Signals & Systems (ECN-203)

Lecture 3
(Some important type of signals)

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July 30, 2020

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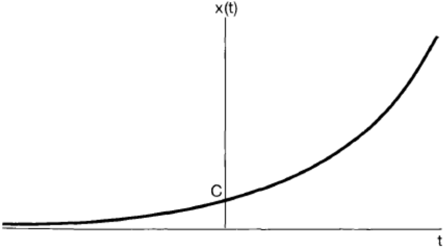
General form

- The general form of a continuous-time exponential signal is:
  - $x(t) = Ce^{at}$
  - C and a are, in general, complex numbers
- Depending upon the values of these parameters, $x(t)$ exhibit several different characteristics

Real exponential signals

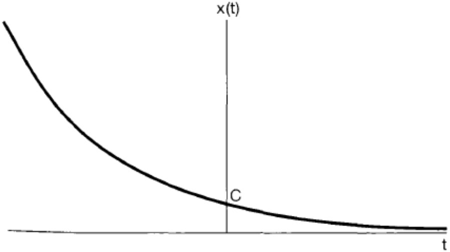
- $C \in \mathbb{R}$ and $a \in \mathbb{R}$

- $a > 0$ (Growing exponential)



- e.g., chain reactions, population growth

- $a < 0$ (Decaying exponential)

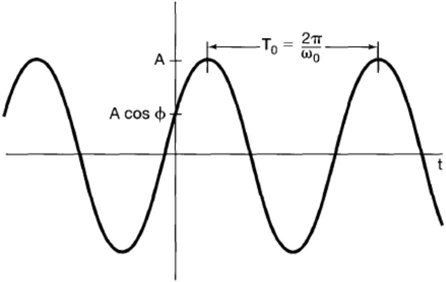


- e.g., radioactive decay, responses of RC circuits, damped mechanical systems

Periodic complex exponential and sinusoidal signals

- a is a purely imaginary number, i.e., $a = j\omega_0$
  - $x(t) = e^{j\omega_0 t}$
- This signal is periodic with fundamental period $T_0 = \frac{2\pi}{|\omega_0|}$
  - $x(t + T_0) = e^{j\omega_0(t+T_0)} = e^{j\omega_0 t}e^{j\omega_0 T_0} = e^{j\omega_0 t}e^{j2\pi} = e^{j\omega_0 t} = x(t)$

- A signal closely related to the periodic complex exponential is the sinusoidal signal: $x(t) = Acos(\omega_0 t + \phi)$



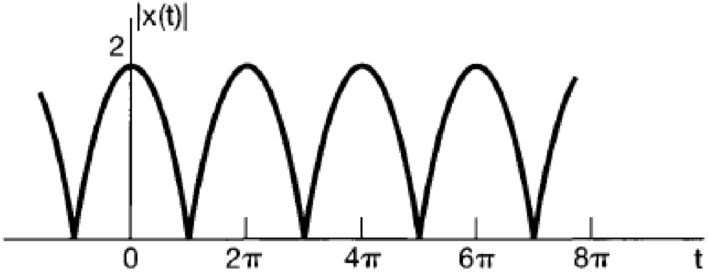
- $Acos(\omega_0 t + \phi) = A\mathbb{R}e\{e^{j(\omega_0 t+\phi)}\}$
- $Asin(\omega_0 t + \phi) = A\mathbb{I}m\{e^{j(\omega_0 t+\phi)}\}$

Energy and power of periodic complex exponential and sinusoidal signals

- These signals have infinite total energy but finite average power
- Total energy in a period, $E_{period} = \int_0^{T_0} |e^{j\omega_0 t}|^2 dt = T_0$
- Total energy in the signal, $E_\infty = \infty$
- Average power in a period, $P_{period} = \frac{E_{period}}{T_0} = 1$
- Total average power of the signal, $P_\infty = 1$

Example

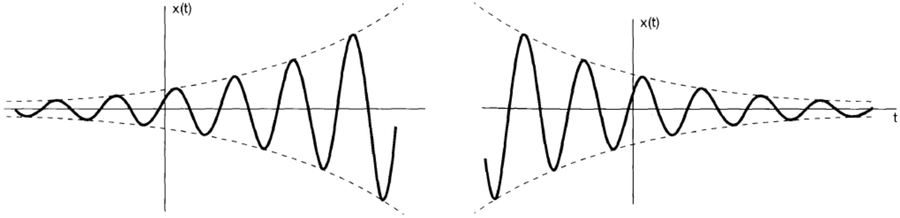
- Plot the magnitude of the signal: $x(t) = e^{j2t} + e^{j3t}$
- Express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid
- $x(t) = e^{j\frac{5}{2}t}(e^{-j\frac{1}{2}t} + e^{j\frac{1}{2}t}) = 2e^{j\frac{5}{2}t}cos(0.5t)$
- $|x(t)| = 2|cos(0.5t)|$
  - Magnitude of the complex exponential is always unity



General complex exponential signals

- C and a are both complex numbers
  - $C = |C|e^{j\theta}$
  - $a = r + j\omega_0$
- $Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)} = |C|e^{rt}cos(\omega_0 t + \theta) + j|C|e^{rt}sin(\omega_0 t + \theta)$
- The real and imaginary part of the complex exponential are:
  - $r = 0 \rightarrow$ sinusoidal signals
  - $r > 0 \rightarrow$ sinusoidal signals multiplied by a growing exponential
  - $r < 0 \rightarrow$ sinusoidal signals multiplied by a decaying exponential
- Commonly referred to as “damped sinusoids”
- Applications:
  - Response of RLC circuits
  - Mechanical systems containing both damping and restoring forces, e.g., automotive suspension systems

Growing/decaying sinusoidal signals



(a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

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General form

- The general form of a discrete-time exponential signal is:
  - $x[n] = C\alpha^n = Ce^{\beta n}$, where $\alpha = e^\beta$

Real exponential signals

- $C, \alpha \in \mathbb{R}$
- $|\alpha| > 1 \rightarrow$ Signal magnitude grows exponentially with n
- $|\alpha| < 1 \rightarrow$ A decaying exponential
- $\alpha > 0 \rightarrow$ All the values of $x[n]$ are of the same sign
- $\alpha < 0 \rightarrow$ The sign of consecutive samples of $x[n]$ alternates
- $\alpha = 1 \rightarrow x[n]$ is a constant
- $\alpha = -1 \rightarrow x[n]$ alternates in value between $+C$ and $-C$
- Applications:
  - Population growth as a function of generation
  - Return on investment as a function of day, month, or quarter

Real exponential signals

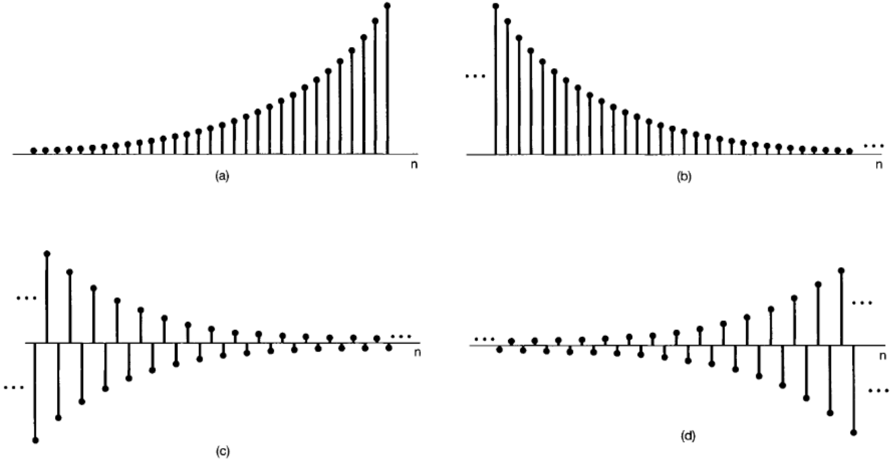
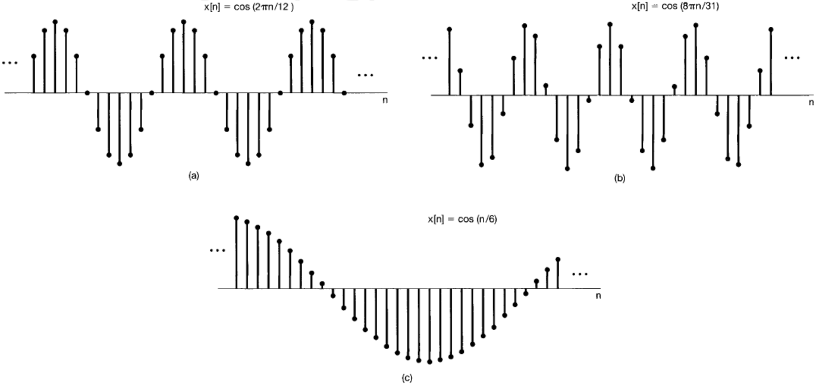


Figure 1.24 The real exponential signal $x[n] = C\alpha^n$: (a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

Sinusoidal signals

- $\beta = j\omega_0$ is a purely imaginary number
  - $|\alpha| = |e^{j\omega_0 n}| = 1$
- $x[n] = e^{j\omega_0 n} = cos(\omega_0 n) + jsin(\omega_0 n)$
- These signals have infinite total energy but finite average power



General complex exponential signals

- C and α are both complex numbers
  - $C = |C|e^{j\theta}$
  - $\alpha = |\alpha|e^{j\omega_0}$
- $C\alpha^n = |C|e^{j\theta}(|\alpha|e^{j\omega_0})^n = |C||\alpha|^n e^{j(\omega_0 n+\theta)} = |C||\alpha|^n cos(\omega_0 n + \theta) + j|C||\alpha|^n sin(\omega_0 n + \theta)$
- The real and imaginary part of the complex exponential are:
  - $\alpha = 1 \rightarrow$ sinusoidal signals
  - $\alpha > 1 \rightarrow$ sinusoidal signals multiplied by a growing exponential
  - $\alpha < 1 \rightarrow$ sinusoidal signals multiplied by a decaying exponential

General complex exponential signals

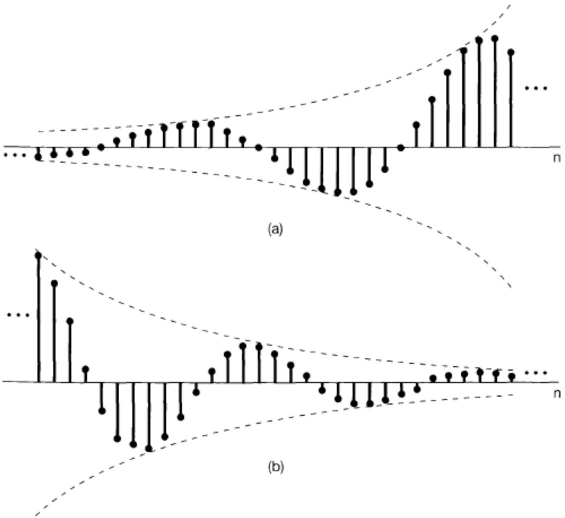


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

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Periodicity: continuous-time vs discrete-time

- Two properties of continuous-time sinusoid: $x(t) = e^{j\omega_0 t}$
  - The larger the magnitude of ω_0, the higher the rate of oscillation
  - $x(t)$ is periodic for any value of ω_0
- However, these properties are not the same for discrete-time sinusoids $x[n] = e^{j\omega_0 n}$

The first property for discrete-time sinusoids

- Consider the discrete-time complex exponential with frequency $\omega_0 + 2\pi$
  - $e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n} = x[n]$
  - Discrete-time complex exponential with frequency $\omega_0 + 2\pi$ is the same as that with frequency ω_0
- Different from the continuous-time case, where the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0
- In discrete-time case, these signals are not distinct as the signal with frequency $\omega_0 \pm 2k\pi$, $k \in \mathbb{I}$ are identical
- For discrete-time complex exponentials, we only consider a frequency interval of length 2π in which to choose ω_0
  - Usually $[0, 2\pi]$ or $[-\pi, \pi]$

Rate of oscillation for discrete-time sinusoids

- The signal $e^{j\omega_0 n}$ does not have a continually increasing rate of oscillation as ω_0 is increased in magnitude
- As ω_0 increases from 0 to π, we get signals that oscillate more and more rapidly
- As ω_0 increases from π to 2π the rate of oscillation decreases
- Low-frequency (slowly varying) discrete-time exponentials: ω_0 close to 0 and 2π
- High-frequency (rapidly varying) discrete-time exponentials: ω_0 close to π

Rate of oscillation for discrete-time sinusoids

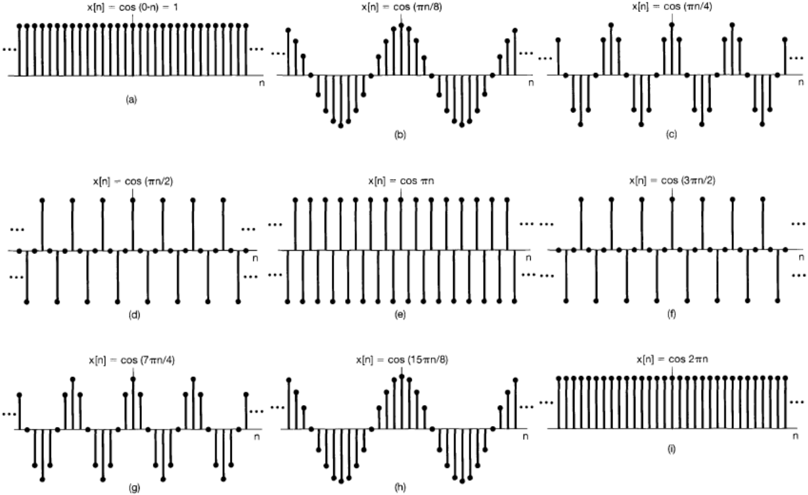
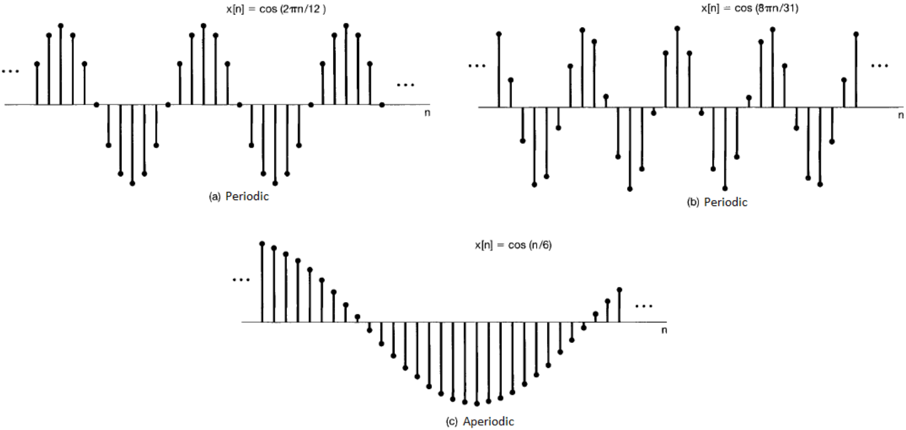


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

The second property for discrete-time sinusoid

- Continuous-time complex exponential signal is always periodic, but can we say the same about the discrete-time exponentials?
- For the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$,
  $e^{j\omega_0 (n+N)} = e^{j\omega_0 n}$
  - $e^{j\omega_0 N} = 1$
  - $\omega_0 N$ must be a multiple of 2π
  - $\omega_0 N = 2m\pi$, for some $m \in \mathbb{I}$
  - $\frac{\omega_0}{2\pi} = \frac{m}{N}$ (a rational number)
- $e^{j\omega_0 N}$ is periodic if $\frac{\omega_0}{2\pi}$ is rational, otherwise, not

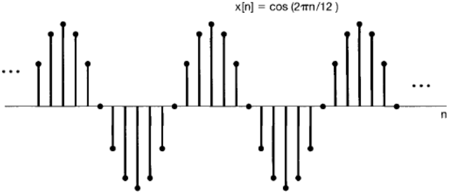
Periodicity of discrete-time sinusoids



Fundamental frequency and period

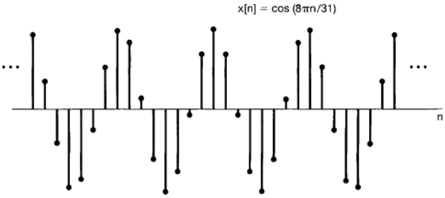
- Consider a periodic complex exponential $x[n] = e^{j\omega_0 n}$ with $\omega_0 \neq 0$
- Fundamental period: N, fundamental frequency: $\frac{2\pi}{N}$
- They must satisfy $\frac{\omega_0}{2\pi} = \frac{m}{N}$
  - m and N have no factor in common
- Fundamental period: $T_0 = N$
- Fundamental frequency, $F_0 = \frac{2\pi}{N} = \frac{\omega_0}{m}$

Fundamental frequency and period example



- $x[n] = cos(\frac{2\pi n}{12})$
- Samples of $x(t) = cos(\frac{2\pi t}{12})$ at integer time points
- $x(t)$ and $x[n]$ are both periodic with fundamental period $T_0 = 12$

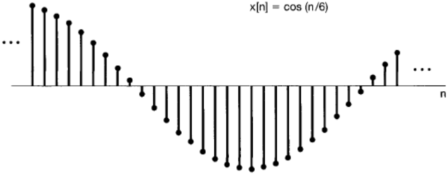
Fundamental frequency and period example



- $x[n] = cos(\frac{8\pi n}{31})$
- Samples of $x(t) = cos(\frac{8\pi t}{31})$ at integer time points
- $x(t)$ is periodic with fundamental period $T_0 = \frac{31}{4}$
- $x[n]$ is periodic with fundamental period $T_0 = 31$

- Discrete-time signal is defined only for integer values of the independent variable
- There is no sample at time $t = \frac{31}{4}, \frac{2\times 31}{4}$, and $\frac{3\times 31}{4}$, when $x(t)$ completes one, two, and three periods (starting from t = 0)
- But there is a sample at $t = \frac{4\times 31}{4} = 31$, when $x(t)$ has completed four periods

Fundamental frequency and period example



- $x[n] = cos(\frac{n}{6})$
- Samples of $x(t) = cos(\frac{t}{6})$ at integer time points
- $x(t)$ is periodic with fundamental period $T_0 = 12\pi$

- Values of $x(t)$ at integer sample points never repeat
  - Sample points never span an interval that is an exact multiple of T_0
- Thus, $x[n]$ is not periodic

Comparison of $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any value of ω_0	Periodic only if $\omega_0 = \frac{2\pi m}{N}$ for some integers $N > 0$ and m
Fundamental frequency is ω_0	Fundamental frequency is $\frac{\omega_0}{m}$
Fundamental period is $\frac{2\pi}{\omega_0}$	Fundamental period is $\frac{2\pi m}{\omega_0}$

Example

- Determine the fundamental period of the discrete-time signal
  - $x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$
- $e^{j\frac{2\pi}{3}n}$: $T_{0,1} = 3$
- $e^{j\frac{3\pi}{4}n}$: $T_{0,2} = 8$
- $T_0 = LCM(3, 8) = 24$

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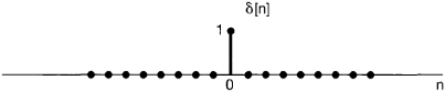
Unit impulse and unit step function

- The unit impulse function is defined as:

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

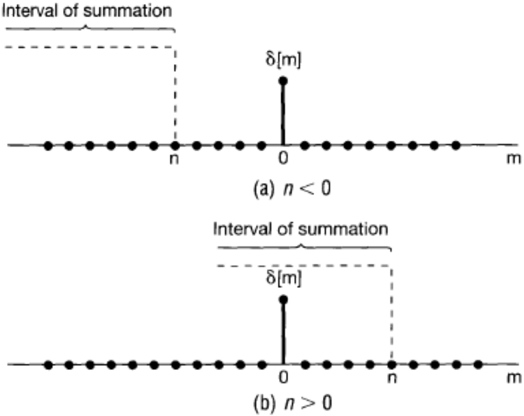


- The unit step function is defined as:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

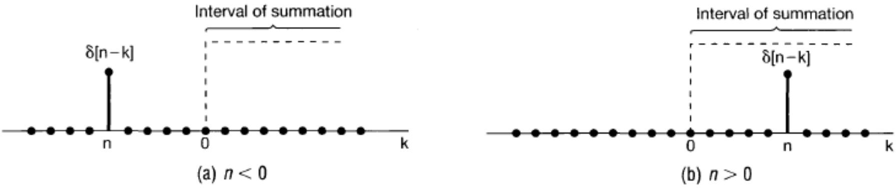
Relation between unit impulse and unit step functions

- Discrete-time unit impulse is the first difference of the discrete-time step
  - $\delta[n] = u[n] - u[n-1]$
- Discrete-time unit step is the running sum of the unit impulse
  - $u[n] = \sum_{m=-\infty}^{n} \delta[m]$



Another way to write step function as a sum of impulses

- $u[n] = \sum_{m=-\infty}^{n} \delta[m]$
  - The summation interval keeps on changing but function to be summed remain same
  - The independent variable for $u[n]$ is in the summation limit
- Change of variable: $k = n - m \rightarrow m = n - k$
  - $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$



- The summation interval remains same but but function to be summed changes

Superposition of delayed impulses

- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
- $u[n]$ can be written as the superposition of delayed impulses ($\delta[n-k]$)
  - $\delta[n]$ at $n = 0$, $\delta[n-1]$ at $n = 1$, $\delta[n-2]$ at $n = 2$, ...
- The unit impulse sequence can be used to sample the value of a signal at $n = 0$
  - $x[n]\delta[n] = x[0]\delta[n]$
- The *shifted* unit impulse sequence can be used to sample the value of a signal at any arbitrary index $n = n_0$
  - $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$

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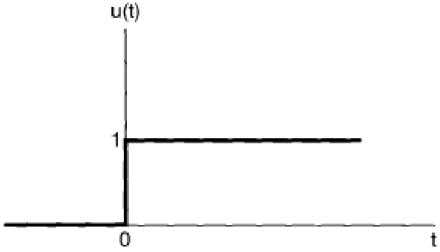
Unit step function

- The continuous-time unit step function $u(t)$ is defined in a manner similar to its discrete-time counterpart
  - $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$
- $u(t)$ is discontinuous at $t = 0$



Unit impulse function

- Continuous-time unit step is the running integral of the unit impulse
  - Analogous to the discrete-time case
- $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \rightarrow \delta(t) = \frac{d}{dt} u(t)$
- However, $u(t)$ is discontinuous at $t = 0$ and consequently, *not* differentiable

Derivation of unit impulse function though approximation

- Consider an approximation $u_\Delta(t)$ to the unit step function $u(t)$
  - $u_\Delta(t)$ rises from 0 to 1 in a short time interval of length Δ
- $u(t) = \lim_{\Delta \to 0} u_\Delta(t)$
- $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & Otherwise \end{cases}$
  - A short pulse of duration Δ and with unit area

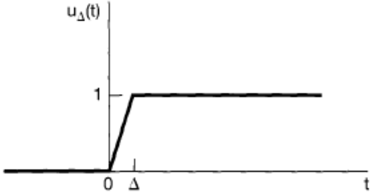


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

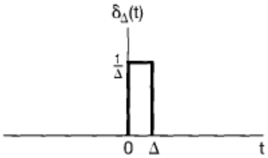


Figure 1.34 Derivative of $u_\Delta(t)$.

Unit impulse function

- $\delta(t)$ can then be thought of as an idealization of $\delta_\Delta(t)$ as Δ becomes insignificant
  - $\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$
  - $\delta(t)$ has, in effect, no duration but unit area

- Graphical notation:
- Arrow at $t = 0$ indicates that the area of the pulse is concentrated at $t = 0$
- Height of the arrow and the "1" next to it represent the area of the impulse

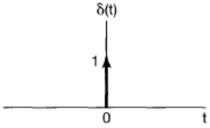


Figure 1.35 Continuous-time unit impulse.

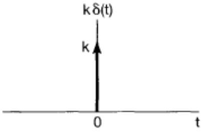


Figure 1.36 Scaled impulse.

Relation between unit impulse and unit step functions

- $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$

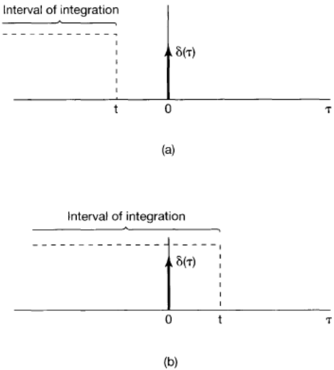


Figure 1.37 Running integral given in eq. (1.71): (a) $t < 0$; (b) $t > 0$.

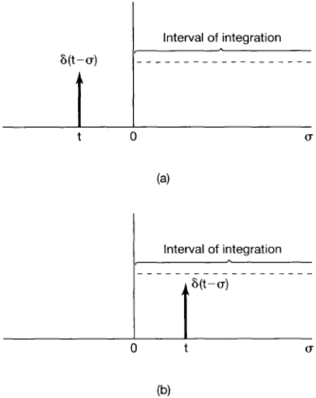


Figure 1.38 Relationship given in eq. (1.75): (a) $t < 0$; (b) $t > 0$.

Sampling property of continuous-time impulse signal

- Consider the product $x_1(t) = x(t) \times \delta_\Delta(t)$
- For small Δ, $x(t)$ is approximately constant over $[0, \Delta]$
- $x(t) \times \delta_\Delta(t) \approx x(0) \times \delta_\Delta(t)$
- $\lim_{\Delta \to 0} x(t) \times \delta_\Delta(t) = x(t) \times \delta(t) = x(0) \times \delta(t)$
- $x(t) \times \delta(t - t_0) = x(t_0) \times \delta(t - t_0)$

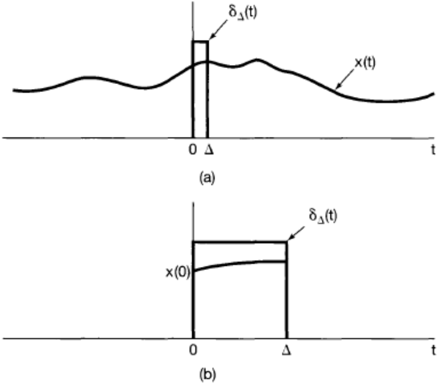


Figure 1.39 The product $x(t)\delta_\Delta(t)$: (a) graphs of both functions; (b) enlarged view of the nonzero portion of their product.

Thanks.