Indian Institute of Technology Roorkee
MAN-001(Mathematics-1), Autumn Semester: 2019-20
Assignment-3: Differential Calculus

(1) Find the following limits, if they exist

(a) $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$ (c) $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin^2(x+y)}{|x|+|y|}$ (f) $\displaystyle\lim_{(x,y)\to(1,1)} f(x,y)$ where $f(x,y) = \begin{cases} 1, & \text{if } x+y \geq 2 \\ -1, & \text{if } x+y < 2 \end{cases}$ (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$

(2) (a) Consider the function $f(x,y) = \dfrac{x+y}{x-y}$ for $(x,y) \in \mathbb{R}^2$ with $x-y \neq 0$. Show that $\displaystyle\lim_{x\to0}[\lim_{y\to0} f(x,y)] = 1$, but $\displaystyle\lim_{y\to0}[\lim_{x\to0} f(x,y)] = -1$. What can you say about the existence of $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$?

(b) Let $f(x,y) = 0$ if $y = 0$, and $f(x,y) = x\sin\left(\dfrac{1}{y}\right)$, if $y \neq 0$. Compute $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ and iterated limits $\displaystyle\lim_{x\to0}[\lim_{y\to0} f(x,y)]$ and $\displaystyle\lim_{y\to0}[\lim_{x\to0} f(x,y)]$ if they exist.

(c) Let $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$ if $x^2y^2+(x-y)^2 \neq 0$. Show that $\displaystyle\lim_{x\to0}[\lim_{y\to0} f(x,y)]$ and $\displaystyle\lim_{y\to0}[\lim_{x\to0} f(x,y)] = 0$. But $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(3) Let $f(x,y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$. Show that for any point (a,b), $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

(4) Examine the continuity of the function $f(x,y)$ at $(0,0)$ in each of the following cases. Also check the existence of $f_x(0,0)$ and $f_y(0,0)$.

(a) $f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \neq (0,0) \\ \dfrac{1}{2}, & (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$

(c) $f(x,y) = \begin{cases} xy\log(x^2+y^2), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y. \end{cases}$

(5) For the function $f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

Compute $f_x(0,y), f_y(x,0), f_x(0,0)$ and $f_y(0,0)$, if they exist.

(6) Show that for the function $f(x,y) = \begin{cases} -xy, & |y| \geq |x| \\ xy, & |y| < |x| \end{cases}$, $f_{xy}(0,0)$ and $f_{yx}(0,0)$ both exist and are unequal.

(7) Prove that $|x|+|y|$ is continuous, but not differentiable at $(0,0)$.

(8) Prove that $f(x,y) = \begin{cases} \frac{(x+y)\{\sqrt{(x^2+y^2)}+xy\}}{\sqrt{(x^2+y^2)}}, & \text{when } x^2+y^2 \neq 0 \\ 0, & \text{when } x^2+y^2 = 0, \end{cases}$ is differentiable at $(0,0)$. Hence, deduce that $f_x(0,0) = f_y(0,0) = 1$.

(9) Show that the function

$$f(x,y) = \begin{cases} x^3 \sin\frac{1}{x^2} + y^3 \sin\frac{1}{y^2}, & \text{when } xy \neq 0 \\ x^3 \sin\frac{1}{x^2}, & \text{when } x \neq 0 \text{ and } y = 0 \\ y^3 \sin\frac{1}{y^2}, & \text{when } x = 0 \text{ and } y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$

is differentiable at $(0,0)$, whereas none of f_x, f_y is continuous at $(0,0)$.

(10) Show that the function $f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is differentiable at $(0,0)$ and that f_x, f_y are not continuous at $(0,0)$.

(11) Determine the values of p for which $f(x,y) = |xy|^p,\ xy \neq 0$, and $f(x,y) = 0,\ xy = 0$, is continuous and differential at $(0,0)$.

Answers:

1. (a) -8 (b) does not exist (c) does not exist (d) 0 (e) 0 (f) does not exist (g) 0
2. (a) does not exist (b) 0, does not exist, 0
4. (a) continuous, 0,0 (b) continuous, does not exists (c) continuous, 0,0 (d) discontinuous, 0,0
5. 0, 1, 0, -1.
11. For continuity $p > 0$ and for differentiability $p \geq 1/2$.