ECE600 Random Variables and Waveforms | Mark R. Bell
Fall 2020 | MSEE 336

Midterm Exam #2
Group 1
Session 18
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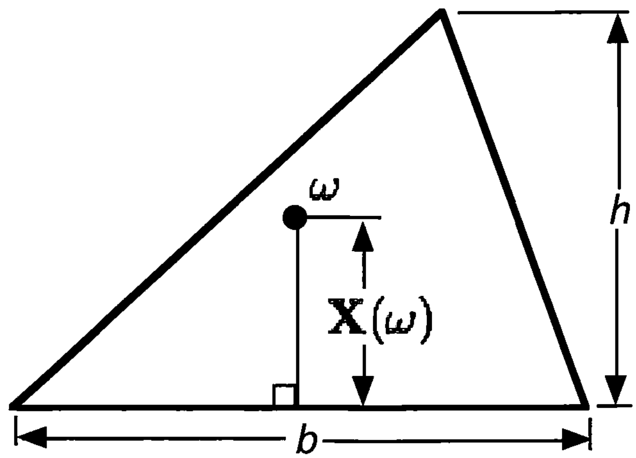
75 minutes

Solutions

1. **(25 pts.)** A point ω is picked at random in the triangle shown below (all points are equally likely.) Let the random variable $\mathbf{X}(\omega)$ be the perpendicular distance from ω to the base as shown in the diagram.



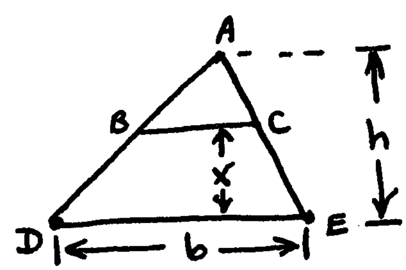
(a) Find the cumulative distribution function (cdf) of $\mathbf{X}$.
(b) Find the probability density function (pdf) of $\mathbf{X}$.
(c) Find the mean of $\mathbf{X}$.
(d) What is the probability that $\mathbf{X} > h/3$?

(Hint: The area of any triangle is $A = \frac{1}{2}bh$, where b is the length of the base of the triangle and h is its height.)

(a) $F_{\mathbf{X}}(x) = P(\{\mathbf{X} \le x\}) = 1 - P(\{\mathbf{X} > x\})$

and

$$P(\{\mathbf{X} > x\}) = \frac{\text{Area}(\Delta ABC)}{\text{Area}(\Delta ADE)}$$



$$= \frac{\frac{1}{2}(h-x)\left[\frac{h-x}{h}b\right]}{\frac{1}{2}bh} = \frac{(h-x)^2}{h^2}$$

where height $(\Delta ABC) = h - x$
base $(\Delta ABC) = \frac{(h-x)}{h} b$ (see figure)

Thus for $x \in [0, h]$, we have $F_{\mathbf{X}}(x) = 1 - \frac{(h-x)^2}{h^2} = \frac{2hx - x^2}{h^2}$

$$\boxed{\therefore\ F_{\mathbf{X}}(x) = \begin{cases} 0, & x < 0 \\ \frac{2hx - x^2}{h^2}, & 0 \le x \le h \\ 1, & x > h. \end{cases}}$$

(Problem 1 Solution Continued)

(b) $f_X(x) = \dfrac{d F_X(x)}{dx} = \dfrac{+2h-2x}{h^2} \cdot 1_{[0,h]}(x) = \boxed{\dfrac{2(h-x)}{h^2} 1_{[0,h]}(x)}$

(c) $E[X] = \displaystyle\int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^h x \frac{2(h-x)}{h^2}\,dx = \frac{2}{h^2}\int_0^h (xh - x^2)\,dx$

$= \dfrac{2}{h^2}\left(\dfrac{hx^2}{2} - \dfrac{x^3}{3}\right)\Bigg|_0^h = \dfrac{2}{h^2}\left[\dfrac{h^3}{2} - \dfrac{h^3}{3}\right] = \dfrac{2h}{6}$

$= \boxed{\dfrac{h}{3}}$

(d) $P(\{X > \frac{h}{3}\}) = 1 - P(\{X \leq \frac{h}{3}\}) = 1 - F_X(\frac{h}{3})$

$= 1 - \left(\dfrac{2h(\frac{h}{3}) - (\frac{h}{3})^2}{h^2}\right)$

$= 1 - \left(\dfrac{\frac{2}{3}h^2 - \frac{1}{9}h^2}{h^2}\right) = 1 - \left(\dfrac{2}{3} - \dfrac{1}{9}\right)$

$= 1 - \dfrac{6}{9} + \dfrac{1}{9} = 1 - \dfrac{5}{9} = \boxed{\dfrac{4}{9}}$

2. **(25 pts.)** Let **N** be a binomially distributed random variable with probability mass function

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots n,$$

where $0 \le p \le 1$.

(a) Find the characteristic function of **N**. (Show your work.)
(b) Find the mean of **N**.
(c) Find the variance of **N**.
(d) The random variable **X** defined as

$$\mathbf{X} = \frac{\mathbf{N}}{n}$$

is often used as an estimator of p. Find the characteristic function and the mean of **X**.

(a) $$\Phi_N(\omega) = E[e^{i\omega N}] = \sum_{k=0}^{n} e^{i\omega k} p_N(k) = \sum_{k=0}^{n} e^{i\omega k} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^{i\omega})^k (1-p)^{n-k}$$

$$\overset{\text{Binomial Theorem}}{=} (pe^{i\omega} + 1 - p)^n = \boxed{(1 + p(e^{i\omega} - 1))^n}$$

(b) $\phi_N(s) = (1 + p(e^s - 1))^n$, so by the moment theorem, we have

$$E[X] = \left.\frac{d\phi_N(s)}{ds}\right|_{s=0} = \left. n(1+p(e^s-1))^{n-1} \cdot pe^s \right|_{s=0} = \boxed{np}$$

(c) $\mathrm{Var}(N) = E[N^2] - (E[N])^2 = E[N^2] - n^2p^2$

By the moment theorem, we have

$$E[N^2] = \left.\frac{d^2\phi_N(s)}{ds^2}\right|_{s=0} = \frac{d}{ds}\left[n(1+p(e^s-1))^{n-1} \cdot pe^s\right]\Bigg|_{s=0}$$

$$= np(1+p(e^s-1))^{n-1} e^s + n(n-1)(1+p(e^s-1))^{n-2} pe^s \cdot pe^s \Big|_{s=0}$$

$$= np + n(n-1)p^2 = np + n^2p^2 - np^2$$

$$\therefore \mathrm{Var}(N) = np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} = np - np^2 = \boxed{np(1-p)}$$

(Problem 2 Solution Continued)

(d) $\mathbb{X} = \frac{\mathbb{N}}{n} \Rightarrow \Phi_{\mathbb{X}}(\omega) = E\left[e^{i\omega \mathbb{X}}\right] = E\left[e^{i\omega \frac{\mathbb{N}}{n}}\right]$

$$= E\left[e^{i\left(\frac{\omega}{n}\right)\mathbb{N}}\right] = \Phi_{\mathbb{N}}\left(\frac{\omega}{N}\right).$$

$$\therefore \quad \Phi_{\mathbb{X}}(\omega) = \boxed{\left(1 + p\left(e^{i\omega/n} - 1\right)\right)^n}$$

$$E[\mathbb{X}] = E\left[\frac{\mathbb{N}}{n}\right] = \frac{1}{n} E[\mathbb{N}] = \frac{1}{n} np = \boxed{p}$$

3. **(25 pts.)** Let **X** be a random variable with pdf

$$f_{\mathbf{X}}(x) = kx^2 \cdot 1_{[0,1]}(x).$$

(a) For what value of k is $f_{\mathbf{X}}(x)$ a valid pdf?
(b) Find the cdf $F_{\mathbf{X}}(x)$.
(c) Find the conditional pdf of **X** given the event $\{\mathbf{X} > a\}$, where $0 < a < 1$.
(d) Find the conditional mean of **X** conditioned on $\{\mathbf{X} > a\}$, where $0 < a < 1$.

(a) For a valid pdf $f_{\mathbf{X}}(x)$, we know that $\int_{-\infty}^{\infty} f_{\mathbf{X}}(x)\,dx = 1$.

$$\therefore \quad 1 = \int_{-\infty}^{\infty} f_{\mathbf{X}}(x)\,dx = \int_{-\infty}^{\infty} kx^2 1_{[0,1]}(x)\,dx = k\int_0^1 x^2 dx$$

$$= k\frac{x^3}{3}\Big|_0^1 = \frac{k}{3}(1^3 - 0^3) = \frac{k}{3} \Rightarrow \boxed{k = 3}$$

(b) $$F_{\mathbf{X}}(x) = \int_{-\infty}^{x} f_{\mathbf{X}}(\alpha)\,d\alpha = \int_{-\infty}^{x} 3\alpha^2 1_{[0,1]}(\alpha)\,d\alpha.$$

For $\underline{x > 1}$: $F_{\mathbf{X}}(x) = 1$. For $\underline{x < 0}$: $F_{\mathbf{X}}(x) = 0$.

For $\underline{0 \le x \le 1}$: $F_{\mathbf{X}}(x) = \int_{-\infty}^{x} 3\alpha^2 \cdot 1_{[0,1]}(\alpha)\,d\alpha = \int_0^x 3\alpha^2 d\alpha$

$$= \alpha^3\Big|_0^x = x^3.$$

$$\therefore \boxed{F_{\mathbf{X}}(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 1, & x > 1. \end{cases}}$$

(c) $$F_{\mathbf{X}}(x \mid \{\mathbf{X} > \alpha\}) = P(\{\mathbf{X} \le x\} \mid \{\mathbf{X} > \alpha\})$$

$$= \frac{P(\{\mathbf{X} \le x\} \cap \{\mathbf{X} > \alpha\})}{P(\{\mathbf{X} > \alpha\})} = \frac{P(\{\alpha < \mathbf{X} \le x\})}{1 - P(\{\mathbf{X} \le \alpha\})} \quad \text{--- } (*)$$

(Problem 3 Solution Continued)

The denominator of $(*)$ is $1-P(\{X \le \alpha\}) = 1 - F_X(\alpha) = 1-\alpha^3$.

The numerator of $(*)$ is

$$P(\{\alpha < X \le x\}) = \begin{cases} 0, & x<\alpha \\ F_X(x) - F_X(\alpha), & \alpha \le x \le 1 \\ 1 - F_X(\alpha), & x > 1. \end{cases}$$

$$= \begin{cases} 0, & x<\alpha \\ x^3-\alpha^3, & \alpha \le x \le 1 \\ 1-\alpha^3, & x>1. \end{cases}$$

$$\therefore \quad F_X(x \mid \{X>\alpha\}) = \begin{cases} 0, & x<\alpha \\ \dfrac{x^3-\alpha^3}{1-\alpha^3}, & \alpha \le x \le 1 \\ 1, & x>1. \end{cases}$$

It thus follows that

$$f_X(x \mid \{X>\alpha\}) = \frac{dF_X(x \mid \{X>\alpha\})}{dx} = \boxed{\frac{3x^2}{1-\alpha^3} 1_{(\alpha,1]}(x)}$$

(d) $$E[X \mid \{X>\alpha\}] = \int_{-\infty}^{\infty} x\, f_X(x \mid \{X>\alpha\})\,dx = \int_{\alpha}^{1} x \frac{3x^2}{1-\alpha^3}\,dx$$

$$= \frac{3}{1-\alpha^3}\int_{\alpha}^{1} x^3\,dx = \frac{3}{1-\alpha^3} \left.\frac{x^4}{4}\right|_{\alpha}^{1} = \frac{3}{1-\alpha^3}\cdot\frac{1}{4}(1-\alpha^4)$$

$$= \boxed{\frac{3(1-\alpha^4)}{4(1-\alpha^3)}}$$

4. **(25 pts.)** Let $f_1(x)$ be a Gaussian pdf with mean μ_1, variance σ_1^2, and corresponding characteristic function

$$\Phi_1(\omega) = e^{i\mu_1\omega}e^{-\frac{1}{2}\sigma_1^2\omega^2},$$

and let $f_2(x)$ be a Gaussian pdf with mean μ_2, variance σ_2^2, and corresponding characteristic function

$$\Phi_2(\omega) = e^{i\mu_2\omega}e^{-\frac{1}{2}\sigma_2^2\omega^2}.$$

Now consider the function $f_3(x)$ defined in terms if $f_1(x)$ and $f_2(x)$ by

$$f_3(x) = \lambda f_1(x) + (1-\lambda)f_2(x), \quad \lambda \in [0,1].$$

Here λ is a real number satisfying $0 \le \lambda \le 1$.
(a) Show that $f_3(x)$ is a valid pdf.
(b) Determine the mean of a random variable having pdf $f_3(x)$.
(c) Determine the characteristic function of a random variable having pdf $f_3(x)$.
(d) Is the random variable with pdf $f_3(x)$ a Gaussian random variable? Justify your answer.

(a) We have that $f_3(x)$ is a valid pdf if

(i) $f_3(x) \geq 0, \ \forall x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f_3(x)\,dx = 1.$

Let's check these two conditions

(i): $f_3(x) = \underbrace{\lambda}_{\geq 0} \cdot \underbrace{f_1(x)}_{\geq 0, \forall x \in \mathbb{R}} + \underbrace{(1-\lambda)}_{\geq 0} \cdot \underbrace{f_2(x)}_{\geq 0, \forall x \in \mathbb{R}} \geq 0, \forall x \in \mathbb{R}.$

(ii): $\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-\infty}^{\infty}(\lambda f_1(x) + (1-\lambda)f_2(x))\,dx$

$= \lambda \underbrace{\int_{-\infty}^{\infty} f_1(x)\,dx}_{1} + (1-\lambda)\underbrace{\int_{-\infty}^{\infty} f_2(x)\,dx}_{1}$

$= \lambda + (1-\lambda) = 1$

$\therefore$ $f_3(x)$ is a valid pdf.

(Problem 4 Solution Continued)

(b) $E_3[X] = \int_{-\infty}^{\infty} x\,(\lambda f_1(x) + (1-\lambda) f_2(x))\,dx$

$$= \lambda \underbrace{\int_{-\infty}^{\infty} x f_1(x)\,dx}_{\mu_1} + (1-\lambda) \underbrace{\int_{-\infty}^{\infty} x f_2(x)\,dx}_{\mu_2}$$

$$= \boxed{\lambda \mu_1 + (1-\lambda)\mu_2}$$

(c) $\Phi_3(\omega) = \int_{-\infty}^{\infty} f_3(x) e^{i\omega x}\,dx = \int_{-\infty}^{\infty} [\lambda f_1(x) + (1-\lambda) f_2(x)]\, e^{i\omega x}\,dx$

$$= \lambda \int_{-\infty}^{\infty} f_1(x) e^{i\omega x}\,dx + (1-\lambda) \int_{-\infty}^{\infty} f_2(x) e^{i\omega x}\,dx$$

$$= \lambda \Phi_1(\omega) + (1-\lambda)\Phi_2(\omega)$$

$$= \boxed{\lambda e^{i\omega\mu_1} e^{-\frac{1}{2}\sigma_1^2\omega^2} + (1-\lambda) e^{i\omega\mu_2} e^{-\frac{1}{2}\sigma_2^2\omega^2}}$$

(d) A random variable with pdf $f_3(x)$ is Gaussian iff $\Phi_3(\omega)$ can be written in the form

$$\Phi_3(\omega) = e^{i\mu_3\omega} e^{-\frac{1}{2}\sigma_3^2\omega^2}$$

for some μ_3, σ_3^2. In general, this cannot be the case, so $f_3(x)$ is <u>not</u> a Gaussian pdf. The only exceptions occur when $\lambda = 0$, $\lambda = 1$, or $\mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$.

So in general, a random variable with pdf $f_3(x)$ is <u>not</u> a Gaussian random variable!