Unit 4- Estimation Theory

Content: Point estimation, Interval estimate and confidence Interval, criteria for good estimates (un-biasedness, consistency), Methods of estimation including maximum likelihood estimation

What is estimation?

. In statistics, estimation (or inference) refers to the process by which one makes inferences (e.g. draws conclusions) about a population, based on information obtained from a sample.

• A statistic is any measurable quantity calculated from a sample of data (e.g. the average). This is a stochastic variable as, for a given population, it will in general vary from sample to sample.

• An estimator is any quantity calculated from the sample data which is used to give information about an unknown quantity in the population (the estimand).

• An estimate is the particular value of an estimator that is obtained by a particular sample of data and used to indicate the value of a parameter.

Basic Terminologies:

Population
Sample
Parameter
Statistic
Estimate
Estimator

Example: Interval Estimation

1. A large number of light bulbs was turned on continuously to determine the average number of days a bulb can last. The study revealed that the average lifetime of a bulb is 120 days with a standard deviation of 10 days. If the lifetimes are assumed to be independent normal random variables, find the confidence limits for a confidence level of 90% on the sample mean that is computed from a sample size of (a) 100 and (b) 25.

Given that for 90% confidence level, k= 1.64.

Sol: 95% confidence interval for the sample mean are given by

$$\overline{X} - 1.96\sigma_X \le E[\overline{X}] \le X + 1.96\sigma_X.$$

Corresponding to k% the confidence level, the formula is

$$\overline{X} - k\sigma_X \le E[\overline{X}] \le X + k\sigma_X.$$

2. A random sample of 50 of the 200 electrical engineering students' grades in applied probability showed a mean of 75% and a standard deviation of 10%.
   a. What are the 95% confidence limits for the estimate of the mean of the 200 grades?
   b. What are the 99% confidence limits for the estimate of the mean of the 200 grades?

MLE Example:

1. For the following sample, determine the likelihood function of θ. Given that $X_i \sim Binomial\ (3, \theta)$ and observations are $(x_1, x_2, x_3, x_4) = (1,3,2,2)$.

. If $X_i \sim Binomial(3, \theta)$, then

$$P_{X_i}(x;\theta) = \binom{3}{x}\theta^x(1-\theta)^{3-x}$$

$$\begin{aligned} L(x_1,x_2,x_3,x_4;\theta) &= P_{X_1X_2X_3X_4}(x_1,x_2,x_3,x_4;\theta) \\ &= P_{X_1}(x_1;\theta)P_{X_2}(x_2;\theta)P_{X_3}(x_3;\theta)P_{X_4}(x_4;\theta) \\ &= \binom{3}{x_1}\binom{3}{x_2}\binom{3}{x_3}\binom{3}{x_4}\theta^{x_1+x_2+x_3+x_4}(1-\theta)^{12-(x_1+x_2+x_3+x_4)}. \end{aligned}$$

observed $(x_1, x_2, x_3, x_4) = (1,3,2,2)$, we have

$$\begin{aligned} L(1,3,2,2;\theta) &= \binom{3}{1}\binom{3}{3}\binom{3}{2}\binom{3}{2}\theta^8(1-\theta)^4 \\ &= 27 \quad \theta^8(1-\theta)^4. \end{aligned}$$

2. For the following sample, determine the likelihood function of θ. Given that $X_i \sim Exponential\ (\theta)$ and observations are $(x_1, x_2, x_3, x_4) =$ (1.23,3.32,1.98,2.12).

If $X_i \sim Exponential(\theta)$, then

$$f_{X_i}(x;\theta) = \theta e^{-\theta x} u(x),$$

For the following random samples, find the maximum likelihood estimate of θ:

1. Given that $X_i \sim Binomial\ (3, \theta)$ and observations are $(x_1, x_2, x_3, x_4) =$ (1,3,2,2).
2. Given that $X_i \sim Exponential\ (\theta)$ and observations are $(x_1, x_2, x_3, x_4) =$ (1.23,3.32,1.98,2.12).

Sol:

1. we found the likelihood function as

$$L(1,3,2,2;\theta) = 27 \qquad \theta^8 (1-\theta)^4$$

To find the value of θ that maximizes the likelihood function, we can take the derivative and set it to zero. We have

$$\frac{dL(1,3,2,2;\theta)}{d\theta} = 27\big[\qquad 8\theta^7 (1-\theta)^4 - 4\theta^8 (1-\theta)^3 \big]$$

$$\hat{\theta}_{ML} = \frac{2}{3}.$$

1. A city dispute center that settles disputes per month follows a Poisson distribution. The following table gives the number of disputes settled in a month. Determine the maximum likelihood estimator.

Number of Disputes	1	2	3	4	5	6	7	8	9	10
Frequency	2	11	18	6	24	3	10	9	7	10

(Ans: 5.45)

2. Assume that the weight of all students is normally distributed. A sample data of 20 student's weight in Kg. is collected.

53.14	45.56	48.18	60.71	53.21	39.86	70.18	51.21	49.54	41.15
56.71	53.35	47.20	46.75	62.34	54.25	52.76	47.87	44.90	49.18

Determine the maximum likelihood estimators of population mean and population variance. (Ans: 51.4025, 52.2331)

3. A coin is tossed 100 times and lands heads 62 times. What is the maximum likelihood estimate for θ the probability of heads?
4. A coin is tossed 100 times and lands heads 55 times. What is the maximum likelihood estimate for θ the probability of heads? (0.55)

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Ans: For a given value of θ, the probability of getting 55 heads is the Binomial probability given by

P(55 heads)= ${}^{100}C_{55}\theta^{55}(1-\theta)^{45} = l(\theta)$ .Hence,

$$MLE(\theta)=\frac{d}{d\theta}(l(\theta))=\frac{d}{d\theta}\left({}^{100}C_{55}\theta^{55}(1-\theta)^{45}\right)=0$$

$$\Rightarrow\frac{d}{d\theta}(Log(\theta))=\frac{d}{d\theta}\left[\log{}^{100}C_{55}+55\log\theta+45\log(1-\theta)\right]=0$$

$$\Rightarrow\frac{55}{\theta}-\frac{45}{1-\theta}=0\Rightarrow\theta=0.55$$

5. A population has a density function given by

$$f(x)=\begin{cases}(k+1)x^k & 0\le x\le 1\\ 0 & otherwise\end{cases}$$

For n observations, $X_1, X_2,........X_n$ made from this population, find the maximum likelihood of k. Ans: $-1-\frac{n}{\ln(x_1*x_2*....*x_n)}$

$$\left[\begin{array}{l} Hint:\ likelihood(k)= f(x_1,k)*f(x_2,k)*....*f(x_n,k)\\ \qquad =(k+1)^n*(x_1*x_2*...*x_n)^n \qquad Take\ \log likelihood\ and\ find\ MLE. \end{array}\right]$$

6. Suppose 15 rats are used in a biomedical study where the rats are injected with cancer cells and given cancer drug that is designed to increase their survival rate. The survival times, in months, are 14,17,27,18,12,8,22,13,19 and 12. Assume that the exponential distribution applies, find maximum likelihood estimate of mean survival.

$$\left[\text{Hint}: Calculate\ln L(x_{1,}x_2,...x_{10};\theta)\ ;\theta=\bar{x}=16.2\right]$$

It is known that a sample of 12, 11.2, 13.5, 12.3, 13.8, and 11.9 comes from a population with the density function $f(x;\theta)=\begin{cases}\frac{\theta}{x^{\theta+1}}, & x>1\\ 0, & otherwise\end{cases}$ $where\ \ \theta>0$. Find the maximum likelihood estimate of θ .

$$\left[Ans:\theta=\frac{n}{\sum_{i=1}^{n}\ln(x_i)}\ ,\text{after putting sample points},\theta=0.3970\right]$$

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MBA.Tech.(AI) III Semester

Random Processes and Estimation Techniques

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