Group 1

Directions: This is an open book, open class notes exam. You may not use internet resources or other materials besides the class notes and textbook. You may use a calculator. You have 75 minutes to work the exam. Write your numbered solutions to the four problems on paper, scan your solutions to a pdf file, and send the scanned solutions to me via email (mrb@ecn.purdue.edu) or Purdue Filelocker. (See "Exam 2 Instructions(Group 1).pdf", sent via email, for complete details.)

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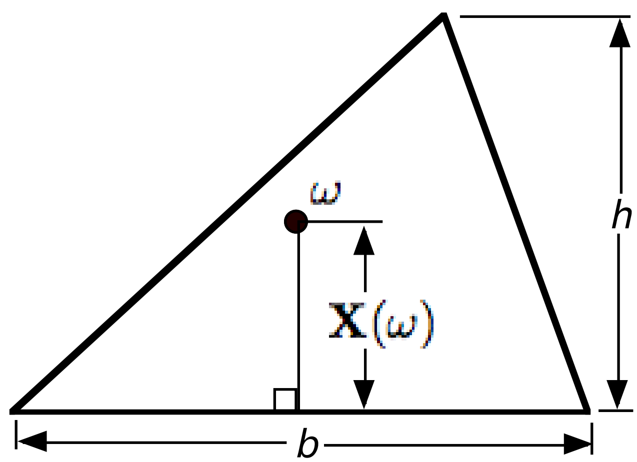
1. **(25 pts.)** A point ω is picked at random in the triangle shown below (all points are equally likely.) Let the random variable $\mathbf{X}(\omega)$ be the perpendicular distance from ω to the base as shown in the diagram.



   (a) Find the cumulative distribution function (cdf) of $\mathbf{X}$.
   (b) Find the probability density function (pdf) of $\mathbf{X}$.
   (c) Find the mean of $\mathbf{X}$.
   (d) What is the probability that $\mathbf{X} > h/3$?

   (Hint: The area of any triangle is $A = \frac{1}{2}bh$, where b is the length of the base of the triangle and h is its height.)

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2. **(25 pts.)** Let $\mathbf{N}$ be a binomially distributed random variable with probability mass function

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots n,$$

   where $0 \leq p \leq 1$.
   (a) Find the characteristic function of $\mathbf{N}$. (Show your work.)
   (b) Find the mean of $\mathbf{N}$.
   (c) Find the variance of $\mathbf{N}$.
   (d) The random variable $\mathbf{X}$ defined as

$$\mathbf{X} = \frac{\mathbf{N}}{n}$$

   is often used as an estimator of p. Find the characteristic function and the mean of $\mathbf{X}$.

3. **(25 pts.)** Let $\mathbf{X}$ be a random variable with pdf

$$f_{\mathbf{X}}(x) = kx^2 \cdot 1_{[0,1]}(x).$$

   (a) For what value of k is $f_{\mathbf{X}}(x)$ a valid pdf?
   (b) Find the cdf $F_{\mathbf{X}}(x)$.
   (c) Find the conditional pdf of $\mathbf{X}$ given the event $\{\mathbf{X} > a\}$, where $0 < a < 1$.
   (d) Find the conditional mean of $\mathbf{X}$ conditioned on $\{\mathbf{X} > a\}$, where $0 < a < 1$.

4. **(25 pts.)** Let $f_1(x)$ be a Gaussian pdf with mean μ_1, variance σ_1^2, and corresponding characteristic function

$$\Phi_1(\omega) = e^{i\mu_1\omega} e^{-\frac{1}{2}\sigma_1^2\omega^2},$$

and let $f_2(x)$ be a Gaussian pdf with mean μ_2, variance σ_2^2, and corresponding characteristic function

$$\Phi_2(\omega) = e^{i\mu_2\omega} e^{-\frac{1}{2}\sigma_2^2\omega^2}.$$

Now consider the function $f_3(x)$ defined in terms if $f_1(x)$ and $f_2(x)$ by

$$f_3(x) = \lambda f_1(x) + (1-\lambda) f_2(x), \quad \lambda \in [0,1].$$

Here λ is a real number satisfying $0 \leq \lambda \leq 1$.
   (a) Show that $f_3(x)$ is a valid pdf.
   (b) Determine the mean of a random variable having pdf $f_3(x)$.
   (c) Determine the characteristic function of a random variable having pdf $f_3(x)$.
   (d) Is the random variable with pdf $f_3(x)$ a Gaussian random variable? Justify your answer.