ECE600 Random Variables and Waveforms
Fall 2020

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MSEE 336

**Midterm Exam #3**
**Group 1**
Session 25
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75 minutes

**Solutions**

1. **(25 pts.)** Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly distributed random variables with joint pdf

$$f_{\mathbf{XY}}(x,y) = kx^2y \cdot 1_{[0,y]}(x)1_{[0,1]}(y).$$

(a) Find the constant $k$ that makes $f_{\mathbf{XY}}(x,y)$ a valid joint pdf.
(b) Find the marginal pdf $f_{\mathbf{Y}}(y)$.
(c) Find the conditional pdf $f_{\mathbf{X}}(x|\{\mathbf{Y}=y\})$.
(d) Find the *minimum mean-square error* (MMS) estimator $\hat{x}_{MMS}(y)$ of $\mathbf{X}$ given $\{\mathbf{Y}=y\}$.
(e) Find the *maximum aposteriori probability* (MAP) estimator $\hat{x}_{MMS}(y)$ of $\mathbf{X}$ given $\{\mathbf{Y}=y\}$.

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(a) $$1 = \iint_{\mathbb{R}^2} f_{XY}(x,y)\,dx\,dy = \int_0^1\int_0^y kx^2y\,dx\,dy = k\int_0^1 y\int_0^y x^2\,dy\,dy$$
$$= k\int_0^1 y\left[\frac{x^3}{3}\Big|_0^y\right]dy = \frac{k}{3}\int_0^1 y^4\,dy = \frac{k}{15}\cdot y^5\Big|_0^1$$
$$= \frac{k}{15}. \quad \therefore \boxed{k = 15}$$

(b) $$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^y 15x^2y\,dx = \frac{15}{3}\cdot y\cdot x^3\Big|_0^y$$
$$= \frac{15}{3}y^4\cdot 1_{[0,1]}(y) = \boxed{5y^4\cdot 1_{[0,1]}(y)}$$

(c) $$f_X(x|\{Y=y\}) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{15x^2y\cdot 1_{[0,y]}(x)\cdot 1_{[0,1]}(y)}{5y^4\cdot 1_{[0,1]}(y)}$$
$$= \boxed{\frac{3x^2}{y^3}\cdot 1_{[0,y]}(x)}$$

(Problem 1 Solution Continued)

(d) $$\hat{x}_{MMS}(y) = E[X \mid \{Y=y\}] = \int_{-\infty}^{\infty} x\, f_{*}(x \mid \{Y=y\})\, dx$$

$$= \int_0^y x\, \frac{3x^2}{y^3}\, dx = \int_0^y \frac{3x^3}{y^3}\, dx = \frac{3}{4y^3} \cdot x^4 \Big|_0^y$$

$$= \frac{3}{4y^3}(y^4 - 0) = \boxed{\frac{3y}{4}}$$

(e) $$\hat{x}_{MAP} = \arg \max_{x \in [0,y]} \left\{ \frac{3x^2}{y^3} \right\} = \boxed{y}$$ (the endpoint)

2. **(25 pts.)** Let **X** and **Y** be two jointly distributed random variables with joint characteristic function

$$\Phi_{\mathbf{XY}}(\omega_1, \omega_2) = \frac{1}{(1 - i\omega_1)(1 - i2\omega_2)}.$$

(a) Calculate the value of E[**X**].
(b) Calculate the value of E[**Y**].
(c) Calculate the value of E[**XY**].
(d) Calculate the value of E[$\mathbf{X}^j\mathbf{Y}^k$] if $j$ and $k$ are positive integers.
(e) Calculate the correlation coefficient between **X** and **Y**.

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We have $\phi_{XY}(s_1, s_2) = \dfrac{1}{(1-s_1)(1-2s_2)}$

$$\phi_X(s) = \phi_{XY}(s,0) = \frac{1}{1-s}$$

$$\phi_Y(s) = \phi_{XY}(0,s) = \frac{1}{1-2s^2}$$

(a) $E[X] = \left.\dfrac{d\phi_X(s)}{ds}\right|_{s=0} = \left.\dfrac{d}{ds}(1-s)^{-1}\right|_{s=0} = \left.-1(1-s)^{-2}(-1)\right|_{s=0}$

$= \boxed{1}$

(b) $E[Y] = \left.\dfrac{d\phi_Y(s)}{ds}\right|_{s=0} = \left.\dfrac{d}{ds}(1-2s)^{-1}\right|_{s=0} = \left.-1(1-2s)^{-2}(-2)\right|_{s=0}$

$= \boxed{2}$

(c) $E[XY] = \left.\dfrac{\partial^2 \phi_{XY}(s_1,s_2)}{\partial s_1 \partial s_2}\right|_{\substack{s_1=0\\ s_2=0}} = \left.\dfrac{\partial^2}{\partial s_1 \partial s_2}(1-s_1)^{-1}(1-2s_2)^{-1}\right|_{\substack{s_1=0\\ s_2=0}}$

$= \left.-1(1-s_1)^{-2}(-1)(-1)(1-2s_2)^{-2}(-2)\right|_{\substack{s_1=0\\ s_2=0}}$

$= (-1)(1)(-1)(-1)(1)(-2)$

$= \boxed{2}$

(Problem 2 Solution Continued)

(d) $E[X^j \cdot Y^k] = \dfrac{\partial^{j+k} \phi_{XY}(s_1, s_2)}{\partial s_1^j \, \partial s_2^k}$

$= \dfrac{\partial^{j+k}}{\partial s_1^j \, \partial s_2^k} \left\{ (1-s_1)^{-1}(1-2s_2)^{-1} \right\}$

$= \underbrace{(-1)(-2)\cdots(-j)(1-s_1)^{-(j+1)}}_{} \cdot \underbrace{(-1)^j \cdot (-1)(-2)\cdots(-k)}_{}(1-2s_2)^{-(k+1)}(-2)^k$

$= (j!)(k!)(2^k) = \boxed{j!\, k!\, 2^k}$

(e) $r_{XY} = \dfrac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y} = \dfrac{E[XY] - E[X] \cdot E[Y]}{\sigma_X \sigma_Y} = \dfrac{2 - (1)(2)}{\sigma_X \sigma_Y}$

$= \dfrac{0}{\sigma_X \sigma_Y} = \boxed{0}$

3. **(25 pts.)** The number of cars $\mathbf{N}$ that pass a point on a highway in one hour is a Poisson random variable with mean $\lambda$. The probability that any car is exceeding the speed limit is $p$, independent of the other cars. Let $\mathbf{M}$ be the number of cars exceeding the speed limit during this one hour period. In this problem, you will find the pmf of $\mathbf{M}$ using characteristic functions and iterated expectation, using the following procedure:
   (a) Compute the characteristic function of the Poisson random variable $\mathbf{N}$.
   (b) Using iterated expectation, compute the chracteristic function of $\mathbf{M}$, using the fact that

$$\Phi_{\mathbf{M}}(\omega) = \mathrm{E}[e^{i\omega\mathbf{M}}] = \mathrm{E}_{\mathbf{N}}[\mathrm{E}_{\mathbf{M}}[e^{i\omega\mathbf{M}}|\mathbf{N}]] = \sum_{n=0}^{\infty} p_{\mathbf{N}}(n)\cdot \mathrm{E}_{\mathbf{M}}[e^{i\omega\mathbf{M}}|\{\mathbf{N}=n\}].$$

   (c) Based on your answer in part (b), write down the pmf of the random variable $\mathbf{M}$.
   (d) What are the mean and variance of $\mathbf{M}$?

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(a) $$\Phi_N(\omega) = E[e^{i\omega N}] = \sum_{n=0}^{\infty} p_N(n)e^{i\omega n} = \sum_{n=0}^{\infty} e^{i\omega n}\cdot\frac{\lambda^n e^{-\lambda}}{n!}$$
$$= e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda e^{i\omega})^n}{n!} = e^{-\lambda}\cdot e^{\lambda e^{i\omega}}$$
$$= \boxed{e^{\lambda(e^{i\omega}-1)}}$$
(The characteristic function of a Poisson RV with mean $\lambda$.)

(b) $$\Phi_M(\omega) = E[e^{i\omega M}] = E_N[E_M[e^{i\omega M}|N]]$$
$$= \sum_{n=0}^{\infty} p_N(n)\cdot E[e^{i\omega M}|N=n] \quad --- (*)$$

Now given that $N=n$, $M$ is a binomial RV corresponding to $n$ trials with probability $p$ of success (speeding in this case.) Thus we have

$$E[e^{i\omega M}|N=n] = \sum_{m=0}^{n} e^{i\omega m}\binom{n}{m}p^m(1-p)^{n-m}$$
$$= \sum_{m=0}^{n}\binom{n}{m}(pe^{i\omega})^m(1-p)^{n-m} \overset{\text{binomial theorem}}{=} (pe^{i\omega}+1-p)^n$$
$$= (1+p(e^{i\omega}-1))^n$$

(Problem 3 Solution Continued)

So
$$\Phi_{M}(\omega) = \sum_{n=0}^{\infty} P_N(n)\left(1+p(e^{i\omega}-1)\right)^n$$
$$= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!}\left(1+p(e^{i\omega}-1)\right)^n$$
$$= e^{-\lambda}\sum_{n=0}^{\infty} \frac{[\lambda(1+p(e^{i\omega}-1))]^n}{n!} = e^{-\lambda}e^{\lambda(1+p(e^{i\omega}-1))}$$
$$= \boxed{e^{\lambda p(e^{i\omega}-1)}}$$
(This is the char ftn. of a Poisson RV with mean $\lambda p$.)

(c) From the char. ftn. in part (b), we have that $M$ is a Poisson RV with mean $\lambda p$. Thus

$$\boxed{P_M(m) = \frac{(\lambda p)^m e^{-\lambda p}}{m!}, \quad m=0,1,2,\ldots}$$

(d) For a Poisson RV, the mean is equal to the variance. Thus we have

$$\boxed{\mathrm{var}(M) = E[M] = \lambda p.}$$

4. **(25 pts.)** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n, \ldots$ be a sequence of independent, identically distributed, exponential random variables, each having mean $\mu$. Define a new random sequence $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n, \ldots$, where

$$\mathbf{Y}_n = \min\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}, \quad n = 1, 2, 3, \ldots.$$

(*i.e.*, $\mathbf{Y}_n$ takes on the minimum value of the first $n$ random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots . \mathbf{X}_n$ in the initial random sequence.)

(a) Find the probability density function of $\mathbf{Y}_n$.

(b) Does the random sequence $\{\mathbf{Y}_n\}$ converge *in probability* as $n \to \infty$? Justify your answer.

(c) Does the random sequence $\{\mathbf{Y}_n\}$ converge *in the mean-square sense* as $n \to \infty$? Justify your answer.

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(a) We will start by finding the cdf of $\mathbf{Y}_n$.

$$F_{\mathbf{Y}_n}(y) = P(\{\mathbf{Y}_n \le y\}) = 1 - P(\{\mathbf{Y}_n > y\})$$
$$= 1 - P(\{\mathbf{X}_1 > y\} \cap \{\mathbf{X}_2 > y\} \cap \cdots \cap \{\mathbf{Y}_n > y\})$$
$$= 1 - P(\{\mathbf{X}_1 > y\}) \cdot P(\{\mathbf{X}_2 > y\}) \cdots P(\{\mathbf{X}_n > y\})$$

Now $P(\{\mathbf{X}_j > y\}) = 1 - F_{\mathbf{X}_j}(y)$, where

$$F_{\mathbf{X}_j}(y) = \int_{-\infty}^{y} \frac{1}{\mu} e^{-y/\mu} \cdot 1_{[0,\infty)}(x)\, dx = \begin{cases} 1 - e^{-y/\mu}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$\therefore \quad F_{\mathbf{Y}_n}(y) = \underline{1 - (1 - F_{\mathbf{X}_j}(y))^n} = \begin{cases} (e^{-y/\mu})^n, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

Thus it follows that

$$f_{\mathbf{Y}_n}(y) = \frac{dF_{\mathbf{Y}_n}(y)}{dy} = \frac{d}{dy}\left(\exp\left(\frac{-y}{\mu/n}\right) 1_{[0,\infty)}(y)\right)$$

$$= \boxed{\frac{1}{\mu/n} \exp\left(\frac{-y}{\mu/n}\right) 1_{[0,\infty)}(y)}$$

which is an exponential pdf with mean $\frac{\mu}{n}$.

(Problem 4 Solution Continued)

(b) Because $Y_n$ is exponential with mean $\mu/n$, I suspect it is converging to 0 in some sense. Lets check to see if it is converging to 0 in probability.

The sequence $\{Y_n\}$ converges to 0 in probability if

$$P(\{|Y_n - 0| > \varepsilon\}) \longrightarrow 0 \text{ as } n \to \infty \text{ for all } \varepsilon > 0.$$

Note that

$$P(\{|Y_n - 0| > \varepsilon\}) = P(\{Y_n > \varepsilon\}) = 1 - F_{Y_n}(\varepsilon)$$

$$= 1 - \left(1 - \exp\left[\frac{-\varepsilon}{\mu/n}\right]\right), \; \varepsilon > 0$$

$$= \exp\left(-\frac{\varepsilon n}{\mu}\right) \xrightarrow{n\to\infty} 0 \text{ for all } \varepsilon > 0 \text{ and } \mu > 0.$$

$$\therefore \; P(\{|Y_n - 0| > \varepsilon\}) \longrightarrow 0 \text{ as } n \to \infty \text{ for } \forall \varepsilon > 0.$$

$\Rightarrow Y_n \xrightarrow{(p)} 0$ as $n \to \infty$ $\therefore$ $\boxed{\{Y_n\} \text{ converges to } 0 \text{ in probability}}$

(c) We now check to see if $\{Y_n\}$ converges to 0 in the mean-square sense.

$$E[|Y_n - 0|^2] = E[Y_n^2]$$

Now from part (a), we know that $Y_n$ is exponentially distributed with mean $\mu_n \triangleq E[Y_n] = \frac{\mu}{n}$.

(Problem 4 Solution Continued)

Thus it follows that $Y_n$ has characteristic function

$$\Phi_{Y_n}(\omega) = \frac{1}{(1 - i\omega\mu_n)}$$

and by the moment theorem, it follows that

$$E[Y_n^2] = \frac{\partial^2}{\partial(i\omega)^2}(1 - i\omega\mu_n)^{-1}\Big|_{i\omega=0} = \frac{\partial}{\partial(i\omega)}\left\{\mu_n(1-i\omega\mu_n)^{-2}\right\}\Big|_{i\omega=0}$$

$$= 2\mu_n^2(1-i\omega\mu_n)^{-3}\Big|_{i\omega=0} = 2\mu_n^2 = \frac{2\mu}{n^2}$$

Thus it follows that

$$E[|Y_n - 0|^2] = \frac{2\mu}{n^2} \to 0 \text{ as } n \to \infty.$$

$$\therefore \quad Y_n \xrightarrow{(m.s)} 0 \quad \text{as } n \to \infty$$

$\boxed{\text{Yes, } \{Y_n\} \text{ converges to } 0 \text{ in mean-square}}$