Why is the dot product of orthogonal vectors zero?

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Apr 1 · 2 min read ★

It is *"by definition"*. Two non-zero vectors are said to be orthogonal when (if and only if) their dot product is zero.

Ok. But why did we define the orthogonality this way?

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The dot product of two vectors is defined **algebraically:**

The dot product of two vectors **a** = [a_1, a_2, …, a_n] and **b** = [b_1, b_2, …, b_n] is defined as:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

The algebraic definition

Yet, there is also a **geometric** definition of the dot product:

```
a • b = ||a|| * ||b|| * cosø
```

which is multiplying **the length of the first vector** with **the length of the second vector** with **the cosine of the angle** between the two vectors.

And the angle between the two perpendicular vectors is 90°.

When we substitute ø with 90° (cos 90°=0), `a•b` becomes zero.

. . .

I want more than just a definition. Show me something real?

Ok… then let me show you how those two definitions (geometric and algebraic) agree with each other through the Pythagorean theorem.

Below is a basic recap of the vector norm. They will be used in expanding Pythagoras theorem to n-dimensions.

① Length in 3D

$$\|X\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

② Length squared in n-dimension

$$\|X\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = X^T X$$

The length of n-dimensional vectors

Now, consider two vectors **[-1, 2]** and **[4, 2]**.

Algebraically, [-1, 2] • [4, 2] = -4 + 4 = 0.

Applying the length formula ② to the Pythagorean theorem:

Pythagorean Theorem in n-dimension

$$\|X\|^2 + \|y\|^2 = \|X - y\|^2$$

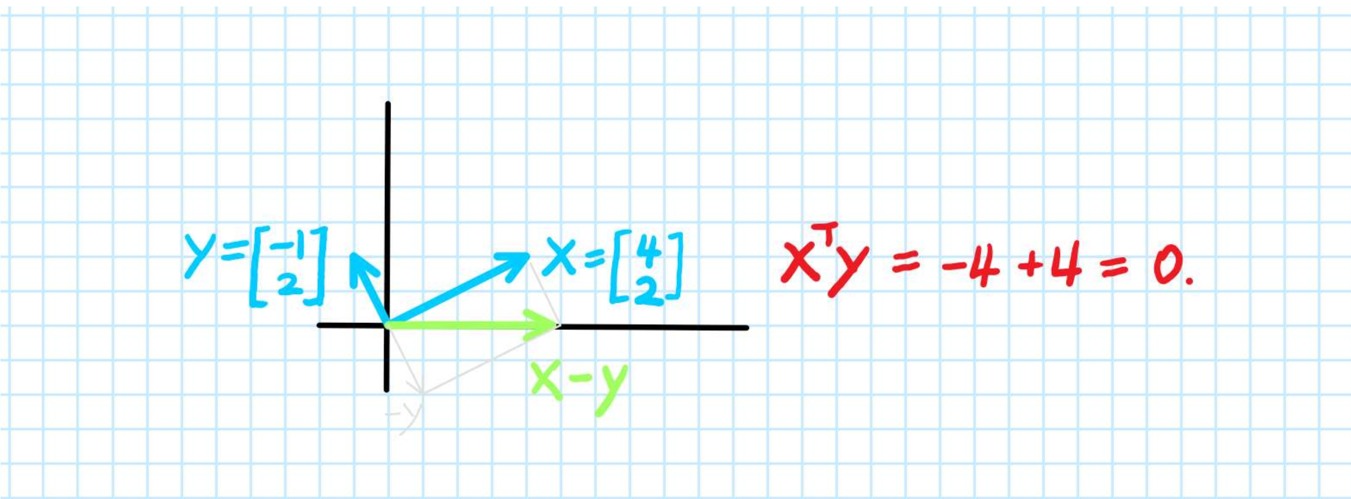
$$(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2) = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2$$

$$0 = -2(x_1y_1 + \cdots + x_ny_n)$$

$$0 = x^Ty \quad \blacksquare$$

We can extend Pythagorean theorem to n-space.

Geometrically, you can see they are perpendicular as well.



Orthogonality shown algebraically and geometrically.

> The geometric definition matches with the algebraic definition!

. . .

> *A Pop Quiz:*
>
> ***Orthogonal vectors are linearly independent.*** *This sound obvious- but can you prove it mathematically?*
>
> ***Hint****: You can use two definitions. 1) The algebraic definition of orthogonality*
> *2) The definition of linear Independence: The vectors {**V1**, **V2**, … , **Vn**} are linearly*

> *independent if the equation $a1 * \mathbf{V1} + a2 * \mathbf{V2} + \ldots + an * \mathbf{Vn} = 0$ can only be satisfied by $ai = 0$ for all i.*

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