RAMANUJAN COLLEGE

(UNIVERSITY OF DELHI)

2022-2023

GE – IV

NUMERICAL METHOD PRACTICAL FILE

Submitted By –

Rahul Chopra

Roll No. = 20211449

BSc (Hons.) Computer Science

Semester – IV

Submitted To -

Mr. Sanyam Gupta

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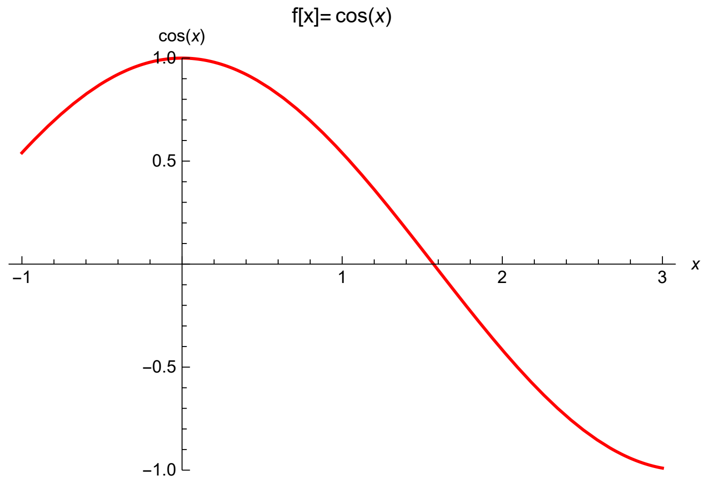
Practical 1
Rahul Chopra | BSc(H) Computer Science | Sem - IV | 20211449

Bisection Method

Question 1 :

```
x0 = 1.0;
x1 = 2.0;
Nmax = 20;
eps = 0.0001;
f[x_] := Cos[x];
If[N[f[x0] * f[x1]] > 0,
  Print["Your values do not satisfy the IVP so change the value."],
  For[i = 1, i ≤ Nmax, i++, m = (x0 + x1) / 2;
   If[Abs[(x1 - x0) / 2] < eps, Return[m],
    Print[i, "th iteration value is:", m];
    Print["Estimated error in", i, "th iteration is:", (x1 - x0) / 2];
    If[f[m] * f[x1] > 0, x1 = m, x0 = m]]];
  Print["Root is:", m] *
   Print["Estimated error in", i, "th iteration is:", (x1 - x0) / 2]];
Plot[f[x], {x, -1, 3}, PlotRange → {-1, 1},
 PlotStyle → Red, PlotLabel → "f[x]=" f[x], AxesLabel → {x, f[x]}]
```

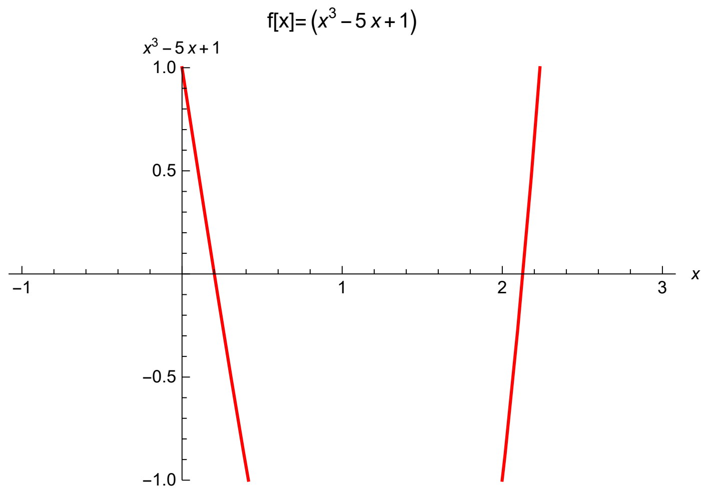
```
1th iteration value is:1.5
Estimated error in1th iteration is:0.5
2th iteration value is:1.75
Estimated error in2th iteration is:0.25
3th iteration value is:1.625
Estimated error in3th iteration is:0.125
4th iteration value is:1.5625
Estimated error in4th iteration is:0.0625
5th iteration value is:1.59375
Estimated error in5th iteration is:0.03125
6th iteration value is:1.57813
Estimated error in6th iteration is:0.015625
7th iteration value is:1.57031
Estimated error in7th iteration is:0.0078125
8th iteration value is:1.57422
Estimated error in8th iteration is:0.00390625
9th iteration value is:1.57227
Estimated error in9th iteration is:0.00195313
10th iteration value is:1.57129
Estimated error in10th iteration is:0.000976563
11th iteration value is:1.5708
Estimated error in11th iteration is:0.000488281
12th iteration value is:1.57056
Estimated error in12th iteration is:0.000244141
13th iteration value is:1.57068
Estimated error in13th iteration is:0.00012207
Return[1.57074]
```



Question 2 :

```
x0 = 0;
x1 = 1.0;
Nmax = 20;
eps = 0.0001;
f[x_] := x^3 - 5 x + 1;
If[N[f[x0] * f[x1]] > 0,
  Print["Your values do not satisfy the IVP so change the value."],
  For[i = 1, i ≤ Nmax, i++, m = (x0 + x1) / 2;
   If[Abs[(x1 - x0) / 2] < eps, Return[m],
    Print[i, "th iteration value is:", m];
    Print["Estimated error in", i, "th iteration is:", (x1 - x0) / 2];
    If[f[m] * f[x1] > 0, x1 = m, x0 = m]]];
  Print["Root is:", m] *
   Print["Estimated error in", i, "th iteration is:", (x1 - x0) / 2]];
Plot[f[x], {x, -1, 3}, PlotRange → {-1, 1},
 PlotStyle → Red, PlotLabel → "f[x]=" f[x], AxesLabel → {x, f[x]}]
```

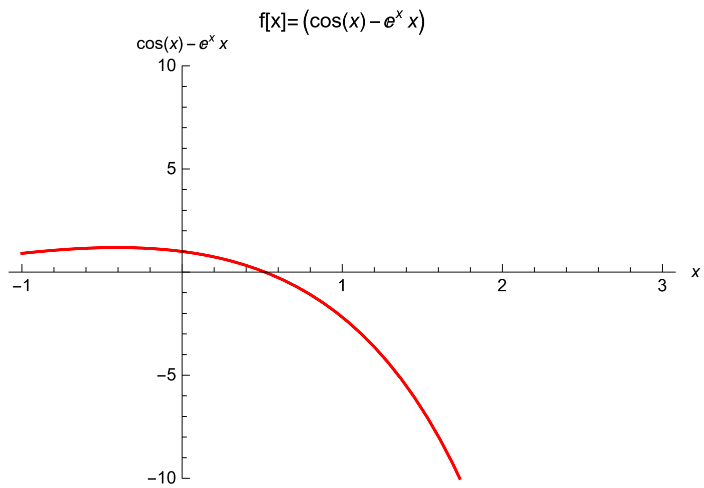
```
1th iteration value is:0.5
Estimated error in1th iteration is:0.5
2th iteration value is:0.25
Estimated error in2th iteration is:0.25
3th iteration value is:0.125
Estimated error in3th iteration is:0.125
4th iteration value is:0.1875
Estimated error in4th iteration is:0.0625
5th iteration value is:0.21875
Estimated error in5th iteration is:0.03125
6th iteration value is:0.203125
Estimated error in6th iteration is:0.015625
7th iteration value is:0.195313
Estimated error in7th iteration is:0.0078125
8th iteration value is:0.199219
Estimated error in8th iteration is:0.00390625
9th iteration value is:0.201172
Estimated error in9th iteration is:0.00195313
10th iteration value is:0.202148
Estimated error in10th iteration is:0.000976563
11th iteration value is:0.20166
Estimated error in11th iteration is:0.000488281
12th iteration value is:0.201416
Estimated error in12th iteration is:0.000244141
13th iteration value is:0.201538
Estimated error in13th iteration is:0.00012207
Return[0.201599]
```



Question 3 :

```
x0 = 0;
x1 = 1.0;
Nmax = 20;
eps = 0.0001;
f[x_] := Cos[x] - x * Exp[x];
If[N[f[x0] * f[x1]] > 0,
  Print["Your values do not satisfy the IVP so change the value."],
  For[i = 1, i ≤ Nmax, i++, m = (x0 + x1) / 2;
   If[Abs[(x1 - x0) / 2] < eps, Return[m],
    Print[i, "th iteration value is:", m];
    Print["Estimated error in", i, "th iteration is:", (x1 - x0) / 2];
    If[f[m] * f[x1] > 0, x1 = m, x0 = m]]];
  Print["Root is:", m] *
   Print["Estimated error in", i, "th iteration is:", (x1 - x0) / 2]];
Plot[f[x], {x, -1, 3}, PlotRange → {-10, 10},
 PlotStyle → Red, PlotLabel → "f[x]=" f[x], AxesLabel → {x, f[x]}]
```

```
1th iteration value is:0.5
Estimated error in1th iteration is:0.5
2th iteration value is:0.75
Estimated error in2th iteration is:0.25
3th iteration value is:0.625
Estimated error in3th iteration is:0.125
4th iteration value is:0.5625
Estimated error in4th iteration is:0.0625
5th iteration value is:0.53125
Estimated error in5th iteration is:0.03125
6th iteration value is:0.515625
Estimated error in6th iteration is:0.015625
7th iteration value is:0.523438
Estimated error in7th iteration is:0.0078125
8th iteration value is:0.519531
Estimated error in8th iteration is:0.00390625
9th iteration value is:0.517578
Estimated error in9th iteration is:0.00195313
10th iteration value is:0.518555
Estimated error in10th iteration is:0.000976563
11th iteration value is:0.518066
Estimated error in11th iteration is:0.000488281
12th iteration value is:0.517822
Estimated error in12th iteration is:0.000244141
13th iteration value is:0.5177
Estimated error in13th iteration is:0.00012207
Return[0.517761]
```


f[x]= (cos(x) − e^x x)
cos(x) − e^x x
10
5
−5
−10
−1
1
2
3
x

Practical 2 (a)

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Secant Method

Question 1 :

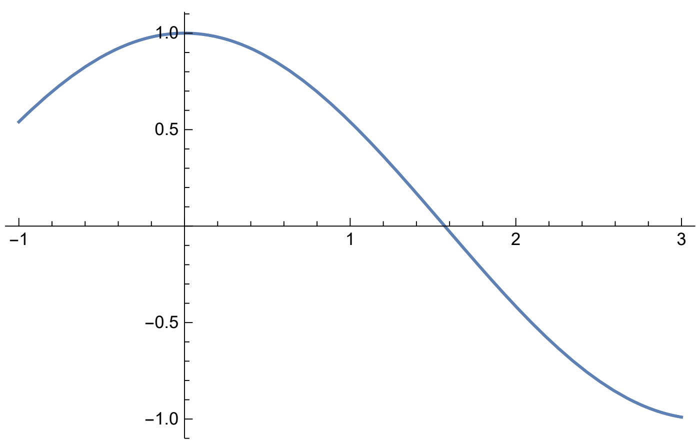
```
x0 = Input["Enter first guess:"];
x1 = Input["Enter second guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["x1=", x1];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := Cos[x];
Print["f[x]:=", f[x]];
For[i = 1, 1 ≤ Nmax, i++,
  x2 = N[x1 - (f[x] /. x → x1) * (x1 - x0) / ((f[x] /. x → x1) - (f[x] /. x → x0))];
  If[Abs[x1 - x2] < eps, Return[x2], x0 = x1; x1 = x2;];
  Print["In ", i, "th number of iterations the root is:", x2];
  Print["Estimated error is:", Abs[x1 - x0]]];
Print["Root is:", x2];
Print["Estimated error is:", Abs[x2 - x1]];
Plot[f[x], {x, -1, 3}]
```

```
x0=1

x1=2

Nmax=20
```

$\text{eps}=\frac{1}{1\,000\,000}$

```
f[x]:=Cos[x]

In 1th number of iterations the root is:1.5649

Estimated error is:0.435096

In 2th number of iterations the root is:1.57098

Estimated error is:0.0060742

In 3th number of iterations the root is:1.5708

Estimated error is:0.000182249

Return[1.5708]

Root is:1.5708
```

Estimated error is:1.02185×10^{-9}



Question 2 :

```
x0 = Input["Enter first guess:"];
x1 = Input["Enter second guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["x1=", x1];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := x^3 - 5 x + 1;
Print["f[x]:=", f[x]];
For[i = 1, 1 ≤ Nmax, i++,
  x2 = N[x1 - (f[x] /. x → x1) * (x1 - x0) / ((f[x] /. x → x1) - (f[x] /. x → x0))];
  If[Abs[x1 - x2] < eps, Return[x2], x0 = x1; x1 = x2;];
  Print["In ", i, "th number of iterations the root is:", x2];
  Print["Estimated error is:", Abs[x1 - x0]]];
Print["Root is:", x2];
Print["Estimated error is:", Abs[x2 - x1]];
Plot[f[x], {x, -1, 3}]
```

x0=1

x1=2

Nmax=20

epsilon=$\frac{1}{1\,000\,000}$

f[x]:=$1 - 5 x + x^3$

In 1th number of iterations the root is:2.5

Estimated error is:0.5

In 2th number of iterations the root is:2.09756

Estimated error is:0.402439

In 3th number of iterations the root is:2.12134

Estimated error is:0.0237786

In 4th number of iterations the root is:2.12859

Estimated error is:0.0072456

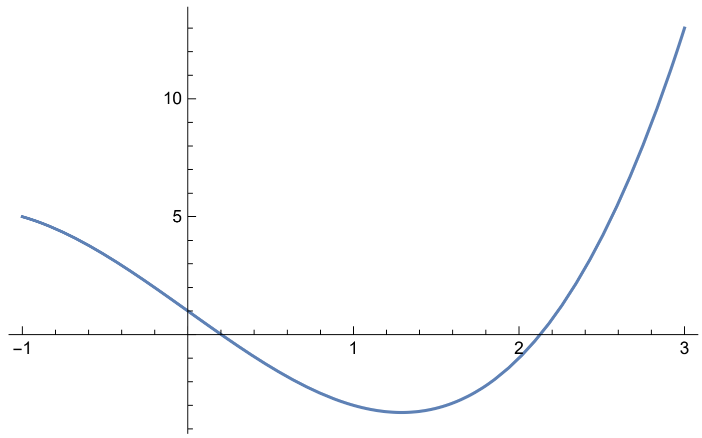
In 5th number of iterations the root is:2.12842

Estimated error is:0.000166952

Return[2.12842]

Root is:2.12842

Estimated error is:8.77361×10^{-7}



Question 3 :

```
x0 = Input["Enter first guess:"];
x1 = Input["Enter second guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["x1=", x1];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := Cos[x] - x * Exp[x];
Print["f[x]:=", f[x]];
For[i = 1, 1 ≤ Nmax, i++,
  x2 = N[x1 - (f[x] /. x → x1) * (x1 - x0) / ((f[x] /. x → x1) - (f[x] /. x → x0))];
  If[Abs[x1 - x2] < eps, Return[x2], x0 = x1; x1 = x2;];
  Print["In ", i, "th number of iterations the root is:", x2];
  Print["Estimated error is:", Abs[x1 - x0]]];
Print["Root is:", x2];
Print["Estimated error is:", Abs[x2 - x1]];
Plot[f[x], {x, -1, 3}]
```

```
x0=1

x1=2

Nmax=20
```

epsilon= $\frac{1}{1\,000\,000}$

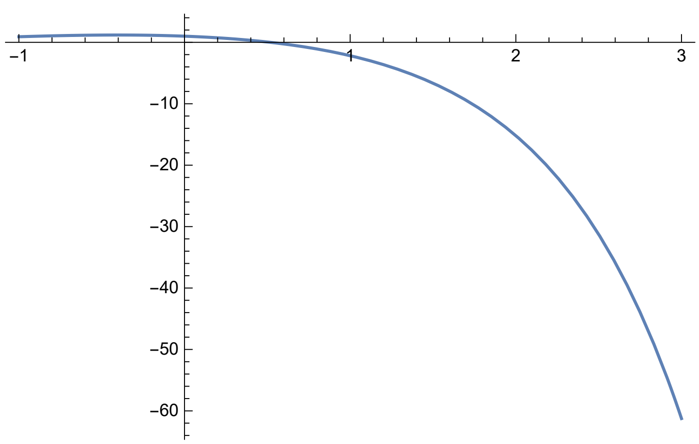
f[x]:= $-e^{x}$ x + Cos[x]

```
In 1th number of iterations the root is:0.832673

Estimated error is:1.16733

In 2th number of iterations the root is:0.728779

Estimated error is:0.103894

In 3th number of iterations the root is:0.562401

Estimated error is:0.166377

In 4th number of iterations the root is:0.524782

Estimated error is:0.0376189

In 5th number of iterations the root is:0.518014

Estimated error is:0.00676874

In 6th number of iterations the root is:0.517759

Estimated error is:0.0002547

In 7th number of iterations the root is:0.517757
```

Estimated error is:1.50138×10^{-6}

```
Return[0.517757]

Root is:0.517757
```

Estimated error is:3.22103×10^{-10}



Practical 2 (b)
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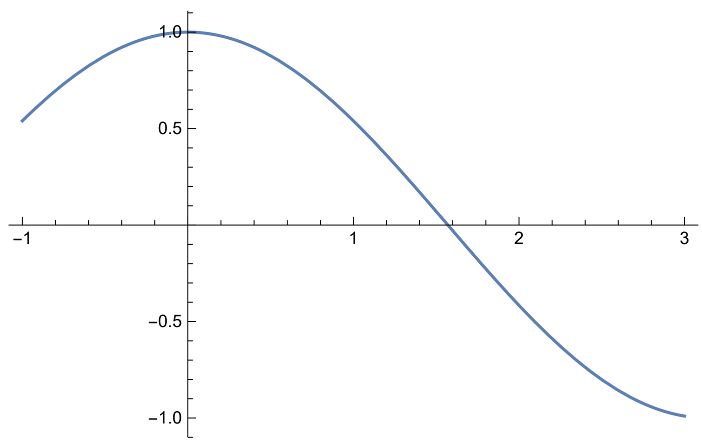
Regula Falsi Method

Question 1 :

```
x0 = Input["Enter first guess:"];
x1 = Input["Enter second guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["x1=", x1];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := Cos[x];
Print["f[x]:=", f[x]]; If[N[f[x0] * f[x1]] > 0,
 Print["These values do not satisfy the IVP so change the values."],
 For[i = 1, i ≤ Nmax, i++, a = N[x1 - f[x1] * (x1 - x0) / (f[x1] - f[x0]), 16];
  If[Abs[(x1 - x0) / 2] < eps,
   Return[N[a, 16]], Print[i, "th iteration value is:", N[a, 16]];
   Print["In ", i, "th number of iterations the root is:", x2];
   Print["Estimated error is:", N[x1 - x0, 16]];
   If[f[a] * f[x1] > 0, x1 = a, x0 = a]]];
 Print["Root is:", N[a, 16]];
 Print["Estimated error is:", N[x1 - x0, 16]]];
Plot[f[x], {x, -1, 3}]
```

```
x0=1
x1=2
Nmax=10
epsilon=0.0001
f[x]:=Cos[x]
1th iteration value is:1.564904375891578
In 1th number of iterations the root is:1.5708
Estimated error is:1.000000000000000
2th iteration value is:1.570978574535018
In 2th number of iterations the root is:1.5708
Estimated error is:0.435095624108422
3th iteration value is:1.570796325773051
In 3th number of iterations the root is:1.5708
Estimated error is:0.006074198643440
Return[1.57079632679490]
```



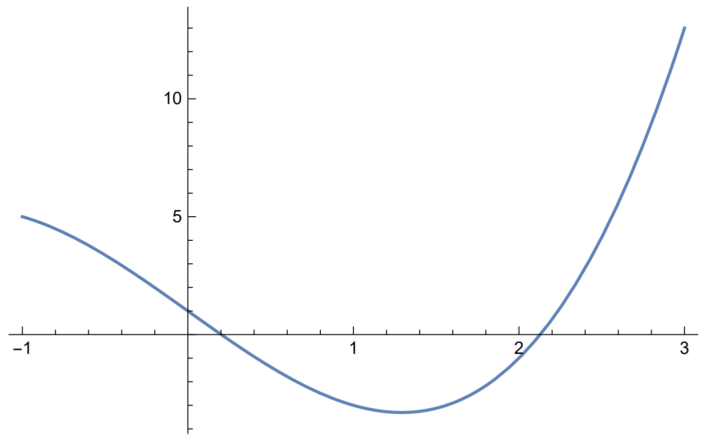
Question 2 :

```
x0 = Input["Enter first guess:"];
x1 = Input["Enter second guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["x1=", x1];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := x^3 - 5 x + 1;
Print["f[x]:=", f[x]]; If[N[f[x0] * f[x1]] > 0,
 Print["These values do not satisfy the IVP so change the values."],
 For[i = 1, i ≤ Nmax, i++, a = N[x1 - f[x1] * (x1 - x0) / (f[x1] - f[x0]), 16];
  If[Abs[(x1 - x0) / 2] < eps,
   Return[N[a, 16]], Print[i, "th iteration value is:", N[a, 16]];
   Print["In ", i, "th number of iterations the root is:", x2];
   Print["Estimated error is:", N[x1 - x0, 16]];
   If[f[a] * f[x1] > 0, x1 = a, x0 = a]]];
 Print["Root is:", N[a, 16]];
 Print["Estimated error is:", N[x1 - x0, 16]]];
Plot[f[x], {x, -1, 3}]
```

```
x0=0
x1=1
Nmax=10
epsilon=0.0001
```

$f[x] := 1 - 5x + x^3$

```
1th iteration value is:0.2500000000000000
In 1th number of iterations the root is:0.517757
Estimated error is:1.000000000000000
2th iteration value is:0.2025316455696203
In 2th number of iterations the root is:0.517757
Estimated error is:0.2500000000000000
3th iteration value is:0.201654334550389
In 3th number of iterations the root is:0.517757
Estimated error is:0.2025316455696203
4th iteration value is:0.201639916089655
In 4th number of iterations the root is:0.517757
Estimated error is:0.201654334550389
5th iteration value is:0.201639679664634
In 5th number of iterations the root is:0.517757
Estimated error is:0.201639916089655
6th iteration value is:0.20163967578803
In 6th number of iterations the root is:0.517757
Estimated error is:0.201639679664634
7th iteration value is:0.20163967572446
In 7th number of iterations the root is:0.517757
Estimated error is:0.20163967578803
8th iteration value is:0.20163967572342
In 8th number of iterations the root is:0.517757
Estimated error is:0.20163967572446
9th iteration value is:0.20163967572340
In 9th number of iterations the root is:0.517757
Estimated error is:0.20163967572342
Return[0.2016396757234]
```



Question 3 :

```
x0 = Input["Enter first guess:"];
x1 = Input["Enter second guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["x1=", x1];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := Cos[x] - x * Exp[x];
Print["f[x]:=", f[x]]; If[N[f[x0] * f[x1]] > 0,
 Print["These values do not satisfy the IVP so change the values."],
 For[i = 1, i ≤ Nmax, i++, a = N[x1 - f[x1] * (x1 - x0) / (f[x1] - f[x0]), 16];
  If[Abs[(x1 - x0) / 2] < eps,
   Return[N[a, 16]], Print[i, "th iteration value is:", N[a, 16]];
   Print["In ", i, "th number of iterations the root is:", x2];
   Print["Estimated error is:", N[x1 - x0, 16]];
   If[f[a] * f[x1] > 0, x1 = a, x0 = a]]];
 Print["Root is:", N[a, 16]];
 Print["Estimated error is:", N[x1 - x0, 16]]];
Plot[f[x], {x, -1, 3}]
```

x0=0

x1=1

Nmax=10

epsilon=0.0001

f[x]:=$-e^{x}$ x + Cos[x]

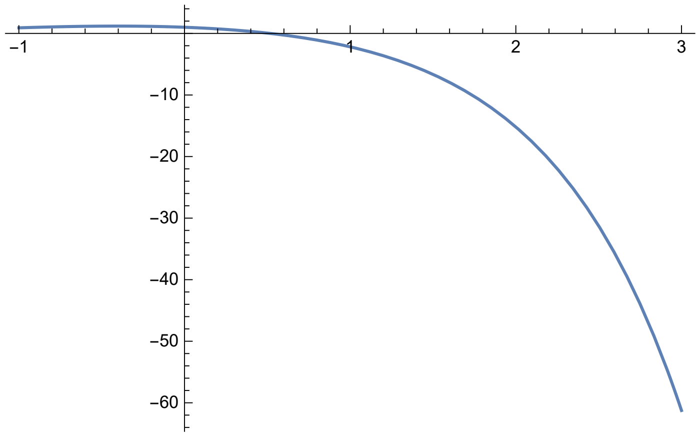
1th iteration value is:0.3146653378007709

In 1th number of iterations the root is:0.517757

Estimated error is:1.000000000000000

2th iteration value is:0.4467281445913339

```
In 2th number of iterations the root is:0.517757

Estimated error is:0.6853346621992291

3th iteration value is:0.4940153365958987

In 3th number of iterations the root is:0.517757

Estimated error is:0.5532718554086661

4th iteration value is:0.509946140365247

In 4th number of iterations the root is:0.517757

Estimated error is:0.5059846634041013

5th iteration value is:0.515201009902250

In 5th number of iterations the root is:0.517757

Estimated error is:0.490053859634753

6th iteration value is:0.516922210010517

In 6th number of iterations the root is:0.517757

Estimated error is:0.484798990097750

7th iteration value is:0.517484676784512

In 7th number of iterations the root is:0.517757

Estimated error is:0.483077789989483

8th iteration value is:0.517668344977730

In 8th number of iterations the root is:0.517757

Estimated error is:0.482515323215488

9th iteration value is:0.51772830527141

In 9th number of iterations the root is:0.517757

Estimated error is:0.482331655022270

10th iteration value is:0.51774787832211

In 10th number of iterations the root is:0.517757

Estimated error is:0.48227169472859

Root is:0.51774787832211

Estimated error is:0.48225212167789
```



-1
1
2
3
-10
-20
-30
-40
-50
-60

Practical 3
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Newton Raphson Method

Question 1 :

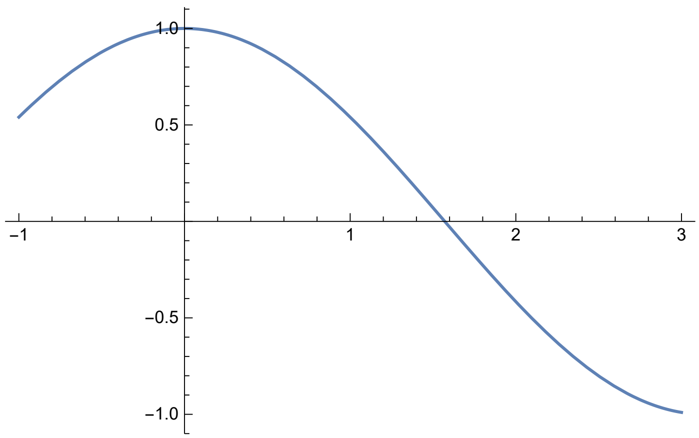
```
x0 = Input["Enter first guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := Cos[x];
Print["f[x]:=", f[x]];
Print["f'[x]:=", D[f[x], x]];
For[i = 1, i ≤ Nmax, i++, x1 = N[x0 - (f[x] /. x → x0) / (D[f[x], x] /. x → x0)];
  If[Abs[x1 - x0] < eps, Return[x1], x0p = x0; x0 = x1;];
  Print["In ", i, "th number of iterations the root is:", x1];
  Print["Estimated error is:", Abs[x1 - x0p]]];
Print["The final approximation of root is:", x1];
Print["Estimated error is:", Abs[x1 - x0]];
Plot[f[x], {x, -1, 3}]
```

```
x0=1.5
Nmax=20
```

$\text{epsilon}=\frac{1}{1000000}$

```
f[x]:=Cos[x]
f'[x]:=-Sin[x]
In 1th number of iterations the root is:1.57091
Estimated error is:0.0709148
In 2th number of iterations the root is:1.5708
Estimated error is:0.000118518
Return[1.5708]
The final approximation of root is:1.5708
```

Estimated error is:5.54889×10^{-13}



Question 2 :

```
x0 = Input["Enter first guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := x^3 - 5 x + 1;
Print["f[x]:=", f[x]];
Print["f'[x]:=", D[f[x], x]];
For[i = 1, i ≤ Nmax, i++, x1 = N[x0 - (f[x] /. x → x0) / (D[f[x], x] /. x → x0)];
  If[Abs[x1 - x0] < eps, Return[x1], x0p = x0; x0 = x1;];
  Print["In ", i, "th number of iterations the root is:", x1];
  Print["Estimated error is:", Abs[x1 - x0p]]];
Print["The final approximation of root is:", x1];
Print["Estimated error is:", Abs[x1 - x0]];
Plot[f[x], {x, -1, 3}]
```

x0=1.5

Nmax=20

epsilon=$\frac{1}{1\,000\,000}$

f[x]:=$1 - 5x + x^3$

f'[x]:=$-5 + 3x^2$

In 1th number of iterations the root is:3.28571

Estimated error is:1.78571

In 2th number of iterations the root is:2.55386

Estimated error is:0.73185

In 3th number of iterations the root is:2.21833

Estimated error is:0.33553

In 4th number of iterations the root is:2.13386

Estimated error is:0.0844789

In 5th number of iterations the root is:2.12844

Estimated error is:0.00541474

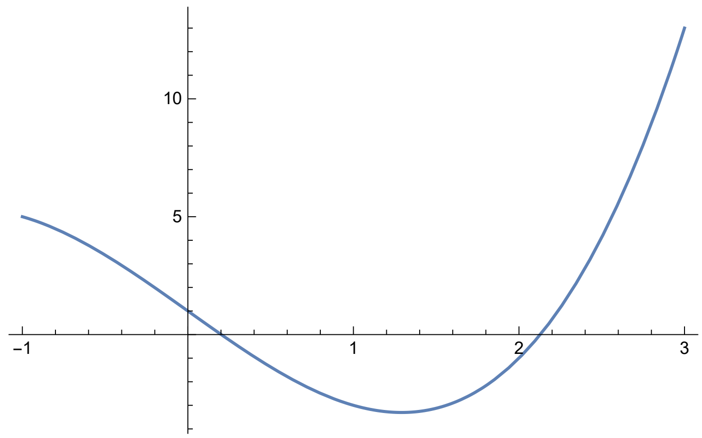
In 6th number of iterations the root is:2.12842

Estimated error is:0.0000218294

Return[2.12842]

The final approximation of root is:2.12842

Estimated error is:3.54197×10^{-10}



Question 3 :

```
x0 = Input["Enter first guess:"];
Nmax = Input["Enter maximum number of iterations:"];
eps = Input["Enter the value of convergence parameter:"];
Print["x0=", x0];
Print["Nmax=", Nmax];
Print["epsilon=", eps];
f[x_] := Cos[x] - x * Exp[x];
Print["f[x]:=", f[x]];
Print["f'[x]:=", D[f[x], x]];
For[i = 1, i ≤ Nmax, i++, x1 = N[x0 - (f[x] /. x → x0) / (D[f[x], x] /. x → x0)];
  If[Abs[x1 - x0] < eps, Return[x1], x0p = x0; x0 = x1;];
  Print["In ", i, "th number of iterations the root is:", x1];
  Print["Estimated error is:", Abs[x1 - x0p]]];
Print["The final approximation of root is:", x1];
Print["Estimated error is:", Abs[x1 - x0]];
Plot[f[x], {x, -1, 3}]
```

```
x0=1.5

Nmax=20
```

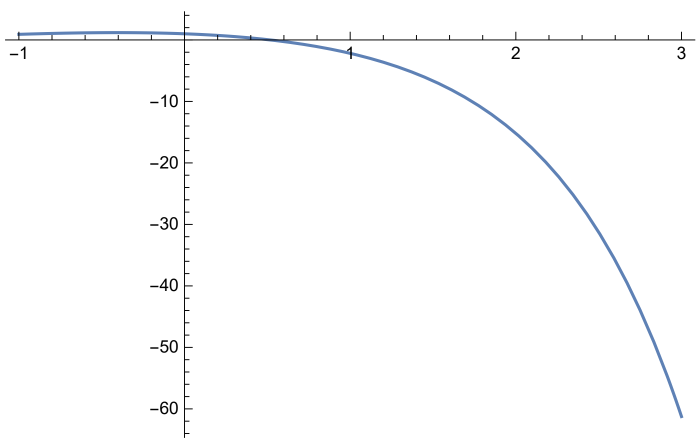
$epsilon=\frac{1}{1000000}$

$f[x]:=-e^{x}x+Cos[x]$

$f'[x]:=-e^{x}-e^{x}x-Sin[x]$

```
In 1th number of iterations the root is:0.954848

Estimated error is:0.545152

In 2th number of iterations the root is:0.632019

Estimated error is:0.322829

In 3th number of iterations the root is:0.527616

Estimated error is:0.104403

In 4th number of iterations the root is:0.517838

Estimated error is:0.00977784

In 5th number of iterations the root is:0.517757

Estimated error is:0.0000806043

Return[0.517757]

The final approximation of root is:0.517757
```

Estimated error is:5.44033×10^{-9}



Practical 4

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1. Gaussian Elimination Method

Q1. Solve the following system of equations by using Gaussian Elimination Method

2x1-3x2+10x3=-2

x1-2x2+3x3=-2

-x1+3x2+x3=4

In[1]:=
```
MatrixForm[A = {{2, -3, 10, -2}, {1, -2, 3, -2}, {-1, 3, 1, 4}}]
```

Out[1]//MatrixForm=

$$\begin{pmatrix} 2 & -3 & 10 & -2 \\ 1 & -2 & 3 & -2 \\ -1 & 3 & 1 & 4 \end{pmatrix}$$

In[2]:=
```
MatrixForm[A = {A[[2]], A[[1]], A[[3]]}]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 2 & -3 & 10 & -2 \\ -1 & 3 & 1 & 4 \end{pmatrix}$$

In[3]:=
```
MatrixForm[A = {A[[1]], A[[2]] - 2 A[[1]], A[[3]] + A[[1]]}]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \end{pmatrix}$$

In[4]:=
```
MatrixForm[A = {A[[1]], A[[2]], A[[3]] - A[[2]]}]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[5]:=
```
Solve[{x1 - 2 x2 + 3 x3 == -2, x2 + 4 x3 == 2}, {x3, x2, x1}]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[5]= {{x2 → 2 - 4 x3, x1 → 2 - 11 x3}}

Q1. Solve the following system of equations by using Gaussian Elimination Method

2x1+x2+x3=10

3x1+2x2+3x3=18

x1+4x2+9x3=16

In[6]:=
```
MatrixForm[A = {{2, 1, 1, 10}, {3, 2, 3, 18}, {1, 4, 9, 16}}]
```

Out[6]//MatrixForm=

$$\begin{pmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$

In[7]:=
```
MatrixForm[A = {A[[1]], A[[2]] - 3/2 A[[1]], A[[3]] - 1/2 A[[1]]}]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & \frac{7}{2} & \frac{17}{2} & 11 \end{pmatrix}$$

In[8]:=
```
MatrixForm[A = {A[[1]], A[[2]], A[[3]] - 7 A[[2]]}]
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 0 & -2 & -10 \end{pmatrix}$$

In[9]:=
```
Solve[{2 x1 + x2 + x3 == 10, 1/2 x2 + 3/2 x3 == 3, -2 x3 == -10}, {x3, x2, x1}]
```

Out[9]= {{x3 → 5, x2 → -9, x1 → 7}}

2. Gauss Jordan Elimination Method

Q1. Solve the following system of equations by using Gauss

Jordan Elimination Method

2x1+x2+x3=10

3x1+2x2+3x3=18

x1+4x2+9x3=16

In[10]:=
```
MatrixForm[B = {{2, 1, 1, 10}, {3, 2, 3, 18}, {1, 4, 9, 16}}]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$

In[11]:=
```
MatrixForm[RowReduce[B]]
```

Out[11]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

In[12]:=
```
Solve[{x1 == 7, x2 == -9, x3 == 5}, {x3, x2, x1}]
```

Out[12]= $\{\{x3 \to 5, x2 \to -9, x1 \to 7\}\}$

Inverse

In[13]:=
```
MatrixForm[B = {{2, 1, 1, 1, 0, 0}, {3, 2, 3, 0, 1, 0}, {1, 4, 9, 0, 0, 1}}]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{pmatrix}$$

In[14]:=
```
MatrixForm[RowReduce[B]]
```

Out[14]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -3 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 12 & -\frac{17}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -5 & \frac{7}{2} & -\frac{1}{2} \end{pmatrix}$$

Practical 5
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Gauss Jacobi/Seidel Method

Question 1 :

```
GaussJacobi[A0_, b0_, X0_, maxiter_] :=
  Module[{A = N[A0], b = N[b0], xk = X0, xk1, i, j, k = 0, n, m, OutputDetails},
   size = Dimensions[A];
   n = size[[1]];
   m = size[[2]];
   If[n ≠ m,
    Print["Not a square matrix, cannot proceed with Gauss Jacobi method"];
    Return[]];
   OutputDetails = {xk};
   xk1 = Table[0, {n}];
   While[k < maxiter,
    For[i = 1, i ≤ n, i++,
     xk1[[i]] = 1/A[[i, i]] (b[[i]] - Sum[A[[i, j]] * xk[[j]], {j, 1, i-1}] - Sum[A[[i, j]] * xk[[j]], {j, i+1, n}]);];
    k++;
    OutputDetails = Append[OutputDetails, xk1];
    xk = xk1;];
   colHeading = Table[X[s], {s, 1, n}];
   Print[NumberForm[TableForm[OutputDetails,
      TableHeadings → {None, colHeading}], 6]];
   Print["No. of iterations performed ", maxiter];];
A = {{5, 1, 2}, {-3, 9, 4}, {1, 2, -7}};
b = {10, -14, -33};
X0 = {0, 0, 0};
GaussJacobi[A, b, X0, 15]
```

X[1]	X[2]	X[3]
0	0	0
2.	-1.55556	4.71429
0.425397	-2.98413	4.55556
0.774603	-3.43845	3.92245
1.11871	-3.04067	3.84253
1.07112	-2.89044	4.00534
0.975953	-2.97867	4.04146
0.979148	-3.02644	4.00266
1.00422	-3.00813	3.98947
1.00584	-2.99391	3.99828
0.99947	-2.99729	4.00257
0.998428	-3.00132	4.0007
0.999985	-3.00083	3.9994
1.00041	-2.99974	3.99976
1.00004	-2.99976	4.00013
0.999898	-3.00004	4.00008

No. of iterations performed 15

Question 2 :

```
GaussJacobi[A0_, b0_, X0_, maxiter_] :=
  Module[{A = N[A0], b = N[b0], xk = X0, xk1, i, j, k = 0, n, m, OutputDetails},
   size = Dimensions[A];
   n = size[[1]];
   m = size[[2]];
   If[n ≠ m,
    Print["Not a square matrix, cannot proceed with Gauss Jacobi method"];
    Return[]];
   OutputDetails = {xk};
   xk1 = Table[0, {n}];
   While[k < maxiter,
    For[i = 1, i ≤ n, i++,
     xk1[[i]] = 1/A[[i, i]] (b[[i]] - Sum[A[[i, j]] * xk[[j]], {j, 1, i-1}] - Sum[A[[i, j]] * xk[[j]], {j, i+1, n}]);];
    k++;
    OutputDetails = Append[OutputDetails, xk1];
    xk = xk1;];
   colHeading = Table[X[s], {s, 1, n}];
   Print[NumberForm[TableForm[OutputDetails,
      TableHeadings → {None, colHeading}], 6]];
   Print["No. of iterations performed ", maxiter];];
A = {{1, 5, 7}, {-2, 5, -8}, {2, 6, -9}};
b = {11, -13, 24};
X0 = {0, 0, 0};
GaussJacobi[A, b, X0, 15]
```

X[1]	X[2]	X[3]
0	0	0
11.	-2.6	-2.66667
42.6667	-2.46667	-1.95556
37.0222	11.3378	5.17037
-81.8815	20.4815	13.119
-183.24	-14.3622	-7.20823
133.268	-87.4294	-52.9616
818.878	-34.0311	-31.3377
400.519	274.811	156.619
-2459.39	408.198	269.545
-3916.8	-555.082	-277.065
4725.87	-2012.62	-1243.12
18776.	-101.249	-294.224
2576.81	7037.03	4102.27
-63890.1	7591.76	5261.31
-74777.	-17140.5	-9139.29

No. of iterations performed 15

Question 3 :

```
GaussJacobi[A0_, b0_, X0_, maxiter_] :=
  Module[{A = N[A0], b = N[b0], xk = X0, xk1, i, j, k = 0, n, m, OutputDetails},
   size = Dimensions[A];
   n = size[[1]];
   m = size[[2]];
   If[n ≠ m,
    Print["Not a square matrix, cannot proceed with Gauss Jacobi method"];
    Return[]];
   OutputDetails = {xk};
   xk1 = Table[0, {n}];
   While[k < maxiter,
    For[i = 1, i ≤ n, i++,
```

$$xk1[[i]] = \frac{1}{A[[i, i]]}\left(b[[i]] - \sum_{j=1}^{i-1} A[[i, j]] * xk[[j]] - \sum_{j=i+1}^{n} A[[i, j]] * xk[[j]]\right);];$$

```
    k++;
    OutputDetails = Append[OutputDetails, xk1];
    xk = xk1;];
   colHeading = Table[X[s], {s, 1, n}];
   Print[NumberForm[TableForm[OutputDetails,
      TableHeadings → {None, colHeading}], 6]];
   Print["No. of iterations performed ", maxiter];];
A = {{2, 4, 6}, {-3, 6, -9}, {4, -6, -7}, {1, 3, 5}};
b = {4, 8, -10};
X0 = {0, 0, 0};
GaussJacobi[A, b, X0, 15]
```

Not a square matrix, cannot proceed with Gauss Jacobi method

Practical 6 (a)

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Lagrange Interpolation Polynomial

P - 1

```
LagrangePolynomial[x0_, f0_] :=
 Module[{xi = x0, fi = f0, n, m, polynomial},
  n = Length[xi];
  m = Length[fi];
  If[n ≠ m,
   Print["List of points and function values are not of same size"];
   Return[];];
  For[i = 1, i ≤ n, i++,
   L[i, x_] = (∏_{j=1}^{i-1} (x - xi[[j]])/(xi[[i]] - xi[[j]])) (∏_{j=i+1}^{n} (x - xi[[j]])/(xi[[i]] - xi[[j]]));];
  polynomial[x_] = ∑_{k=1}^{n} L[k, x] * fi[[k]];
  Return[polynomial[x]];]
```

Q1.

```
nodes = {0, 1, 3};
values = {1, 3, 55};
LagrangePolynomial[x_] = LagrangePolynomial[nodes, values]
```

$$\frac{1}{3}(1-x)(3-x)+\frac{3}{2}(3-x)x+\frac{55}{6}(-1+x)x$$

```
Expand[1/3 (1 - x) (3 - x) + 3/2 (3 - x) x + 55/6 (-1 + x) x]
```

$1 - 6x + 8x^2$

Q2.

```
nodes = {0, 1, 3};
values = {1, 3};
LagrangePolynomial[x_] = LagrangePolynomial[nodes, values]
```

```
List of points and function values are not of same size
```

P - 2

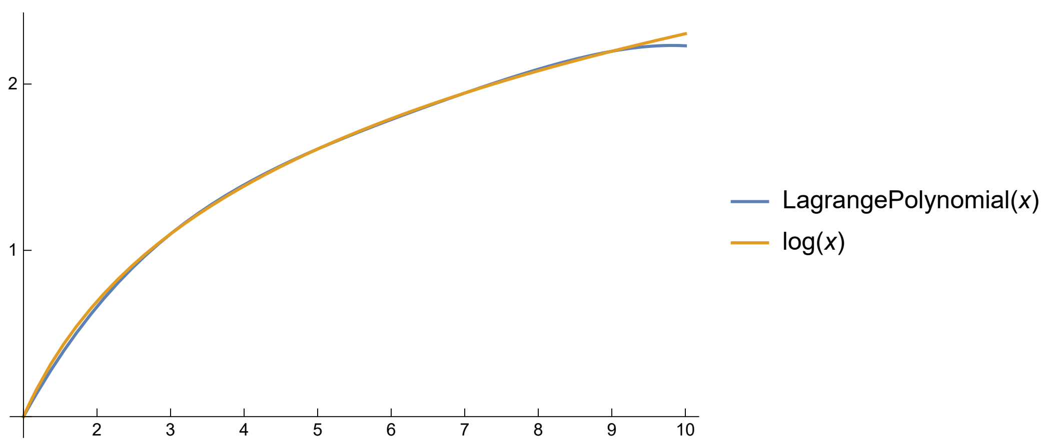
```
nodes = {1, 3, 5, 7, 9};
values = {N[Log[1]], N[Log[3]], N[Log[5]], N[Log[7]], N[Log[9]]};
LagrangePolynomial[x_] = LagrangePolynomial[nodes, values]
```

$0. + 0.0114439 (5 - x) (7 - x) (9 - x) (-1 + x) + 0.0251475 (7 - x) (9 - x) (-3 + x) (-1 + x) + 0.0202699 (9 - x) (-5 + x) (-3 + x) (-1 + x) + 0.00572194 (-7 + x) (-5 + x) (-3 + x) (-1 + x)$

```
Simplify[0.` + 0.011443878006959476` (5 - x) (7 - x) (9 - x) (-1 + x) +
  0.025147467381782817` (7 - x) (9 - x) (-3 + x) (-1 + x) +
  0.020269897385992844` (9 - x) (-5 + x) (-3 + x) (-1 + x) +
  0.005721939003479738` (-7 + x) (-5 + x) (-3 + x) (-1 + x)]
```

$-0.987583 + 1.18991 x - 0.223608 x^2 + 0.0221231 x^3 - 0.000844369 x^4$

```
Plot[{LagrangePolynomial[x], Log[x]}, {x, 1, 10},
 Ticks → {Range[0, 10]}, PlotLegends → "Expressions"]
```



```
nodes = {-1, 0, 1, 2};
values = {5, 1, 1, 11};
LagrangePolynomial[x_] = LagrangePolynomial[nodes, values]
```

$$-\frac{5}{6}(1-x)(2-x)x+\frac{1}{2}(1-x)(2-x)(1+x)+\frac{1}{2}(2-x)x(1+x)+\frac{11}{6}(-1+x)x(1+x)$$

Simplify$\left[-\frac{5}{6}(1-x)(2-x)x+\frac{1}{2}(1-x)(2-x)(1+x)+\frac{1}{2}(2-x)x(1+x)+\frac{11}{6}(-1+x)x(1+x)\right]$

$1-3x+2x^2+x^3$

```
LagrangePolynomial[1.5]
```

4.375

Practical 6 (b)
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Newton Divided Difference Interpolating Polynomial

Q1.

In[1]:=
```
NthDividedDiff[x0_, f0_, startindex_, endindex_] :=
  Module[{x = x0, f = f0, i = startindex, j = endindex, answer},
   If[i == j, Return[f[[i]]],
     answer =
      ((NthDividedDiff[x, f, i + 1, j] - NthDividedDiff[x, f, i, j - 1]) / (x[[j]] - x[[i]]));
     Return[answer]];
  ];
x = {0, 1, 3};
f = {1, 3, 55};
NthDividedDiff[x, f, 1, 2]
```

Out[4]= 2

In[5]:=
```
x = {0, 1, 3};
f = {1, 3, 55};
NthDividedDiff[x, f, 2, 3]
```

Out[7]= 26

In[8]:=
```
NthDividedDiff[x, f, 1, 3]
```

Out[8]= 8

In[9]:=
```
x = {-1, 0, 1, 2};
f = {5, 1, 1, 11};
NthDividedDiff[x, f, 1, 2]
```

Out[11]= -4

In[12]:=
```
NthDividedDiff[x, f, 2, 3]
```
Out[12]= 0

In[13]:=
```
NthDividedDiff[x, f, 1, 3]
```
Out[13]= 2

In[14]:=
```
NthDividedDiff[x, f, 2, 4]
```
Out[14]= 5

In[15]:=
```
NthDividedDiff[x, f, 1, 4]
```
Out[15]= 1

Q2.

In[16]:=
```
NewtonDDPoly[x0_, f0_] :=
  Module[{x1 = x0, f = f0, n, newtonPolynomial, k, j},
   n = Length[x1];
   newtonPolynomial[Y_] = 0;
   For[i = 1, i ≤ n, i++,
    prod[Y_] = 1;
    For[k = 1, k ≤ i - 1, k++,
     prod[Y_] = prod[Y] * (y - x1[[k]])];
    newtonPolynomial[Y_] = newtonPolynomial[Y] + NthDividedDiff[x1, f, 1, i] * prod[Y]];
   Return[newtonPolynomial[Y]];];
nodes = {0, 1, 3};
values = {1, 3, 55};
NewtonDDPoly[nodes, values]
```
Out[19]= $1+2y+8(-1+y)y$

In[20]:=
```
Simplify[1 + 2 y + 8 (-1 + y) y]
```
Out[20]= $1-6y+8y^2$

Practical 7 (a)

Rahul Chopra | BSc(H)

Computer Science | Sem - IV | 20211449

Trapezoidal Method

Q1.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := Log[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Tn = (h / 2) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Trapezoidal estimate is :", Tn]
in = Integrate[Log[x], {x, 4, 5.2}]
Print["True value is ", in]
Print["Absolute error is ", Abs[Tn - in]]
```

```
For n= 6 Trapezoidal estimate is :26.8772

1.82785

True value is 1.82785

Absolute error is 25.0494
```

Q2.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := Sin[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Tn = (h / 2) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Trapezoidal estimate is :", Tn]
in1 = Integrate[Sin[x], {x, 0, π/2}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Tn - in1]]
```

For n= 6 Trapezoidal estimate is :-0.944145

1

True value is 1

Absolute error is 1.94415

Q3.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := Sin[x] - Log[x] + Exp[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Tn = (h / 2) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Trapezoidal estimate is :", Tn]
in1 = Integrate[Sin[x] - Log[x] + Exp[x], {x, 0.2, 1.4}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Tn - in1]]
```

For n= 6 Trapezoidal estimate is :5.92567×10^8

4.05095

True value is 4.05095

Absolute error is 5.92567×10^8

Q4.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := 1/(1 + x^2);
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Tn = (h / 2) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Trapezoidal estimate is :", Tn]
in1 = Integrate[1/(1 + x^2), {x, 0, 1}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Tn - in1]]
```

For n= 6 Trapezoidal estimate is :0.0501042

$\frac{\pi}{4}$

True value is $\frac{\pi}{4}$

Absolute error is 0.735294

Practical 7 (b)
Rahul Chopra | BSc(H) Computer Science | Sem - IV | 20211449

Simpson Method

Q1.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := 1/x;
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Sn = (h / 3) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Simpson estimate is :", Sn]
in1 = Integrate[1/x, {x, 1, 2}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Sn - in1]]
```

```
For n= 6 Simpson estimate is :0.463252
```

```
Log[2]
```

```
True value is Log[2]
```

```
Absolute error is 0.229896
```

Q2.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := Log[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Sn = (h / 3) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Simpson estimate is :", Sn]
in1 = Integrate[Log[x], {x, 4, 5.2}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Sn - in1]]
```

```
For n= 6 Simpson estimate is :17.9182

1.82785

True value is 1.82785

Absolute error is 16.0903
```

Q3.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := Sin[x] - Log[x] + Exp[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Sn = (h / 3) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Simpson estimate is :", Sn]
in1 = Integrate[Sin[x] - Log[x] + Exp[x], {x, 0.2, 1.4}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Sn - in1]]
```

For n= 6 Simpson estimate is :3.95045×10^{8}

4.05095

True value is 4.05095

Absolute error is 3.95045×10^{8}

Q4.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := Sin[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Sn = (h / 3) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Simpson estimate is :", Sn]
in1 = Integrate[Sin[x], {x, 0, π/2}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Sn - in1]]
```

```
For n= 6 Simpson estimate is :-0.62943
1
True value is 1
Absolute error is 1.62943
```

Q5.

```
a = Input["Enter the left end point: "];
b = Input["Enter the right end point: "];
n = Input["Enter the number of sub intervals to be formed: "];
h = (b - a) / n;
y = Table[a + i * h, {i, 1, n}];
f[x] := (x^0.5) * Exp[x];
sumodd = 0;
sumeven = 0;
For[i = 1, i < n, i += 2, sumodd += 2 * f[x] /. x → y[[i]]];
For[i = 2, i < n, i += 2, sumeven += 2 * f[x] /. x → y[[i]]];
Sn = (h / 3) * ((f[x] /. x → a) + N[sumodd] + N[sumeven] + (f[x] /. x → b));
Print["For n= ", n, " Simpson estimate is :", Sn]
in1 = Integrate[(x^0.5) * Exp[x], {x, 1, 2}]
Print["True value is ", in1]
Print["Absolute error is ", Abs[Sn - in1]]
```

For n= 6 Simpson estimate is :1.73692×10^{9}

5.85023

```
True value is 5.85023
```

Absolute error is 1.73692×10^{9}