Tutorial Sheet – 4

1 (i) $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$. $\Rightarrow \tan z = \frac{x^3+y^3}{x-y} = u$

then u is a homogeneous function of x and y of degree $3-1 = 2$

$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ —— ①

$u = \tan z \implies \frac{\partial u}{\partial x} = \sec^2 z \cdot \frac{\partial z}{\partial x}$ & $\frac{\partial u}{\partial y} = \sec^2 z \frac{\partial z}{\partial y}$

putting in ①

$$x\sec^2 z\frac{\partial z}{\partial x} + y\sec^2 z\frac{\partial z}{\partial y} = 2\cdot\tan z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2\tan z}{\sec^2 z} = 2\sin z\cos z = \sin 2z \quad \text{—— (II)}$$

(ii) Diff (II) w.r.to x & y partially

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y} = 2\cos 2z\frac{\partial z}{\partial x} \quad \text{—— (III)}$$

$$x\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = 2\cos 2z\frac{\partial z}{\partial y} \quad \text{—— (IV)}$$

Multiplying (III) by x and (IV) by y and adding

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\cos 2z\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} = \sin 4z - \sin 2z$$

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(2) $z = x^m f\left(\frac{y}{x}\right) + x^n g\left(\frac{y}{x}\right)$ then show that

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Sol. Let $z = z_1 + z_2$

Now z_1 and z_2 are hom. fn. of degree m & n. resp.

So
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\frac{\partial}{\partial x}(z_1+z_2) + y\frac{\partial}{\partial y}(z_1+z_2)$$

$$= \left(x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}\right) + \left(x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y}\right)$$

$$= mz_1 + nz_2 \quad \text{—— ①}$$

Now Diff ① w.r.to x and y respectively

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=m\frac{\partial z_1}{\partial x}+n\frac{\partial z_2}{\partial x} \quad \text{—— ②}$$

$$x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=m\frac{\partial z_1}{\partial y}+n\cdot\frac{\partial z_2}{\partial y} \quad \text{—— ③}$$

multiplying ② and ③ by x and y and adding

$$\underbrace{x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}}_{\text{let } \alpha}+\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=m\left[x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}\right]+n\left[x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}\right] \quad \text{④}$$

$$\alpha+\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=m\left[x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}\right]+n\left[x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}\right]$$

$$\alpha+mz_1+nz_2=m[mz_1)+n(nz_2)$$

$$\alpha=(m^2-m)z_1+(n^2-n)z_2$$

$$\alpha=(m^2-m)z_1+(n^2-n)z_2+mn(z_1+z_2)-mn(z_1+z_2)$$

$$\alpha+mn(z_1+z_2)=(m^2-m)z_1+(n^2-n)z_2+mn(z_1+z_2)$$

$$\alpha+mnz=m^2z_1+mnz_2+mnz_1+n^2-mz_1-nz_2$$

$$\alpha+mnz=(m+n-1)(mz_1+nz_2)$$

$$\boxed{\alpha+mnz=(m+n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)} \quad \text{(from ①)}$$

(3) Compute $\frac{dz}{dt}$; $z = \sin(x^2+y^2)$; $x = t^2+3$, $y = t^3$

Sol. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= [2x\cos(x^2+y^2)][2t] + [2y\cos(x^2+y^2)][3t^2]$

~~$= 6(x+y)t^2\cos(x^2+y^2)$~~

$= 4xt\cos(x^2+y^2) + 6yt^2\cos(x^2+y^2)$

(4) Compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = x^2y^2$, $x = st$, $y = t^2-s^2$

Sol. $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$

$= (2xy^2)(t) + (2x^2y)(-2s)$

$= 2xty^2 - 4x^2sy$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$

$= (2xy^2)(s) + (2x^2y)(2t)$

$= 2xsy^2 + 4x^2ty$

(5) Find all partial derivatives of z w.r.t x and y

$xy + yz + zx = 1$

let $f = xy + yz + zx - 1 = 0$

then $\frac{\partial z}{\partial x} = \frac{-\partial f/\partial x}{\partial f/\partial z} = -\frac{(y+z)}{(x+y)}$

$\frac{\partial z}{\partial y} = \frac{-\partial f/\partial y}{\partial f/\partial z} = -\frac{(x+z)}{(x+y)}$

(6) $z = e^x\sin y + e^y\cos x$

$x^3 + x + e^t + t^2 + t - 1 = 0$

$yt^3 + y^3t + t + y = 0$

find $\frac{dz}{dt}$ at $t=0$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

at $t=0$

we have $\begin{cases} x = 0 \\ y = 0 \end{cases}$

$\frac{\partial z}{\partial x} = e^x\sin y - e^y\sin x$

$= e^0\sin 0 - e^y\sin 0$

$= 0$

$\frac{\partial z}{\partial y} = e^x\cos y + e^y\cos x = 2$

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$$\frac{dx}{dt} = \frac{-\frac{\partial f_1}{\partial t}}{\frac{\partial f_1}{\partial x}} = \frac{-(e^t+2t+1)}{(3x^2+1)} = \frac{-(1+0+1)}{1} = -2$$

$$f_1 \equiv x^3+x+e^t+t^2+t-1=0$$

$$f_2 \equiv yt^3+y^3t+t+y=0$$

$$\frac{dy}{dt} = -\frac{\frac{\partial f_2}{\partial t}}{\frac{\partial f_2}{\partial y}} = \frac{-(3yt^2+y^3+1)}{(t^3+3y^2t+1)} = -1$$

Thus

$$\frac{dz}{dt} = 0(-2)-2(-1) = 2$$

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(7) Find n, so that the eq^n $v = r^n(3\cos^2\theta - 1)$ satisfies

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0$$

Sol. Given Eq^n:

$$2r\frac{\partial v}{\partial r} + r^2\frac{\partial^2 v}{\partial r^2} + \frac{1}{\sin\theta}\cdot\sin\theta\frac{\partial^2 v}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial v}{\partial\theta} = 0$$

$$2r\frac{\partial v}{\partial r} + r^2\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial\theta^2} + \cot\theta\frac{\partial v}{\partial\theta} = 0 \quad \text{—— } (A)$$

Now $\quad v = r^n(3\cos^2\theta - 1)$

$$\frac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$$

$$\frac{\partial^2 v}{\partial r^2} = n(n-1)r^{n-2}(3\cos^2\theta - 1)$$

$$\frac{\partial v}{\partial\theta} = r^n\,(-6\cos\theta\sin\theta) = r^n(-3\sin 2\theta)$$

$$\frac{\partial^2 v}{\partial\theta^2} = r^n\,(-6\cos 2\theta)$$

putting in (A)

$$\cancel{r^n[(3\cos\theta-1)(2n+n(n-1)) - 6\cos\theta] = 0}$$

$$\cancel{r^n[(n^2+n)(3\cos\theta-1) - 6\cos\theta] = 0}$$

$$2r\left(n r^{n-1}(3\cos^2\theta-1)\right) + r^2\left(n(n-1) r^{n-2}(3\cos^2\theta-1)\right) +$$
$$r^n(-6\cos2\theta) + \cot\theta\, r^n(-3\sin2\theta) = 0$$

$$r^n\left[2n(3\cos^2\theta-1) + n(n-1)(3\cos^2\theta-1) - 6\cos2\theta - 3\cot\theta\sin2\theta\right]=0$$

$$r^n\left[(2n+n(n-1))(3\cos^2\theta-1) - 6(\cos2\theta) - 6\cos^2\theta\right] = 0$$

$$r^n\left[(n^2+n)(3\cos^2\theta-1) - 6(2\cos^2\theta-1) - 6\cos^2\theta\right] = 0$$

$$r^n\left[(n^2+n)(3\cos^2\theta-1) - 6(3\cos^2\theta-1)\right] = 0$$

$$r^n\left[((n^2+n) - 6)(3\cos^2\theta-1)\right] = 0$$

$$r^n(n^2+n-6)(3\cos^2\theta-1) = 0$$

$$\Rightarrow \quad n^2+n-6 = 0$$

$$(n+3)(n-2) = 0$$

$$n = -3, 2$$

(8) If $v = v(r)$ where $r^2 = \sum_{i=1}^{n} x_i^2$.

Show that $\sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 v}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial v}{\partial r}$

Sol.

$$r^2 = \sum_{i=1}^{n} x_i^2$$

$$r^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

$$2r\frac{\partial r}{\partial x_1} = 2x_1 \qquad 2r\frac{\partial r}{\partial x_2} = 2x_2 \quad \dots, \quad 2r\frac{\partial r}{\partial x_n} = 2x_n$$

i.e. $$2r\frac{\partial r}{\partial x_i} = 2x_i$$

$$\Rightarrow \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad \text{--- (1)}$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) &= \frac{r - \frac{x_i^2}{r}}{r^2} \\ &= \frac{r^2 - x_i^2}{r^3}\end{aligned}}$$

Now $v = v(r)$

$$\frac{\partial v}{\partial x_i} = \frac{\partial}{\partial x_i}(v(r))$$

$$= \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x_i}$$

$$= \frac{\partial v}{\partial r}\cdot\left(\frac{x_i}{r}\right) \quad \text{from (1)}$$

Diff again

$$\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial v}{\partial r}\cdot\frac{x_i}{r}\right)$$

$$= \frac{\partial^2 v}{\partial r^2}\left(\frac{x_i}{r}\right)^2 + \frac{\partial v}{\partial r}\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right)$$

$$= \frac{\partial^2 v}{\partial r^2}\cdot\frac{x_i^2}{r^2} + \frac{\partial v}{\partial r}\left(\frac{r^2 - x_i^2}{r^3}\right)$$

Now taking Summation

$$\sum_{i=1}^{n}\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 v}{\partial r^2}\sum_{i=1}^{n}\frac{x_i^2}{r^2} + \frac{\partial v}{\partial r}\sum_{i=1}^{n}\left(\frac{r^2 - x_i^2}{r^3}\right)$$

$$= \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r}\left(\frac{nr^2 - r^2}{r^3}\right) = \frac{\partial^2 v}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial v}{\partial r}$$

(10) $w = f(u,v)$: $w_{uu} + w_{vv} = 0$

$u = \dfrac{x^2 - y^2}{2}$

$\dfrac{\partial u}{\partial x} = x \qquad \dfrac{\partial u}{\partial y} = -y$

$\dfrac{\partial^2 u}{\partial x^2} = 1 \qquad \dfrac{\partial^2 u}{\partial y^2} = -1$

$\dfrac{\partial^2 u}{\partial x \partial y} = 0$

$v = xy$

$\dfrac{\partial v}{\partial x} = y \qquad \dfrac{\partial v}{\partial y} = x$

$\dfrac{\partial^2 v}{\partial x^2} = 0 \qquad \dfrac{\partial^2 v}{\partial y^2} = 0$

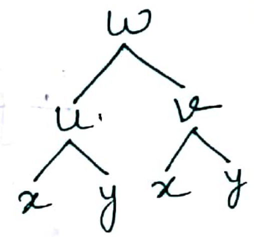
$\dfrac{\partial^2 v}{\partial x \partial y} = 1$

Now

$$w_{xx} + w_{yy} = \frac{\partial^2}{\partial x^2}(w) + \frac{\partial^2}{\partial y^2}(w)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$



$$= \left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}\right)\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}\right)$$

$$= \frac{\partial^2 w}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial u}\right)\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^2 w}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial w}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

$$= w_{uu}(x)^2 + \left(\frac{\partial w}{\partial u}\right)\cdot(1) + w_{vv}(x)^2 + \left(\frac{\partial w}{\partial v}\right)(0)$$

$$= (w_{uu} + w_{vv})x^2 + \frac{\partial w}{\partial u} + 0$$

$$= (0)x^2 + \frac{\partial w}{\partial u}$$

In this step

$$= \frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y}\right]$$

$\cancel{\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right)\right]}$

(11) Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$ for $x=\rho\sin\theta\cos\phi$
$y=\rho\sin\theta\sin\phi$
$z=\rho\cos\theta$

$$J=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi\\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi\\ \cos\theta & -\rho\sin\theta & 0\end{vmatrix}$$

expanding along R_3

$$J=\cos\theta\left[\rho^2\sin\theta\cos\theta\cos^2\phi+\rho^2\sin\theta\cos\theta\sin^2\phi\right]$$

$$+\rho\sin\theta\left[\rho\sin^2\theta\cos^2\phi+\rho\sin^2\theta\sin^2\phi\right]$$

$$=\rho^2\cos^2\theta\sin\theta[1]+\rho^2\sin^3\theta[1]$$

$$=\rho^2\sin\theta\left[\cos^2\theta+\sin^2\theta\right]$$

$$=\rho^2\sin\theta$$

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(12) $J=\frac{\partial(x,y)}{\partial(u,v)}$ $x=\sqrt{2}u-\sqrt{\frac{2}{3}}v$
$y=\sqrt{2}u+\sqrt{\frac{2}{3}}v$

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\sqrt{2} & -\sqrt{2/3}\\ \sqrt{2} & \sqrt{2/3}\end{vmatrix}=\frac{2}{\sqrt{3}}+\frac{2}{\sqrt{3}}=\frac{4}{\sqrt{3}}$$

$$=\frac{4\sqrt{3}}{3}$$

(13) (i) $f(x,y) = \log x - \log y \qquad g(x,y) = \frac{x}{2y} + \frac{3y}{2x}$

$$g(x,y) = \frac{x^2 + 3y^2}{2xy}$$

$3\sin\theta\sin\phi$

$\theta\cos\phi$

0

$$J = \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/x & -1/y \\ \left(\frac{1}{2y} - \frac{3y}{2x^2}\right) & \left(\frac{-x}{2y^2} + \frac{3}{2x}\right) \end{vmatrix}$$

$$= \frac{1}{x}\left[\frac{-x}{2y^2} + \frac{3}{2x}\right] + \frac{1}{y}\left[\frac{1}{2y} - \frac{3y}{2x^2}\right]$$

$$= \frac{-1}{2y^2} + \frac{3}{2x^2} + \frac{1}{2y^2} - \frac{3}{2x^2}$$

$$= 0$$

$\therefore$ f, g are dependent functionally

Now $f(x,y) = \log x - \log y \qquad g(x,y) = \frac{x}{2y} + \frac{3y}{2x}$

$$f(x,y) = \log\left(\frac{x}{y}\right)$$

$$e^{f(x,y)} = \frac{x}{y}$$

$$\therefore \quad g(x,y) = \frac{1}{2} e^{f(x,y)} + \frac{3}{2} e^{-f(x,y)}$$

$$2g(x,y) = e^{f(x,y)} + 3e^{-f(x,y)}$$

(ii) ~~find~~ $J = 0$

$\frac{4}{\sqrt{3}}$

$$f(x,y) = \frac{y}{x} \qquad g(x,y) = \frac{x-y}{x+y}$$

$$g(x,y) = \frac{1 - y/x}{1 + y/x}$$

$$g(x,y) = \frac{1 - f(x,y)}{1 + f(x,y)}$$

(14) $u = x+y+z$

$v = x^2+y^2+z^2$

$w = x^3+y^3+z^3-3xyz$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy) \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ x^2-yz & (y-x)(x+y+z) & (z-x)(x+y+z) \end{vmatrix}$$ by $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

= expanding we get 0.

$\therefore$ u, v, w are dependent

Now to find the relationship.

$$w = x^3+y^3+z^3-3xyz$$

$$= (x+y+z)(x^2+y^2+z^2-yz-zx-xy)$$

$$\cancel{= (x+y+z)[(x+y+z)^2-3(yz+zx+xy)]}$$

$$\cancel{= u[u^2-3}$$

$$= (x+y+z)(x^2+y^2+z^2-(xy+yz+zx))$$

Now $u = (x+y+z)$

$$u^2 = x^2+y^2+z^2+2xy+2yz+2zx$$

$$\Rightarrow xy+yz+zx = \frac{1}{2}(u^2-(x^2+y^2+z^2))$$

$$= \frac{1}{2}(u^2-v)$$

$$\therefore w = (x+y+z)\left((x^2+y^2+z^2)-\frac{1}{2}(u^2-v)\right)$$

$$= u\left(v-\frac{1}{2}(u^2-v)\right)$$

$$= \frac{1}{2}u(3v-u^2)$$

(15) $x_1 = u_1(1-u_2)$ $\quad x_3 = u_1u_2u_3(1-u_4)$

$x_2 = u_1u_2(1-u_3)$ $\quad x_4 = u_1u_2u_3u_4$

T.P. $\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = u_1^2u_2^2u_3^2$

$$J = \begin{vmatrix} (1-u_2) & -u_1 & 0 & 0 \\ u_2(1-u_3) & u_1(1-u_3) & -u_1u_2 & 0 \\ u_2u_3(1-u_4) & u_1u_3(1-u_4) & u_1u_2(1-u_4) & -u_1u_2u_3 \\ u_2u_3u_4 & u_1u_3u_4 & u_1u_2u_4 & u_1u_2u_3 \end{vmatrix}$$

expanding along R_1

$$= (1-u_2)\begin{vmatrix} u_1(1-u_3) & -u_1u_2 & 0 \\ u_1u_3(1-u_4) & u_1u_2(1-u_4) & -u_1u_2u_3 \\ u_1u_3u_4 & u_1u_2u_4 & u_1u_2u_3 \end{vmatrix} + u_1\begin{vmatrix} u_2(1-u_3) & -u_1u_2 & 0 \\ u_2u_3(1-u_4) & u_1u_2(1-u_4) & -u_1u_2u_3 \\ u_2u_3u_4 & u_1u_2u_4 & u_1u_2u_3 \end{vmatrix}$$

$$= (1-u_2)\left[u_1(1-u_3)\left(u_1^2u_2^2u_3(1-u_4) + u_1^2u_2^2u_3u_4\right) + u_1u_2\left(u_1^2u_2u_3^2(1-u_4) + u_1^2u_2u_3^2u_4\right)\right.$$

$$+ u_1\left[\left[u_2(1-u_3)\left(u_1^2u_2^2u_3(1-u_4) + u_1^2u_2^2u_3u_4\right)\right] + u_1u_2\left[u_1u_2^2u_3^2(1-u_4) + u_1u_2^2u_3^2u_4\right]\right]$$

$$= (1-u_2)\left[u_1(1-u_3)\left(u_1^2u_2^2u_3 - \cancel{u_1^2u_2^2u_3u_4} + \cancel{u_1^2u_2^2u_3u_4}\right)\right] + u_1u_2\left[u_1^2u_2u_3^2\right]$$

$$+ u_1\left[u_2(1-u_3)\left(u_1^2u_2^2u_3\right) + u_1u_2\left(u_1u_2^2u_3^2\right)\right]$$

$$= (1-u_2)\left[u_1(1-u_3)u_1^2u_2^2u_3 + u_1u_2\left(u_1^2u_2u_3^2\right)\right] + u_1\left[u_2(1-u_3)u_1^2u_2^2u_3 + u_1^2u_2^3u_3^2\right]$$

$$= (1-u_2)\,u_1^3u_2u_3\left[(1-u_3)u_2 + u_2u_3\right] + u_1^3u_2^3u_3\left[1-u_3+u_3\right]$$

$$= (1-u_2)\,u_1^3u_2u_3\left[u_2\right] + u_1^3u_2^3u_3$$

$$= u_1^3u_2^2u_3 - u_1^3u_2^3u_3 + u_1^3u_2^3u_3$$

$$= u_1^3u_2^2u_3$$

(16) $(t-x)^3 + (t-y)^3 + (t-z)^3 = 0$ roots u, v, w

then P.T. $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = -2\dfrac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

we have

$$(t-x)^3 + (t-y)^3 + (t-z)^3 = 0$$

$$3t^3 - 3t^2(x+y+z) + 3t(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$$

Since u, v, w are roots of this eqn.

So

$$u+v+w = (x+y+z)$$

$$uv+vw+uw = x^2+y^2+z^2$$

$$uvw = \frac{1}{3}(x^3+y^3+z^3)$$

Let

$$F_1 \equiv u+v+w-x-y-z$$

$$F_2 \equiv uv+vw+wu-x^2-y^2-z^2$$

$$F_3 = uvw - \frac{1}{3}(x^3+y^3+z^3)$$

Now

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \cdot \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}$$

evaluating these two Jacobians.

we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

---

(17) Squaring and adding $r^2 = x^2+y^2$

Now $\dfrac{\partial r}{\partial x} = 2x \quad \dfrac{\partial r}{\partial y} = 2y \quad \dfrac{\partial^2 r}{\partial x^2} = 2 \quad \dfrac{\partial^2 r}{\partial y^2} = 2 \quad \dfrac{\partial^2 r}{\partial x \partial y} = 0$

part (ii), (iii) are wrong

(18) If $u = \log_e (x^3+y^3+z^3-3xyz)$

Show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$

Now

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right]$$

$$= \frac{-9}{(x+y+z)^2}$$