Indian Institute of Technology Roorkee
Department of Mathematics
MAN-010 (Optimization Techniques)
Theory of Games

Ex. 7

1. Examine the following payoff matrices for saddle points. In case the saddle point exists, find the optimal strategies and value of the game. In every case verify that

$$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}$$

(i) $\begin{bmatrix} -1 & 3 \\ -2 & 10 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & -2 \\ 1 & 0 & 1 \\ -2 & -1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$ (vii) $\begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$

2. Solve the games with the following payoff matrices.

(i) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$

3. Solve graphically the games whose payoff matrices are the following.

(i) $\begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$

4. Use the notion of dominance to simplify the following payoff matrices and then solve the game.

(i) $\begin{bmatrix} 0 & 5 & -4 \\ 3 & 9 & -6 \\ 3 & -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 \\ 4 & 3 & 1 & 3 & 2 \\ 4 & 3 & 4 & -1 & 2 \end{bmatrix}$

5. Write both the primal and the dual LP problems corresponding to the rectangular games with the following payoff matrices. Solve the game by solving the LP problem by simplex method.

(i) $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

6. Consider a 2×2 two-person zero-sum game. Let $x \textit{ and } (1-x)$ be the probabilities of selecting strategies $A_1 \textit{ and } A_2$ by player A, and $y \textit{ and } (1-y)$ be the probabilities of selecting strategies $B_1 \textit{ and } B_2$ by player B. The payoff matrix $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. Show that the expected value of the game to player A is $E(x, y) = 1 - 2y\left(x - \frac{1}{2}\right)$ and deduce that in the solution of the game, the second player follows a pure strategy while the first has infinite number of mixed strategies.

7. For a 2×2 two-person zero-sum game without any saddle point, having payoff matrix for player A as, $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Find the optimal strategies and value of the game.

8. Find the range of p and q so that the entry (2, 2) is a saddle point in the following games:

(a) $\begin{bmatrix} 2 & q & 4 \\ p & 6 & 11 \\ 7 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 3 \\ 8 & 5 & q \\ 2 & p & 4 \end{bmatrix}$

Answers:

1. (i) $Saddle\ point = -1;\ V = -1;\ Optimal\ strategies: A \to Row\ 1, B \to Column\ 1.$
   (ii) $Saddle\ point = 2;\ V = 2;\ Optimal\ strategies: A \to Row\ 1, B \to Column\ 1, 2.$
   (iii) $Saddle\ point = 0;\ V = 0;\ Optimal\ strategies: A \to Row\ 2, B \to Column\ 2.$
   (iv) $Saddle\ point = 4;\ V = 4;\ Optimal\ strategies: A \to Row\ 2, B \to Column\ 3.$
   (v) $No\ saddle\ point.$ (vi) $No\ saddle\ point.$ (vii) $No\ saddle\ point.$

2. (i) $V = 0;\ Optimal\ strategies: A \to \left(\frac{1}{2}, \frac{1}{2}\right),\ B \to \left(\frac{1}{2}, \frac{1}{2}\right)$ (ii) $V = \frac{5}{2};\ Optimal\ strategies: A \to \left(\frac{1}{2}, \frac{1}{2}\right),\ B \to \left(\frac{1}{4}, \frac{3}{4}\right).$ (iii) $V = \frac{7}{2};\ Optimal\ strategies: A \to \left(\frac{3}{4}, \frac{1}{4}\right),\ B \to \left(\frac{1}{2}, \frac{1}{2}\right).$

3. (i) $V = 5.2143$; $Optimal\ strategies$: $A \to (0.64286,\ \ 0,\ \ 0.35714)$, $B \to (0.35714,\ \ 0.64286)$.

(ii) $V = 1.4$; $Optimal\ strategies$: $A \to (0.4,\ \ 0.6)$, $B \to (0,\ \ 0.2,\ \ 0.8)$.

(iii) $V = 3.4$; $Optimal\ strategies$: $A \to (0, 0.6, 0, 0.4, 0, 0)$, $B \to (0.8, 0,2)$.

4. (i) $V = 0.667$; $Optimal\ strategies$: $A \to (0, 0.1667, 0.8333)$, $B \to (0, 0.444, 0.556)$.

(ii) $V = 1.85714$; $Optimal\ strategies$: $A \to (0, 0, 0.7143, 0.286)$, $B \to (0, 0, 0.5714, 0.429, 0)$.

5. (i) $V = 1$; $Optimal\ strategies$: $B \to (0.333, 0.333, 0.333)$, $A \to (0.333, 0.333, 0.333)$.

(ii) $V = 1$; $Optimal\ strategies$: $B \to (0, 0.5, 0.5)$, $A \to (0.667, 0.333,0)$.

(iii) $V = 1.4$; $Optimal\ strategies$: $B \to (0, 0.2, 0.8)$, $A \to (0.4, 0.6)$

8. (a) $p \geq 6, q \leq 6$; (b) $p \leq 5, q \geq 5$.