Covariance Matching Filter for IMU Error Estimation

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Motivation

- IMUs are extensively used with GPS, visual and magnetometer sensors for accurate navigation.
- MEMS IMU sensors are popular because of their low cost, low power and lightweight.

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- IMUs are extensively used with GPS, visual and magnetometer sensors for accurate navigation.
- MEMS IMU sensors are popular because of their low cost, low power and lightweight.
- However, they cannot compare with the high-end established sensors because of random errors and time varying biases.
- Hence, IMU errors have to be modeled, estimated and compensated for, to be used for navigation.

Modeling IMU errors

Error models can be classified into two types:[1]

- Deterministic: Bias mean, scale factors and misalignment errors.
  - Static and rate tests

[1] Hemerly, E.M., 2017. "MEMS IMU stochastic error modelling". Systems Science & Control Engineering, 5(1), pp.1-8.

Modeling IMU errors

Error models can be classified into two types:[1]

- Deterministic: Bias mean, scale factors and misalignment errors.
  - Static and rate tests
- Stochastic: Bias instability, velocity & angular random walk.
  - Allan Variance methods
  - On-line estimation algorithms

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On-line estimation algorithms

Previous work:

- Sage-Husa adaptive Kalman filters
- Expectation Maximization algorithm
- ARMA error model estimation

[2] Moghe, R., Zanetti R. and Akella M. 2018, December. Covariance Matching Kalman filter for observable LTI systems. In Decision and Control, 2018 57th IEEE Conference (Forthcoming). IEEE.

On-line estimation algorithms

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Our work:

- A recently established adaptive filter[2] to estimate the VRW
  - LTI System
  - Observability
- Either process covariance Q or the measurement covariance R is estimated on-line.

[2] Moghe, R., Zanetti R. and Akella M. 2018, December. Covariance Matching Kalman filter for observable LTI systems. In Decision and Control, 2018 57th IEEE Conference (Forthcoming). IEEE.

Dynamic model

2D kinematic model:

$$\ddot{r}_x = a_x \qquad \ddot{r}_y = a_y$$

The bias is modeled as an Exponentially correlated random variable:

$$\dot{b}_x(t) = -\frac{b_x(t)}{\tau_x} + w_x(t) \qquad \dot{b}_y(t) = -\frac{b_y(t)}{\tau_y} + w_y(t)$$

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The concatenated state $\mathbf{x} = \begin{bmatrix} r_x & \dot{r}_x & b_x & r_y & \dot{r}_y & b_y \end{bmatrix}^T$ and the noise w_x and w_y are zero-mean white Gaussian with known power spectral density.

Measurement model

We assume a position and velocity feedback

$$\mathbf{z}_k = H\mathbf{x}_k + \boldsymbol{\eta}_k$$

The accelerometer measurement is given by

$$\tilde{\mathbf{a}}(t) = \mathbf{a}(t) + \mathbf{b}(t) + \mathbf{v}(t)$$

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Accumulating the accelerometer measurements for the sampling time T,

$$\Delta\tilde{\mathbf{v}}_k = \Delta\mathbf{v}_k + \mathbf{b}_k T + \mathbf{v}_k$$

wherein, $\Delta\mathbf{v}_k = \mathbf{a}_k T$. The covariance of the zero-mean white sequence $\mathbf{v}_k$ is unknown.

Noise Statistics

- The steady state variances of w_{x_k} and w_{y_k} is $\sigma_{ss,x}$ and $\sigma_{ss,y}$ and the bias variances is $Q_{b_x} = (1 - e^{\frac{-2T}{\tau_x}})\sigma^2_{ss,x}$ and $Q_{b_y} = (1 - e^{\frac{-2T}{\tau_y}})\sigma^2_{ss,y}$ respectively.
- $\boldsymbol{\eta}_k$ has a known covariance of R.

$$cov(\boldsymbol{\eta}_k) = Q_a = \begin{bmatrix} \frac{T^3}{3}S_{v_x} & \frac{T^2}{2}S_{v_x} & 0 & 0 & 0 & 0 \\ \frac{T^2}{2}S_{v_x} & TS_{v_x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{T^3}{3}S_{v_y} & \frac{T^2}{2}S_{v_y} & 0 \\ 0 & 0 & 0 & \frac{T^2}{2}S_{v_y} & TS_{v_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The power spectral densities S_{v_x} and S_{v_y} are unknown.

Propagation

The dynamics can be reformulated in terms of the accelerometer measurement.

$$\mathbf{x}_{k+1} = F\mathbf{x}_k + G\Delta\tilde{\mathbf{v}}_k - \mathbf{v}_k + \Lambda\mathbf{w}_k$$

Hence, the state estimate is as follows

$$\hat{\mathbf{x}}_{k+1|k} = F\hat{\mathbf{x}}_k + G\Delta\tilde{\mathbf{v}}_k$$
$$P_{k+1|k} = FP_kF^T + \hat{Q}_a + \Lambda Q_b\Lambda^T$$

Update

The update step is identical to that of the linear Kalman filter. Hence, the state estimate is as follows

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + K_{k+1}(\mathbf{z}_{k+1} - H\hat{\mathbf{x}}_{k+1|k})$$
$$K_{k+1} = P_{k+1|k}H^T(HP_{k+1|k}H^T + R)^{-1}$$
$$P_{k+1} = (I - K_{k+1}H)P_{k+1|k}(I - K_{k+1}H)^T + K_{k+1}RK_{k+1}^T$$

Covariance Estimation

Stacked measurement model:

$$\mathbf{z}_k = HF\mathbf{x}_{k-1} + HG\Delta\tilde{\mathbf{v}}_{k-1} - H\mathbf{v}_{k-1} + H\Lambda\mathbf{w}_{k-1} + \boldsymbol{\eta}_k$$

$$\begin{bmatrix}\mathbf{z}_k \\ \mathbf{z}_{k-1}\end{bmatrix} = \underbrace{\begin{bmatrix}HF \\ H\end{bmatrix}}_{\triangleq M_o}\mathbf{x}_{k-1} - \underbrace{\begin{bmatrix}H \\ \mathbf{0}\end{bmatrix}}_{\triangleq M_v}\mathbf{v}_{k-1} + \underbrace{\begin{bmatrix}HG \\ \mathbf{0}\end{bmatrix}}_{\triangleq M_a}\Delta\tilde{\mathbf{v}}_{k-1} + \underbrace{\begin{bmatrix}H\Lambda \\ \mathbf{0}\end{bmatrix}}_{\triangleq M_w}\mathbf{w}_{k-1} + \underbrace{\begin{bmatrix}\boldsymbol{\eta}_k \\ \boldsymbol{\eta}_{k-1}\end{bmatrix}}_{\triangleq E_k}$$

$$\underbrace{\begin{bmatrix}\mathbf{z}_k \\ \mathbf{z}_{k-1}\end{bmatrix} - M_a\Delta\tilde{\mathbf{v}}_{k-1}}_{\triangleq \mathcal{Y}_k} = M_o\mathbf{x}_{k-1} + M_w\mathbf{w}_{k-1} - M_v\mathbf{v}_{k-1} + E_k$$

Covariance Estimation

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Observability $\implies$ M_o has full column rank.

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Observability $\implies$ M_o has full column rank.

$$M_o^\dagger\mathcal{Y}_k = \mathbf{x}_{k-1} + M_o^\dagger M_w\mathbf{w}_{k-1} - M_o^\dagger M_v\mathbf{v}_{k-1} + M_o^\dagger E_k$$
$$M_o^\dagger\mathcal{Y}_{k-1} = \mathbf{x}_{k-2} + M_o^\dagger M_w\mathbf{w}_{k-2} - M_o^\dagger M_v\mathbf{v}_{k-2} + M_o^\dagger E_{k-1}$$

Covariance Estimation

$$\underbrace{M_o^{\dagger}\mathcal{Y}_k - FM_o^{\dagger}\mathcal{Y}_{k-1} - G\Delta\tilde{\mathbf{v}}_{k-2}}_{\triangleq \mathcal{Z}_k} = \underbrace{\mathbf{w}_{k-2} + M_o^{\dagger}M_w\mathbf{w}_{k-1} - FM_o^{\dagger}M_w\mathbf{w}_{k-2}}_{\triangleq \mathcal{W}_k} +$$
$$\underbrace{FM_o^{\dagger}M_v\mathbf{v}_{k-2} - \mathbf{v}_{k-2} - M_o^{\dagger}M_v\mathbf{v}_{k-1}}_{\triangleq \mathcal{V}_k} + \underbrace{M_o^{\dagger}E_{k+1} - FM_o^{\dagger}E_k}_{\triangleq \mathcal{E}_k}$$

All $\mathcal{W}_k$, $\mathcal{V}_k$, and $\mathcal{E}_k$ are zero mean, uncorrelated with each other and their covariances are time invariant. Hence, their covariances are given as follows.

$$Cov(\mathcal{Z}_k) = Cov(\mathcal{W}_k) + Cov(\mathcal{V}_k) + Cov(\mathcal{E}_k)$$
$$Cov(\mathcal{Z})_k - Cov(\mathcal{W}) - Cov(\mathcal{E}) = A_1\hat{Q}_{a_k}A_1^T + A_2\hat{Q}_{a_k}A_2^T$$

The most recent positive definite $\hat{Q}_{a_k}$ is chosen.

Outline of Proof

$\mathcal{Z}_k$ can be written as a **linear strictly stationary** time series as follows:

$$\mathcal{Z}_k = A_1\mathbf{v}_{k-1} + A_2\mathbf{v}_{k-2} + B_1\mathbf{w}_{k-1} + B_2\mathbf{w}_{k-2} + C_1\boldsymbol{\eta}_k + C_2\boldsymbol{\eta}_{k-1} + C_3\boldsymbol{\eta}_{k-2}$$

Using the **central limit theorem** for linear strictly stationary time series, we get

$$\frac{1}{k}\sum_{i=1}^{k}\mathcal{Z}_i\mathcal{Z}_i^T \xrightarrow{P} Cov(\mathcal{Z}) \implies \hat{Q}_{a_k} \xrightarrow{P} Q_a$$

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Defining the sequences P_k and $\hat{P}_k$ as follows

$$\hat{P}_{k+1|k} = F\hat{P}_kF^T + \hat{Q}_a + \Lambda Q_b\Lambda^T$$
$$P_{k+1|k} = FP_kF^T + Q_a + \Lambda Q_b\Lambda^T$$

We prove that $\hat{P}_k \xrightarrow{P} P_k$.

Simulation Parameters

Assume that $T = 0.1$ sec, $\tau_x = 3600$ sec, and $\tau_y = 3600$ sec. The true noise covariance matrices mentioned above have the following values.

$$S_{v_x} = 10^{-6}\ m^2/s^3,\ S_{v_y} = 10^{-6}\ m^2/s^3$$
$$Q_{b_x} = 5.55 \times 10^{-11}\ m^2/s^4,\ Q_{b_y} = 5.55 \times 10^{-11}\ m^2/s^4$$
$$R = \begin{bmatrix} 10^{-8} m^2 & 0 & 0 & 0 \\ 0 & 10^{-6} m^2/s^2 & 0 & 0 \\ 0 & 0 & 10^{-8} m^2 & 0 \\ 0 & 0 & 0 & 10^{-6} m^2/s^2 \end{bmatrix}$$

Here, the initial estimate is taken to be $\hat{S}_{v_{x_0}} = 2 \times 10^{-6}\ m^2/s^3$ and $\hat{S}_{v_{y_0}} = 2 \times 10^{-6}\ m^2/s^3$.

Position Estimation

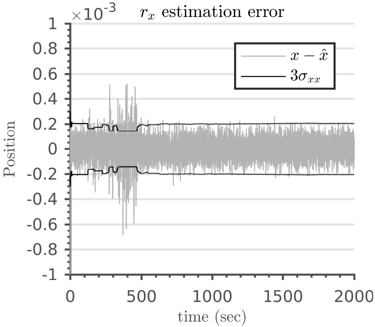

Figure : Position estimation error vs. time and it's 3σ values.

Velocity Estimation

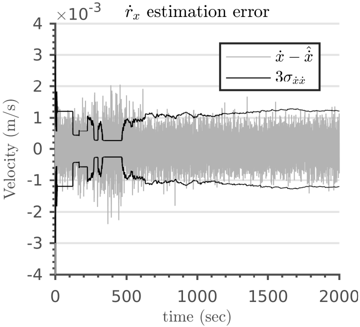

Figure : Velocity estimation error vs. time and it's 3σ values.

Bias Estimation

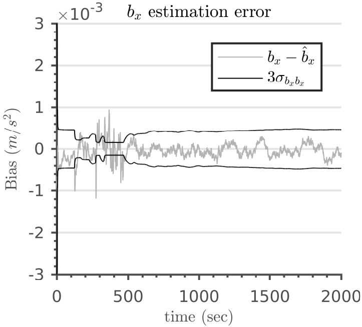

Figure : Bias estimation error vs. time and it's 3σ values.

Covariance Estimation

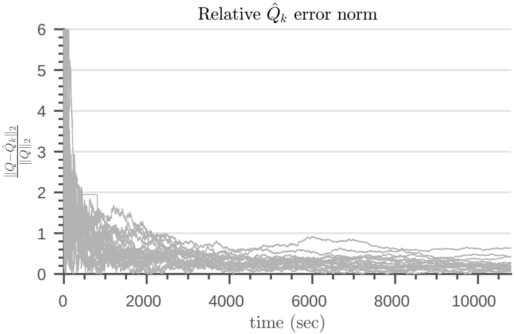

Figure : S_{v_x} and S_{v_y} estimation error for 20 Monte Carlo simulations.

Discussions and Conclusions

- We introduced a Covariance Matching Kalman filter for estimating the velocity random walk of the accelerometer.
- Performed numerical simulations to show that the estimate of the states stay within their 3σ values after the transients have died.
- We showed through Monte Carlo simulations that the covariance estimate converges to its true value.
- Future work:
  - Derive nonlinear adaptive filter for attitude estimation.
  - Experimental validation of the algorithm.

Thank you