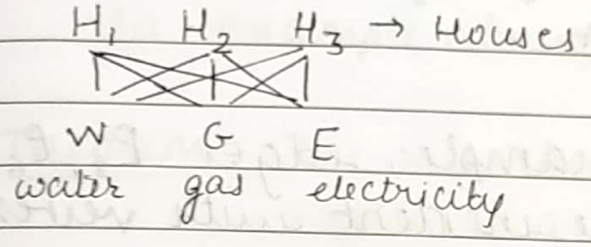
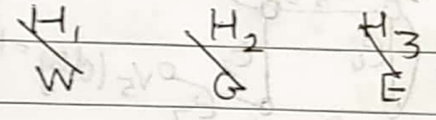
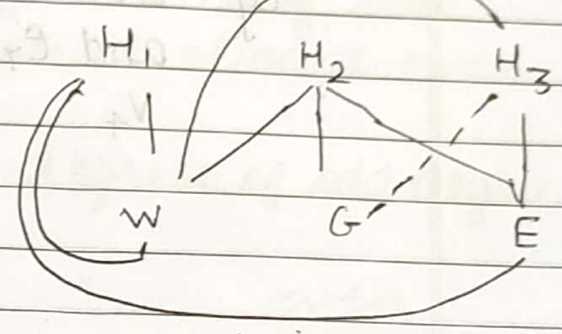
01/07/2020

Q) Is it possible to make the below connections without any crossovers of the conduits?



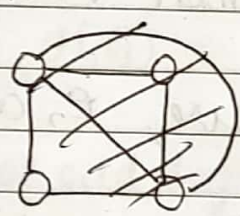


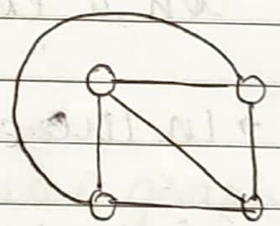


- - - → gets intersected

If we can draw a graph without any crossovers/intersections of the lines, it is a PLANAR GRAPH

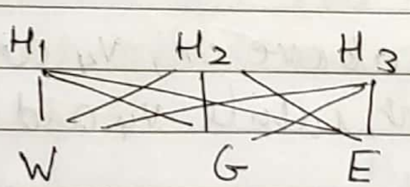
eg:



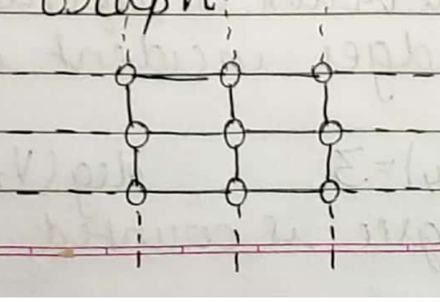


If we cannot draw a graph without any crossovers of lines, it is a non-planar ~~lin~~ graph.

eg:



Infinite Graph:



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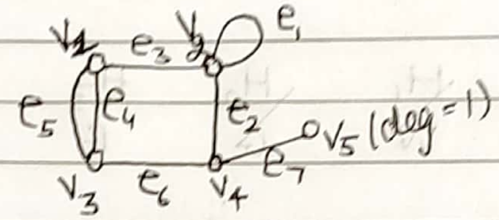
Incidence

When a vertex V_i is an edge vertex of some edge e_j, V_i & e_j are incident with each other.

eg: In this example, edges e_2, e_6 and e_7 are incident with vertex V_4:



Adjacent

Two non-parallel ~~non-p~~ edges are said to be adjacent if they are incident on a common vertex.

→ In the above, e_2 and e_6 are adjacent.

Similarly, two vertices are said to be adjacent if they are end vertices of an edge.

→ In the above ex, V_4 and V_1 are not adjacent, while V_4 and V_2 are.

Degree

The degree of a vertex is defined as the number of edges incident on the vertex

eg: $\deg(V_4)=3$ $\deg(V_2)=4$ $\deg(V_1)=3$

Self loop's degree is counted twice

01/07/2020

$$\sum_{i=1}^{n} d(v_i) = 2e$$ | e = # of edges
LHS = sum of all degrees

In example above, $\sum d = 14$, $e = 7 \Rightarrow 2(7) = 14$

Theorem: The number of vertices of odd degree in a graph is always even.

Proof: Let o = sum of degrees of odd-degree vertices
e = " " even "

$$\sum_{i=1}^{n} d(V_i) = \sum_{odd} d(V_i) + \sum_{even} d(V_i)$$

$$\Rightarrow \text{even} = \sum_{odd} d(V_i) + \text{even}$$

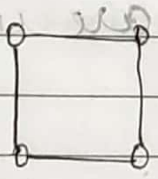
$$\Rightarrow \text{even} = \sum_{odd} d(V_i)$$

$\Rightarrow$ # of vertices of odd degree in a graph is always even.

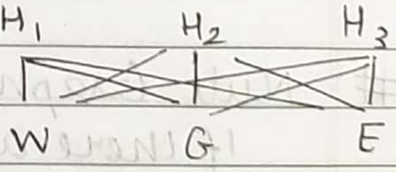
Regular Graph

A graph in which all the vertices are of equal degree is called a regular graph

eg:



$d = 2$



$d = 3$

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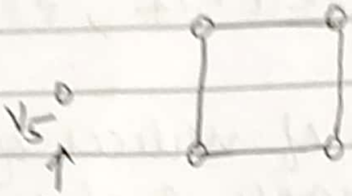
Isolated Vertex

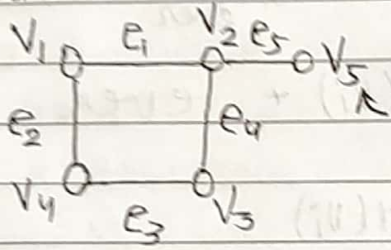
If the degree of the vertex is zero, then the verex is called an isolated vertex



Isolated

Pendant Vertex

If the degree of the vertex is one, then the vertex is called a pendant vertex.



Pendant

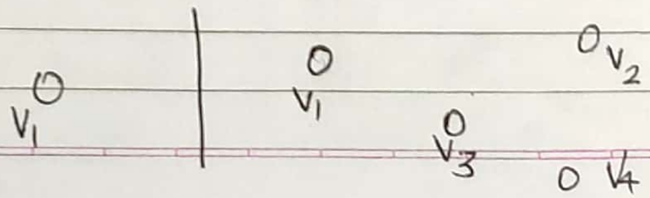
Series

Two adjacent edges are said to be in series if the common vertex is of degree 2.

In above example e_1 & e_2, and e_2 & e_3, and e_3 & e_4 are in series.

Null Graph

If there are no edges in a graph (must have at least one vertex), it is called a null graph



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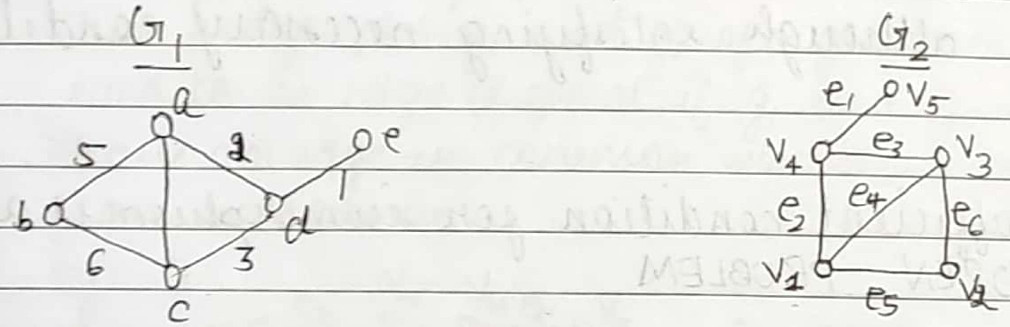
Isomorphism

Two graphs G_1 & G_2 are said to be isomorphic if there is 1 to 1 correspondence between their vertices and between their edges such that the incidence relationship is preserved.

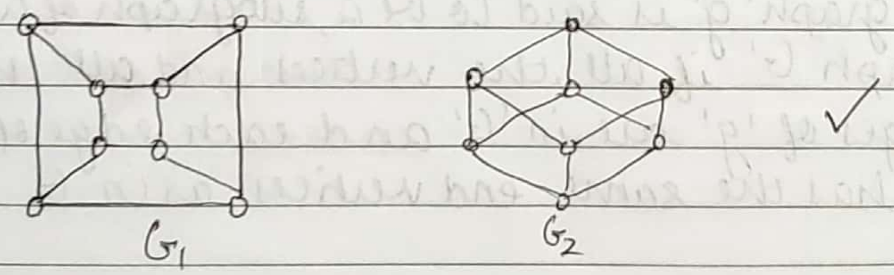
In other words, suppose the edge 'e' is incident on vertices V_1 and V_2 in G_1, then the corresponding edge 'e'' in G_2 must be incident on the vertices V_1' and V_2' that correspond to V_1 & V_2 respectively.



* $G_1 : V_1 = \{a, b, c, d, e\}$ * $E_1 = \{1, 2, 3, 4, 5\}$
  $G_2 : V_2 = \{V_1, V_2, V_3, V_4, V_5\}$ $E_2 = \{e_1, e_2, e_3, e_4, e_5\}$

*degrees of all vertices in each graph should match.

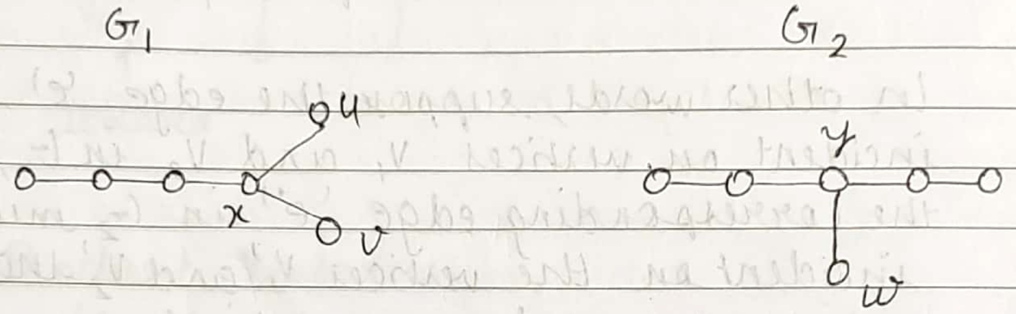
Q) Check for isomorphism



01/07/2019

necessary condit^n, not sufficient — ✱ Two isomorphic graphs must have the same number of vertices, edges and same number of vertices with an equal degree.

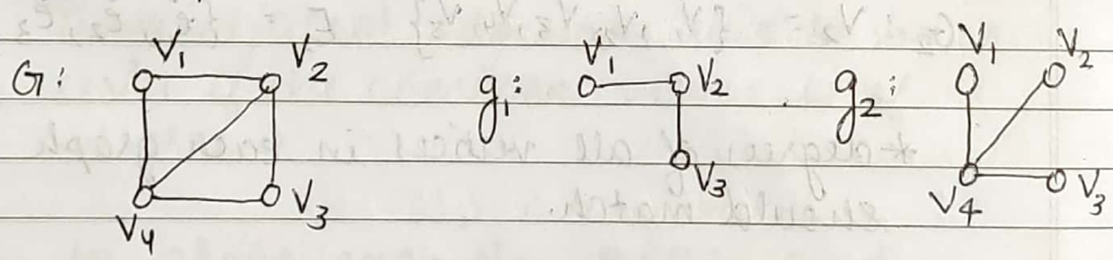
eg:



NOT ISOMORPHIC,
although satisfying necessary conditions.

Sufficient condition for isomorphism is an OPEN PROBLEM

Sub-Graph



A graph 'g' is said to be a subgraph of a graph 'G' if all the vertices and all the edges of 'g' are in 'G' and each edge of 'g' has the same end vertices as in G.

01/09/2020

1. A graph G is a subgraph of itself

2. A subgraph of a subgraph of G is also a subgraph of G.

3. A single vertex can be the subgraph of G.

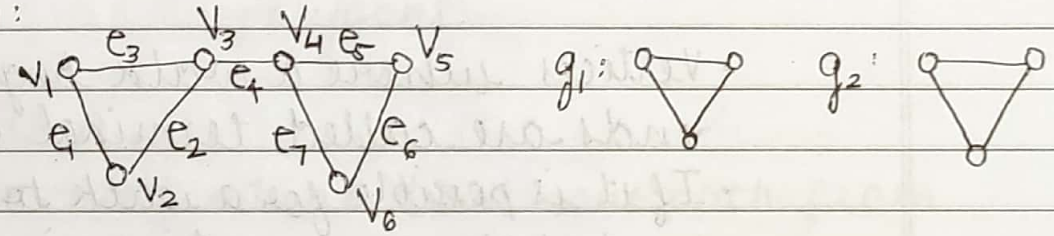
4. A single edge along with its two vertices can also be a subgraph of G.

→ Edge Disjoint Subgraph

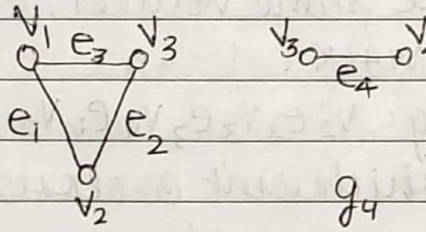
Two subgraphs g_1 and g_2 of a graph G are said to be edge disjoint if g_1 and g_2 do not have an edge in common, whereas, a vertex can be common.

eg: G:

#1



2:



g_3 g_4

→ Vertex Disjoint

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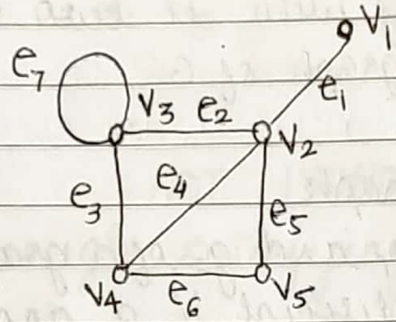
Walk, Path, and Circuits

→ Walk: A walk is defined as the finite alternating sequence of vertices and edges beginning and ending in a vertex such that incident with the vertices and following it. No edge appears more than once in a walk, vertices can be repeated.



terminal vertex ↑ … terminal vertex ↑

eg : $v_1 e_1 v_2 e_2 v_3 e_7 v_3 e_3 v_4 e_4 v_2 e_5 v_5$
→ no edge repeated
→ vertices can be repeated

aka chain, edge train

- Vertices where a walk begins and ends are called terminal vertices.
- If it is possible for a walk to begin and end at the same vertices, it is called a CLOSED WALK.

eg: $v_2 e_2 v_3 e_3 v_4 e_4 v_2$

- A walk which isn't a closed walk is called an open walk.

→ Path: An open walk in which no vertex appears more than once is called a path.

The length of the path is the number of edges encountered during the open walk.

01/09/2020

- The edges which are not self loops are counted once. Self loops are not considered as paths.

- Terminal vertices of paths have degree 1, while the rest have degree 2.

→ Circuit : A closed walk in which no vertex appears more than once except the initial and final vertex.

eg: $v_2 e_2 v_3 e_3 v_4 e_4 v_2$ (on left graph)

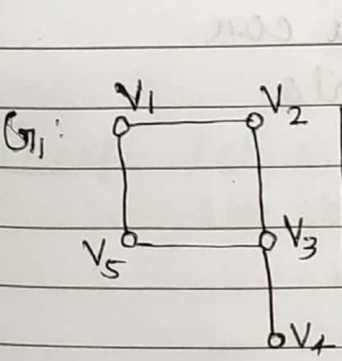
- In circuits, all the vertices are of degree 2.

aka cycles, circular paths, polygons.

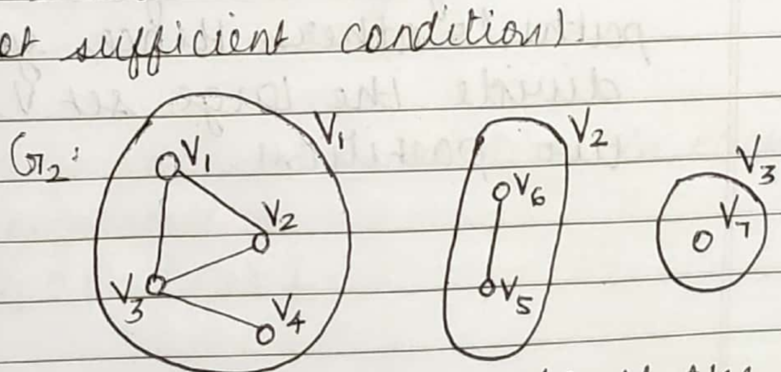
Connectedness

If we can find at least one path from every vertex to every other vertex, we call it a connected graph.

i.e no isolated vertices (necessary, not sufficient condition)



connected graph



V_1, V_2, V_3 are components of the unconnected graph G_2, and each component is a connected subgraph

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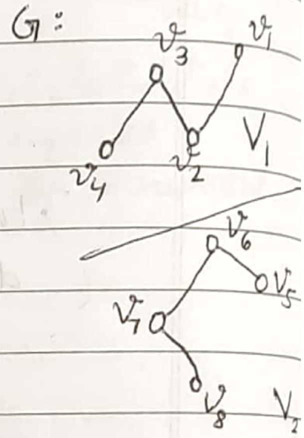
Theorem 1:

A graph G is disconnected iff its vertex set V can be partitioned into two non-empty, disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end is in V_1 and the other in V_2.

Proof:

Take two arbitrary vertices a and b such that a belongs to V_1 and b belongs to V_2. Now, no path exists from a to b, otherwise, there would be at least one end vertex in V_1 and another in V_2. Hence, if partition exists, V is not connected.



Conversely, let's consider G to be a disconnected graph. Take a vertex a in V_1. You are able to find a path to certain points while you cannot find the paths to others. Hence, you can divide the large set V into two partitions.

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Theorem 2:

If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof:

Let G be a graph with all even degree vertices except vertices v_1 and v_2 which have odd degrees. By theorem $\sum d(v_i) = 2e$, which holds for every graph, therefore for every component of the disconnected graph, no graph can have an odd number of odd vertices. ⇒ In a graph G, v_1 and v_2 must be in the same component ⇒ a path must connect v_1 and v_2.

Theorem 3:

A simple graph (a graph without parallel edges and self loops) with n vertices and k components have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let $n_1, n_2, \ldots, n_k$ be the vertices in each of the k components.

$$\sum n_i = n \quad \forall n_i \geq 1$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

01/10/2020

$$\sum_{i=1}^{k} (n_i^2 - 2n_i) + k + \text{non-negative terms} = n^2 + k^2 - 2nk$$

$$\forall (n_i - 1) \geq 0.$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2kn - k + 2n$$
$$= n^2 - (2n-k)(k-1) \quad -(*) \text{ (maximum value)}$$

Now, the maximum number of edges in the i^{th} component of G will be:

$$^{n_i}C_2 = \frac{n_i(n_i - 1)}{2}.$$

Therefore the maximum number of edges =

$$\frac{1}{2} \sum_{i=1}^{k} n_i^2 - n_i$$

$$= \frac{1}{2}\left[n^2 - (2n-k)(k-1) - n\right] \quad \text{From } (*)$$

$$= \frac{1}{2}\left[n^2 + k^2 + n - 2kn - k\right]$$

$$= \frac{1}{2}\left[(n-k) + (n-k)^2\right]$$

$$= \frac{(n-k)(n-k+1)}{2}$$

Therefore, if a disconnected graph with k components has a total of n vertices, it will have a maximum

$$\frac{(n-k)(n-k+1)}{2}$$

$k=1 \Rightarrow \frac{n(n-1)}{2}$ edges } Complete graphs

Page No. 15

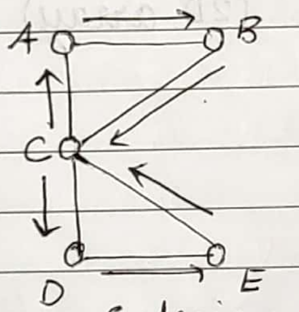
01/10/2020

Note: Every complete graph is connected but not every connected graph is complete.

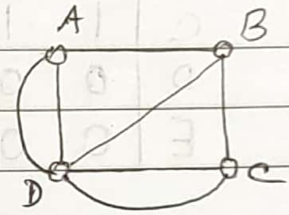
Q) Draw a simple graph with 5 vertices and 11 edges
⇒ NOT POSSIBLE, b/c max # of edges = 10 ($^5C_2$).

Euler Graph

A closed walk in which all the edges of the graph will be covered exactly once is called an Euler Line/ Euler Trail and the graph is called an Euler Graph



Eulerian



Not Eulerian

Theorem:

The following statements are equivalent for a connected graph G:

(1) G is Eulerian

(2) Every vertex of G has an even degree

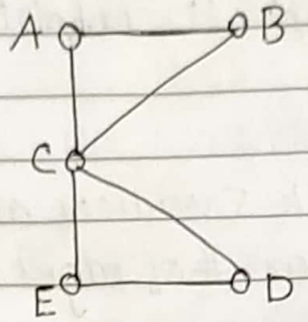
(3) The set of edges of G can be partitioned into cycles.

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

Try to prove before mid-term revision

01/10/2020

Representation of Graphs



1) Adjacency Matrix

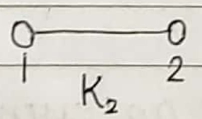
	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	0	0
C	1	1	0	1	1
D	0	0	1	0	1
E	0	0	1	1	0

Basically a 5x5 matrix (2D array)

now, $t = O(n)$

if we fill only upper triangular matrix : $\dfrac{n^2 - n}{2}$] Faster stores enough information

eg:



(i)

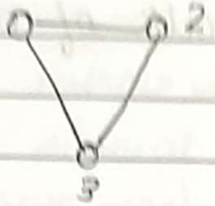
	1	2
1	0	1
2	1	0

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Eigen values of $K_2 = \pm 1$

$= -1, 1$

(ii) K_3:



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array}$$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

K_3: $-1, -1, 2$

$$-(\lambda^3 + 2 - (3\lambda)) = 0$$
$$\lambda^3 - 3\lambda + 2 = 0 \qquad \lambda = 1$$
$$\lambda^3 - \lambda - 2\lambda + 2 = 0$$
$$\lambda(\lambda^2 - 1) - 2(\lambda - 1) = 0$$
$$(\lambda - 2)(\lambda - 1)(\lambda - 1) = 0$$
$$(\lambda - 1)(\lambda(\lambda + 1) - 2)$$
$$(\lambda - 1)(\lambda + 2)(\lambda - 1) = 0$$
$$1, 1, -2$$

(iii) K_4: $-1, -1, -1, 3$

$\Rightarrow$ for K_n: $\underbrace{-1, -1, -1 \ldots}_{n-1 \text{ times}}, (n-1)$

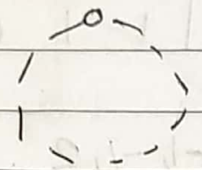
If a graph has a negative value and exactly one positive value, it is complete.

01/4/2020

Q) Prove that a closed walk of odd length contains a circuit.

~~G, n vertices,~~ ~~x~~

~~n =~~

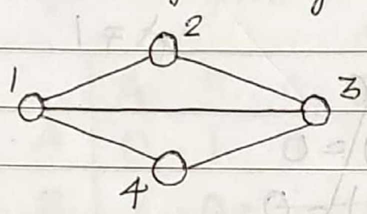


$\#$ edges $= n$

$\Sigma deg = 2n$

---

2) <u>Packed Adjacency List</u>



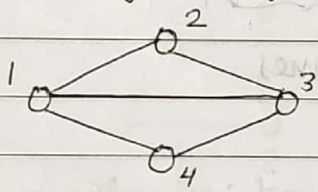
l:

0	1	2	3	4	5	6	7	8	9	10
2	3	4	1	3	1	2	4	1	3	NULL

h: {0, 3, 5, 8, 10}

$$x = \begin{cases} (e-1) & \text{digraph } \times \\ 2e+1 & \text{graph } \checkmark \end{cases}$$

3) <u>Linked Adjacency List</u>



aList [1] → 2 → 3 → 4 → NULL
aList [2] → 1 → 3 → NULL
aList [3] → 2 → 1 → 4 → NULL
aList [4] → 1 → 3 → NULL

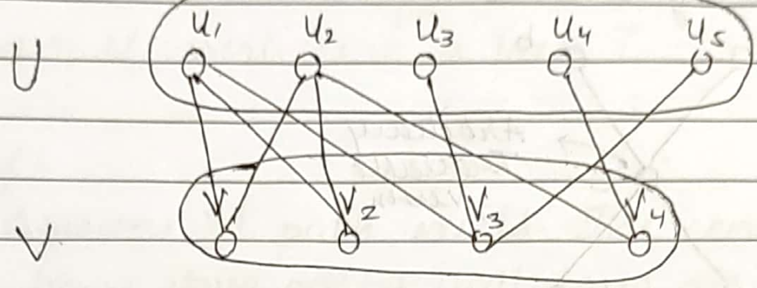
<u>Bucket sort</u>

* $K_5 \rightarrow$ complete graph with 5 vertices

01/17/2020

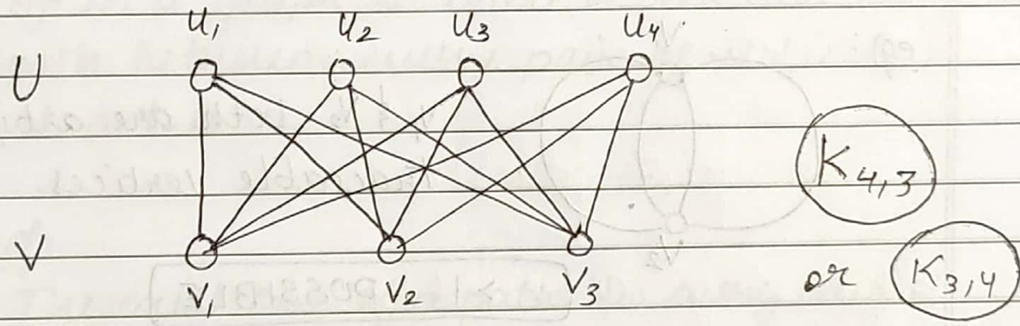
Bipartite Graphs

A graph whose vertices can be partitioned into two disjoint sets U and V (not necessarily of the same size) so that every edge connects a vertex in U to a vertex in V.



A graph is bipartite if an only if it does not have a cycle of odd length.

Complete Bipartite Graph



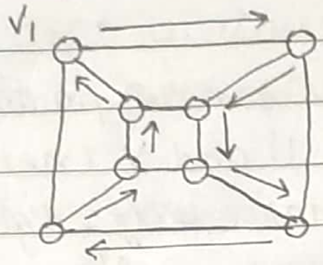
Every vertex in ~~edge~~ U is connected to every vertex in V.

~~Hat~~ Hamiltonian Graph

A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G EXACTLY once ~~expe~~ except the starting vertex at which the walk also terminates.

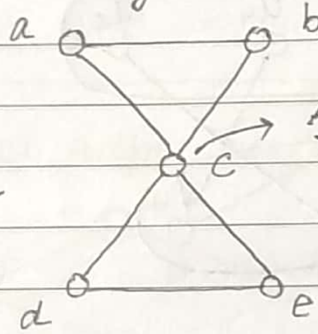
01/17/2020

eg:



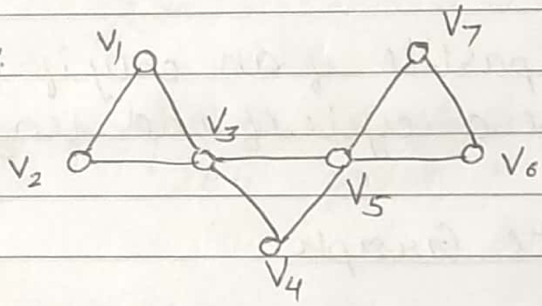
Arbitrary Traceable Vertices

Euler



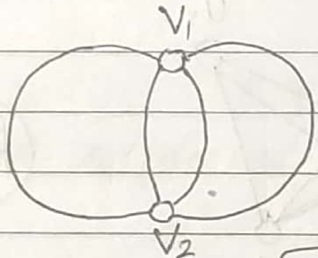
Arbitrary Traceable vertex

eg:



NONE

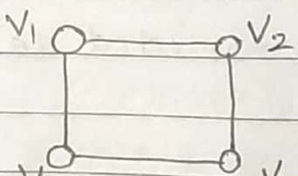
eg:



V_1 & V_2 both are arbitrary traceable vertices.

$\therefore$ >1 POSSIBLE

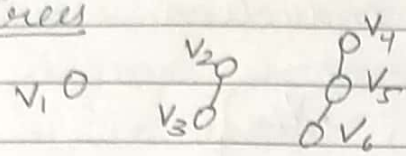
eg:



All are ATVs.

01/7/2020 ~~Theory~~

Trees



A connected graph with no circuits is called a tree.

→ Thm 1:

There is one and only one path between every pair of vertices in a tree, T.

Proof:

Assume ≥1 path exists. The connection of these two paths will form a circuit, which contradicts the definition of the tree. Hence, there is ~~a maximum of~~ one path connecting every pair of vertices in G (tree).

→ Thm 2:

If in a graph G there is one and only path between every pair of vertices, it is a tree.

Proof:

The existence of a path b/w every pair of vertices shows that G is connected. Since G has one and only path between every pair of vertices, it cannot have a circuit. Therefore G is a tree.

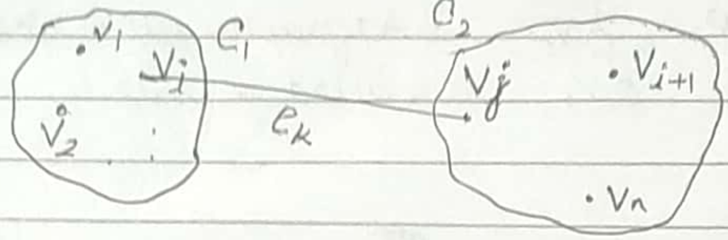
→ Thm 3:

In a tree T with n vertices there are exactly $(n-1)$ edges.

Proof: ⟶

01/17/2020

Proof: By Mathematical Induction.



There is only one edge e_k connecting the both partitions. Assume we remove e_k.
Assume that one component connects n_1 & the other n_2 $(n_1, n_2 < n)$.

In G_1, the number of edges will be n_1-1. In G_2, the number of edges will be n_2-1.

Total number of edges will be: n_1+n_2-2
$n_1+n_2 = n$.

The total number of edges in a tree with n vertices is equal to $n-2+1 = n-1$ (by joining e_k back).

Thm: Any connected graph with n vertices and $(n-1)$ edges is a tree.

Proof: Suppose G is not a tree. Then, G forms a cycle/circuit. Now, try to remove an edge and call the resultant graph G_1.
~~Remove~~
If a circuit still remains, remove another edge, keep removing till when the k^{th} edge is removed, no loop remains.
↳ call graph G_k.

A tree has n vertices wi

01/17/2020

Minimally connected graph

A graph is called a minimally connected graph if, on removal of one edge disconnects the graph.

Thm 5: A graph is called a ~~minimally connected~~ tree ~~grap~~ iff it is minimally connected.

Pf: ⇒ Suppose G is a minimally connected graph, it does not have a cycle.
⇒ G is a tree.

Thm: Conversely, suppose G is a tree. Then by theorem, there exists a unique path between every pair of vertices. If we remove any edge from the tree, it will be disconnected. By definition of a tree, there are no cycles in a tree. Therefore, G is minimally connected.

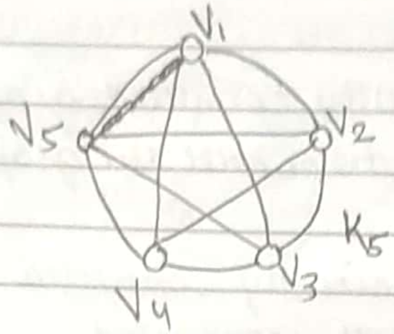
Thm: In a complete graph with n vertices there are $\frac{(n-1)}{2}$ edge disjoint hamiltonian circuits

if n is odd and ≥ 3.

Proof: In the case of a complete graph with n vertices: it has: $\frac{n(n-1)}{2}$ edges.

The hamiltonian cycle has exactly n edges. Therefore, the number of edge disjoint hamiltonian circuits in G cannot have more than $\frac{n(n-1)}{2n} = \frac{n-1}{2}$ edges. Also, the subgraph of a complete graph with n vertices is a Hamiltonian cycle.

01/17/2020



$\frac{2\pi}{n-1}, \frac{2\cdot 2\pi}{n-1}, \ldots \frac{(n-3)2\pi}{2(n-1)}$

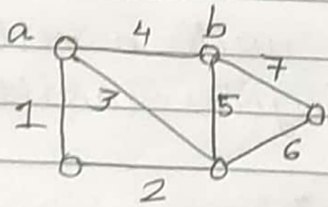
Each rotation (clockwise by the above θs) will produce one hamiltonian circuit. Therefore, totally, we will have $\frac{n-3}{2}+1$ trivial hamiltonian circuit. Totally

$\frac{n-3}{2}+1$

we will have $\frac{n-1}{2}$ edge disjoint Hamiltonian circuits.

Unicursal Graph aka **Open Euler Line**

This is an open walk that traces all the edges of a graph without retracing any edge.

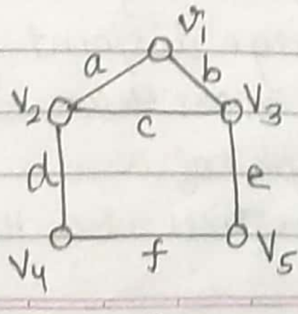


Operations on Graphs

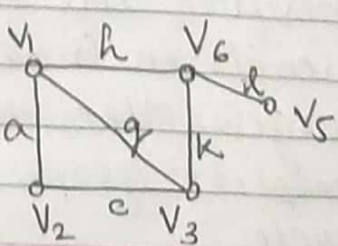
- Ring Sum: G_1, G_2

$$G_1 \oplus G_2 = (G_1 \cup G_2) \setminus (G_1 \cap G_2)$$

$V_1 \cup V_2$



G_1



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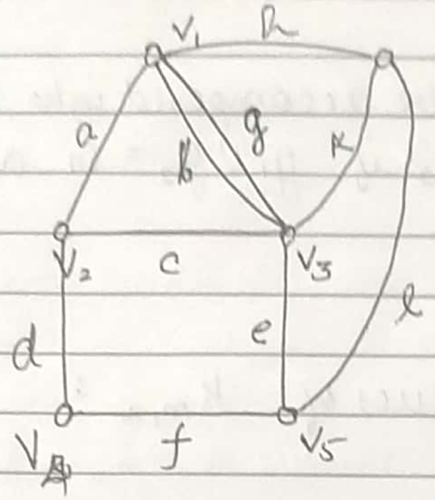
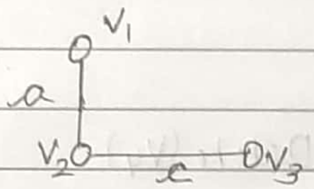
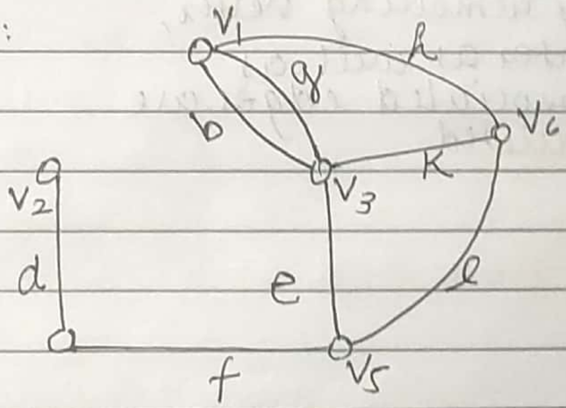


$G_1 \cup G_2$



$G_1 \cap G_2$

↳ Ring sum:



- G Ring sum G is a null graph
- G_1 & G_2 are edge disjoint $\Rightarrow G_1 \oplus G_2 = G_1 \cup G_2$
- g is a subgraph G $\Rightarrow G \oplus g = \{G \setminus g\} \cup \{\text{vertices of } g\}$

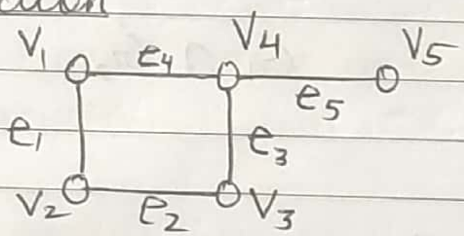
01/17/2020.

Decomposition

A graph G is said to be decomposed into two subgraphs g_1 and g_2 if $g_1 \cup g_2 = G$ and $g_1 \cap g_2$ is a null graph

Find the eigen values of $K_{m,n}$:

Deletion

(i) 

Delete (V_4)

⇒ By removing vertex, vertex as well as associated edges are deleted

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Thm: Every tree has either one or two centers

Proof:

Let T be a tree with n vertices. Now, remove all the pendant vertices. The resultant tree is T'.

In this tree T', the present eccentricities of the vertices will be one less than the previous tree T. Again remove pendant vertices to get a tree T''.

⋮

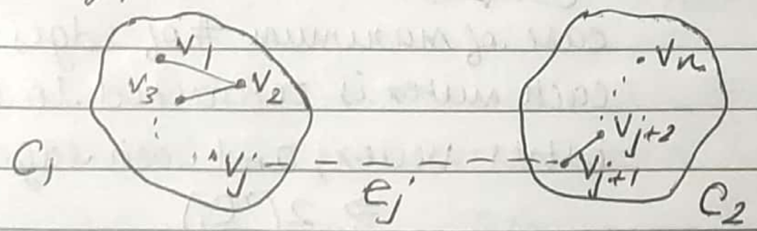
continue this process k times.

⋮

We will end up with either a vertex or one edge, which will be the center(s) of the tree T.

Thm: A graph G with n vertices, $(n-1)$ edges and no circuits is connected

Proof: Suppose G is not connected.



n vertices
$(n-1)$ edges

C_1 is a tree $\Rightarrow C_1 \cup C_2 \cup \{e_j\}$ is a tree
C_2 is a tree with n vertices.
$\Rightarrow$ contradicts original statement

$\therefore$ MUST BE CONNECTED

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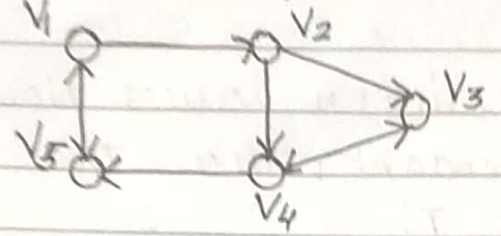
Directed Graphs



$V_1 V_2 V_3$: simple path.

In-degree: d_{in} $\qquad d_{in}(V_1) = 1 \quad d_{in}(V_4) = 2$

Out-degree: d_{out} $\qquad d_{out}(V_1) = 2 \quad d_{out}(V_4) = 2$

$$\sum d(V_i) = 2e \quad \text{for undirected}$$

$$|V| = n \quad |E| = e$$

(i) $\sum_i d_{in}(V_i) = \sum_i d_{out}(V_i) = e$ for directed

* For bidirectional edges in directed graphs, edges are counted twice *

(ii) $0 \leq e \leq \underbrace{n(n-1)}$

case of maximum # of edges where each vertex is connected to every other vertex, and each edge is bidirectional $\Rightarrow 2({}^nC_2)$.

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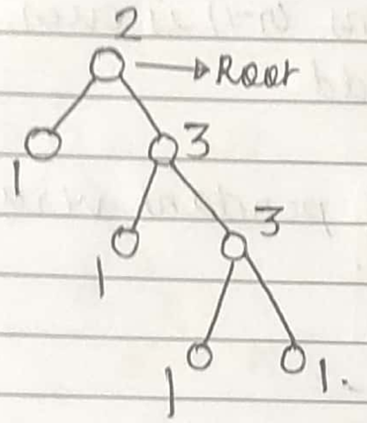
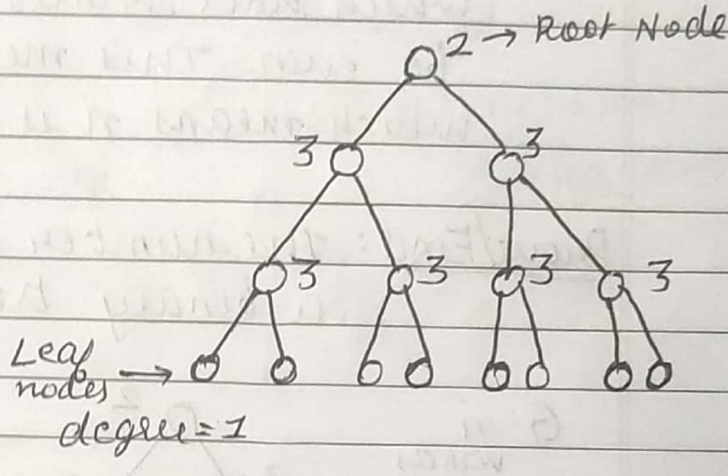
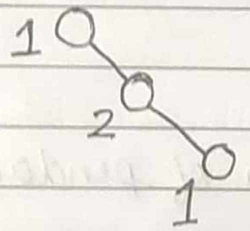
Rooted Tree and Binary Tree (undirected graph)

In a tree, if one vertex has a special property, then the tree is called a rooted tree.

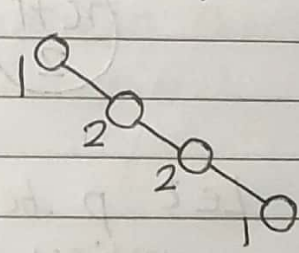




A Binary Tree is defined as a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices are of degrees 3 & 1.



Binary Tree



Not a Binary tree

Prove: The number of vertices in a binary tree is always odd.

$$\sum d(v_i) = \underbrace{2e}_{\text{even}}$$

LHS should also be even.

$d(r) = 2$. The remaining $(n-1)$ vertices should have an ~~odd~~ even. sum of degrees.

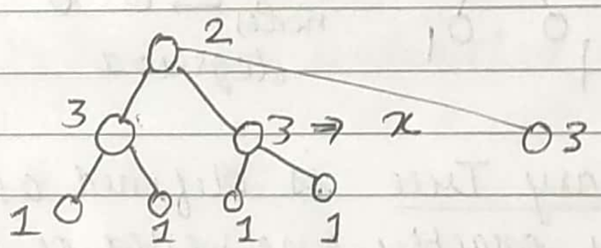
$\Rightarrow (n-1)$ should be even

$\Rightarrow n$ should be odd.

∴ The number of vertices of odd degree is even. Root vertex has a degree 2 (even). Therefore, the remaining $(n-1)$ vertices, which have vertices 3 and 1 (odd) must be even. This means $(n-1)$ is even, which means n is odd.

<u>Prove/Find</u>: The number of pendant vertices in a binary tree.

G: n vertices



$$\frac{n+1}{2}$$

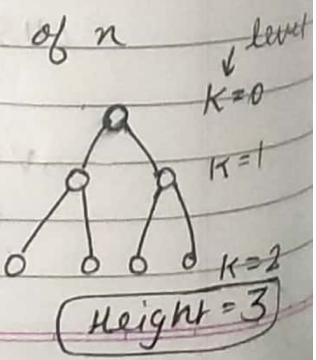
Let p be the number of pendant vertices

$$\Rightarrow (n-p-1)(3) + 2 + p = 2(n-1)$$

$$3n - 1 - 2p = 2n - 2$$

$$\frac{n+1}{2} = p$$

<u>Level number</u>: Given a binary tree of n nodes are given.



$$\frac{l(l+1)}{2} = n$$

$$l^2 + l - 2n = 0$$

$$2^{l-1} = n$$

$$l - 1 = \log_2 n$$

$$l = \log_2 n + 1$$

01/23/2020

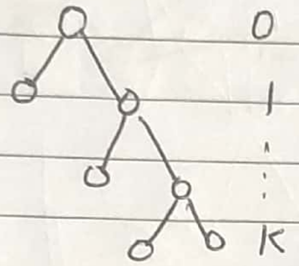
$$N \leq 2^0 + 2^1 + 2^2 + \dots + 2^K$$

$$N \leq 1\left(\frac{1-2^{K+1}}{1-2}\right)$$

$$N \leq \frac{2^{K+1}-1}{1} \Rightarrow N \leq 2^{K+1}-1$$

$$\Rightarrow \log_2(N+1) \leq K+1$$

$$K \geq \log_2(N+1) - 1$$



$$N \geq 1 + 2K$$

$$\frac{N-1}{2} \geq K$$

$$\boxed{O(N) \geq K \geq \log(N)}$$

<u>Spanning Trees</u>

n vertices $\rightarrow n^{n-2}$ spanning trees.
(labelled trees)

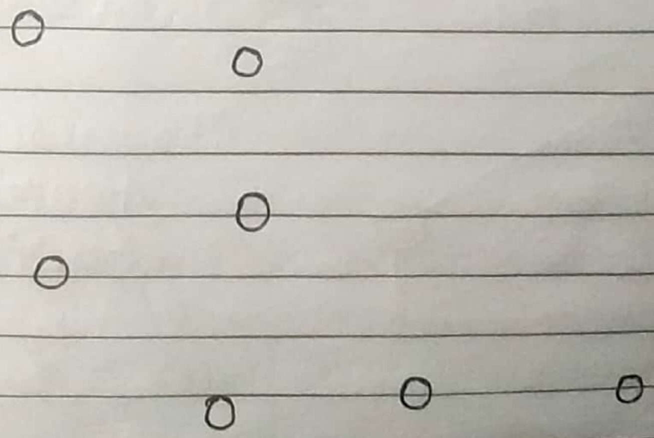
$\boxed{n \geq 2}$

Tree containing least number of edges connecting all edges

Shortest path

<u>Weighted Graph</u>

V, E (weights)



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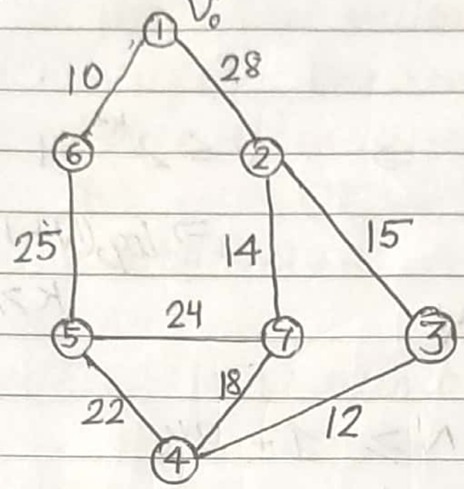
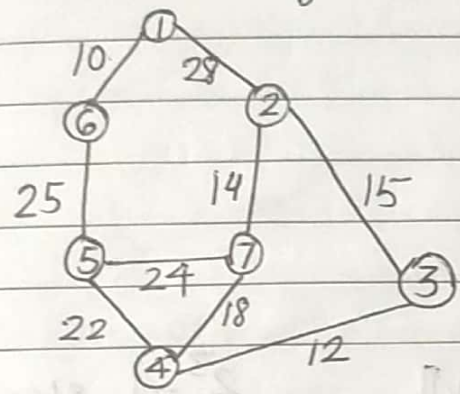
Q) Dijkstra's shortest path problem:

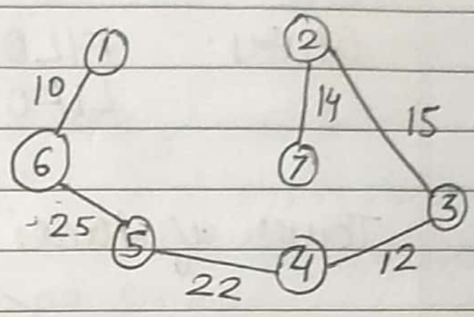


10 28

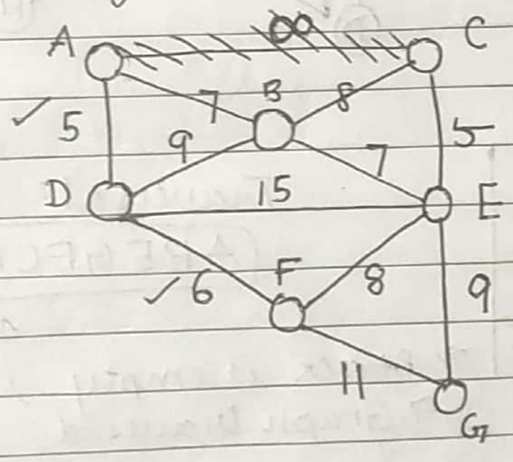
~~2~~ 35, 28

Q) Krushkal's algorithm :

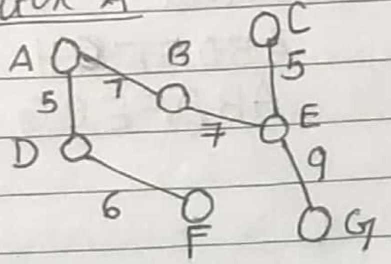




Q) Prim's algorithm :



Vertex A



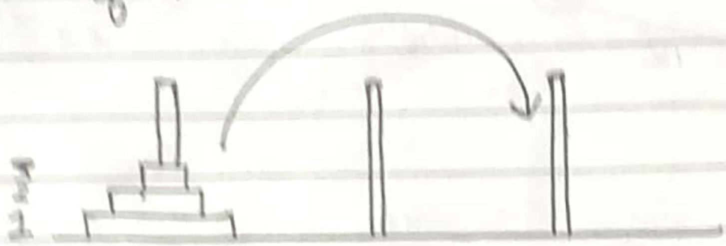
~~AB-7~~ } min
~~DB-9~~
~~DE-15~~
~~FE-8~~
FG-11 POSSIBLE
~~BC-8~~
~~BE-7~~
~~EC-5~~
~~EG-9 POSSIBLE~~
} min

01/24/2020

Depth First search (DFS)

Stacks: FILO
LIFO

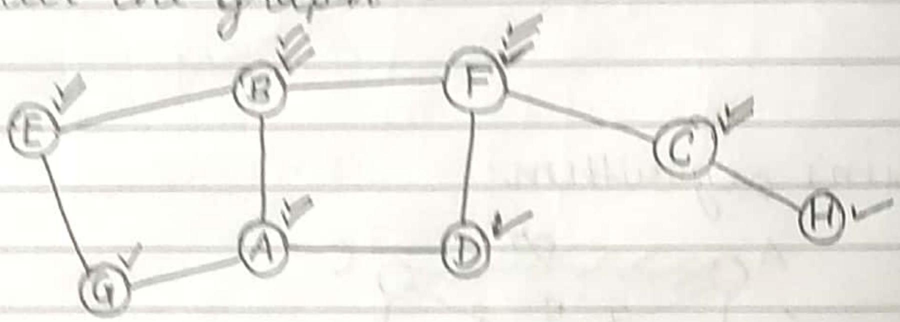
Tower of Hanoi



$2^3 - 1$ steps

Stacks: Data structure we use during DFS.

→ Consider the graph:



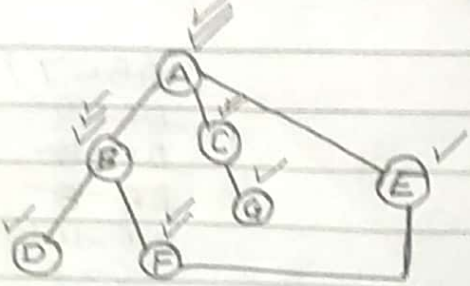
* Drawing stack upside down

A	A	~~A~~
B	B	~~B~~
~~E~~	F	~~F~~
~~G~~	~~C~~	~~D~~
	~~H~~	

→ stack is empty
→ Graph traversed

Traversal:
$\boxed{\text{ABEGFCHD}}$

→

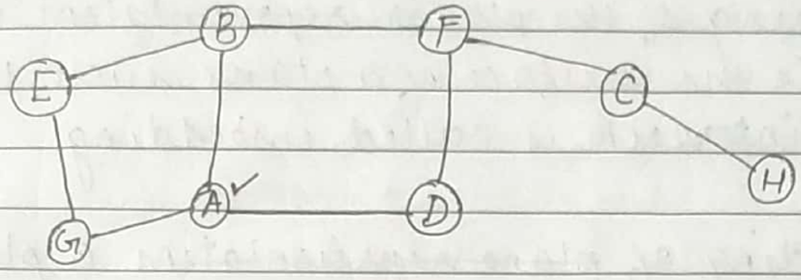


Traversal:
~~ABDFEGH~~
ABDFECG

01/28/2020

Breadth First Search (BFS)

G_1



Queue
FIFO
LILO

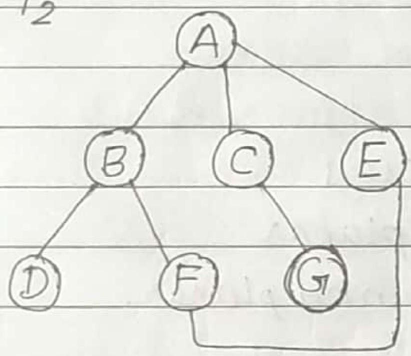
Insertion at back, release at front.

Queue:

R R R
~~A~~ ~~B~~ ~~D~~ ~~G~~ ~~E~~ ~~F~~ ~~C~~ ~~H~~
Fr Fr Fr

Traversal
ABDGEFCH

G_2:

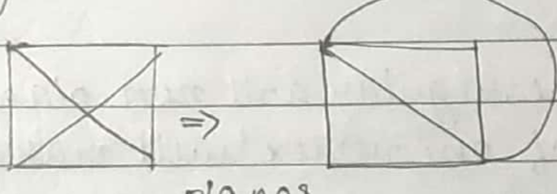


Queue R R
~~A~~ ~~B~~ ~~C~~ ~~E~~ D F G
F F F F F

Traversal
AB CEDFG
↑
BFS
(DFS was ABDFECG)

Planar Graphs

A graph is said to be planar if there exists some geometrical representation of G which can be drawn on a plane such that no two of its edges intersect.



planar

A graph that cannot be drawn on a plane without a crossover b/w its edges is a non-

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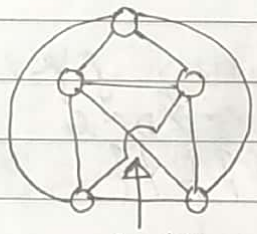
Embedding

A drawing of the ~~planar representation~~ ~~of the~~ of a graph in the surface of a plane such that no edges intersect is called embedding

An Embedding of ~~plane representation~~ a planar graph G on the surface of a plane is called the

Kuratowski's Graphs: K_n (or) $K_{n,m}$

Thm: K_5 is non-planar

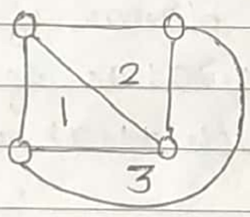


intersection occurs

$\Rightarrow K_n, n \geq 5$ is non planar

Similarly, $K_{3,3}$ is also non planar

Region



$4 \rightarrow \infty$

$1,2,3 \rightarrow$ finite

K_5 & $K_{3,3}$ are regular and non planar.

$\Rightarrow$ Removal of one vertex will make the graph planar

$\Rightarrow$ K_5 is non-planar with smallest number of vertices

$\Rightarrow$ $K_{3,3}$ is a non-planar graph with smallest number of edges

01/28/2020

Euler's Theorem:

A connected planar graph with n vertices, m edges, and f regions, then:

$$n - m + f = 2.$$

Proof: by induction:
Given m is # of edges.

- $m = 0$, $n = 1$, $f = 1$ ✓

$m = 1 \rightarrow$  & 

$n = 1$, $f = 2$ ✓ $n = 2$, $f = 1$ ✓

Suppose result is true for up to $m-1$ edges.

Case (i): Assume that G is a tree with n vertices, m edges, and f regions.

$$\Rightarrow f = 1$$
$$m = n - 1$$
$$n = n$$
$$\therefore f - m + n = 1 + 1 - n + n = 2 \checkmark$$

Case (ii): G is not a tree with n vertices, m edges, and f regions. G contains at least one cycle.

$G \setminus \{e\}$ s/t it becomes a tree
↳ n vertices
↳ $m-1$ edges
↳ $f-1$ regions

$\Rightarrow n - (m-1) + (f-1)$
$= n - m + f$
valid for $(m-1)$ edges anyways.

$n - (m-K) + (f-K) = n - m + f = 2$
↳ K edges removed to make resultant graph a tree

01/30/2020 Set Theory

(1) $U = \mathbb{R}$, the set of real numbers if

$A = \{x : x \in \mathbb{R},\ 0 < x < 2\}$

$B = \{x : x \in \mathbb{R}\ \ 1 < x \leq 3\}$

Find A^c, B^c, $A \cup B$, $A \cap B$ $A \setminus B$

$A^c : \{x : x \in \mathbb{R},\ x \in (-\infty, 0] \cup [2, \infty)\}$

$B^c : \{x : x \in \mathbb{R},\ x \in (-\infty, 1] \cup (3, \infty)\}$

$A \cup B : \{x : x \in \mathbb{R},\ x \in (0, 3]\}$

$A \setminus B : \{x : x \in \mathbb{R},\ x \in (0, 1]\} = A \cap B^c$

$A \cap B : \{x : x \in \mathbb{R},\ x \in (0, 1)\}$

Q) The 2 Finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of m & n ?

$\begin{matrix} 6 & 3 \\ m & n \end{matrix}$ $2^m - 2^n = 56$

Q) $n(U) = 700$ ~~n(AA)~~
$n(A) = 200$
$n(B) = 300$
$n(A \cap B) = 100$
Find $n(A^c \cap B^c)$

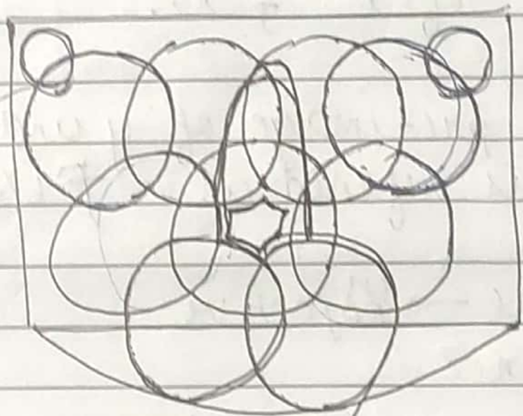
$n(A) + n(B) - n(A \cap B) = n(A \cup B)$

$\Rightarrow n(A^c \cap B^c) = n(U) - n(A \cup B)$

$= 300$

01/30/2020

Q) Of the members of 3 athletic teams in a certain institute, 21 are on the Basketball team, 26 on the hockey team and 27 on the football team. 14 play both hockey and basketball. 15 play hockey and football. 12 play football and basketball, and 8 play all 3 games. How many members are there in all.



$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= (21+26+27) - (14+15+12) + 8$$

$$= 74 - 41 + 8$$

$$= 82 - 41$$

$$= 41$$

$$\begin{array}{r} 29 \\ 12 \\ \hline 41 \end{array}$$

Functions

Consider X and Y be 2 non-empty sets. A subset F of $X \times Y$ is called a function from X to Y iff to each $x \in X$ there exists a unique y in Y such that $(x, y) \in f$.

$$f : \underbrace{X}_{\text{domain}} \rightarrow \underbrace{Y}_{\text{co-domain}}$$

$(1, 2)$ $(2, 4, 6, 8, 10)$

$f: 2x$ $(2, 4)$

01/30/2020

(i) To each $x \in X$, there exi $\uparrow$
(ii) $(x, y_1) \in f$ and $(x, y_2) \in f$ then $y_1 = y_2$

The above to conditions suggest that the image of ~~f~~ x under F or the value of x is defined by $f(x)$.

image of x : $f(x)$.

x is a pre-image of y and the range of x is defined as $f[X]$

one-to-one $\longrightarrow$ injective
many-to-one

onto
into

bijective : one-to-one onto

Composition of Two Functions

$f \circ g$ and $g \circ f$

$f(x) = 2x + 1$
$g(x) = x^2 - 2$

$$\left.\begin{array}{l} g \circ f(x) = g(f(x)) = 4x^2 + 4x - 1 \\ f \circ g(x) = f(g(x)) = 2x^2 - 3 \end{array}\right\} g \circ f \neq f \circ g$$

Q) If $A = \{x : -1 \leq x \leq 1\} = B$

$f : A \rightarrow B$

(a) $f(x) = \dfrac{x}{2}$ $\rightarrow$ one-one
NOT onto $\Rightarrow$ into

(b) $f(x) = |x|$ $\rightarrow$ many-one
into

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Page No. 43

01/30/2020

(c) $f(x) = x|x|$ one-one onto

(d) $f(x) = \sin(\pi x)$ many-one, onto.

Q) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 2x - 3$

$g: \mathbb{R} \to \mathbb{R}$ $g(x) = x^3 + 5$

Find $[f\circ g]^{-1}(x) = g^{-1}(f^{-1}(x))$

$f^{-1}(x)$: $\frac{y+3}{2}$

$g^{-1}(x)$: $(y-5)^{1/3}$

$\therefore \left(\frac{y+3}{2} - 5\right)^{1/3}$

$= \left(\frac{y-7}{2}\right)^{1/3}$

$g^{-1}(f^{-1}(x)) = \left(\frac{x-7}{2}\right)^{1/3}$

$f\circ g = 2x^3 + 7$

$f\circ g^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$

01/31/2020

Countable Sets

Two sets A and B are said to be equivalent if there is a bijection F from A to B.

A set A is said to be countably infinite if A is equivalent to the set of natural numbers.

A is set said to be countable if it is finite or countably infinite.

Q) Now, prove that set of all integers ($\mathbb{Z}$) is countable.

$$f : N \to Z$$

$$f(N) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{1-n}{2} & n \text{ is odd} \end{cases}$$

Q) $A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, --- \right\}$

$f(n) = \frac{n}{n+1}$ one-one onto

$f(x) = f(y) \Rightarrow x = y \Rightarrow$ one-one

Q) Prove (Q) - set of rational numbers is countable.

$\frac{m}{n} \Rightarrow$ $m+n = 1$ Mapped to natural numbers.

$m+n = 2$

$m+n = 3$

$\vdots$

CANTOR DIAGONALIZATION

(0,1) $\{1, 2, 3, \ldots\}$

(0,2) some infinities are larger than other infinities

02/07/2020 OKAY OKAY

Thm: A subset of a countable set is countable.

Proof: Set A is countable and let $B \subseteq A$.

A is a countable infinite.
& Let $A = \{a_1, a_2, a_3, \ldots a_j\}$

Let b_{n_1} is the first element in B which is also in A. & b_{n_2} is the second element in B which is also in A.

$$B = \{b_{n_1}, b_{n_2}, \ldots, b_{n \ldots}\}$$

Thus, all the elements in B can be labelled by the elements of the natural number set. Therefore, B is countable.

Thm: $Q^+ \cup \{0\}$ is countable.

Proof: $\frac{a}{b}$ $\quad a+b=n$

$n=1$, $n=2$, $n=3$, $n=4$, $n=5$

$$\begin{array}{cccccc}
\frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\
 & \frac{2}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\
 & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

02/07/2020

Cantor diagonalization process

$f(1) = 0$ $\leftarrow$ $f(n)$ = number of elements in the n^{th} line

$f(2) = 1$

$f(3) = 1/2$

$f(4) = 2$ $\Leftarrow$ BIJECTIVE FUNCTION

$f(5) = 1/3$

$f(6) = 3$

$\Rightarrow \mathbb{Q}^+ \cup \{0\}$ is countable

$\Rightarrow \mathbb{Q}$ is countable

Countable union of countable sets is countable.

<u>Thm</u> 3: $\mathbb{N} \times \mathbb{N}$ is countable (x, y)

<u>Proof</u>:

$n=2$ $n=3$ $n=4$ $n=5$

$$\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & \dots \\
(2,1) & (2,2) & (2,3) & \dots \\
(3,1) & (3,2) & (3,3) & \dots \\
(4,1) & (4,2) & (4,3) & \dots
\end{array}$$

(a, b) such that $a + b = n$

$n = 1$ $f(1) = (1,1)$

$n = 2$ $f(2) = (2,1)$

$\vdots$ $f(3) = (1,2)$

$\vdots$

$\therefore$ Bijective f^n

$\Rightarrow$ countable.

02/07/2020

Thm: If the sets A and B are countable, then the sets $A \times B$ is also countable.

$n=2$ $n=3$ $n=4$ $(i+j=n \Rightarrow \text{cantor})$

Proof: (a_1, b_1) (a_1, b_2) (a_1, b_3) ... $(a_1, b_{n\infty})$
(a_2, b_1) (a_2, b_2) (a_2, b_3) ... $(a_2, b_{n\infty})$
⋮
(a_{m_∞}, b_1) (a_{m_∞}, b_2) .... $(a_{m_\infty}, b_{n\infty})$

Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$

Now, define:

$f : \mathbb{N} \times \mathbb{N} \to A \times B$
s/t : $f(i, j) = (a_i, b_j)$

and we have to prove f is one-one and onto:

Suppose $x = (p, q) \in \mathbb{N} \times \mathbb{N}$
$y = (u, v) \in \mathbb{N} \times \mathbb{N}$

Now, if $f(x) = f(y)$
$f((p, q)) = f((u, v))$
$\Rightarrow (a_p, b_q) = (a_u, b_v)$
$\Rightarrow a_p = a_u$ & $b_q = b_v$
$\Rightarrow p = u$ & $q = v$
$\Rightarrow x = y$

f is one-one

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To prove: f is onto

$(a_m, a_n) \in A \times B$.

Then $(m, n) \in (\mathbb{N} \times \mathbb{N})$
and $f(m,n) = (a_m, a_n)$.

Therefore, f is onto.

Q) 1) Prove: $\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}\}$ is countable.

Let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}\}$

$A \subseteq \mathbb{Q}$, ~~hen~~ and $\mathbb{Q}$ is a countable, so A is countable. because subset of a countable set is countable.

2) Prove: The set of all points (x, y) in the Euclidean plane with rational coefficients is countable.

The given set above is $\mathbb{P} = A \times B$, where $A = B = \mathbb{Q}$.

$\therefore \mathbb{P} = \mathbb{Q} \times \mathbb{Q}$.

If A and B are countable sets then $A \times B$ is countable.

since $\mathbb{Q}$ is countable, $\mathbb{Q} \times \mathbb{Q}$ is countable, hence, $\mathbb{P}$ is countable.

02/07/2020

Uncountable Sets

Thm: $[0,1]$ is uncountable.

Pf: Assume $[0,1]$ is countable.

$f_1 \to 0.a_{11}a_{21}a_{31}\dots a_{n1}$

$f_2 \to 0.a_{22}a_{22}a_{32}\dots a_{n2}$

$f_3 \to 0.a_{13}a_{23}a_{33}\dots a_{n3}$

We can always place two numbers between any two adjacent numbers. Hence, a mapping cannot be created. Hence, $[0,1]$ is not countable.

Thm: Any subset A of $\mathbb{R}$ which contains $[0,1]$ is uncountable.

Pf: Suppose A is countable. Then, any subset of a countable set must be countable.

$[0,1]$ is a subset of A. $[0,1]$ must be countable, however, it is uncountable. Hence this is uncountable.

Thm: $\mathbb{R}$ is uncountable.

Corollary: Set of irrational numbers is uncountable.

Pf: We know $\mathbb{Q}$ is countable.

$\mathbb{R} = \mathbb{Q} \cup$ irrational

$\mathbb{R}$ → uncountable, $\mathbb{Q}$ → countable, irrational → must be uncountable.

02/11/2020

Metric Space

The metric space is a non empty space M together with a function:

$$d: M \times M \to \mathbb{R} \quad \text{satisfying:}$$

(i) $d(x,y) \geq 0 \quad \forall \{x,y\} \in M$
(ii) $d(x,y) = 0 \quad$ iff $x = y \;\; \forall \{x,y\} \in M$
(iii) $d(x,y) = d(y,x) \quad \forall x,y \in M$
(iv) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in M$
(Triangle Inequality)

Then, d = metric/ distance function
& $d(x,y)$ is called distance between x & y.

Ex 1: In $\mathbb{R}$ with $d(x,y) = |x-y|$ Then d is a metric on $\mathbb{R}$. In fact, this is a usual metric.

modulus

(i) $d(x,y) \geq 0 \quad \forall \{x,y\} \in \mathbb{R}$
(ii) $d(x,y) = 0 \quad$ when $x = y$
$\Rightarrow |x-y| = 0$ when $x = y \quad \forall \{x,y\} \in \mathbb{R}$
(iii) $d(x,y) = |x-y| = |y-x| = d(y,x) \;\; \forall \{x,y\} \in \mathbb{R}$

(iv)
$$\begin{aligned} d(x,z) &= |x-z| \\ &= |x-y+y-z| \\ &\leq |x-y| + |y-z| \end{aligned}$$
$$d(x,z) \leq d(x,y) + d(y,z)$$

02/11/2020

Discrete Metric

On a non-empty set M, we define $d(x,y)$:

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Then d is a metric on M and this is called a discrete metric

To prove d is a metric: First three conditions are straightforward.

Case (i): $x = z$

$$d(x,z) = 0$$
$$d(x,y) + d(y,z) = 2d(x,y) \geq 0$$
$$2d(x,y) \geq d(x,z)$$

Case (ii): $x \neq z$

$$d(x,z) = 1$$
$$\underbrace{d(x,y) + d(y,z)}_{\text{Both cannot be zero at the same time}} \geq d(x,z)$$

\# In $\mathbb{R}^n$, if we define $d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

where $x = (x_1, x_2, \ldots x_n)$
$y = (y_1, y_2, \ldots y_n)$

02/11/2020

d is a metric on $\mathbb{R}^n$

This is a usual metric in $\mathbb{R}^n$

(i) $d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} \geq 0$

(ii) $d(x,y) = 0 \Rightarrow (x_i - y_i)^2 = 0$
$\Rightarrow x_i = y_i$

(iii) $d(x,y) = d(y,x)$ b/c
$(x_i - y_i)^2 = (y_i - x_i)^2$

Minkowski's Thm. Inequality (iv) $\left[\sum_{i=1}^{n} (x_i - z_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n} (y_i - z_i)^2\right]^{1/2}$

When $n=2$, it is called the Euclidean Metric.

Q) Determine whether $d(x,y) = (x-y)^2$ in $\mathbb{R}$ is a metric or not?

(i) $d(x,y) \geq 0$ b/c $a^2 \geq 0$ always

(ii) $d(x,y) = 0 \Rightarrow (x-y)^2 = 0 \Rightarrow (x-y) = 0$
$\Rightarrow x = y$

and vice versa

(iii) $d(x,y) = d(y,x)$ because
$(x-y)^2 = (y-x)^2$

not a metric.

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ not always necessarily true
$(x-z)^2 \leq (x-y)^2 + (y-z)^2$
$-2xz \leq 2y^2 - 2xy - 2yz$
$0 \leq y^2 + xz - xy - yz \Rightarrow 0 \leq (y-x)(y-z)$
$0 \leq y(y-z) - x(y-z)$

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Q) Let M be a non empty set together with the function $d: M \times M \to \mathbb{R}$ such that.

(i) $d(x,y) = 0$ iff $x = y$ $\forall\, x, y \in M$

(ii) $d(x,y) \le d(x,z) + d(\cancel{z,x})$ $\forall\, x, y, z \in M$.

$\cancel{d(z,x)}\ \cancel{d(z,y)}\ d(y,z)$

Prove that d is a metric

~~(i) $d(x,y) \geq 0$~~

① put $y = x$ in (ii)

$\cancel{d(x,x) \le d(x,z) + d(z,x)}$

$\cancel{d(x,y) \le d(x,y) + d(y,y)}$

$\cancel{0 \le d(y,y)}$

$d(x,x) \le d(x,z) + d(x,z)$

$0 \le 2d(x,z)$

$\therefore d(x,z) \geq 0$

② put $z = x$ in (ii)

$d(x,y) \le d(x,x) + d(y,x)$ *

$d(x,y) \le d(y,x)$

Replacing x & y in *, $\Rightarrow d(x,y) \geq d(y,x)$

$\Rightarrow d(x,y) = d(y,x)$.

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Bounded Set in a Metric Space

(M, d). We say that a set A of M is bounded if there exists a positive real number K such that $d(x, y) \leq K$

Thm: Any finite subset of a metric space (M, d) is bounded. Let A be any finite set of M. If A is empty, then it is bounded.

Pf: If A is non-empty:

$d(x, y)$ is a finite set of real numbers.

$$\{d(x,y) \mid x, y \in A\}$$

$$\text{Let } K = \max\{d(x,y) \mid x, y \in A\}$$

$$\Rightarrow d(x,y) \leq K \quad \forall (x, y) \in A$$

$\Rightarrow$ A is bounded.

~~Ex) Any finite subset A of a~~

Q) Let $A = [0, 1]$ in $\mathbb{R}$ and $d(x, y) = |x - y|$

$d(x, y) \leq 1$ (Bdd)

Q) $A = \{1, 3, 5, 7, 9\}$ in $\mathbb{R}$ $d(x, y) = |x - y|$

$d(x, y) \leq 8$

Q) $A = (0, \infty)$ in $\mathbb{R}$ $d(x, y) = |x - y|$

$\Rightarrow$ unbounded

but in $(0, \infty)$ Discrete Metric is bounded.

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Let $p \geq 1$, Let l_p denote the set of all sequence $\langle x_n \rangle$ s/t $\sum_{n=1}^{\infty} |x_n|^p$ is convergent.

Define $d(x,y) = \left[\sum_{n=1}^{\infty} |x_n - y_n|^p \right]^{1/p}$

$x = \langle x_n \rangle \quad y = \langle y_n \rangle$

The d is a metric space on l_p.

Ex: In l_2, let: $e_1 = (1, 0, 0, \dots 0 \dots)$
$e_2 = (0, 1, 0, \dots)$
$e_3 = (0, 0, 1, \dots)$
$\vdots$

let $A = \{e_1, e_2, \dots e_n, \dots\}$

Prove: A is a bounded subset of l_2

Proof:

$$d(e_n, e_m) = \begin{cases} \sqrt{2} & n \neq m \\ 0 & n = m \end{cases}$$

$$d(e_n, e_m) \leq \sqrt{2} \quad \forall e_n, e_m \in A.$$

Diameter

Infimum - Greatest Lower Bound (glb)
Supremum - Least Upper Bound (lub)

(i) $A = \{2, 4, 6, 8, 10\}$
glb (infimum) = 2
lub (supremum) = 10

(ii) $A = \{x \mid 0 < x < 1\}$
glb (infimum) = 0
lub (supremum) = 1

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Diameter: Let (M, d) be a metric space and $A \subset M$, then the diameter of A, denoted by $d(A)$:

$$d(A) = l.u.b \{ d(x,y) \mid x, y \in A \}$$
distance

(D) # A non empty set A is bounded if and only if A is finite.

Let $A, B \subseteq M$ and $A \subset B$, then $d(A) \leq d(B)$

In the case of a discrete metric space, diameter of any non empty subset is equal to 1.

Q) In $\mathbb{R}$, the diameter of any interval is equal to the length of the interval. Find the diameters for the following sets:

(i) $A = \{1, 3, 5, 7, 9\}$
$= 8$

(ii) $A = \{0, 1, 2, \ldots 100\}$
$= 100$

(iii) $A = [-3, 5]$ in $\mathbb{R}$
$= 8$

(iv) $[-\frac{1}{2}, \frac{1}{2}]$ in $\mathbb{R}$
$= 1$

(v) $\mathbb{N}$ infinite

(vi) $\mathbb{Q}$ infinite

(vii) $[1, 2] \cup [5, 6]$ in $\mathbb{R}$
$6 - 1 = 5$

(viii) $[3, 6] \cap [4, 8]$
2

02/13/2020

Open Ball or Open Sphere $a \in M$

Let (M, d) be a metric space. Let ~~a ∈ A~~ and r be a +ve number $\in \mathbb{R}$ then the open sphere is defined with center a and radius r as:

$$B_d(a, r) = \{x \in M \mid d(a, x) < r\}$$

↳ usual metric

- $B(a, r)$ is non empty. Point a will always exist.

- $B(a, r)$ is bounded.

Let $x, y \in B(a, r)$

$d(a, x) < r$ & $d(a, y) < r$

$d(x, y) \leq d(x, a) + d(a, y)$

$\leq 2r$

Ex1) $B(a, r)$ in $\mathbb{R}$ with usual metric

$$B(a, r) = \{x \in \mathbb{R} \mid d(a, x) < r\}$$
$$= \{x \in \mathbb{R} \mid |a - x| < r\}$$
$$= \{x \in \mathbb{R} \mid a - r < x < a + r\}$$

In case of $\mathbb{R}^2$ / complex numbers, this (open ball) is ~~the~~ an open circle.

Q) In case of a discrete metric space M, find the open ball $B(a, r)$.

$$B(a, r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \leq 1 \end{cases} \qquad d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

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Q) Let $M = [0,1]$, with the usual metric $d(x,y) = |x$
Find $B(0, \frac{1}{2})$

$$B(0, \tfrac{1}{2}) = \{ x \mid x \in [0, \tfrac{1}{2}) \}$$

Subspace

Let (M, d) be a M.S. and $M_1 \subseteq M$, $M_1 \neq \phi$, ~~then~~ and M_1 also is a M.S. with the same distance d, then M_1 is called a subspace of M.

Note: If M_1 is a subspace of M, a set which is open in M_1 need not be open in M.

Thm: Let M be a metric space and M_1 is a subspace of M. Let $A_1 \subset M_1$. Then A_1 is open in M_1 iff $\exists$ an open set A in M s/t $A_1 = A \cap M_1$.

Ex 1: Let $M = \mathbb{R}$ and $M_1 = [0,1]$.
Let $A_1 = [0, 1/2) = (-1/2, 1/2) \cap [0,1]$
$\Rightarrow [0, 1/2)$ is open in $[0,1]$

Ex 2: Let $M = \mathbb{R}$ and $M_1 = [1,2] \cup [3,4]$
Let $A_1 = [1,2] = (0, 5/2) \cap M_1$
$\Rightarrow A = (0, 5/2)$
$\Rightarrow A_1$ is open in $[1,2] \cup [3,4]$

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$[0,1] \cup [2,5]$
$(0,1) \cup (2,5)$

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Interior of a Set

Let (M,d) be a metric space. Let A is a subset of M.
$A \subseteq M$.

Let $x \in A$, then x is said to be an interior of A if $\exists\, r > 0,\ r \in \mathbb{R}^+$ s/t

$$B(x,r) \subseteq A.$$

ex: $M = \mathbb{R}$
$A = [0,1]$. Find the set of all interior points in A.
$\hookrightarrow (0,1)$ $\{0,1\}$ are on the edges (not interior).

ex: $M = \mathbb{R}$.
$A = \mathbb{Q}$
Interior points $\Rightarrow \phi$.
~~Set of all Int~~ $Int(A) = \phi$.

ex: $M = \mathbb{R}$
$A = \{0, 1, 1/2, 1/3, 1/4, \ldots 1/n\}$
$Int(A) = ? = \phi$.

ex: Consider $\mathbb{R}$ with a discrete metric.
$A = [0,1]$
$Int(A) = [0,1]$

REVISE

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Thm: Let (M, d) be a metric space. Let $A, B \subseteq M$.

(i) A is open iff $Int(A) = A$.
In particular, $Int(\phi) = \phi$
$Int(M) = M$.

(ii) $Int(A)$ = Union of all the open sets in A

(iii) $Int(A)$ is an open subset of A and if B is any other open set contained in A, then $B \subseteq Int\, A$.

(iv) $Int(A)$ is the largest open set in A.

(v) $A \subseteq B \Rightarrow Int(A) \subseteq Int(B)$

(vi) $Int(A \cap B) = Int(A) \cap Int(B)$

(vii) $Int(A \cup B) - Int(A) \cup Int(B)$.

$(0,1)$

02/20/2020

Closed Set

Let (M,d) be a metric space and $A \subseteq M$. A is said to be closed iff A^c is open.

ex: In $\mathbb{R}$, every $A = [a,b]$ is closed.

$$A^c = (-\infty, a) \cup (b, \infty) \;\Rightarrow\; A \text{ is closed.}$$

$\downarrow$ open $\quad$ open $\cup$ open $=$ open.

ex: $M = \mathbb{R}$

$A = [a,b) \longrightarrow$ neither open nor closed

$$A^c = (-\infty, a) \cup [b, \infty)$$

$\uparrow$ open $\quad$ $\uparrow$ closed $\Rightarrow$ can't say.

$\hookrightarrow$ neither

$$\boxed{A \text{ not closed} \Rightarrow A^c \text{ not open}}$$

ex: $M = \mathbb{R}$

$A = \mathbb{Q}$

$\cancel{A^c = \bar{\mathbb{Q}}}$ $\qquad\qquad$ $A^c = \mathbb{R} \setminus \mathbb{Q}$

irrational $\rightarrow$ ~~closed~~

A^c is not open

$\Rightarrow \mathbb{Q}$ is not closed.

ex: $M = \mathbb{R}$

$A = \mathbb{Z}$

A is closed $\qquad A = \{\dots, -1, 0, 1, 2 \dots\}$

A^c is open $\qquad (-\infty, -\text{int}) \dots (-1, 0)$

$$A^c = \bigcup_{n=-\infty}^{\infty} (n, n+1)$$

M . M } Both open and closed
ϕ . ϕ }

02/20/2020

ex: In $M = \mathbb{R}$, usual metric
$A = \{x\}$.
⇒ Every singleton

ex: Every subset of a discrete metric space is both closed and open

Let (M, d) be a metric space and $A \subseteq M$.
Consider A^c in M.
A^c is a

Closed Ball
⇒ $M = \mathbb{R}$
$\hookrightarrow B_d(x, r) = \{a \in \mathbb{R} \mid d(a, x) \leq r\}$

In $\mathbb{R}^2$
$B_d(x, r) = \{a \in \mathbb{R}^2 \mid d(a, x) \leq r\}$
↳ circle (edge + interior).

Thm: In any MS, a closed ball is a closed set.

- Arbitrary union of open sets is open
- Arbitrary intersection of open sets is open

02/20/2020.

Thm: In any metric space, arbitrary intersection of closed sets is closed.

$$A = \bigcap_{i \in I} A_i$$

$$A^c = \left(\bigcap_{i \in I} A_i\right)^c = \underbrace{\bigcup_{i \in I} A_i^c}_{\text{open}}$$

$\Rightarrow A$ is closed.

Thm: Finite union of closed set is closed.

$$A_n = \left[\frac{1}{n}, 1\right]$$

$$= \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right]$$

$$= (0, 1]$$

A_n is not closed.

* Arbitrary union ~~it~~ may not be closed *