Matrix Tiling $\leq$ and Maximum set of independent Unit disks : L-reduction

Rahul Goraniya

November 2024

Note to the Admission Committee: This research report presents findings from my ongoing project for my undergraduate thesis. While the project is still in progress (this report reflects the work completed till now, the first reduction was completed in the first 8 weeks), the results presented here demonstrate meaningful contributions. Please note that I still have more than half semester remaining to complete this project, and this report represents only about 70% of the overall work. I am currently doing more research for this project, and I will continue to develop and refine it. As such, citations, grammar, and the rigor of certain sections will be addressed in the final version.

1 Abstract

Existing frameworks for establishing lower bounds on computational problems often utilize reductions from the Grid Tiling problem. While direct reductions from Grid Tiling are complex, the introduction of an intermediate problem, Grid Tiling $\leq$, provides a simpler reduction target, enabling streamlined reductions to the desired problems while preserving rigor. In this work, we extend this approach to optimization problems by introducing a reduction framework for the Matrix Tiling problem, the optimization variant of Grid Tiling. Specifically, we reduce the Matrix Tiling problem to its intermediate variant, Matrix Tiling $\leq$, and demonstrate how this can be further reduced to derive lower bounds for other optimization problems. Our results refine the existing methodology, making it more accessible for analyzing the complexity of optimization problems and establishing new lower bounds.

2 Matrix Tiling

2.1 Introduction

We are providing the L-reduction from Matrix Tiling to Matrix Tiling$\leq$.

Matrix Tiling:
Input: Integers k, D, and k^2 nonempty sets $S_{i,j} \subset Z_D \times Z_D (1 \leq i, j \leq k)$.
Find: For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 = b_1$.
2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 = b_2$.

Goal: Maximize the number of pairs (i, j) $(1 \leq i, j \leq k)$ with $s_{i,j} \neq \star$.

Matrix Tiling $\leq$:
Input: Integers k, D, and k^2 nonempty sets $S_{i,j} \subset Z_D \times Z_D (1 \leq i, j \leq k)$.
Find: For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 \leq b_1$.
2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 \leq b_2$.

Goal: Maximize the number of pairs (i, j) $(1 \leq i, j \leq k)$ with $s_{i,j} \neq \star$.

Where L-reduction is defined as follows:

Let A *and* B be optimization problems and c_A *and* c_B their respective cost functions. A pair of *polynomial time-computable functions* R *and* S *is an* L-reduction if all of the following conditions are met:

1. If x is an instance of problem A, then $R(x)$ is an instance of problem B,
2. If y is a solution to $R(x)$, then $S(y)$ is a solution to x,
3. There exists a constant $\alpha > 0$ such that $OPT(R(x)) \leq \alpha\, OPT(x)$,
4. There exists a constant $\beta > 0$ such that $|OPT(x) - c_A(S(y))| \leq \beta |OPT(R(x)) - c_B(y)|$.

Note:

- Here R is polynomial in the size of input instance of A, and S is polynomial in the size of B, but because R is polynomial, we can conclude that the size of S is polynomial in the size of R, which would then imply that R and S are in polynomial in the size of A.
- In the [paper] Marx, for defining L-reduction is requiring R and S to be logspace-computable functions, but here we are constraining R and S to be polytime functions which is slightly weaker requirement than logspace.
- Throughout this paper, we will considering the origin to be the top-left corner of the matrix.

2.2 Reduction and relation between solutions of both the instances.

2.2.1 Construction of R

Given an instance (n, k, S) of the Matrix Tiling problem, we construct an equivalent instance (n', k', S') of the Matrix Tiling $\leq$ problem through the following steps:

Step 1: Shifting Coordinates:

For each cell (i, j) in the matrix, let $S_{i,j} \subseteq [n] \times [n]$ denote the set of coordinate pairs associated with that cell. We shift all coordinates in $S_{i,j}$ by n, producing a new set $S''_{i,j}$:

$$S''_{i,j} = \{(x + n, y + n) \mid (x, y) \in S_{i,j}\}.$$

After the transformation, we define $n'' = 3n$, this new n'' and the transformation will prepare us for adding new coordinates which is described in the next section. .

Step 2: Additional Coordinate Pairs:

To help us prove a the fourth condition of the L-reduction, we will introduce a lemma in the next section and to help us prove that lemma, we introduce eight additional pairs of coordinates for each cell (i, j) in the MT instance, the first four will be used for the horizontal direction constraints of the $MT \leq$ for the lemma and the remaining four will be used for the vertical direction constraints.
The First four coordinates are:

$$(a^{min}_{i,j}, b^{r+}_{i,j}) = (\min(\text{first}(S_{i,j})), \max(\text{second}(S_{i,j+1})) + 1),$$

$$(a^{min}_{i,j}, b^{r-}_{i,j}) = (\min(\text{first}(S_{i,j})), \min(\text{second}(S_{i,j+1})) - 1),$$

$$(a^{min}_{i,j}, b^{l+}_{i,j}) = (\min(\text{first}(S_{i,j})), \max(\max(\text{second}(S_{i,j-1})), b^{r+}_{i,j-1}, b^{r-}_{i,j-1}) + 1),$$

$$(a^{min}_{i,j}, b^{l-}_{i,j}) = (\min(\text{first}(S_{i,j})), \min(\min(\text{second}(S_{i,j-1})), b^{r+}_{i,j-1}, b^{r-}_{i,j-1}) - 1).$$

The remaining four coordinates are:

$$(a^{d+}_{i,j}, b^{min}_{i,j}) = (\max(\text{first}(S_{i+1,j})) + 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{d-}_{i,j}, b^{min}_{i,j}) = (\min(\text{first}(S_{i+1,j})) - 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{u+}_{i,j}, b^{min}_{i,j}) = (\max(\max(\text{first}(S_{i-1,j})), a^{d+}_{i-1,j}, a^{d-}_{i-1,j}) + 1, (\min(\text{second}(S_{i,j}))).$$

$$(a^{u-}_{i,j}, b^{min}_{i,j}) = (\min(\min(\text{first}(S_{i-1,j})), a^{d-}_{i-1,j}, a^{d+}_{i-1,j}) - 1, (\min(\text{second}(S_{i,j}))).$$

Step 3: Constructing the equivalent instance of Matrix Tiling$\leq$(Definition of R):

Now from the (n'', k, S'') of Matrix Tiling problem, we construct the equivalent instance (n', k', S') of Matrix Tiling with $\leq$, defined with:

$$n' = 3n''^2(k+1) + n''^2 + n'', \quad k' = 4k.$$

For each set $S''_{i,j}$ in the original instance, we create 16 corresponding sets $S'_{i',j'}$ in the new instance, where $4i - 3 \leq i' \leq 4i$ and $4j - 3 \leq j' \leq 4j$ (see Fig 1). We call these sets, "gadget" representing $S''_{i,j}$.

1. Inner Dummy Sets: The four inner sets ($S'_{4i-2,4j-2}$, $S'_{4i-2,4j-1}$, $S'_{4i-1,4j-2}$, $S'_{4i-1,4j-1}$) are dummy sets any they have one only pairs for each of them. These sets are placeholders and do not depend on $S''_{i,j}$.

2. Outer Sets: The 12 outer sets are populated using a mapping function $\iota(a, b) = n''a + b$ and a scaling factor $N = 3n''^2$. For each $(a, b) \in S''_{i,j}$ *(note: we do not use newly the added eight pairs for the construction of $S'_{i',j'}$)*, we compute $z = \iota(a, b)$ and introduce pairs into the outer sets as follows:

$S'_{4i-3,4j-3}$: $(iN - z, jN + z)$	$S'_{4i-3,4j-2}$: $(iN + a, jN + z)$	$S'_{4i-3,4j-1}$: $(iN - a, jN + z)$	$S'_{4i-3,4j}$: $(iN + z, jN + z)$
$S'_{4i-2,4j-3}$: $(iN - z, jN + b)$	$S'_{4i-2,4j-2}$: $((i+1)N, (j+1)N)$	$S'_{4i-2,4j-1}$: $(iN, (j+1)N)$	$S'_{4i-2,4j}$: $(iN + z, (j+1)N + b)$
$S'_{4i-1,4j-3}$: $(iN - z, jN - b)$	$S'_{4i-1,4j-2}$: $((i+1)N, jN)$	$S'_{4i-1,4j-1}$: (iN, jN)	$S'_{4i-1,4j}$: $(iN + z, (j+1)N - b)$
$S'_{4i,4j-3}$: $(iN - z, jN - z)$	$S'_{4i,4j-2}$: $((i+1)N + a, jN - z)$	$S'_{4i,4j-1}$: $((i+1)N - a, jN - z)$	$S'_{4i,4j}$: $(iN + z, jN - z)$

Figure 1: The 16 sets of the constructed Matrix Tiling with $\leq$ instance representing a set $S_{i,j}$ of the Matrix Tiling in the reduction in the proof of together with the pairs corresponding to a pair $(a, b) \in S'_{i,j}$(with $z = \iota(a, b)$)

As mentioned earlier, we do not use the newly added coordinates directly to construct $S'_{i',j'}$. Instead, we employ the first four newly added coordinates solely for the construction of the $S'_{4i-2,4j-3}$, $S'_{4i-1,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$ sets within the corresponding gadget, as follows:
Let $B^{\oplus} = \{b^{r+}_{i,j}, b^{l+}_{i,j}, b^{r-}_{i,j}, b^{l-}_{i,j}\}$, The new coordinates of the sets are defined as:

$$S_{4i-2,4j} : (iN + z, (j+1)N + \{B^{\oplus}\}),$$

$$S_{4i-1,4j} : (iN + z, (j+1)N - \{B^{\oplus}\}),$$

$$S_{4i-2,4j-3} : (iN - z, jN + \{B^{\oplus}\}),$$

$$S_{4i-1,4j-3} : (iN - z, jN - \{B^{\oplus}\}).$$

where $z = \iota(a'_{i,j}, b'_{i,j})$, such that $(a'_{i,j}, b'_{i,j})$ is the pair form $S''_{i,j}$ where $a'_{i,j} = \min(\text{first } S_{i,j})$.

Similar to the first four newly added coordinates above, we also do not use these four coordinates directly to construct sets in all the 16 cells of the corresponding gadget of the MT $\leq$ instance. Instead, we employ these for the construction of the $S_{4i-3,4j-2}$, $S_{4i-3,4j-1}$, $S_{4i,4j-2}$, $S_{4i,4j-1}$ sets within the corresponding gadget, as follows:
Let $A^{\oplus} = \{a^{u+}_{i,j}, a^{d+}_{i,j}, a^{u-}_{i,j}, a^{d-}_{i,j}\}$, The new coordinates of the sets are defined as:

$$S_{4i-3,4j-2} : (iN + \{A^{\oplus}\}, jN + z),$$

$$S_{4i-3,4j-1} : (iN - \{A^{\oplus}\}, jN + z),$$

$$S_{4i,4j-2} : ((i+1)N + \{A^{\oplus}\}, jN - z),$$

$$S_{4i,4j-1} : ((i+1)N - \{A^{\oplus}\}, jN - z)$$

where $z = \iota(a'_{i,j}, b'_{i,j})$, such that $(a'_{i,j}, b'_{i,j})$ is the pair form $S''_{i,j}$ where $a'_{i,j} = \min(\text{first } S_{i,j})$.

It can be verified that all coordinates of each pair are positive and bounded by n'.

Note: Typically, coordinates are expressed as numerical values or variables. The inclusion of $\{B^{\oplus}\}$, which is a set, within a coordinate is a non-standard practice. In this context $\{B^{\oplus}\}$ represents all the different four elements of the set, so for example $S_{4i-2,4j} : (iN+z, (j+1)N+\{B^{\oplus}\})$, means we add four new coordinates in the $S_{4i-2,4j}$ which are in this case: $(iN+z, (j+1)N+b^{r+}_{i,j})$, $(iN+z, (j+1)N+b^{r-}_{i,j})$, $(iN+z, (j+1)N+b^{l+}_{i,j})$, and $(iN+z, (j+1)N+b^{l-}_{i,j})$.

These pairs are carefully constructed to help us prove the following lemma.

Lemma 1. *If there is a star in any cell in the solution of the Matrix Tiling instance, we can pick at least 15 non-stars in the corresponding gadget for the solution of the Matrix Tiling with $\leq$ instance.*

Proof. To prove this lemma we will first provide the way to select the pairs for the solution our $MT \leq$ instance from a solution of MT instance. We will have two different situations to mention: first is what to pick if the solution of MT has a non-$\star$ pair for the particular cell, and how we can pick 15 non-$\star$ pairs for the gadget corresponding to the cell which has $\star$ in the solution for MT instance:

1. For every non-star pair $s_{i,j} = (a, b)$, we select the corresponding pairs from the 16 sets in the gadget of $S''_{i,j}$ for $z = \iota(a, b)$, as shown in Figure 1.

2. For every star in the cell $S''_{i,j}$, pick one cell from $S'_{4i-2,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j-3}$, or $S'_{4i-1,4j}$ to be star, for this proof we will pick $S'_{4i-2,4j-3}$ as the star. We will pick the middle 4 sets $S_{4i-2,4j-2}$, $S_{4i-2,4j-1}$, $S_{4i-1,4j-2}$, $S_{4i-1,4j-1}$ in the corresponding gadget (because of the way they are defined there is only one pair corresponding to each of these so pick that pair), for all the corner sets $S'_{4i-3,4j-3}, S'_{4i-3,4j}, S'_{4i,4j}, S'_{4i,4j-3}$. We will pick the pairs formed using the pair $z = \iota(a'_{i,j}, b'_{i,j})$, where $a'_{i,j} = \min(\text{first}(S''_{i,j}))$ (i.e., the minimum of all the a's for that cell).

   We will pick pairs for $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}$ whose second coordinates are formed using $b^{l+}, b^{l-}, b^{r-}, b^{r+}$,(Note: here one from $S'_{4i-2,4j-3}$, $S'_{4i-1,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$, $S'_{4i-2,4j-3}$, $S'_{4i-1,4j-3}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$ these must be picked as star, and as for this proof we are choosing $S'_{4i-2,4j-3}$ to be star then we don't need to mention it but because one can pick any of these four to be a $\star$, for the sake of completeness we have provided the way to pick the pairs whose value is non-$\star$) respectively. We will pick pairs for $S'_{4i-3,4j-2}, S'_{4i-3,4j-1}, S'_{4i,4j-2}, S'_{4i,4j-1}$ whose first coordinates are formed using $a^{u+}, a^{u-}, a^{d-}, a^{d+}$, respectively.

Now we provide the proof that the solution formed using the way described above satisfies all the conditions of $MT \leq$:

1. First, it is easy to verify that the constraints are satisfied between the sets of the same gadget for both these cases.

2. Now we want to prove that the $MT \leq$ condition holds between all the gadgets. We have 4 cases for both the horizontal and vertical directions.

   (a) For the horizontal direction, we have the following 4 cases:

      i. Both $S''_{i,j}$ and $S''_{i,j+1}$ are non-stars,
      ii. $S''_{i,j}$ is a non-star and $S''_{i,j+1}$ is a star,
      iii. $S''_{i,j}$ is a star and $S''_{i,j+1}$ is a non-star,
      iv. Both $S''_{i,j}$ and $S''_{i,j+1}$ are stars.

   Case (i): Both $S''_{i,j}$ and $S''_{i,j+1}$ are non-stars. We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i+1,j}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has second coordinate $jN + z$, while the pair selected from $S'_{4i-3,4(j+1)-3} = S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z$, which is greater than or equal to $jN + z$. Similarly, there is no conflict between the last sets of these columns. If $b_{i,j} = b_{i,j+1}$ are the first coordinates of $S''_{i,j}$ and $S''_{i,j+1}$, then the second coordinates of the sets selected from the second sets of the rows, $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$, are $(j+1)N + b_{i,j}$ and $(j+1)N + b_{i+1,j}$, respectively, and the former is equal to the latter. One can show in a similar way that there is no conflict between the third sets of these columns.

   Case (ii): $S''_{i,j}$ is a non-star and $S''_{i,j+1}$ is a star. We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z'$, where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z$, where z is the value formed using the pair with the minimum a-value. We can observe that $(j+1)N + z \geq jN + z'$. Similarly, there is no conflict between the last sets of these columns. We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a star as defined earlier. Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b'_{i,j}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b^{l-}_{i,j+1}$. From the definition of $b^{l-}_{i,j+1}$, we get the inequality $(j+1)N - b'_{i,j} \leq (j+1)N - b^{l-}_{i,j+1}$.

Case (iii): $S''_{i,j}$ is a star and $S''_{i,j+1}$ is a non-star. We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z$, where z is the value formed using the pair with the minimum a-value, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z'$, where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and we can observe that $(j+1)N + z' \geq jN + z$. Similarly, there is no conflict between the last sets of these columns. Now the pair selected from $S'_{4i-2,4j}$ has the second coordinate $(j+1)N + b^{r-}$ and the pair selected from $S'_{4i-2,4j+1}$ has the second coordinate $(j+1)N + b'_{i,j+1}$. From the definition of b^{r-}, we get the inequality $(j+1)N + b^{r-} \leq (j+1)N + b'_{i,j+1}$. Similarly, the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b^{r+}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b'_{i,j+1}$. From the definition of b^{r+}, we get the inequality $(j+1)N - b^{r+} \leq (j+1)N - b'_{i,j+1}$.

Case (iv): Both $S''_{i,j}$ and $S''_{i,j+1}$ are stars. We look at the last column of the gadget of $S''_{i,j}$ and the first column of the gadget of $S''_{i,j+1}$. For the first sets in these columns, the constraints are satisfied: $S'_{4i-3,4j}$ has the second coordinate $jN + z_{i,j}$ where $z_{i,j}$ is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z_{i,j+1}$ where $z_{i,j+1}$ is the value formed using the pair which has the minimum a value, and we can observe that $(j+1)N + z_{i,j+1} \geq jN + z_{i,j}$.Similarly there is now conflict between last sets of these columns. We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a star as defined earlier. Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b^{r+}_{i,j}$, and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b^{l-}_{i,j+1}$, and from the definition of $b^{r+}_{i,j}$ and $b^{l-}_{i,j+1}$ we get the inequality $(j+1)N - b^{r+}_{i,j} \leq (j+1)N - b^{l-}_{i,j+1}$.

(b) Now for the vertical direction, we have the following 4 cases:

i. Both $S''_{i,j}$ and $S''_{i+1,j}$ are non-stars,
ii. $S''_{i,j}$ is a non-star and $S''_{i+1,j}$ is a star,
iii. $S''_{i,j}$ is a star and $S''_{i+1,j}$ is a non-star,
iv. Both $S''_{i,j}$ and $S''_{i+1,j}$ are stars.

Case (i): Both $S''_{i,j}$ and $S''_{i+1,j}$ are non-stars. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has first coordinate less than iN, while the pair selected from $S'_{4(i+1)-3,4j-3} = S'_{4i+1,4j-3}$ has the first coordinate at least $(i+1)N - (n^2+n) > iN$. Similarly, there is no conflict between the last sets of these rows. If $a_{i,j} = a_{i+1,j}$ are the first coordinates of $S''_{i,j}$ and $S''_{i+1,j}$, then the first coordinates of the sets selected from the second sets of the rows, $S'_{4i,4j-2}$ and $S'_{4i+1,4j-2}$, are $(i+1)N + a_{i,j}$ and $(i+1)N + a_{i+1,j}$, respectively, and the former is equal to the latter. One can show in a similar way that there is no conflict between the third sets of the rows.

Case (ii): $S''_{i,j}$ is a non-star and $S''_{i+1,j}$ is a star. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z'$, where $z' = \iota(a'_{i,j}, b'_{i,j})$ and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z$, where z is the value formed using the pair which has the minimum a value, and we can observe that $(i+1)N - z \geq iN - z'$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i,j+1}$, and from the definition of $a^{u+}_{i,j+1}$, we get the inequality $(i+1)N + a^{u+}_{i,j+1} \geq (i+1)N + a'_{i,j}$. Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i,j+1}$ and from the definition of $a^{u-}_{i,j+1}$, we get the inequality $(i+1)N - a^{u-}_{i,j+1} \geq (i+1)N - a'_{i,j}$.

Case (iii): $S''_{i,j}$ is a star and $S''_{i+1,j}$ is a non-star. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z$,, where z is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z'$ where $z' = \iota(a'_{i+1,j}, b'_{i+1,j})$, and we can observe that $(i+1)N - z' \geq iN - z$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$ and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a'_{i+1,j}$, and from the definition of $a^{d-}_{i,j}$, we get the inequality $(i+1)N + a'_{i+1,j} \geq$ $i+1)N + a^{d-}_{i,j}$. Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a'_{i+1,j}$, and from the definition of $a^{d+}_{i,j}$, we get the inequality $(i+1)N - a'_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

Case (iv): Both $S''_{i,j}$ and $S''_{i+1,j}$ are stars. We look at the last row of the gadget of $S''_{i,j}$ and the first row of the gadget of $S''_{i+1,j}$. For the first sets in these row, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z_{i,j}$, where $z_{i,j}$ is the value formed using the pair which has the minimum a value, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z_{i+1,j}$, where $z_{i+1,j}$ is the value formed using the pair which has the minimum a value, we can observe that $(i+1)N - z_{i+1,j} \geq iN - z_{i,j}$. Similarly there is now conflict between last sets of these rows. Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i+1,j}$, and from the definition of $a^{d-}_{i,j}$ and $a^{u+}_{i+1,j}$, we get the inequality $(i+1)N + a^{u+}_{i+1,j} \geq (i+1)N + a^{d-}_{i,j}$.Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i+1,j}$, and from the definition of $a^{d+}_{i,j}$ and $a^{u-}_{i+1,j}$, we get the inequality $(i+1)N - a^{u-}_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

Note: As mention earlier, for both vertical and horizontal direction cases we have picked $S'_{4i-2,4j-3}$ to be the star in the case when the cell corresponding to this gadget is star in the MT solution, but we can pick any one of the $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}, S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}$ cell to be star and this lemma can be proved in the similar way.
We can also observe that if the cell in the MT instance is a $\star$ then we have to pick at least one of the $S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}, S'_{4i-1,4j}, S'_{4i-2,4j-3}, S'_{4i-1,4j-3}, S'_{4i-2,4j}$ to be $\star$ because if we pick all of these cells to be non-$\star$ then because of the $MT \leq$ condition it will form cycle of inequalities (which will be described in the next section) and the conditions will also be satisfies in the neighboring gadgets such that we would be able to pick a pair (a, b) in the MT solution which will be contradiction.
Thus we have proved that for every non-star in the MT solution we will have all 16 non-stars for it's corresponding gadget in the $MT \leq$ solution and for every star in the MT solution we can pick at least 15 non-stars for it's corresponding gadget in the $MT \leq$ solution. ☐

From the lemma 1, if we have $OPT(x) = k^2 - a$ and $OPT(R(x)) = 16k^2 - b$, we can the following relation between these two:

$$\begin{aligned} OPT(R(x)) = 16k^2 - b &\geq 16(k^2 - a) + 15a \\ \implies 16k^2 - b &\geq 16k^2 - a \\ \implies -b &\geq -a \\ \implies a &\geq b \end{aligned} \tag{1}$$

2.2.2 Construction of S

Consider any solution y of $R(x)$ and the corresponding 4×4 cells. If all 16 cells in the gadget are non-stars, we claim that the outer 12 cells of the gadget are formed using the same pair (a, b). The proof of this claim is deferred to later in this section. Based on this observation, If all 16 cells are non-stars, select the pair (a, b) as the non-star solution in the corresponding cell of the MT instance. Otherwise, select a star.

Proof of the claim: Assume that all 16 cells in the gadget are non-stars. We now show that the outer 12 cells of the gadget are formed using the same pair (a, b).

The 12 outer sets in the gadget correspond to $S''_{i,j}$. The pairs selected in the solution from these sets define 12 values z, denoted as $z_{4i-3,4j-3}, z_{4i-3,4j-2}, \ldots$, representing the values selected from these sets. We claim that all these 12 values are equal.

To see this, consider the second coordinate of the pairs selected from these sets:

- The second coordinate of the set selected from $S''_{4i-3,4j-3}$ is $jN + z_{4i-3,4j-3}$. - Similarly, the second coordinate of the set selected from $S''_{4i-3,4j-2}$ is $jN + z_{4i-3,4j-2}$.

By the definition of Matrix Tiling with $\leq$, it follows that:

$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2}.$$

Continuing this reasoning for the other sets, we establish the following chain of inequalities:

- First row:
$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2} \leq z_{4i-3,4j-1} \leq z_{4i-3,4j}.$$

- Last column:
$$z_{4i-3,4j} \leq z_{4i-2,4j} \leq z_{4i-1,4j} \leq z_{4i,4j}.$$

- Last row (negated):
$$-z_{4i,4j-3} \leq -z_{4i,4j-2} \leq -z_{4i,4j-1} \leq -z_{4i,4j}.$$

- First column (negated):
$$-z_{4i-3,4j-3} \leq -z_{4i-2,4j-3} \leq -z_{4i-1,4j-3} \leq -z_{4i,4j-3}.$$

Combining all these inequalities results in a cycle of equalities, implying that all 12 values are the same.

Let $z_{i,j}$ denote this common value, and let $s''_{i,j} = (a_{i,j}, b_{i,j})$ be the corresponding pair such that $\iota(a_{i,j}, b_{i,j}) = z_{i,j}$. Since $z_{i,j}$ was defined using the pairs appearing in $S''_{i,j}$, it follows that $s''_{i,j} \in S''_{i,j}$.

Since $S''_{i,j}$ was constructed from $S_{i,j}$ by adding n to each coordinate, we can recover $s_{i,j} \in S_{i,j}$ by subtracting n from both coordinates of $s''_{i,j}$. Thus, $s_{i,j} = (a_{i,j} - n, b_{i,j} - n)$.

Consistency across neighboring gadgets: Next, consider the gadgets of neighboring cells:

1. **Vertical consistency:** Let $S''_{i+1,j}$ be the gadget of the cell below $S''_{i,j}$, and assume all 16 cells in these gadgets are non-stars. The pair selected from $S''_{4i,4j-2}$ has first coordinate $(i+1)N + a_{i,j}$, while the pair selected from $S''_{4i+1,4j-2}$ has first coordinate $(i+1)N + a_{i+1,j}$. By the definition of Matrix Tiling with $\leq$, we have:

$$a_{i,j} \leq a_{i+1,j}.$$

Similarly, comparing the first coordinates of the pairs selected from $S'_{4i,4j-1}$ and $S'_{4i+1,4j-1}$ yields $-a_{i,j} \leq -a_{i+1,j}$, which implies:

$$a_{i,j} = a_{i+1,j}.$$

2. **Horizontal consistency:** Let $S''_{i,j+1}$ be the gadget of the cell to the right of $S''_{i,j}$, and assume all 16 cells in these gadgets are non-stars. A similar argument using the last column of $S''_{i,j}$ and the first column of $S''_{i,j+1}$ shows:

$$b_{i,j} = b_{i,j+1}.$$

Case with stars: If one or more cells in the neighboring gadgets are stars, the solution selects a star in the MT instance. In this case, we do not need to prove consistency for $a_{i,j} = a_{i+1,j}$ or $b_{i,j} = b_{i,j+1}$, as the corresponding coordinates contain stars.

Conclusion: We have shown that the constructed $s_{i,j} \in S(y)$ forms a valid solution to the Matrix Tiling instance.

We can also find the relation between $c_A(S(y)) = k^2 - n$ and $c_B(y) = 16k^2 - m$, for every gadget with one or more $\star$ in y, we are selecting $\star$ for the corresponding cell in the $S(y)$, therefore in the best case $c_B(y)$ might have all the $\star$'s which are corresponding the different gadgets which would give $c_B(y) = 16k^2 - n$ (as we have $c_A = k^2 - n$ for the best case all the n stars were corresponding to the n gadgets with one $\star$) following inequality:

$$\begin{aligned} c_B(y) = 16k^2 - m &\leq 16k^2 - n \\ \implies -m &\leq -n \\ \implies m &\geq n \end{aligned} \tag{2}$$

We can also find the relation between $OPT(x) = k^2 - a$ and $OPT(R(x)) = 16k^2 - b$, now for every $\star$ in the optimal solution of x we can have at most 15 non-$\star$'s in its corresponding 4×4 gadget because if we had more than 15 (i.e., 16) then we would get the above chain of inequalities which would then imply that all the outer 12 cells of the gadget are formed using same (a, b)-pair and then we would be able to pick this particular pair it the corresponding cell for the MT solution and would also mean that this pair satisfies all the conditions of MT, This will lead to contradiction because we assumed this we are using optimal solution of MT which means there is no pair in the cells that are $\star$'s in this solution which satisfies the MT conditions. Therefore:

$$\begin{aligned} OPT(R(x)) = 16k^2 - b &\leq 16(k^2 - a) + 15a \\ \implies 16k^2 - b &\leq 16k^2 - a \\ \implies -b &\leq -a \\ \implies b &\geq a \end{aligned} \tag{3}$$

2.3 Relation between the optimal solutions of the original MT instance and the reduced instance of $MT \leq$ instance.

We can notice that the optimum is always at least $\frac{k^2}{4}$: if i and j are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$; otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: k^2, which gives us the following inequalities:

$$k^2/4 \leq OPT(x) \leq k^2 \tag{4}$$

$$k'^2/4 \leq OPT(R(x)) \leq k'^2$$

From the definition of R, we know that $k' = 4k$, therefore:

$$4k^2 \leq OPT(R(x)) \leq 16k^2 \tag{5}$$

Now from the equations (4) and (5), we can find the value of α for the 3^{rd} condition of the L-reduction:

$$\begin{aligned} OPT(R(x)) \leq 16k^2 &= 64k^2/4 = 64OPT(x) \\ \implies OPT(R(x)) &\leq 64OPT(x) \end{aligned} \tag{6}$$

Thus for $\alpha = 64$, we have $OPT(R(x)) \leq \alpha OPT(x)$.

2.4 Relation between the optimal solutions and any approximate solutions of the original MT instance and the reduced instance of $MT \leq$.

We have

$$OPT(x) = k^2 - a,$$
$$c_B(y) = 16k^2 - m,$$
$$c_A(S(y)) = k^2 - n,$$
$$OPT(R(x)) = 16k^2 - b.$$

from (2) and (1) it follows:

$$\begin{aligned} n &\leq m \\ \implies n - a &\leq m - a \\ \implies n - a \leq m - a &\leq m - b \\ \implies n - a &\leq m - b \\ \implies (n - a) &\leq (1)(m - b) \end{aligned} \tag{7}$$

we now look at the 4^{th} condition of L-reduction:

$$\begin{aligned} (OPT(x) - c_A(S(y))) &\leq \beta(OPT(R(x)) - c_B(y)) \\ (k^2 - a - (k^2 - n)) &\leq \beta(16k^2 - b - (16k^2 - m)) \\ (n - a) &\leq \beta(m - b) \end{aligned} \tag{8}$$

from (7) and (8) we can get $\beta = 1$.
Thus for $\beta = 1$, we have $|OPT(x) - c_A(S(y))| \leq \beta|OPT(R(x)) - c_B(y)|$.

Note: Because both the problems MT and $MT \leq$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

This completes L-reduction from Matrix Tiling to Matrix Tiling with $\leq$, where the values of α and β are 64 and 1 respectively.

3 Maximum Independent Set on Unite Disk Graph

3.1 Introduction

Maximum Independent Set on Unit Disk Graph($MISUDG$):
We are given a set S of unit- diameter disks in the plane(described by the coordinates of their centers). The goal is to find a maximum cardinality subset $S' \in S$ of disks, such that the disks in S' are pairwise disjoint.

Note: When dening the Matrix Tiling problem, we imagined the sets $S_{i,j}$ arranged in a matrix, with $S_{i,j}$ being in row i and column j. When reducing Matrix Tiling to a geometric problem, the natural idea is to represent $S_{i,j}$ with a gadget located around coordinate (i, j). However, this introduces an unnatural 90 degrees rotation compared to the layout of the $S_{i,j}$'s in the matrix, which can be confusing in the presentation of a reduction. Therefore, for geometric problems, it is convenient to imagine that $S_{i,j}$ is located at coordinate (i, j). To emphasize this interpretation, we use the notation $S[x, y]$ to refer to the sets; we imagine that $S[x, y]$ is at location (x, y), hence sets with the same x are on a vertical line and sets with the same y are on the same horizontal line (see Fig. 2). The constraints of Matrix Tiling are the same as before: the pairs selected from $S[x, y]$ and $S[x + 1, y]$ agree in the first coordinate, while the pairs selected from $S[x, y]$ and $S[x, y + 1]$ agree in the second coordinate. Matrix Tiling with $\leq$ is defined similarly. With this notation, we can give a very clean and transparent L-reduction to Maximum Independent Set of Unit Disk Graphs.

3.2 Reduction and relation between solutions of both the instances.

3.2.1 Construction of R

It will be convenient to work with open disks of radius $\frac{1}{2}$ (diameter 1). Two disks are nonintersecting if and only if the distance between their centers is at least 1.
Let $I = (n, k, S)$ be an instance of $MT \leq$. We construct a set D of unit disks such that D contains a set formed using the solution r of I.
Let $\epsilon = 1/n^2$. For every $1 \leq x, y \leq k$ and every $(a, b) \in S[x, y] \subset [n] \times [n]$, we introduce into D an open disk of radius $\frac{1}{2}$ centered at $(x + \epsilon a, y + \epsilon b)$; Let $D[x, y]$ be the set of these $|S[x, y]|$ disks introduced for a fixed x and y (see Fig.). Note that the disks in $D[x, y]$ all intersect each other. Therefore, if $D' \subseteq D$ is a set of pairwise nonintersecting disks, then $|D'| \leq k^2$ and $|D'| = k^2$ is possible only if D' contains exactly one disk from each $D[x, y]$. We need the following observation first. Consider two disks centered at $(x + \epsilon a, y + \epsilon b)$ and $(x + 1 + \epsilon a', y + \epsilon b')$ for some $(a, b), (a', b') \in [n] \times [n]$. We claim that they are nonintersecting if and only if $a \leq a'$. Indeed, if $a > a'$, then the square of the distance of the two centers is

$$\begin{aligned}(1 + \epsilon(a' - a))^2 + \epsilon^2(b' - b)^2 &\leq (1 + \epsilon(a' - a))^2 + \epsilon^2 n^2 \\ &\leq (1 - \epsilon)^2 + \epsilon = 1 - \epsilon + \epsilon^2 < 1\end{aligned}$$

(in the first inequality, we have used $b', b \leq n$; in the second inequality, we have used $a \geq a' + 1$ and $\epsilon = 1/n^2$). On the other hand, if $a \leq a'$, then the square of the distance is at least $(1 + \epsilon(a' - a))^2 \geq 1$, hence the two disks do not intersect (recall that the disks are open). This proves our claim. A similar claim shows that disks centered at $(x + \epsilon a, y + \epsilon b)$ and $(x + \epsilon a', y + 1 + \epsilon b')$ are nonintersecting if and only if $b \leq b'$. Moreover, it is easy to see that the disks centered at $(x + \epsilon a, y + \epsilon b)$ and $(x' + \epsilon a', y' + \epsilon b')$ for some $1 \leq a, a', b, b' \leq n$ cannot intersect if $|x - x'| + |y - y'| \geq 2$: the square of the distance between the two centers is at least $2(1 - \epsilon n)^2 > 1$.

$S[1,3]$: (1,1) (2,5) (3,3)	$S[2,3]$: (3,2) (2,3)	$S[3,3]$: (5,4) (3,4)
$S[1,2]$: (5,1) (1,4) (5,3)	$S[2,2]$: (3,1) (2,2)	$S[3,2]$: (1,1) (2,3)
$S[1,1]$: (1,1) (3,1) (2,4)	$S[2,1]$: (2,2) (1,4)	$S[3,1]$: (1,3) (2,3) (3,3)

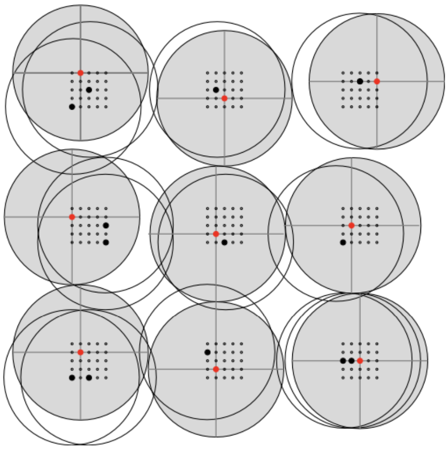

Figure 2: An instance of Matrix Tiling with $\leq$ with $k = 3$ and $n = 5$ and the corresponding instance of Maximum Independent Set on Unit Disk Graph constructed in the reduction. The small dots show the potential positions for the centers, the large dots are the actual centers in the constructed instance. The shaded disks with red centers correspond to the solution of Matrix Tiling with $\leq$ shown on the left

Lemma 2. *If we have some any solution $k^2 - e$ for I we can have at least $k^2 - e$ for our constructed instance of $MISUDG$.*

Proof. If we have $k^2 - e$ as a solution for $MT \leq$, it means we have the number of $\star$'s in the solution is e, for all the cells with the $\star$ in the solution don't pick any disk from that corresponding $D[x, y]$, and for the $k^2 - e$ cells which are non-$\star$ pairs. Let these non-$\star$ pairs be $s[x, y] = (a[x, y], b[x, y])$ for the $S[x, y]$'th cell. For every non-$\star$ cell, we select the disk $d[x, y]$ centered at $(x + \epsilon a[x, y], y + \epsilon b[x, y]) \in D[x, y]$.

Now we prove that all the disks selected this way do not intersect each other:

If the neighboring (right) cell in the solution of I was $\star$ we are not selecting the disk from that particular set ($D[x + 1, 1]$, where cell $S[x + 1, y]$ is $\star$). then we do not need to worry about the intersecting condition as the next closest disk (on the right side) will be in the set $D[x+2, y]$ which will imply that $|x-x'|+|y-y'| \geq 2$ and as proved earlier $d[x, y]$ and $d[x + 2, y]$ cannot intersect.

We can prove in the similar fashion all the remaining three cases:

- where the left cell is $\star$,
- where the above cell is $\star$,
- and where the below cell is $\star$.

Now we will prove that the disks selected in the suggested way will not intersect even in the case where all the four immediate neighboring cells are non-$\star$ in the solution of I. We can prove this is the following way: As have seen, if $|x - x'| + |y - y'| \geq 2$, then $d[x, y]$ and $d[x', y']$ cannot intersect. As the $s[x, y]$'s form a solution of the instance I, we have that $a[x, y] \leq a[x + 1, y]$. Therefore, by our claim above, the disks $d[x, y]$ and $d[x + 1, y]$ do not intersect. Similarly, we have $b[x, y] \leq b[x, y + 1]$, implying that $d[x, y]$ and $d[x, y + 1]$ do not intersect either. Hence there is indeed a set of at least $k^2 - e$ pairwise nonintersecting disks in D. $\square$

Now using the Lemma 2, we can prove the following relation for $OPT(x) = k^2 - a$ and $OPT(R(x)) = k^2 - b$

$$\begin{aligned} OPT(R(x)) &\geq OPT(x) \\ \implies k^2 - b &\geq k^2 - a \\ \implies -b &\geq -a \\ \implies b &\leq a \end{aligned} \tag{9}$$

3.2.2 Construction of S

Let $D' \subseteq D$ be a set of $k^2 - n$ pairwise independent disks. As mentioned earlier, the disks in $D[x, y]$ all intersect each other, which would imply that there at most one disk for each $D[x, y]$. Look at $D[x, y]$ and if there is a disk $d[x, y]$ centered at $(x + \epsilon a[x, y], y + \epsilon b[x, y])$ for some $(a[x, y], b[x, y]) \in [n] \times [n]$, first $d[x, y] \in D[x, y]$ implies that $s[x, y] = (a[x, y], b[x, y]) \in S[x, y]$, select this pair to form the solution for the instance I.

If there is no $d[x, y]$ selected from $D[x, y]$ in D' select $\star$ for the $S[x, y]$ cell to form the solution for the instance I.

We claim that the solution formed this way satisfies the conditions of $MT \leq$. For any a disk $d[x, y]$ centered at $(x + \epsilon a[x, y], y + \epsilon b[x, y])$, we have two cases:

1. The neighbor of $s[x, y] = (a[x, y], b[x, y]) \in S[x, y]$ (i.e. $S[x, y + 1]$) is star:
   In this case, as mentioned earlier we pick $\star$ for the $S[x, y + 1]$ for the solution of I instance, so we do not have to check the $MT \leq$ condition of the pair selected for $s[x, y] = (a[x, y], b[x, y])$.

2. The neighbors $S[x + 1, y]$ and $S[x, y + 1]$ of $s[x, y] = (a[x, y], b[x, y]) \in S[x, y]$ are non-$\star$:
   We know $S[x+1, y], S[x, y]$ and $S[x, y+1]$ are non-$\star$, because there are $d[x, y], d[x+1, y]$ and $d[x, y+1]$ is selected from $D[x, y], D[x + 1, y]$ and $D[x, y + 1]$ in D'. As we have seen above, the fact that $d[x, y]$ and $d[x + 1, y]$ do not intersect implies that $a[x, y] \leq a[x + 1, y]$. Similarly, the fact that $d[x, y]$ and $d[x, y + 1]$ do not intersect each other implies that $b[x, y] \leq b[x, y + 1]$.

Note: In the above proof for neighboring cases we only proved for the below neighbor ($D[x, y+1]$), but all the remaining cases (right neighbor $D[x + 1, y]$, top neighbor $D[x, y - 1]$, and left neighbor $D[x - 1, y]$) can be proved in the similar fashion.
Thus the $s[x, y]$'s selected this way indeed form a solution for the Matrix Tiling With $\leq$ instance I.
Here we can also conclude that if we construct the solution $c_A(S(y)) = k^2 - n$ of the instance I, from $c_B(y) = k^2 - m$ a solution of $MISUDG$ this way, we have the following equality:

$$\begin{aligned} k^2 - n &= k^2 - m \\ \implies n &= m \end{aligned} \tag{10}$$

3.3 Relation between the optimal solutions of the original $MT \leq$ instance and the reduced instance of $MISUDG$.

We can notice that the optimum for $MT \leq$ is always at least $\frac{k^2}{4}$: if i and j are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$; otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: k^2, which gives us the following inequality:

$$k^2/4 \leq OPT(x) \leq k^2 \tag{11}$$

For $MISUDG$ the lower bound can be 1, because it cannot be intersected by anything. The upper bound is k^2 because as mentioned earlier, the disks in $D[x, y]$ all intersect each other, which would imply that there at most one disk for each $D[x, y]$, and if we have one disk for each $D[x, y]$ we will get k^2 disks therefore:

$$1 \leq OPT(R(x)) \leq k^2 \tag{12}$$

Now from the equations (11) and (12), we can find the value of α for the 3^{rd} condition of the L-reduction:

$$\begin{aligned} OPT(R(x)) &\leq k^2 = 4k^2/4 = 4OPT(x) \\ \implies OPT(R(x)) &\leq 4OPT(x) \end{aligned} \tag{13}$$

Thus for $\alpha = 4$, we have $OPT(R(x)) \leq \alpha OPT(x)$.

3.4 Relation between the optimal solutions and any approximate solutions of the original $MT \leq$ instance and the reduced instance of $MISUDG$.

Let

$$\begin{aligned} OPT(x) &= k^2 - a, \\ OPT(R(x)) &= k^2 - b, \\ c_B(y) &= k^2 - m, \\ c_A(S(y)) &= k^2 - n. \end{aligned}$$

from (10) and (9) in follows:

$$\begin{aligned} n &= m \\ \implies n - a &= m - a \\ \implies n - a &= m - a \leq m - b \\ \implies n - a &\leq (1)(m - b). \end{aligned} \tag{14}$$

we now look at the 4^{th} condition of L-reduction:

$$\begin{aligned} OPT(x) - c_A(S(y)) &\leq (\beta)(OPT(R(x)) - c_B(y)) \\ \implies (k^2 - a) - (k^2 - n) &\leq (\beta)((k^2 - b) - (k^2 - m)) \\ \implies (n - a) &\leq (\beta)(m - b) \end{aligned} \tag{15}$$

from (14) and (15) we can get $\beta = 1$.
Thus for $\beta = 1$, we can satisfy: $|OPT(x) - c_A(S(y))| \leq \beta |OPT(R(x)) - c_B(y)|$.

Note: Because both the problems $MT \leq$ and $MISUDG$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

This completes L-reduction from Matrix Tiling with $\leq$ to Maximum Independent Set on Unit Disk Graph, where the values of α and β are 4 and 1 respectively.