Question 2.3

Tuesday, April 21, 2020 6:00 PM

MAP estimate with $p(\theta)$

given: $\theta_i \sim p(\theta_i) = \frac{1}{\alpha} \exp\left\{ -\frac{|\theta_i|}{\alpha} \right\}$

assuming $\forall \theta_i$ are independent & $\underline{\theta} \in \mathbb{R}^m$ $\Rightarrow$ $p(\underline{\theta}) = \prod_{i=0}^{i=m-1} \frac{1}{\alpha} \exp\left\{ -\frac{|\theta_i|}{\alpha} \right\}$

$$\Rightarrow p(\underline{\theta}) = \frac{1}{\alpha^m} \exp\left\{ -\frac{1}{\alpha} \sum_{i=0}^{m-1} |\theta_i| \right\} = \frac{1}{\alpha^m} \exp\left\{ -\frac{1}{\alpha} \|\underline{\theta}\|_1 \right\}$$

$$\Rightarrow \hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left\{ -\log\left(p_\theta(x, y) \cdot p(\theta) \right) \right\}$$

$$= \underset{\theta}{\operatorname{argmin}} \left\{ -\log\left(p_\theta(x|y) \right) - \log(p(\theta)) \right\}$$

$$= \underset{\theta}{\operatorname{argmin}} \left\{ \frac{1}{2\sigma^2} \sum_{k=0}^{K-1} \|x_k - f_\theta(y_k)\|^2 + \frac{1}{\alpha} \|\theta\|_1 \right\}$$

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left\{ \sum_{k=0}^{K-1} \|x_k - f_\theta(y_k)\|^2 + \frac{2\sigma^2}{\alpha} \|\underline{\theta}\|_1 \right\}$$

FONC: $$-2 \sum_{k=0}^{K-1} \left(x_k - f_\theta(y_k) \right)^T \cdot \nabla_\theta f_\theta(y_k) + \frac{2\sigma^2}{\alpha} \operatorname{sign}(\theta) \Big|_{\theta = \hat{\theta}} = \underline{0}$$

Solution to the above equation gives the MAP estimate $\hat{\theta}$