PTAS Lower bound for Covering Points with Squares

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Contents

1 Introduction

We begin by formally defining both the variants of Matrix Tiling.

Matrix Tiling:

Input: Integers k, n, and k^2 nonempty sets $\mathcal{S}_{i,j} \subseteq [n] \times [n]$, for $1 \leq i, j \leq k$.
Goal: For each $1 \leq i, j \leq k$, a value $s_{i,j} \in \mathcal{S}_{i,j} \cup \{\star\}$ such that:

- If $s_{i,j} = (a_1, a_2)$ and $s_{i,j+1} = (b_1, b_2)$, then $a_1 = b_1$.
- If $s_{i,j} = (a_1, a_2)$ and $s_{i+1,j} = (b_1, b_2)$, then $a_2 = b_2$.

The objective is to maximize the number of pairs $s_{i,j} \neq \star$.

Matrix Tiling with $\leq$:

Input: Integers k, n, and k^2 nonempty sets $G_{i,j} \subseteq [n] \times [n]$ for each $1 \leq i, j \leq k$.
Find: For each $1 \leq i, j \leq k$, a value $g_{i,j} \in G_{i,j} \cup \{\star\}$ such that:

- If $g_{i,j} = (a_1, a_2)$ and $g_{i,j+1} = (b_1, b_2)$, then $a_1 \leq b_1$.
- If $g_{i,j} = (a_1, a_2)$ and $g_{i+1,j} = (b_1, b_2)$, then $a_2 \leq b_2$.

The objective is to maximize the number of pairs $g_{i,j} \neq \star$.

We prove the following ETH-based PTAS lower bound for Matrix Tiling with $\leq$:

Theorem 1. *If there are constants $\delta, d > 0$ such that* Matrix Tiling with $\leq$ *has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

Our approach is based on an L-reduction from Matrix Tiling problem to the Matrix Tiling with $\leq$. An L-reduction between problems A and B, with respective cost functions c_A and c_B, is a pair of polynomial-time computable functions R and S satisfying the following:

1. If x is an instance of problem A, then $R(x)$ is an instance of problem B,
2. If y is a solution to $R(x)$, then $S(y)$ is a solution to x,
3. There exists a constant $\alpha > 0$ such that $OPT(R(x)) \leq \alpha\, OPT(x)$,
4. There exists a constant $\beta > 0$ such that $|OPT(x) - c_A(S(y))| \leq \beta |OPT(R(x)) - c_B(y)|$.

Note: In the above theorem n is not the range for the coordinates, but the input size of the problem instance.

1.1 Conventions and Helper Function:

- We denote $[n] = \{1, \ldots, n\}$,
- Throughout this reduction, the origin $(0, 0)$ lies at the top-left corner of the matrix,
- For a pair (a, b), we use $\text{first}(a, b) = a$, and $\text{second}(a, b) = b$,
- Given a cell $\mathcal{S}_{i,j}$, we refer to its neighboring cell as:
    1. Right: $\mathcal{S}_{i,j+1}$,
    2. Left: $\mathcal{S}_{i,j-1}$,
    3. Above: $\mathcal{S}_{i-1,j}$,
    4. Below: $\mathcal{S}_{i+1,j}$.

$\mathcal{S}_{i,j}$	$\mathcal{S}_{i,j+1}$
$\mathcal{S}_{i+1,j}$	$\mathcal{S}_{i+1,j+1}$

- Let the following helper functions compute min/max coordinate values in neighboring cells:
    - Vertical (first coordinate):
        * $a_{\text{min}}^{\text{below}(i,j)} = \min\{\text{first}(s_{i+1,j}) \mid s_{i+1,j} \in \mathcal{S}_{i+1,j}\}$,
        * $a_{\text{max}}^{\text{below}(i,j)} = \max\{\text{first}(s_{i+1,j}) \mid s_{i+1,j} \in \mathcal{S}_{i+1,j}\}$,

* $a_{\text{min}}^{\text{above}(i,j)}$, and $a_{\text{max}}^{\text{above}(i,j)}$ defined analogously from $\mathcal{S}_{i-1,j}$.
  - Horizontal (second coordinate):
    * $b_{\text{min}}^{\text{left}(i,j)}$, $b_{\text{max}}^{\text{left}(i,j)}$ from $\mathcal{S}_{i,j-1}$,
    * $b_{\text{min}}^{\text{right}(i,j)}$, $b_{\text{max}}^{\text{right}(i,j)}$ from $\mathcal{S}_{i,j+1}$.

- Let the pair $(a_{i,j}^{\text{max}}, b_{i,j}^{\text{max}})$ be the pair in the set $\mathcal{S}_{i,j}$ with the largest first coordinate. Similarly, let the pair $(a_{i,j}^{\text{min}}, b_{i,j}^{\text{min}})$ be the pair in the set $\mathcal{S}_{i,j}$ with the smallest first coordinate.

We now proceed to defining the functions R, and S and analyze their properties to prove that it is an L-reductions with respect to all four conditions of an L-reduction.

2 Constructing the instance of Matrix Tiling with $\leq$ (Definition of R):

We now describe a polynomial-time L-reduction from Matrix Tiling to Matrix Tiling with $\leq$. Let $\mathcal{I} = (k, n, \{\mathcal{S}_{i,j}\})$ be an instance of the Matrix Tiling problem. We will construct an instance $\mathcal{M} = (k', n', \{G_{i',j'}\})$ of Matrix Tiling with $\leq$ such that an approximate solution to $\mathcal{M}$ can be efficiently transformed into an approximate solution to $\mathcal{I}$, satisfying the four L-reduction conditions.

2.1 Shifting Coordinates (Step 1):

To allow room for inserting auxiliary pairs, we first apply a uniform shift to all coordinate values.

Let each $\mathcal{S}_{i,j} \subseteq [n] \times [n]$. We define a new set:

$$\mathcal{S}'_{i,j} = \{(x+k, y+k) \mid (x,y) \in \mathcal{S}_{i,j}\}. \tag{1}$$

We update the domain size to $n \leftarrow n+2k$, so that coordinates now lie in $[n+2k]$. This shift ensures:

- The minimum coordinate is at least k, allowing insertion of values less than any existing coordinate (down to 0),
- The maximum coordinate is at most $n+k$, allowing insertion of values greater than any existing coordinate (up to $n+2k$).

This transformation preserves all original pair relations, prepares the instance for the addition auxiliary pairs in order to satisfy the four L-reduction conditions.

2.2 Auxiliary Coordinate Values (Step 2):

To keep our reduction "approximation preserving" and to satisfy all four conditions L-reduction conditions, we introduce seven auxiliary values for each cell $\mathcal{S}'_{i,j}$, derived from adjacent cells.

For each cell $\mathcal{S}_{i,j}$, we define the following eight new values, collectively called as NEW-I:

For Vertical constraints:	**For Horizontal constraints:**
• $a_{i,j}^{\text{d}+} = a_{\text{max}}^{\text{below}(i,j)} + 1$,	• $b_{i,j}^{\text{r}+} = b_{\text{max}}^{\text{right}(i,j)} + 1$,
• $a_{i,j}^{\text{d}-} = a_{\text{min}}^{\text{below}(i,j)} - 1$,	• $b_{i,j}^{\text{r}-} = b_{\text{min}}^{\text{right}(i,j)} - 1$,
• $a_{i,j}^{\text{u}+} = \max\{a_{\text{max}}^{\text{above}(i,j)}, a_{i-1,j}^{\text{d}+}, a_{i-1,j}^{\text{d}-}\} + 1$,	• $b_{i,j}^{\text{l}+} = \max\{b_{\text{max}}^{\text{left}(i,j)}, b_{i,j-1}^{\text{r}+}, b_{i,j-1}^{\text{r}-}\} + 1$,
• $a_{i,j}^{\text{u}-} = \min\{a_{\text{min}}^{\text{below}(i,j)}, a_{i-1,j}^{\text{d}+}, a_{i-1,j}^{\text{d}-}\} - 1$.	• $b_{i,j}^{\text{l}-} = \min\{b_{\text{min}}^{\text{left}(i,j)}, b_{i,j-1}^{\text{r}+}, b_{i,j-1}^{\text{r}-}\} - 1$.

2.3 Constructing the instance of Matrix Tiling with $\leq$ (Step 3):

We now construct $\mathcal{M} = (n', k', G_{i',j'})$ of Matrix Tiling with $\leq$, with

$$n' = 3n^2(k+1) + n^2 + 3n, \quad \text{and} \quad k' = 4k$$

Set $N = 4n^2$, and define a encoding function $\iota(a,b) = n \cdot a + b$, let $z[i,j] = \{\iota(a,b) \mid (a,b) \in \mathcal{S}'_{i,j}\}$, and define:

$$z_{i,j}^{+} = \iota((a_{i,j}^{\text{max}} + 2), b_{i,j}^{\text{max}}), \quad \text{and} \quad z_{i,j}^{-} = ((a_{i,j}^{\text{min}} - 2), b_{i,j}^{\text{min}}).$$

For each cell $\mathcal{S}'_{i,j}$ we construct a gadget which is a 4×4 grid of sets $G_{i',j'}$, indexed by $(4i-3 \leq i' \leq 4i)$, and $(4j-3 \leq j' \leq 4j)$ (see Figure 1). These can be categorized into two groups:

- 4 **inner sets**: ($G_{4i-2,4j-2}$, $G_{4i-2,4j-1}$, $G_{4i-1,4j-2}$, $G_{4i-1,4j-1}$) are dummy sets and they have one only pairs for each of them. These sets are placeholders and do not depend on pairs from $S^{3n}_{i,j}$.
- 12 **outer sets**: are populated using a mapping function $\iota(a_{i,j}, b_{i,j})$ and N. For each $(a_{i,j}, b_{i,j}) \in \mathcal{S}'_{i,j}$, we call them encoded pairs.

$G_{4i-3,4j-3}$: $(iN-z, jN+z)$	$G_{4i-3,4j-2}$: $(iN+a, jN+z)$	$G_{4i-3,4j-1}$: $(iN-a, jN+z)$	$G_{4i-3,4j}$: $(iN+z, jN+z)$
$G_{4i-2,4j-3}$: $(iN-z, jN+b)$	$G_{4i-2,4j-2}$: $((i+1)N, (j+1)N)$	$G_{4i-2,4j-1}$: $(iN, (j+1)N)$	$G_{4i-2,4j}$: $(iN+z, (j+1)N+b)$
$G_{4i-1,4j-3}$: $(iN-z, jN-b)$	$G_{4i-1,4j-2}$: $((i+1)N, jN)$	$G_{4i-1,4j-1}$: (iN, jN)	$G_{4i-1,4j}$: $(iN+z, (j+1)N-b)$
$G_{4i,4j-3}$: $(iN-z, jN-z)$	$G_{4i,4j-2}$: $((i+1)N+a, jN-z)$	$G_{4i,4j-1}$: $((i+1)N-a, jN-z)$	$G_{4i,4j}$: $(iN+z, jN-z)$

Figure 1: The 16 sets of the constructed Matrix Tiling with $\leq$ instance representing a set $S_{i,j}$ of the Matrix Tiling in the reduction in the proof of together with the pairs corresponding to a pair $(a, b) \in S'_{i,j}$(with $z = \iota(a, b)$)

Now we add some pairs to the specific cells in each gadget $G_{i',j'}$ which are created using the NEW-I values introduced in the previous section in the following way, we call these 13 pairs as NEW-A pairs :

1. Add new pairs to the *corner* cells of the gadget as follows:
   (a) $G_{4i-3,4j-3} = (iN - z^+_{i,j}, jN + z^+_{i,j})$,
   (b) $G_{4i-3,4j} \quad = (iN + z^+_{i,j}, jN + z^+_{i,j})$,
   (c) $G_{4i,4j-3} \quad = (iN - z^+_{i,j}, jN - z^+_{i,j})$,
   (d) $G_{4i,4j} \qquad = (iN + z^+_{i,j}, jN - z^+_{i,j})$
2. Use the values $\left(b^{\mathrm{r}+}_{i,j}, b^{\mathrm{r}-}_{i,j}, b^{\mathrm{l}-}_{i,j}\right)$, to construct the pairs and add them to the cells as mentioned below:
   (a) $G_{4i-1,4j-3} = (iN - z^+_{i,j}, jN - b^{\mathrm{l}-})$.
   (b) $G_{4i-2,4j} \quad = (iN + z^+_{i,j}, (j+1)N + b^{\mathrm{r}-})$,
   (c) $G_{4i-1,4j} \quad = (iN + z^+_{i,j}, (j+1)N - b^{\mathrm{r}+})$,
3. Use $\left(a^{\mathrm{u}+}_{i,j}, a^{\mathrm{d}+}_{i,j}, a^{\mathrm{u}-}_{i,j}, a^{\mathrm{d}-}_{i,j}\right)$, to construct the pairs and add them to the cells as mentioned below:
   (a) $G_{4i-3,4j-2} = (iN + a^{\mathrm{u}+}, jN + z^+_{i,j})$,
   (b) $G_{4i-3,4j-1} = (iN - a^{\mathrm{u}-}, jN + z^+_{i,j})$,
   (c) $G_{4i,4j-2} \quad = ((i+1)N + a^{\mathrm{d}-}, jN - z^+_{i,j})$,
   (d) $G_{4i,4j-1} \quad = ((i+1)N - a^{\mathrm{d}+}, jN - z^+_{i,j})$,
4. Finally, we add a pair fp $= (iN - z^-_{i,j}, jN + b^{l+})$ to the cell $G_{4i-2,4j-3}$, we call this pair **FORBIDDEN PAIR**.

We call the first 11 pairs i.e., NEW-A \ FORBIDDEN PAIR (all the pairs from NEW-A excluding the fp pair) as NEW-M pairs.

Claim 1. *For all $z \in z[i,j]$, we have the following relation:*

$$z < z^+_{i,j}$$

Proof. We know that:

- $a_{i,j} \le a_{i,j}^{\max}$, since $a_{i,j}^{\max} = \max\{\text{first}(s_{i,j}) \mid s_{i,j} \in \mathcal{S}'_{i,j}\}$,
- $z = n \cdot a_{i,j} + b_{i,j}$,
- $z_{i,j}^{+} = n \cdot (a_{i,j}^{\max} + 2) + b_{i,j}^{\max}$.

Let us upper-bound the largest value in the set $z[i,j] : z_{a_{i,j},b_{i,j}}$. Since $a_{i,j} \le a_{i,j}^{\max}$, the worst case is $a_{i,j} = a_{i,j}^{\max}$, and since $b_{i,j} \le b_{i,j}^{\max}$. Then:

$$z_{a_{i,j},b_{i,j}} \le n \cdot a_{i,j}^{\max} + b_{i,j}^{\max}.$$

On the other hand:

$$z_{i,j}^{+} = n \cdot (a_{i,j}^{\max} + 2) + b_{i,j}^{\max} = n \cdot a_{i,j}^{\max} + 2n + b_{i,j}^{\max}.$$

Therefore,

$$\begin{aligned} z_{i,j}^{+} - z_{a_{i,j},b_{i,j}} &\ge 2n > 0 \\ \implies z_{i,j}^{+} &> z \quad \text{for all} \quad z \in z[i,j]. \end{aligned}$$

□

Claim 2. *For all $z \in z[i,j]$, we have the following relation:*

$$z, z_{i,j}^{+} > z_{i,j}^{-}$$

Proof. We know that:

- $a_{i,j} \ge a_{i,j}^{\min}$, since $a_{i,j}^{\min} = \min\{\text{first}(s_{i,j}) \mid s_{i,j} \in \mathcal{S}'_{i,j}\}$,
- $z = n \cdot a_{i,j} + b_{i,j}$,
- $z_{i,j}^{-} = n \cdot (a_{i,j}^{\min} - 2) + b_{i,j}^{\min}$.

Let us lower-bound the smallest value in the set $z[i,j] : z_{a_{i,j},b_{i,j}}$. Since $a_{i,j} \ge a_{i,j}^{\min}$, the worst case is $a_{i,j} = a_{i,j}^{\min}$, and since $b_{i,j} \ge b_{i,j}^{\min}$. Then:

$$z_{a_{i,j},b_{i,j}} \ge n \cdot a_{i,j}^{\min} + b_{i,j}^{\min}.$$

On the other hand:

$$z_{i,j}^{-} = n \cdot (a_{i,j}^{\min} - 2) + b_{i,j}^{\min} = n \cdot a_{i,j}^{\min} - 2n + b_{i,j}^{\min}.$$

Therefore,

$$\begin{aligned} z_{a_{i,j},b_{i,j}} - z_{i,j}^{-} &\ge 2n > 0 \\ \implies z &> z_{i,j}^{-} \quad \text{for all} \quad z \in z[i,j]. \end{aligned}$$

Now, since $z_{i,j}^{+} > z$ for all $z \in z[i,j]$ (from 1), it follows that:

$$z(a_{i,j}, b_{i,j}), z_{i,j}^{+} > z_{i,j}^{-}$$

. □

Observation 1. *For all $z \in z[i,j]$, $(1 \le i,j \le k)$ we have:*

$$z, z_{i,j}^{+}, z_{i,j}^{-} \le N = 4n^2 \tag{2}$$

Claim 3. *For any $1 \le i,j \le k$, suppose the assigned pair for the cell $G_{4i-2,4j-3}$ is*

$$(iN - z_{i,j}^{-},\ jN - b_{i,j}^{l+}).$$

Then, the cell $G_{4i-1,4j-3}$ must be assigned $\star$ in any feasible solution.

Proof. Since $G_{4i-2,4j-3}$ is above $G_{4i-1,4j-3}$, the Matrix Tiling with $\le$ constraint requires:

$$\text{first}(g_{4i-1,4j-3}) \ge iN - z_{i,j}^{-}.$$

Since $\text{first}(g_{4i-1,4j-3})$ is of the form $iN - z_{i,j}$, it follows that $z_{i,j}$ less than $z_{i,j}^{-}$.

However by Claim 2, no pair $z \in z[i,j]$ satisfies $z < z_{i,j}^{-}$. Hence, no pair with a feasible first coordinate exists for that cell, and it must be assigned $\star$. □

Lemma 1. *Suppose a NEW-M pair is assigned to any cell of the gadget $G_{i,j}$. Then, the total number of non-$\star$ assignments in $G_{i,j}$ is at most* 15*.*

Proof. Assume that in the $G_{i,j}$, one of the selected pairs is a NEW-M pair. Without loss of generality, suppose the selected pair appears in cell $G_{4i-3,4j-2}$ and is $(iN + a^{\mathrm{u}+}_{i,j}, jN + z^{+}_{i,j})$.

To satisfy the $\leq$ constraint in Matrix Tiling with $\leq$, the pair selected in the next cell in the row, $G_{4i-3,4j-1}$, must have its second coordinate at least $jN + z^{+}_{i,j}$. Since all values $z \in z[i,j]$ satisfy $z < z^{+}_{i,j}$ by 1 Hence,the only feasible option for this cell is the NEW-M pair $(iN - a^{\mathrm{u}-}_{i,j}, jN + z^{+}_{i,j})$.

Proceeding clockwise around the outer sets of the gadget, each cell is similarly forced to be assigned a NEW-M pair to maintain feasibility under the $\leq$ constraints.

Eventually, this propagation reaches a cell (namely $G_{4i-2,4j-3}$) that cannot satisfy the inequality unless it also receives a matching NEW-M pair (because the first coordinate of the pair for this cell must be at most $iN - z^{+}_{i,j}$). However in our construction, no such pair was added to that cell. This cell only received one additional pair why has the first coordinate $iN - z^{-}_{i,j}$, which is strictly greater $iN - z^{+}_{i,j}$ (by claim 2), and therefore there are no pairs which have the first coordinate which is exactly $iN - z^{+}_{i,j}$, which is required to satisfy the condition from the cells above and below it. Thus, it must be assigned $\star$.

Hence, any gadget where a NEW-M pair is selected must contain at least one $\star$, and therefore at most 15 non-$\star$'s. ☐

Observation 2. *From 2 and Lemma 1, we can conclude than if all the* 16 *cells of a gadget $G_{i',j'}$ are non-$\star$ pairs, then none of them is a "NEW-A" pair.*

Lemma 2. *Let $\mathcal{S}'_{i,j} = \star$ in a feasible solution to $\mathcal{I}$. Then, the corresponding 4×4 gadget $G_{i',j'}$ in $\mathcal{M}$ admits a feasible assignment with at least* 15 *non-$\star$ entries.*

Proof. We explicitly construct a feasible assignment for the gadget $G_{i',j'}$ with 15 non-$\star$ pairs, and show that all constraints are satisfied.

Gadget Construction: Select the 15 NEW-M pairs corresponding to the non-$\star$ cells in $G_{i',j'}$. Each pair is chosen to satisfy the local constraints within the gadget:

- In columns 1 and 4, the first coordinates of selected pairs are equal.
- In columns 2 and 3, the vertical consistency is ensured by the ordering of coordinates: for example, in the third row of column 2, we have $iN + a$ above $iN + N$ (as $N = 4n^2$ and $a \leq n$), satisfying the constraint. Horizontal constraints follow similarly.

Inter-Gadget Constraints: We verify that the "$\leq$" constraints hold across gadgets for both horizontal and vertical adjacencies. Each direction is handled via four exhaustive cases depending on whether each adjacent cell is $\star$ or not.

Horizontal Cases:

second$(g_{4i-3,4j})$: $jN + z'_{i,j}$	$\leq$	second$(g_{4i-3,4j+1})$: $(j+1)N + z'_{i,j+1}$
second$(g_{4i-2,4j})$: $(j+1)N + b'_{i,j}$	$\leq$	second$(g_{4i-2,4j+1})$: $(j+1)N + b'_{i,j+1}$
second$(g_{4i-1,4j})$: $(j+1)N - b'_{i,j}$	$\leq$	second$(g_{4i-1,4j+1})$: $(j+1)N - b'_{i,j+1}$
second$(g_{4i,4j})$: $jN - z'_{i,j}$	$\leq$	second$(g_{4i,4j+1})$: $(j+1)N - z'_{i,j+1}$

(a) 1

second$(g_{4i-3,4j})$: $jN + z'_{i,j}$	$\leq$	second$(g_{4i-3,4j+1})$: $(j+1)N + z^{+}_{i,j+1}$
second$(g_{4i-2,4j})$: $(j+1)N + b'_{i,j}$	$\leq$	second$(g_{4i-2,4j+1})$: $\star$
second$(g_{4i-1,4j})$: $(j+1)N - b'_{i,j}$	$\leq$	second$(g_{4i-1,4j+1})$: $(j+1)N - b^{\mathrm{l}-}_{i,j+1}$
second$(g_{4i,4j})$: $jN - z'_{i,j}$	$\leq$	second$(g_{4i,4j+1})$: $(j+1)N - z^{+}_{i,j+1}$

(b) 2

second$(g_{4i-3,4j})$: $jN + z^{+}_{i,j}$	$\leq$	second$(g_{4i-3,4j+1})$: $(j+1)N + z'_{i,j+1}$
second$(g_{4i-2,4j})$: $(j+1)N + b^{\mathrm{r}-}_{i,j}$	$\leq$	second$(g_{4i-2,4j+1})$: $(j+1)N + b'_{i,j+1}$
second$(g_{4i-1,4j})$: $(j+1)N - b^{\mathrm{r}+}_{i,j}$	$\leq$	second$(g_{4i-1,4j+1})$: $(j+1)N - b'_{i,j+1}$
second$(g_{4i,4j})$: $jN - z^{+}_{i,j}$	$\leq$	second$(g_{4i,4j+1})$: $(j+1)N - z'_{i,j+1}$

(c) 3

second$(g_{4i-3,4j})$: $jN + z^{+}_{i,j}$	$\leq$	second$(g_{4i-3,4j+1})$: $(j+1)N + z^{+}_{i,j+1}$
second$(g_{4i-2,4j})$: $(j+1)N + b^{\mathrm{r}-}_{i,j}$	$\leq$	second$(g_{4i-2,4j+1})$: $\star$
second$(g_{4i-1,4j})$: $(j+1)N - b^{\mathrm{r}+}_{i,j}$	$\leq$	second$(g_{4i-1,4j+1})$: $(j+1)N - b^{\mathrm{l}-}_{i,j+1}$
second$(g_{4i,4j})$: $jN - z^{+}_{i,j}$	$\leq$	second$(g_{4i,4j+1})$: $(j+1)N - z^{+}_{i,j+1}$

(d) 4

Figure 2: All four cases for the horizontal constraint.

(i) Both $\mathcal{S}'_{i,j}$ and $\mathcal{S}'_{i,j+1}$ are non-$\star$: As shown in Figure 2 (a), the second coordinates of all pairs selected in each cell satisfy the constraints using the fact that $N > z'_{i,j}$ (from Equation 2), and $b'_{i,j} = b'_{i,j+1}$.

(ii) $\mathcal{S}'_{i,j}$ is non-$\star$, $\mathcal{S}'_{i,j+1}$ is $\star$: See Figure 2 (b), the second coordinates of all pairs selected in each cell satisfy the constraints using the fact that $N > z'_{i,j}$ (from Equation 2), and $b'_{i,j} > b^{l-}_{i,j+1}$.

(iii) $\mathcal{S}'_{i,j}$ is $\star$, $\mathcal{S}'_{i,j+1}$ is non-$\star$: See Figure 2 (c), the second coordinates of all pairs selected in each cell satisfy the constraints using the fact that $N > z^{+}_{i,j}$ (from Equation 2), $b^{r-}_{i,j} < b'_{i,j+1}$, and $b^{r+}_{i,j} > b'_{i,j+1}$.

(iv) Both are $\star$: See Figure 2 (d), the second coordinates of all pairs selected in each cell satisfy the constraints using the fact that $N > z^{+}_{i,j}$ (from Equation 2), and $b^{r+}_{i,j} > b^{l-}_{i,j+1}$.

Vertical Cases:

$\text{first}(g_{4i,4j-3})$: $iN - z'_{i,j}$	$\text{first}(g_{4i,4j-2})$: $(i+1)N + a'_{i,j}$	$\text{first}(g_{4i,4j-1})$: $(i+1)N - a'_{i,j}$	$\text{first}(g_{4i,4j})$: $iN + z'_{i,j}$
$\leq$	$\leq$	$\leq$	$\leq$
$\text{first}(g_{4i+1,4j-3})$: $(i+1)N - z'_{i+1,j}$	$\text{first}(g_{4i+1,4j-2})$: $(i+1)N + a'_{i+1,j}$	$\text{first}(g_{4i+1,4j-1})$: $(i+1)N - a'_{i+1,j}$	$\text{first}(g_{4i+1,4j})$: $(i+1)N + z'_{i+1,j}$

(a) 1

$\text{first}(g_{4i,4j-3})$: $iN - z'_{i,j}$	$\text{first}(g_{4i,4j-2})$: $(i+1)N + a'_{i,j}$	$\text{first}(g_{4i,4j-1})$: $(i+1)N - a'_{i,j}$	$\text{first}(g_{4i,4j})$: $iN + z'_{i,j}$
$\leq$	$\leq$	$\leq$	$\leq$
$\text{first}(g_{4i+1,4j-3})$: $(i+1)N - z^{+}_{i+1,j}$	$\text{first}(g_{4i+1,4j-2})$: $(i+1)N + a^{\text{u}+}_{i+1,j}$	$\text{first}(g_{4i+1,4j-1})$: $(i+1)N - a^{\text{u}-}_{i+1,j}$	$\text{first}(g_{4i+1,4j})$: $(i+1)N + z^{+}_{i+1,j}$

(b) 2

$\text{first}(g_{4i,4j-3})$: $iN - z^{+}_{i,j}$	$\text{first}(g_{4i,4j-2})$: $(i+1)N + a^{\text{d}-}_{i,j}$	$\text{first}(g_{4i,4j-1})$: $(i+1)N - a^{\text{d}+}_{i,j}$	$\text{first}(g_{4i,4j})$: $iN + z^{+}_{i,j}$
$\leq$	$\leq$	$\leq$	$\leq$
$\text{first}(g_{4i+1,4j-3})$: $(i+1)N - z'_{i+1,j}$	$\text{first}(g_{4i+1,4j-2})$: $(i+1)N + a'_{i+1,j}$	$\text{first}(g_{4i+1,4j-1})$: $(i+1)N - a'_{i+1,j}$	$\text{first}(g_{4i+1,4j})$: $(i+1)N + z'_{i+1,j}$

(c) 3

$\text{first}(g_{4i,4j-3})$: $iN - z^{+}_{i,j}$	$\text{first}(g_{4i,4j-2})$: $(i+1)N + a^{\text{d}-}_{i,j}$	$\text{first}(g_{4i,4j-1})$: $(i+1)N - a^{\text{d}+}_{i,j}$	$\text{first}(g_{4i,4j})$: $iN + z^{+}_{i,j}$
$\leq$	$\leq$	$\leq$	$\leq$
$\text{first}(g_{4i+1,4j-3})$: $(i+1)N - z^{+}_{i+1,j}$	$\text{first}(g_{4i+1,4j-2})$: $(i+1)N + a^{\text{u}+}_{i+1,j}$	$\text{first}(g_{4i+1,4j-1})$: $(i+1)N - a^{\text{u}-}_{i+1,j}$	$\text{first}(g_{4i+1,4j})$: $(i+1)N + z^{+}_{i+1,j}$

(d) 4

Figure 3: All four cases for the vertical constraint.

(i) Both $\mathcal{S}'_{i,j}$ and $\mathcal{S}'_{i+1,j}$ are non-$\star$: See Figure 3 (a), the first coordinates of all pairs selected in each cell satisfy the constraints using the fact that $N > z'_{i,j}$ (from Equation 2), and $a'_{i,j} = a'_{i,j+1}$.

(ii) $\mathcal{S}'_{i,j}$ is non-$\star$, $\mathcal{S}'_{i+1,j}$ is $\star$: See Figure 3(b). Constraints are satisfied using the inequalities: $N > z'_{i,j}$, $a^{u+}_{i+1,j} > a'_{i,j}$ and $a^{u-}_{i+1,j} < a'_{i,j}$.

(iii) $\mathcal{S}'_{i,j}$ is $\star$, $\mathcal{S}'_{i+1,j}$ is non-$\star$: See Figure 3(c). Constraints are satisfied using the inequalities: $N > z^{+}_{i,j}$, $a^{d-}_{i,j} < a'_{i+1,j}$ and $a^{d+}_{i,j} > a'_{i+1,j}$.

(iv) Both are $\star$: See Figure 3(d). Constraints are satisfied using the inequalities: $N > z^{+}_{i,j}$, $a^{d-}_{i,j} < a^{u+}_{i+1,j}$ and $a^{d+}_{i,j} > a^{u-}_{i+1,j}$.

In each case, constraints across the gadgets are satisfied due to the definitions and inequalities involving parameters like z^{+}, $a^{u\pm}$, $a^{d\pm}$, b^{l-}, $b^{r\pm}$, ensuring feasibility.

> **Remark 1.** *Here we point out that we have also added the pair* fp $= (iN - z^{-}_{i,j}jN + b^{l+})$ *to the cell $G_{4i-2,4j-3}$. Similar to all the other pairs it satisfies the inter-gadget constraint in all the cases, both it doesn't satisfy the vertical constraint within the gadget that is why it cannot be picked. But this pair can be useful whenever we are doing L-reductions from* Matrix Tiling *and using this constructed instance of* Matrix Tiling with $\leq$ *as an intermediate gadget, because this pair satisfies one of the two constraints of the cell (horizontal), hence during L-reductions to minimization problems, whenever there is a $\star$ in the optimal solution of* Matrix Tiling, *in the corresponding intermediate gadget of* Matrix Tiling with $\leq$, *we only need to worry about one directional constraint (vertical).*

□

3 Constructing a solution of $\mathcal{I}$ given any solution of Matrix Tiling with $\leq$ (Definition of S):

Lemma 3 (Uniform Encoding in Gadgets)**.** *Suppose a gadget $G_{i',j'}$ in $R(x)$ contains 16 non-$\star$ values in a feasible solution y, Then all 12 outer cells of the gadget encode the same pair $(a, b) \in \mathcal{S}'_{i,j}$.*

Proof. Since the gadget has no $\star$ assignment, none of the selected values are NEW-A (see observation 2). Therefore, all selected values come from the encoding of some original pair $(a, b) \in \mathcal{S}'_{i,j}$.

Let the pairs selected in the solution from these sets define 12 values z, denoted as $z_{4i-3,4j-3}, z_{4i-3,4j-2}, \dots$, representing the values selected from these sets. We claim that all these 12 values are equal.

To see this, let us first consider the second coordinate of the pairs selected from the set $G_{4i-3,4j-3}$ which is $jN + z_{4i-3,4j-3}$, and $G_{4i-3,4j-2}$ which is $jN + z_{4i-3,4j-2}$. By the $\leq$ constraint of Matrix Tiling with $\leq$, it follows that:

$$z_{4i-3,4j-3} \leq z_{4i-3,4j-2}.$$

Continuing this reasoning for the other sets, we obtain the following chain of inequalities:

$$\begin{array}{rcrcrcrr}
z_{4i-3,4j-3} & \leq & z_{4i-3,4j-2} & \leq & z_{4i-3,4j-1} & \leq & z_{4i-3,4j} & \text{(first row)} \\
z_{4i-3,4j} & \leq & z_{4i-2,4j} & \leq & z_{4i-1,4j} & \leq & z_{4i,4j} & \text{(last column)} \\
-z_{4i,4j-3} & \leq & -z_{4i,4j-2} & \leq & -z_{4i,4j-1} & \leq & -z_{4i,4j} & \text{(last row)} \\
-z_{4i-3,4j-3} & \leq & -z_{4i-2,4j-3} & \leq & -z_{4i-1,4j-3} & \leq & -z_{4i,4j-3} & \text{(first column)}
\end{array}$$

Combining all these inequalities results in a cycle of equalities, which implies that all the 12 values are the same.

Let $z^{i,j}$ be this common value and let $s_{i,j} = (a_{i,j}, b_{i,j})$ be the corresponding pair, that is, $\iota(a_{i,j}, b_{i,j}) = z^{i,j}$. The fact that $z^{i,j}$ was defined using the pairs appearing in the gadget of $\mathcal{S}'_{i,j}$ implies that $s_{i,j} \in \mathcal{S}'_{i,j}$. We call this step of retrieving the pairs as *decoding the gadget.*

□

Let y be a feasible solution to the Matrix Tiling with $\leq$ instance $\mathcal{M}$, which is the output of our reduction $R(x)$ applied to an instance x of Matrix Tiling. Define the function S as follows:

Definition 1 (Solution Mapping $S(y)$)**.** *For each gadget $G_{i',j'}$ in $\mathcal{M}$, define:*

- *If **all 16 cells** of the gadget are assigned non-$\star$ values, decode the gadget to retrieve the unique encoded pair $(a, b) \in \mathcal{S}'_{i,j}$ using the gadget decoding step from Lemma 3, and assign:*

$$s_{i,j} := (a - k, b - k) \in S_{i,j}. \tag{3}$$

- *If **any cell** in the gadget is assigned $\star$, set:*

$$s_{i,j} := \star$$

Then $S(y) = \{s_{i,j} | 1 \leq i, j \leq k\}$ is a candidate solution to the instance $\mathcal{I}$, of Matrix Tiling.

Lemma 4 (Feasibility of $S(y)$ for $\mathcal{I}$)**.** *Let y be any feasible solution to $R(x)$. Then $S(y)$ is a feasible solution to x.*

Proof. Let $s_{i,j}$ be the pair extracted from gadget $G_{i',j'}$, and let $s_{i,j} = \star$ if any cell in the gadget is $\star$.

We now verify that the equality constraints of Matrix Tiling are satisfied for each pair of adjacent cells:

- Notice we do not have to check the constraints where either of the adjacent cells is a $\star$.
- Therefore, suppose both $s_{i,j} \neq \star$ and $s_{i+1,j} \neq \star$, and they were decoded to values $(a_{i,j}, b_{i,j}) \in \mathcal{S}'_{i,j}$ and $(a_{i+1,j}, b_{i+1,j}) \in \mathcal{S}'_{i+1,j}$ respectively.
- The first coordinates of the pairs selected from the cells $G_{4i,4j-2}$ and $G_{4i+1,4j-2}$ are $(i+1)N + a_{i,j}$ and $(i+1)N + a_{i+1,j}$, and by the $\leq$ constraint of Matrix Tiling with $\leq$, we obtain: $a_{i,j} \leq a_{i+1,j}$.
- Similarly, comparing the first coordinates of the pairs selected from the cells $G_{4i,4j-1}$ and $G_{4i+1,4j-1}$ yields $-a_{i,j} \leq -a_{i+1,j}$.
- Comparing the above two equations, we can conclude:

$$a_{i,j} = a_{i+1,j}.$$

- With the similar argument for the horizontal direction, we get $b_{i,j} = b_{i,j+1}$.

Finally, in first step of defining R, we shifted all the coordinates of all pairs by k (see Equation 1), that is why we have defined our solution mapping function S to subtract k from both the coordinates of the retrieved pair $(a, b) \in \mathcal{S}'_{i,j}$ (see Equation 3) to get the pair which belongs to the original $\mathcal{S}_{i,j}$ set in x. □

4 Relation between the optimal solutions of x and $R(x)$ (Deriving α):

Now without analyzing the function R, we trivially bound the ratio between the optimal values of an instance x of Matrix Tiling and its reduced instance $R(x)$ of Matrix Tiling$\leq$, thereby deriving the constant α in the L-reduction.

Lemma 5. *There exists a constant $\alpha = 64$ such that*

$$\mathrm{OPT}(R(x)) \leq \alpha \cdot \mathrm{OPT}(x),$$

where $\mathrm{OPT}(x)$ and $\mathrm{OPT}(R(x))$ denote the optimal values of the respective instances.

Proof. Since at most one element is assigned per cell, we have $\mathrm{OPT}(x) \leq k^2$. For a lower bound, assign an arbitrary element in cells (i, j) where both i and j are odd, and $\star$ elsewhere. This gives at least $k^2/4$ assignments, so $\mathrm{OPT}(x) \geq k^2/4$ (with similar arguments, same bounds hold for $R(x)$ as well).

In the reduced instance $R(x)$, the grid has size $k' = 4k$, so

$$\mathrm{OPT}(R(x)) \leq (k')^2 = 16k^2 = 64 \cdot (k^2/4) \leq 64 \cdot \mathrm{OPT}(x)$$

Thus, $\alpha = 64$ satisfies the required bound. □

5 Relation between the optimal solutions and any approximate solutions of $\mathcal{I}$ and $\mathcal{M}$ (Deriving β):

Let us first analyze the relation between the optimum solutions of both the instances:

Lemma 6. *If $OPT(x) = k^2 - a$, then $OPT(R(x)) = 16k^2 - a$.*

Proof. Assume that the optimal solution for instance x selects $k^2 - a$ cells, implying that exactly a cells are assigned the symbol $\star$. We construct a corresponding solution for instance $R(x)$ as follows.

For each cell $\mathcal{S}_{i,j}$ such that the selected entry in the optimal solution of x is a valid pair $(a_{i,j}, b_{i,j})$, we include in the solution of $R(x)$ all 16 encoding cells within the corresponding gadget $G_{i,j}$ that encode this pair. For each cell $\mathcal{S}_{i,j}$ where the optimal solution of x contains a $\star$, we apply the construction in Lemma 2 to select exactly 15 non-$\star$ cells from the corresponding gadget $G_{i,j}$.

This yields a total of

$$(k^2 - a) \cdot 16 + a \cdot 15 = 16k^2 - a$$

non-$\star$ cells in the constructed solution for $R(x)$, thus establishing that $\mathrm{OPT}(R(x)) \geq 16k^2 - a$.

To show optimality, suppose there exists a solution for $R(x)$ with more than $16k^2 - a$ non-$\star$ cells. Then there must exist some gadget $G_{i,j}$, corresponding to a $\star$-cell $\mathcal{S}_{i,j}$ in the optimal solution of x, in which all 16 encoding cells are selected. By Lemma 3, this implies that all selected cells correspond to a common pair (a, b), which must satisfy the row and column constraints of x (Lemma 4). This contradicts the assumption that $\mathcal{S}_{i,j}$ is a $\star$-cell in the optimal solution of x. Hence, no such solution exists, and the constructed solution is indeed optimal.

□

Now, based on our definition of the function S, let analyze the relation of the solution to x, based on the definition of our solution mapping function S, given any feasible solution to $R(x)$:

Lemma 7. *If $c_A(y) = 16k^2 - m$ and $c_B(S(y)) = k^2 - n$, then $m \geq n$.*

Proof. By the definition of the mapping function S, an entry $s_{i,j} \neq \star$ only if all 16 vertices in the corresponding gadget $G_{i,j}$ are non-$\star$ in y. Thus, each $\star$ in y can invalidate at most one such gadget

Since $c_A(y) = 16k^2 - m$, the assignment y contains exactly m cell which are $\star$, implying that the number of $\star$ entries in $S(y)$ is at most m. Since $S(y)$ contains exactly n $\star$'s. Therefore, we must have

$$k^2 - n \geq k^2 - m,$$

which implies $m \geq n$, as required.

□

Now, based on Lemma 6 and Lemma 7, it is easy to see that for our L-reduction, the value of β can be 1, which is proved formally below:

Lemma 8. *There exists a constant $\beta = 1$ such that*

$$|OPT(x) - c_A(S(y))| \leq \beta |OPT(R(x)) - c_B(y)|$$

where $\mathrm{OPT}(x)$ *and* $\mathrm{OPT}(R(x))$ *denote the costs of optimal solutions to the respective instances, and* $c_A(S(y))$, $c_B(y)$ *denote the costs of the mapped and original (possibly non-optimal) solutions respectively.*

Proof. Let $c_A(S(y)) = k^2 - n, c_B(y) = 16k^2 - m$ and $\mathrm{OPT}(x) = k^2 - a$, from Lemma 6 $\mathrm{OPT}(R(x)) = 16k^2 - a$, for some $a, n, m \in [0, k^2]$.Substituting into the inequality, we obtain:

$$\begin{aligned} &\mathrm{OPT}(x) - c_A(S(y)) \leq 1 \cdot (\mathrm{OPT}(R(x)) - c_B(y)) \\ \implies &k^2 - a - k^2 + n \leq 16k^2 - a - 16k^2 + m \\ \implies &n - a \leq m - a \end{aligned}$$

By Lemma 7, we have $n \leq m$, and since a is fixed across both sides, the inequality holds. Hence, the claim holds with $\beta = 1$.

□

Note: Because both the problems Matrix Tiling and Matrix Tiling with $\leq$ are maximization optimization problems, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \geq c_B(y)$. So we can ignore the modulus used in the fourth condition in the L-reduction definition.

6 Proof of Theorem 1:

We are now ready to prove our main Theorem 1, which is restated below:

Theorem. *If there are constants $\delta, d > 0$ such that* Matrix Tiling with $\leq$ *has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

Proof. It is easy to verify that the functions R and S in our reduction are computable in polynomial time with respect to the size of the Matrix Tiling instance. From section 4 and section 5, we have established that $\alpha = 64$ and $\beta = 1$. Thus, the reduction from Matrix Tiling to Matrix Tiling with $\leq$ is an L-reduction.

Now by [2, Lemma 2.8(1)], if there exists an L-reduction from Matrix Tiling to a problem X (in our case, Matrix Tiling with $\leq$), then X cannot admit a PTAS with running time of the form $2^{O((1/\epsilon)^d)} \cdot n^{O((1/\epsilon)^{1-\delta})}$ for any constants $d, \delta > 0$, unless the ETH fails.

Applying this lemma to our reduction completes the proof.

□

Note: In the above theorem n is not the range for the coordinates, but the input size of the problem instance. Now we have the following lemma:

Lemma 9. *If there is an L-reduction from* Matrix Tiling with $\leq$ *to Problem X, then there are no $d, \delta > 0$ such that Problem X admits a PTAS with running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, unless ETH fails.*

Remark 2. *When defining the Matrix Tiling problem, we imagined the sets $S_{i,j}$ arranged in a matrix, with $S_{i,j}$ being in row i and column j. When reducing Matrix Tiling to a geometric problem, the natural idea is to represent $S_{i,j}$ with a gadget located around coordinate (i, j). However, this introduces an unnatural 90 degrees rotation compared to the layout of the $S_{i,j}$'s in the matrix, which can be confusing in the presentation of a reduction. Therefore, for geometric problems, it is convenient to imagine that $S_{i,j}$ is located at coordinate (i, j). To emphasize this interpretation, we use the notation $S[x, y]$ to refer to the sets; we imagine that $S[x, y]$ is at location (x, y), hence sets with the same x are on a vertical line and sets with the same y are on the same horizontal line (see Figure* **??***). The constraints of Matrix Tiling are also inverse from before: the first coordinate from pair selected from $S[x, y]$ is $\geq$ than the first coordinate from pair selected from $S[x + 1, y]$, Similar for the second coordinates of pairs selected from $S[x, y]$ and $S[x, y+1]$. Which can be achieved by replacing each number i in the pairs by $k+1-i$, (it is easy to see that* Matrix Tiling with $\geq$ *and* Matrix Tiling with $\leq$*) With this notation, we can give a very clean and transparent L-reduction to* Covering Points with Squares.

7 Introduction:

Let us define the problem we are interested in:

> **Covering Points with Squares:**
> ***Input:*** Set of points
> ***Find:*** Set of unit squares, which can cover all the input points.
> ***Goal:*** Minimize the number of squares.

We will derive the PTAS lower bound for Covering Points with Squares which is stated below:

> **Theorem 2.** *If there are constants $\delta, d > 0$ such that* Covering Points with Squares *has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

We will prove this theorem by providing L-reduction from Matrix Tiling. let us recall the problem definitions and L-reduction definition (Exact version, proved **W-[1]** hard by Marx [1]).

8 Constructing the instance of Covering Points with Squares (Definition of R):

We now reduce the intermediate instance $\mathcal{M}$ of Matrix Tiling with $\geq$ to an instance $\mathcal{C}$ of Covering Points with Squares.

We work in the plane using standard directions: E (east), N (north), NE (northeast), etc. Throughout the construction, we assume squares are closed on their west and south boundaries and open on their east and north boundaries. That is, a unit square whose SW corner is at (a, b) covers the region: $a \leq x < a+1$, $b \leq y < b+1$.

Set $\epsilon := 1/n^2$. Every point constructed in the reduction has coordinates that are integer multiples of ϵ. Hence, we may assume that the southwest (SW) corner of any square used in the solution lies at the integer multiples of ϵ.

In our construction, we will have three types of gadgets: *blocks*, *connectors*, and *testers*.

8.1 Description of different components used in the construction:

8.1.1 Control points:

To enforce that each square corresponds uniquely to a single block, we define five control points per block. These points are arranged so that they can only be simultaneously covered by a square associated with that block.

Let (x, y) denote the position of a block. We add the following 5 control points:

1. Central-*control point* : $(x + 0.5, y + 0.5)$,
2. W-*control point* : $(x + n\epsilon, y + 0.5)$,
3. E-*control point* : $(x + 1 - n\epsilon - \epsilon, y + 0.5)$,
4. S-*control point* : $(x + 0.5, y + n\epsilon)$,
5. N-*control point* : $(x + 0.5, y + 1 - n\epsilon - \epsilon)$.

We define the **horizontal offset** $h_{x,y} \in [-n, n]$ and **vertical offset** $v_{x,y} \in [-n, n]$ of a square corresponding to block (x, y) such that its SW corner lies at $(x + h_{x,y}\epsilon, y + v_{x,y}\epsilon)$.

Lemma 10. *In any feasible solution, each block must be assigned to a unique square that covers its five control points. In particular, if we have k' blocks, any solution must use at least k' squares, at least one square per block.*

Proof. By construction, central control points of different blocks lie at least at a distance of 1 apart in either the horizontal or vertical direction. Given that the unit squares ate half-open on the north and east, no square can cover central points of multiple blocks.

Moreover, the extreme coordinates reachable from a square at (x, y) with offset in $[-n, n]$ lie within $[x - n\epsilon, x + 1 + n\epsilon)$ and $[y - n\epsilon, y + 1 + n\epsilon)$, because of the half open nature of the squares. Thus, the square of block (x, y) cannot cover the W control point of the block $(x + 1, y)$ which lies at $(x + 1 + n\epsilon)$, similarly it cannot cover the E control point of the block $(x - 1, y)$ which lies at $(x - n\epsilon - \epsilon)$. We can similarly see it cannot cover the S control and N control points of the blocks $(x, y+1)$ and $(x, y-1)$ respectively. Hence, each control point set must be covered by a distinct square., implying at least k' squares are required.

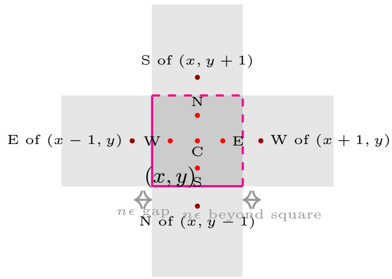

Figure 4: Control points from neighboring blocks lie outside the unit square of block (x, y) due to $n\epsilon$-offsets. No square can cover multiple blocks' points.

□

8.1.2 Boundary points:

Boundary points are introduced to enforce certain constraints on the horizontal and vertical offsets of the blocks. For each block at (x, y), we may include the following boundary points:

1. *N-boundary point:* $(x+0.5, y+1)$
2. *S-boundary point:* $(x+0.5, y)$
3. *W-boundary point:* $(x, y+0.5)$
4. *E-boundary point:* $(x+1, y+0.5)$

These points may only be added if the corresponding neighbor (north, south, west, east) is absent. Each boundary point enforces a constraint on the block's offset, which is explained below.

Lemma 11. *N-boundary point for the block (x, y) enforces that $v_{x,y} > 0$.*

Proof. The square from block (x, y) can only cover the points with vertical coordinate less than $y+1+v_{x,y}\epsilon$. To include N boundary point with vertical coordinate $y+1$, we require:

$$y+1+v_{x,y}\epsilon > y+1 \implies v_{x,y} > 0$$

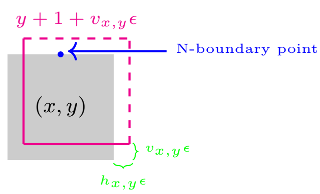

Figure 5: N-boundary point at $(x+0.5, y+1)$ is only covered if $v_{x,y} > 0$, since the unit square must extend above $y+1$.

□

Similarly, the E-boundary point ensures that the horizontal offset of the block is positive.

Lemma 12. *E-boundary point for the block (x, y) enforces that $h_{x,y} > 0$.*

Proof. Similar to Lemma 11, the square can only cover the points with horizontal coordinates less than $x+1+h_{x,y}\epsilon$. To cover the E-boundary point with horizontal coordinate $x+1$, we require:

$$x+1+h_{x,y}\epsilon > x+1 \implies h_{x,y} > 0$$

□

Lemma 13. *S-boundary point for the block (x, y) enforces that $v_{x,y} \leq 0$.*

Proof. Since the square is closed on its southern boundary, it includes all points with vertical coordinate at least $y + v_{x,y}\epsilon$. To cover the S-boundary point with vertical coordinate y, we require:

$$y + v_{x,y}\epsilon \leq y \implies v_{x,y} \leq 0$$

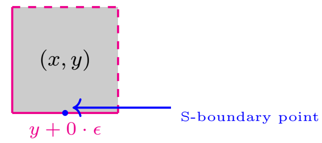

Figure 6: S-boundary point at $(x + 0.5, y)$ is only covered if $v_{x,y} \leq 0$, since the unit square must extend down to or below y.

□

Similarly, the W-boundary point ensures that the horizontal offset of the block is not positive.

Lemma 14. *W-boundary point for the block (x, y) enforces that $h_{x,y} \leq 0$.*

Proof. Similar to Lemma 13, the is closed on its western boundary, and cover all points with horizontal coordinate at least $x + h_{x,y}\epsilon$. To cover the W-boundary point with the horizontal coordinate x, we require:

$$x + h_{x,y}\epsilon \leq x \implies h_{x,y} \leq 0$$

□

8.1.3 Connector points:

We define a **connector points** as a set of points that lies on the shared boundary or corner between adjacent blocks. There are 4 types of connectors, each helps us in enforcing some relation between the offsets of both the blocks:

- **Horizontal connector:**
  - Between: Blocks (x, y) and $(x + 1, y)$
  - Points: $(x + 1 + i\epsilon, y + 0.5)$ for $i \in [-n, n - 1]$
  - Constraint enforced: $h_{x,y} \geq h_{x+1,y}$

Lemma 15. *All the points of the horizontal connector is covered if and only if $h_{x,y} \geq h_{x+1,y}$.*

Proof. The block at (x, y) has its eastern boundary at $x + 1 + h_{x,y}\epsilon$, therefore it can cover points whose horizontal coordinate is up to $x + 1 + h_{x,y}\epsilon$.

Similarly, the block $(x + 1, y)$ has its western edge at $(x + 1 + h_{x+1,y}\epsilon)$, therefore it can cover the points with the horizontal coordinate at least $(x + 1 + h_{x+1,y}\epsilon)$.

For all the points to be covered, the east side of the square at block (x, y) should be either on or left of the west side of the square at block $(x + 1, y)$, which means we require:

$$\begin{aligned} &x + 1 + h_{x+1,y}\epsilon \leq x + 1 + h_{x,y}\epsilon \\ \implies &h_{x,y} \geq h_{x+1,y} \end{aligned} \tag{4}$$

In other words, if $h_{x,y} < h_{x+1,y}$, there will be a gap between the squares where some connector points will not be covered by any of the two squares.

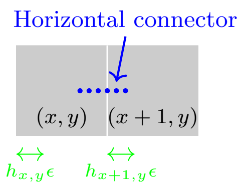

Figure 7: Horizontal connector between adjacent blocks is covered iff $h_{x,y} \geq h_{x+1,y}$.

□

- **Vertical connector:**
  - Between: Blocks (x, y) and $(x, y + 1)$
  - Points: $(x + 0.5, y + 1 + j\epsilon)$ for $j \in [-n, n - 1]$
  - Constraint enforced: $v_{x,y} \geq v_{x,y+1}$

Lemma 16. *All the points of the vertical connector are covered if and only if $v_{x,y} \geq v_{x,y+1}$.*

Proof. The block at (x, y) has its north edge at $y+1+v_{x,y}\epsilon$, therefore it can cover points whose vertical coordinate is up to $y + 1 + v_{x,y}\epsilon$.

Similarly, the block $(x, y+1)$ has its south edge at $(y+1+v_{x,y+1}\epsilon)$, therefore it can cover the points with the vertical coordinate starting from $(y + 1 + v_{x,y+1}\epsilon)$.

For all the points to be covered, the north side of the square at block (x, y) should be either on or above of the south side of the square at block $(x, y + 1)$, which means we require:

$$\begin{aligned} & y + 1 + v_{x,y+1}\epsilon \leq y + 1 + v_{x,y}\epsilon \\ \implies & v_{x,y} \geq v_{x,y+1} \end{aligned} \tag{5}$$

□

- **Right diagonal connector:**
  - Between: Blocks (x, y) and $(x + 1, y + 1)$
  - Points: $(x + 1 + i\epsilon, y + 1 + i\epsilon)$ for $i \in [-n, n - 1]$
  - Constraint enforced: $h_{x+1,y+1}, v_{x+1,y+1} \leq \min(h_{x,y}, v_{x,y})$

Lemma 17. *All the points of the right diagonal connector are covered if and only if:*

$$h_{x+1,y+1}, v_{x+1,y+1} \leq \min(h_{x,y}, v_{x,y})$$

.

Proof. The block at (x, y) can cover the right diagonal point $(x + 1 + h_{x,y}\epsilon, y + 1 + v_{x,y}\epsilon)$ (which is its NE corner) only if:

$$\begin{aligned} x + 1 + i\epsilon \leq x + h_{x,y}\epsilon + 1 \quad &\Rightarrow \quad i \leq h_{x,y}, \\ y + 1 + i\epsilon \leq y + v_{x,y}\epsilon + 1 \quad &\Rightarrow \quad i \leq v_{x,y}. \end{aligned}$$

Therefore, this square can cover the connector point only if $i \leq \min(h_{x,y}, v_{x,y})$.

The SW corner of square at block $(x + 1, y + 1)$ is at $(x + 1 + h_{x+1,y+1}\epsilon,\ y + 1 + v_{x+1,y+1}\epsilon)$, and can cover the connector point only if:

$$\begin{aligned} x + 1 + i\epsilon \geq x + 1 + h_{x+1,y+1}\epsilon \quad &\Rightarrow \quad i \geq h_{x+1,y+1}, \\ y + 1 + i\epsilon \geq y + 1 + v_{x+1,y+1}\epsilon \quad &\Rightarrow \quad i \geq v_{x+1,y+1}. \end{aligned}$$

So it can only cover the point if $i \geq \max(h_{x+1,y+1}, v_{x+1,y+1})$.

The right diagonal connector is only covered by (x, y) and $(x + 1, y + 1)$, therefore, we must have:

$$i \leq \min(h_{x,y}, v_{x,y}) \quad \text{or} \quad i \geq \max(h_{x+1,y+1}, v_{x+1,y+1}).$$

This is only possible if:

$$\max(h_{x+1,y+1}, v_{x+1,y+1}) \leq \min(h_{x,y}, v_{x,y}),$$

which implies:

$$h_{x+1,y+1}, v_{x+1,y+1} \leq \min(h_{x,y}, v_{x,y}).$$

□

- **Left diagonal connector:**
  - Between: Blocks (x, y) and $(x + 1, y - 1)$
  - Points: $(x + 1 + i\epsilon, y - i\epsilon)$ for $i \in [-n, n - 1]$
  - Constraint enforced: $h_{x+1,y-1} \leq \min(h_{x,y}, -v_{x,y})$, and $v_{x+1,y-1} \geq \max(-h_{x,y}, v_{x,y})$

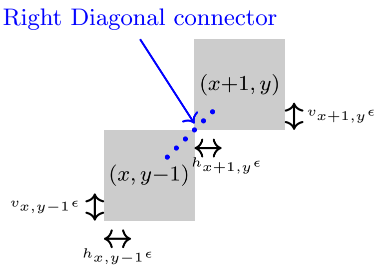

Figure 8: Right diagonal connector is covered iff $h_{x,y-1} + v_{x,y-1} \geq h_{x+1,y} + v_{x+1,y}$.

Lemma 18. *If the square at block $\theta = (x, y)$ has offsets i_1 and j_1, and the square at block $\Delta = (x+1, y-1)$ has offsets i_2 and j_2, then the left diagonal connector between these blocks enforces the following conditions:*

$$i_2 \leq min(i_1, -j_1),$$

$$j_2 \geq max(-i_1, j_1)$$

Proof. Let the SW corner of θ and Δ, be at $(x + i_1\epsilon, y + j_1\epsilon)$, and $(x + 1 + i_2\epsilon, y - 1 + j_2\epsilon)$ respectively, the left diagonal point is the collection of the points: $(x + 1 + i\epsilon, y - i\epsilon)$ for each $-n \leq i \leq n - 1$.

The SW corner of θ is at $(x+1+i_1\epsilon, y+j_1\epsilon)$, it can cover the left diagonal point $(x+1+i\epsilon, y-i\epsilon)$ only if:

$$x + 1 + i\epsilon \leq x + 1 + i_1\epsilon \implies i \leq i_1,$$

$$y - i\epsilon \geq y + j_1\epsilon \implies i \leq -j_1.$$

Therefore, the square at block θ, can cover the connector only if $i \leq \min(i_1, -j_1)$.

The NW corner of Δ is at $(x+1+i_2\epsilon, y+j_2\epsilon)$, it can cover the left diagonal point $(x+1+i\epsilon, y-i\epsilon)$ only if:

$$x + 1 + i\epsilon \geq x + 1 + i_2\epsilon \implies i \geq i_2,$$

$$y - i\epsilon \leq y + j_2\epsilon \implies i \geq -j_2.$$

Therefore, the square at block Δ, can cover the connector only if $i \geq \max(i_2, -j_2)$.

The tester right diagonal points are only covered by θ and Δ, therefore, we must have:

$$i \leq \min(i_1, -j_1) \quad \text{or} \quad i \geq \max(i_2, -j_2).$$

This is only possible if:

$$\min(i_1, -j_1) \geq \max(i_2, -j_2),$$

which implies

$$i_2 \leq \min(i_1, -j_1),$$

and

$$-j_2 \leq \min(i_1, -j_1) \implies j_2 \geq \max(-i_1, j_1)$$

$\square$

8.1.4 Safety points:

Safety points are introduced to enforce constraint that the square for the block cannot cover the diagonal connector placed between the two of its neighbors, for each block at (x, y), we may include the following boundary points:

1. *N-safety point:* $(x + 0.5, y + 1 + (n - 1)\epsilon)$
2. *E-safety point:* $(x + 1 + (n - 1)\epsilon, y + 0.5)$

Lemma 19. *The presence of an N-safety point at block (x, y) ensures that the square selected at this block cannot cover the right diagonal connector between blocks $(x, y - 1)$ and $(x + 1, y)$.*

Proof. The square from block (x, y) can only cover the points with vertical coordinate less than $(y + 1 + v_{x,y}\epsilon)$. To include N-safety points with vertical coordinate $y + 1 + (n-1)\epsilon$, we require:

$$y + 1 + (n-1)\epsilon < y + 1 + v_{x,y}\epsilon \implies v_{x,y} \geq n.$$

The right diagonal connector between blocks $(x, y-1)$ and $(x+1, y)$ consists of the points:

$$(x + 1 + i\epsilon,\ y + i\epsilon) \quad \text{for } -n \leq i \leq n-1.$$

The highest such point is:

$$(x + 1 + (n-1)\epsilon,\ y + (n-1)\epsilon).$$

Observe that:

$$y + (n-1)\epsilon < y + n\epsilon,$$

so the entire diagonal lies strictly below the bottom edge (S-edge) of the square at block (x, y). Thus, none of the diagonal connector points are covered by the square when the N-safety point is present.

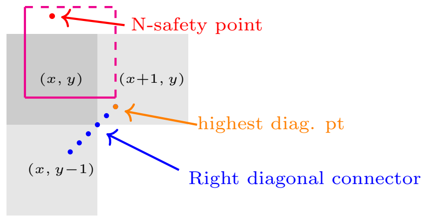

Figure 9: With the N-safety point placed at height $y + 1 + (n{-}1)\epsilon$, the square at (x, y) must have $v_{x,y} \geq n$ to cover it. This pushes the square too far up to cover any point of the right diagonal connector between $(x, y{-}1)$ and $(x{+}1, y)$.

□

Lemma 20. *If the horizontal offset of the square at block (x, y) is positive, the presence of an N-safety point at this block ensures that the square selected from it cannot cover the left diagonal connector between blocks $(x-1, y)$ and $(x, y-1)$.*

Proof. The square from block (x, y) can only cover the points with vertical coordinate less than $(y + 1 + v_{x,y}\epsilon)$. To include N-safety point at height $y + 1 + (n-1)\epsilon$, we require:

$$y + 1 + (n-1)\epsilon < y + 1 + v_{x,y}\epsilon \implies v_{x,y} \geq n.$$

The left diagonal connector between blocks $(x-1, y)$ and $(x, y-1)$ contains the points:

$$(x + i\epsilon,\ y - i\epsilon) \quad \text{for } -n \leq i \leq n-1.$$

with topmost point being: $(x - n\epsilon,\ y + n\epsilon)$.

Since the horizontal offset of the block (x, y) is positive, the SW corner of the square at (x, y) lies at or to the right of x, so it cannot cover the point $(x - n\epsilon,\ y + n\epsilon)$.

Moreover, all other points of the diagonal connector lie strictly below this topmost point, and hence strictly below the S-edge of the square=, which is at height $y + n\epsilon$. That is:

$$y + (n-1)\epsilon < y + n\epsilon.$$

Hence, no point of the diagonal connector is covered by the square selected for block (x, y) when the N-safety point is present.

□

Lemma 21. *If there are three blocks at $(x, y), (x, y+1)$, and $(x-1, y)$ and right diagonal connector connecting $(x-1, y)$ and $(x, y+1)$ with E-safety point for block (x, y), then the square selected at block (x, y) cannot cover the right diagonal points.*

Proof. To cover the E-safety point at $(x + 1 + (n-1)\epsilon,\ y + 0.5)$, the square selected for the block at (x, y) must have a horizontal offset of at least n, since the unit square is open at the East and thus the E-safety point must lie on the left of the E-edge. In particular, a horizontal offset less than n would result in the E-edge of the square lying strictly left of $x + 1 + n\epsilon$, so the E-safety point would not be included. Therefore, the square must have horizontal offset exactly n, which places its W-edge at $x + n\epsilon$.

Now, the diagonal connector consists of the points

$$(x + i\epsilon,\ y + 1 + i\epsilon) \quad \text{for} \ -n \le i \le n - 1.$$

In particular, the rightmost such point is at

$$(x + (n-1)\epsilon,\ y + 1 + (n-1)\epsilon).$$

Since the West edge of the square at block (x, y) lies at $x + n\epsilon$, and since

$$x + (n-1)\epsilon < x + n\epsilon,$$

the entire diagonal connector lies strictly to the left of the square.

Therefore, no point of the diagonal connector lies inside the square selected for block (x, y) when the E-safety point is present, as the square must be shifted rightward enough to include the E-safety point and thus cannot reach leftward to cover the diagonal.

□

Lemma 22. *If there are three blocks at $(x, y), (x, y-1)$, and $(x-1, y)$ and left diagonal connector connecting $(x-1, y)$ and $(x, y-1)$ with E-safety point for block (x, y), then the square selected at block (x, y) cannot cover the left diagonal points.*

Proof. Can Be Done Similarly... To cover the E-safety point at $(x + 1 + (n-1)\epsilon,\ y + 0.5)$, the square selected for the block at (x, y) must have a horizontal offset of at least n, since the unit square is open at the East and thus the E-safety point must lie on the left of the E-edge. In particular, a horizontal offset less than n would result in the E-edge of the square lying strictly left of $x+1+n\epsilon$, so the E-safety point would not be included. Therefore, the square must have horizontal offset exactly n, which places its W-edge at $x + n\epsilon$.

Now, the diagonal connector consists of the points

$$(x + i\epsilon,\ y - i\epsilon) \quad \text{for} \ -n \le i \le n - 1.$$

In particular, the rightmost such point is at

$$(x + (n-1)\epsilon,\ y - (n-1)\epsilon).$$

Since the West edge of the square at block (x, y) lies at $x + n\epsilon$, and since

$$x + (n-1)\epsilon < x + n\epsilon,$$

the entire diagonal connector lies strictly to the left of the square.

Therefore, no point of the diagonal connector lies inside the square selected for block (x, y) when the E-safety point is present, as the square must be shifted rightward enough to include the E-safety point and thus cannot reach leftward to cover the diagonal.

□

8.1.5 Tester points:

1. *tester horizontal points:* Let (x, y) be the coordinates of the SE-corner of a block, then we place these points at $(x + \ell\epsilon, y + \epsilon)$ $(1 \le \ell \le n)$.

2. *tester vertical points:* Let (x, y) be the coordinates of the NW-corner of a block, then we place these points at $(x + \epsilon, y + \ell\epsilon)$ $(1 \le \ell \le n)$.

3. *tester-right diagonal connector:* Are placed between blocks z and u, which is defined as follows: Let (x, y) be the coordinates of the NE-corner of the block z. The connector consists of the points $(x + (\ell+1)\epsilon, y + \ell\epsilon)$ $(-n \le \ell \le n)$.

Lemma 23. *For all the tester horizontal points placed for the block $T = (x-1, y)$ to be covered, the horizontal offset of block T must be at least horizontal offset of the block $Z = (x, y-1)$, if there is a S-boundary point for the S-neighbor of block T which means this block is at $(x-1, y-1)$ we call this block H_{10}.*

Proof. There is a S-boundary point at the block H_{10}, which means the vertical offset $(H_{10}) \le 0$, moreover, the top boundary of square at block H_{10} is at the height at most (y), which implies the square at block H_{10} cannot cover these tester horizontal point, moreover only T and Z can cover them.

The connector points lies between the east side of block T and the west side of the block at Z, at height of $y + \epsilon$, and that the horizontal coordinates from $(x + \epsilon)$ to $(x + n\epsilon)$.

If the horizontal offset of the of block T is i_1, which means the square will have its east edge at $x + i_1\epsilon$, therefore it can cover points whose horizontal coordinate is less than $x + i_1\epsilon$ as east side of squares are open.

Suppose the vertical offset of Z is at least 2 and horizontal offset is i_2. Then its S-edge is at height $y - 1 + 2\epsilon$, so its N edge is at $(y + 2\epsilon)$. Hence, the point $(x + \ell\epsilon,\ y + \epsilon)$ lies within the vertical range of this square. The block Z has its west edge at $(x + i_2\epsilon)$, therefore it can cover the points with the horizontal coordinate starting from $(x + i_2\epsilon)$.

Now to cover all the points, the east side of the square at block T should be either on or right of the west side of the square at block Z, and we must have the following inequality:

$$\begin{aligned} & x + i_2\epsilon \leq x + i_1\epsilon \\ \Longrightarrow\ & i_2 \leq i_1 \end{aligned} \tag{6}$$

In other words, if $i_2 > i_1$, there will be a gap (in the horizontal direction) between the squares where some connector points will not be covered by any of the two squares.

Hence, all the added points are covered if and only if horizontal offset of block T is at least horizontal offset of block Z. □

Lemma 24. *For all the tester vertical points placed for block $S = (x, y - 1)$ to be covered, the vertical offset of block S must be at least vertical offset of the block $W = (x - 1, y)$, if there is a W boundary point for the block W-neighbor of the block S which means this block is at $(x - 1, y - 1)$ we call this V_{10} .*

Proof. Can be done similarly...

There is a W-boundary point at the block V_{10}, which means the horizontal offset $(V_{10}) \leq 0$, moreover, the right boundary of square at block V_{10} is at the at most (x), which implies the square at block V_{10} cannot cover these tester vertical point, moreover only S and W can cover them.

The connector points lies between the North side of block S and the south side of the block at W, with horizontal coordinate of $x + \epsilon$, and that the vertical coordinates from $(y + \epsilon)$ to $(y + n\epsilon)$.

If the vertical offset of the of block S is j_1, which means the square will have its north edge at $y + j_1\epsilon$, therefore it can cover points whose vertical coordinate is less than $y + j_1\epsilon$ as north side of squares are open.

Suppose the horizontal offset of W is at least 2 and vertical offset is j_2. Then its W-edge has horizontal coordinate at $x - 1 + 2\epsilon$, so its E edge is at $(x + 2\epsilon)$. Hence, the point $(x + \epsilon,\ y + \ell\epsilon)$ lies within the horizontal range of this square. The block W has its south edge at $(y + j_2\epsilon)$, therefore it can cover the points with the vertical coordinate starting from $(y + j_2\epsilon)$.

Now to cover all the points, the north side of the square at block S should be either on or above of the south side of the square at block W, and we must have the following inequality:

$$\begin{aligned} & y + j_2\epsilon \leq y + j_1\epsilon \\ \Longrightarrow\ & j_2 \leq j_1 \end{aligned} \tag{7}$$

In other words, if $j_2 > j_1$, there will be a gap (in the vertical direction) between the squares where some connector points will not be covered by any of the two squares.

Hence, all the added points are covered if and only if vertical offset of block S is at least vertical offset of block W.

□

Lemma 25. *For all the tester right diagonal points between $Z = (x - 1, y - 1)$ and $U = (x, y)$ to be covered, the vertical offset of $U \leq$ horizontal offset of $Z - 1$, if there is a S-boundary point for the S neighbor of the block U, which means this block is at $(x, y - 1)$ we call it H_{12}.*

Proof. observation: adding S-boundary point to the S neighbor of u implies that the S neighbor of u cannot cover this points.

There is a S-safety point at the block H_{12}, which means the $\text{vo}(H_{12}) \leq 0$, moreover, only U and Z can cover the tester diagonal points.

Let the SW corner of Z and U, be at $(x - 1 + i_1\epsilon, y - 1 + j_1\epsilon)$, and $(x + i_2\epsilon, y + j_2\epsilon)$ respectively, the tester right diagonal point is the collection of the points: $(x + (\ell + 1)\epsilon, y + \ell\epsilon)$ for each $-n \leq \ell \leq n$.

The NE corner of Z is at $(x + i_1\epsilon, y + j_1\epsilon)$, it can cover the tester right diagonal point $(x + (i + 1)\epsilon, y + i\epsilon)$ only if:

$$x + (i + 1)\epsilon \leq x + i_1\epsilon \implies i \leq i_1 - 1,$$

$$y + i\epsilon \le y + j_1\epsilon \implies i \le j_1.$$

Therefore, the square at block Z, can cover the connector only if $i \le \min(i_1 - 1, j_1)$.

The SW corner of U is at $(x+i_2\epsilon, y+j_2\epsilon)$, it can cover the tester right diagonal point $(x+(i+1)\epsilon, y+i\epsilon)$ only if:

$$x + (i+1)\epsilon \ge x + i_2\epsilon \implies i \ge i_2 - 1,$$

$$y + i\epsilon \ge y + j_2\epsilon \implies i \ge j_2.$$

Therefore, the square at block U, can cover the connector only if $i \ge \max(i_2 - 1, j_2)$.

The tester right diagonal points are only covered by Z and U, therefore, we must have:

$$i \le \min(i_1 - 1, j_1) \quad \text{or} \quad i \ge \max(i_2 - 1, j_2).$$

This is only possible if:

$$\min(i_1 - 1, j_1) \ge \max(i_2 - 1, j_2),$$

which implies

$$j_2 \le \min(i_1 - 1, j_1)$$

Thus, the vertical offset of square at U is at most the horizontal offset $Z - 1$.

□

Lemma 26. *For all the tester right diagonal points between $W = (x-1, y-1)$ and $R = (x, y)$ to be covered, the horizontal offset of $R \le$ vertical offset of $W - 1$, if there is a W-boundary point for the W neighbor of the block R, which means this block is at $(x-1, y)$ we call this V_{12}.*

Proof. observation: adding W-boundary point to the W neighbor of R implies that W neighbor of R cannot cover this points.

Let the SW corner of W and R, be at $(x-1+i_1\epsilon, y-1+j_1\epsilon)$, and $(x+i_2\epsilon, y+j_2\epsilon)$ respectively, the tester right diagonal point is the collection of the points: $(x+(\ell+1)\epsilon, y+\ell\epsilon)$ for each $-n \le \ell \le n$.

The NE corner of R is at $(x+i_1\epsilon, y+j_1\epsilon)$, it can cover the tester right diagonal point $(x+(i+1)\epsilon, y+i\epsilon)$ only if:

$$x + (i+1)\epsilon \le x + i_1\epsilon \implies i \le i_1 - 1,$$

$$y + i\epsilon \le y + j_1\epsilon \implies i \le j_1.$$

Therefore, the square at block W, can cover the connector only if $i \le \min(i_1 - 1, j_1)$.

The SW corner of R is at $(x+i_2\epsilon, y+j_2\epsilon)$, it can cover the tester right diagonal point $(x+(i+1)\epsilon, y+i\epsilon)$ only if:

$$x + (i+1)\epsilon \ge x + i_2\epsilon \implies i \ge i_2 - 1,$$

$$y + i\epsilon \ge y + j_2\epsilon \implies i \ge j_2.$$

Therefore, the square at block R, can cover the connector only if $i \ge \max(i_2 - 1, j_2)$.

The tester right diagonal points are only covered by W and R, therefore, we must have:

$$i \le \min(i_1 - 1, j_1) \quad \text{or} \quad i \ge \max(i_2 - 1, j_2).$$

This is only possible if:

$$\min(i_1 - 1, j_1) \ge \max(i_2 - 1, j_2),$$

which implies

$$j_2 \le \min(i_1 - 1, j_1)$$

Thus, the vertical offset of square at R is at most the horizontal offset $W - 1$.

□

8.1.6 Gadgets:

1. Horizontal wrap around gadget: We define the *horizontal wrap gadget* with bottom-left reference coordinate (x, y) as follows:

- Horizontal row of 7 blocks from $(x+1, y-3)$ to $(x+7, y-3)$, with horizontal connectors placed between neighbors.
- Vertical column of 2 blocks at $(x, y-1)$ and $(x, y-2)$, with a vertical connector placed between them, a right diagonal connector between the blocks $(x, y-1)$ and $(x+1, y)$, and a left diagonal connector between the blocks $(x, y-2)$ to $(x+1, y-3)$.

- Vertical column of 2 blocks at $(x+8, y-1)$ and $(x+8, y-2)$, with a vertical connector placed between them, a right diagonal connector between the blocks $(x+7, y-3)$ to $(x+8, y-2)$, and a left diagonal connector between $(x+7, y)$ to $(x+8, y-1)$.
- Place N-safety points for the block (x, y) and $(x+8, y)$.

> ***Note:*** We have written $(x+1, y)$ and $(x+7, y)$ in red color because, currently there are no blocks at that coordinates, but as we will "attach" this gadget in the construction, we will have these blocks then. Similar for the blocks (x, y) and $(x+8, y)$ where N-safety points are places.

This construction forms a gadget used to "wrap" horizontal connections while enforcing specific offset relationships between surrounding components in our reduction.

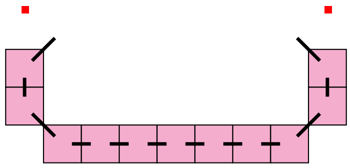

Figure 10: Horizontal wrap around gadget places at (x, y)

Lemma 27. *If there is a row of 9 blocks starting from (x, y) to $(x+8, y)$ (we call them $H_6, \ldots, H_{10}, Z, H_{12}, H_{13}, H_{14}$), and we place the horizontal wrap around gadget at (x, y), then the blocks $H_7, \ldots, Z, \ldots, H_{13}$ has the same horizontal offset.*

Proof. From Lemma 19, square at block H_6 cannot cover the right diagonal between H_7 and R_1, similarly from Lemma 20 square at block H_{14} cannot cover the left diagonal between H_{13} and R_{11}.

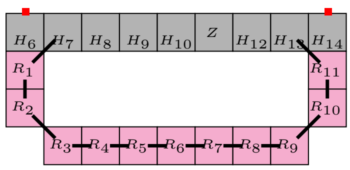

Figure 11: Attached Horizontal Wrap Around Gadget

Assume that the horizontal offset of H_7 is i, i.e., $\text{ho}(H_7) = i$. Since there are horizontal connectors between each of the consecutive blocks $H_7, H_8, \ldots, Z, \ldots, H_{12}$, we know that horizontal offsets can only decrease (or stay the same) across these connections. Thus, we have:

$$\text{ho}(H_7) \geq \text{ho}(H_8) \geq \cdots \geq \text{ho}(H_{12}),$$

which implies in particular that

$$\text{ho}(H_{12}) \leq i.$$

Next, we trace a chain of gadgets that "wraps around" from H_{12} back to H_7, enforcing constraints along the way:

- A left diagonal connector between H_{12} and R_{11} forces:

$$\text{vo}(R_{11}) \geq -\text{ho}(H_{13}) \geq -i.$$

- A vertical connector between R_{11} and R_{10} gives:

$$\text{vo}(R_{10}) \geq \text{vo}(R_{11}) \geq -i.$$

- A right diagonal connector between R_9 and R_{10} implies:

$$\text{ho}(R_9) \geq \text{vo}(R_{10}) \geq -i.$$

- Horizontal connectors from R_3 to R_9 give:

$$\text{ho}(R_3) \geq \text{ho}(R_4) \geq \cdots \geq \text{ho}(R_9) \geq -i.$$

- A left diagonal connector from R_2 to R_3 gives:

$$\text{ho}(R_3) \leq -\text{vo}(R_2) \implies -\text{ho}(R_3) \geq \text{vo}(R_2)$$
$$\implies \text{vo}(R_2) \leq i$$

- A vertical connector from R_1 to R_2 gives:

$$\text{vo}(R_1) \leq \text{vo}(R_2) \leq i.$$

- A right diagonal connector from R_1 to H_7 gives:

$$\text{ho}(H_7) \leq \text{vo}(R_1) \leq i.$$

Now, we compare what we started and ended with. We began with:

$$\text{ho}(H_7) = i,$$

and from the constraints above, we deduced:

$$\text{ho}(H_7) \leq i.$$

So together:

$$i = \text{ho}(H_7) \leq i,$$

which implies that equality must hold at every step. Otherwise, we would conclude $\text{ho}(H_7) < i$, contradicting our assumption.

Therefore, each inequality in the chain must be an equality. It follows that:

$$\text{ho}(H_7) = \text{ho}(H_8) = \cdots = \text{ho}(Z) = \cdots = \text{ho}(H_{12}) = i,$$

as required.

□

2. Vertical Wrap around Gadget: We define the *vertical wrap gadget* with bottom-right reference coordinate (x, y) as follows:

- Vertical column of 7 blocks from $(x-3, y+1)$ to $(x-3, y+7)$, with vertical connectors placed between neighbors.
- Horizontal row of 2 blocks at $(x-1, y)$ and $(x-2, y)$, with a horizontal connector placed between them, a right diagonal connector between $(x-1, y)$ to $(x, y+1)$ and a left diagonal between $(x-2, y)$ to $(x-3, y+1)$.
- Horizontal row of 2 blocks at $(x-1, y+8)$ and $(x-2, y+8)$, with a horizontal connector placed between them, a right diagonal between $(x-3, y+7)$ to $(x-2, y+8)$, and a left diagonal connector between $(x-1, y+8)$ to $(x, y+7)$.
- Place E-safety points for the blocks (x, y) and $(x, y+8)$.

Note: As was the case for the horizontal wrap around gadget, we have written $(x, y+1)$ and $(x, y+7)$ in red color because, currently there are no blocks at these coordinates, but as we will "attach" this gadget in the construction, we will have these blocks then. Similar for the blocks (x, y) and $(x, y+8)$ where E-safety points are places.

This construction forms a gadget used to "wrap" vertical connections while enforcing specific vertical offset constraints in our reduction framework.

Lemma 28. *If there is a column of 9 blocks starting from (x, y) to $(x, y+8)$ (we call them $V_6, \ldots, V_{10}, W, V_{12}, V_{13}, V_{14}$), and we place the vertical wrap around gadget at (x, y), then the blocks $V_7, \ldots, W, \ldots, V_{13}$ has the same vertical offset.*

Proof. Can be proved in the same way as above... From Lemma 21, square at block V_6 cannot cover the right diagonal between V_7 and L_1, similarly from Lemma 22 square at block V_{14} cannot cover the left diagonal between V_{13} and L_{11}.

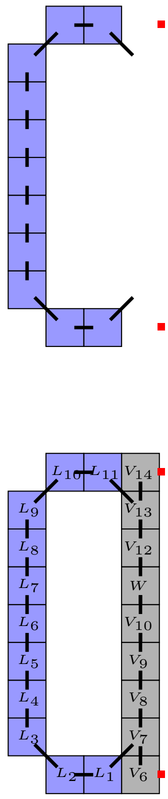

Figure 12: Attached Vertical warp around gadget

Assume that the horizontal offset of V_7 is j, i.e., $\mathrm{ho}(V_7) = j$. Since there are vertical connectors between each of the consecutive blocks $V_7, V_8, \ldots, W, \ldots, V_{13}$, we know that vertical offsets can only decrease (or stay the same) across these connections. Thus, we have:

$$\mathrm{vo}(V_7) \geq \mathrm{vo}(V_8) \geq \cdots \geq \mathrm{vo}(V_{13}),$$

which implies in particular that

$$\mathrm{vo}(V_{13}) \leq j.$$

Next, we trace a chain of gadgets that "wraps around" from H_{12} back to H_7, enforcing constraints along the way:

- A left diagonal connector between V_{12} and L_{11} forces:

$$\mathrm{ho}(L_{11}) \geq -\mathrm{vo}(V_{13}) \geq -j.$$

- A horizontal connector between L_{11} and L_{10} gives:

$$\mathrm{ho}(L_{10}) \geq \mathrm{ho}(L_{11}) \geq -j.$$

- A right diagonal connector between L_9 and L_{10} implies:

$$\mathrm{vo}(L_9) \geq \mathrm{ho}(L_{10}) \geq -j.$$

- vertical connectors from L_3 to L_9 give:

$$\mathrm{vo}(L_3) \geq \mathrm{vo}(L_4) \geq \cdots \geq \mathrm{vo}(L_9) \geq -j.$$

- A left diagonal connector from L_2 to L_3 gives:

$$\mathrm{vo}(L_3) \leq -\mathrm{ho}(L_2) \implies -\mathrm{vo}(L_3) \geq \mathrm{ho}(L_2)$$

$$\implies \mathrm{ho}(L_2) \leq j$$

- A horizontal connector from L_1 to L_2 gives:

$$\mathrm{ho}(L_1) \leq \mathrm{ho}(L_2) \leq j.$$

- A right diagonal connector from L_1 to V_7 gives:

$$\mathrm{vo}(V_7) \leq \mathrm{ho}(L_1) \leq i.$$

Now, we compare what we started and ended with. We began with:

$$\mathrm{vo}(V_7) = j,$$

and from the constraints above, we deduced:

$$\mathrm{vo}(V_7) \leq j.$$

So together:

$$j = \mathrm{vo}(V_7) \leq j,$$

which implies that equality must hold at every step. Otherwise, we would conclude $\mathrm{vo}(V_7) < j$, contradicting our assumption.

Therefore, each inequality in the chain must be an equality. It follows that:

$$\mathrm{vo}(V_7) = \mathrm{vo}(V_8) = \cdots = \mathrm{vo}(W) = \cdots = \mathrm{vo}(V_{13}) = j,$$

as required.

□

3. Tester Gadget: We define the *tester gadget* with reference coordinate (x, y) as follows. This gadget consists of a tight configuration of green-colored blocks arranged in multiple segments to facilitate testing vertical and horizontal offset constraints between connected components in our construction.

Consists of 4 "Transport Gadgets":

1. I_1**-transport:**
   - Horizontal row of 6 blocks, from $(x+1, y+9)$ to $(x+6, y+9)$, with horizontal connectors placed between neighbors, and tester-right diagonal connector between $(x, y+8)$ and $(x+1, y+9)$.
   - Vertical column of 5 blocks from $(x+7, y+8)$ to $(x+7, y+4)$, with vertical connectors placed between neighbors.
   - Left diagonal connector between blocks $(x+6, y+9)$ and $(x+7, y+8)$.
   - Horizontal row of 2 blocks from $(x+6, y+3)$ and $(x+5, y+3)$, with horizontal connector between them.
   - Right diagonal connector between $(x+6, y+3)$ and $(x+7, y+4)$.

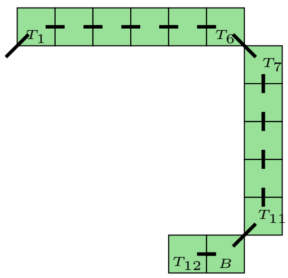

Figure 13: I_1-transport gadget placed at (x, y)

2. I**-transport:**
   - Vertical row of 3 blocks from $(x+1, y+7)$ to $(x+1, y+5)$, with vertical connector between the neighbors, and tester vertical connector between $(x+1, y+7)$ and $(x, y+8)$.
   - Horizontal row of 3 blocks from $(x+2, y+4)$ to $(x+4, y+4)$, with horizontal connector between the neighbors.
   - Left diagonal connector between $(x+1, y+5)$ and $(x+2, y+4)$.
3. J_1**-transport:**
   - Vertical column of 5 blocks from $(x+9, y+1)$ to $(x+9, y+5)$, with vertical connectors between neighbors, and tester-right diagonal between $(x+8, y)$ and $(x+9, y+1)$.

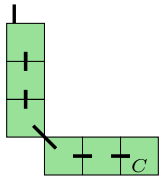

Figure 14: I-transport gadget placed at (x, y)

- Horizontal row of 3 blocks from $(x+8, y+6)$ to $(x+6, y+6)$, with horizontal connector between neighbors.
- Left diagonal connector between $(x+9, y+5)$ and $(x+8, y+6)$.
- Vertical column of 2 blocks from $(x+5, y+5)$ to $(x+5, y+4)$, with vertical connector between them.
- Right-diagonal connector between $(x+5, y+5)$ and $(x+6, y+6)$.

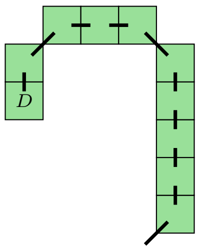

Figure 15: J_1-transport gadget placed at (x, y)

4. ***J*-transport:**

- horizontal column of 3 blocks from $(x+7, y+1)$ to $(x+5, y+1)$, with horizontal connector between the neighbors, and tester horizontal connector between $(x+7, y+1)$ and $(x+8, y)$.
- Vertical column of 2 blocks from $(x+4, y+2)$ to $(x+4, y+3)$, with vertical connector between them.
- Left-diagonal connector between $(x+4, y+2)$ and $(x+5, y+1)$.

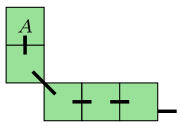

Figure 16: J-transport gadget placed at (x, y)

Putting all together we have the tester gadget (see Figure 17).

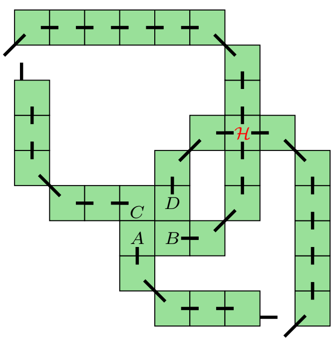

Figure 17: Tester gadget placed at (x, y).

Remark 3. *In the complete tester gadget one block from the the I_1-transport gadget and one block from J_1-transport gadget are overlapping with each other which is marked as "$\mathcal{H}$" in the Figure 17, but this is fine as in the I_1-transport gadget we are only interested about the vertical offset of the block, and in the case of J_1-transport gadget we are only interested in the horizontal offset of the block, so this overlapping is not causing any issues in the working of both the transport gadgets.*

Lemma 29. *If β is the vertical offset of the block $W = (x, y+8)$ and block $V_{12} = (x, y+9)$ has a W-safety point then the NW corner of block B has the horizontal coordinate at least $(x - \beta\epsilon + \epsilon)$*

Proof. From Lemma 26, we know horizontal offset of $R \le \beta - 1$. Now Similarly, since there is a horizontal connector between R and B_5 (see Figure 18), we get

$$\mathrm{ho}(B_5) \le \cdots \le \mathrm{ho}(B_1) \le \mathrm{ho}(R) \le \beta - 1.$$

Next, there is a left-diagonal connector between B_5 and B_6, this implies that

$$\mathrm{vo}(B_6) \ge -\mathrm{ho}(B_5) \ge -\beta + 1.$$

Next, there is a vertical connector between B_6 and B_{10}, which gives

$$\mathrm{vo}(B_{10}) \ge \mathrm{vo}(B_6) \ge -\beta + 1.$$

Finally, there is a right diagonal connector between B_{10} and B_{11}, which gives

$$\mathrm{ho}(B_{11}) \ge \mathrm{vo}(B_{10}) \ge -\beta + 1.$$

Finally, there is a horizontal connector between B and B_{11}, which gives

$$\mathrm{ho}(B) \ge \mathrm{vo}(B_{11}) \ge -\beta + 1.$$

Therefore, the horizontal offset of B is more than $-\beta + 1$, which implies that the x-coordinate of the NE-corner of B is at least $y + (-\beta + 1)\epsilon$. □

Lemma 30. *If β is the vertical offset of the block $W = (x, y+8)$ and block $V_{10} = (x, y+7)$ has a W-safety point then the SE corner of block C has the horizontal coordinate at most $(x - \beta\epsilon)$.*

Proof. From Lemma 24, we know vertical offset of $S \ge \beta$. Now There are vertical connectors between C_2 and S(see Figure 18), therefore the vertical offset of C_1 is at least that of S, hence

$$\mathrm{vo}(C_1) \ge \mathrm{vo}(S) \ge \beta.$$

Similarly, since there is a vertical connector between C_2 and C_1, we get

$$\mathrm{vo}(C_2) \ge \mathrm{vo}(C_1) \ge \beta.$$

Next, there is a left-diagonal connector between C_3 and C_2, this implies that

$$\mathrm{ho}(C_3) \le -\mathrm{vo}(C_2) \le -\beta.$$

Next, there is a horizontal connector between C_4 and C_3, which gives

$$\mathrm{ho}(C_4) \le \mathrm{ho}(C_3) \le -\beta.$$

Finally, there is a horizontal connector between C_4 and C_4, which gives

$$\mathrm{ho}(C) \le \mathrm{ho}(C_4) \le -\beta.$$

Therefore, the horizontal offset of C is at most $-\beta$, which implies that the x-coordinate of the SE-corner of C is at most $y - \beta\epsilon$.

□

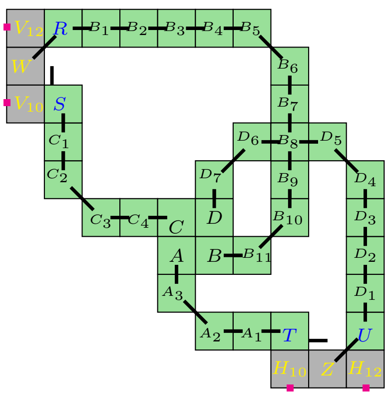

Figure 18: Tester gadget placed at (x, y) attached to some blocks

Lemma 31. *If α is the horizontal offset of the block $Z = (x + 8, y)$ and block $H_{12} = (x + 8, y)$ has a S-safety point then the SW corner of the block D has the vertical coordinate at least $(y - \alpha\epsilon + \epsilon)$.*

Proof. From Lemma 25, we know that the vertical offset of block $U \leq \alpha - 1$. Now, Since there is a vertical connector between $D_4, \ldots D_1$, and U(see Figure 18), we get

$$\mathrm{vo}(D_4) \leq \cdots \leq \mathrm{vo}(D_1) \leq \mathrm{vo}(U) \leq \alpha - 1.$$

Next, there is a left-diagonal connector between D_5 and D_4, this implies that

$$\mathrm{ho}(D_5) \geq -\,\mathrm{vo}(D_4) \geq -\alpha + 1.$$

Next, there is a horizontal connector between D_6, B_8, and D_5 which gives

$$\mathrm{ho}(D_6) \geq \mathrm{ho}(B_8) \geq \mathrm{ho}(D_5) \geq -\alpha + 1.$$

There is a right diagonal connector between D_6 and D_7, which gives

$$\mathrm{vo}(D_7) \geq \mathrm{ho}(D_6) \geq -\alpha + 1.$$

Finally, there is a vertical connector between D and D_7, which gives

$$\mathrm{vo}(D) \geq \mathrm{vo}(D_7) \geq -\alpha + 1.$$

Therefore, the vertical offset of D is at least $-\alpha + 1$, which implies that the x-coordinate of the SW-corner of D is at least $x + (-\alpha + 1)\epsilon$.

□

Lemma 32. *If α is the horizontal offset of the block $Z = (x + 8, y)$ and block $H_{10} = (x + 6, y)$ has a S-safety point then the NE corner of block A has the vertical coordinate at most $(y - \alpha\epsilon)$.*

Proof. From Lemma 23, we know that the horizontal offset of block $T \geq \alpha$. Now, There is a horizontal connector between A_1 and t (see Figure 18). By Lemma 23, the horizontal offset of A_1 is at least that of t, hence

$$\mathrm{ho}(A_1) \geq \mathrm{ho}(t) \geq \alpha.$$

Similarly, since there is a horizontal connector between A_2 and A_1, we get

$$\mathrm{ho}(A_2) \geq \mathrm{ho}(A_1) \geq \alpha.$$

Next, there is a left diagonal connector between A_3 and A_2, this implies that

$$-\,\mathrm{vo}(A_3) \geq \mathrm{ho}(A_2) \geq \alpha \implies \mathrm{vo}(A_3) \leq -\alpha.$$

Finally, there is a vertical connector between A and A_3, which gives

$$\mathrm{vo}(A) \leq \mathrm{vo}(A_3) \leq -\alpha.$$

Therefore, the vertical offset of A is at most $-\alpha$, which implies that the y-coordinate of the NE-corner of A is at most $y - \alpha\epsilon$.

□

4. Core Cell Gadget: For every cell (i, j) in Matrix Tiling with $\geq$, we define the *core cell gadget* with bottom-left reference coordinate (x, y) as a composite structure that integrates three key components—horizontal and vertical propagation paths, and a central tester gadget—to encode and propagate offset constraints. The construction proceeds as follows:

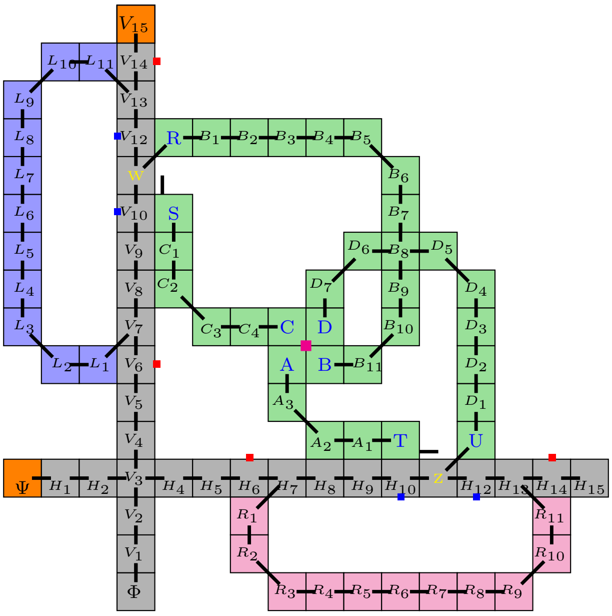

Figure 19: Core cell gadget placed at (x, y)

- **Horizontal Propagation Path.**
  - Place 16 consecutive gray-colored horizontal blocks from $(x, y+3) = \Psi^{i,j}$ through $(x+15, y+3) = H_{15}^{i,j}$.
  - Each pair of consecutive blocks is connected by a horizontal connector.
- **Vertical Propagation Path.**
  - Place 16 consecutive gray-colored vertical blocks from $(x+3, y) = \Phi^{i,j}$ through $(x+3, y+15) = V_{15}^{i,j}$.
  - Each pair of consecutive blocks is connected by a vertical connector.
- **Tester Gadget.**
  - Place the Tester Gadget at coordinate $(x+3, y+3)$, which attaches to the horizontal and vertical propagation paths.
- **Wrap Gadgets.**
  - Place a horizontal wrap gadget at $(x+6, y+3)$.
  - Place a vertical wrap gadget at $(x+3, y+6)$.
- **Tester core points:**
  - Let (x, y) be the common corner of blocks A, B, C, and D, for each $(p, q) \notin S_{i,j} \cap p, q \in [n]$, place the tester core points at the coordinate: $(x - q\epsilon + \epsilon, y - p\epsilon)$.

Note: Until now, we hadn't added anything to encode the Matrix Tiling with $\geq$ into our Covering Points with Squares instance, so now finally we will added, tester core points, which will only be covered by either A, B, C or D, if the $(p, q) \in S_{i,j}$. We will prove this property latter.

This construction enforces a local condition within the cell while also enabling global offset propagation via horizontal and vertical block sequences. The tester gadget acts as the core, and the wrap gadgets ensure same offsets for the blocks.

Notation: We use $\mathrm{vo}(K)$ to denote the vertical offset of block K, and $\mathrm{ho}(K)$ to denote the horizontal offset of block at K

Definition 2 (assignment). *We call the pair $(a_{i,j}, b_{i,j})$ the assignment of the (i,j)-th Core cell gadget, where $a_{i,j}$ is the vertical offset of H_7 (and we know by Lemma 27, $H_7, \ldots, H_{13}$ have same horizontal offset), and $b_{i,j}$ is the horizontal offset of V_7 (and we know by Lemma 28, $V_7, \ldots, V_{13}$ have same horizontal offset).*

Definition 3 (complete assignment). *We call the pair* $(a_{i,j}, b_{i,j})$ *the complete assignment of the* (i, j)*-th Core cell gadget, where* $a_{i,j}$ *is the horizontal offset of all the blocks in horizontal band* $(\Psi, H_1, \cdots, V_3, \cdots, Z, \cdots, H_{15})$*, and* $b_{i,j}$ *is the vertical offset of* $(\Phi, V_1, \cdots, W, \cdots, V_{15})$*, and the coordinates of squares in the tester gadget, vertical wrap around band, and horizontal wrap around gadget, are such that they satisfies the constraints of their corresponding points and connectors.*

Lemma 33. *If* (α, β) *is the pair of the Core cell gadget, and If* $(\alpha, \beta) \in S_{i,j}$*, then all the tester core points are covered.*

Proof. From the Lemma 32, Lemma 29, Lemma 30, and Lemma 31, we know the following:

1. The NE-corner of block a has vertical coordinate which is at most $(y - \alpha)$.
2. The NW-corner of block b has horizontal coordinate which is at least $(x - \beta\epsilon + \epsilon)$.
3. The SE-corner of block c has horizontal coordinate which is at most $(x - \beta\epsilon)$.
4. The SW-corner of block d has vertical coordinate which is at least $(y - \alpha\epsilon + \epsilon)$.
5. We have the *tester core points* with coordinates $(x - q\epsilon + \epsilon, y - p\epsilon)$ for each $(p, q) \notin S_{i,j}$.

We will first look at the α:

Case 1: $\alpha > p$

In this case the square selected from the block d can cover all the points, as points will have the vertical coordinate $(y - p\epsilon)$, and the SW-corner of d has the vertical coordinate as $(y - \alpha\epsilon + \epsilon)$, and because $\alpha > p$, we have the following inequality $(y - p\epsilon) \geq (y - \alpha\epsilon + \epsilon)$.

Case 2: $\alpha < p$

In this case the square selected from the block a can cover all the points, as points will have the vertical coordinate $(y - p\epsilon)$, and the NE-corner of a has the vertical coordinate as $(y - \alpha\epsilon)$, and because $\alpha < p$, we have the following inequality $(y - p\epsilon) < (y - \alpha\epsilon)$.

Case 3: $\alpha = p$

Notice, if $\alpha = p$, β must not be equal to q, as we assumed that $(\alpha, \beta) \in S_{i,j}$. We look at two cases of β here:

Case 3.i: $\beta > q$

In this case square selected from the block b can cover these points as they have horizontal coordinate which is equal to $(x - q\epsilon + \epsilon) > (x - \beta\epsilon + \epsilon)$.

Case 3.ii: $\beta < q$

In this case square selected from the block c can cover these points as they have horizontal coordinate which is equal to $(x - q\epsilon + \epsilon) \leq (x - \beta\epsilon)$.

Now we will look at the β:

Case 1: $\beta > q$

In this case the square selected from the block c can cover all the points, as points will have the horizontal coordinate $(x - q\epsilon + \epsilon)$, and the SE-corner of c has the horizontal coordinate as $(x - \beta\epsilon)$, and because $\beta > q$, we have the following inequality $(x - q\epsilon + \epsilon) \leq (x - \beta\epsilon)$.

Case 2: $\beta < q$

In this case the square selected from the block b can cover all the points, as points will have the horizontal coordinate $(x - q\epsilon + \epsilon)$, and the NW-corner of b has the horizontal coordinate as $(x - \beta\epsilon + \epsilon)$, and because $\beta > q$, we have the following inequality $(x - q\epsilon + \epsilon) > (x - \beta\epsilon + \epsilon)$.

Case 3: $\beta = q$

Notice, if $\beta = q$, α must not be equal to p, as we assumed that $(\alpha, \beta) \in S_{i,j}$. We look at two cases of α here:

Case 3.i: $\alpha > p$

In this case square selected from the block d can cover these points as they have vertical coordinate which is equal to $(y - p\epsilon) \geq (x - \alpha\epsilon + \epsilon)$.

Case 3.ii: $\alpha < p$

In this case square selected from the block a can cover these points as they have vertical coordinate which is equal to $(y - p\epsilon) < (x - \alpha\epsilon)$.

□

Lemma 34. *If* (α, β) *is the pair of the Core cell gadget, and If* $(\alpha, \beta) \in S_{i,j}$*, then all the points in the Core cell gadget can be covered by* 86 *squares.*

Proof. From Lemma 33, we know all the tester core points are covered by square at blocks $A \cup B \cup C \cup D$, which means as there are 86 blocks in the core cell gadget, we only need 86 squares (one per block), to cover all the points.

□

Lemma 35. *If (α, β) is the pair of the Core cell gadget, and If $(\alpha, \beta) \notin S_{i,j}$, then the tester core point at $(x - \beta\epsilon + \epsilon, y - \alpha\epsilon)$ will not be covered by any squares at block A, B, C, and D.*

Proof. From From Lemma 33, we know the coordinates of the corners which are closest to this point thus we can conclude that:

1. The square at block A can only cover points whose vertical coordinate is strictly less than $(y - \alpha\epsilon)$, so it can cover this point.

2. The square at block B can only cover points whose horizontal coordinate is strictly greater than $(x - \beta\epsilon + \epsilon)$, thus it cannot cover this point.

3. The square at block C can only cover the points whose horizontal coordinate is less than $(x - \beta\epsilon)$, thus it cannot cover this point.

4. The square at block D can cover the points whose vertical coordinate is at least $(y - \alpha\epsilon + \epsilon)$, thus it cannot cover this point.

□

Lemma 36. *If (α, β) is the pair of the Core cell gadget, and If $(\alpha, \beta) \notin S_{i,j}$, then to cover all the tester core points are covered we have to select 87 squares from the core cell gadget.*

Proof. From Lemma 35, we know that $(\alpha, \beta) \notin S_{i,j}$ implies the tester core point $(x - \beta\epsilon + \epsilon, y - \alpha\epsilon)$ will not be covered by any square from blocks A, B, C and D, therefore to cover this point we will have to pick an additional square whose SW coordinate is at $(x - \beta\epsilon, y - \alpha\epsilon - \epsilon)$ which will cover this point. □

8.2 Constructing full instance of Covering Points with Squares:

For every cell of $\mathcal{M}$, we construct Core cell gadget as explained in section 8, and for every neighbor of the cell in $\mathcal{M}$, we join the corresponding Core cell gadget as follows:

1. If we have a neighboring cell in the horizontal direction, that is let we have two cell at (i, j) and $(i + 1, j)$, then we place the $\Psi^{i+1,j}$ at $(x + 1, y)$ and we place the $H_{15}^{i,j}$ at (x, y) and place the horizontal connector between them Core cell gadget.

2. If we have a neighboring cell in the vertical direction, that is let we have two cell at (i, j) and $(i, j + 1)$, then we place the $\Phi^{i,j+1}$ at $(x, y + 1)$ and we place the $V_{15}^{i,j}$ at (x, y) and place the vertical connector between them Core cell gadget.

As $\mathcal{M}$ was constructed from $\mathcal{I}$, such that for every cell in $\mathcal{I}$, we have a 16 celled gadget in $\mathcal{M}$, we call the 16 Core cell gadgets attached together according to the 16 celled gadget of $\mathcal{M}$ as "Core cell cluster".

Finally, after constructing the whole instance of Covering Points with Squares by this method we place for each $i \in [k]$, a E-boundary point for the blocks $H_{15}^{i,k}$, and for each $j \in [k]$, a N-boundary point for the blocks $V_{15}^{k,j}$. Adding these boundary points will mean that the by Lemma 11 and Lemma 12 the horizontal offset and vertical offset of any cells in the horizontal propagation path and vertical propagation path respectively, of any core cell gadgets will always be positive.

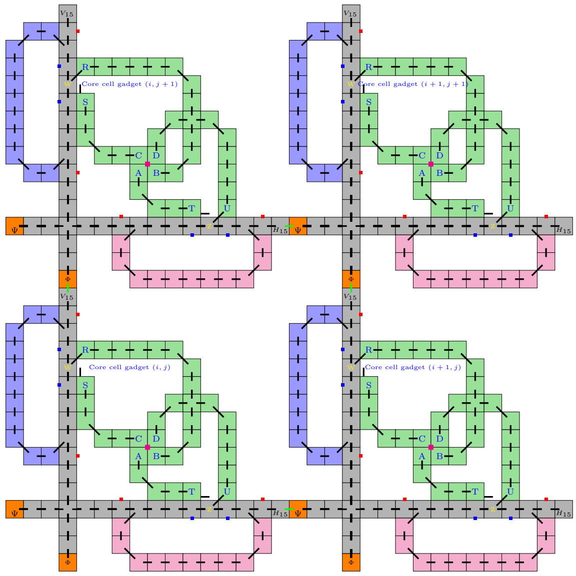

Figure 20: Explanation on how to place the Core cell gadget corresponding to the cells in $\mathcal{M}$

8.2.1 Correspondence between $\mathcal{M}$ and $\mathcal{C}$:

We now describe the correspondence between the constraints on solutions to Matrix Tiling with $\geq$ and the structure of the constructed instance $\mathcal{C}$ of Covering Points with Squares.

1. In Matrix Tiling with $\geq$, the first coordinate of each pair must be non-increasing in the horizontal direction (i.e., as we move right), and the second coordinate must be non-increasing in the vertical direction (i.e., as we move upwards).

2. In $\mathcal{C}$, each pair of neighboring core cell gadgets is connected via a horizontal or vertical connector. These enforce that the horizontal (respectively, vertical) offsets of their total pairs must also be non-increasing in the rightward (respectively, upward) direction.

Consider two horizontally adjacent cells with solution values $s_{[i,j]} = (a_{i,j}, b_{i,j})$ and $s_{[i+1,j]} = (a_{i+1,j}, b_{i+1,j})$. We use for the left core cell gadget to the pair $(a_{i,j}, b_{i,j})$ as the total pair of the core cell gadget and $(a_{i+1,j}, b_{i+1,j})$ for the right gadget. Since $a_{i+1,j} \leq a_{i,j}$, the horizontal connector between them is satisfied. Furthermore, as both pairs are valid elements of $S_{i,j}$, Lemma 34 implies that all tester core points are covered using $2 \times 86 = 172$ squares. An analogous argument applies in the vertical direction.

We now prove the following lemma, which gives us the relationship between the optimal solutions in Matrix Tiling with $\geq$ and Covering Points with Squares.

Lemma 37. *If $OPT(x) = k^2 - r$, then $OPT(R(x)) = 1376k^2 + r$.*

Proof. For each non-$\star$ in the solution to $\mathcal{I}$, assign the corresponding Complete Gadget Mapping as the complete assignment for all 16 core cell gadgets in the corresponding core cell cluster.

For each $\star$ in the solution to $\mathcal{I}$, use the associated Partial Gadget Mapping as the complete assignment for the 16 core cell gadgets. In this partial gadget mapping, we have one cell in the solution of $\mathcal{M}$ as a $\star$, therefore for the corresponding core cell gadget (corresponding to the $\star$ cell), use the pair $(p, q) = (iN - z^{+}_{i,j}, jN + b^{l+})$ as its total pair.

By Lemma 35, since $(p, q) \notin S'_{4i-2,4j-3}$, the tester point at position $(x - (q+1)\epsilon, y - p\epsilon)$ will not be covered by the 86 standard squares. Therefore, an additional square is required to cover it, resulting in a total of 87 squares for this cluster (see Lemma 36).

Hence, each non-$\star$ contributes exactly 1376 squares, and each $\star$ contributes 1377 squares. The total number of squares required is:

$$\mathrm{OPT}(R(x)) = 1376(k^2 - r) + 1377r = 1376k^2 + r.$$

□

9 Constructing solution of $\mathcal{I}$ given a solution of Covering Points with Squares (Definition of S):

For every Core cell cluster, which has 16 core cell gadgets, if the cluster has only 1376 squares, means that each core cell gadget has exactly 86 squares, for each "assignment" (a, b) of each core cell gadget in the cluster we select the pair (a, b) as the solution for the corresponding cell in $\mathcal{M}$, we argue that this forms a valid solution of $\mathcal{M}$ as due to the horizontal connectors between horizontal core cell gadgets and vertical connectors between vertical core cell gadgets, we know that the pair $(a_{i,j}, b_{i,j})$ selected from core cell gadget at (i, j), and the pair $(a_{i+1,j}, b_{i+1,j})$ selected from core cell gadget at $(i+1, j)$, by Lemma 15 have the following property $a_{i,j} \geq a_{i+1,j}$. For vertical direction we will have $b_{i,j} \geq b_{i,j+1}$ by similar argument. And because there were only 1376 squares from this core cell cluster, we can infer from Lemma 33 that $(a_{i,j}, b_{i,j}) \in S^{3n}_{i,j}$.

After selecting the pairs for the intermediate gadgets of $\mathcal{M}$ we use the method stated in **??** to pick the pair for the corresponding cell of $\mathcal{I}$, If the cluster has more than 1376 squares, then pick a $\star$ for the solution for $\mathcal{I}$.

Observation 3. *If $C_B(y) = 1376k^2 + m$ then $C_A(S(y)) \geq k^2 - m$. Because we know any solution needs at last 1376 squares, and if there m more squares, this could mean that all the extra squares are from m different Core cell clusters, which means all the m corresponding cell in the solution of $\mathcal{I}$ will be picked as a $\star$.*

10 Relation between the optimal solutions of $\mathcal{I}$ and Covering Points with Squares (Deriving α):

We can notice that the optimum is always at least $\frac{k^2}{4}$: if i and j are both odd, then let $s_{i,j}$ be an arbitrary element of $S_{i,j}$ (alternative row and columns); otherwise, let $s_{i,j} = \star$. And we have the upper bound on the optimum: k^2, which gives us the following inequalities:

$$k^2/4 \leq OPT(x) \leq k^2 \tag{8}$$

For each Core cell cluster (i, j), we cal always pick 1377 squares as mentioned in Lemma 37, while maintaining the global constraints which will make the OPT(R(x)) $\leq 1377k^2$.

Now we can analyze the Optimum solutions for x and $R(x)$:

$$\begin{aligned} &OPT(R(x)) \leq 1377k^2 = 5508 \cdot k^2/4 = 5508 \cdot OPT(x) \\ \implies &OPT(R(x)) \leq 5508 \cdot OPT(x) \end{aligned} \tag{9}$$

Thus for $\alpha = 5508$, we have $OPT(R(x)) \leq \alpha OPT(x)$.

11 Relation between the optimal solutions and any approximate solutions of $\mathcal{I}$ and Covering Points with Squares (Deriving β):

From Lemma 37, and 3, we have the following equations:

$$\begin{aligned} &OPT(x) - C_A(S(y)) \leq k^2 - r - k^2 + m \\ \implies &OPT(x) - C_A(S(y)) \leq m - r \end{aligned} \tag{10}$$

and

$$\begin{aligned} &C_B(y) - OPT(R(x)) = 1376k^2 + m - 1376k^2 - r \\ \implies &C_B(y) - OPT(R(x)) = m - r \end{aligned} \tag{11}$$

Now from Equation 10, and Equation 11, we get the following relation:

$$\begin{aligned} &OPT(x) - C_A(S(y)) \leq C_B(y) - OPT(R(x)) \\ \implies \ &OPT(x) - C_A(S(y)) \leq 1 \cdot (C_B(y) - OPT(R(x))) \end{aligned} \tag{12}$$

Thus for $\beta = 1$, we have $|OPT(x) - C_A(S(y))| \leq \beta |OPT(R(x)) - C_B(y)|$.

Note: Because Matrix Tiling is a maximization problem, and Covering Points with Squares is a minimization problem, we have $OPT(x) \geq c_A(S(y))$, and $OPT(R(x)) \leq c_B(y)$. So removing the modulus used in the fourth condition in the L-reduction definition lends us $OPT(x) - C_A(S(y)) \leq \beta \cdot (C_B(y) - OPT(R(x)))$ equation to derive the value of β.

12 Proof of Theorem 2:

We are now ready to prove our main theorem, which is restated below:

Theorem. *If there are constants $\delta, d > 0$ such that* Covering Points with Squares *has a PTAS with the running time $2^{O(1/\epsilon)^d} \cdot n^{O(1/\epsilon)^{1-\delta}}$, then ETH fails.*

Proof. It is easy to verify that the functions R and S in our reduction are computable in polynomial time with respect to the size of the Matrix Tiling instance. From section 10 and section 11, we have established that $\alpha = 5508$ and $\beta = 1$. Thus, the reduction from Matrix Tiling to Covering Points with Squares is an L-reduction.

Now by [2, Lemma 2.8(1)], if there exists an L-reduction from Matrix Tiling to a problem X (in our case, Covering Points with Squares), then X cannot admit a PTAS with running time of the form $2^{O((1/\epsilon)^d)} \cdot n^{O((1/\epsilon)^{1-\delta})}$ for any constants $d, \delta > 0$, unless the ETH fails.

Applying this lemma to our reduction completes the proof.

□

References

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