ECE600 Random Variables and Waveforms
Fall 2020

Midterm Exam #2
Session 18
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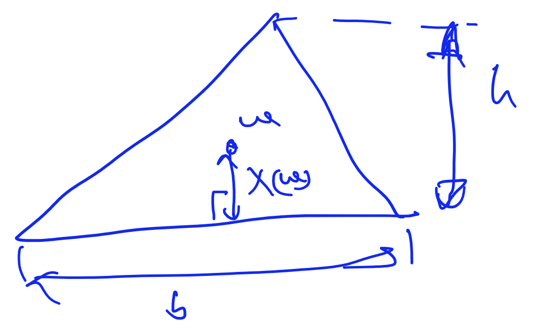
1	23
2	25
3	25
4	24
5	
6	
7	
8	
Total	97

Q1

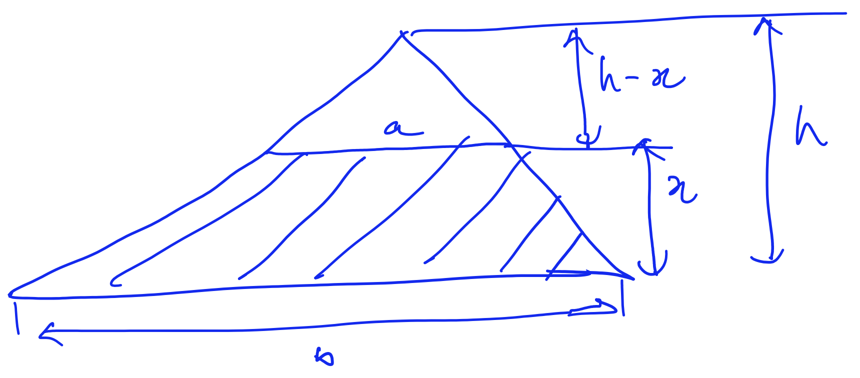
given: ω is a pt. picked in triangle



$\frac{23}{25}$

$X(\omega)$ is the $\perp$ distance of ω from base

(a) To find CDF of X



$$F_X(x) = P(\{X < x\}) \qquad 0 < x < h$$

$$= \frac{\text{area of shaded region}}{\frac{1}{2}bh}$$

by similarity of triangles we have $\frac{a}{b} = \frac{h-x}{h}$

$$\Rightarrow a = \frac{b(h-x)}{h}$$

$$\Rightarrow F_X(x) = \frac{\frac{1}{2}bh - \frac{1}{2}a.(h-x)}{\frac{1}{2}bh} = 1 - \frac{\frac{b(h-x)}{h}.(h-x)}{bh}$$

$$F_X(x) = \left(1 - \frac{(h-x)^2}{h^2}\right)\mathbb{1}_{[0,h]}(x) + \mathbb{1}_{(h,\infty)}(x)$$

$\frac{8}{10}$

(b) To find $f_X(x)$

$$f_X(x) = \frac{d}{dx}F_X(x) = +\frac{2(h-x)}{h^2}\,\mathbb{1}_{[0,h]}(x)$$

$\frac{5}{5}$

(c) $\mu_X = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^h x \cdot \frac{2(h-x)}{h^2}\,dx$

$$= \frac{2}{h^2}\left(\frac{hx^2}{2} - \frac{x^3}{3}\right)\Big|_0^h$$

$$= \frac{2}{h^2}\left(\frac{h^3}{2} - \frac{h^3}{3}\right) = \frac{\cancel{2}}{\cancel{h^2}} \times \frac{h^{\cancel{3}}}{\cancel{6}\,3}$$

$$\boxed{\mu_X = \frac{h}{3}}$$

$\frac{5}{5}$

(d) $P(\{X > h/3\}) = 1 - F_X(h/3)$

$$= \cancel{1} + \left(\cancel{1} + \frac{(h - h/3)^2}{h^2}\right)$$

$$= \frac{4}{9}\frac{\cancel{h^2}}{\cancel{h^2}}$$

$$\Rightarrow \boxed{P(\{X > h/3\}) = 4/9}$$

$\frac{5}{5}$

Q2

given: $p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n \qquad 0 \le p \le 1$

a) $\Phi_N(\omega)$?

N is a discrete R.V. with pdf given by

$$f_N(n) = \sum_{k=0}^{n} p_n(k)\, \delta(n - n_k)$$

$$\Phi_N(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} \cdot f_N(x)\, dx = \sum_{k=0}^{n} e^{i\omega k} p_n(k)$$

$$= \sum_{k=0}^{n} e^{i\omega k} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (e^{i\omega} p)^k (1-p)^{n-k}$$

$$= (e^{i\omega} p + (1-p))^n = (1 + p(e^{i\omega} - 1))^n$$

$\frac{7}{7}$

$$\boxed{\Phi_N(\omega) = (1 + p(e^{i\omega} - 1))^n} \Rightarrow \phi(s) = (1 + p(e^s - 1))^n$$

(b) $\mu = E[N] = \int_{-\infty}^{\infty} x\, f_N(x)\, dx = \sum_{k=0}^{n} \int_{-\infty}^{\infty} x \cdot p_n(k)\, \delta(x - x_k)\, dx$

$$= \sum_{k=0}^{n} k \cdot p_n(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} np \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$\frac{5}{5}$

$$= np\, (1)^{n-1} = \underline{np}$$

(c) $Var(N) = E[N^2] - (E[N])^2$

$$E[N^2] = \frac{d^2}{ds^2} \phi(s) \Big|_{s=0}$$

$$= \frac{d}{ds} \left(n (1 + p(e^s - 1))^{n-1} \times p e^s \right)$$

$$= np \cdot (n-1) \cdot p\, (1 + p(e^s - 1))^{n-2} e^s e^s + np e^s (1 + p(e^s - 1))^{n-1} \Big|_{s=0}$$

$$= np^2 (n-1)(1)^{n-2} + np = n^2 p^2 + np - np^2$$

$\Rightarrow$ $Var(N) = \cancel{n^2 p^2} + np - np^2 - \cancel{(np)^2}$

$$= \underline{np(1-p)}$$

$\frac{7}{7}$

(d) $X = \frac{N}{n}$

Using the property of $\Phi(\omega)$, $Y = aX + b$

$$\Phi_Y(\omega) = e^{i\omega b}\, \Phi_X(a\omega)$$

we get

$$\Phi_X(\omega) = \underbrace{e^{i\omega(0)}}_{1}\, \Phi_N\left(\frac{\omega}{n}\right)$$

$\frac{6}{6}$

$$= \Phi_N\left(\frac{\omega}{n}\right)$$

$$\Phi_X(\omega) = (1 + p(e^{i\omega/n} - 1))^n$$

$$\mu_X = E[X] = E\left[\frac{N}{n}\right] = \frac{1}{n} \cdot E[N] = \frac{1}{n} \cdot np = p$$

$$\boxed{\mu_X = p}$$

Q3

$\frac{25}{25}$

given: $X \sim f_X(x) = Kx^2 \, \mathbb{1}_{[0,1]}(x)$

(a) For $f_X(x)$ to be a valid pdf. it should satisfy the following conditions

(i) $f_X(x) \geq 0 \quad \forall \, x \in [0,1]$

(ii) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

Checking (i) : $f_X(x) = Kx^2 \geq 0 \quad x \in [0,1]$

$\Rightarrow K \geq 0$

Checking (ii) $\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 Kx^2 dx = K\frac{x^3}{3}\Big|_0^1 = \frac{K}{3} = 1$

$\Rightarrow \boxed{K = 3}$ ✓

$\frac{5}{5}$

$\Rightarrow f_X(x) = 3x^2 \, \mathbb{1}_{[0,1]}(x)$

(b) $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha = \int_0^x 3\alpha^2 d\alpha \qquad x \in [0,1]$

$= x^3$

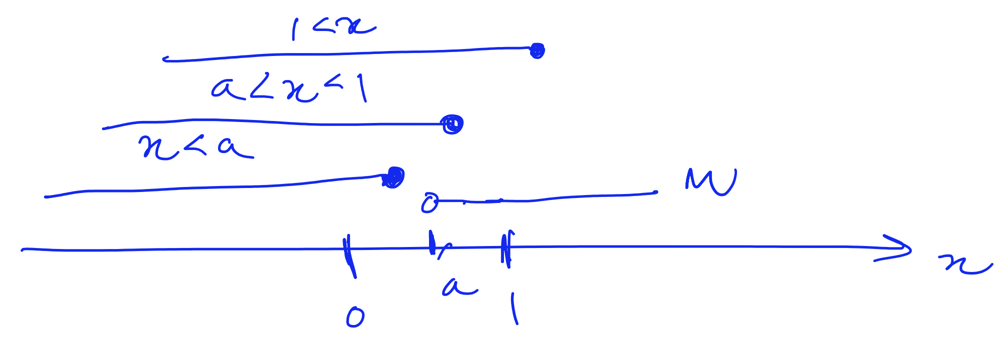
$\Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$ ✓

$\frac{5}{5}$

(c) given: $M = \{X > a\}, \; 0 < a < 1$

To find: $f_{X|M}(x|M)$

Solⁿ: $F_{X|M}(x) = P(\{X < x\} \mid \{X > a\}) = \frac{P(\{X < x\} \cap \{X > a\})}{1 - F_X(a)}$



case (i) $x \leq a$

$F_{X|M}(x) = \frac{P(\phi)}{1 - F_X(a)} = 0$

case (ii) $a < x < 1$

$F_{X|M}(x) = \frac{P(\{a < X \leq x\})}{1 - F_X(a)}$

$= \frac{F_X(x) - F_X(a)}{1 - F_X(a)}$

$= \frac{x^3 - a^3}{1 - a^3}$ ✓

case (iii) $x \geq 1$

$F_{X|M}(x) = \frac{P(\{X > a\})}{P(\{X > a\})} = 1$

$\Rightarrow F_{X|M}(x) = \begin{cases} 0 & x \leq a \\ \frac{x^3 - a^3}{1 - a^3} & a < x < 1 \\ 1 & x \geq 1 \end{cases}$ ✓

$\Rightarrow f_{X|M}(x|M) = \frac{d}{dx} F_{X|M}(x) = \begin{cases} 0 & x \leq a \\ \frac{3x^2}{1 - a^3} & a < x < 1 \\ 0 & x \geq 1 \end{cases}$

$\frac{10}{10}$ ✓

(d) $E[X|M] = \int_{-\infty}^{\infty} x \, f_{X|M}(x|M)\, dx = \int_{-\infty}^{a} 0 \cdot dx + \int_1^{\infty} 0 \cdot dx + \int_a^1 x \cdot \frac{3x^2}{1 - a^3} dx$

$= \frac{1}{1 - a^3} \frac{3x^4}{4}\Big|_a^1 = \frac{3(1 - a^4)}{4(1 - a^3)}$

$\frac{5}{5}$ ✓

$\Rightarrow E[X|M] = \frac{3(1 - a^4)}{4(1 - a^3)} = \frac{3(1 + a^2)(1 + a)}{4(1 + a^2 + a)} = \frac{3(1 + a^3 + a^2 + a)}{4(1 + a^2 + a)}$

Q4

given : $f_1(x) \sim N(\mu_1, \sigma_1)$

$\Phi_1(\omega) = e^{i\mu_1\omega} e^{-\frac{1}{2}\omega^2\sigma_1^2}$

$f_2(x) \sim N(\mu_2, \sigma_2)$, $\Phi_2(\omega) = e^{i\mu_2\omega} e^{-\frac{1}{2}\omega^2\sigma_2^2}$

$f_3(x) = \lambda f_1(x) + (1-\lambda) f_2(x)$, $\lambda \in [0,1]$

(a) To show: $f_3(x)$ is a valid p.d.f.

For $f_3(x)$ to be a valid p.d.f. it should satisfy

(i) $f_3(x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f_3(x)\,dx = 1$

Checking for (i) : $f_3(x) = \lambda f_1(x) + (1-\lambda) f_2(x)$, $\lambda \in [0,1]$

as $\lambda \in [0,1] \Rightarrow \lambda \geq 0$, $1-\lambda \geq 0$

Also as $f_1(x) \sim N(\mu_1, \sigma_1)$, $f_2(x) \sim N(\mu_2, \sigma_2)$

$f_1(x)$ & $f_2(x)$ are valid p.d.f.

& $f_1(x) \geq 0$, $f_2(x) \geq 0 \quad \forall x \in \mathbb{R}$

& $f_3(x) \geq 0 \quad \forall x \in \mathbb{R}$

Checking for (ii) : $\int_{-\infty}^{\infty} f_3(x)\,dx = \lambda \int_{-\infty}^{\infty} f_1(x)\,dx + (1-\lambda) \int_{-\infty}^{\infty} f_2(x)\,dx$

as $f_1(x)$ & $f_2(x)$ are valid p.d.f.

& $\int_{-\infty}^{\infty} f_1(x)\,dx = 1 = \int_{-\infty}^{\infty} f_2(x)\,dx$

& $\int_{-\infty}^{\infty} f_3(x)\,dx = \cancel{\lambda} + 1 - \cancel{\lambda} = \underline{\underline{1}}$ $\quad \frac{6}{6}$

$\therefore f_3(x)$ is a valid p.d.f.

(b) To find : $\mu_3 = \int_{-\infty}^{\infty} x f_3(x)\,dx$

$= \int_{-\infty}^{\infty} \lambda x f_1(x)\,dx + \int_{-\infty}^{\infty} (1-\lambda)\, x f_2(x)\,dx$

Using : Mean of gaussian R.V. $N(\mu, \sigma)$

$E[X] = \mu$

$\boxed{\mu_3 = \lambda \mu_1 + (1-\lambda)\mu_2}$ $\quad \frac{6}{6}$

(c) To find: $\phi_3(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_3(x)\,dx$

$= \lambda \int_{-\infty}^{\infty} e^{i\omega x} f_1(x)\,dx + (1-\lambda) \int_{-\infty}^{\infty} e^{i\omega x} f_2(x)\,dx$

$\frac{6}{6}$

$= \lambda \phi_1(\omega) + (1-\lambda)\phi_2(\omega)$ (by def of $\Phi(\omega)$)

$\Rightarrow \boxed{\phi_3(\omega) = (\lambda) e^{i\omega\mu_1} e^{-\frac{1}{2}\omega^2\sigma_1^2} + (1-\lambda) \cdot e^{i\omega\mu_2} e^{-\frac{1}{2}\omega^2\sigma_2^2}}$

Special case : Gaussian

(d) From (c) we can see that we cannot express $\phi_3(\omega)$ in the form of $e^{i\omega\mu_3} e^{-\frac{1}{2}\omega^2\sigma_3^2}$ $\quad \frac{6}{7}$

therefore by uniqueness of Fourier Transform we can say that $f_3(x)$ is not a gaussian R.V.