ECE600 Random Variables and Waveforms
Fall 2020

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MSEE 336

Midterm Exam #1
Group 1
Session 9
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75 minutes

Solutions

1. **(20 pts.)** A standard die is tossed twice and the number of dots facing up in each toss are counted and noted in the order of occurance.
   (a) Find the sample space $\mathcal{S}$ for this experiment.
   (b) Find a reasonable event space for this experiment.
   (c) Find the set A corresponding to the event "the number of dots in the first toss is greater than or equal to the number of dots in the second toss."
   (d) Find the set B corresponding to the event "the number of dots in the first toss is 6."
   (e) Find $A \cap \overline{B}$ and describe it in words.
   (f) Let C be the set corresponding to the event "the number of dots on the two dice differs by 2." Find $A \cap C$.
   (g) What is the value of $P(A|B)$? Justify your answer.

(a) The outcome of the experiment consists of an ordered pair of numbers (ω, ξ), where $\omega \in \{1,2,3,4,5,6\}$ and $\xi \in \{1,2,3,4,5,6\}$. Thus we have

$$\begin{aligned}\mathcal{S} &= \{(\omega,\xi) : \omega \in \{1,2,3,4,5,6\} \text{ and } \xi \in \{1,2,3,4,5,6\}\} \\ &= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ &\quad (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ &\quad (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ &\quad (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ &\quad (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ &\quad (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}\end{aligned}$$

(b) For a finite sample space such as this, the power set of $\mathcal{S}$ is the obvious event space. Thus

$$\mathcal{F} = \mathcal{P}(\mathcal{S}).$$

(n.b., $|\mathcal{F}| = 2^{36} = 68{,}719{,}476{,}736$.)

(Problem 1 Solution Continued)

(c) $A = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

(d) $B = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

(e) $A \cap \overline{B} = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5)\}$

= the event that the number of dots on the first toss is greater than or equal to that on the second, and the number of dots on the first toss is not 6.

(f) $C = \{(1,3), (2,4), (3,1), (3,5), (4,2), (4,6), (5,3), (6,4)\}$

$\therefore\ A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$

(g) We note that $B \subset A \Rightarrow A \cap B = B$. Thus we have

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = \boxed{1}$$

2. **(20 pts.)** This problem involves two short problems involving the axioms of probability.

   (a) Let $(\mathcal{S}, \mathcal{F}, P)$ be a probability space and let $M \in \mathcal{F}$ be an event with nonzero probability. Show that $P(\cdot|M)$ is a valid probability measure satisfying the axioms of probability.

   (b) Let Let $(\mathcal{S}, \mathcal{F}, P)$ be a probability space having a finite sample space with $|\mathcal{S}| = n$, event space $\mathcal{F}$ equal to the power set of $\mathcal{S}$, and probability measure P given by the classical probability measure. Show that P satisfies the axioms of probability.

In both parts (a) and (b), we must show that the probability measure under consideration satisfies the axioms of probability, which are:

(i) $P(A) \geq 0, \ \forall A \in \mathcal{F}$;

(ii) $P(\mathcal{S}) = 1$;

(iii) If $A, B \in \mathcal{F}$ are disjoint, then $P(A \cup B) = P(A) + P(B)$;

(iv) If $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ are disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

(a) We must show that $P(A|M) = \dfrac{P(A \cap M)}{P(M)}$ satisfies the axioms of probability:

(i): $P(A|M) = \dfrac{P(A \cap M)}{P(M)} \geq 0$, because $P(A \cap M) \geq 0$ and $P(M) > 0$.

(ii): $P(\mathcal{S}|M) = \dfrac{P(\mathcal{S} \cap M)}{P(M)} = \dfrac{P(M)}{P(M)} = 1.$

(iii): $$P(A \cup B|M) = \frac{P((A \cup B) \cap M)}{P(M)} = \frac{P(\underbrace{(A \cap M) \cup (B \cap M)}_{\text{disjoint}})}{P(M)}$$

$$\overset{\text{Ax. 3}}{=} \frac{P(A \cap M) + P(B \cap M)}{P(M)} = \frac{P(A \cap M)}{P(M)} + \frac{P(B \cap M)}{P(M)} = P(A|M) + P(B|M).$$

(iv): Similarly, for $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ and disjoint, ← disjoint union

$$P\left(\bigcup_{i=1}^{\infty} A_i \Big| M\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap M\right)}{P(M)} = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap M)\right)}{P(M)}$$

$$= \frac{\sum_{i=1}^{\infty} P(A_i \cap M)}{P(M)} = \sum_{i=1}^{\infty} \frac{P(A_i \cap M)}{P(M)} = \sum_{i=1}^{\infty} P(A_i|M).$$

(Problem 2 Solution Continued)

(b) Here $P(A) = \frac{|A|}{|\mathcal{S}|} = \frac{|A|}{n}$, $\forall A \in \mathcal{F}$. We now check the axioms:

(i): $P(A) = \frac{|A|}{n} > 0$, because $|A| \geq 0$ and $n > 0$.

(ii): $P(\mathcal{S}) = \frac{|\mathcal{S}|}{n} = \frac{n}{n} = 1$.

(iii): If A and B are disjoint, they have no elements in common, which means that $|A \cup B| = |A| + |B|$. Thus

$$P(A \cup B) = \frac{|A \cup B|}{n} = \frac{|A| + |B|}{n} = \frac{|A|}{n} + \frac{|B|}{n}$$

$$= P(A) + P(B).$$

By induction, it follows that for any finite M and disjoint $A_1, \ldots, A_M$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_M) = \sum_{i=1}^{M} P(A_i).$$

(iv): We don't really need to check this case, as there are only a finite number of disjoint non-empty events. Thus given any countable collection of disjoint events $A_1, \ldots, A_K, \ldots$, we can relabel them as

$$\underbrace{A_1', A_2', \ldots, A_M'}_{(M \text{ non-empty disjoint events})}, \phi, \phi, \phi, \ldots$$

and we then have

$$P\left(\bigcup_{i=1}^{\infty} A_i'\right) = P\left(\bigcup_{i=1}^{M} A_i'\right) \overset{\text{(iii)}}{=} \sum_{i=1}^{M} P(A_i') = \sum_{i=1}^{\infty} P(A_i),$$

where $P(A_i') = 0$, $i = M+1, M+2, \ldots$.

$$\therefore P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

3. **(20 pts.)** Consider a probability space $(\mathcal{S}, \mathcal{F}, P)$ with events $A, B, C \in \mathcal{F}$ and probabilities $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/4$, $P(A \cap B) = 1/6$, $P(B \cap C) = 1/12$, $P(A \cap C) = 1/6$, and $P(A \cap B \cap C) = 1/36$.
   (a) Find the probability $P(A \cup B \cup C)$.
   (b) Find the conditional probability $P(A|C)$.
   (c) Find the conditional probability $P(A|\overline{C})$.
   (d) Find the conditional probability $P(B|\overline{A} \cup \overline{C})$.
   (e) Are events B and C statistically independent? Justify your answer.

(a) $$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C) + P(A\cap B\cap C)$$

$$= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{6}-\frac{1}{6}-\frac{1}{12}+\frac{1}{36}$$

$$= \frac{18+12+9-6-6-3+1}{36} = \boxed{\frac{25}{36}}$$

(b) $$P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{1/6}{1/4} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

(c) $$P(A|\overline{C}) = \frac{P(A\cap\overline{C})}{P(\overline{C})} = \frac{P(A\cap\overline{C})}{1-P(C)}$$

Now $A = (A\cap C)\cup(A\cap\overline{C}) \Rightarrow P(A) = P(A\cap C)+P(A\cap\overline{C})$

disjoint $\Rightarrow P(A\cap\overline{C}) = P(A)-P(A\cap C)$

$$\therefore P(A|\overline{C}) = \frac{P(A)-P(A\cap C)}{1-P(C)} = \frac{\frac{1}{2}-\frac{1}{6}}{3/4} = \frac{\frac{6}{12}-\frac{2}{12}}{\frac{9}{12}} = \boxed{\frac{4}{9}}$$

(d) $$P(B|\overline{A}\cup\overline{C}) = \frac{P(B\cap(\overline{A}\cup\overline{C}))}{P(\overline{A\cap C})} = \frac{P(B\cap(\overline{A\cap C}))}{1-P(A\cap C)}$$

Now $P(B\cap(\overline{A\cap C})) = P(B)-P(B\cap A\cap C)$

$$= P(B)-P(A\cap B\cap C) = \frac{1}{3}-\frac{1}{36} = \frac{11}{36}$$

(Problem 3 Solution Continued)

$$\therefore\ P(B|\bar{A}\cup\bar{C}) = \frac{11/36}{1-1/6} = \frac{11/36}{1-6/36} = \frac{11/36}{30/36} = \boxed{\frac{11}{30}}$$

(e) $P(B\cap C) = P(B)\cdot P(C)$ iff B and C are stat. indep.

Here $P(B\cap C) = \frac{1}{12}$ and $P(B)\cdot P(C) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$

$\therefore\ P(B\cap C) = P(B)\cdot P(C)$

$\Rightarrow$ B and C are statistically independent.

4. **(20 pts.)** Suppose I flip a fair coin until heads occurs. The pmf of the number of flips until a heads occurs is a geometic pmf of the form

$$p(k) = \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} \quad k = 1, 2, 3, \ldots .$$

Next suppose I independently roll a fair die until a "1" occurs. The pmf of the number of rolls until a "1" occurs is a geometic pmf of the form

$$q(n) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \quad n = 1, 2, 3, \ldots .$$

Now suppose I form the the joint experiment made up of these two independent experiments. What is the probability that the number of coin flips until a heads occurs is equal to the number of die rolls until a "1" occurs?

Outcomes of the combined experiment are an ordered pair of positive integers (k, n) representing k coin flips until a Head occurs and n die rolls before a 1 occurs. By the independence of the constituent experiments

$$P(\{\text{First Head on k-th toss}\} \cap \{\text{First 1 on n-th roll}\})$$

$$= P_1(\{\text{First Head on k-th toss}\}) \cdot P_2(\{\text{First 1 on n-th roll}\})$$

$$= p(k) \cdot q(n)$$

It follows that the probability that it takes an equal number ($k = m$ and $n = m$) of flips and rolls to get the first Head and the first "1" is

$$p(m) \cdot q(m) = \frac{1}{2}\left(\frac{1}{2}\right)^{m-1} \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{m-1} = \frac{1}{12}\left(\frac{5}{12}\right)^{m-1}$$

So if we sum over all $m = 1, 2, 3, \ldots$, we get

$$P(E) = \sum_{m=1}^{\infty} \frac{1}{12}\left(\frac{5}{12}\right)^{m-1} = \frac{1}{12}\sum_{r=0}^{\infty}\left(\frac{5}{12}\right)^{r} = \frac{1}{12} \cdot \frac{1}{1 - \frac{5}{12}} = \frac{1}{12-5} = \boxed{\frac{1}{7}}$$

where E = The number of coin flips to the first Heads and the number of die rolls to the first 1 is equal

5. **(20 pts.)** A court is investigating the possible occurence of an unlikely event T. The reliability of two independent witnesses named Art and Bob is known to the court: Art tells the truth with probability α and Bob tells the truth with probability β, and there is no collusion in their answers (i.e., they answer independently.) Let A and B be the events that Art and Bob assert (respectively) that T occured, and the probability that T has occured (without considering Art's and Bob's testimony) is $P(T) = \tau$. What is the probability that T occured given that both Art and Bob declare that T occured?

Let T = event of interest
A = Art asserts T occured
B = Bob asserts T occured

We are to calculate $P(T|A\cap B)$.

Now

$$P(T|A\cap B) = \frac{P(T\cap A\cap B)}{P(A\cap B)} = \frac{P(A\cap B|T)\cdot P(T)}{P(A\cap B)}. \quad (*)$$

We can write the denominator of $(*)$ as

$$P(A\cap B) = P(\mathcal{S}\cap A\cap B) = P((T\cup\bar{T})\cap A\cap B)$$
$$= P(\underbrace{(T\cap A\cap B)\cup(\bar{T}\cap A\cap B)}_{\text{disjoint}})$$
$$= P(T\cap A\cap B) + P(\bar{T}\cap A\cap B)$$
$$= P(A\cap B|T)P(T) + P(A\cap B|\bar{T})P(\bar{T})$$

Now
$$\left.\begin{aligned} P(A\cap B|T) &= P(A|T)P(B|T) = \alpha\beta \\ P(A\cap B|\bar{T}) &= P(A|\bar{T})\cdot P(B|\bar{T}) = (1-\alpha)(1-\beta) \end{aligned}\right\} \text{By independence of Art and Bob's assertions}$$

Thus we have

$$P(A\cap B) = P(A\cap B|T)P(T) + P(A\cap B|\bar{T})P(\bar{T}) = \alpha\beta\tau + (1-\alpha)(1-\beta)(1-\tau)$$

and $P(A\cap B|T)P(T) = \alpha\beta\tau$

Thus from $(*)$, we have

$$P(\bar{T}|A\cap B) = \boxed{\frac{\alpha\beta\tau}{\alpha\beta\tau + (1-\alpha)(1-\beta)(1-\tau)}}$$