Rahul Bhadani

Signals and Systems for Computer Engineers

WORKSHEET

Department of Electrical & Computer Engineering,
The University of Alabama in Huntsville

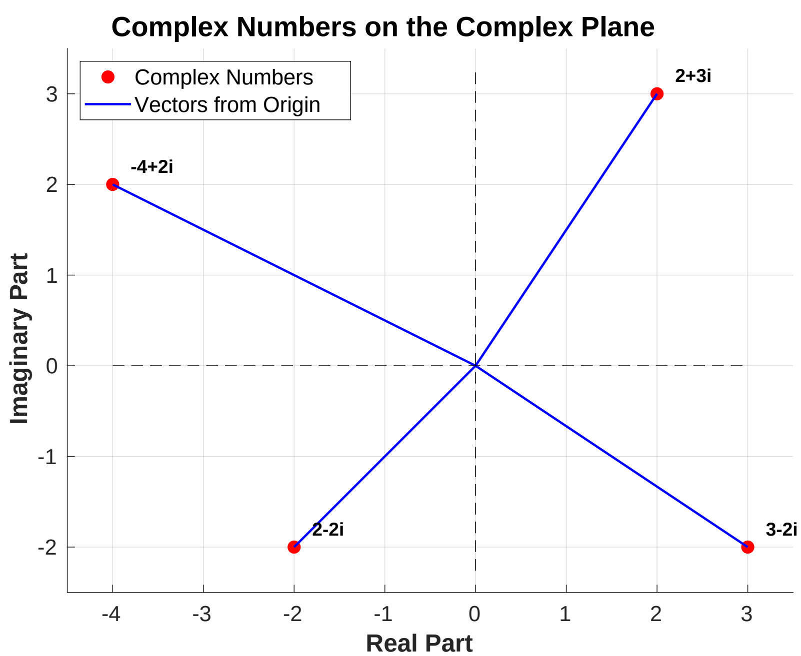
1. *Mathematical Preliminaries*

1.1 *Complex Numbers*

1. **4pts** Plot the following complex numbers on the complex plane:

   - $2 + i3$
   - $3 - i2$
   - $-2 - i2$
   - $-4 + j2$

   Solution:



2. **4pts** Express $\frac{-1+3i}{2+5i}$ in the form $a + ib$.

Solution:

$$\begin{aligned}
\frac{-1+3i}{2+5i} &= \frac{-1+3i}{2+5i} \times \frac{2-5i}{2-5i} \\
&= \frac{-2+5i+6i-15i^2}{2^2+5^2} \\
&= \frac{-2+11i+15}{4+25} \\
&= \frac{13+11i}{29} \\
&= \frac{13}{29} + \frac{11}{29}i
\end{aligned} \tag{1.1}$$

3. **2pts** If $Z = 3 + i5$ is a complex number, what is the value of the modulus $|Z|$?

   Solution:

   $$|Z| = \sqrt{3^2+5^2} = \sqrt{9+25} = \sqrt{34}$$

4. **4pts** Find the roots of the equation $x^2 + x + 1 = 0$.

   Solution:

   For a quadratic equation $ax^2 + bx + c = 0$, the roots are given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. In this case, $a = 1$, $b = 1$, $c = 1$.

   $$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

5. **2pts** Write the following complex numbers in the polar form:

   (a) $z = 1 + i$

   (b) $w = \sqrt{3} - i$

   Solution for (a): $|z| = \sqrt{1^2+1^2} = \sqrt{2}$ $\tan\theta = \frac{1}{1} = 1$, so $\theta = \frac{\pi}{4}$. In polar form: $z = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$.

   Solution for (b): $|w| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$ $\tan\theta = \frac{-1}{\sqrt{3}}$, so $\theta = -\frac{\pi}{6}$ (since the number is in the 4th quadrant). In polar form: $w = 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$.

6. **4pts** Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in the polar form.

   Solution:

   From the previous problem, we have: $z_1 = 1 + i = \sqrt{2}e^{i\pi/4}$

   $z_2 = \sqrt{3} - i = 2e^{-i\pi/6}$

   $z_1 z_2 = (\sqrt{2}e^{i\pi/4})(2e^{-i\pi/6}) = 2\sqrt{2}e^{i(\pi/4-\pi/6)} = 2\sqrt{2}e^{i(\pi/12)}$

   In standard polar form: $2\sqrt{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$.

7. **2pts** Find $(\frac{1}{2} + \frac{1}{2}i)^{10}$.

   Solution:

   First, convert to polar form:

   $\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

   Using De Moivre's Theorem:

   $(\frac{1}{2} + \frac{1}{2}i)^{10} = (\frac{\sqrt{2}}{2})^{10}(\cos(10 \cdot \frac{\pi}{4}) + i\sin(10 \cdot \frac{\pi}{4}))$

   $= (\frac{2^{1/2}}{2})^{10}(\cos(\frac{5\pi}{2}) + i\sin(\frac{5\pi}{2}))$

   $= (\frac{1}{2^{1/2}})^{10}(\cos(\frac{\pi}{2} + 2\pi) + i\sin(\frac{\pi}{2} + 2\pi))$

   $= (\frac{1}{2^5})(0 + i \cdot 1) = \frac{i}{32}$

8. **2pts** Evaluate or Simplify:

   (a) $e^{i\pi}$

   (b) $e^{-1+i\pi/2}$

   Solution for (a): $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$.

   Solution for (b): $e^{-1+i\pi/2} = e^{-1}e^{i\pi/2} = e^{-1}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{e}\left(0 + i(1)\right) = \frac{i}{e}$

9. **2pts** Evaluate the expression below and write your answer in the form $a + ib$.

   (a) $(5 - i6) + (3 + i2)$

   (b) $\frac{3}{4-i3}$

   Solution for (a): $(5 - i6) + (3 + i2) = (5 + 3) + (-6 + 2)i = 8 - 4i$.

   Solution for (b): $\frac{3}{4-i3} = \frac{3}{4-i3} \times \frac{4+i3}{4+i3} = \frac{12+i9}{4^2+3^2} = \frac{12+i9}{16+9} = \frac{12+i9}{25} = \frac{12}{25} + \frac{9}{25}i$.

10. **2pts** Find the complex conjugate and modulus of the number:

    (a) $12 + i5$

    (b) $-1 + 2\sqrt{2}i$

    Solution for (a):

    Complex conjugate: $12 - i5$.

    Modulus: $|12 + i5| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$.

    Solution for (b):

    Complex conjugate: $-1 - 2\sqrt{2}i$.

    Modulus: $|-1 + 2\sqrt{2}i| = \sqrt{(-1)^2 + (2\sqrt{2})^2} = \sqrt{1 + 8} = \sqrt{9} = 3$.

11. **2pts** Apply De Moivre's Theorem to simplify:

    (a) $(1+i)^{20}$

    (b) $(1-\sqrt{3}i)^5$

    (c) $(1-i)^8$

    Solution for (a):

    Convert $1+i$ to polar form: $r = \sqrt{1^2+1^2} = \sqrt{2}$,
    $\theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$.
    $1+i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$.
    $(1+i)^{20} = (\sqrt{2})^{20}(\cos(20\cdot\frac{\pi}{4}) + i\sin(20\cdot\frac{\pi}{4}))$
    $= 2^{10}(\cos(5\pi) + i\sin(5\pi)) = 1024(-1+0i) = -1024$.

    Solution for (b):

    Convert $1-\sqrt{3}i$ to polar form: $r = \sqrt{1^2+(-\sqrt{3})^2} = 2$, $\theta = \tan^{-1}(\frac{-\sqrt{3}}{1}) = -\frac{\pi}{3}$.
    $1-\sqrt{3}i = 2(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$.
    $(1-\sqrt{3}i)^5 = 2^5(\cos(5\cdot(-\frac{\pi}{3})) + i\sin(5\cdot(-\frac{\pi}{3})))$
    $= 32(\cos(-\frac{5\pi}{3}) + i\sin(-\frac{5\pi}{3})) = 32(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}))$
    $= 32(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 16(1+i\sqrt{3})$.

    Solution for (c):

    Convert $1-i$ to polar form: $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$, $\theta = \tan^{-1}(\frac{-1}{1}) = -\frac{\pi}{4}$.
    $1-i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$.
    $(1-i)^8 = (\sqrt{2})^8(\cos(8\cdot(-\frac{\pi}{4})) + i\sin(8\cdot(-\frac{\pi}{4})))$
    $= 2^4(\cos(-2\pi) + i\sin(-2\pi)) = 16(1+0i) = 16$.

12. **3pts** Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

    (a) $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

    (b) $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

    Solution: Euler's formula states: $e^{ix} = \cos x + i\sin x$ $e^{-ix} = \cos x - i\sin x$

    To prove (a), add the two equations: $e^{ix} + e^{-ix} = (\cos x + i\sin x) + (\cos x - i\sin x) = 2\cos x$. Therefore, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$.

    To prove (b), subtract the second equation from the first: $e^{ix} - e^{-ix} = (\cos x + i\sin x) - (\cos x - i\sin x) = 2i\sin x$. Therefore, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.