Indian Institute of Technology Roorkee
MAN-001(Mathematics-1), Autumn Semester: 2019-20
Assignment-1: Matrix Algebra I

(1) Reduce each of the following matrices into row echelon form and then find their ranks:

$$(a)\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&7&8&9\end{bmatrix}\quad(b)\begin{bmatrix}2&4&6\\-1&3&2\\1&4&6\\2&8&7\end{bmatrix}\quad(c)\begin{bmatrix}1&-2&5&-3\\2&3&1&-4\\3&8&-3&-5\end{bmatrix}\quad(d)\begin{bmatrix}1&1&2&0&-4\\1&2&0&-4&1\\2&0&-4&1&1\\0&-4&1&1&2\\-4&1&1&2&0\end{bmatrix}$$

(2) Examine the following set of vectors over $\mathbb{R}$ for linear dependence:

(a) $\{(1,1,2),\ (1,2,5),\ (5,3,4)\}$ (b) $\{(1,-1,1),\ (2,1,1),\ (8,1,5)\}$

(c) $\{(1,1,-1,1),\ (1,-1,2,-1),\ (3,1,0,1)\}$ (d) $\{(1,2,1),\ (2,1,0),\ (1,-1,2)\}$

(3) (a) Find the conditions on α and β for which the matrix
$\begin{pmatrix}\alpha&1&2\\0&2&\beta\\1&3&6\end{pmatrix}$ has (i) rank $=1$ (ii) rank $=2$ (iii) rank $=3$.

(b) For what values of α and β is the following system consistent?
$2x+4y+(\alpha+3)z=2,\ \ x+3y+z=2,\ \ (\alpha-2)x+2y+3z=\beta.$

(4) Solve the following system of linear equations by Gauss elimination method:

(a) $x+4y-z=4,\ x+y-6z=-4,\ 3x-y-z=1$

(b) $x+2y-z=1,\ 2x+y+5z=2,\ 3x+3y+4z=1$

(c) $x+2y+z=2,\ 3x+y-2z=1,\ 2x+4y+2z=4$

(5) Consider the following systems of linear equations:

(a) $-2x+y+z=a,\ x-2y+z=b,\ x+y-2z=c.$

(b) $x+y-z=1,\ 2x+3y+\lambda z=3,\ x+\lambda y+3z=2.$

(c) $\lambda x+y+z=p,\ x+\lambda y+z=q,\ x+y+\lambda z=r.$

Find the values of unknown constant(s) such that each of the above systems has
(i) no solution (ii) a unique solution (iii) infinitely many solutions .

(6) Use Gauss elimination method to show that following system has no solution:

$$(a)\ \begin{aligned}2\sin x-\cos y+3\tan z&=3\\4\sin x+2\cos y-2\tan z&=10\\6\sin x-3\cos y+\tan z&=9.\end{aligned}\ ,\quad (b)\ \begin{aligned}2x_2+2x_3+3x_4&=b_1\\2x_1+4x_2+6x_3+7x_4&=b_2\\x_1+x_2+2x_3+2x_4&=b_3,\end{aligned}\ \text{ for some } (b_1,b_2,b_3)\in\mathbb{R}^3.$$

(7) Let P_2 be the set of all polynomials of degree 2 or less. Use Gauss elimination method to find all polynomials $f\in P_2:\ f(1)=2$ and $f(-1)=6$.

(8) Find the values of k for which the following system of equations has
(i) trivial solution (ii) non-trivial solution.

$$(a)\ \begin{aligned}(3k-8)x+3y+3z&=0\\3x+(3k-8)y+3z&=0\\3x+3y+(3k-8)z&=0\end{aligned}\qquad (b)\ \begin{aligned}(k-1)x+(3k+1)y+2kz&=0\\(k-1)x+(4k-2)y+(k+3)z&=0\\2x+(3k+1)y+3(k-1)z&=0\end{aligned}$$

(9) By employing elementary row operations, find the inverse of the following matrices:

$$(a)\begin{pmatrix}1&-1&2\\2&1&0\\3&4&1\end{pmatrix},\qquad (b)\begin{pmatrix}1&1&1&1\\2&1&3&0\\3&0&2&5\\2&1&1&3\end{pmatrix}$$

(10) If p is a solution of a non-homogeneous system $AX=Y$, then show that $v+p$ is also a solution of $AX=Y$, for every solution v of the homogeneous system $AX=0$.

(11) (a) Let A be an $n \times n$ matrix. Prove the following two statements:

(i) If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.

(ii) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

(b) If $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & 1 & 0 & 1 \\ 0 & 5 & -2 & 7 \\ -1 & 3 & -1 & 4 \end{bmatrix}$, find a 4×4 matrix $B \neq 0$ such that $AB = 0$.

(12) Consider a 4×5 matrix $A = \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 3 & 21 & 0 & 9 & 0 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$.

(a) Find the row-reduced echelon form of A.

(b) Find an invertible matrix P such that $PA = \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix}$.

(c) Find the locus of the point $(x, y, z) \in R^3$ such that for each column vector $Y = (x, y, z, 5)^T$, the equation $AX = Y$ has a solution.

(d) If $X = (x_1, x_2, x_3, x_4, x_5)^T$, then find the conditions on x_1, x_2, x_3, x_4, x_5 such that $AX = 0$.

ANSWERS

(1) (a) 4 (b) 3 (c) 2 (d) 4

(2) (a) LD (b) LD (c) LI (d) LI

(3) (a) (i) Not possible (ii) $\alpha = \frac{1}{3}$ or $\beta = 4$ (iii) $\alpha \neq \frac{1}{3}$, $\beta \neq 4$
(b) $\alpha = 3$ and $\beta = 1$; or $\alpha = -2$ and $\beta = 6$; or $\alpha \neq 3, -2$.

(4) (a) (1,1,1) (b) No solution (c) (1,0,1)

(5) [(a)]
(i) $a + b + c \neq 0$ (ii) Not possible (iii) $a + b + c = 0$
(b) (i) $\lambda = -3$ (ii) $\lambda \neq -3, 2$ (iii) $\lambda = 2$
(c) (i) $\lambda = 1$ and $p + q - 2r \neq 0$ OR $\lambda = 1$ and $q \neq r$
OR $\lambda = -2$ and $p + q + r \neq 0$ and $q \neq r$
(ii) $\lambda \neq 1, -2$
(iii) $\lambda = 1$ and $p = q = r$ OR $\lambda = -2$ and $p + q + r = 0$

(7) $f = (4 - k)x^2 - 2x + k, \ k \in \mathbb{R}$

(8) (a) (i) $k \neq \frac{2}{3}, \frac{11}{3}$ (ii) $k = \frac{2}{3}$ or $\frac{11}{3}$ (b) (i) $k \neq 0, \ 3$ (ii) $k = 0$ or 3

(9) (a) $\frac{1}{13} \begin{bmatrix} 1 & 9 & -2 \\ -2 & -5 & 4 \\ 5 & -7 & 3 \end{bmatrix}$ (b) $\frac{1}{4} \begin{bmatrix} -16 & 4 & -4 & 12 \\ 5 & -1 & -1 & 0 \\ 9 & -1 & 3 & -8 \\ 6 & -2 & 2 & -4 \end{bmatrix}$

(11) (b) $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -7 & 2 & -5 \\ 5 & 0 & 5 & 5 \\ 0 & 5 & 0 & 5 \end{bmatrix}$ (This is just one solution. The matrix B is not unique).

(12) (a) $\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \end{bmatrix}$

(c) $x + y + z = 5$

(d) $x_1 + 7x_2 + 3x_4 = 0, \quad x_3 + 5x_4 = 0, \quad x_5 = 0$.