Exam 3: cover sheet

ECE600 Random Variables and Waveforms
Fall 2020

Midterm Exam #3
Session 25
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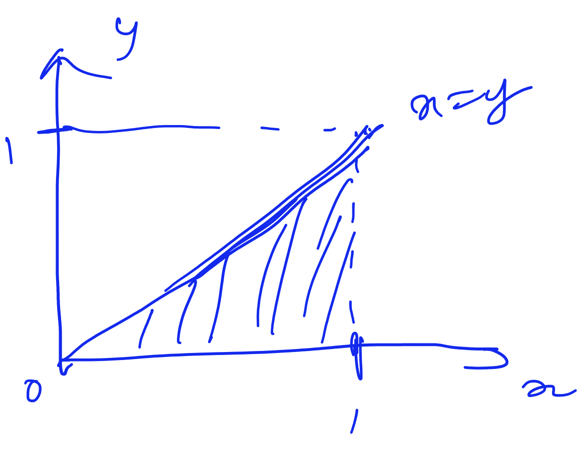
1	20
2	25
3	25
4	16
5	
6	
7	
8	
Total	86

Q1

Given: $f_{XY}(x,y) = Kx^2y \; \mathbb{1}_{[0,y]}^{(x)} \; \mathbb{1}_{[0,1]}^{(y)}$



$\frac{20}{25}$

(a) Finding K, For valid j-dist^n pdf we have:

(i) $Kx^2y \geq 0$, $x>0$, $y>0$

$\Rightarrow K \geq 0$

(ii) $\int_0^1 \int_0^y f_{XY}(x,y)\,dx\,dy = 1$

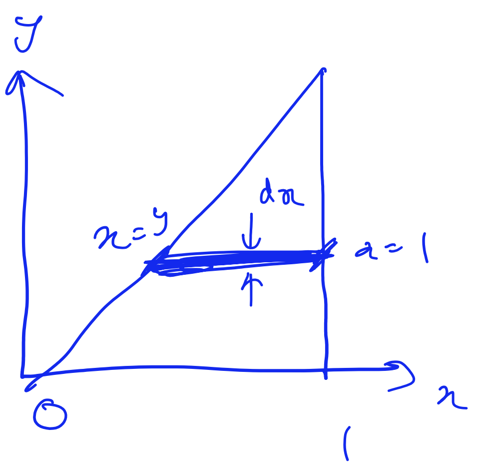
$\Rightarrow \int_0^1 \int_0^y Kx^2y \, dx\,dy = \int_0^1 y\left[\int_0^y Kx^2 dx\right] dy = 1$

$\Rightarrow \int_0^1 y \left. \frac{Kx^3}{3} \right|_0^y dy = 1$

$\Rightarrow \frac{K}{3}\int_0^1 y^4 dy = \frac{K}{3\times 5} \left. y^5 \right|_0^1 = 1$

$\Rightarrow \boxed{K = 15}$ ✓ $\frac{5}{5}$

(b) $f_Y(y) = \int_{0\; y}^{y\; 1} f_{XY}(x,y)\,dx$



$= \int_y^1 15x^2y\,dx = 15y \left. \frac{x^3}{3} \right|_y^1$

$f_Y(y) = 5y(1-y^3)\; \mathbb{1}_{[0,1]}^{(y)}$ $\frac{2}{5}$

(c) $f_X(x \mid \{Y=y\}) = \frac{f_{XY}(x,y)}{f_Y(y)}$ ✓

$= \frac{15x^2y}{5y(1-y^3)} \mathbb{1}_{[0,y]}^{(x)} = \frac{3x^2}{(1-y^3)} \cdot \mathbb{1}_{[0,y]}^{(x)}$ ✓ ✗ $\frac{4}{5}$

(d) $\hat{X}_{MMS}(y) = E_X[X \mid \{Y=y\}]$

$= \int_{-\infty}^{\infty} x \cdot f_X(x \mid \{Y=y\})\,dx$ ✓

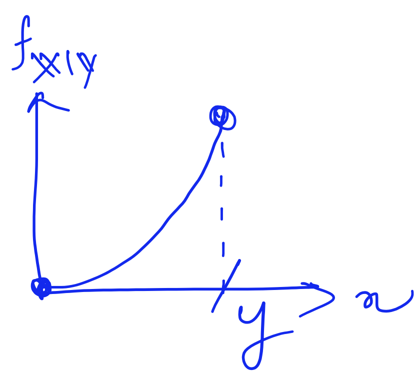
$= \int_0^y x \cdot \frac{3x^2}{(1-y^3)}\,dx$

$= \frac{3}{(1-y^3)} \left. \frac{x^4}{4} \right|_0^y$

$\Rightarrow \boxed{\hat{X}_{MMS}(y) = \frac{3}{4}\frac{y^4}{(1-y^3)}}$ ✗ $\frac{4}{5}$

(e) $\hat{X}_{MAP}(y) = \underset{X}{\operatorname{argmax}} \{ f_X(\cancel{x} \mid \{Y=y\}) \}$

$= \underset{x}{\operatorname{argmax}} \left\{ \frac{3x^2}{(1-y^3)} \mathbb{1}_{[0,y]}^{(x)} \right\}$



The function $\frac{3x^2}{(1-y^3)}$ (with y as constant) is max at end points

$\Rightarrow \boxed{\hat{X}_{MAP}(y) = y}$ ✓ $\frac{5}{5}$

Q2

given: $\Phi_{XY}(\omega_1, \omega_2) = \frac{1}{(1-i\omega_1)} \cdot \frac{1}{(1-2i\omega_2)}$ $\frac{25}{25}$

(a) $E[X] = \frac{\partial}{\partial(i\omega_1)} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\omega_1=0,\,\omega_2=0}$

$= \frac{1}{(1-2i\omega_2)} \cdot \frac{-1 \times -1}{(1-i\omega_1)^2}\Big|_{\omega_1=0,\,\omega_2=0}$

$= \frac{1}{(1-0)} \times \frac{1}{(1-0)^2} = 1$

$\Rightarrow E[X] = 1$ ✓ $\frac{4}{4}$

(b) $E[Y] = \frac{\partial}{\partial(i\omega_2)} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\omega_1=0\,\omega_2=0}$

$= \frac{1}{(1-i\omega_1)} \cdot \frac{-1 \times -2}{(1-2i\omega_2)^2}\Big|_{\omega_1=0,\,\omega_2=0}$

$E[Y] = 2$ ✓ $\frac{4}{4}$

(c) $E[XY] = \frac{\partial^2}{\partial(i\omega_1)\,\partial(i\omega_2)} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\substack{\omega_1=0\\ \omega_2=0}}$

$= \frac{-1 \times -1}{(1-i\omega_1)^2} \cdot \frac{-1 \times -2}{(1-2i\omega_2)^2}\Big|_{\substack{\omega_1=0\\ \omega_2=0}}$

$E[XY] = 2$ ✓ $\frac{5}{5}$

(d) $E[X^j Y^k] = \frac{\partial^{j+k}}{\partial^k(i\omega_2)\,\partial^j(i\omega_1)} \Phi_{XY}(\omega_1,\omega_2)\Big|_{\substack{\omega_1=0\\ \omega_2=0}}$

$= \frac{\partial^k}{\partial^k(i\omega_2)}\left(\frac{1}{(1-2i\omega_2)} \frac{\partial^j}{\partial^j(i\omega_1)} \frac{1}{(1-i\omega_1)}\right)\Big|_{\substack{\omega_1=0\\ \omega_2=0}}$

$= \frac{2^k k!}{(1-2i\omega_2)^{k+1}} \cdot \frac{j!}{(1-i\omega_1)^{j+1}}\Big|_{\substack{\omega_1=0\\ \omega_2=0}}$

$E[X^j Y^k] = 2^k k!\, j!$ ✓ $\frac{8}{8}$

(e) $\rho_{XY} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{2 - 2\times 1}{\sigma_X \sigma_Y} = 0$ ✓

$\boxed{\rho_{XY} = 0}$ ✓ $\frac{4}{4}$

Q3

$\frac{25}{25}$

given: $\mathbb{N}$ is a poisson variable with mean λ

$\Rightarrow P_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}$, $n = 0, 1, \dots \infty$

Probability of car exceeding speed limit : p (Independent of $\mathbb{N}$)

$\mathbb{M}$ ~ no. of cars exceeding speed limit

a) $\Phi_{\mathbb{N}}(\omega) = E[e^{j\omega \mathbb{N}}] = \sum_{K=0}^{\infty} e^{j\omega K} e^{-\lambda} \frac{\lambda^K}{K!}$

$= e^{-\lambda} \sum_{K=0}^{\infty} \frac{(e^{j\omega}\lambda)^K}{K!}$

$= e^{-\lambda} e^{\lambda e^{j\omega}}$

$\Phi_{\mathbb{N}}(\omega) = e^{-\lambda(1-e^{j\omega})}$ ✓ $\frac{5}{5}$

b) $\Phi_{\mathbb{M}}(\omega) = E[e^{j\omega \mathbb{M}}]$

$= E_{\mathbb{N}}[E_{\mathbb{M}}[e^{j\omega \mathbb{M}} | \mathbb{N}]]$

$= \sum_{n=0}^{\infty} P_N(n) E_{\mathbb{M}}[e^{j\omega \mathbb{M}} | \{\mathbb{N}=n\}]$

We need to find $f_{\mathbb{M}|\mathbb{N}}(m | \{N = n\})$

which translates to, given there are n cars on the road we need to find the probability that m cars out of n are speeding

Clearly the conditional distribution is a Binomial Distribution with the success probability p

$\Rightarrow f_{\mathbb{M}|\mathbb{N}}(m | \{N=n\}) = \binom{n}{m} p^m (1-p)^{n-m}$

$\Rightarrow E_{\mathbb{M}}[e^{j\omega \mathbb{M}} | \{\mathbb{N}=n\}] = \sum_{m=0}^{n} e^{j\omega m} \binom{n}{m} p^m (1-p)^{n-m}$

$= \sum_{m=0}^{n} \binom{n}{m} (pe^{j\omega})^m (1-p)^{n-m}$

$= (pe^{j\omega} + 1 - p)^n$ ✓

$\Rightarrow \Phi_{\mathbb{M}}(\omega) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \cdot (pe^{j\omega} + 1 - p)^n$

$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda(pe^{j\omega} + 1 - p))^n}{n!}$

$= \exp(-\lambda + \lambda(pe^{j\omega} + 1 - p))$

$\Phi_{\mathbb{M}}(\omega) = \exp(\lambda p(e^{j\omega} - 1))$

$= \exp(-\lambda p(1 - e^{j\omega}))$ ✓ $\frac{10}{10}$

c) On comparing result from (a) & (b) we get:

M is a poisson dist^n with mean λp

$P_{\mathbb{M}}(m) = e^{-\lambda p} \frac{(\lambda p)^m}{m!}$, $m = 0, 1, \dots$ ✓ $\frac{5}{5}$

(d) $\mu_{\mathbb{M}} = E[\mathbb{M}] = \frac{\partial}{\partial (j\omega)} \Phi_{\mathbb{M}}(\omega) \Big|_{\omega=0}$

$= +\lambda p e^{j\omega} \cdot \exp(-\lambda p(1 - e^{j\omega})) \Big|_{\omega=0}$

$\mu_{\mathbb{M}} = \lambda p$ ✓

$\sigma_{\mathbb{M}}^2 = E[\mathbb{M}^2] - (E[\mathbb{M}])^2$

$E[\mathbb{M}^2] = \frac{\partial^2}{\partial (j\omega)^2} \Phi_{\mathbb{M}}(\omega) \Big|_{\omega=0}$

$= \frac{\partial}{\partial (j\omega)} (\lambda p e^{j\omega} \cdot \exp(-\lambda p (1 - e^{j\omega}))) \Big|_{\omega=0}$

$= \lambda p e^{j\omega} \exp(-\lambda p(1 - e^{j\omega})) + \lambda p e^{j\omega} \cdot (\lambda p) \cdot \exp(-\lambda p(1 - e^{j\omega})) \Big|_{\omega=0}$

$= \lambda p + (\lambda p)^2$

$\Rightarrow \sigma_{\mathbb{M}}^2 = \lambda p + \cancel{(\lambda p)^2} - \cancel{(\lambda p)^2}$ $\frac{5}{5}$

$\boxed{\sigma_{\mathbb{M}}^2 = \lambda p}$ ✓

Q4

$\frac{16}{25}$

given: $X_1, X_2, X_3, \ldots, X_n$ is a sequence of i.i.d. exponential R.V. with mean μ

Another sequence $\{Y_n\}$ defined as:

$$Y_n = \min\{X_1, X_2, \ldots, X_n\} \quad n = 1, 2, 3, \ldots$$

(a) As the sequence $\{X_n\}$ is i.i.d. exponential R.V. with mean μ

$\therefore$ Their joint distⁿ is given by:

$$f_{X_1,X_2,\ldots X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \frac{1}{\mu} e^{\frac{-x_i}{\mu}} \quad x_i \geq 0, \mu \geq 0$$

$$= \frac{1}{\mu^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i)}{\mu}\right)$$

Also since $X_i \perp\!\!\!\perp X_j \; \forall i \neq j$ we can compute any marginal as:

$$f_{X_1,X_2,X_j}(x_1, x_2, \ldots x_j) = \frac{1}{\mu^j} \exp\left(-\frac{\sum_{i=1}^{j}(x_i)}{\mu}\right) \quad \text{— Eq (1)}$$

For $n = 1$

$Y_1 = \min\{X_1\} = X_1$ & Y_1 is exponential distⁿ

& $Y_1 \sim \frac{1}{\mu} e^{-\frac{y_1}{\mu}} \quad y \geq 0, \mu \geq 0$

For any $n > 1$

$$Y_n = \min\{X_1, X_2 \ldots X_n\}$$

$$F_{Y_n}(y) = P\left(\{Y_n \leq y\} \cap \left\{\bigcup_{i \geq 1}^{n} X_i = \min\{X_1, X_2 \ldots X_n\}\right\}\right)$$

$$= \sum_{i=1}^{n} P\left(\{X_i \leq y\} \cap \{X_i = \min\{X_1, X_2, \ldots, X_n\}\}\right)$$

$$= 1 - P(\{Y_n > y\})$$

$$= 1 - P(\{X_i > y, \forall i\})$$

$$= 1 - \prod_{i=1}^{n} P(\{X_i > y\}) \checkmark$$

$$= 1 - \prod_{i=1}^{n} \left(\cancel{1 - e^{-y/\mu}}\right) = 1 - \left(\cancel{1 - e^{-y/\mu}}\right)^n$$

$e^{-y/\mu}$ $\quad$ $e^{-y/\mu}$

& $f_{Y_n}(y) = \frac{\partial}{\partial y} F_{Y_n}(y)$

$$= n\left(\cancel{1 - e^{-y/\mu}}\right)^{n-1} \times \frac{1}{\mu} \cdot e^{-y/\mu}$$

$e^{-y/\mu}$

$\frac{8}{13}$

$$= \frac{n}{\mu}\left(1 - e^{-y/\mu}\right)^{n-1} \mathbb{1}_{[0,\infty)}(y) \times = \frac{n}{\mu} e^{-\frac{yn}{\mu}} \cdot \mathbb{1}(y)_{[0,\infty)}$$

$\times$

(b) $P(\{|Y_n - Y| > \epsilon\}) \to 0$ as $n \to \infty$? ✓

See solutions

$$E[Y_n^2] = \int_0^\infty y^2 \frac{n}{\mu}\left(1 - e^{-y/\mu}\right)^{n-1} dy$$

$4/6$

See solutions $4/6$

(c) $E[|Y_n - Y|^2] \to 0$ as $n \to \infty$? ✓