Assignment - 9

Deepak Singh

1) ⟨i⟩ $u(t) = u_1(t)\hat{i} + u_2(t)\hat{j} + u_3(t)\hat{k}$

$u(t) = \text{Const.} \iff u_1(t) = \text{const}, \ u_2(t) = \text{Const}, \ u_3 = \text{Const}.$

$\iff \frac{du_1(t)}{dt} = 0, \ \frac{du_2(t)}{dt} = 0, \ \frac{du_3(t)}{dt} = 0$

$\iff \frac{du_1(t)}{dt}\hat{i} + \frac{du_2(t)}{dt}\hat{j} + \frac{du_3(t)}{dt}\hat{k} = 0$

$\iff \frac{du(t)}{dt} = 0$

⟨ii⟩ $u(t)$ = constant magnitude

$|u(t)| = \text{const} \iff |u(t)|^2 = \text{const}$

$\iff u(t) . u(t) = \text{const}$

$\iff u(t) . \frac{du(t)}{dt} + \frac{du(t)}{dt} . u(t) = 0$

$\iff u(t) . \frac{du(t)}{dt} + u(t) . \frac{du(t)}{dt} = 0$

$\iff 2\, u(t) . \frac{du(t)}{dt} = 0$

$\iff u(t) . \frac{du(t)}{dt} = 0$

$\iff u . \frac{du}{dt} = 0$

(iii)

(iii) Let $u(t) = |u(t)|\, \hat{n}(t)$ $\quad$ $\hat{n}(t)$ = Unit vector in direction of $u(t)$

$$\frac{du(t)}{dt} = \frac{d(|u(t)|)}{dt}\hat{n}(t) + |u(t)|\frac{d\hat{n}(t)}{dt}$$

taking cross product with $u(t)$

$$u(t) \times \frac{du(t)}{dt} = \frac{d|u(t)|}{dt}\left[u(t) \times \hat{n}(t)\right] + |u(t)|\left[u(t) \times \frac{d\hat{n}(t)}{dt}\right]$$

$$= \frac{d|u(t)|}{dt}\left[|u(t)|\ \hat{n}(t) \times \hat{n}(t)\right] + |u(t)|\left[u(t) \times \frac{d\hat{n}(t)}{dt}\right]$$

$$= 0 + |u(t)|\left[u(t) \times \frac{d\hat{n}(t)}{dt}\right]$$

Now if $u(t)$ have constant direction then $\hat{n}(t)$ is not a function of t $\Rightarrow \frac{d\hat{n}(t)}{dt} = 0$

Thus $\boxed{u(t) \times \frac{du(t)}{dt} = 0}$

Scanned by CamScanner

2(i) If $r = (\sinh t)a + (\cosh t)b$ where a, b are constant

Show that $\frac{d^2r}{dt^2} = r$.

$r = (\sinh t)a + (\cosh t)b$

$\frac{dr}{dt} = (\cosh t)a + (\sinh t)b$

$\frac{d^2r}{dt^2} = (\sinh t)a + (\cosh t)b = r$

(ii) If $r = ae^{nt} + be^{-nt}$ where a, b are constant

Show that $\frac{d^2r}{dt^2} = n^2 r$.

$r = ae^{nt} + be^{-nt}$

$\frac{dr}{dt} = an \cdot e^{nt} - bne^{-nt}$

$\frac{d^2r}{dt^2} = an^2 e^{nt} + bn^2 e^{-nt}$

$= n^2 (ae^{nt} + be^{-nt})$

$= n^2 r$

(iii) $r = (\cos nt)\hat{i} + (\sin nt)\hat{j}$ s.t. $r \times \frac{dr}{dt} = nk$

$\frac{dr}{dt} = (-n\sin nt)\hat{i} + (n\cos nt)\hat{j}$

$$r \times \frac{dr}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix}$$

$= \hat{i}[0] - \hat{j}[0] + \hat{k}[n\cos^2 nt + n\sin^2 nt]$

$= n\hat{k}(\cos^2 nt + \sin^2 nt)$

$= n\hat{k}$

(3) $r = \cos(t-1)\,\hat{i} + \sinh(t-1)\,\hat{j} + \alpha t^3\,\hat{k}$

acceleration is normal to position vector

$\Rightarrow \quad \frac{d^2r}{dt^2}\cdot r = 0 \quad$ at $t=1$

$$\frac{dr}{dt} = -\sin(t-1)\,\hat{i} + \cosh(t-1)\,\hat{j} + 3\alpha t^2\,\hat{k}$$

$$\frac{d^2r}{dt^2} = -\cos(t-1)\,\hat{i} + \sinh(t-1)\,\hat{j} + 6\alpha t\,\hat{k}$$

$$\frac{d^2r}{dt^2}\cdot r = -\cos^2(t-1) + \sinh^2(t-1) + 6\alpha^2 t^4$$

at $t=1$ $\quad \frac{d^2r}{dt^2}\cdot r = 0$

$$-\cos^2(0) + \sinh^2(0) + 6\alpha^2 = 0$$

$$-1 + 0 + 6\alpha^2 = 0$$

$$6\alpha^2 = 1$$

$$\alpha = \pm 1/6$$

---

(4) $[a\ b\ c] = a\cdot(b\times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ $\quad a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$r = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j} + (bt)\,\hat{k}$

(i) $r^2 = r\cdot r = a^2\cos^2 t + a^2\sin^2 t + b^2t^2$

$$= a^2 + b^2t^2$$

(ii) $(r'\times r'')^2$ $\quad r' = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}$

$\quad r'' = -a\cos t\,\hat{i} - a\sin t\,\hat{j} + 0\,\hat{k}$

$$r'\times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \hat{i}\,[ab\sin t) - \hat{j}\,[ab\cos t) + \hat{k}\,[a^2]$$

$$(r'\times r'')^2 = (ab\sin t)^2 + (ab\cos t)^2 + a^4$$

$$= a^2b^2 + a^4$$

$$= a^2(a^2 + b^2$$

(iii) $r = (a\cos t)\hat{i} + (a\sin t)\hat{j} + (bt)\hat{k}$

$r' = (-a\sin t)\hat{i} + (a\cos t)\hat{j} + (b)\hat{k}$

$r'' = (-a\cos t)\hat{i} + (-a\sin t)\hat{j} + 0\hat{k}$

$r''' = (a\sin t)\hat{i} + (-a\cos t)\hat{j} + 0\hat{k}$

$$[r'\ r''\ r'''] = \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= b[a^2\cos^2 t + a^2\sin^2 t]$$

$$= a^2 b$$

---

(5). $\frac{d}{dt}[f\ f'\ f''] = [f\ f'\ f''']$

$$\frac{d}{dt}[f\ f'\ f''] = [\tfrac{df}{dt}\ f'\ f''] + [f\ \tfrac{df'}{dt}\ f''] + [f\ f'\ \tfrac{df''}{dt}]$$

$$= [f'\ f'\ f''] + [f\ f''\ f''] + [f\ f'\ f''']$$

$$= 0 + 0 + [f\ f'\ f''']$$

$$= [f\ f'\ f''']$$

---

(6) (i) If $\phi = 2xz^4 - x^2y$ find $\nabla\phi$ and $|\nabla\phi|$ at $(2,-2,1)$

$\nabla\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = (2z^4 - 2xy)\hat{i} - x^2\hat{j} + 8xz^3\hat{k}$

$\nabla\phi =$

$\nabla\phi|_{(2,-2,1)} = (2\cdot1^4 + 2(2)(+2))\hat{i} - 4\hat{j} + 8(2)(1)^3\hat{k}$

$= 10\hat{i} - 4\hat{j} + 16\hat{k}$

$|\nabla\phi| = \sqrt{10^2 + 4^2 + 16^2} = \sqrt{100 + 16 + 256} = \sqrt{372}$

$= 2\sqrt{93}$

(ii) If $\nabla\phi = (y+y^2+z^2)\hat{i} + (x+z+2xy)\hat{j} + (y+2xz)\hat{k}$

find ϕ s.t. $\phi(1,1,1) = 3$

$$\nabla\phi = \phi_1 \hat{i} + \phi_2 \hat{j} + \phi_3 \hat{k}$$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = y+y^2+z^2, \quad \frac{\partial\phi}{\partial y} = x+z+2xy, \quad \frac{\partial\phi}{\partial z} = y+2xz$$

$$\cancel{\phi = yx + y^2x + z^2x + c_1, \ \phi = xy + zy + xy^2 + c_2, \ \phi = zy + xz^2 + c_3}$$

$$\cancel{\phi(1,1,1) = 3}$$

$$\cancel{\Rightarrow 1+1+1+c_1 = 3 \Rightarrow c_1 = 0, \ c_2 = 0, \ c_3 = 1}$$

$$\cancel{\phi = xy + xy^2 + xz^2, \ \phi = xy + xy^2 + yz, \ \phi = yz + xz^2 + 1}$$

$$\cancel{xy + xy^2 + yz = yz + xz^2 + 1}$$

$$\cancel{xy + xy^2 = xz^2 + 1}$$

$$\Rightarrow \phi = yx + xy^2 + xz^2 + f_1(y,z) \quad \text{(i)}$$

Also $\phi = xy + zy + xy^2 + f_2(x,z)$ — (ii)

Also $\phi = yz + xz^2 + f_3(x,y)$ — (iii)

$\phi(1,1,1) = 3$

$\Rightarrow 3 = 1+1+1 + f_1(1,1)$

$\Rightarrow f_1(1,1) = 0$

$\Rightarrow 3 = 1+1+1 + f_2(1,1)$

$\Rightarrow f_2(1,1) = 0$

$3 = 1+1+ f_3(1,1)$

$\Rightarrow f_3(1,1) = 1$

<u>Comparing</u>

$$\phi = xy + xy^2 + xz^2 + yz - 1$$

(7) If $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad |\mathbf{r}| = r$ Show that

⟨i⟩ $\nabla r^n = n r^{n-1} \mathbf{r}$. $\quad |\mathbf{r}| = \sqrt{x^2+y^2+z^2} = r$

$$r^n = (x^2+y^2+z^2)^{n/2}$$

$$\nabla r^n = \left(\hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}\right)$$

$$= \left(\hat{i}\, n.r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, n.r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, n r^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= n r^{n-1}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$$

$$= n r^{n-2}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$= n r^{n-2}.\mathbf{r}$$

(ii) $$\nabla\left(\frac{1}{r}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \frac{-1}{r^2}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right] = \frac{-1}{r^3}\left[x\hat{i}+y\hat{j}+z\hat{k}\right]$$

$$= -\frac{\mathbf{r}}{r^3}$$

(iii) $\nabla f(r) = \frac{f'(r)}{r}\mathbf{r}$, $\quad \nabla f(r) \times \mathbf{r} = 0$

$$\nabla f(r) = \hat{i}\frac{\partial}{\partial x}f(r) + \hat{j}\frac{\partial}{\partial y}f(r) + \hat{k}\frac{\partial}{\partial z}f(r)$$

$$= \hat{i}f'(r)\frac{\partial r}{\partial x} + \hat{j}f'(r)\frac{\partial r}{\partial y} + \hat{k}f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\hat{i}.\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] = \frac{f'(r)}{r}\left[x\hat{i}+y\hat{j}+z\hat{k}\right]$$

$$= \frac{f'(r)}{r}\mathbf{r}$$

$$\nabla f(r) \times \mathbf{r} = \left(\frac{f'(r)}{r}\mathbf{r}\right)\times\mathbf{r} = \left(\frac{f'(r)}{r}\right)\mathbf{r}\times\mathbf{r}$$

$$= \left(\frac{f'(r)}{r}\right)\cdot(0) = 0$$

(IV) $\nabla[r \; a \; b] = \nabla(r \cdot (a\times b))$ $\qquad \nabla(r.(a\times b))$

$\cancel{\sum \hat{i}\frac{\partial}{\partial x}(r.(a\times b))} =$

$\hat{i}\cancel{\frac{\partial}{\partial x}(r.(a\times b)) + \hat{j}\frac{\partial}{\partial y}}(r.(a\times b))$

$\cancel{+\hat{k}\frac{\partial}{\partial z}(r.(a\times b))}$

$$= \nabla(r.(a\times b))$$

$$= \hat{i}\frac{\partial}{\partial x}(r.(a\times b)) + \hat{j}\frac{\partial}{\partial y}(r.(a\times b)) + \hat{k}\frac{\partial}{\partial z}(r.(a\times b))$$

$$= \hat{i}\left[\frac{\partial r}{\partial x}.(a\times b) + r.\frac{\partial}{\partial x}(a\times b)\right] + \hat{j}\left[\frac{\partial r}{\partial y}.(a\times b) + r.\frac{\partial}{\partial y}(a\times b)\right]$$
$$+ \hat{k}\left[\frac{\partial r}{\partial z}.(a\times b) + r.\frac{\partial}{\partial z}(a\times b)\right]$$

$$= \hat{i}\left[\hat{i}.(a\times b) + 0\right] + \hat{j}\left[\hat{j}.(a\times b) + 0\right] + \hat{k}\left[\hat{k}.(a\times b) + 0\right]$$

$$= \hat{i}\left[(a\times b).\hat{i}\right] + \hat{j}\left[(a\times b).\hat{j}\right] + \hat{k}\left[(a\times b).\hat{k}\right]$$

$$= (a\times b)$$

~~vector~~

---

8 (i) $\phi = x^2 - 2y^2 + 4z^2$ at $(1,1,-1)$ in the direction $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$

$$\nabla\phi = 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

at $(1,1,-1)$ $\nabla\phi = 2\hat{i} - 4\hat{j} - 8\hat{k}$

Directional derivative in the direction of $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$

$$= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (2\hat{i} - 4\hat{j} - 8\hat{k}) \cdot \frac{(2\hat{i} + 2\hat{j} + \hat{k})}{3}$$

$$= \frac{(4 - 8 - 8)}{3} = -4$$

(ii) $\phi = x^2(y+z)$ at $(1,1,0)$ in $\vec{a} = (2-0)\hat{i} + (-1-0)\hat{j} + (2-0)\hat{k}$

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$$\nabla\phi = 2x(y+z)\hat{i} + x^2\hat{j} + x^2\hat{k}$$

at $(1,1,0)$

$$\nabla\phi = 2\hat{i} + \hat{j} + \hat{k}$$

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (2\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(2\hat{i} - \hat{j} + 2\hat{k})}{3} = \frac{4 - 1 + 2}{3} = \frac{5}{3}$$

(iii) $\phi = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ $Q(5,0,4)$,

$\overrightarrow{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

$\nabla\phi = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$\nabla\phi$ at $(1,2,3) = 2\hat{i} - 4\hat{j} + 12\hat{k}$

$$\nabla\phi \cdot \frac{\overrightarrow{PQ}}{|PQ|} = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

(iv) $\phi = xy + 2yz + 3xz$ at $(1,1,1)$

$\nabla\phi = (y+3z)\hat{i} + (x+2z)\hat{j} + (2y+3x)\hat{k}$

$\nabla\phi(1,1,1) = 4\hat{i} + 3\hat{j} + 5\hat{k}$ —— (1)

Let $\vec{a}$ be the vector along which the directional derivative is maximum then

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \nabla\phi \cdot \vec{b} \qquad \vec{b} = \frac{\vec{a}}{|\vec{a}|}$$

$$= |\nabla\phi||\vec{b}| \cos\theta$$

$=$ maximum if $\cos\theta$ is max.

$\Rightarrow \theta = 0.$

$\Rightarrow \vec{b}$ and $\nabla\phi$ are along same line.

$\Rightarrow \vec{a}$ and $\nabla\phi$ are along same line

$\Rightarrow \vec{a} = (4\hat{i} + 3\hat{j} + 5\hat{k})$

Directional Derivative

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (4\hat{i} + 3\hat{j} + 5\hat{k}) \cdot \frac{(4\hat{i} + 3\hat{j} + 5\hat{k})}{\sqrt{16+9+25}} = \frac{50}{\sqrt{50}}$$

$$= \sqrt{50} = 5\sqrt{2}$$

(9) (1) $\phi \equiv xy + y^2 - z^2 - 5$

Unit vector normal to surface at (1,2,1) $= \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = y\hat{i} + (x+2y)\hat{j} - 2z\hat{k}$$

$$\nabla\phi\,(1,2,1) = 2\hat{i} + 5\hat{j} - 2\hat{k}$$

$$|\nabla\phi| = \sqrt{4+25+4} = \sqrt{33}$$

So $\dfrac{(2\hat{i} + 5\hat{j} - 2\hat{k})}{\sqrt{33}}$ is unit normal to surface at (1,2,1)

---

(11). Angle b/w the surfaces at a point is same as the angle b/w their normals at that point $\phi_1 \equiv x^2+y^2+z^2-9$ $\phi_2 = z-x^2-y^2+3$

Normal to surface ϕ_1 $= \nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Normal to surfae ϕ_2 $= \nabla\phi_2 = -2x\hat{i} - 2y\hat{j} + \hat{k}$

at point $(2,-1,2)$

$$\nabla\phi_1 = 4\hat{i} - 2\hat{j} + 4\hat{k} \qquad \nabla\phi_2 = -4\hat{i} + 2\hat{j} + \hat{k}$$

Angle: $\theta = \cos^{-1}\left(\dfrac{(\nabla\phi_1 \cdot \nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}\right)$

$$= \cos^{-1}\left(\frac{-16-4+4}{\sqrt{36}\,\sqrt{21}}\right) = \cos^{-1}\left(\frac{-16}{6\sqrt{21}}\right)$$

$$= \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

(17) (i) curl (curl f) = grad (div f) − $\nabla^2 f$

$$\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f$$

Let $f = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore \nabla \times f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$\therefore \nabla \times (\nabla \times f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) & \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) & \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial^2 f_2}{\partial y \partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x}\right] + \hat{j}\left[\frac{\partial^2 f_3}{\partial z \partial y} - \frac{\partial^2 f_2}{\partial z^2} - \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_1}{\partial x \partial y}\right] + \hat{k}\left[\frac{\partial^2 f_1}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_2}{\partial y \partial z}\right]$$

adding and subtracting $\frac{\partial^2 f_1}{\partial x^2}, \frac{\partial^2 f_2}{\partial y^2}, \frac{\partial^2 f_3}{\partial z^2}$ in all three brackets respectively.

$$= \hat{i}\left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2}\right] +$$

$$\hat{j}\left[\frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial y \partial z} - \frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial y^2} - \frac{\partial^2 f_2}{\partial z^2}\right] +$$

$$\hat{k}\left[\frac{\partial^2 f_1}{\partial z \partial x} + \frac{\partial^2 f_2}{\partial z \partial y} + \frac{\partial^2 f_3}{\partial z^2} - \frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2} - \frac{\partial^2 f_3}{\partial z^2}\right]$$

$$= \hat{i}\left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_3}{\partial x \partial z}\right] + \hat{j}\left[\frac{\partial^2 f_1}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_3}{\partial y \partial z}\right]$$

$$+ \hat{k}\left[\frac{\partial^2 f_1}{\partial z \partial x} + \frac{\partial^2 f_2}{\partial z \partial y} + \frac{\partial^2 f_3}{\partial z^2}\right] - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_1 \hat{i}$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_2 \hat{j} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f_3 \hat{k}$$

$$= \hat{i}\,\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + \hat{j}\,\frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)$$

$$+ \hat{k}\,\frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}\right)$$

$$= \nabla(\nabla \cdot f) - \nabla^2(\hat{i} f_1 + \hat{j} f_2 + \hat{k} f_3)$$

$$= \nabla(\nabla \cdot f) - \nabla^2 f$$

So $\boxed{\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f}$