PHY315 Term Project

Acoustic Analogue of Galactic Scale Gravitational Wave Detector

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Aim: To demonstrate an acoustic analogue of a galactic scale gravitational gravitational wave detectors, i.e., a pulsar array.

Theory: A pulsar is a highly magnetic rotating compact star. The magnetic field of a pulsar is billions of times stronger than that of the earth. Due to this strong magnetic field, pulsars emit radio waves along their magnetic axis as they rotate about their spinning axis. If it so happens that the earth is in the region where the pulsar's emitted radiation is visible, this appears to us as a pulsing object in the sky, with a pulse frequency equal to the rotational frequency of the pulsar. Millisecond pulsars have a time period of one millisecond, and thus a pulse frequency of 1 KHz. The pulsars spin extremely consistently, with their temporal consistency rivalling that of the best atomic clocks.

However, a gravitational wave passing between the Earth and a pulsar will stretch and squeeze space transverse to its motion, slightly advancing or retarding the arrival times of the individual pulses. Since a gravitational wave is not local to a pulsar but rather affects the entire vicinity of the earth, all the pulsars in a pulsar array will have their arrival times slightly advanced or retarded. If we plot the timing residuals, i.e. the difference between measured and expected arrival times of the pulsar, it will behave like a high frequency wave modulated by a low frequency wave, the high frequency corresponding to the pulse frequency and the low frequency corresponding to the gravitational wave frequency. If we compute the cross correlation of two such timing residual plots from two different pulsars, it has very well defined dependence on the angle between the lines joining the earth to the pulsars, the earth pulsar baselines, given by the Hellings and Downs correlation curve. A figure describing such a pulsar timing array is given below.

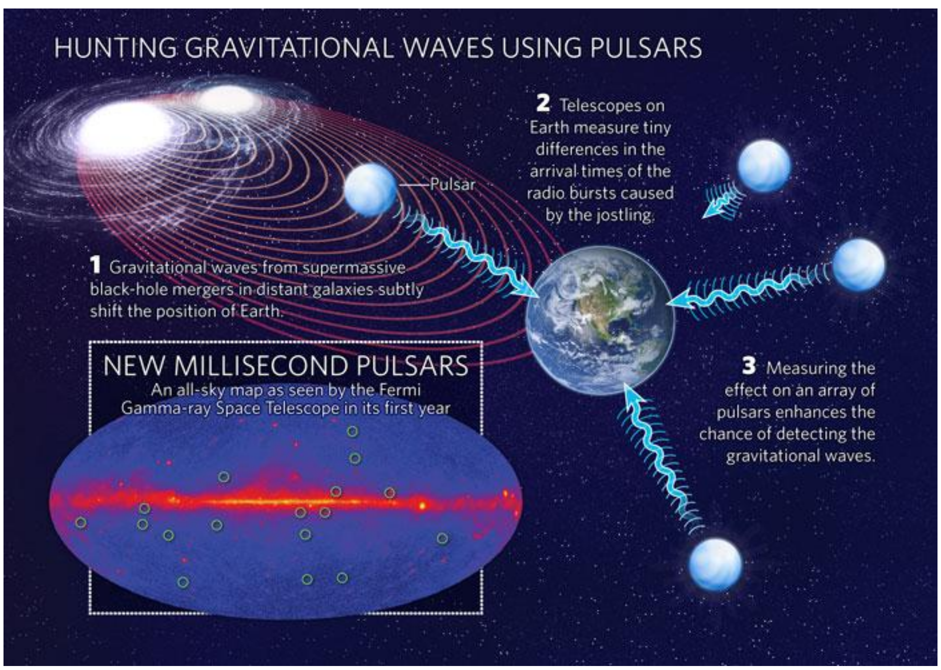

Fig. 1: Pulsar Timing Array [1]

In this experiment, we use a microphone-metronome system to simulate a pulsar timing array and demonstrate how, using a sufficiently detailed model, the existence of gravitational waves, or it's acoustic analogue, can be predicted.

It is sufficient to use one microphone and two metronomes, since we only need to observe the cross correlation as a function of the angle between two microphone-metronome baselines. In our analysis, we used metronomes of 200 bpm and 120 bpm frequencies respectively. First, we discuss the process to obtain pulse profiles and time periods, which is done using single metronome analysis.

(i) Single Metronome Analysis: In this setup, we place a metronome about 1 metre away from a microphone. Then, we turn the metronome on and record a time series audio signal. The time series audio signal will have some periodic background noise due to the motor, which can be removed using a noise reduction algorithm called noisereduce [(2)]. The single metronome signals of 200 bpm and 120 bpm metronomes that we used in our analysis are given below:

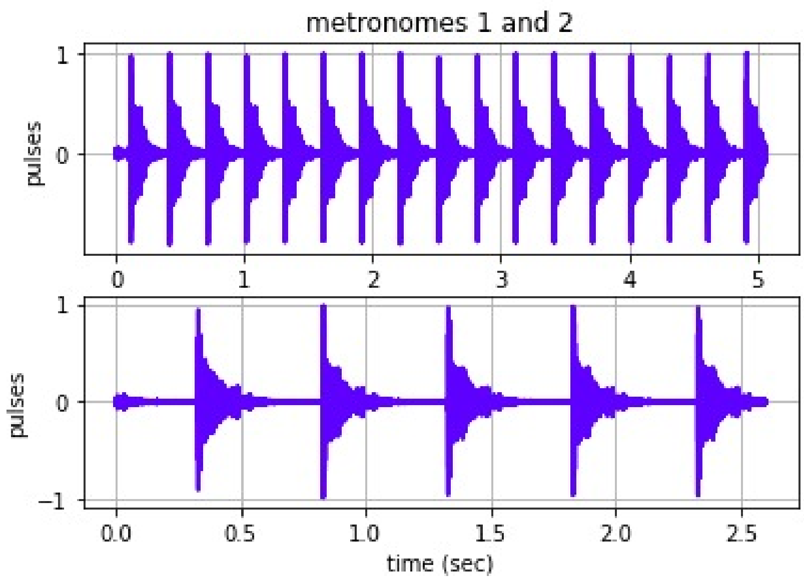

Fig. 2: Single Metronome Signals of (Top) 200 bpm and (Bottom) 120 bpm

From the single metronome signal data, we need to extract the time period and pulse profile of the metronome. This is done by a technique called folding data. Suppose we record data for a time T and guess a time period T_p. Then, we split the T second audio into bins of size T_p, and take the time average of all such bins. Now, if the bin size is not equal to the actual time period of the metronome, the resulting average will look just like a noise signal with no distinct local maximum. However, if the bin size is close the the actual time period of the metronome, the average will have a distinct peak at the pulse location. An example of this as done in the reference paper is given below:

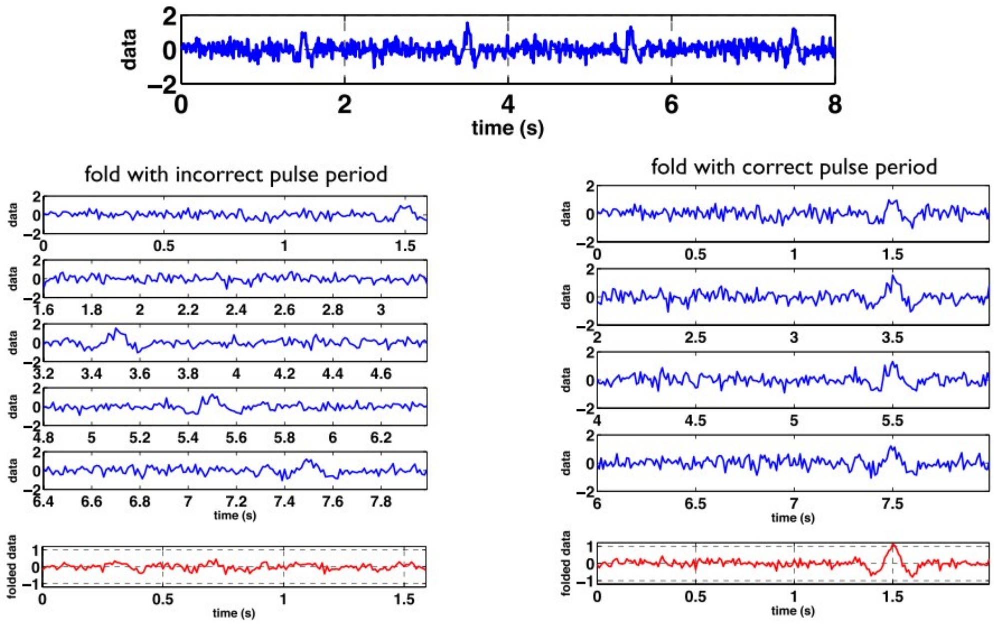

Fig. 3: (Top) Input data, (Bottom Left) Profile with incorrect pulse period, (Bottom Right) Profile with correct pulse period [3]

Using an algorithm that plots the amplitude at the local maximum of the average profile as a function of bin size, we locate the largest amplitude at local maximum. The bin size at which this occurs is the time period of the metronome pulses, and the profile so obtained (restricted to the pulse itself) is the pulse profile of the metronome. Thus, we have obtained both the pulse profile and time period using data folding. The pulse profiles that we used in our analysis are given below:

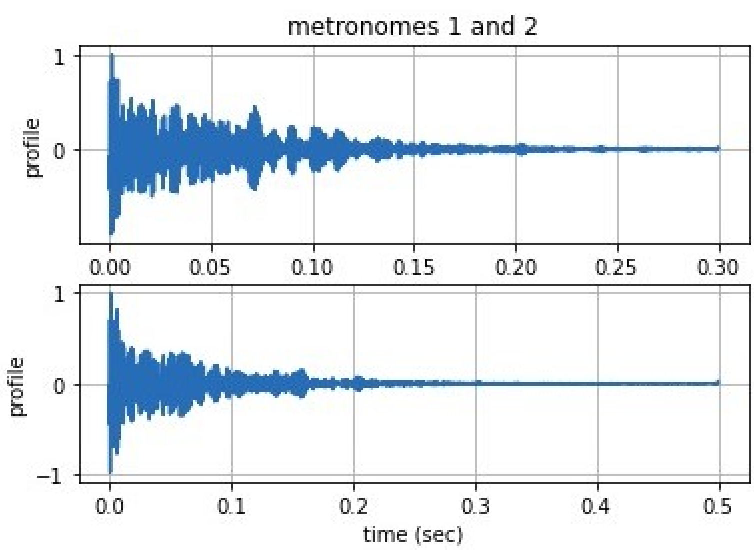

Fig. 4: Pulse Profiles of (Top) 200 bpm and (Bottom) 120 bpm

Note that since the algorithm only looks at a finite number of possible time periods, the above time period is subject to an error. This error will be dealt with later, in the double metronome analysis.

Now suppose we place the microphone on a turntable with A = 10 cm from the rotating axis of the turntable, which for our analysis shall be fixed as the origin. Then, we place a metronome 1 m from the origin, and rotate the microphone. The path length that needs to be travelled by the pulse profiles changes with time as the microphone rotates. Due to this, in some cases a pulse arrives more than T_p after the previous pulse, and in other cases the pulse arrives less than T_p after the previous pulse. After a pulse, the expected time of arrival of the next pulse is T_p. Suppose our reference pulse is the indexed by i_0 and the pulses are indexed as i. If τ_e is the expected pulse arrival time and τ_m is the measured pulse arrival time, we define the ith timing residual $\delta\tau[i]$ as:

$$\delta\tau[i] = \tau_m[i] - \tau_e[i]$$

But, if t = 0 when the i_0th pulse arrives,

$$\tau_e[i] = T_p(i - i_0)$$

Understanding how $\tau_m[i]$ behaves allows us to predict the timing residuals precisely. A diagram of the situation is shown below:

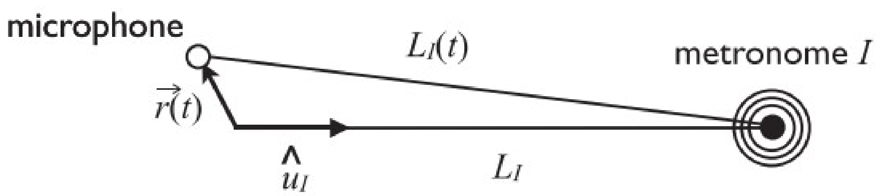

Fig. 5: Metronome placed in front of a rotating microphone[3]

Although in the experiment we can only obtain a discrete amount of pulses, for the analysis we may assume that the metronome produces a continuous amount of pulses (so that the residues can be plotted as continuous functions of time). Then, the residue at the time t is given as

$$\delta\tau = \frac{\Delta L_I}{c_s} = \frac{L_I(t) - L_I}{c_s}$$

where c_s is the speed of sound. Since we have that $\frac{L_I}{A} \approx 0.1$, we may assume that $L_I(t) - L_I = -\vec{r}(t).\hat{u_I}$. If the frequency of microphone rotations is f_0, then

$$\vec{r}(t) = A\cos(2\pi f_0 t + \phi_0)\hat{x} + A\sin(2\pi f_0 t + \phi_0)\hat{y}$$

If the metronome is placed at an angle θ_I w.r.t. the x-axis, then $\hat{u_I} = \cos\theta_I\hat{x} + \sin\theta_I\hat{y}$. Thus

$$\Delta L \approx -\vec{r}(t).\hat{u_I} = -A\left(\cos(2\pi f_0 t + \phi_0)\cos\theta_I + \sin(2\pi f_0 t + \phi_0)\sin\theta_I\right)$$

$$\therefore \Delta L = -A\cos(2\pi f_0 t + \phi_0 - \theta_I)$$

Hence, we get that

$$\delta\tau(t) = -\frac{A}{c_s}\cos(2\pi f_0 t + \phi_0 - \theta_I)$$

Suppose we have two metronomes at angles θ_1 and θ_2 respectively such that $\theta_1 - \theta_2 = \alpha$. If we denote the timing residuals for these two as $\delta\tau_1(t)$ and $\delta\tau_2(t)$ respectively, then

$$\delta\tau_1(t) = -\frac{A}{c_s}\cos(2\pi f_0 t + \phi_0 - \theta_1)$$

$$\delta\tau_2(t) = -\frac{A}{c_s}\cos(2\pi f_0 t + \phi_0 - \theta_2)$$

The normalised cross correlation of the two ρ_{12}, given as

$$\rho_{12} = \frac{\langle\delta\tau_1\delta\tau_2\rangle}{\sqrt{\langle\delta\tau_1^2\rangle\langle\delta\tau_2^2\rangle}}$$

where the time averaging is done over a period T, such that

$$\langle\delta\tau_1^2\rangle = \frac{1}{T}\int_0^T \delta\tau_1^2 dt$$

$$\langle\delta\tau_2^2\rangle = \frac{1}{T}\int_0^T \delta\tau_2^2 dt$$

$$\langle\delta\tau_1\delta\tau_2\rangle = \frac{1}{T}\int_0^T \delta\tau_1\delta\tau_2 dt$$

Thus, ρ_{12} takes values between -1 and 1. In this specific case, computing the integrals we get that

$$\langle\delta\tau_1^2\rangle = \langle\delta\tau_2^2\rangle = \frac{A^2}{2c_s^2}$$

Since a cosine function averages out to $\frac{1}{2}$. Using the identity

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

we get

$$\delta\tau_1\delta\tau_2 = \frac{A^2}{2c_s^2}\left(\cos(4\pi f_0 t + 2\phi_0 - \theta_1 - \theta_2) + \cos(\theta_1 - \theta_2)\right)$$

Taking the time average, we get

$$\langle\delta\tau_1\delta\tau_2\rangle = \frac{A^2}{2c_s^2}\cos(\theta_1 - \theta_2)$$

Thus, we get the cross correlation coefficient ρ_{12} as

$$\rho_{12} = \cos(\theta_1 - \theta_2) = \cos\alpha$$

This elegant dependence of the cross correlation of timing residuals on the angle between microphone metronome baselines is what we measure through this experiment. With the theory built up, the final part of the experiment is the double metronome analysis.

(ii) Double Metronome Analysis: In this setup, we place the microphone on a rotating turntable, with $A \approx 10cm$, and we fix the axis of rotation of the turntable as the origin. Then, we place a metronome, metronome 1, at a distance of 1 m from the origin. The metronome 1 - microphone baseline is the x-axis. Then, we place another metronome making an angle α w.r.t. the x-axis. A diagram of the setup is shown below:

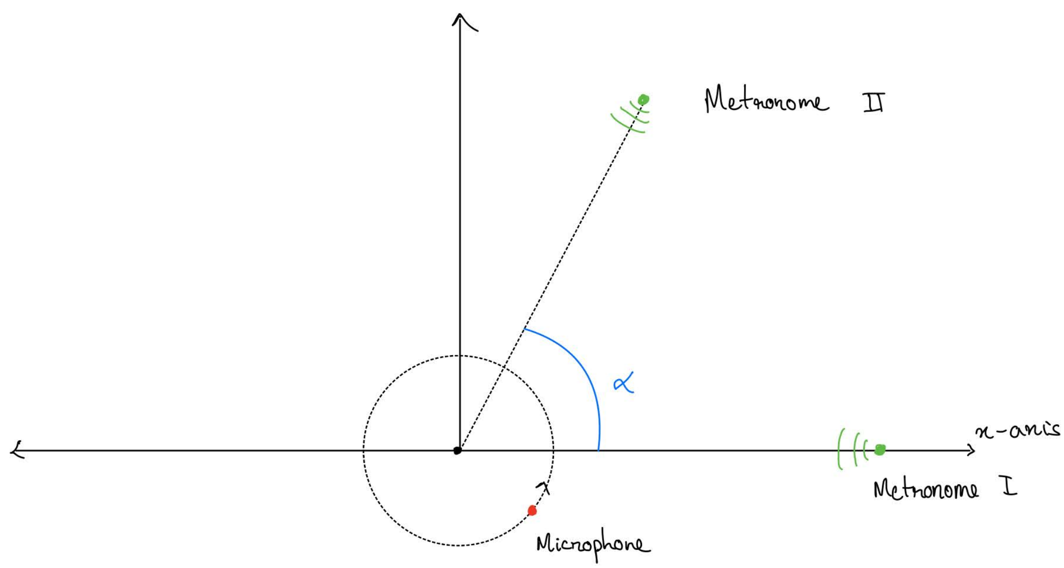

Fig. 6: Double Metronome Analysis Setup

Then, we turn on the turntable and rotate the microphone, and turn on both the metronomes simultaneously. We then record the double metronome input signal, and use it to deduce the measured times of arrival of pulses. Note that it is better if we use metronomes of two different frequencies, and also two different sound types, since then the pulse profiles will be distinct, and thus can be resolved. A double metronome signal for $\alpha = 135°$ that we used in our analysis is given below:

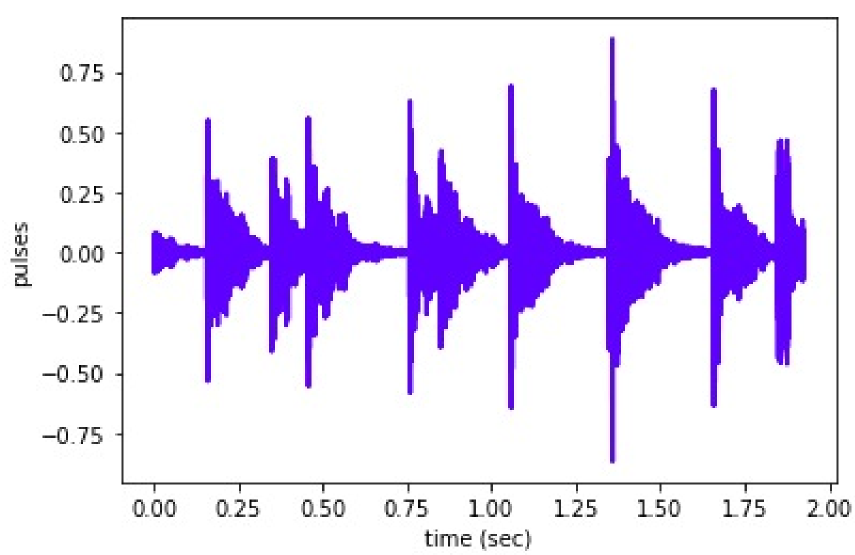

Fig. 7: Double Metronome Analysis Signal, $\alpha = 135°$

In order to locate the measured times of arrival of metronome 1, we cross-correlate the pulse profile of metronome 1 propagated forward in time with the double metronome signal. Suppose T is the length of the recorded signal. Thus, if p(t) is the pulse profile time series and y(t) is the double metronome time series, we define for any ΔT such that $0 \leq \Delta T \leq T$, we get

$$C(\Delta T) = N \int_0^T y(t)p(t - \Delta T)dt$$

We plot $C(\Delta T)$ as a function of time, and the local maxima of $C(\Delta T)$ correspond to points where the pulse profile has a high correlation with the sound signal. These points of local maxima correspond to the measured times of arrival of the pulses from metronome 1. Similarly, by cross correlating the pulse profile of metronome 2 propagated forward in time with the double metronome signal, we can obtain the measured times of arrival of the pulses from metronome 2.

In order to measure the timing residuals, in case of metronome 1, we set the expected time of arrival of the first pulse in the signal equal to the measured time of arrival. Then, for each consecutive pulse, we set the expected time of arrival to be $(i-1)T_p$, where i is the index of the pulse and T_p is the time period of metronome 1 obtained through single metronome analysis. Then, we compute the timing residuals as

$$\delta\tau[i] = \tau_m[i] - \tau_e[i]$$

However, since the single metronome analysis produces errors in the time period, this error propagates into the timing residuals. The timing residuals are themselves only about 1% of the time period error. However, this error, if assumed to be small, propagates linearly into the timing residuals. If the measured time period is $T'_p = T_p(1+\epsilon)$, then

$$\delta\tau'[i] = \tau_m[i] - (i-1)T_p - \epsilon(i-1)T_p = \delta\tau[i] - \epsilon(i-1)T_p$$

We can fit a line to the timing residuals, and remove the linear trend. This will produce an improved estimate of the time periods, and the timing residuals will now behave as expected. Finally, by the above analysis we expect the timing residuals to fit a sinusoid. Thus, we fit a sinusoid to the timing residuals. The above analysis is then identical for metronome 2. A plot of the fitted timing residuals when $\alpha = 90°$ used in our analysis is given below:

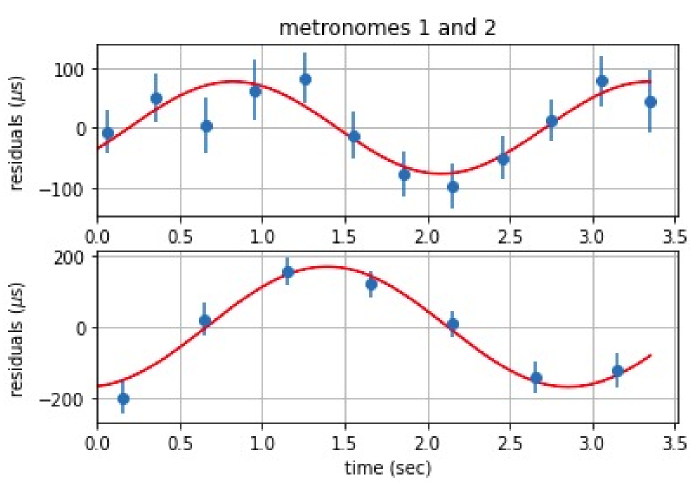

Fig. 8: Fitted residuals at $\alpha = 90°$, (Top) 200 bpm and (Bottom) 120 bpm

We can cross correlate the fitted sinusoids for different values of the angle between the metronome-microphone baselines in order to obtain a plot of ρ_{12} w.r.t. the angle α.

Procedure:

(i) Single Metronome Analysis:

1. Place the microphone in a stationary position.
2. Place the metronome about 1 m from the microphone.
3. Turn on the metronome and record single metronome analysis signal.
4. Proceed with the analysis as above to produce the plot profile and time period of the metronome.
5. Repeat the above procedure for both the 200 bpm and 120 bpm metronomes.

(ii) Double Metronome Analysis:

1. Construct a turntable of radius 10 cm using a motor and other available resources. The motor is powered by household DC batteries. We have observed that the rpm of the motor does not affect the analysis as long as the metronome frequency to motor frequency ratio is greater than 10.
2. Place a metronome at a distance of about 1 m from the axis of rotation of the turntable.
3. Place the second metronome at a known angle α w.r.t. the first metronome- microphone baseline, again at a distance of about 1 m from the axis of rotation.
4. At this angle, record the double metronome analysis signal.
5. Proceed with the analysis as above to obtain the cross correlation ρ_{12} at this angle α.
6. Repeat the above process for different values of α. We have chosen the values $\alpha = 0°$, $45°$, $90°$, $135°$ and $180°$.
7. Plot the Cross correlation ρ_{12} as a function of the angle α.

Results:
A few plots used in the analysis have already been presented above. We show the relevant plots for $\alpha = 0°$.

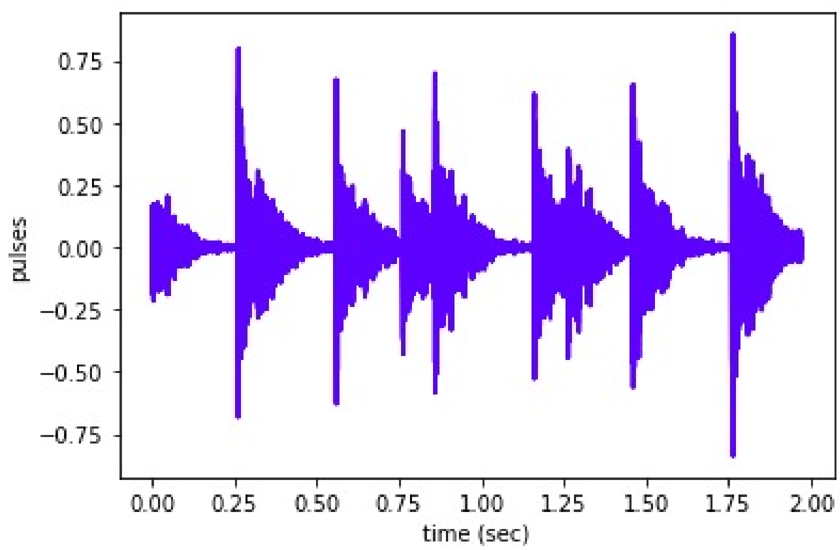

Fig. 9: Double Metronome Analysis Signal, $\alpha = 0°$

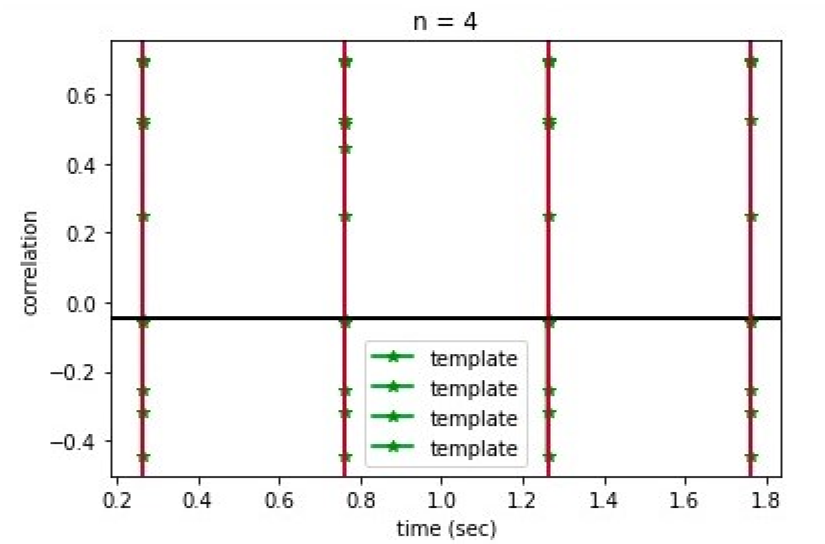

Fig. 10: Measured time of arrivals at $\alpha = 0°$, 120bpm

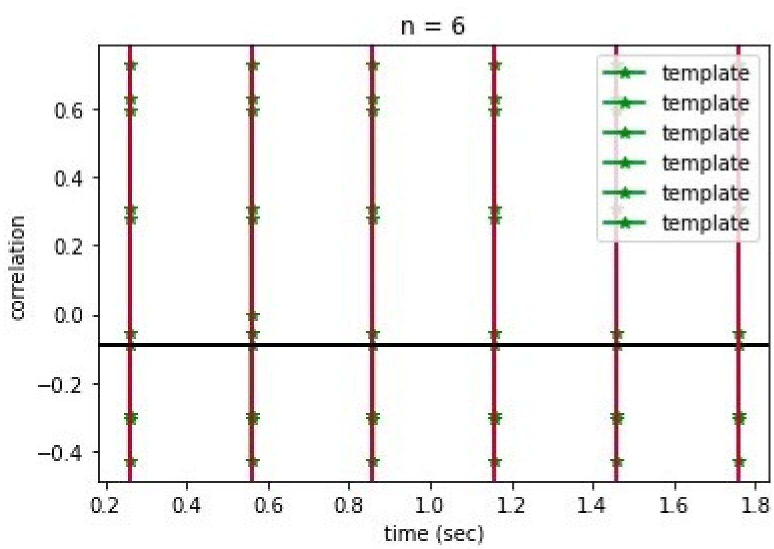

Fig. 11: Measured time of arrivals at $\alpha = 0°$, 200bpm

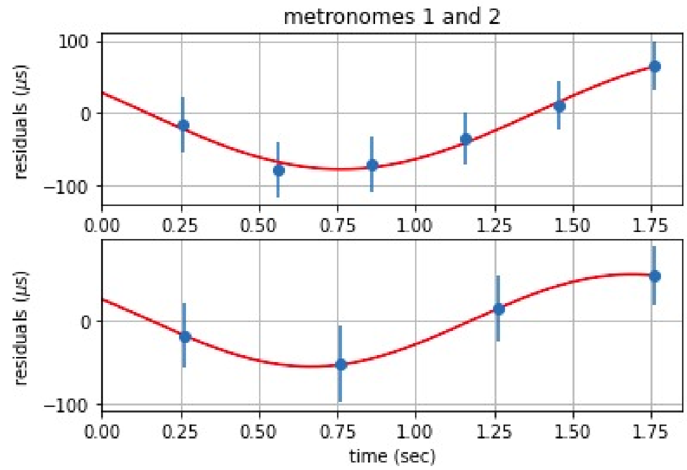

Fig. 12: Fitted residuals at $\alpha = 0°$, (Top) 200 bpm and (Bottom) 120 bpm

Using the analysis methods described above, we get the plot of the normalized correlation coefficient ρ_{12} w.r.t. the angle α as:

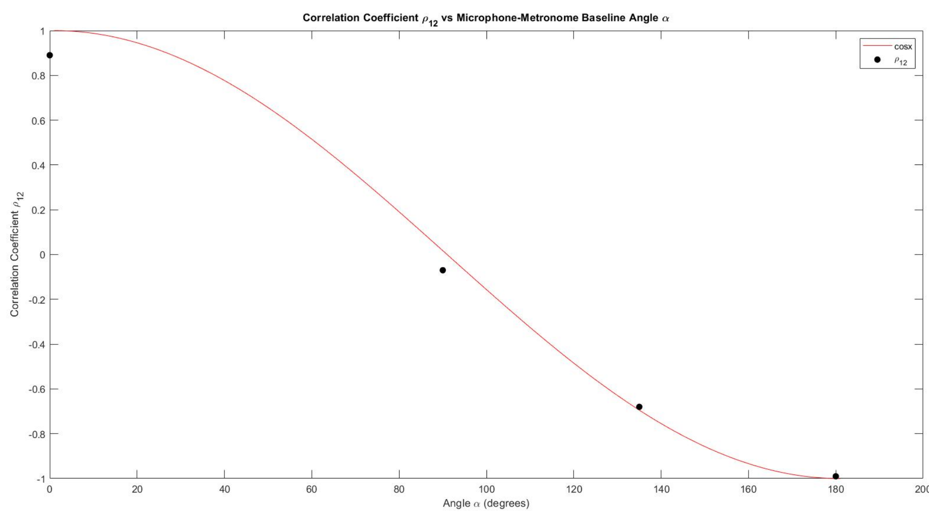

Fig. 13: Cross Correlation Coefficient ρ_{12} versus angle α

Clearly, the values of ρ_{12} obtained experimentally fit well with the expected values through the above theory.

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Bibliography:

(1) NASA/DOE/Fermi LAT Collaboration via Nature.
(2) https://github.com/timsainb/noisereduce
(3) American Journal of Physics 86, 755 (2018); doi: 10.1119/1.5050190