Neural Network By hand

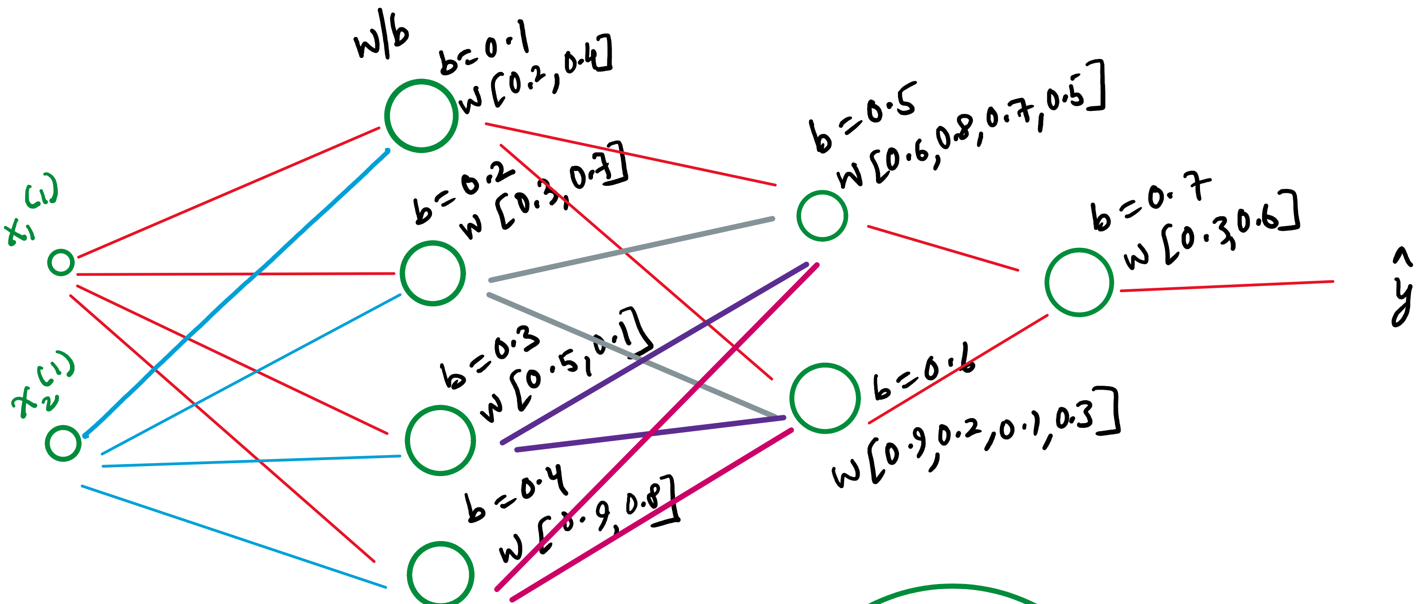
Monday, February 24, 2025 6:42 PM

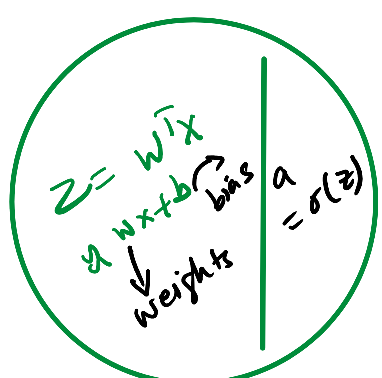
$x^{(1)}$	[1, 4]	$y^{(1)}$	-1
$x^{(2)}$	[5, 6]	$y^{(2)}$	0
$x^{(3)}$	[9, 12]	$y^{(3)}$	$+1$

Neural Network: Two hidden layer

Hidden layer 1 : 4 units

" " 2 : 2 units





a single neuron

Forward Propagation

$z_1^{[1]}$ → layer index

index of the neuron for the given layer

$$z_1^{[1]} = (1 \times 0.2 + 4 \times 0.4) + 0.1$$

$$= 1.9$$

$$\left[b + \underbrace{\begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} [1 \quad 4]}_{w^T x} \right.$$

$z_2^{[1]} = (1 \times 0.3 + 4 \times 0.7) + 0.2 = 3.3$

$z_3^{[1]} = (1 \times 0.5 + 4 \times 0.1) + 0.3 = 1.2$

$z_4^{[1]} = (1 \times 0.9 + 4 \times 0.8) + 0.4 = 4.5$

Use activation function $a = \sigma(x) = \frac{1}{1+e^{-x}}$

$a_1^{[1]} = \frac{1}{1+e^{-1.9}} \approx 0.87$

$a_2^{[1]} = \frac{1}{1+e^{-3.3}} \approx 0.96$

$a_3^{[1]} = \frac{1}{1+e^{-1.2}} \approx 0.77$

$a_4^{[1]} = \frac{1}{1+e^{-4.5}} \approx 0.99$

— Out put from the first layer and input to the Se cond layer.

For the Second layer,

$z_1^{[2]} = 0.87 \times 0.6 + 0.96 \times 0.8 + 0.77 \times 0.0.7 + 0.99 \times 0.5 + 0.5$

$= 2.824$

$z_2^{[2]} = 0.87 \times 0.9 + 0.96 \times 0.2 + 0.77 \times 0.1 + 0.99 \times 0.3 + 0.6$

$= 1.949$

$a_1^{[2]} = \frac{1}{1+e^{-2.824}} \approx 0.94$

$a_2^{[2]} = \frac{1}{1+e^{-1.949}} \approx 0.88$

Cal culate O/P at the O/P layer

$z_1^{[3]} = 0.94 \times 0.3 + 0.88 \times 0.6 + 0.7$

$= 1.51$

$$z_1^{[3]} = \ldots$$
$$= 1.51$$
$$a_1^{[3]} = \frac{1}{1+e^{-1.51}} = 0.82 = \hat{y}$$

Now in this case, the loss function is

$$\mathcal{L}(y, \hat{y}) = -\left[y \log(\hat{y}) + (1-y)\log(1-\hat{y}) \right]$$
$$= -\left[(-1)\log(0.82) + (1-(-1))\log(1-0.82)\right]$$
$$= +3.23$$

Backward Propagation

We know:

① $$\frac{\partial \mathcal{L}}{\partial \hat{y}} = \frac{-y}{\hat{y}} + \frac{1-y}{\hat{y}}$$

② $$\frac{\partial \hat{y}}{\partial z} = \hat{y}(1-\hat{y})$$ (Here z is $z_1^{[3]}$)

③ Input to the o/p layer

$$h = [a_1^{[2]}, a_2^{[2]}]$$

o/p of the o/p layer

$$z = z_1^{[3]} = W^{[3]T} h + b^{[3]}$$
$$\frac{\partial z}{\partial W^{[3]}} = h$$

In backward propagation, we will update the weight of the last layer first based on loss incurred:

of the last layer first based on loss incurred:

$$\boxed{W = W - \eta \frac{\partial L}{\partial W}}$$ — SGD.

$$W^{[3]} = W^{[3]} - \eta \frac{\partial L}{\partial W^{[3]}}$$

$$\frac{\partial L}{\partial W^{[3]}} = \frac{\partial L}{\partial \hat{y}} \times \frac{\partial \hat{y}}{\partial z} \times \frac{\partial z}{\partial W^{[3]}}$$

$\frac{\partial z}{\partial W^{[3]}}$: $[0.94,\ 0.88]$

$\frac{\partial \hat{y}}{\partial z}$: $0.82(1-0.82) = 0.15$

$\frac{\partial L}{\partial \hat{y}}$: $\left(-\frac{1}{0.82} + \frac{(1-(-1))}{1-0.82}\right) = 9.89$

$$= 9.89 \times 0.15 \times [0.94,\ 0.88]$$
$$= [1.39,\ 1.30]$$

$$W^{[3]} = W^{[3]} - \eta \times \frac{\partial L}{\partial W^{[3]}}$$

Let $\eta = 0.05$ (learning rate)

$$W^{[3]} = [1.3,\ 0.6] - 0.05 \times [1.39,\ 1.30]$$
$$= [6.23,\ 0.54]$$

So far, we only considered on training sample to compute the cross entropy loss.

Same for hidden layer 2.

$$W^{[2]} = W^{[2]} - \eta \frac{\partial L}{\partial W^{[2]}}$$