# ETH-based Hardness for Approximation via L-Reductions from Matrix Tiling with $\leq$

**Rahul Goraniya**

Supervisor: Dr. Rajesh Chitnis

School of Computer Science

College of Engineering and Physical Sciences

University of Birmingham

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# Abstract

Existing frameworks for establishing lower bounds on parameterized problems often utilize reductions from the GRID TILING problem. While direct parameterized reductions from GRID TILING are complex, the introduction of an intermediate problem, GRID TILING WITH $\leq$, provides a simpler parameterized reduction to the desired problems while preserving the same bounds. In this work, we extend this approach to optimization problems. Specifically, we L-reduce the MATRIX TILING problem to its intermediate variant, MATRIX TILING WITH $\leq$, and demonstrate how this can be further reduced to derive PTAS lower bounds for other optimization problems. Our results refine the existing methodology, making it more accessible for establishing PTAS lower bounds.

# Acknowledgements

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# Preface

This report consists of six chapters.

- Chapter 1 provides the necessary background and literature review.

- Chapter 2 introduces the motivation and specific problems studied in this project.

- Chapter 3 presents the main contribution, where we give an L-reduction from MATRIX TILING to MATRIX TILING WITH $\leq$ deriving PTAS lower bound, inspired by the known reduction from GRID TILING to GRID TILING WITH $\leq$.

- Chapters 4 and 5 revisit the Independent Set and Scattered Set problems, respectively. Although PTAS lower bounds for these are already known, we rederive them using our reduction, thereby reinforcing the broader utility of the approach.

- Chapter 6 discusses future work that can be done based on this project

Readers already familiar with the definitions and context may choose to skip Chapters 1 and 2. Similarly, those primarily interested in the new contributions may focus on Chapter 3 and skip the final two chapters.

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Introduction and Overview

In this chapter, we introduce the basic terminology and notations used throughout the report. Unless otherwise stated, the definitions and theorems are adapted from [9].

## 1.1 NP hardness and P vs NP

In theoretical computer science there are classes of problems which are defined based on the time it takes for algorithms to solve them, below we provide some of the basic definitions.

**Definition 1.** *P: the class of problems which we can solve in polynomial time in the size od the input instance of the problem.*

**Definition 2.** *NP: the class of problems where we can verify a potential solution/answer in polynomial time in the size od the input instance of the problem.*

Clearly, $\mathbf{P} \subseteq \mathbf{NP}$, meaning solving is a (hard) way of verifying. But what about the other direction $\mathbf{P} \subseteq^? \mathbf{NP}$ i.e., Is $\mathbf{P} = \mathbf{NP}$, or $\mathbf{P} \neq \mathbf{NP}$?. We don't know the answer at this moment.

**Definition 3.** *Polytime reduction: We say that $Y \leq_P X$ if*

- *The reduction taked polynomial time*

- *$Y$ can be solved using (a black-box which solves) $X$.*

*We say that $Y$ is polytime reducible to $X$.*

**Definition 4.** *NP-hard: A problem $X$ is **NP-hard** if for all problems $Y \in NP$ we have $X \leq_P Y$.*

Intuitively, the above definition means that a problem is **NP-hard** if it is at least as hard as any other problem in NP. Assuming $P \neq NP$, a problem $X$ being NP-hard implies that we cannot have an algorithm $ALG$ for it which satisfies both of the following properties:

- $ALG$ is correct

- $ALG$ runs in polynomial time

This has led to development of new algorithmic paradigms such as:

- (Exact) Exponential Algorithms

- (Polytime) Approximation Algorithms

- (Polytime) Randomization Algorithms

- Parameterized Algorithms

## 1.2 Introduction to Parameterized Complexity and Why It Is Useful

In classical complexity, we can think of it as analysis in 1-D, we analyze in terms of input size $n$. But in Parameterized complexity we analyze complexity in terms on input size $n$ as well as solution size $k$, i.e., we can think of this as analysis in 2-D, as we not only analyze in terms of input size $n$, but also in terms of size of input $k$.

In classical complexity the problems: VERTEX COVER and INDEPENDENT SET are equivalent: If $S$ is an independent set, $V \setminus S$ is a vertex cover. But in Parameterized complexity the problems: K-VERTEX COVER and K-INDEPENDENT SET are not equivalent

- There is an **FPT** $\left(f(k) \cdot n^{O(1)}\right)$ algorithm for $k$-VC.

- But, there is no function $f$ such that the k-IS problem has an algorithm which runs in $f(k) \cdot n^{O(1)}$ time.

The above example is a basic example which shows that the parameterized complexity is much more refined way of analyzing computational complexity of problems. Now we formally define parameterized complexity and parameterized algorithms.

Algorithms with running time $f(k) \cdot n^c$, for a constant $c$ independent of both $n$ and $k$, are called *fixed-parameter algorithms*, or FPT algorithms. Typically the goal in parameterized algorithmics is to design FPT algorithms, trying to make both the $f(k)$ factor and the constraint $c$ in the bound on the running time as small as possible. FPT algorithms can be put in contrast with less efficient XP algorithms (for *slice-wise polynomial*), where the running time is of the form $f(k) \cdot n^{g(k)}$, for some functions $f, g$. There is a tremendous difference in the running times $f(k) \cdot n^{g(k)}$ and $f(k) \cdot n^c$.

In parameterized algorithmics, $k$ is simply a *relevant secondary measurement* that encapsulates some aspect of the input instance, be it the size of the solution sought after, or a number describing how "structured" the input instance is.

Any algorithmic theory is incomplete without an accompanying complexity theory that establishes intractability of certain problems. There is such a complexity theory providing lower bounds on the running time required to solve parameterized problems which we will describe later sections.

For the same problem there can be multiple choices of parameters. Selecting the right parameter(s) for a particular problem is an art.

**Definition 5** ([9])**.** *A parameterized problem is a language $L \subseteq \sum^* \times \mathbb{N}$, where $\sum$ is a fixed, finite alphabet. For an instance $(x, k) \in \sum \times \mathbb{N}$, $k$ is called the parameter.*

**Definition 6** ([9])**.** *A parameter problem $L \subseteq \sum^* \times \mathbb{N}$ is called fixed-parameter tractable (FPT) if there exists an algorithm $\mathcal{A}$ (called a fixed-parameter algorithm), a computable function $f : \mathbb{N} \to \mathbb{N}$, and a constant c such that, given $(x, k) \in \sum^* \times \mathbb{N}$, the algorithm $\mathcal{A}$ correctly decides whether $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^c$. The complexity class containing all fixed-parameter tractable problems is called FPT.*

**Definition 7** ([9])**.** *A parameterized problem $L \subseteq \sum^* \times \mathbb{N}$ is called slice-wise polynomial (XP) if there exists an algorithm $\mathcal{A}$ correctly decided whether $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^{g(k)}$. The complexity class containing all slice-wise polynomial problems is called XP.*

## 1.3   W[1] hardness

The theory of NP-completeness at least shows that there is one common underlying reason for the lack of polynomial-time algorithms, and therefore conditional lower bounds based on NP-completeness are the best we can have at this point of time.

Similar to NP-completeness theory of polynomial-time computation, a lower bound theory for parameterized problems has also been developed, which helps algorithm designers to present evidence for as many problems as possible that algorithms with certain specifications do not exist. Being aware of and being able to produce such negative results saves algorithm designers countless hours trying to prove results that contradict commonly accepted assumptions.

As we have no proof of $P \neq NP$, we cannot rule out the possibility that problems such as CLIQUE and DOMINATING SET are polynomial-time solvable and hence FPT. Therefore, the lower bound theory for parameterized problems has to be conditional: we are proving the statements of the form "if problem A has a certain type of algorithm, then problem B has a certain type of algorithm as well". If we have accepted as a working hypothesis that B has no such algorithms (or we have already proved that such an algorithm for B would contradict our working hypothesis), then this provides evidence that problem A does not have this kind of algorithm either.

The standard notion of polynomial-time reduction used in NP-completeness theory is not sufficient for our purposes. There a notion of parameterized reductions has been developed, which has slightly different flavor that NP-hardness proofs.

If we accept as a working hypothesis that CLIQUE is not fixed-parameter tractable, then the reductions from CLIQUE to other problems provide practical evidence that these problems are not fixed-parameter tractable either.

The remarkable aspect of NP-completeness is that there are literally thousands of natural hard problems that are equally hard in the sense that they are reducible to each other. The situation is different in the case of parameterized problems: there seems to be different levels of hardness in the case of parameterized problems: there seem to be different levels of hardness and even basic problems such as CLIQUE and DOMINATING SET seem to occupy different levels.

Downey and Fellows introduced the W-hierarchy in an attempt to classify parameterized problems according to their hardness. The CLIQUE problem is W[1]-complete, that is, complete for the first level of the W-hierarchy. Therefore, CLIQUE not being fixed-parameter tractable is equivalent to FPT $\neq$ W[1]. This is the basic assumption of parameterized complexity; we interpret W[1]-hardness as evidence that a problem is not fixed-parameter tractable.

**Definition 8 (Parameterized reduction, [9]).** *Let $A, B \subseteq \sum^* \times \mathbb{N}$ be two parameterized problems. A parameterized reduction form $A$ to $B$ is an algorithm that, given an instance $(n, k)$ of $A$, outputs an instance $(x', k')$ of $B$ such that*

1. *$(x, k)$ is a yes-instance of $A$ if and only if $(x', k')$ is an yes-instance of $B$,*

2. *$k' \leq g(k)$ for some computable function $g$, and*

3. *the running time is $f(k) \cdot |x|^{\mathcal{O}(1)}$ for some computable function $f$.*

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A *Boolean circuit* is a directed acyclic graph where the nodes are labeled in the following way:

- every node of indegree 0 is an *input node*,

- every node of indegree 1 is a *negation node*,

- every node of indegree $\geq 2$ is either an *and-node* or an *or-node*.

Additionally, exactly one of the nodes with outdegree 0 is labeled as *output node* (in addition to being, for example, an and-node). The *depth* of the circuit is the maximum length of a path from an input node to the output node.

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Deciding if a circuit has a satisfying assignment is clearly an NP-complete problem: for example, 3-SAT is its special case. The parameterized version of this problem is defined in the following way. The *weight* of an assignment is the number of input gates receiving value 1. In the WEIGHTED CIRCUIT SATISFIABILITY (WCS) problem, we are given a circuit $C$ and an integer $k$, the task is to decide if $C$ has a satisfying assignment of weight exactly $k$ and then checking whether it satisfies $C$. This problem does not seem to be fixed-parameter tractable.

The levels of the W-hierarchy are defined by restricting WEIGHTED CIRCUIT SATISFIABILITY to various classes of circuits. Formally, if $\mathcal{C}$ is a class of circuits, then we define WCS[$\mathcal{C}$] to be the restriction of the problem where the input circuit $C$ belongs to $\mathcal{C}$. To define what kind of restriction we are going to use, we first distinguish between *small nodes,* which have indegree at most 2, and *large nodes*, which have indegree $> 2$. The *weft* of a circuit is the maximum number of large nodes on a path from an input node to the output node. We denote by $\mathcal{C}_{t,d}$ the class of circuits with weft at most $t$ and depth at most $d$.

**Definition 9** (W-hierarchy [9])**.** *For $t \geq 1$, a parameterized problem $P$ belongs to the class $W[t]$ if there is a parameterized reduction from $P$ to $WCS[\mathcal{C}_{t,d}]$ for some $d \geq 1$.*

## 1.4 E.T.H. and classical complexity

The Exponential Time Hypothesis (ETH) is a conjecture stating that, roughly speaking, 3-SAT has no algorithms subexponential in the number of variables. This conjecture implies that FPT$\neq$W[1], hence it can also be used to give conditional evidence that certain problems are not fixed-parameter tractable. We can for example prove results saying that (assuming ETH) a problem cannot be solved in time $2^{o(n)}$, or a parameterized problem can not be solved in time $f(k)n^{o(k)}$, or a fixed-parameter tractable problem does not admit a $2^{o(n)}n^{\mathcal{O}(1)}$-time algorithm. In many cases, the lower bounds obtained this way match (up to small factors) the best known algorithm.

For $q \geq 3$, let $\delta_q$ be the infimum of the set of constants $c$ for which there exists an algorithm solving $q$-SAT in time $\mathcal{O}^*(2^{cn})$. The *Exponential-Time Hypothesis* is then defined as follows.

**Conjecture 1** (Exponential-Time Hypothesis [9])**.**

$$\delta_3 > 0$$

Intuitively, ETH states that any algorithm for 3-SAT needs to search through an exponential number of alternatives. Note that ETH implies that 3-SAT cannot be solved in time $2^{o(n)}$.

**Theorem 1** ([9])**.** *Unless ETH fails, there exists a constant $c > 0$ such that no algorithm for 3-SAT can achieve running time $\mathcal{O}^*(2^{c(n+m)})$. In particular, 3-SAT cannot be solved in time $2^{o(n+m)}$.*

The CNF-SAT and 3-SAT problems lie at the very foundations of the theory of NP-completeness. The problems around satisfiability of propositional formulas were the first problems whose NP-completness has been settled, and the standard approach to prove NP-hardness of a given problem is to try to find a polynomial-time reduction from 3-SAT, or from some problem whose NP-hardness is already known. Therefore, it is not surprising that by making a stronger assumption about the complexity of 3-SAT, we can infer stronger corollaries about all the problems that can be reached via polynomial time reductions from 3-SAT.

Consider, for instance, a problem $A$ that admits admits a *linear* reduction from 3-SAT, i.e., a polynomial-time algorithm that takes an instance of 3-SAT on $n$ variables and $m$ clauses, and outputs an equivalent instance of $A$ whose size is bounded by $\mathcal{O}(n + m)$. Then if $A$ admitted an algorithm with running time $2^{o(|x|)}$, where $|x|$ is the size of the input instance , then composing the reduction with such an algorithm would yield an algorithm for 3-SAT running in time $2^{o(n+m)}$, which contradicts ETH by Theorem 1.
There are many problems that are proved to be W[1]-hard based on ETH. Some examples are presented in the book Theorem 14.11, Theorem 14.12, Theorem 14.14 etc. in [9, Section 14.4].

Lower bounds based on ETH and motivation of this project

## 2.1 Framework for lower bounds

Clique was shown to be a W[1]-*hard*, which appears as Theorem 14.21 in [9, Section 14.4]. But it is well known fact that planar graphs do not contain a clique of size 5 or more. So reductions from this problem is not that useful to show W[1]-*hardness* in planar graph problems.

In his 2007 paper, Marx proposed a new problem called the MATRIX TILING problem [12].

**Note:** We denote by $Z_D$ the set $\{0, 1, ..., D-1\}$ throughout the paper.

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MATRIX TILING

**Input:** Integers $k, D$, and $k^2$ nonempty sets $S_{i,j} \subseteq Z_D \times Z_D$ for $1 \leq i, j \leq k$.

**Find:** For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 = b_1$.

2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 = b_2$.

**Goal:** Maximize the number of pairs $(i, j)$ with $s_{i,j} \neq \star$.

---

| $S_{1,1}$: | $S_{1,2}$: | $S_{1,3}$: |
|---|---|---|
| (1,1) | (5,1) | (1,1) |
| (3,1) | (1,4) | (2,5) |
| (2,4) | (5,3) | (3,3) |
| $S_{2,1}$: | $S_{2,2}$: | $S_{2,3}$: |
| (2,2) | (3,1) | (3,2) |
| (1,4) | (2,2) | (2,3) |
| | | |
| $S_{3,1}$: | $S_{3,2}$: | $S_{3,3}$: |
| (1,3) | (1,1) | (5,4) |
| (2,3) | (2,3) | (3,4) |
| (3,3) | | |

Figure 2.1: An instance of Matrix Tiling with $\leq$ with $k = 3$ and $n = 5$. The red pairs form a solution [9, Figure 14.4]

In the same paper, Marx derived a PTAS lower bound for the MATRIX TILING problem [12].

**Definition 10** ([15], Definition 1.2). *A polynomial-time approximation scheme (PTAS) is a family of algorithms $\{A_\epsilon\}$, where there is an algorithm for each $\epsilon > 0$, such that $A_\epsilon$ is a $(1 + \epsilon)$-approximation algorithm (for minimization problems) or a $(1 - \epsilon)$-approximation algorithm (for maximization problems).*

**Theorem 2** ([12], Theorem 2.3). *If there are constants $\delta, d > 0$ such that MATRIX TILING has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

We can use the above Theorem 2 to derive PTAS lower bounds for new other problems using the following a reduction known as L-reduction and the lemma accompanying it.

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L-reduction:

Let A *and* B be optimization problems and $c_A$ *and* $c_B$ their respective cost functions. A pair of *polynomial time-computable functions* R *and* S *is an* L-reduction if all of the following conditions are met:

1. If $x$ is an instance of problem $A$, then $R(x)$ is an instance of problem $B$,

2. If $y$ is a solution to $R(x)$, then $S(y)$ is a solution to $x$,

3. There exists a constant $\alpha > 0$ such that $OPT(R(x)) \leq \alpha\, OPT(x)$,

4. There exists a constant $\beta > 0$ such that $|OPT(x) - c_A(S(y))| \leq \beta|OPT(R(x)) - c_B(y)|$.

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**Lemma 1** ([12], Lemma 2.8 (i)). *If there is an L-reduction from* MATRIX TILING *to Problem X, then there are no $d, \delta > 0$ such that Problem X admits a PTAS with running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, unless ETH fails.*

This problem turns out to be very useful especially for planar graphs in $\mathbb{R}^2$, because of its planar structure. So Marx then modified this to be a yes/no problem [14], which is known as GRID TILLNG problem.

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GRID TILING

**Input:** Integers $k, n$, and $k^2$ nonempty sets $S_{i,j} \subseteq Z_n \times Z_n (1 \leq i, j \leq k)$.
**Find:** For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 = b_1$.

2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 = b_2$.

---

But in this problem we have for rows and columns constraint of equality so for reductions we have to check two conditions $\leq$ and $\geq$ for two things to be equal.

So to make reductions simpler Marx and Sidiropoulos simplified the GRID TILING to GRID TILING WITH $\leq$. Now we only have to check one inequality. GRID TILING WITH $\leq$ is very useful in deriving lower bounds for many graph theory problems [3, 10, 1, 13, 7, 6, 2, 11, 8, 5, 4].

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GRID TILING WITH $\leq$

**Input:** Integers $k, n$, and $k^2$ nonempty sets $S_{i,j} \subseteq Z_n \times Z_n (1 \leq i, j \leq k)$.
**Find:** For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 \leq b_1$.

2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 \leq b_2$.

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## 2.2 Motivation of the project

The question we explored for this project is that can we do the same thing for optimization version MATRIX TILING and derive the PTAS lower bound for MATRIX TILING WITH $\leq$, which would make L-reductions to derive Lower bounds for other problems comparatively easier.
In next chapter, we prove the PTAS lower bound for MATRIX TILING WITH $\leq$, which is inspired from the exact reduction Theorem 14.30 in [9, Section 14.4] and in chapter 4 and chapter 5, we explore two problems: MAXIMUM INDEPENDENT SET ON UNIT DISK GRAPHS and MAXIMUM SCATTERED SET.

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MAXIMUM INDEPENDENT SET ON UNIT DISK GRAPH
We are given a set $S$ of unit-diameter disks in the plane(described by the coordinates of their centers). The goal is to find a maximum cardinality subset $S' \in S$ of disks, such that the disks in $S'$ are pairwise disjoint.

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SCATTERED SET
Given a graph $G$ and integer $d$, the task is to find a maximum sized set $S$ of vertices in $G$ such that the distance between any two distinct $u, v \in S$ is at least $d$.

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The Lower bounds for the above two problems were already derived but in this project we derive same lower bounds using L-reduction from MATRIX

TILING WITH $\leq$ which are inspired from the exact reductions Theorem 14.33 and Theorem 14.32 in [9, Section 14.4], providing evidence that deriving lower bounds using this framework requires simpler reductions compared to existing techniques.

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# Lower bound for MATRIX TILING with $\leq$

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## 3.1 Introduction

Let us recall the problem definitions and L-reduction definition.
***Note:*** We denote by $Z_D$ the set $\{0, 1, ..., D-1\}$ throughout the paper.

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**Matrix Tiling:**
***Input:*** Integers $k, n$, and $k^2$ nonempty sets $S_{i,j} \subseteq Z_n \times Z_n (1 \leq i, j \leq k)$.
***Find:*** For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 = b_1$.

2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 = b_2$.

***Goal:*** Maximize the number of pairs $(i, j)$ $(1 \leq i, j \leq k)$ with $s_{i,j} \neq \star$.

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**Matrix Tiling $\leq$:**
***Input:*** Integers $k, n$, and $k^2$ nonempty sets $S_{i,j} \subseteq Z_n \times Z_n (1 \leq i, j \leq k)$.
***Find:*** For each $1 \leq i, j \leq k$, a value $s_{i,j} \in S_{i,j} \cup \{\star\}$ such that:

1. If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 \leq b_1$.

2. If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 \leq b_2$.

***Goal:*** Maximize the number of pairs $(i, j)$ $(1 \leq i, j \leq k)$ with $s_{i,j} \neq \star$.

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Where L-reduction is defined as follows:

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Let A *and* B be optimization problems and $c_A$ *and* $c_B$ their respective cost functions. A pair of *polynomial time-computable functions* R *and* S *is an* L-reduction if all of the following conditions are met:

---

1. If $x$ is an instance of problem $A$, then $R(x)$ is an instance of problem $B$,

2. If $y$ is a solution to $R(x)$, then $S(y)$ is a solution to $x$,

3. There exists a constant $\alpha > 0$ such that $OPT(R(x)) \leq \alpha\, OPT(x)$,

4. There exists a constant $\beta > 0$ such that $|OPT(x) - c_A(S(y))| \leq \beta|OPT(R(x)) - c_B(y)|$.

***Note:***

- Here $R$ is polynomial in the size of input instance of $A$, and $S$ is polynomial in the size of $B$, but because $R$ is polynomial, we can conclude that the size of $S$ is polynomial in the size of $R$, which would then imply that $R$ and $S$ are in polynomial in the size of $A$.

- In [12] Marx, for defining L-reduction is requiring $R$ and $S$ to be logspace-computable functions, but here we are constraining $R$ and $S$ to be polytime functions which is slightly weaker requirement than logspace.

- Throughout this paper, we will considering the origin to be the top-left corner of the matrix.

***Notations:***

1. Let $\text{first}(a, b) = a$, and $\text{second}(a, b) = b$.

2. In a matrix or grid, for some cell $S_{i,j}$, the cell to the right is $S_{i,j+1}$, the cell to the left is $S_{i,j-1}$, the cell to the top is $S_{i-1,j}$, and the cell to the down is $S_{i+1,j}$.

| | |
|---|---|
| $S_{i,j}$ | $S_{i,j+1}$ |
| $S_{i+1,j}$ | $S_{i+1,j+1}$ |

3. Let $b_{\min}^{\text{left}(i,j)}$ be the value of the second coordinate of the pair which has the minimum second coordinate among all the pairs from the cell to the left of the current cell ($S_{i,j-1}$) (i.e., $b_{\min}^{\text{left}(i,j)} = \min(\text{second}(S_{i,j-1}))$ for each $1 \leq i \leq k$, $1 < j \leq k$).

4. Let $b_{\max}^{\text{left}(i,j)}$ be the value of the second coordinate of the pair which has the maximum second coordinate among all the pairs from the cell to the left of the current cell ($S_{i,j-1}$) (i.e., $b_{\max}^{\text{left}(i,j)} = \max(\text{second}(S_{i,j-1}))$ for each $1 \leq i \leq k$, $1 < j \leq k$).

5. Let $b_{\min}^{\text{right}(i,j)}$ be the value of the second coordinate of the pair which has the minimum second coordinate among all the pairs from the cell to the right of the current cell ($S_{i,j+1}$) (i.e., $b_{\min}^{\text{right}(i,j)} = \min(\text{second}(S_{i,j+1}))$ for each $1 \leq i \leq k$, $1 \leq j < k$).

6. Let $b_{\max}^{\mathrm{right(i,j)}}$ be the value of the second coordinate of the pair which has the maximum second coordinate among all the pairs from the cell to the right of the current cell $(S_{i,j+1})$ (i.e., $b_{\max}^{\mathrm{right(i,j)}} = \max(\mathrm{second}(S_{i,j+1}))$ for each $1 \leq i \leq k$, $1 \leq j < k$).

7. Let $a_{\min}^{\mathrm{top(i,j)}}$ be the value of the first coordinate of the pair which has the minimum first coordinate among all the pairs from the cell to the top of the current cell $(S_{i-1,j})$ (i.e., $a_{\min}^{\mathrm{top(i,j)}} = \min(\mathrm{first}(S_{i-1,j}))$ for each $1 < i \leq k$, $1 \leq j \leq k$).

8. Let $a_{\max}^{\mathrm{top(i,j)}}$ be the value of the first coordinate of the pair which has the maximum first coordinate among all the pairs from the cell to the top of the current cell $(S_{i-1,j})$ (i.e., $a_{\max}^{\mathrm{top(i,j)}} = \max(\mathrm{first}(S_{i-1,j}))$ for each $1 < i \leq k$, $1 \leq j \leq k$).

9. Let $a_{\min}^{\mathrm{down(i,j)}}$ be the value of the first coordinate of the pair which has the minimum first coordinate among all the pairs from the cell to the down of the current cell $(S_{i+1,j})$ (i.e., $a_{\min}^{\mathrm{down(i,j)}} = \min(\mathrm{first}(S_{i+1,j}))$ for each $1 \leq i < k$, $1 \leq j \leq k$).

10. Let $a_{\max}^{\mathrm{down(i,j)}}$ be the value of the first coordinate of the pair which has the maximum first coordinate among all the pairs from the cell to the down of the current cell $(S_{i+1,j})$ (i.e., $a_{\max}^{\mathrm{down(i,j)}} = \max(\mathrm{first}(S_{i+1,j}))$ for each $1 \leq i < k$, $1 \leq j \leq k$).

11. Let $(a_{i,j}^{z_{\max}}, b_{i,j}^{z_{\max}})$ be the pair which has the maximum first coordinate from all the pairs of the cell $S_{i,j}$.

We are providing the L-reduction from Matrix Tiling to Matrix Tiling$\leq$. And by doing so from Theorem 2 and Lemma 1, we can derive the PTAS lower bound for MATRIX TILING WITH $\leq$ which is described in the following theorem:

**Theorem 3.** *If there are constants $\delta, d > 0$ such that* MATRIX TILING WITH $\leq$ *has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

**Note:** In the above theorem $n$ is not the range for the coordinated, but the input size of the problem instance.

## 3.2 Reduction and relation between solutions of both the instances.

Given an instance $\mathcal{I} = (n, k, S)$ of the Matrix Tiling problem, we construct an equivalent instance $\mathcal{M} = (n', k', S')$ of the Matrix Tiling $\leq$ problem through the following steps:

### Step 1: Shifting Coordinates:

First for each cell $(i, j)$ in the matrix, let $S_{i,j} \subseteq [n] \times [n]$ denote the set of coordinate pairs associated with that cell. We shift all coordinates in $S_{i,j}$ by $n$,

producing a new set $S_{i,j}^{3n}$:

$$S_{i,j}^{3n} = \{(x+n, y+n) \mid (x,y) \in S_{i,j}\}. \tag{3.1}$$

After this we change $n \to 3n$, this new range $(3n)$ and the following transformation will prepare us for adding new coordinates which is described in the next section.

We performed this step to prepare for the addition of seven new coordinates in the following step. Some of these new coordinates will be strictly smaller than all the pair in from the neighboring cells, while others will be strictly larger. For instance, if the largest value of the coordinate from right is $n$, keeping $n$ as the upper bound would prevent the addition of any new coordinates greater than $n$. By changing $n$ to $3n$, we effectively create space for adding coordinates greater than $n$.

In addition, we shifted all the coordinates by $n$ to ensure that we can also add values smaller than the current ones. Without this shift, if there is a coordinate at 0 on the left, no smaller values could be added. By shifting all the coordinates by $n$, we move 0 to $n$, which allows us to add, a values strictly less than $n$ as well and so on.

## Step 2: Additional Coordinate Pairs:

We introduce seven additional pairs of coordinates for each cell $(i,j)$ in the $\mathcal{I}$ instance, which will help in proving necessary condition to satisfy the fourth condition of the L-reduction, the first three will be used for the horizontal direction constraints of the MATRIX TILING WITH $\leq$, and the remaining four will be used for the vertical direction constraints.
The First three coordinates are:

$$a_{i,j}^{d+} = a_{max}^{down(i,j)} + 1 \tag{3.2}$$

Note that $a_{i,j}^{d+}$ is strictly greater than all the values of first coordinates from all pairs present in the cell the the down of the current cell $S_{i+1,j}$. For the border case that is when $i = k$, there is no row below this row so we can just set $a_{i,j}^{d+} = \max(\text{first}(S_{k,j}))$, for all $1 \leq j \leq k$.

$$a_{i,j}^{d-} = a_{min}^{down(i,j)} - 1 \tag{3.3}$$

Note that $a_{i,j}^{d-}$ is strictly less than all the values of first coordinates from all pairs present in the cell the the down of the current cell $S_{i+1,j}$. For the border case that is when $i = k$, there is no row below this row so we can just set $a_{i,j}^{d-} = \min(\text{first}(S_{k,j}))$, for all $1 \leq j \leq k$.

$$a_{i,j}^{u+} = \max(a_{max}^{top(i,j)}, a_{i-1,j}^{d-}, a_{i-1,j}^{d+}) + 1 \tag{3.4}$$

Note that $a_{i,j}^{u+}$ is strictly greater than all the values of first coordinates from all pairs present in the cell the the top of the current cell $S_{i-1,j}$. It is also strictly greater than the new two values added to the cell which are $a_{i-1,j}^{d-}$ and $a_{i-1,j}^{d+}$ For the border case that is when $i = 1$, there is no row above this row so we can just set $a_{i,j}^{u-} = \min(\text{first}(S_{1,j}))$, for all $1 \leq j \leq k$..

$$a_{i,j}^{u-} = \min(a_{min}^{top(i,j)}, a_{i-1,j}^{d-}, a_{i-1,j}^{d+}) - 1 \tag{3.5}$$

Note that $a_{i,j}^{u-}$ is strictly less than all the values of first coordinates from all pairs present in the cell the the top of the current cell $S_{i-1,j}$. It is also strictly less than the new two values added to the cell which are $a_{i-1,j}^{d-}$ and $a_{i-1,j}^{d+}$. For the border case that is when $i = 1$, there is no row above this row so we can just set $a_{i,j}^{u+} = \max(\text{first}(S_{1,j}))$, for all $1 \leq j \leq k$.

$$b_{i,j}^{r-} = b_{min}^{right(i,j)} - 1 \tag{3.6}$$

Note that $a_{i,j}^{r-}$ is strictly less than all the values of second coordinates from all pairs present in the cell the the right of the current cell $S_{i,j+1}$. For the border case that is when $j = k$, there is no column right to this column so we can just set $b_{i,k}^{r-} = \min(\text{first}(S_{i,k}))$, for all $1 \leq i \leq k$.

$$b_{i,j}^{r+} = b_{max}^{right(i,j)} + 1 \tag{3.7}$$

Note that $a_{i,j}^{r+}$ is strictly greater than all the values of second coordinates from all pairs present in the cell the the right of the current cell $S_{i,j+1}$. For the border case that is when $j = 1$, there is no column right to this column so we can just set $b_{i,k}^{r+} = \max(\text{first}(S_{i,k}))$, for all $1 \leq i \leq k$.

$$b_{i,j}^{l-} = \min(b_{min}^{left(i,j)}, a_{i,j-1}^{r+}, a_{i,j-1}^{r-}) - 1 \tag{3.8}$$

Note that $b_{i,j}^{l-}$ is strictly less than all the values of second coordinates from all pairs present in the cell the the left of the current cell $S_{i,j-1}$. It is also strictly less than the new two values added to the cell which are $b_{i,j-1}^{r-}$ and $b_{i,j-1}^{r+}$.For the border case that is when $j = 1$, there is no column left to this column so we can just set $b_{i,1}^{l-} = \min(\text{first}(S_{i,1}))$, for all $1 \leq i \leq k$.

Using the above defined seven values we add the following seven pairs to each cell in the instance $I$:

$$(a_{i,j}^{u+}, b_{i,j}^{z_{max}}), (a_{i,j}^{u-}, b_{i,j}^{z_{max}}), (a_{i,j}^{d+}, b_{i,j}^{z_{max}}), (a_{i,j}^{d-}, b_{i,j}^{z_{max}}),$$
$$(a_{i,j}^{z_{max}}, b_{i,j}^{r-}), (a_{i,j}^{z_{max}}, b_{i,j}^{r+}), (a_{i,j}^{z_{max}}, b_{i,j}^{l+})$$

**Definition 11.** *Let us call the above added seven pairs collectively as* NEW-I *pairs.*

We introduce these coordinate pairs to help us establish a condition, which states that for every $\star$ in the optimal solution of the $\mathcal{I}$ instance, we can select at least 15 non-$\star$ points in its corresponding gadget in the $\mathcal{M}$ instance. This property provides a crucial connection between the optimal solutions of $\mathcal{I}$ and $\mathcal{M}$, which is necessary for determining the value of $\beta$ in the fourth condition of the L-reduction.

To illustrate this, consider a case where a $\star$ appears in the optimal solution of $\mathcal{I}$ due to the absence of equal neighboring values in the coordinate pairs. By introducing additional values that are strictly greater and strictly smaller than all corresponding coordinate pairs on the right, top, left, and bottom, we ensure that these newly introduced values can be selected in the corresponding gadget's cells. Since these values maintain the necessary ordering constraints, they naturally satisfy the $\leq$ condition in the $\mathcal{M}$ instance.

## Step 3: Constructing the instance of Matrix Tiling with $\leq$ (Definition of R):

Now from the $\mathcal{I} = (n, k, S^{3n})$ of MATRIX TILING, we construct the instance $\mathcal{M} = (n', k', S')$ of MATRIX TILING WITH $\leq$, defined with:

$$n' = 3n^2(k+1) + n^2 + 3n, \quad k' = 4k.$$

For each set $S_{i,j}^{3n}$ in the instance $\mathcal{I}$, we create 16 corresponding sets $S_{i',j'}'$ in the instance $\mathcal{M}$, where $4i - 3 \leq i' \leq 4i$ and $4j - 3 \leq j' \leq 4j$ (see Figure 3.1). We call these sets, "gadget" representing $S_{i,j}^{3n}$.

**1. Inner Dummy Sets:** The four inner sets $(S_{4i-2,4j-2}', S_{4i-2,4j-1}', S_{4i-1,4j-2}', S_{4i-1,4j-1}')$ are dummy sets and they have one only pairs for each of them. These sets are placeholders and do not depend on pairs from $S_{i,j}^{3n}$.

**2. Outer Sets:** The 12 outer sets are populated using a mapping function $\iota(a_{i,j}, b_{i,j}) = na_{i,j} + b_{i,j}$ and a scaling factor $N = 3n^2$. For each $(a_{i,j}, b_{i,j}) \in S_{i,j}^{3n}$ (note: we do not use NEW-I pairs for the construction of $S_{i',j'}'$, we only use the original pairs), we compute $z = \iota(a,b)$ and introduce pairs into the outer sets as follows:

**Definition 12.** Let $z_{i,j}^+ = \iota(a_{i,j}^{z_{max}} + 2, b_{i,j}^{z_{max}})$.

**Claim 1.** No value $z$ created from any pair $(a_{i,j}, b_{i,j}) \in S_{i,j}^{3n}$ can be greater or equal to $z_{i,j}^+$.

*Proof.* Because we have the constraint the $0 \leq b_{i,j} \leq n$ and $0 \leq a_{i,j} \leq n$ from the definition of MATRIX TILING problem. Let $(a, b)$ is the pair form $S_{i,j}^{3n}$ where $a = \max(\text{first } S_{i,j}^{3n})$, which implies that $a \geq a_{i,j}$ From the Definition 12 of $z_{i,j}^+$, if we have any $z' \geq z_{i,j}^+$, this would imply that

$$a_{i,j}n + b_{i,j} \geq an + n + b$$
$$\implies b_{i,j} \geq n(a - a_{i,j} + 2) + b$$
$$\implies b_{i,j} \geq 2n$$

which is not possible. Therefore no value $z$ created from any pair $(a_{i,j}, b_{i,j}) \in S_{i,j}^{3n}$ can be greater or equal to $z_{i,j}^+$.

$\square$

As mentioned earlier, we do not use the NEW-I pairs directly to construct $S_{i',j'}'$. Instead we add specific pairs to specific cells as explained below:

1. We use the newly created pairs $\left(b_{i,j}^{r+}, b_{i,j}^{r-}, b_{i,j}^{l-}\right)$, to construct the pairs for the following specific cells of the gadgets in the following way:

   (a)
   $$S_{4i-2,4j}' = (iN + z_{i,j}^+, (j+1)N + b^{r-}),$$

   (b)
   $$S_{4i-1,4j}' = (iN + z_{i,j}^+, (j+1)N - b^{r+}),$$

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| | | | |
|---|---|---|---|
| $S'_{4i-3,4j-3}$ $(iN-z, jN+z)$ | $S'_{4i-3,4j-2}$ $(iN+a, jN+z)$ | $S'_{4i-3,4j-1}$ $(iN-a, jN+z)$ | $S'_{4i-3,4j}$ $(iN+z, jN+z)$ |
| $S'_{4i-2,4j-3}$ $(iN-z, jN+b)$ | $S'_{4i-2,4j-2}$ $((i+1)N, (j+1)N)$ | $S'_{4i-2,4j-1}$ $(iN, (j+1)N)$ | $S'_{4i-2,4j}$ $(iN+z, (j+1)N+b)$ |
| $S'_{4i-1,4j-3}$ $(iN-z, jN-b)$ | $S'_{4i-1,4j-2}$ $((i+1)N, jN)$ | $S'_{4i-1,4j-1}$ $(iN, jN)$ | $S'_{4i-1,4j}$ $(iN+z, (j+1)N-b)$ |
| $S'_{4i,4j-3}$ $(iN-z, jN-z)$ | $S'_{4i,4j-2}$ $((i+1)N+a, jN-z)$ | $S'_{4i,4j-1}$ $((i+1)N-a, jN-z)$ | $S'_{4i,4j}$ $(iN+z, jN-z)$ |

Figure 3.1: The 16 sets of the constructed Matrix Tiling with $\leq$ instance representing a set $S_{i,j}$ of the Matrix Tiling in the reduction in the proof of together with the pairs corresponding to a pair $(a,b) \in S'_{i,j}$ (with $z = \iota(a,b)$)

(c)
$$S'_{4i-1,4j-3} = (iN - z^+_{i,j}, jN - b^{l-}).$$

2. We use the newly created pairs $\left(a^{u+}_{i,j}, a^{d+}_{i,j}, a^{u-}_{i,j}, a^{d-}_{i,j}\right)$,to construct the pairs for the following specific cells of the gadgets in the following way:

   (a)
   $$S'_{4i-3,4j-2} = (iN + a^{u+}, jN + z^+_{i,j}),$$

   (b)
   $$S'_{4i-3,4j-1} = (iN - a^{u-}, jN + z^+_{i,j}),$$

   (c)
   $$S'_{4i,4j-2} = ((i+1)N + a^{d-}, jN - z^+_{i,j}),$$

   (d)
   $$S'_{4i,4j-1} = ((i+1)N - a^{d+}, jN - z^+_{i,j})$$

3. We also add new pairs to the *corner* cells of the gadget as follows:

   (a)
   $$S'_{4i-3,4j-3} = (iN - z^+_{i,j}, jN + z^+_{i,j})$$

   (b)
   $$S'_{4i-3,4j} = (iN + z^+_{i,j}, jN + z^+_{i,j}),$$

   (c)
   $$S'_{4i,4j-3} = (iN - z^+_{i,j}, jN - z^+_{i,j}),$$

   (d)
   $$S'_{4i,4j} = (iN + z^+_{i,j}, jN - z^+_{i,j})$$

**Definition 13.** *Let us call the above newly added 15 pairs to the gadget as* NEW-M *pairs.*

Note that all the NEW-M pairs have used the $z^+_{i,j}$ from the Definition 12 , this is important because we want to enforce that once any cell of a particular gadget has these NEW-M pairs selected for the solution, then that gadget can only have 15 non-$\star$'s in the solution, which is proved below.

**Claim 2.** *If any of the cell from a particular gadget has* NEW-M *pair selected as part of solution then that gadget can at most 15 non-$\star$'s in that solution.*

*Proof.* If any of the outer cells has NEW-M pair selected in the solution (let for example say that $S'_{4i-3,4j-2}$ is the cell where the selected pair is $(iN + a^{u+}_{i,j}, jN + z^+_{i,j})$, which is a NEW-M pair), to satisfy the condition of MATRIX TILING WITH $\leq$, the second coordinate $jN + z'_{i,j}$of the pair $(iN - a'_{i,j}, jN + z'_{i,j})$ selected from the cell $S'_{4i-3,4j-1}$ has to be $\geq$ than $jN + z^+_{i,j}$ as the first part $(jN)$ is same we need to have $z'_{i,j} \geq z^+_{i,j}$, and because the way $z^+_{i,j}$ is defined in Definition 12 and from Claim 1 there is no other option for the pair selected from $S'_{4i-3,4j-1}$ other than to be NEW-M pair $(iN - a^{u-}_{i,j}, jN + z^+_{i,j})$, we can similarly show in the clockwise circular direction that $S'_{4i-3,4j}$, will have to be $(iN + z^+_{i,j}, jN + z^+_{i,j})$, $S'_{4i,4j}$, will have to be $(iN + z^+_{i,j}, jN - z^+_{i,j})$, $S'_{4i,4j-3}$ will

have to be $(iN - z_{i,j}^+, jN - z_{i,j}^+)$, $S'_{4i-1,4j-3}$ will have to be $(iN - z_{i,j}^+, jN - b_{i,j}^{l-})$. Because we did not anything new to the $S'_{4i-2,4j-3}$ cell and if $S'_{4i-1,4j-3}$ has NEW-M pair as part of solution, from Claim 1 this cell will have to be a $\star$ in the solution, because the constraint of MATRIX TILING WITH $\leq$ which is:

$$iN - z_{i,j} \leq iN - z_{i,j}^+$$
$$\implies z_{i,j} \geq z_{i,j}^+$$

cannot be satisfied by any pair present in this cell by Claim 1, and therefore this cell has to be $\star$.

---

***Note.*** Here for this proof we assumed that $S'_{4i-3,4j-2}$ had NEW-M pair as the part of solution and we worked our way in clockwise direction satisfying conditions of MATRIX TILING WITH $\leq$ and proving than all the pairs selected from cells in this path before $S'_{4i-2,4j-3}$ have to be the NEW-M pairs, but if we have any solution, and if the cell $S'_{4i-3,4j-2}$ has the pair which is not newly added, this is not the problem, but all the cells in the clockwise direction after the first cell which has the newly added pairs as the part of solution and before $S'_{4i-2,4j-3}$ will have to be newly added pairs and this would imply as proved above that the $S'_{4i-2,4j-3}$ cell will have to be a $\star$. Which is sufficient to prove our claim.

---

$\square$

These pairs are carefully constructed to help us prove the lemma which is stated below. It can also be verified easily that all coordinates of each pair are positive and bounded by $n'$ (we defined the value of $n'$ so that it can satisfy this).

**Lemma 2.** *If there is a $\star$ in any cell in the solution of the instance $\mathcal{I}$, we can pick at least 15 non-$\star$'s in the corresponding gadget for the solution of the Matrix Tiling with $\leq$ instance $\mathcal{M}$.*

*Proof.* To prove this lemma we will first provide the way to select the pairs for the solution our instance $\mathcal{M}$ from a solution of the instance $\mathcal{I}$. We will have two different situations to mention: First is what to pick if the solution of $\mathcal{I}$ has a non-$\star$ pair for the particular cell, and how we can pick 15 non-$\star$ pairs for the gadget corresponding to the cell which has $\star$ in the solution for instance $\mathcal{I}$:

1. For every non-$\star$ pair $s_{i,j} = (a, b)$, we select the corresponding pairs from the 16 sets in the gadget of $S_{i,j}^{3n}$ for $z = \iota(a, b)$, as shown in Figure 3.1.

2. For every $\star$ in the cell $S_{i,j}^{3n}$ we can pick non-$\star$ pairs in the following way:

   - $\star$ **Selection:** Choose $S'_{4i-2,4j-3}$ to be a $\star$.
   - **Middle Cell Selection:** The middle four cells in the gadget are:

$$S'_{4i-2,4j-2} = ((i+1)N, (j+1)N),$$
$$S'_{4i-2,4j-1} = (iN, (j+1)N),$$
$$S'_{4i-1,4j-2} = ((i+1)N, jN),$$
$$S'_{4i-1,4j-1} = (iN, jN)$$

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- **Corner Cells Selection:** For the corner sets:

$$S'_{4i-3,4j-3} = (iN - z^+_{i,j}, jN + z^+_{i,j}),$$
$$S'_{4i-3,4j} = (iN + z^+_{i,j}, jN + z^+_{i,j}),$$
$$S'_{4i,4j-3} = (iN - z^+_{i,j}, jN - z^+_{i,j}),$$
$$S'_{4i,4j} = (iN + z^+_{i,j}, jN - z^+_{i,j})$$

- **Pair Selection for $\star$-related Sets:** We pick pairs for

$$S'_{4i-1,4j-3} = (iN - z^+_{i,j}, jN - b^{l-}),$$
$$S'_{4i-2,4j} = (iN + z^+_{i,j}, (j+1)N + b^{r-}),$$
$$S'_{4i-1,4j} = (iN + z^+_{i,j}, (j+1)N - b^{r+})$$

- **Pair Selection for Remaining Sets:** We pick pairs for

$$S'_{4i-3,4j-2} = (iN + a^{u+}, jN + z^+_{i,j}),$$
$$S'_{4i-3,4j-1} = (iN - a^{u-}, jN + z^+_{i,j}),$$
$$S'_{4i,4j-2} = ((i+1)N + a^{d-}, jN - z^+_{i,j}),$$
$$S'_{4i,4j-1} = ((i+1)N - a^{d+}, jN - z^+_{i,j})$$

Now we provide the proof that the solution formed using the way described above satisfies all the conditions of MATRIX TILING WITH $\leq$:

1. First, it is easy to verify that the constraints are satisfied between the sets of the same gadget for both these cases (<u>idea:</u> in first and last columns first coordinates are same, and in middle two columns the first coordinate of the second and third cells are same and the last cell is $iN + N + a$ which is always greater or equal to $iN + N$ which is the cell above it, and the second coordinate is $iN + N$ is always greater or equal to $iN + a$ as $N = 3n$ and $a \leq n$, similar argument can prove the horizontal condition for rows).

2. Now we want to prove that the "$\leq$" condition holds between all the gadgets. We have 4 cases for both the horizontal and vertical directions.

   (a) For the horizontal direction, we have the following 4 cases:

      i. Both $S^{3n}_{i,j}$ and $S^{3n}_{i,j+1}$ are non-$\star$'s,
      ii. $S^{3n}_{i,j}$ is a non-$\star$ and $S^{3n}_{i,j+1}$ is a $\star$,
      iii. $S^{3n}_{i,j}$ is a $\star$ and $S^{3n}_{i,j+1}$ is a non-$\star$,
      iv. Both $S^{3n}_{i,j}$ and $S^{3n}_{i,j+1}$ are $\star$'s.

| $S'_{4i-3,4j}:$ $(iN + z_{i,j}, jN + z_{i,j})$ | $S'_{4i-3,4j+1}:$ $(iN - z_{i,j+1}, (j+1)N + z_{i,j+1})$ |
|---|---|
| $S'_{4i-2,4j}:$ $(iN + z_{i,j}, (j+1)N + b_{i,j})$ | $S'_{4i-2,4j+1}:$ $(iN - z_{i,j+1}, (j+1)N + b_{i,j+1})$ |
| $S'_{4i-1,4j}:$ $(iN + z_{i,j}, (j+1)N - b_{i,j})$ | $S'_{4i-1,4j+1}:$ $(iN - z_{i,j+1}, (j+1)N - b_{i,j+1})$ |
| $S'_{4i,4j}:$ $(iN + z_{i,j}, jN - z_{i,j})$ | $S'_{4i,4j+1}:$ $(iN - z_{i,j+1}, (j+1)N - z_{i,j+1})$ |

Figure 3.2: The last column of the gadget of $S_{i,j}$ and the first column of the gadget of $S_{i,j+1}$ with the pairs that are picked when both $S_{i,j}$ and $S_{i,j+1}$ are non-$\star$'s.

**Case (i): Both $S^{3n}_{i,j}$ and $S^{3n}_{i,j+1}$ are non-$\star$'s.** We look at the last column of the gadget of $S^{3n}_{i,j}$ and the first column of the gadget of $S^{3n}_{i+1,j}$ (see Figure 3.2).

- For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has second coordinate $jN + z_{i,j}$, while the pair selected from $S'_{4i-3,4(j+1)-3} = S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z_{i,j+1}$, which is greater than or equal to $jN + z_{i,j}$.
- Similarly, there is no conflict between the last sets of these columns.
- If $b_{i,j} = b_{i,j+1}$ are the first coordinates of $S^{3n}_{i,j}$ and $S^{3n}_{i,j+1}$, then the second coordinates of the sets selected from the second sets of the rows, $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$, are $(j+1)N + b_{i,j}$ and $(j+1)N + b_{i+1,j}$, respectively, and the former is equal to the latter.
- One can show in a similar way that there is no conflict between the third sets of these columns.

| | |
|---|---|
| $S'_{4i-3,4j}$ : $\quad (iN + z'_{i,j}, jN + z'_{i,j})$ | $S'_{4i-3,4j+1}$ : $\quad (iN - z^+_{i,j+1}, (j+1)N + z^+_{i,j+1})$ |
| $S'_{4i-2,4j}$ : $\quad (iN + z'_{i,j}, (j+1)N + b'_{i,j})$ | $S'_{4i-2,4j+1}$ : $\quad \star$ |
| $S'_{4i-1,4j}$ : $\quad (iN + z'_{i,j}, (j+1)N - b'_{i,j})$ | $S'_{4i-1,4j+1}$ : $\quad (iN - z^+_{i,j+1}, (j+1)N - b^{l-}_{i,j+1})$ |
| $S'_{4i,4j}$ : $\quad (iN + z'_{i,j}, jN - z'_{i,j})$ | $S'_{4i,4j+1}$ : $\quad (iN - z^+_{i,j+1}, (j+1)N - z^+_{i,j+1})$ |

Figure 3.3: The last column of the gadget of $S_{i,j}$ and first column of the gadget of $S_{i,j+1}$ with the pairs that are picked when $S_{i,j}$ is a non-$\star$ and $S_{i,j+1}$ is a $\star$'s.

**Case (ii): $S^{3n}_{i,j}$ is a non-$\star$ and $S^{3n}_{i,j+1}$ is a $\star$.** We look at the last column of the gadget of $S^{3n}_{i,j}$ and the first column of the gadget of $S^{3n}_{i,j+1}$ (see Figure 3.3).

- For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN+z'$, where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N+z^+_{i,j+1}$. We can observe that $(j+1)N + z^+_{i,j+1} \geq jN + z'$.
- Similarly, there is no conflict between the last sets of these columns.
- We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a $\star$ as defined earlier.
- Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N-b'_{i,j}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b^{l-}_{i,j+1}$. From the Equation (3.8) of $b^{l-}_{i,j+1}$, we get the inequality $b'_{i,j} \geq b^{l-}_{i,j+1}$, which implies $(j+1)N-b'_{i,j} \leq (j+1)N - b^{l-}_{i,j+1}$.

| $S'_{4i-3,4j}:$ $(iN + z^+_{i,j}, jN + z^+_{i,j})$ | $S'_{4i-3,4j+1}:$ $(iN - z'_{i,j+1}, (j+1)N + z'_{i,j+1})$ |
|---|---|
| $S'_{4i-2,4j}:$ $(iN + z'_{i,j}, (j+1)N + b^{r-}_{i,j})$ | $S'_{4i-2,4j+1}:$ $(iN - z'_{i,j+1}, (j+1)N + b'_{i,j+1})$ |
| $S'_{4i-1,4j}:$ $(iN + z^+_{i,j}, (j+1)N - b^{r+}_{i,j})$ | $S'_{4i-1,4j+1}:$ $(iN - z'_{i,j+1}, (j+1)N - b'_{i,j+1})$ |
| $S'_{4i,4j}:$ $(iN + z^+_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j+1}:$ $(iN - z'_{i,j+1}, (j+1)N - z'_{i,j+1})$ |

Figure 3.4: The last column of the gadget of $S_{i,j}$ and first column of the gadget of $S_{i,j+1}$ with the pairs that are picked when $S_{i,j}$ is a $\star$ and $S_{i,j+1}$ is a non-$\star$'s.

**Case (iii): $S^{3n}_{i,j}$ is a $\star$ and $S^{3n}_{i,j+1}$ is a non-$\star$.** We look at the last column of the gadget of $S^{3n}_{i,j}$ and the first column of the gadget of $S^{3n}_{i,j+1}$ (see Figure 3.4).

- For the first sets in these columns, the constraints are satisfied: the pair selected from $S'_{4i-3,4j}$ has the second coordinate $jN + z^+_{i,j}$, and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N + z'$, where $z' = \iota(a'_{i,j+1}, b'_{i,j+1})$, and we can observe that $(j+1)N + z^+_{i,j} \geq jN + z$.
- Similarly, there is no conflict between the last sets of these columns.
- Now the pair selected from $S'_{4i-2,4j}$ has the second coordinate $(j+1)N + b^{r-}_{i,j}$ and the pair selected from $S'_{4i-2,4j+1}$ has the second coordinate $(j+1)N + b'_{i,j+1}$. From the Equation (3.6) of $b^{r-}_{i,j}$, we get the inequality $b^{r-}_{i,j} \leq b'_{i,j+1}$ which implies $(j+1)N + b^{r-} \leq (j+1)N + b'_{i,j+1}$.
- Similarly, the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N - b^{r+}_{i,j}$ and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N - b'_{i,j+1}$. From the Equation (3.7) of $b^{r+}_{i,j}$, we get the inequality $b^{r+}_{i,j} \geq b'_{i,j+1}$ which implies $(j+1)N - b^{r+} \leq (j+1)N - b'_{i,j+1}$.

| | |
|---|---|
| $S'_{4i-3,4j}:$ <br> $(iN+z^+_{i,j}, jN+z^+_{i,j})$ | $S'_{4i-3,4j+1}:$ <br> $(iN-z^+_{i,j+1}, (j+1)N+z^+_{i,j+1})$ |
| $S'_{4i-2,4j}:$ <br> $(iN+z^+_{i,j}, (j+1)N+b_{i,j})$ | $S'_{4i-2,4j+1}:$ <br> $\star$ |
| $S'_{4i-1,4j}:$ <br> $(iN+z^+_{i,j}, (j+1)N-b^{r+}_{i,j})$ | $S'_{4i-1,4j+1}:$ <br> $(iN-z^+_{i,j+1}, (j+1)N-b^{l-}_{i,j+1})$ |
| $S'_{4i,4j}:$ <br> $(iN+z^+_{i,j}, jN-z^+_{i,j})$ | $S'_{4i,4j+1}:$ <br> $(iN-z^+_{i,j+1}, (j+1)N-z^+_{i,j+1})$ |

Figure 3.5: The last column of the gadget of $S_{i,j}$ and the first column of the gadget of $S_{i,j+1}$ with the pairs that are picked when both $S_{i,j}$ and $S_{i,j+1}$ are $\star$'s.

**Case (iv): Both $S^{3n}_{i,j}$ and $S^{3n}_{i,j+1}$ are $\star$'s.** We look at the last column of the gadget of $S^{3n}_{i,j}$ and the first column of the gadget of $S^{3n}_{i,j+1}$ (see Figure 3.5).

- For the first sets in these columns, the constraints are satisfied: $S'_{4i-3,4j}$ has the second coordinate $jN+z^+_{i,j}$ and the pair selected from $S'_{4i-3,4j+1}$ has the second coordinate $(j+1)N+z^+_{i,j+1}$, and we can observe that $(j+1)N+z^+_{i,j+1} \geq jN+z^+_{i,j}$.
- Similarly there is now conflict between last sets of these columns.
- We will not have to check the condition for second coordinates of the pair selected from $S'_{4i-2,4j}$ and $S'_{4i-2,4j+1}$ because $S'_{4i-2,4j+1}$ will be a $\star$ as defined earlier.
- Now the pair selected from $S'_{4i-1,4j}$ has the second coordinate $(j+1)N-b^{r+}_{i,j}$, and the pair selected from $S'_{4i-1,4j+1}$ has the second coordinate $(j+1)N-b^{l-}_{i,j+1}$, and from the Equation (3.7), and Equation (3.8) of $b^{r+}_{i,j}$ and $b^{l-}_{i,j+1}$, we get the inequality $b^{r+}_{i,j} \geq b^{l-}_{i,j+1}$ which implies $(j+1)N-b^{r+}_{i,j} \leq (j+1)N-b^{l-}_{i,j+1}$.

(b) Now for the vertical direction, we have the following 4 cases:

   i. Both $S^{3n}_{i,j}$ and $S^{3n}_{i+1,j}$ are non-$\star$'s,

  ii. $S^{3n}_{i,j}$ is a non-$\star$ and $S^{3n}_{i+1,j}$ is a $\star$,

 iii. $S^{3n}_{i,j}$ is a $\star$ and $S^{3n}_{i+1,j}$ is a non-$\star$,

 iv. Both $S^{3n}_{i,j}$ and $S^{3n}_{i+1,j}$ are $\star$'s.

| $S'_{4i,4j-3}:$ | $S'_{4i,4j-2}:$ | $S'_{4i,4j-1}:$ | $S'_{4i,4j}:$ |
|---|---|---|---|
| $(iN - z'_{i,j}, jN - z'_{i,j})$ | $((i+1)N + a'_{i,j}, jN - z'_{i,j})$ | $((i+1)N - a'_{i,j}, jN - z'_{i,j})$ | $(iN + z'_{i,j}, jN - z'_{i,j})$ |
| $S'_{4i+1,4j-3}:$ | $S'_{4i+1,4j-2}:$ | $S'_{4i+1,4j-1}:$ | $S'_{4i+1,4j}:$ |
| $((i+1)N - z'_{i+1,j}, jN - z'_{i+1,j})$ | $((i+1)N + a'_{i+1,j}, jN + z'_{i+1,j})$ | $((i+1)N - a'_{i+1,j}, jN + z'_{i+1,j})$ | $((i+1)N + z'_{i+1,j}, jN + z'_{i+1,j})$ |

Figure 3.6: The last row of the gadget of $S_{i,j}$ and first row of the gadget of $S_{i+1,j}$ with the pairs that are picked when both $S_{i,j}$ and $S_{i+1,j}$ are non-$\star$'s.

**Case (i): Both $S_{i,j}^{3n}$ and $S_{i+1,j}^{3n}$ are non-$\star$'s.** We look at the last row of the gadget of $S_{i,j}^{3n}$ and the first row of the gadget of $S_{i+1,j}^{3n}$ (see Figure 3.6).

- For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has first coordinate less than $iN$, while the pair selected from $S'_{4(i+1)-3,4j-3} = S'_{4i+1,4j-3}$ has the first coordinate at least $(i+1)N - (n^2 + n) > iN$.
- Similarly, there is no conflict between the last sets of these rows.
- If $a_{i,j} = a_{i+1,j}$ are the first coordinates of $S_{i,j}^{3n}$ and $S_{i+1,j}^{3n}$, then the first coordinates of the sets selected from the second sets of the rows, $S'_{4i,4j-2}$ and $S'_{4i+1,4j-2}$, are $(i+1)N + a_{i,j}$ and $(i+1)N + a_{i+1,j}$, respectively, and the former is equal to the latter.
- One can show in a similar way that there is no conflict between the third sets of the rows.

| $S'_{4i,4j-3}:$ | $S'_{4i,4j-2}:$ | $S'_{4i,4j-1}:$ | $S'_{4i,4j}:$ |
|---|---|---|---|
| $(iN - z'_{i,j}, jN - z'_{i,j})$ | $((i+1)N + a'_{i,j}, jN - z'_{i,j})$ | $((i+1)N - a'_{i,j}, jN - z'_{i,j})$ | $(iN + z'_{i,j}, jN - z'_{i,j})$ |
| $S'_{4i+1,4j-3}:$ | $S'_{4i+1,4j-2}:$ | $S'_{4i+1,4j-1}:$ | $S'_{4i+1,4j}:$ |
| $((i+1)N - z^+_{i+1,j}, jN - z^+_{i+1,j})$ | $((i+1)N + a^{u+}_{i+1,j}, jN + z^+_{i+1,j})$ | $((i+1)N - a^{u-}_{i+1,j}, jN + z^+_{i+1,j})$ | $((i+1)N + z^+_{i+1,j}, jN + z^+_{i+1,j})$ |

Figure 3.7: The last row of the gadget of $S_{i,j}$ and first row of the gadget of $S_{i+1,j}$ with the pairs that are picked when $S_{i,j}$ is a non-$\star$ and $S_{i+1,j}$ is a $\star$'s.

**Case (ii): $S_{i,j}^{3n}$ is a non-$\star$ and $S_{i+1,j}^{3n}$ is a $\star$.** We look at the last row of the gadget of $S_{i,j}^{3n}$ and the first row of the gadget of $S_{i+1,j}^{3n}$ (see Figure 3.7).

- For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z'_{i,j}$, where $z'_{i,j} = \iota(a'_{i,j}, b'_{i,j})$ and the pair selected from $S'_{4i+1,4j-3}$ has

the first coordinate $(i+1)N - z^+_{i+1,j}$, and we can observe that $(i+1)N - z^+_{i+1,j} \geq iN - z'_{i,j}$.

- Similarly there is now conflict between last sets of these rows.
- Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i+1,j}$, and from the Equation (3.4),of $a^{u+}_{i+1,j}$, we get the inequality $a^{u+}_{i+1,j} \geq a'_{i,j}$ which implies $(i+1)N + a^{u+}_{i+1,j} \geq (i+1)N + a'_{i,j}$.
- Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a'_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i+1,j}$ and from the Equation (3.5) of $a^{u-}_{i+1,j}$, we get the inequality $a^{u-}_{i+1,j} \leq a'_{i,j}$ which implies $(i+1)N - a^{u-}_{i,j+1} \geq (i+1)N - a'_{i,j}$.

| $S'_{4i,4j-3}$ :  $(iN - z^+_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j-2}$ :  $((i+1)N + a^{d-}_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j-1}$ :  $((i+1)N - a^{d+}_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j}$ :  $(iN + z^+_{i,j}, jN - z^+_{i,j})$ |
|---|---|---|---|
| $S'_{4i+1,4j-3}$ :  $((i+1)N - z'_{i+1,j}, jN - z'_{i+1,j})$ | $S'_{4i+1,4j-2}$ :  $((i+1)N + a'_{i+1,j}, jN + z'_{i+1,j})$ | $S'_{4i+1,4j-1}$ :  $((i+1)N - a'_{i+1,j}, jN + z'_{i+1,j})$ | $S'_{4i+1,4j}$ :  $((i+1)N + z'_{i+1,j}, jN + z'_{i+1,j})$ |

Figure 3.8: The last row of the gadget of $S_{i,j}$ and first row of the gadget of $S_{i+1,j}$ with the pairs that are picked when $S_{i,j}$ is a $\star$ and $S_{i+1,j}$ is a non-$\star$.

**Case (iii):** $S^{3n}_{i,j}$ **is a $\star$ and** $S^{3n}_{i+1,j}$ **is a non-$\star$.** We look at the last row of the gadget of $S^{3n}_{i,j}$ and the first row of the gadget of $S^{3n}_{i+1,j}$ (see Figure 3.8).

- For the first sets in these rows, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z^+_{i,j}$, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z'_{i+1,j}$ where $z'_{i+1,j} = \iota(a'_{i+1,j}, b'_{i+1,j})$, and we can observe that $(i+1)N - z'_{i+1,j} \geq iN - z^+_{i,j}$.
- Similarly there is now conflict between last sets of these rows.
- Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$ and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a'_{i+1,j}$, and from the Equation (3.3) of $a^{d-}_{i,j}$, we get the inequality $a^{d-}_{i,j} \leq a'_{i+1,j}$ which implies $(i+1)N + a'_{i+1,j} \geq (i+1)N + a^{d-}_{i,j}$.
- Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a'_{i+1,j}$, and from the Equation (3.2) of $a^{d+}_{i,j}$, we get the inequality $a^{d+}_{i,j} \geq a'_{i+1,j}$ which implies $(i+1)N - a'_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

| | | | |
|---|---|---|---|
| $S'_{4i,4j-3}:$ $(iN - z^+_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j-2}:$ $((i+1)N + a^{d-}_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j-1}:$ $((i+1)N - a^{d+}_{i,j}, jN - z^+_{i,j})$ | $S'_{4i,4j}:$ $(iN + z^+_{i,j}, jN - z^+_{i,j})$ |
| $S'_{4i+1,4j-3}:$ $((i+1)N - z^+_{i+1,j}, jN - z^+_{i+1,j})$ | $S'_{4i+1,4j-2}:$ $((i+1)N + a^{u+}_{i+1,j}, jN + z^+_{i+1,j})$ | $S'_{4i+1,4j-1}:$ $((i+1)N - a^{u-}_{i+1,j}, jN + z^+_{i+1,j})$ | $S'_{4i+1,4j}:$ $((i+1)N + z^+_{i+1,j}, jN + z^+_{i+1,j})$ |

Figure 3.9: The last row of the gadget of $S_{i,j}$ and first row of the gadget of $S_{i+1,j}$ with the pairs that are picked when both $S_{i,j}$ and $S_{i+1,j}$ are $\star$'s.

**Case (iv): Both $S^{3n}_{i,j}$ and $S^{3n}_{i+1,j}$ are $\star$'s.** We look at the last row of the gadget of $S^{3n}_{i,j}$ and the first row of the gadget of $S^{3n}_{i+1,j}$ (see Figure 3.9).

- For the first sets in these row, the constraints are satisfied: the pair selected from $S'_{4i,4j-3}$ has the first coordinate $iN - z^+_{i,j}$, and the pair selected from $S'_{4i+1,4j-3}$ has the first coordinate $(i+1)N - z^+_{i+1,j}$, and we can observe that $(i+1)N - z^+_{i+1,j} \geq iN - z^+_{i,j}$.
- Similarly there is now conflict between last sets of these rows.
- Now the pair selected from $S'_{4i,4j-2}$ has the first coordinate $(i+1)N + a^{d-}_{i,j}$, and the pair selected from $S'_{4i+1,4j-2}$ has the first coordinate $(i+1)N + a^{u+}_{i+1,j}$, and from the Equation (3.3) and Equation (3.4) of $a^{d-}_{i,j}$ and $a^{u+}_{i+1,j}$, we get the inequality $a^{d-}_{i,j} \leq a^{u+}_{i+1,j}$ which implies $(i+1)N + a^{u+}_{i+1,j} \geq (i+1)N + a^{d-}_{i,j}$.
- Similarly now the pair selected from $S'_{4i,4j-1}$ has the first coordinate $(i+1)N - a^{d+}_{i,j}$, and the pair selected from $S'_{4i+1,4j-1}$ has the first coordinate $(i+1)N - a^{u-}_{i+1,j}$, and from the Equation (3.2) and Equation (3.5) of $a^{d+}_{i,j}$ and $a^{u-}_{i+1,j}$, we get the inequality $a^{d+}_{i,j} \geq a^{u-}_{i+1,j}$ which implies $(i+1)N - a^{u-}_{i+1,j} \geq (i+1)N - a^{d+}_{i,j}$.

Thus we have proved that for every non-$\star$ in the solution of $\mathcal{I}$ we will have all 16 non-$\star$'s for it's corresponding gadget in the solution of $\mathcal{M}$, and for every $\star$, we can pick at least 15 non-$\star$'s for it's corresponding gadget in the solution of $\mathcal{M}$.

$\square$

## Definition of S

Consider any solution $y$ of $R(x)$ and the corresponding $4 \times 4$ gadgets.

**Claim 3.** *For each $(i,j) \in [k] \times [k]$, if the all the 16 cells $\big($in the gadget of the cell $(i,j)\big)$ $S'_{i',j'}$ where $4i - 3 \leq i' \leq 4i$ and $4j - 3 \leq j' \leq 4j$ are non-$\star$'s, we claim that the outer 12 cells of the gadget $\big($i.e., $S'_{4i-3,4j-3}$, $S'_{4i-3,4j-2}$, $S'_{4i-3,4j-1}$, $S'_{4i-3,4j}$, $S'_{4i-2,4j}$, $S'_{4i-1,4j}$, $S'_{4i,4j}$, $S'_{4i,4j-1}$, $S'_{4i,4j-2}$, $S'_{4i,4j-3}$, $S'_{4i-1,4j-3}$, and $S'_{4i-2,4j-3}\big)$ contains the same $z^{a_{i,j},b_{i,j}}_{i,j}$ as the part for their construction.*

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*Proof.* Assume that all 16 cells in the gadget are non-$\star$'s. We now show that the outer 12 cells of the gadget are formed using the same pair $(a, b)$.

Firstly, from Claim 2, We can argue that because all the 16 cells are non-$\star$'s none of the pairs are NEW-M pairs, and it follows that these are the pairs which are created using the original $(a_{i,j}, b_{i,j}) \in S_{i,j}^{3n}$ pairs.

The 12 outer sets in the gadget correspond to $S_{i,j}^{3n}$. The pairs selected in the solution from these sets define 12 values $z$, denoted as $z_{4i-3,4j-3}, z_{4i-3,4j-2}, \cdots,$ representing the values selected from these sets. We claim that all these 12 values are equal.

To see this, consider the second coordinate of the pairs selected from these sets:

- The second coordinate of the set selected from $S_{4i-3,4j-3}^{3n}$ is $jN + z_{4i-3,4j-3}$.
- Similarly, the second coordinate of the set selected from $S_{4i-3,4j-2}^{3n}$ is $jN + z_{4i-3,4j-2}$.

By the definition of Matrix Tiling with $\leq$, it follows that:

$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2}.$$

Continuing this reasoning for the other sets, we establish the following chain of inequalities:

- First row:

$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2} \leq z_{4i-3,4j-1} \leq z_{4i-3,4j}.$$

- Last column:
$$z_{4i-3,4j} \leq z_{4i-2,4j} \leq z_{4i-1,4j} \leq z_{4i,4j}.$$

- Last row (negated):

$$-z_{4i,4j-3} \leq -z_{4i,4j-2} \leq -z_{4i,4j-1} \leq -z_{4i,4j}.$$

- First column (negated):

$$-z_{4i-3,4j-3} \leq -z_{4i-2,4j-3} \leq -z_{4i-1,4j-3} \leq -z_{4i,4j-3}.$$

Combining all these inequalities results in a cycle of equalities, implying that all 12 values are the same.

$\square$

Let $z_{i,j}$ denote this common value, and let $s_{i,j}^{3n} = (a_{i,j}, b_{i,j})$ be the corresponding pair such that $\iota(a_{i,j}, b_{i,j}) = z_{i,j}$. Since $z_{i,j}$ was defined using the pairs appearing in $S_{i,j}^{3n}$, it follows that $s_{i,j}^{3n} \in S_{i,j}^{3n}$. Since $S_{i,j}^{3n}$ was constructed from $S_{i,j}$ by adding $n$ in the Step 3.1 to each coordinate, we can recover $s_{i,j} \in S_{i,j}$ by subtracting $n$ from both coordinates of $s_{i,j}^{3n}$. Thus, $s_{i,j} = (a_{i,j} - n, b_{i,j} - n)$.

We now define $S$ by saying that if all 16 cells are non-$\star$'s, select the pair $s_{i,j}$ (described above) as the non-$\star$ solution in the corresponding cell of the $\mathcal{I}$. Otherwise, select a $\star$.

**Consistency across neighboring gadgets:** Now consider the gadgets of neighboring cells, we have two cases, first is the case where at least of neighboring gadgets is $\star$, and the other os where both the neighboring gadgets are non-$\star$'s:

1. **Case where at least on of the neighbor gadget is a $\star$:** If one or more cells in the neighboring gadgets are $\star$'s, the solution selects a $\star$ in the $\mathcal{I}$. In this case, we do not need to prove consistency for $a_{i,j} = a_{i+1,j}$ or $b_{i,j} = b_{i,j+1}$, as the corresponding coordinates contain $\star$'s.

2. **Case where both the neighboring gadgets are non-$\star$'s:**

   (a) Vertical consistency: Let $S_{i+1,j}^{3n}$ be the gadget of the cell below $S_{i,j}^{3n}$, and assume all 16 cells in these gadgets are non-$\star$'s. The pair selected from $S_{4i,4j-2}^{3n}$ has first coordinate $(i+1)N + a_{i,j}$, while the pair selected from $S_{4i+1,4j-2}^{3n}$ has first coordinate $(i+1)N + a_{i+1,j}$. By the definition of Matrix Tiling with $\leq$, we have:

   $$a_{i,j} \leq a_{i+1,j}.$$

   Similarly, comparing the first coordinates of the pairs selected from $S'_{4i,4j-1}$ and $S'_{4i+1,4j-1}$ yields $-a_{i,j} \leq -a_{i+1,j}$, which implies:

   $$a_{i,j} = a_{i+1,j}.$$

   (b) Horizontal consistency: Let $S_{i,j+1}^{3n}$ be the gadget of the cell to the right of $S_{i,j}^{3n}$, and assume all 16 cells in these gadgets are non-$\star$'s. A similar argument using the last column of $S_{i,j}^{3n}$ and the first column of $S_{i,j+1}^{3n}$ shows:
   $$b_{i,j} = b_{i,j+1}.$$

**Conclusion:** We have shown that the constructed $s_{i,j} \in S(y)$ forms a valid solution to the instance $\mathcal{I}$.

From this, we also infer the relation between $c_A(S(y)) = k^2 - n$ and $c_B(y) = 16k^2 - m$, that for every gadget with one or more $\star$ in $y$, we are selecting $\star$ for the corresponding cell in the $S(y)$, therefore because $c_A(S(y)) = k^2 - n$ in the best case $c_B(y)$ would have all the $n$-$\star$'s which are corresponding the different gadgets which would give $c_B(y) = 16(k^2 - n) + 15n = 16k^2 - n$ (as we have $c_A(S(y)) = k^2 - n$ for the best case all the $n$ $\star$'s were corresponding to the $n$ gadgets with one $\star$) and we can have the following inequality:

$$\begin{aligned} c_B(y) = 16k^2 - m &\leq 16k^2 - n \\ &\implies -m \leq -n \\ &\implies m \geq n \end{aligned} \tag{3.9}$$

We can also observe that if the cell in the optimum solution of the instance $\mathcal{I}$ is $\star$, then we have to pick at least one of the $S'_{4i-2,4j-3}$ to be $\star$ because if we pick the pairs of all the 16 cells in the gadget then by Claim 2 these will not be NEW-M pairs, and will form cycle of inequalities (which we described above) and the conditions will also be satisfied in the neighboring gadgets such that we would be able to pick a pair that pair which created the values of these gadget

pairs: $(a, b)$ in the solution of $\mathcal{I}$, which will be a contradiction, on the other hand, if we pick any cell of the gadget from the NEW-M pairs we can argue from Claim 12 that at least on of the 16 cells of the gadget will have to be a $\star$.

Therefore, if we have $OPT(x) = k^2 - a$ and $OPT(R(x)) = 16k^2 - b$, then from Lemma 2, we can the following relation between these two:

$$
\begin{aligned}
OPT(R(x)) = 16k^2 - b &\geq 16(k^2 - a) + 15a \\
&\implies 16k^2 - b \geq 16k^2 - a \\
&\implies -b \geq -a \\
&\implies a \geq b
\end{aligned}
\tag{3.10}
$$

## 3.3 Relation between the optimal solutions of $\mathcal{I}$ and $\mathcal{M}$

We can notice that the optimum is always at least $\frac{k^2}{4}$: if $i$ and $j$ are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$ (alternative row and columns); otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: $k^2$, which gives us the following inequalities:

$$
k^2/4 \leq OPT(x) \leq k^2
\tag{3.11}
$$

$$
k'^2/4 \leq OPT(R(x)) \leq k'^2
$$

From the definition of $R$, we know that $k' = 4k$, therefore:

$$
4k^2 \leq OPT(R(x)) \leq 16k^2
\tag{3.12}
$$

Now from the equation 3.11 and equation 3.12, we can find the value of $\alpha$ for the $3^{rd}$ condition of the L-reduction:

$$
\begin{aligned}
OPT(R(x)) &\leq 16k^2 = 64k^2/4 = 64 \cdot OPT(x) \\
&\implies OPT(R(x)) \leq 64 \cdot OPT(x)
\end{aligned}
\tag{3.13}
$$

Thus for $\alpha = 64$, we have $OPT(R(x)) \leq \alpha OPT(x)$.

## 3.4 Relation between the optimal solutions and any approximate solutions of $\mathcal{I}$ and $\mathcal{M}$

We have
$$
\begin{aligned}
OPT(x) &= k^2 - a, \\
c_B(y) &= 16k^2 - m, \\
c_A(S(y)) &= k^2 - n, \\
OPT(R(x)) &= 16k^2 - b.
\end{aligned}
$$

from equation 3.9 and equation 3.10 it follows:

$$
\begin{aligned}
n &\leq m \\
&\implies n - a \leq m - a \\
&\implies n - a \leq m - a \leq m - b \\
&\implies n - a \leq m - b \\
&\implies (n - a) \leq (1)(m - b)
\end{aligned}
\tag{3.14}
$$

we now look at the $4^{th}$ condition of L-reduction:

$$
\begin{aligned}
(OPT(x) - c_A(S(y))) &\leq \beta(OPT(R(x)) - c_B(y)) \\
(k^2 - a - (k^2 - n)) &\leq \beta(16k^2 - b - (16k^2 - m)) \\
(n - a) &\leq \beta(m - b)
\end{aligned}
\tag{3.15}
$$

from equations 3.14 and equations 3.15 we can get $\beta = 1$.
Thus for $\beta = 1$, we have $|OPT(x) - c_A(S(y))| \leq \beta|OPT(R(x)) - c_B(y)|$.

***Note:*** Because both the problems MATRIX TILING and MATRIX TILING WITH $\leq$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

This completes L-reduction from MATRIX TILING to MATRIX TILING WITH $\leq$, where the values of $\alpha$ and $\beta$ are 64 and 1 respectively. Completing the proof of Theorem 3 which is restated below:

**Theorem.** *If there are constants $\delta, d > 0$ such that MATRIX TILING WITH $\leq$ has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

***Note:*** In the above theorem $n$ is not the range for the coordinates, but the input size of the problem instance. Now we have the following lemma:

**Lemma 3.** *If there is an L-reduction from MATRIX TILING WITH $\leq$ to Problem X, then there are no $d, \delta > 0$ such that Problem X admits a PTAS with running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, unless ETH fails.*

Maximum Independent Set on Unite Disk Graph

## 4.1 Introduction

> **Maximum Independent Set on Unit Disk Graph:**
> We are given a set $S$ of unit- diameter disks in the plane(described by the coordinates of their centers). The goal is to find a maximum cardinality subset $S' \in S$ of disks, such that the disks in $S'$ are pairwise disjoint.

**Note:** When dening the Matrix Tiling problem, we imagined the sets $S_{i,j}$ arranged in a matrix, with $S_{i,j}$ being in row $i$ and column $j$. When reducing Matrix Tiling to a geometric problem, the natural idea is to represent $S_{i,j}$ with a gadget located around coordinate $(i, j)$. However, this introduces an unnatural 90 degrees rotation compared to the layout of the $S_{i,j}$'s in the matrix, which can be confusing in the presentation of a reduction. Therefore, for geometric problems, it is convenient to imagine that $S_{i,j}$ is located at coordinate $(i, j)$. To emphasize this interpretation, we use the notation $S[x, y]$ to refer to the sets; we imagine that $S[x, y]$ is at location $(x, y)$, hence sets with the same $x$ are on a vertical line and sets with the same $y$ are on the same horizontal line (see Figure 4.1). The constraints of Matrix Tiling are the same as before: the pairs selected from $S[x, y]$ and $S[x + 1, y]$ agree in the first coordinate, while the pairs selected from $S[x, y]$ and $S[x, y + 1]$ agree in the second coordinate. Matrix Tiling with $\leq$ is defined similarly. With this notation, we can give a very clean and transparent L-reduction to Maximum Independent Set of Unit Disk Graphs.

## 4.2 Reduction and relation between solutions of both the instances.

**Construction of $R$**

It will be convenient to work with open disks of radius $\frac{1}{2}$ (diameter 1). Two disks are nonintersecting if and only if the distance between their centers is at least 1.

Let $I = (n, k, S)$ be an instance of $MT \leq$. We construct a set $\mathcal{D}$ of unit disks. Let $\epsilon = 1/n^2$. For every $1 \leq x, y \leq k$ and every $(a, b) \in S[x, y] \subset [n] \times [n]$, we introduce into $\mathcal{D}$ an open disk of radius $\frac{1}{2}$ centered at $(x + \epsilon a, y + \epsilon b)$; Let $D[x, y]$ be the set of these $|S[x, y]|$ disks introduced for a fixed $x$ and $y$ (see Fig. ). Note that the disks in $D[x, y]$ all intersect each other. Therefore, if $D' \subseteq \mathcal{D}$ is a set of pairwise nonintersecting disks, then $|D'| \leq k^2$ and $|D'| = k^2$ is possible only if $D'$ contains exactly one disk from each $D[x, y]$. We need the following observation first. Consider two disks centered at $(x + \epsilon a, y + \epsilon b)$ and $(x + 1 + \epsilon a', y + \epsilon b')$ for some $(a, b), (a', b') \in [n] \times [n]$. We claim that they are nonintersecting if and only if $a \leq a'$. Indeed, if $a > a'$, then the square of the distance of the two centers is

$$(1 + \epsilon(a' - a))^2 + \epsilon^2(b' - b)^2 \leq (1 + \epsilon(a' - a))^2 + \epsilon^2 n^2$$
$$\leq (1 - \epsilon)^2 + \epsilon = 1 - \epsilon + \epsilon^2 < 1$$

(in the first inequality, we have used $b', b \leq n$; in the second inequality, we have used $a \geq a' + 1$ and $\epsilon = 1/n^2$). On the other hand, if $a \leq a'$, then the square of the distance is at least $(1 + \epsilon(a' - a))^2 \geq 1$, hence the two disks do not intersect (recall that the disks are open). This proves our claim. A similar claim shows that disks centered at $(x + \epsilon a, y + \epsilon b)$ and $(x + \epsilon a', y + 1 + \epsilon b')$ are nonintersecting if and only if $b \leq b'$. Moreover, it is easy to see that the disks centered at $(x + \epsilon a, y + \epsilon b)$ and $(x' + \epsilon a', y' + \epsilon b')$ for some $1 \leq a, a', b, b' \leq n$ cannot intersect if $|x - x'| + |y - y'| \geq 2$: the square of the distance between the two centers is at least $2(1 - \epsilon n)^2 > 1$.
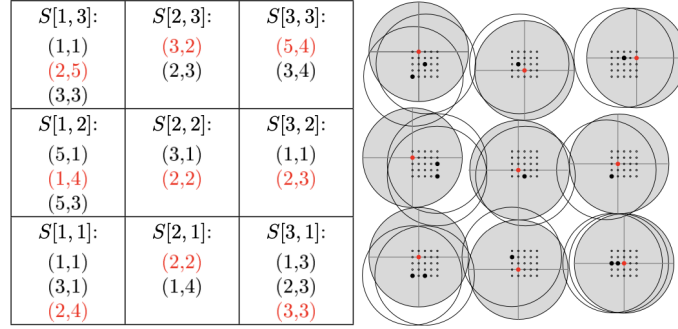
| $S[1,3]$: | $S[2,3]$: | $S[3,3]$: |
|---|---|---|
| (1,1) | (3,2) | (5,4) |
| (2,5) | (2,3) | (3,4) |
| (3,3) | | |
| $S[1,2]$: | $S[2,2]$: | $S[3,2]$: |
| (5,1) | (3,1) | (1,1) |
| (1,4) | (2,2) | (2,3) |
| (5,3) | | |
| $S[1,1]$: | $S[2,1]$: | $S[3,1]$: |
| (1,1) | (2,2) | (1,3) |
| (3,1) | (1,4) | (2,3) |
| (2,4) | | (3,3) |



Figure 4.1: An instance of Matrix Tiling with $\leq$ with $k = 3$ and $n = 5$ and the corresponding instance of Maximum Independent Set on Unit Disk Graph constructed in the reduction. The small dots show the potential positions for the centers, the large dots are the actual centers in the constructed instance. The shaded disks with red centers correspond to the solution of Matrix Tiling with $\leq$ shown on the left [9, Figure 14.7]

**Lemma 4.** *If we have some any solution $k^2 - e$ for $I$ we can have at least $k^2 - e$ for our constructed instance of $\mathcal{D}$.*

*Proof.* If we have $k^2 - e$ as a solution for $I$, it means we have the number of $\star$'s in the solution is $e$, for all the cells with the $\star$ in the solution don't pick any disk from that corresponding $D[x, y]$, and for the $k^2 - e$ cells which are non-$\star$ pairs. Let these non-$\star$ pairs be $s[x, y] = (a[x, y], b[x, y])$ for the $S[x, y]$'th cell. For every non-$\star$ cell, we select the disk $d[x, y]$ centered at $(x + \epsilon a[x, y], y + \epsilon b[x, y]) \in D[x, y]$.

Now we prove that all the disks selected this way do not intersect each other:

If the neighboring (right) cell in the solution of $I$ was $\star$ we are not selecting the disk from that particular set ($D[x + 1, 1]$, where cell $S[x + 1, y]$ is $\star$). then we do not need to worry about the intersecting condition as the next closest disk (on the right side) will be in the set $D[x + 2, y]$ which will imply that $|x - x'| + |y - y'| \geq 2$ and as proved earlier $d[x, y]$ and $d[x + 2, y]$ cannot intersect.

We can prove in the similar fashion all the remaining three cases:

- where the left cell is $\star$,

- where the above cell is $\star$,

- and where the below cell is $\star$.

Now we will prove that the disks selected in the suggested way will not intersect even in the case where all the four immediate neighboring cells are non-$\star$ in the solution of $I$. We can prove this is the following way: As have seen, if $|x - x'| + |y - y'| \geq 2$, then $d[x, y]$ and $d[x', y']$ cannot intersect. As the $s[x, y]$'s form a solution of the instance $I$, we have that $a[x, y] \leq a[x + 1, y]$. Therefore, by our claim above, the disks $d[x, y]$ and $d[x + 1, y]$ do not intersect. Similarly, we have $b[x, y] \leq b[x, y + 1]$, implying that $d[x, y]$ and $d[x, y + 1]$ do not intersect either. Hence there is indeed a set of at least $k^2 - e$ pairwise nonintersecting disks in $\mathcal{D}$.

$\square$

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Now using the Lemma 4, we can prove the following relation for $OPT(x) = k^2 - a$ and $OPT(R(x)) = k^2 - b$

$$OPT(R(x)) \geq OPT(x)$$
$$\implies k^2 - b \geq k^2 - a$$
$$\implies -b \geq -a$$
$$\implies b \leq a \tag{4.1}$$

**Construction of $S$**

Let $D' \subseteq \mathcal{D}$ be a set of $k^2 - n$ pairwise independent disks. As mentioned earlier, the disks in $D[x, y]$ all intersect each other, which would imply that there at most one disk for each $D[x, y]$. Look at $D[x, y]$ and if there is a disk $d[x, y]$ centered at $(x + \epsilon a[x, y], y + \epsilon b[x, y])$ for some $(a[x, y], b[x, y]) \in [n] \times [n]$, first $d[x, y] \in D[x, y]$ implies that $s[x, y] = (a[x, y], b[x, y]) \in S[x, y]$, select this pair to form the solution for the instance $I$.
If there is no $d[x, y]$ selected from $D[x, y]$ in $D'$ select $\star$ for the $S[x, y]$ cell to form the solution for the instance $I$.
We claim that the solution formed this way satisfies the conditions of MATRIX TILING WITH $\leq$. For any a disk $d[x, y]$ centered at $(x + \epsilon a[x, y], y + \epsilon b[x, y])$, we have two cases:

1. The neighbor of $s[x, y] = (a[x, y], b[x, y]) \in S[x, y]$ (i.e. $S[x, y + 1]$) is star: In this case, as mentioned earlier we pick $\star$ for the $S[x, y + 1]$ for the solution of $I$ instance, so we do not have to check the MATRIX TILING WITH $\leq$ condition of the pair selected for $s[x, y] = (a[x, y], b[x, y])$.

2. The neighbors $S[x + 1, y]$ and $S[x, y + 1]$ of $s[x, y] = (a[x, y], b[x, y]) \in S[x, y]$ are non-$\star$:
   We know $S[x + 1, y], S[x, y]$ and $S[x, y + 1]$ are non-$\star$, because there are $d[x, y], d[x + 1, y]$ and $d[x, y + 1]$ is selected from $D[x, y], D[x + 1, y]$ and $D[x, y+1]$ in $D'$. As we have seen above, the fact that $d[x, y]$ and $d[x+1, y]$ do not intersect implies that $a[x, y] \leq a[x + 1, y]$. Similarly, the fact that $d[x, y]$ and $d[x, y + 1]$ do not intersect each other implies that $b[x, y] \leq b[x, y + 1]$.

   ***Note:*** In the above proof for neighboring cases we only proved for the below neighbor ($D[x, y + 1]$), but all the remaining cases (right neighbor $D[x + 1, y]$, top neighbor $D[x, y - 1]$, and left neighbor $D[x - 1, y]$) can be proved in the similar fashion.
Thus the $s[x, y]$'s selected this way indeed form a solution for the instance $I$.
Here we can also conclude that if we construct the solution $c_A(S(y)) = k^2 - n$ of the instance $I$, from $c_B(y) = k^2 - m$ a solution of $\mathcal{D}$ this way, we have the following equality:

$$k^2 - n = k^2 - m$$
$$\implies n = m \tag{4.2}$$

### 4.2.1 Relation between the optimal solutions and any approximate solutions of $I$ and $\mathcal{D}$

We can notice that the optimum for $I$ is always at least $\frac{k^2}{4}$: if $i$ and $j$ are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$; otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: $k^2$, which gives us the following inequality:

$$k^2/4 \leq OPT(x) \leq k^2 \tag{4.3}$$

For $\mathcal{D}$ the lower bound can be 1, because it cannot be intersected by anything. The upper bound is $k^2$ because as mentioned earlier, the disks in $D[x, y]$ all intersect each other, which would imply that there at most one disk for each $D[x, y]$, and if we have one disk for each $D[x, y]$ we will get $k^2$ disks therefore:

$$1 \leq OPT(R(x)) \leq k^2 \tag{4.4}$$

Now from the equation 5.7 and equation 5.8 equations, we can find the value of $\alpha$ for the $3^{rd}$ condition condition of the L-reduction:

$$OPT(R(x)) \leq k^2 = 4k^2/4 = 4OPT(x)$$
$$\implies OPT(R(x)) \leq 4OPT(x) \tag{4.5}$$

Thus for $\alpha = 4$, we have $OPT(R(x)) \leq \alpha OPT(x)$.

### 4.2.2 Relation between the optimal solutions and any approximate solutions of $I$ and $\mathcal{D}$.

Let
$$OPT(x) = k^2 - a,$$
$$OPT(R(x)) = k^2 - b,$$
$$c_B(y) = k^2 - m,$$
$$c_A(S(y)) = k^2 - n.$$

from equation 4.2 and equation 4.1 in follows:

$$n = m$$
$$\implies n - a = m - a$$
$$\implies n - a = m - a \leq m - b$$
$$\implies n - a \leq (1)(m - b). \tag{4.6}$$

we now look at the $4^{th}$ condition condition of L-reduction:

$$OPT(x) - c_A(S(y)) \leq (\beta)(OPT(R(x)) - c_B(y))$$
$$\implies (k^2 - a) - (k^2 - n) \leq (\beta)((k^2 - b) - (k^2 - m))$$
$$\implies (n - a) \leq (\beta)(m - b) \tag{4.7}$$

from equation 4.6 and equation 4.7 we can get $\beta = 1$.
Thus for $\beta = 1$, we can satisfy: $|OPT(x) - c_A(S(y))| \leq \beta|OPT(R(x)) - c_B(y)|$.

**Note:** Because both the problems $I$ and $\mathcal{D}$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

This completes L-reduction from Matrix Tiling with $\leq$ to Maximum Independent Set on Unit Disk Graph, where the values of $\alpha$ and $\beta$ are 4 and 1 respectively. Providing the PTAS lower bound such that there are no $d, \delta > 0$ such that MAXIMUM INDEPENDENT SET ON UNIT DISK GRAPHS has PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$ unless ETH fails.

Maximum Scattered Set

## 5.1 Introduction

**Scattered Set:**
Given a graph $G$ and integer $d$, the task is to find a maximum sized set $S$ of vertices in $G$ such that the distance between any two distinct $u, v \in S$ is at least $d$.

## 5.2 Reduction and relation between solutions of both the instances.

### 5.2.1 Construction of a $SS$ instance from a given $MT \leq$ instance.

Let $I = (n, k, S)$ be an instance of Matrix Tiling With $\leq$; without loss of generality assume that $k \leq 2$. We represent each $S_{i,j}$ with an $n \times n$ grid $R_{i,j}$; let $v_{a,b}^{i,j}$ be the vertex of this grid in a row $a$ and column $b$. (Note that the distance between the first and last columns or rows of an $n \times n$ grid is $n - 1$.) Let $L = 100n$ and $d = 4L + n - 1$. We construct a graph $G'$ in the following way(see Figure 4.11).

(i) For every $1 \leq i, j \leq k$ and $(a, b) \in S_{i,j}$, we introduce a vertex $w_{a,b}^{i,j}$ and connect it to the vertex $v_{a,b}^{i,j}$ with a path of length $L$.

(ii) For every $1 \leq i \leq k, 1 \leq j < k$, we introduce a new vertex $\alpha_{i,j}$ and connect $\alpha_{i,j}$ to $v_{a,n}^{i,j}$ and $v_{a,1}^{i,j+1}$ for every $a \in [n]$ with a path of length $L$.

(iii) For every $1 \leq i < k, 1 \leq j \leq k$, we introduce a new vertex $\gamma_{i,j}$ and connect $\gamma_{i,j}$ to $v_{n,b}^{i,j}$ and $v_{1,b}^{i+1,j}$ for every $b \in [n]$ with a path of length $L$.

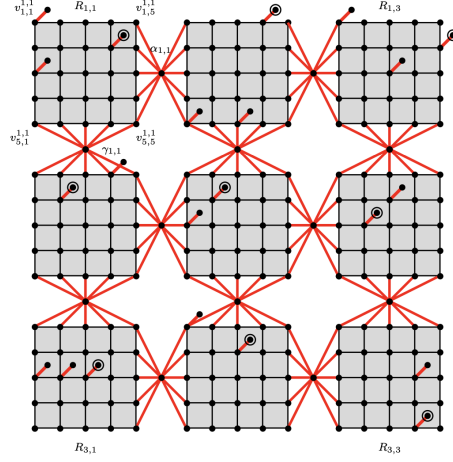Note that the resulting graph is planar, (see Figure 4.11).



Figure 5.1: Reducing the instance of Matrix Tiling with $\leq$ in with $k = 3$ and $n = 5$ to an instance of Scattered Set. The red edges represent paths of length L, the circled vertices correspond to the solution [9, Figure 14.6]

### 5.2.2 Construction of $R$

Let $R_{i,j} \bigcup w_{a,b}^{i,j} \bigcup \alpha_{i,j} \bigcup \gamma_{i,j} = C_{i,j}$ for every $1 \leq i, j \leq k$. For each $(i,j) \in [k] \times [k]$, if there is a non-$\star$ in the solution for $I$, say $s_{i,j} = (a_{i,j}, b_{i,j}) \in S_{i,j}$. Then let $Z$ contain $z_{i,j} = w_{a_{i,j}, b_{i,j}}^{i,j}$, if there is a $\star$ in the solution for some cell (i.e., $S_{i,j} = \star$) don't pick anything for $z_{i,j}$ in $Z$.

Some remarks on the solution of SCATTERED SET constructed this way:

Let $OPT(x) = k^2 - a$ i.e., number of $\star$'s is $a$. We will categorize the stars into the following categories:

1. The star whose neighbors are all non-$\star$'s: $t$.

2. The star who has one neighbor as a $\star$ and that $\star$ also has only one neighbor as star which is this one: $u$.

3. The star who has one neighbor as a $\star$ and that $\star$ has two neighbor as star which is this one: $v$.

4. The star who has one neighbor as a $\star$ and that $\star$ has three neighbor as star which is this one: $i$.

5. The star who has two $\star$'s as neighbors: $z$.

6. The star who has three $\star$'s as neighbors: $l$.

7. The star who has four $\star$'s as neighbors: $j$.

Note that a star cannot have all 4 neighbors as stars in the optimum as then we can pick arbitrary pair for this cell in the solution.

***Remarks:***

1. For every star whose neighbors are all non-$\star$'s, there is no vertices in its corresponding $C_{i,j}$ which is at a distance at least $d$ from any vertex which is picked in the solution set. <span style="color:red">t</span>

2. For every star who has one neighbor as a $\star$ and that $\star$ also has only one neighbor as star which is this one, there is exactly one vertex among corresponding $C_{i,j}$ and $C_{i+1,j}$ (or $C_{i,j+1}$) which is at a distance at least $d$ from any vertex which is picked in the solution set. For analysis we can here think that these two $\star$'s contribute half vertex each (i.e., one between them).

3. For star who has one neighbor as a $\star$ and that $\star$ has two neighbor as star which is this one, we can pick exactly one vertex among corresponding $C_{i,j}$, $C_{i',j'}$ and $C_{i'',j''}$ which is at a distance at least $d$ from any vertex which is picked in the solution set, and once we pick this vertex in the solution set we cannot pick anything from this as everything will be at a distance less than d from this vertex. For analysis we can here think that these three $\star$'s contribute $\frac{1}{3}$ vertex each (i.e., one between them).

4. For star who has one neighbor as a $\star$ and that $\star$ has three neighbor as star which is this one, we can pick exactly one vertex among corresponding $C_{i,j}$, $C_{i',j'}$, $C_{i'',j''}$ and $C_{i''',j'''}$ which is at a distance at least $d$ from any vertex which is picked in the solution set, and once we pick this vertex in the solution set we cannot pick anything from this as everything will be at a distance less than d from this vertex. For analysis we can here think that these four $\star$'s contribute $\frac{1}{4}$ vertex each (i.e., one between them).

5. For star who has two $\star$'s as neighbors, we can pick exactly one vertex among corresponding $C_{i,j}$, $C_{i',j'}$ and $C_{i'',j''}$ which is at a distance at least $d$ from any vertex which is picked in the solution set, and once we pick this vertex in the solution set we cannot pick anything from this as everything will be at a distance less than d from this vertex. For analysis we can here think that these three $\star$'s contribute $\frac{1}{3}$ vertex each (i.e., one between them).

6. For star who has three $\star$'s as neighbors, we can pick exactly one vertex among corresponding $C_{i,j}$, $C_{i',j'}$, $C_{i'',j''}$ and $C_{i''',j'''}$ which is at a distance at least $d$ from any vertex which is picked in the solution set, and once we pick this vertex in the solution set we cannot pick anything from this as everything will be at a distance less than d from this vertex. For analysis we can here think that these four $\star$'s contribute $\frac{1}{4}$ vertex each (i.e., one between them).

7. For star who has four $\star$'s as neighbors, we can pick exactly one vertex among corresponding $C_{i,j}$, $C_{i',j'}$, $C_{i'',j''}$, $C_{i''',j'''}$ and $C_{i'''',j''''}$ which is at a distance at least $d$ from any vertex which is picked in the solution

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> set, and once we pick this vertex in the solution set we cannot pick anything from this as everything will be at a distance less than d from this vertex. For analysis we can here think that these five $\star$'s contribute $\frac{1}{5}$ vertex each (i.e., one between them).

The intuitive idea for the above remarks is the if for example we have two non-stars neighboring each other, then while we can't pick anything from $w$'s vertices from set $C$'s corresponding to both of these cells, but we can still pick $\alpha/\gamma$ which is connecting them while still satisfying the conditions for scattered set.

Now from the above remarks we can analyze the relation between optimum solutions of both the instances:

$$OPT(x) = k^2 - a = k^2 - t - u - v - i - z - l - j \tag{5.1}$$

and

$$OPT(R(x)) = k^2 - b = k^2 - t - u - v - i - z - l - j + \frac{u}{2} + \frac{v}{3} + + \frac{i}{4}\frac{z}{3} + \frac{l}{4} + \frac{j}{5}$$

$$= k^2 - t - \frac{u}{3} - \frac{2v}{3} - - \frac{3i}{4} - \frac{2z}{3} - \frac{3l}{4} - \frac{4j}{5} \tag{5.2}$$

### 5.2.3 Construction of $S$

Let $R_{i,j} \bigcup w_{a,b}^{i,j} \bigcup \alpha_{i,j} \bigcup \gamma_{i,j} = C_{i,j}$ for every $1 \le i, j \le k$, we say that for some pair $(i,j) \in [k] \times [k]$ and some $(a_{i,j}, b_{i,j}) \in S_{i,j}$ if $C_{i,j} \bigcap Z = w_{a_{i,j}, b_{i,j}}^{i,j}$ then pick the $(a_{i,j}, b_{i,j})$ pair for the cell $S_{i,j}$ for the solution $S(y)$ of the $I$, otherwise pick a star in all other $S_{i',j'}$ cells.

We argue that the solution $S(y)$ of $I$, constructed in the above way form a valid solution of $I$. For every non-star cell, we need to consider two case for its neighbor: first when the neighbor is picked as a star, and the second when the neighbor is picked as a non-star:

1. Neighbor is a star: Without loss of generality let say the cell on the right of the cell we are considering is a star (i.e., $S_{i+1,j} = \star$), in the case we don't need to check any condition for $MT \le$, as from the definition of $MT \le$ we only require to satisfy the condition of the pairs which are immediate neighbor and are both non-$\star$'s

2. Neighbor is a non-star: Let $z_{i,j} = w_{a_{i,j}, b_{i,j}}^{i,j}, z_{i+1,j} = w_{a_{i+1,j}, b_{i+1,j}}^{i+1,j}$ be the vertices of $Z$ contained in $C_{i,j}$ and $C_{i+1,j}$ respectively, and let $s_{i,j} = (a_{i,j}, b_{i,j})$ and $s_{i+1,j} = (a_{i+1,j}, b_{i+1,j})$. We claim that the pairs $s_{i,j}, s_{i+1,j}$ are a part of a valid solution for the instance $I$. First $s_{i,j} = (a_{i,j}, b_{i,j}) \in S_{i,j}$ follows from the way the graph is defined. To see that $a_{i,j} \le a_{i+1,j}$, let us compute the distance between $z_{i,j}$ and $z_{i+1,j}$. The distance between $z_{i+1,j} = w_{a_{i+1,j}, b_{i+1,j}}^{i+1,j}$ and $\gamma_{i,j}$ is $L + a_{i+1,j} - 1$, hence the distance between $z_{i,j}$ and $z_{i+1,j}$ is $4L + n - 1 + (a_{i+1,j} - a_{i,j}) = d + (a_{i+1,j} - a_{i,j})$. As we know that this distance is at least $d$, the inequality $a_{i,j} \le a_{i+1,j}$ follows. By a similar argument if the $S_{i,j}$ ans $S_{i,j+1}$ both are non-$\star$'s in $I$, computing the distance between $z_{i,j}$ and $z_{i,j+1}$ shows that $b_{i,j} \le b_{i,j+1}$ holds.

Thus the pairs $s_{i,j}$ indeed form a solution of instance $I$.

Now from the remarks in the previous section we can analyze the relation between solutions of both the instances:

$$C_A(S(y)) = k^2 - n = k^2 - t' - u' - v' - i' - z' - l' - j' \qquad (5.3)$$

and

$$C_B(y) = k^2 - m \leq k^2 - t' - u' - v' - i' - z' - l' - j' + \frac{u'}{2} + \frac{v'}{3} + \frac{i'}{4} + \frac{z'}{3} + \frac{l'}{4} + \frac{j'}{5}$$
$$\leq k^2 - t' - \frac{u'}{3} - \frac{2v'}{3} - \frac{3i'}{4} - \frac{2z'}{3} - \frac{3l'}{4} - \frac{4j'}{5} \qquad (5.4)$$

Let us calculate from equation 5.1 and equation 5.3 $OPT(x) - C_A(S(y))$:

$$OPT(x) - C_A(S(y)) = k^2 - t - u - v - i - z - l - j - k^2 + t' + u' + v' + i' + z' + l' + j'$$
$$\implies = t' - t + u' - u + v' - v + i' - i + z' - z + l' - l + j' - j$$
$$\implies OPT(x) - C_A(S(y)) \leq 2\left(t' - t + \frac{u'}{2} - \frac{u}{2} + \frac{2v'}{3} - \frac{2v}{3} + \frac{3i'}{4} - \frac{3i}{4} + \frac{2z'}{3} - \frac{2z}{3} + \frac{3l'}{4} - \frac{3l}{4} + \frac{4j'}{5} - \frac{4j}{5}\right)$$
$$(5.5)$$

Let us calculate equation 5.2 and equation 5.4 $OPT(R(X)) - C_B(y)$:

$$k^2 - m + k^2 - t - \frac{u}{2} - \frac{2v}{3} - \frac{3i}{4} - \frac{2z}{3} - \frac{3l}{4} - \frac{4j}{5} \leq k^2 - b + k^2 - t' - \frac{u'}{2} - \frac{2v'}{3} - \frac{3i'}{4} - \frac{2z'}{3} - \frac{3l'}{4} - \frac{4j'}{5}$$
$$\implies -m - t - \frac{u}{2} - \frac{2v}{3} - \frac{3i}{4} - \frac{2z}{3} - \frac{3l}{4} - \frac{4j}{5} \leq -b - t' - \frac{u'}{2} - \frac{2v'}{3} - \frac{3i'}{4} - \frac{2z'}{3} - \frac{3l'}{4} - \frac{4j'}{5}$$
$$\implies t' - t + \frac{u'}{2} - \frac{u}{2} + \frac{2v'}{3} - \frac{2v}{3} + \frac{3i'}{4} - \frac{3i}{4} + \frac{2z'}{3} - \frac{2z}{3} + \frac{3l'}{4} - \frac{3l}{4} + \frac{4j'}{5} - \frac{4j}{5} \leq m - b + k^2 - k^2$$
$$\implies t' - t + \frac{u'}{2} - \frac{u}{2} + \frac{2v'}{3} - \frac{2v}{3} + \frac{3i'}{4} - \frac{3i}{4} + \frac{2z'}{3} - \frac{2z}{3} + \frac{3l'}{4} - \frac{3l}{4} + \frac{4j'}{5} - \frac{4j}{5} \leq k^2 - b - (k^2 - m)$$
$$\implies \left(t' - t + \frac{u'}{2} - \frac{u}{2} + \frac{2v'}{3} - \frac{2v}{3} + \frac{3i'}{4} - \frac{3i}{4} + \frac{2z'}{3} - \frac{2z}{3} + \frac{3l'}{4} - \frac{3l}{4} + \frac{4j'}{5} - \frac{4j}{5}\right) \leq OPT(R(x)) - C_B(y)$$
$$(5.6)$$

### 5.2.4 Relation between the optimal solutions and any approximate solutions of $I$ and $SS$.

We can notice that the optimum for $MT \leq$ is always at least $\frac{k^2}{4}$: if $i$ and $j$ are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$; otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: $k^2$, which gives us the following inequality:

$$k^2/4 \leq OPT(x) \leq k^2 \qquad (5.7)$$

Because all the vertices in $R_{i,j} \bigcup \alpha_{i,j} \bigcup \gamma_{i,j} w_{a_{i,j}, b_{i,j}^{i,j}}$ are at the distance less than $d$, we can argue that the maximum number of vertices in the $SS$ instance that

are at a distance $\geq d$ is at most $k^2$, and therefore we get the following condition for the optimum solution of $SS$ instance:

$$1 \leq OPT(R(x)) \leq k^2 \tag{5.8}$$

Now from the equations equation 5.7 and equation 5.8, we can find the value of $\alpha$ for the $3^{rd}$ condition of the L-reduction:

.

$$OPT(R(x)) \leq k^2 = 4k^2/4 = 4OPT(x)$$
$$\implies OPT(R(x)) \leq 4OPT(x) \tag{5.9}$$

Thus for $\alpha = 4$, we have $OPT(R(x)) \leq \alpha OPT(x)$. $\qquad\square$

### 5.2.5 Relation between the optimal solutions and any approximate solutions of $I$ and $SS$.

From equation 5.6, and equation 5.5, we get:

$$OPT(x) - C_A(S(y)) \leq 2\left(OPT(R(x)) - C_B(y)\right) \tag{5.10}$$

**Note:** Because both the problems $MT \leq$ and $SS$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

This completes L-reduction from Matrix Tiling with $\leq$ to Maximum Scattered Set, where the values of $\alpha$ and $\beta$ are 4 and 2 respectively. Providing the PTAS lower bound such that there are no $d, \delta > 0$ such that Maximum Scatterd Set has PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, unless ETH fails.

CHAPTER 6

Future Work

For future work, one can look at existing exact reductions from GRID TILING WITH ≤ and can try to get L-reduction, which would be from MATRIX TILING WITH ≤, and from Theorem 3 and Lemma 3 can derive the PTAS lower bound for the optimization version of these problems.

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