Signals & Systems (ECN-203)

Lecture 5
(Linear time-invariant systems)

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Introduction

- In last chapter, we introduced and discussed several basic system properties
- Two very important properties: Linearity and time invariance $\rightarrow$ linear time-invariant (LTI) systems
  - Many physical processes possess these properties and can be modeled as LTI systems
  - LTI systems can be analyzed in considerable detail
    - Provide insight into their properties
    - Powerful tools that form the core of signal and system

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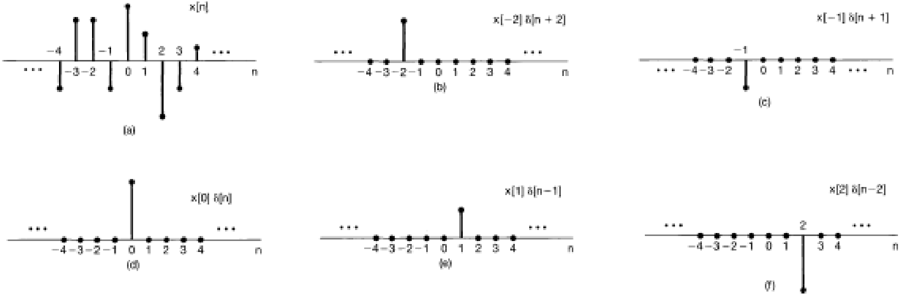
Discrete-time signals as a sum of shifted impulses

- Discrete-time unit impulse can be used to construct any discrete-time signal
  - Using scaling and shifting
  - $x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$
  - $x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$
  - $x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}$
- $x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots$

Sifting property of discrete-time unit impulse

- $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
  - Any arbitrary sequence can be written as a linear combination of shifted unit impulses $\delta[n-k]$
  - Weights in this linear combination are $x[k]$



Convolution-sum representation of LTI systems

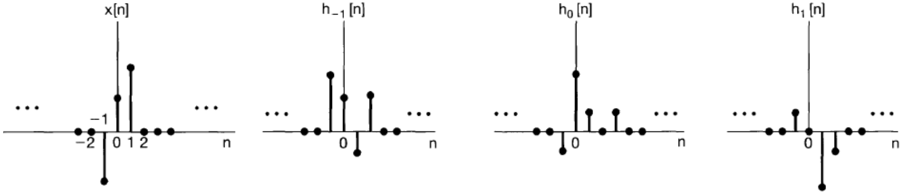
- Sifting property represents $x[n]$ as a superposition of scaled versions of shifted unit impulses $\delta[n-k]$
- $x[n]$ as an input to a linear system
  - Response is the superposition of the scaled responses of the system to each of these shifted impulses
- $x[n]$ as an input to a time-invariant system
  - Responses to the time-shifted unit impulses are time-shifted versions of one another
- $x[n]$ as an input to a LTI system
  - Time-shifted superimposition = convolution-sum

Impulse response

- Response of a linear (but possibly time-varying) system to an arbitrary input $x[n]$
  - $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- Let $h_k[n]$ denote the response of the linear system to the shifted unit impulse $\delta[n-k]$
  - Impulse response
- Superposition property: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$

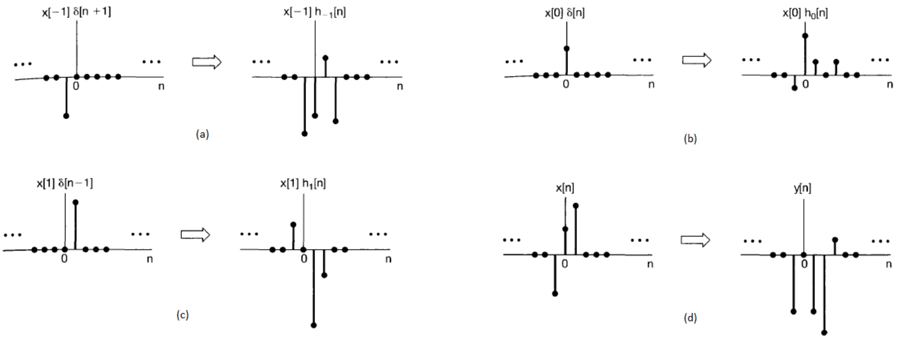
Example

- Input signal $x[n]$ and impulse responses $h_k[n]$



Example

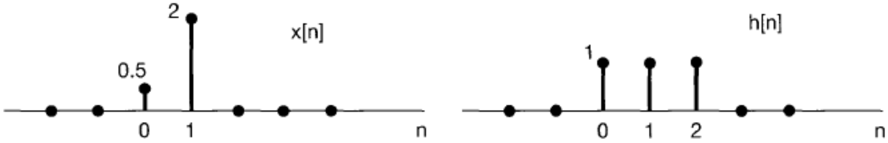
- $y[n]$ as sum of $x[k]h_k[n]$ for $-\infty < k < \infty$



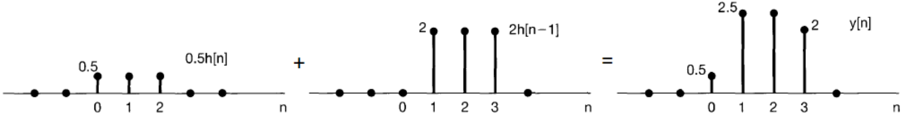
Response of a LTI system

- In general, $h_k[n]$ need not be related to each other for different values of k
- However, for time-invariant systems, responses to time-shifted unit impulses ($h_k[n]$) are time-shifted versions of each other
  - $h_k[n] = h_0[n-k]$
- $h[n] = h_0[n] =$ unit impulse response
  - $h[n]$ is the output of the LTI system when $\delta[n]$ is the input
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$
  - convolution sum or superposition sum
- LTI system is completely characterized by its impulse response

LTI system Example 1



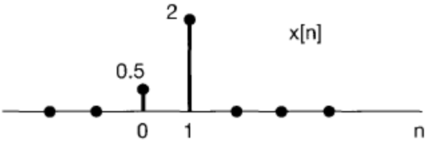
- $y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$

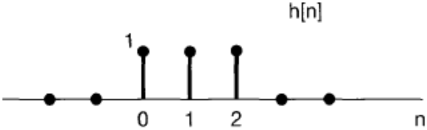


An alternative way of finding convolution sum

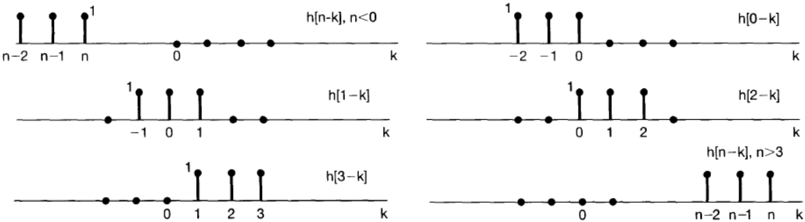
- To calculate the output value at some specific time n
- Consider $x[k]$ and $h[n-k]$ viewed as functions of k
- $g[k] = x[k]h[n-k]$ at each time k represent the contribution of $x[k]$ to the output at time n
- $\sum_{k=-\infty}^{\infty} g[k]$ yields the output value at the selected time n
- Repeat this procedure for each value of n to calculate $y[n]$ for all values of n

LTI system Example 2



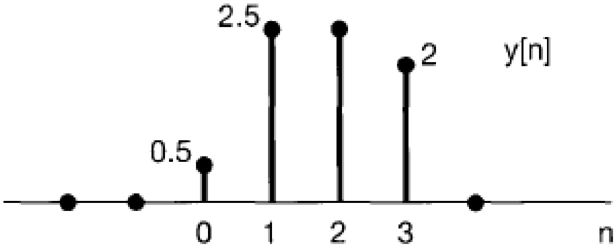


- Sequence $h[n-k]$ for fixed n viewed as a function of k
  - Time-reversed and shifted version of the impulse response $h[k]$
  - First time reverse $h[k]$ to obtain $h[-k]$
  - Shift $h[-k]$ to the right/left (by n) if n is positive/negative



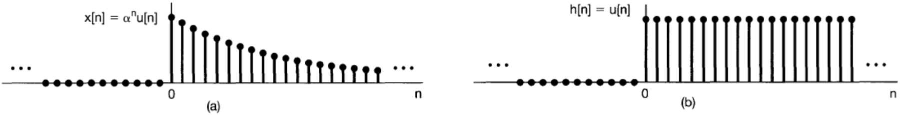
LTI system Example 2

- $y[n] = 0$ for $n < 0$
- $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k] = 0.5 \times 1 = 0.5$
- $y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k] = 2 \times 1 = 2$
- $y[n] = 0$ for $n > 3$

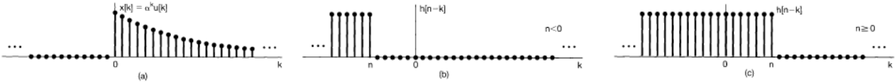


LTI system Example 3

- $x[n] = \alpha^n u[n]$, and $h[n] = u[n]$, with $0 < \alpha < 1$



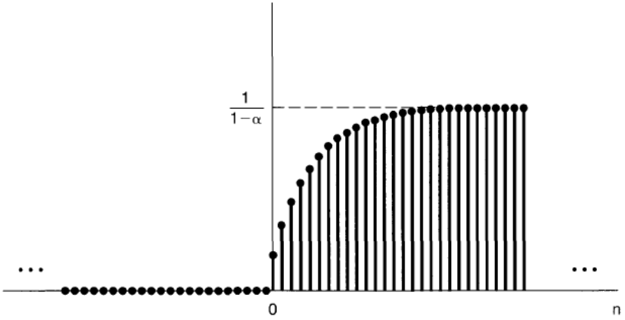
- $x[k]$ and $h[n-k]$ for $n < 0$ and $n \geq 0$



- $x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & Otherwise \end{cases}$

LTI system Example 3

- $y[n] = \begin{cases} 0, & n < 0 \\ \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}, & n \geq 0 \end{cases}$
- $y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n]$



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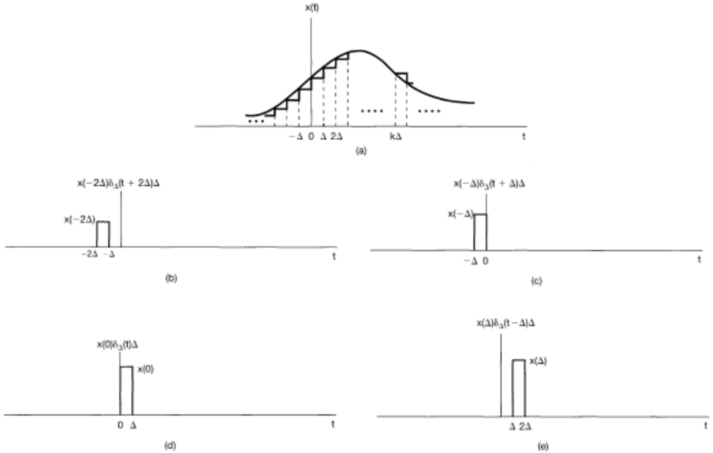
Continuous-time signals as an integral of shifted impulses

- Staircase approximation, $\hat{x}(t)$, to a continuous-time signal $x(t)$



Continuous-time signals as an integral of shifted impulses

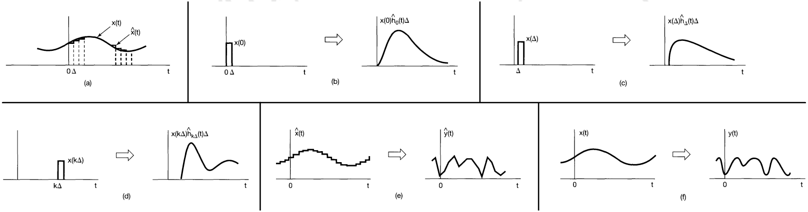
- Time shifted pulse $\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & Otherwise \end{cases}$
- $\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$
-

$$\begin{aligned} x(t) &= \lim_{\Delta \to 0} \hat{x}(t) \\ &= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \end{aligned}$$

- Sifting property of the continuous-time impulse

Unit impulse response

- Any arbitrary continuous-time signal can be expressed as the superposition of scaled and shifted impulses
  - $\hat{x}(t)$ is the sum of scaled and shifted versions of the basic pulse signal $\delta_\Delta(t)$
- Let $\hat{h}_{k\Delta}(t)$ is the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$
- Response $\hat{y}(t)$ of an LTI system to $\hat{x}(t)$ will be the superposition of scaled versions of $\hat{h}_{k\Delta}(t)$
  - $\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$

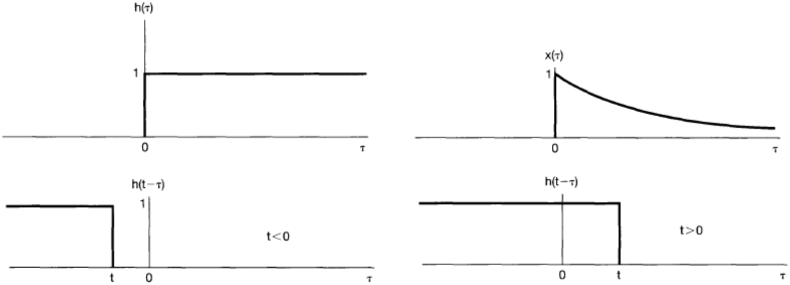


Convolution integral

- $$y(t) = \lim_{\Delta \to 0} \hat{y}(t)$$
  $$= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$
  $$= \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$
  - $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ
  - For LTI system, $h_\tau(t) = h(t-\tau)$, i.e., shifted version of unit impulse at $t = 0$
- $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$
  - Convolution integral
- A continuous-time LTI system is completely characterized by its impulse response

Example 1

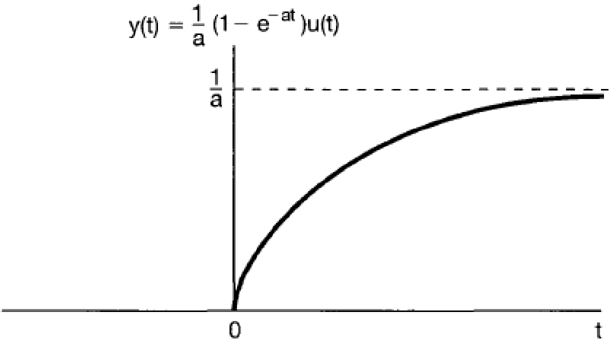
- $x(t) = e^{-at}u(t)$, $a > 0$, and $h(t) = u(t)$



- $x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & Otherwise \end{cases}$

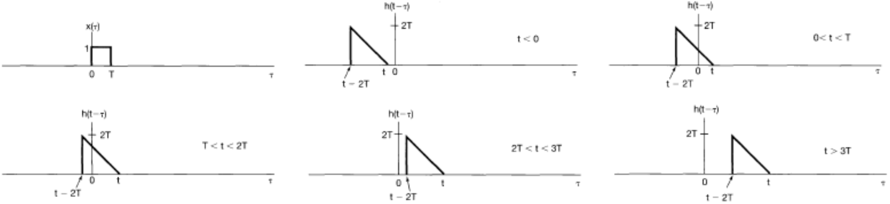
Example 1

- $y(t) = \begin{cases} \int_0^t e^{-a\tau} d\tau = \frac{1}{a}(1 - e^{-at}), & t \geq 0 \\ 0, & t < 0 \end{cases}$
- $y(t) = \frac{1}{a}(1 - e^{-at})u(t)$

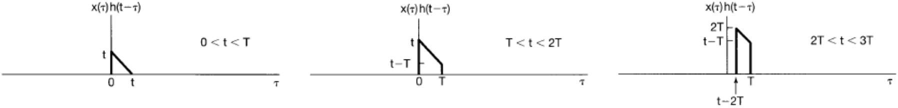


Example 2

- $x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & Otherwise \end{cases}$ $\quad h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & Otherwise \end{cases}$

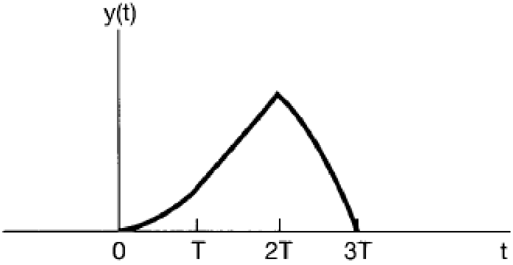


- $x(\tau)h)(t-\tau)$



Example 2

- $$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t (t-\tau)d\tau = \frac{t^2}{2}, & 0 < t < T \\ \int_0^T (t-\tau)d\tau = Tt - \frac{1}{2}T^2, & T < t < 2T \\ \int_{t-2T}^T (t-\tau)d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & t > 3T \end{cases}$$



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LTI system input-output relationship

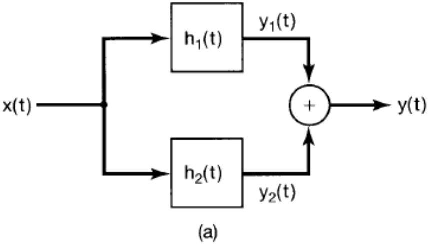
- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[k]$
- $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$
- Characteristics of an LTI system are completely determined by its impulse response and the convolution operation

Commutative property

- Output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$
  - $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
  - Substitute $r = n - k \rightarrow k = n - r$
  - $k \approx -\infty \rightarrow r \approx \infty$, $k \approx \infty \rightarrow r \approx -\infty$
  - $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n]$
  - Roles of $x[n]$ and $h[n]$ are interchangeable
- Output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI system with input $h(t)$ and unit impulse response $x(t)$
  - $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$

Distributive property

- Convolution distributes over addition
  - $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
  - $x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$



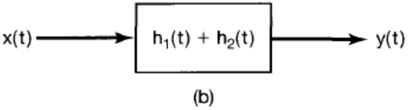
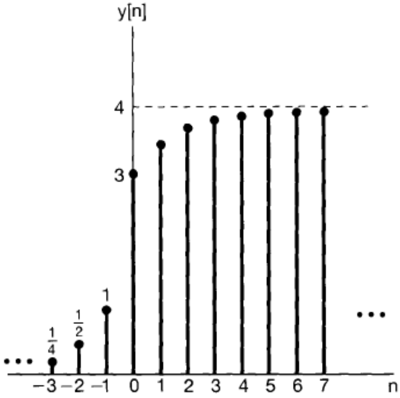


Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

- $y_1(t) = x(t) * h_1(t)$, $y_2(t) = x(t) * h_2(t)$
- $y(t) = x(t) * (h_1(t) + h_2(t))$

Distributive property application

- $x[n] = (\frac{1}{2})^n u[n] + 2^n u[-n]$, $h[n] = u[n]$
  - $x[n]$ is nonzero along the entire time axis $\rightarrow$ Computation of convolution is tedious
- Express $x[n]$ as the sum of two simpler sequences
  - $x[n] = x_1[n] + x_2[n]$
  - $x_1[n] = (\frac{1}{2})^n u[n]$
  - $x_2[n] = 2^n u[-n]$
- $y[n] = x[n] * h[n] = (x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n] = y_1[n] + y_2[n]$



Associative property

- $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- $x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$
- Series interconnection of the two systems having impulse responses $h_1[n]$ and $h_2[n]$ is equivalent to the single system having impulse response $h_1[n] * h_2[n]$

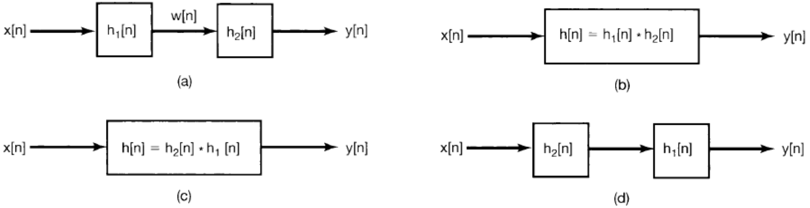


Figure 2.25 Associative property of convolution and the implication of this and the commutative property for the series interconnection of LTI systems.

- Note: This property holds only for LTI systems, not for nonlinear systems

With memory/memory less property

- A system is memory less if its output at any time depends only on the value of the input at that same time
  - True for a discrete-time LTI system if $h[n] = 0$ for $n \neq 0$
    - $h[n] = K\delta[n]$, where $K = h[0]$ is a constant
  - $y[n] = Kx[n]$
- If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$
  - Then the system has memory
- If $K = 1$, memory less LTI system is identity system
  - $h[n] = \delta[n] \rightarrow y[n] = x[n] * h[n] = x[n] * \delta[n] = x[n]$

Invertibility of a LTI system

- If an LTI system is invertible, then its inverse is also LTI

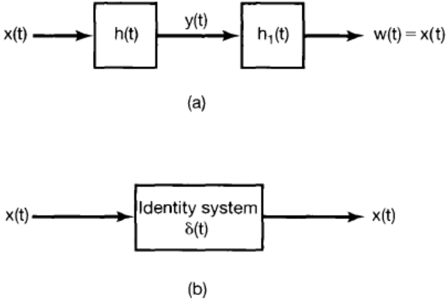


Figure 2.26 Concept of an inverse system for continuous-time LTI systems. The system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h(t)$ if $h(t) * h_1(t) = \delta(t)$.

- Given system impulse response: $h[n]$
- Inverse system impulse response: $h_1[n]$
- $h[n] * h_1[n] = \delta[n]$
- $h(t) * h_1(t) = \delta(t)$

Invertible system example 1

- Time-shift system: $y(t) = x(t - t_0)$
  - $t_0 > 0$: Delay system
  - $t = 0$: Identity system
  - $t_0 < 0$: Advance system
- $h(t) = \delta(t - t_0) \rightarrow x(t - t_0) = x(t) * \delta(t - t_0)$
  - $h_1(t) = \delta(t + t_0)$ (shifts the output back)
- $h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$

Invertible system example 2

- Summer or accumulator system: $h[n] = u[n]$
- $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k]$
  - $u[n-k]$ is 0 for $n-k < 0$ and 1 for $n-k > 0$
- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- Inverse system: $y[n] = x[n] - x[n-1]$ (first difference)
  - $h_1[n] = \delta[n] - \delta[n-1]$
-

$$
\begin{aligned}
h[n] * h_1[n] &= u[n] * (\delta[n] - \delta[n-1]) \\
&= (u[n] * \delta[n]) - (u[n] * \delta[n-1]) \\
&= u[n] - u[n-1] \\
&= \delta[n]
\end{aligned}
$$

Causality of a LTI system

- For a discrete-time LTI system to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$
- All of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero
  - $h[n] = 0$ for $n < 0$
  - Impulse response must be zero before the impulse occurs
    - The property of "initial rest"
    - If the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time
    - Equivalency of causality and "initial rest" applies only to linear systems
- $y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$
- A continuous-time LTI system is causal if $h(t) = 0$ for $t < 0$
- $y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$

Causal system example

- Both the accumulator system ($h[n] = u[n]$) and its inverse, first difference system ($h_1[n] = \delta[n] - \delta[n-1]$) are causal
- Time-shift system $h(t) = \delta(t - t_0)$ is causal if $t_0 \geq 0$ (Delay/identity system)
  - Noncausal for $t_0 < 0$ (Advance system)

Unit step response of an LTI System

- Unit step response ($s[n]$ or $s(t)$) is the output of an LTI system when $x[n] = u[n]$ or $x(t) = u(t)$
  - $s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$
    - $h[n] = s[n] - s[n-1]$
  - $s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$
    - $h(t) = \frac{d}{dt}s(t)$

Thanks.