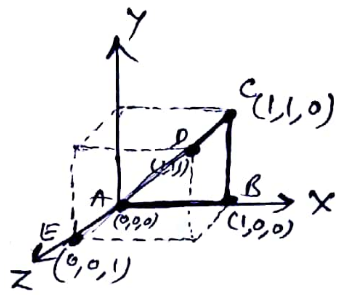
Assignment – 10 Deepak Singh

① $\int_C \vec{F}\cdot d\vec{r}$ $\vec{F} = x^2\hat{i} - xz\hat{j} + y^2\hat{k}$



$$\int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot dr + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DE} \vec{F}\cdot d\vec{r}$$

A (0,0,0)
B (1,0,0)
C (1,1,0)
D (1,1,1)
E (0,0,1)

along AB, $dy=0, dz=0$, $y=0, z=0$

along BC : $x=1$, $z=0$
$dx=0$ $dz=0$

along CD : (parallel to z) $dx=0, dy=0$, $y=1$

along DE : (parallel to XY plane) $dz=0$ & $(x=y)$
$z=1$ $dx=dy$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{AB} (x^2\hat{i} - xz\hat{j} + y^2z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{x=0}^{1} x^2\,dx = \frac{1}{3}$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{BC} (x^2\hat{i} - xz\hat{j} + y^2z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{y=0}^{1} (-xz)\,dy = -\int_0^1 -(1\times 0)\,dy = 0$$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_{CD} (x^2\hat{i} - xz\hat{j} + y^2z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{z=0}^{1} (y^2z)\,dz = \int_0^1 (1\cdot z)\,dz = \frac{1}{2}$$

$$\int_{DE} \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} (x^2\,dx - xz\,dy) = \int_0^1 (x^2 - x)\,dx = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \frac{1}{3} + 0 + \frac{1}{2} - \frac{1}{6} = \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

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(2). $\vec{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)\hat{k}$

T.P. $\vec{F}$ is conservative field. And find a function ϕ s.t. $\vec{F} = \nabla\phi$. Also w.d. form $(0,1,-1)$ to $(\frac{\pi}{2},-1,2)$

Sol. $\vec{F}$ is conservative if curl $\vec{F} = 0$

$$\text{curl}F = \nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2+2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right] + \hat{j}\left[\frac{\partial}{\partial z}(y^2\cos x + z^3) - \frac{\partial}{\partial x}(3xz^2+2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= \hat{i}[0-0] + \hat{j}[3z^2 - 3z^2] + \hat{k}[2y\cos x - 2y\cos x]$$

$$= 0$$

$\therefore \nabla\times\vec{F} = 0 \Rightarrow \vec{F}$ is conservative.

Now consider the function ϕ s.t. $F = \nabla\phi$

Then total differential coefficient

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \nabla\phi\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= F\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz$$

$$\Rightarrow \phi = \int(y^2\cos x + z^3)dx + \int(2y\sin x - 4)dy + \int(3xz^2+2)dz$$

$$\phi = y^2\sin x + xz^3 + y^2\sin x - 4y + xz^3 + 2z$$

$$\phi = 2y^2\sin x + 2xz^3 - 4y + 2z$$

$$\cancel{\text{Workdone} = \int_{AB} F.dr \qquad A(0,1,-1) \text{ to } B(\pi/2,-1,2)}$$

$$\cancel{= \int_0^{\pi/2} (y^2\cos x + z^3)\,dx + \int_1^{-1} (2y\sin x - 4)\,dy + \int_{-1}^{2} (3xz^2+2)\,dz}$$

$$\text{Work done} = \int_C F.dr \qquad C: \underset{(0,1,-1)}{A} \text{ to } \underset{(\frac{\pi}{2},-1,2)}{B}$$

$$= \int_C \nabla\phi \cdot dr$$

$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_C d\phi$$

$$= [\phi]_B - [\phi]_A$$

$$= \left[2y^2\sin x + 2xz^3 - 4y + 2z\right]_{(\pi/2,-1,2)} - \left[2y^2\sin x + 2xz^3 - 4y + 2z\right]_{(0,1,-1)}$$

$$= \left[2 + 2\cdot\frac{\pi}{2}\cdot 8 + 4 + 4\right] - \left[0 + 0 - 4 - 2\right]$$

$$= \left[2 + 8\pi + 8\right] - \left[-6\right]$$

$$= 8\pi + 16$$

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(24) $\iint_S \vec{F}\cdot\hat{n}\, ds$

(i) $\vec{F} = yz\,\hat{\imath} + zx\,\hat{\jmath} + xy\,\hat{k}$ and $S: x^2+y^2+z^2=1$ in first octant

Here $\phi = x^2+y^2+z^2-1$

Vector normal to the surface $= \nabla\phi = \hat{\imath}\dfrac{\partial\phi}{\partial x} + \hat{\jmath}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$$= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$\vec{F}\cdot\hat{n} = xyz + xyz + xyz = 3xyz$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|} = \iint_S (3xyz)\frac{dx\,dy}{z}$$

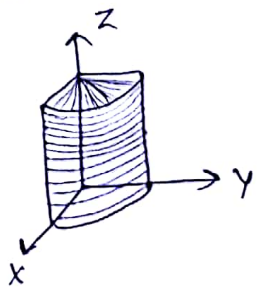
$$= \int_0^1\int_0^{\sqrt{1-x^2}} 3xy\,dx\,dy = 3\int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x(1-x^2)\,dx = \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right]$$

$$= \frac{3}{8}$$

(ii) $\vec{F} = z\hat{\imath} + x\hat{\jmath} - 3y^2z\,\hat{k}$ $\quad S: x^2+y^2=16$ in first octant $z=0$ to $z=5$.

$$\iint \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv \quad \text{(Gauss Div.)}$$



$$= \int_{z=0}^{5}\int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}} (-3y^2)\,dy\,dx\,dz$$

$$= -5\int_{x=0}^{4}\left[\frac{y^3}{3}\right]_0^{\sqrt{16-x^2}} dx = -5\int_{x=0}^{4}(16-x^2)^{3/2}dx =$$

(iii) $\vec{F} = \dfrac{\vec{r}}{r^3}$ $\quad S: x^2+y^2+z^2 = a^2$

$$\vec{F} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}} \qquad \phi = x^2+y^2+z^2-a^2$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$\vec{F} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a^3}$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint (\vec{F}\cdot\hat{n})\frac{dx\,dy}{(\hat{n}\cdot\hat{k})} = \iint \frac{(x^2+y^2+z^2)}{a^4}\,\frac{dx\,dy}{\left(\frac{z}{a}\right)}$$

$$= \frac{1}{a^3}\iint \left(\frac{x^2+y^2+z^2}{z}\right) dx\,dy$$

$$= \frac{1}{a^3}\iint \frac{a^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$= \frac{1}{a}\iint \frac{1}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$x = r\cos\theta \quad y = r\sin\theta$

$$= \frac{1}{a}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{1}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$= \frac{-1}{2a}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{-2r}{2\sqrt{a^2-r^2}}\,dr\,d\theta$$

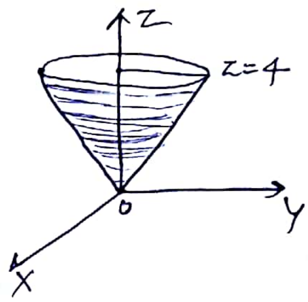
$$= \frac{-1}{2a}(2\pi)\left[\sqrt{a^2-r^2}\right]_{r=0}^{a}$$

$$= \frac{-1}{2a}(2\pi)\left[0-a\right]$$

$$= \pi$$

5 $\cancel{\iint_S \vec{F}\cdot\hat{n}\,ds}$ $\quad \vec{F} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$

$z^2 = x^2+y^2$



$$\iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \iiint_V (4z + xz^2 + 3)\,dx\,dy\,dz$$

Limits of z: $\sqrt{x^2+y^2}$ to 4

$$= \iint \int_{z=\sqrt{x^2+y^2}}^{4} (4z + \cancel{x}z^2 + 3)\,dz\,dy\,dx$$

$$= \iint \left[2z^2 + \frac{xz^3}{3} + 3z\right]_{z=\sqrt{x^2+y^2}}^{4} dy\,dx$$

$$= \iint \left[\left(32 + \frac{64x}{3} + 12\right) - \left[2(x^2+y^2) + \frac{x(x^2+y^2)^{3/2}}{3} + 3\sqrt{x^2+y^2}\,\right]\right] dy\,dx$$

$$= \iint \left[44 + \frac{64x}{3} - 2(x^2+y^2) - \frac{x(x^2+y^2)^{3/2}}{3} - 3\sqrt{x^2+y^2}\right] dy\,dx$$

$x = r\cos\theta$, $y = r\sin\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} \left[44 + \frac{64r\cos\theta}{3} - 2r^2 - \frac{r\cos\theta\, r^3}{3} - 3r\right) r\,dr\,d\theta$$

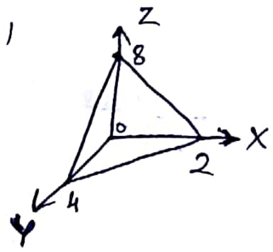
$$= \int_0^{2\pi}\int_0^{4} \left[44r + \frac{64r^2\cos\theta}{3} - 2r^3 - \frac{r^5\cos\theta}{3} - 3r^2\right) d\theta\,dr$$

(Solving)

$$= 320\pi$$

(6) $\iiint_V \phi \, dv \qquad \phi = 45x^2y \qquad V: 4x+2y+z=8, \; x=0, y=0, z=0$

$$\frac{x}{2}+\frac{y}{4}+\frac{z}{8}=1$$



$$\iiint_V 45\,(2u)^2(4v)\,64\,du\,dv\,dw$$

$$\left\{\begin{array}{l} \frac{x}{2}=u \quad \frac{y}{4}=v \quad \frac{z}{8}=w \\ x=2u \\ y=4v \qquad dx\,dy\,dz = 64\,du\,dv\,dw \\ z=8w \end{array}\right.$$

$$u+v+w=1$$

$$u, v, w \geq 0$$

$$= 45\times4\times4\times64 \iiint_V u^2 v w^0 \, du\,dv\,dw$$

$$= 45\times4^5 \iiint_V u^{3-1} v^{2-1} w^{1-1} \, du\,dv\,dw$$

$$= 45\times4^5 \frac{\Gamma 3\,\Gamma 2\,\Gamma 1}{\Gamma(3+2+1+1)}$$

$$= 45\times4^5 \frac{2!\,1!\,0!}{6!} = 45\cdot4^5 \cdot \frac{2}{6\cdot5\cdot4\cdot3\cdot2}$$

$$= \frac{45\cdot4^4}{6\cdot5\cdot3} = \frac{3\cdot4^4}{6} = \frac{4^4}{2} = 4\times4\times4\times2 \quad \cancel{=128}$$

$$= 128$$

7 $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds$, $\quad \vec{F} = y^2\hat{i} + y\hat{j} - xz\hat{k}$

$S : x^2+y^2+z^2=a^2$ above xy plane

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & y & -xz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(-xz)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(y^2)\right]$$

$$= \hat{i}[0-0] + \hat{j}[0-(-z)] + \hat{k}[0-2y]$$

$$= 0\hat{i} + z\hat{j} - 2y\hat{k}$$

$\phi = x^2+y^2+z^2-a^2 \qquad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$

$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a} \qquad \hat{n}\cdot\hat{k} = \frac{z}{a}$

$$(\nabla\times\vec{F})\cdot\hat{n} = (0\hat{i} + z\hat{j} - 2y\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)$$

$$= \frac{zy-2yz}{a} = -\frac{yz}{a}$$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \iint \left(-\frac{yz}{a}\right)\frac{dx\,dy}{n\cdot k}$$

$$= \iint \frac{(-yz)}{a}\frac{dx\,dy}{\left(\frac{z}{a}\right)}$$

$$= \iint (-y)\, dx\, dy$$

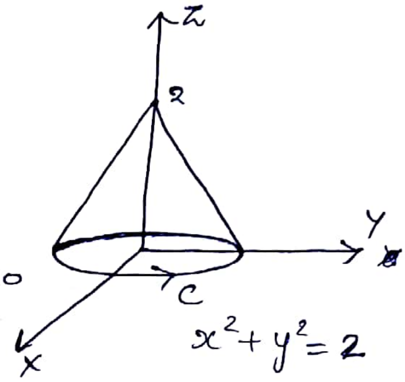
$$= -\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y\, dx\, dy \quad \text{(odd function)}$$

$$= 0$$

⑨ $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds \qquad F = (x-z)\hat{i} + (x^3 + yz)\hat{j} - 3xy^2\hat{k}$

S: $z = 2 - \sqrt{x^2+y^2}$
above xy plane

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \oint_c \vec{F} \cdot d\vec{r}$$



$\cancel{\vec{F} \cdot d\vec{r} = (x-z)dx + (x^3+yz)dy - 3xy^2 dz}$

$dr = dx\hat{i} + dy\hat{j}$

C is the boundary of circle $x^2+y^2 = 2, z=0$

$$= \oint_c (x-z)dx + (x^3+yz)dy$$

$x = \sqrt{2}\cos\theta \qquad dx = \sqrt{2}(-\sin\theta)d\theta$

$y = \sqrt{2}\sin\theta \qquad dy = \sqrt{2}\cos\theta\, d\theta$

$$= \oint_c x\,dx + x^3 dy$$

$$= \int_0^{2\pi} (\sqrt{2}\cos\theta)\sqrt{2}(-\sin\theta)\,d\theta + \int_0^{2\pi} (\sqrt{2})^3 \cos^3\theta\,(\sqrt{2}\cos\theta)\,d\theta$$

$$= 2\int_0^{2\pi} (-\sin\theta\cos\theta)\,d\theta + 4\int_0^{2\pi} \cos^4\theta\,d\theta$$

$$= -2\int_0^{2\pi} \sin\theta\cos\theta\,d\theta + 4\int_0^{2\pi} \cos^4\theta\,d\theta$$

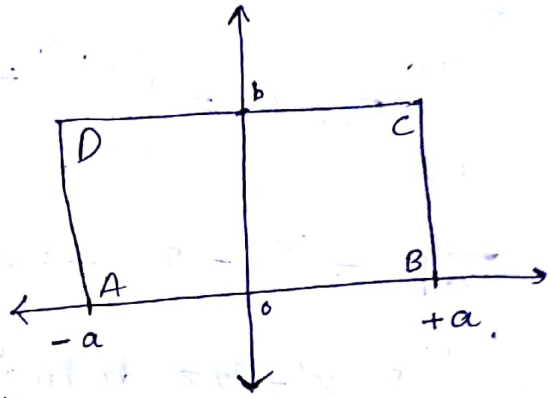
$$= 0 + 4 \cdot 2 \cdot 2 \cdot \int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 16 \cdot \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{4+0+2}{2}} = 8\,\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\pi}{\Gamma 3} = \frac{8 \cdot 3 \cdot \pi}{2 \cdot 2 \cdot 2} = 3\pi$$

10 Verify Stoke's theorem. $F = (x^2+y^2)\hat{i} - 2xy\,\hat{j}$

$x = \pm a,\ y = 0,\ y = b.$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$



$$= \hat{i}[0-0] + \hat{j}[0-0] + \hat{k}[-2y-2y]$$

$$= -4y\hat{k}$$

$$\iint_S \vec{F}\cdot d\vec{r} = \iint (\nabla\times F)\cdot\hat{n}\, ds \qquad ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$\cancel{= \iint (-4y)\, dx\, dy}$$

$$= \iint (-4y\hat{k})\cdot\hat{k}\, dx\, dy = -4\int_{x=-a}^{a}\int_{y=0}^{b} y\, dx\, dy.$$

$$= -4\int_{x=-a}^{a}\left[\frac{y^2}{2}\right]_0^b dx = -2b^2\int_{-a}^{a} dx = -4ab^2.$$

Now $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CD}\vec{F}\cdot d\vec{r} + \int_{DA}\vec{F}\cdot d\vec{r}$

AB: $y = 0,\ dy = 0$

$$\int_{AB}\vec{F}\cdot d\vec{r} = \int_{-a}^{a}(x^2+y^2)\,dx - 2xy\,dy = \int_{-a}^{a} x^2\,dx = 2\left[\frac{a^3}{3}\right] = \frac{2a^3}{3}$$

BC: $x = a,\ dx = 0$

$$\int_{BC}\vec{F}\cdot d\vec{r} = \int_0^b -2ay\,dy = -a\left[y^2\right]_0^b = -ab^2$$

CD: $y = b,\ dy = 0$

$$\int_{CD}\vec{F}\cdot d\vec{r} = \int_a^{-a}(x^2+b^2)\,dx = \left[\frac{x^3}{3}+b^2x\right]_a^{-a} = \left(-\frac{a^3}{3} - b^2a\right) - \left(\frac{a^3}{3} + b^2a\right)$$

$$= -\frac{2a^3}{3} - 2b^2a$$

DA: $x = -a\quad dx = 0$

$$\int_{DA}\vec{F}\cdot d\vec{r} = \int_b^0 -2(-a)y\,dy = a\int_b^0 2y\,dy = a\left[y^2\right]_b^0 = -ab^2$$

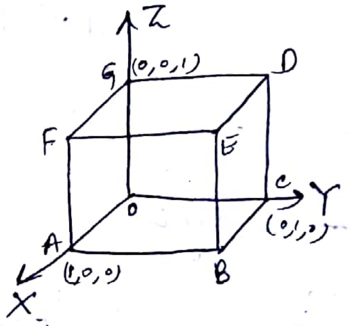
So $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2b^2a - ab^2 = -4ab^2$

(11) Verify Gauss Div. Theorem:

(1) $\vec{F} = (2x - z)\hat{i} - x^2 y\,\hat{j} + 4xz^2\hat{k}$

$x = 0,\ x = 1$
$y = 0,\ y = 1$
$z = 0,\ z = 1$

$$\iint_S F\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot F)\,dv$$



$$\nabla\cdot F = (2 - x^2 + 8xz)$$

$$\iiint_V (2 - x^2 + 8xz)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (2 - x^2 + 8xz)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[2z - x^2 z + 4xz^2\right]_0^1 dy\,dx$$

$$= \int_0^1\int_0^1 \left[2 - x^2 + 4x\right] dy\,dx$$

$$= \int_0^1 \left[2y - x^2 y + 4xy\right]_0^1 dx$$

$$= \int_0^1 (2 - x^2 + 4x)\,dx$$

$$= \left[2x - \frac{x^3}{3} + 2x^2\right]_0^1$$

$$= 2 - \frac{1}{3} + 2 = \frac{11}{3}$$

$$\iint_S F\cdot\hat{n}\,ds = \iint_{ABCO} F\cdot\hat{n}\,ds + \iint_{FEDG} F\cdot\hat{n}\,ds + \iint_{ABEF} F\cdot\hat{n}\,ds + \iint_{OCDG} F\cdot\hat{n}\,ds + \iint_{BCDE} F\cdot\hat{n}\,ds + \iint_{OGFA} F\cdot\hat{n}\,ds$$

ABCO	FEDG	ABEF	OCDG	BCDE	OGFA
$\hat{n} = -\hat{k}$	$\hat{n} = \hat{k}$	$\hat{n} = \hat{i}$	$\hat{n} = -\hat{i}$	$\hat{n} = \hat{j}$	$\hat{n} = -\hat{j}$
$z = 0$	$z = 1$	$x = 1$	$x = 0$	$y = 1$	$y = 0$
$dz = 0$	$dz = 0$	$dx = 0$	$dx = 0$	$dy = 0$	$dy = 0$
$dx\,dy$	$dx\,dy$	$dy\,dz$	$dy\,dz$	$\cancel{dy}\,dz$ dx	$\cancel{dy}\,dz$

$$\iint_{ABCO} \vec{F}\cdot\hat{n}\,ds = \iint(-4xz^2)\,dx\,dy = \iint -(4x\times 0)\,dx\,dy = 0$$

$$\iint_{FEDG} (4xz^2)\,dx\,dy = \int_0^1\int_0^1 4x\,dx\,dy = (2x^2)_0^1 = 2.$$

$$\iint_{ABEF} (2x-z)\,dz\,dy = \int_0^1\int_0^1 (2-z)\,dz\,dy = \left(2z-\frac{z^2}{2}\right)_0^1 = 2-\frac{1}{2} = \frac{3}{2}$$

$$\iint_{OCDG} (-(2x-z))\,dz\,dy = -\int_0^1\int_0^1 (0-z)\,dz\,dy = \frac{1}{2}$$

$$\iint_{BCDE} (-x^2y)\,dx\,dz = \int_0^1\int_0^1 -x^2\,dx\,dz = -\int_0^1 \left[\frac{x^3}{3}\right]_0^1 dz = -\frac{1}{3}$$

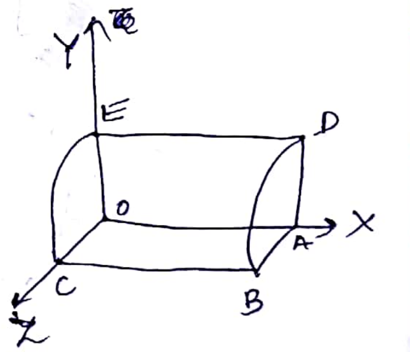
$$\iint_{OGFA} (-x^2y)\,dx\,dz = 0$$

adding all integrals

$$\iint F\cdot\hat{n}\,ds = 0+2+\frac{3}{2}+\frac{1}{2}-\frac{1}{3} = 2+2-\frac{1}{3} = 4-\frac{1}{3} = \frac{11}{3}$$

(ii) $F = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ first octant $x^2+y^2=9$, $x=2$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iiint (4xy-2y+8xz)\,dx\,dy\,dz$$



$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy-2y+8xz)\,dz\,.\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{3} \left[4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 4x(9-y^2)\right] dy\,dx$$

$$= \int_{x=0}^{2} \left[-\frac{4x}{2}\,\frac{2}{3}(9-y^2)^{3/2} + \frac{2}{3}(9-y^2)^{3/2} + 36xy - \frac{4xy^3}{3}\right]_0^3 dx$$

$$= \int_0^2 [0+0+108x-36x+36x-18)\,dx$$

$$= \int_0^2 (108x-18)\,dx = \left[108\frac{x^2}{2} - 18x\right]_0^2$$

$$= 216-36$$

$$= 180$$

$$\iint_S F.\hat{n}ds = \iint_{OABC} F.\hat{n}ds + \iint_{OCE} F.\hat{n}ds + \iint_{OADE} F.\hat{n}ds + \iint_{ABD} F.\hat{n}ds$$

$$+ \iint_{BCED} F.\hat{n}ds$$

$$\iint_{BDEC} F.\hat{n}ds = \iint_{BDEC} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2).\hat{n}\,ds$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2y\hat{j} + 2z\hat{k}}{\sqrt{4y^2+4z^2}} = \frac{y\hat{j}+z\hat{k}}{3} \qquad ds = \frac{dx\,dy}{\left(\frac{z}{3}\right)}$$

$$= \frac{1}{3}\iint (-y^3 + 4xz^3)\frac{dx\,dy}{(z/3)}$$

$$= \frac{1}{3}\int_{x=0}^{2}\int_{y=0}^{3}\left(-\frac{y^3}{z} + 4xz^2\right)dy\,dx \qquad \begin{array}{l} y = 3\sin\theta \\ z = 3\cos\theta \end{array}$$

$$= \cancel{\tfrac{1}{3}}\int_{x=0}^{2}\int_{\theta=0}^{\pi/2}\left[\frac{-27\sin^3\theta}{3\cos^3\theta} + 4x(9\cos^2\theta)\right]d\theta\,dx$$

$$= \int_0^2 \left(-27\frac{2}{3} + 108x + \frac{2}{3}\right)dx$$

$$= \int_0^2 (-18 + 172x)dx$$

$$= 108$$

$$\iint_{OABC} F.\hat{n}ds = \iint 4xz^2 ds = 0 \qquad (\because z = 0)$$

$$\iint_{OADE} F.\hat{n}ds = \iint y^2 ds = 0 \qquad (\because y = 0)$$

$$\iint_{OCE} F.\hat{n}ds = \iint -2x^2y\,ds = 0 \qquad (\because x = 0)$$

$$\iint_{ABD} \vec{F}\cdot\hat{n}\, ds = \iint 2x^2 y\, ds$$

$$= \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}} 2(2)^2 y\, dy\, dz$$

$$= 8\int_{z=0}^{3} \left[\frac{y^2}{2}\right]_0^{\sqrt{9-z^2}} dz$$

$$= 4\left[9z - \frac{z^3}{3}\right]_0^3 = 4(27-9) = 72$$

Adding all integrals

$$\iint_S F.\hat{n}\, ds = 0+0+0+108+72$$

$$= 180$$

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(12). Evaluate $\iint \vec{F}.\hat{n}\, ds$ $\vec{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$

$S: 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$

$$\iint F.\hat{n}\, ds = \iiint (\nabla . F)\, dv$$

$$= \iiint (2x+2y+2z)\, dx\, dy\, dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x+y+z)\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b} \left[cx + cy + \frac{c^2}{2}\right] dy\, dx$$

$$= 2\int_{x=0}^{a} \left[bcx + \frac{cb^2}{2} + \frac{c^2 b}{2}\right] dx$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= (a+b+c)\, abc$$