**Indian Institute of Technology Roorkee**
**MAN-001(Mathematics I)**
**Autumn Semester 2019–20**
**Assignment 6: (Multiple Integrals)**

1. Sketch the region $R$ in the $xy$-plane bounded by the curves $y^2 = 2x$ and $y = x$, and find its area.

2. Evaluate the following integrals by interchanging the order of integration:

   (a) $\int_0^1 \int_{4y}^4 e^{x^2} dxdy.$ (b) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx.$ (c) $\int_0^\infty \int_0^x xe^{-\frac{x^2}{y}} dydx.$

   (d) $\int_0^8 \int_{y^{\frac{1}{3}}}^2 \sqrt{(x^4+1)} dxdy.$ (e) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y) dxdy.$ (f) $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}} dydx.$

3. Evaluate the following double integrals:

   (a) $\int\int_D (4x+2) dA$, where $D$ is a region enclosed by the curves $y = x^2$ and $y = 2x$.

   (b) $\int\int_R (x^2+y^2) dA$, where $R$ is the region of the plane given by $x^2 + y^2 \le a^2$.

   (c) $\int_0^1 \int_{y^2}^1 (ye^{x^2}) dxdy.$ (d) $\int_0^1 \int_{\sqrt{3}\,y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2} dxdy.$

4. Evaluate the following triple integrals:

   (a) $\int\int\int_E 2x dV$, where $E$ is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

   (b) $\int\int\int_E \sqrt{3x^2+3z^2} dV$, where $E$ is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

   (c) $\int\int\int_E xyz dV$, where $E$ is the solid bounded by the sphere of radius 2 in the first octant.

   (d) $\int\int\int_E z dV$, where $E$ be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$.

   (e) $\int\int\int_E dV$, where $E$ is the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 1$ and $x + z = 5$.

5. Evaluate the following integrals by changing the variables into cylindrical coordinates:

   (a) $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{16-x^2-y^2} \sqrt{x^2+y^2} dzdydx.$

   (b) $\int\int\int_E \sqrt{(x^2+y^2)} dV$, where $E$ is the region lying above the $xy$-plane and below the cone $z = 4 - \sqrt{x^2+y^2}$.

   (c) $\int\int\int_E dV$, where $E$ is the region bounded above by the sphere $x^2+y^2+z^2 = a^2$ and below by the plane $z = b$, where $a > b > 0$.

6. By using spherical coordinates evaluate the following triple integrals:

   (a) $\int\int\int_E (x^2+y^2+z^2)^{\frac{1}{2}} dV$, where $E$ is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$.

(b) $\int\int\int_E (x^2+y^2+z^2)^{-\frac{3}{2}}dV$, where $E$ is the region bounded by the spheres of radius 2 and 3.

7. Show the followings by changing the order of integration:

   (a) $\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)drd\theta = \int_0^{2a}\int_0^{\cos^{-1}(r/2a)} f(r,\theta)d\theta dr.$

   (b) $\int_0^{\pi/3}\int_{a\sec^2(\theta/2)}^{(8a/3)\cos\theta} f(r,\theta)drd\theta = \left[\int_a^{4a/3}\int_0^{2\cos^{-1}(\sqrt{a/r})} + \int_{4a/3}^{8a/3}\int_0^{\cos^{-1}(3r/8a)}\right] f(r,\theta)d\theta dr.$

   (c) $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)dydx = \left[\int_0^{\frac{a}{2}}\int_{\frac{y^2}{a}}^{\frac{a}{2}-\sqrt{\frac{a^2}{4}-y^2}} + \int_0^{\frac{a}{2}}\int_{\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a} + \int_{\frac{a}{2}}^{a}\int_{\frac{y^2}{a}}^{a}\right] f(x,y)dxdy.$

8. Evaluate $\int\int_R (\frac{x-y}{x+y+2})^2 dxdy$, where $R$ is the region bounded by the lines $x+y=\pm 1,\ x-y=\pm 1$. (Use the transformation $u=x+y,\ v=x-y$ and integrate over an appropriate region in $uv-$plane.)

9. Evaluate $\int\int_R (x+y)dA$, where $R$ is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $(5/2,5/2)$ and $(5/2,-5/2)$, using the transformation $x=2u+3v$ and $y=2u-3v$.

10. Evaluate $\int\int_R e^{x^2-y^2}dA$, where $R$ is the region in the first quadrant bounded by $x^2-y^2=1,\ x^2-y^2=4, y=0$ and $y=(3/5)x$, by using the transformation $u=x^2-y^2$ and $v=x+y$.

11. Evaluate $\int_0^3\int_0^4\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1}(\frac{2x-y}{2}+\frac{z}{3})dxdydz$ by applying the transformation $u=\frac{2x-y}{2},\ v=\frac{y}{2},\ w=\frac{z}{3}$, and integrating over an appropriate region in $uvw-$plane.

12. Prove that

    (a) $\int_0^a\int_0^x \frac{f'(y)dydx}{\sqrt{(a-x)(x-y)}} = \pi(f(a)-f(0)).$

    (b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{bdydx}{(x^2+y^2+b^2)^{3/2}(x^2+y^2+a^2)^{1/2}} = \frac{2\pi}{a+b}$. (By changing into polar coordinates.)

## Answers:

**1.** $\frac{2}{3}$.

**2.** (a) $\frac{1}{8}(e^{16}-1)$. (b) 1. (c) $\frac{1}{2}$. (d) $\frac{1}{6}(17^{\frac{3}{2}}-1)$. (e) $\frac{241}{60}$. (f) $\frac{\pi a^2}{6}$.

**3.** (a) 8. (b) $\frac{\pi a^4}{2}$. (c) $\frac{1}{4}(e-1)$. (d) $\frac{4\pi}{9}$.

**4.** (a) 9. (b) $\frac{256\sqrt{3}\ \pi}{15}$. (c) $\frac{4}{3}$. (d) $\frac{1}{8}$. (e) $36\pi$.

**5.** (a) $\frac{1024(\pi)}{15}$. (b) $\frac{64}{3}(2\pi)$. (c) $2\pi(\frac{a^2}{3}-\frac{a^2b}{2}+\frac{b^3}{6})$.

**6.** (a) $\frac{27\pi}{2}(2\sqrt{2}-1)$, (b) $4\pi\log(\frac{3}{2})$.

**8.** $\frac{2}{9}$. **9.** $\frac{125}{4}$.

**10.** $\frac{\log 2}{2}(e^4-e)$. **11.** 12.