ECE600 Random Variables and Waveforms
Fall 2020

Final Exam #4
Session 30
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Q1

Given : $A \parallel B$

$$P(A \cup B) = 1/3 = P(A) + P(B) - P(A)P(B) \quad -①$$

$$P(A \cap \bar{B}) = 1/9 \Rightarrow P(A)P(\bar{B}) = 1/9$$

$$\Rightarrow P(A)(1 - P(B)) = 1/9$$

$$\Rightarrow P(A) - P(A)P(B) = 1/9 \quad -②$$

$$\Rightarrow ① \& ② \rightarrow 1/9 + P(B) = 1/3$$

$$P(B) = \frac{2}{9}$$

Ans $\boxed{C}$

Q2

Twins = Identical Twins ∪ Fraternal Twins

$(T) = (IT) \cup (FT)$

$IT \cap FT = \phi$

$P(FT) = p$, $P(IT) = 1-p = q$

SS = Same sex

$P(SS|IT) = 1$

$P(SS|FT) = 1/2$

$$P(IT|SS) = \frac{P(SS|IT)\,P(IT)}{P(SS)}$$

$$= \frac{1 \cdot q}{P(SS|IT)\,P(IT) + P(SS|FT)\,P(FT)}$$

$$= \frac{q}{1 \cdot q + 1/2 \cdot p}$$

$$= \frac{2q}{p+2q} = \frac{2q}{1+q}$$

Ans $\boxed{E}$

Q3

8 W 2 R   Assuming Identical Ws & R's

Total perms $\rightarrow \frac{10!}{8!\,2!} = \frac{\cancel{10}^{5} \times 9}{2} = 45$

A = (7 or more W Together) = (8 W Together) ∪ (7 W Together)

(8 W) → RR8W, R8WR, 8WRR → $\underline{\underline{3}}$

(7 W Together) → 7WRWR, R7WRW, WR7WR, WRR7W, 7WRRW, RWR7W → 6

$P(A) = \frac{3+6}{45} = \frac{9}{45} = \underline{\underline{\frac{1}{5}}}$

Ans $\underline{\boxed{E}}$

Q4

$$X \sim \mathbb{1}_{(0,1)}(x)$$

$$Y = \sqrt{-\alpha \ln(X)} \quad \& \quad y = \sqrt{-\alpha \ln(x)} \qquad \alpha > 0$$

$$y^2 = -\alpha \ln(x)$$

$$x = e^{-\frac{y^2}{\alpha}}$$

$$\left| \frac{\partial x(y)}{\partial y} \right| = \left| \frac{-2y}{\alpha} e^{-y^2/\alpha} \right|$$

$$= \frac{2y}{\alpha} e^{-y^2/\alpha} \quad \text{as } y \geq 0$$

$$f_Y(y) = f_X\left(e^{-y^2/\alpha}\right) \cdot \frac{2y}{\alpha} e^{-y^2/\alpha} \, \mathbb{1}_{[0,\infty)}(y)$$

$$= \frac{2y}{\alpha} e^{-y^2/\alpha} \, \mathbb{1}_{[0,1)} e^{-y^2/\alpha}$$

$$= \frac{2y}{\alpha} e^{-y^2/\alpha} \, \mathbb{1}_{[0,\infty)}(y)$$

Ans $\boxed{D}$

Q5

$A \cap B = A \cap C = \phi \Rightarrow (A \cap C) \cap B = \phi \cap B = \phi$

$B \cap C \neq \phi$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(\cancel{A \cap B}) - P(\cancel{A \cap C}) - P(B \cap C) + P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(B \cap C) + P(\cancel{A \cap B \cap C})$

$= P(A) + P(B) + P(C) - P(B \cap C)$

Ans $\boxed{E}$

Q6

$X \sim$ Poisson $\sigma_X = 4 \Rightarrow \sigma_X^2 = 16 = \lambda$

$$p_K(n) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, \ldots$$

$$P(\{X=1\} \mid \{X \geq 1\}) = \frac{P(\{X=1\})}{1 - P(\{X<1\})}$$

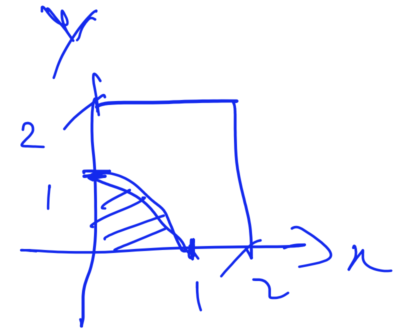
$$= \frac{P(\{X=1\})}{1 - P(\{X=0\})}$$

$$= \frac{\frac{\lambda}{1!} e^{-\lambda}}{1 - \frac{\lambda^0}{1} e^{-\lambda}}$$

$$= \frac{16 e^{-\lambda}}{1 - e^{-\lambda}} = \frac{16}{e^{\lambda} - 1} = \frac{16}{e^{16} - 1}$$

Ans $\boxed{E}$

Q7



$X \perp\!\!\!\perp Y$ iid $\sim \frac{1}{2}\mathbb{1}_{[0,2]}(\cdot)$

$$P(\{X^2+Y^2 \le 1\}) = \iint f_{XY}(x,y)\,dx\,dy$$

$$= \iint \frac{1}{4}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^1 \frac{1}{4}\, r\,dr\,d\theta = \frac{\pi}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}$$

$$= \frac{\pi}{16}$$

Ans $\boxed{D}$

Q8

$$f_{XY}(x,y) = \begin{cases} x+y, & 0<x<1 \text{ and } 0<y<1 \\ 0, & \text{e.w.} \end{cases}$$

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1$$

$$f_X(x) = \left(x + 1/2\right) \mathbb{1}_{[0,1]}(x)$$

$$\hat{Y}_{MMS}(1/3) = E_Y\left[Y \mid \{X = 1/3\}\right]$$

$$f_Y(y \mid x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{(x+y)}{(x+1/2)}$$

$$\hat{Y}_{mms}(1/3) = \int_0^1 y \cdot \frac{(x+y)}{(x+1/2)}\,dy \Big|_{x=1/3}$$

$$= \int_0^1 y \,\frac{(y+1/3)}{5/6}\,dy$$

$$= \frac{\frac{y^3}{3} + \frac{1}{6}y^2\Big|_0^1}{5/6}$$

$$= \frac{1/3 + 1/6}{5/6} = \frac{3/\cancel{6}}{5/\cancel{6}}$$

$$\hat{Y}_{mms}(1/3) = 3/5$$

Ans $\boxed{E}$

Q9

$X \perp\!\!\!\perp Y \sim \text{iid}$ & $\rho_{XY} = 0$

& $E[X] = E[Y] = \mu$ $\quad E(XY) = E[X]\,E[Y] = \mu^2$

$E[(X-\mu)^2] = E[(Y-\mu)^2] = \sigma^2$

$Z = aX + Y \quad , a > 0$

$\rho_{XZ} = 1/3$

$E[Z] = aE[X] + E[Y] = (a+1)\mu$

$E[Z^2] = a^2 E[X^2] + E[Y^2] + 2aE[XY] = a^2(\mu^2+\sigma^2) + (\mu^2+\sigma^2) + 2a.\mu^2$

$\sigma_Z^2 = E[Z^2] - (E(Z))^2 = a^2\sigma^2 + \sigma^2 = (a^2+1)\sigma^2$

$\rho_{XZ} = \dfrac{E[XZ] - E[X]\,E[Z]}{\sigma_X \sigma_Z} = 1/3$

$\Rightarrow \dfrac{E[X(aX+Y)] - \mu\cdot(a+1)\mu}{\sigma_X \sigma_Z} = 1/3$

$\Rightarrow \dfrac{a(\sigma^2 + \cancel{\mu^2}) + \cancel{\mu^2} - \cancel{a\mu^2} - \cancel{\mu^2}}{\sigma_X \sigma_Z} = 1/3$

$\Rightarrow \dfrac{a\cancel{\sigma^2}}{\cancel{\sigma^2}\sqrt{a^2+1}} = 1/3$

$\Rightarrow \dfrac{a^2}{a^2+1} = \dfrac{1}{9} \Rightarrow 9a^2 = a^2 + 1$

$8a^2 = 1$

$a = \dfrac{1}{2\sqrt{2}}$

Ans $\boxed{A}$

Q10

$X(t) \sim$ W.S.S. with $R_X(\tau)$

$Y(t) = -X(t-t_0)$ $\quad$ t_0 is a constant

$$R_{YY}(t_1,t_2) = E[Y(t_1)Y(t_2)]$$

$$= E[-X(t_1-t_0)\,{-X}(t_2-t_0)]$$

$$= R_X(t_1 \cancel{-t_0} - t_2 \cancel{+t_0})$$

$$= R_X(t_1-t_2)$$

Ans $\boxed{A}$

Q11

Saturday, December 12, 2020 8:51 AM

ECE600Final Exam

11. Problem 11 is made up of 8 True/False questions, worth 2 points each. Label your answer sheet for problem 11 with the letters A through H for each of the statements below, and write T (true) or F (false) next to each letter, depending on whether the corresponding statements are true or false, respectively.

A. F If two events A and B are statistically independent, then they are disjoint.

$P(A \cap B) = P(A)\,P(B) \neq 0 \quad \forall A, B \in \mathcal{F}$

B. F If the autocorrelation function $R_X(t_1, t_2)$ of a random process $\mathbf{X}(t)$ can be written as a function of the time difference $t_1 - t_2$, then $\mathbf{X}(t)$ is wide-sense stationary.

$E[X(t)] =$ constant too!

C. T If two random variables $\mathbf{X}$ and $\mathbf{Y}$ are statistically independent, then they are uncorrelated.

$r_{XY} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = 0$ ↳ uncorr. $E(XY) = E[X]E[Y]$ ← stat Ind

D. T Let A, B, and C be three events defined on a random experiment. If $P(A \cap B \cap C) = P(A)P(B)P(C)$, then A, B, and C are statistically independent. By defn

E. T If two Gaussian random variables $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated, then they are statistically independent. only for Gaussian

F. F If a random process is wide-sense stationary, then it is strict-sense stationary.

G. T Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be jointly distributed continuous random variables. Then the random variables $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are statistically independent if and only if $f_{\mathbf{XYZ}}(x, y, z) = f_{\mathbf{X}}(x) f_{\mathbf{Y}}(y) f_{\mathbf{Z}}(z)$.

H. T Let A and B be two events defined on a random experiment. Then if A and B are statistically independent, then $P(B|A) = P(B)$.

$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)\cancel{P(A)}}{\cancel{P(A)}} = P(B)$

Q12

Given: $P(\{\mathbb{N}=n\} | \{\mathbb{R}=r\}) = \dfrac{r^n e^{-r}}{n!}$; $n = 0, 1, 2 \ldots$

$$f_{\mathbb{R}}(r) = \frac{1}{\mu} e^{-r/\mu} \, 1_{[0,\infty)}(r) \qquad \mu > 0 \;\&\; r > 0$$

(a) Using Total Probability we have

$$P(\{\mathbb{N}=n\}) = \int_{-\infty}^{\infty} P(\{\mathbb{N}=n\} | \{\mathbb{R}=r\}) \cdot f_{\mathbb{R}}(r)\, dr$$

$$= \int_0^{\infty} \frac{r^n e^{-r}}{n!} \cdot \frac{1}{\mu} e^{-r/\mu}\, dr$$

$$= \frac{1}{n!\mu} \int_0^{\infty} r^n e^{-r(1+1/\mu)}\, dr$$

Let $t = r(1+1/\mu)$ & $dt = (1+1/\mu)\, dr$

$$\& \; P(\{\mathbb{N}=n\}) = \frac{1}{n!\mu} \int_0^{\infty} \frac{t^n}{(1+1/\mu)^n} e^{-t} \frac{dt}{(1+1/\mu)}$$

$$= \frac{1}{(1+1/\mu)^{n+1} n! \mu} \cdot \underbrace{\int_0^{\infty} t^n e^{-t}\, dt}_{\Gamma(n+1) = n!}$$

$$= \frac{\cancel{n!}}{\cancel{n!} \frac{(\mu+1)^{n+1}}{\mu^{n+1}} \mu} = \frac{\mu^n}{(\mu+1)^{n+1}}$$

$$\boxed{P(\{\mathbb{N}=n\}) = \frac{\mu^n}{(\mu+1)^{n+1}}}$$

(b) $\mu_{\mathbb{N}} = E[\mathbb{N}] = \sum_{K=0}^{\infty} K \cdot P(\{\mathbb{N}=K\})$

$$= \sum_{K=0}^{\infty} K \cdot \frac{\mu^K}{(\mu+1)^{K+1}}$$

$$= \frac{1}{(\mu+1)} \sum_{K=0}^{\infty} K \left(\frac{\mu}{\mu+1}\right)^K$$

Let $a = \dfrac{\mu}{\mu+1}$ as $\mu > 0$ & $0 < a < 1$

$$\& \; \mu_{\mathbb{N}} = \frac{1}{\mu+1} \left[\sum_{K=0}^{\infty} K a^K \right] \times \frac{(1-a)}{(1-a)}$$

$$= \frac{1}{(\mu+1)(1-a)} \cdot \Big[\underbrace{\sum_{K=0}^{\infty} K (1-a) a^K}_{E[\mathbb{A}]} \Big]$$

$\mathbb{A}$ is Geometric R.V.

Using Mean and Expectation of Geometric R.V. we get

$$\sigma_{\mathbb{A}}^2 = \frac{1-a}{a^2}, \quad \mu_{\mathbb{A}} = \frac{1-a}{a}$$

$$\mu_{\mathbb{N}} = \frac{1}{(\mu+1)(1-a)} \cdot \left[\mu_{\mathbb{A}} \right]$$

using $a = \dfrac{\mu}{\mu+1}$ & $1 - a = \dfrac{1}{\mu+1}$

$$\& \; \sigma_{\mathbb{A}}^2 = \frac{\frac{1}{(\mu+1)}}{\left(\frac{\mu}{\mu+1}\right)^2} = \frac{\mu+1}{\mu^2}, \quad \mu_{\mathbb{A}} = \frac{1}{\mu}$$

$$\mu_{\mathbb{N}} = \frac{1}{\cancel{(\mu+1)} \cdot \frac{1}{\cancel{(\mu+1)}}} \left[\frac{1}{\mu} \right]$$

$$\boxed{\mu_{\mathbb{N}} = \frac{1}{\mu}}$$

(c)

$$\sigma_{\mathbb{N}}^2 = E[N^2] - (E[\mathbb{N}])^2$$

$$E[\mathbb{N}^2] = \sum_{K=0}^{\infty} K^2 P(\{\mathbb{N}=K\})$$

$$= \sum_{K=0}^{\infty} K^2 \frac{\mu^K}{(\mu+1)^{K+1}}$$

$$= \frac{1}{(\mu+1)} \sum_{K=0}^{\infty} K^2 \left(\frac{\mu}{\mu+1}\right)^K$$

$$= \frac{1}{(\mu+1)(1-a)} \Big[\underbrace{\sum_{K=0}^{\infty} K^2 (1-a) a^K}_{E[\mathbb{A}^2]} \Big]$$

$$= \frac{1}{(\mu+1)(1-a)} \left\{ \sigma_{\mathbb{A}}^2 + \mu_{\mathbb{A}}^2 \right]$$

$$= \frac{1}{\cancel{(\mu+1)} \frac{1}{\cancel{(\mu+1)}}} \left[\frac{\mu+1}{\mu^2} + \frac{1}{\mu^2} \right]$$

$$E[\mathbb{N}^2] = \frac{\mu+2}{\mu^2}$$

$$\& \; \sigma_{\mathbb{N}}^2 = \frac{\mu+2}{\mu^2} - \frac{1}{\mu^2}$$

$$\boxed{\sigma_{\mathbb{N}}^2 = \frac{\mu+1}{\mu^2}}$$

Q13

Given: $X(t) \sim$ W.S.S. Gaussian white Noise

$$\mu_X = E[X(t)] = 0$$

$$R_{XX}(\tau) = \delta(\tau)$$

$$h(t) = \alpha e^{-\alpha t} \mathbb{1}_{[0,\infty)}(t) \qquad \alpha > 0$$

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

(a) $\mu_Y = \cancel{\mu_X}^{\,0} \displaystyle\int_{-\infty}^{\infty} h(t-\alpha)\, d\alpha = 0$

(b)
$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$
$$= \int_{-\infty}^{\infty} \delta(\tau)\, e^{-i\omega\tau}\, d\tau = 1$$

$$\boxed{S_{XX}(\omega) = 1}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$$
$$= \int_{-\infty}^{\infty} \alpha e^{-\alpha t} \mathbb{1}_{[0,\infty)}(t)\, e^{-i\omega t}\, dt$$
$$= \int_{0}^{\infty} \alpha e^{-(\alpha+i\omega)t}\, dt$$
$$= \frac{\alpha}{-(\alpha+i\omega)} e^{-(\alpha+i\omega)t}\Big|_0^{\infty}$$

$$H(\omega) = \frac{\alpha}{\alpha+i\omega} \quad \& \quad |H(\omega)|^2 = \frac{\alpha^2}{\alpha^2+\omega^2}$$

$$S_{YY}(\omega) = S_{XX}(\omega)\, |H(\omega)|^2$$
$$S_{YY}(\omega) = 1 \cdot \frac{\alpha^2}{\alpha^2+\omega^2}$$

$$R_Y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega) \cdot e^{i\omega\tau}\, d\omega$$

$$= \frac{\alpha}{2\times 2\pi}\int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2+\omega^2}\, e^{i\omega\tau}\, d\omega$$

$$\boxed{R_Y(\tau) = \frac{\alpha}{4\pi} \cdot e^{-\alpha|\tau|}}$$

(c)
$$f_Y(y_1, y_2; t_1, t_2) = f_X(x_1(y_1,y_2), x_2(y_1,y_2); t_1, t_2) \left| \frac{\partial(x_1(y_1,y_2), x_2(y_1,y_2))}{\partial(y_1,y_2)} \right|$$

$$R_Y(\tau) = E[Y(\underbrace{t+\tau}_{t_1})\, Y(\underbrace{t}_{t_2})] = \frac{\alpha}{4\pi} e^{-\alpha|\tau|}$$

$$\& \; R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1-t_2-\beta+\gamma)\, h(\beta)h(\gamma)\, d\beta\, d\gamma = \iint y_1 y_2\, f_Y(y_1,y_2;t_1,t_2)\, dy_1\, dy_2 = E[Y(t_1)\,Y(t_2)]$$

$$\& \; f_Y(y_1,y_2;t_1,t_2) = \frac{1}{y_1 y_2} \cdot \delta(t_1-t_2-\alpha+\beta) \cdot \alpha e^{-\alpha\beta} \mathbb{1}_{[0,\infty)}(\beta) \cdot \alpha e^{-\alpha\gamma} \mathbb{1}_{[0,\infty)}(\gamma)$$

$$= \frac{\alpha^2 e^{-\alpha(\beta+\gamma)}}{y_1 y_2}\, \delta(t_1-t_2-\alpha+\beta), \quad \beta, \gamma \geq 0$$

(d) $Y(t)$ is WSS. as $X(t)$ is WSS.

& $Y(t)$ is not SSS.