Gomory's Cutting plane method for all integer LPP

We have already discussed 'Branch and Bound technique' for solving all integer / mixed integer LPP. This is a graphical approach & valid for two variable LPP only. Gomory's Cutting plane method is a simplex - algorithm approach for solving integer type LPP. First, we will discuss all integer LPP.

Consider the following LPP:

$$\text{Max } z = x_1 - x_2$$

s/t

$$x_1 + 2x_2 \leq 4,$$

$$6x_1 + 2x_2 \leq 9, \quad x_1, x_2 \geq 0 \text{ \& integers.}$$

The optimal table of the above LPP is:

	$c_j \rightarrow$	1	-1	0	0	
C_B	x_B	x_1	x_2	S_1	S_2	b
$c_j - z_j \rightarrow$		0	$-4/3$	0	$-1/6$	
	S_1	0	$5/3$	1	$-1/6$	$5/2$
	x_1	1	$1/3$	0	$1/6$	$3/2$

The optimal solution is NOT an all integer solution. [$x_1 = 3/2$, $S_1 = 5/2$ → Not integers]. To find an integer solution, we introduce a cut (an additional constraint) to the above problem. Let us introduce a cut for x_1 variable. [usually we prefer that variable having largest fractional part, but here fractional part of both the variables x_1 & S_1 are same, so choose any-one].

From the table, x_1 row is:

$$x_1 + \frac{1}{3} x_2 + \frac{1}{6} S_2 = \frac{3}{2}.$$

In general, we may write:

$$x_1 + \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_p y_p = \alpha, \quad \text{—(1)}$$

y_i's are non-basic-variables. [Here, in the given table, $y_1 = x_2$, $y_2 = S_2$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{6}$, $\alpha = 3/2$ & remaining α_i's are zero].

(1) $\Rightarrow$
$$x_1 + \sum_{i=1}^{p} (\beta_i + f_i) y_i = \beta + f \quad \text{—(2)}$$

where
$$[\alpha_i] = \beta_i + f_i, \quad 0 \le f_i < 1, \quad i = 1, 2, \ldots, p$$
$$[\alpha] = \beta + f, \quad 0 < f < 1,$$

$[\alpha]$ denotes greatest integer ~~is~~ $\le \alpha$.

[Here $f > 0$ since the variable is not an integer].

Therefore, (2) $\Rightarrow$

$$x_1 = (\beta - \sum_{i=1}^{p} \beta_i y_i) + (f - \sum_{i=1}^{p} f_i y_i) \quad \text{—(3)}$$

Now, the aim is to make x_1 integer. For this, ofcourse the solution will change. That is, one of the non-basic variable (i.e. y_i) must become greater than 0. (Since there is a unique solution with $y_i = 0$, $i = 1, 2, \ldots, p$). Now, the first branch of RHS of (3) is an integer (positive or negative), therefore for x_1 to be integer, the expression

$$f - \sum_{i=1}^{p} f_i y_i$$

must be an integer.

Since $0 \le f_i < 1$, $y_i \ge 0$, therefore

$$\sum_{i=1}^{p} f_i y_i \ge 0.$$

Also $f < 1$, therefore

$$f - \sum_{i=1}^{p} f_i y_i < 1.$$

Now,

Suppose $f > \sum_{i=1}^{p} f_i y_i$

$\Rightarrow 0 < f - \sum_{i=1}^{p} f_i y_i < 1$

$\Rightarrow f - \sum_{i=1}^{p} f_i y_i$ is always a fraction

$\Rightarrow x_1$ will never be an integer

Hence,

$$f \le \sum_{i=1}^{p} f_i y_i$$

or

$$\boxed{f - \sum_{i=1}^{p} f_i y_i \le 0.}$$

This is an additional constraint (or cut).

In the above problem, the cut will be

$$\frac{1}{2} - \frac{1}{3}x_2 - \frac{1}{6}S_2 \le 0$$

or

$$2x_2 + S_2 \ge 3. \quad \rightarrow 2x_2 + S_2 - S_3 = 3$$
$$\Rightarrow -2x_2 - S_2 + S_3 = -3.$$

Now, we will apply sensitivity analysis, to add this constraint in the problem.

	x_1	x_2	S_1	S_2	S_3	b
	0	$-4/3$	0	$-1/6$	0	
S_1	0	$5/3$	1	$-1/6$	0	$5/2$
x_1	1	$1/3$	0	$1/6$	0	$3/2$
S_3	0	-2	0	$\boxed{-1}$	1	-3
	0	-2	0	-1	1	-3
S_1						3
x_1						1
S_2						2

optimal solution is: $x_1 = 1$, $x_2 = 0$, $z = 1$.

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