UNIT 4
Assignment Problem

Ramakant Soni
Assistant Professor, Computer Science Department
B K Birla Institute of Engineering & Technology, Pilani, Rajasthan
ramakant.soni@bkbiet.ac.in

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Assignment Problem Definition

- Assignment problem is a special type of linear programming problem which deals with the allocation of the various resources to the various activities on one to one basis.

- Assignment is done in such a way that the cost or time involved in the process of assignment is minimum.

- The problem of assignment arises because available resources such as men, machines etc. have varying degrees of efficiency for performing different activities, therefore, cost, profit or loss of performing the different activities is different.

- The objective of assignment problem is to assign a number of jobs to an equal number of machines so as to minimize the total assignment cost or to minimize the total consumed time for execution of all the jobs.

Assignment Problem Formulation

Suppose there are **n** facilitates (e.g. Machines, workers etc.) and **n** jobs
There will be **n** assignments. Each facility can perform each job, one at a time.

C_{ij} =

Machine \ Job		Job				
		J_1	J_2	J_3	j^{th}	J_n
Machine	M_1	C_{11}	C_{12}	C_{13}	…	C_{1n}
	M_2	C_{21}	C_{22}	C_{23}	…	C_{2n}
	M_3	C_{31}	C_{32}	C_{33}	…	C_{3n}
	i^{th}	⋮	⋮	⋮	C_{ij}	⋮
	M_n	C_{n1}	C_{n2}	C_{n3}		C_{nn}

In the table, C_{ij} is defined as the cost when j^{th} job is assigned to i^{th} facility.

Assignment Problem Formulation

imp

Suppose x_{jj} is a variable which is defined as:

1 if the i^{th} job is assigned to j^{th} machine or facility

0 if the i^{th} job is not assigned to j^{th} machine or facility.

Now as the problem forms one to one basis or one job is to be assigned to one facility or machine.

$$\sum_{i=1}^{n} X_{ij} = 1$$

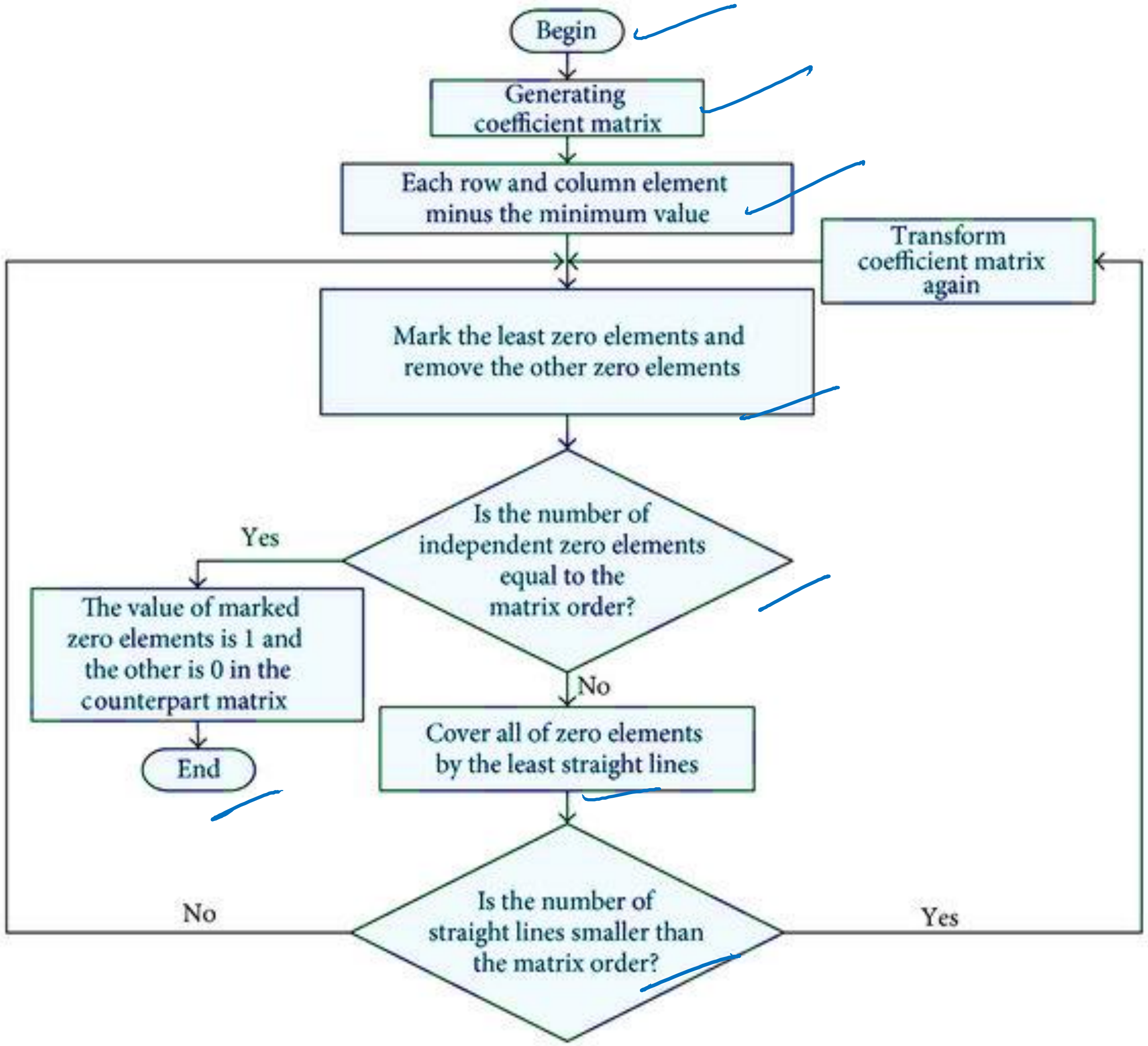
The total assignment cost will be given by

$$U = \sum_{j=1}^{n} \sum_{i=1}^{n} X_{ij} C_{ij}$$

The above definition can be developed into mathematical model as follows:

Determine X_{ij} > 0 (i, j = 1,2, 3…n) in order to Minimize $U = \sum_{j=1}^{n} \sum_{i=1}^{n} X_{ij} C_{ij}$

Hungarian method to solve Assignment Problem



Assignment Problem Formulation

Consider the objective function of minimization type. Following steps are involved in solving this Assignment problem:

Cost Matrix Reduction < Row, Column

1. Locate the smallest cost element in each row of the given cost table starting with the first row. Now, this smallest element is subtracted form each element of that row. So, we will be getting at least one zero in each row of this new table.

2. Having constructed the table (as by step-1) take the columns of the table. Starting from first column locate the smallest cost element in each column. Now subtract this smallest element from each element of that column. Having performed the step 1 and step 2, we will be getting at least one zero in each column in the reduced cost table.

Assignment Problem Formulation

3. Now, the assignments are made for the reduced table in following manner.

(i) Rows are examined successively, until the row with exactly single (one) zero is found. Assignment is made to this single zero by putting square □ around it and in the corresponding column, all other zeros are crossed out (x) because these will not be used to make any other assignment in this column. Step is conducted for each row.

(ii) Step 3 (i) in now performed on the columns as follow:- columns are examined successively till a column with exactly one zero is found. Now , assignment is made to this single zero by putting the square around it and at the same time, all other zeros in the corresponding rows are crossed out (x) step is conducted for each column.

(iii) Step 3, (i) and 3 (ii) are repeated till all the zeros are either marked or crossed out. Now, if the number of marked zeros or the assignments made are equal to number of rows or columns, optimum solution has been achieved. There will be exactly single assignment in each or columns without any assignment. In this case, we will go to step 4.

Assignment Problem Formulation

4. At this stage, draw the minimum number of lines (horizontal and vertical) necessary to cover all zeros in the matrix obtained in step 3, Following procedure is adopted:

(i) Tick mark ( ) all rows that do not have any assignment.

(ii) Now tick mark( ) all these columns that have zero in the tick marked rows.

(iii) Now tick mark all the rows that are not already marked and that have assignment in the marked columns.

(iv) All the steps i.e. (4(i), 4(ii), 4(iii) are repeated until no more rows or columns can be marked.

(v) Now draw straight lines which pass through all the un marked rows and marked columns. It can also be noticed that in an n x n matrix, always less than ‘n’ lines will cover all the zeros if there is no solution among them.

Assignment Problem Formulation

5. In step 4, if the number of lines drawn are equal to n or the number of rows, then it is the optimum solution if not, then go to step 6.

6. Select the smallest element among all the uncovered elements. Now, this element is subtracted from all the uncovered elements and added to the element which lies at the intersection of two lines. This is the matrix for fresh assignments.

7. Repeat the procedure from step (3) until the number of assignments becomes equal to the number of rows or number of columns.

Assignment Problem Example

Example

	J_1	J_2	J_3	J_4	J_5
M_1	150	120	165	180	190
M_2	125	110	120	150	165
M_3	130	100	145	160	175
M_4	40	40	70	70	100
M_5	45	25	60	70	95

Step 1: Row Reduction

Reduce minimum value from each row

	J1	J2	J3	J4	J5	Column Min.
M1	150	120	165	180	190	– 120
M2	125	110	120	150	165	– 110
M3	130	100	145	160	175	– 100
M4	40	40	70	70	100	– 40
M5	45	25	60	70	95	– 25

Step 2: Column Reduction

Then, reduce minimum value from each column

	J1	J2	J3	J4	J5
M1	30	0	45	60	70
M2	15	0	10	40	55
M3	30	0	45	60	75
M4	0	0	30	30	60
M5	20	0	35	45	70
			↑	↑	↑
Row Minimum			-10	-30	-55

Assignment Problem Example

Step 3: Use Hangarian Method to marks assignments

	J1	J2	J3	J4	J5
M1	30	0	35	30	15
M2	15	0	0	10	0
M3	30	0	35	30	20
M4	0	0	20	0	5
M5	20	0	25	15	15

Step 1

Row assignment :

look for exactly one zero in each row. Mark the zero and draw a line covering all the elements of the column in which zero is present.

Step 2

Column assignment :

look for exactly one zero in each column Mark that zero and draw a line covering all the elements of the row in which zero is present

step 3 :-

when no more assignment possible count zeroes.

Assignment Problem Example

Step 3: Use Hangarian Method to marks assignments

	J1	J2	J3	J4	J5
M1	30	0	35	30	15
M2	15	0	0	10	0
M3	30	0	35	30	20
M4	0	0	20	0	5
M5	20	0	25	15	15

intersection point

Min. value

if no of assignment = no. of machines we get the optimal assignment.

Like in our example,

no. of assignment = 03 ≠ no. of machines

∴ — we select the minimum value from the uncovered elements.

— Reduce it from all the uncovered elements

— Add it to the elements which lye on the intersection.

Assignment Problem Example

Step 4: Assignment Count ! = Machine count

	J1	J2	J3	J4	J5
M1	~~15~~	~~0~~	~~20~~	~~15~~	0 5
M2	~~15~~	~~15~~	0 3	~~10~~	~~0~~
M3	15	0 1	20	15	5
M4	2 0	~~15~~	~~20~~	~~0~~	~~5~~
M5	~~5~~	~~0~~	~~10~~	0 4	~~0~~

Again do the assignment

Here, no. of assignment = 5

no. of machines = 5

Both equal ∴ we here got the optimal assignment.

Assignment Problem Example

Step 5: Assignment Count = Machine count

	J1	J2	J3	J4	J5
M1	0	0	0	0	190
M2	0	0	120	0	0
M3	0	100	0	0	0
M4	40	0	0	0	0
M5	0	0	0	70	0

Answer **Assignment Cost= 520**

	J_1	J_2	J_3	J_4	J_5
M_1	150	120	165	180	190
M_2	125	110	120	150	165
M_3	130	100	145	160	175
M_4	40	40	70	70	100
M_5	45	25	60	70	95

Quadratic Assignment Problem

Quadratic Assignment Problem

The objective of the Quadratic Assignment Problem (QAP) is to assign *n* facilities to *n* locations in such a way as to minimize the assignment cost.

The assignment cost is the sum, over all pairs, of the flow between a pair of facilities multiplied by the distance between their assigned locations.

QAP Statement

The quadratic assignment problem (QAP) was introduced by Koopmans and Beckman in 1957. The objective of the problem is to assign a set of facilities to a set of locations in such a way as to minimize the total assignment cost.

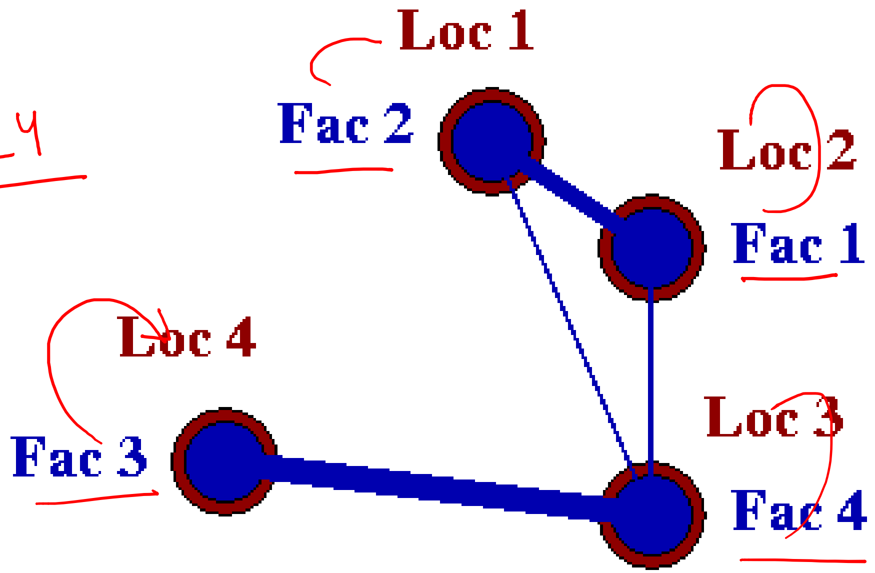
The assignment cost for a pair of facilities is a function of the flow between the facilities and the distance between the locations of the facilities.

Consider a facility location problem with four facilities (and four locations). One possible assignment is shown in the figure:

- facility 2 is assigned to location 1,
- facility 1 is assigned to location 2,
- facility 4 is assigned to location 3,
- and facility 4 is assigned to location 3.

$\underline{|n}$

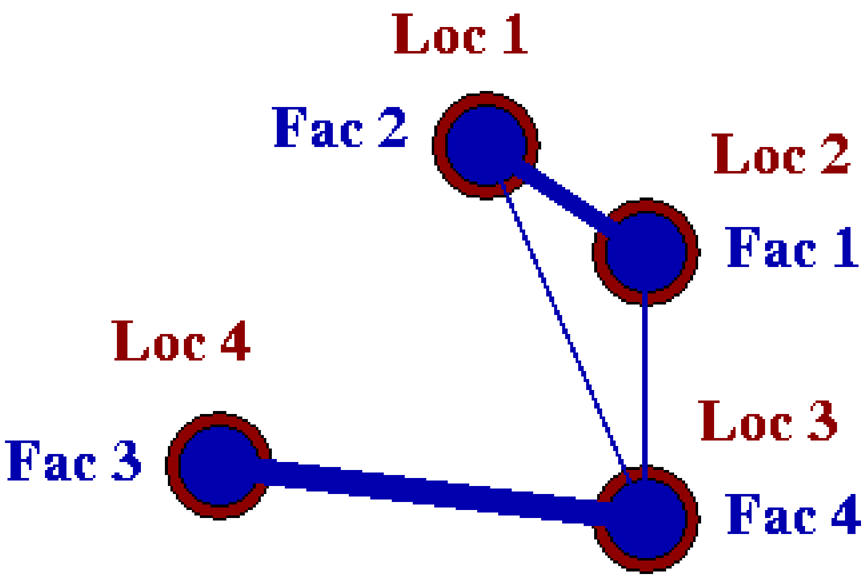
$\underline{|4} = \underline{24}$



This assignment can be written as the permutation *p*={2,1,4,3}, which means that facility 2 is assigned to location 1, facility 1 is assigned to location 2, facility 4 is assigned to location 3, and facility 4 is assigned to location 3.

In the figure, the line between a pair of facilities indicates that there is required flow between the facilities, and the thickness of the line increases with the value of the flow

QAP Example:



To calculate the assignment cost of the permutation, the required flows between facilities and the distances between locations are needed.

Distances between locations

location i	location j	distance(i,j)
1	3	53
2	1	22
2	3	40
3	4	55

Flows between facilities

facility i	facility j	flow(i,j)
1	2	3
1	4	2
2	4	1
3	4	4

Then, the assignment cost of the permutation can be computed as:

$f(1,2)\cdot d(2,1)+f(1,4)\cdot d(2,3)+f(2,4)\cdot d(1,3)+f(3,4)\cdot d(3,4)$
$= 3\cdot 22+2\cdot 40+1\cdot 53+4\cdot 55 = 419.$

Note that this permutation is not the optimal solution.

Mathematical Formulation

Here we present the Koopmans-Beckmann formulation of the QAP. Given a set of facilities and locations along with the flows between facilities and the distances between locations, the objective of the Quadratic Assignment Problem is to assign each facility to a location in such a way as to minimize the total cost.

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Sets

$N = \{1, 2, \cdots, n\}$

$S_n = \phi : N \to N$ is the set of all permutations

Parameters

$F = (f_{ij})$ is an $n \times n$ matrix where f_{ij} is the required flow between facilities i and j

$D = (d_{ij})$ is an $n \times n$ matrix where d_{ij} is the distance between locations i and j

Optimization Problem

$\min_{\phi \in S_n} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \cdot d_{\phi(i)\phi(j)}$

The assignment of facilities to locations is represented by a permutation ϕ, where $\phi(i)$ is the location to which facility i is assigned. Each individual product $f_{ij} \cdot d_{\phi(i)\phi(j)}$ is the cost of assigning facility i to location $\phi(i)$ and facility j to location $\phi(j)$.

Example : QAP 4 – Assignment 1

Given the distance matrix and the required flows. Assign each facility (1, 2, 3, 4) to one location (A, B, C, D).

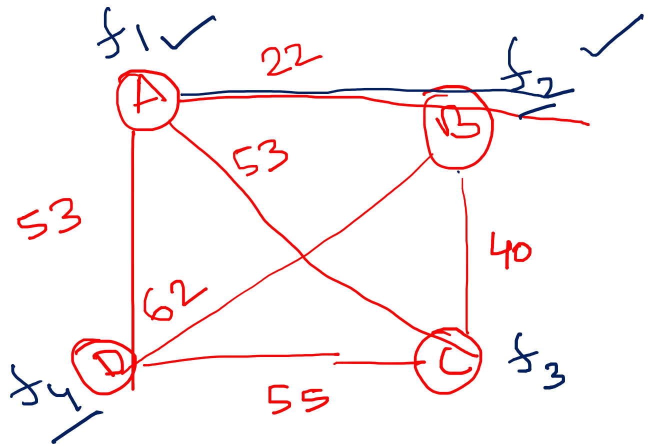
Distance Matrix

	A	B	C	D
A	0	22	53	53
B	22	0	40	62
C	53	40	0	55
D	53	62	55	0

Flows

	1	2	3	4
1	0	3	0	2
2	3	0	0	1
3	0	0	0	4
4	2	1	4	0

A 1 B 2 C 3 D 4



$\{1, 2, 3, 4\}$

$4! = 24$

Location	Distance (D)	Facility	Flow (F)	Assignment Cost D * F
A, B	22	f_1 f_2	3	66
A D	53	f_1 f_4	2	106
B D	62	f_2 f_4	1	62
A C	53	f_1 f_3	0	0
B C	40	f_2 f_3	0	0
C D	55	f_3 f_4	4	220
Total Cost of Assignment =				454

Example : QAP 4 – Optimal Assignment (Minimum Cost)

Given the distance matrix and the required flows. Assign each facility (1, 2, 3, 4) to one location (A, B, C, D).

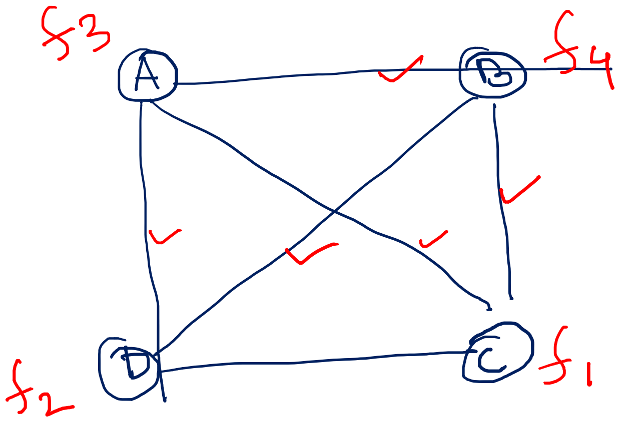
Distance Matrix

	A	B	C	D
A	0	22	53	53
B	22	0	40	62
C	53	40	0	55
D	53	62	55	0

Flows

	1	2	3	4
1	0	3	0	2
2	3	0	0	1
3	0	0	0	4
4	2	1	4	0

A 3 B 4 C 1 D 2



$\frac{24}{1}$

S = {3, 4, 1, 2}

Location	Distance (D)	Facility	Flow (F)	Assignment Cost D * F
AB	22	f3 f4	4	88
AC	53	f1 f3	0	0
AD	53	f2 f3	0	0
BC	40	f1 f4	2	80
BD	62	f2 f4	1	62
CD	55	f1 f2	3	165
Total Cost of Assignment =				395

Thanks