Assignment - 5

(1) $f(x) = (x-a)^{5/2}$ find θ as $x \to a$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h) \quad \text{--- (1)}$$

Now $f'(x) = \frac{5}{2}(x-a)^{3/2}$ $\qquad f''(x) = \frac{15}{4}(x-a)^{1/2}$

putting in (1)

$$(x+h-a)^{5/2} = (x-a)^{5/2} + h \cdot \frac{5}{2}(x-a)^{3/2} + \frac{h^2}{2!} \frac{15}{4}(x+\theta h-a)^{1/2}$$

as $x \to a$, we have

$$h^{5/2} = 0 + 0 + \frac{15}{8} h^2 (\theta h)^{1/2}$$

$$h^{5/2} = \frac{15}{8} h^2 \cdot \theta^{1/2} h^{1/2}$$

$$\frac{8}{15} = \theta^{1/2} \implies \boxed{\theta = \frac{64}{225}}$$

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(2) (a). T.P. $1+x+\frac{x^2}{2} < e^x < 1+x+\frac{x^2}{2}e^x$, $x>0$

$$f(x) = e^x$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x) \qquad 0<\theta<1$$

$$e^x = 1 + x + \frac{x^2}{2} e^{\theta x} \quad \text{--- (1)}$$

Now $0<\theta<1 \implies 0<\theta x<x$ $\quad (\because x>0)$

$\implies 1 < e^{\theta x} < e^x$ $\quad (\because e^x$ is increasing fn.)

Now $1 < e^{\theta x} \implies \frac{x^2}{2} < \frac{x^2}{2} e^{\theta x}$

$\implies 1+x+\frac{x^2}{2} < 1+x+\frac{x^2}{2}e^{\theta x} = e^x$ (from (1))

i.e. $1+x+\frac{x^2}{2} < e^x$ --- (2)

Again

$e^{\theta x} < e^x \implies \frac{x^2}{2}e^{\theta x} < \frac{x^2}{2}e^x$

$\implies 1+x+\frac{x^2}{2}e^{\theta x} < 1+x+\frac{x^2}{2}e^x$

$\implies e^x < 1+x+\frac{x^2}{2}e^x$ --- (3)

from (2), (3)

$$1+x+\frac{x^2}{2} < e^x < 1+x+\frac{x^2}{2}e^x$$

(b) $x - \frac{x^3}{3!} < \sin x < x \quad , x > 0$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x) \qquad 0 < \theta < 1$

$\sin x = x - \frac{x^3}{3!} \cancel{\sin}\cos(\theta x)$

Now $\because \cos\theta x \leq 1 \implies -\frac{x^3}{3!}\cos\theta x \geq -\frac{x^3}{3!} \qquad (\because x > 0)$

$\implies x - \frac{x^3}{3!}\cos\theta x \geq x - \frac{x^3}{3!} \implies \sin x \geq x - \frac{x^3}{3!}$ —— ①

Again $f(x) = f(0) + x f'(\theta x)$

$\sin x = 0 + x\cos(\theta x)$

$\because \cos\theta x \leq 1 \implies x\cos\theta x \leq x \qquad (\because x > 0)$

$\implies \sin x \leq x$ —— ②

from ①, ② we get

$x - \frac{x^3}{3!} < \sin x < x$

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③ $f(x,y) = x^2 y + 3y - 2$ about $(1,-2)$

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y}\right]$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f(a,b)$$

$$f(x,y) = f(1,-2) + \left[(x-1)\,2(1)(-2) + (y+2)((1)^2+3)\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2\,2(-2) + 2(x-1)(y+2)\,2(1) + (y+2)^2 \cdot 0\right] + \frac{1}{3!}\left[3\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial f}{\partial y}\right]$$

$$= ((1)^2(-2) + 3(-2) - 2) + \left[-4(x-1) + 4(y+2)\right] + \frac{1}{2!}\left[-4(x-1)^2 + 4(x-1)(y+2)\right]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

(4) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ up to second degree also find $f(1.1, 0.9)$

We have by Taylor's theorem

$$f(x,y) = f(1,1) + \left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]f(1,1) + \frac{1}{2!}\left[(x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^2 f(1,1)$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{(x^2+y^2)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{(x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$f(1,1) = \pi/4$

$f_x(1,1) = -1/2$

$f_y(1,1) = 1/2$

$f_{xx}(1,1) = 1/2$

$f_{yy}(1,1) = -1/2$

$f_{xy}(1,1) = 0$

thus

$$f(x,y) = \frac{\pi}{4} + \left[(x-1)\left(\frac{-1}{2}\right) + (y-1)\frac{1}{2}\right] + \frac{1}{2!}\left[(x-1)^2\frac{1}{2} - (y-1)^2\frac{1}{2} + 2(x-1)(y-1).0\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Now at $(1.1, 0.9)$

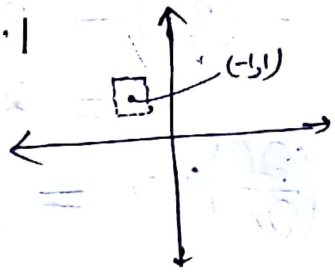
$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(1.1-1) + \frac{1}{2}(0.9-1) + \frac{1}{4}(1.1-1)^2 - \frac{1}{4}(0.9-1)^2$$

$$= 0.685$$

(5). (a) $f(x,y) = 2x^2 - xy + y^2 + 3x - 4y + 1$

at $P_0 = (-1,1)$ and $R: |x+1| < 0.1$

$|y-1| < 0.1$



$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right]_{(a,b)} + R_n$$

where $R_n = \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(\theta, \varphi)$

$$f(a,b) = f(-1,1) = 2(-1)^2 - (-1)(1) + (1)^2 + 3(-1) - 4(1) + 1$$
$$= 2 + 1 + 1 - 3 - 4 + 1$$
$$= -2$$

$\frac{\partial f}{\partial x} = 4x - y + 3$ $\qquad$ $\frac{\partial f}{\partial y} = -x + 2y - 4$

$\left(\frac{\partial f}{\partial x}\right)_{(-1,1)} = -4 - 1 + 3 = -2$ $\qquad$ $\left(\frac{\partial f}{\partial y}\right)_{(-1,1)} = 1 + 2 - 4 = -1$

thus

$$f(x,y) = -2 + \left[(x+1)(-2) + (y-1)(-1)\right]$$
$$= -2 - 2(x+1) - (y-1)$$

Max. absolute error in Linear appx. is given by

$$|R_1| \le \frac{B}{2}\left[|x+1| + |y-1|\right]^2 \le \frac{B}{2}\left[(0.1) + (0.1)\right]^2$$
$$\le \frac{B}{2}[0.04] = 0.02\,B$$

where $B = \max\left[|f_{xx}|, |f_{yy}|, |f_{xy}|\right]$ in the given region $|x+1| < 0.1$, $|y-1| < 0.1$

Now $\max|f_{xx}| = \max|4| = 4 \leftarrow B$

$\max|f_{yy}| = \max|2| = 2$

$\max|f_{xy}| = \max|-1| = 1$

hence $|R_1| \le 0.02 \times 4$ ie $|R_1| \le 0.08$

(b). $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at $P_0 = (3,2)$

$R: |x-3| < 0.1$
$|y-2| < 0.1$

$f(3,2) = 3^2 - 3.2 + \frac{1}{2}2^2 + 3$
$= 8$

$\left(\frac{\partial f}{\partial x}\right)_{(3,2)} = [2x - y]_{(3,2)} = 6-2 = 4$

$\left(\frac{\partial f(3,2)}{\partial y}\right) = [-x+y]_{(3,2)} = -3+2 = -1$

thus $f(x,y) = 8 + 4(x-3) - 1(y-2)$.

Now Max. absolute error

$|R_2| \leq \frac{B}{2}[|x-3| + |y-2|]^2$

$\leq \frac{B}{2}[(0.1) + (0.1)]^2 = \frac{B}{2} \times 0.04$ ~~$= \frac{B}{2}(0.2)$~~

$\leq B \times 0.02$

where
$B = \max[|f_{xx}|, |f_{yy}|, |f_{xy}|]$ in R

$\max|f_{xx}| = \max|2| = 2 \leftarrow B$

$\max|f_{yy}| = \max|1| = 1$

$\max|f_{xy}| = \max|-1| = 1$

thus $|R_2| \leq 0.02 \times 2$

$|R_2| \leq 0.04$

6) $f(x,y) = \sin x \sin y$ at origin.

Quadratic appx. if $|x| \le 0.1$, $|y| \le 0.1$

$$f(x,y) = f(0,0) + \left[x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right] + \frac{1}{2!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f(0,0)$$

$f(0,0) = 0$

$\frac{\partial f}{\partial x}(0,0) = 0$ $\quad \frac{\partial f}{\partial y}(0,0) = 0$

$\frac{\partial^2 f}{\partial x^2}(0,0) = 0$ $\quad \frac{\partial^2 f(0,0)}{\partial x \partial y} = 1$ $\quad \frac{\partial^2 f(0,0)}{\partial y^2} = 0$

thus

$$f(x,y) = \frac{1}{2!}\left[2xy\frac{\partial^2 f(0,0)}{\partial x \partial y}\right]$$

$$= \frac{1 \cdot 2xy}{2!} = xy$$

Now Max. absolute error

$$|E(x,y)| \le \frac{B}{2}[|x| + |y|]^2$$

$$\le \frac{B}{2}[(0.1) + (0.1)]^2 = 0.02 \cdot B$$

where $B = \max\left[|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\right]$ in $|x| < 0.1$, $|y| < 0.1$

$\max|f_{xxx}| = \max|-\cos(0.1)\sin(0.1)| =$ ~~0.0998~~ ~~0.001745~~

$|f_{xxy}| = \max|-\sin(0.1)\cos(0.1)| =$ ~~0.0993~~ ~~0.001745~~

$|f_{xyy}| = \max|\cos(0.1)\sin(0.1)| =$ ~~0.0993~~ ~~0.001745~~

$|f_{yyy}| = \max|-\sin(0.1)\cos(0.1)| =$ ~~0.0993~~ ~~0.001745~~ 0.0993

thus

$$|E(x,y)| \le 0.02 \times 0.0993$$

$$\le 0.00198$$

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(7) $f(x) = \left(\frac{1}{x}\right)^x, \ x > 0$ has maximum at $x = e^{-1}$.

$f'(x) = 0 \Rightarrow f'(x) = -\left(\frac{1}{x}\right)^x (1 + \log x) = 0 \Rightarrow 1 + \log x = 0$

$x = e^{-1}$.

$f''(x) = -\left(\frac{1}{x}\right)^x (1+\log x)^2 - \left(\frac{1}{x}\right)^x \frac{1}{x} < 0$ at $x = e^{-1}$.

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(8) (a) $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$\frac{\partial f}{\partial x} = y - 2x - 2 \qquad \frac{\partial f}{\partial y} = x - 2y - 2 \qquad (-2, -2)$

$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0 \Rightarrow \ -2x + y - 2 = 0$

$x - 2y - 2 = 0$

$-3x - 6 = 0 \qquad \begin{pmatrix} x = -2 \\ y = -2 \end{pmatrix}$

$r = \frac{\partial^2 f}{\partial x^2} = -2, \ t = \frac{\partial^2 f}{\partial y^2} = -2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 1 \qquad r < 0$ maxima

$rt - s^2 = (-2)(-2) - 1^2 = 3 > 0$

Max. value: $f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$

$= 4 - 4 - 4 + 4 + 4 + 4$

$= 8$

(b). $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay \qquad f_y = 3y^2 - 3ax$

$\left.\begin{array}{l} 3x^2 - 3ay = 0 \\ 3y^2 - 3ax = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x^2 - ay = 0 \\ y^2 - ax = 0 \end{array} \rightarrow y = x^2/a$

So $\left(\frac{x^2}{a}\right)^2 - ax = 0 \Rightarrow x^4 - a^3 x = 0 \Rightarrow x(x^3 - a^3) = 0$

$x = 0$ or $x = a$

So (a,a) or $(0,0)$

Now, $r = f_{xx} = 6x \qquad s = f_{xy} = -3a \qquad t = f_{yy} = 6y$

$rt - s^2 = (6x)(6y) - (3a)^2$

$= 36xy - 9a^2$

at $(0,0)$ $\quad rt - s^2 = -9a^2 < 0 \ \forall a \leftarrow$ Saddle point

at (a,a) $\quad rt - s^2 = 36a^2 - 9a^2$

$= 27a^2 > 0 \ \forall a$

Now $\quad r = 6x = 6a > 0$ for $a > 0 \leftarrow$ Minima

< 0 for $a < 0 \leftarrow$ Maxima

(c). $f(x,y) = x^2y^2 - 5x^2 - 8xy - 5y^2$

$f_x = 2xy^2 - 10x - 8y = 0, \quad f_y = 2x^2y - 8x - 10y = 0$

$$\left.\begin{array}{l} xy^2 - 5x - 4y = 0 \\ x^2y - 4x - 5y = 0 \end{array}\right\}$$

(d). $f(x,y) = 2(x-y)^2 - x^4 - y^4$

$f_x = 4(x-y) - 4x^3 = 0 \qquad f_y = -4(x-y) - 4y^3 = 0$

$(x-y) - x^3 = 0$ — (i) $\Rightarrow x^3 = (x-y), \; y^3 = -(x-y)$

$-(x-y) - y^3 = 0$ — (ii)

$\Rightarrow x^3 + y^3 = 0$

$y = x - x^3$ then put in (ii)

$-(x-(x-x^3)) - (x-x^3)^3 = 0$

$-x^3 - (x-x^3)^3 = 0$

$x^3\left[-1 - (1-x^2)^3\right] = 0$

$x = 0$ & $-1-(1-x^2)^3 = 0 \Rightarrow (1-x^2)^3 = -1$

$1 + (1-x^2)^3 = 0$

$\left[1+(1-x^2)\right]\left[1+(1-x^2)^2 - (1-x^2)\right] = 0$

$(2-x^2)(1+1+x^4-2x^2-1+x^2) = 0$

$(2-x^2)(x^4-x^2+1) = 0$

$\Rightarrow x = \pm\sqrt{2}$

So the points are

$(0,0)$

$(\sqrt{2}, -\sqrt{2})$

$(-\sqrt{2}, +\sqrt{2})$

Now

$r = f_{xx} = 4 - 12x^2$

$s = f_{xy} = 4 \qquad t = 4 - 12y^2$

$rt - s^2 = (4-12x^2)(4-12y^2) - (4)^2$

at $(\sqrt{2}, -\sqrt{2})$ & $(-\sqrt{2}, \sqrt{2})$

$r = 4 - 12(\sqrt{2})^2 = 4 - 24 = -20 < 0 \leftarrow$ Maxima

$rt - s^2 = (4 - 12(\sqrt{2})^2)(4 - 12(\sqrt{2})^2) - 4^2$

$= (-20)(-20) - 16 = 400 - 16 = 384 > 0$

Now at $(0,0)$

$rt - s^2 = 0$ further investigation needed

$f(x,y) - f(0,0) = 2(x-y)^2 - x^4 - y^4$

In nbd of $(0,0)$ along the line $y = x$

$f(x,y) - f(0,0) = 2(0) - x^4 - x^4 = -2x^4 < 0 \; \forall x \in R$

Along the line $y = -x$ $\quad f(x,y) - f(0,0) = 8x^2 - 2x^4 = 2x^2(4 - x^2) > 0 \; \forall x \in (-2,2)$

$\Rightarrow$ Saddle point

(e) $f(x,y) = y \sin x$

$f_x = y \cos x$, $f_y = \sin x$ $\quad t = 0$

$r = f_{xx} = -y \sin x$ $\quad s = f_{xy} = \cos x$

$f_x = 0, f_y = 0 \Rightarrow y \cos x = 0 \Rightarrow y = 0$, $\sin x = 0$, $x = n\pi \quad n \in \mathbb{Z}$

thus $(n\pi, 0)$, $n \in \mathbb{Z}$

$rt - s^2 = (-y \sin x) \cdot 0 - (\cos x)^2$

$= -\cos^2 x < 0 \quad \forall x = n\pi \quad n \in \mathbb{Z}$

thus $(n\pi, 0)$ is a saddle point.

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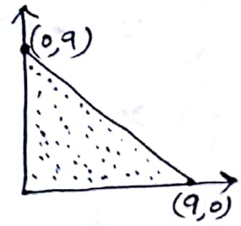
9 (a) $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ $\quad R: x = 0, y = 0, y = 9 - x$

$f_x = 2 - 2x^2 = 0$ $\quad f_y = 2 - 2y^2 = 0$

$x^2 - 1 = 0$ $\quad y^2 - 1 = 0$

$x = \pm 1$ $\quad y = \pm 1$

$(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$



$r = -4x \quad s = 0 \quad t = -4y$

$rt - s^2 = (-4x)(-4y) - 0 = 16xy$

$r = -4x$

	r	$rt - s^2$
(1,1)	-4	16
(1,-1)	-4	-16
(-1,1)	4	-16
(-1,-1)	4	16

(1,1): Local Max.

(-1,-1): local min.

Max. value: $f(1,1) = 2 + 2 + 2 - 1 - 1 = 4$

Min value: $f(-1,-1) = 2 - 2 - 2 - 1 - 1 = -4$

Now on the boundary:

$x=0$ $f(0,y) = 2+2y-y^2$

$\frac{df}{dy} = 2-2y = 0,\ y=1$

$\frac{d^2f}{dy^2} = -2<0$ max.

Max. value $f(0,1) = 2+2-1 = 3$

$y=0$ $f(x,0) = 2+2x-x^2$

$\frac{df}{dx} = 2-2x = 0$ $x=1$

$\frac{d^2f}{dx^2} = -2<0$ max.

Max. value: $f(1,0) = 2+2-1 = 3.$

on third boundary. $y = 9-x$

$f(x,y) = 2+2x+2(9-x) - x^2 - (9-x)^2$
$= 2 + \cancel{2x} + 18 - \cancel{2x} - x^2 - (9-x)^2$
$= 20 - x^2 - (9-x)^2$

$\frac{df}{dx} = -2x + 2(9-x)$
$= -4x + 18 = 0$ $\left(x = \frac{9}{2}\right)$ so $y = 9/2$

$\frac{d^2f}{dx^2} = -4 < 0$ maxima at $\left(\frac{9}{2}, \frac{9}{2}\right)$

at corner points $(9,0)$ $f(0,9)$ $= 2+18-81 = -61$ ← abs. min value at $(9,0)$ $f(0,9)$

we hat $f(9,0) = 2+2\times9+0-9^2$
$f(0,9) = -61$

(b). $f(x,y) = 3x^2+y^2-x$ $R: 2x^2+y^2 \le 1$

$f_x = 6x-1$ $f_y = 2y$ critical points $\left(\frac{1}{6}, 0\right)$

Now $rt - s^2 = 6.2 - 0 = 12 > 0$

and $r = 6 > 0 \longrightarrow$ Point of minima.

Min. value $f(x,y) = f\left(\frac{1}{6}, 0\right) = -1/12$

On the boundary, we have $y^2 = 1-2x^2 \Rightarrow x \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

So $f(x,y) = 3x^2 + (1-2x^2) - x$
$= x^2 - x + 1 = g(x)$

$\frac{dg}{dx} = 2x-1 = 0 \Rightarrow \left(x = 1/2\right)$ also $\frac{d^2g}{dx^2} = 2 > 0$

for $x = 1/2$ $\cdot y^2 = 1 - 2\left(\frac{1}{2}\right)^2$ $y = \pm\frac{1}{\sqrt{2}}$

Hence points are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{2}, \frac{-1}{\sqrt{2}}\right)$ are point of minima.

Min value of f $f\left(\frac{1}{2}, \frac{\pm1}{\sqrt{2}}\right) = \frac{3}{4}$

at vertices we have $f\left(\frac{1}{\sqrt{2}}, 0\right) = (3-\sqrt{2})/2$ $f(0,\pm1) = 1$

$f\left(-1/\sqrt{2}, 0\right) = (3+\sqrt{2})/2$

$\therefore$ absolute min $-1/12$ at $(1/6, 0)$ and absolute max. $\frac{3+\sqrt{2}}{2}$ at $(-1/\sqrt{2}, 0)$

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(10) (a). $f(x,y) = 3x + 4y$ s.t. $x^2+y^2=1$

$g = 3x+4y + \lambda(x^2+y^2-1)$

$g_x = 3 + 2\lambda x = 0 \quad g_y = 4 + 2\lambda y = 0$

$x = \frac{-3}{2\lambda} \qquad y = \frac{-2}{\lambda}$

then $x^2+y^2=1 \Rightarrow \left(\frac{-3}{2\lambda}\right)^2 + \left(\frac{-2}{\lambda}\right)^2 = 1 \Rightarrow \frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$

$\Rightarrow \frac{25}{4\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{25}{4} \Rightarrow \lambda = \pm 5/2$

So $x = \frac{-3}{2(5/2)} = \frac{-3}{5}$ and $x = \frac{-3}{2(-5/2)} = \frac{3}{5}$

$y = \frac{-2}{(5/2)} = \frac{-4}{5}$ and $y = \frac{-2}{(-5/2)} = \frac{4}{5}$

thus critical points are $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$

$g_{xx} = 2\lambda \qquad g_{xy} = 0 \qquad g_{yy} = 2\lambda$

$(g_{xx} \cdot g_{yy}) - (g_{xy})^2 = (2\lambda)(2\lambda) - 0 = 4\lambda^2 > 0$ for $\lambda = \pm 5/2$

Also $g_{xx} = 2\lambda \quad > 0$ if $\lambda = 5/2$

$\qquad\qquad\qquad < 0$ if $\lambda = -5/2$

thus $\left(\frac{-3}{5}, \frac{-4}{5}\right)$ is Point of ~~maxima~~ minima & $\left(\frac{3}{5}, \frac{4}{5}\right)$ is point of maxima

Min value: $f\left(\frac{-3}{5}, \frac{-4}{5}\right) = 3\left(\frac{-3}{5}\right) + 4\left(\frac{-4}{5}\right) = \frac{-25}{5} = -5$

Max. value: $f\left(\frac{3}{5}, \frac{4}{5}\right) = 3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5$

(b). $f(x,y,z) = x^m y^n z^p$ s.t. $x+y+z=a$

$g = x^m y^n z^p + \lambda(x+y+z-a)$

$g_x = m x^{m-1} y^n z^p + \lambda = 0$ — (I)

$g_y = n x^m y^{n-1} z^p + \lambda = 0$ — (II)

$g_z = p x^m y^n z^{p-1} + \lambda = 0$ — (III)

from (I),(II) $\frac{m x^{m-1} y^n z^p}{n x^m y^{n-1} z^p} = \frac{-\lambda}{-\lambda}$

$\Rightarrow \frac{my}{nx} = 1 \Rightarrow \frac{x}{m} = \frac{y}{n}$

Similarily we get $\frac{x}{m} = \frac{y}{n} = \frac{z}{p} = k$ (say)

then $x = km$, $y = kn$, $z = kp$

So $x+y+z=a \Rightarrow km + kn + kp = a \Rightarrow k = \frac{a}{m+n+p}$

thus $x = \frac{am}{m+n+p}$, $y = \frac{an}{m+n+p}$, $z = \frac{ap}{m+n+p}$ So $f(x,y,z) = \frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}$ is the extrema.

(c). $f(x,y) = xy$ s.t. $\frac{x^2}{8} + \frac{y^2}{2} = 1$

$$g = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$$

$g_x = y + \frac{\lambda x}{4} = 0 \longrightarrow y = -\frac{\lambda x}{4}$

$g_y = x + \lambda y = 0$ putting here $x + \lambda\left(\frac{-\lambda x}{4}\right) = 0$

So $x = -\lambda y$

$\left(x - \frac{\lambda^2 x}{4}\right) = 0$

$x\left(1 - \frac{\lambda^2}{4}\right) = 0 \Rightarrow x = 0$ or $\lambda = \pm 2$

then $\left[\begin{matrix} x = -2y & \text{or } 2y \\ \text{for } (\lambda=2) & \text{for } (\lambda=-2) \end{matrix}\right]$

$\frac{x^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \frac{(-2y)^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \frac{y^2}{2} + \frac{y^2}{2} = 1$

~~Also~~ then for $y = 1$ $\Rightarrow y^2 = 1$ $\boxed{y = \pm 1}$

$(x = 2, -2)$ & $y = -1$, $x = -2, 2$

thus we have critical points as:

$(\pm 2, \pm 1) \rightarrow \lambda = -2$ and $(\mp 2, \pm 1) \rightarrow \lambda = 2$

$g_{xx} = \lambda/4$ $\quad (g_{xx} g_{yy}) - (g_{xy})^2 = \frac{\lambda}{4} \cdot \lambda - 1^2 = \frac{\lambda^2}{4} - 1 = 0$ at $\lambda = \pm 2$

$g_{xy} = 1$ $\quad g_{xx} = \frac{\lambda}{4}$ > 0 for $\lambda = +2 \rightarrow$ min

$g_{yy} = \lambda$ $\quad < 0$ for $\lambda = -2 \rightarrow$ max

Max. value $f(\pm 2, \pm 1) = (\pm 2)(\pm 1) = 2$

Min value $f(\mp 2, \pm 1) = (\mp 2)(\pm 1) = -2$

further investigation of $g(x,y) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$

at $(\pm 2, \pm 1)$, $\lambda = -2$: $g(x,y) - g(\pm 2, \pm 1) = xy - 2\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) - \left(2 - 2\left(\frac{1}{2} + \frac{1}{2} - 1\right)\right)$

$= xy - 2\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) - 2$

$= xy - 2\left(\frac{x^2}{8} + \frac{y^2}{2} - 2\right)$

$= xy - 2\left(\frac{x^2 + 4y^2 - 16}{8}\right)$

$= \frac{4xy - x^2 - 4y^2}{4} + \frac{16}{4}$

Similarly $= -\frac{(x-2y)^2}{4} + 4 = 4 - \frac{(x-2y)^2}{4} > 0$ in nbd of $(\pm 2, \pm 1)$ $\Rightarrow$ max

$g(x,y) - g(\mp 2, \pm 1)$, $\lambda = 2$: $= xy + 2\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right) - 2$

$= \frac{(x+2y)^2}{4} - 4 < 0$ in nbd of $(\mp 2, \pm 1) \Rightarrow$ min

(11) Perimeter = constant

So $x+y+z = K$ (const)

$\Rightarrow \quad \dfrac{x+y+z}{2} = s$ (also const)

Now $\quad z = 2s-(x+y).$

Area

$A = \sqrt{s(s-x)(s-y)(s-z)}$ A is max if A^2 is max.

So $f = A^2 = s(s-x)(s-y)(s-z)$

$\quad = s(s-x)(s-y)(s-(2s-(x+y)))$

$f = s(s-x)(s-y)(x+y-s)$

$f_x = s(s-x)(s-y) - s(s-y)(x+y-s) = 0$

$s(s-y)[(s-x)-(x+y-s)) = 0$

$s(s-y)[2s-2x-y] = 0$

$\Rightarrow \quad 2x+y-2s = 0$ —— (I)

$f_y = s(s-x)(s-y) - s(s-x)(x+y-s) = 0$

$\Rightarrow s(s-x)[(s-y)-(x+y-s)] = 0$

$s(s-x)[2s-x-2y] = 0$

$\Rightarrow \quad x+2y-2s = 0$ —— (II)

Solving (I) & (II)

$2x+y-2s = 0$

$x+2y-2s = 0$

$3x-2s = 0$

$\left(x = \dfrac{2s}{3}\right)$ so $y = \dfrac{2s}{3}$ and $z = \dfrac{2s}{3}$

So the triangle is equilateral.

Now to show f is max. at $x=y=z=\dfrac{2s}{3}$

$r = f_{xx} = -2s(s-y) = -2s\left(s-\dfrac{2s}{3}\right) = -\dfrac{8s^2}{3} < 0 \quad \forall\ s$

$t = f_{yy} = -2s(s-x) = -\dfrac{8s^2}{3}$

$s = f_{xy} = -s(s-x) = -\dfrac{4s^2}{3}$

$rt - s^2 = \left(-\dfrac{8s^2}{3}\right)\left(-\dfrac{8s^2}{3}\right) - \left(\dfrac{4s^2}{3}\right)^2$

$= \dfrac{48s^2}{9} > 0$

$\Rightarrow$ f has max. at $x=y=z=2s/3$

(12). Shortest distance from $(1,2,-1)$ to $x^2+y^2+z^2=24$

$d=\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$ d is min if d^2 is min

Let $f=(x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)$

$f_x=2(x-1)+2\lambda x=0 \Rightarrow x=\frac{1}{1+\lambda}$

$f_y=2(y-2)+2\lambda y=0$ $\qquad y=\frac{2}{1+\lambda}$

$f_z=2(z+1)+2\lambda z=0$ $\qquad z=\frac{-1}{1+\lambda}$

So $x^2+y^2+z^2=24$

$\Rightarrow \left(\frac{1}{1+\lambda}\right)^2+\left(\frac{2}{1+\lambda}\right)^2+\left(\frac{-1}{1+\lambda}\right)^2=24$

$\frac{1+4+1}{(1+\lambda)^2}=24 \Rightarrow \frac{6}{(1+\lambda)^2}=24 \Rightarrow \frac{1}{(1+\lambda)^2}=4$

$(1+\lambda)^2=\frac{1}{4} \Rightarrow (1+\lambda)=\pm\frac{1}{2}$ $\qquad \lambda=-1\pm\frac{1}{2}$ $\left(\lambda=\frac{-1}{2},\frac{-3}{2}\right)$

$\Rightarrow \quad x=\frac{1}{1-\frac{1}{2}}=2$ & $\frac{1}{1-\frac{3}{2}}=-2$

$y=\frac{2}{1-\frac{1}{2}}=4$ & $\frac{2}{1-\frac{3}{2}}=-4$

$z=\frac{-1}{1-\frac{1}{2}}=-2$ & $\frac{-1}{1-\frac{3}{2}}=2$

Critical points $(2,4,-2)$ & $(-2,-4,2)$

So $d=\sqrt{(2-1)^2+(4-2)^2+(-2+1)^2}$

$=\sqrt{1^2+2^2+1^2}=\sqrt{6}$

Also

$f_{xx}=2+2\lambda$ >0 for $\lambda=\frac{1}{2}$ & $-\frac{1}{2}$ both

$f_{yy}=2+2\lambda$ $\qquad$ So it is point of minima

$f_{zz}=2+2\lambda$ $\qquad$ if Hessian matrix is positive definite.

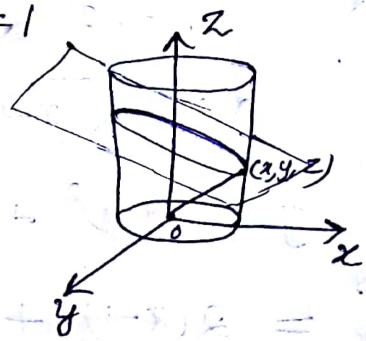
$$H=\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}=\begin{bmatrix} 2+2\lambda & 0 & 0 \\ 0 & 2+2\lambda & 0 \\ 0 & 0 & 2+2\lambda \end{bmatrix}$$

which is (+ve) definite for $\lambda=\frac{1}{2},\frac{-1}{2}$

thus the shortest distance is $\sqrt{6}$

(13) $x+y+z=1$ cuts the cylinder $x^2+y^2=1$

$d=\sqrt{x^2+y^2+z^2}$ d is max/min if d^2 is max/min



$f=(x^2+y^2+z^2)+\lambda(x+y+z-1)+\mu(x^2+y^2-1)$

$f_x = 2x+\lambda+2\mu x = 0 \longrightarrow x = \dfrac{-\lambda}{2(1+\mu)}$

$f_y = 2y+\lambda+2\mu y = 0 \longrightarrow y = \dfrac{-\lambda}{2(1+\mu)}$ $\Big\}$ $x=y$ $\underline{\mu \neq -1}$

$f_z = 2z+\lambda = 0$

$x+y+z=1$

So $x^2+y^2=1 \Rightarrow 2x^2=1 \Rightarrow x=\pm 1/\sqrt{2} \Rightarrow z = 1-\sqrt{2}$ & $z=1+\sqrt{2}$

Now for $\underline{\mu=-1} \Rightarrow \lambda=0 \Rightarrow z=0$

& $x+y+z=1$ & $x^2+y^2=1$

thus critical points $\Rightarrow x=1, y=0$ or $x=0, y=1$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right)$, $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$, $(1,0,0)$, $(0,1,0)$

Clearly $(-1/\sqrt{2}, -1/\sqrt{2}, 1+\sqrt{2})$ is the farthest point from origin and $(1,0,0)$ & $(0,1,0)$ are nearest to origin.

as $d=\sqrt{x^2+y^2+z^2}$.