Signals & Systems (ECN-203)

Lecture 6
(Fourier series representation of periodoc signals)

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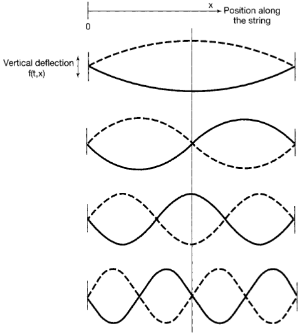
Introduction

- Convolution sum representation and analysis of LTI systems:
  - Represent signals as linear combinations of shifted impulses
- In this chapters, we explore an alternative representation for signals and LTI systems
  - Complex exponentials
    - Fourier series and transform

A historical perspective

- Development of Fourier analysis has a long history involving several individuals and study of various physical phenomena
- Babylonians use "trigonometric sums" to describe periodic phenomena in order to predict astronomical events
  - 2000-1600 BC

- Modern history: L. Euler examined the motion of a vibrating string in 1748
- The "normal modes" are harmonically related sinusoidal functions of x
- Euler predicted configuration of a vibrating string as a linear combination of these normal modes



A historical perspective

- This property isn't particularly useful unless it is generalized
  - i.e., unless a large class of interesting functions could be represented by linear combinations of complex exponentials
- Bernoulli supported this on physical grounds
  - But did not pursue this mathematically
- Lagrange strongly criticized this
  - Based on his belief that it was impossible to represent signals with corners (discontinuous slopes) using trigonometric series
- Fourier presented his ideas in 1768
  - Based on the properties of heat propagation and diffusion
- Fourier took this type of representation one very large step farther:
  - Represented aperiodic signals as weighted integrals of sinusoids that are *not* all harmonically related

Applications of Fourier’s work

- Fourier’s work has enormous impact on so many disciplines within the fields of mathematics, science, and engineering
  - Theory of integration
  - Point-set topology
  - Eigen function expansions
- Application areas:
  - Motion of the planets and the periodic behavior of the earth’s climate
  - Alternating-current sources
  - Waves in the ocean
  - Signals transmitted by radio and television stations

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Why represent a signal as a sum of other signa

- Advantageous to represent signals as linear combinations of basic signals that possess the following two properties:
  - The set of basic signals can be used to construct a broad and useful class of signals
  - Response of an LTI system to each signal should be simple
    - Convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals
- Both of these properties are fulfilled by the set of complex exponential signals in continuous and discrete time
  - e^{st} in continuous time and z^n in discrete time
  - s and z are complex numbers

The second property

- Response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude
  - $e^{st} \rightarrow H(s)e^{st}$
  - $z^n \rightarrow H(z)z^n$
  - $H(s)$ and $H(z)$ are complex functions of s and z respectively
- This is the Eigenvalue-Eigenfunction problem
  - e^{st} and z^n are Eigenfunctions
  - $H(s)$ and $H(z)$ are Eigenvalues

Complex exponential as an input to continuous-time LTI system

- Impulse response: $h(t)$
- Input: $x(t) = e^{st}$
- Output:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\
&= H(s)e^{st}
\end{aligned}
$$

- $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$
- Assumption: This integral converges

Complex exponential as an input to discrete-time LTI system

- Impulse response: $h[n]$
- Input: $x[n] = z^n$
- Output:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \\
&= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} \\
&= H(z)z^n
\end{aligned}
$$

- $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$
- Assumption: This sum converges

Sum of complex exponentials as an input to a LTI system

- Let the input to a continuous-time LTI system is represented as a linear combination of complex exponentials
  - $x(t) = \sum_k a_k e^{s_k t}$
- Output: $y(t) = \sum_k a_k H(s_k) e^{s_k t}$
- If the input to a discrete-time LTI system is represented as a linear combination of complex exponentials
  - $x[n] = \sum_k a_k z_k^n$
- Output: $y[n] = \sum_k a_k H(z_k) z_k^n$

Example

- Consider a time-shift LTI system
  - $y(t) = x(t-3)$
- $x(t) = e^{j2t} \rightarrow y(t) = e^{j2(t-3)} = e^{-6j}e^{j2t}$
  - Eigen function = e^{j2t}
  - Eigen value, $H(2j) = e^{-6j}$
- Impulse response, $h(t) = \delta(t-3) \rightarrow$
  $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}$
- $x(t) = cos(4t) + cos(7t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$
-

$$
\begin{aligned}
y(t) &= \frac{1}{2}e^{-3\times j4}e^{j4t} + \frac{1}{2}e^{-3\times -j4}e^{-j4t} + \frac{1}{2}e^{-3\times j7}e^{j7t} + \frac{1}{2}e^{-3\times -j7}e^{-j7t} \\
&= \frac{1}{2}e^{j4(t-3)} + \frac{1}{2}e^{-j4(t-3)} + \frac{1}{2}e^{j4(t-7)} + \frac{1}{2}e^{-j4(t-7)} \\
&= cos(4(t-3)) + cos(7(t-3))
\end{aligned}
$$

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Linear combinations of harmonically related complex exponentials

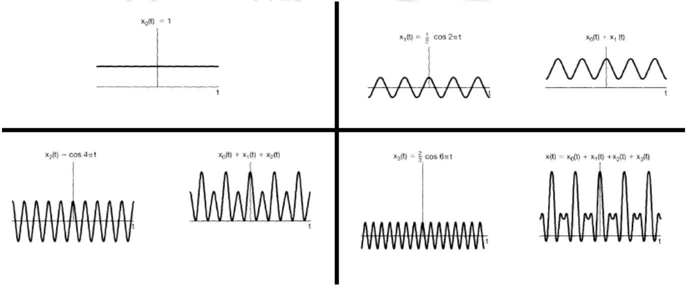
- A signal $x(t)$ is periodic with period T if:
  - $x(t+T) = x(t), \forall\, t$
  - $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency
- Complex exponential $e^{j\omega_0 t}$ is periodic with fundamental frequency ω_0 and fundamental period $T = \frac{2\pi}{\omega_0}$
- Harmonically related complex exponentials of $e^{j\omega_0 t}$ are $\phi_k(t) = e^{jk\omega_0 t} = e^{jk\frac{2\pi}{T}t}, k = 0, \pm 1, \pm 2, ...$
  - Fundamental frequency of these exponentials is a multiple of ω_0
  - Each is periodic with period T (although not the fundamental period)
  - $k = \pm N \rightarrow N^{th}$ harmonic components
- Linear combination of harmonically related complex exponentials: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$
  - Periodic with period T
  - This is the Fourier series representation

Example

- $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$
  - $a_0 = 1,\ a_1 = a_{-1} = \frac{1}{4},\ a_2 = a_{-2} = \frac{1}{2},\ a_3 = a_{-3} = \frac{1}{3}$
-

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

$$= 1 + \frac{1}{2}cos(2\pi t) + cos(4\pi t) + \frac{2}{3}cos(6\pi t)$$



Fourier series of real periodic signals

- Let signal $x(t)$ is real and periodic with period T
  - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- $x(t) = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$
- Substitute $k = -k$
  - $x^*(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
  - $a_{-k}^* = a_k \rightarrow a_{-k} = a_k^*$
- $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty}(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega+0t})$$
  $$= a_0 + \sum_{k=1}^{\infty}(a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega+0t}) = a_0 + \sum_{k=1}^{\infty} 2\mathbb{R}e(a_k e^{jk\omega_0 t})$$
- $a_k = A_k e^{j\theta_k}$
  - $x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathbb{R}e(A_k e^{j\theta_k} e^{jk\omega_0 t}) = a_0 + \sum_{k=1}^{\infty} 2A_k cos(k\omega_0 t + \theta_k)$

Determining Fourier series coefficients

- $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- $x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$
- $$\begin{aligned}
\int_0^T x(t)e^{-jn\omega_0 t} dt &= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T (cos((k-n)\omega_0 t) dt \right. \\
&\quad \left. + j \int_0^T sin((k-n)\omega_0 t)) dt \right]
\end{aligned}$$

Determining Fourier series coefficients

- If $k \neq n$
  - $cos(k-n)\omega_0 t$ and $sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period $\frac{T}{|k-n|}$
  - Integral from 0 to T consists of integer number of periods
  - $\int_0^T (cos((k-n)\omega_0 t)dt = \int_0^T sin((k-n)\omega_0 t)dt = 0$
- If $k = n$
  - $\int_0^T (cos((k-n)\omega_0 t)dt = \int_0^T 1dt = T$
  - $\int_0^T (sin((k-n)\omega_0 t)dt = \int_0^T 0dt = 0$
-
$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$
- $a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$

Fourier series summary

- Synthesis equation
  - $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$
- Analysis equation:
  - $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}dt$

Example 1

- $x(t) = sin(\omega_0 t)$
  - $x(t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$
  - $$a_k = \begin{cases} \frac{1}{2j}, & k = 1 \\ -\frac{1}{2j}, & k = -1 \\ 0, & Otherwise \end{cases}$$

Example 2

- $x(t) = 1 + sin(\omega_0 t) + 2cos(\omega_0 t) + cos(2\omega_0 t + \frac{\pi}{4})$
- $$x(t) = 1 + \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) + (e^{j\omega_0 t} + e^{-j\omega_0 t})$$
  $$+ \frac{1}{2j}(e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})})$$
- $x(t) = 1 + (1 + \frac{1}{2j})e^{j\omega_0 t} + (1 - \frac{1}{2j})e^{-j\omega_0 t} + \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j(2\omega_0 t)} + \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j(2\omega_0 t)}$
- $$a_k = \begin{cases} 1, & k = 0 \\ 1 + \frac{1}{2j} = 1 - \frac{1}{2}j, & k = 1 \\ 1 - \frac{1}{2j} = 1 + \frac{1}{2}j, & k = -1 \\ \frac{1}{2j}e^{j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1 + j), & k = 2 \\ \frac{1}{2j}e^{-j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1 - j), & k = -2 \\ 0, & Otherwise \end{cases}$$

Example 2

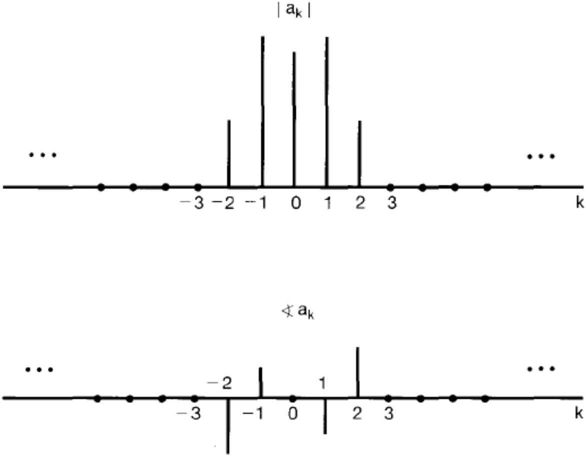
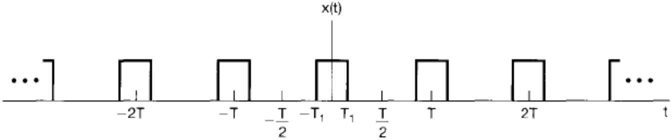


Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

Example 3

- $x(t)$ ia a periodic square wave defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$



- $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}$

- $a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{-1}{jk\omega_0 T}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right) = \frac{sin(k\omega_0 T_1)}{k\pi}$

Example 3

- $T = 4T_1 \rightarrow x(t)$ is a square wave that is unity for half the period and zero for half the period
-

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{\pi}, & k = 1, -1 \\ \frac{-1}{3\pi}, & k = 3, -3 \\ \frac{1}{5\pi}, & k = 5, -5 \\ \ldots & \\ 0, & Otherwise \end{cases}$$

Example 3

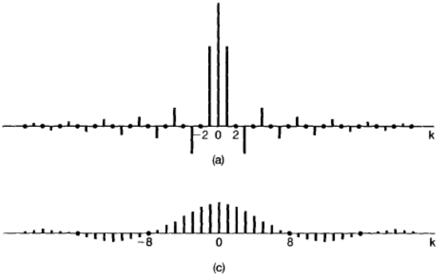


Figure 3.7 Plots of the scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: (a) $T = 4T_1$; (b) $T = 8T_1$; (c) $T = 16T_1$. The coefficients are regularly spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

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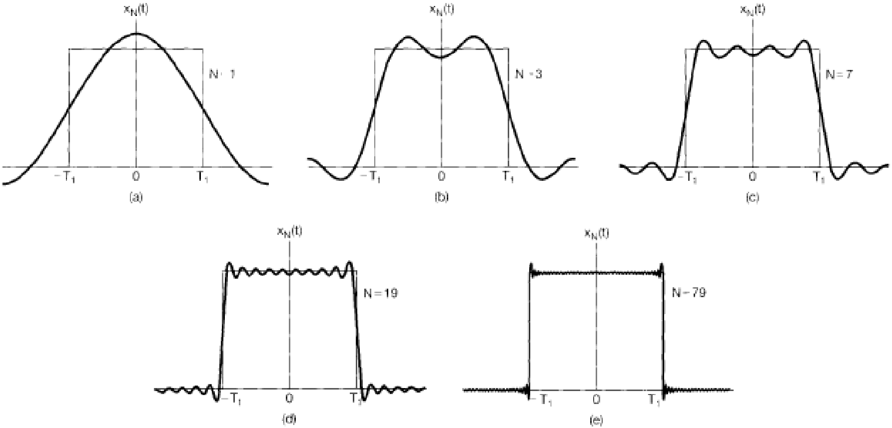
Square wave Fourier series example

- $x(t)$ is discontinuous while each of its harmonic components is continuous
  - Is it possible to represent a discontinuous function as a sum of continuous functions?
- Consider the problem of approximating a given periodic signal by a linear combination of *finite* number of harmonically related complex exponentials
  - $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$
  - Approximation error, $e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$
  - Energy in the error over one period, $E_N = \int_T |e_N(t)|^2 dt$
- It can be shown that use of Fourier coefficients as a_ks minimizes E_N

Square wave Fourier series example



Convergence of Fourier series

- A periodic signal $x(t)$ have a Fourier series representation if all $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ are finite
  - i.e., the integral does not diverge
- In some cases, even if all a_ks are finite, the resulting infinite series may not converge to the original signal $x(t)$
- If $\lim_{N\to\infty} E_N = 0$, $x(t)$ has a Fourier series representation
- Periodic signals having finite energy over a single period, $\int_T |x(t)|^2 dt < \infty$ are representable through Fourier series

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Linearity

- Fourier series coefficients of $x(t)$ are denoted by a_k
  - Notation: $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k$
- Let $x(t)$ and $y(t)$ are two periodic signals, both with period T
  - $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k$, and $y(t) \overset{\textbf{FS}}{\longleftrightarrow} b_k$
- $z(t) = Ax(t) + By(t) \overset{\textbf{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$

Time shift

- If $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k$, then $y(t) = x(t - t_0) \overset{\textbf{FS}}{\longleftrightarrow} = b_k = ?$
- $b_k = \frac{1}{T}\int_T y(t)e^{-jk\omega_o t}dt = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_o t}dt$
- Substitute $\tau = t - t_0$
  - $b_k = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_o(\tau + t_0)}d\tau = e^{-jk\omega_o t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_o \tau}d\tau = e^{-jk\omega_o t_0}a_k$
- $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k \rightarrow x(t - t_0) \overset{\textbf{FS}}{\longleftrightarrow} e^{-jk\omega_o t_0}a_k$
  - When a periodic signal is shifted in time, the magnitudes of its Fourier series coefficients remain unaltered

Time reversal

- If $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k$, then $y(t) = x(-t) \overset{\textbf{FS}}{\longleftrightarrow} = b_k = ?$
- $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$
- Substitute $k = -m$
  - $x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$
  - $b_k = a_{-k}$
- $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k \rightarrow x(-t) \overset{\textbf{FS}}{\longleftrightarrow} a_{-k}$
- If $x(t)$ is even, i.e., $x(-t) = x(t) \rightarrow a_{-k} = a_k$
- If $x(t)$ is odd, i.e., $x(-t) = -x(t) \rightarrow a_{-k} = -a_k$

Time scaling

- Time scaling changes the period of the underlying signal
- $x(t) \rightarrow$ period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$
  - $x(\alpha t) \rightarrow$ period $\frac{T}{\alpha}$ and fundamental frequency $\alpha\omega_0$
- Time-scaling applies directly to each of the harmonic components of $x(t)$
  - Fourier coefficients for each of those components remain the same
- $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$
  - Fourier series representation has changed because of the change in the fundamental frequency

Multiplication of two signals

- Let $x(t)$ and $y(t)$ are both periodic with period T
  - $x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$, and $y(t) \overset{\text{FS}}{\longleftrightarrow} b_k$
- $x(t) \times y(t) \overset{\text{FS}}{\longleftrightarrow} c_k = ?$
- $$x(t) \times y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \times \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}$$
  $$= \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_l b_{k-l} \right) e^{jk\omega_0 t}$$
- $c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$
  - Discrete-time convolution of the sequence representing the Fourier coefficients of $x(t)$ and $y(t)$

Conjugation and Conjugate Symmetry

- If $x(t) \overset{\textbf{FS}}{\longleftrightarrow} a_k$, then $y(t) = x^*(t) \overset{\textbf{FS}}{\longleftrightarrow} = b_k = ?$
- $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$
- Substitute $k = -m$
  - $x^*(t) = \sum_{m=-\infty}^{\infty} a_{-m}^* e^{jm\omega_0 t}$
  - $b_k = a_{-k}^*$
- If $x(t)$ is real
  - $x^*(t) = x(t) \rightarrow a_k = a_{-k}^* \rightarrow a_{-k} = a_k^*$
  - Fourier series coefficients will be conjugate symmetric
- if $x(t)$ is real and even
  - $a_{-k} = a_k^*$ and $a_{-k} = a_k \rightarrow a_k = a_k^*$ (real and even coefficients)
- if $x(t)$ is real and odd
  - $a_{-k} = a_k^*$ and $a_{-k} = -a_k \rightarrow a_k = -a_k^*$ (purely imaginary and odd coefficients)

Parseval's relation

- Average power (i.e., energy per unit time) of the periodic signal $x(t)$ is given by:
  -

$$P = \frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt$$

$$= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} a^*_{-m} e^{jm\omega_0 t}\right) dt$$

$$= \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} a_l a^*_{k+l}\right) e^{jk\omega_0 t} dt$$

  - $e^{jk\omega_0 t}$ is periodic with period T for all values of k, hence $\int_T e^{jk\omega_0 t} = 0$
  - Except for $k = 0$, for which it is constant
  - $P = \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty} a_l a_l^* dt = \sum_{l=-\infty}^{\infty} |a_l|^2$
- $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

Summary of properties

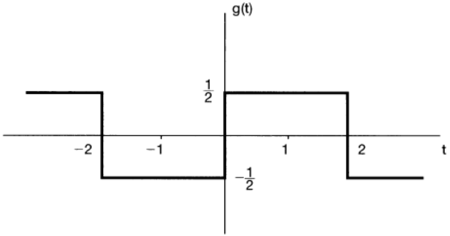
Property	Periodic signal	Fourier series coefficients
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0}$
Frequency shifting	$e^{jM\omega_0 t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t)$, $\alpha > 0$ (periodic with period $\frac{T}{\alpha}$)	a_k
Periodic convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$

Summary of properties

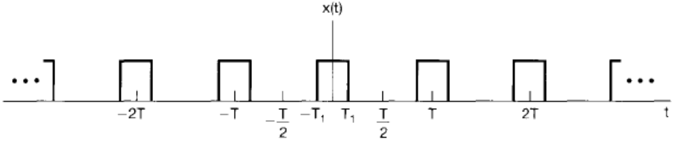
Property	Periodic signal	Fourier series coefficients
Differentiation	$\frac{d}{dt}x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$ (finite valued and periodic only if $a_0 = 0$)	$\frac{1}{jk\omega_0} a_k$
Conjugate symmetry for real signals	$x(t)$ is real	$a_k = a^*_{-k}$
	$x(t)$ is real and even	a_k real and even
	$x(t)$ is real and odd	a_k is purely imaginary and odd
Even-odd decomposition	$\begin{cases} x_e(t) = \frac{x(t)+x(-t)}{2} \\ x_0(t) = \frac{x(t)-x(-t)}{2} \end{cases}$	$\begin{cases} \mathbb{R}e\{a_k\} \\ j\mathbb{I}m\{a_k\} \end{cases}$
Parseval's relation	$\frac{1}{T}\int_T \|x(t)\|^2 dt = \sum_{k=-\infty}^{\infty} \|a_k\|^2$	

Example 1

- Consider the signal $g(t)$ with a fundamental period of 4
- Determine the Fourier series representation of $g(t)$



- Remember $x(t)$ from Slide 25? $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$



- $a_0 = \frac{2T_1}{T}$, $a_k = \frac{sin(k\omega_0 T_1)}{k\pi}$ for $k \neq 0$

Example 1

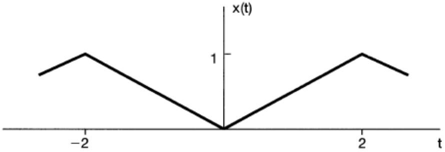
- Can we relate $g(t)$ with $x(t)$?
  - With $T = 4(\omega_0 = \frac{2\pi}{4})$ and $T_1 = 1$, $g(t) = x(t-1) - \frac{1}{2}$
- Fourier Series coefficients of $x(t-1)$ are $b_k = a_k e^{-jk\frac{\pi}{2}}$
  - Time shift property
- Fourier series coefficients of the dx offset term $\frac{-1}{2}$ are:

$$c_k = \begin{cases} 0, & k \neq 0 \\ \frac{-1}{2}, & k = 0 \end{cases}$$

- Using linearity property, coeffecients of $g(t) = x(t-1) - \frac{1}{2}$ are:
  - $$d_k = \begin{cases} a_k e^{-jk\frac{\pi}{2}}, & k \neq 0 \\ a_0 - \frac{1}{2}, & k = 0 \end{cases} = \begin{cases} \frac{\sin(k\frac{2\pi}{4})}{k\pi} e^{-jk\frac{\pi}{2}}, & k \neq 0 \\ 0, & k = 0 \end{cases}$$

Example 2

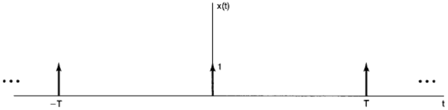
- Consider the triangular wave signal $x(t)$ with period $T = 4$
- Determine its Fourier series representation



- $x(t)$ is the integral of $g(t)$ in the previous example
  - $x(t) = \int_{-\infty}^{t} g(\tau)d\tau$
- Fourier series coefficients of $x(t)$, $e_k = \frac{1}{jk\frac{\pi}{2}} d_k$
- $e_k = \frac{2sin(k\frac{\pi}{2})}{jk^2\pi^2} e^{-jk\frac{\pi}{2}}, k \neq 0$
- $e_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{T} \times \frac{1}{2} \times T \times 1 = \frac{1}{2}$

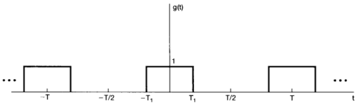
Example 3

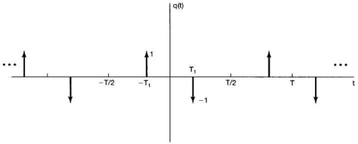
- Consider a periodic train of impulses (impulse train)
  - $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$



- $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$
  - All Fourier series coefficients of the impulse train are identical

- Consider the square-wave signal $g(t)$ and its derivative $q(t)$
- $q(t) = x(t + T_1) - x(t - T_1)$





Example 3

- Fourier series coefficients b_k of $q(t)$ are:
  $b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k = (e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}) a_k = \frac{2jsin(k\omega_0 T_1)}{T}$
  - Using time-shifting and linearity properties
- Since $g(t)$ is integral of $q(t)$, the Fourier series coefficients c_k of $g(t)$ are:
  - $c_k = \frac{1}{jk\omega_0} b_k = \frac{2jsin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{sin(k\omega_0 T_1)}{\pi k}, K \neq 0$
- c_0 is the average value of $g(t)$ over one period
  - $c_0 = \frac{2T_1}{T}$

Example 4

- Following facts about a signal $x(t)$ are known
  1. $x(t)$ is a real signal
  2. $x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k
  3. $a_k = 0$ for $|k| > 1$
  4. The signal with Fourier coefficients $b_k = e^{-jk\frac{\pi}{2}} a_{-k}$ is odd
  5. $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$
- Determine the signal $x(t)$
- Fact 2 $\rightarrow \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$
- Fact 3 $\rightarrow x(t) = a_0 + a_1 e^{j\frac{\pi}{2}t} + a_{-1} e^{-j\frac{\pi}{2}t}$
- Fact 1 $\rightarrow a_1 = a_{-1}^* \rightarrow x(t) = a_0 + 2\mathbb{R}e\{a_1 e^{j\frac{\pi}{2}t}\}$

Example 4

- Fact 4 $\rightarrow$
  - a_{-k} corresponds to the signal $x(-t)$
  - $b_k = e^{-jk\frac{\pi}{2}} a_{-k} = e^{-jk\omega_0} a_{-k}$ correspond to the signal $x(-(t-1)) = x(-t+1)$
  - According to fact 4, $x(-t+1)$ is odd
  - Since $x(t)$ is real, $x(-t+1)$ should also be real
  - Real and odd $x(-t+1) \rightarrow$ purely imaginary and odd b_ks
    - $b_0 = 0$, and $b_{-1} = -b_1$
- Fact 5 $\rightarrow$
  - Time-reversal and time-shift does not change average power
  - $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{4}\int_4 |x(-t+1)|^2 dt = \frac{1}{2}$
  - Using Parseval's relation: $|b_1|^2 + |b_{-1}|^2 = |b_1|^2 + |-b_1|^2 = 2|b_1|^2 = \frac{1}{2} \rightarrow b_1 = \frac{j}{2}$ or $\frac{-j}{2}$
- $a_0 = 0$, $a_1 = e^{-j\frac{\pi}{2}} b_{-1} = -jb_{-1} = jb_1$
- $b_1 = \frac{j}{2} \rightarrow x(t) = -cos(\frac{\pi t}{2})$
- $b_1 = \frac{-j}{2} \rightarrow x(t) = cos(\frac{\pi t}{2})$

Table of Contents

complex exponentials as an input to a LTI syst

- Response of an LTI system to a linear combination of complex exponentials takes a particularly simple form
  - $x(t) = e^{st} \rightarrow y(t) = H(s)e^{st}$
  - $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$
    - $h(t)$ is the impulse response
    - $H(s)$ is called system function
- Consider the specific case where $\mathbb{R}e\{s\} = 0$, i.e., $s = j\omega$
- $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$ is called frequency response of the system

Response of a LTI system to a periodic signal

- Let $x(t)$ is a periodic signal: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
- Output of a LTI system with $x(t)$ as input:
  $y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$
  - Since each of the complex exponentials is an eigenfunction of the system
- $y(t)$ is also periodic with period T and Fourier series coefficients $a_k H(jk\omega_0)$

Example

- Input $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$
  - $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$
- Impulse response: $h(t) = e^{-t}u(t)$
- Impulse response:

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau$$
$$= \int_{0}^{\infty} e^{-(1+j\omega)\tau}d\tau = \frac{1}{1+j\omega}$$

- Output $y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t}$
  - $b_k = a_k H(jk2\pi) = \frac{a_k}{1+jk2\pi}$
  - $b_0 = 1$, $b_1 = \frac{1}{4(1+j2\pi)}$, $b_{-1} = \frac{1}{4(1-j2\pi)}$, $b_2 = \frac{1}{2(1+j4\pi)}$, $b_{-2} = \frac{1}{2(1-j4\pi)}$, $b_3 = \frac{1}{3(1+j6\pi)}$, $b_{-3} = \frac{1}{3(1-j6\pi)}$

Thanks.