MT L-reduction

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Note to the Admission Committee: This research report presents findings from my ongoing project for my undergraduate thesis. While the project is still in progress (this report reflects the work completed during the first 8 weeks), the results presented here demonstrate meaningful contributions. Please note that I still have one full semester remaining to complete this project, and this report represents only about 40% of the overall work. There is still a significant portion of the research to be done, and I will continue to develop and refine it. As such, citations, grammar, and the rigor of certain sections will be addressed in the final version.

1 Introduction

We are providing the L-reduction from Matrix Tiling to Matrix Tiling$\leq$.

Matrix Tiling:
Input: Integers k, D, and k^2 nonempty sets $S_{i,j} \subset Z_D \times Z_D (1 \leq i, j \leq k)$.
Find: For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 = b_1$.
2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 = b_2$.

Goal: Maximize the number of pairs (i, j) $(1 \leq i, j \leq k)$ with $s_{i,j} \neq \star$.

Matrix Tiling $\leq$:
Input: Integers k, D, and k^2 nonempty sets $S_{i,j} \subset Z_D \times Z_D (1 \leq i, j \leq k)$.
Find: For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 \leq b_1$.
2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 \leq b_2$.

Goal: Maximize the number of pairs (i, j) $(1 \leq i, j \leq k)$ with $s_{i,j} \neq \star$.

Where L-reduction is defined as follows:

Let A *and* B be optimization problems and c_A *and* c_B their respective cost functions. A pair of *polynomial time-computable functions* R *and* S *is an* L-reduction if all of the following conditions are met:

1. If x is an instance of problem A, then $R(x)$ is an instance of problem B,
2. If y is a solution to $R(x)$, then $S(y)$ is a solution to x,
3. There exists a constant $\alpha > 0$ such that $OPT(R(x)) \leq \alpha \, OPT(x)$,
4. There exists a constant $\beta > 0$ such that $|OPT(x) - c_A(S(y))| \leq \beta |OPT(R(x)) - c_B(y)|$.

Note:

- Here R is polynomial in the size of input instance of A, and S is polynomial in the size of B, but because R is polynomial, we can conclude that the size of S is polynomial in the size of R, which would then imply that R and S are in polynomial in the size of A.

- In the [paper] Marx, for defining L-reduction is requiring R and S to be logspace-computable functions, but here we are constraining R and S to be polytime functions which is slightly weaker requirement than logspace.

- Throughout this paper, we will considering the origin to be the top-left corner of the matrix.

2 Reduction and relation between solutions of both the instances.

2.1 Construction of a $MT \leq$ instance from a given MT instance. (Definition of R)

Given an instance (n, k, S) of the Matrix Tiling problem, we construct an equivalent instance (n', k', S') of the Matrix Tiling $\leq$ problem through the following steps:

Step 1: Shifting Coordinates

For each cell (i, j) in the matrix, let $S_{i,j} \subseteq [n] \times [n]$ denote the set of coordinate pairs associated with that cell. We shift all coordinates in $S_{i,j}$ by n, producing a new set $S''_{i,j}$:

$$S''_{i,j} = \{(x + n, y + n) \mid (x, y) \in S_{i,j}\}.$$

After the transformation, we define $n'' = 3n$, this new n'' and the transformation will prepare us for adding new coordinates which is described in the next section. .

Step 2: Additional Coordinate Pairs

To help us prove a the fourth condition of the L-reduction, we will introduce a lemma in the next section and to help us prove that lemma, we introduce eight additional pairs of coordinates for each cell (i, j) in the MT instance, the first four will be used for the horizontal direction constraints of the $MT \leq$ for the lemma and the remaining four will be used for the vertical direction constraints.
The First four coordinates are:

$$(a^{+-}_{i,j}, b^{r+}_{i,j}) = (\min(\text{first}(S_{i,j})), \max(\text{second}(S_{i,j+1})) + 1),$$

$$(a^{+-}_{i,j}, b^{r-}_{i,j}) = (\min(\text{first}(S_{i,j})), \min(\text{second}(S_{i,j+1})) - 1),$$

$$(a^{+-}_{i,j}, b^{l+}_{i,j}) = (\min(\text{first}(S_{i,j})), \max(\max(\text{second}(S_{i,j-1})), b^{r+}_{i,j-1}, b^{r-}_{i,j-1}) + 1),$$

$$(a^{+-}_{i,j}, b^{l-}_{i,j}) = (\min(\text{first}(S_{i,j})), \min(\min(\text{second}(S_{i,j-1})), b^{r+}_{i,j-1}, b^{r-}_{i,j-1}) - 1).$$

The remaining four coordinates are:

$$(a^{d+}_{i,j}, b^{+-}_{i,j}) = (\max(\text{first}(S_{i+1,j})) + 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{d-}_{i,j}, b^{+-}_{i,j}) = (\min(\text{first}(S_{i+1,j})) - 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{u+}_{i,j}, b^{+-}_{i,j}) = (\max(\max(\text{first}(S_{i-1,j})), a^{d+}_{i-1,j}, a^{d-}_{i-1,j}) + 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{u-}_{i,j}, b^{+-}_{i,j}) = (\min(\min(\text{first}(S_{i-1,j})), a^{d-}_{i-1,j}, a^{d+}_{i-1,j}) - 1, (\min(\text{second}(S_{i,j}))).$$

Step 3: Constructing the Equivalent Instance of Matrix Tiling with $\leq$

Now from the (n'', k, S'') of Matrix Tiling problem, we construct the equivalent instance (n', k', S') of Matrix Tiling with $\leq$, defined with:

$$n' = 3n''^2(k + 1) + n''^2 + n'', \quad k' = 4k.$$

For each set $S''_{i,j}$ in the original instance, we create 16 corresponding sets $S'_{i',j'}$ in the new instance, where $4i - 3 \leq i' \leq 4i$ and $4j - 3 \leq j' \leq 4j$ (see Fig 1). We call these sets, "gadget" representing $S''_{i,j}$.

1. Inner Dummy Sets: The four inner sets are defined as follows:

$$S'_{4i-2,4j-2} = S'_{4i-2,4j-1} = S'_{4i-1,4j-2} = S'_{4i-1,4j-1} = \{((i + 1)N, (j + 1)N)\}.$$

These sets are placeholders and do not depend on $S''_{i,j}$.

2. Outer Sets: The 12 outer sets are populated using a mapping function $\iota(a,b) = n''a + b$ and a scaling factor $N = 3n''^2$. For each $(a,b) \in S''_{i,j}$ *(note: we do not use newly the added eight pairs for the construction of $S'_{i',j'}$)*, we compute $z = \iota(a,b)$ and introduce pairs into the outer sets as follows:

$S'_{4i-3,4j-3}$: $(iN - z, jN + z)$	$S'_{4i-3,4j-2}$: $(iN + a, jN + z)$	$S'_{4i-3,4j-1}$: $(iN - a, jN + z)$	$S'_{4i-3,4j}$: $(iN + z, jN + z)$
$S'_{4i-2,4j-3}$: $(iN - z, jN + b)$	$S'_{4i-2,4j-2}$: $((i+1)N, (j+1)N)$	$S'_{4i-2,4j-1}$: $(iN, (j+1)N)$	$S'_{4i-2,4j}$: $(iN + z, (j+1)N + b)$
$S'_{4i-1,4j-3}$: $(iN - z, jN - b)$	$S'_{4i-1,4j-2}$: $((i+1)N, jN)$	$S'_{4i-1,4j-1}$: (iN, jN)	$S'_{4i-1,4j}$: $(iN + z, (j+1)N - b)$
$S'_{4i,4j-3}$: $(iN - z, jN - z)$	$S'_{4i,4j-2}$: $((i+1)N + a, jN - z)$	$S'_{4i,4j-1}$: $((i+1)N - a, jN - z)$	$S'_{4i,4j}$: $(iN + z, jN - z)$

Figure 1: The 16 sets of the constructed Matrix Tiling with $\leq$ instance representing a set $S_{i,j}$ of the Grid Tiling in the reduction in the proof of together with the pairs corresponding to a pair $(a,b) \in S'_{i,j}$(with $z = \iota(a,b)$)

As mentioned earlier, we do not use the newly added coordinates directly to construct $S'_{i',j'}$.
Instead, we employ the first four newly added coordinates solely for the construction of the $S'_{4i-2,4j-3}$, $S'_{4i-1,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$ sets within the corresponding gadget, as follows:
Let $b^{new} = \{b^{r+}_{i,j}, b^{l+}_{i,j}, b^{r-}_{i,j}, b^{l-}_{i,j}\}$, The new coordinates of the sets are defined as:

$$S_{4i-2,4j} : (iN + z, (j+1)N + \{b^{new}\}),$$

$$S_{4i-1,4j} : (iN + z, (j+1)N - \{b^{new}\}),$$

$$S_{4i-2,4j-3} : (iN - z, jN + \{b^{new}\}),$$

$$S_{4i-1,4j-3} : (iN - z, jN - \{b^{new}\}).$$

where $z = \iota(a'_{i,j}, b'_{i,j})$, such that $(a'_{i,j}, b'_{i,j})$ is the pair form $S''_{i,j}$ where $a'_{i,j} = \min(\text{first } S_{i,j})$.

Similar to the first four newly added coordinates above, we also do not use these four coordinates directly to construct sets in all the 16 cells of the corresponding gadget of the MT $\leq$ instance. Instead, we employ these for the construction of the $S_{4i-3,4j-2}$, $S_{4i-3,4j-1}$, $S_{4i,4j-2}$, $S_{4i,4j-1}$ sets within the corresponding gadget, as follows:
Let $a^{new} = \{a^{u+}_{i,j}, a^{d+}_{i,j}, a^{u-}_{i,j}, a^{d-}_{i,j}\}$, The new coordinates of the sets are defined as:

$$S_{4i-3,4j-2} : (iN + \{a^{new}\}, jN + z),$$

$$S_{4i-3,4j-1} : (iN - \{a^{new}\}, jN + z),$$

$$S_{4i,4j-2} : ((i+1)N + \{a^{new}\}, jN - z),$$

$$S_{4i,4j-1} : ((i+1)N - \{a^{new}\}, jN - z)$$

where $z = \iota(a'_{i,j}, b'_{i,j})$, such that $(a'_{i,j}, b'_{i,j})$ is the pair form $S''_{i,j}$ where $a'_{i,j} = \min(\text{first } S_{i,j})$.

It can be verified that all coordinates of each pair are positive and bounded by n'.

Note: Typically, coordinates are expressed as numerical values or variables. The inclusion of b^{new}, which is a set, within a coordinate is a non-standard practice. In this context b^{new} represents all the different four elements of the set, so for example $S_{4i-2,4j} : (iN+z, (j+1)N+\{b^{new}\})$, means we add four new coordinates in the $S_{4i-2,4j}$ which are in this case: $(iN+z, (j+1)N+b^{r+}_{i,j})$, $(iN+z, (j+1)N+b^{r-}_{i,j})$, $(iN+z, (j+1)N+b^{l+}_{i,j})$, and $(iN+z, (j+1)N+b^{l-}_{i,j})$.

These pairs are carefully constructed to help us prove the following lemma.

Lemma 1. *If there is a star in any cell in the solution of the Matrix Tiling instance, we can pick exactly 15 non-stars in the corresponding gadget for the solution of the Matrix Tiling with $\leq$ instance.*

Proof. For every non-star pair $s_{i,j} = (a, b)$, we select the corresponding pairs from the 16 sets in the gadget of $S''_{i,j}$ for $z = \iota(a, b)$, as shown in Figure 1.

For every star in the cell $S''_{i,j}$, pick one cell from $S'_{4i-2,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j-3}$, or $S'_{4i-1,4j}$ to be star, for this proof we will pick $S'_{4i-2,4j-3}$ as the star. We will pick the middle 4 sets $S_{4i-2,4j-2}$, $S_{4i-2,4j-1}$, $S_{4i-1,4j-2}$, $S_{4i-1,4j-1}$ in the corresponding gadget (because of the way they are defined there is only one pair corresponding to each of these so pick that pair), for all the corner sets $S'_{4i-3,4j-3}, S'_{4i-3,4j}, S'_{4i,4j}, S'_{4i,4j-3}$. We will pick the pairs formed using the pair $z = \iota(a'_{i,j}, b'_{i,j})$, where $a'_{i,j} = \min(\text{first}(S''_{i,j}))$ (i.e., the minimum of all the a's for that cell).

We will pick pairs for $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}$ whose second coordinates are formed using $b^{l+}, b^{l-}, b^{r-}, b^{r+}$,(Note: here one from $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}, S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}$ these must be picked as star, and as for this proof we are choosing $S'_{4i-2,4j-3}$ to be star then we don't need to mention it but because one can pick any of these four to be a $\star$, for the sake of compeletness we have provided the way to pick the pairs whose value is non-$\star$) respectively. We will pick pairs for $S'_{4i-3,4j-2}, S'_{4i-3,4j-1}, S'_{4i,4j-2}, S'_{4i,4j-1}$ whose first coordinates are formed using $a^{u+}, a^{u-}, a^{d-}, a^{d+}$, respectively.

First, it is easy to verify that the constraints are satisfied between the sets of the same gadget for both these cases.

Now we want to prove that the $MT \leq$ condition holds between all the gadgets. We have 4 cases for both the horizontal and vertical directions.

For the horizontal direction, we have the following 4 cases:

1. Both $S''_{i,j}$ and $S''_{i,j+1}$ are non-stars,
2. $S''_{i,j}$ is a non-star and $S''_{i,j+1}$ is a star,
3. $S''_{i,j}$ is a star and $S''_{i,j+1}$ is a non-star,
4. Both $S''_{i,j}$ and $S''_{i,j+1}$ are stars.

Case 1: Both $S''_{i,j}$ and $S''_{i,j+1}$ are non-stars.

We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i+1,j}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has second coordinate $jN+z$, while the pair selected from $S'_{4i-3,4(j+1)-3} = S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N+z$, which is greater than or equal to $jN + z$. Similarly, there is no conflict between the last sets of these columns. If $b_{i,j} = b_{i,j+1}$ are the first coordinates of $S''_{i,j}$ and $S''_{i,j+1}$, then the second coordinates of the sets selected from the second sets of the rows, $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$, are $(j + 1)N + b_{i,j}$ and $(j + 1)N + b_{i+1,j}$, respectively, and the former is equal to the latter. One can show in a similar way that there is no conflict between the third sets of these columns.

Case 2: $S''_{i,j}$ is a non-star and $S''_{i,j+1}$ is a star.

We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z'$, where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j + 1)N + z$, where z is the value formed using the pair with the minimum a-value. We can observe that $(j + 1)N + z \geq jN + z'$. Similarly, there is no conflict between the last sets of these columns. We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a star as defined earlier. Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j + 1)N - b'_{i,j}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j + 1)N - b^{l-}_{i,j+1}$. From the definition of $b^{l-}_{i,j+1}$, we get the inequality $(j + 1)N - b'_{i,j} \leq (j + 1)N - b^{l-}_{i,j+1}$.

Case 3: $S''_{i,j}$ is a star and $S''_{i,j+1}$ is a non-star.

We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z$, where z is the value formed using the pair with the minimum a-value, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j + 1)N + z'$, where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and we can observe that $(j + 1)N + z' \geq jN + z$. Similarly, there is no conflict between the last sets of these columns. Now the pair selected from $S'_{4i-2,4j}$ has the second coordinate $(j + 1)N + b^{r-}$ and the pair selected from $S'_{4i-2,4j+1}$ has the second coordinate $(j + 1)N + b'_{i,j+1}$. From the definition of b^{r-}, we get the inequality $(j + 1)N + b^{r-} \leq (j + 1)N + b'_{i,j+1}$. Similarly, the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j + 1)N - b^{r+}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j + 1)N - b'_{i,j+1}$. From the definition of b^{r+}, we get the inequality $(j + 1)N - b^{r+} \leq (j + 1)N - b'_{i,j+1}$.

Case 4: Both $S''_{i,j}$ and $S''_{i,j+1}$ are stars.

We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: $S'_{4i-3,4j}$ has the second coordinate $jN + z_{i,j}$ where $z_{i,j}$ is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z_{i,j+1}$ where $z_{i,j+1}$ is the value formed using the pair which has the minimum a value, and we can observe that $(j+1)N + z_{i,j+1} \geq jN + z_{i,j}$.Similarly there is now conflict between last sets of these columns. We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a star as defined earlier. Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b^{r+}_{i,j}$, and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b^{l-}_{i,j+1}$, and from the definition of $b^{r+}_{i,j}$ and $b^{l-}_{i,j+1}$ we get the inequality $(j+1)N - b^{r+}_{i,j} \leq (j+1)N - b^{l-}_{i,j+1}$.

Now for the vertical direction, we have the following 4 cases:

1. Both $S''_{i,j}$ and $S''_{i+1,j}$ are non-stars,
2. $S''_{i,j}$ is a non-star and $S''_{i+1,j}$ is a star,
3. $S''_{i,j}$ is a star and $S''_{i+1,j}$ is a non-star,
4. Both $S''_{i,j}$ and $S''_{i+1,j}$ are stars.

Case 1: Both $S''_{i,j}$ and $S''_{i+1,j}$ are non-stars.

We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has first coordinate less than iN, while the pair selected from $S'_{4(i+1)-3,4j-3} = S'_{4i+1,4j-3}$ has the first coordinate at least $(i+1)N - (n^2+n) > iN$. Similarly, there is no conflict between the last sets of these rows. If $a_{i,j} = a_{i+1,j}$ are the first coordinates of $S''_{i,j}$ and $S''_{i+1,j}$, then the first coordinates of the sets selected from the second sets of the rows, $S'_{4i,4j-2}$ and $S'_{4i+1,4j-2}$, are $(i+1)N + a_{i,j}$ and $(i+1)N + a_{i+1,j}$, respectively, and the former is equal to the latter. One can show in a similar way that there is no conflict between the third sets of the rows.

Case 2: $S''_{i,j}$ is a non-star and $S''_{i+1,j}$ is a star.

We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z'$, where $z' = \iota(a'_{i,j}, b'_{i,j})$ and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z$, where z is the value formed using the pair which has the minimum a value, and we can observe that $(i+1)N - z \geq iN - z'$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i,j+1}$, and from the definition of $a^{u+}_{i,j+1}$, we get the inequality $(i+1)N + a^{u+}_{i,j+1} \geq (i+1)N + a'_{i,j}$. Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i,j+1}$ and from the definition of $a^{u-}_{i,j+1}$, we get the inequality $(i+1)N - a^{u-}_{i,j+1} \geq (i+1)N - a'_{i,j}$.

Case 3: $S''_{i,j}$ is a star and $S''_{i+1,j}$ is a non-star.

We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z$,, where z is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z'$ where $z' = \iota(a'_{i+1,j}, b'_{i+1,j})$, and we can observe that $(i+1)N - z' \geq iN - z$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$ and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a'_{i+1,j}$, and from the definition of $a^{d-}_{i,j}$, we get the inequality $(i+1)N + a'_{i+1,j} \geq i+1)N + a^{d-}_{i,j}$. Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a'_{i+1,j}$, and from the definition of $a^{d+}_{i,j}$, we get the inequality $(i+1)N - a'_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

Case 4: Both $S''_{i,j}$ and $S''_{i+1,j}$ are stars.

We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these row, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z_{i,j}$, where $z_{i,j}$ is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z_{i+1,j}$, where $z_{i+1,j}$ is the value formed using the pair which has the minimum a value, we can observe that $(i+1)N - z_{i+1,j} \geq iN - z_{i,j}$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i+1,j}$, and from the definition of $a^{d-}_{i,j}$ and $a^{u+}_{i+1,j}$, we get the inequality $(i+1)N + a^{u+}_{i+1,j} \geq (i+1)N + a^{d-}_{i,j}$.Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i+1,j}$, and from the definition of $a^{d+}_{i,j}$ and $a^{u-}_{i+1,j}$, we get the inequality $(i+1)N - a^{u-}_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

Note: As mention earlier, for both vertical and horizontal direction cases we have picked $S'_{4i-2,4j-3}$ to be the star in the case when the cell corresponding to this gadget is star in the MT solution, but we can pick any one of the $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}, S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}$ cell to be star and this lemma can be proved in the similar way.
We can also obeserve that if the cell in the MT instance is a $\star$ then we have to pick atleast one of the $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}, S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}$ to be $\star$ because if we pick all of these cells to be non-$\star$ then because of the $MT \leq$ condition it will form cycle of inequalities (which will be described in the next section) and the conditions will also be satisfies in the neighbouring gadgets such that we would be able to pick a pair (a, b) in the MT solution which will be contradiction.
Thus we have proved that for every non-star in the MT solution we will have all 16 non-stars for it's corresponding gadget in the $MT \leq$ solution and for every star in the MT solution we can pick atmost 15 non-stars for it's corresponding gadget in the $MT \leq$ solution.

□

2.2 Construction of S

Consider any solution y of $R(x)$ and the corresponding 4×4 cells. If all 16 cells in the gadget are non-stars, we claim that the outer 12 cells of the gadget are formed using the same pair (a, b). The proof of this claim is deferred to later in this section. Based on this observation, If all 16 cells are non-stars, select the pair (a, b) as the non-star solution in the corresponding cell of the MT instance. Otherwise, select a star.

Proof of the claim: Assume that all 16 cells in the gadget are non-stars. We now show that the outer 12 cells of the gadget are formed using the same pair (a, b).

The 12 outer sets in the gadget correspond to $S''_{i,j}$. The pairs selected in the solution from these sets define 12 values z, denoted as $z_{4i-3,4j-3}, z_{4i-3,4j-2}, \ldots$, representing the values selected from these sets. We claim that all these 12 values are equal.

To see this, consider the second coordinate of the pairs selected from these sets:

- The second coordinate of the set selected from $S''_{4i-3,4j-3}$ is $jN + z_{4i-3,4j-3}$. - Similarly, the second coordinate of the set selected from $S''_{4i-3,4j-2}$ is $jN + z_{4i-3,4j-2}$.

By the definition of Matrix Tiling with $\leq$, it follows that:

$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2}.$$

Continuing this reasoning for the other sets, we establish the following chain of inequalities:

- First row:
$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2} \leq z_{4i-3,4j-1} \leq z_{4i-3,4j}.$$
- Last column:
$$z_{4i-3,4j} \leq z_{4i-2,4j} \leq z_{4i-1,4j} \leq z_{4i,4j}.$$
- Last row (negated):
$$-z_{4i,4j-3} \leq -z_{4i,4j-2} \leq -z_{4i,4j-1} \leq -z_{4i,4j}.$$
- First column (negated):
$$-z_{4i-3,4j-3} \leq -z_{4i-2,4j-3} \leq -z_{4i-1,4j-3} \leq -z_{4i,4j-3}.$$

Combining all these inequalities results in a cycle of equalities, implying that all 12 values are the same.

Let $z_{i,j}$ denote this common value, and let $s''_{i,j} = (a_{i,j}, b_{i,j})$ be the corresponding pair such that $\iota(a_{i,j}, b_{i,j}) = z_{i,j}$. Since $z_{i,j}$ was defined using the pairs appearing in $S''_{i,j}$, it follows that $s''_{i,j} \in S''_{i,j}$.

Since $S''_{i,j}$ was constructed from $S_{i,j}$ by adding n to each coordinate, we can recover $s_{i,j} \in S_{i,j}$ by subtracting n from both coordinates of $s''_{i,j}$. Thus, $s_{i,j} = (a_{i,j} - n, b_{i,j} - n)$.

Consistency across neighboring gadgets: Next, consider the gadgets of neighboring cells:

1. **Vertical consistency:** Let $S''_{i+1,j}$ be the gadget of the cell below $S''_{i,j}$, and assume all 16 cells in these gadgets are non-stars. The pair selected from $S''_{4i,4j-2}$ has first coordinate $(i+1)N + a_{i,j}$, while the pair selected from $S''_{4i+1,4j-2}$ has first coordinate $(i+1)N + a_{i+1,j}$. By the definition of Matrix Tiling with $\leq$, we have:
$$a_{i,j} \leq a_{i+1,j}.$$
Similarly, comparing the first coordinates of the pairs selected from $S'_{4i,4j-1}$ and $S'_{4i+1,4j-1}$ yields $-a_{i,j} \leq -a_{i+1,j}$, which implies:
$$a_{i,j} = a_{i+1,j}.$$

2. **Horizontal consistency:** Let $S''_{i,j+1}$ be the gadget of the cell to the right of $S''_{i,j}$, and assume all 16 cells in these gadgets are non-stars. A similar argument using the last column of $S''_{i,j}$ and the first column of $S''_{i,j+1}$ shows:
$$b_{i,j} = b_{i,j+1}.$$

Case with stars: If one or more cells in the neighboring gadgets are stars, the solution selects a star in the MT instance. In this case, we do not need to prove consistency for $a_{i,j} = a_{i+1,j}$ or $b_{i,j} = b_{i,j+1}$, as the corresponding coordinates contain stars.

Conclusion: We have shown that the constructed $s_{i,j} \in S(y)$ forms a valid solution to the Matrix Tiling instance.

2.3 Relation between the optimal solutions of the original MT instance and the reduced instance of $MT\leq$ instance.

We can notice that the optimum is always at least $\frac{k^2}{4}$: if i and j are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$; otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: k^2, which gives us the following inequalities:

$$k^2/4 \leq OPT(x) \leq k^2 \tag{1}$$

$$k'^2/4 \leq OPT(R(x)) \leq k'^2$$

From the definition of R, we know that $k' = 4k$, therefore:

$$4k^2 \leq OPT(R(x)) \leq 16k^2 \tag{2}$$

Now from the equations (1) and (2), we can find the value of α for the 3^{rd} condition of the L-reduction:

.

$$\begin{aligned} &OPT(R(x)) \leq 16k^2 = 64k^2/4 = 64OPT(x) \\ &\implies OPT(R(x)) \leq 64OPT(x) \end{aligned} \tag{3}$$

Thus for $\alpha = 64$, we have $OPT(R(x)) \leq \alpha OPT(x)$. □

2.4 Relation between the optimal solutions and any approximate solutions of the original MT instance and the reduced instance of $MT\leq$ instance.

Let

$$OPT(x) = k^2 - b, \tag{4}$$

$$c_B(y) = 16k^2 - r, \tag{5}$$

$$c_A(S(y)) = k^2 - s. \tag{6}$$

from the definition of S, we can observe that:

$$c_B(y) \geq 16c_A(S(y)). \tag{7}$$

If we have $c_A(S(y)) = k^2 - s$, we have upper bound for $c_B(y) \leq 16(k^2 - s) + 15s$ (for every non-star in solution of MT we have 16 non-stars in $MT\leq$ and for every star we will have at most 15 non-stars in it's gadget). Thus we will get the inequality:

$$\begin{aligned} c_B(y) = 16k^2 - r &\leq 16(k^2 - s) + 15s \\ \implies 16k^2 - r &\leq 16k^2 - 16s + 15s \\ \implies -r &\leq -s \\ \implies r &\geq s. \end{aligned} \tag{8}$$

now we will use (8), to derive the 4^{th} condition of L-reduction:

.

$$\begin{aligned} &s \leq r \\ &\implies s - b \leq r - b \\ &\implies k^2 - k^2 + s - b \leq 16k^2 - 16k^2 + r - b \\ &\implies (k^2 - b) - (k^2 - s) \leq (16k^2 - b) - (16k^2 - r) \\ &\implies (k^2 - b) - (k^2 - s) \leq (1)((16k^2 - b) - (16k^2 - r)) \\ &\implies OPT(x) - c_A(S(y)) \leq (1)(OPT(R(x)) - c_B(y)). \end{aligned} \tag{9}$$

Thus for $\beta = 1$, we have $|OPT(x) - c_A(S(y))| \leq \beta|OPT(R(x)) - c_B(y)|$.

□

Note: Because both the problems MT and $MT \leq$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

Therefore, we have completed L-reduction from Matrix Tiling to Matrix Tiling with $\leq$, where the values of α and β are 64 and 1 respectively.