Introduction to FVM in two dimensions

In this section, to illustrate the underlying concepts of the FVM in higher dimensions, we consider the two-dimensional Poisson equation:

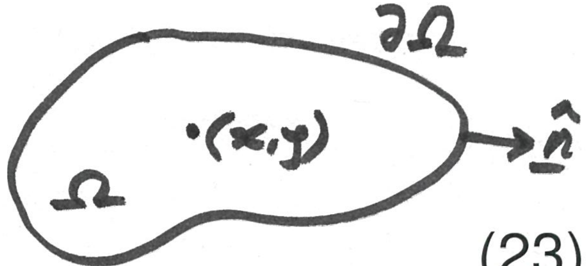
$\nabla \cdot \underline{J} = f$

$\underline{\underline{D}} = \begin{pmatrix} D_{xx} & 0 \\ 0 & D_{yy} \end{pmatrix}$

$$\nabla \cdot (D(x, y)\nabla\varphi) = f(x, y), \quad (x, y) \in \Omega \subset \mathbb{R}^2 \tag{22}$$

$\underline{J} = \underline{\underline{D}}\, \underline{\nabla}\varphi$

subject to the general boundary conditions:



$\frac{\partial \varphi}{\partial n} = \nabla\varphi \cdot \underline{\hat{n}}$

$$A_b\varphi + B_b\frac{\partial\varphi}{\partial n} = C_b, \text{ on } \partial\Omega \tag{23}$$

where $\frac{\partial\varphi}{\partial n}$ denotes the directional derivative normal (pointing outwards) to the boundary $\partial\Omega$ and A_b, B_b, C_b are constants.

Both the diffusivity $D(x, y)$ and the function $f(x, y)$ are assumed to vary spatially.

$$\underline{J} = \begin{pmatrix} D_{xx} & 0 \\ 0 & D_{yy} \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} D_{xx} \frac{\partial \varphi}{\partial x} \\ D_{yy} \frac{\partial \varphi}{\partial y} \end{pmatrix}.$$

How would we deal with Heterogeniety?

$$\underline{\underline{D}} = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix} \quad \text{Diagonalise}$$

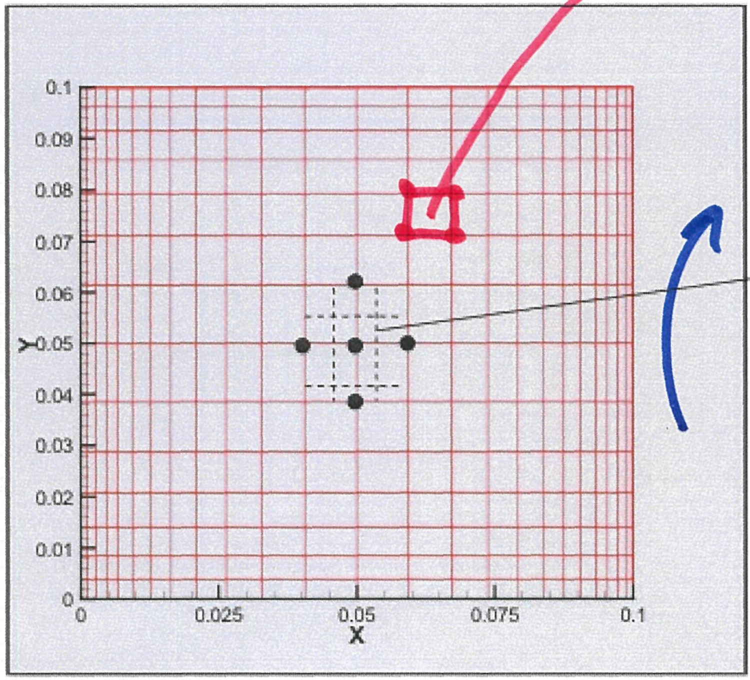
$$\underline{\underline{D}} = P \Lambda P^T$$

$$\Lambda = \mathrm{diag}(\lambda_x, \lambda_y).$$

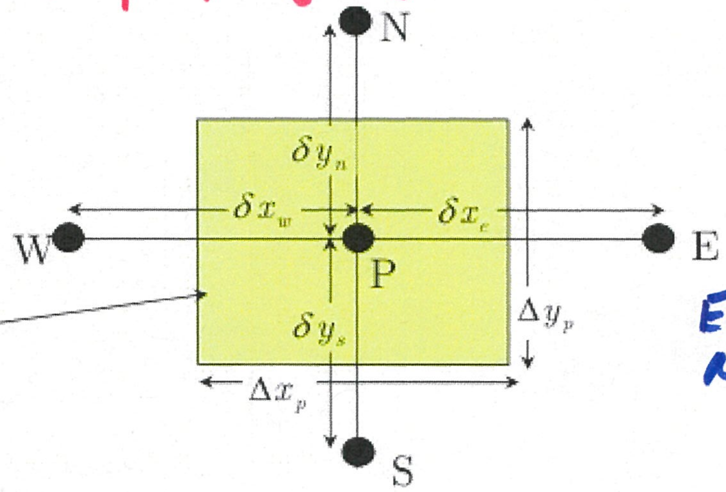
Rearrange the general B.C.:

$$\cancel{B_x} \boxed{\frac{\partial \varphi}{\partial n} = \frac{C_b}{B_b} - \frac{A_b}{B_b}\varphi}$$

Assign the element a property type. i.e, Sandstone.



(a)



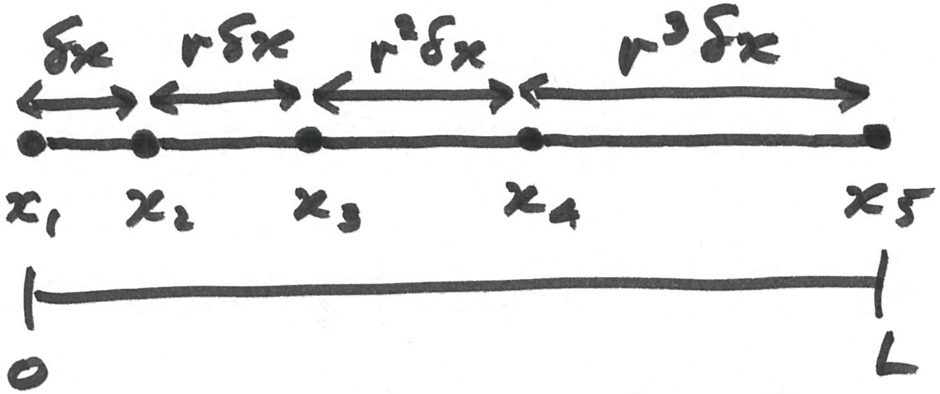
Extension of notation from 1D.

(b)

Figure: Structured mesh in two-dimensions: (a) mesh with irregular node spacing; (b) The control volume V_p is shown in yellow depicting the usual control volume notation.

Mesh Refinement: Geometric ~~ratio~~ factor "r" ← given

$r = 1.1$



How do we find δx?

$$\delta x + r\,\delta x + r^2\delta x + r^3\delta x = L$$

$$\delta x\,(1 + r + r^2 + r^3) = L$$

$$\therefore \quad \boxed{\delta x = \frac{(1-r)}{(1-r^4)}\,L}$$

$x_2 = x_1 + \delta x$

$x_3 = x_2 + r\,\delta x$

$x_4 = x_3 + r^2\delta x$

$\vdots$

check:

$$(1-r)(1+r+r^2+r^3) = 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} - \cancel{r} - \cancel{r^2} - \cancel{r^3} - r^4$$

$$\sum_{k=0}^{3} r^k = \frac{1-r^4}{1-r}$$

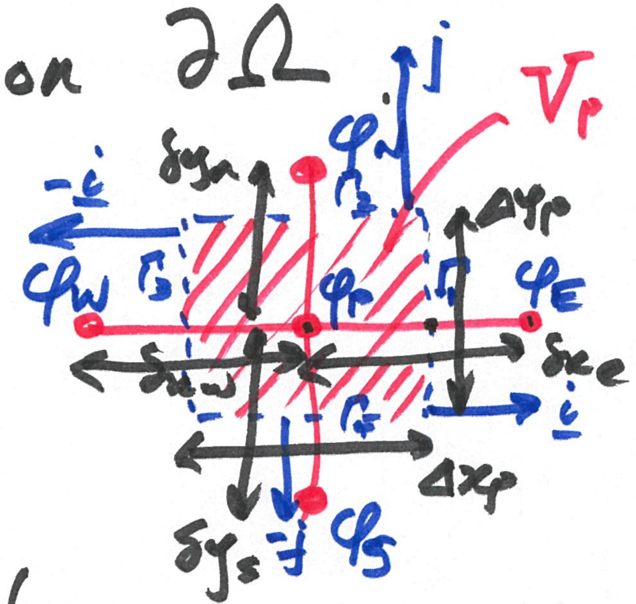
Model Equation : $\underline{J} = \underline{\underline{D}} \nabla \varphi = \begin{pmatrix} D_{xx} \frac{\partial \varphi}{\partial x} \\ D_{yy} \frac{\partial \varphi}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\underline{\nabla} \cdot \underline{J} = f(x,y) \quad \text{for} \quad (x,y) \in \Omega \quad \text{---} (1)$$

Boundary conditions :

$$\frac{\partial \varphi}{\partial n} = \frac{C_b}{B_b} - \frac{A_b}{B_b} \varphi \quad \text{on } \partial \Omega$$

USE FVM for discretisation :



Integrate (1) over V_P :

$$\iint_{V_P} \underline{\nabla} \cdot \underline{J} \, dx\, dy = \iint_{V_P} f(x,y) \, dx\, dy$$

Use Gauss Divergence Th^m : $\iiint_V \nabla \cdot f \, dV = \iint_A f \cdot \underline{\hat{n}} \, d\sigma$

$$\int_\Gamma \underline{J} \cdot \underline{\hat{n}} \, d\sigma = \Delta x_P \Delta y_P \bar{f} \quad , \quad \bar{f} = \frac{1}{\Delta x_P \Delta y_P} \iint_{V_P} f(x,y) \, dx\, dy \, .$$

i.e., $$\sum_{j=1}^{4} \int_{\Gamma_i} \underline{J}\cdot\hat{\underline{n}}\, d\sigma = \Delta x_p \Delta y_p \overline{f} \simeq f_p$$ (Exact at this point in the derivation)

Q: what's the error?

We now introduce approximations:

① $\overline{f} \simeq f_p$, which would be second order if evaluated at cell center.

② $$\int_{\Gamma_i} \underline{J}\cdot\hat{\underline{n}}\, d\sigma \simeq \underbrace{(\underline{J}\cdot\hat{\underline{n}})_{m_i}}_{\text{Evaluate flux at the midpoint of face}} l_i$$ one-point (mid-point) quadrature rule.

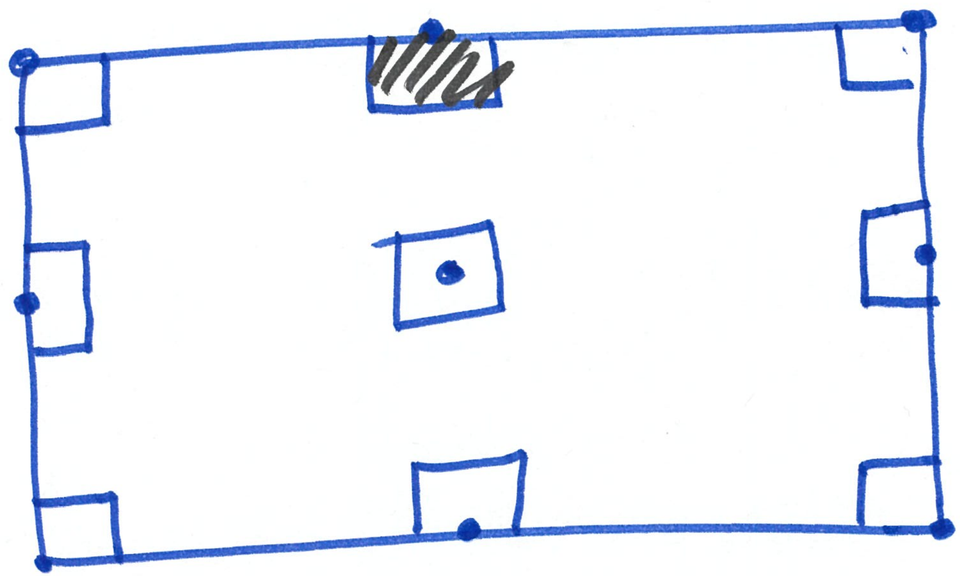
l_i: face length.

i.e., $$\overbrace{\frac{D_{xx_e}}{\delta x_e}(\varphi_E-\varphi_P)\Delta y_P}^{\int_{\Gamma_1}(\underline{J}\cdot\underline{i})\,d\sigma,\ \underline{J}\cdot\underline{i} = D_{xx}\frac{\partial\varphi}{\partial x}} + \overbrace{\frac{D_{yy_n}}{\delta y_n}(\varphi_N-\varphi_P)\Delta x_P}^{\int_{\Gamma_2}(\underline{J}\cdot\underline{j})\,d\sigma,\ \underline{J}\cdot\underline{j} = D_{yy}\frac{\partial\varphi}{\partial y}}$$

$$+ \underbrace{\frac{D_{xx_w}}{\delta x_w}(\varphi_W-\varphi_P)\Delta y_P}_{\int_{\Gamma_3}(\underline{J}\cdot(-\underline{i}))\,d\sigma} + \underbrace{\frac{D_{yy_s}}{\delta y_s}(\varphi_S-\varphi_P)\Delta x_P}_{\int_{\Gamma_4}(\underline{J}\cdot(-\underline{j}))\,d\sigma} = \Delta x_P \Delta y_P f_P . \quad \cdots (2)$$

In terms of "FSOLVE"

$$\mathcal{F}_P(\underline{u}) := \frac{Dxx_e}{\delta x_e}(\varphi_E - \varphi_P)\Delta y_P + \frac{Dyy_n}{\delta y_n}(\varphi_N - \varphi_P)\Delta x_P + \frac{Dxx_w}{\delta x_w}(\varphi_W - \varphi_P)\Delta y_P + \frac{Dyy_s}{\delta y_s}(\varphi_S - \varphi_P)\Delta x_P - \Delta x_P \Delta y_P f_P .$$



We need to consider nine different cases.

To obtain

$$D_{xx_e} \simeq \frac{1}{2}\left(D_{xx_P} + D_{xx_E}\right)$$

$$D_{xx_w} \simeq \frac{1}{2}\left(D_{xx_P} + D_{xx_W}\right)$$

$$D_{yy_n} \simeq \frac{1}{2}\left(D_{yy_P} + D_{yy_N}\right)$$

$$D_{yy_s} \simeq \frac{1}{2}\left(D_{yy_P} + D_{yy_S}\right)$$

We use arithmetic averaging.

Next, dealing with boundary cells:
As an example, consider north Boundary

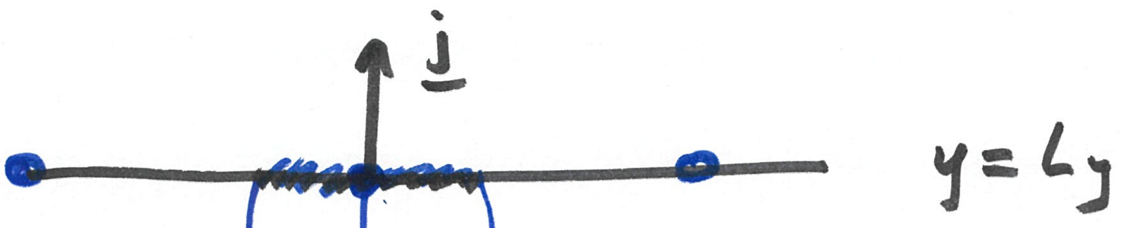
$$\boxed{D_{yy}\frac{\partial \varphi}{\partial y} = \frac{D_{yy} C_b}{B_b} - \frac{D_{yy} A_b}{B_b}\varphi}$$

— North face B.C. at $y = L_y$

Replace this expression into general FVE (2) for the North flux:

$$\frac{D_{xx_e}}{\delta x_e}(\varphi_E - \varphi_P)\Delta y_P + \left(\frac{D_{yy_n} C_b}{B_b} - \frac{D_{yy_n} A_b}{B_b}\varphi_P\right)\Delta x_P$$

$$+ \frac{D_{xx_w}}{\delta x_w}(\varphi_E - \varphi_P)\Delta y_P + \frac{D_{yy_s}}{\delta y_s}(\varphi_S - \varphi_P)\Delta x_P = \Delta x_P \Delta y_P f_P$$



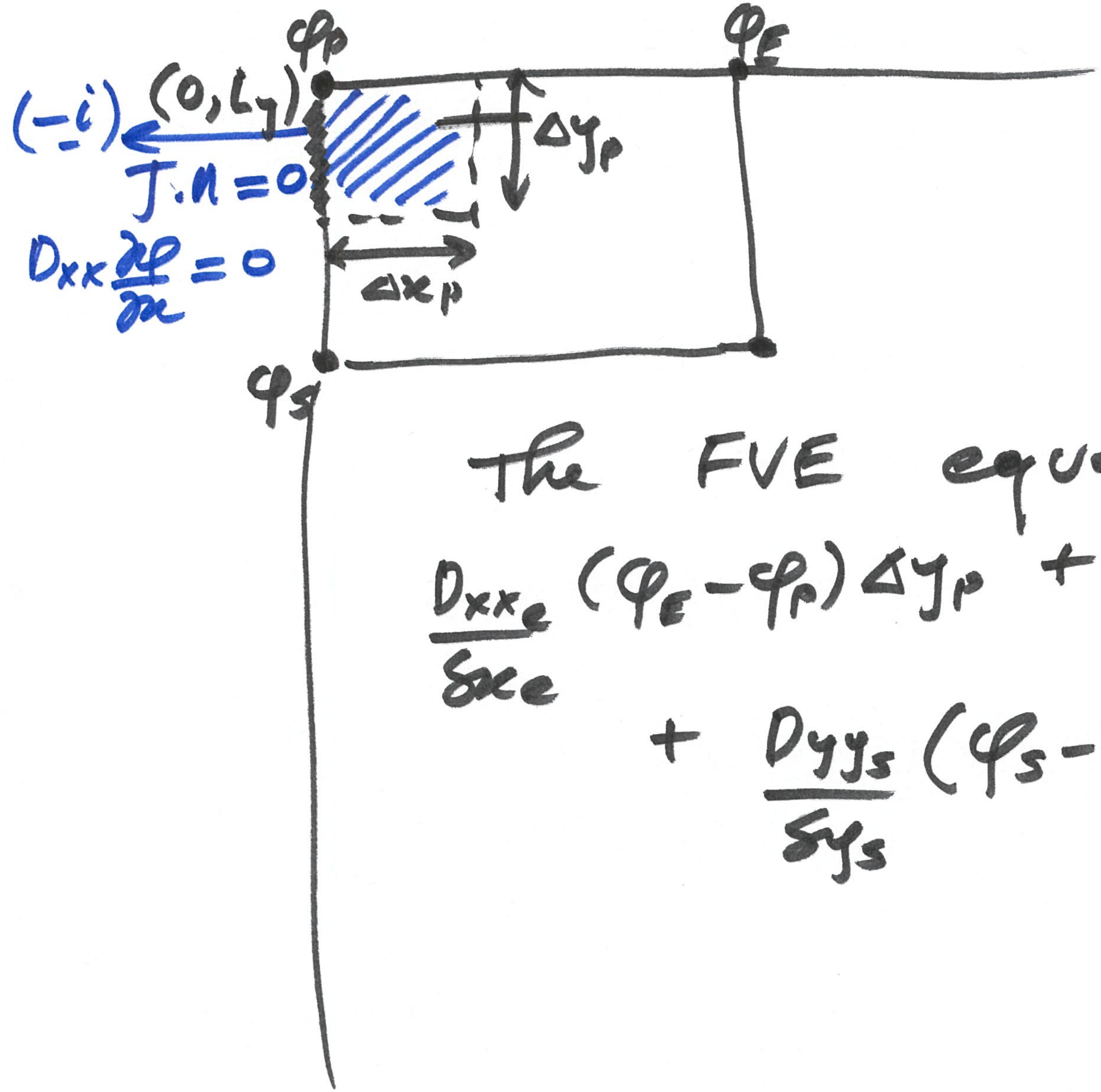
At $y = L_y$ is $A_b \varphi(x, y_L) + B_b \frac{\partial \varphi}{\partial y}(x, y_L) = C_b$

Recall $\underline{J} = \begin{pmatrix} D_{xx} & 0 \\ 0 & D_{yy} \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} D_{xx} \frac{\partial \varphi}{\partial x} \\ D_{yy} \frac{\partial \varphi}{\partial y} \end{pmatrix}$

$\therefore \underline{J} \cdot \underline{j} = D_{yy} \frac{\partial \varphi}{\partial y}(x, y_L)$

Rearrange : $\cancel{B_b} \frac{\partial \varphi}{\partial y}(x, y_L) = \frac{C_b}{B_b} - \frac{A_b}{B_b} \varphi(x, y_L)$

$\therefore \underline{D_{yy} \frac{\partial \varphi}{\partial y}(x, y_L)} = D_{yy} \left(\frac{C_b}{B_b} - \frac{A_b}{B_b} \varphi(x, y_L) \right)$

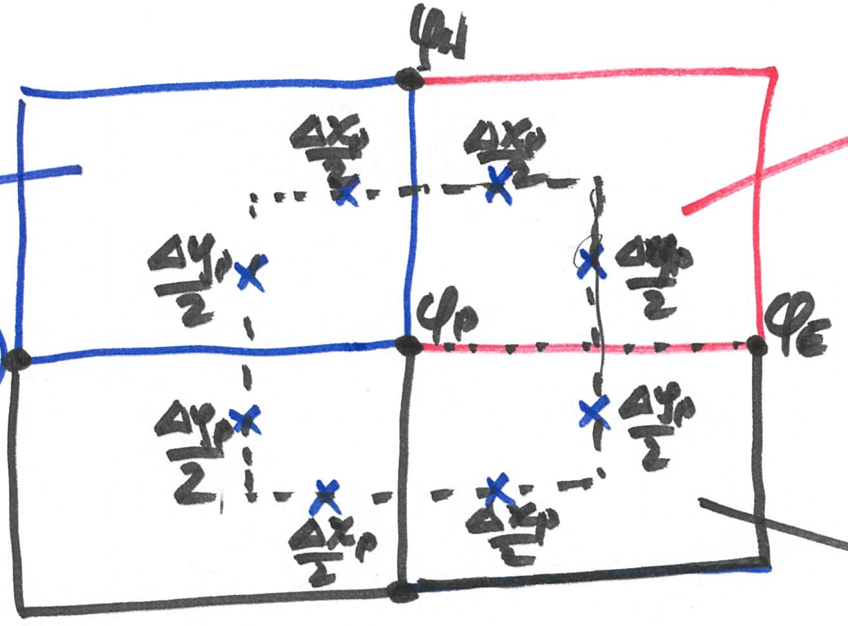


B.C.

The same north boundary condition as before, but no flux on the $x = 0$ boundary. (West face)

The FVE equation becomes

$$\frac{D_{xx_e}}{\delta x_e}(\varphi_E - \varphi_P)\Delta y_P + \left(\frac{D_{yy_n} C_b}{B_b} - \frac{D_{yy_n} A_b}{B_b}\varphi_P\right)\Delta x_P + \frac{D_{yy_s}}{\delta y_s}(\varphi_S - \varphi_P)\Delta x_P = \Delta x_P \Delta y_P f_P .$$



$\underline{\underline{D}}(x,y) = \begin{pmatrix} D_{xx} & 0 \\ 0 & D_{yy} \end{pmatrix}$

Material type must be recorded.

$\underline{\underline{D}}(x,y)$ "red type" $= \begin{pmatrix} D_{xx} & 0 \\ 0 & D_{YY} \end{pmatrix}$

$\underline{\underline{D}}(x,y)$ "black type" $= \begin{pmatrix} D_{xx} & 0 \\ 0 & D_{YY} \end{pmatrix}$.

$$\underbrace{\sum_{j=1}^{8} \int_{\Gamma_j} J \cdot \underline{\hat{n}} \, d\sigma}_{} = \Delta x_p \Delta y_p f_p$$

$$\frac{D_{xx_e}^{(R)}}{\delta x_e}(\varphi_E - \varphi_P)\frac{\Delta y_p}{2} + \frac{D_{yy_n}^{(R)}}{\delta y_n}(\varphi_N - \varphi_P)\frac{\Delta x_p}{2} + \frac{D_{yy_n}^{(B)}}{\delta y_n}(\varphi_N - \varphi_P)\frac{\Delta x_p}{2}$$

$$+ \frac{D_{xx}^{(B)}}{\delta x_w}(\varphi_W - \varphi_P)\frac{\Delta y_p}{2} + \frac{D_{xx}^{(B)}}{\delta x_w}(\varphi_W - \varphi_P)\frac{\Delta y_p}{2} + \cdots$$

$$= \Delta x_p \Delta y_p f_p.$$