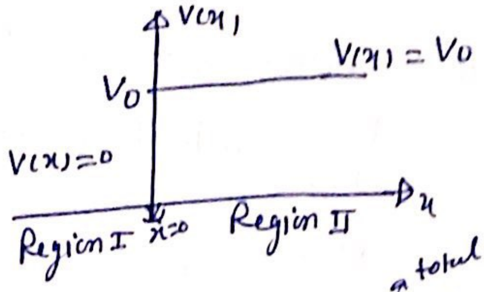
(E) P.C. Srivastav

1-D Potential Step:-

Ex-3

$V(x) = 0 \quad \text{for } x<0$

$\quad\;\; = V_0 \quad \text{for } x \geq 0$



(Potential increases sharply i.e. it becomes repulsive or attractive)

We consier the incidence of a particle (of total energy E) from the le of the barrier.

$E > V_0$ total

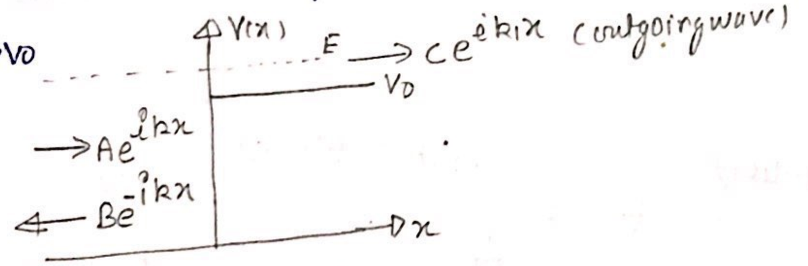
- Classically the particle penetrates the barrier and keeps on moving (with reduced kinetic energy) in the positive x direction.
- Quantum mechanically, the particle may undergo reflection.

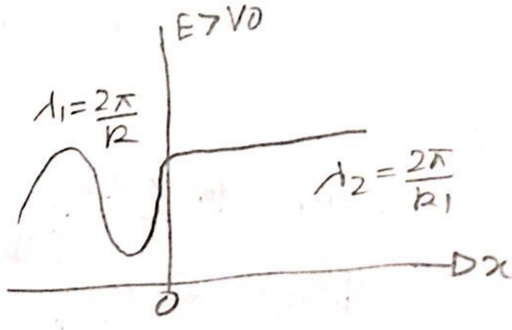
$E < V_0$

- Classically particle is reflected back at $x=0$, because the partic does not have sufficient energy to be in the region $x>0$.

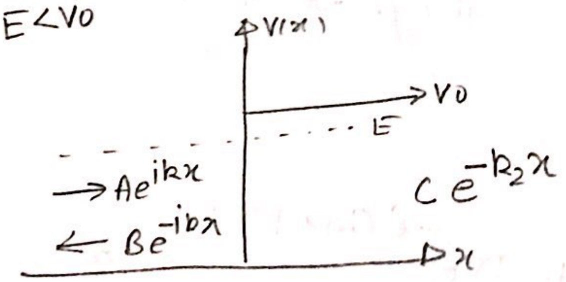
QM • Although the reflection coefficient is unity ($R+T=1$) (or $T=0$), the wave function is non-zero in the region $x>0$, implying that there is a finite probability of finding the particle in the region $x>0$.

E>V0





E<V0



(evanescent wave)
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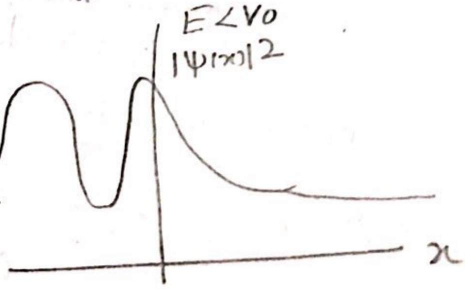
क्षणिक



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Case 1:- $E > V_0$

The time-independent Schrödinger eqn takes the form

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0 \quad \text{for } x<0 \quad \cdots (2)$$

and

$$\frac{d^2\psi(x)}{dx^2} + k_1^2\psi(x) = 0 \quad \text{for } x>0 \quad \cdots (3)$$

where $k = \left[\frac{2mE}{\hbar^2}\right]^{1/2}$ and $k_1 = \left[\frac{2m}{\hbar^2}(E-V_0)\right]^{1/2}$ -- (4)

The solns are

$$\psi(x) = A e^{ikx} + B e^{-ikx} \quad x<0 \quad \cdots (5)$$

$$= C e^{ik_1x} + D e^{-ik_1x} \quad x>0 \quad \cdots (6)$$

Since we are dealing with stationary states,

The complete (time-dependent) wavefunctions is obtained by multiplying eqn (5) and (6) by

$$e^{-iEt/\hbar}$$

$$\left| \frac{E}{\hbar} = \frac{h\nu}{h} \cdot 2\pi = 2\pi\nu = \omega \right.$$

Therefore for $x<0$

$$\psi(x,t) = A e^{i(kx-\omega t)} + B e^{-i(kx+\omega t)} \quad \cdots (7)$$

$A e^{i(kx-\omega t)}$: Incident wave; propagating along $+x$ direction

$B e^{-i(kx+\omega t)}$: Reflected wave; propagating along $-x$ direction

where $\omega = \frac{E}{\hbar}$

Similary for $x>0$

$$\psi(x,t) = C e^{i(k_1x-\omega t)} + D e^{-i(k_1x+\omega t)} \quad \cdots (8)$$

$C e^{i(k_1x-\omega t)}$: (Transmitted wave propagating along $+x$ direction,

$D e^{-i(k_1x+\omega t)}$: (wave propagating along $-x$ direction)

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(2)

Since we are considering the incidence of a particle from the left of the barrier at $x=0$, there cannot be a wave propagating in the $-x$ direction in the region $x>0$ and hence $D=0$.

* We now return to eqn (5) and (6), Continuity of ψ and $\frac{d\psi}{dx}$ at $x=0$ gives us

Linear momentum must be continuous

$\hat{p}\psi(x) = -i\hbar \frac{d\psi}{dx}$ must be continuous as particle moves from left to right

$$A+B = C$$

and

$$ik(A-B) = ik_1 C$$

From which we readily obtain

$ik(A-B) = ik_1(A+B)$

$B = \frac{(k-k_1)A}{(k+k_1)}$

substracting
$$B = \left(\frac{k-k_1}{k+k_1}\right)A \quad \text{--- (9)}$$

adding
$$C = \left(\frac{2k}{k+k_1}\right)A \quad \text{--- (10)}$$

Now, the current density is given by

$$\bar{J} = \frac{i\hbar}{2m}\left[\psi \nabla \psi^* - \psi^* \nabla \psi\right]$$

$$\bar{J} = Re\left[\psi^* \frac{\hbar}{im}\frac{\partial \psi}{\partial x}\right]\hat{x} \quad \text{--- (11)}$$

If J_i, J_r, and J_t represents the magnitude of current densities associated with the incident wave (corresponding to Be^{-ikx}) and the Ae^{ikx}), reflected wave (corresponding to ~~[illegible]~~ Ce^{ik_1x}), respectively. transmitted wave (corresponding to

$J_i = \frac{i\hbar}{2m}\left(\psi_i(x)\frac{d\psi_i^*(x)}{dx} - \psi_i^*(x)\frac{d\psi_i(x)}{dx}\right)$

Thus from (11),

$$J_i = \frac{\hbar k}{m}|A|^2 \quad \text{--- (12)}$$

$$J_r = -\frac{\hbar k}{m}|B|^2 \quad \text{--- (13)}$$

and

$$J_t = \frac{\hbar k_1}{m}|C|^2 \quad \text{--- (14)}$$

The Reflection and transmission coefficients are given by

$$R = \left|\frac{J_r}{J_i}\right| = \frac{|B|^2}{|A|^2} = \frac{(k-k_1)^2}{(k+k_1)^2} \quad \text{--- (15)}$$

$$T = \left|\frac{J_t}{J_i}\right| = \frac{k_1|C|^2}{k|A|^2} = \frac{4kk_1}{(k+k_1)^2} \quad \text{--- (16)}$$

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Thus $R+T=1.$

Case 2:- $E < V_0$

The Schrödinger eq^n becomes.

$$\frac{d^2\psi}{dx^2} - k_2^2\,\psi(x) = 0 \quad . \quad x>0 \quad -- (17)$$

where $$k_2 = \left[\frac{2m}{\hbar^2}(V_0-E)\right]^{1/2} \quad -- (18)$$

Thus, in the region $x>0$, the soln is given by

$$\psi(x) = C e^{-k_2 x} + \underset{\downarrow 0}{D} e^{k_2 x} \quad -- (19)$$

(Because it increases indefinitely with x).

The remaining analysis would ~~be the~~ remains the same except k_1 has to be replace by ik_2:

$$k_1 \to ik_2 = i\left[\frac{2m}{\hbar^2}(V_0-E)\right]^{1/2} \quad -- (20)$$

Continuity of ψ and $\frac{d\psi}{dx}$ leads to

$$B = \left(\frac{k - ik_2}{k+ik_2}\right)A = e^{-2i\gamma} A \quad -- (21)$$

and $$C = \left(\frac{2k}{k+ik_2}\right)A \quad -- (22)$$

where $\tan\gamma = \frac{k_2}{k}$

$B = \left(\frac{k-k_1}{k+k_1}\right)A$

$\frac{1-i\tan x}{1+i\tan x}$

$\frac{\cos x - i\sin x}{\cos x + i\sin x}$

$= \frac{(\cos x - i\sin x)(\cos x - i\sin x)}{(\cos x + i\sin x)(\cos x - i\sin x)}$

$= \cos^2 x - \sin^2 x - 2i\sin x\cos x$

$= \cos 2x - i\sin 2x$

$= e^{-2ix}$

Eq^n (21) tells us that there is a phase change on reflection but the amplitude of the reflected wave is the same as that of the incident wave. Thus the reflection coefficient is given by

$$R = \frac{|B|^2}{|A|^2} = 1 \quad -- (23)$$

In the region $x>0$, since $\psi(x)$ is now real, the transmitted current vanishes and one has $T=0$

$\psi^*(x) = \psi(x)$

$= \frac{\hbar}{2im}\left(\psi(x)\frac{d\psi(x)}{dx} - \psi(x)\frac{d\psi(x)}{dx}\right)$

$= 0$

Thus $\boxed{R+T=1}$

It may be noted that although the transmission coefficient vanishes, the wavefunction is not zero in the region $x>0$, i.e. there is a finite probability of finding the particle in the classically forbidden region.

However, in order to observe that the particle in the region $x>0$, $\Delta x \sim \frac{1}{k_2}$ cd, therefore, the uncertainty in the momentum should be given by.

$$\Delta p \gtrsim \frac{\hbar}{\Delta x} \sim \hbar k_2 \sim \sqrt{2m(V_0-E)}$$

Thus uncertainty in the kinetic energy $\sim \frac{(\Delta p)^2}{2m} \sim (V_0-E)$

↓ high

⇓

Thus, if we try to observe the particle in the region $x>0$, we necessarily impart so much of kinetic energy to it that the total energy is greater than V_0.

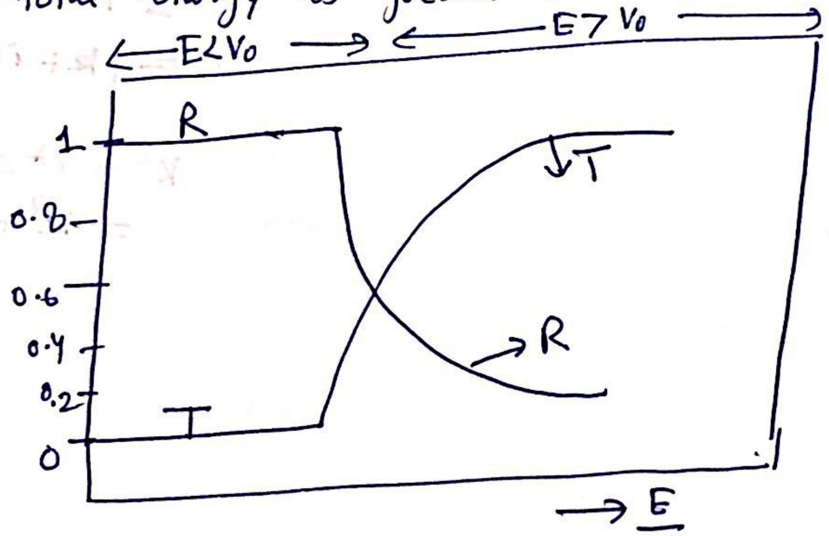

Fig → The variation of reflection and transmission coefficient with $\frac{E}{V_0}$.

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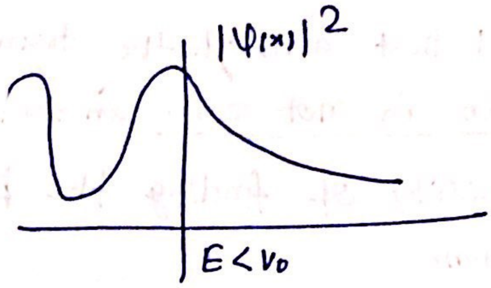

$$P(x) = |\psi(x)|^2 = c^2 e^{-2k_2 x}$$

$$= \frac{4k^2|A|^2}{k^2+k_2^2} e^{-2k_2 x}$$

$\psi = c e^{ik_1 x}$
$\psi^* = c e^{-k_1 x}$
$\psi\psi^* = c^2$

is appreciable near $x=0$ and falls exponentially to small values as x becomes large

$c^* = k - ik_2$
$c = |k + ik_2|$

$k^2 = c^* c$
$= k^2 + k_2^2$

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