Signals & Systems (ECN-203)

Lecture 4
(Systems and their properties)

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2. Interconnections of systems
3. Basic system properties

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1 **Continuous-time and discrete-time systems**

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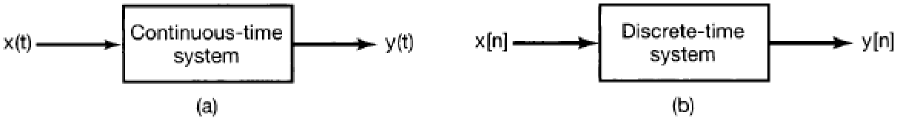
3 Basic system properties

Physical systems

- Physical systems are an interconnection of components, devices, or subsystems
- Examples:
  - Signal processing and communications
  - Electro-mechanical motors
  - Automotive vehicles
  - Chemical-processing plants
- A system transforms input signals, resulting in other signals as outputs
- The classification of input and output signals determine the type of system

Continuous-time and discrete-time systems

- Continuous-time system: A system in which inputs and outputs are continuous-time signals
  - $x(t) \to y(t)$
- Discrete-time system: A system in which inputs and outputs are discrete-time signals
  - $x[n] \to y[n]$



Continuous-time system example (RC circuit)

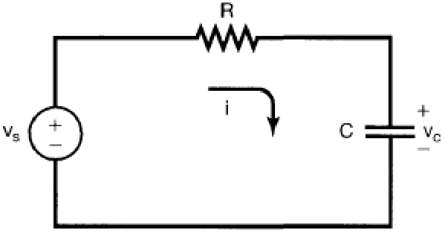


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

- Input signal: $V_s(t)$
- Output signal: $v_c(t)$
- $i(t) = \frac{V_s(t) - v_c(t)}{R} = C\frac{dv_c(t)}{dt}$

- Relationship between input and output signals:
  - $\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}V_s(t)$
  - First-order linear differential equation

Continuous-time system example (moving car)

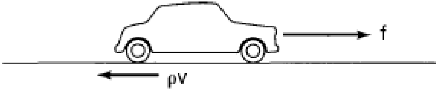


Figure 1.2 An automobile responding to an applied force f from the engine and to a retarding frictional force ρv proportional to the automobile's velocity v.

- Input signal: Force, $f(t)$
- Output signal: Velocity, $v(t)$
- $\frac{dv(t)}{dt} = \frac{1}{m}(f(t) - \rho v(t))$

- Relationship between input and output signals:
  - $\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}v(t)$
  - First-order linear differential equation

Discrete-time system example (balance in a bank account from month to month)

- $x[n]$: Net deposit (i.e., deposits minus withdrawals) during the n^{th} month
- $y[n]$: Balance at the end of the n^{th} month
- $y[n]$ evolves from month to month according to:
  - $y[n] = 1.01y[n-1] + x[n]$
  - The account accrue 1% interest each month
- $y[n] - 1.01y[n-1] = x[n]$
- First-order linear difference equation

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A complex physical systems

- Many real systems are built as interconnections of several subsystems
- Audio system: radio receiver + compact disc player + tape deck + amplifier + one or more speakers
- Digitally controlled aircraft
  - The aircraft described by its equations of motion and the aerodynamic forces affecting it
  - The sensors which measure various aircraft variables such as accelerations, rotation rates, and heading
  - A digital autopilot which responds to the measured variables and to command inputs from the pilot
    - Desired course, altitude, and speed
  - The aircraft's actuators which respond to inputs provided by the autopilot
    - Control rudder, tail, and ailerons to change the aerodynamic forces on the aircraft

Analyzing a complex physical system

- By visualizing a complex system as an interconnection of its components
  - Component system description
  - How they are connected

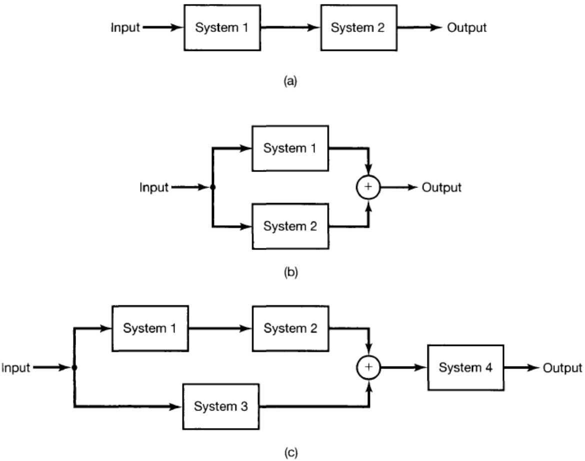
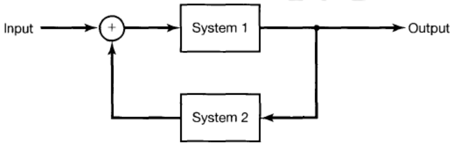


Figure 1.42 Interconnection of two systems: (a) series (cascade) interconnection; (b) parallel interconnection; (c) series-parallel interconnection.

Feedback interconnection



- Output of System 1 is the input to System 2
- Output of System 2 is fed back to modify the input to System 1

- Application: cruise control system
  - Senses vehicle's velocity and adjusts the fuel flow to keep the speed at the desired level

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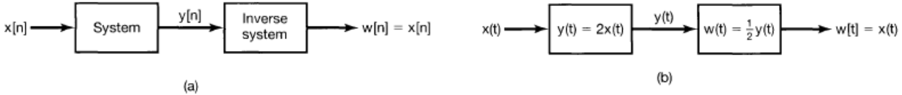
Systems with and without memory

- Memoryless system: Output for each value of the independent variable at a given time is dependent only on the input at that same time
  - $y[n] = (2x[n] - x^2[n])^2$
  - Example: A resistor ($v(t) = R \times i(t)$)
- Systems with memory: Retains information about input values at times other than the current time
  - An accumulator or summer: $y[n] = \sum_{k=-\infty}^{n} x[k]$
  - A delay system: $y[n] = x[n-1]$
  - A capacitor: $v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau$
- For most physical systems: memory = storage of energy
  - Capacitor stores energy by accumulating electrical charge (integral of the current)
  - Accumulator: Memory is associated with the storage registers

Invertibility and inverse systems

- Invertible systems: Distinct inputs lead to distinct outputs
  - Then an inverse system exists that, when cascaded with the original system, yields an identity system



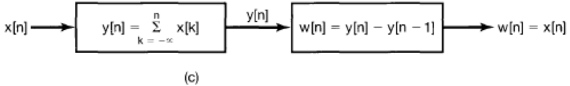


Figure 1.45 Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

- Non-invertible systems:
  - $y[n] = 0$ (zero output sequence for any input sequence)
  - $y(t) = x^2(t)$ (cannot determine the sign of $x(t)$ from $y(t)$)
- Applications:
  - Encoding/decoding in communication systems
  - Cryptography

Causality

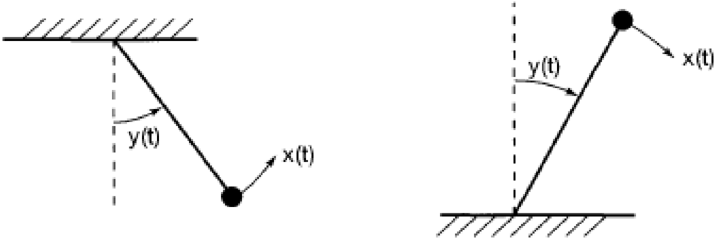
- Causal system: Output at any time depends only on values of the input at the present time and in the past
  - Nonanticipative: System output does not anticipate future values of the input
  - e.g., RC circuit: capacitor voltage responds only to the present and past values of the source voltage
  - e.g., Motion of an automobile: does not anticipate future actions of the driver
- Causality is not an essential constraint in applications where:
  - Independent variable is not time, e.g., image processing
  - Processing data that have been recorded previously, e.g., speech, geophysical, or meteorological signals
- Non-causal systems:
  - $y[n] = x[n] - x[n+1]$
  - $y[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$
  - $y(t) = x(t+1)$

Causality example

- $y[n] = x[-n]$: is this system causal?
  - Output at times $n < 0$ depends on a future value of the input
  - Not causal
- $y(t) = x(t) \times cos(t + 1)$: is this system causal?
  - Only the current value of the input $x(t)$ influences the current value of the output $y(t)$
  - Causal

Stability

- A stable system is one in which small inputs lead to responses that do not diverge



(a) A stable pendulum (b) an unstable inverted pendulum

- Unstable systems: chain reactions with infinite fuel, population growth with unlimited food supplies and no predators

Stability (mathematical formulation)

- Stable system: If the input is bounded, then the output must also be bounded

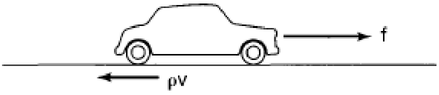


Figure 1.2 An automobile responding to an applied force f from the engine and to a retarding frictional force ρv proportional to the automobile's velocity v.

- Vehicle initially at rest
- Input force $f(t) = F$ (constant)
- Velocity of the car will increase
- Retarding frictional force also increases with velocity

- Velocity stabilizes when frictional force balances the applied force
  - $F = \rho V \rightarrow V = \frac{F}{\rho}$

Stability examples

- $y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$ (Stable or unstable?)
  - Average of a finite set of values of the input
  - If $x[n]$ is bounded (say by B), then $y[n]$ is also bounded by B
- $y[n] = \sum_{k=-\infty}^{n} x[k]$ (Stable or unstable?)
  - Sums all of the past values of the input
  - The sum can grow continually even if $x[n]$ is bounded
    - $x[n] = 1 \rightarrow y[n] = \infty$

Stability examples

- $S_1 : y(t) = tx(t)$
  - $x(t) = 1 \rightarrow y(t) = t$
  - No matter what finite constant bound B we pick, $y(t)$ will exceed that constant for some t
  - Unstable
- $S_2 : y(t) = e^{x(t)}$
  - Let $x(t)$ is an arbitrary signal bounded by B, i.e., $-B < x(t) < B$
  - $e^{-B} < y(t) < e^{B} \rightarrow$ Bounded
  - Stable

Time invariance

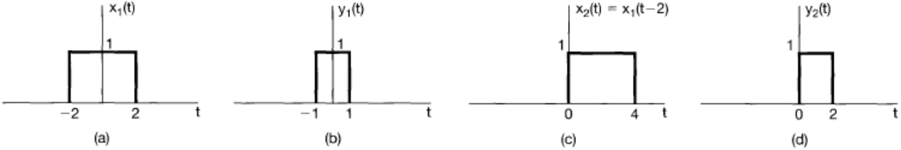
- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal
  - $x(t) \rightarrow y(t)$ –> $x(t - t_0) \rightarrow y(t - t_0)$
- e.g., consider a RC circuit, where the R and C values are constant over time
  - We would expect to get the same results from an experiment with this circuit at any time t

Time invariance examples

- $y(t) = sin(x(t))$
  - $y_1(t) = sin(x_1(t))$
  - $x_2(t) = x_1(t - t_0) \rightarrow y_2(t) = sin(x_2(t)) = sin(x_1(t - t_0))$
  - $y_1(t - t_0) = sin(x_1(t - t_0)) = y_2(t) \rightarrow$ Time-invariant
- $y[n] = nx[n]$
  - $y_1[n] = nx_1[n]$
  - $x_2[n] = x_1[n - n_0] \rightarrow y_2[n] = nx_2[n] = nx_1[n - n_0]$
  - $y_1[n - n_0] = (n - n_0)x_1[n - n_0] \neq y_2[n] \rightarrow$ Time-variant
- Alternatively, $x_1[n] = \delta[n] \rightarrow y_1[n] = nx_1[n] = n\delta[n] = 0$
- $x_2[n] = x_1[n - 1] = \delta[n - 1] \rightarrow$
  $y_2[n] = nx_2[n] = n\delta[n - 1] = \delta[n - 1] \neq y_1[n - 1]$
  - Time-variant

Time invariance examples

- Time scaling system: $y(t) = x(2t)$
  - $y(t)$ is a time-compressed (by a factor of 2) version of $x(t)$
  - Any time shift in the input will also be compressed by a factor of 2
    - Hence, this system is not time invariant



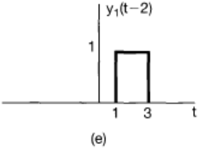


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

Linearity

- A linear system possesses the property of superposition
  - If $x_1(t) \to y_1(t)$, and $x_2(t) \to y_2(t)$
  - Then $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$ (additivity)
  - And $ax_1(t) \to ay_1(t)$ (scaling or homogeneity)
  - Or combining them together: $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$
    - a and b are any complex constants
- Superimposition property:
  - If $x_k[n] \to y_k[n]$ for $k = 1, 2, \ldots$ then $\sum_k a_k x_k[n] \to \sum_k a_k y_k[n]$
  - A zero input to a linear system results in a zero output
    - If $x[n] \to y[n]$, then $0 = 0 \times x[n] \to 0 \times y[n] = 0$
- Note: A system can be linear without being time invariant and it can be time invariant without being linear

Linearity example

- $y(t) = tx(t)$ (is this system linear?)
  - $x_1(t) \to y_1(t) = tx_1(t)$
  - $x_2(t) \to y_2(t) = tx_2(t)$
  - $ax_1(t) + bx_2(t) \to y(t) = t(ax_1(t) + bx_2(t)) = atx_1(t) + btx_2(t) = ay_1(t) + by_2(t)$ (linear system)
- $y(t) = x^2(t)$ (is this system linear?)
  - $x_1(t) \to y_1(t) = x_1^2(t)$
  - $x_2(t) \to y_2(t) = x_2^2(t)$
  - $ax_1(t) + bx_2(t) \to y(t) = (ax_1(t) + bx_2(t))^2 = a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \neq ay_1(t) + by_2(t)$ (Not linear system)

Linearity example

- A linear system must satisfy both the additivity and homogeneity properties
- Signals and scaling constants are allowed to be complex
- $y[n] = \mathbb{R}e\{x[n]\}$
  - $x_1[n] = r[n] + js[n] \rightarrow y_1[n] = r[n]$
  - $x_2[n] = jx_1[n] = j(r[n] + js[n]) = -s[n] + jr[n] \rightarrow$ $y_2[n] = -s[n] \neq jy_1[n]$ (Not linear system)
- $y[n] = 2x[n] + 3$
  - $x_1[n] = 2 \rightarrow y_1[n] = 2x_1[n] + 3 = 7$
  - $x_2[n] = 3 \rightarrow y_2[n] = 2x_2[n] + 3 = 9$
  - $x_1[n] + x_2[n] = 5 \rightarrow y[n] = 2(x_1[n] + x_2[n]) + 3 = 13 \neq y_1[n] + y_2[n]$ (Not linear system)
  - Alternatively, $x[n] = 0 \rightarrow y[n] = 3 \neq 0$ (violates the zero-input zero-output property)

Incremental linear system

- $y[n] = \mathbb{R}e\{x[n]\}$ is surprisingly non-linear system
  - Since its input-output equation is linear

- $y[n]$ can be represented as the sum of a linear system ($x[n] \to 2x[n]$) and a zero-input response ($y_0[n] = 3$)

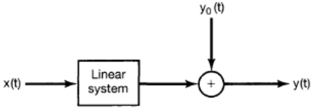


Figure 1.48 Structure of an incrementally linear system. Here, $y_0[n]$ is the zero-input response of the system.

- A large classes of systems can be represented using this model
  - Superposition of the response of a linear system with a zero-input response
  - Incremental linear system: Respond linearly to changes in the input

Thanks.