Signals & Systems (ECN-203)

Lecture 7

(Continuous time Fourier transform)

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September 29, 2020

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Introduction

- In previous lecture, we developed a representation of periodic signals as linear combinations of complex exponentials
- But what happens if the signal is not periodic?
  - Can we extend the concept of Fourier series to aperiodic signals?
- As it turns out a large class of signals can be represented as a linear combination of complex exponentials
  - In fact, all signals with finite energy

Periodic vs aperiodic signal

- For periodic signals, the complex exponential building blocks are harmonically related (frequency f, 2f, 3f, ...)
- For aperiodic signals they are infinitesimally close in frequency
- Linear combination takes the form of an integral rather than a sum
- Resulting spectrum of coefficients is called the Fourier transform

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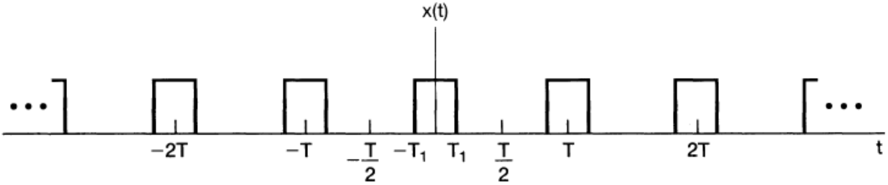
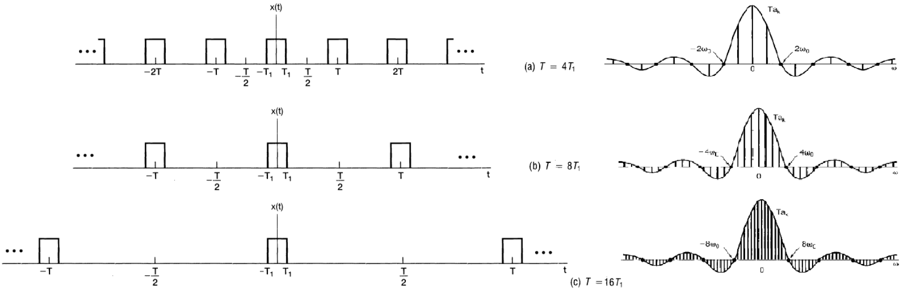
Fourier series of a periodic square wave

- Consider a periodic square wave with period T, whose one period is given as: $x(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & T_1 \leq |t| \leq \frac{T}{2} \end{cases}$



- Its Fourier series coefficients are: $a_k = \frac{2sin(k\omega_0 T_1)}{k\omega_0 T}$
  - $\omega_0 = \frac{2\pi}{T}$
  - Alternative interpretation: Samples of an envelope function $Ta_k = \frac{2sin(\omega T_1)}{\omega}$ at $\omega = k\omega_0$ (equally spaced samples)
  - For a fixed T_1, this envelope is independent of T

Periodic square wave and its Fourier series representation for different T



- $T = 4T_1 \rightarrow \omega_{01} = \frac{2\pi}{4T_1}$ (samples at $k\omega_{01}$, $k \in I$)
- $T = 8T_1 \rightarrow \omega_{02} = \frac{2\pi}{8T_1} = \frac{\omega_{01}}{2}$ (samples at $k\omega_{02} = \frac{k\omega_{01}}{2}$, $k \in I$)
- $T = 16T_1 \rightarrow \omega_{0_4} = \frac{2\pi}{4T_1} = \frac{\omega_{0_1}}{4}$ (samples at $k\omega_{04} = \frac{k\omega_{01}}{4}$, $k \in I$)

What happens as $T \to \infty$

- As the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases, the envelope is sampled with a closer and closer spacing
- As $T \to \infty$, the periodic square wave approaches a rectangular pulse (aperiodic)
- $\omega_0 \to 0$: Samples of the envelope are more and more closely spaced
  - Fourier series coefficients approaches the envelope function as $T \to \infty$
  - This envelope function is called "Fourier Transform"

Fourier transform synthesis equation

- Consider a signal $x(t)$ of finite duration
  - For some T_1, $x(t) = 0$ for $|t| > T_1$
- Construct a periodic signal $\tilde{x}(t)$ of period $T > T_1$ for which $x(t)$ is one period

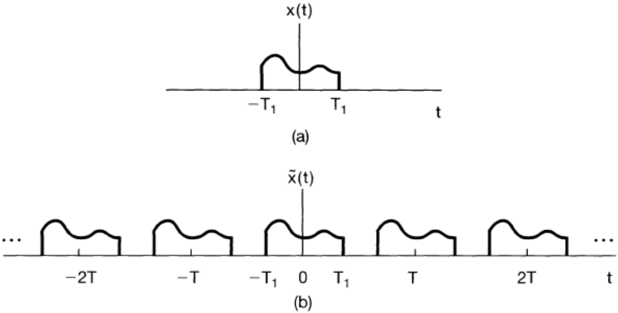


Figure 4.3 (a) Aperiodic signal $x(t)$; (b) periodic signal $\tilde{x}(t)$, constructed to be equal to $x(t)$ over one period.

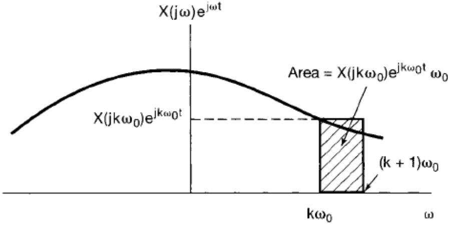
- As $T \to \infty$, $\tilde{x}(t)$ is equal to $x(t)$ for any finite value of t

Fourier series representation of $\tilde{x}(t)$

- $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, $a_k = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt$
  - $\omega_0 = \frac{2\pi}{T}$
- $x(t) = \begin{cases} \tilde{x}(t), & |t| \leq \frac{T}{2} \\ 0, & Otherwise \end{cases}$
  - $a_k = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$
- Envelope of Ta_k is $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
  - Coefficients a_k's are given by $a_k = \frac{1}{T} X(jk\omega_0)$
- $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$

Fourier transform representation of $x(t)$

- As $T \to \infty$, $\tilde{x}(t) \to x(t)$
  - $\omega_0 \to 0$ and the sum $\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$ is approximated by an integral



- $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
- $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

- For aperiodic signals, the complex exponentials occur at a continuum of frequencies and have amplitude of $X(j\omega)\frac{d\omega}{2\pi}$
  - $X(j\omega)$ is referred to as the spectrum of $x(t)$

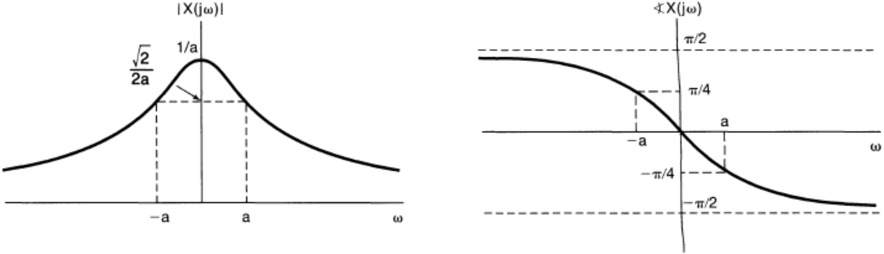
Example 1

- Aperiodic square pulse: $x(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & Otherwise \end{cases}$
-

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$= \int_{-T_1}^{T_1} e^{-j\omega t}dt$$
$$= \frac{2}{\omega} sin(\omega T_1)$$

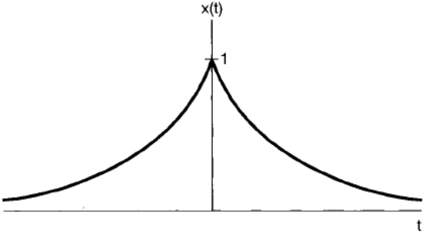
Example 2

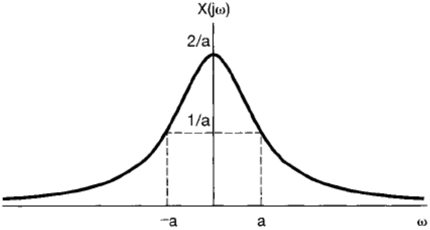
- $x(t) = e^{-at}u(t),\ a > 0$
- $X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty = \frac{1}{a+j\omega}$
- $|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}},\ \angle X(j\omega) = -tan^{-1}(\frac{\omega}{a})$



Example 3

- $x(t) = e^{-a|t|}$, $a > 0$
- $X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$





Example 4

- $x(t) = \delta(t)$ (Delta function)
- $X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega 0} = 1$
- Unit impulse has a Fourier transform consisting of equal contributions at all the frequencies

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Can we find Fourier transform for periodic signals too?

- In last section, we derived Fourier transform representation for aperiodic signals
  - $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
  - No constrain in this expression for $x(t)$ to be aperiodic
- What will happen if $x(t)$ is periodic?
- What would be the relation between Fourier series coefficients and Fourier transform?

Inverse Fourier transform of an impulse functio

- Let $x(t)$ has a Fourier transform $X(j\omega) = 2\pi\delta(\omega - \omega_0)$
-

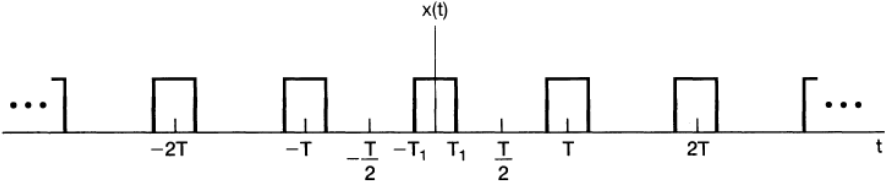
$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega \\
&= e^{j\omega t}
\end{aligned}
$$

  - Fourier transform of a sinusoidal is an impulse function

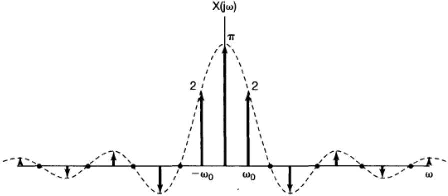
Inverse Fourier transform of an impulse train

- Generalization: $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency
  - $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$
  - The inverse Fourier transform is $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$
    - Fourier series representation
- Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$
  - Train of impulses occurring at the harmonically related frequencies
  - Area of the impulse at the k^{th} harmonic frequency $k\omega_0$ is 2π times the k^{th} Fourier series coefficient a_k
- This representation allows us to consider both periodic and aperiodic signals within a unified context

Fourier transform of a periodic square wave

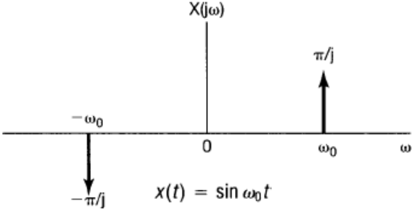


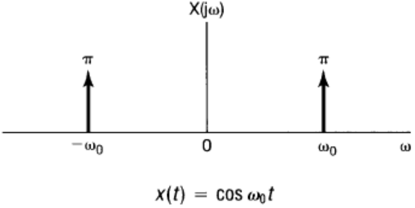
- Its Fourier series coefficients are: $a_k = \frac{2sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{sin(k\omega_0 T_1)}{k\pi}$
- Fourier transform of this signal is:
  $X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$



Fourier transform of sin and cos

- $x(t) = sin(\omega_0 t)$
  - Fourier series coefficients: $a_1 = \frac{1}{2j}$, $a_{-1} = \frac{-1}{2j}$, $a_k = 0$, for $k \neq 1, -1$
  - $X(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$
- $x(t) = cos(\omega_0 t)$
  - Fourier series coefficients: $a_1 = \frac{1}{2}$, $a_{-1} = \frac{1}{2}$, $a_k = 0$, for $k \neq 1, -1$
  - $X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$





Fourier transform of an impulse train

- Impulse train is an extremely useful signal
  - $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$
    - Periodic with period T
  - Its Fourier series coefficients are: $a_k = \frac{1}{T}$ for all k
- Fourier transform:
  $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$
  - Periodic impulse train with period $\frac{2\pi}{T}$

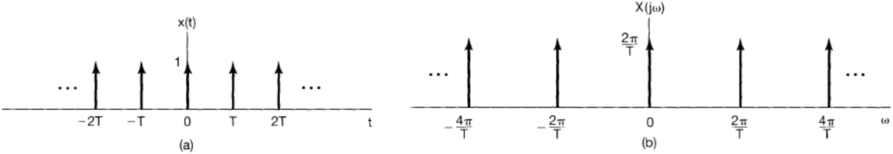


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

- Inverse relationship between the time and the frequency domains
  - If period of $x(t)$ increases, that of $X(j\omega)$ decreases, and vice versa

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Notation

- Fourier transform synthesis and analysis equations
  - $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
  - $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
- Fourier transform, $X(j\omega) = \mathcal{F}\{x(t)\}$
- Inverse Fourier transform, $x(t) = \mathcal{F}^{-1}\{X(j\omega)\}$
- Fourier transform pair: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

Linearity

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$
- Then $ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$
- Proof: Apply the analysis to $ax(t) + by(t)$
- Linearity property can be extended to a linear combination of an arbitrary number of signals

Time-shifting

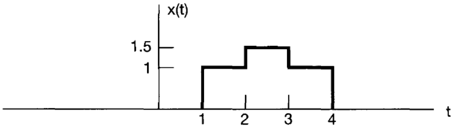
- If $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$
  - $x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega)$
- Proof: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
-

$$
\begin{aligned}
x(t-t_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega(t)}d\omega
\end{aligned}
$$

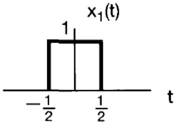
- $\mathcal{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$
- $\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = e^{-j\omega t_0}|X(j\omega)|e^{j\angle X(j\omega)} = |X(j\omega)|e^{j(\angle X(j\omega)-\omega t_0)}$
  - Time shift $\rightarrow$ phase shift in the transform (of $-\omega t_0$, a linear function of ω)

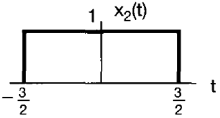
Time-shift example

- Find the Fourier transform of the signal shown below:



- We can represent $x(t)$ as as sum of two time-shifted signals for which we already know the Fourier transform
- $x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$
  - $x_1(t)$ and $x_2(t)$ are the rectangular pulse signals shown below





Time-shift example

- $x(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & Otherwise \end{cases} \xleftrightarrow{\mathcal{F}} \frac{2}{\omega} sin(\omega T_1)$
- $\mathcal{F}\{x_1(t)\} = \frac{2}{\omega} sin(\frac{1}{2}\omega)$
  - $\mathcal{F}\{x_1(t-2.5)\} = e^{-j\frac{5}{2}\omega} \times \frac{2}{\omega} sin(\frac{1}{2}\omega)$
- $\mathcal{F}\{x_2(t)\} = \frac{2}{\omega} sin(\frac{3}{2}\omega)$
  - $\mathcal{F}\{x_2(t-2.5)\} = e^{-j\frac{5}{2}\omega} \times \frac{2}{\omega} sin(\frac{3}{2}\omega)$
- $\mathcal{F}\{x(t)\} = e^{-j\frac{5}{2}\omega} \frac{sin(\frac{\omega}{2}) + 2sin(\frac{3\omega}{2})}{\omega}$

Conjugation and conjugate symmetry

- If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ then $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$
- Proof:

$$X^*(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^*$$
$$= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt$$

- replace ω by $-\omega$
  - $X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \mathcal{F}\{x^*(t)\}$
- If $x(t)$ is real (i.e., $x(t) = x^*(t)$)
  - $X(j\omega) = X^*(-j\omega) \rightarrow X(-j\omega) = X^*(j\omega)$ (conjugate symmetric)

Fourier transform of even-odd decomposition

- Let $x(t)$ is both, a real and even signal
  - $X(-j\omega) = X^*(j\omega)$ (since $x(t)$ is real)
-

$$\begin{aligned}
X(-j\omega) &= \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(-\tau)e^{-j\omega\tau}d\tau \text{ (by substituting } t \text{ with } -\tau) \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \text{ (since } x(-\tau) = x(\tau)) \\
&= X(j\omega)
\end{aligned}$$

  - $X(j\omega)$ is an even function
  - $X(-j\omega) = X^*(j\omega) = X(j\omega)$ ($X(j\omega)$ is also a real function)
- Real and even $x(t) \rightarrow$ Real and even $X(j\omega)$

Fourier transform of even-odd decomposition

- Similarly it can also be shown that: Real and odd $x(t) \rightarrow$ Purely imaginary and odd $X(j\omega)$
- Even-odd decomposition: $x(t) = x_e(t) + x_o(t)$
- Using linearity property: $\mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\}$
  - $\mathcal{F}\{x_e(t)\}$ is a real function
  - $\mathcal{F}\{x_o(t)\}$ is a purely imaginary function
- $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$
  - $\mathbb{EV}\{x(t)\} \xleftrightarrow{\mathcal{F}} \mathbb{RE}\{X(j\omega)\}$
  - $\mathbb{OD}\{x(t)\} \xleftrightarrow{\mathcal{F}} j\mathbb{IM}\{X(j\omega)\}$

Fourier transform of even-odd decomposition example

- Consider $x(t) = e^{-a|t|}$, where $a > 0$ (Example 3, slide 15)
- From example 2 (slide 14): $e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}$
-

$$
\begin{aligned}
x(t) = e^{-a|t|} &= e^{-at}u(t) + e^{at}u(-t) \\
&= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\
&= 2\mathbb{EV}\{e^{-at}u(t)\}
\end{aligned}
$$

- Since $e^{-at}u(t)$ is real, $\mathbb{EV}\{e^{-at}u(t)\} \xleftrightarrow{\mathcal{F}} \mathbb{RE}\{\frac{1}{a+j\omega}\}$
- $x(t) \xleftrightarrow{\mathcal{F}} 2\mathbb{RE}\{\frac{1}{a+j\omega}\} = \frac{2a}{a^2+\omega^2}$
  - Same as in slide 15

Differentiation and integration

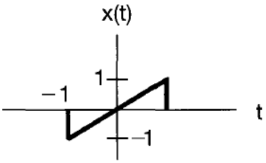
- $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$
- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
- Differentiate both sides with respect to t
  - $\frac{d}{dt}(x(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$
  - $\frac{d}{dt}(x(t)) \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$
  - Differentiation in time domain translates to multiplication by $j\omega$ in the frequency domain
- Similarly, $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
  - The impulse term ($\pi X(0)\delta(\omega)$) is due to the dc or average value as a result of integration

Example 1

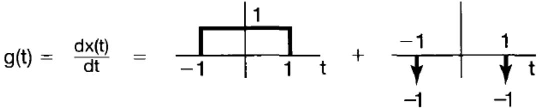
- Find Fourier transform of the step function using that of the impulse function
  - $g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1$
- $x(t) = u(t) = \int_{-\infty}^{t} g(\tau)d\tau \xleftrightarrow{\mathcal{F}} X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega)$
- $X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$
  - Since $G(j\omega) = 1$
- $\delta(t) = \frac{d}{dt}u(t) \xleftrightarrow{\mathcal{F}} j\omega(\frac{1}{j\omega} + \pi\delta(\omega)) = 1$
  - $\omega\delta(\omega) = 0$

Example 2

- Find the Fourier transform of the following signal:



- The derivative of $x(t)$ can be shown to be the sum of rectangular pulse and two impulses



Example 2

- $G(j\omega) = \frac{2}{\omega} sin(\omega) - e^{j\omega} - e^{-j\omega}$
- As $\omega \to 0$, $G(0) = 0$
- $$X(j\omega) = \frac{1}{j\omega} G(j\omega) + \pi G(0)\delta(\omega)$$
$$= \frac{1}{j\omega}\left(\frac{2}{\omega} sin(\omega) - e^{j\omega} - e^{-j\omega}\right)$$
$$= \frac{2sin(\omega)}{j\omega^2} - \frac{2cos(\omega)}{j\omega}$$

Time and frequency scaling

- $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \rightarrow x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}X(\frac{j\omega}{a}),\ a \neq 0$
- Proof: $\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$
- Substitute $\tau = at$
  - $\mathcal{F}\{x(\tau)\} = \begin{cases} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\frac{\tau}{a}}\frac{1}{a}d\tau, & a > 0 \\ \int_{\infty}^{-\infty} x(\tau)e^{-j\omega\frac{\tau}{a}}\frac{1}{a}d\tau, & a < 0 \end{cases}$
  - $\mathcal{F}\{x(\tau)\} = \begin{cases} \frac{1}{a}X(\frac{j\omega}{a}), & a > 0 \\ \frac{-1}{a}X(\frac{j\omega}{a}), & a < 0 \end{cases} = \frac{1}{|a|}X(\frac{j\omega}{a})$
- Linear scaling in time by a factor of a corresponds to a linear scaling in frequency by a factor of $\frac{1}{a}$, and vice versa

Example

- An audio recorded at one speed and played back at a different speed
- If playback speed is higher than the recording speed
  - Compression in time ($a > 1$)
  - Spectrum is expanded in frequency
  - Playback frequencies are higher
    - High pitch sound
- If playback speed is less than the recording speed
  - Signal is expanded in time ($0 < a < 1$)
  - Spectrum is compressed in frequency
  - Playback frequencies are smaller
    - Deep sound effect

Duality

- The analysis and synthesis equation of Fourier transform are similar, but not quite identical
  - $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
  - $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
- Let $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$, then what is the Fourier transform of $X(t)$?
- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\rho)e^{j\rho t}d\rho$
- Replace t by $-j\omega$ and multiply both sides by 2π
  - $2\pi x(-j\omega) = \int_{-\infty}^{\infty} X(j\rho)e^{j\rho(-j\omega)}d\rho$
- Replace $j\rho$ by t
  - $2\pi x(-j\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt = \mathcal{F}\{X(t)\}$
- $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \rightarrow X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-j\omega)$

Duality example 1

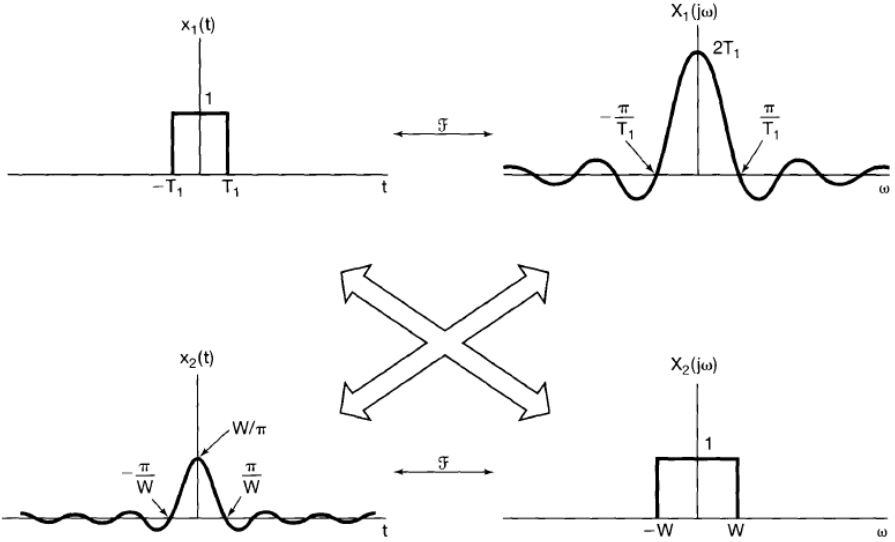
- Consider the aperiodic square pulse signal as shown in slide 13
  - $x(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & Otherwise \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{\omega} sin(\omega T_1)$
- Now, find the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| \leq W \\ 0, & Otherwise \end{cases}$$

  -

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega \\ &= \frac{sin(Wt)}{\pi t} \end{aligned}$$

Duality example 1



Duality example 2

- Use duality property to find the Fourier transform $G(j\omega)$ of the signal $g(t) = \frac{2}{1+t^2}$
- From slide 15, $x(t) = e^{-a|t|} \xleftrightarrow{\mathcal{F}} X(j\omega) = \frac{2a}{a^2+\omega^2}$
- Substitute $a = 1$
  - $x(t) = e^{-|t|} \xleftrightarrow{\mathcal{F}} X(j\omega) = \frac{2}{1+\omega^2}$
- Using duality, $X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-j\omega)$
  - $\frac{2}{1+t^2} \xleftrightarrow{\mathcal{F}} 2\pi e^{-|\omega|}$

Deriving other properties using duality

- Differentiation in frequency domain
  - $\frac{d}{dt}(x(t)) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega) \rightarrow -jtx(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega}(X(j\omega))$
    - Proof: Differentiate the analysis equation
- Frequency shift
  - $x(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega) \rightarrow e^{j\omega_0 t}x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_0))$
- Integration in frequency domain
  - $\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) \rightarrow$
    $\frac{-1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} x(\eta)d\eta$

Parseval's relation

- $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$
- Proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t) x^*(t) dt \\
&= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}
$$

Parseval's relation

- Relation between total energy of the signal in time and frequency domain
- Left hand side: $\int_{-\infty}^{\infty} |x(t)|^2 dt$
  - Energy per unit time $|x(t)|^2$ integrated over all time
- Right hand side: $\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$
  - Energy per unit frequency $\frac{1}{2\pi}|X(j\omega)|^2$ integrated over all frequencies
- $|X(j\omega)|^2$: Energy-density spectrum of the signal $x(t)$

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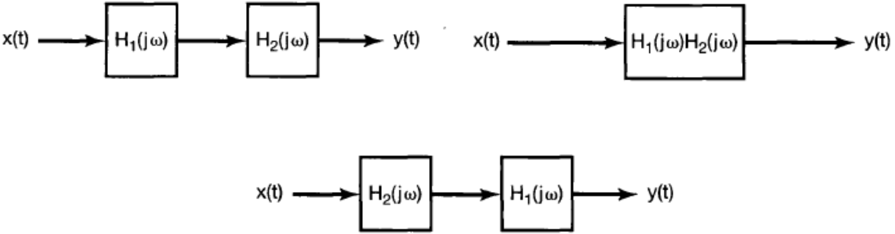
Input-output relation for LTI system

- Let $x(t)$ be the input to a LTI system having impulse response $h(t)$, generating the output $y(t)$
  - $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
- $$\begin{aligned} Y(j\omega) = \mathcal{F}\{y(t)\} &= \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right] e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau \\ &= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = H(j\omega)X(j\omega) \end{aligned}$$

Input-output relation for LTI system

- $y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega)X(j\omega)$
- Fourier transform maps the convolution of two signals into the product of their Fourier transforms



Impulse response $\xleftrightarrow{\mathcal{F}}$ Frequency response

- Time shift system: $h(t) = \delta(t - t_0) \xleftrightarrow{\mathcal{F}} H(j\omega) = e^{-j\omega t_0}$
  - $Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega)$
    - $y(t) = x(t - t_0)$ (Time-shift property)
- Differentiation system: $y(t) = \frac{dx(t)}{dt}$
  - $Y(j\omega) = j\omega X(j\omega) \to H(j\omega) = j\omega$
    - Frequency response of differentiator
- Integration system: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
  - Impulse response of this system is unit step: $h(t) = u(t)$
  - Frequency response: $H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$
  -

$$Y(j\omega) = H(j\omega)X(j\omega) = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)X(j\omega)$$

$$= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

    - Integration property

Example 1

- Find the response of an LTI system with impulse response $h(t) = e^{-at}u(t)$, $a > 0$, to the input signal $x(t) = e^{-bt}u(t)$, $b > 0$
  - $h(t) = e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} H(j\omega) = \frac{1}{a+j\omega}$
  - $x(t) = e^{-bt}u(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{1}{b+j\omega}$
- $Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{a+j\omega} \times \frac{1}{b+j\omega}$
  - For $b \neq a$, $Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$
    - $y(t) = \frac{1}{b-a}(e^{-at}u(t) - e^{-bt}u(t))$
  - For $b = a$, $Y(j\omega) = \frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$
    - $y(t) = te^{-at}u(t)$

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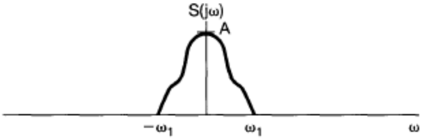
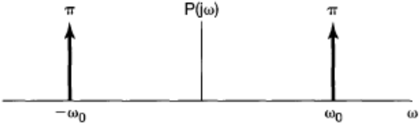
Dual of the convolution property

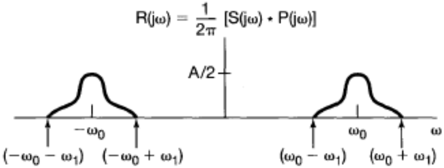
- Convolution property: Convolution in the time domain corresponds to multiplication in frequency domain
- Because of duality between the time and frequency domains, we would expect a dual property to hold too
- $r(t) = s(t)p(t) \xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta$

Example

- Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is shown below
- $p(t) = cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
- Let $r(t) = p(t) \times s(t) \overset{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega - \theta))d\theta = \frac{1}{2}(S(j(\omega - \omega_0)) + S(j(\omega + \omega_0)))$





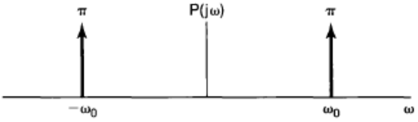


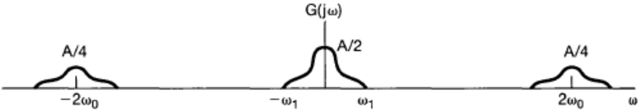
Example

- Here we have assumed $\omega_0 > \omega_1$
  - So that the two nonzero portions of $R(j\omega)$ do not overlap
- All of the information in the signal $s(t)$ is preserved
  - Shifted to higher frequencies
  - Basis for amplitude modulation

Example

- How do we recover $s(t)$ from $r(t)$?
  - Multiply $r(t)$ again by $p(t)$
  - $g(t) = r(t)p(t)$





- Then select only that part of $G(j\omega)$ for which $-\omega_1 \leq \omega \leq \omega_1$

Thanks.