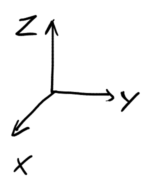
Spatial Descriptors and Transformations

Descriptions:
- **Position**
- **Orientation**
- **Frames**



A position vector P on a frame A can be represented as ^{A}P

$$^{A}P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

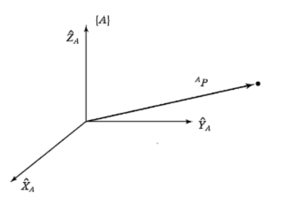
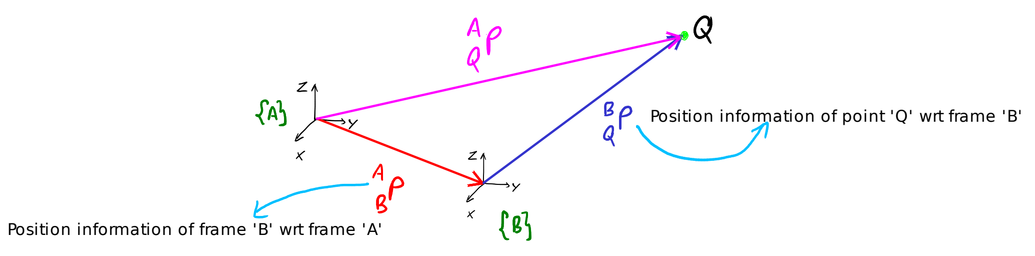


FIGURE 2.1: Vector relative to frame (example).

Mapping involving Translated Frames:



* There are two frames {A} and {B} each consit of three orthonormal vectors 'X', 'Y' and 'Z'
* There is a point Q in 3D space, which can be described from frame {B} as a positional vector of point 'Q' wrt 'B' $^{B}_{Q}P$
* The positional information of {B} wrt {A} can be represented as $^{A}_{B}P$

Now, the positional information of 'Q' wrt {A} can be represented as $^{A}_{Q}P = ^{B}_{Q}P + ^{A}_{B}P$

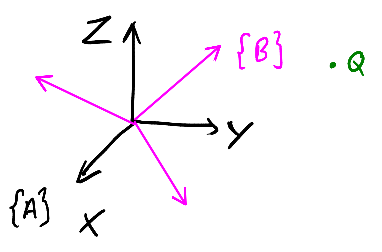
Description of Orientation

Orientation describes how the axes (XYZ) of one frame is aligned wrt other frame (XYZ)

$$^{A}_{B}R = \left[\, ^{A}\hat{X}_B \;\; ^{A}\hat{Y}_B \;\; ^{A}\hat{Z}_B \,\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$^{A}_{B}R = \left[\, ^{A}\hat{X}_B \;\; ^{A}\hat{Y}_B \;\; ^{A}\hat{Z}_B \,\right] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

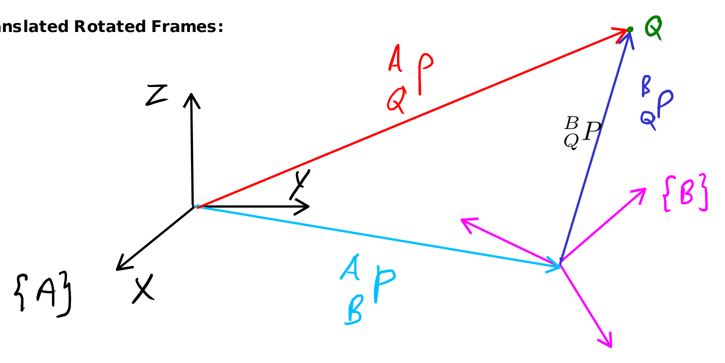
Mapping involving Rotated Frame:



* Two frames {A} and {B} having no translation but oriented differently
* The positional information of point 'Q' wrt 'B' is known, the poisitonal info of 'Q' wrt 'A' can be represented as

$^{A}_{Q}P = ^{A}_{B}R ^{B}_{Q}P$ where, $^{A}_{B}R$ is the Rotation information of frame {B} wrt {A}

Mapping involving Translated Rotated Frames:



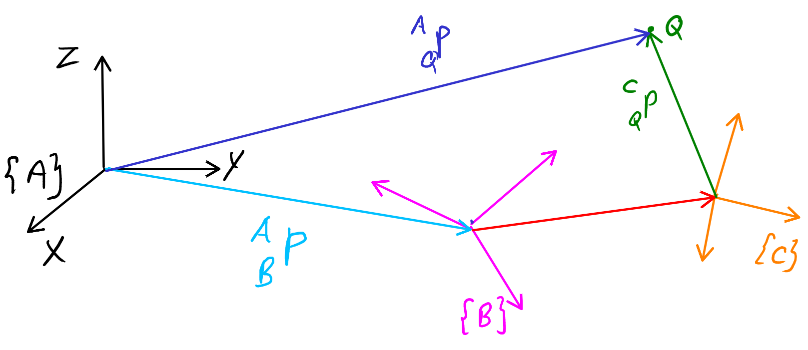
Here, frame {B} is oriented differently from frame {A}, the point Q wrt {B} is known

The Position information of point Q wrt A can be identified by:
- Orienting the frame {B} similar to {A}, It can be done by multiplying $^{A}_{B}R\, ^{B}_{Q}P$
- Add the transformed point Q wrt {B} and positional information of {B} wrt {A}

$$^{A}_{Q}P = ^{A}_{B}P + ^{A}_{B}R ^{B}_{Q}P$$

Exercise: Suppose there are three bodies which are associated with frames {A}, {B} and {C} and there is a point Q in space.
The positional information of point Q wrt C is known. The positional inforamtion of Q wrt A can be identified as

$$^{A}_{Q}P = \, ^{A}_{B}P + \, ^{A}_{B}R \, ^{B}_{C}P + \, ^{A}_{B}R \, ^{B}_{C}R \, ^{C}_{Q}P$$



Note: Here, if the number of bodies are increased the complexity of multiplying the rotation and translation information gets complex.

This complexity can be reduced by using Transformation Matrix, which maps one frame to another frame in matrix form.

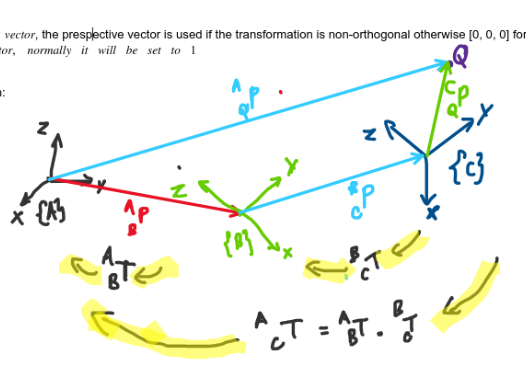
Homogeneous Transformation Matrix: It is a 4x4 matrix, which maps a homogeneous position vector from one frame to another.

Where, homogeneous position vector is added with 1 at the end. Which will make it to $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

$$^{A}_{B}\mathbf{T} = \begin{bmatrix} ^{A}_{B}\mathbf{R} & ^{A}_{B}\mathbf{P} \\ \eta & \sigma \end{bmatrix}$$

Where,
η is the Prespective vector, the prespective vector is used if the transformation is non-orthogonal otherwise [0, 0, 0] for orthogonal projection
σ is the scaling factor, normally it will be set to 1

Compound Transformation:



$$^{A}_{Q}\mathbf{P} = ^{A}_{C}\mathbf{T} ^{C}_{Q}\mathbf{P} = ^{A}_{B}\mathbf{T} ^{B}_{C}\mathbf{T} ^{C}_{Q}\mathbf{P}$$

$$\Rightarrow ^{A}_{C}\mathbf{T} = ^{A}_{B}\mathbf{T} ^{B}_{C}\mathbf{T}$$

$$^{A}_{C}\mathbf{T} = \begin{bmatrix} ^{A}_{B}\mathbf{R} ^{B}_{C}\mathbf{R} & ^{A}_{B}\mathbf{P} + ^{A}_{B}\mathbf{R} ^{B}_{C}\mathbf{P} \\ 0_{1\times 3} & 1 \end{bmatrix}$$