Indian Institute of Technology Roorkee
MAN-001(Mathematics-1)
Autumn Semester: 2019-20
Assignment-7: (Gamma and Beta Functions)

1. Evaluate: (i) $\Gamma(7)$, (ii) $\Gamma(\frac{7}{2})$.

2. Show that (i) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$; (ii) $\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2\mathrm{m}+1)}{2^{2m}\,\Gamma(\mathrm{m}+1)}$;

   (iii) $2^{2m-1}\Gamma(\mathrm{m})\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2\mathrm{m})$,m is an integer in both (ii) and (iii).

3. For $s > 0, p > 0$, show that

   (i) $\int_0^\infty x^{p-1}e^{-sx}\,dx = \Gamma(p)/s^p$ (ii) $\int_0^\infty e^{-s^2x^2}\,dx = \sqrt{\pi}/2s.$

4. Show that $\Gamma(p) = \int_0^1 (\log\left(1/y\right))^{p-1}dy,\ \ p > 0$; using this

   evaluate $\int_0^1 (\log\left(1/y\right))^{-1/2}dy.$

5. Show that for integer m $> -1, n > 0$,

$$\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

6. Show that for c > 1,

$$\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}.$$

7. Show that for r > -1,

$$\int_0^\infty x^r e^{-s^2x^2}dx = \frac{1}{2\,s^{r+1}}\Gamma(\frac{r+1}{2}).$$

8. Using reflection property show that $\int_0^{\pi/2}\tan^n\theta\,d\theta = \frac{\pi}{2}\sec\frac{n\pi}{2}.$

9. Prove the following:

   (i) $\mathrm{B}(x,y) = 2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta$, (ii) $\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$,

   (iii) $\mathrm{B}(x,y) = \mathrm{B}(x+1,y) + \mathrm{B}(x,y+1)$,

(iv) $\frac{1}{x+y}\mathrm{B}(x,y) = \frac{1}{x}\mathrm{B}(x+1,y) = \frac{1}{y}\mathrm{B}(x,y+1)$,

(v) $\int_0^1 t^{m-1}(1-t^2)^{n-1}dt = \frac{1}{2}\mathrm{B}\left(\frac{m}{2},n\right)$, (vi) $\int_0^1 (1-t^6)^{-1/6}dt = \frac{\pi}{3}$.

10. Show that, for any positive integer m

$$\mathrm{B}(m,m) = \frac{\sqrt{\pi}\Gamma(\mathrm{m})}{2^{2m-1}\Gamma(\mathrm{m}+1/2)}.$$

11. Evaluate following integrals in terms of Gamma or Beta functions;

(i) $\int_0^\infty e^{-x^4}dx$, (ii) $\int_0^\infty x^{-7/4}e^{-\sqrt{x}}dx$, (iii) $\int_0^1 x^5(1-x^3)^{10}dx$,

(iv) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx$, (v) $\int_0^a x^9\sqrt[3]{(a^6-x^6)}dx$, (vi) $\int_0^a x^3(a^5-x^5)^3dx$.

ANSWERS:

1. (i) 720, (ii) $\frac{15}{8}\sqrt{\pi}$ 4. $\sqrt{\pi}$

11. (i) $\Gamma(\frac{5}{4})$,

(ii) $\frac{8}{3}\sqrt{\pi}$,

(iii) $\frac{1}{3}\mathrm{B}(2,11) = \frac{1}{396}$,

(iv) $\frac{1}{4.2^{1/4}}\mathrm{B}\left(\frac{1}{4},\frac{7}{4}\right)$,

(v) $\frac{a^6}{6}\mathrm{B}\left(\frac{5}{3},\frac{4}{3}\right)$

(vi) $\frac{a^{19}}{65}\mathrm{B}\left(\frac{4}{5},4\right)$.