CPE 490 590: Machine Learning for Engineering Applications

03 Mathematical Preliminaries: Probability and Statistics Review

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Outline

Logistics

Announcement

- Homework 1 Due Date: Jan 22, 2025, 11:59 PM
- Quiz 1 Deadline: Jan 26, 2025, 11:59 PM

Statistics and Probability Refresher

Probability Theory

1. Probability is a branch of mathematics that deals with uncertainty.
2. Probability is the likelihood or chance that something will occur.
3. Probability describes things whose outcomes are uncertain or random.

Sample Space

Definition

A sample space $\mathcal{S}$ is the possible outcome of an experiment. A point s in $\mathcal{S}$ is called sample outcome, realization, or outcome. Subsets of $\mathcal{S}$ are called events.

Example:

In an experiment of tossing two coins, our sample space is $\mathcal{S} = \{HH, HT, TH, TT\}$. An event that at least one of the coins is heads is $\mathcal{E} = \{HH, HT, TH\}$.

Probability

Definition

For each event $\mathcal{E}$ in the sample space $\mathcal{S}$, the probability is a function which associates with $\mathcal{E}$ a number between 0 and 1, i.e, $P(\mathcal{E}) \in [0, 1]$.

Probability

Definition

It is technically hard to describe the probability of each single event. In that case, we consider a good collection $\mathcal{B}$ of events which is large enough to contain all the useful events including $\emptyset$, and $\mathcal{S}$, and is closed under all possible countable set operations. This collection $\mathcal{B}$ is called a sigma-algebra or Borel field. Probability is a set function defined only on this collection, i.e.,

$$P : \mathcal{B} \to [0, 1]$$

it means 0 will be included in the set.

[→ inclusion or closed set

(0,1] → 0 is excluded.

Probability: Example

Example: Tossing a fair die.

1. One possible σ-algebra $\mathcal{B}_1$ is $\{\emptyset, \mathcal{S}\}$. → An example of Borel field
2. Another σ-algebra is the power set of $\mathcal{S}$.
3. Another σ-algebra is $\mathcal{B} = \{\emptyset, \mathcal{S}, \{0\}, \{2, 3, 4\}\}$.

$\{1, 2\}$ → Outcome or event

$\{\phi, \{1\}, \{2\}, \{1,2\}\}$

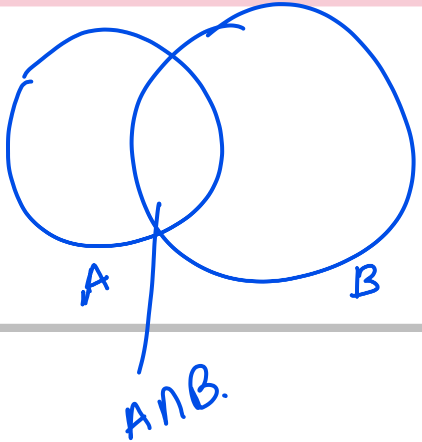
Conditional Probability

Definition

The probability of an event A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1}$$

given $P(B) > 0$.

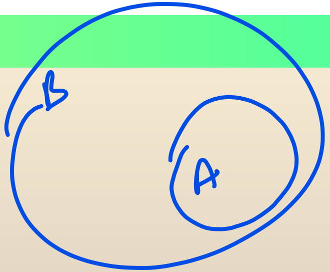


$P(B) > 0$

Conditional Probability

Some facts

1. $P(B|B) = 1$
2. $P(A|B) = \dfrac{P(A)}{P(B)}$ for $A \subset B$
3. $P(B|A) = 1$ for $A \subset B$
4. Multiplication formula is useful when it is easier to obtain conditional probability: $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$.
5. $P(A) = P(A \cap B) + P(A \cap B^c) = P(B)P(A|B) + P(B^c)P(A|B^c)$, where B^c is the complement of B.



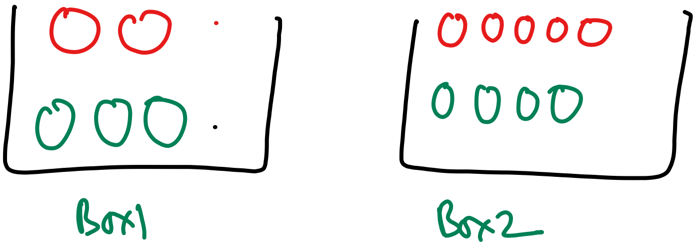
$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(B)\,P(A|B)$

Conditional Probability

Example 1

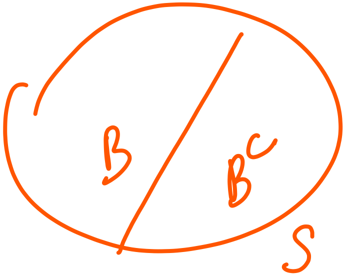
Let box 1 contain 2 red balls and 3 green balls and box 2 contain 5 red balls and 4 green balls. One box is chosen at random and one ball is drawn randomly from the chosen box. What is the probability of getting a red ball?



Conditional Probability

A = Choosing a red ball

B = Choosing the first box



Solution: Blank space for calculation

$P(B) = \frac{1}{2}$

$P(B^c) = \frac{1}{2}$

$P(A|B) = \frac{2}{5}$

$P(A|B^c) = \frac{5}{9}$

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c)$$

$$= \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{5}{9} = \frac{43}{90}$$

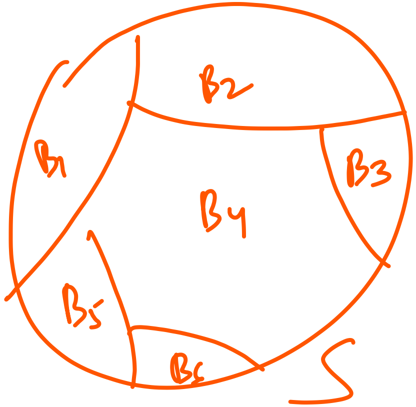
Bayes' Rule as Inversion of Conditional Probability

Conditional probability.

Assume that we can partition sample space $\mathcal{S}$ into B_1, B_2, B_3. The conditional probabilities can be $P(A|B_i)$, $i = 1, 2, \cdots$. Then, the posterior probabilities can be written as

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i)P(A|B_i)}{\sum_j P(B_j)P(A|B_j)} \tag{2}$$

$i=1$



Posterior Probability

$$P(B_1|A) = \frac{P(B_1)\ P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \cdots}$$

Bayes' Rule as Inversion of Conditional Probability

Example 1 Extension

Consider the example 1 from the previous section. Given that we obtained a red ball, what is the probability that box 1 was selected?

Bayes' Rule as Inversion of Conditional Probability

$$P(B|A) = \frac{P(B)\,P(A|B)}{P(B)\,P(A|B) + P(B^c)\,P(A|B^c)}$$

$B^c = B_2$

$B = B_1$

Solution: Blank space for calculation

$$= \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{18}{43}$$

Bayes' Rule as Inversion of Conditional Probability

Example 2: Rare disease probability

Consider that the probability of occurrence of a disease is 1/1000. A test accurately predicts the occurrence with 99% accuracy and negates the disease with 98% accuracy. Given that the test has shown positive, what is the probability of actually having the disease?

Bayes' Rule as Inversion of Conditional Probability

B_1 = event of the occurrence of disease
B_2 = event that the patient is healthy.
A = the event that the patient has been tested positive.

Solution: Blank space for calculation

$P(B_1) = \frac{1}{1000} = 0.001$

$P(B_2) = 1 - 0.001 = 0.999$

$P(A|B_1) = 0.99$

$P(A|B_2) = 1 - 0.98 = 0.02$

According to the Bayes Rule

$$P(B_1|A) = \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.02}$$

$$= 0.047210$$

Independence

Definition

Two events are independent if the occurrence of one of the events gives us no information about whether or not the other event will occur. In other words, events do not influence each other. Formally, we write

$$P(A|B) = P(A) \tag{3}$$

which is still applicable even if $P(A)$ or $P(B)$ is 0.

Independence

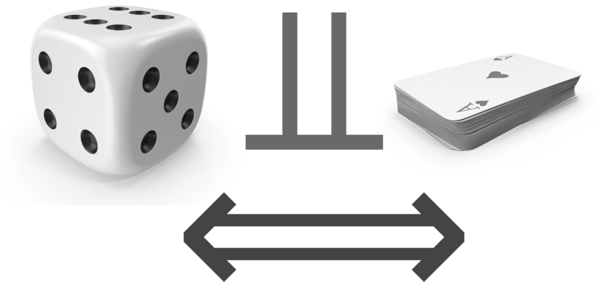
$P(A|B) = P(A)$ is equivalent to $P(A \cap B) = P(A)P(B)$, that is, the probability that they both occur is equal to the product of the probabilities of the two individual events.

$P(A|B) = P(A)$

$P(A \cap B) = P(A)\,P(B) \longrightarrow$ Product Rule of probabilities

Independence: Symmetric Relationship and Unrelated Events

Symmetric relationship: A is independent of B implies B is independent of A.



$\perp\!\!\!\perp$

Unrelated events must be independent.

Independence vs. Mutual Exclusivity

Independent and mutually exclusive events are not the same. If $P(A) > 0$, and $P(B) > 0$, then:

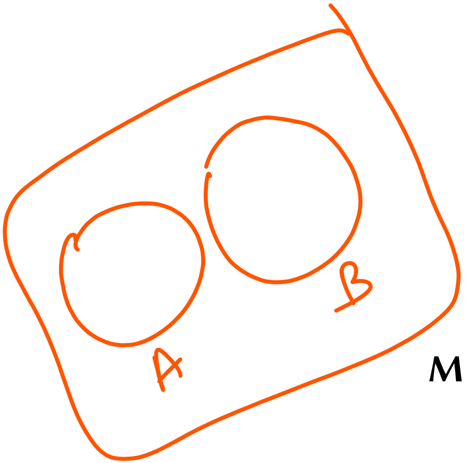
1. If A and B are mutually exclusive, they cannot be independent.
2. If A and B are independent, they cannot be mutually exclusive.

A = Event of tail

$P(A) = \frac{3}{4}$

B = event of head

$P(B) = \frac{1}{4}$



A: Head B: Tail

$$A \cap B = \emptyset$$

Mutually exclusive but not independent

$P(A) + P(B) = 1$

$A \cap B = \phi$

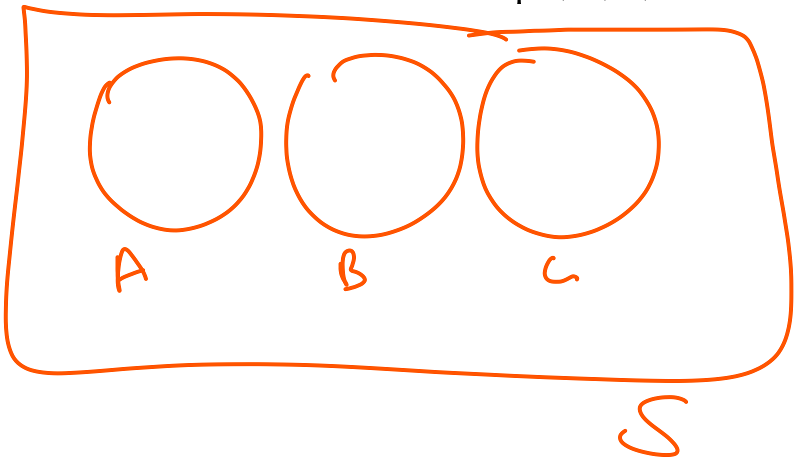
Independence: Complementary Events

If A and B are independent, so are A, B^c ; A^c, B; and A^c, B^c.

Independence of Multiple Events

The idea of independence can be extended to more than two events. For example, A, B, and C are independent if:

1. A and B are independent,
2. A and C are independent, and
3. B and C are independent; and
4. $P(A \cap B \cap C) = P(A)P(B)P(C)$.



Independence

Example 3

Let's say that a man and a woman each have a pack of 52 playing cards. Each draws a card from his/her pack. Find the probability that they each draw the ace of clubs ♣.

Independence

$$P(\text{Ace of Clubs}) = \frac{1}{52}$$

$E_1 =$ Man gets the ace of clubs.

$E_2 =$ Woman gets the ace of clubs

Solution: Blank space for calculation

$$P(E_1)P(E_2) = P(\text{Ace of Clubs from both}) = \frac{1}{52} \times \frac{1}{52}.$$

Independence

Example 4

What is the chance of getting at least one six in 4 throws of a dice?

Independence

We first calculate the probability of no six in 4 throws of a dice which is given by $\left(\frac{5}{6}\right)^4$

Solution: Blank space for calculation $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$$= 1 - \left(\frac{5}{6}\right)^4$$

Random Variable

Random Variable

Definition

A random variable is a real-valued function defined on the sample space $\mathcal{S}$, that is a rule which assigns a number to each outcome. Random variables can be thought of as a stochastic function. Random variables are denoted by capital letters such as X, Y, Z. The outcome of random variables is represented with corresponding small letters such as x, y, z.

Random Variable

Example

For example, the probability that a random variable takes the value of 2 would be expressed as $P(X = 2)$ where $x = 2$.

- A random variable is a function $\mathcal{S} \to \mathbb{R}$.

- A random variable is called discrete if it takes only a finite or countably many values.
- A random variable is called continuous if it can assume all the values in an interval.

Probability Mass Function

Definition

At this point, we can transition from probability to probability law which will help us in writing computer programs at a later stage.
The probability law of a discrete random variable can be described by the function defined as

$$\boxed{p(x) = P(X = x)} \tag{4}$$

Probability mass function or pmf.

Probability Mass Function

Let $x_1, x_2, \cdots$ be the points where p gives positive masses (or values) that we can call $p_1, p_2, \cdots$. Thus $P(X = x_j) = p_j, j = 1, 2, \cdots$

- $p_j \geq 0$, and $\sum_{j=1}^{\infty} p_j = 1$.

Probability Mass Function: Example

Example 5

If a coin is tossed twice, the sample space $\mathcal{S} = \{HH, HT, TH, TT\}$. Let X be the random variable that denotes the number of heads that can come up. With each sample point, we can associate a number for X, as shown below:

	HH	HT	TH	TT
X	2	1	1	0

$P(X=x_2) = p_2 \Rightarrow P(X=2) = \frac{1}{4}$

Note: We can also define some other random variable on the same sample space, such as the square of the number of heads, i.e. $Y = X^2$ or the number heads minus the number of tails $Z = X - W$, where W is the random variable the denotes the number of tails that can come up.

$x_0 = \#\text{head} = 0$

$x_1 = \#\text{head} = 1$

$x_2 = \#\text{head} = 2$

$P(X=x_1) = P(X=1) = \frac{2}{4}$

$P(Y=y_1) = P(X^2=y_1) \neq (P(X))^2$

Probability Mass Function: Example

Example 6

$$p_k = P(X = k) = q^{k-1}p, \text{ where } 0 < p < 1, q = 1 - p, \quad k = 1, 2, \cdots \tag{5}$$

This is pmf as it satisfies $p_k \geq 0$, and $\sum_{k=0}^{\infty} p_k = 1$.

$$\sum_{k=1}^{\infty} q^{k-1} p = p \sum_{k=1}^{\infty} q^{k-1}$$

$$= p \cdot \frac{1}{1-q} = p \cdot \frac{1}{p} = 1$$

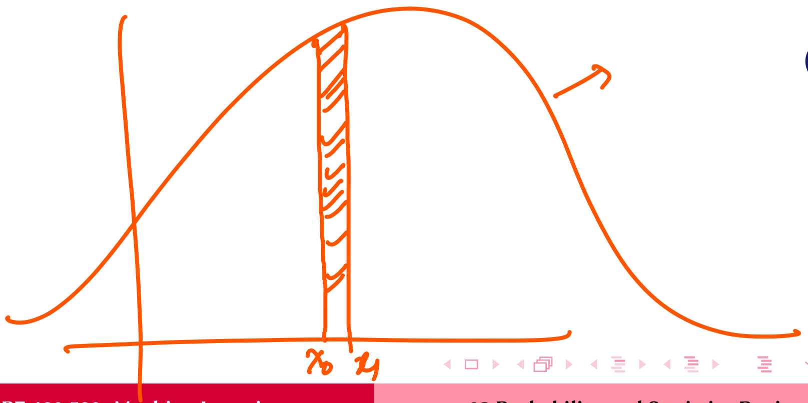
$q = 1 - p$

$\Rightarrow p = 1 - q$

Probability Density Function

When it comes to the continuous random variable X that takes real values, then it is not possible to define the probability for a single point, as they are real values. At what real value we should define a probability – there can be a possibility of an infinite number after a decimal point!!!. Hence, we definite the probability over a range. In this case, we get the probability density function (PDF) $p(x)$ such that

$$P(x_0 < x < x_1) = \int_{x_0}^{x_1} p(x)dx. \tag{6}$$



Probability Density Function

Endpoints x_0 and x_1 may not be included. They do not matter for continuous random variable X. In this case, $p(x)$ satisfies the following property:

- $p(x) \geq 0 \quad \forall x$
- $\int_{-\infty}^{\infty} p(x)dx = 1$

Probability Distribution

Cumulative Distribution Function (CDF)

Definition

For any $x \in \mathbb{R}$, we define cumulative distribution function as

$$F(x) = P(X \leq x) \tag{7}$$

They are sometimes to referred just as **distribution** or **probability distribution**. Shorthand notation for CDF is F_X.

Random Variable

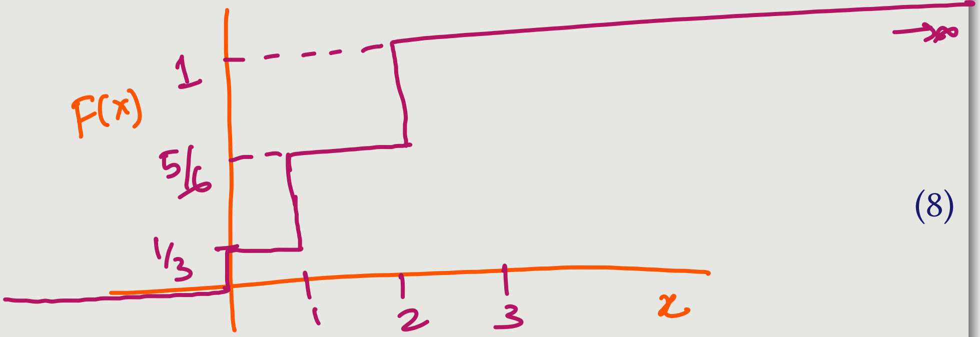
CDF:Example

Example 7: CDF of a Discrete Random Variable

Consider a discrete random variable X such that $P(X=0)=1/3$, $P(X=1)=1/2$, $P(X=2)=1/6$. Then pmf of X, $p(x)$ is written as $p(0)=1/3$, $p(1)=1/2$, $p(2)=1/6$ and $p(x)=0$ for all other x.

The CDF of X, $F(X)$ is given by

$$F(x)=\begin{cases} 0, & x<0 \\ 1/3, & 0 \leq x \leq 1 \\ 5/6, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases} \tag{8}$$



In this case, the graph of F is a step function, having jumps at the point where X has mass.

CDF: Example

$$P_k = P(X=k) = q^{k-1}p \quad \text{where } 0<p<1$$
$$q = 1-p$$
$$p = 1-q$$
$$\text{and } k = 1, 2, \ldots$$

What would be the CDF from Example 6?

Solution: Blank space for calculation

$$F(x) = P(X \le k) = P(X=0) + P(X=1) + \cdots P(X=k)$$

$$= \overbrace{q^0 p + q^1 p + q^2 p + \cdots q^{k-1} p}^{k \text{ terms}}$$

$$= p\,(\underline{q^0 + q^1 + q^2 \cdots q^{k-1}})$$

$$= p \cdot \frac{1 \cdot (1-q^k)}{1-q} = \frac{\cancel{p}(1-q^k)}{\cancel{p}} = 1-q^k$$

we apply G.P. sum

$$S_{GP} = \frac{a(1-r^n)}{1-r}$$

a = first term

r = the common ratio

CDF of a Continuous Random Variable (RV)

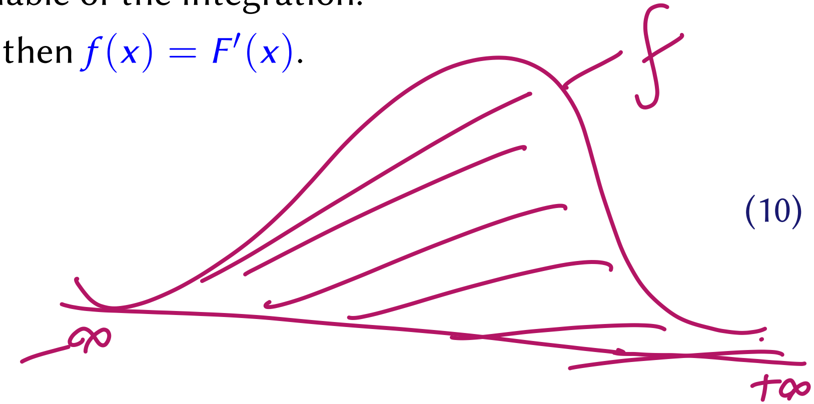
For a continuous RV, we can define the CDF as

$$F(x) = \int_{-\infty}^{x} f(t)dt \tag{9}$$

for the probability density function (PDF) $f(x)$. We also write f_X to denote that f is the PDF of the random variable X. Here, t is the dummy variable of the integration.

1. If $F(x)$ is differentiable at a given point, then $f(x) = F'(x)$.
2. f must satisfy

$$\boxed{f(t) \geq 0, \quad \int_{-\infty}^{\infty} f(t)dt = 1.} \tag{10}$$



Identically Distributed

$P(X = x_i)$ or $P(x_1 \le X \le x_2)$

discrete r.v.   continuous r.v

If X and Y are two random variables with $P(X \in A) = P(Y \in A)$ for all sets A. Then, they are called identically distributed. This is equivalent to $F_X(x) = F_Y(x)$ for all x, i.e. two CDFs are equal.

$$\underbrace{P(X \in A) = P(Y \in A)}_{\text{identically distributed}}$$

$$F_X(x) = F_Y(x)$$

Independent and Identically Distributed (IID)

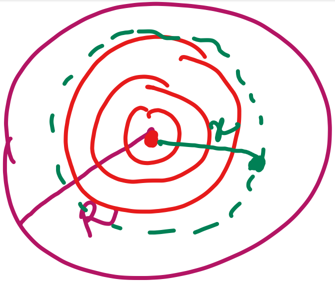
In addition to identically distributed, if two random variables are mutually independent, then they are called independent and identically distributed or (IID). It means sample items are independent events; the knowledge of the value of one variable will not give any information about the value of the other and vice versa.

Example

Example 8

Hit a dartboard of radius R randomly. Let X be the random variable denoting the distance of the chosen point from the center. In this case, probabilities are proportional to area, so

$$F(x) = P(X \leq x) = \frac{\pi x^2}{\pi R^2} = (x/R)^2, \quad 0 \leq x \leq R.$$



$$F(x) = P(X \leq x) = \frac{\pi x^2}{\pi R^2}$$

$$= \frac{x^2}{R^2}$$

Expected Values or Expectations Also known as

average or mean

Consider a random variable X with pdf or pmf $f(x)$. If we have a function $g(X)$ of a random variable, then we can define the expectation as

$Y = g(X)$

$$\mathbb{E}(g(X)) = \begin{cases} \int_{-\infty}^{\infty} g(x) f(x)\, dx, & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} g(x) f(x), & \text{if } X \text{ is discrete} \end{cases} \tag{11}$$

$\boxed{g(X)} = X \cdot 1$

$g(X) = X^2, X^3 \ldots$

provided that $|\mathbb{E}(g(X)| < \infty$. If $|\mathbb{E}(g(X)| = \infty$, we say that the expectation doesn't exist. Here $\mathcal{X}$ is the sample space of the random variable X.

Some literature also write E instead of $\mathbb{E}$ for the expectation.

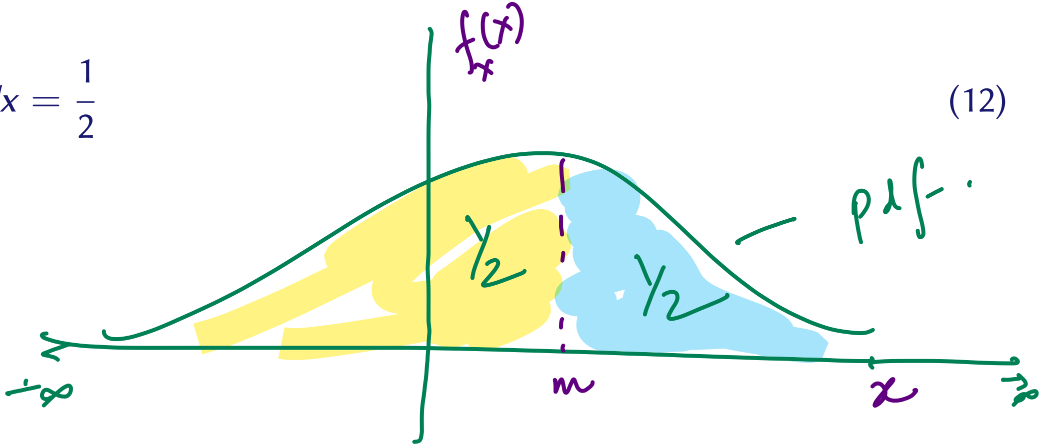
Expectation is also referred to as mean or average.

Median

If X is a continuous random variable and has CDF F_X. Its median m is the value that satisfies $F_X(m) = 1/2$, that is,

$$\int_{-\infty}^{m} f_X(x)\,dx = \int_{m}^{\infty} f_X(x)\,dx = \frac{1}{2} \tag{12}$$

Equivalently, $m = F_X^{-1}(1/2)$.



Variance and Standard Deviation

The variance of a random variable X can be defined as

$$var(X) = \mathbb{E}(X - \mathbb{E}(X))^2 \tag{13}$$

x_1
x_2
x_3
$\vdots$
x_n

$\bar{x}$

$\text{Variance} = \frac{1}{n} \Sigma (x_i - \bar{x})^2$

We also denote $\sigma^2 = var(X)$. The standard deviation of X is the square root of $var(X)$,i.e.,

$$\sigma = \sqrt{var(X)} \tag{14}$$

$\text{variance} = \frac{1}{n-1} \Sigma (x_i - \bar{x})^2$

Variance and standard deviation measure the degree of spread of a distribution around its mean $\mathbb{E}(X)$.

Textbook: Statistical Inference by George Casellas.

Common Families of Distribution: Discrete Distribution

Discrete Uniform Distribution

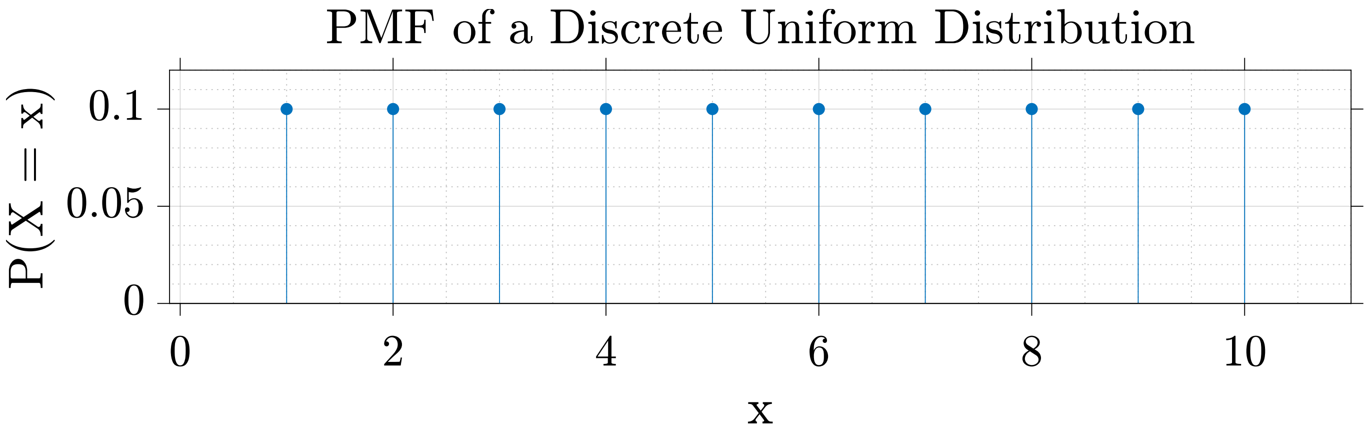
Each distribution is specified by some parameters that is a design time parameter. One of the goal of machine learning algorithms is to estimate those parameters from data.

1. Discrete Uniform

X: possible values $1, 2, 3, \cdots, N$. Here N is the parameter.

$$ \tag{15} $$

Discrete Uniform Distribution



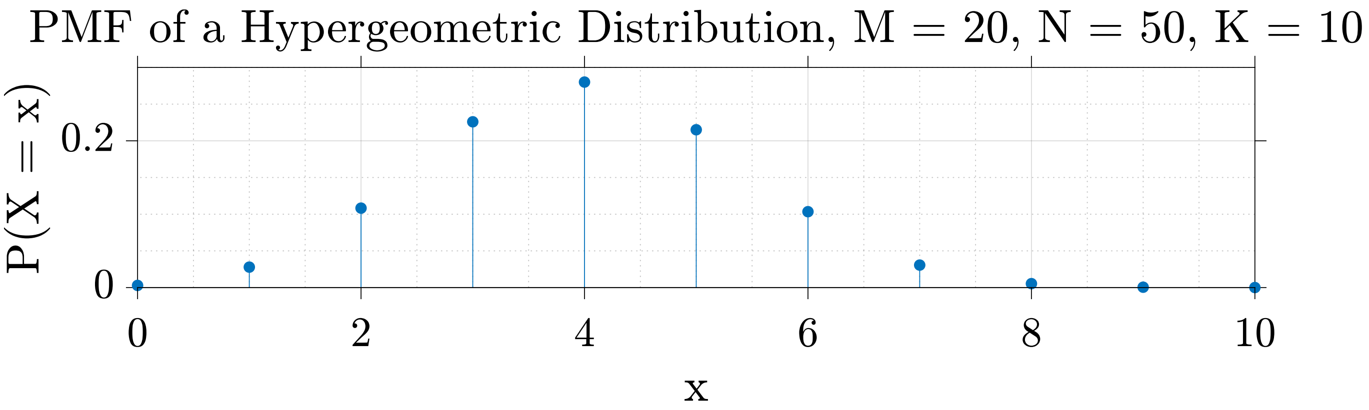
Hypergeometric Distribution

2. Hypergeometric

$$ \tag{16} $$

The following problem exhibits Hypergeometric distribution: there is a large urn filled with N balls, M red, and $N - M$ green balls. Draw K balls at random without replacement.

Hypergeometric Distribution



Binomial Distribution

3. Binomial

Repeat a random experiment n times that satisfies the following conditions

1. Only two possible outcomes: success and failure.
2. The probability of success p is the same for each trial.
3. The experiments are independent of each other.
4. $X =$ the number of total number of success in n trials.

$P(X=8)$

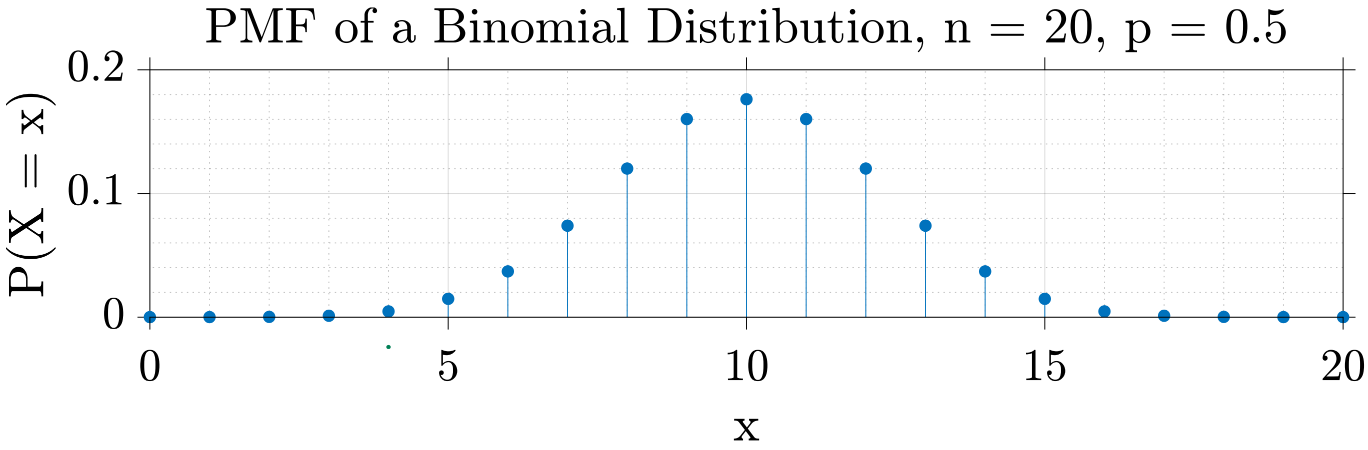
$x=8$

We call these Bernoulli trials. We can then write as $X \sim Bin(n, p)$ with pmf:

$$P(X=x) \quad (17)$$

where $x=8$

Binomial Distribution



Poisson Distribution

4. Poisson

The random variable X takes non-negative integer values such that

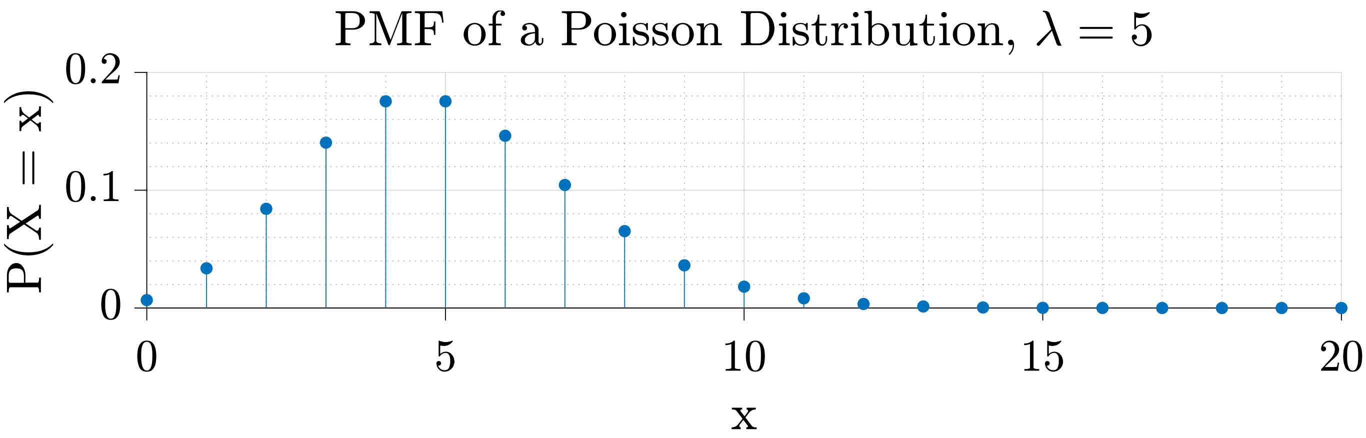
$$P(X=k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda} \qquad (18)$$

λ = Parameter.

Poisson Distribution

Poisson distribution is often used for describing the number of occurrences of a certain event in a very large number of observations, the probability for the event to occur in each observation being very small. Some examples: (i) Nuclear decay of atoms; (ii) Mutation of DNA; (iii) Photon counting by a photodetector.

Poisson Distribution



Common Families of Distribution: Continuous Distribution

Continuous Uniform Distribution

5. Continuous Uniform

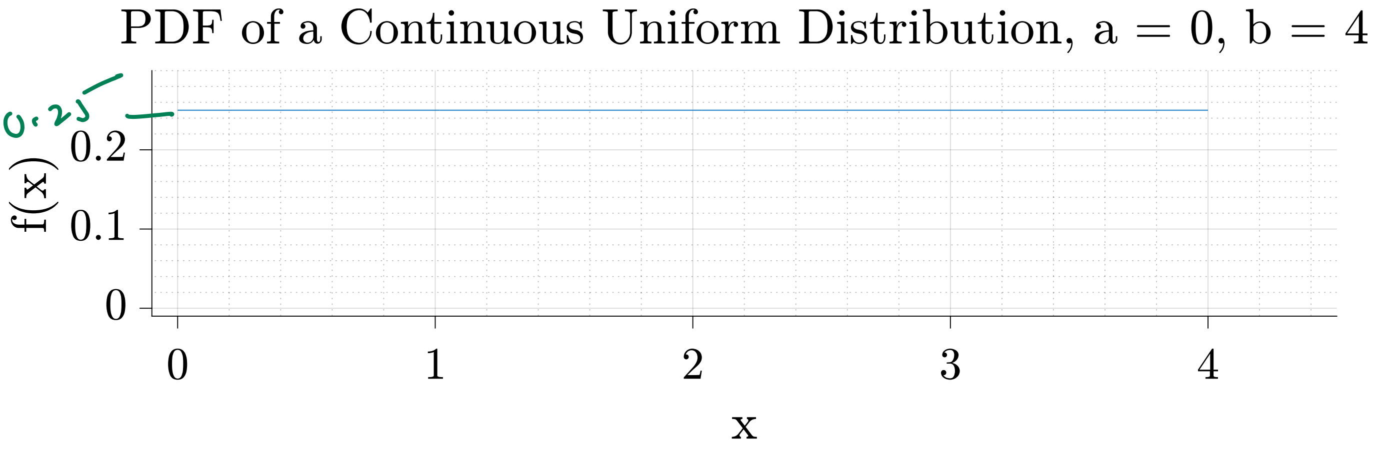
We say $X \sim Unif(a, b)$ if it has PDF

$$f(x) = \frac{1}{b-a} \tag{19}$$

a = lower bound

b = upper bound.

Continuous Uniform Distribution



$$\frac{1}{4-0} = \frac{1}{4} = 0.25$$

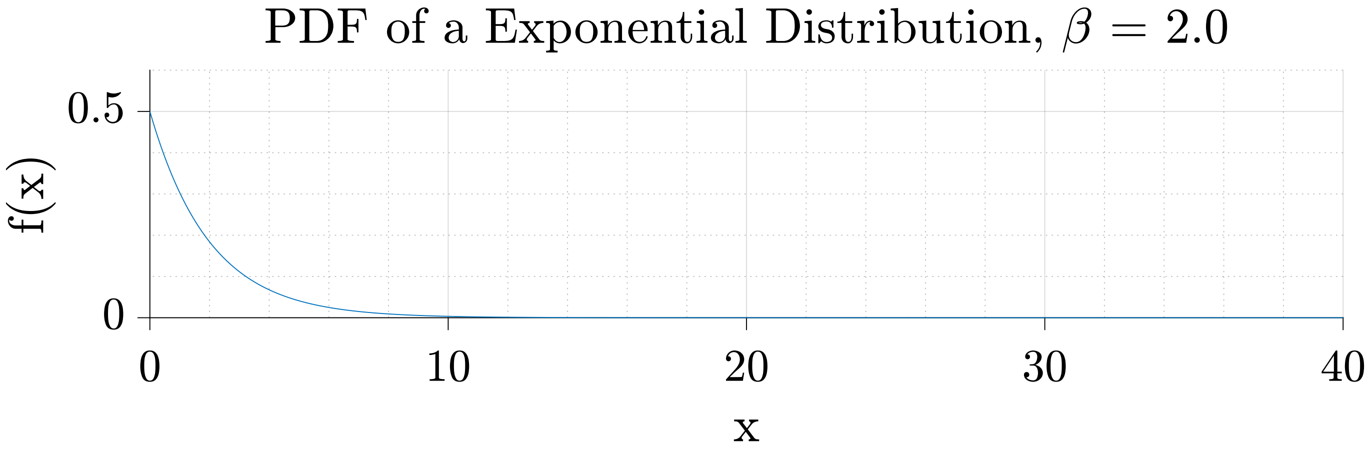
Exponential Family Distribution

6. Exponential Family

We say $X \sim Exp(\beta)$ if it has PDF

$$f(x|\beta) = \beta e^{-\beta x} \qquad \beta = \text{rate Parameter.} \tag{20}$$

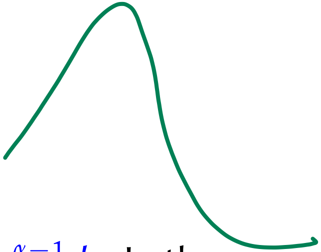
Exponential Family Distribution



Gamma Family Distribution

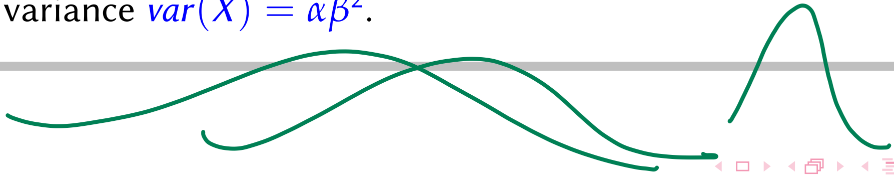
6. Gamma Family

For a random variable X, Gamma distribution is characterized by two parameters: α and β. Its PDF is given by

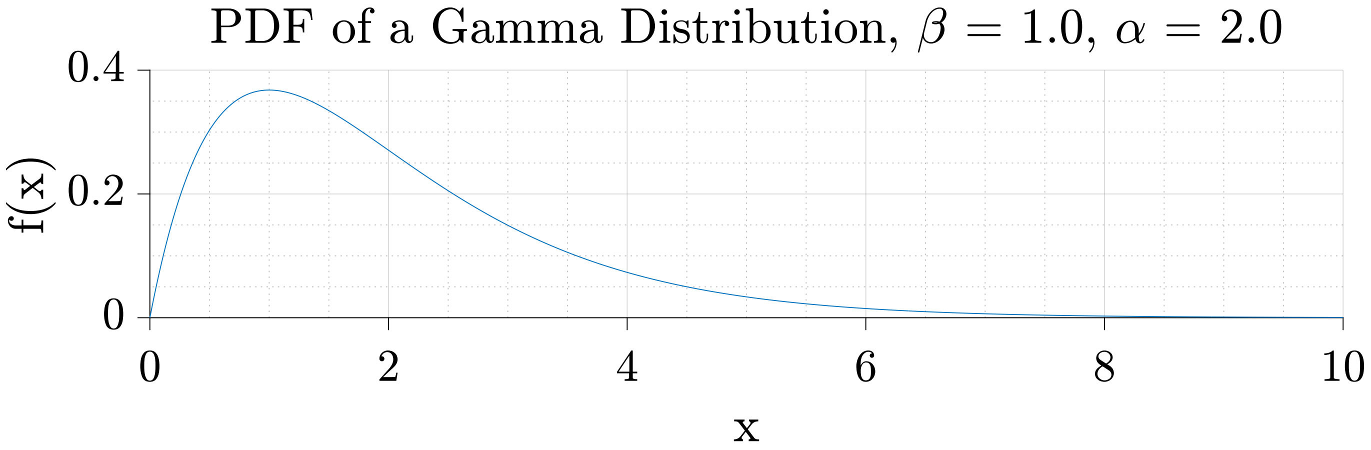


$$f(x|\alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1} \tag{21}$$

where gamma function $\Gamma(\alpha)$ is given by $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$. In the gamma distribution, α is a shape parameter that defines the peakedness of the distribution while β is a scale parameter that influences the spread of the distribution. The mean of the gamma distribution, $\mathbb{E}(X) = \alpha\beta$ while the variance $var(X) = \alpha\beta^2$.



Gamma Family Distribution



Weibull Distribution

7. Weibull

exponential distribution

If we have a random variable $X \sim Exp(\beta)$, and another random variable $Y \sim X^{1/\gamma}$, then in this case the Y follows a distribution called as Weibull distribution. The PDF of Weibull distribution is given by

$$f(y|\beta,\gamma) = \frac{\gamma}{\beta} y^{\gamma-1} e^{-y^\gamma/\beta}, \quad y > 0 \tag{22}$$

$X \sim Exp(\beta)$

$Y \sim (Exp(\beta))^{1/\gamma}$

Weibull Distribution

For the Weibull distribution, the mean and variance are given by

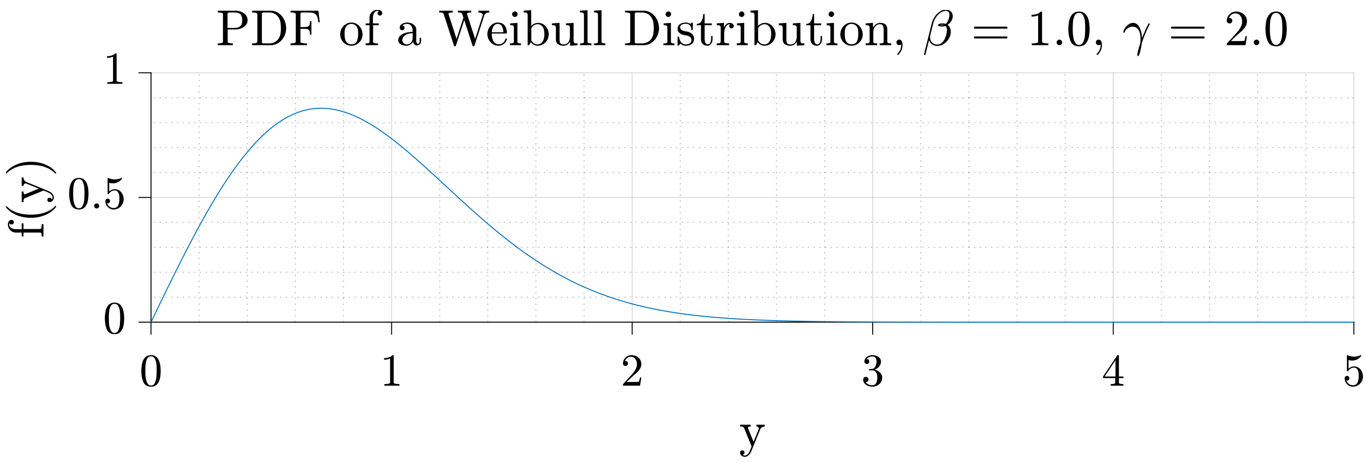
$$E(Y) = \beta^{1/\gamma}\Gamma(1+\frac{1}{\gamma})$$ = Variance.

Weibull Distribution

Remark

The difference between the Weibull and Gamma distributions is that in the Gamma distribution, y has a linear term in the exponential while in the Weibull distribution, there is y to the power γ.

Weibull Distribution



Normal Distribution/Gaussian Distribution

8. Gaussian

Gaussian distribution is the most common distribution everyone is familiar with. For a random variable X following a Gaussian distribution, we write $\boxed{X \sim N(\mu, \sigma^2)}$. Its PDF is given by

$$f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{23}$$

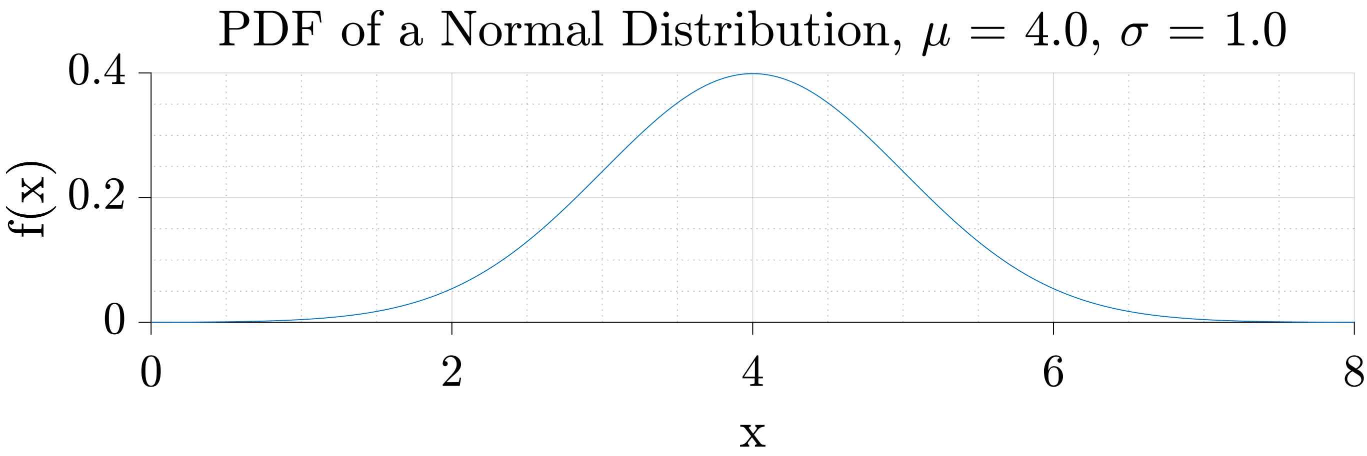
μ = mean

σ^2 = variance

Its expectations and variance are given by $\boxed{\mathbb{E}(X) = \mu}$, and $\boxed{var(X) = \sigma^2}$ respectively. A standard normal distribution has a mean of 0 and a variance of 1 denoted as $X \sim N(0, 1)$.

$$\boxed{X \sim N(0, 1)}$$

Normal Distribution/Gaussian Distribution



Laplace Distribution

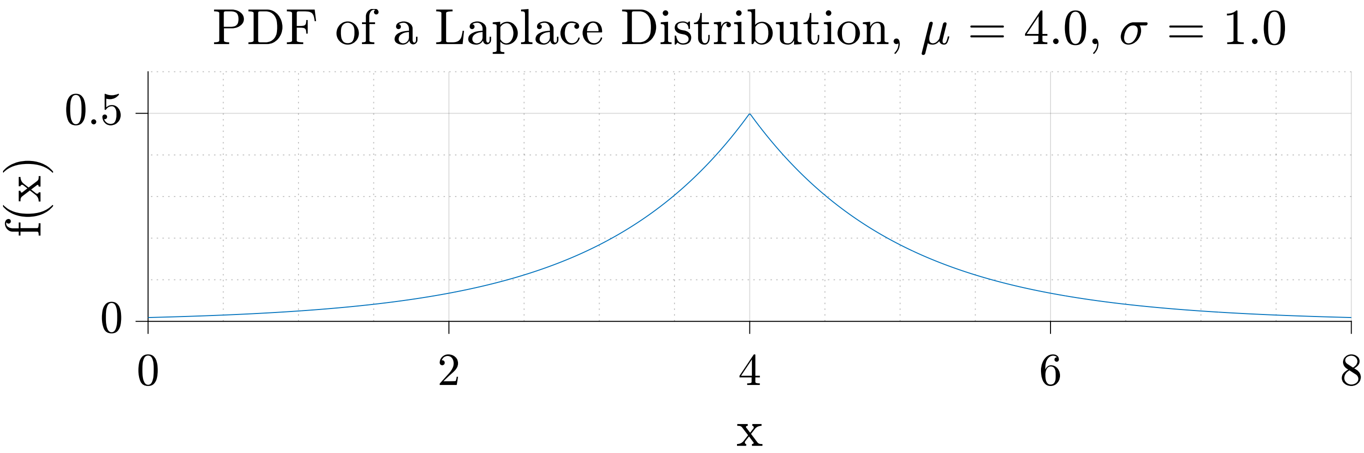
9. Laplace

A random variable X following the Laplace distribution, also known as the double exponential distribution has the following PDF:

$$f(x|\mu,\sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad -\infty < x < \infty \tag{24}$$

Its mean and variance are given by $\mathbb{E}(X) = \mu$, $var(X) = 2\sigma^2$ respectively.

Laplace Distribution



Synthetic Data Generation from Common Families of Distribution

Generating Data from Discrete Distribution

`scipy.stats.rv_discrete`

Synthetic datasets from distributions discussed above can be generated using `scipy.stats.rv_discrete`. Below is the code snippet that requires Python 3.8 or above followed by a stem plot showing the PMF of each distribution.

Generating Data from Discrete Distribution

rv_discrete is a base class to construct specific distribution classes and instances for discrete random variables. It can also be used to construct an arbitrary distribution defined by a list of support points and corresponding probabilities.
It allows you to sample a random number of that particular distribution you are specifying, thereby generating synthetic datasets that follow the given distribution.

2.0	2.4	4.5	Value
0.35	0.10	0.25	Probabilities

Generating Data from Hypergeometric Distribution

```python
# hypergeometric
N = 100
M = 50
K = 10
x2k = np.arange(min(M,K))
y2k = np.zeros(min(M,K))
for i, x in enumerate(x2k):
    y2k[i] = (math.comb(M, x)*math.comb(N-M, K-x))/math.comb(N, K)
# because of numerical round off and how many samples
# we choose, sum of y2k is less than 1. So just normalize it
y2k =y2k/sum(y2k)

# Create a probability Mass Function
pmf2 = rv_discrete(name='hypergeometric', values=(x2k, y2k))
```

0 to min(M,k).

$y = f(x; N, M, k)$

or $f(x \mid N, M, k)$

Normalization

follows the rule of probability.

Plotting Probability Mass Functions

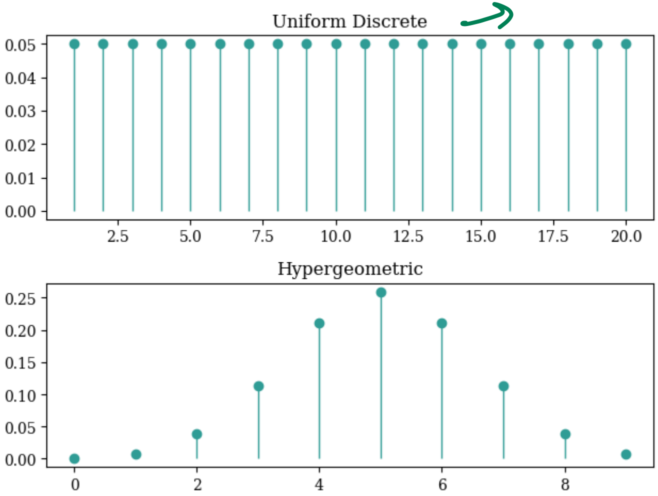
```
import matplotlib.pyplot as plt
fig, ax = plt.subplots(2, 1)
ax=np.ravel(ax)

ax[0].plot(x1k, pmf1.pmf(x1k), 'o', ms=6, mec='#2F9C95', markerfacecolor="#2F9C95")
ax[0].vlines(x1k, 0, pmf1.pmf(x1k), colors='#2F9C95', lw=1)
ax[0].set_title('Uniform Discrete')

ax[1].plot(x2k, pmf2.pmf(x2k), 'o', ms=6, mec='#2F9C95', markerfacecolor="#2F9C95")
ax[1].vlines(x2k, 0, pmf2.pmf(x2k), colors='#2F9C95', lw=1)
ax[1].set_title('Hypergeometric')

plt.tight_layout()
plt.show()
```

Plotting Probability Mass Functions



Hypergeometric.

Generating Data from Continuous Distribution

`scipy.stats.rv_continuous`

Synthetic datasets from continuous distributions discussed in the lecture can be generated using `scipy.stats.rv_continuous`. Below is the code snippet that requires Python 3.8. Code is followed by a histogram plot of the probability density function of some distributions discussed here.

Generating Data from Exponential Distribution

```
class Exponential(rv_continuous):
    "Exponential"
    def __init__(self, beta, **kwargs):
        super().__init__(**kwargs)
        self.beta = beta
    def _pdf(self, x):
        if(x <=0):
            return 0
        y = (1.0/self.beta)*np.exp(-x/self.beta)
        return y
P2 = Exponential(name='Exponential', beta = 2.0)
# Sample 1000 numbers
B2 = P2.rvs(size = 1000)
```

Python Class

Constructor in Python

variable arguments.

abstract member function that needs to be defined in the inherited class

1000 random numbers that follow exponential distribution.

Generating Data from Gamma Distribution

```
class Gamma(rv_continuous):
    "Gamma"
    def __init__(self, alpha, beta, **kwargs):
        super().__init__(**kwargs)
        self.alpha = alpha
        self.beta = beta
    def _pdf(self, x):
        if(x <=0):
            return 0
        y = (1.0/((self.beta**self.alpha)*gamma(self.alpha)))*
                    np.exp(-x/self.beta)*(x**(self.alpha-1))
        return y
P3 = Gamma(name='Gamma', alpha = 1, beta = 0.5)
B3 = P3.rvs(size = 1000)
```

actual function of pdf of Gamma distribution.

Plotting Probability Density Functions

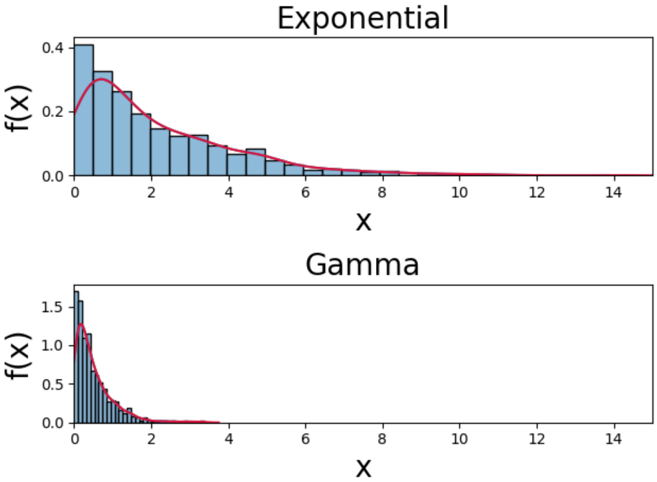
Plotting PDFs

```
fig, ax = plt.subplots(2, 1)
ax=np.ravel(ax)
a= s.histplot(B2, ax = ax[0], stat = 'density', kde=True)
s.kdeplot(B2, color='crimson', ax=a)
ax[0].set_xlim([0, 15])
ax[0].set_xlabel('x',  fontsize = 20)
ax[0].set_ylabel('f(x)', fontsize =20)
ax[0].set_title('Exponential', fontsize =20)
a = s.histplot(B3, ax = ax[1], stat = 'density', kde=True)
s.kdeplot(B3, color='crimson', ax=a)
ax[1].set_xlim([0, 15])
ax[1].set_xlabel('x',  fontsize = 20)
ax[1].set_ylabel('f(x)', fontsize =20)
ax[1].set_title('Gamma', fontsize =20)
plt.tight_layout()
```

kdeplot

= kernel density estimation plot.

Plotting Probability Density Functions



Python Notebook for Synthetic Data Generation

https://github.com/rahulbhadani/CPE490_590_Sp2025/blob/master/Code/CPE490590_SyntheticDataGeneration.ipynb

The End