

ECE 580: Homework 5

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April 29, 2020

Exercise 1

In my Canonical GA, I am using the following parameter settings:

- Number of bits used to represent each variable: 10 (resolution=0.0098)
- Population size: 40
- Number of iterations: 30
- Probability for cross-over: 0.9
- Probability of Mutation: 0.01
- Selection Method: tournament selection method-2

After carrying out several trials, I obtain an optimal function value of $1.7928e - 05$ for the optimal solution as $x^* = [-0.0049 \quad -0.0049]^T$. The plot for best, average, and the worst objective function values in the population for every generation is at Figure 1.

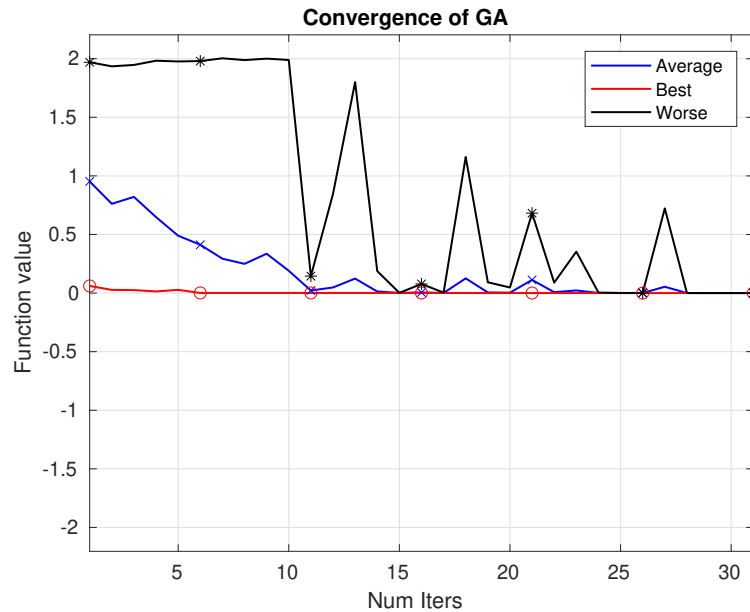


Figure 1: Plot of Average, Best and Worse function values for GA

For main file for GA refer to Listing 1 at page 19. The fitness function can be referred at Listing 2 at page 20 & Listing 3 at page 20. The encoding and decoding functions can be found at Listing 4 at page 21 & Listing 5 at page 21 respectively. The function for Tournament selection is at Listing 7 at page 22. The function for crossover, mutation and elitism can be referred at Listing 8 at page 22, Listing 9 at page 23 & Listing 10 at page 23 respectively.

Exercise 2

In my Real-Number GA, I am using the following parameter settings:

- Population size: 40
- Number of iterations: 30
- Probability for cross-over: 0.9
- Crossover-Method: Convex Combination
- Probability of Mutation: 0.01
- Selection Method: tournament selection method-2

After carrying out several trials, I obtain an optimal function value of $1.7333e - 11$ for the optimal solution as $x^* = 1.0e - 05 * [0.0195 \ 0.8318]^T$. The plot for best, average, and the worst objective function values in the population for every generation is at Figure 2.

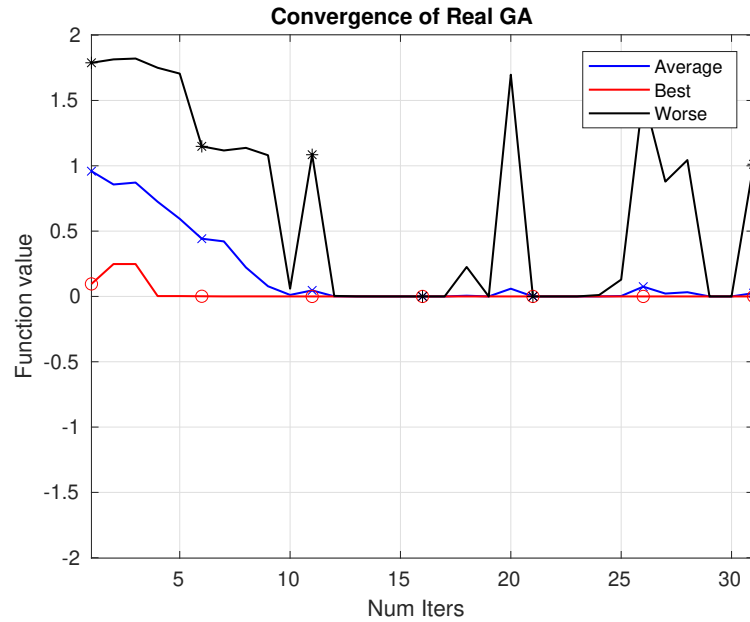


Figure 2: Plot of Average, Best and Worse function values for GA

For main file for GA refer to Listing 12 at page 24. The fitness function can be referred at Listing 2 at page 20 & Listing 3 at page 20. The function for Tournament selection is at Listing 7 at page 22. The function for crossover, mutation and elitism can be referred at Listing 13 at page 25, Listing 14 at page 26 & Listing 10 at page 23 respectively.

Exercise 3

The given LP problem is:

$$\begin{aligned}
 & \text{maximize} && -4x_1 - 3x_2 \\
 & \text{subject to} && 5x_1 + x_2 \geq 11 \\
 & && 2x_1 + x_2 \geq 8 \\
 & && x_1 + 2x_2 \geq 7 \\
 & && x_1, x_2 \geq 0
 \end{aligned}$$

We first convert the above problem to standard form:

$$\begin{aligned}
 & \text{minimize} && 4x_1 + 3x_2 \\
 & \text{subject to} && 5x_1 + x_2 - x_3 = 11 \\
 & && 2x_1 + x_2 - x_4 = 8 \\
 & && x_1 + 2x_2 - x_5 = 7 \\
 & && x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

We then solve the above problem using Two-phase simplex method. The computations (with pivot elements in boxes) are as follows:

Phase 1:

$$\begin{aligned}
 \begin{bmatrix} \mathbf{A} & \mathbf{I} & \mathbf{b} \\ \mathbf{0}^T & \mathbf{1}^T & 0 \end{bmatrix} &= \begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \boxed{5} & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ -8 & -4 & 1 & 1 & 1 & 0 & 0 & 0 & -26 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1/5 & -1/5 & 0 & 0 & 1/5 & 0 & 0 & 11/5 \\ 0 & 3/5 & 2/5 & -1 & 0 & -2/5 & 1 & 0 & 18/5 \\ 0 & \boxed{9/5} & 1/5 & 0 & -1 & -1/5 & 0 & 1 & 24/5 \\ 0 & -12/5 & -3/5 & 1 & 1 & 8/5 & 0 & 0 & -42/5 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & -2/9 & 0 & 1/9 & 2/9 & 0 & -1/9 & 5/3 \\ 0 & 0 & \boxed{1/3} & -1 & 1/3 & -1/3 & 1 & -1/3 & 2 \\ 0 & 1 & 1/9 & 0 & -5/9 & -1/9 & 0 & 5/9 & 8/3 \\ 0 & 0 & -1/3 & 1 & -1/3 & 4/3 & 0 & 4/3 & -2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 & -2/3 & 1/3 & 0 & 2/3 & -1/3 & 3 \\ 0 & 0 & 1 & -3 & 1 & -1 & 3 & -1 & 6 \\ 0 & 1 & 0 & 1/3 & -2/3 & 0 & -1/3 & 2/3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

Phase 2:

$$\begin{aligned} \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^T & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & -2/3 & 1/3 & 3 \\ 0 & 0 & 1 & -3 & 1 & 6 \\ 0 & 1 & 0 & 1/3 & -2/3 & 2 \\ 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & -2/3 & 1/3 & 3 \\ 0 & 0 & 1 & -3 & 1 & 6 \\ 0 & 1 & 0 & 1/3 & -2/3 & 2 \\ 0 & 0 & 0 & 5/3 & 2/3 & -18 \end{bmatrix} \end{aligned}$$

The optimal solution is given by $x_1^* = 3, x_2^* = 2$ with maximum function value as $-4x_1^* - 3x_2^* = -18$

For MATLAB function for this problem refer to Listing 16 at page 26 & Listing 17 at page 28 and the call to the function can be referred at Listing 15 at page 26 with corresponding output at Listing 18 at page 29.

Exercise 4

The dual problem is given by:

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{\lambda}^T \mathbf{b} \\ &\text{subject to} \quad \boldsymbol{\lambda}^T \mathbf{A} \leq \mathbf{c}^T \end{aligned}$$

that is:

$$\begin{aligned} &\text{maximize} \quad 11\lambda_1 + 8\lambda_2 + 7\lambda_3 \\ &\text{subject to} \quad [\lambda_1 \quad \lambda_2 \quad \lambda_3]^T \begin{bmatrix} 5 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \leq [4 \quad 3] \\ &\quad \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

Since we have already obtained the optimal BFS $\mathbf{x}^* = [3 \quad 2 \quad 0 \quad 0 \quad 0]^T$ corresponding to the optimal basis $\mathbf{B} = \begin{bmatrix} 5 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ and cost coefficients $\mathbf{c}_B = [4 \quad 3 \quad 0]^T$. From theorem of duality, we have

$$\begin{aligned} \boldsymbol{\lambda}^T \mathbf{b} &= \mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} \\ \Rightarrow \boldsymbol{\lambda}^T &= \mathbf{c}_B^T \mathbf{B}^{-1} \\ &= [4 \quad 3 \quad 0]^T \begin{bmatrix} 5 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \\ &= \frac{1}{-3} [4 \quad 3 \quad 0]^T \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & -2 \\ 3 & -9 & 3 \end{bmatrix}^T \\ \boldsymbol{\lambda}^{*T} &= [0 \quad 5/3 \quad 2/3] \\ \boldsymbol{\lambda}^{*T} \mathbf{b} &= 18 \end{aligned}$$

Exercise 5

We have to solve the following constrained optimization problem:

$$\begin{aligned} & \text{maximize} && f(x) = 4x_1 + x_2^2 \\ \text{eqv minimize} &&& -f(x) = -4x_1 - x_2^2 \\ & \text{subject to} && h_1(x) = x_1^2 + x_2^2 - 9 = 0 \end{aligned}$$

$$\text{lagrangian} \quad l(\mathbf{x}, \lambda_1) = -4x_1 - x_2^2 + \lambda_1(x_1^2 + x_2^2 - 9)$$

Using FONC:

$$\begin{aligned} \nabla_{\mathbf{x}} l(\mathbf{x}, \lambda_1) &= \begin{bmatrix} -4 & -2x_2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} = \mathbf{0}^T \\ &\Rightarrow \begin{bmatrix} -4 + 2\lambda_1 x_1 \\ 2x_2(\lambda_1 - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \nabla_{\lambda_1} l(\mathbf{x}, \lambda_1) &= x_1^2 + x_2^2 - 9 = 0 \end{aligned}$$

The solution to the above set of equations is:

$$\mathbf{x}_a^* = \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} \pm 3 \\ 0 \\ \pm \frac{2}{3} \end{bmatrix}, \quad \mathbf{x}_b^* = \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 2 \\ \pm\sqrt{5} \\ 1 \end{bmatrix}$$

Computing the Hessian of lagrangian:

$$\begin{aligned} L(\mathbf{x}, \lambda_1) &= 2 \begin{bmatrix} \lambda_1 & 0 \\ 0 & (\lambda_1 - 1) \end{bmatrix} \\ \Rightarrow L(\mathbf{x}_b^*, \lambda_1^* = 1) &= 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \succcurlyeq 0 \Rightarrow \mathbf{x}_b^* \text{ is a possible minimizer of } -f(\mathbf{x}) \\ L(\mathbf{x}_{a2}^*, \lambda_1^* = -\frac{2}{3}) &= 2 \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{5}{3} \end{bmatrix} \preccurlyeq 0 \Rightarrow \mathbf{x}_b^* \text{ is a possible maximizer of } -f(\mathbf{x}) \end{aligned}$$

For SONC we need to find the Tangent Space $T(\mathbf{x}^*)$:

$$\begin{aligned} T(\mathbf{x}_b^*) &= \{y : \nabla_{\mathbf{x}} h_1^T(x)y = 0\} \\ &= \{y : \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix} y = 0\} \\ &= \{y : \begin{bmatrix} 2 & \pm\sqrt{5} \end{bmatrix} y = 0\} \\ &= \{y : y = t \begin{bmatrix} 1 \\ \mp \frac{\sqrt{5}}{2} \end{bmatrix}, \quad t \in \mathbb{R} - \{0\}\} \\ y^T L(\mathbf{x}_b^*) y &= t^2 \begin{bmatrix} 1 & \mp \frac{\sqrt{5}}{2} \end{bmatrix} 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \mp \frac{\sqrt{5}}{2} \end{bmatrix} \\ &= 2t^2 > 0 \quad \forall \quad t \in \mathbb{R} - \{0\} \end{aligned}$$

Therefore $\mathbf{x}_b^* = \begin{bmatrix} 2 \\ \pm\sqrt{5} \\ 1 \end{bmatrix}$ is a strict minimizer of $-f(x)$ and thus is a maximizer of $f(x)$

Exercise 6

We have to solve the following optimization problem:

$$\text{maximize } r(x) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}$$

Suppose the above problem has a solution x^* which maximizes the function. Then $x = tx^*, t \in \mathbb{R}$ can be another possible solution. Therefore we convert the above problem to a constrained maximization problem with unique solution by enforcing the denominator to be 1. The converted problem is given by :

$$\begin{aligned} &\text{maximize } f(x) = 18x_1^2 - 8x_1x_2 + 12x_2^2 \\ &\text{subject to } h(x) = 1 - (2x_1^2 + 2x_2^2) = 0 \end{aligned}$$

Converting to minimization problem:

$$\begin{aligned} &\text{minimize } -f(x) = -\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 36 & -8 \\ -8 & 24 \end{bmatrix} \mathbf{x} \\ &\text{subject to } h(x) = 1 - \frac{1}{2}\mathbf{x}^T \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{x} \\ &\text{lagrangian } l(x, \lambda) = -\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 36 & -8 \\ -8 & 24 \end{bmatrix} \mathbf{x} + \lambda(1 - \frac{1}{2}\mathbf{x}^T \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{x}) \end{aligned}$$

FONC:

$$\begin{aligned} \nabla_{\lambda} l(\mathbf{x}, \lambda) &= 2x_1^2 + 2x_2^2 - 1 = 0 \\ \nabla_{\mathbf{x}} l(\mathbf{x}, \lambda) &= \begin{bmatrix} -36 & 8 \\ 8 & -24 \end{bmatrix} \mathbf{x} - \lambda \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{x} = \mathbf{0} \\ &\Rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -36 & +8 \\ +8 & -24 \end{bmatrix} \mathbf{x} = \lambda \mathbf{x} \\ &\Rightarrow \begin{bmatrix} -9 & +2 \\ +2 & -6 \end{bmatrix} \mathbf{x} = \lambda \mathbf{x} \\ &\Rightarrow \det \left(\begin{bmatrix} -9 & +2 \\ +2 & -6 \end{bmatrix} - \lambda \mathbf{I}_2 \right) = 0 \Rightarrow \lambda = -10, -5 \\ &\mathbf{x} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

The solution to the above equations is:

$$\mathbf{x}_a^* = \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} \mp \frac{2}{\sqrt{10}} \\ \pm \frac{1}{\sqrt{10}} \\ -10 \end{bmatrix}, \quad \mathbf{x}_b^* = \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} \pm \frac{1}{\sqrt{10}} \\ \pm \frac{2}{\sqrt{10}} \\ -5 \end{bmatrix}$$

SONC:

$$\begin{aligned}
L(\mathbf{x}, \lambda) &= \begin{bmatrix} -36 & 8 \\ 8 & -24 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\
&= \left(\begin{bmatrix} -9 - \lambda & +2 \\ +2 & -6 - \lambda \end{bmatrix} \right) \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\
\Rightarrow L(\mathbf{x}, \lambda = -10) &= \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} \succcurlyeq 0 \text{ is a possible minimizer of } -f(x) \\
\Rightarrow L(\mathbf{x}, \lambda = -5) &= \begin{bmatrix} -16 & +8 \\ +8 & -4 \end{bmatrix} \preccurlyeq 0 \text{ is a possible maximizer of } -f(x)
\end{aligned}$$

SOSC: We need to find the Tangent Space $T(\mathbf{x}^*)$

$$\begin{aligned}
T(\mathbf{x}_a^*) &= \{y : \nabla_{\mathbf{x}} h_1^T(x)y = 0\} \\
&= \{y : \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \mp \frac{2}{\sqrt{10}} \\ \pm \frac{1}{\sqrt{10}} \end{bmatrix} y = 0\} \\
&= \{y : y = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R} - \{0\}\} \\
\Rightarrow y^T L(\mathbf{x}_a^*)y &= 100t^2 > 0 \quad \forall \quad t \in \mathbb{R} - \{0\}
\end{aligned}$$

Therefore the solution corresponding to $\lambda = -10$, $\mathbf{x}_a^* = \begin{bmatrix} \mp \frac{2}{\sqrt{10}} \\ \pm \frac{1}{\sqrt{10}} \end{bmatrix}$ is a strict minimizer of $-f(x)$ and thus is a maximizer of $f(x)$.

Therefore all possible maximizers of

$$r(x) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}$$

are of the form $\mathbf{x}^* = t \begin{bmatrix} \mp \frac{2}{\sqrt{10}} \\ \pm \frac{1}{\sqrt{10}} \end{bmatrix} \quad \forall \quad t \in \mathbb{R}$

Exercise 7

We need to find the extremizers for problems (a)-(c)

Problem (a)

The given problem is:

$$\begin{aligned} f(\mathbf{x}) &= x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26 \\ \text{subject to } g(\mathbf{x}) &= \begin{bmatrix} \frac{x_2}{5} - x_1^2 \\ 5x_1 + \frac{x_2}{2} - 5 \end{bmatrix} \leq \mathbf{0} \end{aligned}$$

The gradients are given by:

$$\begin{aligned} \nabla f(x) &= \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{bmatrix} \\ \nabla g(x) &= \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix} \\ F(x) &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ G_1(x) &= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \\ G_2(x) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The lagrangian is given by:

$$\begin{aligned} l(x, \lambda, \mu) &= f(x) + \lambda^T h(x) + \mu^T g(x) \\ \nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{bmatrix} + \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \\ L(x, \lambda, \mu) &= F(x) + \sum_{i=1}^m \lambda_i H_i(x) + \sum_{i=1}^p \mu_i G_i(x) \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

FONC: Using KKT:-

Case 1: $\mu_1 = 0, \mu_2 = 0$

$$\begin{aligned} \nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{bmatrix} = 0 \\ \text{subject to } g_1(x) &= \frac{x_2}{5} - x_1^2 \leq 0 \\ g_2(x) &= 5x_1 + \frac{x_2}{2} - 5 \leq 0 \end{aligned}$$

The solution to the above system of equations is:

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ g_1(x) &= 0 \leq 0 \quad \checkmark \\ g_2(x) &= 5 + 5/2 \leq 0 \quad \times\end{aligned}$$

Therefore the above solution is infeasible

Case 2: $\mu_1 = 0, \mu_2 > 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{bmatrix} + \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix}^T \begin{bmatrix} 0 \\ \mu_2 \end{bmatrix} = 0 \\ \text{subject to } g_1(x) &= \frac{x_2}{5} - x_1^2 \leq 0 \\ g_2(x) &= 5x_1 + \frac{x_2}{2} - 5 = 0\end{aligned}$$

The solution to the above system of equations is:

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 51/101 \\ 500/101 \end{bmatrix} \\ \mu_2^* &= 20/101 \\ g_1(x) &= \frac{x_2}{5} - x_1^2 = (100 * 101 - 51^2)/101^2 \leq 0 \quad \times\end{aligned}$$

There is no feasible solution.

Case 3: $\mu_1 > 0, \mu_2 > 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{bmatrix} + \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = 0 \\ \text{subject to } g_1(x) &= \frac{x_2}{5} - x_1^2 = 0 \\ g_2(x) &= 5x_1 + \frac{x_2}{2} - 5 = 0\end{aligned}$$

The solution to the above system of equations is:

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \pm \sqrt{3} \\ 20 \mp \sqrt{3} \end{bmatrix} \\ \mu_1^* &= \frac{304 - 202\sqrt{3}}{-2\sqrt{3}} \quad \checkmark \\ \mu_2^* &= (-300 + 200\sqrt{3} - 2\mu_1)/5 > 0 \quad \checkmark\end{aligned}$$

SONC:

$$\begin{aligned}
L(\mathbf{x}^*, \mu^*) &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \\
T(\mathbf{x}^*) &= \{y : \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix} y = 0\} \\
&= \{y : \begin{bmatrix} -2(-1 \pm \sqrt{3}) & 1/5 \\ 5 & 1/2 \end{bmatrix} y = 0\} \\
&= \{y : y = \mathbf{0}\} \\
&\Rightarrow y^T L(\mathbf{x}^*, \mu^*) y = 0 \geq 0
\end{aligned}$$

Therefore $\mathbf{x}^* = \begin{bmatrix} -1 \pm \sqrt{3} \\ 20 \mp \sqrt{3} \end{bmatrix}$ is an extremizer.

SOSC:

$$\begin{aligned}
\tilde{T}(\mathbf{x}^*) &= \{y : \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix} y = 0\} \\
&= \{y : \begin{bmatrix} -2(-1 \pm \sqrt{3}) & 1/5 \\ 5 & 1/2 \end{bmatrix} y = 0\} \\
&= \{y : y = \mathbf{0}\} \\
&\Rightarrow y^T L(\mathbf{x}^*, \mu^*) y = 0 > 0 \quad \text{✗}
\end{aligned}$$

As \mathbf{x}^* does not satisfy SONC, it is not a minimizer. It is a saddle point.

Case 4: $\mu_1 > 0, \mu_2 = 0$

$$\begin{aligned}
\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 10 \end{bmatrix} + \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ 0 \end{bmatrix} = 0 \\
\text{subject to } g_1(x) &= \frac{x_2}{5} - x_1^2 = 0 \\
g_2(x) &= 5x_1 + \frac{x_2}{2} - 5 \leq 0
\end{aligned}$$

The solution to the above system of equations is:

$$\begin{aligned}
\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -48 + 10\sqrt{23} \\ 23020 - 4800\sqrt{23} \end{bmatrix} \\
\mu_1^* &= \frac{52 + 10\sqrt{23}}{4} > 0 \quad \text{✓} \\
g_2(x) &= 5x_1 + \frac{x_2}{2} - 5 = 11265 - 2350\sqrt{23} \leq 0 \quad \text{✓}
\end{aligned}$$

Therefore $\mathbf{x}^* = \begin{bmatrix} -48 + 10\sqrt{23} \\ 23020 - 4800\sqrt{23} \end{bmatrix}$ is a possible extremizer as it satisfies the KKT conditions.

SONC:

$$\begin{aligned}
L(\mathbf{x}^*, \mu^*) &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \\
T(\mathbf{x}^*) &= \{y : \begin{bmatrix} -2x_1 & 1/5 \\ 5 & 1/2 \end{bmatrix} y = 0\} \\
&= \{y : \begin{bmatrix} -2(-48 + 10\sqrt{23}) & 1/5 \\ 5 & 1/2 \end{bmatrix} y = 0\} \\
&= \{y : y = \mathbf{0}\} \\
&\Rightarrow y^T L(\mathbf{x}^*, \mu^*) y = 0 \geq 0
\end{aligned}$$

Therefore \mathbf{x}^* is an extremizer as it satisfies SONC.

SOSC:

$$\begin{aligned}
\tilde{T}(\mathbf{x}^*) &= \{y : [-2x_1 \quad 1/5] y = 0\} \\
&= \{y : y = t \begin{bmatrix} 1 \\ 10x_1 \end{bmatrix} \quad \forall \quad t \in \mathbb{R} - \{0\}\} \\
&\Rightarrow y^T L(\mathbf{x}^*, \mu^*) y = 50x_1(1 + x_1)t^2 \\
&= 50 * (-48 + 10\sqrt{23})(1 - 48 + 10\sqrt{23})t^2 < 0 \quad \forall \quad t \in \mathbb{R} - \{0\}
\end{aligned}$$

Therefore $\mathbf{x}^* = \begin{bmatrix} -48 + 10\sqrt{23} \\ 23020 - 4800\sqrt{23} \end{bmatrix}$ is a strict maximizer.

Problem (b)

$$\begin{aligned} f(\mathbf{x}) &= x_1^2 + x_2^2 \\ \text{subject to } g(\mathbf{x}) &= \begin{bmatrix} -x_1 \\ -x_2 \\ -x_1 - x_2 + 5 \end{bmatrix} \leq \mathbf{0} \end{aligned}$$

The gradients are given by:

$$\begin{aligned} \nabla f(x) &= \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \\ \nabla g(x) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix} \\ F(x) &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ G_1(x) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ G_2(x) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ G_3(x) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The lagrangian is given by:

$$\begin{aligned} l(x, \lambda, \mu) &= f(x) + \lambda^T h(x) + \mu^T g(x) \\ \nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \\ L(x, \lambda, \mu) &= F(x) + \sum_{i=1}^m \lambda_i H_i(x) + \sum_{i=1}^p \mu_i G_i(x) \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \mu_3 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \succ 0 \quad \text{in all of } \mathbb{R}^2 \end{aligned}$$

Thus as $L(x, \lambda, \mu)$ is PD in all of \mathbb{R}^2 , the SONC is always satisfied for any solution satisfying KKT.

FONC: Using KKT:-

Case 1: $\mu_1 > 0, \mu_2 > 0, \mu_3 > 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

subject to $g_1(x) = -x_1 = 0$
 $g_2(x) = -x_2 = 0$
 $g_3(x) = -x_1 - x_2 + 5 = 0$

There is no feasible solution to the above system of equations.

Case 2: $\mu_1 > 0, \mu_2 > 0, \mu_3 = 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \\ 0 \end{bmatrix}$$

subject to $g_1(x) = -x_1 = 0$
 $g_2(x) = -x_2 = 0$
 $g_3(x) = -x_1 - x_2 + 5 = 5 \leq 0$ ✗

There is no feasible solution to the above system of equations.

Case 3: $\mu_1 > 0, \mu_2 = 0, \mu_3 > 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ 0 \\ \mu_3 \end{bmatrix}$$

subject to $g_1(x) = -x_1 = 0$
 $g_2(x) = -x_2 \leq 0$
 $g_3(x) = -x_1 - x_2 + 5 = 0$

The solution to the above set of equations is :

$$\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\mu_3^* = 10$$

$$\mu_1^* = -10 > 0$$
 ✗

There is no feasible solution.

Case 4: $\mu_1 = 0, \mu_2 > 0, \mu_3 > 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

subject to $g_1(x) = -x_1 \leq 0$
 $g_2(x) = -x_2 = 0$
 $g_3(x) = -x_1 - x_2 + 5 = 0$

The solution to the above set of equations is :

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\ \mu_3^* &= 10 \\ \mu_2^* &= -10 > 0 \quad \text{✗}\end{aligned}$$

There is no feasible solution.

Case 5: $\mu_1 > 0, \mu_2 = 0, \mu_3 = 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ 0 \\ 0 \end{bmatrix} \\ \text{subject to } g_1(x) &= -x_1 = 0 \\ g_2(x) &= -x_2 \leq 0 \\ g_3(x) &= -x_1 - x_2 + 5 \leq 0\end{aligned}$$

The solution to the above set of equations is :

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ g_2(x) &= -x_2 = 0 \leq 0 \quad \text{✓} \\ g_3(x) &= -x_1 - x_2 + 5 = 5 \leq 0 \quad \text{✗} \\ \mu_1^* &= 0 > 0 \quad \text{✗}\end{aligned}$$

There is no feasible solution.

Case 6: $\mu_1 = 0, \mu_2 > 0, \mu_3 = 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ \mu_2 \\ 0 \end{bmatrix} \\ \text{subject to } g_1(x) &= -x_1 \leq 0 \\ g_2(x) &= -x_2 = 0 \\ g_3(x) &= -x_1 - x_2 + 5 \leq 0\end{aligned}$$

The solution to the above set of equations is :

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ g_1(x) &= -x_1 = 0 \leq 0 \quad \text{✓} \\ g_3(x) &= -x_1 - x_2 + 5 = 5 \leq 0 \quad \text{✗} \\ \mu_2^* &= 0 > 0 \quad \text{✗}\end{aligned}$$

There is no feasible solution.

Case 7: $\mu_1 = 0, \mu_2 = 0, \mu_3 = 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{subject to } g_1(x) &= -x_1 \leq 0 \\ g_2(x) &= -x_2 \leq 0 \\ g_3(x) &= -x_1 - x_2 + 5 \leq 0 \end{aligned}$$

The solution to the above set of equations is :

$$\begin{aligned} \mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ g_1(x) &= -x_1 = 0 \leq 0 \quad \checkmark \\ g_2(x) &= -x_2 = 0 \leq 0 \quad \checkmark \\ g_3(x) &= -x_1 - x_2 + 5 = 5 \leq 0 \quad \times \end{aligned}$$

There is no feasible solution.

Case 8: $\mu_1 = 0, \mu_2 = 0, \mu_3 > 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \mu_3 \end{bmatrix}$$

$$\begin{aligned} \text{subject to } g_1(x) &= -x_1 \leq 0 \\ g_2(x) &= -x_2 \leq 0 \\ g_3(x) &= -x_1 - x_2 + 5 = 0 \end{aligned}$$

The solution to the above set of equations is :

$$\begin{aligned} \mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}, \mu_3^* = 5 \\ g_1(x) &= -x_1 = -5/2 \leq 0 \quad \checkmark \\ g_2(x) &= -x_2 = -5/2 \leq 0 \quad \checkmark \end{aligned}$$

Therefore $\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$ is a possible extremizer. Also since $L(x, \lambda, \mu) \succ 0$ for all of \mathbb{R}^2 , thus \mathbf{x}^* a strict minimizer.

Problem (c)

$$f(\mathbf{x}) = x_1^2 + 6x_1x_2 - 4x_1 - 2x_2$$
$$\text{subject to } g(\mathbf{x}) = \begin{bmatrix} x_1^2 + 2x_2 - 1 \\ 2x_1 - 2x_2 - 1 \end{bmatrix} \leq \mathbf{0}$$

The gradients are given by:

$$\nabla f(x) = \begin{bmatrix} 2x_1 + 6x_2 - 4 \\ 6x_1 - 2 \end{bmatrix}$$

$$\nabla g(x) = \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix}$$

$$F(x) = \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix}$$

$$G_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$G_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The lagrangian is given by:

$$l(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$
$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 + 6x_2 - 4 \\ 6x_1 - 2 \end{bmatrix} + \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$
$$L(x, \lambda, \mu) = F(x) + \sum_{i=1}^m \lambda_i H_i(x) + \sum_{i=1}^p \mu_i G_i(x)$$
$$= \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix} + \mu_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

FONC: Using KKT:-

Case 1: $\mu_1 = 0, \mu_2 = 0$

$$\nabla_x l(x, \lambda, \mu) = \begin{bmatrix} 2x_1 + 6x_2 - 4 \\ 6x_1 - 2 \end{bmatrix} + \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\text{subject to } g_1(x) = x_1^2 + 2x_2 - 1 \leq 0$$
$$g_2(x) = 2x_1 - 2x_2 - 1 \leq 0$$

The solution to the above set of equations is :

$$\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5/9 \end{bmatrix}$$
$$g_1(x) = x_1^2 + 2x_2 - 1 = 2/9 \leq 0 \quad \text{✗}$$
$$g_2(x) = 2x_1 - 2x_2 - 1 = -13/9 \leq 0 \quad \text{✓}$$

There is no feasible solution.

Case 2: $\mu_1 > 0, \mu_2 = 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 + 6x_2 - 4 \\ 6x_1 - 2 \end{bmatrix} + \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ 0 \end{bmatrix} \\ \text{subject to } g_1(x) &= x_1^2 + 2x_2 - 1 = 0 \\ g_2(x) &= 2x_1 - 2x_2 - 1 \leq 0\end{aligned}$$

The solution to the above set of equations gives:

$$\begin{aligned}x_1 &= \frac{1 - \mu_1}{3} \\ x_2 &= \frac{2}{3} - \frac{1}{9}(1 - \mu_1^2) \\ g_1(x) &= x_1^2 + 2x_2 - 1 = 0 \\ &\Rightarrow 3\mu_1^2 - 2\mu_1 + 2 = 0 \\ &\Rightarrow \mu_1^* = \frac{2 \pm \sqrt{4 - 24}}{2} \notin \mathbb{R}\end{aligned}$$

There is no feasible solution.

Case 3: $\mu_1 > 0, \mu_2 > 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 + 6x_2 - 4 \\ 6x_1 - 2 \end{bmatrix} + \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix}^T \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \\ \text{subject to } g_1(x) &= x_1^2 + 2x_2 - 1 = 0 \\ g_2(x) &= 2x_1 - 2x_2 - 1 = 0\end{aligned}$$

The solution to the above set of equations gives :

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \pm \sqrt{3} \\ -3/2 \pm \sqrt{3} \end{bmatrix} \\ \mu_1^* &= \pm \frac{1}{\sqrt{3}}(23 \mp 14\sqrt{3}) > 0 \quad \text{✗} \\ \mu_2^* &= \mu_1 + (-4 \pm 3\sqrt{3})\end{aligned}$$

There is no feasible solution.

Case 4: $\mu_1 = 0, \mu_2 > 0$

$$\begin{aligned}\nabla_x l(x, \lambda, \mu) &= \begin{bmatrix} 2x_1 + 6x_2 - 4 \\ 6x_1 - 2 \end{bmatrix} + \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix}^T \begin{bmatrix} 0 \\ \mu_2 \end{bmatrix} \\ \text{subject to } g_1(x) &= x_1^2 + 2x_2 - 1 \leq 0 \\ g_2(x) &= 2x_1 - 2x_2 - 1 = 0\end{aligned}$$

The solution to the above set of equations is :

$$\begin{aligned}\mathbf{x}^* &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9/14 \\ 1/7 \end{bmatrix} \\ \mu_2^* &= 13/14 > 0 \quad \text{✓} \\ g_1(x) &= x_1^2 + 2x_2 - 1 = -59/196 \leq 0 \quad \text{✓}\end{aligned}$$

Therefore $\mathbf{x}^* = \begin{bmatrix} 9/14 \\ 1/7 \end{bmatrix}$ is a possible extremizer.

SONC:

$$\begin{aligned}
L(\mathbf{x}^*, \mu^*) &= \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix} \\
T(\mathbf{x}^*) &= \{y : \begin{bmatrix} 2x_1 & 2 \\ 2 & -2 \end{bmatrix} y = \mathbf{0}\} \\
&= \{y : \begin{bmatrix} 9/7 & 2 \\ 2 & -2 \end{bmatrix} y = \mathbf{0}\} \\
&= \{y : y = \mathbf{0}\} \\
\Rightarrow y^T L(\mathbf{x}^*, \mu^*) y &= 0 \geq 0
\end{aligned}$$

Thus \mathbf{x}^* satisfies SONC and therefore is an extremizer.

SOSC:

$$\begin{aligned}
\tilde{T}(\mathbf{x}^*) &= \{y : \begin{bmatrix} 2 & -2 \end{bmatrix} y = 0\} \\
&= \{y : y = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \forall \quad t \in \mathbb{R} - \{0\}\} \\
\Rightarrow y^T L(\mathbf{x}^*, \mu^*) y &= 14t^2 > 0 \quad \forall \quad t \in \mathbb{R} - \{0\}
\end{aligned}$$

Therefore $\mathbf{x}^* = \begin{bmatrix} 9/14 \\ 1/7 \end{bmatrix}$ is a strict minimizer.

MATLAB Code

Canonical GA Code

Listing 1: Canonical GA Main Code

```
1 % ECE 580 HW5: Problem 1
2 % Rahul Deshmukh
3 % deshmun5@purdue.edu
4 clc; clear all; close all;
5 format short;
6 %% Include paths
7 addpath('.../OptimModule/optimizers/global/GA/');
8 save_dir = './pix/';
9 %% set seed
10 rng(7); % tried 0:10, 7 gave the smallest result, can jump between global and ...
    local min
11 %% Canonical GA problem setup
12 lb = -5*ones(1,2);
13 ub = 5*ones(1,2);
14 Num_vars = length(lb);
15 bits = 10;
16 coded_lens = bits*ones(1,Num_vars);
17 resolution = (ub-lb)./(2.^coded_lens-1)
18
19 %% GA: solver params
20 N_pop = 40; % cant be odd integer
21 p_xover = 0.9;
22 p_mut = 0.01;
23 N_iters = 30;
24 selection_method = 'tournament_method2';
25
26 %% GA starts
27
28 % initialize collectors
29 best_f = [];
30 av_f = [];
31 worse_f = [];
32
33 % choose type of selector
34 if strcmp(selection_method, 'roulette')
35     selection = @(x,f) roulette(x,f);
36 elseif strcmp(selection_method, 'tournament_method1')
37     selection = @(x,f) tournament_selection(x,f,1);
38 elseif strcmp(selection_method, 'tournament_method2')
39     selection = @(x,f) tournament_selection(x,f,2);
40 end
41
42 % draw initial population
43 X = rand(N_pop, Num_vars);
44 % scale to domain
45 X = (X.*(ub-lb) + lb);
46 % discretize to resolution
47 X = floor((X - lb)./resolution).*resolution + lb;
48
49 %encode X
50 parents = encode(X, lb, ub, coded_lens, resolution);
```

```

51 % evaluate fitness of parents
52 f_parent = -1*fitness_griewank(parents, lb, coded_lens, resolution);
53 [best_f, av_f, worse_f] = log_f(f_parent, best_f, av_f, worse_f);
54 for i=1:Niters
55     % generate mating pool using selection
56     mating_pool = selection(parents, f_parent);
57     %perform crossover
58     parents = two_point_crossover(mating_pool, p_xover);
59     %perform mutation
60     parents = mutation(parents, p_mut);
61     %perform elitism
62     parents = elitism(parents, f_parent);
63     %evaluate fitness of offspring
64     f_parent = -1*fitness_griewank(parents, lb, coded_lens, resolution);
65     [best_f, av_f, worse_f] = log_f(f_parent, best_f, av_f, worse_f);
66 end
67 % find the best offspring
68 [f_star, k_star] = max(f_parent);
69 fprintf(strcat('best fval: \t', num2str(-1*f_star)))
70 x_star_coded = parents(k_star,:);
71 x_star = decode(x_star_coded, lb, coded_lens, resolution)
72
73 %% Convergence Plotting
74 fig1 = figure(1);
75 hold on; grid on;
76 x = 1:Niters+1;
77 h1 = plot(x, -1*av_f, '-b', 'LineWidth', 1);
78 h2 = plot(x, -1*best_f, '-r', 'LineWidth', 1);
79 h3 = plot(x, -1*worse_f, '-k', 'LineWidth', 1);
80 v = 1:5:Niters+1;
81 plot(x(v), -1*av_f(v), 'bx');
82 plot(x(v), -1*best_f(v), 'ro');
83 plot(x(v), -1*worse_f(v), 'k*');
84 legend([h1, h2, h3], {'Average', 'Best', 'Worse'}, 'Location', 'northeast');
85 hold off;
86 box('on');
87 xlabel('Num Iters'); ylabel('Function value');
88 xlim([1, Niters+1])
89 ylim(max(abs(worse_f))*(1.1)*[-1, 1]);
90 title('Convergence of GA');
91 saveas(fig1, strcat(save_dir, 'ga_canon_conv'), 'eps');

```

Listing 2: Fitness function

```

1 function f = fitness_griewank(X_coded, lb, code_lens, resolution)
2 X = decode(X_coded, lb, code_lens, resolution);
3 f = griewank_fun(X);
4 end

```

Listing 3: Griewank function

```

1 function y = griewank_fun(X_swarm, min_bool)
2 % d dimensionalfriewank function
3 switch nargin
4     case 1
5         min_bool=1;

```

```

6 end
7
8 [x_dim ,Nswarm] = size(X_swarm);
9 y = zeros(Nswarm,1);
10 for k=1:Nswarm
11     sum = 0;
12     prod = 1;
13     x = X_swarm(:,k);
14     for i=1:x_dim
15         x_i = x(i);
16         sum = sum + x_i^2/4000;
17         prod = prod * cos(x_i/sqrt(i));
18     end
19     y(k) = sum - prod +1;
20 end
21 if ~min_bool
22     y = -1*y;
23 end
24 end

```

Listing 4: Encoding function

```

1 function X_coded = encode(X, lb, ub, code_lens, resolution)
2 [N_pop,~] = size(X);
3 L = sum(code_lens);
4 cumsum_code_lens = [0, cumsum(code_lens)];
5 X_coded = zeros(N_pop,L);
6 Num_var = length(lb);
7 % convert discretized X to integers for encoding
8 X = round((X - lb)./resolution);
9 for i = 1:N_pop
10     x = X(i,:);
11     x_coded = zeros(1,L);
12     for j = 1:Num_var
13         xj = x(j);
14         x_coded(cumsum_code_lens(j) + 1 : cumsum_code_lens(j+1)) = de2bi(xj ...
15             ,code_lens(j));
16     end
17     X_coded(i,:) = x_coded;
18 end

```

Listing 5: Decoding function

```

1 function X = decode(X_coded, lb, code_lens, resolution)
2 [N_pop,~] = size(X_coded);
3 L = sum(code_lens);
4 cumsum_code_lens = [0, cumsum(code_lens)];
5 Num_var = length(lb);
6 X = zeros(N_pop,Num_var);
7 for i=1:N_pop
8     x_coded = X_coded(i,:);
9     x = zeros(1,Num_var);
10    for j=1:Num_var
11        xj_coded = x_coded(cumsum_code_lens(j) + 1 : cumsum_code_lens(j+1));
12        x(j) = resolution(j)*bi2de(xj_coded);
13    end

```

```
14     X(i,:) = x + lb;
15 end
16 end
```

Listing 6: Roulette-wheel selection function

```
1 function mating_pool = roulette(parent, f_parent)
2 [N_pop,~] = size(parent);
3 f_min = min(f_parent);
4 f = f_parent - f_min;
5 F = sum(f);
6 p = f/F;
7 q = cumsum(p);
8 rand_nums = rand(N_pop,1);
9 mating_idx = zeros(N_pop,1);
10 temp = q' - rand_nums;
11 for k=1:N_pop
12     mating_idx(k) = find(temp(k,:) > 0, 1);
13 end
14 mating_pool = parent(mating_idx, :);
15 end
```

Listing 7: Tournament selection function

```
1 function mating_pool = tournament_selection(parent, f_parent, method)
2 [N_pop,~] = size(parent);
3 mating_idx = zeros(N_pop,1);
4 if method == 1
5     a = randi([1, N_pop], 1, N_pop) ;
6     b = randi([1, N_pop], 1, N_pop) ;
7     fa = f_parent(a);
8     fb = f_parent(b);
9     for k=1:N_pop
10         if fa(k)>fb(k)
11             mating_idx(k) = a(k);
12         else
13             mating_idx(k) = b(k);
14         end
15     end
16 elseif method == 2
17     a = randi([1, N_pop], 1, N_pop);
18     fa = f_parent(a);
19     for k=1:N_pop
20         if fa(k)>f_parent(k)
21             mating_idx(k) = a(k);
22         else
23             mating_idx(k) = k;
24         end
25     end
26 end
27 mating_pool = parent(mating_idx, :);
28 end
```

Listing 8: Cross-over function

```

1 function offspring = two_point_crossover(mating_pool, p_xover)
2
3 [N_pop,L] = size(mating_pool);
4 % shuffle parents
5 mating_pool = mating_pool(randperm(N_pop),:);
6 % generate rand nums for deciding if do crossover?
7 rand_nums = rand(1,round(N_pop/2));
8 do_xover = rand_nums > (1-p_xover);
9 offspring = zeros(N_pop, L);
10
11 for k = 1:round(N_pop/2)
12     parents = mating_pool([2*k-1 ,2*k],:);
13     if do_xover(k)
14         % find crossover point
15         xover_pt = randi(L,1);
16         % switch genes
17         offspring(2*k-1,:)= [parents(1,1:xover_pt), parents(2,xover_pt+1:end)];
18         offspring(2*k,:) = [parents(2,1:xover_pt), parents(1,xover_pt+1:end)];
19     else
20         offspring([2*k-1, 2*k],:) = parents;
21     end
22 end
23
24 end

```

Listing 9: Mutation function

```

1 function mutated = mutation(parents, p_mut)
2 [N_pop,~] = size(parents);
3 rand_nums = rand(N_pop,1);
4 do_mut_idx = find(rand_nums < p_mut);
5 mutated = parents;
6 % complement each bit in parent
7 mutated(do_mut_idx,:) = 1-parents(do_mut_idx,:);
8 end

```

Listing 10: Elitism function

```

1 function new_pop = elitism(pop, fitness)
2 new_pop = pop;
3 temp_fit = fitness;
4 [~, max_fit_idx] = max(temp_fit);
5 temp_fit(max_fit_idx) = min(temp_fit);
6 [~, other_max_fit_idx] = max(temp_fit);
7 new_pop([1,2],:) = pop([max_fit_idx, other_max_fit_idx],:);
8 end

```

Listing 11: Logging function

```

1 function [best_f, av_f, worse_f] = log_f(f_parent, best_f, av_f, worse_f)
2 best_f = [best_f, max(f_parent)];
3 av_f = [av_f, mean(f_parent)];
4 worse_f = [worse_f, min(f_parent)];
5 end

```

Real GA Code

Listing 12: Real GA Main Code

```
1 % ECE 580 HW5: Problem 2
2 % Rahul Deshmukh
3 % deshmun5@purdue.edu
4 clc; clear all; close all;
5 format short;
6 %% Include paths
7 addpath('../OptimModule/optimizers/global/GA/');
8 addpath('../OptimModule/optimizers/global/GA/Real_Num_GA/');
9 save_dir = './pix/';
10 %% set seed
11 rng(6); % tried 0:10 ,6 was the best and always converges to global min
12 %% Real GA problem setup
13 lb = -5*ones(1,2);
14 ub = 5*ones(1,2);
15 Num_vars = length(lb);
16
17 %% GA: solver params
18 N_pop = 40; % cant be odd integer
19 p_xover = 0.9;
20 p_mut = 0.01;
21 Niters = 30;
22 selection_method = 'tournament_method2';
23 xover_method = 'conv-combo';
24
25 %% GA starts
26
27 % initialize collectors
28 best_f = [];
29 av_f = [];
30 worse_f = [];
31
32 % choose type of selector
33 if strcmp(selection_method, 'roulette')
34     selection = @(x,f) roulette(x,f);
35 elseif strcmp(selection_method, 'tournament_method1')
36     selection = @(x,f) tournament_selection(x,f,1);
37 elseif strcmp(selection_method, 'tournament_method2')
38     selection = @(x,f) tournament_selection(x,f,2);
39 end
40
41 % draw initial population
42 X = rand(N_pop, Num_vars);
43 % scale to domain
44 X = (X.*(ub-lb) + lb);
45
46 parents = X;
47 % evaluate fitness of parents
48 f_parent = -1*griewank_fun(parents);
49 [best_f, av_f, worse_f] = log_f(f_parent, best_f, av_f, worse_f);
50 for i=1:Niters
51     % generate mating pool using selection
52     mating_pool = selection(parents, f_parent);
53     %perform crossover
54     parents = crossover(mating_pool, p_xover, xover_method);
```



```

55     %perform mutation
56     parents = mutation(parents, p_mut, lb, ub);
57     %perform elitism
58     parents = elitism(parents, f_parent);
59     %evaluate fitness of offspring
60     f_parent = -1*griewank_fun(parents');
61     [best_f, av_f, worse_f] = log_f(f_parent, best_f, av_f, worse_f);
62 end
63 % find the best offspring
64 [f_star, k_star] = max(f_parent);
65 fprintf(strcat('best fval: \t', num2str(-1*f_star)))
66 x_star = parents(k_star,:);
67
68 %% Convergence Plotting
69 fig1 = figure(1);
70 hold on; grid on;
71 x = 1:Niters+1;
72 h1 = plot(x, -1*av_f, '-b', 'LineWidth', 1);
73 h2 = plot(x, -1*best_f, '-r', 'LineWidth', 1);
74 h3 = plot(x, -1*worse_f, '-k', 'LineWidth', 1);
75 v = 1:5:Niters+1;
76 plot(x(v), -1*av_f(v), 'bx');
77 plot(x(v), -1*best_f(v), 'ro');
78 plot(x(v), -1*worse_f(v), 'k*');
79 legend([h1, h2, h3], {'Average', 'Best', 'Worse'}, 'Location', 'northeast');
80 hold off;
81 box('on');
82 xlabel('Num Iters'); ylabel('Function value');
83 xlim([1, Niters+1])
84 ylim(max(abs(worse_f))*(1.1)*[-1, 1]);
85 title('Convergence of Real GA');
86 saveas(fig1, strcat(save_dir, 'ga-real-conv'), 'eps');

```

Listing 13: Cross-over function

```

1 function offspring = crossover(mating_pool, p_xover, method)
2 switch nargin
3     case 2
4         method='av';
5 end
6 %shuffle mating pool
7 [Npop, ~] = size(mating_pool);
8 mating_pool = mating_pool(randperm(Npop), :);
9 rand_num = rand(round(Npop/2), 1);
10 do_xover = rand_num < p_xover;
11 do_xover_idx = find(do_xover);
12
13 alpha = rand(length(do_xover_idx), 1);
14 offspring = mating_pool;
15 if strcmp(method, 'av')
16     for i = 1:length(do_xover_idx)
17         k = do_xover_idx(i);
18         parents = mating_pool([2*k-1 2*k], :);
19         offspring([2*k-1 2*k], :) = [sum(parents, 1)/2.0;
20                                     parents((alpha(i) > 0.5)+1, :)]];
21     end
22 elseif strcmp(method, 'conv_combo')
23     for i = 1:length(do_xover_idx)

```

```

24         k = do_xover_idx(i);
25         parents = mating_pool([2*k-1 2*k],:);
26         offspring([2*k-1 2*k], :) = [alpha(i), 1-alpha(i);
27                                     1-alpha(i), alpha(i)]*parents;
28     end
29 else
30     error('Method for crossover not implemented');
31 end
32
33 end

```

Listing 14: Mutation function

```

1 function mutated = mutation(parents, p_mut, lb, ub)
2 [N_pop,~] = size(parents);
3 rand_nums = rand(N_pop,1);
4 do_mut_idx = find(rand_nums < p_mut);
5 mutated = parents;
6 alpha = rand(length(do_mut_idx),1);
7 w = rand(length(do_mut_idx),length(lb));
8 % scale and translate w to domain
9 w = w.*(ub-lb) + lb;
10 mutated(do_mut_idx,:) = parents(do_mut_idx,:).*alpha + w.*(1-alpha);
11 end

```

Linear programming Code

Listing 15: Linprog Main Code

```

1 % ECE 580 HW5: Problem 3
2 % Rahul Deshmukh
3 % deshmun5@purdue.edu
4 clc; clear all;
5 format rat;
6 %% include paths
7 addpath('.../OptimModule/optimizers/linprog/');
8 %%
9 verbose=0;
10 c = [4; 3];
11 A = [-5, -1;
12      -2, -1;
13      -1, -2];
14 b = [-11; -8; -7];
15 Aeq = [];
16 beq = [];
17 LB=[];
18 UB=[];
19 [x_str, fval] = mylinprog(c,A,b,Aeq,beq,LB,UB,verbose)

```

Listing 16: Two Phase Simplex

```

1 function [x_str, fval] = mylinprog(c,A,b,Aeq,beq,LB,UB,verbose)
2 % Two Phase Simplex LP solver

```

```

3 % the problem is not given in std form
4 % Given: min c'*x
5 % st:      Aeq*x = beq
6 %          Ax ≤ b
7 %          0 ≤ x or LB ≤ x ≤ UB
8 if nargin==7
9     verbose=0; %by default
10 end
11 % transform variables: z = x - LB if LB is not 0
12 if ~isempty(LB)
13     error('Not implemented');
14     b = b - A*LB;
15     if ~isempty(Aeq)
16         beq = beq - Aeq*LB;
17     end
18 end
19 % stack inequality conditions Iz ≤ UB-LB
20 if ~isempty(UB)
21     A = [A; eye(length(UB))];
22     b = [b; UB-LB];
23 end
24 %% Convert to std form using slack and surplus variables st b_std ≥ 0
25 A_std = []; b_std = []; c_std = [];
26 A_std = [A_std; A, eye(size(A,1))];
27 b_std = [b_std; abs(b)];
28 c_std = [c_std; c; zeros(size(A,1),1)];
29 % make inequality b's positive
30 neg_b_idx = find(b < 0);
31 A_std(neg_b_idx,:) = -1*A_std(neg_b_idx,:);
32 % stack equality equations
33 if ~isempty(Aeq)
34     neg_b_idx = find(beq < 0);
35     Aeq(neg_b_idx,:) = -1*Aeq(neg_b_idx,:);
36     A_std = [A_std; Aeq, zeros(size(Aeq,1),size(A,1))];
37     b_std = [b_std; abs(beq)];
38 end
39 %% Solve std form using two-phase simplex method
40 if verbose
41     fprintf('\text{Phase 1:}&\nonumber\\\\n%%\n');
42 end
43 % Phase1: Find initial basis using artificial variables
44 A1 = [A_std, eye(size(A_std,1))];
45 c1 = [zeros(size(A_std,2),1); ones(size(A_std,1),1)];
46 basis_idx = size(A_std,2) + (1:size(A_std,1));
47 [¬, ¬, basis_idx_p1, tab] = simplex(A1, b_std, c1, basis_idx, verbose);
48 if verbose
49     fprintf('\text{Phase 2:}&\nonumber\\\\n%%\n') ;
50 end
51 % Phase2: Find optimal solution
52 A2 = tab(1:end-1, 1:size(A_std,2));
53 b2 = tab(1:end-1, end);
54 [x_str_p2, fval, ¬, ¬] = simplex(A2, b2, c_std, basis_idx_p1, verbose);
55 x_str = x_str_p2(1:length(c));
56 if ~isempty(LB)
57     % transform from z_str to x_str, x_str = z_str + LB
58     x_str = x_str + LB;
59     fval = fval + c'*LB;
60 end
61 fprintf('\n** Optimum Solution found using mylinprog **\n');

```

```

62 if verbose
63     display(x_str);
64     display(fval);
65 end
66 end

```

Listing 17: Simplex Method

```

1 function [x_str, fval, basis_idx, tab]= simplex(A,b,c, basis_idx, verbose)
2 % simplex method to solve LP in std form:
3 % Given: min c^Tx
4 % st:      Ax = b
5 %          x ≥ 0
6 % basis_idx the indices are in order of std cartesian basis 1...n
7 tol = 1e-6;
8 tab = [A, b;
9        c', 0];
10 if verbose
11     print_tab(tab);
12 end
13
14 % make cost coeffs zero for basis idx
15 for i=1:length(basis_idx)
16     % r = find(tab(1:end-1,basis_idx(i))>0);
17     tab(end,:) = tab(end,:) - tab(end,basis_idx(i))*tab(i,:);
18 end
19
20 % till all cost coeffs are non-negative do:
21 while ~isempty(find(tab(end,1:end-1)< 0))
22     % choose pivot column
23     [r,p] = min(tab(end,1:end-1));
24     % find pivot element
25     % using only positive a_p's
26     pos_ap_idx = find(tab(1:end-1,p)>0);
27     [r,q_idx] = min(tab(pos_ap_idx,end)./tab(pos_ap_idx, p));
28     q = pos_ap_idx(q_idx);
29     % update basis idx
30     basis_idx(q) = p;
31     % print tab with pivot element
32     if verbose
33         print_tab(tab,[q,p]);
34     end
35     % make pivot 1
36     tab(q,:) = tab(q,+)/tab(q,p);
37     % make pivot column
38     for r=1:size(tab,1)
39         if r≠q
40             tab(r,:) = tab(r,:) - tab(r,p)*tab(q,:);
41         end
42     end
43     tab(tab>-tol & tab<tol) = 0;
44 end
45 if verbose
46     print_tab(tab)
47 end
48 %return x_str and basis_idx
49 x_str = zeros(length(c),1);
50 x_str(basis_idx) = tab(1:end-1,end);

```

```

51 fval= c'*x_str;
52 end
53
54 function print_tab(tab, pivot)
55 if nargin==1
56     pivot = 0;
57 end
58 % display(tab)
59 % print latex bmatrix
60 [n_row, n_col] = size(tab);
61 fprintf('&=\begin{bmatrix}\n');
62 for r=1:n_row
63     for c=1:n_col-1
64         if r==pivot(1) && c== pivot(2)
65             % this is the pivot
66             fprintf('\fbox{{\color{red}%s}}&t',strtrim(rats(tab(r,c))));
67         else
68             fprintf('%s&t',strtrim( rats(tab(r,c)) ));
69         end
70     end
71     fprintf('%s\\\\\n',strtrim( rats(tab(r,c+1)) ));
72 end
73 fprintf('\end{bmatrix}\\nonumber\n%\n');
74 end

```

Listing 18: LP output

```

1  ** Optimum Solution found using mylinprog **
2
3  x_str =
4
5          3
6          2
7
8
9  fval =
10
11          18

```