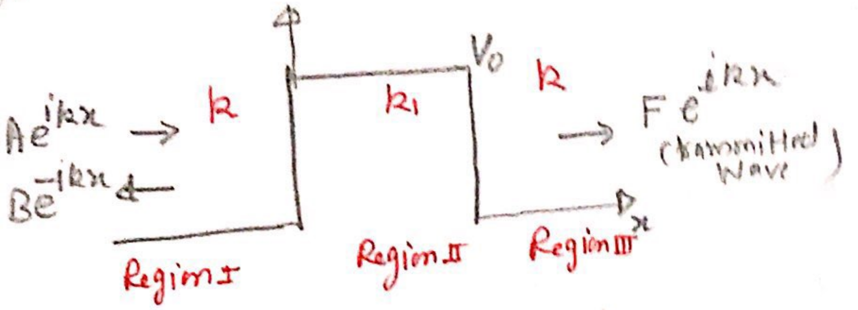
Ex-4 (F) The Rectangular Potential Barrier ① P.C.Srivastava



$\Longrightarrow$ $V(x) = V_0$ for $0<x<a$
$= 0$ for $x<0$, $x>a$

$E < V_0$ Classically
- Particle gets reflected

QM. There is a finite probability of the particle tunnelling through the barrier
e.g. emission of α-particle from nucleus.

$E > V_0$ Classically
- Particle gets transmitted.

QM. Finite probability for reflection.

Case 1:- $E<V_0$ (Tunneling)

For such a case the solutions of the Schrödinger eqn in the three regions are given by:

$$\begin{aligned}\psi &= Ae^{ikx} + Be^{-ikx} & x<0 \\ &= Ce^{k_1x} + De^{-k_1x} & 0<x<a \\ &= Fe^{ikx} + Ge^{-ikx} & x>a\end{aligned} \quad \text{--- ①}$$

where $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ and $k_1 = \left(\frac{2m(V_0-E)}{\hbar^2}\right)^{1/2}$ --- ②

Here $G=0$, because there cannot be a wave propagating in the $-x$ direction in the region $x>a$.

Continuity of ψ and $\frac{d\psi}{dx}$ at $x=0$ gives

ψ $$A+B = C+D$$

$\frac{d\psi}{dx} \rightarrow$ $$\frac{ik}{k_1}(A-B) = C-D$$

Thus $$C = \frac{1}{2}\left(1+\frac{ik}{k_1}\right)A + \frac{1}{2}\left(1-\frac{ik}{k_1}\right)B \quad \cdots (3)$$

and $$D = \frac{1}{2}\left(1-\frac{ik}{k_1}\right)A + \frac{1}{2}\left(1+\frac{ik}{k_1}\right)B \quad \cdots (4)$$

Continuity Conditions at $x=a$, give us

$\psi \rightarrow \psi(a)$ $$Ce^{k_1 a} + De^{-k_1 a} = Fe^{ika}$$

$\frac{d\psi(a)}{dx}$ and $$Ce^{k_1 a} - De^{-k_1 a} = \frac{ik}{k_1} Fe^{ika}$$

thus $$C = \frac{1}{2}\left(1+\frac{ik}{k_1}\right)Fe^{-k_1 a}e^{ika} \quad \cdots (5)$$

$$D = \frac{1}{2}\left(1-\frac{ik}{k_1}\right)Fe^{k_1 a}e^{ika} \quad \cdots (6)$$

Therefore $$\frac{C}{D} = \frac{\left(1+\frac{ik}{k_1}\right)}{\left(1-\frac{ik}{k_1}\right)}e^{-2k_1 a} = \frac{\left(1+\frac{ik}{k_1}\right)A + \left(1-\frac{ik}{k_1}\right)B}{\left(1-\frac{ik}{k_1}\right)A + \left(1+\frac{ik}{k_1}\right)B} \quad \cdots (7)$$

Simple manipulations give us

$$\frac{B}{A} = \frac{(k^2+k_1^2)\sinh k_1 a}{(k^2-k_1^2)\sinh k_1 a + 2ikk_1\cosh k_1 a} \quad \cdots (8)$$

We also have

$$\frac{F}{A} = \frac{2ikk_1 e^{-ika}}{(k^2-k_1^2)\sinh k_1 a + 2ikk_1\cosh k_1 a} \quad \cdots (9)$$

(N)
$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

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(2)

The above equations will lead to the following expressions for the reflection and transmission coefficients:

$$R = \left|\frac{J_r}{J_i}\right| = \frac{\frac{\hbar k}{m}|B|^2}{\frac{\hbar k}{m}|A|^2} = \left[1 + \frac{4\xi(1-\xi)}{\sinh^2(\alpha\sqrt{1-\xi})}\right]^{-1} \quad \text{--- (10)}$$

$$T = \left|\frac{J_t}{J_i}\right| = \frac{\frac{\hbar k}{m}|F|^2}{\frac{\hbar k}{m}|A|^2} = \left[1 + \frac{\sinh^2(\alpha\sqrt{1-\xi})}{4\xi(1-\xi)}\right]^{-1} \quad \text{--- (11)}$$

ξ, Xi
$\zeta \rightarrow$ zeta

where $\xi = \frac{E}{V_0}$ --- (12)

and $\alpha = \left(\frac{2mV_0a^2}{\hbar^2}\right)^{1/2}$ --- (13)

Is a dimensionless variable characterizing the potential.

In writing eq^n (10) and (11), we use the fact that

$$k_1a = \alpha\sqrt{1-\xi}$$

Here, $R + T = 1$

Further for $k_1a \gg 1$

$$\boxed{T \approx 16\xi(1-\xi)e^{-2k_1a}}$$

Approximate transmission probability

$T = e^{-2k_1a}$

where $k_1 = \left(\frac{2m(V_0-E)}{\hbar^2}\right)^{1/2}$

$a \rightarrow$ barrier width

This shows that the transmission coefficient is not zero, as it would be classically, but has a finite value. So, quantum mechanically, there is finite tunneling beyond the barrier $x > a$.

Case 2 $E > V_0$:—

The analysis is very similar, excepting that in the region $0<x<a$, instead of the solution $e^{\pm k_1 x}$ we will have solns $e^{\pm i k_2 x}$ where

$$k_2 = \left[\frac{2m(E-V_0)}{\hbar^2}\right]^{1/2} \quad \text{--- (15)}$$

Thus instead of eqn (1), we will have

$$\left.\begin{aligned} \psi &= Ae^{ikx} + Be^{-ikx} && x<0 \\ &= Ce^{ik_2x} + De^{-ik_2x} && 0<x<a \\ &= Fe^{ikx} && x>a \end{aligned}\right\} \rightarrow (16)$$

Implying that in the entire analysis we have to replace k_1 by ik_2 everywhere

$$k_1 \rightarrow ik_2 = i\left[\frac{2m(E-V_0)}{\hbar^2}\right]^{1/2} \quad \text{--- (17)}$$

The final results are

$$R = \left[1 + \frac{4\xi(\xi-1)}{\sin^2(\alpha\sqrt{\xi-1})}\right]^{-1} \quad \text{--- (18)}$$

$$T = \left[1 + \frac{\sin^2(\alpha\sqrt{\xi-1})}{4\xi(\xi-1)}\right]^{-1} \quad \text{--- (19)}$$

once again $R+T=1$

In writing eqn (18) and (19), we use the fact that

$$k_2 a = \alpha\sqrt{\xi-1}$$

From eqn (19), we may readily see that the transmission coefficient is unity for

$$k_2 a \left[= \alpha\sqrt{\xi-1}\right] = \pi, 2\pi, 3\pi, \ldots \quad (20)$$

or $$a = \frac{\lambda_2}{2}, \frac{2\lambda_2}{2}, \frac{3\lambda_2}{2}, \ldots \quad (21)$$

Where $$\lambda_2 = \frac{2\pi}{k_2} \quad \text{--- (22)}$$

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Thus, whenever the barrier width is multiple of $\frac{\lambda_2}{2}$, perfect transmission occurs.

$$k_2 a = \sqrt{\frac{2m(E-V_0)}{\hbar^2} a^2} = \alpha \sqrt{\left(\frac{E}{V_0}-1\right)} \quad \text{--- (22)}$$

Here $\alpha = \left(\frac{2mV_0 a^2}{\hbar^2}\right)^{1/2}$

Thus, using eq^n (21), the transmission coefficient is unity for

$$\boxed{\frac{E}{V_0} = 1 + \frac{n^2\pi^2}{\alpha^2}} \quad \text{--- (23)}$$

$$\alpha^2\left(\frac{E}{V_0}-1\right) = n^2\pi^2$$

For $\alpha = 3.4709$, the values are

$\frac{E}{V_0} \approx 1.819, 4.277, 8.373 \ldots$

$M = m_p = 1836\, m_e$

$a = 0.5\, Å$

$V_0 = 0.1\, eV$

$\alpha = 3.4709$

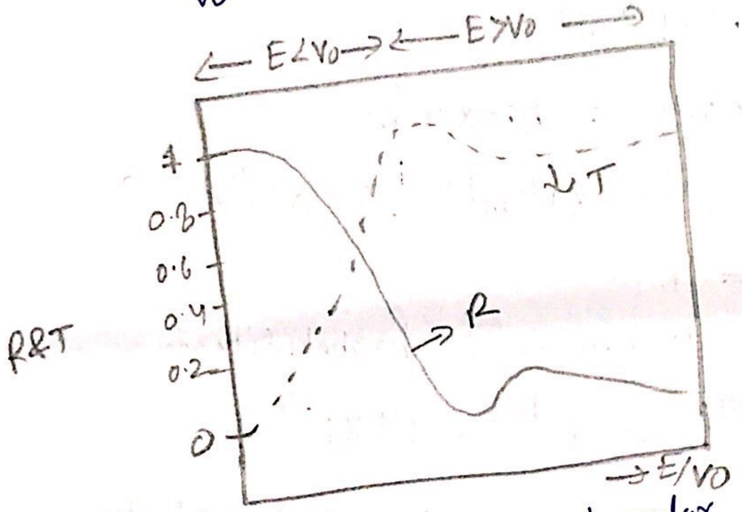
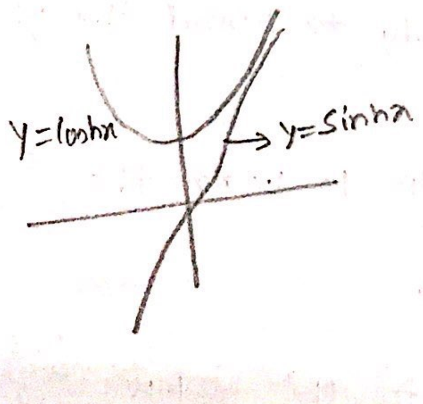
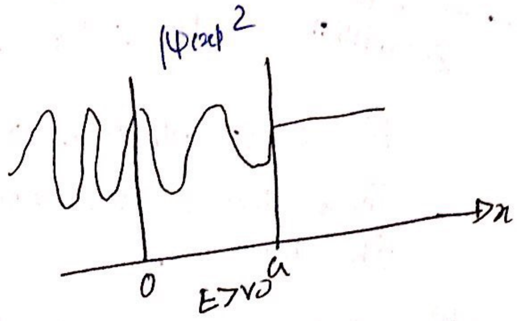
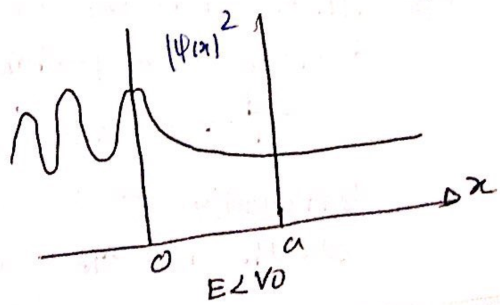


Fig:- R&T for a rectangular potential barrier with $\alpha = 3.4709$.







Q:- Electrons with energies of 1.0 eV and 2.0 eV are incident on a barrier 10.0 eV high and 0.50 nm wide.

ⓐ Find their respective transmission probabilities.

ⓑ How are these affected if the barrier is doubled in width?

Soln.

ⓐ For 1.0 eV

$$T = e^{-2k_1 a}$$

$$T = 16\xi(1-\xi)e^{-2k_1 a}$$

$$\xi = \frac{E}{V_0}$$

Where $k_1 = \left(\frac{2m(V_0-E)}{\hbar^2}\right)^{1/2}$

$$k_1 = \frac{\sqrt{2m(V_0-E)}}{\hbar} = \frac{\sqrt{2\times(9.1\times10^{-31}\,kg)(10-1.0)eV(1.6\times10^{-19}\,J/eV)}}{1.054\times10^{-34}\,J\text{-}sec}$$

$$= 1.6\times10^{10}\,m^{-1}$$

Since $a = 0.50\,nm = 5.0\times10^{-10}\,m$

$$2k_1 a = 2(1.6\times10^{10}\,m^{-1})(5.0\times10^{-10}\,m)$$

$$= 16$$

and the approximate transmission probability is

$$T_1 = e^{-2k_1 a} = e^{-16} = 1.1\times10^{-7}$$

$$\frac{1}{1.1\times10^{-7}} = 8886110 = 8.9\text{ million}$$

One out of 8.9 million can tunnel through the 10 eV barrier on the average.

For 2 eV electrons a similar calculation gives $T_2 = 2.4\times10^{-7}$. (5.25×10^{-7} ?)
These electrons are over twice as likely to tunnel through the barrier.

ⓑ If the barrier is doubled in width to 1.0 nm, the transmission probabilities become

$$T_1' = 1.3\times10^{-14},\ T_2' = 5.1\times10^{-14}$$

Evidently T is more sensitive to the width of the barrier than to the particle energy here.

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