ECN-203
19114079-Shreyas Dodamani
Assignment 3

1) a) $y(t) = t^2 x(t-1)$

Linearity:
$$y_1(t) = t^2 x_1(t-1)$$
$$y_2(t) = t^2 x_2(t-1)$$

$$\Rightarrow a x_1(t) + b(x_2(t)) \rightarrow y(t) = t^2(a x_1(t-1) + b x_2(t-1))$$

$$y(t) = a(t^2 x_1(t-1)) + b(t^2 x_2(t-1))$$
$$= a y_1(t) + b y_2(t)$$

$$\Rightarrow a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t)$$

$\boxed{\text{Hence, system is linear}}$

Time Invariance:
$$\cancel{y_1(t) = t^2 x_1(t) t_0}$$
$$y_1(t) = t^2 x_1(t-1)$$
$$x_2(t) = x_1(t-t_0)$$
$$\Rightarrow y_2(t) = t^2 x_2(t-1)$$
$$y_2(t) = t^2 x_1(t-t_0-1)$$

$$y_1(t-t_0) = (t-t_0)^2 x_1(t-t_0-1)$$
$$\neq t^2 x_1(t-t_0-1)$$

$$\Rightarrow y_2(t) \neq y_1(t-t_0)$$
when $x_2(t) = x_1(t-t_0)$

$\boxed{\text{Hence, system is NOT time-invariant}}$

b) $y[n] = (x[n-2])^2$

Linearity : $y_1[n] = (x_1[n-2])^2$
$y_2[n] = (x_2[n-2])^2$

$ax_1[n] + bx_2[n] \rightarrow y(\,)$
$\Rightarrow y[n] = a^2(x_1[n-2])^2 + b^2(x_2[n-2])^2$
$+ 2ab\, x_1[n-2]\, x_2[n-2]$

however, $ay_1[n] + by_2[n]$
$= a(x_1[n-2])^2 + b(x_2[n-2])^2$

$\Rightarrow ax_1[n] + bx_2[n] \not\rightarrow ay_1[n] + by_2[n]$

$\boxed{\text{Hence, system is NOT Linear}}$

Time variance $\quad y_1[n] = (x_1[n-2])^2$
Let $x_2[n] = x_1[n-n_0]$

$\Rightarrow y_2[n] = \cancel{(x_2[n-2])^2}\ (x_2[n-2])^2$
$= (x_1[n-n_0-2])^2$

$y_1[n-n_0] = (x_1[n-n_0-2])^2$

Hence, $y_2[n] = y_1[n-n_0]$ when
$x_2[n] = x_1[n-n_0]$

$\boxed{\text{Thus, system is time invariant}}$

Page:
Date: / /

c) $y[n] = x[n+1] - x[n-1]$

Linearity: $y_1[n] = x_1[n+1] - x_1[n-1]$
$y_2[n] = x_2[n+1] - x_2[n-1]$

$ax_1[n] + bx_2[n] \rightarrow ?$

$y[n] = (ax_1[n+1] + bx_2[n+1]) - (ax_1[n-1] + bx_2[n-1])$
$= a(x_1[n+1] - x_1[n-1]) + b(x_2[n+1] - x[n-1])$
$y[n] = a y_1[n] + b y_2[n]$

$\Rightarrow ax_1[n] + bx_2[n] \rightarrow a y_1[n] + b y_2[n]$

$\boxed{\text{Hence, system is linear}}$

Time Invariance: $y_1[n] = x_1[n+1] - x_1[n-1]$
Let $x_2[n] = x_1[n-n_0]$
$y_2[n] = x_2[n+1] - x_2[n-1]$
$y_2[n] = x_1[n-n_0+1] - x_1[n-n_0-1]$

$y_1[n-n_0] = x_1[n-n_0+1] - x_1[n-n_0-1]$

Hence, $y_2[n] = y_1[n-n_0]$
when $x_2[n] = x_2[n-n_0]$

$\boxed{\text{The system is time-invariant}}$

d) $y(t) = EV(x(t)) = \dfrac{x(t) + x(-t)}{2}$

<u>Linearity</u>: $y_1(t) = \frac{1}{2}(x_1(t) + x_1(-t))$

$y_2(t) = \frac{1}{2}(x_2(t) + x_2(-t))$

$ax_1(t) + bx_2(t) \rightarrow ?$

$y(t) = \frac{1}{2}\left((ax_1(t) + bx_2(t)) + (ax_1(-t) + bx_2(-t))\right)$

$= \frac{1}{2}(a)(x_1(t) + x_1(-t)) + \frac{1}{2}(b)(x_2(t) + x_2(-t))$

$= a(y_1(t)) + b(y_2(t))$

$\Rightarrow ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$\boxed{\text{Hence, system is linear}}$

<u>Time invariance</u>: $y_1(t) = \frac{1}{2}(x_1(t) + x_1(-t))$

Let $x_2(t) = x_1(t - t_0)$

$\Rightarrow y_2(t) = \frac{1}{2}(x_2(t) + x_2(-t)\cancel{-t_0})$

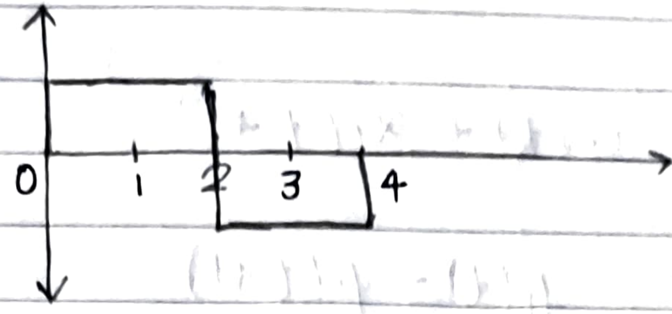
$= \frac{1}{2}(x_1(t - t_0) + x_1(-t + t_0))$

$y_1(t - t_0) = \frac{1}{2}(x_1(t - t_0) + x_1(-t - t_0))$

$\Rightarrow y_2(t) \neq y_1(t - t_0)$ when $x_2(t) = x_1(t - t_0)$

$\boxed{\text{Hence, system is time-variant}}$

2. $x_2(t) =$



$\Rightarrow x_2(t) = x_1(t) - x_1(t-2)$

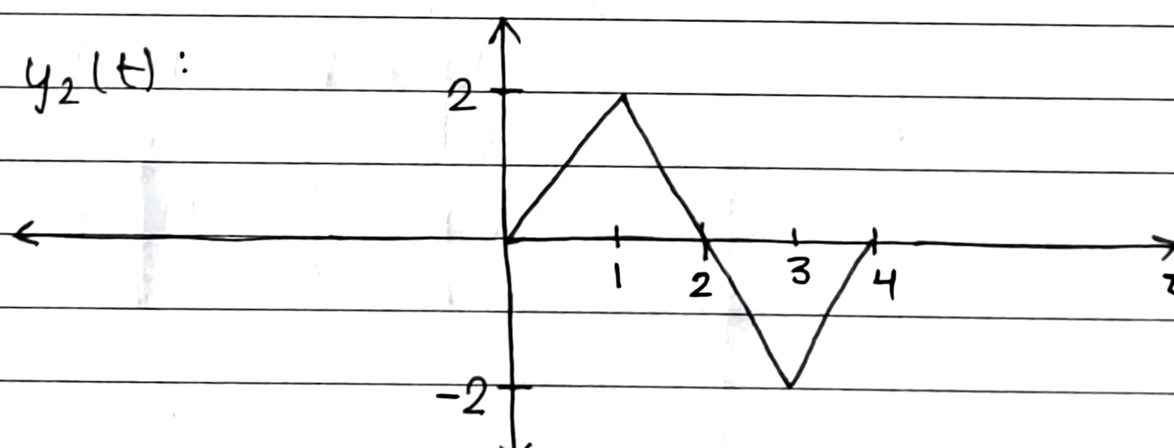
since system is LTI:

$y_2(t) = y_1(t) - y_1(t-1)$

$\cancel{\Rightarrow y_2(t) = \{ \; y_1(t) - 0}$

$$y_2(t) = \begin{cases} 2t & 0 \le t \le 1 \\ 4-2t & 1 \le t \le 2 \\ 4-2t & 2 \le t \le 3 \\ -8+2t & 3 \le t \le 4 \end{cases}$$

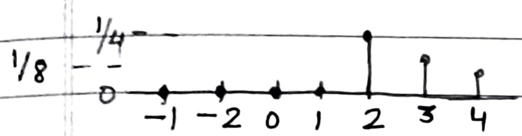
$y_2(t)$:

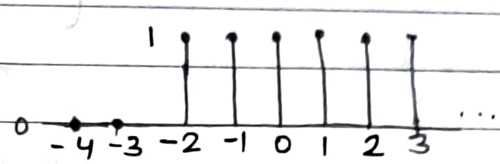


3) $x[n] = (1/2)^n u[n-2]$ $\qquad$ $h[n] = u[n+2]$

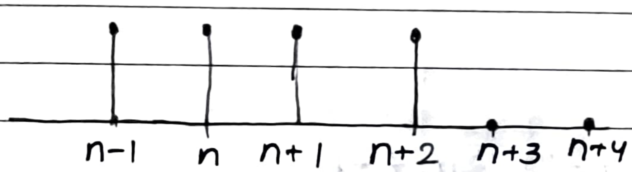
$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

$x[K]$: $\qquad\qquad$ $h[K]$





$h[n-K] = u[n-K+2]$



$$y[0] = \sum_{K=-\infty}^{\infty} x[K]\, h[-K] = \underbrace{\sum_{K=-\infty}^{1} 0}_{\text{b/c of } x[K]} + \underbrace{\sum_{K=2}^{\infty} 0}_{\text{b/c of } h[-K]} = 0$$

FOR $x[K] > 0$: $\quad K-2 \geq 0 \Rightarrow K >= 2 \Rightarrow$

FOR $h[n-K] > 0$: $\quad n-K+2 \geq 0 \Rightarrow K \leq n+2$

$\therefore \quad 2 \leq K \leq n+2$

$\boxed{\Rightarrow n \geq 0}$

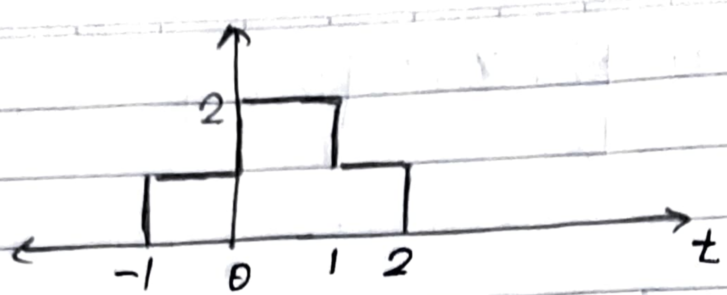
$$x[K] = \begin{cases} (1/2)^K, & K \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow y[n] = \sum_{K=2}^{\infty} (1/2)^K \left(u[n+2-K]\right)$$

$$\Rightarrow y[n] = \begin{cases} \displaystyle\sum_{K=2}^{n+2} (1/2)^K & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Page :
Date : / /

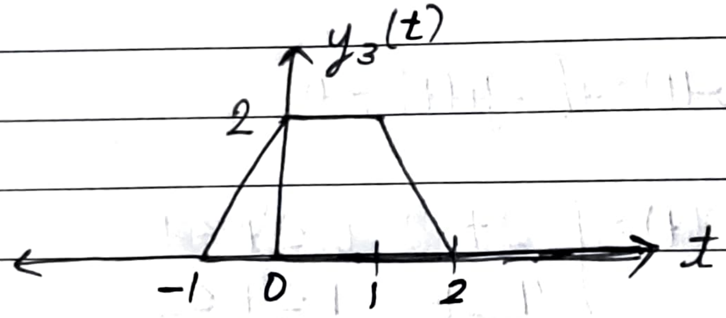
$x_3(t)$:



$$x_3(t) = x_1(t) + x_1(t+1)$$

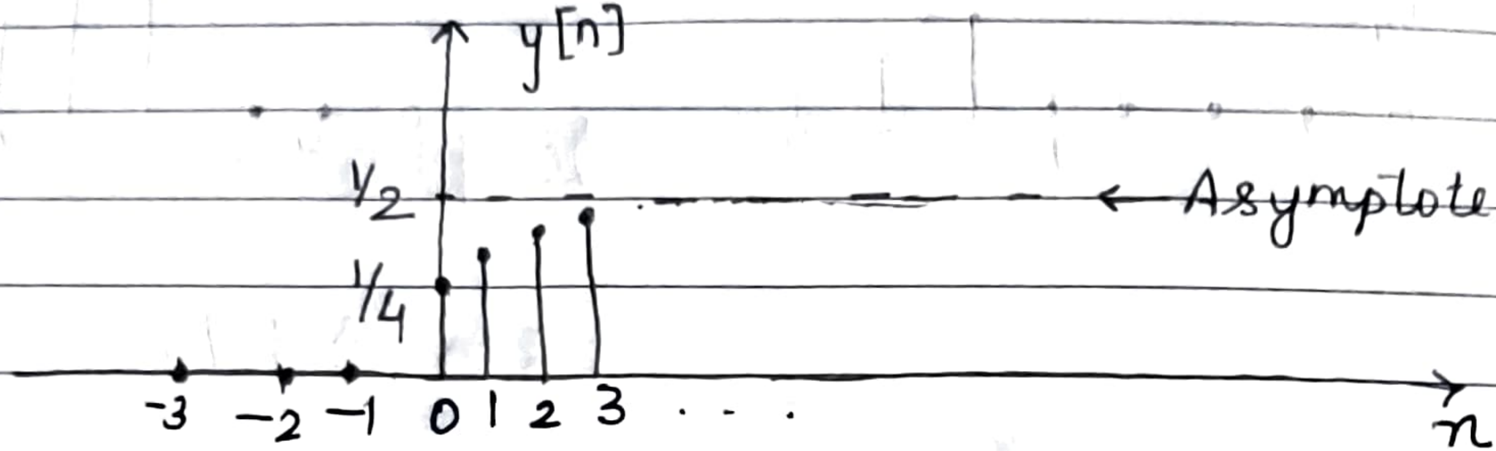
$$\Rightarrow y_3(t) = y_1(t) + y_1(t+1)$$

$$\Rightarrow y_3(t) = \begin{cases} 2t+2 & t \in [-1,0] \\ 2 & t \in [0,1] \\ 4-2t & t \in [1,2] \end{cases}$$



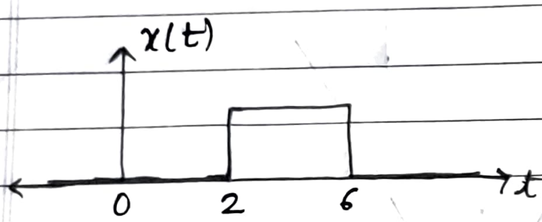
$$\Rightarrow y[n] = \begin{cases} \left(\frac{1}{4}\right)\left[\frac{1-(1/2)^{(n+1)}}{1-1/2}\right] & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

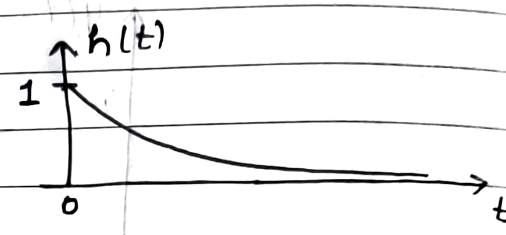
$$\Rightarrow y[n] = \begin{cases} \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n+1}\right) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



4) $x(t) = u(t-2) - u(t-6)$

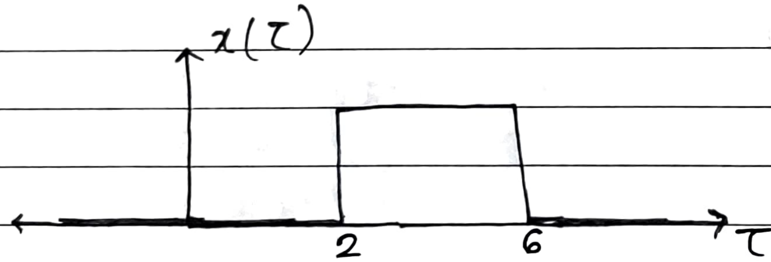
$h(t) = e^{-2t} u(t)$



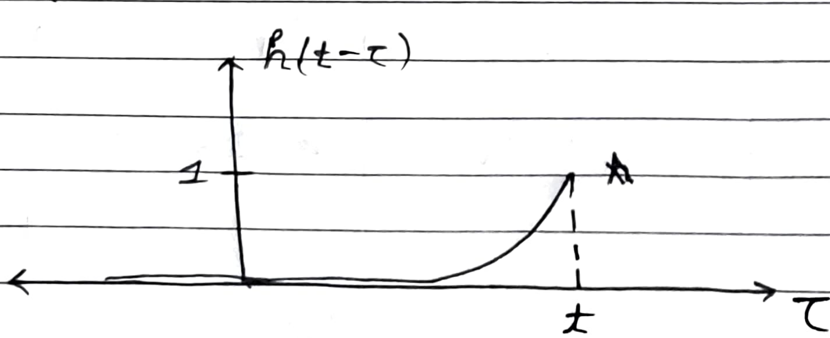


(a) $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

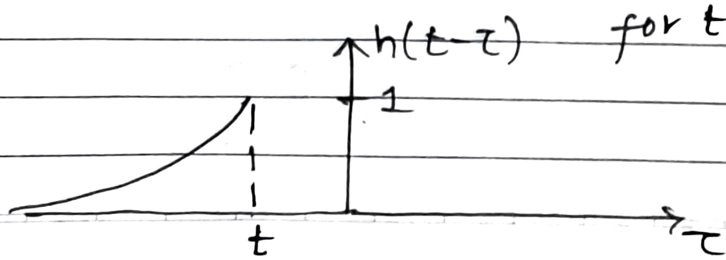
$x(\tau)$



$h(t-\tau)$



for $t > 0$



$t < 0$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

since $x(t) = 0$ for $-\infty < t < 2$ and $6 < t < \infty$

$$y(t) = \int_{-\infty}^{2} (0)(h(t-\tau)\,d\tau + \int_{2}^{6} x(\tau)(h(t-\tau)\,d\tau)$$

$$+ \int_{6}^{\infty} (0)(h(t-\tau)\,d\tau)$$

$$\therefore\ y(t) = \int_{2}^{6} h(t-\tau)\, d\tau$$

Let $t - \tau = \theta \Rightarrow d\tau = -d\theta$

$$\boxed{y(t) = \int_{t-6}^{t-2} h(\theta)\, d\theta}$$

$$y(t) = \int_{t-6}^{t-2} h(\theta)\,d\theta \qquad h(\theta) = \begin{cases} 0 & \theta < 0 \\ e^{-2\theta} & \theta \geq 0 \end{cases}$$

① FOR $t<2$: $h(\theta) = 0$

$$y(t) = \int_{t-6}^{t-2} 0\,d\theta = 0$$

② FOR $2<t<6$ $h(\theta) = \begin{cases} 0 & \theta<0 \\ e^{-2\theta} & \theta \geq 0 \end{cases}$

$$y(t) = \int_{t-6}^{0} 0\,d\theta + \int_{0}^{t-2} e^{-2\theta}\,d\theta$$

$$= \frac{-1}{2}\left(e^{4-2t} - 1\right) = \frac{1-e^{4-2t}}{2}$$

③ FOR $t>6$ $h(\theta) = e^{-2\theta}$

$$y(t) = \int_{t-6}^{t-2} e^{-2\theta}\,d\theta$$

$$= \frac{-1}{2}\left(e^{-2(t-2)} - e^{-2(t-6)}\right)$$

$$= \frac{e^{12-2t} - e^{4-2t}}{2}$$

$$\therefore \quad y(t) = \begin{cases} 0 & t<2 \\ \dfrac{1-e^{4-2t}}{2} & 2<t<6 \\ \dfrac{e^{12-2t}-e^{4-2t}}{2} & t>6 \end{cases}$$

(b) $g(t) = \left(\frac{d(x(t))}{dt} \right) * h(t)$

$$\frac{d(x(t))}{dt} = \frac{d(u(t-2) - u(t-6))}{dt}$$

$$= \delta(t-2) - \delta(t-6)$$

$$g(t) = (\delta(t-2) - \delta(t-6)) * h(t)$$

$$= h(t-2) - h(t-6)$$

$$h(t) = e^{-2t} u(t)$$

$$g(t) = e^{-2(t-2)} u(t-2) - e^{-2(t-6)} u(t-6)$$

$$= e^{-2t} (e^{4} u(t-2) - e^{12} u(t-6))$$

① $t-2<0 \Rightarrow t<2 \quad \Rightarrow u(t-2)=0, \; u(t-6)=0$

$g(t) = 0$

② $2<t<6 \Rightarrow \quad u(t-2)=1 \quad u(t-6)=0$

$$\Rightarrow g(t) = e^{-2t}(e^{4}) = e^{4-2t}$$

③ $t > 6$ $\quad u(t-2) = u(t-6) = 1$

$$g(t) = e^{-2t}(e^{4} - e^{12})$$
$$= e^{4-2t}(1 - e^{8})$$

$$\therefore g(t) = \begin{cases} 0 & t < 2 \\ e^{4-2t} & 2 \leq t \leq 6 \\ e^{4-2t}(1-e^{8}) & t > 6 \end{cases}$$

$$g(t) = e^{4-2t}(u(t-2)) - e^{4-2t}(1-e^{8})(u(t-6))$$

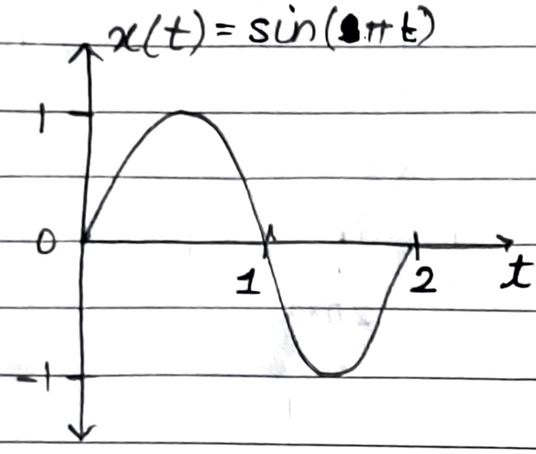
(c) $$y(t) = \begin{cases} 0 & t < 2 \\ 1/2\,(1 - e^{4-2t}) & 2 < t < 6 \\ 1/2\,(e^{12-2t} - e^{4-2t}) & t > 6 \end{cases}$$

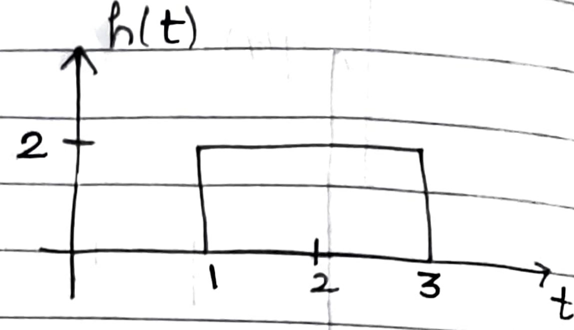
$$g(t) = \begin{cases} 0 & t < 2 \\ e^{4-2t} & 2 < t < 6 \\ e^{4-2t} - e^{12-2t} & 6 < t \end{cases}$$

$$\therefore y(t) = \begin{cases} g(t) & t < 2 \\ 1/2(1 - g(t)) & 2 < t < 6 \\ -1/2\, g(t) & t > 6 \end{cases}$$

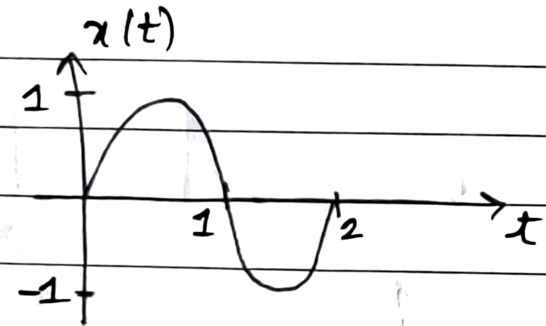
$$\boxed{\Rightarrow g(t) = \frac{d(y(t))}{dt} = \frac{d(x(t))}{dt} * h(t)}$$

5) a) $x(t)$

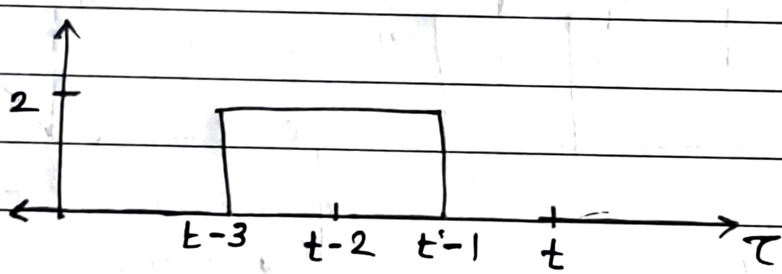




$x(\tau)$



$h(t-\tau)$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

since $x(\tau)$ is 0 everywhere except for $0 \leq \tau \leq 2$.

$$y(t) = \int_{0}^{2} \sin(\pi\tau)\, h(t-\tau)\, d\tau$$

Let $\theta = t - \tau$ $\qquad \tau \to 0 \Rightarrow \theta \to t$

$\Rightarrow d\theta = -d\tau$ $\qquad \tau \to 2 \Rightarrow \theta \to t-2$

Page :
Date : / /

$$\Rightarrow y(t) = \int_{t-2}^{t} \sin(\pi(t-\theta))\, h(\theta)\, d\theta$$

① $t<1$ and $t-2>3 \Rightarrow t>5$

$\Rightarrow h(\theta) = 0 \Rightarrow y(t) = 0$

② $1<t<3 \Rightarrow$ $$y(t) = \int_{t-2}^{1} 0\, d\tau + \int_{1}^{t} \sin(\pi(t-\theta))(2\, d\theta)$$

$$= \int_{1}^{t} \sin(\pi t - \pi\theta)(2\, d\theta)$$

$$= \frac{2}{\pi}\left(\cos(\pi t - \pi\theta)\right)_{1}^{t}$$

$$= \frac{2}{\pi}\left[\cos(\pi(t-t)) - \cos(\pi t - \pi)\right]$$

$$= \frac{2}{\pi}\left(1 - \cos(\pi(t-1))\right) = \frac{2}{\pi}\left[1 + \cos(\pi t)\right]$$

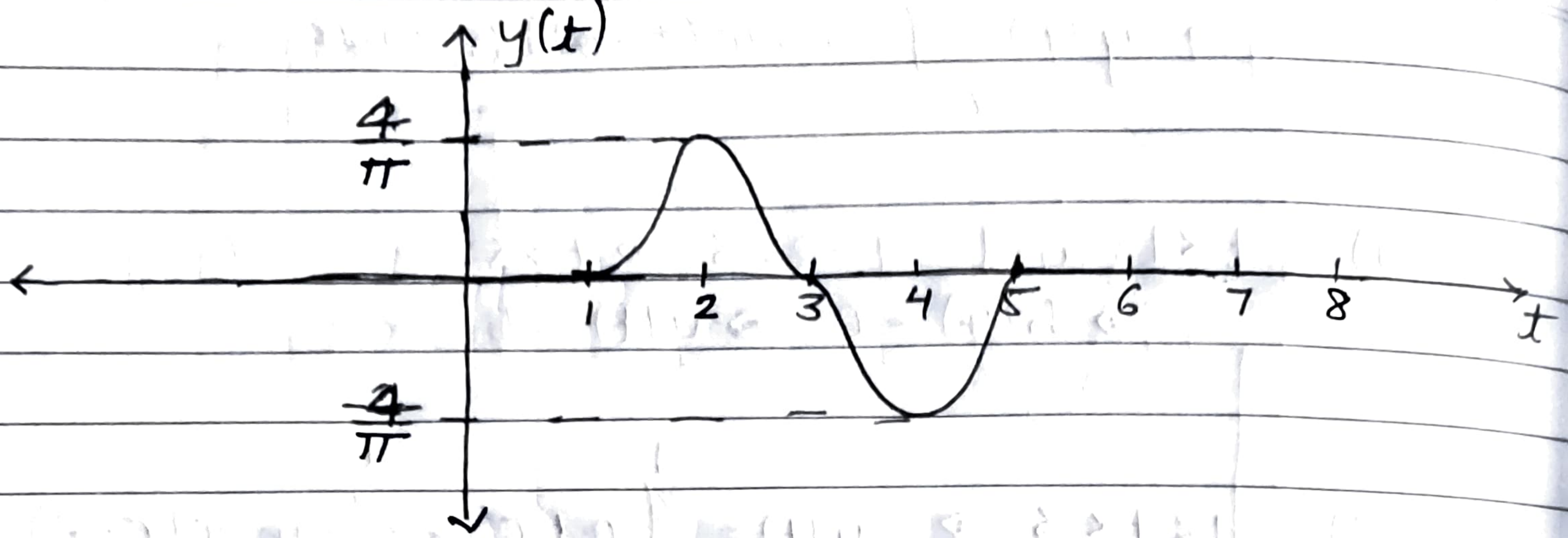
③ $1<t-2<3$

$3<t<5$ $$y(t) = \int_{t-2}^{3} 2\sin(\pi(t-\theta))\, d\theta + \int_{3}^{t} 0\, d\theta$$

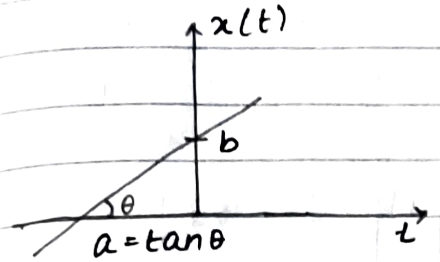
$$= \frac{2}{\pi}\left[\cos(\pi(t-\theta))_{t-2}^{3}\right]$$

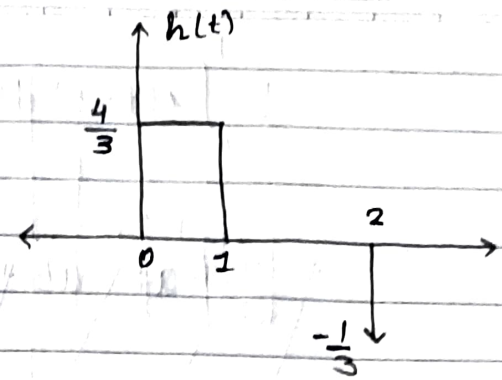
$$= \frac{2}{\pi}\left[-\cos(\pi t) - 1\right]$$

$$y(t) = \begin{cases} \frac{2}{\pi}(1+\cos(\pi t)) & 1<t<3 \\ \frac{-2}{\pi}(1+\cos(\pi t)) & 3<t<5 \\ 0 & \text{otherwise} \end{cases}$$


y(t)
4/π
-4/π
1
2
3
4
5
6
7
8
t

b)





$$x(\tau) = a\tau + b$$

Let $h(\tau) = h_1(\tau) + \left(\frac{-1}{3}\delta(t-2)\right)$

$$\therefore \quad y(t) = x(t) * h(t)$$
$$= x(t) * \left(h_1(t) - \tfrac{1}{3}\delta(t-2)\right)$$
$$= x(t) * h_1(t) - \tfrac{1}{3}x(t) * \delta(t-2)$$
$$= x(t) * h_1(t) - \frac{1}{3}x(t-2)$$

$$h_1(t) = \begin{cases} 4/3 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) * h_1(t) = \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau$$

since $h_1(\tau) = \begin{cases} 4/3 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

$x(t) * h_1(t) = \int_0^t$ Let $t - \tau = \theta \Rightarrow \tau = t - \theta$

$\Rightarrow d\tau = -d\theta$

$$x(t) * h_1(t) = \int_{-\infty}^{\infty} x(t-\theta)\, h(\theta)\, d\theta$$
$$= \int_0^1 x(t-\theta)(4/3)\, d\theta + 0$$

$$= \frac{4}{3}\int_0^1 x(t-\theta)\,d\theta$$

$$= \frac{4}{3}\int_0^1 (at - a\theta + b)\,d\theta$$

$$= \frac{4}{3}(at+b) - \frac{4}{3}a\left(\frac{1}{2}\right)$$

$$\therefore \quad x(t) * h_1(t) = \frac{4(at+b)}{3} - \frac{2a}{3}$$

$$\frac{1}{3}x(t-2) = \frac{1}{3}\left(a(t-2)+b\right)$$

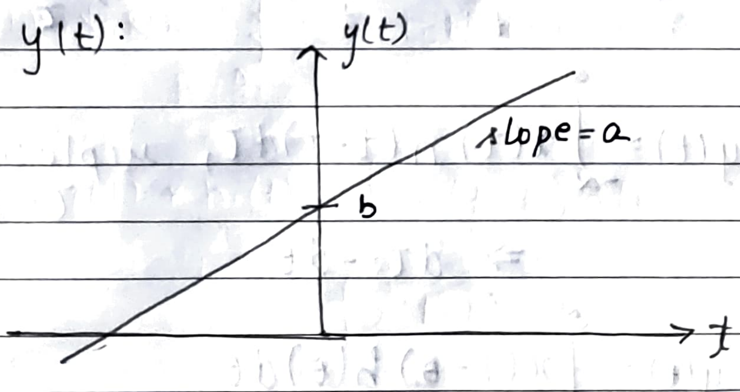
$$y(t) = x(t) * h_1(t) - \frac{1}{3}x(t-2)$$

$$= \frac{4(at+b)}{3} - \frac{2a}{3} - \frac{1}{3}\left(a(t-2)+b\right)$$

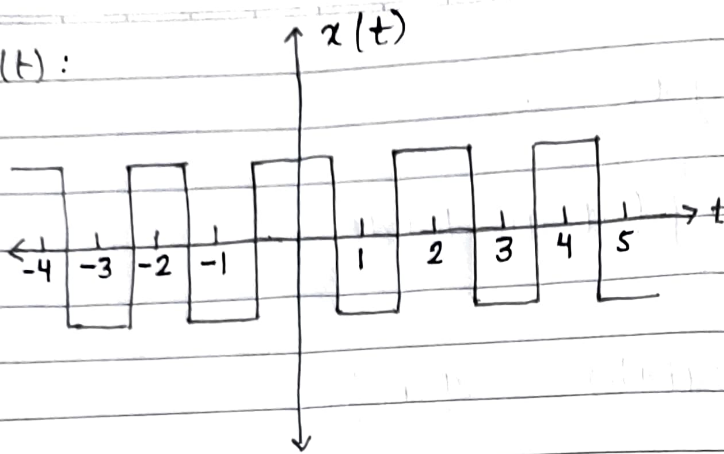
$$y(t) = at+b$$

$$\boxed{y(t) = x(t)}$$

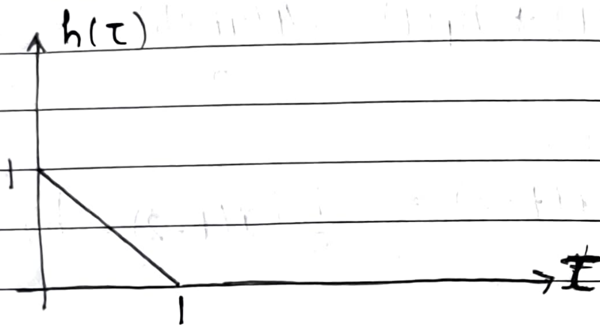
$y(t)$:



5) c) $x(t)$:

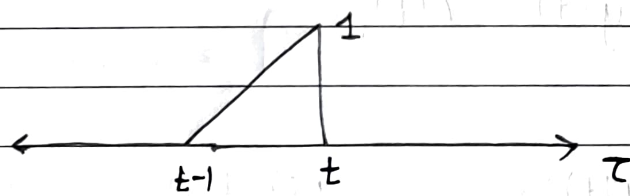


$h(\tau)$:



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$h(t-\tau)$:



In $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$, replace: $\tau = t - \theta$

$$\Rightarrow d\tau = -d\theta$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\theta)\, h(\theta)\, d\theta$$

since $h(\theta)$ is only non-zero b/w $0 \le \theta \le 1$:

$$y(t) = \int_0^1 x(t-\theta)(h(\theta))\,d\theta + \int_{-\infty}^{0} 0\,d\theta + \int_1^{\infty} 0\,d\theta$$

$$h(\theta) = \begin{cases} 1-\theta & 0 \le \theta \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore y(t) = \int_0^1 x(t-\theta)(1-\theta)\,d\theta$$

since $\tau = t-\theta$

replace again

$$\cancel{x} \; y(t) = \int_{t-1}^{t} x(\tau)(1-t+\tau)\,d\tau$$

$\theta \to 0 \Rightarrow \tau \to t$

$\theta \to 1 \Rightarrow \tau \to t-1$

$$x(\tau) = \begin{cases} -1 & 2n+\tfrac{1}{2} \le \tau \le 2n+\tfrac{3}{2} \\ +1 & 2n-\tfrac{1}{2} \le \tau \le 2n+\tfrac{1}{2} \end{cases} \quad (n \in \mathbb{I})$$

Page :

Date : / /

① $2n+1/2 \le t \le 2n+3/2 \qquad n \in I$

② $2n-1/2 \le t \le 2n+1/2$

To solve this, we can simply take $n=0$ because the function $x(t)$ is periodic, and same output will hold for all values of n.

① $n=0 \Rightarrow 1/2 \le t \le 3/2$

$$y(t) = \int_{t-1}^{1/2} 1(1-t+\tau)\,d\tau + \int_{1/2}^{t} (-1)(1-t+\tau)\,d\tau$$

$$= (1-t)\left(\tfrac{1}{2}-t+1\right) + \tfrac{1}{8} - \tfrac{(t-1)^2}{2} + (-1)\left[(1-t)\left(t-\tfrac{1}{2}\right) + \tfrac{t^2}{2} - \tfrac{1}{8}\right]$$

$$= \tfrac{1}{2} - t + 1 - \tfrac{t}{2} + t^2 - t + \tfrac{1}{4} + t^2 - 2t + 1 + (t-1)\left(t-\tfrac{1}{2}\right) + \tfrac{1}{8} - \tfrac{t^2}{2}$$

$$= \tfrac{7}{4} - \tfrac{5t}{2} + t^2 + t^2 - 2t + 1 + t^2 - \tfrac{3t}{2} + \tfrac{1}{2} + \tfrac{t^2}{2} - \tfrac{1}{8}$$

$$= t^2 - 3t + 7/4 \qquad \text{for: } 2n+1/2 \le t \le 2n+3/2 \quad n \in I$$

② $n=0 \Rightarrow -1/2 \le t \le 1/2$

$$\int_{t-1}^{-1/2} (-1)(1-t+\tau)\,d\tau + \int_{-1/2}^{t} (1-t+\tau)\,d\tau$$

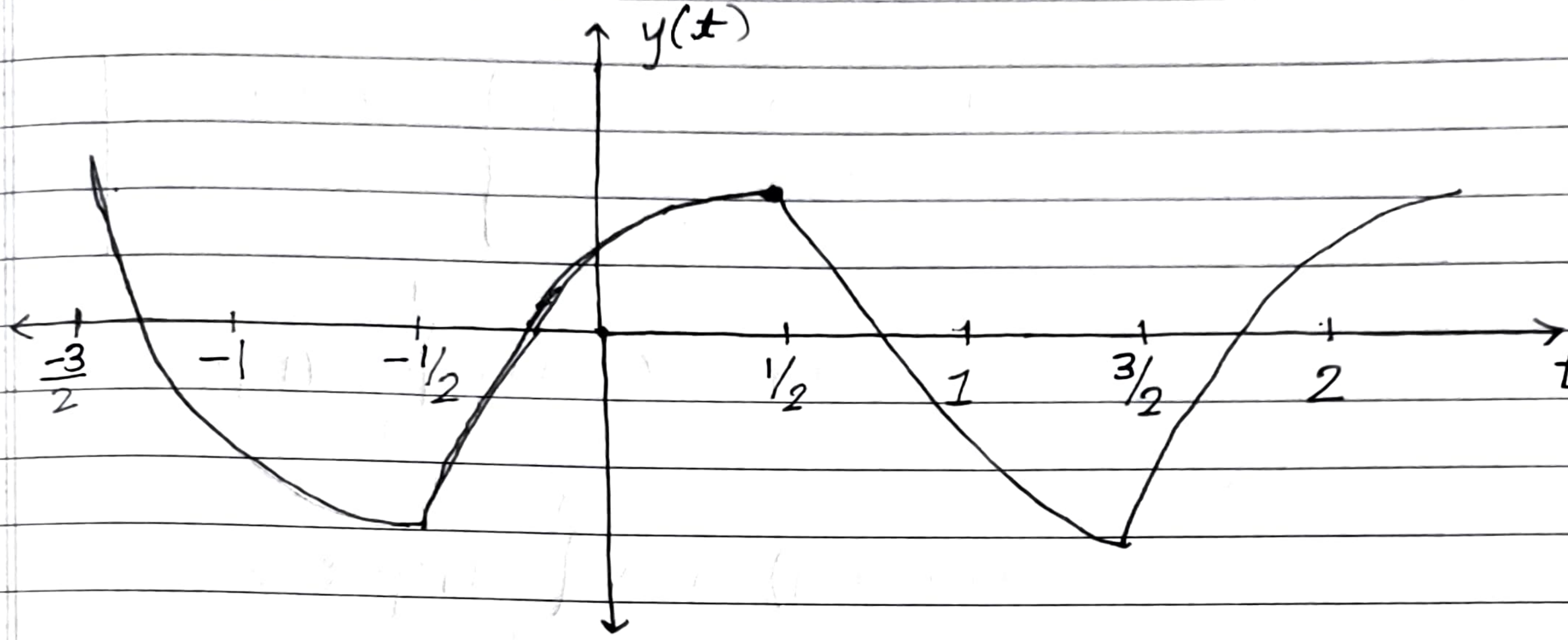
$$= \int_{t-1}^{-1/2} (t-1-\tau)\,d\tau + \int_{-1/2}^{t} (1-t+\tau)\,d\tau$$

$$= (t-1)\left(\tfrac{1}{2}-t\right) + \tfrac{(t-1)^2}{2} - \tfrac{1}{8} + (1-t)\left(t+\tfrac{1}{2}\right) + \tfrac{t^2}{2} - \tfrac{1}{8}$$

$$= -t^2 + \tfrac{3t}{2} - \tfrac{1}{2} + \tfrac{t^2}{2} - t + \tfrac{1}{2} - \tfrac{1}{8} + t + \tfrac{1}{2} - t^2 - \tfrac{t}{2} + \tfrac{t^2}{2} - \tfrac{1}{8}$$

$$= -t^2 + t + \tfrac{1}{4} \Rightarrow \text{for: } 2n - \tfrac{1}{2} \le t \le 2n + \tfrac{1}{2} \quad n \in I$$

since this function is periodic, the pattern that we observe between $[\frac{-1}{2}, \frac{3}{2}]$ will repeat itself. with a period of 2



$$y(t) = \begin{cases} (t-2n)^2 - 3(t-2n) + 7/4 & 2n+1/2 \le t \le 2n+3/2 \\ -(t-2n)^2 + (t-2n) + 1/4 & 2n-1/2 \le t \le 2n+1/2 \end{cases}$$