UNIT II : Probability

Probability:

The chance of getting an event in a random experiment from a finite sample space is called Probability

Classical definition of Probability:

In a random experiment, let there are 'n' mutually exclusive and exhaustive events and 'm' are favourable to the any event 'E', then the probability is defined as:

$$P(E) = \frac{m}{n} = \frac{\text{No. of favourable events}}{\text{Exhaustive events}}$$

If 'm' are favourable to the any event 'E', then "n-m" are non-favourable to the event 'E', then the probability is: $P(\bar{E}) = \frac{n-m}{n}$

$$P(\bar{E}) = 1 - \frac{m}{n}$$

$$P(\bar{E}) = 1 - P(E)$$

$$P(E) + P(E) = 1$$

Set:

A collection of well-defined objects is called "set". We use capital letters to represent set and small letters to represent elements in the set.

Ex: $A = \{a, b, c, \ldots\}$

Sample space:

A set of all possible outcomes of an experiment is called a sample space

The sample

Experiment:

An activity which gives results is known as an "Experiment."

Random Experiment:

If the result is varying, then the experiments are called Random experiments.

Ex: Tossing of a coin, Throwing of a die.

Trial:

Each performance in a random experiment is called a Trial.

Outcome:

The result of a trial in a random experiment is called outcome.

Event:

Every non-empty sub-set of sample space of a random experiment is called an Event. An Event is denoted by 'E' and size of an event is denoted by $n(E)$

Types of Events:

1. Singleton event / simple event:

If there is only one element in the event is called simple event

Ex: $E_1 = \{1\}$

2. Composite event:

If there are two or more elements in the event is called composite event

Ex: $E_1 = \{1, 2\}$, $E_2 = \{3, 4, 5\}$

3. Mutually Exclusive events: / Disjoint events

If E_1 and E_2 are any two

Experiment:

An activity which gives results is known as an "Experiment."

Random Experiment:

If the result is varying; then the experiments are called Random experiments.

Ex: Tossing of a coin, Throwing of a die.

Trial:

Each performance in a random experiment is called a Trial.

Outcome:

The result of a trial in a random experiment is called outcome.

Event:

Every non-empty sub-set of sample space of a random experiment is called an Event. An Event is denoted by 'E' and size of an event is denoted by $n(E)$

Types of Events:

1. Singleton event / simple event:

If there is only one element in the event is called simple event.

Ex: $E_1 = \{1\}$

2. Composite event:

If there are two or more elements in the event is called composite event

Ex: $E_1 = \{1,2\}$, $E_2 = \{3,4,5\}$

3. Mutually Exclusive events: / Disjoint events

If E_1 and E_2 are any two

if $E_1 \cap E_2 = \phi$, then E_1 and E_2 are Disjoint event / Mutually exclusive events

Ex: $E_1 = \{1,2\}$ & $E_2 = \{3,4,5\}$ & $E_1 \cap E_2 = \phi$

4. Exhaustive event:

The two events E_1 and E_2 are said to be exhaustive events if $E_1 \cup E_2 = S$

Ex: $E_1 = \{1,2,3\}$ & $E_2 = \{3,4,5,6\}$ & $E_1 \cup E_2 = \{1,2,3,4,5,6\}$

5. Equally - Likely events:

The two events E_1 and E_2 are said to be Equally- Likely events if $n(E_1) = n(E_2)$

Ex: $E_1 = \{1,2,3\}$ & $E_2 = \{3,4,5\}$

6. Independent event:

If E_1 and E_2 are said to be Independent events that do not depends one upon another

i.e, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

7. Null event:

There is no chance of getting an event then it is called Null event or impossible event

i.e, $P(E) = 0$

8. Sure event:

The event E is said to be sure event if it is compulsory occurs

i.e, $P(E) = 1$

Axiomatic definition of probability:

Let E be the event defined on a sample space 'S'

2. $P(S) = 1$ (certaintty / sure)
$\underset{\text{sample space}}{S}$ property

3. If E_1 & E_2 are two events which are mutually exclusive

i.e., $E_1 \cap E_2 = \phi$,

$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2)$ (Additive property)

Theorems:

Theorem 1:

To show that $P(\phi) = 0$.

proof:

Let 'ϕ' be the event defined on a sample space 'S'

$\phi \subseteq S = S$

$\phi \cup S = S$

Taking prob. on both sides

$P(\phi \cup S) = P(S)$ ($\because$ by using axiom II & III)

$P(\phi) + P(S) = P(S)$

$P(\phi) + 1 = 1$

$P(\phi) = 0$

Theorem 2:

To show that $P(\bar{E}) = 1 - P(E)$

proof:

Let E be the event and let $\bar{E}$ also be the event which are defined on a sample space 'S'

$E \cup \bar{E} = S$

$P(E \cup \bar{E}) = P(S)$

$P(E) + P(\bar{E}) = 1$

$P(\bar{E}) = 1 - P(E)$

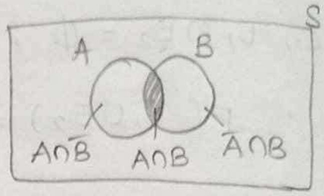
Theorem 3:

For any two events 'A' and 'B' defined on a sample space 'S' $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

proof:

Let A and B are two events which are mutually exclusive events defined on a sample space 'S'

from the diagram,



$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Similarly,

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

Addition Theorem on probability for 2 events:

Statement:

For any 2 events A & B defined on a sample space 'S':

$$P(\underset{E_1}{A} \cup \underset{E_2}{B}) = P(A) + P(B) - P(A \cap B)$$

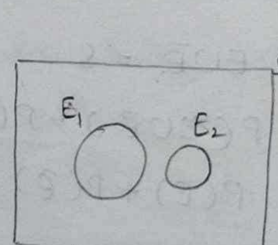
Proof:

Let $\underset{E_1}{A}$ & $\underset{E_2}{B}$ are 2 events defined on a sample space 'S'

case

i) The two events are disjoint events.

from the diagram,



$$E_1 \cap E_2 = \phi$$

Taking prob. on both sides.

$$P(E_1 \cap E_2) = P(\phi)$$

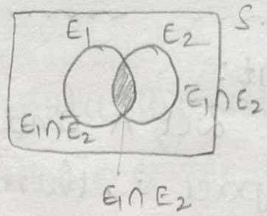
$$P(E_1 \cap E_2) = 0$$

We know that axiom

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

ase(ii)

from the diagram,



$E_1 \cup E_2 = E_1 \cup (\bar{E_1} \cap E_2)$

$P(E_1 \cup E_2) = P(E_1 \cup (\bar{E_1} \cap E_2))$

$P(E_1 \cup E_2) = P(E_1) + P(\bar{E_1} \cap E_2)$ — ①

from the diagram,

$E_2 = (E_1 \cap E_2) \cup (\bar{E_1} \cap E_2)$

$P(E_2) = P[(E_1 \cap E_2) \cup (\bar{E_1} \cap E_2)]$

$P(E_2) = P(E_1 \cap E_2) + P(\bar{E_1} \cap E_2)$

$P(\bar{E_1} \cap E_2) = P(E_2) - P(E_1 \cap E_2)$ — ②

Sub. eq'n ② in eq'n ①

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Hence proved

Conditional probability:

Let A and B are two events defined on the sample space 'S'. If $P(B) > 0$, then

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad ; P(B) > 0$$

Here probability of A given B $[P(A/B)]$ represent the conditional probability of occurance of A where B has already occured.

Similarly probability of B given A:

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad ; P(A) > 0$$

Here $P(B/A)$ represent the conditional probability of occurance of B where A has already occured.

Multiplication Theorem on probability for 2 events:

Statement:

Let A and B are two events defined on the sample space S, then the probability of occurance of A and B is equal to the product of conditional probability of A and unconditional probability of B

i.e, $P(A \cap B) = P(A/B) \cdot P(B)$

(or)

$P(A \cap B) = P(B/A)\, P(A)$

Proof:

Let A and B are two events defined on the sample space S, and let $n(A)$, $n(B)$ are the favourable cases to the events A & B

From the definition of probability:

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

by dividing & multiplying with $n(A)$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} \times \frac{n(A)}{n(A)}$$

$$= \frac{n(A \cap B)}{n(A)} \times \frac{n(A)}{n(S)}$$

$$P(A \cap B) = P(B/A) \cdot P(A)$$

Illy, $P(A \cap B) = \frac{n(A \cap B)}{n(S)}$

Multiplying & dividing with $n(B)$

$$= \frac{n(A \cap B)}{n(S)} \times \frac{n(B)}{n(B)}$$

$$= \frac{n(A \cap B)}{n(B)} \times \frac{n(B)}{n(S)}$$

Baye's Theorem:

statement:

If A_i's $(i=1,2,\ldots n)$ are mutually exclusive and exhaustive events defined on the sample space 'S' and let 'B' be the another event which is subset of finite union of A_i's

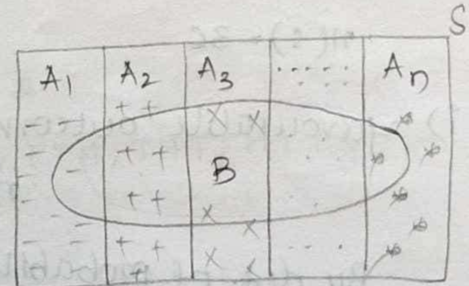
$$P(A_i/B) = \frac{P(B/A_i)\cdot P(A_i)}{\sum_{i=1}^{n} P(B/A_i)\cdot P(A_i)} \quad ; i=1,2,\ldots n$$

proof:

Let A_i's are mutually exclusive and exhaustive events $(i=1,2,\ldots n)$ defined on the sample space's' and let B be the another event which is subset of finite union of A_i's

i.e, $B \subset \bigcup_{i=1}^{n} A_i$

$A_i \cap B$ = mutually exclusive events



from the diagram,

$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \ldots \ldots \cup (A_n \cap B)$

$$B = \bigcup_{i=1}^{n} (A_i \cap B)$$

Taking 'prob' on both sides.

$$P(B) = P\left[\bigcup_{i=1}^{n} (A_i \cap B)\right]$$

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B)$$

$$P(B) = \sum_{i=1}^{n} P(B/A_i)\cdot P(A_i) \quad \text{--- (1)}$$

We know that,

$$P(A_i \cap B) = P(B/A_i)\cdot P(A_i) = P(A_i/B)\cdot P(B)$$

$$P(B/A_i)\cdot P(A_i) = P(A_i/B)\cdot P(B)$$

Sub. eq'n ① in eq'n ②, we get

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{\sum_{i=1}^{n} P(B/A_i)\cdot P(A_i)} \quad ; i = 1, 2, \ldots n$$

Hence proved.

Problems:

1. Two dice are thrown at a time, what is the probability of getting a sum.
   i) 10
   ii) which is perfect square.
   iii) prime numbers.
   iv) even

Sol: Two dice are thrown at a time, we get $6^2 = 36$

$S = \{(1,1)(1,2)(1,3)(1,4)(1,5)(1,6)(2,1) \ldots (6,6)\}$

$n(S) = 36$

i) favourable outcomes = $\{(4,6)(5,5)(6,4)\}$

$n(E) = 3$

By def. of probability : $P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

ii) favourable events = $\{(1,3)(2,2)(3,1)(3,6)(4,5)(5,4)(6,3)\}$

$n(E) = 7$

By def. of prob: $P(E) = \frac{n(E)}{n(S)} = \frac{7}{36}$

iii) favourable events = $\{(1,1)(1,2)(2,1)(2,3)(3,2)(1,4)(4,1)(1,6)(6,1)(2,5)(5,2)(3,4)(4,3),(5,6)(6,5)\}$

$n(E) = 15$

By. def. of prob: $P(E) = \frac{n(E)}{\ } = \frac{15}{\ }$

iv) favourable events = { (1,1) (1,3) (3,1) (2,2) (2,4) (4,2) (1,5) (5,1) (3,3) (4,4) (2,6) (6,2) (3,5) (5,3) (5,5) (6,6) (6,4) (4,6) }

$n(E) = 18$

$$P(E) = \frac{n(E)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

2. Three coins are tossed at a time. what is the probability of getting :

i) atleast 2 heads

ii) atmost 1 head.

iii) no head.

Sol: Three coins are tossed at a time, we get $2^3 = 8$

S = { HHH, HHT, HTH, THH, TTT, TTH, THT, HTT }

$n(S) = 8$

i) favourable event E = { HHH, HHT, HTH, THH }

$n(E) = 4$

$P(E) = \frac{4}{8} = \frac{1}{2}$

ii) favourable event E = { TTT, TTH, THT, HTT }

$n(E) = 4$

$P(E) = \frac{4}{8} = \frac{1}{2}$

iii) favourable event E = { TTT }

$n(E) = 1$

$P(E) = \frac{1}{8}$

3. A card is drawn from a well shuffled pack of cards. what is probability that it is either a spade or an ace

Sol: Let n be the sample space.

Let A_1 denote the event of getting a spade

" A_2 " " " an ace

$n(S) = 52$

$P(A_1) = \frac{13}{52}$

$P(A_2) = \frac{4}{52}$

$A \cap A_2 = \frac{1}{52}$

$P(S \cup A) = P(S) + P(A) - P(S \cap A)$

$P(\text{Spade or Ace}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$

$= \frac{16}{52}$

$= \frac{4}{13}$

4. 3 students A, B, C are in running race. A & B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins the race

We know that,

$A \cup B \cup C = S$

$P(A) = P(B) = 2P(C)$

$P(A) = 2P(C)$ | $P(B) = 2P(C)$

$P(A \cup B \cup C) = 1$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) = 1$

$\Rightarrow 2P(C) + 2P(C) + P(C) = 1$

$\Rightarrow 5P(C) = 1$

$\Rightarrow P(C) = \frac{1}{5}$

$P(A) = 2P(C)$ | Hly. $P(B) = 2P(C)$

$= 2(\frac{1}{5})$ | $= 2(\frac{1}{5})$

$$P(B\cup C) = P(B) + P(C) - \underbrace{P(B\cap C)}_{0}$$

$$= \frac{2}{5} + \frac{1}{5}$$

$$P(B\cup C) = \frac{3}{5}$$

5. Determine:

(i) $P(B/A)$ (ii) $P(A/B^c)$

If A and B are events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$

$P(A\cup B) = \frac{1}{2}$

(i) $P(B/A) = \dfrac{P(A\cap B)}{P(A)}$

We know that,

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} - P(A\cap B)$$

$$P(A\cap B) = \frac{1}{12}$$

$$P(B/A) = \frac{1}{12} \times \frac{3}{1} = \frac{1}{4}$$

(ii) $P(A/B^c) = \dfrac{P(A\cap B^c)}{P(B^c)} = \dfrac{P(A) - P(A\cap B)}{1 - P(B)}$

$$= \frac{\frac{1}{3} - \frac{1}{12}}{1 - \frac{1}{4}}$$

$$P(A/B^c) = \frac{1}{3}$$

6. Two aeroplanes bomb a target in succession. The probability of each correctly scoring a hit is 0.3 and 0.2 respectively. The second will bomb only if the first misses the target. Find the probability that:

i) target is hit

ii) both fails to score hits.

$P(A) = 0.3$, $P(B) = 0.2$

$P(\bar{A}) = 1 - P(A) = 0.7$

i) $P(A\cup B) = P(A \cup (\bar{A}\cap B))$ $= 0.3 + 0.7\times 0.2$

$= P(A) + P(\bar{A}\cap B)$ $= 0.44$

$= P(A) + P(\bar{A})\cdot P(B)$

ii) $P(\bar{A}\cap\bar{B}) = P(\bar{A})\cdot P(\bar{B}) = 0.7\times 0.8 = 0.56$

Starting:

Let A be the event of first plane hitting the target and B be the event of second plane hitting the target.

7. In a Bolt factory, machines A, B, C manufacture 20%, 30% & 50% of the total of their output and 6%, 3% and 2% are defective. A bolt is drawn at random and found to be defective. Find the probabilities that it is manufactured from

i) machine A ii) machine B iii) machine C

Sol: Let A_1 be the event which is output of machine A

" A_2 " " B

" A_3 " " C

$P(A_1) = \frac{20}{100}$

$P(A_2) = \frac{30}{100}$

$P(A_3) = \frac{50}{100}$

Let 'D' be the defective bolt

probability of defective bolt from A_1 ; $P(D/A_1) = \frac{6}{100}$

" A_2 : $P(D/A_2) = \frac{3}{100}$

" A_3 : $P(D/A_3) = \frac{2}{100}$

We know that, baye's theorem:

$$P(A_i/B) = \frac{P(B/A_i)\cdot P(A_i)}{\sum_{i=1}^{n} P(B/A_i)\cdot P(A_i)}$$

i) prob. of A_1 is defective : $P(A_1/D) = \dfrac{P(D/A_1) \cdot P(A_1)}{\sum_{i=1}^{3} P(D/A_i) \cdot P(A_i)}$

$$= \frac{P(D/A_1) \cdot P(A_1)}{P(D/A_1)\, P(A_1) + P(D/A_2) \cdot P(A_2) + P(D/A_3) \cdot P(A_3)}$$

$$P(A_1/D) = \frac{\frac{6}{100} \times \frac{20}{100}}{\frac{6}{100} \times \frac{20}{100} + \frac{3}{100} \times \frac{30}{100} + \frac{2}{100} \times \frac{50}{100}}$$

$$= \frac{\frac{120}{\cancel{100 \times 100}}}{\frac{120+90+100}{\cancel{100 \times 100}}} = \frac{12\cancel{0}}{31\cancel{0}}$$

$$P(A_1/D) = \frac{12}{31}$$

Illy,

$$P(A_2/D) = \frac{9}{31}, \quad P(A_3/D) = \frac{10}{31}$$

8. Of the 3 men, the chances that a politician, a businessman or an academician will be appointed as a vice chancellor of a university are 0.5, 0.3 & 0.2 respectively. Probability that research is promoted by these persons if they are appointed as vice chancellor are 0.3, 0.7, 0.8 respectively.

   i) Determine the probability that research is promoted

   ii) if research is promoted what is the probability that v.c is an academician.

Sol: Let A_1 be the event that the selected person is a politician

& " A_2 " " businessman

" A_3 " " academician

$P(A_1) = 0.5$

$P(A_2) = 0.3$

Let R be the research is promoted

The probability of research is promoted from A_1

i.e, $P(R/A_1) = 0.3$

Illy, $P(R/A_2) = 0.7$

$P(R/A_3) = 0.8$

We know that, baye's theorem

i) The research is promoted: by using total prob. theorem :

i.e, $P(R) = P(R/A_1)P(A_1) + P(R/A_2)P(A_2) + P(R/A_3)P(A_3)$

$= 0.3 \times 0.5 + 0.7 \times 0.3 + 0.8 \times 0.2$

$P(R) = 0.52$

ii) $P(A_3/R) = \dfrac{P(R/A_3).P(A_3)}{P(R)}$

$= \dfrac{0.8 \times 0.2}{0.52}$

$P(A_3/R) = \dfrac{4}{13}$

9. A business man goes to hotel X, Y, Z are 20%, 30% & 50% of the time respectively. It is known that 5%, 4% & 8% of the rooms in X, Y, Z hotels have faulty plumbing

What is the probability that: bussinessman's room having faulty plumbing is assigned to hotel Z

Sol: $P(A_1) = 20/100$ $\quad P(D/A_1) = \dfrac{5}{100}$

$P(A_3/D) = \dfrac{P(D/A_3).P(A_3)}{}$

Random Variables:

A real number 'x', whose values is determined by the outcome of a random experiment is called random variable. If $X(s)$ is a set of real number and it is a subset of real number

i.e., $X(s) : S \to R$

$X(s) \subset R$ (subset)

Types of Random variables:

There are two types of Random variables. They are:

1. Discrete Random variables
2. Continuous Random variable

1. Discrete Random variables:

A Random variable X which can take only a finite number of discrete values in an interval of domain is called Discrete Random variable.

Ex: In tossing a coin 2 times, then the sample space consisting of 4 events. $S\{HH, HT, TH, TT\}$

Let x be assigned to the no. of heads

X:	HH	HT	TH	TT
	2	1	1	0

2. Continuous Random Variables:

A Random variable X which can take values continuously i.e, which takes all possible values in a given interval is called continuous Random variable.

Ex: The age, temperature, height, weight etc.,

Discrete probability Distribution: (or) Probability Mass function:

If each event in a sample space 'S' has a certain probability of occurance, a formula representing all these probabilities with a discrete random variable is called Discrete probability Distribution function (or) Probability Mass function

A Discrete Random Variable 'X' is a function $P(x)$ satisfy the following 2 conditions:

1) $P(x) \geq 0$
2) $\sum_{i=1}^{n} P(X = x_i) = 1$

Probability Density function:

Let 'X' be a continuous random variable and let $f(x)$ be a function of x to take values of x, the function $f(x)$ is said to be probability Density function, if it satisfy the following 2 conditions, They are:

1) $f(x) \geq 0$
2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Probability Distribution function:

Let 'X' be the random variable, it's distribution function denoted by $F_X(x)$ (or) $F(x)$ and is given by:

$$F(x) = P(X \leq x)$$

Properties of Distribution function:

1. If F is the distribution function of a random variable 'X' and if $a < b$, then

ii) $P(a \le x \le b) = P(X=a) + [F(b)-F(a)]$

iii) $P(a < x < b) = [F(b)-F(a)] - P(X=b)$

iv) $P(a \le x < b) = [F(b)-F(a)] - P(X=b) + P(X=a)$

2. All distribution functions are increasing and lie between 0 and 1

i) $0 \le F(x) \le 1$

ii) $F(x) < F(y)$, when $x < y$

3. i) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

ii) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

Mean / Expectation:

Suppose a random variable x is assumed values $x_1, x_2, \dots x_n$ with probabilities $P_1, P_2, \dots P_n$, then the expectation of x is denoted by $E(x)$.

$$\text{Mean} = E(x) = \sum_{i=1}^{n} P_i x_i \quad \text{(discrete series)}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx \quad \text{(continuous series)}$$

Variance:

$$V(x) = \sigma^2 = E(x^2) - [E(x)]^2$$

$$\sigma^2 = \Sigma P_i x_i^2 - [\Sigma P_i x_i]^2 \quad \text{(discrete series)}$$

$$\text{Variance} = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) \cdot dx - \left[\int_{-\infty}^{\infty} x f(x) \cdot dx\right]^2$$

(continuous series)

Properties of expectation / Mean:

1. Expectation of constant : $E(C) = C$
2. $E(ax+b) = a E(x) + b$

4. $E(xy) = E(x) \cdot E(y)$

Properties of variance:

1. $V(c) = 0$
2. $V(ax+b) = a^2 \cdot V(x)$
3. $V(x \cdot y) = V(x) \cdot V(y)$

Problems:

1. A random variable X has the following probability function:

x :	0	1	2	3	4	5	6	7
$P(x)$:	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$
	0	0.1	0.3	0.5	0.8	0.81	0.83	0.17

i) Determine K

ii) $P(x<6)$, $P(x \geq 6)$, $P(0<x<5)$

iii) $P(x \leq K) > 0.5$

Find the minimum value of K

iv) Distribution function of x

v) Mean

vi) Variance

vii) $E(5x^2+7)$

viii) $V(2x+3)$

i) $\sum_{i=1}^{n} P(X=x_i) = 1$

$\Rightarrow \sum_{i=0}^{7} P(X=x_i) = 1$

$\Rightarrow P(X=0) + P(X=1) + P(X=2) + \dots + P(X=7) = 1$

$\Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow 10K^2 + 9K - 1 = 0$

$\Rightarrow 10K^2 + 10K - K - 1 = 0$

$\Rightarrow 10K(K+1) - (K+1) = 0$

ii) $p(X<6) = P(X=0)+P(X=1)+\cdots+P(X=5)$

$= 0+0.1+0.2+0.2+0.3+0.01$

$p(X<6) = 0.81$

$p(X \geq 6) = 1-P(X<6)$

$= 1-0.81$

$= 0.19$

$p(0<x<5) = p(X=1)+\cdots+P(X=4)$

$= 0.1+0.2+0.2+0.3$

$= 0.8$

iii) $p(x \leq K) > 0.5$

if $K=0$, $p(X \leq 0) > 0.5$

$= p(X=0) > 0.5$

$0 > 0.5$ (cond'n false)

$K=1$; $P(x \leq 1) > 0.5$

$= P(X=0)+P(X=1) > 0.5$

$= 0+0.1 > 0.5$ (false)

$K=2$, $P(X \leq 2) > 0.5$

$= P(X=0)+P(X=1)+P(X=2) > 0.5$

$= 0+0.1+0.2 > 0.5$

$K=3$; $P(X \leq 3) > 0.5$

$= P(X \leq 2)+P(X=3) > 0.5$

$= 0.3+0.2 + 0.5$ (false)

$K=4$, $P(X \leq 4) > 0.5$

$= P(X \leq 3)+P(X=4)$

$= 0.5+0.3 > 0.5$

$= 0.8 > 0.5$ (condition is true)

$K=4$

iv) $F(x) = P(X \le x)$

if $x=0$, $F(0) = P(X \le 0)$

$P(X=0) = 0$

$F(0) = 0$

$x=1$, $F(1) = P(X \le 1) \rightarrow P(X=0) + P(X=1)$

$= 0 + 0.1$

$F(1) = 0.1$

$x=2$, $F(2) = P(X \le 2) = P(X=0) + P(X=1), P(X=2)$

$= 0 + 0.1 + 0.2$

$F(2) = 0.3$

v) Mean $= E(x) = \Sigma p_i x_i$

$= 0 \times 0 + 0.1 \times 1 + 0.2 \times 2 + 0.2 \times 3 + 0.3 \times 4 +$
$0.01 \times 5 + 0.02 \times 6 + 0.17 \times 7$

$= 3.66$

vi) Variance $= E(X^2) - [E(X)]^2 = 16.8 - (3.66)^2 = 3.4$

$E(X^2) = 0 \times 0^2 + 0.1 \times 1^2 + 0.2 \times 2^2 + 0.2 \times 3^2 + 0.3 \times 4^2 +$
$0.01 \times 5^2 + 0.02 \times 6^2 + 0.17 \times 7^2$

$E(X^2) = 16.8$

vii) $E(5x^2+7) = 5E(x^2) + 7$

$= 5 \times 16.8 + 7 = 91$

viii) $V(2x+3) = 4V(x)$

$= 4(3.4)$

$= 13.6$

i) $P(1<x<3) = \int_1^3 f(x) \cdot dx$

$= \int_1^3 2 \cdot e^{-2x} \cdot dx$

$= 2\int_1^3 e^{-2x} \cdot dx$

$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3$

$= -\left[e^{-2(3)} - e^{-2(1)}\right]$

$P(1<x<3) = e^{-2} - e^{-6}$

$= 0.1353 - 0.0002$

$= 0.1357$

2. A continuous random variable x has the distribution function:

$$F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ K(x-1)^4, & \text{if } 1 < x \le 3 \\ 1, & \text{if } x > 3 \end{cases}$$

Determine i) $f(x)$
ii) K value
iii) Mean

Sol: We know that,

$$f(x) = \frac{d}{dx} F(x)$$

i) $$f(x) = \begin{cases} 0, & \text{if } x \le 1 \\ 4K(x-1)^3, & \text{if } 1 < x \le 3 \\ 0, & \text{if } x > 3 \end{cases}$$

ii) $$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$$

$$\Rightarrow \underbrace{\int_{-\infty}^{1} f(x) \cdot dx}_{0} + \int_{1}^{3} f(x) \cdot dx + \underbrace{\int_{3}^{\infty} f(x) \cdot dx}_{0} = 1$$

$$\Rightarrow \int_{1}^{3} 4K(x-1)^3 \cdot dx = 1$$

$$\Rightarrow \cancel{4}K \left[\frac{(x-1)^4}{\cancel{4}}\right]_1^3 = 1$$

$$\Rightarrow K[16-0] = 1$$

$$K = \frac{1}{16}$$

iii) $$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_{1}^{3} x \cdot 4K(x-1)^3 \cdot dx$$

$$= 4K \int_{1}^{3} x (x-1)^3 \cdot dx$$

Let $x-1 = t$, $x = 1+t$

$$= 4K \int_0^2 (1+t)\, t^3 \cdot dt$$

$$= 4K \int_0^2 (t^3 + t^4) \cdot dt$$

$$= 4K \left[\frac{t^4}{4} + \frac{t^5}{5} \right]_0^2$$

$$= 4 \cdot \frac{1}{16} \left[\left(\frac{2^4}{4} + \frac{2^5}{5} \right) - (0) \right]$$

$$= \frac{2^4}{4} \cdot \left[\frac{1}{4} + \frac{2}{5} \right]$$

$$= 4 \left[\frac{5+8}{20} \right]$$

$$E(x) = \frac{13}{5}$$

$$\boxed{E(x) = 2.6}$$

3. A continuous random variable x if defined :

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, & -3 \le x < -1 \\ \frac{1}{16}(6-2x^2), & \text{if } -1 \le x < 1 \\ \frac{1}{16}(3-x)^2, & \text{if } 1 \le x \le 3 \\ 0, & \text{o/w otherwise} \end{cases}$$

then verify that: i) $f(x)$ is a probability density function (PDF)

ii) Mean

Sol: we know that, $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$$\Rightarrow \underbrace{\int_{-\infty}^{-3} f(x) \cdot dx}_{0} + \int_{-3}^{-1} f(x) \cdot dx + \int_{-1}^{1} f(x) \cdot dx + \int_{1}^{3} f(x) \cdot dx + \underbrace{\int_{3}^{\infty} f(x) \cdot dx}_{0} = 1$$

$$\Rightarrow \int_{-3}^{-1} \frac{1}{16}(3+x)^2 \cdot dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2)\, dx +$$

$$\Rightarrow \frac{1}{16}\int_{-3}^{-1}(3+x)^2 dx + \frac{1}{\cancel{16}_8}\cancel{2}\int_0^1 (6-2x^2)\,dx + \frac{1}{16}\int_1^3 (3-x)^2 dx = 1$$

$$\Rightarrow \frac{1}{16}\left[\frac{(3+x)^3}{3(1)}\right]_{-3}^{-1} + \frac{1}{8}\left[6x - \frac{2x^3}{3}\right]_0^1 + \frac{1}{16}\left[\frac{(3-x)^3}{3(-1)}\right]_1^3 = 1$$

$$\Rightarrow \frac{1}{\cancel{16}_2}\left[\frac{\cancel{8}}{3}\right] + \frac{1}{\cancel{8}}\left[\frac{\cancel{16}^2}{3}\right] - \frac{1}{\cancel{16}_2}\left[0 - \left(\frac{\cancel{8}^1}{3}\right)\right] = 1$$

$$\Rightarrow \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1$$

$$\Rightarrow \frac{1}{3} + \frac{2}{3} = 1$$

$$\Rightarrow \frac{3}{3} = 1$$

ii) Mean $= E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$$= \underbrace{\int_{-\infty}^{-3} x \cdot f(x) \cdot dx}_{0} + \int_{-3}^{-1} x \cdot f(x) \cdot dx + \int_{-1}^{1} x \cdot f(x) \cdot dx +$$

$$\int_1^3 x \cdot f(x) \cdot dx + \underbrace{\int_3^{\infty} x \cdot f(x) \cdot dx}_{0}$$

$$= \int_{-3}^{-1} x \cdot \frac{1}{16}(3+x)^2 \cdot dx + \underbrace{\int_{-1}^{1} x \cdot \frac{1}{16}(6-2x^2) \cdot dx}_{0} +$$

$$\int_1^3 x \cdot \frac{1}{16}(3-x)^2 \cdot dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(9+x^2+6x) \cdot dx + \frac{1}{16}\int_1^3 x(9+x^2-6x) \cdot dx$$

$$= \frac{1}{16}\int_{-3}^{-1}(9x+x^3+6x^2) \cdot dx + \frac{1}{16}\int_1^3 (9x+x^3-6x^2) \cdot dx$$

$$= \frac{1}{16}\left[\frac{9x^2}{2} + \frac{x^4}{4} + \frac{\cancel{6}^2 x^3}{\cancel{3}_1}\right]_{-3}^{-1} + \frac{1}{16}\left[\frac{9x^2}{2} + \frac{x^4}{4} - \frac{\cancel{6}^2 x^3}{\cancel{3}_1}\right]_1^3$$

$$= \frac{1}{16}\left[\left(\frac{9}{2} + \frac{1}{4} - 2\right) - \left(\frac{81}{2} + \frac{81}{4} - 54\right)\right]$$

$$+ \frac{1}{16}\left[\left(\frac{81}{2} + \frac{81}{4} - 54\right) - \left(\frac{9}{2} + \frac{1}{4} - 2\right)\right]$$

Mean $= E(x) = 0$

4) A continuous random variable has the PDF

$$f(x) = \begin{cases} Kxe^{-\lambda x} ; & \text{for } x \geq 0, \lambda > 0 \\ 0 ; & \text{o/w} \end{cases}$$

Determine: i) K, ii) Mean, iii) Variance

i) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$$\Rightarrow \int_0^{\infty} K \cdot x e^{-\lambda x} \cdot dx = 1$$

$$\Rightarrow K \cdot \int_0^{\infty} \underset{u}{x} \cdot \underset{v}{e^{-\lambda x}} \cdot dx = 1$$

$$\Rightarrow K \left[x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{-\lambda \cdot -\lambda} \right]_0^{\infty} = 1$$

$$\Rightarrow K \left[0 - \left(-\frac{1}{\lambda^2} \right) \right] = 1$$

$$\Rightarrow K/\lambda^2 = 1$$

$$\boxed{K = \lambda^2}$$

ii) Mean $= E(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

$$= \int_0^{\infty} \lambda^2 \underset{u}{x^2} \cdot \underset{v}{e^{-\lambda x}} \cdot dx$$

$$= \lambda^2 \left[x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} - 2x \cdot \frac{e^{-\lambda x}}{\lambda^2} + 2(1) \frac{e^{-\lambda x}}{-\lambda^3} \right]_0^{\infty}$$

$$= \cancel{\lambda^2} \left[0 - \frac{2(1)}{-\lambda^{\cancel{3}}} \right]$$

$$E(x) = \frac{2}{\lambda}$$

(iii) Variance $= V(x) = E(x^2) - [E(x)]^2$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - \left[\int_{-\infty}^{\infty} x f(x) dx \right]^2$$

$$= \lambda^2 \int_0^{\infty} x^2 \cdot x e^{-\lambda x} dx - \left(\frac{2}{\lambda} \right)^2$$

$$= \lambda^2 \left[x^3 \cdot \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \cdot \frac{e^{-\lambda x}}{\lambda^2} + 6x \frac{e^{-\lambda x}}{-\lambda^3} - 6 \cdot \frac{e^{-\lambda x}}{\lambda^4} \right]_0^{\infty} - \frac{4}{\lambda^2}$$

Bivariate Random variable:

A Bivariate Random variable is a real valued function which takes the values in R^2 corresponding to the outcome of a random experiment. It is also called a two-dimensional random variable (x,y) is a Bivariate Random variable such that $(x,y): S \to R^2$

Ex: Throwing of 2 dice, the sample space

$S = \{(1,1)(1,2) \ldots (6,6)\}$

Let x = the no. of first die

y = the sum of the numbers on 2 dice

$x = 1, 2, 3, 4, 5, 6$

$y = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

$(x,y) = \{(1,2)(1,3)(1,4) \ldots \ldots (6,12)\}$

Functions under Bivariate Random variables:

Discrete

1) Joint p.m.f (probability mass func'n)

$P(x,y)$

(or)

$P(X = x_i, Y = y_i)$

2) Conditions:

$P(x,y) \geq 0$

$\sum_x \sum_y P(x,y) = 1$

3) Marginal p.m.f

$P(x) = \sum_y P(x,y)$

$P(y) = \sum_x P(x,y)$

4) Conditional p.m.f

$P(x/y) = \frac{P(x,y)}{\ldots}$; $P(y) \neq 0$

Continuous.

1) Joint p.d.f (prob. density func'n)

$f(x,y)$

2) conditions:

$f(x,y) \geq 0$

$\int_x \int_y f(x,y)\, dx\, dy = 1$

3) Marginal p.d.f

$f(x) = \int_y f(x,y)\, dy$

$f(y) = \int_x f(x,y)\, dx$

4) Conditional p.d.f

$f(x \ldots) = \frac{f(x, y)}{\ldots}$; $f(y) \neq 0$

Problems:

* The two random variables 'X' and 'Y' are said to be independent if $P(x,y) = P(x)\cdot P(y)$
$$f(x,y) = f(x)\cdot f(y)$$

1. The joint probability mass function (X, Y) is given by

$x \backslash y$	1	2	3	4	5	6	$P(x)$
0	0	0	$1/2$	$2/32$	$2/32$	$3/32$	$P(X=0)=1/4$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$	$P(X=1)=5/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$	$P(X=2)=1/8$
$P(y)$	$3/32$	$3/32$	$11/64$	$13/64$	$6/32$	$16/64$	

i) Find Marginal p.m.f's of X, Y

ii) Find the conditional probability of x given y.

iii) $P(X=1, Y=3)$

iv) $P(X\leq 1, Y=2)$

v) $P(X\leq 1)$

vi) $P(X<3, y\leq 4)$

vii) $P(X<3/y=2)$

Sol: i) Marginal p.m.f of 'x'

$$P(x) = \sum_y P(x,y)$$

if $x=0$, $P(0) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$
$+ P(X=0, Y=4) + P(X=0, Y=5) + P(X=0, Y=6)$

$$= 0 + 0 + \frac{1}{2} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32} = \frac{8}{32} = \frac{1}{4} = P(X=0)$$

$$P(X=1) = \frac{2}{16} + \frac{4}{8} = \frac{10}{16} = \frac{5}{8}$$

$$P(X=2) = \frac{2}{32} + \frac{4}{64} = \frac{4+4}{64} = \frac{8}{64} = \frac{1}{8}$$

ii) conditional p.m.f of x/y

$$P(x/y) = \frac{P(x,y)}{P(y)}$$

$$P(X=0, y=1) = \frac{P(x=0, y=1)}{P(y=1)} = \frac{0}{3/32} = 0$$

$$P(X=1, y=1) = \frac{P(X=1, y=1)}{P(y=1)} = \frac{1}{\cancel{16}} \times \frac{\cancel{32}^{\,2}}{3} = \frac{2}{3}$$

$$P(X=2, y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1}{\cancel{32}} \times \frac{\cancel{32}}{3} = \frac{1}{3}$$

— — 18 values.

iii) $P(X=1, Y=3) = \frac{1}{8}$

iv) $P(X \leq 1, Y=2) = P(X=0, Y=2) + P(X=1, Y=2)$

$$= 0 + \frac{1}{16} = \frac{1}{16}$$

v) $P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$

vi) $P(X<3, y \leq 4) = P(X=0, Y\leq 4) + P(X=1, Y \leq 4) + P(X=2, Y \leq 4)$

$= P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=0, Y=4)$
$+ P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$
$+ P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3) + P(X=2, Y=4)$

$$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}$$

$$= \frac{9}{16}$$

vii) $P(X<3 / Y=2) = P(X=0/Y=2) + P(X=1/Y=2) + P(X=2/Y=2)$

$$= \frac{P(X=0, Y=2)}{P(Y=2)} + \frac{P(X=1, Y=2)}{P(Y=2)} + \frac{P(X=2, Y=2)}{P(Y=2)}$$

$$= 0 + \frac{\frac{1\times 2}{16\times 2}}{\frac{3}{\cancel{32}}} + \frac{\frac{1}{\cancel{32}}}{\frac{3}{\cancel{32}}}$$

$$= \frac{2}{3} + \frac{1}{3}$$

2. The Joint p.m.f Of X and Y is $P(x,y) = \frac{x+y}{21}$,

if $x = 1,2,3,$ & $y = 1,2$, find the marginal probability mass functions x and y

Sol:

$x \backslash y$	1	2	$P(x) = \sum_y P(x,y)$
1	2/21	3/21	5/21
2	3/21	4/21	7/21
3	4/21	5/21	9/21
$P(y) = \sum_x P(x,y)$	9/21	12/21	

3. The Joint p.d.f of X and Y is given by:

$$f(x,y) = \begin{cases} Kxy & ; 0<x<1, 0<y<1 \\ 0 & ; o/w \end{cases}$$

i) Find K value

ii) Find marginal p.d.f's of X and Y

iii) Find conditional p.d.f of X and Y

iv) check whether X and Y are independent?

Sol: We Know that,

(i) $$\int_x \int_y f(x,y)\, dx\, dy = 1$$

$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{1} Kxy \cdot dy\, dx = 1$$

$$\Rightarrow K\left(\int_{x=0}^{1} x \int_{y=0}^{1} y \cdot dy\right) dx = 1$$

$$\Rightarrow K \int_{x=0}^{1} x \left[\frac{y^2}{2}\right]_0^1 \cdot dx = 1$$

$$\Rightarrow \frac{K}{2} \int_{x=0}^{1} x \cdot dx = 1$$

$$\Rightarrow \frac{K}{4}\left[x^2\right]_0^1 = 1$$

$$\Rightarrow \frac{K}{4} = 1$$

$$K = 4$$

(ii) Marginal p.d.f of 'x'

$$f(x) = \int_y f(x,y)\cdot dy$$

$$= \int_{y=0}^{1} 4xy\cdot dy$$

$$= 4x\int_{y=0}^{1} y\cdot dy$$

$$= \cancel{4}^{2}x\left[\frac{y^2}{\cancel{2}}\right]_0^1$$

$$= 2x$$

$$f(x) = 2x$$

Marginal p.d.f of 'y'

$$f(y) = \int_x f(x,y)\cdot dx$$

$$= \int_{x=0}^{1} 4xy\cdot dx$$

$$= 4y\int_{x=0}^{1} x\cdot dx$$

$$= \cancel{4}^{2}y\left[\frac{x^2}{\cancel{2}}\right]_0^1$$

$$= 2y$$

$$f(y) = 2y$$

(iii) conditional p.d.f of x/y

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{\cancel{4}^{2}x\cancel{y}}{\cancel{2y}}$$

$$f(x/y) = 2x$$

conditional p.d.f of y/x

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{\cancel{4}^{2}\cancel{x}y}{\cancel{2x}}$$

$$= 2y$$

(iv) $f(x,y) = f(x)\cdot f(y)$

$$4xy = 2x\cdot 2y$$

$$4xy = 4xy.$$

$\therefore$ x & Y are independent

4. The joint p.d.f is given by:

$$f(x,y) = \frac{x^3y^3}{16}, \quad 0<(x,y)<1$$

Find the marginal pdf

Sol: $f(x) = \int_y f(x,y)\cdot dy$

$$= \frac{1}{16}\int_{y=0}^{1} x^3y^3\cdot dy$$

$f(y) = \int_x f(x,y)\cdot dx$

$$= \frac{1}{16}\int^{1} x^3y^3\cdot dy$$

5. The joint p.d.f of X and Y is given by:

$$f(x,y) = e^{-(x+y)} \; ; \; x \geq 0, \; y \geq 0.$$

Find the marginal p.d.f's.

Sol:

$$\int_x \int_y f(x,y) \cdot dy \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x+y)} \cdot dy \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-x} \cdot e^{-y} \cdot dy \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} e^{-x} \int_{y=0}^{\infty} e^{-y} \cdot dy \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^{\infty} \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} e^{-x} - \left(e^{-(\infty)} - e^{-(0)}\right) \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} e^{-x} (1) \cdot dx = 1$$

$$\Rightarrow \int_{x=0}^{\infty} e^{-x} \cdot dx = 1$$

$$\Rightarrow \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow 1 = 1$$

$\therefore$ The given function is a joint p.d.f

Marginal p.d.f 'x'

$$f(x) = \int_y f(x,y) \cdot dy.$$

$$f(x) = \int_{y=0}^{\infty} e^{-(x+y)} \cdot dy$$

$$= \int_0^{\infty} e^{-x} \cdot e^{-y} \cdot dy$$

$$= e^{-x} \int e^{-y} \cdot dy$$