Indian Institute of Technology Roorkee
MAN-001(Mathematics-1):
Autumn Semester: 2019-20
Assignment-9: Vector Calculus I (Gradient, Divergence, Curl)

Notation: $\mathbf{i} = \vec{i}$, $\mathbf{j} = \vec{j}$ and $\mathbf{k} = \vec{k}$ are the unit vectors along x, y and z axis respectively. Boldface letters represent vectors.

1. Show that
   (i) the necessary and sufficient condition for the vector function $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$ to be a constant is that $\dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{0}$.
   (ii) the necessary and sufficient condition for the vector function $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$ to have constant magnitude is that $\mathbf{u} \cdot \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = 0$.
   (iii) the necessary and sufficient condition for the vector function $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$ to have constant direction is $\mathbf{u} \times \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{0}$.

2. (i) If $\mathbf{r} = (\sinh t)\mathbf{a} + (\cosh t)\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that $\dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t} = \mathbf{r}$.
   (ii) If $\mathbf{r} = \mathbf{a}e^{nt} + \mathbf{b}e^{-nt}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that $\dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t} = n^2\mathbf{r}$.
   (iii) If $\mathbf{r} = (\cos nt)\mathbf{i} + (\sin nt)\mathbf{j}$, show that $\mathbf{r} \times \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = n\mathbf{k}$.

3. The position vector of a particle at time t is $\mathbf{r} = \cos(t-1)\mathbf{i} + \sinh(t-1)\mathbf{j} + \alpha t^3\mathbf{k}$. Find the condition imposed on α by requiring that at time $t = 1$, the acceleration is normal to the position vector.

4. Let $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Given $\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$, show that
   (i) $\mathbf{r}^2 = \mathbf{r} \cdot \mathbf{r} = a^2 + b^2t^2$,
   (ii) $(\mathbf{r}' \times \mathbf{r}'')^2 = a^2(a^2 + b^2)$,
   (iii) $[\mathbf{r}'\ \mathbf{r}''\ \mathbf{r}'''] = a^2b$,

   where $\mathbf{r}' = \dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, $\mathbf{r}'' = \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ and $\mathbf{r}''' = \dfrac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}$.

5. If $\mathbf{f}$ is a vector function of the scalar variable t, show that
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{f}\ \mathbf{f}'\ \mathbf{f}''] = [\mathbf{f}\ \mathbf{f}'\ \mathbf{f}'''].$$

6. (i) If $\varphi = 2xz^4 - x^2y$, find $\nabla\varphi$ and $|\nabla\varphi|$ at the point $(2, -2, 1)$.
   (ii) If $\nabla\varphi = (y+y^2+z^2)\mathbf{i} + (x+z+2xy)\mathbf{j} + (y+2zx)\mathbf{k}$, find φ such that $\varphi(1, 1, 1) = 3$.

7. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{r}| = r$, then show that
   (i) $\nabla r^n = nr^{n-2}\mathbf{r}$,
   (ii) $\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}$,
   (iii) $\nabla f(r) = \frac{f'(r)}{r}\mathbf{r}, \ \ \nabla f(r) \times \mathbf{r} = \mathbf{0}$,
   (iv) $\nabla[\mathbf{r}\ \mathbf{a}\ \mathbf{b}] = \mathbf{a} \times \mathbf{b}$,

   where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

8. (i) Find the directional derivative of $\varphi = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ in the direction of $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
   (ii) Find the directional derivative of $\varphi = x^2(y + z)$ at $(1, 1, 0)$ in the direction of the line joining the origin to the point $(2, -1, 2)$.
   (iii) Find the directional derivative of the function $\varphi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q is the point $(5, 0, 4)$.
   (iv) Find the direction along which the directional derivative of the function $\varphi = xy + 2yz + 3xz$ is greatest at the point $(1, 1, 1)$. Also find the greatest directional derivative.

9. (i) Find the unit vector normal to the level surface $xy + y^2 - z^2 = 5$ at $(1, 2, 1)$.
   (ii) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

10. If $\mathbf{a}$ is a constant vector and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $r = |\mathbf{r}|$, then show that
   (i) $\text{div}(\mathbf{r} \times \mathbf{a}) = 0$, i.e., $\mathbf{r} \times \mathbf{a}$ is solenoidal,
   (ii) $\text{curl}(\mathbf{r} \times \mathbf{a}) = -2\mathbf{a}$ or $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$,
   (iii) $\text{grad}(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$,
   (iv) $\nabla \cdot (r^2\mathbf{a}) = 2\mathbf{a} \cdot \mathbf{r}$.

11. (i) Determine a so that the vector $\mathbf{F} = (z+3y)\mathbf{i} + (x-2z)\mathbf{j} + (x+az)\mathbf{k}$ is soleniodal.
   (ii) Find the value of a if $\mathbf{F} = (axy - z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 - axz)\mathbf{k}$ is irrotational.
   (iii) A field $\mathbf{F}$ is of the form $\mathbf{F} = (6xy + z^3)\mathbf{i} + (3x^2 - z)\mathbf{j} + (3xz^2 - y)\mathbf{k}$. Show that $\mathbf{F}$ is a conservative filed (i.e., $\mathbf{F}$ is irrotational) and find its scalar potential.

12. If $\mathbf{F}$ is a differentiable vector function and φ is a differentiable scalar function, then prove that
   (i) $\text{div}(\varphi\mathbf{F}) = \text{grad}\ \varphi \cdot \mathbf{F} + \varphi\text{div}\ \mathbf{F}$ or $\nabla \cdot (\varphi\mathbf{F}) = \nabla\varphi \cdot \mathbf{F} + \varphi\nabla \cdot \mathbf{F}$,

(ii) $\operatorname{curl}(\varphi\mathbf{F}) = \varphi \operatorname{curl} \mathbf{F} + \operatorname{grad} \varphi \times \mathbf{F}$ or $\nabla \times (\varphi\mathbf{F}) = \varphi(\nabla \times \mathbf{F}) + (\nabla\varphi) \times \mathbf{F}$.

13. For $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show that

    (i) $\nabla \cdot \left(\dfrac{\mathbf{r}}{r^3}\right) = 0$,

    (ii) $\nabla \cdot (r^3\mathbf{r}) = 6r^3$,

    (iii) $\nabla \cdot \left\{ r\nabla\left(\dfrac{1}{r^3}\right)\right\} = 3r^{-4}$,

    (iv) $\nabla \cdot \{r^n(\mathbf{a} \times \mathbf{r})\} = 0$,

    where $r = |\mathbf{r}|$ and $\mathbf{a}$ is a constant vector.

14. If $\mathbf{r}$ is the position vector of a variable point (x, y, z) and $|\mathbf{r}| = r$, then show that

$$\nabla \cdot \{f(r)\mathbf{r}\} = rf'(r) + 3f(r).$$

    Also, if $\nabla \cdot \{f(r)\mathbf{r}\} = 0$, then show that $f(r) = \dfrac{C}{r^3}$, where C is a constant.

15. (i) Show that $r^n\mathbf{r}$ is an irrotational vector for any value of n, but is solenoidal only if $n = -3$.

    (ii) Prove that the vector $f(r)\mathbf{r}$ is irrotational.

16. If $\mathbf{a}$ is a constant vector, then prove that

$$\operatorname{curl}\left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\frac{\mathbf{a}}{r^3} + \frac{3\mathbf{r}}{r^5}(\mathbf{a} \cdot \mathbf{r}).$$

17. (i) If $\mathbf{F}$ is a vector function, prove that $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2\mathbf{F}$, where $\nabla^2 = \nabla \cdot \nabla$.

    (ii) Use the above result to establish that curl curl curl curl $\mathbf{F} = \mathbf{0}$ if $\mathbf{F}$ is solenoidal.

18. Prove that

    (i) $\nabla^2\left(\dfrac{1}{r}\right) = 0$,

    (ii) $\nabla^2(r^n\mathbf{r}) = n(n+3)r^{n-2}\mathbf{r}$,

    (iii) $\nabla^2 f(r) = f''(r) + \dfrac{2}{r}f'(r)$,

    where $r = |\mathbf{r}|$.

19. If $\nabla^2 f(r) = 0$, then show that $f(r) = a + \dfrac{b}{r}$, where $r^2 = x^2 + y^2 + z^2$ and a and b are constants.

20. Show that

(i) $\varphi = x^2 - y^2$ satisfies the Laplace equation $\nabla^2\varphi = 0$.

(ii) $\nabla^2 \left\{\nabla \cdot \left(\frac{\mathbf{r}}{r^2}\right)\right\} = \frac{2}{r^4}$,

(iii) if $\varphi = \frac{x}{r^3}$, then $\nabla^2\varphi = 0$.

Answers.

3. $\alpha = \pm\frac{1}{\sqrt{6}}$.

6. (i) $\nabla\varphi\big|_{(2,-2,1)} = 10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$, $|\nabla\varphi| = 2\sqrt{93}$. (ii) $\varphi = xy + xy^2 + xz^2 + yz - 1$.

8. (i) -4. (ii) $\frac{5}{3}$. (iii) $\frac{4}{3}\sqrt{21}$. (iv) $4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $5\sqrt{2}$.

9. (i) $\frac{2\mathbf{i}+5\mathbf{j}-2\mathbf{k}}{\sqrt{33}}$. (ii) $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

11. (i) $a = 0$. (ii) $a = 2$. (iii) $\varphi = 3x^2y + xz^3 - yz + C$.