Indian Institute of Technology Roorkee
MAN-001(Mathematics-1)
Autumn Semester: 2019-20
Assignment-10: Vector Calculus II
(Line and surface integrals, Green's, Gauss and Stokes's theorem and their applications)

1. Evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = x^2\,\hat{i} - xz\,\hat{j} + y^2\,\hat{k}$ along the path C joining the points $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1) \to (0,0,1)$ via straight lines.

2. Show that $\vec{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\,\hat{j} + (3xz^2 + 2)\,\hat{k}$ is a conservative vector field and find a function ϕ such that $\vec{F} = \nabla\phi$. Also, find the work done by a moving particle from $(0, 1, -1)$ to $(\pi/2, -1, 2)$.

3. If $\vec{F} = \left(\dfrac{x\,\hat{j}}{x^2+y^2} - \dfrac{y\,\hat{i}}{x^2+y^2}\right)$, evaluate $\int_C \vec{F}.d\vec{r}$ for the various curves C from $(0, 1)$ to $(1, 0)$ along

   (i) the arc of $x^2 + y^2 = 1$ lying in the second, third and fourth quadrant.

   (ii) $x + y = 1$.

   (iii) the arc of $x^2 + y^2 = 1$ lying in the first quadrant.

   Is the vector field $\vec{F}$ conservative? If so, find ϕ such that $\nabla\phi = \vec{F}$. Why is the line integral not path independent?

4. Evaluate the surface integral $\iint_S \vec{F}.\hat{n}\; dS$, if

   (i) $\vec{F} = yz\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and S is the surface of $x^2 + y^2 + z^2 = 1$ in the firstoctant.

   (ii) $\vec{F} = z\hat{i} + x\,\hat{j} - 3y^2z\,\hat{k}$ and S is the surface of $x^2 + y^2 = 16$ in the first octant between $z = 0$ and $z = 5$.

   (iii) $\vec{F} = \dfrac{\vec{r}}{r^3}$ and S is the surface of $x^2 + y^2 + z^2 = a^2$.

5. If $\vec{F} = 4xz\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$, evaluate the volume integral $\iiint_V \nabla.\vec{F}\,dV$ over the entire surface of the region above the *xy*- plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$.

6. Evaluate $\iiint_V \phi\; dV$, where $\phi = 45x^2y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$ and $z = 0$.

7. Evaluate $\iint_S (\nabla\times\vec{F}).\hat{n}\, dS$, where $\vec{F} = y^2\,\hat{i} + y\,\hat{j} - xz\,\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above xy- plane.

8. Verify Green's theorem for
   (i) $\oint_C [(xy^2-2xy)\,dx+(x^2y+3)\,dy]$ around the boundary curve C of the regionenclosed by $y^2=8x$ and $x=2$.
   (ii) $\oint_C [(xy+y^2)\,dx+x^2\,dy]$, C bounds the region enclosed by $y=x$ and $y=x^2$.
   (iii) $\oint_C [(3x^2-8y^2)\,dx+(4y-6xy)\,dy]$ and C bounds the region enclosed by $x=0$, $y=0$ and $x+y=1$.

9. By converting into the line integral, evaluate $\iint_S (\nabla\times\vec{F}).\hat{n}\, dS$,

   where $\vec{F}=(x-z)\hat{i}+(x^3+yz)\hat{j}-3xy^2\hat{k}$ and
   S is the surface of the cone $z=2-\sqrt{x^2+y^2}$ above xy-plane.

10. Verify Stoke's theorem for $\vec{F}=(x^2+y^2)\hat{i}-2xy\,\hat{j}$ taken around the rectangle bounded by the lines $x=\pm a$, $y=0$ and $y=b$.

11. Verify Gauss's divergence theorem for
    (i) $\vec{F}=(2x-z)\hat{i}-x^2y\,\hat{j}+4xz^2\,\hat{k}$ taken over the region bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.
    (ii) $\vec{F}=2x^2y\hat{i}-y^2\,\hat{j}+4xz^2\,\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$.

12. Evaluate $\iint_S \vec{F}.\vec{n}\,ds$ where $\vec{F}=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ and S is a rectangular parallelepiped $0\le x\le a, 0\le y\le b$ and $0\le z\le c$.

Answers

(1) 3/2 (2). $\phi = y^2\sin x + xz^3 - 4y + 2z$, $4\pi+15$

(3). $\dfrac{3\pi}{2}, \dfrac{\pi}{2}, -\dfrac{\pi}{2}$, $\phi=\tan^{-1}(y/x)$(4). (i) 3/8 (ii) 90(iii) 4π

(5). 320 π (6). 128(7). 0(9). 12π

(12) $abc(a+b+c)$