ECE 580: Homework 1

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February 3, 2020

Exercise 1

Proof by contraposition

$$\mathbf{S}: \quad A \implies B$$

$$\mathbf{S}: (\text{if } 7x+9 \text{ is even} \implies x \text{ is odd} \quad | \quad \forall \quad x \in \mathbf{Z})$$

In a proof by contraposition we set-out to prove that (not B) $\implies$ (not A)

$$\begin{aligned} &\text{not } B: x = 2n \text{ is even} \quad \forall \quad n \in \mathbb{Z} \\ &\quad \Rightarrow 7x = 14n \text{ is also even} \\ &\quad \Rightarrow 7x + 9 = 14n + 9 \text{ is odd as sum of even and odd is odd} \end{aligned}$$

Thus we have:

$$\text{not } B \implies \text{not } A$$

Therefore, by a proof by contraposition the statement **S** is true. Hence proved.

Exercise 2

Proof by contradiction:

$$\mathbf{S}: \quad A \implies B$$

$$A: x \in \big[0, \quad \pi/2\big] \quad B: sin(x) + cos(x) \geq 1$$

In proof by contradiction we set-out to prove : not(A and (not B))

$$(A \text{ and } (\text{not } B)): \quad x \in \big[0, \quad \pi/2\big] \implies sin(x) + cos(x) < 1$$

Lets examine the equation $sin(x) + cos(x)$ closely:

$$\begin{aligned} sin(x) + cos(x) &= \sqrt{2}(\frac{1}{\sqrt{2}}sin(x) + \frac{1}{\sqrt{2}}cos(x)) \\ &= \sqrt{2}sin(x + \pi/4) \\ \text{for } x \in \big[0, \quad \pi/2\big]: &\quad (x + \pi/4) \in \big[\pi/4, \quad 3\pi/4\big] \\ \text{We know that } sin(y) &\geq \frac{1}{\sqrt{2}} \quad \forall \quad y \in \big[pi/4, \quad 3\pi/4\big] \\ \Rightarrow sin(x) + cos(x) &\geq \sqrt{2}\frac{1}{\sqrt{2}} = 1 \end{aligned}$$

This is a contradiction

Therefore, the statement **S** is true. Hence proved.

Exercise 3

Proof by induction:
Given: for any natural number $n \in \mathbb{N}$

$$\mathbf{S}: \quad \sum_{i=1}^{n} i = f(n) = \frac{n(n+1)}{2}$$

Let's assume the statement **S** is true, then the sum of $n+1$ natural numbers should be given by $f(n+1)$. Also it should be same as $f(n)+(n+1)$ by induction. Let's verify:

$$f(n+1) = \frac{(n+1)(n+2)}{2} = \frac{n^2+3n+2}{2} \tag{1}$$

$$f(n)+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n^2+n+2(n+1)}{2} = \frac{n^2+3n+2}{2} \tag{2}$$

From Eq. 1 and Eq. 2 we can see that $f(n+1) = f(n)+(n+1)$. Therefore the statement **S** is true. Hence proved.

Exercise 4

Solution for system of linear of equations:

(a)

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Finding Rank($\mathbf{A}$) and Rank($\mathbf{A}|\mathbf{b}$)

$$\begin{aligned}
[\boldsymbol{A}|\boldsymbol{b}] &= \begin{bmatrix} 1 & 1 & 2 & 1 & | & 1 \\ 1 & 2 & 4 & -2 & | & 0 \end{bmatrix} \\
R_2 &\leftarrow R_2 - R_1 \\
&= \begin{bmatrix} 1 & 1 & 2 & 1 & | & 1 \\ 0 & 1 & 2 & -3 & | & -1 \end{bmatrix} \\
R_1 &\leftarrow R_1 - R_2 \\
&= \begin{bmatrix} 1 & 0 & 0 & 4 & | & 2 \\ 0 & 1 & 2 & -3 & | & -1 \end{bmatrix}
\end{aligned}$$

Since Rank($\mathbf{A}$) = Rank($\mathbf{A}|\mathbf{b}$) $= 2 \neq (n = 4)$ we have infinitely many solutions
we can have arbitrary values for non-pivot columns

$$\text{Let } x_3 = r \quad \& \quad x_4 = s \quad \forall \quad r, s \in \mathbb{R}$$

By substitution we get:

$$x_1 = 2 - 4s$$
$$x_2 = -1 - 2r + 3s$$

The general solution for the system is given by:

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & 3 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Finding Rank($\mathbf{A}$) and Rank($\mathbf{A}|\mathbf{b}$)

$$\left[\boldsymbol{A}|\boldsymbol{b}\right] = \begin{bmatrix} 2 & 1 & 2 & 1 & | & 0 \\ 6 & 3 & 6 & 3 & | & 1 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - 3R_1$$

$$= \begin{bmatrix} 2 & 1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 1 \end{bmatrix}$$

Since (Rank($\mathbf{A}$) $= 1$) $\neq$ (Rank($\mathbf{A}|\mathbf{b}$) $= 2$) we have no solution

Exercise 5

We have to find the condition for which the system of linear equations has a unique solution and also find the solution. We can use Cramer's Rule for this problem.

The system of linear equations is:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{bmatrix} c & 0 & a \\ 0 & c & b \\ b & a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$

$$D = det(\boldsymbol{A}) = -2abc$$

Therefore to have unique solution $D = det(\boldsymbol{A}) = -2abc \neq 0$ which means $a \neq 0, b \neq 0, c \neq 0$

Now the solution is given by:

$$\begin{aligned}
x_1 &= D_{x_1}/D \\
&= \frac{det(\begin{bmatrix} b & 0 & a \\ a & c & b \\ c & a & 0 \end{bmatrix})}{(-2abc)} = \frac{a(a^2-c^2)-ab^2}{(-2abc)} = \frac{b^2+c^2-a^2}{2bc} \\
x_2 &= D_{x_2}/D \\
&= \frac{det(\begin{bmatrix} c & b & a \\ 0 & a & b \\ b & c & 0 \end{bmatrix})}{(-2abc)} = \frac{-c^2b+b^3-a^2b}{-2abc} = \frac{a^2+c^2-b^2}{2ac} \\
x_3 &= D_{x_3}/D \\
&= \frac{det(\begin{bmatrix} c & 0 & b \\ 0 & c & a \\ b & a & c \end{bmatrix})}{(-2abc)} = \frac{c^3-a^2c-b^2c}{-2abc} = \frac{a^2+b^2-c^2}{2ab}
\end{aligned}$$

Exercise 6

Given $\boldsymbol{A} = \begin{bmatrix} cos(x) & sin(x) \\ -sin(x) & cos(x) \end{bmatrix}$. We need to find $\boldsymbol{A}^{57}$.

Note that:

$$\boldsymbol{A}^2 = AA = \begin{bmatrix} cos^2(x) - sin^2(x) & 2sin(x)cos(x) \\ -2sin(x)cos(x) & cos^2(x) - sin^2(x) \end{bmatrix}$$

Using trigonometric identities:

$sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$ &

$cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$

$$\boldsymbol{A}^2 = \begin{bmatrix} cos(2x) & sin(2x) \\ -sin(2x) & cos(2x) \end{bmatrix}$$

Therefore by induction we can say that: $A^n = \begin{bmatrix} cos(nx) & sin(nx) \\ -sin(nx) & cos(nx) \end{bmatrix}$

Therefore $\boldsymbol{A}^{57} = \begin{bmatrix} cos(57x) & sin(57x) \\ -sin(57x) & cos(57x) \end{bmatrix}$

Exercise 7

Given : $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $\quad f_1(x) = x^2 - 2x + 5, \quad$ and $\quad f_2(x) = 7x + 5$

To Find: $5f_1(\boldsymbol{A}) - 3f_2(\boldsymbol{A})$

$$\begin{aligned} 5f_1(\boldsymbol{A}) - 3f_2(\boldsymbol{A}) &= 5(\boldsymbol{A}^2 - 2\boldsymbol{A} + 5\boldsymbol{I}) - 3(7\boldsymbol{A} + 5\boldsymbol{I}) \\ &= 5\boldsymbol{A}^2 - 31\boldsymbol{A} + 10\boldsymbol{I} \\ &= 5\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} - 31\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 5\begin{bmatrix} 7 & 2 \\ 3 & 6 \end{bmatrix} - 31\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 35 & 10 \\ 15 & 30 \end{bmatrix} - \begin{bmatrix} 31 & 62 \\ 93 & 0 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} 14 & -52 \\ -78 & 10 \end{bmatrix} \end{aligned}$$

Exercise 8

Finding basis in which matrix A is diagonal

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 5 & -2 & 1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$$

We can diagonalize the matrix by computing its eigenvectors. Thus we will get $\boldsymbol{A} = \boldsymbol{VDV}^{-1}$. We first compute the eigenvalues using the characteristic equation

$$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

$$det(\begin{bmatrix} -1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 2 & 5 & -2-\lambda & 1 \\ -1 & 1 & 0 & 3-\lambda \end{bmatrix}) = 0$$

$$\Rightarrow (-1-\lambda)(1-\lambda)(-2-\lambda)(3-\lambda) = 0$$

The Eigen values are: $\lambda = 3, 1, -1, -2$

The corresponding Eigen vectors are:

For $\lambda_1 = 3$

$$(\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0}$$
$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 2 & 5 & -5 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{v}_1 = \boldsymbol{0}$$
$$\boldsymbol{v}_1 = k \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \Rightarrow \hat{\boldsymbol{v}}_1 = \frac{1}{\sqrt{26}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

For $\lambda_2 = 1$

$$(\boldsymbol{A} - 1\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0}$$
$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 5 & -3 & 1 \\ -1 & 1 & 0 & 2 \end{bmatrix} \boldsymbol{v}_2 = \boldsymbol{0}$$
$$\boldsymbol{v}_2 = k \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} \Rightarrow \hat{\boldsymbol{v}}_2 = \frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

For $\lambda_3 = -1$

$$(\boldsymbol{A} + 1\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{0}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 5 & -1 & 1 \\ -1 & 1 & 0 & 4 \end{bmatrix} \boldsymbol{v}_3 = \boldsymbol{0}$$
$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & 0 & 4 \end{bmatrix} \boldsymbol{v}_3 = \boldsymbol{0}$$
$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \boldsymbol{v}_3 = \boldsymbol{0}$$
$$\boldsymbol{v}_3 = k \begin{bmatrix} -2 \\ 1 \\ \frac{1}{4} \\ \frac{-3}{4} \end{bmatrix} \Rightarrow \hat{\boldsymbol{v}}_3 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 8 \\ -4 \\ -1 \\ 3 \end{bmatrix}$$

For $\lambda_4 = -2$

$$(\boldsymbol{A} + 2\boldsymbol{I})\boldsymbol{v}_4 = \boldsymbol{0}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 5 & 0 & 1 \\ -1 & 1 & 0 & 5 \end{bmatrix} \boldsymbol{v}_4 = \boldsymbol{0}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & 5 \end{bmatrix} \boldsymbol{v}_4 = \boldsymbol{0}$$

$$\boldsymbol{v}_4 = k \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \hat{\boldsymbol{v}}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore with the new basis as $\boldsymbol{V} = \begin{bmatrix} \hat{\boldsymbol{v}}_1^T & \hat{\boldsymbol{v}}_2^T & \hat{\boldsymbol{v}}_3^T & \hat{\boldsymbol{v}}_4^T \end{bmatrix}^T$ we can diagonalize matrix $\boldsymbol{A}$ and the diagonal matrix in the new basis is given by:

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Exercise 9

Given $f = f(x_1, x_2) = x_1^2 + x_2^2 + 4x_1x_2 + \frac{2}{3}x_2^3 - 2x_2 + 7$

(a) Points that satisfy FONC:

$$\nabla f = \begin{bmatrix} 2x_1 + 4x_2 \\ 2x_2 + 4x_1 + 2x_2^2 - 2 \end{bmatrix}$$

$$\begin{aligned} \text{FONC: } \nabla f = \boldsymbol{0} & \\ \Rightarrow & x_1 = -2x_2 \\ \Rightarrow & 2x_2 - 8x_2 + 2x_2^2 - 2 = 0 \\ & 2x_2^2 - 6x_2 - 2 = 0 \\ & x_2^2 - 3x_2 - 1 = 0 \\ \Rightarrow x_2 &= \frac{3 \pm \sqrt{14}}{2} \\ x_1 &= -2x_2 = -(3 \pm \sqrt{14}) \end{aligned}$$

The two points that satisfy the FONC are: $\boldsymbol{x}_a = \begin{bmatrix} -3 - \sqrt{14} \\ \frac{3+\sqrt{14}}{2} \end{bmatrix}$ $\quad \boldsymbol{x}_b = \begin{bmatrix} -3 + \sqrt{14} \\ \frac{3-\sqrt{14}}{2} \end{bmatrix}$

(b) SONC check using leading principal minors:

$$\nabla^2 f = \begin{bmatrix} 2 & 2 \\ 4 & 4x_2 + 2 \end{bmatrix}$$

$$\Rightarrow Q = \frac{(F + F^T)}{2} = \begin{bmatrix} 2 & 3 \\ 3 & 4x_2 + 2 \end{bmatrix}^T$$

The Leading principal minors of Q are:

$$\Delta_1 = 2 > 0 \quad \forall \quad \boldsymbol{x} \in \mathbb{R}^2$$
$$\Delta_2 = 8x_2 - 5$$
$$\text{for } \boldsymbol{x}_a : \Delta_2(\boldsymbol{x}_a) = 7 + 4\sqrt{14} > 0$$
$$\text{for } \boldsymbol{x}_b : \Delta_2(\boldsymbol{x}_b) = 7 - 4\sqrt{14} < 0$$

For $\boldsymbol{x} = \boldsymbol{x}_a$ we have $\Delta_i > 0 \quad \forall \quad i \in [1, 2]$, which means that $Q(\boldsymbol{x}_a) \succ 0$. Therefore by SONC, $\boldsymbol{x}_a = \begin{bmatrix} -3 - \sqrt{14}, & \frac{3+\sqrt{14}}{2} \end{bmatrix}^T$ is a strict local minimizer.

Whereas for $\boldsymbol{x} = \boldsymbol{x}_b$ we have $\Delta_1(\boldsymbol{x}_b) > 0, \quad \Delta_2(\boldsymbol{x}_b) < 0$, which means $Q(\boldsymbol{x}_b)$ is indefinite.

Exercise 10

Finding quadratic form of f such that $f = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ with $\boldsymbol{Q} = \boldsymbol{Q}^T$:

Note that:

$$\nabla f = \frac{1}{2}(Q + Q^T)x = Qx$$
$$\nabla^2 f = F = Q$$

(a) $f(x_1, x_2, x_3, x_4) = 7x_1^2 + x_3^2 - 2x_1x_3 + x_1x_4$

$$\nabla f = \begin{bmatrix} 14x_1 - 2x_3 + x_4 & 0 & 2x_3 - 2x_1 & x_1 \end{bmatrix}^T$$
$$\nabla^2 f = Q = \begin{bmatrix} 14 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the quadratic form is given by:

$$f = \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 14 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(b) $f(x_1, x_2, x_3) = x_2^2 - 3x_1x_2$

$$\nabla f = \begin{bmatrix} -3x_2 & 2x_2 - 3x_1 & 0 \end{bmatrix}^T$$
$$\nabla^2 f = Q = \begin{bmatrix} 0 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the quadratic form is given by:

$$f = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 0 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) $f(x_1, x_2, x_3) = 2x_1^2 - 5x_2^2 + 2x_1x_2$

$$\nabla f = \begin{bmatrix} 4x_1 + 2x_2 & -10x_2 + 2x_1 & 0 \end{bmatrix}^T$$

$$\nabla^2 f = Q = \begin{bmatrix} 4 & 2 & 0 \\ 2 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the quadratic form is given by:

$$f = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 4 & 2 & 0 \\ 2 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$