Assignment - 7

1. (i) $\Gamma 7 \quad (\because \Gamma n = \lfloor n-1)$

$= \lfloor 6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

(ii) $\Gamma \frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma \frac{1}{2} = \frac{15}{8} \sqrt{\pi}$

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2. (i) $\Gamma \frac{1}{3} \Gamma \frac{2}{3}$ we have $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$

so $\Gamma \frac{1}{3} \Gamma \frac{2}{3} = \Gamma \frac{1}{3} \, \Gamma 1-\frac{1}{3} = \frac{\pi}{\sin(\frac{1}{3}\pi)} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$

(iii) T.P. $2^{2m-1} \Gamma m \, \Gamma m+\frac{1}{2} = \sqrt{\pi} \, \Gamma 2m$

We have

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma \frac{p+1}{2} \Gamma \frac{q+1}{2}}{2 \Gamma \frac{p+q+2}{2}}$$

Let $\frac{p+1}{2} = m \qquad \frac{q+1}{2} = n$

$p = 2m-1 \qquad q = 2n-1$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2 \Gamma m+n} \quad \text{——— (A)}$$

putting $2n-1 = 0 \Rightarrow n = \frac{1}{2}$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma \frac{1}{2}}{2 \Gamma m+\frac{1}{2}}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m \sqrt{\pi}}{2 \Gamma m+\frac{1}{2}} \quad \text{——— (1)}$$

Again putting $m = n$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma m}{2 \Gamma m+m}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$2\theta = t \qquad 2d\theta = dt \qquad \theta \to 0 \to \pi/2$

$t \to 0 \to \pi$

$$\frac{1}{2^{2m-1}}\int_0^{\pi} (\sin t)^{2m-1} \frac{dt}{2} = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\frac{1}{2^{2m}}\int_0^{\pi} \sin^{2m-1} t \, dt = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

change the limit

$$\frac{2}{2^{2m}}\int_0^{\pi/2} \sin^{2m-1} t \, dt = \frac{(\Gamma m)^2}{2\Gamma 2m}$$

$$\int_0^{\pi/2} \sin^{2m-1} t \, dt = \frac{(\Gamma m)^2}{2\Gamma 2m} \cdot 2^{2m-1} \quad \text{——— (II)}$$

Comparing (I) & (II)

$$\frac{(\Gamma m)^2}{2\Gamma 2m} \cdot 2^{2m-1} = \frac{\Gamma m \sqrt{\pi}}{2\Gamma (m+\frac{1}{2})}$$

$$\boxed{\Gamma m \, \Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m}$$

(ii) from(iii) $\Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{\Gamma 2m}{\Gamma m}$$

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m}} \cdot \frac{2\,\Gamma 2m}{\Gamma m}$$

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m}} \cdot \frac{2m\,\Gamma 2m}{m\,\Gamma m}$$

$$\boxed{\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m+1)}{2^{2m}\,\Gamma(m+1)}}$$

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(3) For $s>0, p>0$ show that

(i) $\int_0^\infty x^{p-1} e^{-sx}\, dx = \frac{\Gamma p}{s^p}$

we have $\int_0^\infty y^{p-1} e^{-y}\, dy = \Gamma p$

putting $y = sx \Rightarrow dy = s\,dx$

$$\int_0^\infty (sx)^{p-1} e^{-sx}\, s\,dx = \Gamma p$$

$$\int_0^\infty s^{p-1} x^{p-1} e^{-sx}\, s\,dx = \Gamma p$$

$$\int_0^\infty s^p x^{p-1} e^{-sx}\, dx = \Gamma p$$

$$\boxed{\int_0^\infty x^{p-1} e^{-sx}\, dx = \frac{\Gamma p}{s^p}}$$

(ii) $\int_0^\infty e^{-s^2x^2}\, dx = \sqrt{\pi}/2s$

we have $\int_0^\infty y^{p-1} e^{-y}\, dy = \Gamma p$

take $p = 1/2$

$$\int_0^\infty y^{\frac{1}{2}-1} e^{-y}\,dy = \Gamma\tfrac{1}{2}$$

$$\int_0^\infty y^{-1/2} e^{-y}\,dy = \sqrt{\pi}$$

put $y = s^2x^2$

$dy = 2s^2x\,dx$

$$\int_0^\infty (s^2x^2)^{-1/2} e^{-s^2x^2} \cdot (2s^2x\,dx) = \sqrt{\pi}$$

$$\int_0^\infty s^{-1}x^{-1} e^{-s^2x^2}\, 2 \cdot s^2 x\,dx = \sqrt{\pi}$$

$$\boxed{\int_0^\infty e^{-s^2x^2}\,dx = \frac{\sqrt{\pi}}{2s}}$$

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(4) T.P. $\Gamma p = \int_0^1 \left(\log\frac{1}{y}\right)^{p-1} dy \; ; \; p > 0$

We have $\int_0^\infty x^{p-1} e^{-x}\,dx = \Gamma p$

$e^{-x} = y \implies$ ~~$x = \log y$~~ $x = \log\left(\frac{1}{y}\right)$

$-e^{-x}dx = dy \implies e^{-x}dx = -dy$ <u>limit</u>: $x \to 0 \;\; t \to 1$, $x \to \infty \;\; t \to 0$

$$\Gamma p = \int_1^0 \left[\log\frac{1}{y}\right]^{p-1} (-dy) = -\int_1^0 \left(\log\frac{1}{y}\right)^{p-1} dy$$

So $\boxed{\int_0^1 \left(\log\frac{1}{y}\right)^{p-1} dy = \Gamma p}$

putting $p = 1/2$

$$\int_0^1 \left(\log\frac{1}{y}\right)^{-1/2} dy = \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

(5). $m > -1,\ n > 0$

$$\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Let $\log x = -t$

$x = e^{-t} \longrightarrow x^m = e^{-mt}$

$dx = -e^{-t} dt$ limit: $x \to 0$ to 1, $t \to \infty$ to 0

$$\int_0^1 x^m (\log x)^n dx = \int_\infty^0 e^{-mt} (-t)^n (-e^{-t} dt)$$

$$= \int_0^\infty (-1)^n e^{-(m+1)t} t^n dt$$

$(m+1)t = y \Rightarrow (m+1)\, dt = dy$

$$= \int_0^\infty (-1)^n e^{-y} \left(\frac{y}{m+1}\right)^n \frac{1}{(m+1)} dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n dy$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} n!$$

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(6) For $c > 0$ $\displaystyle\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$

Put $c^x = e^t \Rightarrow x \log c = t \Rightarrow x = \dfrac{t}{\log c} \Rightarrow dx = \dfrac{dt}{\log c}$

So $$\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty \left(\frac{t}{\log c}\right)^c e^{-t} \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-t} t^c dt$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^{(c+1)-1} e^{-t} dt = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$$

⑦ For $r > -1$ $\int_0^\infty x^r e^{-s^2x^2} dx = \frac{1}{2s^{r+1}} \Gamma\frac{r+1}{2}$

we have

$$\int_0^\infty y^{p-1} e^{-y} dy = \Gamma p$$

put $y = s^2x^2 \Rightarrow dy = 2s^2x\, dx$

$$\int_0^\infty (s^2x^2)^{p-1} e^{-s^2x^2} \cdot 2s^2x\, dx = \Gamma p$$

$$\int_0^\infty s^{2(p-1)} x^{2(p-1)} e^{-s^2x^2} 2 \cdot s^2 x\, dx = \Gamma p$$

$$\int_0^\infty 2 \cdot s^{2p} \cdot x^{2p-1} e^{-s^2x^2} dx = \Gamma p$$

Let $2p-1 = r \Rightarrow p = \frac{r+1}{2}$

$$\int_0^\infty 2 \cdot s^{2\left(\frac{r+1}{2}\right)} x^r e^{-s^2x^2} dx = \Gamma\frac{r+1}{2}$$

$$\int_0^\infty x^r e^{-s^2x^2} dx = \frac{1}{2 \cdot s^{r+1}} \Gamma\frac{r+1}{2}$$

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⑧ $\int_0^{\pi/2} \tan^n\theta\, d\theta = \frac{\pi}{2} \sec\left(\frac{n\pi}{2}\right)$

$$\int_0^{\pi/2} \tan^n\theta\, d\theta = \int_0^{\pi/2} \sin^n\theta \cos^{-n}\theta\, d\theta = \frac{\Gamma\frac{n+1}{2}\Gamma\frac{-n+1}{2}}{2\Gamma\frac{n-n+2}{2}} = \frac{\Gamma\frac{n+1}{2}\Gamma\frac{-n+1}{2}}{2}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma 1-\left(\frac{n+1}{2}\right)}{2} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma 1-\left(\frac{n+1}{2}\right)$$

$$= \frac{1}{2}\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} = \frac{1}{2}\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{\pi}{2\cos\frac{n\pi}{2}}$$

$$= \frac{\pi}{2}\sec\left(\frac{n\pi}{2}\right).$$

(9) (i) $B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$

We have $B(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}\, dz$

Let $z = \sin^2\theta$

$dz = 2\sin\theta \cos\theta \, d\theta$

<u>limits</u>: $z \to 0$ to 1

$\theta \to 0$ to $\pi/2$

So

$$B(x,y) = \int_0^{\pi/2} (\sin^2\theta)^{x-1} (1-\cancel{\cos^2\theta}\sin^2\theta)^{y-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta)^{2x-2} (\cos\theta)^{2y-2} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= 2 \cdot \int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$$

(ii) $B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt$

We have

$B(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}\, dz$

putting $z = \frac{1}{1+t} \Rightarrow dz = \frac{-1}{(1+t)^2}\, dt$ <u>limit</u> $z \to 0$ to 1, $t \to \infty$ to 0

then $1 - z = \frac{t}{1+t}$

$$B(x,y) = \int_\infty^0 \left(\frac{1}{1+t}\right)^{x-1} \left(\frac{t}{1+t}\right)^{y-1} \left(\frac{-1}{(1+t)^2}\right) dt$$

$$= \int_0^\infty \frac{t^{y-1}}{(1+t)^{x-1+y-1+2}}\, dt$$

x, y can be interchanged ($\because B(x,y) = B(y,x)$)

$$= \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, dt$$

(iii) $B(x,y) = B(x+1,y) + B(x,y+1)$

RHS: $\cancel{B(x+1,y)+B(x,y+1) = \int_0^1 x^{m}(1-x)^{n-1}dx}$

$$B(x+1,y) + B(x,y+1) = \int_0^1 t^{(x+1)-1}(1-t)^{y-1}dt + \int_0^1 t^{x-1}(1-t)^{(y+1)-1}dt$$

$$= \int_0^1 t^{x}(1-t)^{y-1}dt + \int_0^1 t^{x-1}(1-t)^{y}dt$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}\left[t + (1-t)\right]dt$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

$$= B(x,y)$$

(iv) $\frac{1}{(x+y)}B(x,y) = \frac{1}{x}B(x+1,y) = \frac{1}{y}B(x,y+1)$

$$\frac{B(x,y)}{(x+y)} = \frac{1}{(x+y)}\cdot\frac{\Gamma x\,\Gamma y}{\Gamma x+y} = \frac{\Gamma x\,\Gamma y}{(x+y)\Gamma x+y} = \frac{\Gamma x\,\Gamma y}{\Gamma x+y+1} = \frac{x\,\Gamma x\,\Gamma y}{x\,\Gamma x+1+y}$$

$$= \frac{\Gamma x+1\,\Gamma y}{x\,\Gamma (x+1)+y} = \frac{1}{x}B(x+1,y)$$

Similarily

$$\frac{B(x,y)}{(x+y)} = \frac{\Gamma x\,\Gamma y}{(x+y)\,\Gamma x+y} = \frac{\Gamma x\,\Gamma y}{\Gamma x+y+1} = \frac{\Gamma x\,y\,\Gamma y}{y\,\Gamma x+y+1}$$

$$= \frac{\Gamma x\,\Gamma y+1}{y\,\Gamma x+(y+1)} = \frac{1}{y}B(x,y+1)$$

(V) $\int_0^1 t^{m-1}(1-t^2)^{n-1}\,dt = \frac{1}{2}B\left(\frac{m}{2}, n\right)$

LHS $\qquad$ Let $t^2 = x$, $t = \sqrt{x} \Rightarrow dt = \frac{1}{2\sqrt{x}}dx$

$$\int_0^1 t^{m-1}(1-t^2)^{n-1}\,dt = \int_0^1 (\sqrt{x})^{m-1}(1-x)^{n-1}\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$= \int_0^1 x^{\frac{m}{2}-\frac{1}{2}}(1-x)^{n-1}\frac{1}{2x^{1/2}}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m}{2}-\frac{1}{2}-\frac{1}{2}}(1-x)^{n-1}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{m}{2}-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{2}B\left(\frac{m}{2}, n\right)$$

(VI) $\int_0^1 (1-t^6)^{-1/6}\,dt = \frac{\pi}{3}$

We have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$$

put $x = t^6$ and $n-1 = -1/6 \Rightarrow n = 5/6$

$dx = 6t^5\,dt$

$$\int_0^1 (t^6)^{m-1}(1-t^6)^{-1/6}\cdot 6t^5\,dt = B\left(m, \frac{5}{6}\right)$$

$$6\int_0^1 t^{6m-6}(1-t^6)^{-1/6}\,t^5\,dt = B\left(m, \frac{5}{6}\right)$$

$$6\int_0^1 t^{6m-1}(1-t^6)^{-1/6}\,dt = B\left(m, \frac{5}{6}\right)$$

put $6m-1 = 0$

$m = 1/6$

$$\int_0^1 t^0 (1-t^6)^{-1/6}\, dt = \frac{1}{6} B\left(\frac{1}{6}, \frac{5}{6}\right)$$

$$\int_0^1 (1-t^6)^{-1/6}\, dt = \frac{1}{6} \frac{\Gamma\frac{1}{6}\,\Gamma\frac{5}{6}}{\Gamma\left(\frac{5}{6}+\frac{1}{6}\right)}$$

$$= \frac{1}{6} \cdot \frac{\Gamma\frac{1}{6}\,\Gamma\left(1-\frac{1}{6}\right)}{\Gamma 1}$$

$$= \frac{1}{6} \Gamma\tfrac{1}{6}\,\Gamma\left(1-\tfrac{1}{6}\right) \qquad \left(\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

$$= \frac{1}{6} \frac{\pi}{\sin\left(\frac{\pi}{6}\right)}$$

$$= \frac{1}{6} \frac{\pi}{\left(\frac{1}{2}\right)} = \frac{\pi}{3}$$

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(10). $$B(m,m) = \frac{\sqrt{\pi}\,\Gamma m}{2^{2m-1}\,\Gamma\left(m+\frac{1}{2}\right)}$$

We have

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

put $\frac{p+1}{2} = m \qquad \frac{q+1}{2} = n$

$p = 2m-1 \qquad q = 2n-1$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m\,\Gamma n}{2\,\Gamma(m+n)} \quad \text{—— (A)}$$

put $n = \frac{1}{2}$ $\qquad \cancel{= \frac{1}{2} B(m,n)}$

So

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m\,\Gamma\frac{1}{2}}{2\,\Gamma\left(m+\frac{1}{2}\right)}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma m\,\sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)} \quad \text{—— (1)}$$

Again putting $n=m$ in (A)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma m}{2\,\Gamma m+m}$$

$$\int_0^{\pi/2} \frac{(2\sin\theta\cos\theta)^{2m-1}}{2^{2m-1}} \, d\theta = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

$$\int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta = 2^{2m-2}\frac{(\Gamma m)^2}{\Gamma 2m}$$

$2\theta = t$ limits

$d\theta = dt/2$ $\theta \to 0$ to $\pi/2$

$t \to 0$ to π

$$\int_0^{\pi} \sin^{2m-1} t \, \frac{dt}{2} = 2^{2m-2}\frac{(\Gamma m)^2}{\Gamma 2m}$$

$$2\int_0^{\pi/2} \sin^{2m-1} t \, \frac{dt}{2} = 2^{2m-2}\frac{(\Gamma m)^2}{\Gamma 2m}$$

$$\int_0^{\pi/2} \sin^{2m-1} t \, dt = 2^{2m-2}\frac{(\Gamma m)^2}{\Gamma 2m} \quad \text{—— (II)}$$

from (I) & (II) we get

$$2^{2m-2}\frac{(\Gamma m)^2}{\Gamma 2m} = \frac{\Gamma m \sqrt{\pi}}{2\,\Gamma m+\frac{1}{2}}$$

$$\frac{(\Gamma m)^2}{\Gamma 2m} = \frac{\Gamma m \sqrt{\pi}}{2^{2m-1}\Gamma m+\frac{1}{2}}$$

$$\frac{\Gamma m \, \Gamma m}{\Gamma m+m} = \frac{\Gamma m \sqrt{\pi}}{2^{2m-1}\Gamma m+\frac{1}{2}}$$

$$\boxed{B(m,m) = \frac{\sqrt{\pi}\,\Gamma m}{2^{2m-1}\Gamma m+\frac{1}{2}}}$$

(11) (i) $\int_0^\infty e^{-x^4}\,dx$ Let $x^4 = t$

$x = t^{1/4}$ $\quad dx = \frac{1}{4} t^{-3/4}\,dt$

limits are same

$x \to 0$ to ∞ $\quad t \to 0$ to ∞

$$= \int_0^\infty e^{-t} \frac{1}{4} t^{-3/4}\,dt$$

$$= \frac{1}{4}\int_0^\infty t^{\frac{1}{4}-1} e^{-t}\,dt$$

$$= \frac{1}{4}\Gamma\left(\tfrac{1}{4}\right) = \Gamma\left(1+\tfrac{1}{4}\right) = \Gamma\left(\tfrac{5}{4}\right)$$

(ii) $\int_0^\infty x^{-7/4} e^{-\sqrt{x}}\,dx$ let $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

limits

$x \to 0$ to ∞

$t \to 0$ to ∞

$$= \int_0^\infty (t^2)^{-7/4} e^{-t}\, 2t\,dt$$

$$= 2\int_0^\infty t^{-7/2+1} e^{-t}\,dt$$

$$= 2\int_0^\infty t^{-5/2} e^{-t}\,dt$$

$$= 2\int_0^\infty t^{-3/2-1} e^{-t}\,dt$$

$$\cancel{= 2\cdot\Gamma\left(\tfrac{7}{2}\right) = 2\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{15\sqrt{\pi}}{4}}$$

$$= 2\cdot\Gamma\left(\tfrac{-3}{2}\right)$$

Now we have

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$$

put $n = -3/2$

$$\Gamma\left(\tfrac{-3}{2}\right)\Gamma\left(1+\tfrac{3}{2}\right) = \frac{\pi}{\sin\left(-\frac{3\pi}{2}\right)}$$

$$\Gamma\left(\tfrac{-3}{2}\right)\Gamma\left(\tfrac{5}{2}\right) = \frac{\pi}{-\sin\frac{3\pi}{2}}$$

$$\Gamma\left(\tfrac{-3}{2}\right)\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi} = \frac{\pi}{-(-1)} \Rightarrow \Gamma\left(\tfrac{-3}{2}\right) = \frac{4\sqrt{\pi}}{3}$$

$$= 2\cdot\frac{4}{3}\sqrt{\pi}$$

$$= \frac{8\sqrt{\pi}}{3}$$

(III) $\int_0^1 x^5 (1-x^3)^{10} \, dx$

let $x^3 = t \Rightarrow x = t^{1/3}$

So $dx = \frac{1}{3} t^{-2/3} dt$

$$I = \int_0^1 (t^{1/3})^5 (1-t)^{10} \cdot \frac{1}{3} t^{-2/3} dt$$

$$= \frac{1}{3} \int_0^1 t^{\frac{5}{3} - \frac{2}{3}} (1-t)^{10} dt$$

$$= \frac{1}{3} \int_0^1 t^{2-1} (1-t)^{11-1} dt$$

$$= \frac{1}{3} B(2,11) = \frac{1}{3} \cdot \frac{\Gamma 2 \, \Gamma 11}{\Gamma 13} = \frac{1}{3} \frac{\Gamma 2 \, \Gamma 11}{12 \cdot 11 \cdot \Gamma 11}$$

$$= \frac{1}{3} \frac{\lfloor 1}{12 \cdot 11} = \frac{1}{396}$$

(IV) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2} dx$

(V) $\int_0^a x^9 \sqrt[3]{a^6 - x^6}\, dx$

$$I = \int_0^a x^9 \left(a^6 - x^6\right)^{1/3} dx$$

$$I = \int_0^a x^9\, a^2 \left(1 - \left(\frac{x}{a}\right)^6\right)^{1/3} dx$$

Let $\left(\frac{x}{a}\right)^6 = t$ $\quad x = at^{1/6} \Rightarrow$ $\cancel{dx = a\,dt}$ $\quad dx = a \cdot \frac{1}{6} t^{-5/6} dt$

So

$$I = \int_0^1 \left(at^{1/6}\right)^9 a^2 (1-t)^{1/3} \cdot a \cdot \frac{1}{6} t^{-5/6} dt$$

$$= \frac{1}{6} \int_0^1 a^9 \cdot t^{\frac{9}{6}} a^2 (1-t)^{1/3}\, a\, t^{-5/6}\, dt$$

$$= \frac{1}{6} a^{12} \int_0^1 t^{\frac{3}{2}} (1-t)^{1/3}\, t^{-5/6}\, dt$$

$$= \frac{a^{12}}{6} \int_0^1 t^{2/3} (1-t)^{1/3}\, dt$$

$$= \frac{a^{12}}{6} \int_0^1 t^{\frac{5}{3}-1} (1-t)^{\frac{4}{3}-1}\, dt$$

$$= \frac{a^{12}}{6} B\left(\frac{5}{3}, \frac{4}{3}\right)$$

(VI) $\int_0^a x^3 \left(a^5 - x^5\right)^3 dx$

$$I = \int_0^a x^3\, a^{15} \left(1 - \left(\frac{x}{a}\right)^5\right)^3 dx$$

Let $\left(\frac{x}{a}\right)^5 = t \Rightarrow x = a \cdot t^{1/5} \Rightarrow dx = a \cdot \frac{1}{5} t^{-4/5} dt$

$$I = \int_0^1 \left(at^{1/5}\right)^3 \cdot a^{15} (1-t)^3 \cdot a \cdot \frac{1}{5} t^{-4/5} dt$$

$$I = \int_0^1 a^3 t^{3/5} a^{15} (1-t)^3 \frac{a}{5} t^{-4/5} \, dt$$

$$= \frac{a^{19}}{5} \int_0^1 t^{3/5 - 4/5} (1-t)^3 \, dt$$

$$= \frac{a^{19}}{5} \int_0^1 t^{-1/5} (1-t)^3 \, dt$$

$$= \frac{a^{19}}{5} \int_0^1 t^{4/5 - 1} (1-t)^{4-1} \, dt$$

$$= \frac{a^{19}}{5} B\left(\frac{4}{5}, 4\right)$$

$$= \frac{a^{19}}{5} B\left(\frac{4}{5}, 4\right)$$