Note

(i) Radian is the unit to measure angle and **it does not means that π stands for 180°**, π is a real number. Where as π^c **stands for 180°**.
Remember the relation **π radians = 180 degrees = 200 grade**.

(ii) The number of radians in an angle subtended by an arc of a circle at the centre is equal to $\frac{\text{arc}}{\text{radius}}$.

$$\Rightarrow \qquad \theta = \frac{s}{r}$$

| Example 1. Convert $40^\circ 21'$ into radian measure.

Sol. We know that $180^\circ = \pi$ radian.

Hence $\quad 40^\circ 21' = 40\frac{1}{3}$ degree

$$= \frac{\pi}{180} \times \frac{121}{3} \text{ radian} = \frac{121\pi}{540} \text{ radian.}$$

Therefore $40^\circ 21' = \frac{121\pi}{540}$ radian.

| Example 2. Express the following angle in degrees.

(i) $\left(\frac{5\pi}{12}\right)^c$ $\qquad$ (ii) $-\left(\frac{7\pi}{12}\right)^c$

(iii) $\frac{1^c}{3}$ $\qquad$ (iv) $-\frac{2\pi^c}{9}$

Sol. (i) $\left(\frac{5\pi}{12}\right)^c = \left(\frac{5\pi}{12} \times \frac{180}{\pi}\right)^\circ = (5 \times 15)^\circ = 75^\circ$

(ii) $-\left(\frac{7\pi}{12}\right)^c = -\left(\frac{7\pi}{12} \times \frac{180}{\pi}\right)^\circ$

$= -(7 \times 15)^\circ = -105^\circ$

(iii) $\left(\frac{1}{3}\right)^c = -\left(\frac{1}{3} \times \frac{180}{\pi}\right)^\circ = -\left(\frac{60}{\pi}\right)^\circ = 19^\circ 5' 27''$

(iv) $-\frac{2\pi^c}{9} = -\left(\frac{2\pi}{9} \times \frac{180}{\pi}\right)^\circ = -(2 \times 20)^\circ = -40^\circ$

| Example 3. Express the following angle in degrees, minutes and seconds form

$$(321.9)^\circ$$

Sol. $(321.9)^\circ = 321^\circ + 0.9^\circ$

$= 321^\circ + (0.9^\circ \times 60)'$

$= 321^\circ + 54' = 321^\circ 54'$

Example 4. In ΔABC, $m\angle A = \dfrac{2\pi^c}{3}$ and $m\angle B = 45°$. Find $m\angle C$ in both the systems.

Sol. $m\angle A = \dfrac{2\pi^c}{3} = \left(\dfrac{2\pi}{3} \times \dfrac{180}{\pi}\right)^\circ = 120°$

$$m\angle B = 45°$$
$$= \left(45 \times \frac{\pi}{180}\right)^c = \frac{\pi^c}{4}$$

In ΔABC, $m\angle A + m\angle B + m\angle C = 180°$

$\because$ The sum of angles of a triangle is 180°

$\Rightarrow \quad 120° + 45° + m\angle C = 180°$

$\Rightarrow \quad 165° + m\angle C = 180°$

$\Rightarrow \quad m\angle C = 180° - 165°$

$\Rightarrow \quad m\angle C = 15°$

$\Rightarrow \quad m\angle C = \left(15 \times \dfrac{\pi}{180}\right)^\circ$

$\therefore \quad m\angle C = \dfrac{\pi^c}{12}$

Example 5. The sum of two angles is $5\pi^c$ and their difference is 60°. Find the angles in degrees.

Sol. Let the angles be x and y in degrees.

Then, $\quad x + y = 5\pi^c \Rightarrow x + y = \left(5\pi \times \dfrac{180}{\pi}\right)^\circ$

$\therefore \quad x + y = 900° \quad$ ...(i)

$\quad x - y = 60° \quad$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2x = 960°$$

$\therefore \quad x = 480°$

On putting $x = 480°$ in Eq. (i), we get

$$480° + y = 900°$$

$\therefore \quad y = 420°$

$\therefore$ Hence, the angles are 480° and 420°.

Example 6. One angle of a quadrilateral has measure $\dfrac{2\pi^c}{5}$ and the measures of other three angles are in the ratio 2 : 3 : 4. Find their measures in radians and in degrees.

Sol. One angle $= \dfrac{2\pi^c}{5} = \left(\dfrac{2\pi}{5} \times \dfrac{180}{\pi}\right)^\circ = 72°$

Since, measures of other three angles are in the ratio 2 : 3 : 4. Let the angle be $2k$, $3k$ and $4k$ measured in degree.

$\therefore$ Sum of all angles of quadrilateral = 360°

$\Rightarrow \quad 72° + 2k + 3k + 4k = 360°$

$\Rightarrow \quad 9k = 288° \Rightarrow k = 32°$

$\therefore$ The other three angles are

$$2k = 2 \times 32 = 64°$$
$$3k = 3 \times 32 = 96°$$
$$4k = 4 \times 32 = 128°$$

$\therefore$ The other three angles measured in degree are 64°, 96° and 128°.

The angles in radians are

$$64° = \left(64 \times \frac{\pi}{180}\right)^c = \frac{16\pi^c}{45}$$
$$96° = \left(96 \times \frac{\pi}{180°}\right)^c = \frac{8\pi^c}{15}$$
$$128° = \left(128 \times \frac{\pi}{180}\right)^c = \frac{32\pi^c}{45}$$

$\therefore$ The other three angles measured in radian are

$$\frac{16\pi^c}{45}, \frac{8\pi^c}{15} \text{ and } \frac{32\pi^c}{45}.$$

Example 7. Express the following angles in radians.

(i) 120° (ii) −600°

(iii) −144°

Sol. (i) $120° = \left(120 \times \dfrac{\pi}{180}\right)^c = \dfrac{2\pi^c}{3}$

(ii) $-600° = -\left(600 \times \dfrac{\pi}{180}\right)^c = -\dfrac{10\pi}{3}$

(iii) $-144° = \left(-144 \times \dfrac{\pi}{180}\right) = -\dfrac{4\pi^c}{5}$

Example 8. If the three angles of a quadrilateral are $60°$, 60^g and $\dfrac{5\pi}{6}$. Then, find the fourth angle.

Sol. First angle = 60°

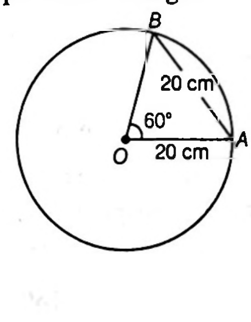
Second angle $= 60^g = 60 \times \dfrac{90}{100}$ degrees $= 54°$

Third angle $= \dfrac{5\pi}{6}$ radian $= \dfrac{5 \times 180}{6} = 150°$

$\therefore \quad$ Fourth angle $= 360° - (60° + 54° + 150°) = 96°$

Example 9. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.

Sol. Let arc $AB = S$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, ΔOAB is an equilateral triangle.



Hence, $\angle AOB = 60°$

$$= \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$$

Now, $\quad \theta = \dfrac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \dfrac{\pi}{3} = \dfrac{S}{20} \Rightarrow S = \dfrac{20\pi}{3}$ cm

Example 10. In the circle of 5 cm. radius, what is the length of the arc which subtends and angle of 33°15′ at the centre.

Sol. Here, $r = 5$ cm; $15' = \frac{15}{60} = \left(\frac{1}{4}\right)$

$\therefore \quad \theta = 33°15' = 33 + \frac{1}{4} = \frac{133}{4}$ degrees

$$= \frac{133}{4} \times \frac{\pi}{180} = \frac{133}{4} \times \frac{22}{7 \times 180} = \frac{1463}{2520} \text{ radians}$$

Now, $\theta = \frac{l}{r}$

$\therefore \quad l = \theta r = \frac{1463}{2520} \times 5 = 2\frac{65}{72}$ cm (approx.)

Example 11. The minute hand of a watch is 35 cm long. How far does its tip move in 18 minutes? $\left(\text{use } \pi = \frac{22}{7}\right)$

Sol. The minute hand of a watch completes one revolution in 60 minutes. Therefore the angle traced by a minute hand in 60 minutes = 360° = 2π radians.

$\therefore$ Angle traced by the minute hand in 18 minutes

$$= 2\pi \times \frac{18}{60} \text{ radians} = \frac{3\pi}{5} \text{ radians}$$

Let the distance moved by the tip in 18 minutes be l, then $l = r\theta$

$$= 35 \times \frac{3\pi}{5} = 21\pi = 21 \times \frac{22}{7} = 66 \text{ cm}$$

Example 12. The wheel of a railway carriage is 40 cm. in diameter and makes 6 revolutions in a second; how fast is the train going?

Sol. Diameter of the wheel = 40 cm

$\therefore$ radius of the wheel = 20 cm

Circumference of the wheel = $2\pi r = 2\pi \times 20 = 40\pi$ cm

Number of revolutions made in 1 second = 6

$\therefore$ Distance covered in 1 second = $40\pi \times 6 = 240\pi$ cm

$\therefore$ Speed of the train = 240π cm/sec.

Example 13. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of the letters that he can read at a distance of 12 metres.

Sol. Let the height of the letters be h metres.

Now, h many be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

$\therefore \quad \theta = 5' = \left(\frac{5}{60} \times \frac{\pi}{180}\right) \text{radians} = \left(\frac{\pi}{12 \times 180}\right) \text{radian}$

and $\quad r = 12$ m

$\therefore \quad h = r\theta = 12 \times \frac{\pi}{12 \times 180} = \left(\frac{\pi}{180}\right) \text{metres} \approx 1.7 \text{ cm}$

Exercise for Session 1

1. The difference between two acute angles of a right angle triangle is $\frac{3\pi}{10}$ rad. Find the angles in degree.
2. Find the length of an arc of a circle of radius 6 cm subtending an angle of 15° at the centre.
3. A horse is tied to post by a rope. If the horse moves along circular path always keeping the tight and describes 88 m, when it has traced out 72° at centre, find the length of rope.
4. Find the angle between the minute hand and hour hand of a clock, when the time is 7 : 30 pm.
5. If OQ makes 4 revolutions in 1s, find the angular velocity in radians per second.
6. If a train is moving on the circular path of 1500 m radius at the rate of 66 km/h, find the angle in radian, if it has in 10 second.
7. Find the distance from the eye at which a coin of 2.2 cm diameter should be held so as to conceal the full moon with angular diameter 30′.
8. The wheel of a railway carriage is 40 cm in diameter and makes 7 revolutions in a second, find the speed of train.
9. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of letters that he can read a distance of 12 m.
10. For each natural number k, let C_K denotes the circle with radius k cm and centre at origin. On the circle C_K, a particle moves k cm in the counter-clockwise direction. After completing its motion on C_K, the particle moves on C_{K+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then n is equal to

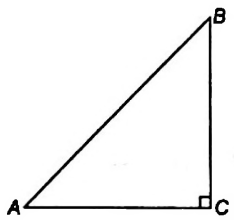
Session 2

Definition of Trigonometric Functions

Definition of Trigonometric Functions

An angle whose measure is greater than 0° but less than 90° is called an acute angle.

In a right angled triangle ABC, $\angle CAB = A$ and $\angle BCA = 90^\circ = \pi/2$. AC is the base, BC the altitude and AB is the hypotenuse. We refer to the base as the adjacent side and to the altitude as the opposite side. There are six trigonometric ratios, also called **trigonometric functions or circular functions** with reference to $\angle A$, the six ratio are



$\frac{BC}{AB} = \frac{\text{opposite side}}{\text{hypotenuse}}$, is called sine of A, and written as $\sin A$.

$\frac{AC}{AB} = \frac{\text{adjacent side}}{\text{hypotenuse}}$, is called the cosine of A, and written as $\cos A$.

$\frac{BC}{AC} = \frac{\text{opposite side}}{\text{adjacent side}}$, is called the tangent of A, and written as $\tan A$.

$\frac{AB}{BC} = \frac{\text{hypotenuse}}{\text{opposite side}}$, is called cosecant of A, and written as $\operatorname{cosec} A$.

$\frac{AB}{AC} = \frac{\text{hypotenuse}}{\text{opposite side}}$, is called secant of A, and written as $\sec A$.

$\frac{AC}{BC} = \frac{\text{adjacent side}}{\text{opposite side}}$, is called cotangent of A, and written as $\cot A$.

Since, the hypotenuse is the greatest side in a right angle triangle, $\sin A$ and $\cos A$ can never be greater than unity and $\operatorname{cosec} A$ and $\sec A$ can never be less than unity.

Hence, $|\sin A| \le 1$, $\cos A| \le 1|$, $|\operatorname{cosec} A| \ge 1$, $|\sec A| \ge 1$, while $\tan A$ and $\cot A$ may have any numerical value lying between $-\infty$ to $+\infty$.

Note

Student must remember the following results

(i) $-1 \le \sin A \le 1$ (ii) $-1 \le \cos A \le 1$

(iii) $\operatorname{cosec} A \ge 1$ or $\operatorname{cosec} A \le -1$ (iv) $\sec A \ge 1$ or $\sec A \le -1$

(v) $\tan A \in R$ (vi) $\cot A \in R$

Some values of Trigonometrical Ratios

Students are already familiar with the values of sin, cos, tan, cot, sec and cosec of angles 0°, 30°, 45°, 60° and 90° which have been given in the following table

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cot	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cosec	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Trigonometric Identities

Trigonometric identities are equalities that involve trigonometric functions that are true for every single value of the occurring variables. In other words, they are equations that hold true regardless of the value of the angles being chosen.

Trigonometric identities are as follows

1. $\sin^2 A + \cos^2 A = 1 \Rightarrow \cos^2 A = 1 - \sin^2 A$

   or $\quad \sin^2 A = 1 - \cos^2 A$

2. $1 + \tan^2 A = \sec^2 A \Rightarrow \sec^2 A - \tan^2 A = 1$

3. $\cot^2 A + 1 = \operatorname{cosec}^2 A$

   $\Rightarrow \quad \operatorname{cosec}^2 A - \cot^2 A = 1$

4. $\tan A = \dfrac{\sin A}{\cos A}$ and $\cot A = \dfrac{\cos A}{\sin A}$

5. Fundamental inequalities: For $0 < A < \pi / 2$;

$$0 < \cos A < \frac{\sin A}{A} < \frac{1}{\cos A}$$

6. It is possible to express trigonometrical ratios in terms of any one of them as,

$$\sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}},$$

$$\cos\theta = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}}, \tan\theta = \frac{1}{\cot\theta},$$

$$\operatorname{cosec}\theta = \sqrt{1+\cot^2\theta}, \sec\theta = \frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$$

i.e. all trigonometrical functions have been expressed in terms of $\cot\theta$.

Similarly, we can express all trigonometric function in other trigonometric ratios.

| Example 14. Show that $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0.$

Sol. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

$$= 2[(\sin^2 x)^3 + (\cos^2 x)^3] - 3(\sin^4 x + \cos^4 x) + 1$$

$$= 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x$$

$$(\sin^2 x + \cos^2 x) - 3[\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x] + 1$$

$$= 2[1 + 3\sin^2 x \cos^2 x] - 3[1 - 2\sin^2 x \cos^2 x] + 1 = 0$$

| Example 15. Show that

(i) $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cdot \cos^2 A)$

(ii) $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

Sol. (i) L.H.S. $= \sin^8 A - \cos^8 A = (\sin^4 A)^2 - (\cos^4 A)^2$

$$= (\sin^4 A - \cos^4 A)(\sin^4 A + \cos^4 A)$$

$$= (\sin^2 A - \cos^2 A)(\sin^2 A + \cos^2 A)$$

$$[(\sin^2 A + \cos^2 A)^2 - 2\sin^2 A \cos^2 A]$$

$$= (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A \cos^2 A)$$

$$[\because \sin^2 A + \cos^2 A = 1]$$

(ii) Given, $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

or $\dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A} = \dfrac{1}{\cos A} + \dfrac{1}{\cos A}$

Here, R.H.S. $= \dfrac{2}{\cos A}$

Now L.H.S. $= \dfrac{1}{\sec A - \tan A} \dfrac{1}{\sec A + \tan A}$

$$= \frac{\sec A + \tan A + \sec A - \tan A}{(\sec A - \tan A)(\sec A + \tan A)} = \frac{2}{\cos A}$$

Thus, L.H.S. = R.H.S.

| Example 16. If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$ and $\sec\theta$.

Sol. Given, $\sec\theta + \tan\theta = \dfrac{3}{2}$ ...(i)

Now, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{2}{3}$ ...(ii)

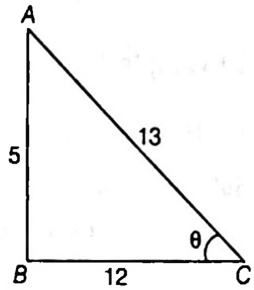
Adding Eqs. (i) and (ii), we get

$$2\sec\theta = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}$$

$\therefore$ $\sec\theta = \dfrac{13}{12}$

$\therefore$ $\tan\theta = \dfrac{5}{12}$

and $\sin\theta = \dfrac{5}{13}$



| Example 17. If $\dfrac{\cos^4 A}{\cos^2 B} + \dfrac{\sin^4 A}{\sin^2 B} = 1$, then prove that

(i) $\sin^4 A + \sin^4 B = 2\sin^2 A \sin^2 B$

(ii) $\dfrac{\cos^4 B}{\cos^2 A} + \dfrac{\sin^4 B}{\sin^2 A} = 1$

Sol. Given, $\dfrac{\cos^4 A}{\cos^2 B} + \dfrac{\sin^4 A}{\sin^2 B} = 1(\cos^2 A + \sin^2 A)$

or $\dfrac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \dfrac{\sin^4 A}{\sin^2 B}$

or $\dfrac{\cos^2 A(\cos^2 A - \cos^2 B)}{\cos^2 B} = \sin^2 A \dfrac{(\sin^2 B - \sin^2 A)}{\sin^2 B}$

or $\dfrac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \dfrac{\sin^2 A}{\sin^2 B}$

$$[(1 - \cos^2 B) - (1 - \cos^2 A)]$$

or $\dfrac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \dfrac{\sin^2 A}{\sin^2 B}(\cos^2 A - \cos^2 B)$

or $(\cos^2 A - \cos^2 B)\left(\dfrac{\cos^2 A}{\cos^2 B} - \dfrac{\sin^2 A}{\sin^2 B}\right) = 0$

When $\cos^2 A - \cos^2 B = 0$, we have

$$\cos^2 A = \cos^2 B \quad \text{...(i)}$$

When $\dfrac{\cos^2 A}{\cos^2 B} - \dfrac{\sin^2 A}{\sin^2 B} = 0$, we have

$$\cos^2 A \sin^2 B = \sin^2 A \cos^2 B$$

or $\cos^2 A(1-\cos^2 B)=(1-\cos^2 A)\cos^2 B$

or $\cos^2 A-\cos^2 A\cos^2 B=\cos^2 B-\cos^2 A\cos^2 B$

or $\cos^2 A=\cos^2 B$ ...(ii)

Thus, in both the cases, $\cos^2 A=\cos^2 B$. Therefore,

$\therefore\quad 1-\sin^2 A=1-\sin^2 B$ or $\sin^2 A=\sin^2 B$ ...(iii)

(i) L.H.S. $=\sin^4 A+\sin^4 B$

$=(\sin^2 A-\sin^2 B)^2+2\sin^2 A\sin^2 B$

$=2\sin^2 A\sin^2 B=$ R.H.S. $[\because \sin^2 A=\sin^2 B]$

(ii) L.H.S. $=\dfrac{\cos^4 B}{\cos^2 A}+\dfrac{\sin^4 B}{\sin^2 A}=\dfrac{\cos^4 B}{\cos^2 B}+\dfrac{\sin^4 B}{\sin^2 B}$

$=\cos^2 B+\sin^2 B=1=$ R.H.S.

Example 18. If $\tan^2\theta=1-e^2$, prove that $\sec\theta+\tan^3\theta\operatorname{cosec}\theta=(2-e^2)^{\frac{3}{2}}$

Sol. Given, $\tan^2\theta=1-e^2$

Now, L.H.S. $=\sec\theta+\tan^3\theta\operatorname{cosec}\theta$

$=\sec\theta\left(1+\tan^3\theta\dfrac{\operatorname{cosec}\theta}{\sec\theta}\right)$

$=\sec\theta(1+\tan^3\theta\cdot\cot\theta)=\sec\theta(1+\tan^2\theta)=\sec\theta\sec^2\theta$

$=\sec^3\theta=(\sec^2\theta)^{\frac{3}{2}}=(1+\tan^2\theta)^{\frac{3}{2}}=(1+1-e^2)^{\frac{3}{2}}=(2-e^2)^{\frac{3}{2}}$

Example 19. For what real values of x and y is the equation $\sec^2\theta=\dfrac{4xy}{(x+y)^2}$ possible?

Sol. Here, $\sec^2\theta=\dfrac{4xy}{(x+y)^2}$

We know $\sec^2\theta\ge 1$ and $\dfrac{4xy}{(x+y)^2}\le 1$ [as AM $\ge$ GM]

$\Rightarrow\quad \sec^2\theta=\dfrac{4xy}{(x+y)^2}$ is only possible if $\sec^2\theta=1$

i.e. $\dfrac{4xy}{(x+y)^2}=1,\ \forall\ x,y\in R^+$

or $4xy=(x+y)^2\quad \forall\ x,y\in R^+$

$\Rightarrow\quad x^2+y^2+2xy-4xy=0,\quad \forall\ x,y\in R^+$

$\Rightarrow\quad (x-y)^2=0,\quad \forall\ x,y\in R^+$

or $x=y;\quad \forall\ x,y\in R^+$

Example 20. Show that the equation $\sin\theta=x+\dfrac{1}{x}$ is impossible if x is real.

Sol. Given, $\sin\theta=x+\dfrac{1}{x}$

$\therefore\quad \sin^2\theta=x^2+\dfrac{1}{x^2}+2x\cdot\dfrac{1}{x}$

$=x^2+\dfrac{1}{x^2}+2\ge 2$

which is not possible since $\sin^2\theta\le 1$

Exercise for Session 2

1. Prove that $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$.
2. If $\cos^2\alpha-\sin^2\alpha=\tan^2\beta$, then show that $\tan^2\alpha=\cos^2\beta-\sin^2\beta$.
3. If $\sin^6\theta+\cos^6\theta-1=\lambda\sin^2\theta\cos^2\theta$, find the value of λ.
4. If $a\cos\theta-b\sin\theta=c$, then find the value of $a\sin\theta+b\cos\theta$.
5. Find the value of $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$.
6. If $\sin\theta+\operatorname{cosec}\theta=2$, then find the value of $\sin^{20}\theta+\operatorname{cosec}^{20}\theta$.
7. Let $F_k(x)=\dfrac{1}{k}(\sin^k x+\cos^k x)$, where $x\in R$ and $k\ge 1$, then find the value of $F_4(x)-F_6(x)$.
8. If $\dfrac{\sin^4 x}{2}+\dfrac{\cos^4 x}{3}=\dfrac{1}{5}$, then show that $\dfrac{\sin^8 x}{8}+\dfrac{\cos^8 x}{27}=\dfrac{1}{125}$.
9. If $\cot\theta+\tan\theta=x$ and $\sec\theta-\cos\theta=y$, then show that $\sin\theta\cdot\cos\theta=\dfrac{1}{x}$ or $\sin\theta\cdot\tan\theta=y$ or $(x^2y)^{2/3}-(xy^2)^{2/3}=1$.
10. If $\sin A+\sin^2 A+\sin^3 A=1$, then find the value of $\cos^6 A-4\cos^4 A+8\cos^2 A$.

Session 3

Application of Basic Trigonometry on Eliminating Variables or Parameters and Geometry

Application of Basic Trigonometry on Eliminating Variables or Parameters

As we know, parameter are those values which could vary, e.g. θ if parameter could take any value as;

$$\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, \ldots$$

Thus, to eliminate these parameter, we have to use basic trigonometric formulae, it could be more clear by some examples :

| Example 21. If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, eliminate θ.

Sol. Given, $\operatorname{cosec}\theta - \sin\theta = m$ or, $\frac{1}{\sin\theta} - \sin\theta = m$

or, $\frac{1-\sin^2\theta}{\sin\theta} = m$ or $\frac{\cos^2\theta}{\sin\theta} = m$ ...(i)

Again $\sec\theta - \cos\theta = n$

or $\frac{1}{\cos\theta} - \cos\theta = n$

or $\frac{1-\cos^2\theta}{\cos\theta} = n$ or $\frac{\sin^2\theta}{\cos\theta} = n$ ...(ii)

From Eq. (i) $\sin\theta = \frac{\cos^2\theta}{m}$ ...(iii)

Putting in (ii), we get $\frac{\cos^4\theta}{m^2\cos\theta} = n$ or, $\cos^3\theta = m^2 n$

$\therefore \cos\theta = (m^2n)^{\frac{1}{3}}$ or, $\cos^2\theta = (m^2n)^{\frac{2}{3}}$ ...(iv)

From Eq. (iii), $\sin\theta = \frac{\cos^2\theta}{m} = \frac{(m^2n)^{\frac{2}{3}}}{m}$

$$= \frac{m^{\frac{4}{3}}n^{\frac{2}{3}}}{m} = m^{\frac{1}{3}}n^{\frac{2}{3}} = (mn^2)^{\frac{1}{3}}$$

$\therefore \quad \sin^2\theta = (mn^2)^{\frac{2}{3}}$ ...(v)

Adding Eqs. (iv) and (v), we get

$$(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta$$

or, $(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = 1$

| Example 22. If $3\sin\theta + 4\cos\theta = 5$, then find the value of $4\sin\theta - 3\cos\theta$.

Sol. Let $4\sin\theta - 3\cos\theta = a$ ...(i)

Thus, we want to eliminate θ from both $3\sin\theta + 4\cos\theta = 5$ and $4\sin\theta - 3\cos\theta = a$, i.e. squaring and adding these equations, we get

$$(3\sin\theta + 4\cos\theta)^2 + (4\sin\theta - 3\cos\theta)^2 = 25 + a^2$$

$$9\sin^2\theta + 16\cos^2\theta + 24\sin\theta\cos\theta + 16\sin^2\theta + 9\cos^2\theta - 24\cos\theta\sin\theta = 25 + a^2$$

$$9 + 16 = 25 + a^2 \quad \text{or} \quad a^2 = 0$$

$$a = 0$$

$$\therefore \quad 4\sin\theta - 3\cos\theta = 0$$

| Example 23. If $a\sec\alpha - c\tan\alpha = d$ and $b\sec\alpha + d\tan\alpha = c$, then eliminate α from above equations.

Sol. Here, $a\sec\alpha - c\tan\alpha = d$ and $b\sec\alpha + d\tan\alpha = c$ could be written as

$a = d\cos\alpha + c\sin\alpha$ ...(i)

and $b = c\cos\alpha - d\sin\alpha$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$a^2 + b^2 = (d\cos\alpha + c\sin\alpha)^2 + (c\cos\alpha - d\sin\alpha)^2$$

$$\Rightarrow \quad a^2 + b^2 = d^2\cos^2\alpha + c^2\sin^2\alpha + 2dc\cos\alpha\sin\alpha + c^2\cos^2\alpha + d^2\sin^2\alpha - 2cd\cos\alpha\sin\alpha.$$

$$= d^2(\cos^2\alpha + \sin^2\alpha) + c^2(\sin^2\alpha + \cos^2\alpha)$$

$$\therefore \quad a^2 + b^2 = c^2 + d^2$$

| Example 24. Eliminate θ between the equations $a\sec\theta + b\tan\theta + c = 0$ and $p\sec\theta + q\tan\theta + r = 0$.

Sol. Given $a\sec\theta + b\tan\theta + c = 0$ ...(i)

and $p\sec\theta + q\tan\theta + r = 0$ ...(ii)

Solving Eqs. (i) and (ii) by cross multiplication method, we have

$$\underset{\text{(i)}}{\frac{\sec\theta}{br - qc}} = \underset{\text{(ii)}}{\frac{\tan\theta}{pc - ar}} = \underset{\text{(iii)}}{\frac{1}{aq - pb}}$$

From Eqs. (i) and (iii), we get

$$\sec\theta = \frac{br - qc}{aq - pb} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$\tan\theta = \frac{pc - ar}{pc - pb} \quad \text{...(iv)}$$

$\because \quad \sec^2\theta - \tan^2\theta = 1$

$\therefore \quad \left(\frac{br - qc}{aq - pb}\right)^2 - \left(\frac{pc - ar}{aq - pb}\right)^2 = 1$

or $\quad (br - qc)^2 - (pc - ar)^2 = (aq - pb)^2$

Example 25. If $x = \sec\theta - \tan\theta$ and $y = \operatorname{cosec}\theta + \cot\theta$, then prove that $xy + 1 = y - x$.

Sol. $xy + 1 = \left(\frac{1 - \sin\theta}{\cos\theta}\right)\left(\frac{1 + \cos\theta}{\sin\theta}\right) + 1 = \frac{1 - \sin\theta + \cos\theta}{\sin\theta\cos\theta}$

$= \frac{(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta} - \frac{(\sin\theta - \cos\theta)}{\sin\theta\cos\theta}$

$= (\tan\theta + \cot\theta) - (\sec\theta - \operatorname{cosec}\theta)$

$= (\operatorname{cosec}\theta + \cot\theta) - (\sec\theta - \tan\theta) = y - x$

Example 26. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. Find the value of $x^2 + y^2 + z^2$.

Sol. Here,

$x^2 + y^2 + z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$

$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta$

$= r^2\sin^2\theta + r^2\cos^2\theta$

$= r^2$

$\therefore \; x^2 + y^2 + z^2 = r^2$

Example 27. If $0 < \theta < \frac{\pi}{2}$, $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\cdot\sin^{2n}\theta$, then show $xyz = xy + z$.

Sol. Here, $x = \sum_{n=0}^{\infty}\cos^{2n}\theta = 1 + \cos^2\theta + \cos^4\theta + \cos^6\theta + \dots\infty$

$= \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta}$

[using, $S_\infty = \frac{a}{1 - r}$ sum of infinite GP]

Similarly, $\quad y = \frac{1}{1 - \sin^2\theta} = \frac{1}{\cos^2\theta}$

and $\quad z = \frac{1}{1 - \sin^2\theta\cdot\cos^2\theta}$

$\therefore \quad xyz = \frac{1}{\sin^2\theta\cos^2\theta\,(1 - \sin^2\theta\cos^2\theta)}$

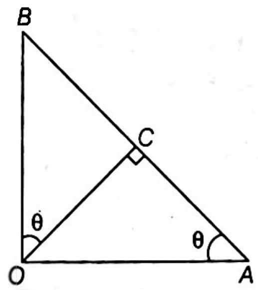
$= \frac{(1 - \sin^2\theta\cos^2\theta) + (\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^2\theta\,(1 - \sin^2\theta\cos^2\theta)}$

$= \frac{1}{\sin^2\theta\cos^2\theta} + \frac{1}{1 - \sin^2\theta\cos^2\theta}$

$= xy + z$

Application of Basic Trigonometry in Geometry

Example 28. If in given fig, $\tan(\angle BAO) = 3$, then find the ratio $BC : CA$.



Sol. From Fig., we have

$\tan\theta = 3$

In ΔOCA and ΔOCB respectively, we get

$\therefore \quad \frac{OC}{AC} = \tan\theta, \; \frac{OC}{BC} = \cot\theta$

On dividing, we get

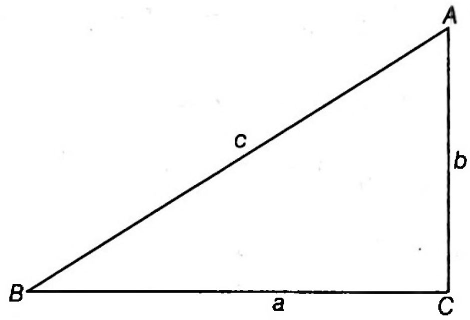
or $\quad \frac{BC}{AC} = \frac{\tan\theta}{\cot\theta} = \tan^2\theta = 9$

$\Rightarrow \quad BC : AC = 9 : 1$

Example 29. If angle C of triangle ABC is $90°$, then prove that $\tan A + \tan B = \frac{c^2}{ab}$ (where, a, b, c are sides opposite to angles A, B, C, respectively).

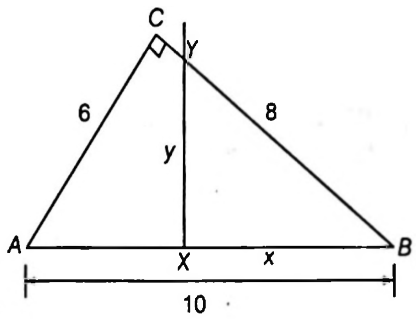
Sol. Draw ΔABC with $\angle C = 90°$. We have

$$\tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$$



Example 30. In triangle ABC, $BC = 8$, $CA = 6$, and $AB = 10$. A line dividing the triangle ABC into two regions of equal area is perpendicular to AB at point X. Find the value of $\frac{BX}{\sqrt{2}}$.

Sol.



We have area of $\Delta XYB = \frac{1}{2}$ area of ΔABC

$\therefore$ $$\frac{1}{2}(XY)\cdot(XB) = \frac{1}{2}\times\frac{1}{2}\times AC\times BC$$

$$2\left(\frac{x\times y}{2}\right) = \frac{8\times 6}{2} = 24$$

or $$x\times x\tan B = 24 \qquad [\because y = x\tan B]$$

or $$x^2\times\frac{3}{4} = 24 \qquad [\because \tan B = \frac{AC}{BC}]$$

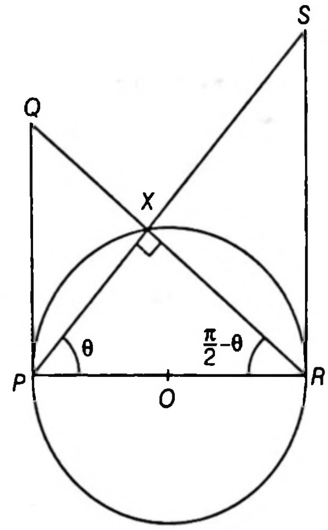
or $$x^2 = 32 \text{ or } x = 4\sqrt{2}$$

| Example 31. Let PQ and RS be tangents at the extremities of the diameter PR or a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then prove that $2r = \sqrt{PQ\times RS}$.

Sol. From Fig. we have

$$\frac{PQ}{PR} = \tan\left(\frac{\pi}{2}-\theta\right)$$

$$= \cot\theta \text{ and } \frac{RS}{PR} = \tan\theta$$



$\therefore$ $$\frac{PQ}{PR}\times\frac{RS}{PR} = 1$$

or $$(PR)^2 = PQ\times PS$$

or $$(2r)^2 = PQ\times PS$$

or $$2r = \sqrt{PQ\times PS}$$

Exercise for Session 3

1. If $\sec\theta + \tan\theta = k$, find the value of $\cos\theta$.

2. If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta = y\cos\theta$,
Find the value of $x^2 + y^2$.

3. If $\sin A + \cos A = m$ and $\sin^3 A + \cos^3 A = n$, prove that $m^3 - 3m + 2n = 0$.

4. If $\sin^2\theta = \frac{x^2 + y^2 + 1}{2x}$. Find the value of x and y.

5. If $\sin\theta - \sqrt{6}\cos\theta = \sqrt{7}\cos\theta$. Prove that $\cos\theta + \sqrt{6}\sin\theta - \sqrt{7}\sin\theta = 0$.

6. If $\sin x + \sin y + \sin z = 3$. Find the value of $\cos x + \cos y + \cos z$.

7. If $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, $\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta = 1$, then eliminate θ.

8. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$, then prove that $\frac{a^2}{b^2} = \frac{(d-a)(c-a)}{(b-c)(b-d)}$.

9. If $a + b\tan\theta = \sec\theta$ and $b - a\tan\theta = 3\sec\theta$, then find the value of $a^2 + b^2$.

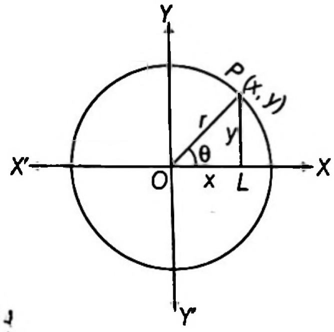
10. Two circles of radii 4 cm and 1 cm touch each other externally and 8 is the angle contained by their direct common tangents. Find the value of $\sin\frac{\theta}{2} + \cos\frac{\theta}{2}$.

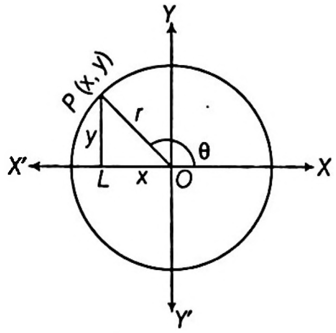
Session 4

Signs and Graph of Trigonometric Functions

Signs of Trigonometric Functions

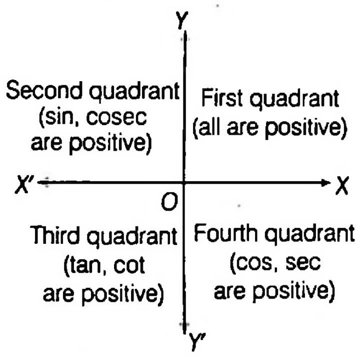
The signs of the trigonometric ratios of an angle depend on the quadrant in which the terminal side of the angle lies. We always take $OP = r$ to be positive (see figure). Thus the signs of all the trigonometric ratios depend on the signs of x and/or y.





An angle is said to be in that quadrant in which its terminal ray lies

For positive acute angles this definition gives the same result as in case of a right angled triangle since x and y are both positive for any point in the first quadrant and consequently they are the length of base and perpendicular of the angle θ.



1. Clearly in first quadrant $\sin\theta, \cos\theta, \tan\theta, \cot\theta, \sec\theta$ and $\operatorname{cosec}\theta$ are all positive as x, y are positive.
2. In second quadrant, x is negative and y is positive, therefore, only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive.
3. In third quadrant, x and y are both negative, therefore, only $\tan\theta$ and $\cot\theta$ are positive.
4. In fourth quadrant, x is positive and y is negative, therefore, only $\cos\theta$ and $\sec\theta$ are positive.

Quadrant →	I	II	III	IV
$\sin\theta$	+	+	–	–
$\cos\theta$	+	–	–	+
$\tan\theta$	+	–	+	–
$\operatorname{cosec}\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

Variation in the Values of Trigonometric Functions in Different Quadrants

We observe that in the first quadrant, as x increases from 0 to $\frac{\pi}{2}$, $\sin x$ increases from 0 to 1 and in the second quadrant as x increases from $\frac{\pi}{2}$ to π, $\sin x$ decreases from 1 to 0.

In the third quadrant, as x increases from π to $\frac{3\pi}{2}$, $\sin x$ decreases from 0 to -1 and finally, in the fourth quadrant, $\sin x$ increases from -1 to 0 as x increase from $\frac{3\pi}{2}$ to 2π.

Function	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	↑ from 0 to 1	↓ from 1 to 0	↓ from 0 to -1	↑ from -1 to 0
$\cos\theta$	↓ from 1 to 0	↓ from 0 to -1	↑ from -1 to 0	↑ from 0 to 1
$\tan\theta$	↑ from 0 to ∞	↑ from $-\infty$ to 0	↑ from 0 to ∞	↑ from $-\infty$ to 0
$\cot\theta$	↓ from ∞ to 0	↓ from 0 to $-\infty$	↓ from ∞ to 0	↓ from 0 to $-\infty$
$\sec\theta$	↑ from 1 to ∞	↑ from $-\infty$ to -1	↓ from -1 to $-\infty$	↓ from ∞ to 1
$\operatorname{cosec}\theta$	↓ from ∞ to 1	↑ from 1 to ∞	↑ from $-\infty$ to -1	↓ from -1 to $-\infty$

Note

$+\infty$ and $-\infty$ are two symbols. These are not real numbers. When we say that $\tan\theta$ increases from 0 to ∞ as θ varies from 0 to $\frac{\pi}{2}$, it means that $\tan\theta$ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains arbitrarily large positive values as θ tends to $\frac{\pi}{2}$. This rule applies to other trigonometric functions also.

Graphs of Trigonometric Functions

As in case of algebraic function, we can have some idea about the nature of a trigonometric function by its graph. Graph has many important applications in mathematical problems. We shall discuss the graphs of trigonometrical functions. We know that $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π and $\tan x$ and $\cot x$ are trigonometric functions of period π. Also if the period of function $f(x)$ is T, then period of $f(ax+b)$ is $\frac{T}{|a|}$.

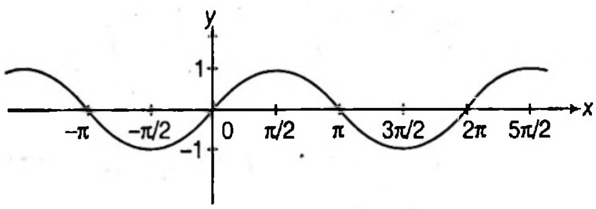
Graph and Other Useful Data of Trigonometric Functions

1. $y = f(x) = \sin x$

   Domain $\to R$,

   Range $\to [-1, 1]$

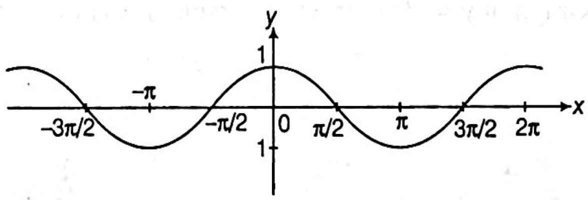
   Period $\to 2\pi$



2. $y = f(x) = \cos x$

   Domain $\to R$, Range $\to [-1, 1]$

   Period $\to 2\pi$

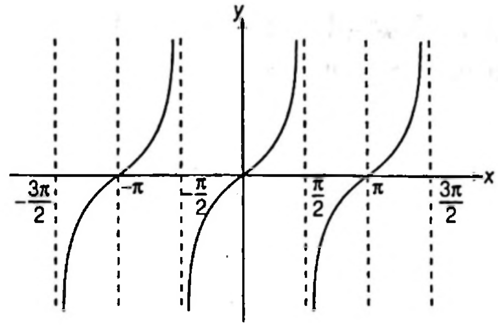


3. $y = f(x) = \tan x$

   Domain $\to R \sim (2n+1)\frac{\pi}{2}, n \in I$

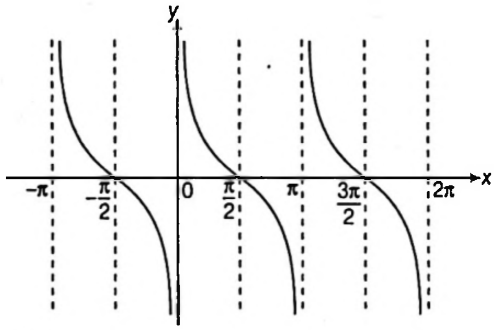
   Range $\to (-\infty, \infty)$

   Period $\to \pi$



4. $y = f(x) = \cot x$

   Domain $\to R \sim n\pi, n \in I$; Range $\to (-\infty, \infty)$; Period $\to \pi$,

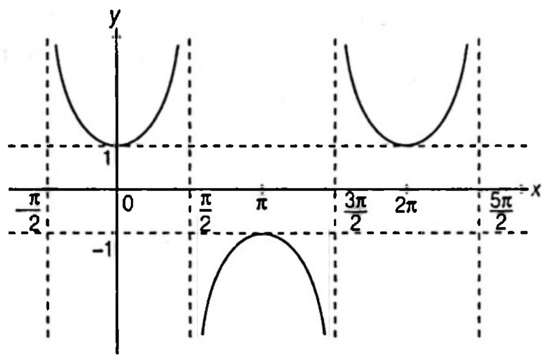


5. $y = f(x) = \sec x$

   Domain $\to R \sim (2n+1)\frac{\pi}{2}, n \in I$

   Range $\to (-\infty, -1] \cup [1, \infty)$

   Period $\to 2\pi$, $\sec^2 x$, $|\sec x| \in [1, \infty)$

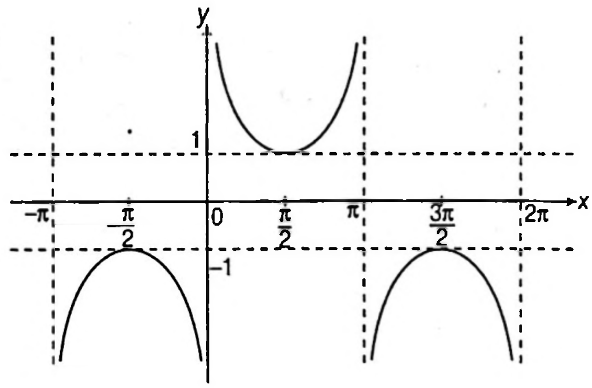


6. $y = f(x) = \operatorname{cosec} x$

   Domain $\to R \sim n\pi, n \in I$;

   Range $\to (-\infty, -1] \cup [1, \infty)$

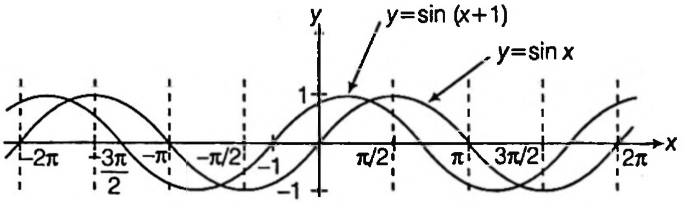
   Period $\to 2\pi$, $\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$



Transformation of the Graphs of Trigonometric Functions

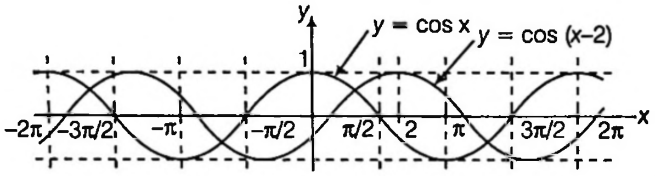
1. To draw the graph of $y = f(x + a)$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units left along the x-axis.

   Consider the following illustration.



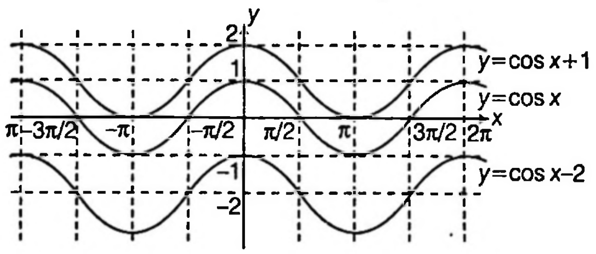
   To draw the graph of $y = f(x - a)$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units right along the x-axis.

   Consider the following illustration.

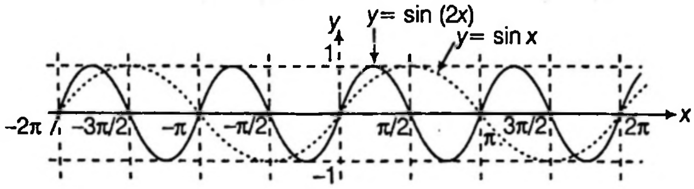


2. To draw the graph of $y = f(x) + a$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units upwards along the y-axis.

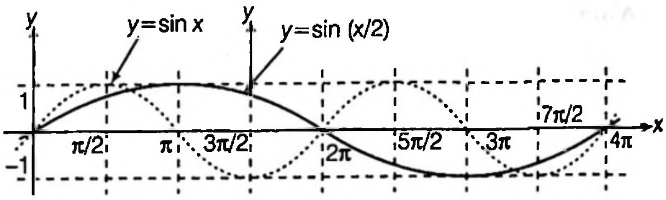
   To draw the graph of $y = f(x) - a$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units downward along the y-axis.



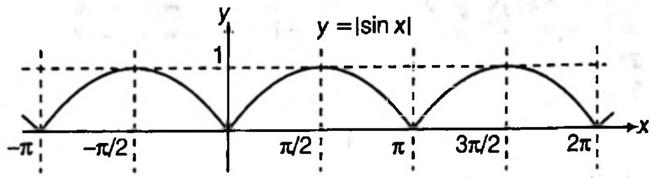
3. If $y = f(x)$ has period T, then period of $y = f(ax)$ is $\dfrac{T}{|a|}$.



Period of $y = \sin(2x)$ is $\dfrac{2\pi}{2} = \pi$



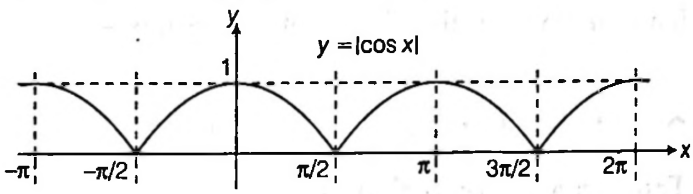
Period of $y = \sin\left(\dfrac{x}{2}\right)$ is $\dfrac{2\pi}{\left(\dfrac{1}{2}\right)} = 4\pi$



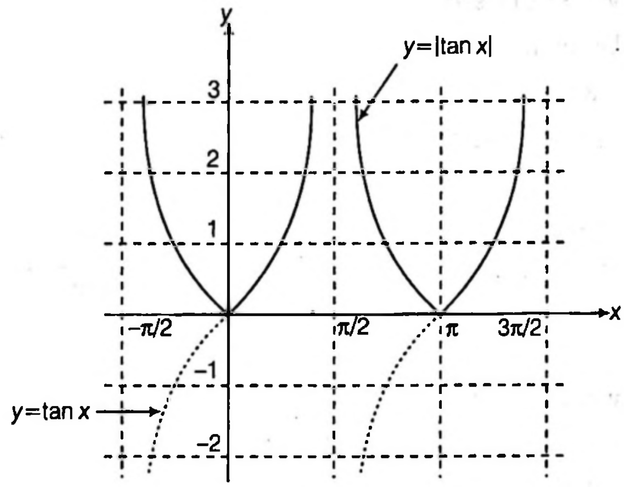
4. Since $y = |f(x)| \geq 0$, to draw the graph of $y = |f(x)|$, take the mirror of the graph of $y = f(x)$ in the x-axis for $f(x) < 0$, retaining the graph for $f(x) > 0$.

   Consider the following illustrations.

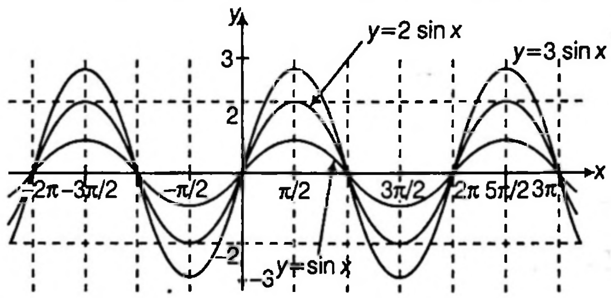
   Here, period of $f(x) = |\sin x|$ is π.

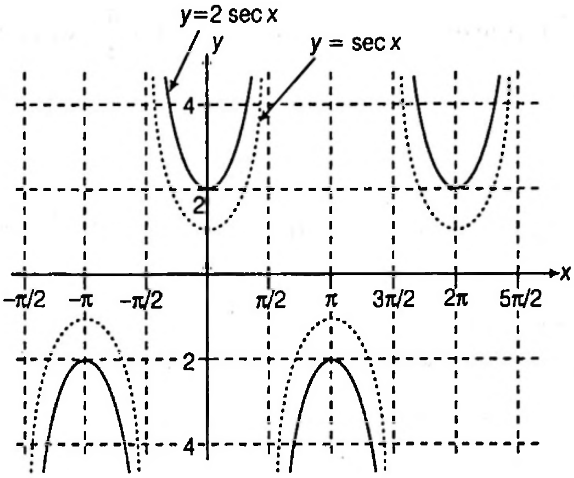


   Here, period of $f(x) = |\cos x|$ is π.



5. Graph of $y = af(x)$ from the graph of y $f(x)$

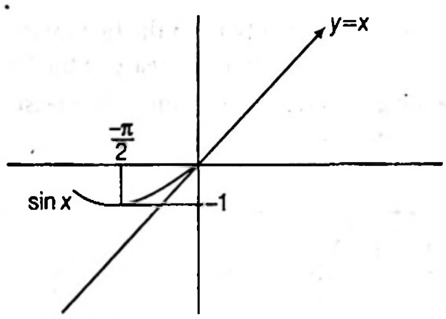




Some Important Graphical Deductions

To find relation between $\sin x$, x and $\tan x$

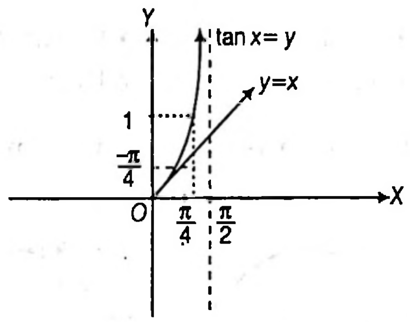
(i)



Thus, when $-\infty < x < 0$

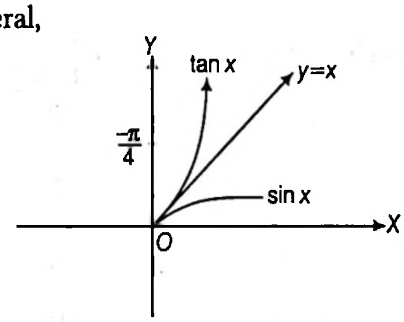
$\Rightarrow \sin x > x$

(ii)



$\therefore \quad \tan x > x$, when $0 > x > \dfrac{\pi}{2}$

(iii) In general,



Thus, $\tan x > x > \sin x, \forall\, x \in \left(0, \dfrac{\pi}{2}\right)$

and $\sin x > x > \tan x, \forall\, x \in \left(-\dfrac{\pi}{2}, 0\right)$.

| Example 32. Find the values of the other five trigonometric functions in each of the following questions

(i) $\tan\theta = \dfrac{5}{12}$, where θ is in third quadrant.

(ii) $\sin\theta = \dfrac{3}{5}$, where θ is in second quadrant.

Sol. (i) Since θ is in third quadrant,

$\therefore$ Only $\tan\theta$ and $\cot\theta$ are positive

Now, $\quad \tan\theta = \dfrac{5}{12}$

Therefore, $\quad \cot\theta = \dfrac{12}{5}$,

$$\sin\theta = -\frac{5}{13},$$

$$\text{cosec} = -\frac{13}{5}$$

$$\cos\theta = -\frac{12}{13} \text{ and } \sec\theta = -\frac{13}{12}.$$

(ii) Since θ is in the second quadrant,

$\therefore$ Only $\sin\theta$ and $\text{cosec}\,\theta$ will be positive.

Now, $\quad \sin\theta = \dfrac{3}{5}$.

Therefore,

$$\text{cosec}\,\theta = \frac{5}{3}, \cos\theta = -\frac{4}{5},$$

$$\sec\theta = -\frac{5}{4}, \tan\theta = -\frac{3}{4}$$

and $\quad \cot\theta = -\dfrac{4}{3}$.

| Example 33. If $\sin\theta = \dfrac{12}{13}$ and θ lies in the second quadrant, find the value of $\sec\theta + \tan\theta$.

Sol. We have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

In the second quadrant, $\cos\theta$ is negative

$$\therefore \quad \cos\theta = -\sqrt{1-\sin^2\theta}$$

Now, $\sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1+\sin\theta}{\cos\theta}$

$$= \frac{1+\sin\theta}{-\sqrt{1-\sin^2\theta}} = \frac{1+\dfrac{12}{13}}{-\sqrt{1-\left(\dfrac{12}{13}\right)^2}}$$

$$= \frac{\dfrac{25}{13}}{-\sqrt{\dfrac{25}{169}}} = \frac{\dfrac{25}{13}}{-\dfrac{5}{13}} = -5$$

| Example 34. Draw the graph of $y = 3\sin 2x$.

Sol. $\sin x$ is a periodic function with period 2π, therefore,

$\sin 2x$ will be a periodic function of period $\frac{2\pi}{|2|} = \pi$

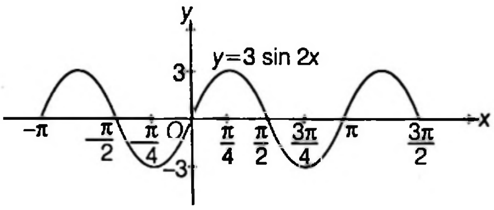
Also $\quad -1 \le \sin 2x \le 1$

$\therefore \quad -3 \le 3\sin 2x \le 3$

In order to draw the graph of $y = 3\sin 2x$, draw the graph of $y = \sin x$ and on X-axis change k to $\frac{k}{2}$, i.e. write $\frac{k}{2}$ wherever it is k. For example, write 15° in place of 30°, 45° in place of 90° etc.

On Y-axis change k to $3k$, i.e. write $3k$ wherever it is k for example, write 3 in place of 1, – 3 in place of – 1, 1.5 in place of 0.5 etc.

The graph of $y = 3\sin 2x$ will be as given in the figure.



| Example 35. Draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$

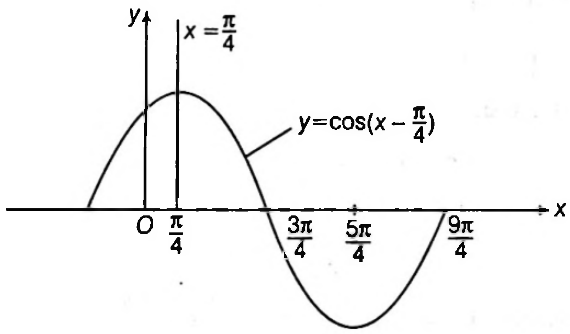
Sol. Given function is $y = \cos\left(x - \frac{\pi}{4}\right)$ ...(i)

Given function is $Y = \cos X$, where

$$X = x - \frac{\pi}{4} \text{ and } Y = y$$

or $\quad Y = 0 \Rightarrow y = 0 \text{ and } X = 0$

$\Rightarrow \quad x - \frac{\pi}{4} = 0 \Rightarrow x = \frac{\pi}{4}$



In order to draw the graph of $y = \cos\left(x - \frac{\pi}{4}\right)$, we draw the graph of $y = \cos x$ and shift it on the right side through a distance of $\frac{\pi}{4}$ unit.

| Example 36. Which of the following is the least?

(a) $\sin 3$ (b) $\sin 2$

(c) $\sin 1$ (d) $\sin 7$

Sol. (a) $\sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$

$\sin 2 = \sin[\pi - (\pi - 2)]$

$= \sin(\pi - 2) = \sin(1.14)$

$\sin 7 = \sin[2\pi + (7 - 2\pi)]$

$= \sin(7 - 2\pi) = \sin(0.72)$

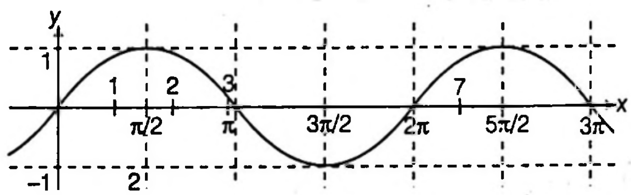
Now, $\quad 1.14 > 1 > 0.72 > 0.14$

$\Rightarrow \quad \sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$

[as 1.14, 0.72, 0.14 lie in the first quadrant and sine functions increase in the first quadrant]

Hence, among the given values, $\sin 3$ is the least.

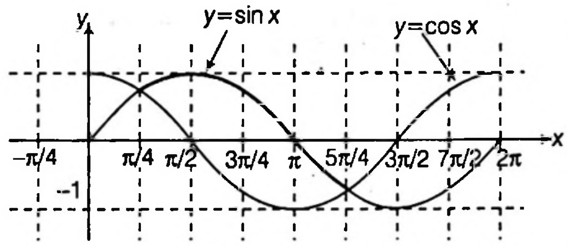
Alternate solution



From the graph, obviously $\sin 3$ is the least.

| Example 37. Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.

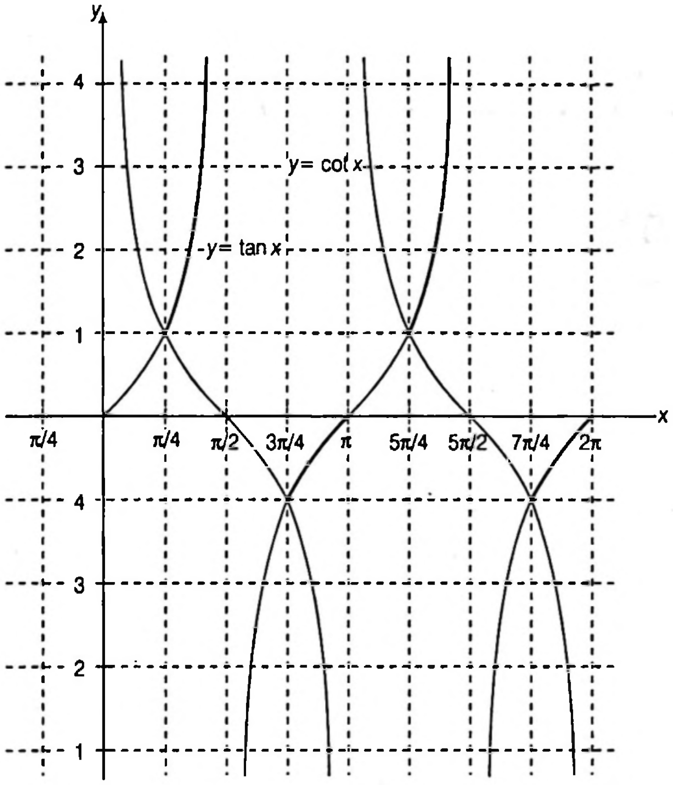
Sol. $f(x) = \sqrt{\sin x - \cos x}$ is defined if $\sin x \ge \cos x$.



From the graph, $\sin x \ge \cos x$, for $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

| Example 38. Solve $\tan x > \cot x$, where $x \in [0, 2\pi]$.

Sol.



We find that $\tan x \geq \cot x$. Therefore, the values of $\tan x$ are more than the value of $\cot x$.

That is, the value of x for which graph of $y = \tan x$ is above the graph of $y = \cot x$.

From the graph, it is clear that

$$x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right) \cup \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right).$$

Exercise for Session 4

1. If $\tan x = -\frac{4}{3}, \frac{3\pi}{2} < x < 2\pi$, find the value of $9 \sec^2 x - 4 \cot x$.

2. Show that $\sin^2 x = p + \frac{1}{p}$ is impossible if x is real.

3. If $\cos x = \frac{3}{5}$ and x lies in the fourth quadrant find the values of $\operatorname{cosec} x + \cot x$.

4. Draw the graph of $y = \sin x$ and $y = \sin \frac{x}{2}$.

5. Draw the graph of $y = \sec^2 x - \tan^2 x$. Is $f(x)$ periodic? If yes, what is its fundamental period?

6. Prove that $\sin \theta < \theta < \tan \theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$.

7. Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.

8. Draw the graph of $y = \sin x$ and $y = \cos x$, $0 \leq x \leq 2\pi$.

9. Draw the graph of $y = \tan(3x)$.

10. If $\cos x = -\frac{\sqrt{15}}{4}$ and $\frac{\pi}{2} < x < \pi$, find the value of $\sin x$.

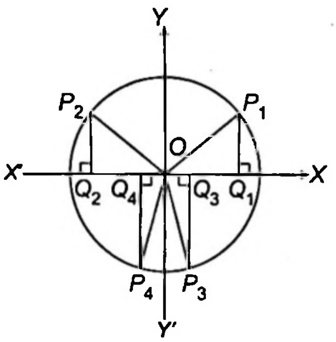
Session 5

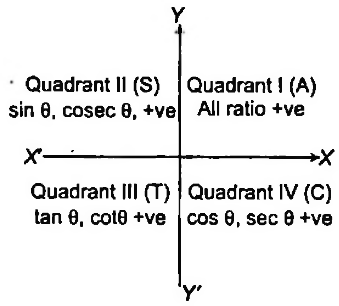
Trigonometric Ratios of any Angle

Trigonometric Ratios of any Angle

Consider the system of rectangular coordinates axis dividing plane into four quadrants. A line OP makes angle θ with the positive x-axis. The angle θ is said to be positive if measured in counter clockwise direction from the positive x-axis and is negative if measured in clockwise direction.

The positive values of the trigonometric ratios in the various quadrants are shown, the signs of the other ratios may be derived.





Note that $\angle XOY = \frac{\pi}{2}$, $\angle XOY' = \pi$, $\angle XOY' = \frac{3\pi}{2}$

P_iQ_i is positive if above the x-axis, negative if below the x-axis, OP_i is always taken positive. OQ_i is positive if along x-axis, negative if in oppositive direction.

$$\sin \angle Q_iOP_i = \frac{P_iQ_i}{OP_i}.$$

$$\cos \angle Q_iOP_i = \frac{O_iQ_i}{OP_i}.$$

$$\tan \angle Q_iOP_i = \frac{P_iQ_i}{OQ_i} \qquad [i = 1, 2, 3]$$

Thus, depending on signs of OQ_i and P_iQ_i, the various trigonometrical ratios will have different signs given

$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\qquad$ $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan(-\theta) = -\tan\theta$ $\qquad$ $\cot(-\theta) = -\cot\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ $\qquad$ $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

$\text{cosec}(-\theta) = -\text{cosec}\,\theta$ $\qquad$ $\sec(-\theta) = \sec\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \text{cosec}\,\theta$ $\qquad$ $\text{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$ $\qquad$ $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

$\sin(\pi - \theta) = \sin\theta$ $\qquad$ $\cos(\pi - \theta) = -\cos\theta$

$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$ $\qquad$ $\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$

$\tan(\pi - \theta) = -\tan\theta$ $\qquad$ $\cot(\pi - \theta) = -\cot\theta$

$\sec\left(\frac{\pi}{2} + \theta\right) = -\text{cosec}\,\theta$ $\qquad$ $\text{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta$

$\sec(\pi - \theta) = -\sec\theta$ $\qquad$ $\text{cosec}(\pi - \theta) = \text{cosec}\,\theta$

$\sin(\pi + \theta) = -\sin\theta$ $\qquad$ $\cos(\pi + \theta) = -\cos\theta$

$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$ $\qquad$ $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$\tan(\pi + \theta) = \tan\theta$ $\qquad$ $\cot(\pi + \theta) = \cot\theta$

$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$ $\qquad$ $\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$

$\sec(\pi + \theta) = -\sec\theta$ $\qquad$ $\text{cosec}(\pi + \theta) = -\text{cosec}\,\theta$

$\sec\left(\frac{3\pi}{2} - \theta\right) = -\text{cosec}\,\theta$ $\qquad$ $\text{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta$

$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$ $\qquad$ $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

$\sin(2\pi - \theta) = -\sin\theta$ $\qquad$ $\cos(2\pi - \theta) = \cos\theta$

$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$ $\qquad$ $\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta$

$\tan(2\pi - \theta) = -\tan\theta$ $\qquad$ $\cot(2\pi - \theta) = -\cot\theta$

$\sec\left(\frac{3\pi}{2} + \theta\right) = \text{cosec}\,\theta$ $\qquad$ $\text{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta$

$\sec(2\pi - \theta) = \sec\theta$ $\qquad$ $\text{cosec}(2\pi - \theta) = -\text{cosec}\,\theta$

$\sin(2\pi + \theta) = \sin\theta$ $\qquad$ $\cos(2\pi + \theta) = \cos\theta$

$\tan(2\pi + \theta) = \tan\theta$ $\qquad$ $\cot(2\pi + \theta) = \cot\theta$

$\sec(2\pi + \theta) = \sec\theta$ $\qquad$ $\text{cosec}(2\pi + \theta) = \text{cosec}\,\theta$

Allied angles (or numbers)

Two angles (or numbers) are called allied iff their sum or difference is a multiple of $\frac{\pi}{2}$. For example, $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are allied, $\frac{5\pi}{6}$ and $-\frac{\pi}{6}$ are allied.

AID TO MEMORY

You must have been overwhelmed by large number of formulae for allied angles (or numbers). Instead of memorising all of them, use the following rules

1. Any trigonometric function of a real number $n\pi \pm x (n \in I)$, treating x as $0 < x < \frac{\pi}{2}$, is numerically equal to the same function of x, with sign depending upon the quadrant in which the arc length (on the unit circle) terminates. The proper sign can be ascertained by 'All – Sin – Tan – Cos' rule. For example, $\sin(\pi + x) = -\sin x$; –ve sign was chosen because $\pi + x$ lies in the third quadrant and sin is –ve in the third quadrant.
2. Any trigonometric function of a real number $(2n+1)\frac{\pi}{2} \pm xm$ treating x as $0 < x < \frac{\pi}{2}$, is numerically equal to cofunction of x, with sign depending upon the quadrant in which the arc length (on the unit circle) terminates. Note that sin and cos are cofunctions of each other; tan and cot are cofunctions of each other; sec and cosec are cofunctions of each other. For example, $\sec\left(\frac{\pi}{2} + x\right) = -\operatorname{cosec} x$, –ve sign was chosen because $\frac{\pi}{2} + x$ lies in the second quadrant and sec is –ve in the second quadrant.

I. Method

To prove $\cos\left(\frac{\pi}{2} \pm \theta\right) = \mp \sin\theta$ and $\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta$

Proof

$$e^{i\left(\frac{\pi}{2} \pm \theta\right)} = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad e^{i\frac{\pi}{2}} \cdot e^{\pm i\theta} = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad i \cdot e^{i(\pm\theta)} = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad i \cdot (\cos\theta \pm i\sin\theta) = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

$$\Rightarrow \quad i \cdot \cos\theta \mp \sin\theta = \cos\left(\frac{\pi}{2} \pm \theta\right) + i\sin\left(\frac{\pi}{2} \pm \theta\right)$$

On comparing real and imaginary part of LHS and RHS, we get

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

II. Method

To prove $\cos(\pi \pm \theta) = -\cos\theta$ and $\sin(\pi \pm \theta) = \mp\sin\theta$

Since, $\quad e^{i(\pi \pm \theta)} = \cos(\pi \pm \theta) + i\sin(\pi \pm \theta)$

$$\Rightarrow \quad e^{i\pi} \cdot e^{i(\pm\theta)} = \cos(\pi \pm \theta) + i\sin(\pi \pm \theta)$$

$$\Rightarrow \quad -(\cos(\pm\theta) + i\sin(\pm\theta)) = \cos(\pi \pm \theta) + i\sin(\pi \pm \theta)$$

On comparing real and imaginary part, we get

$$\cos(\pi + \theta) = -\cos\theta$$
$$\cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi + \theta) = -\sin\theta$$
$$\sin(\pi - \theta) = \sin\theta$$

| Example 39. Prove that

(i) $\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$

(ii) $2\sin^2\frac{\pi}{6} + \operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3} = \frac{3}{2}$

(iii) $\cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{5\pi}{6} + 3\tan^2\frac{\pi}{6} = 6$

(iv) $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

Sol. (i) We have,

$$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} - \tan^2\frac{\pi}{4} = -\frac{1}{2}$$

$$= \left(\sin\frac{\pi}{6}\right)^2 + \left(\cos\frac{\pi}{3}\right)^2 - \left(\tan\frac{\pi}{4}\right)^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2$$

$$= \frac{1}{4} + \frac{1}{4} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

(ii) We have,

$$2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cdot\cos^2\frac{\pi}{3}$$

$$=2\left(\sin\frac{\pi}{6}\right)^2+\left(\operatorname{cosec}\frac{7\pi}{6}\right)^2\cdot\left(\cos\frac{\pi}{3}\right)^2$$

$$=2\left(\sin\frac{\pi}{6}\right)^2+\left\{\operatorname{cosec}\left(\pi+\frac{\pi}{6}\right)\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$=2\left(\sin\frac{\pi}{6}\right)^2+\left\{-\operatorname{cosec}\frac{\pi}{6}\right\}^2\left(\cos\frac{\pi}{3}\right)^2$$

$$[\because \operatorname{cosec}(\pi+\theta)=-\operatorname{cosec}\theta]$$

$$=2\left(\frac{1}{2}\right)^2+(-2)^2\times\left(\frac{1}{2}\right)^2=\frac{1}{2}+1=\frac{3}{2}$$

(iii) We have,

$$\cot^2\frac{\pi}{6}+\operatorname{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}$$

$$=\left(\cot\frac{\pi}{6}\right)^2+\operatorname{cosec}\left(\pi-\frac{\pi}{6}\right)+3\left(\tan\frac{\pi}{6}\right)^2$$

$$=(\sqrt{3})^2+2+3\left(\frac{1}{\sqrt{3}}\right)^2$$

$$=3+2+1=6$$

(iv) We have,

$$2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}$$

$$=2\left(\sin\frac{3\pi}{4}\right)^2+2\left(\cos\frac{\pi}{4}\right)^2+2\left(\sec\frac{\pi}{3}\right)^2$$

$$=2\left(\sin\frac{\pi}{4}\right)^2+2\left(\cos\frac{\pi}{4}\right)^2-2\left(\sec\frac{\pi}{3}\right)^2$$

$$\left[\because \sin\frac{3\pi}{4}=\sin\left(\pi-\frac{\pi}{4}\right)=\sin\frac{\pi}{4}\right]$$

$$=2\left(\frac{1}{\sqrt{2}}\right)^2+2\left(\frac{1}{\sqrt{2}}\right)^2+2(2)^2$$

$$=1+1+8=10$$

| Example 40. Prove that

$$\frac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}=-1.$$

Sol. L.H.S. $=\dfrac{\cos(90^\circ+\theta)\sec(-\theta)\tan(180^\circ-\theta)}{\sec(360^\circ-\theta)\sin(180^\circ+\theta)\cot(90^\circ-\theta)}$

$$=\frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)}$$

$$=-1$$

= R.H.S.

| Example 41. Show that $\tan 1^\circ \tan 2^\circ \dots \tan 89^\circ = 1$

Sol. L.H.S. $=(\tan 1^\circ \tan 89^\circ)(\tan 2^\circ\cdot\tan 88^\circ)\dots$

$$=[\tan 1^\circ \tan(90^\circ-1^\circ)]\cdot[\tan 2^\circ \tan(90^\circ-2^\circ)]\dots[\tan 44^\circ \tan(90^\circ-44^\circ)]\tan 45^\circ$$

$$=(\tan 1^\circ\cdot\cot 1^\circ)(\tan 2^\circ\cdot\cot 2^\circ)\dots(\tan 44^\circ\cdot\cot 44^\circ)\tan 45^\circ$$

$$=1 \qquad [\because \tan\theta\cot\theta=1 \text{ and } \tan 45^\circ=1]$$

| Example 42. Show that

$$\sin^2 5^\circ+\sin^2 10^\circ+\sin^2 15^\circ+\dots+\sin^2 90^\circ=9\frac{1}{2}$$

Sol. L.H.S. $=(\sin^2 5^\circ+\sin^2 85^\circ)+(\sin^2 10^\circ+\sin^2 80^\circ)+\dots+(\sin^2 40^\circ+\sin^2 50^\circ)+\sin^2 45^\circ+\sin^2 90^\circ$

$$=(\sin^2 5^\circ+\cos^2 5^\circ)+(\sin^2 10+\cos^2 10^\circ)+\dots+(\sin^2 40^\circ+\cos^2 40^\circ)+\sin^2 45^\circ+\sin^2 90^\circ$$

$$=(1+1+1+1+1+1+1+1)+\left(\frac{1}{\sqrt{2}}\right)^2+1$$

$$=9\frac{1}{2}$$

| Example 43. Find the value of

$$\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{5\pi}{16}+\cos^2\frac{7\pi}{16}$$

Sol. L.H.S. $=\cos^2\dfrac{\pi}{16}+\cos^2\dfrac{3\pi}{16}+\cos^2\left(\dfrac{\pi}{2}-\dfrac{3\pi}{16}\right)+\cos^2\left(\dfrac{\pi}{2}-\dfrac{\pi}{16}\right)$

$$=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{\pi}{16}$$

$$=\left(\cos^2\frac{\pi}{16}+\sin^2\frac{\pi}{16}\right)+\left(\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}\right)$$

$$=1+1=2$$

Exercise for Session 5

1. Find the value of $\tan \frac{19\pi}{3}$.
2. Find the sign of $\sec 2000°$.
3. The value of $\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 180°$.
4. Find the value of $\cos(270° + \theta)\cos(90° - \theta) - \sin(270° - \theta)\cos\theta$.
5. If $S_n = \cos^n \theta + \sin^n \theta$, find the value of $3S_4 - 2S_6$.
6. $\sin^2 \theta = \frac{x^2 + y^2 + 1}{2x}$, then x must be.
7. If $\sin x + \operatorname{cosec} x = 2$, then find the value of $\sin^{10} x + \operatorname{cosec}^{10} x$.
8. $e^{\sin x} - e^{-\sin x} = 4$ then find the number of real solutions.
9. If $\pi < \alpha < \frac{3\pi}{2}$, then find the value of expression $\sqrt{4\sin^4 \alpha + \sin^2 2\alpha} + 4\cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$.
10. If $\sum_{i=1}^{n} \cos \theta_i = n$, then the value of $\sum_{i=1}^{n} \sin \theta_i$.

Session 6

Trigonometric Ratios of Compound Angles

Trigonometric Ratios of Compound Angles

Algebraic sum of two or more angles is called a compound angle. If A, B, C are any angles then $A + B$, $A - B$, $A + B + C$, $A - B + C$, $A - B - C$, $A + B - C$, etc., are all compound angles.

Till now, we have learnt the values of trigonometric ratios between 0° to 360°. Now, we are going to learn the values of trigonometric ratios of compound angles.

Note

Trigonometric ratios if i.e. sine, cosine, tan, cot, sec and cosec are not distributed over addition and substraction of 2 angles.

i.e. $\sin(A + B) \neq \sin A + \sin B$

Proof : $A = 60°$, $B = 30°$

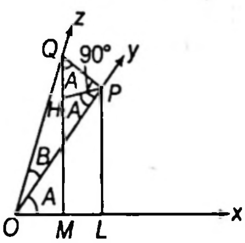
$\sin(90°) \neq \sin 60° + \sin 30°$

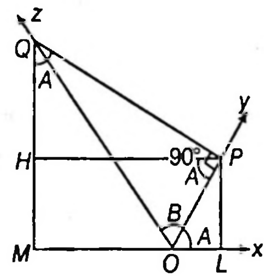
The Addition Formula

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(iii) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$





Let the revolving line starting from the position OX describe first $\angle XOY = A$ and then proceed further so as to describe $\angle YOZ = B$ in its position OZ.

Then, $\angle XOZ = A + B$

In figure 6.1 $A + B < 90°$ and in figure 6.2 $A + B > 90°$

Let Q be a point on OZ. From Q draw $QM \perp OX$ and $QP \perp OY$. From P draw $PH \perp QM$.

Now, $\angle HPO = \angle POX = A$