Classwork 03: Simple Linear Regression
CPE 490/590 ST

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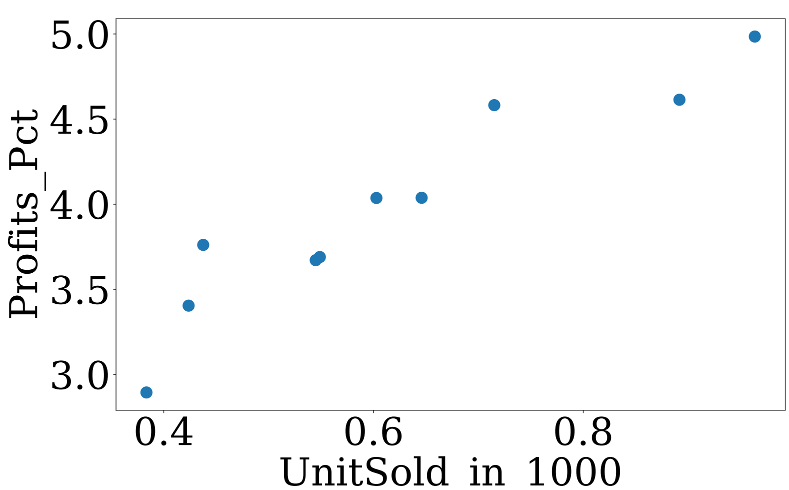
50 points

1 Simple Linear Regression

Consider the dataset that exhibits linear relationship between Units Sold in 1000 (predictor variable x), and profit in percentage (y):

UnitSold_in_1000 x_i	**Profits_Pct** y_i
0.54	3.69
0.72	4.58
0.6	4.03
0.54	3.67
0.42	3.40
0.64	4.03
0.43	3.76
0.89	4.61
0.96	4.98
0.38	2.89

$n = 10$.



Calculate the following quantity ($\sum_{i=1}^{10}$ is shorthanded as $\sum$) (10 points):

1. $\sum x_i =$ 6.12

2. $\sum y_i =$ 39.64

3. $\overline{x} =$ 0.612

4. $\overline{y} =$ 3.964

5. $\sum (x_i - \overline{x}) =$ 0.0

6. $\sum (y_i - \overline{y}) =$ $\approx$ 0.0

7. $\sum (x_i - \overline{x})^2 =$ 0.34516

8. $\sum (x_i y_i) =$ 25.2959

9. $\sum x_i^2 =$ 4.0906

10. $\sum y_i^2 =$ 160.6454

$\sum (x_i - \bar{Y})(Y_i - \bar{Y}) =$ 1.03622

In an ideal scenario, the data was obtained from a linear model $Y = Xw_1 + w_0$. The given data comes follows the $y_i = x_i w_1 + w_0 + \epsilon_i$ where ϵ_i is the error introduced by gather data such that $\epsilon_i \sim \mathcal{N}(\mu = 0, \sigma = 0.3)$.

We will fit our data into the linear model $y_i = x_i w_1 + w_0$ with the hope of minimizing the cost function (or error):

$$Q = \frac{1}{n} \sum_i (y_i - \hat{y})^2 \tag{1}$$

where $\hat{y} = x_i \hat{w}_1 + \hat{w}_0$.

The least-square solution gives the normal equation by solving:

$$\begin{aligned} \sum y_i &= n\hat{w}_0 + \hat{w}_1 \sum x_i \\ \sum x_i y_i &= \hat{w}_0 \sum x_i + \hat{w}_1 \sum x_i^2 \end{aligned} \tag{2}$$

which gives

$$\hat{w}_1 = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sum_i (x_i - \overline{x})^2} = \frac{1.03622}{0.34516} = \boxed{3.002143} \tag{3}$$

(4 Points)

and

$\hat{w}_0 = 3.964 - 3.002143 \times 0.612$
$= 2.1267$

$$\hat{w}_0 = \overline{y} - \hat{w}_1 \overline{x} = \boxed{2.1267} \tag{4}$$

(2 Points)

Alternatively, we can take the derivative of the cost function which can be re-written as

$$Q = \frac{1}{n} \sum (y_i - (w_1 x_i + w_0))^2 \tag{5}$$

Taking the partial derivative with respect to w_1 and w_0:

$$\frac{\partial Q}{\partial w_1} = -\frac{2}{n} \sum x_i \left(y_i - (w_1 x_i + w_0)\right) \tag{6}$$

set $\dfrac{\partial Q}{\partial w_1} = 0$

$$-\frac{2}{n} \sum x_i \left(y_i - (w_1 x_i + w_0)\right) = 0$$

(5 Points)

$$\frac{\partial Q}{\partial w_0} = -\frac{2}{n} \sum \left(y_i - (w_1 x_i + w_0)\right) \tag{7}$$

$\dfrac{\partial Q}{\partial w_0} = 0$

$$\Rightarrow -\frac{2}{n} \sum \left(y_i - (w_1 x_i + w_0)\right) = 0$$

(5 Points)

Setting above to 0 gives the following set of equations:

$$
\begin{aligned}
w_1 &= \frac{n(\sum_{i=1}^{n} x_i y_i) - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n(\sum_{i=1}^{n} x_i^2) - (\sum_{i=1}^{n} x_i)^2} \\
w_0 &= \frac{1}{n}(\sum_{i=1}^{n} y_i) - w_1 \frac{1}{n}(\sum_{i=1}^{n} x_i)
\end{aligned}
\tag{8}
$$

Putting the value in the above equations should give an estimated $\hat{w}_0$ and $\hat{w}_1$. Write down what you got in that Equation.

$$\hat{w}_1 = \frac{10 \times (25.2959) - (6.12 \times 39.64)}{10 \times 4.0906 - (6.12)^2} = 3.00214 \tag{9}$$

$$\hat{w}_0 = \left(\frac{1}{10} \times 39.64\right) - 3.00214 \times \frac{1}{10} \times 6.12 \tag{10}$$

$$= 3.964 - 0.300214 \times 6.12$$

$$= 3.964 - 1.8373 = 2.1267$$

(5 Points)

Using $\hat{w}_0$ and $\hat{w}_1$, we can obtain $\hat{y}_i$, the estimated response. Using estimated response and actual response, we can calculate Sum of Square Error (SSE) and Total Square Sum (SSTO) as:

$$
\begin{aligned}
\mathrm{SSE} &= \sum (y_i - \hat{y}_i)^2 = 0.4015 \\
\mathrm{MSE} &= \frac{\mathrm{SSE}}{n-2} = 0.00509875 \\
\mathrm{SSTO} &= \sum (y_i - \bar{y})^2 = 3.51244
\end{aligned}
\tag{11}
$$

(2 Points)

Calculate the goodness of fit using the coefficient of determination R^2:

$$R^2 = 1 - \frac{\text{SSE}}{\text{SSTO}} = \quad 0.885692 \tag{12}$$

(2 Points)

Based on the value of R^2 how good fit is the linear model on the given dataset?

Pretty Good fit

(2 Points)

Next, we calculate the standard deviation on the estimated coefficients or weight ($\hat{w}_1$, and $\hat{w}_0$).

$$s^2[\hat{w}_1] = \frac{\text{MSE}}{\sum(x_i - \overline{x})^2} = \quad 0.1450 \tag{13}$$

(2 Points)

$$s^2[\hat{w}_0] = \text{MSE}\left(\frac{1}{n} + \frac{\overline{x}^2}{\sum(x_i - \overline{x})^2}\right) = \quad 0.0594 \tag{14}$$

(2 Points)

Now, we need to calculate the 95% confidence interval on the estimated $\hat{w}_1$ and $\hat{w}_0$ for which $\alpha = 0.05$ (use $t(0.975, 8) = 2.306$):

$$\hat{w}_1 \pm t(1 - 0.05/2, 10 - 2)s^2[\hat{w}_1] \tag{15}$$

[2.667 , 3.336]

(2 Points)

$$\hat{w}_0 \pm t(1 - 0.05/2, 10 - 2)s^2[\hat{w}_0] \tag{16}$$

$2.1267 \pm (2.306 \times 0.0594)$

$= [1.9897, 2.2636]$

(2 Points)

Now, consider a new data point $x_{new} = 0.47$ Units Sold (in 1000). Based on the linear regression model that we developed, what is the estimated/predicted profit percent for the given quantity **(5 Points)**?

$\hat{y} = x_{new}\hat{w}_1 + \hat{w}_0$

$= 0.47 \times 3.00214 + 2.1267$

$= 3.5377$