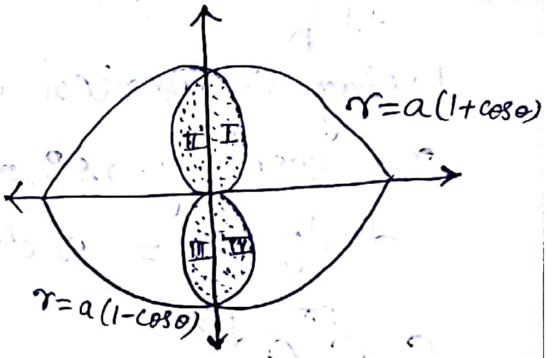
Assignment - 8

(1) $r = a(1+\cos\theta)$, $r = a(1-\cos\theta)$

The required region is divided into four equal regions.



Area = 4 × area of region I

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \frac{a^2(1-\cos\theta)^2}{2}\,d\theta = 2a^2\int_0^{\pi/2}\left(1+\cos^2\theta - 2\cos\theta\right)d\theta$$

$$= 2a^2\left[\frac{\pi}{2}+\frac{\pi}{4}-2\right] = \frac{a^2}{2}(3\pi-8)$$

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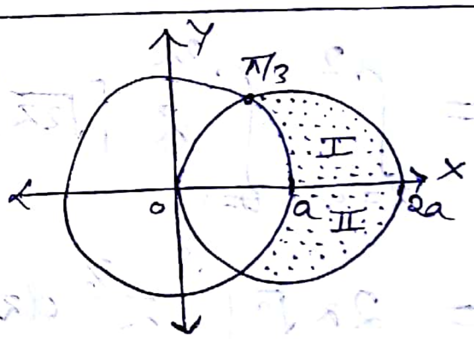
(2) $r = 2a\cos\theta$, $r = a$

Region is symmetric about x-axis

Required area = 2 × area of region I



$$= 2\cdot\int_{\theta=0}^{\pi/3}\int_{r=a}^{2a\cos\theta} r\,dr\,d\theta$$

$r = a$ $\quad r = 2a\cos\theta$

$a = 2a\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3$

$$= 2\cdot\int_{\theta=0}^{\pi/3} \frac{4a^2\cos^2\theta}{2}\,d\theta$$

$$= 4a^2\int_{\theta=0}^{\pi/3}\cos^2\theta\,d\theta = 4a^2\int_{\theta=0}^{\pi/3}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2a^2\int_0^{\pi/3}(1+\cos 2\theta)\,d\theta = 2a^2\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\pi/3}$$

$$= 2a^2\left[\frac{\pi}{3}+\frac{\sin\frac{2\pi}{3}}{2}\right] = 2a^2\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right] = a^2\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

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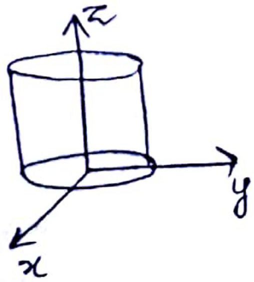
(3) $\iiint_R xy\,dx\,dy\,dz$ where $R: x^2+y^2 \le 1$ with $0 \le z \le 1$



$$\int_{z=0}^{1}\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,dx\,dy\,dz = \int_{z=0}^{1}\int_{x=-1}^{1} x\left[y^2\right]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx\,dz = 0$$

$\longrightarrow$ (~~even~~ odd function)

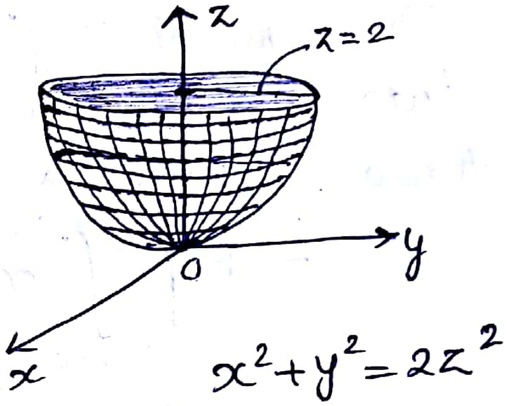
(4). $\iiint_D z(x^2+y^2)^{-1/2}\,dx\,dy\,dz$ D: $z=2$ $2z=x^2+y^2$

Bounded above by the plane $z=2$ and below by $2z=x^2+y^2$

Using cylindrical coordinate

$x=r\cos\theta,\ y=r\sin\theta,\ z=z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$



$$\int_{z=0}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2z}} z(r^2)^{-1/2}\,r\,dr\,d\theta\,dz$$

$$=\int_{z=0}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2z}} z\,\frac{1}{r}\cdot r\,dr\,d\theta\,dz$$

$$=\int_{z=0}^{2}\int_{\theta=0}^{2\pi} z\left[\sqrt{2z}\right]d\theta\,dz$$

$$=2\pi\sqrt{2}\int_{z=0}^{2} z^{3/2}\,dz = 2\sqrt{2}\pi\left[\frac{2z^{5/2}}{5}\right]_0^2$$

$$=2\sqrt{2}\pi\left[\frac{2}{5}\cdot 2^{5/2}\right] = 2\cdot 2\cdot 2^3\cdot\frac{\pi}{5} = \frac{32\pi}{5}$$

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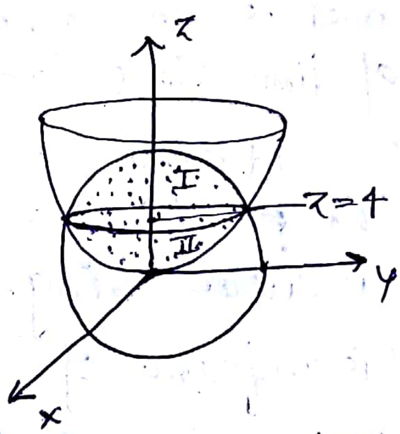
(5). $f(x,y,z)$ avg value $=\dfrac{1}{\text{Vol. of } D}\iiint_D f(x,y,z)\,dv$

$f(x,y,z)=x+y+z$ over $x^2+y^2+z^2=4$.

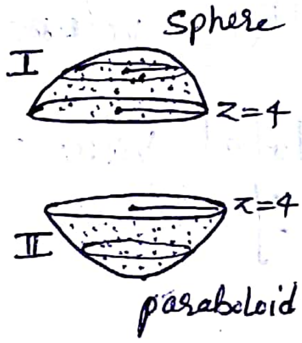
(6) Vol. bdd above by the sphere $x^2+y^2+z^2=32$ and below the paraboloid $x^2+y^2=4z$.

The plane of intersection of the two surfaces :

$$z^2+4z-32=0$$
$$z^2+8z-4z-32=0$$
$$z(z+8)-4(z+8)=0$$
$$(z-4)(z+8)=0$$
$$z=4, \quad \cancel{z=-8}$$



Now the region is divided into two half. Region I & II



$$\text{Vol.} = \iiint_{I} dv + \iiint_{II} dv$$

Using cylindrical coordinates in both region

$$= \int_{z=4}^{\sqrt{32}} \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{32-z^2}} r\,dr\,d\theta\,dz + \int_{z=0}^{4} \int_{\theta=0}^{2\pi} \int_{r=0}^{2\sqrt{z}} r\,dr\,d\theta\,dz$$

$$= 2\pi \int_{z=4}^{\sqrt{32}} \left[\frac{32-z^2}{2}\right] dz + 2\pi \int_{0}^{4} \left[\frac{4z}{2}\right] dz$$

$$= \pi \left[32z - \frac{z^3}{3}\right]_4^{\sqrt{32}} + \pi \left[2z^2\right]_0^4$$

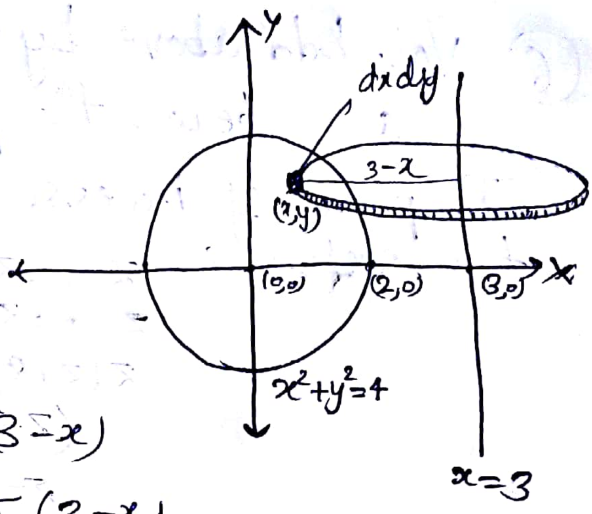
$$= \pi \left[\left[32\sqrt{32} - \frac{32\sqrt{32}}{3}\right] - \left[32\cdot 4 - \frac{8}{3}\right]\right] + 32\pi$$

$$= \pi\left[256\sqrt{2}\right] - \frac{376}{3}\pi + 32\pi$$

$$= \frac{8\pi}{3}\left[128\sqrt{2} - 35\right]$$

Scanned by CamScanner

⑦ Torus: $x^2+y^2=4$
about line $x=3$.



Consider an element at (x,y) of dimensions $dxdy$.
On rotating the element about the line $x=3$, we get a circular ring of radius $=(3-x)$
Circumference of ring $= 2\pi(3-x)$
volume of the elementary torus $= 2\pi(3-x)\,dx\,dy$

Total volume of Torus $= \iint_{R \to \text{circle}} 2\pi(3-x)\,dx\,dy$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\pi(3-x)\,dx\,dy$$

$$= 2\pi\int_{x=-2}^{2} 2(3-x)\sqrt{4-x^2}\,dx$$

$$= 4\pi\int_{x=-2}^{2} 3\sqrt{4-x^2} - 4\pi\int_{x=-2}^{2} x\sqrt{4-x^2}\,dx$$

↳ odd function

$$= 12\pi\int_{-2}^{2}\sqrt{4-x^2}\,dx - 0$$

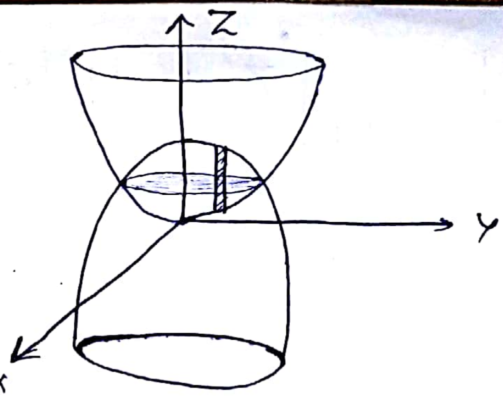
$$= 24\pi\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 24\pi\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 24\pi\left[0 + 2\cdot\frac{\pi}{2} - 0\right]$$

$$= 24\pi^2$$

(8). $z = 4 - x^2 - \frac{y^2}{4}$, $z = 3x^2 + \frac{y^2}{4}$



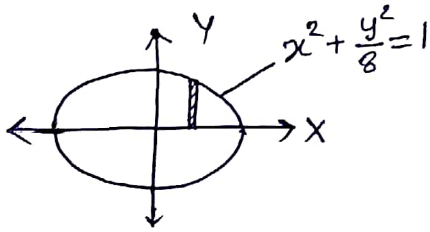
$z + x^2 + \frac{y^2}{4} = 4$ (Paraboloid)

$z = 3x^2 + \frac{y^2}{4}$ (Paraboloid)

To find the equation of intersection of the two paraboloid

$$4 - x^2 - \frac{1}{4}y^2 = 3x^2 + \frac{y^2}{4} \Rightarrow 4x^2 + \frac{y^2}{2} = 4 \Rightarrow x^2 + \frac{y^2}{8} = 1$$

The crossection is an ellipse in four octants.



$$V = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{8(1-x^2)}}\int_{z=3x^2+\frac{y^2}{4}}^{4-x^2-(y^2/4)} dz\, dy\, dx$$

$$= 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{8(1-x^2)}} \left[4 - x^2 - \frac{y^2}{4} - 3x^2 - \frac{y^2}{4}\right] dy\, dx$$

$$= 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{8(1-x^2)}} \left[4 - 4x^2 - \frac{y^2}{2}\right] dy\, dx = 4\int_{x=0}^{1} \left[4y - 4x^2y - \frac{y^3}{6}\right]_0^{\sqrt{8(1-x^2)}} dx$$

$$= 4\int_0^1 \left[4(1-x^2)\sqrt{8}\sqrt{1-x^2} - \frac{1}{6} 8\sqrt{8}\,(1-x^2)^{3/2}\right] dx$$

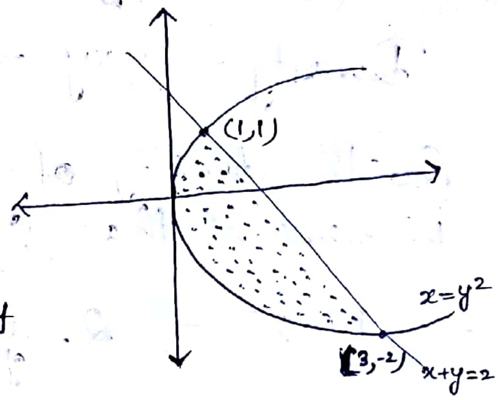
$$= 4\int_0^1 \left(4\sqrt{8} - \frac{4\sqrt{8}}{3}\right)(1-x^2)^{3/2} dx$$

$$= \frac{64\sqrt{2}}{3}\int_0^1 (1-x^2)^{3/2} dx \qquad x = \sin\theta \qquad dx = \cos\theta\, d\theta$$

$$= \frac{64\sqrt{2}}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{64\sqrt{2}}{3}\, \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 4\sqrt{2}\pi$$

(10) C.G. of the area bdd by parabola $y^2 = x$ and $x+y=2$, density = constant.



C.G. $\bar{x} = \dfrac{\int x\,dm}{\int dm}$ $\quad \bar{y} = \dfrac{\int y\,dm}{\int dm}$

Now

$dm = \rho \cdot dA$

So $\bar{x} = \dfrac{\int x \cdot \rho \, dA}{\int \rho \, dA} = \dfrac{\rho \iint x \, dx\,dy}{\rho \iint dx\,dy}$

$$= \frac{\int_{y=-2}^{1}\int_{x=y^2}^{2-y} x\,dx\,dy}{\int_{y=-2}^{1}\int_{x=y^2}^{2-y} dx\,dy} = \frac{\int_{y=-2}^{1}\left(\frac{(2-y)^2}{2} - \frac{y^4}{2}\right)dy}{\int_{y=-2}^{2}\left[(2-y)-y^2\right]dy}$$

$$= \frac{\frac{1}{2}\int_{y=-2}^{1}\left[4+y^2-4y-y^4\right]dy}{\int_{y=-2}^{1}\left[2-y-y^2\right]dy} = \frac{1}{2}\,\frac{\left[4y+\frac{y^3}{3}-2y^2-\frac{y^5}{5}\right]_{-2}^{1}}{\left[2y-\frac{y^2}{2}-\frac{y^3}{3}\right]_{-2}^{1}}$$

$$= \frac{1}{2}\,\frac{\left[\left(4+\frac{1}{3}-2-\frac{1}{5}\right)-\left(-8-\frac{8}{3}-8+\frac{32}{5}\right)\right]}{\left[\left(2-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-\frac{4}{2}+\frac{8}{3}\right)\right]}$$

$$= \frac{\frac{1}{2}\left[\frac{72}{5}\right]}{\left[\frac{9}{2}\right]} = \frac{36}{5} \times \frac{2}{9} = \frac{4\times 2}{5} = \frac{8}{5}$$

$$\bar{y} = \frac{\int_{y=-2}^{1}\int_{x=y^2}^{2-y} y\,dx\,dy}{\int_{y=-2}^{1}\int_{y^2}^{2-y} dy\,dx} = \frac{\int_{-2}^{1} y\left[2-y-y^2\right]dy}{9/2} = \frac{2}{9}\int_{-2}^{1}(2y-y^2-y^3)\,dy$$

$$= \frac{2}{9}\left[y^2-\frac{y^3}{3}-\frac{y^4}{4}\right]_{-2}^{1} = \frac{2}{9}\left[\left(1-\frac{1}{3}-\frac{1}{4}\right)-\left(4+\frac{8}{3}-\frac{16}{4}\right)\right]$$

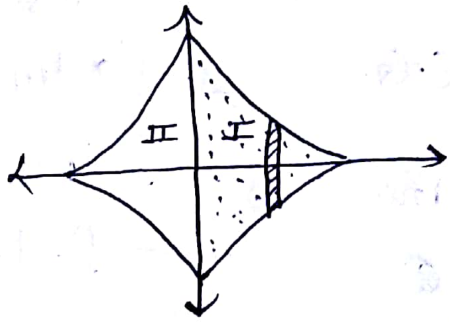
$$= \frac{2}{9}\left[\frac{5}{12}-\frac{8}{3}\right] = \frac{-1}{2}$$

Centre of gravity $= \left(\frac{8}{5}, -\frac{1}{2}\right)$

(11) $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ density $\rho = \mu xy$

Density $= \mu xy \Rightarrow$ symmetric about y axis

So the density of the right half is same as that of left half



So Mass $= 2 \cdot \underset{\text{of I}}{\text{area}} \times \text{density}$

$$= 2 \cdot \int_{x=0}^{a} \mu xy \cdot (y\,dx)$$

$$= 2\mu \int_{x=0}^{a} x y^2\,dx \qquad y = b\left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^{3/2}$$

$$= 2\mu \int_0^a x\, b^2 \left[1 - \left(\frac{x}{a}\right)^{2/3}\right]^3 dx$$

let $x = a\sin^3\theta \qquad dx = 3a\sin^2\theta\cos\theta\,d\theta$

$$= 2\mu b^2 \int_0^{\pi/2} (a\sin^3\theta)\left[1 - \sin^2\theta\right]^3 (3a\sin^2\theta)\cos\theta\,d\theta$$

$$= 6\mu a^2 b^2 \int_0^{\pi/2} \sin^5\theta \cos^7\theta\,d\theta$$

$$= 6\mu a^2 b^2 \frac{\Gamma\frac{5+1}{2}\,\Gamma\frac{7+1}{2}}{2\,\Gamma\frac{5+7+2}{2}} = 6\mu a^2 b^2 \frac{\Gamma 3\,\Gamma 4}{2\,\Gamma 7}$$

$$= 6\mu a^2 b^2 \cdot \frac{\cancel{2}\,\lfloor 3}{2\cdot 6\cdot 5\cdot 4\cdot \lfloor 3} = \frac{6\mu a^2 b^2}{120}$$

$$= \frac{\mu a^2 b^2}{20}$$