Assignment – 6

(1). $y^2 = 2x$, $y = x$

$A = \int\limits_{x=0}^{2}\int\limits_{y=x}^{\sqrt{2x}} dx\,dy = \int_{0}^{2}(\sqrt{2x}-x)\,dx$

$= \left[\frac{2\sqrt{2}}{3}x^{3/2}-\frac{x^2}{2}\right]_{0}^{2} = \left[\frac{2\sqrt{2}\cdot 2^{3/2}}{3}-\frac{2^2}{2}\right] = \left[\frac{8}{3}-\frac{4}{2}\right] = \frac{2}{3}$
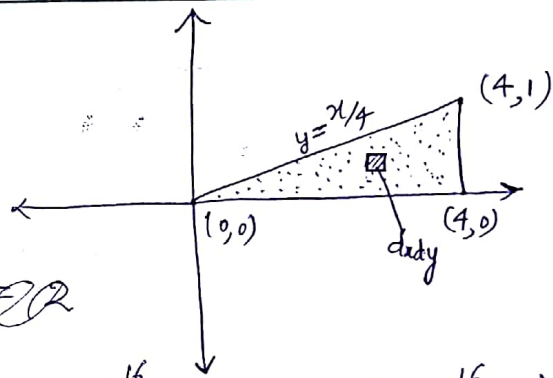
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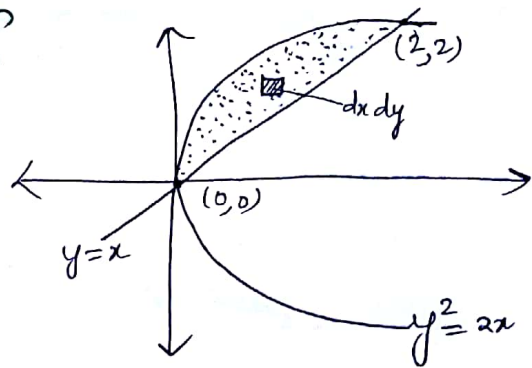
(2) (a) $\int_{0}^{1}\int_{4y}^{4} e^{x^2}\,dx\,dy$

$y=0 \qquad x=4y$
$y=1, \qquad x=4.$

$A = \int\limits_{x=0}^{4}\int\limits_{y=0}^{x/4} e^{x^2}\,dx\,dy$

$= \int_{0}^{4}\frac{x}{4}\,e^{x^2}\,dx = \frac{1}{8}\int_{0}^{4} 2x\,e^{x^2}\,dx = \frac{1}{8}\int_{0}^{16} e^{t}\,dt = \frac{1}{8}(e^{16}-1)$

(b) $\int_{0}^{\infty}\int_{x}^{\infty}\frac{e^{-y}}{y}\,dy\,dx$

$x=0 \qquad y=x$
$x=\infty \qquad y=\infty$

$A = \int\limits_{y=0}^{\infty}\int\limits_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty}\frac{e^{-y}}{y}[y-0]\,dy$

$= \int_{0}^{\infty}e^{-y}\,dy = \left[-e^{-y}\right]_{0}^{\infty} = [-0+1] = 1$
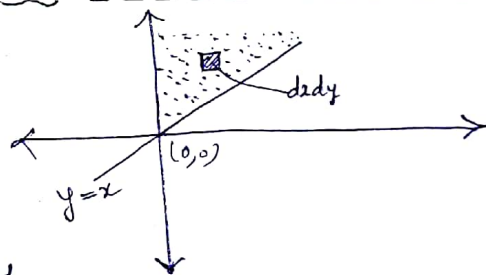
(c). $\int_{0}^{\infty}\int_{0}^{x} x\,e^{-x^2/y}\,dy\,dx$

$x=0,\infty \qquad y=0,x$

$A = \int\limits_{y=0}^{\infty}\int\limits_{x=y}^{\infty} x\,e^{-x^2/y}\,dx\,dy \qquad \begin{array}{l}x^2=t\\ 2x\,dx=dt\end{array}$

$= \int\limits_{y=0}^{\infty}\int\limits_{y^2}^{\infty} e^{-t/y}\,\frac{dt}{2}\,dy = \frac{1}{2}\int_{0}^{\infty}\left[\frac{-e^{-t/y}}{1/y}\right]_{y^2}^{\infty} dy = -\frac{1}{2}\int_{0}^{\infty} y\,[0-e^{-y}]\,dy$
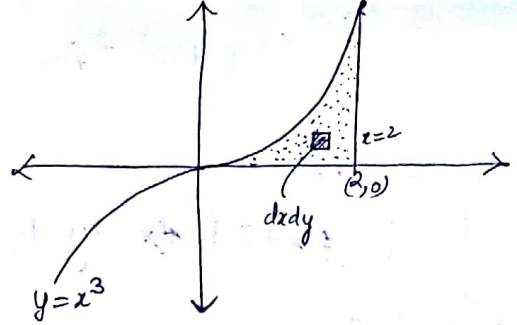
$= \frac{1}{2}$
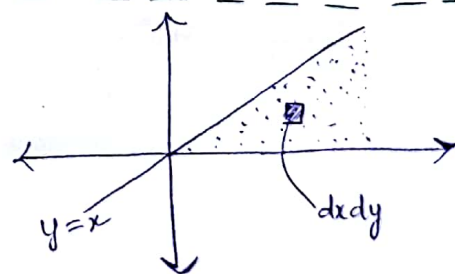
(d) $\int_0^8 \int_{y^{1/3}}^2 \sqrt{x^4+1}\, dx\, dy$

$y=0,8 \quad x=y^{1/3}, 2$

$x^3=y$

$A = \int_{x=0}^2 \int_{y=0}^{x^3} \sqrt{x^4+1}\, dy\, dx = \int_{x=0}^2 x^3\sqrt{x^4+1}\,dx \qquad y=x^3$

$= \frac{1}{4}\int_0^2 4x^3\sqrt{x^4+1}\,dx = \frac{1}{4}\left[\frac{2}{3}(x^4+1)^{3/2}\right]_0^2 = \frac{1}{4}\left[\frac{2}{3}(17^{3/2}-1)\right] = \frac{(17^{3/2}-1)}{6}$

(e) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$
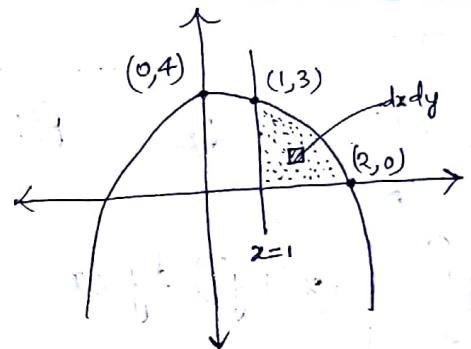
$y=0,3 \quad x=1, \sqrt{4-y} \longrightarrow x^2=4-y$

$A = \int_{x=1}^2 \int_{y=0}^{4-x^2} (x+y)\,dy\,dx$

$= \int_1^2\left[x(4-x^2)+\frac{(4-x^2)^2}{2}\right]dx = \int_1^2\left[4x-x^3+\frac{(16+x^4-8x^2)}{2}\right]dx$

$= \frac{1}{2}\int_1^2\left[8x-2x^3+16+x^4-8x^2\right]dx = \frac{1}{2}\left[\frac{x^5}{5}-\frac{x^4}{2}-\frac{8x^3}{3}+4x^2+16x\right]_1^2$

$= \frac{1}{2}\left[\left(\frac{32}{5}-\frac{16}{2}-\frac{64}{3}+16+32\right)-\left(\frac{1}{5}-\frac{1}{2}-\frac{8}{3}+4+16\right)\right]$
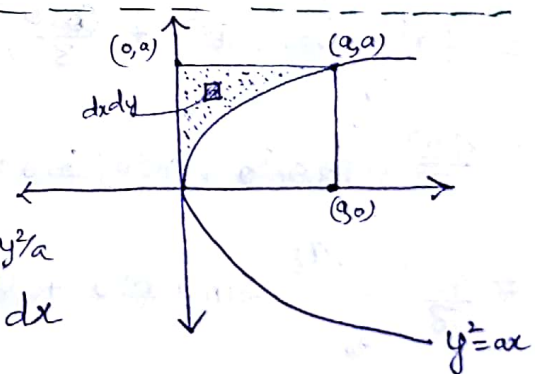
$= \frac{241}{60}$

(f) $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4-a^2x^2}}\,dy\,dx$

$x=0,a \quad y=\sqrt{ax}, a$
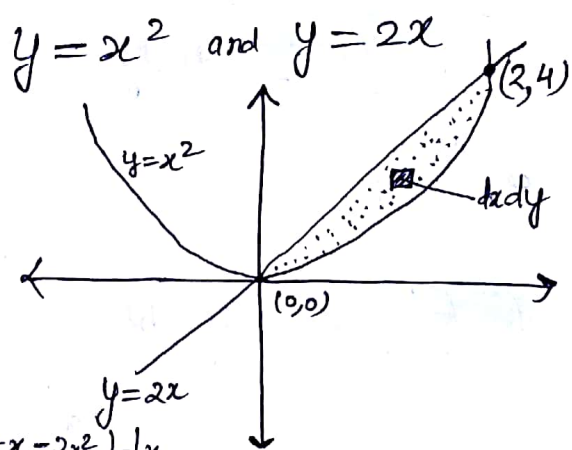
$y^2=ax$

$A = \int_{y=0}^a \int_{x=0}^{y^2/a} \frac{y^2}{\sqrt{y^4-a^2x^2}}\,dx\,dy = \int_{y=0}^a\left[\frac{y^2}{a}\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a}dx$

$= \int_{y=0}^a\left[\frac{y^2}{a}\sin^{-1}(1)-0\right]dx = \int_{y=0}^a \frac{\pi}{2a}y^2\,dx = \frac{\pi}{2a}\left[\frac{a^3}{3}\right] = \frac{\pi a^2}{6}$

③ (a). $\iint_D (4x+2)\,dA$ D: $y=x^2$ and $y=2x$
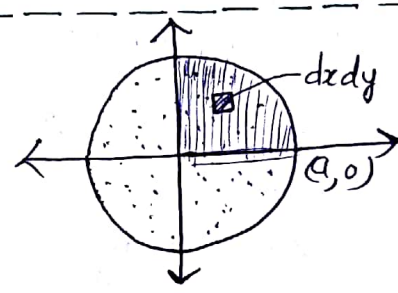
$$= \int_{x=0}^{2} \int_{y=x^2}^{2x} (4x+2)\, dy\, dx$$

$$= \int_{x=0}^{2} (4x+2)\left[2x-x^2\right] dx = \int_0^2 \left(8x^2-4x^3+4x-2x^2\right)dx$$

$$= \int_0^2 (6x^2-4x^3+4x)\, dx. = \left[2x^3 - x^4 + 2x^2\right]_0^2 = \left[16-16+8\right] = 8$$

(b). $\iint_R (x^2+y^2)\,dA$ R: $x^2+y^2 \le a^2$

$$= 4\int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx$$

$$= 4\int_{x=0}^{a} \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx = 4\int_{x=0}^{a} \left[x^2\sqrt{a^2-x^2} + \frac{(a^2-x^2)^{3/2}}{3}\right] dx$$

$x = a\sin\theta$ $dx = a\cos\theta\, d\theta$ $\theta \to 0$ to $\pi/2$

$$= 4\int_0^{\pi/2} \left[a^2\sin^2\theta\, a\cos\theta + \frac{a^3\cos^3\theta}{3}\right] a\cos\theta\, d\theta = 4a^4\int_0^{\pi/2}\left[\sin^2\theta\cos\theta + \frac{\cos^3\theta}{3}\right]\cos\theta\, d\theta$$

$$= 4a^4\int_0^{\pi/2}\left[\sin^2\theta\cos^2\theta + \frac{\cos^4\theta}{3}\right] d\theta = \frac{4a^4}{3}\int_0^{\pi/2}\left(3\sin^2\theta\cos^2\theta + \cos^4\theta\right) d\theta$$

$$= \frac{4a^4}{3}\int_0^{\pi/2}\left[3\sin^2\theta + \cos^2\theta\right]\cos^2\theta\, d\theta = \frac{4a^4}{3}\int_0^{\pi/2}\left[2\sin^2\theta + 1\right]\cos^2\theta\, d\theta$$

$$= \frac{4a^4}{3}\int_0^{\pi/2}\left(2\sin^2\theta\cos^2\theta + \cos^2\theta\right)d\theta = \frac{4a^4}{3}\left[2\cdot\frac{\sqrt{3/2}\,\sqrt{3/2}}{2\sqrt{\frac{2+2+2}{2}}} + \frac{\pi}{4}\right]$$

$$= \frac{4a^4}{3}\left[\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\pi}{\sqrt{3}} + \frac{\pi}{4}\right] = \frac{4a^4}{3}\left[\frac{\pi}{8} + \frac{\pi}{4}\right] = \frac{4a^4}{3}\left[\frac{3\pi}{8}\right] = \frac{\pi a^4}{2}$$
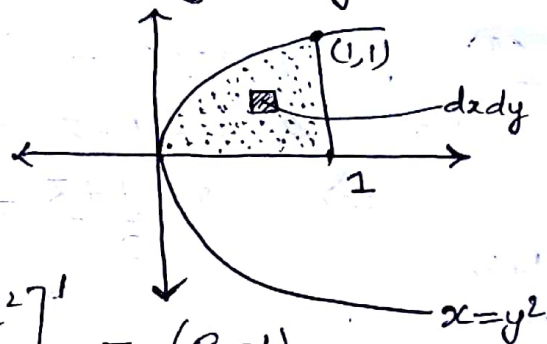
(c) $\int_0^1 \int_{y^2}^1 y e^{x^2} \, dx \, dy$. Difficult to integrate e^{x^2} w.r to x so changing the order of integration.

$y = 0, 1$ $x = y^2, 1$

$\int_{x=0}^1 \int_{y=0}^{\sqrt{x}} y e^{x^2} \, dy \, dx = \int_{x=0}^1 \frac{x}{2} e^{x^2} \, dx$

$= \frac{1}{4} \int_0^1 e^{x^2} 2x \, dx = \frac{1}{4} \int_0^1 e^{x^2} \, dx = \frac{1}{4} \left[e^{x^2} \right]_0^1 = \frac{(e-1)}{4}$

- -

(d). $\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2} \, dx \, dy$

$1 \le y \le 1$

$\sqrt{3}y \le x \le \sqrt{4-y^2}$

$\sqrt{3}y = x$ $x = \sqrt{4-y^2} \longrightarrow x^2 = 4 - y^2$

$x^2 + y^2 = 4$

~~Changing the Order of integration~~

~~$\int_{x=0}^{\sqrt{3}} \int_{y=0}^{x/\sqrt{3}} \sqrt{x^2+y^2} \, dy \, dx + \int_{x=\sqrt{3}}^2 \int_{y=0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2} \, dy \, dx$~~
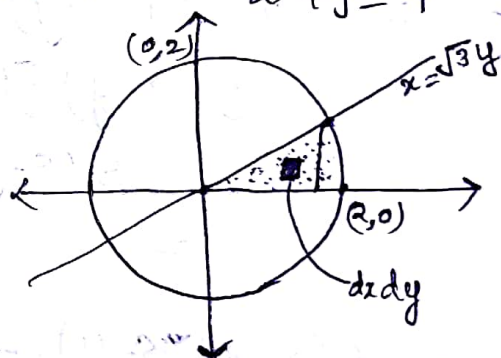

Changing to polar form

$x = r\cos\theta$, $y = r\sin\theta$

$dx \, dy = r \, dr \, d\theta$

$\int_{\theta=0}^{\pi/6} \int_{r=0}^2 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \, r \, dr \, d\theta = \int_{\theta=0}^{\pi/6} \int_{r=0}^2 r^2 \, dr \, d\theta$ ~~$\oint_{\theta=0}$~~

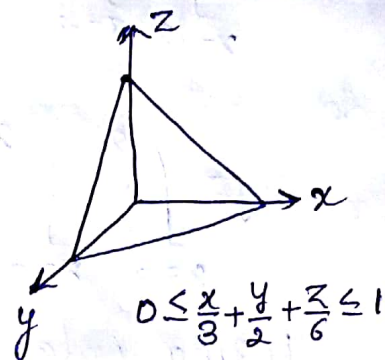$= \int_{\theta=0}^{\pi/6} \left[\frac{r^3}{3} \right]_0^2 \, d\theta = \frac{8}{3} \cdot \frac{\pi}{6} = \frac{4\pi}{9}$

4 (a) $\iiint_E 2x\,dv$ E: $2x+3y+z=6$ first octant

ie $\frac{x}{3}+\frac{y}{2}+\frac{z}{6}=1$

take $\frac{x}{3}=u$, $\frac{y}{2}=v$, $\frac{z}{6}=w$

$dx=3du$, $dy=2dv$, $dz=6dw$

we have, from Dritchlet Integral

if $0\le x+y+z\le 1$, then

$$\iiint x^{l-1}y^{m-1}z^{n-1}\,dz\,dy\,dx = \frac{\lceil l \lceil m \lceil n}{\lceil l+m+n+1}$$

$$0\le \frac{x}{3}+\frac{y}{2}+\frac{z}{6}\le 1$$

$$\iiint_E 2x\,dv = \iiint_E 2(3u)\cdot(3\,du)(2\,dv)(6\,dw)$$

$$= 216\iiint u^{2-1}v^{1-1}w^{1-1}\,du\,dv\,dw$$

$$= 216\frac{\lceil 2\,\lceil 1\,\lceil 1}{\lceil 2+1+1+1} = 216\frac{\lfloor 1\,\lfloor 0\,\lfloor 0}{\lfloor 4} = \frac{216}{4\cdot 3\cdot 2} = 9$$

(b) $\iiint_E \sqrt{3x^2+3z^2}\,dv$ E: $y=2x^2+2z^2$, $y=8$

$$2x^2+2z^2\le y\le 8$$
$$-2\le z\le 2$$
$$-2\le x\le 2$$

$$\int_{x=-2}^{2}\int_{z=-2}^{2}\int_{y=2x^2+2z^2}^{8}\sqrt{3x^2+3z^2}\,dy\,dz\,dx$$

$$\int_{x=-2}^{2}\int_{z=-2}^{2}[y]^{8}_{2x^2+2z^2}\sqrt{3}\sqrt{x^2+y^2}\,dz\,dx$$

$$\sqrt{3}\int_{x=-2}^{2}\int_{z=-2}^{2}\left[(2x^2+2z^2)-8\right]\sqrt{x^2+z^2}\,dz\,dx$$

$$2\sqrt{3}\int_{x=-2}^{2}\int_{z=-2}^{2}(x^2+z^2-4)\sqrt{x^2+z^2}\,dz\,dx$$

$$= 2\sqrt{3} \int_{x=-2}^{2} \int_{z=-2}^{2} (x^2+z^2-4)\sqrt{x^2+z^2} \; dz \, dx$$

$$\underset{\text{even function}}{\underbrace{\hspace{3cm}}}$$

So

$$= 8\sqrt{3} \int_{x=0}^{2} \int_{z=0}^{2} (x^2+z^2-4)\sqrt{x^2+z^2} \; dz \, dx$$

using polar coordinates in x-z plane

$$x = r\cos\theta \qquad z = r\sin\theta$$

$$= 8\sqrt{3} \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} (r^2-4)\, r \cdot r \, dr \, d\theta$$

$$= 8\sqrt{3} \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} (r^4 - 4r^2) \, dr \, d\theta$$

$$= 8\sqrt{3} \int_{\theta=0}^{\pi/2} \left[\frac{r^5}{5} - \frac{4r^3}{3} \right]_{0}^{2} d\theta = 8\sqrt{3} \int_{\theta=0}^{\pi/2} 2^5 \left(\frac{1}{5} - \frac{1}{3} \right) d\theta$$

$$= \frac{8\sqrt{3} \cdot 2^5}{15} (-2) \left(\frac{\pi}{2} \right) = \frac{2^8 \sqrt{3} \, \pi}{15}$$

4(c). $\iiint\limits_{E} xyz\,dV$ $\quad E: x^2+y^2+z^2=4 \quad x,y,z \geq 0$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

$$u+v+w=1$$

$$\left(\frac{x}{2}\right)^2 = u, \quad \left(\frac{y}{2}\right)^2 = v, \quad \left(\frac{z}{2}\right)^2 = w$$

$x=2\sqrt{u}$
$y=2\sqrt{v}$ $\quad z=2\sqrt{w}$

$dx = u^{-\frac{1}{2}}du, \quad dy = v^{-\frac{1}{2}}dv, \quad dz = w^{-\frac{1}{2}}dw$

Now $\quad 0 \leq u+v+w \leq 1 \quad u,v,w \geq 0$

Apply Dirichlet theorem

$$\iiint\limits_{v} 2\sqrt{u}\,2\sqrt{v}\,2\sqrt{w} \cdot u^{-\frac{1}{2}}du\,v^{-\frac{1}{2}}dv\,w^{-\frac{1}{2}}dw$$

$$=8\iiint\limits_{v} du\,dv\,dw$$

$$=8 \iiint\limits_{v} u^{1-1}v^{1-1}w^{1-1}du\,dv\,dw = 8\,\frac{\lceil 1\,\lceil 1\,\lceil 1}{\lceil 1+1+1+1} = 8\,\frac{1}{\lceil 4}$$
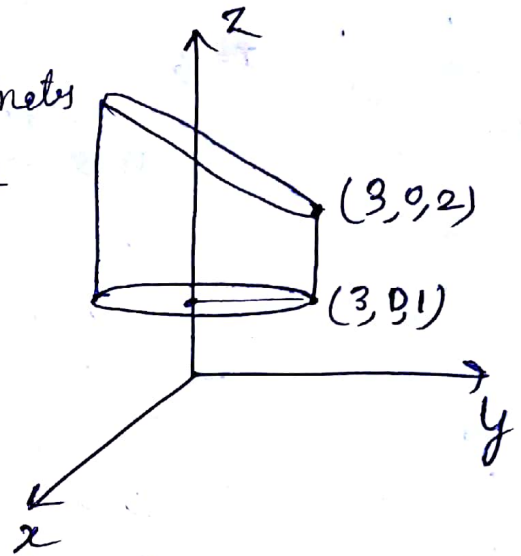
$$=\frac{8}{\lceil 3} = \frac{8}{6} = \frac{4}{3}$$
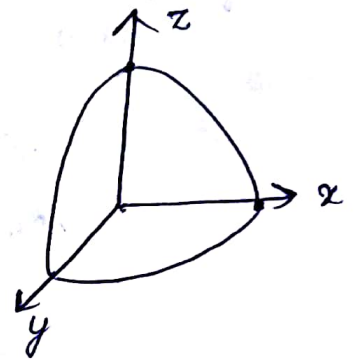
4(e). $\iiint\limits_E dv$ $E : x^2 + y^2 = 9$ $z=1$ $x+z=5$
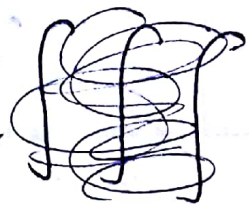
Cylindrical coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$z = z$

Points: $(3,0,2)$, $(3,0,1)$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{3} \int_{z=1}^{5-r\cos\theta} r \, dz \, d\theta \, dr$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{3} r \left[5 - r\cos\theta - 1 \right] dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} \left[4r - r^2 \cos\theta \right] dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[2r^2 - \frac{r^3}{3}\cos\theta \right]_{r=0}^{3} d\theta \; = \; \int_{\theta=0}^{2\pi} \left[18 - 9\cos\theta \right] d\theta$$

$$= 9 \int_{\theta=0}^{2\pi} \left[2 - \cos\theta \right] d\theta \; = \; 9 \left[2\theta - \sin\theta \right]_{0}^{2\pi}$$

$$= 9 \left[(4\pi - 0) - (0) \right] \; = \; 36\pi$$

5 (a) $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\ dz\ dy\ dx$
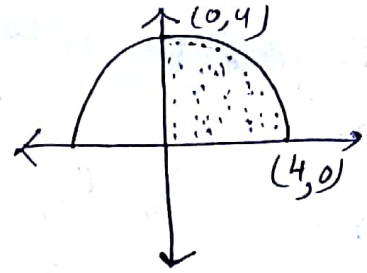
$y = \sqrt{16-x^2}$ $0 \le x \le 4$

$x^2 + y^2 = 16$

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$\displaystyle\int_{\theta=0}^{\pi/2} \int_{r=0}^{4} \int_{z=0}^{16-r^2} r\ dz\ r\ dr\ d\theta$

$= \displaystyle\int_{\theta=0}^{\pi/2} \int_{r=0}^{4} r^2(16-r^2)\ dr\ d\theta$

$= \displaystyle\int_{\theta=0}^{\pi/2} \left[\frac{16r^3}{3} - \frac{r^5}{5}\right]_{r=0}^{4} d\theta = 4^5 \frac{2}{15} \cdot \frac{\pi}{2} = \frac{1024\pi}{15}$

(b). $\displaystyle\iiint_E \sqrt{x^2+y^2}\ dV$ E: above xy-plane & below the cone $z = 4 - \sqrt{x^2+y^2}$

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$\displaystyle\int_{\theta=0}^{2\pi} \int_{r=0}^{4} \int_{z=0}^{4-r} r\ r\,dz\,d\theta\,dr$

$= \displaystyle\int_{\theta=0}^{2\pi} \int_{r=0}^{4} r^2[4-r]\ dr\ d\theta$

$= \displaystyle\int_{\theta=0}^{2\pi} \left[\frac{4r^3}{3} - \frac{r^4}{4}\right]_0^4 d\theta$

$= \displaystyle\int_{\theta=0}^{2\pi} \left[\frac{4\cdot 4^4}{12} - \frac{3\cdot 4^4}{12}\right] d\theta = \frac{4^4}{12}\cdot 2\pi = \frac{64(2\pi)}{3}$
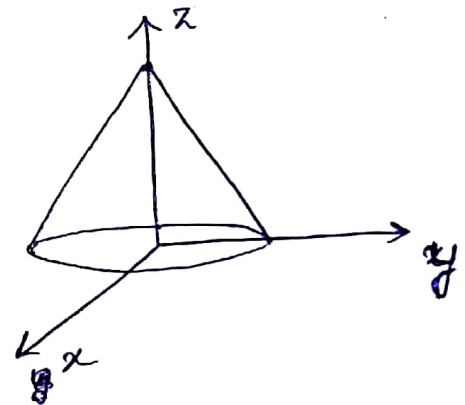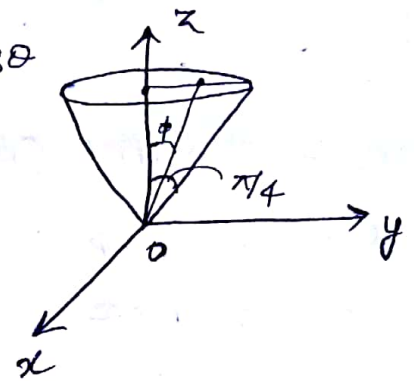
6 (a). $\iiint_E (x^2+y^2+z^2)^{\frac{1}{2}}\,dv$ $z=3$ and $z=\sqrt{x^2+y^2}$

$x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$

$|J| = \left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = r^2\sin\phi$

$r = \dfrac{3}{\cos\phi} = 3\sec\phi$ $0 \le \phi \le \dfrac{\pi}{4}$

                              $0 \le \theta = 2\pi$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{r=0}^{3\sec\phi} r^2(r^2\sin\phi)\,dr\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{r=0}^{3\sec\phi} r^3\sin\phi\,dr\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\left[\frac{3^4\sec^4\phi}{4}\right]\sin\phi\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\frac{3^4\sin\phi}{4\cos^4\phi}\,d\phi\,d\theta$$

$$= \frac{3^4}{4}\cdot 2\pi\int_{\phi=0}^{\pi/4}\frac{\sin\phi}{\cos^4\phi}\,d\phi$$

$$= \frac{3^4}{4}2\pi\left[\frac{1}{3\cos^3\phi}\right]_0^{\pi/4} = \frac{3^4\cdot2\pi}{5\cdot3}\left[\frac{1}{(1/\sqrt{2})^3} - \frac{1}{1}\right]$$

$$= \frac{3^4\cdot2\pi}{4\cdot3}\left[2\sqrt{2}-1\right] = \frac{3^3\cdot2\pi}{4}\cdot\left(2\sqrt{2}-1\right)$$

$$= \frac{27}{2}\left(2\sqrt{2}-1\right)$$

(b). $\iiint\limits_{E} (x^2 + y^2 + z^2)^{-3/2} dv$

$E: \quad x^2 + y^2 + z^2 = 4$
$\qquad x^2 + y^2 + z^2 = 9$

$x = r \sin\phi \cos\theta$
$y = r \sin\phi \sin\theta$
$z = r \cos\theta$

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=2}^{3} r^{-3} \overbrace{r^2 \sin\phi}\, dr\, d\theta\, d\phi$$
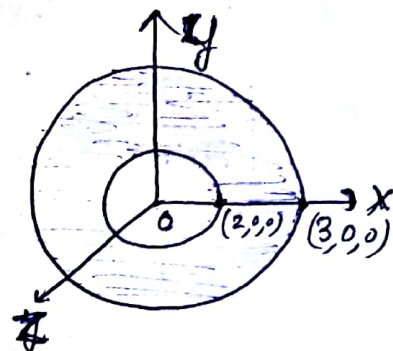
$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=2}^{3} r^{-1} \sin\phi\, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \left[\log r \right]_{2}^{3} \sin\phi\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \left(\log \tfrac{3}{2} \right) \sin\phi\, d\phi\, d\theta$$

$$= \left(2\pi \log \tfrac{3}{2} \right) \left[-\cos\phi \right]_{0}^{\pi} = 4\pi \log \tfrac{3}{2}$$

⑦ (a)

$$\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r,\theta)\, dr\, d\theta$$

Upper half
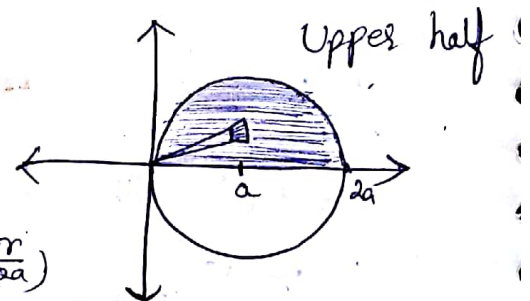
$0 \le \theta \le \pi/2 \qquad 0 \le r \le 2a\cos\theta$

$\theta = 0 \qquad\qquad r = 0 \qquad r = 2a\cos\theta \to \theta = \cos^{-1}\left(\frac{r}{2a}\right)$

$\theta = \pi/2$

changing the order of integration

$\theta = 0$ to $\cos^{-1}\left(\frac{r}{2a}\right)$ and $r = 0$ to $2a$

$$\int_{r=0}^{2a} \int_{\theta=0}^{\cos^{-1}\left(\frac{r}{2a}\right)} f(r,\theta)\, d\theta\, dr.$$

(b)

$$\int_0^{\pi/3} \int_{a\sec^2(\theta/2)}^{\left(\frac{8a}{3}\right)\cos\theta} f(r,\theta)\, dr\, d\theta$$

to $\theta = 0 \qquad r = a\sec^2(\theta/2) \qquad\qquad r = \frac{8a}{3}\cos\theta$

$\theta = \pi/3 \qquad \quad \hookrightarrow r\cos^2(\theta/2) = a$
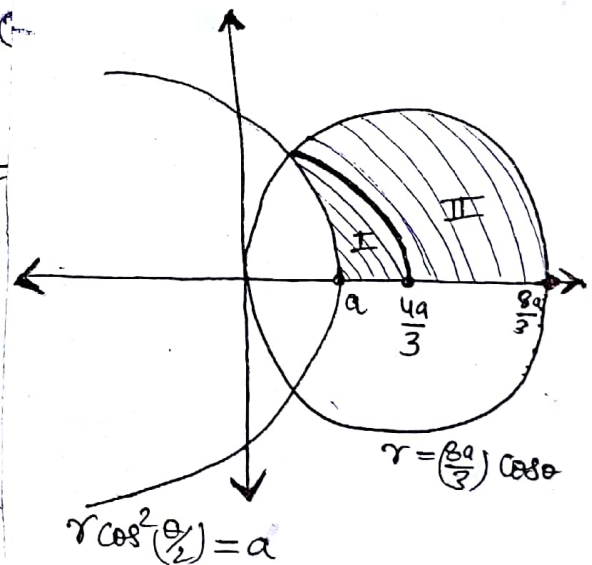
$\qquad\qquad\qquad\qquad \theta = 2\cos^{-1}\left(\sqrt{\frac{a}{r}}\right)$

and

$\left(r = \frac{8a}{3}\cos\theta\right) \to \theta = \cos^{-1}\left(\frac{3r}{8a}\right)$

$r = \left(\frac{8a}{3}\right)\cos\theta$

$r\cos^2(\theta/2) = a$

For region I:

θ varies from 0 to $2\cos^{-1}\left(\sqrt{\frac{a}{r}}\right)$

So

$$I = \int_{r=a}^{\frac{4a}{3}} \int_{\theta=0}^{2\cos^{-1}\left(\sqrt{\frac{a}{r}}\right)} f(r,\theta)\, d\theta\, dr$$

For region II

θ varies from 0 to $\cos^{-1}\left(\frac{3r}{8a}\right)$

So

$$II = \int_{r=\frac{4a}{3}}^{\frac{8a}{3}} \int_{\theta=0}^{\cos^{-1}\left(\frac{3r}{8a}\right)} f(r,\theta)\, d\theta\, dr$$

thus

$$\int_0^{\pi/3} \int_{a\sec^2\frac{\theta}{2}}^{\left(\frac{8a}{3}\right)\cos\theta} f(r,\theta)\, dr\, d\theta = I + II$$

(c). $\displaystyle\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)\, dy\, dx$

$x=0,\,a$ $\qquad y=\sqrt{ax-x^2}$ $\qquad y=\sqrt{ax}$

$\qquad\qquad\qquad x^2+y^2=ax$ $\qquad y^2=ax$

$\qquad\qquad\qquad x^2-ax+y^2=0$

$\qquad\qquad\qquad x=\dfrac{a\pm\sqrt{a^2-4y^2}}{2}$

$\qquad\qquad\qquad x=\dfrac{a}{2}\pm\sqrt{\dfrac{a^2}{4}-y^2}$

for region

I: $\quad x=\dfrac{a}{2}-\sqrt{\dfrac{a^2}{4}-y^2}$

II: $\quad x=\dfrac{a}{2}+\sqrt{\dfrac{a^2}{4}-y^2}$

__Integration over region I__

$I=\displaystyle\int_{y=0}^{\frac{a}{2}}\int_{x=\frac{y^2}{a}}^{\frac{a}{2}-\sqrt{\frac{a^2}{4}-y^2}} f(x,y)\, dx\, dy$

__Integration over region II__

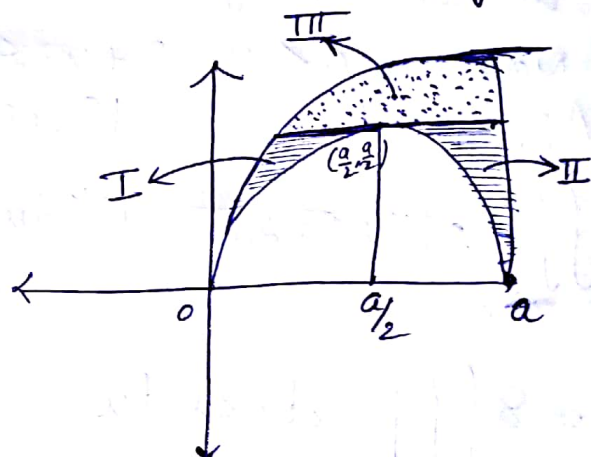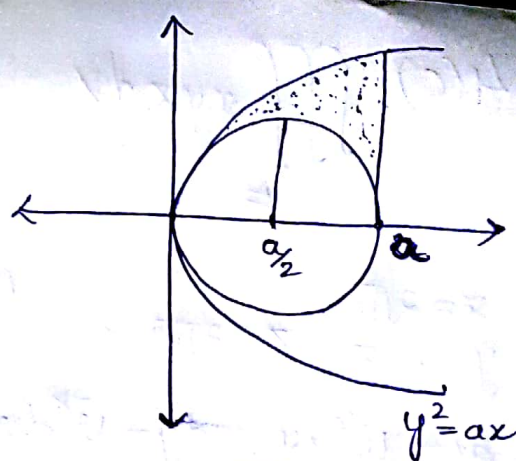$II=\displaystyle\int_{y=0}^{a/2}\int_{x=\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a} f(x,y)\, dx\, dy$

__Integration over the region IV__

$III=\displaystyle\int_{y=a/2}^{a}\int_{x=\frac{y^2}{a}}^{a} f(x,y)\, dx\, dy$

thus

$\displaystyle\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x,y)\, dy\, dx = I+II+III$
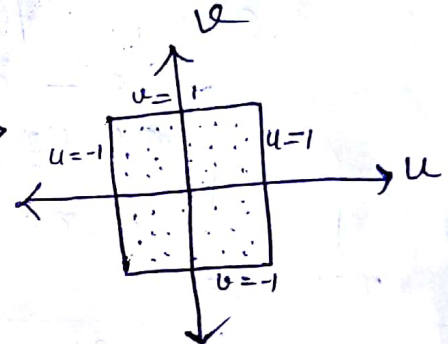
⑧ $\iint_R \left(\dfrac{x-y}{x+y+2}\right)^2 dx\,dy$ $R:\ x+y = \pm 1,\ x-y = \pm 1$

$u = x+y \qquad v = x-y$

$u = \pm 1 \qquad v = \pm 1$

$\iint \left(\dfrac{v}{u+2}\right)^2 J\,du\,dv$

Now $\begin{array}{l} u = x+y \\ v = x-y \end{array} \Rightarrow \begin{array}{l} x = \dfrac{u+v}{2} \\[4pt] y = \dfrac{u-v}{2} \end{array}$
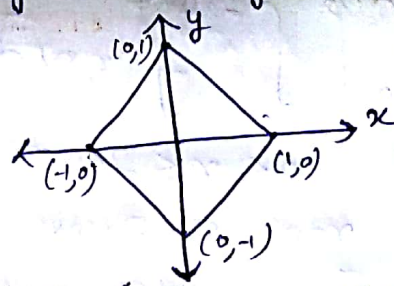
$J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \tfrac12 & \tfrac12 \\ \tfrac12 & -\tfrac12 \end{vmatrix} = -\dfrac12$

So $\displaystyle \int_{u=-1}^{1}\int_{v=-1}^{1} \dfrac{v^2}{(u+2)^2}\left(-\tfrac12\right) dv\,du = \dfrac{-1}{2}\int_{u=-1}^{1}\dfrac{1}{(u+2)^2}\left[\dfrac{v^3}{3}\right]_{-1}^{1} du$

$= \dfrac{-1}{2}\int_{u=-1}^{1}\dfrac{1}{(u+2)^2}\left[\dfrac23\right]du = \dfrac{-1}{3}\int_{u=-1}^{1}(u+2)^{-2}du = \dfrac{-1}{3}\left[\dfrac{(u+2)^{-1}}{-1}\right]_{-1}^{1}$

$= \dfrac{1}{3}\left[\dfrac{1}{u+2}\right]_{-1}^{1} = \dfrac13\left[\dfrac13 - \dfrac11\right] = \dfrac{-2}{9}$

⑨ $\iint_R (x+y)\,dA$ $R:\ (0,0),\ (5,0),\ \left(\tfrac52,\tfrac52\right),\ \left(\tfrac52,-\tfrac52\right)$

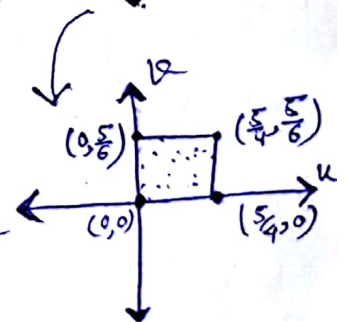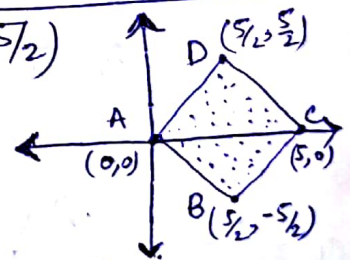$\begin{array}{ll} x = 2u + 3v & u = \dfrac{x+y}{4} \\[6pt] y = 2u - 3v & v = \dfrac{x-y}{6} \end{array}$

$dx\,dy = J\,du\,dv$

$J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12$

$\displaystyle \iint_R (x+y)\,dx\,dy = \int_{v=0}^{5/6}\int_{u=0}^{5/4} 4u\,(-12)\,du\,dv = (-12)\cdot\int_{v=0}^{5/6}\left[2u^2\right]_{u=0}^{5/4}dv$

$= (-12)2\int_{v=0}^{5/6}\left(\dfrac54\right)^2 dv = \dfrac{-24\cdot25}{16}\cdot\dfrac56 = \dfrac{-125}{4}$

(11) $\displaystyle\int_0^3 \int_0^4 \int_{x=y/2}^{x=y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right)$

$(4,4,w)$
(x,y,z)

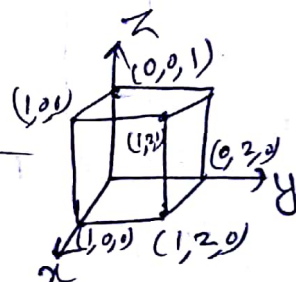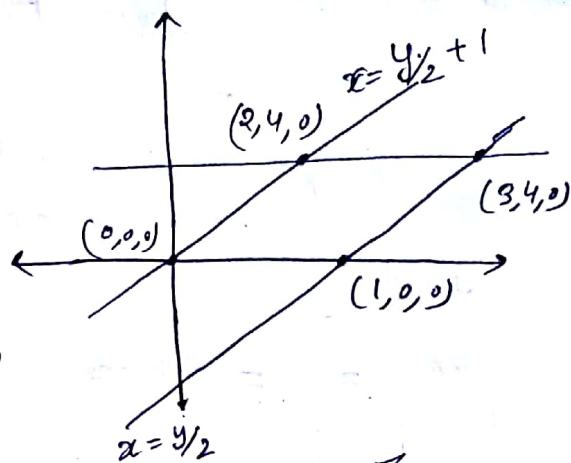| $(0,0,0)$	$(0,0,0)$
$(1,0,0)$	$(1,0,0)$
$(1,2,0)$	$(3,4,0)$
$(0,2,0)$	$(2,4,0)$
$(1,0,1)$	$(1,0,3)$
$(1,2,1)$	$(3,4,3)$
$(0,2,1)$	$(2,4,3)$
$(0,0,1)$	$(0,0,3)$

$u = \dfrac{2x-y}{2}$

$v = y/2$

$w = z/3$

$x = u+v$
$y = 2v$
$z = 3w$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$
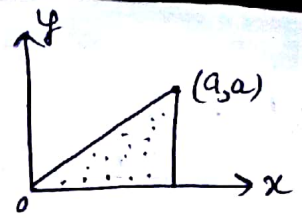
$$\int_{w=0}^{1} \int_{v=0}^{2} \int_{u=0}^{1} (u+w)\,6\,du\,dv\,dw$$

$$= 6 \int_{w=0}^{1} \int_{v=0}^{2} \left[\frac{u^2}{2} + wu\right]_0^1 dv\,dw$$

$$= 6 \int_{w=0}^{1} \left[\frac{v}{2} + wv\right]_0^2 dw = 6\int_{w=0}^{1} \left[1+2w\right] dw$$

$$= 6\left[w + w^2\right]_0^1 = 6\left[1 + 1^2\right] = 12$$

(12) (a) $\int_0^a \int_0^x \dfrac{f'(y)\,dy\,dx}{\sqrt{(a-x)(x-y)}} = \pi\{f(a) - f(0)\}$

Changing the order of integration, we get

$= \int_{y=0}^a \int_{x=y}^a \dfrac{f'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}} \longrightarrow \sqrt{ax - ay - x^2 + xy} \longrightarrow \sqrt{-x^2 + (a+y)x - ay}$

$= \int_{y=0}^a \int_{x=y}^a \dfrac{f'(y)}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x - \frac{(a+y)}{2}\right)^2}}\,dx\,dy$

$= \int_{y=0}^a f'(y) \int_{x=y}^a \dfrac{dx}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x - \frac{(a+y)}{2}\right)^2}}\,dy$

$= \int_{y=0}^a f'(y) \left[\sin^{-1}\left(\dfrac{x - \left(\frac{a+y}{2}\right)}{\left(\frac{a-y}{2}\right)} \right) \right]_y^a dy$

$= \int_{y=0}^a f'(y) \left[\sin^{-1}(1) - \sin^{-1}(-1) \right] dy$

$= \int_{y=0}^a f'(y) \left[\dfrac{\pi}{2} - \left(-\dfrac{\pi}{2}\right) \right] dy$

$= \pi \int_{y=0}^a f'(y)\,dy$

$= \pi\, [f(y)]_0^a$

$= \pi\, [f(a) - f(0)]$

(b). $\displaystyle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{b\, dy\, dx}{(x^2+y^2+b^2)^{3/2}\, (x^2+y^2+a^2)^{1/2}} = \frac{2\pi}{a+b}$

$x = r\cos\theta \quad y = r\sin\theta$

$= \displaystyle \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} \frac{b\; r\, dr\, d\theta}{(r^2+b^2)^{3/2}\, (r^2+a^2)^{1/2}}$

$= \displaystyle \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} \frac{b\; r\, dr\, d\theta}{\dfrac{(r^2+b^2)^{3/2}}{(r^2+a^2)^{3/2}} \cdot (r^2+a^2)^2}$

$= \displaystyle \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} \frac{b\; r\, dr\, d\theta}{\left(\dfrac{r^2+b^2}{r^2+a^2}\right)^{3/2} \cdot (r^2+a^2)^2}$

Let $\quad \dfrac{r^2+b^2}{r^2+a^2} = t \quad \Longrightarrow \quad \dfrac{(r^2+a^2)\, 2r - (r^2+b^2)\, 2r}{(r^2+a^2)^2}\, dr = dt$

$\Longrightarrow \quad \dfrac{2r\,(a^2-b^2)}{(r^2+a^2)^2}\, dr = dt \qquad\qquad$ limits $r=0$ to ∞
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t = \dfrac{b^2}{a^2}$ to 1

$= \displaystyle \int_{\theta=0}^{2\pi} \int_{t=\frac{b^2}{a^2}}^{1} \frac{b\, dt\, d\theta}{2(a^2-b^2)\, t^{3/2}}$

$= \dfrac{1}{2(a^2-b^2)} \displaystyle \int_{\theta=0}^{2\pi} b \left[\dfrac{t^{-3/2+1}}{-3/2+1} \right]_{t=b^2/a^2}^{1} d\theta$

$= \dfrac{b}{2(a^2-b^2)} \displaystyle \int_{\theta=0}^{2\pi} \left[\dfrac{t^{-1/2}}{-1/2} \right]_{t=b^2/a^2}^{1} d\theta$

$$= \frac{b}{2(a^2-b^2)} \; 2\pi \left[\frac{t^{1/2}}{-\frac{1}{2}}\right]'_{t=b^2/a^2}$$

$$= \frac{b}{2(a^2-b^2)} \frac{2\pi}{(-\frac{1}{2})} \left[1 - \frac{1}{\sqrt{b^2/a^2}}\right]$$

$$= \frac{b}{2(a^2-b^2)} (-2\pi) \left[1 - \frac{1}{b/a}\right]$$

$$= \frac{b}{(a^2-b^2)} (-2\pi) \left[\frac{b-a}{b}\right]$$

$$= \frac{-2\pi b}{(a+b)(a-b)} \left(\frac{b-a}{b}\right)$$

$$= \frac{2\pi b}{(a+b)} \cdot \frac{1}{b}$$

$$= \frac{2\pi}{(a+b)}$$