Exam 1: cover sheet

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ECE600 Random Variables and Waveforms
Fall 2020

Midterm Exam #1
Session 9
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Q1

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(a) For a dice toss we will have six outcomes in the sample space S_0

$$S_0 = \{1, 2, 3, 4, 5, 6\}$$

For two dice toss experiment we have

$$S_1 = S_2 = S_0$$

and the sample space of the experiment is given by

$$\begin{aligned} S &= S_1 \times S_2 \\ &= \{ (1,1)\ (2,1)\ (3,1)\ (4,1)\ (5,1)\ (6,1) \\ &\quad (1,2)\ (2,2)\ (3,2)\ (4,2)\ (5,2)\ (6,2) \\ &\quad (1,3)\ (2,3)\ (3,3)\ (4,3)\ (5,3)\ (6,3) \\ &\quad (1,4)\ (2,4)\ (3,4)\ (4,4)\ (5,4)\ (6,4) \\ &\quad (1,5)\ (2,5)\ (3,5)\ (4,5)\ (5,5)\ (6,5) \\ &\quad (1,6)\ (2,6)\ (3,6)\ (4,6)\ (5,6)\ (6,6) \} \end{aligned}$$

(b) A reasonable event space for this experiment will be the sigma field:

$$F = \sigma(\{A \times B : A \in S_1, B \in S_2\})$$

(c) A = No. of dots in first toss $\geq$ No. of dots in second toss

$$\begin{aligned} A = \{ &(1,1)\ (2,1)\ (3,1)\ (4,1)\ (5,1)\ (6,1) \\ &(2,2)\ (3,2)\ (4,2)\ (5,2)\ (6,2) \\ &(3,3)\ (4,3)\ (5,3)\ (6,3) \\ &(4,4)\ (5,4)\ (6,4) \\ &(5,5)\ (6,5) \\ &(6,6) \} \end{aligned}$$

(d) B = No. of dots in first toss is 6

$$B = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

(e)

$$\begin{aligned} A \cap \bar{B} = \{ &(1,1)\ (2,1)\ (3,1)\ (4,1)\ (5,1) \\ &(2,2)\ (3,2)\ (4,2)\ (5,2) \\ &(3,3)\ (4,3)\ (5,3) \\ &(4,4)\ (5,4) \\ &(5,5) \} \end{aligned}$$

$A \cap \bar{B}$ = The event where No. of dots in first toss is greater than or equal to the no. of dots in the second toss and No. of dots in first toss is not 6

(f) C = No. of dots on two dice differs by 2

$$\Rightarrow C = \{(1,3), (3,1), (2,4), (4,2), (3,5), (5,3), (4,6), (6,4)\}$$

$$A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$$

(g)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Now $A \cap B = B$ as $B \subset A$

$$\therefore P(A|B) = \frac{P(B)}{P(B)} = 1$$

Q2

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(a) given: S, F, P is the probability space & $M \in F$

s.t. $P(M) \neq 0$ i.e. $M \neq \phi$

To show: $P(\cdot \mid M)$ is a valid probability measure

Proof: For a probability measure to be valid it needs to satisfy the axioms of probability

also we assume that the probability measure $P(\cdot)$ satisfies these axioms

(i) $P(A \mid M) \geq 0 \quad \forall A \in F$

$$P(A \mid M) = \frac{P(A \cap M)}{P(M)}$$

$P(A \cap M) \geq 0$ & $P(M) \geq 0$ as $P(\cdot)$ satisfies axioms

$\Rightarrow \boxed{P(A \mid M) \geq 0}$ i.e. satisfies axiom 1

(ii) $P(S \mid M) = 1$

$$P(S \mid M) = \frac{P(S \cap M)}{P(M)} = \frac{P(M)}{P(M)} \quad (\text{as } M \subset S)$$

$\Rightarrow \boxed{P(S \mid M) = 1}$ i.e. satisfies axiom 2

(iii) $P\left(\bigcup_{i=1}^{n} A_i \mid M\right) = \sum_{i=1}^{n} P(A_i \mid M) \quad \forall A_i \cap A_j = \phi, \; i \neq j$

$$P\left(\bigcup_{i=1}^{n} A_i \mid M\right) = \frac{P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap M\right)}{P(M)}$$

$$= \frac{P\left(\bigcup_{i=1}^{n} (A_i \cap M)\right)}{P(M)}$$

Now as $A_i \cap A_j = \phi$ & $(A_i \cap M) \cap (A_j \cap M) = \phi$, $i \neq j$

$$\text{\&} \quad P\left(\bigcup_{i=1}^{n} A_i \mid M\right) = \sum_{i=1}^{n} \frac{P(A_i \cap M)}{P(M)} = \sum_{i=1}^{n} P(A_i \mid M)$$

i.e. satisfies axiom 3

(iv) $P\left(\bigcup_{i=1}^{\infty} A_i \mid M\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap M\right)}{P(M)}, \quad A_i \cap A_j = \phi, \; i \neq j$

$$= \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap M)\right)}{P(M)}$$

$$= \sum_{i=1}^{\infty} \frac{P(A_i \cap M)}{P(M)}$$

$$= \sum_{i=1}^{\infty} P(A_i \mid M)$$ i.e. satisfies axiom 4

(b) given: (S, F, P) with finite $|S| = n$, $F = P(S)$

$$P(A) = \frac{|A|}{|S|} = \frac{|A|}{n}$$

Testing for axioms

(i) $P(A) = \frac{|A|}{n} \qquad |A| \geq 0, \; n \geq 0$

& $P(A) \geq 0$ satisfies axiom 1

(ii) $P(S) = \frac{|S|}{|S|} = 1$

i.e. satisfies axiom 2

(iii) $P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{\left|\bigcup_{i=1}^{n} A_i\right|}{|S|} \qquad A_i \cap A_j = \phi \quad i \neq j$

since A_i's are disjoint therefore size of union would be sum of individual terms

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \frac{|A_i|}{|S|}$$

$$= \sum_{i=1}^{n} P(A_i)$$ i.e. satisfies axiom 3

(iv) $F = P(S)$ $\therefore$ we have only 2^n possible events and don't need to check for countable unions

Q3

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given: $P(A) = 1/2$ $P(B) = 1/3$ $P(C) = 1/4$

$P(A\cap B) = 1/6$ $P(B\cap C) = 1/12$ $P(A\cap C) = 1/6$

$P(A\cap B\cap C) = 1/36$

(a) $P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cup B\cup C)$

$= (1/2 + 1/3 + 1/4) - (1/6 + 1/12 + 1/6) + 1/36$

$= \frac{6+4+3}{12} - \frac{5}{12} + \frac{1}{36}$

$= \frac{8}{12} + \frac{1}{36} = \frac{25}{36} = 0.694$

(b) $P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{1/6}{1/4} = \frac{2}{3} = 0.666$

(c) $P(A|\bar{C}) = \frac{P(A\cap \bar{C})}{P(\bar{C})}$ (using $P(A) = P(A\cap C) + P(A\cap \bar{C})$)

$= \frac{P(A) - P(A\cap C)}{1 - P(C)} = \frac{1/2 - 1/6}{1 - 1/4} = \frac{2/6}{3/4}$

$\boxed{P(A|\bar{C}) = 4/9 = 0.444}$

(d) $P(B | \bar{A}\cup\bar{C}) = \frac{P(B\cap(\overline{A\cap C}))}{P(\overline{A\cap C})}$ (using DeMorgan's law)

$P(B\cap(\overline{A\cap C})) = P(B) - P(B\cap(A\cup C))$

$= P(B) - P((B\cap A)\cup(B\cap C))$

$= P(B) - (P(B\cap A) + P(B\cap C) - P(A\cap B\cap C))$

$= 1/3 - (1/6 + 1/12 - 1/36)$

$= 1/3 - \left(\frac{6+3-1}{36}\right) = \frac{4}{36} = \frac{1}{9} = 0.111$

$\Rightarrow P(B|\bar{A}\cup\bar{C}) = \frac{1/9}{1 - P(A\cap C)} = \frac{1/9_{3}}{5/6_{2}} = \frac{2}{15} = 0.133$

(e) $P(A)\,P(B) = 1/2 \times 1/3 = 1/6 = P(A\cap B)$

Even though $P(A)P(B) = P(A\cap B)$ is true that does not imply the events A & B are independent

Q4

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given : S_1 - coin toss - $p(k) = \frac{1}{2}\left(\frac{1}{2}\right)^{k-1}$

S_2 - Die toss - $q(k) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$

$S = S_1 \times S_2$

A = event when no. of coin toss until heads appears is equal to the no. of die rolls until 1 appears

Event A can happen on the first tosses or second tosses and so on

$\therefore A = B_1 \cup B_2 \cup \dots \cup B_K \cup \dots B_\infty$

where B_K is the event when heads appear at K^{th} coin toss & 1 appear on K^{th} Die toss

These events are mutually exclusive

$\therefore P(A) = \sum_{K=1}^{n} P(B_K)$

also the coin toss & Die toss are independent events

$\therefore \quad P(B_K) = P_1(B_K)\, P_2(B_K)$

$$\therefore P(A) = \sum_{K=1}^{\infty} \frac{1}{12}\left(\frac{5}{12}\right)^{K-1}$$

$$= \frac{1}{12} \cdot \left(\frac{1}{1 - 5/12}\right) = \frac{1}{\cancel{12}} \cdot \left(\frac{1}{\frac{7}{\cancel{12}}}\right)$$

$$\boxed{P(A) = \frac{1}{7}}$$

Q5

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given $P(A|T) = \alpha \quad P(B|T) = \beta$

$P(A) = 1/2 = P(B)$

$P(T) = \tau$

$$P(T|A\cap B) = \frac{P(T\cap A\cap B)}{P(A\cap B)} = \frac{P(T\cap A\cap B)}{P(A)\,P(B)}$$

$$= \frac{P(T\cap A\cap B)}{1/4}$$

$$= \frac{P(A\cap B|T)\,P(T)}{1/4}$$

$$= \frac{P(A|T).P(B|T)\,P(T)}{1/4}$$

$$\boxed{P(T|A\cap B) = 4\alpha\beta\tau}$$