ECE600 Random Variables and Waveforms — Mark R. Bell
Fall 2020 — MSEE 336

Final Exam
9:00–11:00am EST (UTC-5:00), December 12, 2020

TABLE 9-1

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}\,d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\,d\tau$$

$$\delta(\tau) \leftrightarrow 1 \qquad 1 \leftrightarrow 2\pi\delta(\omega)$$

$$e^{j\beta\tau} \leftrightarrow 2\pi\delta(\omega-\beta) \qquad \cos\beta\tau \leftrightarrow \pi\delta(\omega-\beta)+\pi\delta(\omega+\beta)$$

$$e^{-\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2} \qquad e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$$

$$e^{-\alpha|\tau|}\cos\beta\tau \leftrightarrow \frac{\alpha}{\alpha^2+(\omega-\beta)^2}+\frac{\alpha}{\alpha^2+(\omega+\beta)^2}$$

$$2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}\left[e^{-(\omega-\beta)^2/4\alpha}+e^{-(\omega+\beta)^2/4\alpha}\right]$$

$$\begin{cases} 1-\dfrac{|\tau|}{T} & |\tau|<T \\ 0 & |\tau|>T \end{cases} \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$$

$$\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega|<\sigma \\ 0 & |\omega|>\sigma \end{cases}$$

Name: ______________________________

ID #: ______________________________

City/Time-Zone: ______________________________

Start Time/Stop Time/Date: ______________________________

Technical Difficulties:

ECE600 Final Exam Instructions

The ECE600 Final Exam is scheduled for Saturday, December 12, 2020 at 9:00–11:00am EST (UTC-5:00). The Final Exam is an open book, open notes, no internet access exam. You are allowed to use a calculator. You are not to discuss the exam with anyone except me until Wednesday, December 16, 2020. You are on your honor to follow these rules. As engineers, who must carry out their work with the highest ethical standards, I trust you to do so.

Before the Exam: Make out a cover sheet with your name, student number, the city and time zone you are taking the exam in, and space to describe any technical difficulties you may have had in taking the exam.

The following are the detailed instructions to take the exam:

1. At 8:50am EST (UTC-5:00), retrieve the pdf file of your final exam from the email sent to you at your on record email address for ECE600 a few minutes before this time.
2. Begin working the exam at 9:00am EST (UTC-5:00). There is no need to print out the exam. Just neatly write out your solutions, carefully numbering the problems, and **starting each problem on a new page**. *On the multiple choice questions, please show your work, as partial credit is possible*.
3. 2 hours after you started your exam, at 11:00am EST, the exam is over. Stop writing.
4. Scan your exam using either a smartphone scanning app or standard scanner. Export the final scan as a pdf document. (include the cover sheet you prepared before the exam.) Name the file with *your* last name and first letter of your first name (e.g., if I were taking the exam, my scanned file would be named "BellM.pdf".)
5. Send me your scanned pdf exam solution using either email (to mrb@ecn.purdue.edu) or upload it to your Purdue Filelocker account and share it with me (https://itap.purdue.edu/service/catalog/security/filelocker.html). If you are sending it via email, please put "ECE600 Final Exam" in the subject line. **The file should be sent or uploaded within 30 minutes of completing your exam.** However, if there are technical difficulties, let me know.

Thank you for your effort in making this online Final Exam work. If you have any technical problems in taking the exam, please let me know. Send them to me at mrb@ecn.purdue.edu after the exam with the subject line "ECE600 Final Exam Technical Issues".

With the highest regard and respect,
Prof. Mark R. Bell

Problems 1–10 are multiple choice problems worth 5 points each. **Start your solution to each problem on a new page**. Number each problem. For each problem, write the letter corresponding to the best answer next to the problem number on your answer sheet and circle it. For each problem, use the remaining space on the page to work out your solution. Please show your work! If your final grade is near a borderline, the quality of your written solutions could significantly impact your final grade.

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1. Let $(\mathcal{S}, \mathcal{F}, P)$ be a probability space, and let $A, B \in \mathcal{F}$ be two independent events such that the probability that at least one of them occurs is $1/3$, and the probability that A occurs but that B does not occur is $1/9$. What is $P(B)$?

   A. $4/9$,

   B. $1/3$,

   C. $2/9$,

   D. $1/7$,

   E. $1/9$.

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2. Identical twins come from the same egg and hence are of the same sex. Fraternal twins have a probability $1/2$ of being of the same sex. Among twins, the probability of a fraternal set is p and of an identical set is $q = 1 - p$. Given that a set of twins selected at random are of the same sex, what is the probability they are identical twins?

   A. $\frac{q}{p}$,

   B. $\frac{p}{1+q}$,

   C. 1,

   D. p,

   E. $\frac{2q}{1+q}$.

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3. Mr. Flowers plants 10 rose bushes in a row. Eight of the bushes are white and two are red, and he plants them in random order. What is the probability he will consecutively place 7 or more white bushes?

   A. $\frac{1}{10}$,

   B. $\frac{1}{9}$,

   C. $\frac{2}{15}$,

   D. $\frac{7}{45}$,

   E. $\frac{1}{5}$.

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4. Let $\mathbf{X}$ be a random variable uniformly distributed on the interval $(0, 1)$. Let $\mathbf{Y}$ be a new random variable defined as

$$\mathbf{Y} = \sqrt{-\alpha \ln \mathbf{X}},$$

where $\alpha > 0$. The probability density function of $\mathbf{Y}$ is given by:

A. $f_{\mathbf{Y}}(y) = \alpha e^{-\alpha y} \cdot 1_{(0,\infty)}(y)$,

B. $f_{\mathbf{Y}}(y) = \frac{1}{\alpha} e^{-y/\alpha} \cdot 1_{(0,\infty)}(y)$,

C. $f_{\mathbf{Y}}(y) = 2y\alpha e^{-\alpha y^2} \cdot 1_{(0,\infty)}(y)$,

D. $f_{\mathbf{Y}}(y) = \frac{2y}{\alpha} e^{-y^2/\alpha} \cdot 1_{(0,\infty)}(y)$,

E. None of the above.

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5. Let $A, B, C \in \mathcal{F}$ be events in probability space $(\mathcal{S}, \mathcal{F}, P)$ such that

$$A \cap B = A \cap C = \emptyset,$$

and

$$B \cap C \neq \emptyset.$$

Then $P(A \cup B \cup C)$ must equal

A. $P(A) + P(B) + P(C)$,

B. $P(A) + P(B) + P(A \cap B \cap C)$,

C. $P(A) + P(B) + P(C) - 2P(A \cap B \cap C)$,

D. $P(A) + P(B) + P(C) - 2P(B \cap C)$,

E. $P(A) + P(B) + P(C) - P(B \cap C)$.

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6. Suppose $\mathbf{X}$ is a Poisson random variable with standard deviation 4. What is the condition probability that $\mathbf{X} = 1$ given that $\mathbf{X} \geq \mathbf{1}$?

   A. $\dfrac{4}{e^4 - 1}$,
   B. $\dfrac{2}{e^2 - 1}$,
   C. $16e^{-16}$,
   D. $4e^{-4}$,
   E. $\dfrac{16}{e^{16} - 1}$.

7. Let $\mathbf{X}$ and $\mathbf{Y}$ be two independent, identically distributed random variables, both uniformly distributed on the interval $[0, 2]$. What is the probability of the event $\{\mathbf{X}^2 + \mathbf{Y}^2 \leq 1\}$?

   A. $\dfrac{\pi}{4}$,
   B. $\dfrac{16 - \pi}{16}$,
   C. $\dfrac{\pi}{8}$,
   D. $\dfrac{\pi}{16}$,
   E. $\dfrac{4 - \pi}{4}$.

8. Let $\mathbf{X}$ and $\mathbf{Y}$ be two jointly distributed random variables with joint p.d.f. given by

$$f_{\mathbf{XY}}(x, y) = \begin{cases} x + y, & \text{for } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   What is the minimum mean-square error estimator of $\mathbf{Y}$ given that $\{\mathbf{X} = 1/3\}$:

   A. $\dfrac{2 + 6y}{5}$,
   B. $\dfrac{1}{3}$,
   C. $\dfrac{5}{12}$,
   D. $\dfrac{1}{2}$,
   E. $\dfrac{3}{5}$.

9. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are two independent, identically distributed random variables. Define a new random variable $\mathbf{Z}$ as

$$\mathbf{Z} = a\mathbf{X} + \mathbf{Y},$$

where $a > 0$. If the correlation coefficient between $\mathbf{X}$ and $\mathbf{Z}$ is equal to $1/3$, what is the value of a?

A. $\frac{1}{2\sqrt{2}}$,

B. $\frac{1}{3}$,

C. $\frac{1}{2}$,

D. $\frac{1}{\sqrt{2}}$,

E. $\frac{1}{\sqrt{3}}$.

10. Assume $\mathbf{X}(t)$ is a wide-sense stationary random process with autocorrelation function $R_X(\tau)$, and that $\mathbf{Y}(t)$ is a random process defined by

$$\mathbf{Y}(t) = -\mathbf{X}(t - t_0),$$

where t_0 is a constant. The autocorrelation function $R_{YY}(t_1, t_2)$ of $\mathbf{Y}(t)$ is given by

A. $R_X(t_1 - t_2)$,

B. $-R_X(t_1 - t_2 - t_0)$,

C. $-R_X(t_1 - t_2 - 2t_0)$,

D. $R_X(t_1 - t_2 - 2t_0)$,

E. None of the above.

11. Problem 11 is made up of 8 True/False questions, worth 2 points each. Label your answer sheet for problem 11 with the letters A through H for each of the statements below, and write T (true) or F (false) next to each letter, depending on whether the corresponding statements are true or false, respectively.

A. ____ If two events A and B are statistically independent, then they are disjoint.

B. ____ If the autocorrelation function $R_X(t_1, t_2)$ of a random process $\mathbf{X}(t)$ can be written as a function of the time difference $t_1 - t_2$, then $\mathbf{X}(t)$ is wide-sense stationary.

C. ____ If two random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent, then they are uncorrelated.

D. ____ Let A, B, and C be three events defined on a random experiment.
If $P(A \cap B \cap C) = P(A)P(B)P(C)$, then A, B, and C are statistically independent.

E. ____ If two Gaussian random variables $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated, then they are statistically independent.

F. ____ If a random process is wide-sense stationary, then it is strict-sense stationary.

G. ____ Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be jointly distributed continuous random variables. Then the random variables $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are statistically independent if and only if $f_{\mathbf{XYZ}}(x, y, z) = f_{\mathbf{X}}(x)f_{\mathbf{Y}}(y)f_{\mathbf{Z}}(z)$.

H. ____ Let A and B be two events defined on a random experiment. Then if A and B are statistically independent, then $P(B|A) = P(B)$.

Problems 12 and 13 are "work out" problems for which partial credit will be awarded for correctly reasoned work. It is important that you coherently present your thinking in the solution of these problems if you wish to receive partial credit (or full credit for that matter.)

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12. **(17 pts.)** The number of defects in a VLSI chip is a Poisson random variable $\mathbf{N}$ with parameter r conditioned on the event $\{\mathbf{R} = r\}$, where $\mathbf{R}$ is a random parameter characterizing the VLSI fabrication line:

$$P(\{\mathbf{N} = n\}|\{\mathbf{R} = r\}) = \frac{r^n e^{-r}}{n!}, \quad n = 0, 1, 2, \ldots .$$

The random parameter $\mathbf{R}$ is exponentially distributed with mean μ:

$$f_R(r) = \begin{cases} (1/\mu)\exp\{-r/\mu\}, & \text{for } r \geq 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\mu > 0$.

(a) Find the probability mass function $P(\{\mathbf{N} = n\})$ of $\mathbf{N}$, the number of defects per VLSI chip.
(b) Find the mean of $\mathbf{N}$.
(c) Find the variance of $\mathbf{N}$.

Note: You may find the following integral useful:

$$\int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1),$$

where $\Gamma(\cdot)$ is the gamma function, and $\Gamma(k) = (k-1)!$ for integers $k = 1, 2, 3, \ldots$.

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13. **(17 pts.)** Let $\mathbf{X}(t)$ be a zero-mean wide-sense stationary Gaussian white noise process with autocorrelation function $R_{XX}(\tau) = \delta(\tau)$. Suppose that $\mathbf{X}(t)$ is the input to a linear time-invariant system with an impulse response

$$h(t) = \alpha e^{-\alpha t} \cdot 1_{[0,\infty)}(t).$$

Where $\alpha > 0$. Let $\mathbf{Y}(t)$ be the output of the system, and assume that the input has been applied to the system for all time.

(a) What is the mean of $\mathbf{Y}(t)$?
(b) What is the autocorrelation function of $\mathbf{Y}(t)$?
(c) Write an expression for the second-order pdf $f_{\mathbf{Y}(t_1)\mathbf{Y}(t_2)}(y_1, y_2)$ of $\mathbf{Y}(t)$.
(d) Is $\mathbf{Y}(t)$ strict-sense stationary? Justify your answer.
