Gomory's cutting plane method for mixed ILPP.

Consider the following problem:

$$\left.\begin{aligned} \text{Max } z &= 7x_1 + 6x_2 \\ \text{s/t} \quad & -x_1 + 3x_2 \le 6, \\ & 7x_1 + x_2 \le 35, \\ & x_1, x_2 \ge 0 \ \& \ x_1 \text{ integer} \end{aligned}\right] \quad (A)$$

[Here, the requirement is only x_1 integer, the other variables may / may not integer, such problem are called Mixed integer LPP].

The cut (additional constraint) of such problem is ofcourse different from the cut of all ILPP. How to find cut for these problems? Let us discuss it by considering the optimal table of (A).

The optimal table of (A) is:

	C_B	x_B	x_1	x_2	S_1	S_2	b
$c_j - z_j \rightarrow$			0	0	$-28/11$	$-15/11$	
		x_2	0	1	$7/22$	$\frac{1}{22}$	$7/2$
		x_1	1	0	$-\frac{1}{22}$	$3/22$	$9/2$

x_1 is NOT an integer. The x_1-row is:

$$x_1 + \frac{3}{22} S_2 - \frac{1}{22} S_1 = \frac{9}{2}. \quad \text{—— (1)}$$

Now, in general, let $\alpha_1^+, \alpha_2^+ \ldots \alpha_p^+$ denote the terms of non-basic variables with ≥ 0 coefficient.

Let $\alpha_{p+1}^-, \alpha_{p+2}^-, \ldots \alpha_s^-$ denote the terms of non-basic variables with < 0 coefficient.

Hence, the x_1-row can be written as:

$$x_1 + \sum_{i=1}^{p} \alpha_i^+ y_i + \sum_{i=p+1}^{s} \alpha_i^- y_i = b \quad \text{—— (2)}$$

Here, y_i's $(i=1,2,\ldots,s)$ are non-basic variable.

[for ex. in example (A), $\alpha_1^+ = 3/22$, $\alpha_1^- = -1/22$
$y_1 = S_2$, $y_2 = s_1$,
$b = 9/2$].

Let $b = \bar{b} + \beta$, $[b] = \bar{b}$, $0 < \beta < 1$.

(2) $\Rightarrow$

$$\sum_{i=1}^{p} \alpha_i^+ y_i + \sum_{i=p+1}^{s} \alpha_i^- y_i = \beta + (\bar{b} - x_4). \quad \text{—} (3).$$

The RHS of (3) is either ≥ 0 or < 0. Let us first take

$$\beta + (\bar{b} - x_4) \geq 0.$$

Since $\beta \in (0,1)$, therefore $\bar{b} - x_4 = 0$ or 1 or $2 \ldots$
or $\bar{b} - x_4 \geq 0$.

(3) $\Rightarrow$

$$\sum_{i=1}^{p} \alpha_i^+ y_i + 0 \geq \underbrace{\sum_{i=1}^{p} \alpha_i^+ y_i + \sum_{i=p+1}^{s} \alpha_i^- y_i} = \beta + \bar{b} - x_4$$
$$\geq \beta + 0$$

$(\because \alpha_i^- \leq 0,\ y_i \geq 0,\ i = p+1, \ldots, s).$

$$\Rightarrow \boxed{\sum_{i=1}^{p} \alpha_i^+ y_i \geq \beta} \quad \text{—} (4)$$

If $\beta + \bar{b} - x_4 < 0 \Rightarrow \bar{b} - x_4 = -1$ or -2 or $\ldots$

$$\Rightarrow 0 + \sum_{i=p+1}^{s} \alpha_i^- y_i \leq \underbrace{\sum_{i=1}^{p} \alpha_i^+ y_i + \sum_{i=p+1}^{s} \alpha_i^- y_i} = \beta + \bar{b} - x_4$$
$$\leq \beta - 1$$

$$\Rightarrow \sum_{i=p+1}^{s} \alpha_i^- y_i \leq \beta - 1 \Rightarrow \boxed{\frac{\beta}{\beta - 1} \sum_{i=p+1}^{s} \alpha_i^- y_i \geq \beta} \quad \text{—} (5)$$

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So, either (4) or (5) must happen, hence

$$\boxed{\sum_{i=1}^{p} \alpha_i^+ y_i + \frac{\beta}{\beta-1} \sum_{i=p+1}^{s} \alpha_i^- y_i \geq \beta}$$

This is the required additional constraint.

In the problem (A): the cut will be:

$$\frac{3}{22} S_2 + \frac{1}{\left(\frac{1}{2}-1\right)}\left(-\frac{1}{22} S_1\right) \geq \frac{1}{2}$$

$$\Rightarrow \quad -3S_2 - S_1 + S_3 = -11 \qquad \text{—— (6)}$$

<u>optimal table</u>

		x_1	x_2	S_1	S_2	S_3	b
		0	0	$-28/11$	$-15/11$	0	
	x_2	0	1	$7/22$	$1/22$	0	$7/2$
	x_1	1	0	$-1/22$	$3/22$	0	$9/2$
	S_3	0	0	-1	$\boxed{-3}$	1	-11

Add (6) here.

			b
	x_2		$10/3$
	x_1		4
	S_2		$\frac{11}{3}$

optimal solution: $x_1 = 4$, $x_2 = 10/3$.