(A-0)

PHN-006

The Simple random walk problem in one dimension:-

A particle performing successive steps, or displacements, in one dimension. After a total of N such steps, each of length l, the particle is located at

$$x = ml$$

$m \downarrow$ integer

$$-N \leq m \leq N$$

we want to calculate the probability $P_N(m)$ of finding the particle at the position $x = ml$ after N such steps.

Let $n_1 \rightarrow$ number of steps towards right

$n_2 \rightarrow$ " " left.

total no. of steps $N = n_1 + n_2 \longrightarrow (1)$

the net displacement (measured to the right in units of a step length) is given by

$$m = n_1 - n_2 \quad \text{-- (2)}$$

$$= n_1 - (N - n_1)$$

$$= 2n_1 - N$$

Each step is characterize by the respective probabilities

p = probability that step is to the right.

$q = 1 - p$ = " " left

The probability of any one given sequence of n_1 steps to the right and n_2 steps to the left is given by multiplying the respective probabilities i.e. by

$$\underbrace{pp \cdots p}_{n_1 \text{ factors}} \; \underbrace{q.q \cdots q}_{n_2 \text{ factors}} = p^{n_1} q^{n_2}$$

The number of distinct possibilities is given by

$$\frac{N!}{n_1! \, n_2!}$$

No. of distinct ways N objects place to N possible places

1st at $\cdots$ N

2nd $\cdots$ N-1

N $\cdots$ 1

$N.(N-1) \cdots 1 \equiv N!$

$n_1 \rightarrow$ indistinguishable $n_1!$

Hence the probability $W_N(n_1)$ of taking n_1 steps to the right and $n_2 = N - n_1$ steps to the left, in any order, is obtained by multiplying the probability of this sequence by the no. of possible seq. of such steps.

$$\boxed{W_N(n_1) = \frac{N!}{n_1!\, n_2!} p^{n_1} q^{n_2}}$$

↓ Binomial distribution.

Because Binomial theorem

$$(p+q)^N = \sum_{n=0}^{N} \frac{N!}{n!\, (N-n)!} p^n q^{N-n}$$

Thus the probability $P_N(m)$ that the particle is found at position m after N steps is the same as $W_N(n_1)$ given by

$$P_N(m) = W_N(n_1)$$

$$n_1 = \frac{1}{2}(N+m), \quad n_2 = \frac{1}{2}(N-m)$$

$$P_N(m) = \frac{N!}{\left[\left(\frac{N+m}{2}\right)!\right]\left[\left(\frac{N-m}{2}\right)!\right]} p^{\left(\frac{N+m}{2}\right)} (1-p)^{\left(\frac{N-m}{2}\right)}$$

in the special case $p = q = \frac{1}{2}$

$$P_N(m) = \frac{N!}{\left[\left(\frac{N+m}{2}\right)!\right]\left[\left(\frac{N-m}{2}\right)!\right]} \left(\frac{1}{2}\right)^N$$

e.g. Suppose $p = q = \frac{1}{2}$ and $N = 3$,

possible number of steps to the right are

$n_1 = 0, 1, 2$ or 3;

the corresponding displacements are

$m = -3, -1, 1$ or 3

the probabilities

$$W_3(n) = P_3(m) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$$

$$W(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

$n << N$

$p << 1$

$$\boxed{Np = \mu = \text{Const.}}$$

$$\boxed{\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - - - \\ \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - - - \end{aligned}}$$

$\ln(1-p) \approx -p \quad \because p << 1$

$\Rightarrow \quad e^{-p} = (1-p)$

$$(1-p)^{N-n} = e^{-p(N-n)} \approx e^{-Np}$$

$$\frac{N!}{(N-n)!} = \frac{N.(N-1).(N-2)\ldots(N-n+1)\cancel{(N-n)!}}{\cancel{(N-n)!}}$$

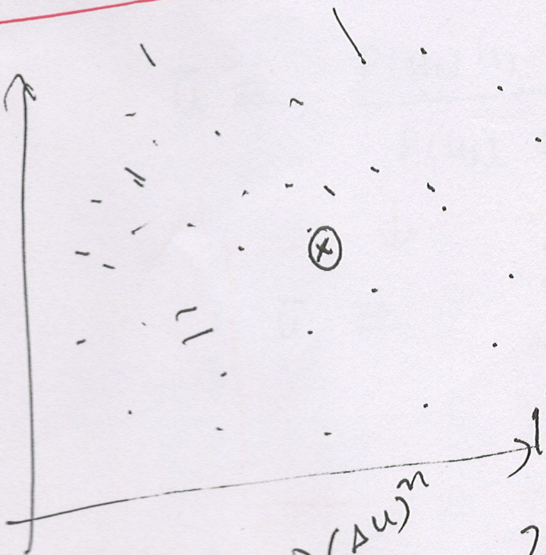
$(N-1) \to N$, $(N-2) \to N$

$$\approx \underbrace{N \ldots\ldots N}_{n \text{ times}} = N^n$$

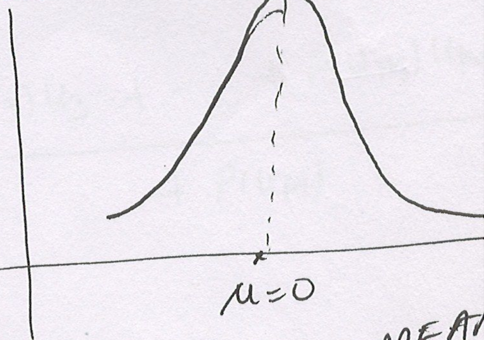
Poisson's Distribution

$$\boxed{W(n) = \frac{N^n}{n!} p^n e^{-Np} = \frac{\mu^n}{n!} e^{-\mu}}$$

$N = 10{,}000$
$n = 100$
$p = 0.01$
$\mu = 100$



W(n)



$\mu = 0$

e.g. Distribution of population of Uttarakhand in comparison to population of India.
$N \to$ millions $\to$ Thus use Poisson's

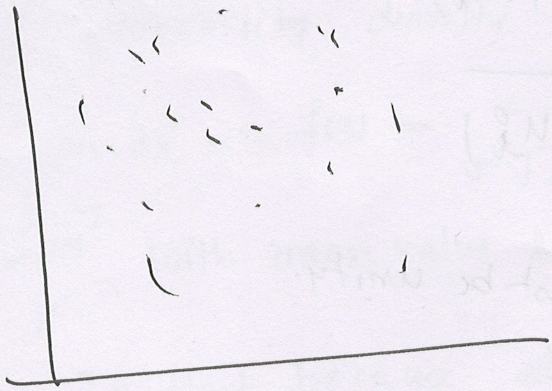
1^{st} moment gives CENTROID. $\to$ MEAN

2^{nd} moment gives DISPERSION around centroid
$\downarrow$
STD. DEVIATION
σ

$$(\Delta u)^n = \sum_{i=1}^{N} P(u_i)(\Delta u)^n$$

with constraint $\sum P(u_i) = 1$

General discussion of Mean Values :-



1^{st} moment gives CENTROID → Mean value

2^{nd} moment gives DISPERSION around Centroid → STD. DEVIATION (σ)

1^{st} Moment :-

Let u be a variable which can assume any of the M discrete values

$u_1, u_2 \ldots, u_M$

with respective probabilities

$P(u_1), P(u_2) \ldots, P(u_M)$

The mean (or average) value of u is denoted by $\bar{u}$ and is defined by

$$\bar{u} \equiv \frac{P(u_1)u_1 + P(u_2)u_2 + P(u_3)u_3 + \cdots + P(u_M)u_M}{P(u_1) + P(u_2) + \cdots + P(u_M)}$$

$$\boxed{\bar{u} \equiv \frac{\sum_{i=1}^{M} P(u_i)\, u_i}{\sum_{i=1}^{M} P(u_i)}}$$

In generally, if $f(u)$ is any function of u, then mean value of $f(u)$ is defined by

$$\overline{f(u)} \equiv \frac{\sum_{i=1}^{M} P(u_i)\, f(u_i)}{\sum_{i=1}^{M} P(u_i)}$$

$$\sum_{i=1}^{M} P(u_i) \longrightarrow \text{must be unity.}$$

$$= 1$$

$$\overline{f(u)} = \sum_{i=1}^{M} P(u_i)\, f(u_i)$$

Second moment of u about its mean :-

$$\overline{(\Delta u)^n} = \sum_{i=1}^{N} P(u_i)\, (\Delta u)^n$$

if $i=1$

$\sum P(u_i) = 1$

$$\overline{(\Delta u)^2} \equiv \sum_{i=1}^{M} P(u_i)\, (u_i - \overline{u})^2 \geq 0$$

↓

'Second moment of u about its mean'

or more simply the "dispersion of u".

↓

This can never negative, since $(\Delta u)^2 \geq 0$

↓

The larger the spread of values of u_i about $\overline{u}$ (mean), the larger the dispersion.

66 The dispersion thus measures the amount of scatter of values of the variable about its mean value (e.g. scatter in grades about the mean grade of the students)

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Gauss' Normal Distribution:-

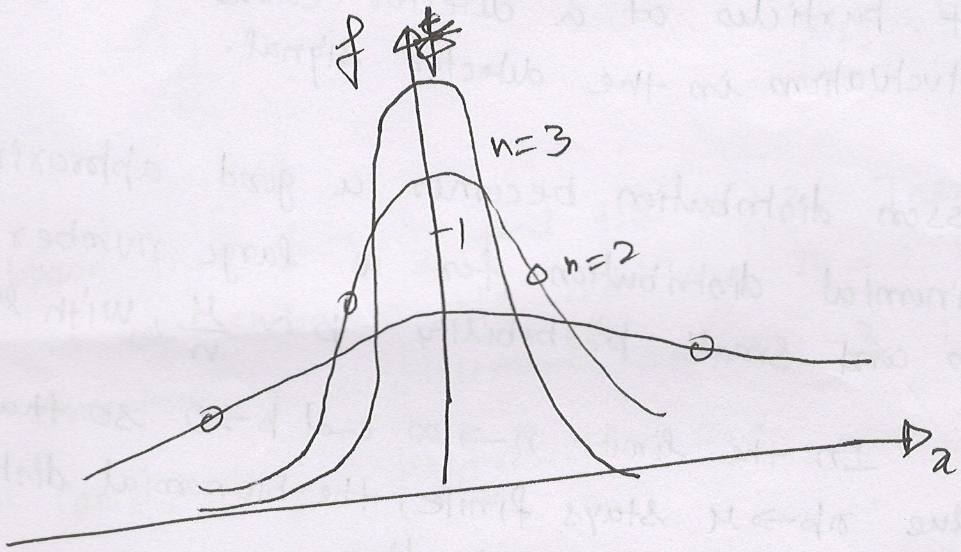
The bell-shaped Gauss distribution is defined by the probability density

$$p(x)\,dx = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{[x-\mu]^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

Probability density

with mean value μ and variance σ^2.

In part because ~~to~~ it represents continuous limits of both the binomial and Poisson distributions, it is by far the most important continuous probability distribution.



$$\mu \equiv (p-q)Nl$$
$$\sigma \equiv 2\sqrt{Npq}\,l$$

→ (N) Standard deviation

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

also $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$.

$$\frac{3!}{3!}\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$n_1 = 0$ $\quad m = -3$ $\quad n_1 + n_2 = N$

$n_1 = 1$ $\quad m = -1$ $\quad n_1 - n_2 = m$

$n_1 = 2$ $\quad m = 1$ $\quad m = 2n_1 - N$

$n_1 = 3$ $\quad m = 3$

$$\frac{3!}{2!}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$\boxed{P_N(m) = \cancel{\frac{N!}{\left[\frac{(N+m)}{2}\right]!\left[\frac{(N-m)}{2}\right]!}}\left(\frac{1}{2}\right)^N}$$

$$\frac{3!}{2!}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$\boxed{P_N(m) = \frac{N!}{n_1!\,n_2!}\,p^{n_1}q^{n_2}}$$

$$\frac{3!}{3!} = \frac{1}{8}$$

<u>Q:-</u> Suppose that $p = q = \frac{1}{2}$ and that $N = 3$. Then the possible numbers of steps to the right are $n_1 = 0, 1, 2$ or 3; the corresponding displacements are $m = -3, -1, 1$ or 3; the corresponding probabilities are

$$W_3(n_1) = P_3(m) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$$

<u>Ans</u>

Stirling's Formula:-

The calculation of $n!$ becomes very laborious for large values of n.

$\Downarrow$

Thus we should like to find a simple approximation formula by which n can be calculated in the limit when n is large.

by defⁿ

$$n! \equiv 1 \times 2 \times 3 \cdots \times (n-1) \times n$$

$$\ln n! = \ln 1 + \ln 2 + \cdots + \ln n = \sum_{m=1}^{n} \ln m \quad \text{--- (1)}$$

When n is large, we can replace summation by integral.

$$\ln n! \simeq \int_1^n \ln x \, dx = [x \ln x - x]_1^n$$

$$\boxed{\ln n! \simeq n \ln n - n}$$

since lower limit negligible when $n >> 1$.