MT L-reduction

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Note to the Admission Committee: This research report presents findings from my ongoing project for my undergraduate thesis. While the project is still in progress (this report reflects the work completed during the first 8 weeks), the results presented here demonstrate meaningful contributions. Please note that I still have one full semester remaining to complete this project, and this report represents only about 40% of the overall work. There is still a significant portion of the research to be done, and I will continue to develop and refine it. As such, citations, grammar, and the rigor of certain sections will be addressed in the final version

1 Abstract

Existing frameworks for establishing lower bounds on computational problems often utilize reductions from the Grid Tiling problem. While direct reductions from Grid Tiling are complex, the introduction of an intermediate problem, Grid Tiling $\leq$, provides a simpler reduction target, enabling streamlined reductions to the desired problems while preserving rigor. In this work, we extend this approach to optimization problems by introducing a reduction framework for the Matrix Tiling problem, the optimization variant of Grid Tiling. Specifically, we reduce the Matrix Tiling problem to its intermediate variant, Matrix Tiling $\leq$, and demonstrate how this can be further reduced to derive lower bounds for other optimization problems. Our results refine the existing methodology, making it more accessible for analyzing the complexity of optimization problems and establishing new lower bounds.

2 Introduction

We are providing the L-reduction from Matrix Tiling to Matrix Tiling$\leq$. Where L-reduction is defined as follows:
Let A *and* B be optimization problems and c_A*and* c_B their respective cost functions. A pair of *polynomial time-computable functions* R *and* S *is an* L-reduction if all of the following conditions are met:

1. If x is an instance of problem A, then $R(x)$ is an instance of problem B.
2. If y is a solution to $R(x)$, then $S(y)$ is a solution to x.
3. There exists a constant $\alpha > 0$ such that $OPT(R(x)) \leq \alpha\, OPT(x)$.
4. There exists a constant $\beta > 0$ such that $|OPT(x) - c_A(S(y))| \leq |OPT(R(x)) - c_B(y)|$.

3 Construction of R

Given an instance (n, k, S) of Matrix Tiling, we first shift all the coordinates in S. Let $S_{i,j} \subseteq [m] \times [m]$ denote the set of coordinate pairs associated with the cell in the i-th row and j-th column of the matrix.

To shift all coordinates in $S_{i,j}$ by $2n$, we define a new set $S''_{i,j}$ as follows:

$$S''_{i,j} = \{(x + n, y + n) \mid (x, y) \in S_{i,j}\}.$$

This transformation is applied for all $1 \leq i, j \leq n$. After the transformation, each $S'_{i,j}$ represents the shifted version of $S_{i,j}$, where every coordinate pair (x, y) has been translated by n along both axes. This ensures that the entire instance is shifted uniformly, preparing it for further steps in the construction. We also define new $n'' = 3n$.

Now, we construct an equivalent instance (n', k', S') of Matrix Tiling with $\leq$ with $n' = 3n''^2(k + 1) + n''^2 + n'' + 2k^2$ and $k' = 4k$. For every set $S''_{i,j}$ of the input instance, there are 16 corresponding sets $S'_{i',j'}(4i - 3 \leq i' \leq 4i, 4j - 3 \leq j' \leq 4j)$ in the constructed instance (see Fig). We call these sets the gadget representing $S_{i,j}$. The four inner sets $S_{4i-2,4j-2}$, $S_{4i-2,4j-1}$, $S_{4i-1,4j-2}$, $S_{4i-1,4j-1}$ are dummy sets containing only the single pair shown in the figure. The 12 outer sets are defined as follows. We define mapping $\iota(a, b) = n''a + b$ and let $N = 5n''^2$. For every $(a, b) \in S''_{i,j}$, we let $z = \iota(a, b)$ and introduce the pairs shown in the Figure 1.2 into the 12 outer sets. It can be easily verified that both coordinates of each pair are positive and at most n'.

$S'_{4i-3,4j-3}$: $(iN - z, jN + z)$	$S'_{4i-3,4j-2}$: $(iN + a, jN + z)$	$S'_{4i-3,4j-1}$: $(iN - a, jN + z)$	$S'_{4i-3,4j}$: $(iN + z, jN + z)$
$S'_{4i-2,4j-3}$: $(iN - z, jN + b)$	$S'_{4i-2,4j-2}$: $((i + 1)N, (j + 1)N)$	$S'_{4i-2,4j-1}$: $(iN, (j + 1)N)$	$S'_{4i-2,4j}$: $(iN + z, (j + 1)N + b)$
$S'_{4i-1,4j-3}$: $(iN - z, jN - b)$	$S'_{4i-1,4j-2}$: $((i + 1)N, jN)$	$S'_{4i-1,4j-1}$: (iN, jN)	$S'_{4i-1,4j}$: $(iN + z, (j + 1)N - b)$
$S'_{4i,4j-3}$: $(iN - z, jN - z)$	$S'_{4i,4j-2}$: $((i + 1)N + a, jN - z)$	$S'_{4i,4j-1}$: $((i + 1)N - a, jN - z)$	$S'_{4i,4j}$: $(iN + z, jN - z)$

We also introduce four new pairs of coordinates $(a^{+-}_{i,j}, b^{r+}_{i,j})$,$(a^{+-}_{i,j}, b^{l+}_{i,j})$,$(a^{+-}_{i,j}, b^{r-}_{i,j})$ and $(a^{+-}_{i,j}, b^{l-}_{i,j})$ to each cell in the original instance of MT. We will define these values as follows:

$$(a^{+-}_{i,j}, b^{r+}_{i,j}) = (\min(\text{first}(S_{i,j})), \max(\text{second}(S_{i,j+1})) + 1).$$

$$(a^{+-}_{i,j}, b^{r-}_{i,j}) = (\min(\text{first}(S_{i,j})), \min(\text{second}(S_{i,j+1})) - 1).$$

$$(a^{+-}_{i,j}, b^{l+}_{i,j}) = (\min(\text{first}(S_{i,j})), \max(\max(\text{second}(S_{i,j-1})), b^{r+}_{i,j-1}, b^{r-}_{i,j-1}) + 1).$$

$$(a^{+-}_{i,j}, b^{l-}_{i,j}) = (\min(\text{first}(S_{i,j})), \min(\min(\text{second}(S_{i,j-1})), b^{r-}_{i,j-1}, b^{r+}_{i,j-1}) - 1).$$

But we do not use these values to get the new coordinates in the $MT \leq$ instance in the way we did for the original pairs, we will only use these new added coordinates to create the new coordinates for $S'_{4i-2,4j-3}$, $S'_{4i-1,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$ cells of the gadget of the corresponding cell as follows:

Let $b^{new} = \{b^{r+}_{i,j}, b^{l+}_{i,j}, b^{r-}_{i,j}, b^{l-}_{i,j}\}$, $S_{4i-2,4j} : (iN + z, (j+1)N + \{b^{new}\})$, $S_{4i-1,4j} : (iN + z, (j+1)N - \{b^{new}\})$, $S_{4i-2,4j-3} : (iN - z, jN + \{b^{new}\})$, and $S_{4i-1,4j-3} : (iN - z, jN - \{b^{new}\})$ where $z = \iota(a'_{i,j}, b'_{i,j})$, where $a'_{i,j} = \min(\text{first } S_{i,j})$.

We also introduce four new pairs of coordinates $(a^{u+}_{i,j}, b^{+-}_{i,j})$, $(a^{d+}_{i,j}, b^{+-}_{i,j})$, $(a^{u-}_{i,j}, b^{+-}_{i,j})$ and $(a^{d-}_{i,j}, b^{+-}_{i,j})$ to each cell in the original instance of MT. We will define these values as follows:

$$(a^{u+}_{i,j}, b^{+-}_{i,j}) = (\max(\max(\text{first}(S_{i-1,j})), a^{d+}_{i-1,j}, a^{d-}_{i-1,j}) + 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{u-}_{i,j}, b^{+-}_{i,j}) = (\min(\min(\text{first}(S_{i-1,j})), a^{d-}_{i-1,j}, a^{d+}_{i-1,j}) - 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{d+}_{i,j}, b^{+-}_{i,j}) = (\max(\text{first}(S_{i+1,j})) + 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{d-}_{i,j}, b^{+-}_{i,j}) = (\min(\text{first}(S_{i+1,j})) - 1, (\min(\text{second}(S_{i,j}))).$$

But we do not use these values to get the new coordinates in the $MT \leq$ instance in the way we did for the original pairs, we only use these new added coordinates to create new coordinates for $S_{4i-3,4j-2}$, $S_{4i-3,4j-1}$, $S_{4i,4j-2}$, $S_{4i,4j-1}$ cells of the gadget of the corresponding cell as follows:

Let $a^{new} = \{a^{u+}_{i,j}, a^{d+}_{i,j}, a^{u-}_{i,j}, a^{d-}_{i,j}\}$, $S_{4i-3,4j-2} : (iN + \{a^{new}\}, jN + z)$, $S_{4i-3,4j-1} : (iN - \{a^{new}\}, jN + z)$, $S_{4i,4j-2} : ((i+1)N + \{a^{new}\}, jN - z)$, and $S_{4i,4j-1} : ((i+1)N - \{a^{new}\}, jN - z)$, where $z = \iota(a'_{i,j}, b'_{i,j})$, where $a'_{i,j} = \min(\text{first } S_{i,j})$.

Lemma 1. *If there is a star any cell in the solution of Matrix Tiling instance, we can pick exactly 15 non-stars in the corresponding gadget for the solution of Matrix Tiling with $\leq$ instance.*

Proof. For every non-star pair $s_{i,j} = (a, b)$, we select the corresponding pairs from the 16 sets in the gadget of $S''_{i,j}$ for $z = \iota(a, b)$, as shown in Figure.

For every star in the cell $S''_{i,j}$, we will pick $S'_{4i-2,4j-3}$ as the star, we will pick the middle 4 sets $S_{4i-2,4j-2}$, $S_{4i-2,4j-1}$, $S_{4i-1,4j-2}$, $S_{4i-1,4j-1}$ in the corresponding gadget (because of the way they are defined in the R), for all the corner sets $S'_{4i-3,4j-3}$, $S'_{4i-3,4j}$, $S'_{4i,4j}$, $S'_{4i,4j-3}$ will pick the pairs which were formed using the pair of $z = \iota(a'_{i,j}, b'_{i,j})$, where $a'_{i,j} = \min(\text{first}\, S''_{i,j})$ (a' is the minimum of all the a's for that cell), we will pick pairs for $S'_{4i-1,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$whose second coordinate are formed using b^{l-}, b^{r-}, b^{r+} respectively, we will pick pairs for $S'_{4i-3,4j-2}$, $S'_{4i-3,4j-1}$, $S'_{4i,4j-2}$, $S'_{4i,4j-1}$ whose first coordinate are formed using a^{u+}, a^{u-}, a^{d-}, a^{d+} respectively.

First, it is easy to verify that the constraints are satisfied between the sets of the same gadget for both these cases.

Now we want to prove that the $MT \leq$ condition holds in between all the gadgets, where we have 4 cases for both the horizontal and vertical directions.

. For the horizontal direction, we have the following 4 cases:

1. Both $S''_{i,j}$ and $S''_{i,j+1}$ are non-stars,
2. $S''_{i,j}$ is a non-star and $S''_{i,j+1}$ is a star,
3. $S''_{i,j}$ is a star and $S''_{i,j+1}$ is a non-star,
4. Both $S''_{i,j}$ and $S''_{i,j+1}$ are stars.

1. Consider now that the cell $S''_{i,j+1}$ on the right of the cell we are considering $S''_{i,j}$ was also a non-star, we look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i+1,j}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has second coordinate $jN + z$, while the pair selected from $S'_{4i-3,4(j+1)-3} = S'_{4i-3,4j+1}$ has the second coordinate $(j + 1)N + z >= jN + z$. Similarly, there is no conflict between the last sets of these columns. If $b_{i,j} = b_{i,j+1}$ are the first coordinates of $s''_{i,j}$ and $s''_{i,j+1}$, then the second coordinates of the sets selected from the second sets of the rows, $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$, are $(j + 1)N + b_{i,j}$ and $(j + 1)N + b_{i+1,j}$, respectively, and the former is equal to the latter. One can show in a similar way that there is no conflict between the third sets of the columns.

2. Now consider in the solution of MT , that the $S''_{i,j}$ is a non-star which is $s''_{i,j} = (a'_{i,j}, b'_{i,j})$ and $S''_{i,j+1}$ is a star. We look at the last column of the gadget of $S_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z'$ where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j + 1)N + z$ where z is the value formed using the pair which has the minimum a value, and we can observe that $(j + 1)N + z \geq jN + z'$. Similarly there is now conflict between last sets of these columns. We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a star as defined earlier. Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j + 1)N - b'_{i,j}$ and the pair selected from $S'_{4i-1,4j+1}$ has the

second coordinate $(j+1)N - b^{l-}_{i,j+1}$, and from the definition of $b^{l-}_{i,j+1}$ we get the inequality $(j+1)N - b'_{i,j} \le (j+1)N - b^{l-}_{i,j+1}$.

3. Now consider in the solution of MT , that the $S''_{i,j}$ is star and $S''_{i,j+1}$ is a non-star which is $s''_{i,j+1} = (a'_{i,j+1}, b'_{i,j+1})$. We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z$ where z is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z'$ where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and we can observe that $(j+1)N + z' \ge jN + z$. Similarly there is now conflict between last sets of these columns. Now the pair selected from $S'_{4i-2,4j}$ has the second coordinate $(j+1)N + b^{r-}$ and the pair selected from $S'_{4i-2,4j+1}$ has the second coordinate $(j+1)N + b'_{i,j+1}$ and from the definition of b^{r-} we get the inequality $(j+1)N + b^{r-} \le (j+1)N + b'_{i,j+1}$.Similarly now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b^{r+}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b'_{i,j+1}$ and from the definition of b^{r+}, we get the inequality $(j+1)N - b^{r+} \le (j+1)N - b'_{i,j+1}$.

4. Now consider in the solution of MT , both $S''_{i,j}$ and $S''_{i,j+1}$ are star. We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: $S'_{4i-3,4j}$ has the second coordinate $jN + z_{i,j}$ where $z_{i,j}$ is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z_{i,j+1}$ where $z_{i,j+1}$ is the value formed using the pair which has the minimum a value, and we can observe that $(j+1)N + z_{i,j+1} \ge jN + z_{i,j}$.Similarly there is now conflict between last sets of these columns. We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a star as defined earlier. Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b^{r+}_{i,j}$, and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b^{l-}_{i,j+1}$, and from the definition of $b^{r+}_{i,j}$ and $b^{l-}_{i,j+1}$ we get the inequality $(j+1)N - b^{r+}_{i,j} \le (j+1)N - b^{l-}_{i,j+1}$.

Now for the vertical direction, we have the following 4 cases:

1. Both $S''_{i,j}$ and $S''_{i+1,j}$ are non-stars,
2. $S''_{i,j}$ is a non-star and $S''_{i+1,j}$ is a star,
3. $S''_{i,j}$ is a star and $S''_{i+1,j}$ is a non-star,
4. Both $S''_{i,j}$ and $S''_{i+1,j}$ are stars.

1. Consider now that the cell $S''_{i+1,j}$ below the cell we are considering $S''_{i,j}$ was also a non-star, we look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has first coordinate less than iN, while the pair selected from $S'_{4(i+1)-3,4j-3} = S'_{4i+1,4j-3}$ has the first coordinate at

least $(i+1)N - (n^2+n) > iN$. Similarly, there is no conflict between the last sets of these rows. If $a_{i,j} = a_{i+1,j}$ are the first coordinates of $S''_{i,j}$ and $S''_{i+1,j}$, then the first coordinates of the sets selected from the second sets of the rows, $S'_{4i,4j-2}$ and $S'_{4i+1,4j-2}$, are $(i+1)N + a_{i,j}$ and $(i+1)N + a_{i+1,j}$, respectively, and the former is equal to the latter. One can show in a similar way that there is no conflict between the third sets of the rows.

2. Now consider in the solution of MT , that the $S''_{i,j}$ is a non-star which is $s''_{i,j} = (a'_{i,j}, b'_{i,j})$ and $S''_{i+1,j}$ is a star. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z'$, where $z' = \iota(a'_{i,j}, b'_{i,j})$ and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z$, where z is the value formed using the pair which has the minimum a value, and we can observe that $(i+1)N - z \geq iN - z'$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i,j+1}$, and from the definition of $a^{u+}_{i,j+1}$, we get the inequality $(i+1)N + a^{u+}_{i,j+1} \geq (i+1)N + a'_{i,j}$. Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i,j+1}$ and from the definition of $a^{u-}_{i,j+1}$, we get the inequality $(i+1)N - a^{u-}_{i,j+1} \geq (i+1)N - a'_{i,j}$.

3. Now consider in the solution of MT , that the $S''_{i,j}$ is star and $S''_{i+1,j}$ is a non-star which is $s''_{i+1,j} = (a'_{i+1,j}, b'_{i+1,j})$. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z$,, where z is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z'$ where $z' = \iota(a'_{i+1,j}, b'_{i+1,j})$, and we can observe that $(i+1)N - z' \geq iN - z$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$ and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a'_{i+1,j}$, and from the definition of $a^{d-}_{i,j}$, we get the inequality $(i+1)N + a'_{i+1,j} \geq i+1)N + a^{d-}_{i,j}$. Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a'_{i+1,j}$, and from the definition of $a^{d+}_{i,j}$, we get the inequality $(i+1)N - a'_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

4. Now consider in the solution of MT , both $S''_{i,j}$ and $S''_{i+1,j}$ are star. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these row, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z_{i,j}$, where $z_{i,j}$ is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z_{i+1,j}$, where $z_{i+1,j}$ is the value formed using the pair which has the minimum a value, we can observe that $(i+1)N - z_{i+1,j} \geq iN - z_{i,j}$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate

$(i+1)N+a^{u+}_{i+1,j}$, and from the definition of $a^{d-}_{i,j}$ and $a^{u+}_{i+1,j}$, we get the inequality $(i+1)N+a^{u+}_{i+1,j} \geq (i+1)N+a^{d-}_{i,j}$.Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i+1,j}$, and from the definition of $a^{d+}_{i,j}$ and $a^{u-}_{i+1,j}$, we get the inequality $(i+1)N - a^{u-}_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

Thus we have proved that for every non-star in the MT solution we will have all 16 non-stars for it's corresponding gadget in the $MT\leq$ solution and for every star in the MT solution we will have **exactly(not sure if this is the correct way of writing the conclusion)**15 non-stars for it's corresponding gadget in the $MT\leq$ solution.

□

4 Construction of S and proof of the second condition of L-reduction

Look at y any solution of $R(x)$ and 4×4 cells. If all of these 16 cells are non-stars then we can prove that the outer 12 of the gadget are formed using the same (a, b) we will see this proof later. and if all of 16 are non-stars then pick that pair (a, b) as the non-star solution in the corresponding cell of MT instance otherwise pick star.

Now we prove the assertion that if all of the 16 cells of the gadget are non-stars then all of the outer 12 cells of the gadget are formed using the same (a, b): Consider the 12 outer sets in the gadget representing $S''_{i,j}$. The 12 pairs selected in the solution from these sets dene 12 values z; let $z_{4i-3,4j-3}$, etc., be these 12 values. We claim that all these 12 values are the same. The second coordinate of the set selected from $S'_{4i-3,4j-3}$ is $jN + z_{4i-3,4j-3}$, the second coordinate of the set selected from $S'_{4i-3,4j-2}$ is $jN + z_{4i-3,4j-2}$; hence the definition of Matrix Tiling with $\leq$implies $z_{4i-3,4j-3} \leq z_{4i-3,4j-2}$. With similar reasoning, by going around the outer sets, we get the following inequalities.

$z_{4i_3,4j-3} \leq z_{4i-3,4j-2} \leq z_{4i-3,4j-1} \leq z_{4i-3,4j}$ (first row)

$z_{4i-3,4j} \leq z_{4i-2,4j} \leq z_{4i-1,4j} \leq z_{4i,4j}$ (last column)

$-z_{4i,4j-3} \leq -z_{4i,4j-2} \leq -z_{4i,4j-1} \leq -z_{4i,4j}$ (last row)

$-z_{4i-3,4j-3} \leq -z_{4i-2,4j-3} \leq -z_{4i-1,4j-3} \leq -z_{4i,4j-3}$ (first column)

Putting everything together, we get a cycle of inequalities showing that these 12 values are all equal; let $z_{i,j}$ be this common value and let $s''_{i,j} = (a_{i,j}, b_{i,j})$ be the corresponding pair, that is, $\iota(a_{i,j}, b_{i,j}) = z_{i,j}$. The fact that $z_{i,j}$ was defined using the pairs appearing in the gadget of $S''_{i,j}$ implies that $s''_{i,j} \in S''_{i,j}$. and because we created $S''_{i,j}$ from $S_{i,j}$ by adding n to each coordinates we can get $s_{i,j} \in S_{i,j}$ by substracting n to both the coordinates of $s''_{i,j}$, so $s_{i,j} = (a_{i,j} - n, b_{i,j} - n)$.

Consider now that the gadget of the cell $S''_{i+1,j}$, below the gadgets of the cell we are considering $S''_{i,j}$ are all 16 non-stars Then, let us show that $a_{i,j} = a_{i+1,j}$. The pair selected from $S'_{4i,4j-2}$ has first coordinate $(i+1)N+a_{i,j}$, while the pair selected from $S'_{4(i+1)-3,4j-2} = S'_{4i+1,4j-2}$ has first coordinate $(i+1)N + a_{i+1,j}$. The definition of Matrix Tiling with $\leq$ implies now that $a_{i,j} \leq a_{i+1,j}$ holds. Similarly, comparing the first coordinates of the pairs selected from $S'_{4i,4j-1}$ and $S'_{4i+1,4j-1}$ shows $-a_{i,j} \leq -a_{i+1,j}$; hence $a_{i,j} = a_{i+1,j}$ follows.

Consider now that the gadget of the cell $S_{i,j+1}$, right the gadgets of the cell we are considering $S_{i,j}$ are all 16 non-stars. A similar argument, by looking at the last column of the gadget representing $S_{i,j}$ and the first column of the gadget representing $S_{i,j+1}$ shows $b_{i,j} = b_{i,j+1}$.

In the case where the gadgets of neighboring cells are one or more non-stars, where are selecting the star in the MT solution so we don't have to prove anything for $a_{i,j} = a_{i+1,j}$ or $b_{i,j} = b_{i,j+1}$ in this case as there will be stars in the corresponding coordinates of the pairs.

Therefore, we have proved that the constructed $s_{i,j}$'s $\in S(y)$ indeed form a solution of the Matrix Tiling instance.

5 Proving the third condition of the L-reduction

We know [from the paper] that the optimum is always at least $\frac{k^2}{4}$: if i and j are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$; otherwise, let $s_{i,j} = \star$. And we have optimum is at most k^2, which gives us the following inequalities:

$$k^2/4 \leq OPT(x) \leq k^2 \tag{1}$$

$$k'^2/4 \leq OPT(R(x)) \leq k'^2 \tag{2}$$

from eq 2:

$$4k^2 \leq OPT(R(x)) \leq 16k^2 \tag{3}$$

now we have:

$$OPT(R(x)) \leq 16k^2 = 64k^2/4 = 64OPT(x) \tag{4}$$

Hence for $\alpha = 64$, we get the inequality $OPT(R(x)) \leq \alpha OPT(x)$, which proves the third condition for the L-reduction.

6 Proving the fourth condition of the L-reduction

Because both MT and MT with $\leq$ are maximizing optimization problems, $OPT(x) - c_A(S(y))$ and $OPT(R(x)) - c_B(y)$ will always be ≥ 0.
Let

$$OPT(x) = k^2 - b \tag{5}$$

from the definition of R we get:

$$OPT(R(x)) \geq 16(k^2 - b) + 15b = 16k^2 - b \tag{6}$$

Let

$$c_B(y) = 16k^2 - r \tag{7}$$

and

$$c_A(S(y)) = k^2 - s \tag{8}$$

from the definition of S, we can observe that:

$$c_B(y) \geq 16c_A(S(y)) \tag{9}$$

if we have $c_A(S(y)) = k^2 - s$, we have upper bound for $c_B(y) \leq 16(k^2 - s) + 15s$ (for every non-star in solution of MT we have 16 non-stars in $MT \leq$ and for every star we will have at most 15 non-stars in it's gadget). Thus we will get the inequality

$$c_B(y) = 16k^2 - r \leq 16(k^2 - s) + 15s$$

$$r \geq s \tag{10}$$

now in the equation:

$$OPT(x) + \beta c_B(y) \leq \beta OPT(R(x)) + c_A(S(y))$$

from eq 5 6 7 and 8 we get:

$$(k^2 - b) + \beta(16k^2 - r) \leq \beta(16k^2 - b) + (k^2 - s)$$

$$(r - b) \leq \beta(s - b) \tag{11}$$

If we choose $\beta = 1$ we will get

$$r - b \geq s - b$$

which will leave us at

$$r \geq s$$

which is already proves equation 10. Hence for $\beta = 1$, we get the inequality $|OPT(x) - c_A(S(y))| \leq |OPT(R(x)) - c_B(y)|$, which proves the fourth condition for the L-reduction.

This completes the L-reduction from Matrix Tiling to Matrix Tiling with $\leq$.