Game-theory

Game theory deals with the situations where there are conflicts of interests ~~btw~~ between persons. This is game of strategies that is which strategy a person should adopt for a possible move of other persons. Here, persons may be ~~people~~, group of people or companies. Each person/group is called a player.

In this, we will deal with ~~zero-sum~~ two person zero-sum game. [It is a game with only two players in which losses of one player are equal to the gain of another so that the sum of their net gain is zero.

→ strategy : list of all possible course of action (more) by a player.

(The first player may have different strategies for the same move of the second player).

→ Pay-off : The pay-off shows the gain (+ve or -ve) for player-I that would result from each combination of strategies ~~from~~ the two player.

'Two-person zero sum games' are also called 'rectangular games' because their pay-off's are given by a matrix.

→ Pure-strategy : If ~~change~~ chance does not determine any move & both players choose a particular strategy, then we have a deterministic situation. Strategies of such situation are called 'Pure-strategy'.

→ Mixed-strategy : when some of the moves are determined by chance the situation is probabilistic & the objective is to maximise the expected gain. In this case, 'mixed strategy'.

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← (Player B).

Player A's strategy ↓

$$\begin{array}{c|cccc} & B_1 & B_2 & \cdots & B_n \\ \hline A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & & \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}$$

A's – pay-off matrix
(Gain or loss of A).

$a_{ij} \rightarrow$ If Player A chooses ith strategy & Player B chooses jth strategy, then the pay-off of A from B is a_{ij}. (if $a_{ij} > 0 \rightarrow$ Gain to A & loss of B
$a_{ij} < 0 \rightarrow$ Loss to A & gain to B)
$a_{ij} = 0 \rightarrow$ (neither loss nor gain to both player A & B)

Minimax (Maximin) Criterion :- Consider the pay-off matrix of the player A. If player A chooses ith strategy, he is sure of getting $\min_j a_{ij}$, j varies over the strategies of B. Then, naturally player A will choose that strategy which will maximize the mini. ~~game~~ gain. With this, in case there is loss to A, it is least. Thus A will choose or opt

$$\max_i \min_j a_{ij} = \underline{a} \text{ (say)}$$

On the other side, B will choose the strategy to minimize the maxi. gain to A. (he is sure that he will not loose more then $\max_i a_{ij}$ (which he will try to minimize). Therefore, the player B will choose or opt :

$$\min_j \max_i a_{ij} = \overline{a} \text{ (say)}.$$

If $\underline{a} = \overline{a} = a_{st}$, then the given game is said to have a saddle pt. 'a_{st}' is called 'value of the game'. (Pure-strategy).
(This is the case of pure strategy).

Example 1

	B_1	B_2	B_3	B_4	
A_1	0	0	-2	5	-2
A_2	1	2	3	1	1
A_3	-3	-4	2	0	-4
A_4	1	3	-2	4	-2
A_5	0	1	-1	2	-1
	1	3	3	5	

maximin: $\max_i \min_j a_{ij}$

minimax: $\min_j \max_i a_{ij}$

$a_{21} = 1$ = value of the game.

(The best strategy for player A is A_2

" " " B is B_1)

RESULT:

$$a_{pq} = \max_i \min_j a_{ij} \leq \min_j \max_i a_{ij} = a_{rs}$$

Proof: $\max_i a_{ij} \geq a_{ij}$, for any i & j fixed.

$\min_j (a_{ij}) \leq a_{ij}$, for any j & i fixed,

let $\max_i a_{ij} = a_{rj}$ & $\min_j a_{ij} = a_{iq}$

$\therefore \; a_{rj} \geq a_{ij} \geq a_{iq} \quad \forall i \,\&\, j$

Thus, $\min_j (a_{rj}) \geq a_{ij} \geq \max_i (a_{iq}) \quad \forall i \,\&\, j$

Hence Proved. □

Ex. 2

0	0	-2	5	-2
4	2	3	5	2
-3	-4	4	0	-4
1	3	-2	4	-2
0	1	-1	2	-1
4	3	4	5	

...le of dominance : It helps to reduce the size of the matrix.

- The principle of dominance states that if one strategy of a player dominates over the other strategy in all conditions then the later strategy can be ignored.
~~A strategy dominates over the others only if it is preferable over other in all conditions.~~

- If all the elements of a column (say i^{th} col.) are greater than or equal to the corresponding elements of any other column (say j^{th} col.), then the i^{th} column is dominated by j^{th} column & can be deleted from the matrix.

- If all the element of a raw (say i^{th} raw) are less than or equal to the corresponding elements of any other raw (say j^{th} raw), then the i^{th} raw is dominated by j^{th} raw & can be deleted from the matrix.

		B_1	B_2	B_3	B_4	← Player B ↓
Player A ↓	A_1	3	5	4	2	2
	A_2	5	6	2	4	2
	A_3	2	1	4	0	0
	A_4	3	3	5	2	2
		5	6	5	4	

$2 \leq v \leq 4$

v = value of the game.

No-saddle point

All the elements of Raw A_4 are $\geq$ corresponding elements of Raw A_3

$\Rightarrow$ Raw A_3 can be deleted. Now, we get

	B_1	B_2	B_3	B_4
A_1	3	5	4	2
A_2	5	6	2	4
A_4	3	3	5	2

All the elements of B_2 are $\geq$ corresponding elements of B_1 $\Rightarrow$ Column B_2 can be deleted. Hence,

	B_1	B_3	B_4
A_1	3	4	2
A_2	5	2	4
A_4	3	5	2

Again, $A_4 \geq A_1$ (A_1 can be deleted).

	B_1	B_3	B_4
A_2	5	2	4
A_4	3	5	2

Again $B_1 \geq B_4$ $\Rightarrow$ delete B_4

	B_3	B_4
A_2	2	4
A_4	5	2

...ebric - Method

		B_3	B_4
A_2	p	2	4
A_4	$1-p$	5	2
		q	$1-q$

$p \to$ probability with which player A play strategy A_2

$q \to$ prob. with which the player B play strategy B_3

$E_i \to$ expected return to player A for i^{th} strategy.

$A_2 \to E_1 = 2p + 5(1-p) = -3p+5 \to$ Expected pay-off.

$A_4 \to E_2 = 4p + 2(1-p) = 2p+2.$

$-3p+5 = 2p+2 \Rightarrow 5p = 3 \Rightarrow \underline{p = 3/5} \to$ for A_2

$\left(0, \frac{3}{5}, 0, \frac{2}{5}\right) \to$ A player.

Value of the game: $-3 \times \frac{3}{5} + 5 = \frac{25-9}{5} = \frac{16}{5}.$

$2q + 4(1-q) = \frac{16}{5} \Rightarrow -2q + 4 = \frac{16}{5}$

$-2q = \frac{16}{5} - 4 = -\frac{4}{5}$

$q = 2/5$

B-Player : $\left(0, 0, \frac{2}{5}, \frac{3}{5}\right).$

2×2

	q	$1-q$
p	a	b
$1-p$	c	d

$ap + c(1-p) = bp + d(1-p)$

$p(a-c-b+d) = -c+d$

$$\boxed{p = \frac{d-c}{a-b-c+d}}$$

$aq + b(1-q) = cq + d(1-q)$

$q(a-b-c+d) = d-b$

$$\boxed{q = \frac{d-b}{a-b-c+d}}$$

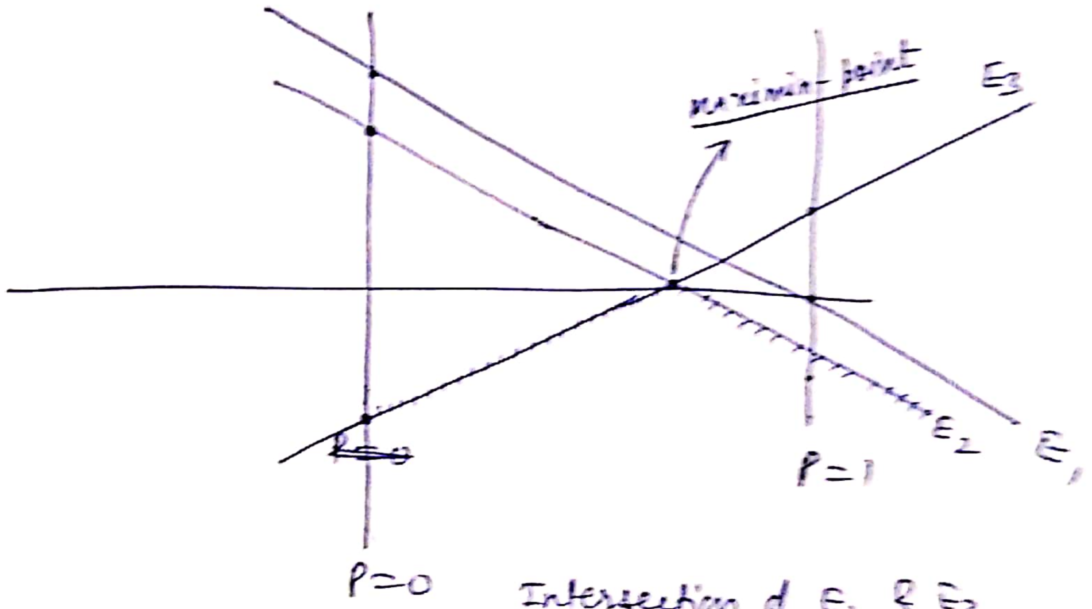
ical Approach (for 2 x m or n x 2 rectangular games).

		Player 2		
		q_1	q_2	q_3
Player 1	p	0	-2	2
	$1-p$	~~4~~ 3	4	-3

$E_1 = 3(1-p) = 3-3p \rightarrow$ B_1 (Pure strategy)
$E_2 = -2p+4(1-p) = -6p+4 \rightarrow$ B_2 (P
$E_3 = 2p-3(1-p) = 5p-3 \rightarrow$ B_3 (Pure str)

Expected pay-off for Player 1 (w.r.t different strategies for Player 2).

$0 \le p \le 1.$



Intersection of E_2 & E_3

$-6p+4 = 5p-3$

$11p = 7$

$\boxed{p = \frac{7}{11}}$

A's strategy

$\left(\frac{7}{11}, \frac{4}{11}\right).$

B's strategy.

$\left(0, \frac{5}{11}, \frac{6}{11}\right)$

$q_1 = 0$; $-2q_2 + 2q_3 = -6 \times \frac{7}{11} + 4 = \frac{2}{11}$

$-q_2 + q_3 = \frac{1}{11}$ — (1)

$4q_2 - 3q_3 = \frac{2}{11}$ — (2)

solving q_2 & q_3 can be find.

Programming approach : $m \times n$ Rectangular two person zero-sum game

Let p_i ($i = 1$ to m) be the probability that player A selects the strategy A_i ($i = 1$ to m) & let q_j ($j = 1$ to n) be the prob. that player B selects the strategy B_j ($j = 1$ to n).

$$\sum_{i=1}^{m} p_i = 1, \quad p_i \geq 0, \quad i = 1 \text{ to } m$$

$$\sum_{j=1}^{n} q_j = 1; \quad q_j \geq 0, \quad j = 1 \text{ to } n.$$

If B selects pure strategy B_j, then expected pay-off to A is : $\sum_{i=1}^{m} a_{ij} p_i$

$$\begin{array}{c|cccc|} & q_1 & q_2 & \cdots & q_n \\ \hline p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ p_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & & \\ p_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline \end{array}$$

Player B can select any of the pure strategies B_j ($j = 1$ to n), hence A will like to select the p_i's which maximize the smallest expected pay off to A. Thus, A's problem is :

$$\underset{p_1, p_2 \ldots p_m}{\text{Max}} \left\{ \min \left(\sum_{i=1}^{m} a_{i1} p_i, \sum_{i=1}^{m} a_{i2} p_i, \ldots, \sum_{i=1}^{m} a_{in} p_i \right) \right\},$$

s/t $\quad \sum_{i=1}^{m} p_i = 1, \quad p_i \geq 0, \quad i = 1$ to m.

Similarly, the player B will select q_j's which minimize the largest expected pay-off to A. Thus,

B's - problem is :

$$\underset{q_1, q_2 \ldots q_n}{\text{Min}} \left\{ \max \left(\sum_{j=1}^{n} a_{1j} q_j, \sum_{j=1}^{n} a_{2j} q_j, \ldots, \sum_{j=1}^{n} a_{mj} q_j \right) \right\}$$

s/t $\quad \sum_{j=1}^{n} q_j = 1, \quad q_j \geq 0, \quad j = 1$ to n.

It is to be noted that these problems are NOT LPP. So, how to convert these problems into LPP ??

Let $\min \left\{ \sum_{i=1}^{m} a_{i1} p_i, \sum_{i=1}^{m} a_{i2} p_i, \ldots, \sum_{i=1}^{m} a_{im} p_i \right\} = y$

Then, A's problem will be :

$$\begin{aligned} \text{Max} \quad & y \\ \text{s/t} \quad & \sum_{i=1}^{m} a_{ij} p_i \geq y, \quad j = 1, 2, \ldots, n. \\ & p_1 + p_2 + \cdots + p_m = 1, \\ & p_i \geq 0, \quad \forall i \end{aligned}$$

Let $y > 0$. & assume $\frac{p_i}{y} = x_i$. Then, the above problem will be :

$$\left. \begin{aligned} \text{Min} \quad & \sum_{i=1}^{m} x_i \\ \text{s/t} \quad & \sum_{i=1}^{m} a_{ij} x_i \geq 1, \quad j = 1, 2, \ldots, n \\ & x_i \geq 0, \quad \forall i \end{aligned} \right] \quad (A)$$

Similarly B's problem will be :

$$\left. \begin{aligned} \text{Max} \quad & \sum_{i=1}^{n} z_i \\ \text{s/t} \quad & \sum_{j=1}^{n} a_{ij} z_j \leq 1, \quad i = 1, 2, \ldots, m \\ & z_j \geq 0, \quad \forall j \end{aligned} \right] \quad (B)$$

Problems (A) & (B) are dual to each other. So, now the problem can be solved by using any LPP algorithms.