Exam 3: cover sheet

ECE600 Random Variables and Waveforms
Fall 2020

Midterm Exam #3
Session 25
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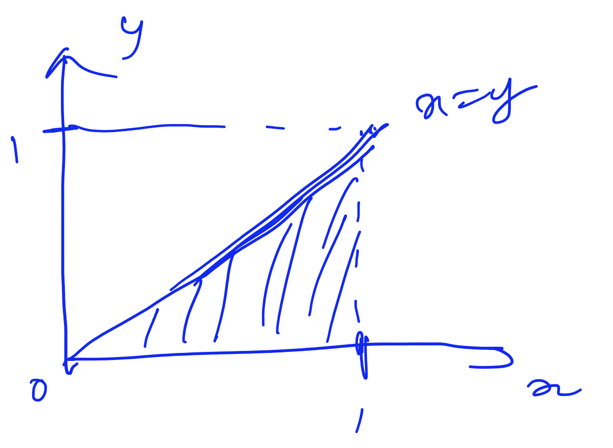
Q1

Given: $f_{XY}(x,y) = Kx^2y\, \mathbb{1}_{[0,y]}^{(x)}\, \mathbb{1}_{[0,1]}^{(y)}$



(a) Finding K, For valid j-dist^n pdf we have:

(i) $Kx^2y \geq 0 \quad, x>0,\ y>0$

$\Rightarrow K \geq 0$

(ii) $\int_0^1\int_0^y f_{XY}(x,y)\,dx\,dy = 1$

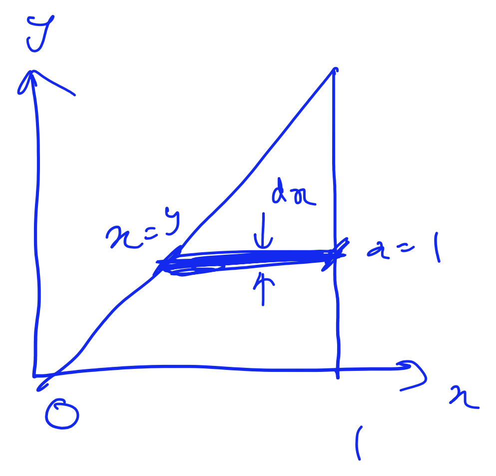
$\Rightarrow \int_0^1\int_0^y Kx^2y\,dx\,dy = \int_0^1 y\left[\int_0^y Kx^2\,dx\right]dy = 1$

$\Rightarrow \int_0^1 y\, \frac{Kx^3}{3}\Big|_0^y dy = 1$

$\Rightarrow \frac{K}{3}\int_0^1 y^4\,dy = \frac{K}{3\times5}\, y^5\Big|_0^1 = 1$

$\Rightarrow \boxed{K = 15}$

(b) $f_Y(y) = \int_y^1 f_{XY}(x,y)\,dx$



$= \int_y^1 15x^2y\,dx = 15y\,\frac{x^3}{3}\Big|_y^1$

$f_Y(y) = 5y(1-y^3)\,\mathbb{1}_{[0,1]}^{(y)}$

(c) $f_X(x\,|\,\{Y=y\}) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

$= \dfrac{15x^2y}{5y(1-y^3)}\,\mathbb{1}_{[0,y]}^{(x)} = \dfrac{3x^2}{(1-y^3)}\cdot\mathbb{1}_{[0,y]}^{(x)}$

(d) $\hat{X}_{MMS}(y) = E_X[X\,|\,\{Y=y\}]$

$= \int_{-\infty}^{\infty} x\cdot f_X(x\,|\,\{Y=y\})\,dx$

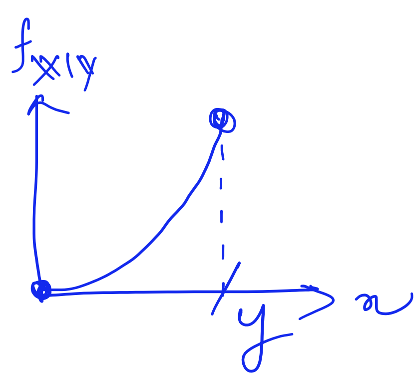
$= \int_0^y x\cdot\frac{3x^2}{(1-y^3)}\,dx$

$= \frac{3}{(1-y^3)}\,\frac{x^4}{4}\Big|_0^y$

$\Rightarrow \boxed{\hat{X}_{MMS}(y) = \frac{3}{4}\frac{y^4}{(1-y^3)}}$

(e) $\hat{X}_{MAP}(y) = \underset{X}{\operatorname{argmax}}\{f_X(\cancel{x}\,|\,\{Y=y\})\}$

$= \underset{x}{\operatorname{argmax}}\left\{\frac{3x^2}{(1-y^3)}\,\mathbb{1}_{[0,y]}^{(x)}\right\}$



The function $\frac{3x^2}{(1-y^3)}$ (with y as constant) is max at end points

$\Rightarrow \boxed{\hat{X}_{MAP}(y) = y}$

Q2

given: $\Phi_{XY}(\omega_1, \omega_2) = \frac{1}{(1-i\omega_1)} \cdot \frac{1}{(1-2i\omega_2)}$

(a) $E[X] = \frac{\partial}{\partial(i\omega_1)} \Phi_{XY}(\omega_1,\omega_2) \Big|_{\omega_1=0,\, \omega_2=0}$

$= \frac{1}{(1-2i\omega_2)} \cdot \frac{-1 \times -1}{(1-i\omega_1)^2} \Big|_{\omega_1=0,\, \omega_2=0}$

$= \frac{1}{(1-0)} \times \frac{1}{(1-0)^2} = 1$

$\Rightarrow E[X] = 1$

(b) $E[Y] = \frac{\partial}{\partial(i\omega_2)} \Phi_{XY}(\omega_1,\omega_2) \Big|_{\omega_1=0\, \omega_2=0}$

$= \frac{1}{(1-i\omega_1)} \cdot \frac{-1 \times -2}{(1-2i\omega_2)^2} \Big|_{\omega_1=0,\, \omega_2=0}$

$E[Y] = 2$

(c) $E[XY] = \frac{\partial^2}{\partial(i\omega_1)\,\partial(i\omega_2)} \Phi_{XY}(\omega_1,\omega_2) \Big|_{\substack{\omega_1=0 \\ \omega_2=0}}$

$= \frac{-1 \times -1}{(1-i\omega_1)^2} \cdot \frac{-1 \times -2}{(1-2i\omega_2)^2} \Big|_{\substack{\omega_1=0 \\ \omega_2=0}}$

$E[XY] = 2$

(d) $E[X^j Y^k] = \frac{\partial^{j+k}}{\partial^k(i\omega_2)\,\partial^j(i\omega_1)} \Phi_{XY}(\omega_1,\omega_2) \Big|_{\substack{\omega_1=0 \\ \omega_2=0}}$

$= \frac{\partial^k}{\partial^k(i\omega_2)} \left[\frac{1}{(1-2i\omega_2)} \frac{\partial^j}{\partial^j(i\omega_1)} \frac{1}{(1-i\omega_1)} \right] \Big|_{\substack{\omega_1=0 \\ \omega_2=0}}$

$= \frac{2^k k!}{(1-2i\omega_2)^{k+1}} \cdot \frac{j!}{(1-i\omega_1)^{j+1}} \Big|_{\substack{\omega_1=0 \\ \omega_2=0}}$

$E[X^j Y^k] = 2^k k!\, j!$

(e) $\rho_{XY} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{2 - 2\times 1}{\sigma_X \sigma_Y} = 0$

$\boxed{\rho_{XY} = 0}$

Q3

given: $\mathbb{N}$ is a poisson variable with mean λ

$$\therefore p_{\mathbb{N}}(n) = \frac{e^{-\lambda}\lambda^n}{n!}, \quad n = 0, 1, \dots \infty$$

Probability of car exceeding speed limit : p (Independent of $\mathbb{N}$)

$\mathbb{M}$ ~ no. of cars exceeding speed limit

a) $$\Phi_{\mathbb{N}}(\omega) = E\left[e^{i\omega \mathbb{N}}\right] = \sum_{K=0}^{\infty} e^{i\omega K} e^{-\lambda}\frac{\lambda^K}{K!}$$

$$= e^{-\lambda}\sum_{K=0}^{\infty}\frac{(e^{i\omega}\lambda)^K}{K!}$$

$$= e^{-\lambda} e^{\lambda e^{i\omega}}$$

$$\Phi_{\mathbb{N}}(\omega) = e^{-\lambda(1-e^{i\omega})}$$

b) $$\Phi_{\mathbb{M}}(\omega) = E\left[e^{i\omega \mathbb{M}}\right]$$

$$= E_{\mathbb{N}}\left[E_{\mathbb{M}}\left[e^{i\omega \mathbb{M}} \mid \mathbb{N}\right]\right]$$

$$= \sum_{n=0}^{\infty} p_{\mathbb{N}}(n)\, E_{\mathbb{M}}\left[e^{i\omega \mathbb{M}} \mid \{\mathbb{N}=n\}\right]$$

We need to find $f_{\mathbb{M}|\mathbb{N}}(m \mid \{\mathbb{N}=n\})$

which translates to, given there are n cars on the road we need to find the probability that m cars out of n are speeding

Clearly the conditional distribution is a Binomial Distribution with the success probability p

$$\therefore f_{\mathbb{M}|\mathbb{N}}(m \mid \{\mathbb{N}=n\}) = \binom{n}{m} p^m (1-p)^{n-m}$$

$$\therefore E_{\mathbb{M}}\left[e^{i\omega \mathbb{M}} \mid \{\mathbb{N}=n\}\right] = \sum_{m=0}^{n} e^{i\omega m}\binom{n}{m} p^m (1-p)^{n-m}$$

$$= \sum_{m=0}^{n}\binom{n}{m}(pe^{i\omega})^m(1-p)^{n-m}$$

$$= \left(pe^{i\omega} + 1 - p\right)^n$$

$$\Rightarrow \Phi_{\mathbb{M}}(\omega) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\cdot\left(pe^{i\omega}+1-p\right)^n$$

$$= e^{-\lambda}\sum_{n=0}^{\infty}\frac{\left(\lambda(pe^{i\omega}+1-p)\right)^n}{n!}$$

$$= \exp\left(-\lambda + \lambda(pe^{i\omega}+1-p)\right)$$

$$\Phi_{\mathbb{M}}(\omega) = \exp\left(\lambda p(e^{i\omega}-1)\right)$$

$$= \exp\left(-\lambda p(1-e^{i\omega})\right)$$

c) On Comparing result from (a) & (b) we get:

$\mathbb{M}$ is a poisson dist^n with mean λp

$$p_{\mathbb{M}}(m) = e^{-\lambda p}\frac{(\lambda p)^m}{m!}, \quad m = 0, 1, \dots$$

(d) $$\mu_{\mathbb{M}} = E[\mathbb{M}] = \frac{\partial}{\partial(i\omega)}\Phi_{\mathbb{M}}(\omega)\Big|_{\omega=0}$$

$$= +\lambda p e^{i\omega}\cdot\exp\left(-\lambda p(1-e^{i\omega})\right)\Big|_{\omega=0}$$

$$\mu_{\mathbb{M}} = \lambda p$$

$$\sigma_{\mathbb{M}}^2 = E[\mathbb{M}^2] - (E[\mathbb{M}])^2$$

$$E[\mathbb{M}^2] = \frac{\partial^2}{\partial(i\omega)^2}\Phi_{\mathbb{M}}(\omega)\Big|_{\omega=0}$$

$$= \frac{\partial}{\partial(i\omega)}\left(\lambda p e^{i\omega}\cdot\exp(-\lambda p(1-e^{i\omega}))\right)\Big|_{\omega=0}$$

$$= \lambda p e^{i\omega}\exp\left(-\lambda p(1-e^{i\omega})\right) + \lambda p e^{i\omega}\cdot(\lambda p)\cdot\exp\left(-\lambda p(1-e^{i\omega})\right)\Big|_{\omega=0}$$

$$= \lambda p + (\lambda p)^2$$

$$\therefore \sigma_{\mathbb{M}}^2 = \lambda p + \cancel{(\lambda p)^2} - \cancel{(\lambda p)^2}$$

$$\boxed{\sigma_{\mathbb{M}}^2 = \lambda p}$$

Q4

given: $X_1, X_2, X_3, \dots, X_n$ is a sequence of i.i.d. exponential R.V. with mean μ

Another sequence $\{Y_n\}$ defined as:

$$Y_n = \min\{X_1, X_2, \dots, X_n\} \quad n = 1, 2, 3 \dots$$

(a) As the sequence $\{X_n\}$ is i.i.d. exponential R.V. with mean μ

$\therefore$ Their joint distⁿ is given by:

$$f_{X_1, X_2, \dots X_n}(x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \frac{1}{\mu} e^{-\frac{x_i}{\mu}} \quad x_i \geq 0,\ \mu \geq 0$$

$$= \frac{1}{\mu^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i)}{\mu}\right)$$

Also since $X_i \perp\!\!\!\perp X_j \ \forall i \neq j$ we can compute any marginal as:

$$f_{X_1, X_2, X_j}(x_1, x_2, \dots x_j) = \frac{1}{\mu^j} \exp\left(-\frac{\sum_{i=1}^{j}(x_i)}{\mu}\right) \quad \text{— Eq } 1$$

For $\quad n = 1$

$Y_1 = \min\{X_1\} = X_1$ & Y_1 is exponential distⁿ

& $Y_1 \sim \frac{1}{\mu} e^{-\frac{y_1}{\mu}} \quad y \geq 0, \mu \geq 0$

For any $n \geq 1$

$$Y_n = \min\{X_1, X_2 \dots X_n\}$$

$$F_{Y_n}(y) = P\left(\{Y_n \leq y\} \cap \{\bigcup_{i=1}^{n} X_i = \min\{X_1, X_2 \dots X_n\}\}\right)$$

$$= \sum_{i=1}^{n} P\left(\{X_i \leq y\} \cap \{X_i = \min\{X_1, X_2, \dots, X_n\}\}\right)$$

$$= 1 - P(\{Y_n > y\})$$

$$= 1 - P(\{X_i > y, \forall i\})$$

$$= 1 - \prod_{i=1}^{n} P(\{X_i > y\})$$

$$= 1 - \prod_{i=1}^{n}\left(1 - e^{-y/\mu}\right) = 1 - \left(1 - e^{-y/\mu}\right)^n$$

& $$f_{Y_n}(y) = \frac{\partial}{\partial y} F_{Y_n}(y)$$

$$= n\left(1 - e^{-y/\mu}\right)^{n-1} \times \frac{1}{\mu}$$

$$= \frac{n}{\mu}\left(1 - e^{-y/\mu}\right)^{n-1} \mathbb{1}_{\{0,\infty\}}(y)$$

(b) $P(\{|Y_n - Y| > \epsilon\}) \to 0$ as $n \to \infty$?

$$E[Y_n^2] = \int_0^\infty y^2 \frac{n}{\mu}\left(1 - e^{-y/\mu}\right)^{n-1} dy$$

(c) $E[|Y_n - Y|^2] \to 0$ as $n \to \infty$?