MXB324
SHORT LECTURE 1

MONOTONICITY

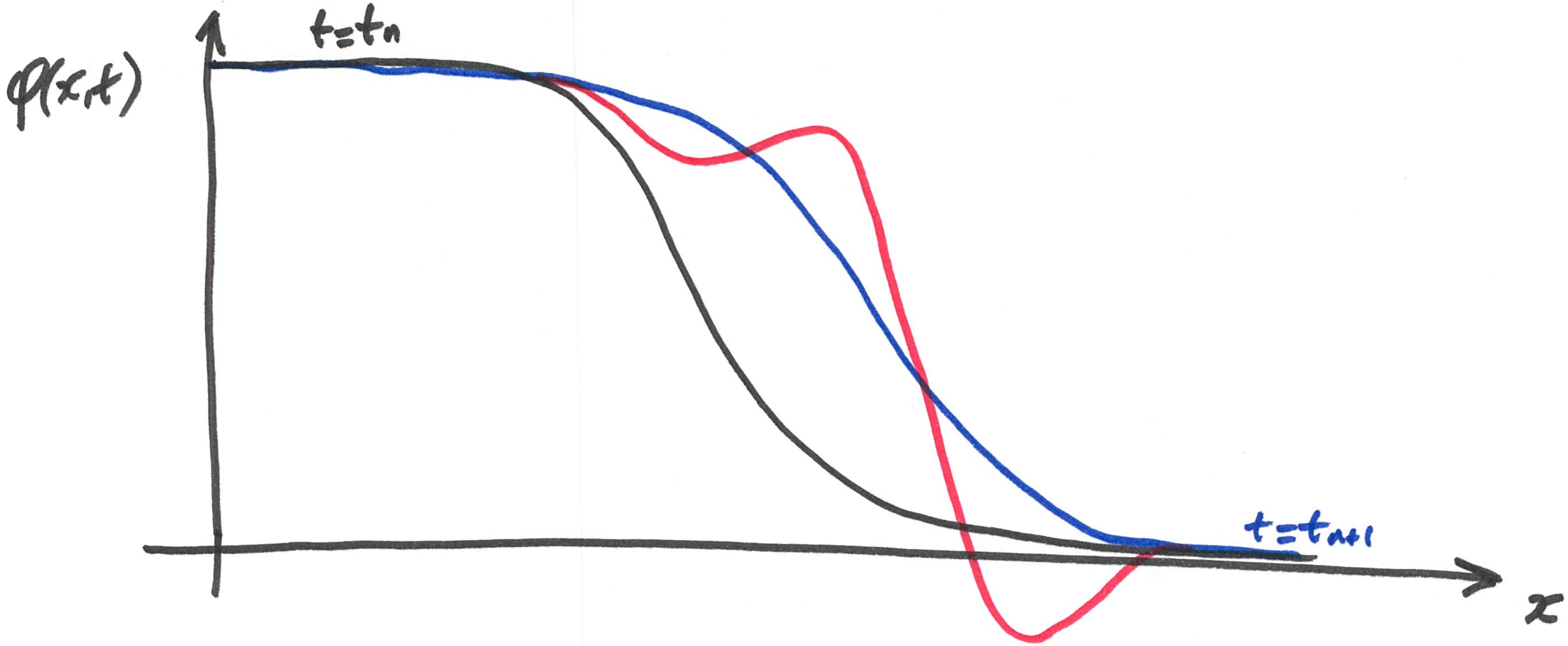
Advection-diffusion Equation

We now consider the linear *advection-diffusion equation with source term*:

$$\int_{x_w}^{x_e} \frac{\partial \varphi}{\partial t}\,dx + \int_{x_w}^{x_e} \frac{\partial}{\partial x}\underbrace{\left(u\varphi - D\frac{\partial \varphi}{\partial x}\right)}_{J} dx = \cancel{S} \quad S=0 \qquad 0 < x < L, \quad t > 0 \qquad (14)$$

subject to the same initial and boundary conditions as before.

$$\Delta x_P \frac{d\varphi_P}{dt} + (J_e - J_w) = 0, \qquad J = \underbrace{u\varphi}_{\text{advective component}}^{+ve} - \underbrace{\boxed{D\frac{\partial \varphi}{\partial x}}}_{\text{diffusive component}}$$



$\varphi(x,t)$
$t=t_n$
$t=t_{n+1}$
x

Illustration of Cell–Centred & Vertex–Centred Schemes in 1D

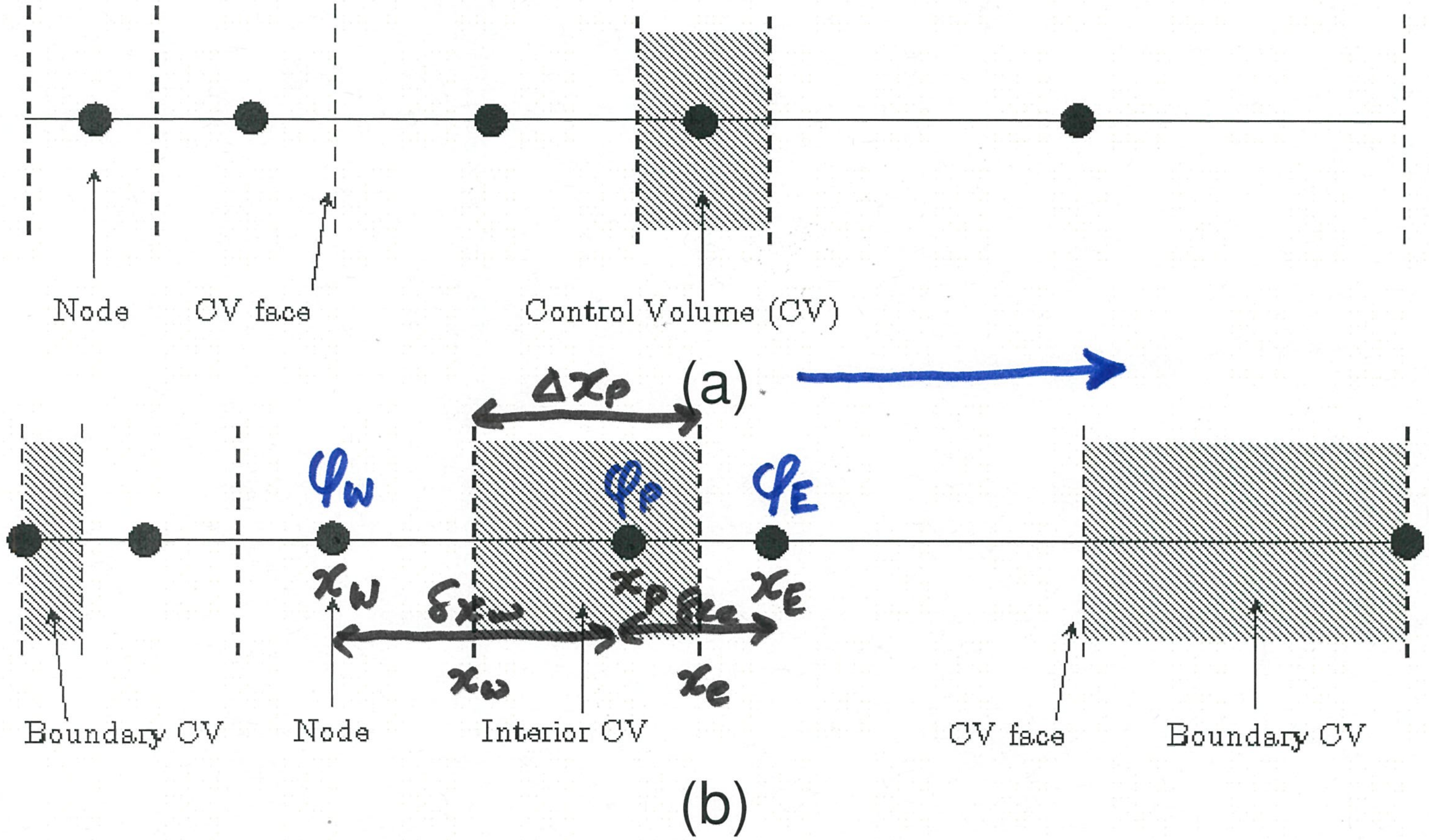


Figure: Constructing control volumes in one dimension: (a) cell-centred; (b) vertex-centred.

Averaging

We require approximations for J_w and J_e. The approximation for the diffusive component of the flux has been described in the previous section. For the advective component, we can approximate φ_w by $(\varphi_W + \varphi_P)/2$ and φ_e by $(\varphi_P + \varphi_E)/2$, to give

$$J_w = \left(u\varphi - D\frac{\partial \varphi}{\partial x}\right)_w \approx u\left(\frac{\varphi_P + \varphi_W}{2}\right) - D\left(\frac{\varphi_P - \varphi_W}{\delta x_w}\right)$$

and

$$J_e = \left(u\varphi - D\frac{\partial \varphi}{\partial x}\right)_e \approx u\left(\frac{\varphi_P + \varphi_E}{2}\right) - D\left(\frac{\varphi_E - \varphi_P}{\delta x_e}\right).$$

Upstream Weighting (continued)

This gives rise to a discretisation known as **upstream weighting** or **upwinding**, where *only* information from upstream (or "upwind") is used to approximate advection terms.

Thus, we use the approximations

$$\varphi_w \approx \begin{cases} \varphi_W, & u > 0 \\ \varphi_P, & u < 0 \end{cases}$$

and

$$\varphi_e \approx \begin{cases} \varphi_P, & u > 0 \\ \varphi_E, & u < 0 \end{cases}.$$

$$\varphi_w = \varphi_W + \frac{\sigma}{2}(\varphi_P - \varphi_W) \quad ; \quad \sigma = 0 \rightarrow \text{upwinding}$$

$$\varphi_e = \varphi_P + \frac{\sigma}{2}(\varphi_E - \varphi_P) \quad ; \quad \sigma = 1 \rightarrow \text{averaging}$$

$$\Delta x_p \frac{d\varphi_P}{dt} = J_w - J_e \quad ;$$

$$J_w = u\left[\varphi_W + \frac{\sigma}{2}(\varphi_P - \varphi_W)\right] - D\frac{(\varphi_P - \varphi_W)}{\delta x_w}$$

$$J_e = u\left[\varphi_P + \frac{\sigma}{2}(\varphi_E - \varphi_P)\right] - D\frac{(\varphi_E - \varphi_P)}{\delta x_e}$$

Hence, we have that

$$\frac{d\varphi_P}{dt} = \frac{1}{\Delta x_p}\left\{ u\left[\varphi_W + \frac{\sigma}{2}(\varphi_P - \varphi_W)\right] - D\frac{(\varphi_P - \varphi_W)}{\delta x_w} - u\left[\varphi_P + \frac{\sigma}{2}(\varphi_E - \varphi_P)\right] + D\frac{(\varphi_E - \varphi_P)}{\delta x_e} \right\}$$

$$\overset{\frac{\varphi_P^{n+1} - \varphi_P}{\delta t}}{\parallel}$$

$$= \frac{1}{\Delta x_p}\left\{ \left[u\left(1 - \frac{\sigma}{2}\right) + \frac{D}{\delta x_w}\right]\varphi_W^{n+1} + \left[-\frac{u\sigma}{2} + \frac{D}{\delta x_e}\right]\varphi_E^{n+1} - \left[u(1-\sigma) + \frac{D}{\delta x_w} + \frac{D}{\delta x_e}\right]\varphi_P^{n+1} \right\}$$

USING THE BACKWARD EULER METHOD

$$\left[1 + \frac{\delta t}{\Delta x_p}\left\{\overset{+ve.}{u(1-\sigma) + D\left(\frac{1}{\delta x_w} + \frac{1}{\delta x_e}\right)}\right\}\right]\varphi_p^{n+1} \qquad 0 \le \sigma \le 1$$

$$= \frac{\delta t}{\Delta x_p}\left\{\overset{+ve.}{u\left(1-\frac{\sigma}{2}\right) + \frac{D}{\delta x_w}}\right\}\varphi_W^{n+1} + \frac{\delta t}{\Delta x_p}\left\{\underbrace{-\frac{u\sigma}{2} + \frac{D}{\delta x_e}}_{-ve\,?}\right\}\varphi_E^{n+1} \quad \ldots (1)$$

$$+ \varphi_p^n .$$

Define $\varphi_p^{max} = \max\left\{\varphi_p^n, \varphi_W^{n+1}, \varphi_E^{n+1}\right\}$

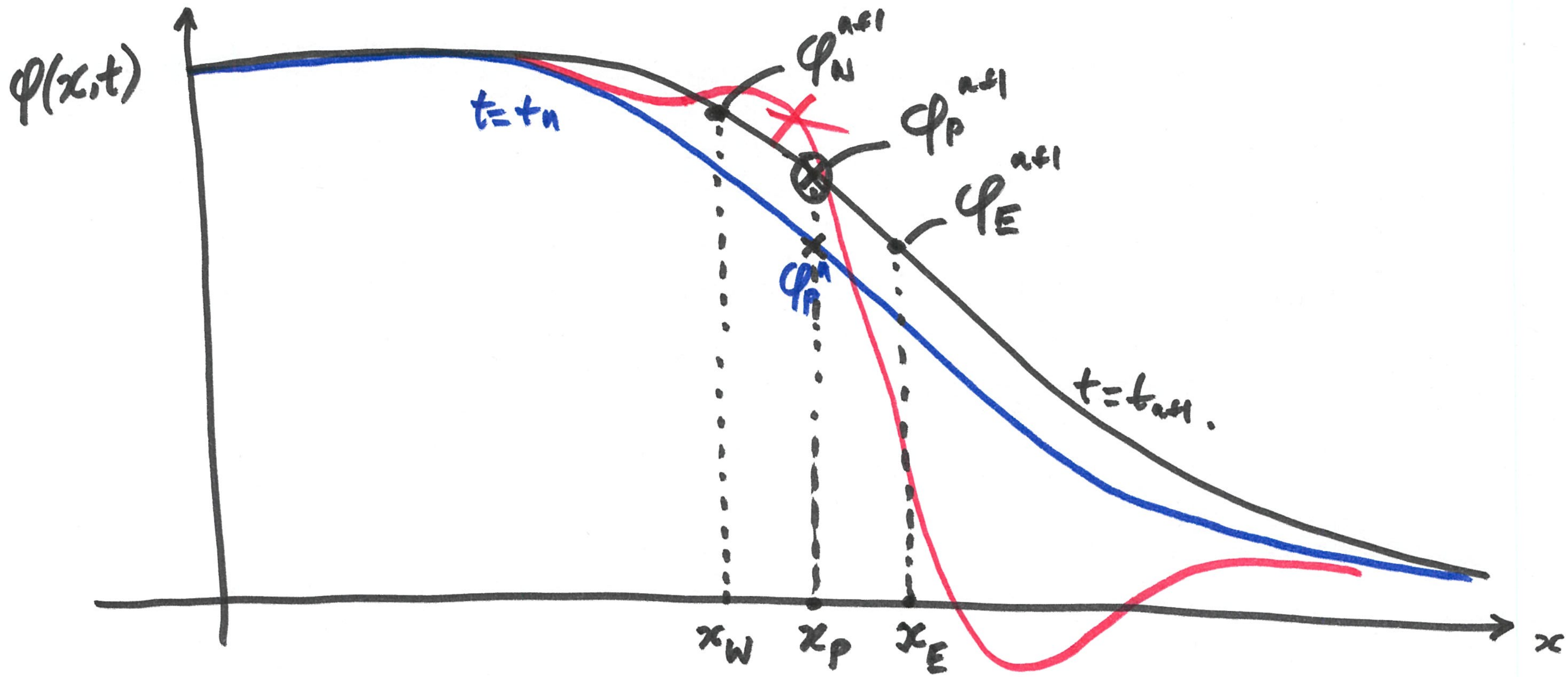
Suppose that all the coefficients in (1) are positive:

$$\left[1 + \frac{\delta t}{\Delta x_p}\{\ldots\}\right]\varphi_p^{n+1} \leqslant \frac{\delta t}{\Delta x_p}\{\ldots\}\varphi_p^{max} + \frac{\delta t}{\Delta x_p}\{\ldots\}\varphi_p^{max} + \varphi_p^{max}.$$

$$\left[1 + \cancel{\frac{\delta t}{\Delta x_p}\{\ldots\}}\right]\varphi_p^{n+1} \leqslant \left[1 + \cancel{\frac{\delta t}{\Delta x_p}\{\ldots\}}\right]\varphi_p^{max}.$$

$\therefore\ \varphi_p^{n+1} \le \varphi_p^{max}$.

Maximum principle.



$$\min\left\{\varphi_P^n, \varphi_W^{n+1}, \varphi_E^{n+1}\right\} \leqslant \varphi_P^{n+1} \leqslant \max\left\{\varphi_P^n, \varphi_W^{n+1}, \varphi_E^{n+1}\right\}$$
$$= \varphi_P^{max}$$

One can also invoke the so-called minimum principle:

Define $\varphi_P^{min} = \min\left\{\varphi_P^{n}, \varphi_W^{n+1}, \varphi_E^{n+1}\right\}$

$$\left[1 + \cancel{\frac{\delta t}{\Delta x_P}\left\{\cdots\right\}}\right]\varphi_P^{n+1} \geqslant \frac{\delta t}{\Delta x_P}\left\{\cdots\right\}\varphi_P^{min} + \frac{\delta t}{\Delta x_P}\left\{\cdots\right\}\varphi_P^{min} + \varphi_P^{min}$$

$$= \left[1 + \cancel{\frac{\delta t}{\Delta x_P}\left\{\cdots\right\}}\right]\varphi_P^{min}.$$

$$\therefore \quad \varphi_P^{n+1} \geqslant \varphi_P^{min}$$

Hence, $\varphi_P^{min} \leqslant \varphi_P^{n+1} \leqslant \varphi_P^{max}$.

$\boxed{\text{KEY}}$ We must have the coefficients of the FVE (for interior & boundary nodes) must remain positive.

MONOTONICITY CONSTRAINT:

We must have that

$$-\frac{u\sigma}{2} + \frac{D}{\delta x_e} \geq 0$$

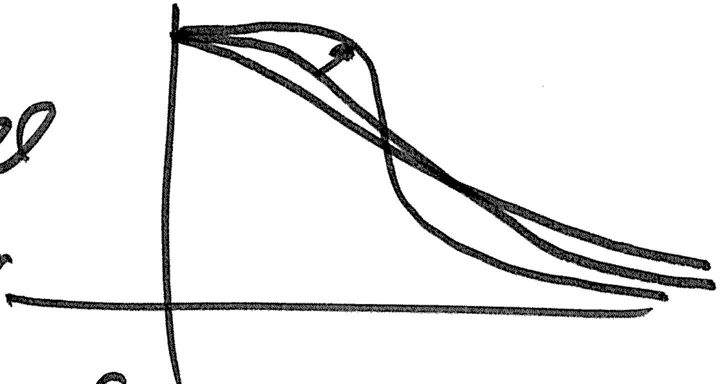
i.e., $$\frac{D}{\delta x_e} \geq \frac{u\sigma}{2}$$

if $\sigma = 0$ always true "upwinding" completely monotone FV discretisation.

what about $\sigma = 1$

$$\boxed{\frac{\delta x_e u}{D} \leq 2}$$

We call $\frac{\delta x_e u}{D} = P_e$ [the cell Peclet number



For a monotone discretisation we require $P_e \leq 2$