UNIT III : Probability Distributions

Bernoulli's Distribution	Normal distribution
Binomial "	Exponential "
Poisson "	

Distributions can be classified into 2 types:

1. Discrete distribution
2. Continuous distribution.

Discrete distribution:

In which distribution, discrete random variable involving, such distribution is called Discrete distribution.

They are mainly

1. Bernoulli's Distribution
2. Binomial Distribution
3. Poisson Distribution

1. Bernoulli's Distribution:

A random experiment is one that may result either of two possible outcomes is known as Bernoulli experiment.

Ex: Tossing of a coin, performance of a student in an exam.

Let 'x' be the discrete random variable which takes only 2 values '0' and '1' probabilities 'q' and 'p' respectively.

i.e, $P(X=0) = P(\text{failure}) = q$

$P(X=1) = P(\text{success}) = p$

Here,

p = probability of success and $q = 1-p$ = probability of failure

$\therefore$ The probability mass function of Bernoulli distribution

$P(x) = p^x (1-p)^{1-x}$

Mean of Bernoulli Distribution:
/ Expectation of x [E(x)]

$$Mean = E(x) = \sum x \cdot p(x)$$
$$= \sum_{x=0}^{1} x \cdot p^x q^{1-x}$$
$$= 0 + p$$
$$E(x) = p$$

Variance of Bernoulli Distribution:

$$V(x) = \sigma^2 = E(x^2) - [E(x)]^2$$

$$\left[\because E(x^2) = \sum x^2 \cdot p(x) = \sum_{x=0}^{1} x^2 \cdot p^x q^{1-x} = 0 \cdot P = P\right]$$

$$V(x) = P - (P)^2$$
$$= P - P^2$$
$$= P(1-P) \quad (\because p+q = 1$$
$$q = 1-p)$$
$$V(x) = pq$$

Moment generating function: $M_x(t)$

It is a tool to calculate higher order moment. The moment generating function of random variable 'x' about origin is probability mass function $p(x)$ and probability density function $f(x)$ is given by:

$$M_x(t) = E(e^{tx})$$
$$M_x(t) = \sum e^{tx} \cdot p(x) \quad (\text{Discrete})$$
$$M_x(t) = \int e^{tx} \cdot f(x) \cdot dx \quad (\text{continuous})$$

Moment generating function of Bernoulli Distribution:

The p.m.f of Bernoulli Distribution:

$$p(x) = p^x \cdot q^{1-x} \; ; \; x = 0,1$$

Moment generating function (M.g.f):

Let $x=0$: $= \sum_{x=0}^{1} (p \cdot e^t)^x q^{1-x}$

$= q + pe^t$

$\therefore M_x(t) = (q + pe^t)$

* By using M.g.f. Find mean and variance.

Raw moments (or) Non-central moments

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

if $r=1$, $\mu_1' = E(x) = \left[\frac{d}{dt}(q + pe^t)\right]_{t=0}$

$$= \left[0 + p \cdot e^t\right]_{t=0}$$

$$E(x) = p$$

if $r=2$, $\mu_2' = E(x^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$

$$= \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[p \cdot e^t\right]_{t=0}$$

$$= \left[p \cdot e^t\right]_{t=0}$$

$$\mu_2' = p$$

$\mu_2 = \mu_2' - (\mu_1')^2$

$= p - (p)^2$

$= p - p^2$

$= p(1-p)$

$\mu_2 = pq$

Variance $= V(x) = \sigma^2 = \mu_2 = pq$

2. Binomial Distribution:

Binomial Distribution was introduced by James Bernoulli in 1700. Binomial experiment is such that:

1. There should be n trials
2. The result of each trial can be classified into two mutually exclusive outcomes say success or failure. Let p be the probability of success and q be the probability of failure

$\therefore p+q=1$

3. The outcome of all trials are independent to each other.

Let x denotes the no. of success in n trials. x can take values $0,1,2,\dots,n$ if there are exactly x success, then the remaining $n-x$ are failures in a specific order of n trials by the compound probability, the expression is $p^x q^{n-x}$ for x success out of n trials is arranged in ^nC_x ways

$\therefore$ The required probability of x no. of success is given by:

$$P(x) = P(X=x) = B(n,p) = {}^nC_x p^x q^{n-x}; \quad x=0,1,\dots n, \quad 0\leq p\leq 1, \quad 0\leq q\leq 1$$

which is the pmf of binomial distribution, n and p are constants.

Mean and variance of Binomial Distribution:

The p.m.f of BD is $p(x) = P(X=x) = B(n,p)$

$$= {}^nC_x p^x q^{n-x}; \; x=0,1,\dots,n$$

Mean $= E(x) = \Sigma x p(x)$

$$= 0 + {}^nC_1 p \cdot q^{n-1} + 2 \cdot {}^nC_2 \cdot p^2 q^{n-2} + 3 \cdot {}^nC_3 \cdot p^3 q^{n-3} + \cdots \cdots + n \cdot {}^nC_n \cdot p^n \cdot q^{n-n}$$

$$= \frac{n!}{(n-1)!\,1!} p \cdot q^{n-1} + \cancel{2} \cdot \frac{n!}{(n-2)!\,\cancel{2!}} p^2 q^{n-2} + \cancel{3} \frac{n!}{(n-3)!\,\cancel{3!}_{2!}} p^3 q^{n-3}$$

$$+ \cdots + np^n$$

$$= \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} p \cdot q^{n-1} + \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} p^2 q^{n-2} +$$

$$\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}\,2!} p^3 q^{n-3} + \cdots\cdots + np^n$$

$$= np\left[q^{n-1} + {}^{(n-1)}C_1\, p \cdot q^{n-2} + {}^{(n-1)}C_2\, p^2 q^{n-3} + \cdots + p^{n-1}\right]$$

$$= np\,(q+p)^{n-1}$$

$$\left[\because (q+p)^n = q^n + {}^nC_1 \cdot p \cdot q^{n-1} + {}^nC_2 \cdot p^2 \cdot q^{n-2} + \cdots\cdots p^n\right]$$

$$= np\,(1)^{n-1}$$

$$E(x) = np$$

The p.m.f of BD is $p(x) = P(X=x) = B(n,p) = {}^nC_x \cdot p^x \cdot q^{n-x}$; $x = 0, 1, \ldots, n$

Variance $= E(x^2) - [E(x)]^2$

$$E(x^2) = \Sigma x^2 p(x)$$

$$= \Sigma\left[x(x-1) + x\right] \cdot p(x)$$

$$= \Sigma x(x-1) \cdot p(x) + \Sigma x p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^nC_x \cdot p^x \cdot q^{n-x} + np.$$

$$= 0 + 0 + 2\,{}^nC_2\, p^2 q^{n-2} + 3(2) \cdot {}^nC_3 \cdot p^3 q^{n-3} +$$

$$4(3) \cdot {}^nC_4 \cdot p^4 \cdot q^{n-4} + \cdots\cdots + n(n-1)\,{}^nC_n \cdot p^n \cdot q^{n-n} + np$$

$$= \cancel{2} \frac{n!}{(n-2)!\,\cancel{2!}} p^2 q^{n-2} + \cancel{6} \frac{n!}{(n-3)!\,\cancel{3!}} p^3 q^{n-3} + \cancel{12} \frac{n!}{(n-4)!\,\cancel{4!}} p^4 q^{n-4}$$

$$= \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} p^2 q^{n-2} + \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} p^3 q^{n-3} +$$

$$\frac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}\ 2!} p^4 . q^{n-4} + \dots + n(n-1)p^n + np.$$

$$= n(n-1)p^2 \left[q^{n-2} + {}^{(n-2)}C_1\, p . q^{n-3} + {}^{(n-2)}C_2\, p^2 q^{n-4} + \dots + p^{n-2} \right] + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)p^2 (1) + np$$

$$= n^2p^2 - np^2 + np$$

$$E(x^2) = n^2p^2 - np^2 + np$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$V(x) = npq$$

Moment generating function of Binomial Distribution:

The pmf of B.D is $p(x) = P(X=x) = B(n,p)$

$$= {}^nC_x . p^x . q^{n-x} \ ; x = 0, 1, \dots, n$$

The M.g.f $M_x(t) = E(e^{tx})$

$$M_x(t) = \Sigma e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x . p^x . q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x . (p . e^t)^x q^{n-x}$$

$$= {}^nC_0 (p . e^t)^0 . q^{n-0} + {}^nC_1 (p . e^t)^1 . q^{n-1} + {}^nC_2 . (p . e^t)^2 . q^{n-2} +$$

$$\dots + {}^nC_n (p e^t)^n q^{n-n}$$

$$= q^n + {}^nC_1 (p . e^t) q^{n-1} + {}^nC_2 (p . e^t) . q^{n-2} + \dots + (pe^t)^n$$

$$M_x(t) = (q + pe^t)^n$$

Mean and Variance of B.D using M.g.f:

Raw moments $\mu_r' = E(X^r) = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$

if $r=1$, $\mu_1' = E(x) = \frac{d}{dt}\left[(q+pe^t)^n\right]_{t=0}$

$$= \left[n(q+pe^t)^{n-1}\frac{d}{dt}(q+pe^t)\right]_{t=0}$$

$$= \left[n(q+p.e^t)^{n-1}(0+p.e^t)\right]_{t=0}$$

$$E(x) = np$$

if $r=2$, $\mu_2' = E(x^2) = \frac{d^2}{dt^2}[M_X(t)]_{t=0}$

$$= \frac{d}{dt}\left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$= np\left[\frac{d}{dt}(q+p.e^t)^{n-1}.e^t\right]_{t=0}$$

$$= np\left[(q+p.e^t)^{n-1}e^t + e^t(n-1)(q+pe^t)^{n-2}(0+pe^t)\right]_{t=0}$$

$$= np\left[1+(n-1)p\right]$$

$E(x^2) = np + n^2p^2 - np^2$

$V(x) = \sigma^2 = \mu_2 = E(x^2) - [E(x)]^2$ (or $= \mu_2' - (\mu_1')^2$.

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p)$$

$$V(X) = npq$$

Mode of Binomial Distribution:

Mode of B.D is the value of x at which $p(x)$ has maximum value. then, Case 1 :

If $(n+1)p$ is not an integer, we have to take integral part of $(n+1)p$ is the mode value

Ex: If $n=6$, $p=\frac{1}{2}$; Mode $=(6+1)\frac{1}{2} = \frac{7}{2} = 3.5$

$\therefore$ Mode $= 3$

Case 2 :

If $(n+1)p$ is an integer, we have to take $(n+1)p$ & $[(n+1)p-1]$ are the modal values, which is called Bimodal class.

Ex: If $n=7$, $p=\frac{1}{2}$; $(n+1)p = (7+1)\frac{1}{2} = 4$

$(n+1)p - 1 = 4-3 = 3$

$\therefore$ Mode $= 4, 3$

Recurrence Relation of Binomial Distribution:

$$P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(x)$$

Expected frequency :

$E(x) = N \cdot P(x)$; where, $N =$ total of frequency

Problems:

1. The mean and variance of Binomial distribution are 4 and 4/3 respectively. Find i) mode

ii) $P(X \geq 1)$

Sol: $np = 4$ — ①

$npq = \frac{4}{3}$ — ②

$$\frac{②}{①} = \frac{\cancel{np}q}{\cancel{np}} = \frac{\frac{\cancel{4}}{3}}{\cancel{4}}$$

$\therefore q = \frac{1}{3}$

We know that,

$np = 4$

$n\left(\frac{\cancel{2}}{3}\right) = \cancel{4}^{2}$

$\therefore n = 6$

$\therefore p+q = 1$

$\quad\quad = 1 - \frac{1}{3} = \frac{2}{3}$

i) Mode $= (n+1)p = (6+1)\frac{2}{3} = (7)\frac{2}{3} = 4.67$

$\therefore$ Mode $= 4$

ii) $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

The p.m.f $P(X = x) = n_{C_x} \cdot p^x q^{n-x}$

$= 1 - {}^6C_0 \cdot \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0}$

$= 1 - \frac{1}{3^6}$

$P(X \geq 1) = 0.9899$

2. 10 coins are thrown simultaneously. Find the probability of getting atleast 7 heads.

Sol: $n = 10$

The prob. of getting head $p = \frac{1}{2}$

$q = \frac{1}{2}$

$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}$

$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$

$= \frac{1}{2^{10}} \left[120 + 45 + 10 + 1\right]$

$= \frac{176}{1024}$

$= 0.1718$

*3. 4 coins are tossed 160 times, the no. of times x heads occur is given below

x	:	0	1	2	3	4
No. of times	:	8	34	69	43	6

i) coins are unbiased

ii) the nature of the coin is unknown (means biased)

Sol: $N = 160$, $n = 4$.

i) The coin is unbiased

$p = \frac{1}{2}$, $q = \frac{1}{2}$

The p.m.f of B.D $P(x) = {}^nC_x \cdot p^x q^{n-x}$;

$x = 0, 1, \ldots, n$

if $x = 0$, $P(0) = {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0}$

$P(0) = \frac{1}{16}$

By using recurrence relation,

$$P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(x)$$

if $x = 0$, $P(0+1) = \frac{4-0}{0+1} \cdot \frac{1/2}{1/2} \cdot P(0)$

$= 4 \times \frac{1}{16} = \frac{1}{4} = P(1)$

If $x = 1$, $P(1+1) = \frac{4-1}{1+1} \cdot \frac{1/2}{1/2} \cdot P(1)$

$= \frac{3}{2} \times \frac{1}{4} = \frac{3}{8} = P(2)$

If $x = 2$, $P(2+1) = \frac{4-2}{2+1} \cdot \frac{1/2}{1/2} \cdot P(2)$

$= \frac{2}{3} \times \frac{3}{8} = \frac{1}{4} = P(3)$

if $x = 3$, $P(3+1) = \frac{4-3}{3+1} \cdot \frac{1/2}{1/2} \times P(3)$

$= \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} = P(4)$

Expected frequency $E(x) = N \cdot P(x)$

If $x = 0$, $E(0) = 160 \times P(0) = 160 \times \frac{1}{16} = 10$

" $x = 1$, $E(1) = 160 \times \frac{1}{4} = 40$

" $x = 2$, $E(2) = 160 \times \frac{3}{8} = 60$

" $x = 3$, $E(3) = 160 \times \frac{1}{4} = 40$

" $x = 4$, $E(4) = 160 \times \frac{1}{16} = 10$

ii) $\bar{x} = np = \frac{\sum fx}{N}$

$= \frac{8\times0 + 34\times1 + 69\times2 + 43\times3 + 6\times4}{160}$

$= \frac{325}{160} = 2.03$

$np = 2.03$

$4p = 2.03$

$p = \frac{2.03}{4} = 0.507$

$q = 1 - 0.507 = 0.493$

Expected frequency $E(x) = 160 \times {}^nC_x \cdot p^x \cdot q^{n-x}$

if $x=0$, $E(0) = 160 \times {}^4C_0 (0.507)^0 (0.493)^{4-0} = 9.4516 \simeq 9$

" $x=1$, $E(1) = 160 \times {}^4C_1 (0.507)^1 (0.493)^{4-1} = 38.88 \simeq 39$

" $x=2$, $E(2) = 160 \times {}^4C_2 (0.507)^2 (0.493)^{4-2} = 59.97 \simeq 60$

" $x=3$, $E(3) = 160 \times {}^4C_3 (0.507)^3 (0.493)^{4-3} = 41.11 \simeq 41$

" $x=4$, $E(4) = 160 \times {}^4C_4 (0.507)^4 (0.493)^{4-4} = 10.57 \simeq 11$

4. 7 coins are tossed and the number of heads are noted. The experiment is repeated 128 times and the following distribution is obtained

No. of heads :	0	1	2	3	4	5	6	7
frequency :	7	6	19	35	30	23	7	1

Fit a Binomial distribution assuming:

i) the coin is unbiased

ii) the nature of the coin is unknown/biased

Sol: i) the coin is unbiased.

$p = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 7$, $N = 128$

The p.m.f of BD is $P(x) = {}^nC_x p^x q^{n-x}$; $x = 0, 1, \ldots, 7$

if $x=0$, $P(0) = {}^7C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{7-0}$

Using recurrence relation:

$$P(x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(x)$$

$x=0,\ P(0+1) = \frac{n-0}{0+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} P(0) = \frac{n}{1} \cdot \frac{1}{128} = \frac{7}{128} = P(1)$

$x=1,\ P(1+1) = \frac{n-1}{1+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} P(1) = \frac{7-1}{1+1} \cdot \frac{7}{128} = \frac{6}{2} \cdot \frac{7}{128} = \frac{21}{128}$

$P(2) = \frac{21}{128}$

$x=2,\ P(2+1) = \frac{n-2}{2+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} P(2) = \frac{7-2}{2+1} \cdot \frac{21}{128} = \frac{5}{3} \cdot \frac{21}{128} = \frac{35}{128}$

$P(3) = \frac{35}{128}$

$x=3,\ P(3+1) = \frac{7-3}{3+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} P(3) = \frac{\cancel{4}}{\cancel{4}} \frac{35}{128} = \frac{35}{128} = P(4)$

$x=4,\ P(4+1) = \frac{7-4}{4+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} \frac{35}{128} = \frac{3}{\cancel{5}} \times \frac{\overset{7}{\cancel{35}}}{128} = \frac{21}{128} = P(5)$

$x=5,\ P(5+1) = \frac{7-5}{5+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} \frac{21}{128} = \frac{\cancel{2}}{\cancel{6}_3} \frac{\overset{7}{\cancel{21}}}{128} = \frac{7}{128} = P(6)$

$x=6,\ P(6+1) = \frac{7-6}{6+1} \cdot \frac{\cancel{1/2}}{\cancel{1/2}} \frac{7}{128} = \frac{1}{\cancel{7}} \cdot \frac{\cancel{7}}{128} = \frac{1}{128} = P(7)$

Expected frequency $E(x) = N \cdot P(x)$.

$x=0,\ E(0) = 128 \times P(0) = \cancel{128} \times \frac{1}{\cancel{128}} = 1$

$x=1,\ E(1) = 128 \times P(1) = \cancel{128} \times \frac{7}{\cancel{128}} = 7$

$x=2,\ E(2) = 128 \times P(2) = \cancel{128} \times \frac{35}{\cancel{128}} = 35$

lly,

$E(3) = 35,\ E(4) = 35,\ E(5) = 21,\ E(6) = 7,\ E(7) = 1$

ii) $\bar{x} = np = \frac{\Sigma fx}{N}$

$$= \frac{7\times 0 + 6\times 1 + 19\times 2 + 35\times 3 + 30\times 4 + 23\times 5 + 7\times 6}{128}$$

$$\bar{x} = \frac{433}{128}$$

$q = 1-p$

$= 1-0.48$

$= 0.52$

$q = 0.52$

$N = 128$, $n = 7$, $p = 0.48$, $q = 0.52$

Expected frequency $E(x) = N \cdot P(x)$

if $x=0$, $E(0) = 128 \times P(0)$

$= 128 \times {}^7C_0 (0.48)^0 (0.52)^{7-0}$

$= 128 \times (0.52)^7$

$= 1.31 \simeq 1$

if $x=1$, $E(1) = 128 \times {}^7C_1 (0.48)^1 (0.52)^{7-1} = 8.5 \simeq 8$

if $x=2$, $E(2) = 128 \times {}^7C_2 (0.48)^2 (0.52)^5 = 23.54 \simeq 24$

if $x=3$, $E(3) = 128 \times {}^7C_3 (0.48)^3 (0.52)^4 = 36.2255 \simeq 36$

if $x=4$, $E(4) = 128 \times {}^7C_4 (0.48)^4 (0.52)^3 = 33.4389 \simeq 33$

if $x=5$, $E(5) = 128 \times {}^7C_5 (0.48)^5 (0.52)^2 = 18.5200 \simeq 19$

if $x=6$, $E(6) = 128 \times {}^7C_6 (0.48)^6 (0.52)^1 = 5.6984 \simeq 6$

if $x=7$, $E(7) = 128 \times {}^7C_7 (0.48)^7 (0.52)^0 = 0.7514 \simeq 1$

(or)

$E(x) = N (q+p)^n$

$\left[\because (q+p)^n = q^n + {}^nC_1\, p \cdot q^{n-1} + {}^nC_2\, p^2 q^{n-2} + {}^nC_3\, p^3 q^{n-3} + \dots p^n\right]$

3. Poisson Distribution:

A random variable 'x' is said to follow a poisson Distribution if it assumes only non negative values and its probablity mass function $p(x)$ is given by:

$$p(x) = p(x, \lambda) = P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!} \; ; x = 0, 1, \dots \infty$$

Here $\lambda > 0$ and

Ex:1) The no of defective electric bulbs manufactured by a company.

2) The no. of printing mistakes per page in a large text

3) The no. of persons born blind per year in a large city

physical
Conditions of poisson distribution:

*. The poisson distribution is used under the following conditions.

1. The variable (no. of occurances) is a discrete variable
2. The occurances are rare
3. The no. of trials 'n' is large
4. The probability of success 'P' is very small
5. $np = \lambda$ is finite

Mean and variance of poisson distribution:

The p.m.f of p.d is $p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$, $x = 0, 1, \dots \infty$

$$\text{Mean} = E(x) = \sum x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=1}^{\infty} \cancel{x} \cdot \frac{\lambda^x}{\cancel{x}(x-1)!}$$

$$= e^{-\lambda}\left[\frac{\lambda}{1} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \dots\right]$$

$$= e^{-\lambda}\lambda\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots\right]$$

$$= \cancel{e^{-\lambda}} \cdot \lambda \cdot \cancel{e^{\lambda}}$$

$$E(x) = \lambda$$

$$\text{Variance} = V(x) = \sigma^2 = E(x^2) - [E(x)]^2$$

$$E(x^2) = \Sigma.\, x^2\, P(x)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \cancel{x}^{\,2} \frac{\lambda^x}{\cancel{x}(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} [(x-1)+1] \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} (x-1) \frac{\lambda^x}{(x-1)!} + e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} (x-1) \frac{\lambda^x}{(x-1)(x-2)!} + e^{-\lambda}\left[\frac{\lambda}{1} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left[\frac{\lambda^2}{1} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots\right] + e^{-\lambda}\cdot \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= e^{-\lambda} \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right] + \cancel{e^{-\lambda}}\, \lambda \cdot \cancel{e^{\lambda}}$$

$$= \cancel{e^{-\lambda}} \cdot \lambda^2 \cdot \cancel{e^{\lambda}} + \lambda$$

$$= \lambda^2 + \lambda$$

$$E(x^2) = \lambda^2 + \lambda$$

$$V(x) = \sigma^2 = E(x^2) - [E(x)]^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{[\lambda]^2}$$

$$= \lambda$$

$$\therefore V(x) = \sigma^2 = \lambda$$

Moment Generating Function of P.d:

The p.m.f of p.d is $p(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$, $x = 0, 1, \ldots \infty$

The m.g.f. $M_x(t) = E(e^{tx})$

$M_x(t) = \Sigma\, e^{tx}\, p(x)$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots \right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$M_x(t) = e^{\lambda(e^t - 1)} \cdot (\text{or } e^{-\lambda(1-e^t)})$$

Mean and Variance of p.d using M.G.F

The m.g.f of poisson distribution $M_x(t) = e^{\lambda(e^t-1)}$

Raw moments $\mu_r' = E(X^r) = \frac{d^r}{dt^r}[M_x(t)]$

if $r=1$, $\mu_1' = E(x) = \frac{d}{dt}\left[e^{\lambda(e^t-1)}\right]_{t=0}$

$$= \left[e^{\lambda(e^t-1)} \frac{d}{dt} \lambda(e^t - 1)\right]_{t=0}$$

$$= \left[e^{\lambda(e^t-1)} \cdot \lambda \cdot e^t\right]_{t=0}$$

$$\therefore E(x) = \lambda$$

if $r=2$, $\mu_2' = E(X^2) = \frac{d^2}{dt^2}[M_x(t)]_{t=0}$

$$= \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \lambda \cdot \frac{d}{dt}\left[\underbrace{e^{\lambda(e^t-1)}}_{u} \cdot \underbrace{e^t}_{v}\right]_{t=0}$$

$$\left(\because \frac{d}{dt}(uv) = u \cdot \frac{d}{dt}(v) + v \cdot \frac{d}{dt}(u)\right)$$

$$= \lambda \cdot \left[e^{\lambda(e^t-1)} \cdot e^t + e^t \cdot e^{\lambda(e^t-1)} \lambda \cdot e^t\right]_{t=0}$$

$$= \lambda\,[1 + \lambda]$$

$$E(x^2) = \lambda + \lambda^2$$

$$V(x) = \sigma^2 = \text{var}(x) = \mu_2 = \mu_2' - (\mu_1')^2$$

Recurrence Relation of Poisson distribution:

$$P(x+1) = \frac{\lambda}{x+1} \cdot P(x)$$

Problems:

1) If the mean of a poisson variate is 1.8. Then find

i) $P(x>1)$

ii) $P(0<x<5)$

Sol: $\lambda = 1.8$

i) $P(X>1) = 1 - P(X \le 1)$

$= 1 - [P(X=0) + P(X=1)]$

$P(x) = P(X=x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$; $x = 0$

$= 1 - \left[e^{-1.8} \frac{(1.8)^0}{0!} + e^{-1.8} \frac{(1.8)^1}{1!} \right]$

$= 1 - \left[e^{-1.8} + 1.8 \cdot e^{-1.8} \right]$

$= 1 - [0.165 + 0.297]$

$P(X>1) = 0.537$

ii) $P(0<x<5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= e^{-1.8}(1.8) + e^{-1.8} \frac{(1.8)^2}{2!} + e^{-1.8} \frac{(1.8)^3}{3!} + e^{-1.8} \frac{(1.8)^4}{4!}$

$= e^{-1.8} [1.8 + 1.62 + 0.972 + 0.437]$

$= e^{-1.8} [4.829]$

$= 0.165 [4.829]$

$= 0.796$

2) If the variance of a poisson variate is 3. Then find probability that:

i) $x = 0$ ii) $0 < x <$

Sol: We know that

$$\text{variance} = \lambda = 3$$

i) The p.m.f of P.D is $P(X=x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$

$$P(X=0) = e^{-3}\frac{(3)^0}{0!}$$

$$= e^{-3}$$

$$= 0.0497$$

ii) $P(0 < x \leq 3) = P(X=1) + P(X=2) + P(X=3)$

$$= e^{-3}\frac{(3)^1}{1!} + e^{-3}\frac{(3)^2}{2!} + e^{-3}\frac{(3)^3}{3!}$$

$$= e^{-3}\left[3 + 4.5 + 4.5\right]$$

$$= 12 \cdot e^{-3}$$

$$= 12\,(0.0497)$$

$$= 0.597$$

3. If a poisson distribution is such that $P(X=1)\frac{3}{2} = P(X=3)$. Find

i) $P(X \geq 1)$ ii) $P(X \leq 3)$

Sol: Given $P(X=1)\frac{3}{2} = P(X=3)$

$$\cancel{e^{-\lambda}}\frac{\cancel{\lambda}}{1!}\cdot\frac{3}{\cancel{2}} = \cancel{e^{-\lambda}}\frac{\lambda^{\cancel{3}\,2}}{3\cancel{!}}$$

$$\lambda^2 = 9$$

$$\lambda = \pm 3$$

$$\therefore \lambda = 3$$

i) $P(X \geq 1) = 1 - P(X < 1)$

$$= 1 - P(X=0)$$

$$= 1 - e^{-3}\frac{3^0}{0!}$$

$$= 1 - e^{-3}$$

$$= 0.9502$$

ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= e^{-3}\frac{(3)^0}{0!} + e^{-3}\frac{(3)^1}{1!} + e^{-3}\frac{(3)^2}{2!} + e^{-3}\frac{(3)^3}{3!}$$

4. It has been found that 2% of the tools produced by a certain machine are defective. What is the probability that in a shipment of 400 such tools

i) 3% or more

ii) 2% or less

will prove defective
be

Sol: We know that,

Sample size $n = 400$

probability of defective tools $p = 2\%$ or $p = 0.02$

$np = \lambda$

$400 \times 0.02 = \lambda$

$\therefore \lambda = 8$

i) $P(X \geq 3) = 1 - [P(X < 3)]$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left[e^{-8}\frac{(8)^0}{0!} + e^{-8}\frac{(8)^1}{1!} + e^{-8}\frac{(8)^2}{2!}\right]$

$= 1 - [e^{-8}(1+8+32)]$

$= 1 - e^{-8}(41)$

$= 0.9863$

ii) $P(x \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= e^{-8}(41)$

$= 0.01375$

5. Fit a poisson distribution for the following data and calculate the expected frequency/cies.

x:	0	1	2	3	4
f:	109	65	22	3	1

Sol: $\bar{X} = \lambda = \frac{\Sigma fx}{N} = \frac{109 \times 0 + 65 \times 1 + 22 \times 2 + 3 \times 3 + 1 \times 4}{200}$

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$P(x) = e^{-\lambda} \frac{\lambda^x}{x!}$; $x = 0, 1, 2, 3, 4$

If $x = 0$, $P(0) = e^{-0.61} \cdot \frac{(0.61)^0}{0!}$

$P(0) = e^{-0.61}$

$P(0) = 0.5433$

Recurrence Relation: $P(x+1) = \frac{\lambda}{x+1} P(x)$

if $x = 0$, $P(0+1) = \frac{0.61}{0+1} P(0) = 0.61 \times 0.5433 = 0.3314$

if $x = 1$, $P(1+1) = \frac{0.61}{1+1} P(1) = \frac{0.61}{2} \times 0.3314$

$P(2) = 0.1010$

if $x = 2$, $P(2+1) = \frac{0.61}{2+1} P(2) = \frac{0.61}{3} \times 0.1010$

$P(3) = 0.0205$

if $x = 3$, $P(3+1) = \frac{0.61}{3+1} P(3) = \frac{0.61}{4} \times 0.0205$

$P(4) = 0.0031$

Expected frequency $= N \cdot P(x)$

If $x = 0$ $E(0) = 200 \times P(0) = 200 \times 0.5433 = 108.66 \simeq 109$

if $x = 1$, $E(1) = 200 \times P(1) = 200 \times 0.3314 = 66.28 \simeq 66$

$x = 2$, $E(2) = 200 \times P(2) = 200 \times 0.1010 = 20.2 \simeq 20$

$x = 3$, $E(3) = 200 \times P(3) = 200 \times 0.0205 = 4.1 \simeq 4$

$x = 4$, $E(4) = 200 \times P(4) = 200 \times 0.0031 = 0.62 \simeq 1$

200

x :	0	1	2	3	4
f :	109	65	22	3	1
$E(x)$:	109	66	20	4	1

6. Fit a poisson distribution to the following data

x :	0	1	2	3	4	5
f :	42	33	14	6	4	1

$\bar{x} = \lambda$ 100

$$P(x) = e^{-\lambda} \frac{\lambda^x}{x!} \; ; x = 0,1,2,3,4,5$$

If $x=0$, $P(0) = e^{-1} \cdot \frac{(-1)^0}{0!} = e^{-1} = 0.3678$

Recurrence Relation: $P(x+1) = \frac{\lambda}{x+1} P(x)$

If $x=0$, $P(0+1) = \frac{+1}{0+1} P(0) = +1\,(0.3678)$

$$P(1) = 0.3678$$

$x=1$, $P(1+1) = \frac{1}{1+1} P(1) = \frac{1}{2}(0.3678)$

$$P(2) = 0.1839$$

$x=2$, $P(2+1) = \frac{1}{2+1} P(2) = \frac{1}{3}(0.1839)$

$$P(3) = 0.0613$$

$x=3$, $P(3+1) = \frac{1}{3+1} P(3) = \frac{1}{4}(0.0613)$

$$P(4) = 0.0153$$

$x=4$, $P(4+1) = \frac{1}{4+1} \cdot P(4) = \frac{1}{5}(0.0153)$

$$P(5) = 0.00306$$

Expected frequency $= N \cdot P(x)$

$E(0) = 100 \times P(0) = 36.78 \simeq 37$

$E(1) = 36.78 \simeq 37$

$E(2) = 18.39 \simeq 18$

$E(3) = 6.13 \simeq 6$

$E(4) = 1.53 \simeq 2$

$E(5) = 0.306 \simeq 0$

x :	0	1	2	3	4	5
f :	42	33	14	6	4	1
$E(x)$:	37	37	18	6	2	0

Normal distribution:

A continuous random variable x is said to have follow a normal distribution, if its probability density function f(x) is given by-

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$\Rightarrow (-\infty < \mu < \infty)$

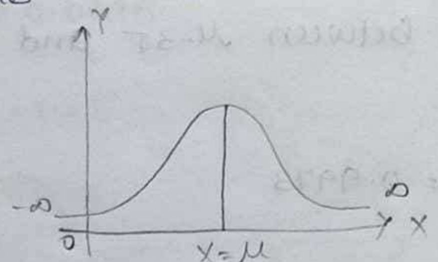
$\Rightarrow -\infty < x < \infty$

$\Rightarrow \sigma > 0$

here, μ and σ are parameters of the normal distribution

Chief characteristics of Normal distribution:

1. The graph of the normal distribution $y = f(x)$ in the xy plane is known as the Normal curve



Mean = Median = Mode

$\bar{X} = M = Z$

at $X = \mu$

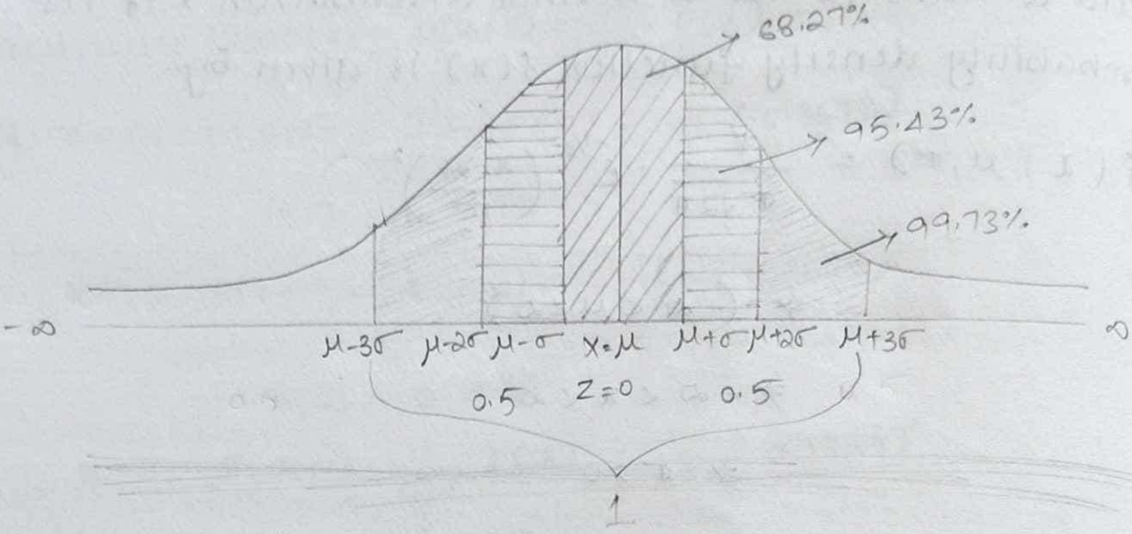
2. The curve is a bell shaped curve and symmetrical with respect to mean i.e, about the line $X=\mu$ and the two tails on the right and left sides of the mean μ extends to infinity. The top of the bell is directly above the mean μ

3. Area under the normal curve represent the total population

4. Mean, median and mode of the distribution coincide at $X=\mu$ as the distribution is symmetrical. So, normal curve is unimodal

6. Linear combination of independent normal variates is also a normal variate

7. Area under the normal curve is distributed



i) Area of normal curve between $\mu-\sigma$ and $\mu+\sigma$ is 68.27%

$$P(\mu-\sigma < x < \mu+\sigma) = 0.6827$$

ii) Area of normal curve between $\mu-2\sigma$ and $\mu+2\sigma$ is 95.43%

$$P(\mu-2\sigma < x < \mu+2\sigma) = 0.9543$$

iii) Area of normal curve between $\mu-3\sigma$ and $\mu+3\sigma$ is 99.73%.

$$P(\mu-3\sigma < x < \mu+3\sigma) = 0.9973$$

Standard normal distribution:

The normal distribution with mean $\mu=0$ and standard deviation $\sigma=1$ is known as standard normal distribution

The standard normal distribution is denoted by z

$$\therefore z = \frac{x-\mu}{\sigma}$$

(also known as standard normal variate)

Problems:

1. If X is a normal variate, find the area 'A'

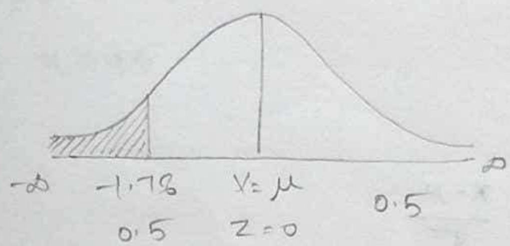
i) to the left of $Z = -1.78$

ii) to the right of $Z = -1.45$

iii) corresponding to $-0.8 \leq Z \leq 1.53$

iv) to the left of $Z = -2.52$ and to the right of $Z = 1.83$

Sol: i) $Z = -1.78$



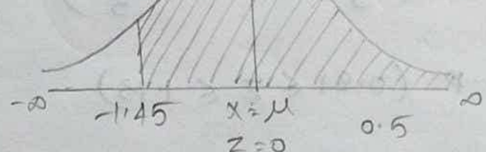
$P(-\infty \leq Z \leq 0) - P(-1.78 \leq Z \leq 0)$

$= 0.5 - A(1.78)$

$= 0.5 - 0.4625$

$= 0.0375$

ii) $Z = -1.45$



$P(-1.45 \leq Z \leq 0) + P(0 \leq Z \leq \infty)$

$= A(1.45) + 0.5$

$= 0.4265 + 0.5$

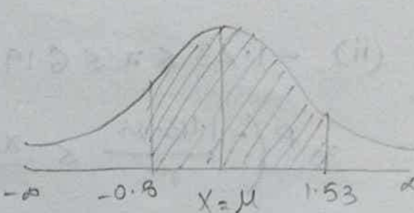
$= 0.9265$

iii) $P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 1.53)$

$= A(0.8) + A(1.53)$

$= 0.2881 + 0.4370$

$= 0.7251$

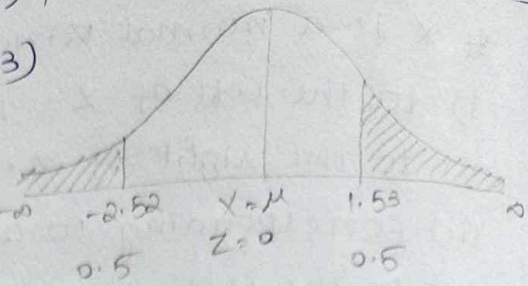


iv) $= P(-\infty \le Z \le 0) - P(-2.52 \le Z \le 0) + P(0 \le Z \le \infty) - P(0 \le Z \le 1.83)$

$= 0.5 - A(2.52) + 0.5 - A(1.83)$

$= 0.5 - 0.4941 + 0.5 - 0.4664$

$= 0.0395$



2. For a normally distributed variate with mean 1 and standard deviation 3, Find the probabilities that

i) $3.43 \le x \le 6.19$

ii) $-1.43 \le x \le 6.19$

Sol: We know that, $Z = \dfrac{x-\mu}{\sigma}$

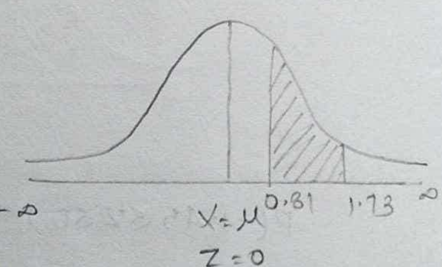
(i) $3.43 \le x \le 6.19$

$P\left(\dfrac{3.43-\mu}{\sigma} \le \dfrac{x-\mu}{\sigma} \le \dfrac{6.19-\mu}{\sigma}\right)$

$= P\left(\dfrac{3.43-1}{3} \le Z \le \dfrac{6.19-1}{3}\right)$

$= P\left(\dfrac{2.43}{3} \le Z \le \dfrac{5.19}{3}\right)$

$= P(0.81 \le Z \le 1.73)$



$= P(0 \le Z \le 1.73) - P(0 \le Z \le 0.81)$

$= A(1.73) - A(0.81)$

$= 0.4582 - 0.2910$

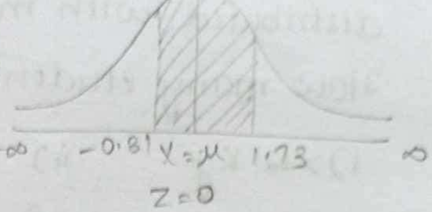
$= 0.1672$

(ii) $-1.43 \le x \le 6.19$

$= P\left(\dfrac{-1.43-\mu}{\sigma} \le \dfrac{x-\mu}{\sigma} \le \dfrac{6.19-\mu}{\sigma}\right)$

$= P(-0.81 \leq z \leq 1.73)$

$= P(-0.81 \leq z \leq 0) + P(0 \leq z \leq 1.73)$



$= A(0.81) + A(1.73)$

$= 0.2910 + 0.4582$

$= 0.7492$

3. If x is a normal variate with mean 30 and standard deviation 5, find the probabilities that

i) $26 \leq x \leq 40$

ii) $x \geq 45$

4. If the masses of 300 students are normally distributed with mean(μ) 68 kg and s.d (σ) 3 kg. How many students have masses

i) >72 kg ii) ≤ 64 kg iii) $65 < x < 71$ kg.

Sol: $\mu = 68$, $\sigma = 3$

i) $X > 72$

WKT $Z = \frac{X-\mu}{\sigma}$

$P\left(\frac{X-\mu}{\sigma} > \frac{72-\mu}{\sigma}\right)$

$P\left(Z > \frac{72-68}{3}\right)$

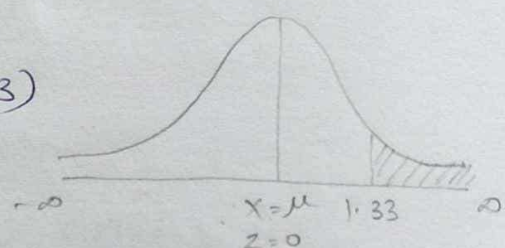
$P\left(Z > \frac{4}{3}\right)$

$P(Z > 1.33)$

$P(0<Z<\infty) - \underbrace{P(0<Z<1.33)}_{A(1.33)}$

$= 0.5 - 0.4082$

$= 0.0918$



$\therefore$ Number of students greater than 72 kg

i.e, 300×0.0918

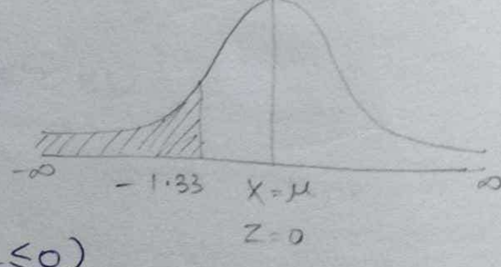
$= 27.54 \simeq 28$

ii) $X \leq 64$

$P\left(\frac{X-\mu}{\sigma} \leq \frac{64-\mu}{\sigma}\right)$

$P\left(Z \leq \frac{64-68}{3}\right)$

$P(Z \leq -1.33)$



$P(-\infty \leq Z \leq 0) - P(-1.33 \leq Z \leq 0)$

$= 0.5 - A(1.33)$

$= 0.5 - 0.4082$

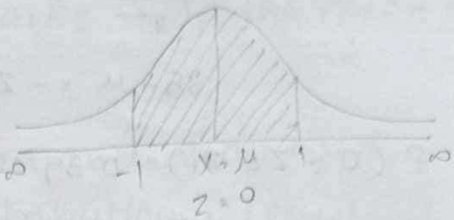
$= 0.0918$

$\therefore$ Number of students less than or equal to 64 kgs

iii) $65 < X < 71$

$= P\left(\frac{65-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{71-\mu}{\sigma}\right)$

$= P\left(\frac{65-68}{3} < Z \leq \frac{71-68}{3}\right)$



$= P(-1 < Z < 1)$

$= P(-1 \leq Z \leq 0) + P(0 \leq Z \leq 1)$

$= A(1) + A(1)$

$= 0.3413 + 0.3413$

$= 0.6826$

$\therefore$ Number of students lies b/w 65 & 71 kgs

i.e, 300×0.6826

$= 204.78 \simeq 205$

5. In a normal distribution 7% of the items are under 35 and 89% are under 63. Determine the mean and variance of the distribution.

Sol: Let μ and σ be the mean and standard deviation of the normal distribution respectively.

Given,

$X < 35 = 7\%$, $X < 63 = 89\%$

$P(X < 35) = 0.07$, $P(X < 63) = 0.89$

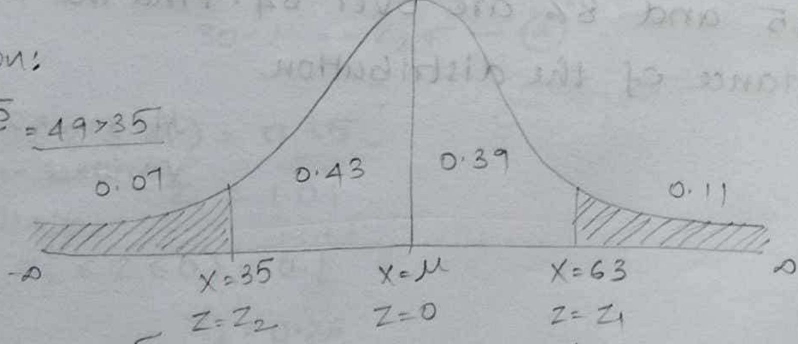
$P(X \geq 63) = 1 - P(X < 63)$

$= 1 - 0.89$

$= 0.11$

Assumption:

$\frac{63+35}{2} = 49 > 35$



from the diagram,

Standard Normal variate : $Z = \frac{X-\mu}{\sigma}$

$$63-\mu = z_1\sigma \quad \text{--- (1)}$$

if $X = 35$, $\dfrac{35-\mu}{\sigma} = -z_2$

$$35-\mu = -z_2\sigma \quad \text{--- (2)}$$

$P(0 \le z \le z_1) = 0.39$

$$z_1 = 1.23$$

$P(z_2 \le z \le 0) = 0.43$

$$z_2 = 1.48$$

Sub. z_1 & z_2 values in eq'n (1) & (2)

$$63-\mu = 1.23\sigma \quad \text{--- (3)}$$

$$35-\mu = -1.48\sigma \quad \text{--- (4)}$$

$$28 = 2.71\sigma$$

$$\sigma = \frac{28}{2.71} = 10.33, \quad \sigma^2 = 106.7$$

Sub. 'σ' value in eq'n (3)

$$63-\mu = 1.23(10.33)$$

$$63-\mu = 12.7$$

$$-\mu = 12.7-63$$

$$\therefore \mu = 50.3$$

$\therefore$ Mean = 50.3 & Variance = 106.7

6. In a normal distribution 31% of items are under 45 and 8% are over 64. Find the mean and variance of the distribution

Mean = 50
variance = 100
$\sigma = 10$

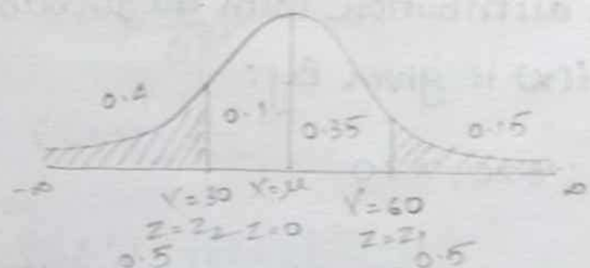
7. The marks obtained in statistics in a certain examination found to be normally distributed. If 15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find the mean and standard deviation

Sol: Let μ and σ be the mean and standard deviation of the normal distribution respectively.

Given,

$X \geq 60 = 15\%$ $\qquad$ $X < 30 = 40\%$

$P(X \geq 60) = 0.15$ $\qquad$ $P(X < 30) = 0.4$



from this diagram,

WKT $\quad Z = \dfrac{X-\mu}{\sigma}$

if $X = 60$,

$$\frac{60-\mu}{\sigma} = Z_1$$

$$60 - \mu = Z_1 \sigma \quad \text{— ①}$$

If $X = 30$,

$$\frac{30-\mu}{\sigma} = -Z_2$$

$$30 - \mu = -Z_2 \sigma \quad \text{— ②}$$

$P(0 \leq Z \leq Z_1) = 0.35$

$$Z_1 = 1.04$$

$P(Z_2 \leq Z \leq 0) = 0.1$

$$Z_2 = 0.26$$

Sub. Z_1 & Z_2 values in eq'n ① & ②

$$60 - \mu = 1.04\sigma$$

$$30 - \mu = -0.26\sigma$$

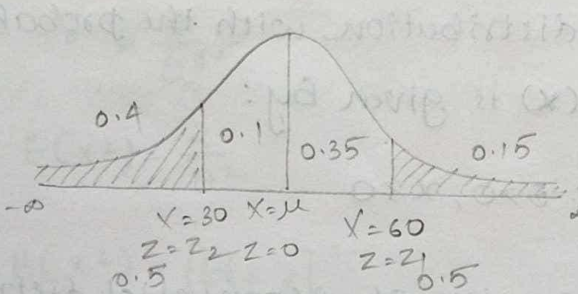
7. The marks obtained in statistics in a certain examination found to be normally distributed. If 15% of the students greater than or equal to 60 marks, 40% less than 30 marks. Find the mean and standard deviation.

Sol: Let μ and σ be the mean and standard deviation of the normal distribution respectively.

Given,

$X \geq 60 = 15\%$ $\qquad$ $X < 30 = 40\%$

$P(X \geq 60) = 0.15$ $\qquad$ $P(X < 30) = 0.4$



from this diagram,

WKT $\quad Z = \dfrac{X-\mu}{\sigma}$

if $X = 60$,

$$\frac{60-\mu}{\sigma} = Z_1$$

$$60 - \mu = Z_1\sigma \quad \text{--- (1)}$$

if $X = 30$,

$$\frac{30-\mu}{\sigma} = -Z_2$$

$$30 - \mu = -Z_2\sigma \quad \text{--- (2)}$$

$P(0 \leq Z \leq Z_1) = 0.35$

$$Z_1 = 1.04$$

$P(Z_2 \leq Z \leq 0) = 0.1$

$$Z_2 = 0.26$$

Sub. Z_1 & Z_2 values in eq'n (1) & (2)

$$60 - \mu = 1.04\sigma$$

$$\sigma = \frac{30}{1.3} = 23.07$$

Sub 'σ' value in eq'n ①

$$60 - \mu = 1.04\,(23.07)$$

$$\mu = 36.01$$

$\therefore$ Mean $= 36.01$ and $s.d = 23.07$.

Exponential Distribution:

A continuous random variable 'x' assumes non-negative values is said to have follow exponential distribution with the probability density function $f(x)$ is given by:

$$f(x,\theta) = \theta . e^{-\theta x} \quad ; \theta > 0, x \geq 0$$

Here 'θ' is the parameter of exponential distribution

Mean and Variance of Exponential Distribution:

The p.d.f of exp. distribution $f(x) = \theta . e^{-\theta x}$; $x \geq 0$, $\theta > 0$

$$\text{Mean} = E(x) = \int x . f(x) . dx$$

$$= \int_0^\infty x . \theta . e^{-\theta x} . dx$$

$$= \theta \int_0^\infty \underset{u}{x} . \underset{v}{e^{-\theta x}} . dx \qquad \int uv = uv_1 - u'v_2 + \ldots$$

inter

diff

$$= \theta \left[x . \frac{e^{-\theta x}}{-\theta} - 1 . \frac{e^{-\theta x}}{\theta^2} \right]_0^\infty$$

$$= \theta \left[0 - \left(0 - \frac{1}{\theta^2} \right) \right]$$

$$= \cancel{\theta} \frac{1}{\theta^{\cancel{2}}}$$

$$= \frac{1}{\theta}$$

$$\therefore E(x) = \frac{1}{\theta}$$

$$Variance = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int x^2 \cdot f(x)\, dx$$

$$= \int_0^{\infty} x^2 \cdot \theta \cdot e^{-\theta x} \cdot dx$$

$$= \theta \int_0^{\infty} \underset{u}{x^2} \cdot \underset{v}{e^{-\theta x}} \cdot dx$$

$$= \theta \left[x^2 \cdot \frac{e^{-\theta x}}{-\theta} - 2x \cdot \frac{e^{-\theta x}}{\theta^2} + 2 \frac{e^{-\theta x}}{-\theta^3} \right]_0^{\infty}$$

$$= \theta \left[0 - \left(0 - 0 - \frac{2}{\theta^3} \right) \right]$$

$$= \cancel{\theta} \frac{2}{\theta^{\cancel{3}\, 2}}$$

$$E(X^2) = \frac{2}{\theta^2}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2$$

$$= \frac{2}{\theta^2} - \frac{1}{\theta^2}$$

$$= \frac{1}{\theta^2}$$

$$\therefore V(X) = \frac{1}{\theta^2}$$

Moment generating function of exponential dis'n :

The p.d.f of exp dis'n $f(x) = \theta \cdot e^{-\theta x}$; $x \geq 0,\ \theta > 0$

The M.g.f $M_X(t) = E(e^{tx})$

$$= \int e^{tx} \cdot f(x) \cdot dx$$

$$= \int_0^{\infty} e^{tx} \cdot \theta \cdot e^{-\theta x} \cdot dx$$

$$= \theta \int_0^{\infty} e^{-(\theta - t)x} \cdot dx$$

$$= \theta \left[\frac{e^{-(\theta - t)x}}{-(\theta - t)} \right]_0^{\infty}$$

$$= \frac{\theta}{-(\theta-t)}\left[e^{-(\theta-t)x}\right]_0^{\infty}$$

$$= \frac{\theta}{\cancel{-}(\theta-t)}\left[0\cancel{-}1\right]$$

$$= \frac{\theta}{\theta-t}$$

$$= \frac{\cancel{\theta}}{\cancel{\theta}\left(1-\frac{t}{\theta}\right)}$$

$$M_X(t) = \left(1-\frac{t}{\theta}\right)^{-1}$$

Memoryless property of exponential distribution:

Statement:

If x follows exponential dis'n with parameter 'θ', then probability:

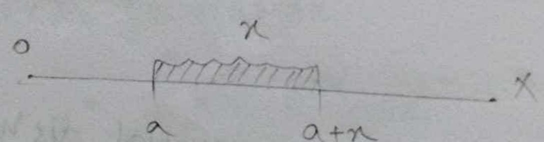
$$P\left[X \geq a+x \,/\, X \geq a\right] = P\left[X \geq x\right]$$

where a and x are the real numbers greater than '0'

proof:

The p.d.f of exp. dis'n is

$$f(x) = \theta \cdot e^{-\theta x} \;;\; x \geq 0,\ \theta > 0$$



$$LHS = P\left[\underbrace{X \geq a+x}_{a} \,/\, \underbrace{X \geq a}_{b}\right]$$

$$= \frac{P\left[X \geq a+x \cap X \geq a\right]}{P(X \geq a)}$$

$$= \frac{P\left[X \geq a+x\right]}{P(X \geq a)}$$

$$P\left[X \geq a+x\right] = \int_{a+x}^{\infty} f(x) \cdot dx$$

$$\int^{\infty} \quad e^{-\theta x}$$

$$= \theta \left[\frac{e^{-\theta x}}{-\theta} \right]_{a+x}^{\infty}$$

$$= -\left[e^{-\theta x} \right]_{a+x}^{\infty}$$

$$= -\left[0 - e^{-\theta(a+x)} \right]$$

$$= e^{-\theta(a+x)}$$

$$P(X \geqslant a+x) = e^{-\theta a} e^{-\theta x}$$

Illy, $P(X \geqslant a) = \int_a^{\infty} f(x) \cdot dx$

$$= \int_a^{\infty} \theta e^{-\theta x} \cdot dx$$

$$= \theta \left[\frac{e^{-\theta x}}{-\theta} \right]_a^{\infty}$$

$$= -\left[0 - e^{-\theta a} \right]$$

$$P(X \geqslant a) = e^{-\theta a}$$

$$\frac{P(X \geqslant a+x)}{P(X \geqslant a)} = \frac{\cancel{e^{-\theta a}} \cdot e^{-\theta x}}{\cancel{e^{-\theta a}}} = e^{-\theta x}$$

$LHS = e^{-\theta x}$

$RHS = P[X \geqslant x]$

$$= \int_x^{\infty} f(x) \cdot dx$$

$$= \int_x^{\infty} \theta \cdot e^{-\theta x} \cdot dx$$

$$= \cancel{\theta} \left[\frac{e^{-\theta x}}{-\cancel{\theta}} \right]_x^{\infty}$$

$$= -\left[0 - e^{-\theta x} \right]$$

$P[X \geqslant x] = e^{-\theta x}$