INVESTMENT ANALYSIS USING MODERN PORTFOLIO THEORY

CLUSTER INNOVATION CENTRE

(UNIVERSITY OF DELHI)

B.TECH IV.1

DOES NATURE PLAY DICE? : THE AMAZING WORLD OF PROBABILITY AND STATISTICS

GUIDE

Dr. Shashi Aggarwal, Associate Professor
Dr. Shivantika Sharad, Assistant Professor
Mr. Umesh Kumar, Assistant Professor

SUBMITTED BY

Sahil Nakul Mathur (101)
Nikita Garg (1667)
Rahul Yadav(1387)

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Mathematical finance is a field of applied mathematics, concerned with financial markets. Generally, mathematical finance will derive and extend the mathematical or numerical models without necessarily establishing a link to financial theory, taking observed market prices as input. The term portfolio refers to any collection of financial assets such as stocks, bonds, and cash. Portfolios may be held by individual investors and/or managed by financial professionals, hedge funds, banks and other financial institutions. It is a generally accepted principle that a portfolio is designed according to the investor's risk tolerance, time frame and investment objectives. The monetary value of each asset may influence the risk/reward ratio of the portfolio and is referred to as the asset allocation of the portfolio.

Source: www.wikipedia.org

The aim of the project is to build a virtual portfolio consisting of real world assets. Before constructing a portfolio, various concepts of probability and finance are understood. The real world data of three assets are taken to calculate the expected rate of return of the individual asset and then, the covariance is computed so as to combine the three assts. After that, random values of weights are generated which represents the fraction of money to be invested in each of the assets to compute a random value of expected return of the portfolio and expected risk of the portfolio. A curve of these random portfolios is plotted on the mean-standard deviation graph along with the efficient frontier. The point of intersection of the two curves, where the portfolio risk is minimum, is obtained and the corresponding value of portfolio return is calculated for the given portfolio.

CHAPTER 1: MODERN PORTFOLIO THEORY

INTRODUCTION

Modern portfolio theory is a theory of finance that attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently minimize risk for a given level of expected return, by carefully choosing the proportions of various assets. Before constructing a portfolio using real data, it is important to understand various concepts used for constructing an efficient portfolio. Thus, this chapter explains various concepts of portfolio management theory.

Generally, when an investment is made, the initial amount of money invested is known but the amount of money to be returned is uncertain. This uncertain amount of money can be treated using probabilistic theory of mean-variance analysis. Before understanding the mean variance approach, it is important to understand few basic concepts of probability and financial market.

1. ASSET

An investment instrument that can be bought and sold frequently is called an asset. [1]

1.1.1 Asset Return

For an asset purchased at time $t = 0$ and sold at time $t = t$, the total return of the asset is defined to be: [2]

$$total\ return\ (R) = \frac{amount\ received}{amount\ invested}$$

And the rate of return is: [3]

$$rate\ of\ return\ (r) = \frac{amount\ received - amount\ invested}{amount\ invested}$$

If X_0 and X_1 are, respectively, the amount of money invested and received in an investment, then [4]

$$R = \frac{X_1}{X_0}$$

And

$$r = \frac{X_1 - X_0}{X_0}$$

- ⇨ $R = 1 +$
- ⇨ $X_1 = (1 + r)X_0$

2. SHORT SALES

Selling an asset by someone not owned by that person is known as short selling. Suppose you borrow an asset from someone and sell the borrowed asset to receive an amount X_0. Later, you repay your loan by purchasing the asset for X_1 and returning it to the lender. If the later amount X_1 is less than the amount received initially, X_0, then a profit of X_0 - X_1 is made. [5]

1.2.1 Total Return

Let X_0 be the amount received at $t = 0$ and X_1 be the amount of money invested at $t = t$, then total return is defined to be: [6]

$$R = \frac{-X_1}{-X_0}$$

$$= \frac{X_1}{X_0}$$

3. PORTFOLIO RETURN

Portfolio return is defined as the monetary return experienced by the owner of the portfolio. [7] Suppose there are n assets available in the market. A master asset or portfolio can be constructed with these n assets by apportioning an amount X_0 among these assets. Let X_{oi}, I = 1, 2,…, n denote the amount of money invested in the i^{th} asset such that $\sum_{i=1}^{n} X_{0i} = X_0$. In case of short selling, some of the X_{oi} are negative as money is received and not invested at $t = 0$.

The amount of money invested in the i^{th} asset can be written as fractions of the total investments. Thus, [8]

$$X_{0i} = w_i X_0$$

I = 1, 2… n

Where w_i is the weight of the i^{th} asset in the portfolio such that,

$$\sum_{i=1}^{n} w_i = 1$$

Let R_i denote the total retun from the i^{th} asset. Thus, the amount of money received from the i^{th} asset at the end of the period is $R_i X_{0i} = R_i w_i X_0$. The total amount received from the portfolio is therefore, $\sum_{i=1}^{n} R_i w_i X_0$. Hence, the overall total return of the portfolio is given by:

$$R = \frac{\sum_{i=1}^{n} R_i w_i X_0}{X_0}$$
$$= \sum_{i=1}^{n} R_i w_i$$

Equivalently,

$$r = \sum_{i=1}^{n} r_i w_i$$

4. RANDOM VARIABLE

Random variable is a variable whose value is unknown or a function that assigns values to each of an experiment's outcomes. Random variables are often designated by letters and can be classified as discrete, which are variables that have specific values, or continuous, which are variables that can have any values within a continuous range. [9]

5. EXPECTED VALUE

The expected value of a random variable *x* is the average value obtained by regarding the probabilities as frequencies. For a case of finite number of possibilities, it is defined as: [10]

$$E(x) = \sum_{i=1}^{m} x_i p_i$$

E(x) is often denoted by $\bar{x}$ and often called as **mean** or **mean value**.

1.5.1 Properties of expected value [11]

- **Certain value**
  - If y is a known value, then $E(y) = y$
  - This states that the expected value of a non-random quantity is equal to the quantity itself.
- **Linearity**
  - If y and z are random, then $E(\alpha y + \beta z) = \alpha E(y) + \beta E(z)$ for any real values of α and β.
- **Non-negativity**
  - If x is a random variable but never less than zero, then $E(x) \geq 0$.

6. VARIANCE

Variance is a measure of the dispersion of a set of data points around their mean value. It is a mathematical expectation of the average squared deviations from the mean. [12]

In general, for any random variable *y* the variance of *y* is defined as: [13]

$$var(y) = E[(y - \bar{y})^2]$$

In mathematical expressions, variance is represented by the symbol σ^2.
The square root of variance, denoted by σ, is called the standard deviation. Thus,

$$\sigma_y = \sqrt{E[(y - \bar{y})^2]}$$

7. COVARIANCE

Covariance is a measure of the degree to which returns on two risky assets move in tandem. A positive covariance means that asset returns move together. A negative covariance means returns move inversely. [14]

Let x_1 and x_2 be two random variables with expected values $\overline{x_1}$ and $\overline{x_2}$. The covariance of these variables is defined to be: [15]

$$cov(x_1, x_2) = E[(x_1 - \overline{x_1})(x_2 - \overline{x_2})]$$

8. CORRELATION

In the world of finance, correlation is defined as a statistical measure of how two securities move in relation to each other. [16]

If two random variables x_1 and x_2 have the property that $\sigma_{12} = 0$, then they are said to be **uncorrelated**. This is the situation when the knowledge of one asset gives no information about the other. If $\sigma_{12} > 0$, then the two variables are said to be **positively correlated**. In this case, if one variable is above its mean, the other is likely to be above its mean as well. On the other hand, if $\sigma_{12} < 0$, then the two variables are **negatively correlated**. [17]

1.8.1 Correlation Coefficient

The correlation coefficient of two variables is defined as: [18]

$$r_{12} = \frac{\sigma_{12}}{\sigma_1 \ \sigma_2}$$

9. MEAN RETURN OF PORTFOLIO

In securities analysis, mean return is the expected value, or mean, of all the likely returns of investments comprising a portfolio and is also known as "expected return". [19]

Let there be n assets with random rate of return $r_1, r_2, \ldots, r_n$. these have the expected values

$E(r_1) = \overline{r_1}, E(r_2) = \overline{r_2}, \ldots, E(r_n) = \overline{r_n}$.

Let w_i , i = 1, 2,…, n, denote the weight of the i^{th} asset. The rate of return of the portfolio in terms of the individual returns is: [20]

$$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n$$

Taking the expected value on both the sides and using the linearity property, we obtain:

$$E(r) = w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n)$$

Thus, the expected rate of return of the portfolio is given by the weighted sum of the individual expected rate of returns.

10. VARIANCE OF THE PORTFOLIO

Variance of the portfolio is the measurement of how the actual returns of a group of securities making up a portfolio fluctuate. It looks at the standard deviation of each security in the portfolio as well as how those individual securities correlate with the others in the portfolio. [21]

Let the variance of the return of the i^{th} asset be denoted by σ_i^2, the variance of the return of the portfolio be σ^2 and the covariance of the return of asset i with asset j be σ_{ij}. Then, [22]

$$\sigma^2 = E[(r - \bar{r}\,)^2]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \overline{r_i}\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} w_i(r_i - \overline{r_i})\right)\left(\sum_{j=1}^{n} w_i(r_j - \overline{r_j})\right)\right]$$

$$= E\left[\sum_{i,j=1}^{n} w_i w_j (r_i - \overline{r_i})(r_j - \overline{r_j})\right]$$

$$= \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

11. DIVERSIFICATION

A risk management technique that mixes a wide variety of investments within a portfolio is known as diversification. The rationale behind this technique contends that a portfolio of different kinds of investments will, on average, yield higher returns and pose a lower risk than any individual investment found within the portfolio. [23]

The effects of diversification can be quantified by using the formula for combining variances. Let there be *n* uncorrelated assets with mean *m* and variance σ^2. A portfolio is constructed by taking equal portion of these assets, that is, $w_i = \frac{1}{n}$ for each *i*. The overall rate of return of the portfolio is: [24]

$$r = \frac{1}{n}\sum_{i=1}^{n} r_i$$

And the corresponding variance is:

$$var(r) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}$$

As observed from the equation, the variance decreases rapidly as *n* increases.
In case of correlated assets, it is possible to reduce the value of variance of the portfolio only to a certain point and not beyond it. Thus, if assets are uncorrelated, it is possible to reduce portfolio variance essentially to zero by taking *n* large and this process is known as diversification.

12. EFFICIENT FRONTIER

A set of optimal portfolios that offers the highest expected return for a defined level of risk or the lowest risk for a given level of expected return. Portfolios that lie below the efficient frontier are sub-optimal, because they do not provide enough return for the level of risk. Portfolios that cluster to the right of the efficient frontier are also sub-optimal, because they have a higher level of risk for the defined rate of return. [25]

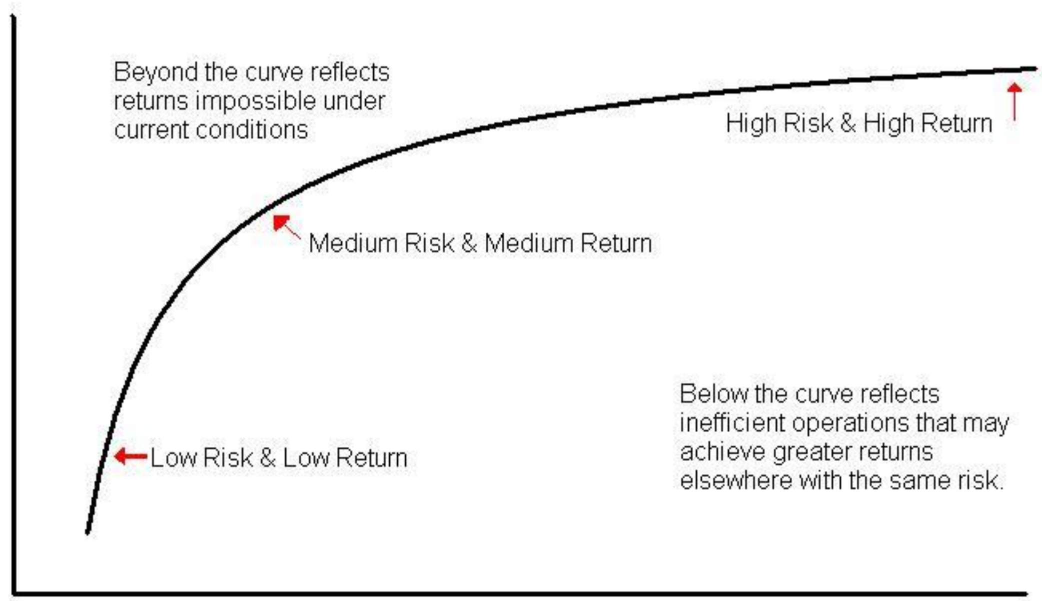

FIGURE 1.1 EFFICIENT FRONTIER CURVE SOURCE: http://fxwords.com/e/efficient-frontier.html

CHAPTER 2: CONSTRUCTION OF A PORTFOLIO USING REAL DATA

INTRODUCTION

After understanding the concepts involved in portfolio theory, a virtual portfolio can be constructed using real data of assets from the market.

2.1 ASSUMPTIONS

- Short selling of assets is not allowed.
- Investment is made only once and returns are gained at the end of the specified period.
- Currency is in INR.
- Stock is non-dividend paying.
- The price of stock changes after one day.

2.2 METHODOLOGY

- Three assets (stocks) are chosen, namely, IOCL, BPCL and HPCL. Weekly prices (Closing price, opening price, highest price and lowest price) for ten years of these assets are taken from Yahoo Finance.
- The rate of return is computed for each asset.
- Expected rate of return is computed for each asset using MATLAB (R2012a) with built in toolbox.
- Covariance matrix is computed to combine the three assets.
- Efficient frontier is plotted for a portfolio consisting of these three assets.
- 100 random weights are generated to build random portfolios with different portfolio return and portfolio risk.
- All random portfolios are plotted on the mean – standard deviation graph.
- The value of minimum portfolio risk is computed and the corresponding value of portfolio return is obtained and conclusions are drawn.

2.3 **VIRTUAL SCENERIO**

PROBLEM STATEMENT: Let there be an investor who wants to invest Rs. 1,00,000 in a portfolio consisting of stocks of IOCL, HPCL and BPCL. The task is to find out how much money should he invest in each asset so as to minimize the risk and maximize the profit. The investor is a risk averse person and thus, prefers minimum portfolio risk.

SOLUTION: The given three assets are positively correlated to each other. All the assets are risky assets.

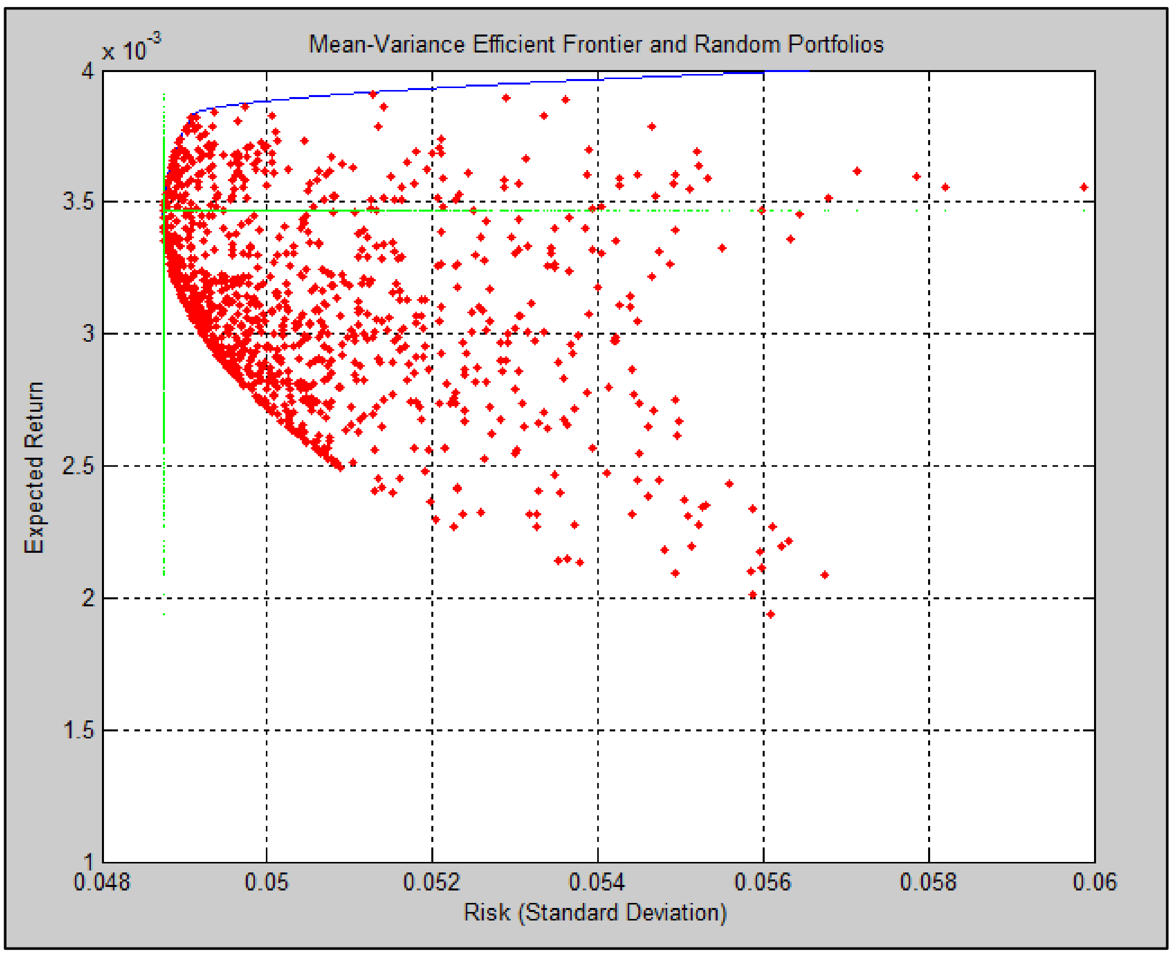

Figure 2.1 Graph of random portfolios generated and efficient frontier

Data computed from MATLAB gives:

$$\textbf{Maximum expected return} = \mathbf{0.003474}$$

$$\textbf{Minimum expected risk} = \mathbf{0.048741}$$

The above data is implied from the plotting of the efficient frontier. The co-ordinate where the efficient frontier meets the two perpendicular axes is the point for maximum expected return and minimum expected risk.

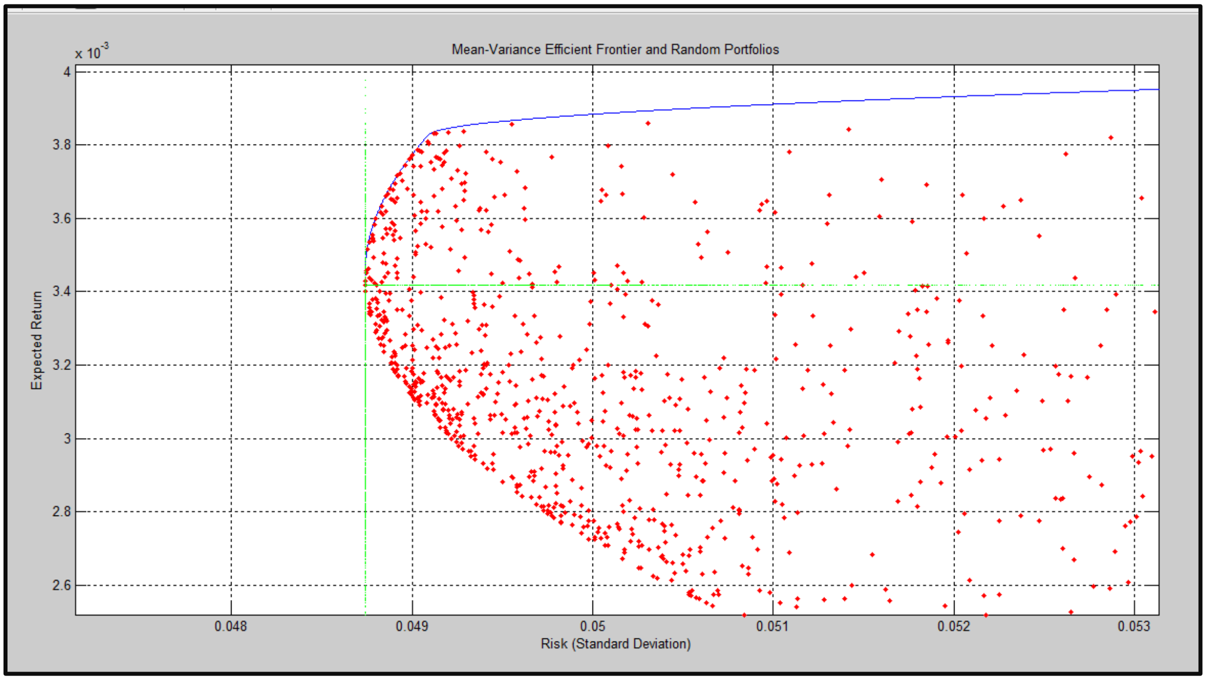

Figure 3 Plotting showing the min expected risk and max expected return

Fraction of money to be invested in IOCL $= \mathbf{0.413353}$

Fraction of money to be invested in HPCL $= \mathbf{0.164034}$

Fraction of money to be invested in BPCL $= \mathbf{0.422613}$

2.3.1 Calculations

Amount of money to be invested in IOCL $= \mathbf{0.413353} \times \mathbf{1,00,000} \approx \mathbf{41,335}$

Amount of money to be invested in HPCL $= \mathbf{0.164034} \times \mathbf{1,00,000} \approx \mathbf{16,403}$

Amount of money to be invested in BPCL $= \mathbf{0.422613} \times \mathbf{1,00,000} \approx \mathbf{42261}$

Expected total rate of return on portfolio $= 0.003474$

As,

$$X_1 = (1 + r)X_0$$

⇨ $\mathrm{X}_1 = (1 + 0.003474\,) \times 1{,}00{,}000 = 1{,}00{,}347$

Thus, the money received by the investor at the end of the specified period is Rs. 1,00,347.

2.4 CONCLUSIONS

- A portfolio can be build based on various concepts probability.
- It is possible to reduce the risk of portfolio by having large number of assets.
- Based on the given problem, the fraction of money to be invested in various assets can be calculated and thus, the expected return of the portfolio can be computed.

REFERENCES

[1], [2], [3], [4], [5], [6], [8], [10], [11], [13], [15], [17], [18], [20], [22], [24]Luenberger, D.G., Investment Science, Oxford University Press: New York, 2009

[7] http://www.investopedia.com/terms/p/portfolio-return.asp

[9] http://www.investopedia.com/terms/r/random-variable.asp

[12] http://www.investopedia.com/terms/v/variance.asp

[14] http://www.investopedia.com/terms/c/covariance.asp

[16] http://www.investopedia.com/terms/c/correlation.asp

[19] http://www.investopedia.com/terms/m/meanreturn.asp

[21] http://www.investopedia.com/terms/p/portfolio-variance.asp

[23] http://www.investopedia.com/terms/d/diversification.asp

[25] http://www.investopedia.com/terms/e/efficientfrontier.asp

APPENDIX

```
%****************************************
%********UNIVERSITY OF DELHI*************
%*****CLUSTER INNOVATION CENTRE**********
****************************************
% PROGRAM TO GENERATE RANDOM*************
% PORTFOLIOS AND EFFICIENT***************
% FRONTIER FOR THE***********************
% GIVEN ASSETS***************************
%****************************************
clc
clear all
%Expected return and Covariance matrix for the given assets

ExpReturn = [0.0036; 0.0018; 0.004];
Covariance = [0.0037 0.0016 0.0014; 0.0016 0.0034
0.0027;0.0014 0.0027 0.0032];

%Number of portfolios to be generated
NumPorts=1000;

%Generating efficient frontier
frontcon(ExpReturn,Covariance, NumPorts);

%Generating random weights for the given assets
Weights = rand(1000, 3);
Total = sum(Weights, 2);
Total = Total(:,ones(3,1));
Weights = Weights./Total;
PortReturn=zeros(1000,1);
PortRisk=zeros(1000,1);
m=1;
for i=1:1000
    %Calculating expected rate of return of the portfolio
PortReturn(i,1)=Weights(i,:)*ExpReturn;
end
```

```
for i=1:1000
    for j=1:3
        for k=1:3

%Calculating expected risk of the portfolio
PortRisk(i,1)=PortRisk(i,1)+(Weights(m,k)*Weights(m,j)*Cova
riance(j,k));
        end
    end
    m=m+1;
end
PortRisk=sqrt(PortRisk);
ExpReturn1 = [0.0036 0.0018 0.004];

%Finding minimum risk of portfolio
[C,E]=min(PortRisk);
hold on

%Plotting random portfolios
scatter (PortRisk, PortReturn, '.y', 'LineWidth',0.2);
title('Random Portfolios and Efficient Frontier');
xlabel('PortfolioRisk')
ylabel('PortfoilioReturn')

%Plotting minimum portfolio
% risk and maximum portfolio
%return corresponding
%to minimum portfolio risk
plot(C,PortReturn,'Color','black');
plot(PortRisk, PortReturn(E),'Color','black');
hold off

%Displaying the values of required data
fprintf('Minimum portfolio risk = %f \n',C);
fprintf('Maximum portfolio return = %f \n',PortReturn(E));
%Calculating expected risk of the portfolio
```

```
%Displaying on the console
%the fraction of money invested
%across various companies
fprintf('Fraction of money to be invested in IOCL = %f
\n',Weights(E,1));
fprintf('Fraction of money to be invested in HPCL = %f
\n',Weights(E,2));
fprintf('Fraction of money to be invested in BPCL = %f
\n',Weights(E,3));


%*************************************
%********Sahil Nakul Mathur*************
%********Nikita Garg*******************
%********Rahul Yadav*******************
%*************************************
```

console