**ECE 600, Exam #3** **Prof. Mark R. Bell** **November 17, 2020**

Group 1

*Directions:* This is an open book, open class notes exam. You may not use internet resources or other materials besides the class notes and textbook. You may use a calculator. You have 75 minutes to work the exam. Write your numbered solutions to the **four problems** on paper, scan your solutions to a pdf file, and send the scanned solutions to me via email (mrb@ecn.purdue.edu) or Purdue Filelocker. (See "Exam 3 Instructions(Group 1).pdf", sent via email, for complete details.)

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1. **(25 pts.)** Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly distributed random variables with joint pdf

$$f_{\mathbf{XY}}(x,y) = kx^2y \cdot 1_{[0,y]}(x)1_{[0,1]}(y).$$

   (a) Find the constant $k$ that makes $f_{\mathbf{XY}}(x,y)$ a valid joint pdf.
   (b) Find the marginal pdf $f_{\mathbf{Y}}(y)$.
   (c) Find the conditional pdf $f_{\mathbf{X}}(x|\{\mathbf{Y}=y\})$.
   (d) Find the *minimum mean-square error* (MMS) estimator $\hat{x}_{MMS}(y)$ of $\mathbf{X}$ given $\{\mathbf{Y}=y\}$.
   (e) Find the *maximum aposteriori probability* (MAP) estimator $\hat{x}_{MMS}(y)$ of $\mathbf{X}$ given $\{\mathbf{Y}=y\}$.

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2. **(25 pts.)** Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly distributed random variables with joint characteristic function

$$\Phi_{\mathbf{XY}}(\omega_1,\omega_2) = \frac{1}{(1-i\omega_1)(1-i2\omega_2)}.$$

   (a) Calculate the value of $\mathrm{E}[\mathbf{X}]$.
   (b) Calculate the value of $\mathrm{E}[\mathbf{Y}]$.
   (c) Calculate the value of $\mathrm{E}[\mathbf{XY}]$.
   (d) Calculate the value of $\mathrm{E}[\mathbf{X}^j\mathbf{Y}^k]$ if $j$ and $k$ are positive integers.
   (e) Calculate the correlation coefficient between $\mathbf{X}$ and $\mathbf{Y}$.

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3. **(25 pts.)** The number of cars $\mathbf{N}$ that pass a point on a highway in one hour is a Poisson random variable with mean $\lambda$. The probability that any car is exceeding the speed limit is $p$, independent of the other cars. Let $\mathbf{M}$ be the number of cars exceeding the speed limit during this one hour period. In this problem, you will find the pmf of $\mathbf{M}$ using characteristic functions and iterated expectation, using the following procedure:
   (a) Compute the characteristic function of the Poisson random variable $\mathbf{N}$.
   (b) Using iterated expectation, compute the chracteristic function of $\mathbf{M}$, using the fact that

$$\Phi_{\mathbf{M}}(\omega) = \mathrm{E}[e^{i\omega\mathbf{M}}] = \mathrm{E}_{\mathbf{N}}[\mathrm{E}_{\mathbf{M}}[e^{i\omega\mathbf{M}}|\mathbf{N}]] = \sum_{n=0}^{\infty} p_{\mathbf{N}}(n) \cdot \mathrm{E}_{\mathbf{M}}[e^{i\omega\mathbf{M}}|\{\mathbf{N}=n\}].$$

   (c) Based on your answer in part (b), write down the pmf of the random variable $\mathbf{M}$.
   (d) What are the mean and variance of $\mathbf{M}$?ional mean of $\mathbf{X}$ conditioned on $\{\mathbf{X}>a\}$, where $0<a<1$.

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4. **(25 pts.)** Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n, \ldots$ be a sequence of independent, identically distributed, exponential random variables, each having mean $\mu$. Define a new random sequence $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n, \ldots$, where

$$\mathbf{Y}_n = \min\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}, \quad n = 1, 2, 3, \ldots,$$

(*i.e.*, $\mathbf{Y}_n$ takes on the minimum value of the first $n$ random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots.\mathbf{X}_n$ in the initial random sequence.)

(a) Find the probability density function of $\mathbf{Y}_n$.

(b) Does the random sequence $\{\mathbf{Y}_n\}$ converge *in probability* as $n \to \infty$? Justify your answer.

(c) Does the random sequence $\{\mathbf{Y}_n\}$ converge *in the mean-square sense* as $n \to \infty$? Justify your answer.