UNIT 2: FOURIER SERIES

Class Work Problems

Fourier Series in $(0,2\pi)$

1. Find the Fourier series for $f(x)=x$ in $\left(0,2\pi\right)$.

$$(\textbf{Ans: } x=\pi-2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+...\right])$$

2. Obtain the Fourier series for $f(x)=x\sin x$ in $\left(0,2\pi\right)$.

$$(\textbf{Ans: } x\sin x=-1+\pi\sin x-\frac{1}{2}\cos x+\frac{2}{2^2-1}\cos 2x+\frac{2}{3^2-1}\cos 3x+...)$$

3. Find the Fourier series to represent $f\left(x\right)=x^2$ in $\left(0,2\pi\right)$ and hence deduce
$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...$

$$(\textbf{Ans: } x^2=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx-4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx)$$

4. Find the Fourier series to represent $f(x)=e^{-x}$ in $\left(0,2\pi\right)$ and $f(x+2\pi)=f(x)$ and hence deduce $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$

$$(\textbf{Ans: } e^{-x}=\frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1+n^2}\left(\cos nx+n\sin nx\right)\right]$$

5. Find the Fourier series to represent $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in $\left(0,2\pi\right)$ and $f(x+2\pi)=f(x)$ and hence deduce

   a. $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...$
   b. $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...$

$$(\textbf{Ans: } \left(\frac{\pi-x}{2}\right)^2=\frac{\pi^2}{12}+\frac{1}{1^2}\cos x+\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+...)$$

Fourier Series in $(-\pi,\pi)$

1. Find the Fourier series to represent $f(x)=e^x$ in $\left(-\pi,\pi\right)$.

$$(\textbf{Ans: } e^x=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\frac{1}{2}\cos x+\frac{1}{5}\cos 2x-\frac{1}{10}\cos 3x+...+\frac{1}{2}\sin x-\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x-...\right])$$

2. Find the Fourier series for $f(x)=x+x^2$ in $-\pi<x<\pi$. Hence deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

(**Ans:** $f(x)=\frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+..\right]-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots\right]$)

3. Find the Fourier series of the function $f(x)=\begin{cases}-1; & -\pi<x<-\pi/2\\ 0; & -\pi/2<x<\pi/2\\ +1; & \pi/2<x<\pi\end{cases}$

(**Ans:** $f(x)=\frac{1}{\pi}\left[2\sin x-2\sin 2x+\frac{2}{3}\sin 3x+\ldots\right]$)

4. Find the Fourier expansion of $f(x)=\begin{cases}0, & -\pi\le x\le 0\\ x, & 0\le x\prec\pi\end{cases}$. Hence deduce that

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

(**Ans:** $f(x)=\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$)

5. Find the Fourier expansion of $f(x)=\begin{cases}1/2, & -\pi<x<0\\ x/\pi, & 0<x<\pi\end{cases}$.

(**Ans:** $f(x)=\frac{1}{2}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}$)

Fourier Series in $(c,c+2l)$, $(-l,l)$

1. Find the Fourier series for $f(x)=4-x^2$ in the interval $(0,2)$

(**Ans:** $\begin{aligned}f(x)&=\frac{8}{3}-\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x+\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x+\ldots\right]\\&-\frac{4}{\pi}\left[\frac{1}{1}\sin\pi x+\frac{1}{2}\sin 2\pi x+\frac{1}{3}\sin 3\pi x+\ldots\right]\end{aligned}$).

2. Find the Fourier expansion of $f(x)=\begin{cases}2; & -2<x<0\\ \text{x}; & 0<x<2\end{cases}$.

(**Ans:** $\begin{aligned}f(x)&=\frac{3}{2}-\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\ldots\right]\\&-\frac{2}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2}+\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\ldots\right]\end{aligned}$).

3. Expand $f(x)=e^{-x}$ as a Fourier series in the interval $(-l,l)$

(Ans: $e^{-x} = \sinh l \left[\begin{array}{l} \frac{1}{l} - 2l\left(\frac{1}{1^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{1^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{1^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \dots \right) \\ -2\pi\left(\frac{1}{1^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{1^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{1^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \dots \right) \end{array} \right]$).

4. Obtain the Fourier expansion of $f(x) = |x|$, in $-2 < x < 2$.

(**Ans:** $f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi x}{n^3}$)

5. Find the Fourier expansion of $f(x) = x - x^3$ in $(-1,1)$

(**Ans:** $f(x) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}$)

Half range sine and cosine expansions:

1. Express $f(x) = x$ as a cosine half range series in $0 < x < 2$.

(**Ans:** $f(x) = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \dots\right]$).

2. Obtain half range sine series for $f(x) = \begin{cases} x, 0 < x < a \\ a, a < x < \pi - a \\ \pi - x, \pi - a < x < \pi \end{cases}$

(**Ans:** $f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin a \sin x + \frac{1}{3^2}\sin 3a \sin 3x + \frac{1}{5^2}\sin 5a \sin 5x + \dots\right]$

3. Obtain half range cosine series for $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$.

(**Ans:** $f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \dots\right]$)

4. Obtain the half range sine series for $f(x) = e^x$ in $0 < x < 1$.

(**Ans:** $f(x) = 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + \dots\right]$).

5. If $f(x) = \begin{cases} \frac{x}{a}, & 0 < x < a \\ \frac{l-x}{l-a}, & a < x < l \end{cases}$, prove that $f(x) = \frac{2l^2}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$.

On Parseval's Identity, Orthogonality, Orthonormality

1. Find the Fourier series for $f(x) = \begin{cases} x, & 0 < x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$. Hence deduce that

   $\frac{\pi^2}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$

   (**Ans:** $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x$).

2. It is given that for $-\pi < x < \pi$, $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$. By using Parseval's identity prove that $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

3. Show that the set of functions $\cos nx$, $n = 1, 2, 3, \dots$ is orthogonal on $(0, 2\pi)$.

4. Show that the set of functions $\{\sin(2n+1)x\}$, $n = 0, 1, 2, \dots$ is orthogonal over $\left(0, \pi/2\right)$. Hence construct orthonormal set of functions.

   (**Ans:** $\frac{2}{\sqrt{\pi}}\sin x$, $\frac{2}{\sqrt{\pi}}\sin 3x$, $\frac{2}{\sqrt{\pi}}\sin 5x$, ...)

5. Show that the set of functions $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, ... is orthogonal on $(0, 2\pi)$. Write down the orthonormal set.

   (**Ans:** $\frac{1}{\sqrt{2\pi}}$, $\frac{1}{\sqrt{\pi}}\sin x$, $\frac{1}{\sqrt{\pi}}\cos x$, $\frac{1}{\sqrt{\pi}}\sin 2x$, $\frac{1}{\sqrt{\pi}}\cos 2x$ ...)

Complex form of Fourier Series and Fourier Integral representation

Classroom Problems

1. Find complex form of Fourier series for the function $f(x) = e^{ax}$, $-\pi < x < \pi$

   Ans. $e^{ax} = \frac{\sinh a\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{a+in}{a^2+n^2}e^{inx}$

2. Find complex form of Fourier series for the function $f(x) = \cos ax$, $-\pi < x < \pi$

   Ans. $\cos ax = \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2-n^2}e^{inx}$

3. Find complex form of Fourier series for the function $f(x) = \sinh ax$, $-l < x < l$

   Ans. $\sinh ax = \sinh al\sum_{n=-\infty}^{\infty}\frac{(-1)^n in\pi}{a^2l^2+n^2\pi^2}e^{i\left(\frac{n\pi x}{l}\right)}$

4. Find complex form of Fourier series for the function $f(x)=e^{-x},\ -1<x<1$

Ans. $e^{-x}=\sum_{n=-\infty}^{\infty}(-1)^n(\sinh 1)\left(\frac{1-in\pi}{1+n^2\pi^2}\right)e^{in\pi x}$

5. Express the function $f(x)=\begin{cases}1, & for\ |x|<1\\ 0, & for\ |x|>1\end{cases}$ as Fourier Integral and hence evaluate

$\int_0^{\infty}\frac{\sin\varpi\cos\varpi x}{\varpi}d\varpi$. **Ans.** $\frac{\pi}{2}f(x)$

6. Express the function $f(x)=\begin{cases}-e^{kx}, & for\ x<0\\ e^{-kx}, & for\ x>0\end{cases}$ as Fourier Integral and hence show that

$\int_0^{\infty}\frac{\varpi\cos\varpi x}{\varpi^2+k^2}d\varpi=\frac{\pi}{2}e^{-kx}, x>0, k>0$.

7. Using Fourier Integral representation, show that $\int_0^{\infty}\frac{\varpi\sin\varpi x}{1+\varpi^2}d\varpi=\frac{\pi}{2}e^{-x}\,(x>0)$.

8. Express $e^{-x}\cos x$ as a Fourier cosine integral and show that

$e^{-x}\cos x=\frac{2}{\pi}\int_0^{\infty}\frac{(\varpi^2+2)}{\varpi^4+4}\cos\varpi x\,d\varpi,(x\geq 0)$

9. Express the function $f(x)=\begin{cases}\pi/2, & for\ 0<x<\pi\\ 0, & for\ x>\pi\end{cases}$ as Fourier sine integral. Hence show that

$\int_0^{\infty}\frac{1-\cos\varpi\pi}{\varpi}\sin\varpi x\,d\varpi=\frac{\pi}{2}, 0<x<\pi$

10. Find Fourier sine integral of $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x>2\end{cases}$.

Ans. $f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\varpi x\left(\frac{2\sin\varpi-\sin 2\varpi}{\varpi^2}\right)d\varpi$

UNIT 3: FOURIER SERIES

Tutorial 1

Topic: (Fourier series for the interval $[0,2\pi]$ and $[-\pi,\pi]$

1. Find the Fourier series to represent $f(x)=e^x$ in $(0,2\pi)$.

(**Ans:** $e^x = \dfrac{e^{2\pi}-1}{\pi}\left[\dfrac{1}{2}+\sum_{n=1}^{\infty}\dfrac{1}{1+n^2}(\cos nx - n\sin nx)\right]$

2. Find the Fourier series to represent $f(x)=\pi-x$ for $0<x<2\pi$.

(**Ans:** $\pi - x = 2\left[\sin x+\dfrac{1}{2}\sin 2x+\dfrac{1}{3}\sin 3x+...+\dfrac{1}{n}\sin nx+...\right]$)

3. Obtain the Fourier series for $f(x)=\begin{cases}\sin x; & 0\le x\le\pi\\ 0; & \pi\le x\le 2\pi\end{cases}$. Hence deduce that

$$\frac{1}{2}=\frac{1}{1\bullet 3}+\frac{1}{3\bullet 5}+\frac{1}{5\bullet 7}+...$$

(**Ans:** $f(x)=\dfrac{1}{\pi}-\dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2-1}\cos 2nx+\dfrac{1}{2}\sin x$)

4. Find the Fourier series for $f(x)=\begin{cases}x+\pi; & 0\le x\le\pi\\ \text{-x-}\pi; & -\pi\le x<0\end{cases}$ and $f(x+2\pi)=f(x)$.

(**Ans:** $f(x)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\left[\dfrac{1}{1^2}\cos x+\dfrac{1}{3^2}\cos 3x+...\right]+4\left[\dfrac{\sin x}{1}+\dfrac{\sin 3x}{3}+...\right]$)

5. Obtain a Fourier series to represent the following periodic function $f(x)=\begin{cases}0; & 0<x<\pi\\ 1; & \pi<x<2\pi\end{cases}$

(**Ans:** $f(x)=\dfrac{1}{2}-\dfrac{2}{\pi}\left[\sin x+\dfrac{1}{3}\sin 3x+\dfrac{1}{5}\sin 5x+...\right]$)

6. Find the Fourier series for $f(x)=\begin{cases}-\pi; & -\pi<x<0\\ x; & 0<x<\pi\\ \dfrac{-\pi}{2}; & x=0\end{cases}$ in $-\pi<x<\pi$. Hence deduce that

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$$

(**Ans:**

$f(x)=\dfrac{-\pi}{4}-\dfrac{2}{\pi}\left[\cos x+\dfrac{1}{3^2}\cos 3x+\dfrac{1}{5^2}\cos 5x+..\right]+3\sin x-\dfrac{1}{2}\sin 2x+\dfrac{1}{3}\sin 3x-\dfrac{1}{4}\sin 4x+...$)

7. An alternating current after passing through a rectifier has the form $i = \begin{cases} I\sin\theta; & 0<\theta<\pi \\ 0 \quad ; & \pi<\theta<2\pi \end{cases}$. Find the Fourier series of the function.

(**Ans:** $f\left(x\right)=\frac{I}{\pi}-\frac{2I}{\pi}\left[\frac{\cos 2\theta}{3}+\frac{\cos 4\theta}{15}+...\right]+\frac{I}{2}\sin\theta$)

8. Obtain the Fourier expression for $f(x)=x^3$ for $-\pi<x<\pi$.

(**Ans:** $f\left(x\right)=2\left[-\left(-\frac{\pi^2}{1}+\frac{6}{1^3}\right)\sin x+\left(-\frac{\pi^2}{2}+\frac{6}{2^3}\right)\sin 2x-\left(-\frac{\pi^2}{3}+\frac{6}{3^3}\right)\sin 3x...\right]$)

9. Expand $f(x)=x\cos x$ in the interval $0\le x\le 2\pi$.

(**Ans:** $f(x)=-\frac{1}{2}+\sum_{n=2}^{\infty}\left(\frac{-2n}{n^2-1}\right)\sin nx$)

10. Obtain the Fourier expansion of $f(x)=\begin{cases}\cos x; & -\pi<x<0 \\ \text{-cosx}; & 0<x<\pi\end{cases}$ and $f\left(x+2\pi\right)=f\left(x\right)$.

(**Ans:** $f(x)=-\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{2n}{4n^2-1}\right)\sin 2nx$)

Tutorial 2

Topic: Fourier series for the interval $(c, c+2l)$, $(-l,l)$ and half range series

1. Find the Fourier series for $f(x)=x^2$ in the interval $(0,a)$. Hence deduce that

   $$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$$

   (**Ans:** $f(x)=\frac{a^2}{3}+\frac{a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{a}+\frac{1}{2^2}\cos\frac{4\pi x}{a}+\dots\right]-\frac{a^2}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{a}+\frac{1}{2}\sin\frac{4\pi x}{a}+\dots\right]$).

2. Find the Fourier expansion of $f(x)=\begin{cases}\pi x; & 0\le x\le 1\\ \pi(2-x); & 1\le x\le 2\end{cases}$ with period 2, show that .

   (**Ans:** $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$).

3. Expand $f(x)=2-\frac{x^2}{2}$ as a Fourier series in the interval $0\le x\le 2$

   (**Ans:** $f(x)=\frac{4}{3}-\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi x$).

4. Find the Fourier series expansion of the periodic function of period 1

   $$f(x)=\begin{cases}\frac{1}{2}+x; & -\frac{1}{2}<x\le 0\\ \frac{1}{2}-x; & 0<x<\frac{1}{2}\end{cases}$$, in $-2<x<2$.

   (**Ans:** $f(x)=\frac{1}{4}+\frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2}+\frac{\cos 6\pi x}{3^2}+\frac{\cos 10\pi x}{5^2}+\dots\right]$).

5. A periodic square wave has a period 4. The function generating the square is

   $$f(t)=\begin{cases}0; & -2<t<-1\\ k; & -1<t<1\\ 0; & 1<t<2\end{cases}$$. Find the Fourier series of the function.

   (**Ans:** $f(x)=\frac{k}{2}+\frac{2k}{\pi}\left[\cos\frac{\pi t}{2}-\frac{1}{3}\cos\frac{3\pi t}{2}+\dots\right]$).

6. Find the Fourier half-range cosine series of the function

   $$f(t)=\begin{cases}2t, & 0<t<1\\ 2(2-t), & 1<t<2\end{cases}$$

   (**Ans:** $f(t)=1-\left(\frac{8}{\pi^2}+\frac{4}{\pi}\right)\cos\frac{\pi t}{2}-\frac{4}{\pi^2}\cos\frac{2\pi t}{2}+\left(-\frac{8}{9\pi^2}+\frac{4}{3\pi}\right)\cos\frac{3\pi t}{2}+\dots$).

7. Find the Fourier half range even expansion of the function $f(x)=\left(-{x}/{l}\right)+1,\ 0\le x\le 1$

(**Ans:** $f\left(x\right)=\frac{1}{2}+\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+...\right]$

8. Obtain half range sine series for $f(x)=\begin{cases}\frac{1}{4}-x, & 0<x<{1}/{2}\\ x-\frac{3}{4}, & {1}/{2}<x<1\end{cases}$.

(**Ans:** $f\left(x\right)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}-\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+...$)

9. Obtain the half range sine series for $f(x)=2x-1$ in $0<x<1$.

(**Ans:** $f\left(x\right)=-\frac{2}{\pi}\left[\sin 2\pi x+\frac{1}{2}\sin 4\pi x+\frac{1}{3}\sin 6\pi x+...\right]$).

10. Obtain half range cosine series for $f\left(x\right)=\left(x-1\right)^2$ in $0<x<1$. Hence find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n^2}$.

(**Ans:** $f\left(x\right)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$. Now put $x=0$ and then $x=1$).

Tutorial 3

Topic: Parseval's Identity, Orthogonality, Orthonormality

1. Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

   (**Ans:** $1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$).

2. Prove that in the range $(0, l)$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{l}$ and deduce that

   $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^2} + \ldots$.

3. If $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0, l)$ then show that $\int_0^l \left[f(x)\right]^2 dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n{}^2$.

4. If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ in $0 < x < l$ then show that $\int_0^l \left[f(x)\right]^2 dx = \frac{l}{2}\left[\frac{a_0{}^2}{2} + \sum_{n=1}^{\infty} a_n{}^2\right]$

5. Is $S = \left\{\sin\left(\frac{\pi x}{4}\right), \sin\left(\frac{3\pi x}{4}\right), \sin\left(\frac{5\pi x}{4}\right), \ldots\right\}$ orthogonal in $(0,1)$

   (**Ans:** S is not orthogonal).

6. Show that the set of functions $\sin\left(\frac{\pi x}{2L}\right)$, $\sin\left(\frac{3\pi x}{2L}\right)$, $\sin\left(\frac{5\pi x}{2L}\right)$, ... is orthogonal over $(0, L)$

7. Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = \left(\frac{3x^2 - 1}{2}\right)$ are orthogonal over $(-1,1)$.

8. Show that the set of functions $\{\sin(2n+1)x\}$, $n = 0,1,2,\ldots$ is orthogonal over $\left(0, \pi/2\right)$.

9. Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on $(-1,1)$. Determine the constants a and b such that the function $f_2(x) = -1 + ax + bx^2$ is orthogonal to both f_1 and f_2 on that interval.

   (**Ans:** $a = 0$, $b = 3$)

10. Show that the set of functions $\cos x$, $\cos 2x$, $\cos 3x$, ... is a set of orthogonal functions over $(-\pi, \pi)$. Hence construct a set of orthonormal functions.

    (**Ans:** $\frac{\cos x}{\sqrt{\pi}}$, $\frac{\cos 2x}{\sqrt{\pi}}$, $\frac{\cos 3x}{\sqrt{\pi}}$,

Tutorial 4

1. Find complex form of Fourier series for the function $f(x)=e^{x},\ -\pi<x<\pi$

   Ans. $e^{x}=\dfrac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^{n}\dfrac{1+in}{1+n^{2}}e^{inx}$

2. Find complex form of Fourier series for the function $f(x)=e^{ax},\ -l<x<l$

   Ans. $e^{ax}=\sum_{n=-\infty}^{\infty}(-1)^{n}\sinh al\left(\dfrac{al+in\pi}{a^{2}l^{2}+n^{2}\pi^{2}}\right)e^{i\left(\frac{n\pi x}{l}\right)}$

3. Find complex form of Fourier series for the function $f(x)=\sin x,\ -\pi<x<\pi$

   Ans. $\sin x=\dfrac{\sin\pi}{\pi}\sum_{n=-\infty}^{\infty}\dfrac{n}{1-n^{2}}e^{inx}$

4. Find complex form of Fourier series for the function $f(x)=\cosh ax,\ -l<x<l$

   Ans. $\cosh ax=al\sinh al\sum_{n=-\infty}^{\infty}\dfrac{(-1)^{n}}{a^{2}l^{2}+n^{2}\pi^{2}}e^{i\left(\frac{n\pi x}{l}\right)}$

5. Find complex form of Fourier series for the function $f(x)=\cos x,\ -\pi<x<\pi$

   Ans. $\cos x=\dfrac{\sin\pi}{\pi}\sum_{n=-\infty}^{\infty}\dfrac{(-1)^{n}}{1-n^{2}}e^{inx}$

6. Find complex form of Fourier series for the function $f(x)=2x,\ 0<x<2\pi$

   Ans. $2x=2\pi+2i\sum_{n=-\infty}^{\infty}\dfrac{1}{n}e^{inx},\ n\neq 0$

7. Using Fourier Integral representation, show that $\displaystyle\int_{0}^{\infty}\frac{\cos\varpi x}{1+\varpi^{2}}d\varpi=\frac{\pi}{2}e^{-x}\,(x\geq 0)$.

8. Express the function $f(x)=\begin{cases}e^{kx}, & for\ \ x<0\\ e^{-kx}, & for\ \ x>0\end{cases}$ as Fourier Integral and hence show that

   $\displaystyle\int_{0}^{\infty}\frac{\cos\varpi x}{\varpi^{2}+k^{2}}d\varpi=\frac{\pi}{2k}e^{-kx},x>0,k>0$.

9. Express $e^{-x}\sin x$ as a Fourier sine integral and show that

   $e^{-x}\sin x=\dfrac{4}{\pi}\displaystyle\int_{0}^{\infty}\frac{\varpi\sin\varpi x}{\varpi^{4}+4}d\varpi,(x\geq 0)$

10. Find Fourier cosine integral representation for $f(x)=\begin{cases}1-x^2, 0\le x\le 1\\ 0, \qquad x>1\end{cases}$.

Ans. $1-x^2=\dfrac{4}{\pi}\int_0^\infty \cos\varpi x\left(\dfrac{\sin\varpi-\varpi\cos\varpi}{\varpi}\right)d\varpi$

* Unit-3 Fourier Transform :-

It is a mathematical tool used for frequency analysis of signals (L.T. is used for analysis of systems & circuits).

Fourier Transform a non-periodic function $f(t)$ is time domain into a function of frequency domain. These transforms are highly useful in the study of conduction of heat, wave propogation, Communication etc.

Def^n:- If $f(x)$ is defined on $(-\infty,\infty)$ which is continuous & integrable in $(-\infty,\infty)$ then the fourier transform of $f(x)$ is denoted by $F\{f(x)\}$ or $F(s)$ & is defined as

$$\boxed{F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)}$$

Ex 1) Find the fourier transform of $f(x)$ if

$$f(x) = \begin{cases} e^{i\omega x}, & a<x<b \\ 0, & x<a \ \& \ x>b \end{cases}$$

⇒ We know that

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{i\omega x} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(\omega+s)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(\omega+s)x}}{i(\omega+s)} \right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i(\omega+s)} \left[e^{i(s+\omega)b} - e^{i(\omega+s)a} \right]$$

$$= \frac{1}{\sqrt{\pi}} \frac{i}{\omega+s} \left[e^{i(\omega+s)a} - e^{i(s+\omega)b} \right]$$

Ex 2) Find fourier transform of $f(x) = \begin{cases} 1-|x| , & |x|<1 \\ 0 , & |x|>1 . \end{cases}$

$\Rightarrow$ Given $f(x) = \begin{cases} 1-|x| , & |x|<1 \text{ ie. } -1<x<1 \\ 0 , & |x|>1 \text{ ie } x<-1 \ \& \ x>1 .. \end{cases}$

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{isx} dx$$

but $|x| = \begin{cases} x , & x>0 \\ -x , & x<0 \end{cases}$

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (1-(-x)) e^{isx} dx + \int_{0}^{1} (1-(x)) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{0} (1+x) e^{isx} dx + \int_{0}^{1} (1-x) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \Bigg[\left[(1+x) \frac{e^{isx}}{(is)} - (1) \frac{e^{isx}}{(is)^2} \right]_{-1}^{0} + \left[(1-x) \frac{e^{isx}}{(is)} - (-1) \frac{e^{isx}}{(is)^2} \right]_{0}^{1} \Bigg]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{\frac{1}{is}} + \frac{1}{s^2} - 0 - \frac{e^{-is}}{s^2} + 0 + \frac{e^{is}}{-s^2} - \cancel{\frac{1}{is}} + \frac{1}{s^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{s^2} - \frac{e^{-is}}{s^2} - \frac{e^{is}}{s^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{s^2} - \frac{2}{s^2} \left[\frac{e^{is} + e^{-is}}{2} \right] \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{1}{s^2} - \frac{1}{s^2} \cos s \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

* Inverse Fourier Transform:-

If $F(s)$ is the Fourier transform of $f(x)$ & if $f(x)$ satisfies Dirichlet's conditions in $(-l, l)$ then

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds}$$ is called as inverse Fourier transform of $F(s)$

Ex1) find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases}$ & hence evaluate a) $\int_{-\infty}^{\infty} \frac{\sin sk \cos sx}{s} ds$, b) $\int_{-\infty}^{\infty} \frac{\sin ks}{s} ds$, c) $\int_{-\infty}^{\infty} \frac{\sin s}{s} ds$.

→ We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} e^{isx} dx$$

$\{\because f(x) = 1, -k < x < k\}$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-k}^{k} = \frac{1}{\sqrt{2\pi}} \frac{1}{is} \left[e^{isk} - e^{-isk}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{s} \left[\frac{e^{isk} - e^{-isk}}{2i}\right] \cdot 2$$

$$\boxed{F(s) = \sqrt{\frac{2}{\pi}} \frac{\sin sk}{s}}$$

* Inverse fourier transform:-

a) $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin sk}{s} e^{-isx} ds.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sk}{s} [\cos sx - i \sin sx]\, ds.$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{\sin sk \cos sx}{s}}_{\text{even function}} ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \cancel{\underbrace{\frac{\sin sk \sin sx}{s}}_{\text{odd function}}}^{\,0} ds.$$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin sk \cos sx}{s} ds$$

$\because \int_{-\infty}^{\infty} \text{odd function} = 0.$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin sk \cos sx}{s} ds = \pi f(x) = \pi \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases} = \begin{cases} \pi, & |x| < k \\ 0, & |x| > k \end{cases}$$

b) put $x = 0$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin sk}{s} ds = \pi$$

c) put $k=1$ in (b)

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi$$

Ex 2) Find the Fourier transform of $f(x)=\begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$

& hence evaluate $\int_0^\infty \frac{x\cos x-\sin x}{s^3}\cos\frac{x}{2}\,dx$.

$\Rightarrow$ Given $f(x)=\begin{cases}1-x^2, & |x|\le 1 \text{ ie } -1\le x\le 1\\ 0, & |x|>1 \text{ ie } x<-1 \text{ \& } x>1.\end{cases}$

Now $F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}dx.$

$=\frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is}-(-2x)\frac{e^{isx}}{(is)^2}+(-2)\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$

$=\frac{1}{\sqrt{2\pi}}\left[0-2\frac{e^{is}}{s^2}+\frac{2e^{is}}{is^3}-0-2\frac{e^{-is}}{s^2}-\frac{2e^{-is}}{is^3}\right]$

$=\frac{1}{\sqrt{2\pi}}\left[-\frac{2}{s^2}\left[e^{is}+e^{-is}\right]+\frac{2}{s^3}\left[\frac{e^{is}-e^{-is}}{i}\right]\right]$

$=\frac{1}{\sqrt{2\pi}}\left[-\frac{4}{s^2}\cos s+\frac{4}{s^3}\sin s\right]$

$F(s)=-2\sqrt{\frac{2}{\pi}}\left[\frac{s\cos s-\sin s}{s^3}\right]$

Inverse Fourier transform :-

$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}-2\sqrt{\frac{2}{\pi}}\left[\frac{s\cos s-\sin s}{s^3}\right]e^{-isx}ds$

$=\frac{-2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)(\cos sx-i\sin sx)\,ds$

$=\frac{-2}{\pi}\left[\underbrace{\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,dx}_{\text{even function}}-i\underbrace{\cancel{\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\sin sx\,ds}}_{\text{odd function}}\right]$

$f(x)=-\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,dx$

$\therefore\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos sx\,dx=-\frac{\pi}{2}f(x)=-\frac{\pi}{2}\begin{cases}(1-x^2), & |x|\le 1\\ 0, & |x|>1\end{cases}$

$=\begin{cases}-\frac{\pi}{2}(1-x^2), & |x|\le 1\\ 0, & |x|>1\end{cases}$

put $x=\frac{1}{2}$

$\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)\cos\frac{s}{2}\,ds=-\frac{\pi}{2}\left(1-\frac{1}{4}\right)=-\frac{3\pi}{8}.$

replace $s\to x$.

$\int_{-\infty}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx=-\frac{3\pi}{8}$

Note :- Some authors define fourier & inverse fourier transform in different ways.

e.g.

1) $F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$ & $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$.

2) $F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$ & $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$.

3) $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$ & $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{isx}\, ds$.

* **Fourier Sine Transform :-**

fourier sine transform OR infinite fourier sine transform of $f(x)$, $0<x<\infty$ is denoted by $F_s(s)$ & is defined as

$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx}$$

Ex1) Find fourier sine transform of $f(x) = \begin{cases} \sin kx, & 0<x<a \\ 0, & x>a \end{cases}$.

$\Rightarrow$ we know that $F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^{a} \sin kx \sin sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^{a} \frac{1}{2}\left[\cos(k-s)x - \cos(k+s)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(k-s)x}{k-s} - \frac{\sin(k+s)x}{k+s}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(k-s)a}{k-s} - \frac{\sin(k+s)a}{k+s}\right]$$

Ex2) Find fourier sine transform of $f(x) = \frac{1}{x}$, $x>0$.

$\Rightarrow$ $F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\sin sx}{x}\, dx$.

put $sx = t \Rightarrow s\,dx = dt \Rightarrow dx = \frac{dt}{s}$

when $x=0 \Rightarrow t=0$ & $x=\infty \Rightarrow t=\infty$

$$\therefore F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\sin t}{t/\cancel{s}}\,\frac{dt}{\cancel{s}} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{\sin t}{t}\, dt$$

but $\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

$$\therefore F_s(s) = \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

* Inverse fourier Sine Transform :-

If $F_s(s)$ is the fourier Sine transform of $f(x)$ then

$$\boxed{f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds}$$ is called as inverse fourier Sine transform of $F_s(s)$.

Ex1) Find fourier sine transform of $e^{-|x|}$ & hence show that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m},\ m>0.$$

$\Rightarrow$ $F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-|x|}\sin sx\,dx$

but $|x|=x$ in $(0,\infty)$

$$\therefore F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-1\sin sx-s\cos sx)\right]_0^\infty$$

$$\boxed{\because \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)}$$

$$=\sqrt{\frac{2}{\pi}}\left[0-\frac{1}{1+s^2}(0-s)\right]$$

$$F_s(s)=\sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}$$

consider inverse sine transform is

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{s}{1+s^2}\sin sx\,ds$$

$$=\frac{2}{\pi}\int_0^\infty\frac{s\sin sx}{1+s^2}ds$$

$$\Rightarrow \int_0^\infty \frac{s\sin sx}{1+s^2}ds=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-|x|}.$$

replace $x\to m$

$$\int_0^\infty\frac{s\sin ms}{1+s^2}ds=\frac{\pi}{2}e^{-|m|}$$

replace $s\to x$

$$\int_0^\infty\frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-|m|}$$

$$=\frac{\pi}{2}e^{-m},\ m>0$$

Ex12) find the fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ & hence evaluate $\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx$. (7)

$\Rightarrow$ consider $F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Differentiating w.r.t s on both sides.

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \frac{d}{ds}(\sin sx) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos sx \cdot \cancel{x} \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} \left(-a\cos sx + s \sin sx \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (-a+0) \right]$$

$$\therefore \frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

Integrating w.r.t. s, we get

$$\int \frac{d}{ds} F_s(s) ds = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2+s^2} ds.$$

$$\Rightarrow F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \qquad \because \underline{\underline{\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)}}$$

Inverse fourier sine transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \tan^{-1}\left(\frac{s}{a}\right) \sin sx \, ds$$

$$\therefore \int_0^\infty \tan^{-1}\left(\frac{s}{a}\right) \sin sx \, ds = \frac{\pi}{2} f(x) = \frac{\pi}{2} \frac{e^{-ax}}{x}.$$

put $x = 1$

$$\therefore \int_0^\infty \tan^{-1}\left(\frac{s}{a}\right) \sin s \, ds = \frac{\pi}{2} e^{-a}.$$

replace $s \to x$

$$\therefore \underline{\underline{\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi}{2} e^{-a}}}$$

* Fourier Cosine Transform :-

The infinite fourier cosine transform of $f(x)$, $0<x<\infty$ is denoted by $F_c(s)$ & is defined as

$$\boxed{F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx}$$

Ex 1) find the fourier cosine transform of $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

$\Rightarrow$ $$F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\,dx+\int_1^2(2-x)\cos sx\,dx+0\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\left[x\left(\frac{\sin sx}{s}\right)-(1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(-\frac{\cos sx}{s^2}\right)\right]_1^2\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\cancel{\frac{\sin s}{s}}+\frac{\cos s}{s^2}-0-\frac{1}{s^2}+0-\frac{\cos 2s}{s^2}-\cancel{\frac{\sin s}{s}}+\frac{\cos s}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos s-\cos 2s-1}{s^2}\right]$$

Ex 2) find fourier cosine transform of e^{-x^2}.

$\Rightarrow$ $$F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\,dx$$

Differentiating w.r.t. s

$$\frac{d}{ds}F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}(-\sin sx\cdot x)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \underbrace{\sin sx}_{u}\ \underbrace{(-e^{-x^2}\cdot x)}_{v}\,dx$$

but $\int uv\,dx=u\int v\,dx-\int\left(\frac{du}{dx}\int v\,dx\right)dx$

$$\therefore \frac{d}{ds}F_c(s)=\sqrt{\frac{2}{\pi}}\left[\left[\sin sx\int -e^{-x^2}x\,dx\right]_0^\infty-\int_0^\infty\left(\frac{d}{dx}\sin sx\right)\left(\int -e^{-x^2}x\,dx\right)dx\right]\to (1)$$

Consider $I=\int -e^{-x^2}x\,dx$ put $x^2=t \Rightarrow 2x\,dx=dt \Rightarrow x\,dx=\frac{dt}{2}$.

When $x=0 \Rightarrow t=0$

$x=\infty \Rightarrow t=\infty$.

$\therefore I = \int -e^{-t}\frac{dt}{2} = \frac{1}{2}\frac{e^{-t}}{-1} = \frac{e^{-t}}{2} = \frac{e^{-x^2}}{2}$

$\therefore$ Equation (1) becomes.

$$\frac{d}{ds}F_c(s) = \sqrt{\frac{2}{\pi}}\left[\left[\sin sx\left(\frac{e^{-x^2}}{2}\right)\right]_0^\infty - \int_0^\infty \cos sx \cdot s\,\frac{e^{-x^2}}{2}dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(0-0) - \frac{s}{2}\int_0^\infty e^{-x^2}\cos sx\,dx\right]$$

$$= -\frac{s}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos sx\,dx\right]$$

$\Rightarrow \frac{d}{ds}F_c(s) = -\frac{s}{2}F_c(s)$

$\Rightarrow \frac{\frac{d}{ds}F_c(s)}{F_c(s)} = -\frac{s}{2}$

Integrating w.r.t. s

$\int \frac{\frac{d}{ds}F_c(s)}{F_c(s)}ds = -\frac{1}{2}\int s\,ds$ $\quad\Big|\quad \because \int \frac{F'(x)}{F(x)}dx = \log F(x)$

$\Rightarrow \log F_c(s) = -\frac{s^2}{4} + \log c$

When $s=0 \Rightarrow F_c(s) = F_c(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}dx = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$

$\therefore \log\frac{1}{\sqrt{2}} = \log c \Rightarrow \boxed{c = \frac{1}{\sqrt{2}}}$

$\therefore \log F_c(s) = -\frac{s^2}{4} + \log\frac{1}{\sqrt{2}}$

$\Rightarrow \log\left(\frac{F_c(s)}{1/\sqrt{2}}\right) = -\frac{s^2}{4}$

$\Rightarrow \frac{F_c(s)}{1/\sqrt{2}} = e^{-\frac{s^2}{4}}.$

$\Rightarrow F_c(s) = \frac{1}{\sqrt{2}}e^{-\frac{s^2}{4}}$

* Inverse fourier cosine Transform:-

If $F_c(s)$ is fourier cosine transform of $f(x)$ then $\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds}$ is called as inverse fourier cosine transform of $F_c(s)$

Ex1) Find fourier cosine transform of e^{-ax} & hence evaluate $\int_0^\infty \frac{\cos \lambda x}{x^2+\alpha^2} dx$.

$\Rightarrow$ We know that

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{s^2+a^2} \left(-a \cos sx + s \sin sx \right) \right]_0^\infty$$

$$\left\{ \because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{s^2+a^2} (-a+0) \right]$$

$$\therefore F_c(s) = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Consider inverse fourier cosine transform:-

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \cos sx \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2} ds$$

$$\therefore \int_0^\infty \frac{\cos sx}{s^2+a^2} ds = \frac{\pi}{2a} f(x)$$

$$= \frac{\pi}{2a} e^{-ax}$$

put $x = \lambda$

$$\int_0^\infty \frac{\cos s\lambda}{s^2+a^2} ds = \frac{\pi}{2a} e^{-a\lambda}$$

replace $s \to x$ & $a \to \alpha$

$$\int_0^\infty \frac{\cos \lambda x}{x^2+\alpha^2} dx = \frac{\pi}{2\alpha} e^{-\lambda\alpha}.$$

(10)

Q.no 2) Solve the integral equation $\int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

& hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2} dt$.

$\Rightarrow$ Fourier cosine Transform is

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx$$

put $x = \theta$ & $s = \alpha$

$$\therefore F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\theta)\cos\alpha\theta\, d\theta = \sqrt{\frac{2}{\pi}} \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

$$\therefore F_c(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\alpha), & 0 \le \alpha \le 1 \\ 0, & \alpha > 1. \end{cases} \longrightarrow (1)$$

By Inverse Fourier cosine transform, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s)\cos sx\, ds$$

put $x = \theta$ & $s = \alpha$

$$\therefore f(\theta) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha)\cos\alpha\theta\, d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\frac{2}{\pi}} (1-\alpha)\cos\alpha\theta\, d\alpha \quad \because \text{by } (1)$$

$$= \frac{2}{\pi}\left[(1-\alpha)\left(\frac{\sin\alpha\theta}{\theta}\right) - (-1)\left(-\frac{\cos\alpha\theta}{\theta^2}\right) \right]_0^1$$

$$= \frac{2}{\pi}\left[0 - \frac{\cos\theta}{\theta^2} - 0 + \frac{1}{\theta^2} \right]$$

$$\therefore f(\theta) = \frac{2}{\pi}\left[\frac{1-\cos\theta}{\theta^2}\right] \rightarrow \text{sol}^n \text{ of integral equation.}$$

consider

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\theta)\cos\alpha\theta\, d\theta$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2}{\pi}\left[\frac{1-\cos\theta}{\theta^2}\right]\cos\alpha\theta\, d\theta.$$

$$= \frac{2}{\pi}\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\cos\alpha\theta\, d\theta.$$

put $\alpha = 0$

$$\therefore F_c(0) = \frac{4}{\pi}\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin^2\theta/2}{\theta^2} d\theta. \quad \text{but } F_c(0) = \sqrt{\frac{2}{\pi}}(1-0)$$

$$\therefore \cancel{\sqrt{\frac{2}{\pi}}} = \frac{4}{\pi}\cancel{\sqrt{\frac{2}{\pi}}} \int_0^\infty \frac{\sin^2\theta/2}{\theta^2} d\theta.$$

$$\therefore \int_0^\infty \frac{\sin^2\theta/2}{\theta^2} d\theta = \frac{\pi}{4} \qquad \text{put } \frac{\theta}{2} = t \Rightarrow \theta = 2t$$
$$\Rightarrow d\theta = 2dt$$

when $\theta = 0 \Rightarrow t = 0$
$\theta = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \frac{\sin^2 t}{4t^2} 2dt = \frac{\pi}{4}.$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

* Properties of Fourier Transform:-

① Linearity property:- If $f(x)$ & $g(x)$ are two functions &
$F\{f(x)\} = F(s)$, $F\{g(x)\} = G(s)$ & a, b are constants then

$$F\{a f(x) \pm b g(x)\} = a F\{f(x)\} \pm b F\{g(x)\}$$
$$= a F(s) \pm b G(s)$$

2) Change of scale property:-

If $F\{f(x)\} = F(s)$ then $F\{f(ax)\} = \frac{1}{a} F\{\frac{s}{a}\}$, $a \neq 0$

ie. $\boxed{F\{f(ax)\} = \frac{1}{a}\left[F\{f(x)\}\right]_{s \to \frac{s}{a}}}$

Similarly:- for fourier cosine & sine transforms, we have

$$F_s\{f(ax)\} = \frac{1}{a}\left[F_s\{f(x)\}\right]_{s\to\frac{s}{a}} \quad \&$$

$$F_c\{f(ax)\} = \frac{1}{a}\left[F_c\{f(x)\}\right]_{s\to\frac{s}{a}} .$$

3) Shifting property:-

If $F\{f(x)\} = F(s)$ then $F\{f(x-a)\} = e^{isa} F(s)$

ie $\boxed{F\{f(x-a)\} = e^{isa}\left[F\{f(x)\}\right]}$

Ex1) Given $F\{e^{-x^2}\} = \sqrt{\pi}\, e^{-s^2/4}$, find fourier transform of
i) $e^{-\frac{x^2}{3}}$, ii) $e^{-4(x-3)^2}$

⇒ Let $f(x) = e^{-x^2}$ ⇒ $F\{f(x)\} = F\{e^{-x^2}\} = \sqrt{\pi}\, e^{-\frac{s^2}{4}} = F(s)$.

i) for $e^{-\frac{x^2}{3}} = e^{-\left(\frac{x}{\sqrt{3}}\right)^2} = f\left(\frac{x}{\sqrt{3}}\right) = f(ax)$ where $a = \frac{1}{\sqrt{3}}$.

but

$\therefore$ $F\{f(ax)\} = \frac{1}{a}\left[F\{f(x)\}\right]_{s\to\frac{s}{a}}$.

put $a = \frac{1}{\sqrt{3}}$

$$\therefore F\{e^{-\frac{x^2}{3}}\} = F\{f(\tfrac{x}{\sqrt{3}})\} = \frac{1}{1/\sqrt{3}} F\{f(x)\}_{s\to\frac{s}{1/\sqrt{3}}}$$
$$= \sqrt{3}\left[\sqrt{\pi}\, e^{-\frac{s^2}{4}}\right]_{s\to\sqrt{3}s}$$
$$= \sqrt{3}\,\sqrt{\pi}\, e^{-\frac{3s^2}{4}}$$
$$= \sqrt{3\pi}\, e^{-\frac{3s^2}{4}}$$

ii) Let $e^{-4(x-3)^2} = g(x-3) \Rightarrow g(x) = e^{-4x^2} = e^{-(2x)^2} = f(2x)$

but $F\{f(ax)\} = \frac{1}{a}\left[F\{f(x)\}\right]_{s\to \frac{s}{a}}$ where $f(x) = e^{-x^2}$

put $a = 2$

$$\therefore F\{f(2x)\} = F\{e^{-4x^2}\} = \frac{1}{2}\left[\sqrt{\pi}\, e^{-\frac{s^2}{4}}\right]_{s\to\frac{s}{2}}$$

$$= \frac{1}{2}\left[\sqrt{\pi}\, e^{-\frac{s^2}{16}}\right]$$

$$\therefore F\{e^{-4(x-3)^2}\} = F\{g(x-3)\} = e^{3is} F\{g(x)\}$$

$$= e^{3is} F\{e^{-4x^2}\}$$

$$= e^{3is} \frac{\sqrt{\pi}}{2} e^{-\frac{s^2}{16}}$$

$$= \frac{\sqrt{\pi}}{2} e^{\left(3is - \frac{s^2}{16}\right)}$$

Ex2) a) find the fourier transform of $e^{-a^2x^2}$, $a<0$. Hence deduce that $e^{-\frac{x^2}{2}}$ is self reciprocal in respect of fourier transform.

b) find fourier transform of e^{-2x^2} using change of scale property

c) find fourier transform of $e^{-2(x-3)^2}$ using shifting property.

$\Rightarrow$ Let $f(x) = e^{-a^2x^2}$, $a<0$

a) $$\therefore F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2 + isx} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{isx}{a^2}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2\left(x^2 - \frac{isx}{a^2} + \left(\frac{is}{2a^2}\right)^2 - \left(\frac{is}{2a^2}\right)^2\right)} dx$$ $\left(\frac{1}{2}\left(\frac{is}{a^2}\right)\right)^2$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2\left[\left(x - \frac{is}{2a^2}\right)^2 - \left(\frac{is}{2a^2}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2\left(x - \frac{is}{2a^2}\right)^2} \cdot e^{a^2\left(\frac{is}{2a^2}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} e^{a^2\left(\frac{is}{2a^2}\right)^2} \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{is}{2a^2}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-a^2\left(x-\frac{is}{2a^2}\right)^2} dx$$

put $a\left(x-\frac{is}{2a^2}\right) = t$

$\Rightarrow a\,dx = dt \Rightarrow dx = \frac{dt}{a}$

when $x = -\infty \Rightarrow t = -\infty$

$x = \infty \Rightarrow t = \infty$.

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{s^2}{4a^2}}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt \quad \text{but} \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2\cancel{\pi}}} \frac{e^{-\frac{s^2}{4a^2}}}{a} \cdot \sqrt{\cancel{\pi}}$$

$$\Rightarrow \boxed{F\{f(x)\} = \frac{1}{\sqrt{2}} \frac{e^{-\frac{s^2}{4a^2}}}{a}}$$

$\therefore$ for $e^{-\frac{x^2}{2}}$ put $a^2 = \frac{1}{2} \Rightarrow a = \frac{1}{\sqrt{2}}$

$$\therefore F\{e^{-\frac{x^2}{2}}\} = \frac{1}{\cancel{\sqrt{2}}} \frac{e^{-\frac{s^2}{4(\frac{1}{2})}}}{1/\cancel{\sqrt{2}}} \Rightarrow F\{e^{-\frac{x^2}{2}}\} = e^{-\frac{s^2}{2}}$$

$\therefore e^{-\frac{x^2}{2}}$ is self reciprocal under fourier transform.

(b) Now Let $f(x) = e^{-\frac{x^2}{2}}$ then $e^{-2x^2} = e^{-\frac{(2x)^2}{2}} = f(2x)$

but $F\{f(ax)\} = \frac{1}{a} F\{f(x)\}_{s \to \frac{s}{a}}$.

$$\therefore F\{f(2x)\} = F\{e^{-2x^2}\} = \frac{1}{2} F\{f(x)\}_{s \to \frac{s}{2}} = \frac{1}{2}\left[e^{-\frac{s^2}{2}}\right]_{s \to \frac{s}{2}}$$

$$= \frac{1}{2} e^{-\frac{s^2}{8}}$$

(c) Let $f(x) = e^{-2x^2}$ then $e^{-2(x-3)^2} = f(x-3)$

but $F\{f(x-a)\} = e^{ias} F\{f(x)\}$

$$\therefore F\{f(x-3)\} = F\{e^{-2(x-3)^2}\} = e^{3is} F\{f(x)\}$$

$$= e^{3is} F\{e^{-2x^2}\}$$

$$= e^{3is} \cdot \frac{1}{2} e^{-\frac{s^2}{8}}$$

$$= \frac{1}{2} e^{\left(3is - \frac{s^2}{8}\right)}$$

* Finite fourier sine, cosine transforms & their inverses :-

These transforms are useful for such a boundary value problem in which at least two of the boundaries are parallel & separated by a finite distance.

The finite fourier sine transform of $f(x)$ in $0<x<c$ is defined as

$$\boxed{F_s(n) = \int_0^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx}$$

where n is +ve integer.

The function $f(x)$ is then called the inverse finite fourier sine transform of $F_s(n)$ which is given by

$$\boxed{f(x) = \frac{2}{c} \sum_{n=1}^{\infty} F_s(n) \sin\left(\frac{n\pi x}{c}\right)}$$

The finite fourier cosine transform of $f(x)$ in $0<x<c$ is defined as

$$\boxed{F_c(n) = \int_0^c f(x) \cos\left(\frac{n\pi x}{c}\right) dx.}$$

where n is +ve integer

The function $f(x)$ is then called as inverse finite fourier cosine transform of $F_c(n)$ which is given by

$$\boxed{f(x) = \frac{1}{c} F_c(0) + \frac{2}{c} \sum_{n=1}^{\infty} F_c(n) \cos\left(\frac{n\pi x}{c}\right)}$$

Note :- The finite fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

Ex1) Find the finite Fourier cosine & sine transforms of $f(x)=x^2$, $0<x<\pi$.

$\Rightarrow$ Here $c=\pi$, Finite Fourier Sine transform.

$$F_s(n)=\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx=\int_0^\pi x^2\sin nx\,dx.$$

$$=\left[x^2\left(\frac{-\cos nx}{n}\right)-(2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$=\left[-\pi^2\frac{\cos n\pi}{n}+2\pi\frac{\cancel{\sin n\pi}^{0}}{n^2}+2\frac{\cos n\pi}{n^3}+0-0-\frac{2}{n^3}\right]$$

$$=\left[\frac{-\pi^2(-1)^n}{n}+\frac{2(-1)^n}{n^3}-\frac{2}{n^3}\right]$$

$$\therefore\quad F_s(n)=\frac{2}{n^3}\left[(-1)^n-1\right]-\frac{\pi^2(-1)^n}{n}$$

* Finite Fourier cosine transform:-

$$F_c(n)=\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx=\int_0^\pi x^2\cos nx\,dx$$

$$=\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$=\pi^2\frac{\cancel{\sin n\pi}^{0}}{n}+2\pi\frac{\cos n\pi}{n^2}-2\frac{\cancel{\sin n\pi}^{0}}{n^3}-0-0+0$$

$$F_c(n)=\frac{2\pi(-1)^n}{n^2}$$

Ex2) Find $F_s(n)$ & $F_c(n)$ for $f(x)=e^{ax}$, $0<x<l$.

$\Rightarrow$ Here $c=l$ $\Rightarrow$ $F_s(n)=\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx=\int_0^l e^{ax}\sin\left(\frac{n\pi x}{l}\right)dx.$

$$=\left[\frac{e^{ax}}{a^2+\left(\frac{n\pi}{l}\right)^2}\left[a\sin\left(\frac{n\pi x}{l}\right)-\frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)\right]\right]_0^l$$

$$=\frac{l^2e^{al}}{a^2l^2+n^2\pi^2}\left[a\cancel{\sin n\pi}^{0}-\frac{n\pi}{l}\cos n\pi\right]-\frac{l^2}{a^2l^2+n^2\pi^2}\left[0-\frac{n\pi}{l}\right].$$

$$=\frac{n\pi l}{a^2l^2+n^2\pi^2}\left[1-e^{al}(-1)^n\right]$$

* $F_c(n) = \int_0^c f(x) \cos\left(\frac{n\pi x}{c}\right) dx = \int_0^l e^{ax} \cos\left(\frac{n\pi x}{l}\right) dx$

$$= \left[\frac{e^{ax}}{a^2 + \left(\frac{n\pi}{l}\right)^2}\left[a\cos\left(\frac{n\pi x}{l}\right) - \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right]\right]_0^l$$

$$= \frac{l^2 e^{al}}{a^2 l^2 + n^2\pi^2}\left(a\cos n\pi - \frac{n\pi}{l}\cancel{\sin n\pi}\right) - \frac{l^2}{a^2l^2+n^2\pi^2}(a - 0)$$

$$= \frac{a l^2}{a^2 l^2 + n^2\pi^2}\left[e^{al}(-1)^n - 1\right]$$

Qn3) find $f(x)$ if it's finite fourier cosine transform is given by

$$F_c(n) = \frac{1}{(2n+1)^2}\cos\left(\frac{2n\pi}{3}\right), \quad 0 < x < \pi.$$

⇒ Inverse finite fourier cosine transform is

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\cos\left(\frac{n\pi x}{c}\right)$$

here $c = \pi$

$$\therefore f(x) = \frac{1}{\pi}F_c(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n)\cos nx.$$

$$\boxed{f(x) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos\left(\frac{2n\pi}{3}\right)\cos nx}$$

* Applications of Fourier Transform to solve Differential Equations: -

formulae: -

① If $F\{f(x)\} = F(s)$ then $F\{f'(x)\} = is F(s)$

i.e $F\{f'(x)\} = is\, F\{f(x)\}$

$\therefore \boxed{F\{y'\} = is\, F\{y\}}$

similarly $\boxed{F\{y''\} = (is)^2 F\{y\} = -s^2 F\{y\}}$

In general

$\boxed{F\{y^n\} = (is)^n F\{y\} = i^n s^n F\{y\}}$

② $\boxed{F\{e^{-a|t|}\} = \frac{2a}{s^2+a^2}} \Rightarrow \boxed{F^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{2a} e^{-a|t|}}$

③ $F\{e^{-at} H(t)\} = \frac{1}{a+is} \Rightarrow F^{-1}\left\{\frac{1}{a+is}\right\} = e^{-at} H(t)$

④ $F\{\delta(t-a)\} = e^{-ais} \Rightarrow F\{\delta(t)\} = 1$

Ex1) solve the following D.E. Using Fourier transform: -

$$\frac{dy}{dt} - 4y = e^{-4t} H(t)$$

$\Rightarrow$ Given $y' - 4y = e^{-4t} H(t)$

Taking Fourier Transform on both sides

$$F\{y'\} - 4F\{y\} = F\{e^{-4t} H(t)\}$$

$$is\, F\{y\} - 4F\{y\} = \frac{1}{4+is} \quad \{\because \text{formula ③}\}$$

$$(is-4)\, F\{y\} = \frac{1}{4+is}$$

$$\therefore F\{y\} = \frac{1}{(4+is)(is-4)}$$

$$= \frac{1}{(is)^2 - (4)^2}$$

$$= \frac{1}{-s^2 - 4^2}$$

$$\therefore F\{y\} = \frac{-1}{s^2+4^2}$$

$\therefore$ Taking Inverse F.T. on both sides.

$$\therefore y = F^{-1}\left\{\frac{-1}{s^2+4^2}\right\} = -F^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$\therefore \boxed{y = -\frac{1}{8} e^{-4|t|}}$ $\{\because$ by formula ② $\}$

Ex 2) solve by fourier transform : $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+5y=\delta(t)$

$\Rightarrow$ Given $y''+6y'+5y=\delta(t)$

$\therefore \; F\{y''\}+6F\{y'\}+5F\{y\}=F\{\delta(t)\}$

$(is)^2F\{y\}+6is\,F\{y\}+5F\{y\}=1$

$(-s^2+6is+5)F\{y\}=1$

$\Rightarrow \; F\{y\}=\frac{1}{-s^2+6is+5}=\frac{-1}{s^2-6is-5}$

$=\frac{-1}{s^2-is-5is-5}=\frac{-1}{s(s-i)-5i\left(s+\frac{1}{i}\right)}$

but $\frac{1}{i}=\frac{i}{i^2}=-i$

$=\frac{-1}{s(s-i)-5i(s-i)}$

$F\{y\}=\frac{-1}{(s-i)(s-5i)}$

consider $\frac{-1}{(s-i)(s-5i)}=\frac{A}{s-i}+\frac{B}{s-5i}$ by Method of partial fractions

$\therefore \; -1=A(s-5i)+B(s-i)$

for A $\Rightarrow$ put $s=i \Rightarrow -1=-4iA \Rightarrow A=\frac{1}{4i}\times\frac{i}{i}\Rightarrow \boxed{A=\frac{-i}{4}}$

for B $\Rightarrow$ put $s=5i \Rightarrow -1=4iB \Rightarrow B=\frac{-1}{4i}\times\frac{i}{i}\Rightarrow \boxed{B=\frac{i}{4}}$

$\therefore F\{y\}=\frac{-1}{(s-i)(s-5i)}=\frac{-i}{4}\frac{1}{s-i}+\frac{i}{4}\frac{1}{s-5i}$

$\Rightarrow F\{y\}=\frac{-i}{4}\frac{1}{i\left(\frac{s}{i}-1\right)}+\frac{i}{4}\frac{1}{i\left(\frac{s}{i}-5\right)}$

$=-\frac{1}{4}\frac{1}{(-is-1)}+\frac{1}{4}\frac{1}{(-is-5)}$

$=\frac{1}{4}\frac{1}{1+is}-\frac{1}{4}\frac{1}{5+is}$

$\therefore$ Taking I.F.T., we get

$y=\frac{1}{4}F^{-1}\left\{\frac{1}{1+is}\right\}-\frac{1}{4}F^{-1}\left\{\frac{1}{5+is}\right\}$

$=\frac{1}{4}\left[e^{-t}H(t)\right]-\frac{1}{4}\left[e^{-5t}H(t)\right]$

$\Rightarrow \boxed{y=\frac{1}{4}\left[e^{-t}-e^{-5t}\right]H(t)}$

Qn 3) solve by Fourier Transform :- $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=e^{-3t}H(t)$

$\Rightarrow$ Given $\quad y''+3y'+2y=e^{-3t}H(t)$

$\therefore \quad F\{y''\}+3F\{y'\}+2F\{y\}=F\{e^{-3t}H(t)\}$

$-s^2F\{y\}+3is\,F\{y\}+2F\{y\}=\frac{1}{3+is}$

$\therefore F\{y\}\left(-s^2+3is+2\right)=\frac{1}{3+is}$

$\Rightarrow \quad F\{y\}=\frac{1}{(3+is)(-s^2+3is+2)}$

$=\frac{-1}{(3+is)(s^2-3is-2)}$

$=\frac{-1}{(3+is)(s^2-2is-is-2)}$

$F\{y\}=\frac{-1}{(3+is)(s-i)(s-2i)}$

but $\frac{-1}{(3+is)(s-i)(s-2i)}=\frac{A}{3+is}+\frac{B}{s-i}+\frac{C}{s-2i}$

$\therefore -1=A(s-i)(s-2i)+B(3+is)(s-2i)+C(3+is)(s-i)$

put $s=3i \Rightarrow -1=A(2i)(i) \Rightarrow \boxed{A=\frac{1}{2}}$

put $s=i \Rightarrow -1=2B(-i) \rightarrow \boxed{B=-\frac{i}{2}}$

put $s=2i \Rightarrow -1=C(1)(i) \Rightarrow \boxed{C=i}$

$\therefore F\{y\}=\frac{1}{2}\frac{1}{3+is}-\frac{i}{2}\frac{1}{s-i}+i\frac{1}{s-2i}$

$=\frac{1}{2}\frac{1}{3+is}-\frac{i}{2}\frac{1}{i(-is-1)}+i\frac{1}{i(-is-2)}$

$=\frac{1}{2}\frac{1}{3+is}+\frac{1}{2}\frac{1}{1+is}-\frac{1}{2+is}$

$\therefore y=\frac{1}{2}F^{-1}\left\{\frac{1}{3+is}\right\}+\frac{1}{2}F^{-1}\left\{\frac{1}{1+is}\right\}-F^{-1}\left\{\frac{1}{2+is}\right\}$

$=\frac{1}{2}e^{-3t}H(t)+\frac{1}{2}e^{-t}H(t)-e^{-2t}H(t)$

$\Rightarrow \boxed{y=\frac{1}{2}\left\{e^{-3t}+e^{-t}-2e^{-2t}\right\}H(t)}$

Fourier Transforms

Classroom Problems

1. Find the Fourier Transform of $f(x) = e^{-x^2/2}, \ -\infty < x < \infty$. **Ans.** $F(s) = \sqrt{2\pi}\, e^{-s^2/2}$
2. Find the Fourier Transform of $f(u) = \begin{cases} 1, & for\ |u| < u_0 \\ 0, & for\ |u| > u_0 \end{cases}$. **Ans.** $F(s) = \frac{2}{s}\sin su_0$
3. Find the Fourier sine transform of $f(x) = \begin{cases} 1, & 0 \le x < a \\ 0, & x \ge a \end{cases}$. **Ans.** $F_s(s) = \frac{1-\cos sa}{s}$
4. Find the Fourier sine transform of $f(x) = e^{-ax}, \ a > 0$. **Ans.** $F_s(s) = \frac{s}{a^2+s^2}$
5. Find the finite Fourier cosine transform of $f(x) = 2x, \ 0 < x < 4$.

   Ans. $F_c(n) = \frac{32\left[(-1)^n - 1\right]}{n^2\pi^2}$
6. Find the finite Fourier sine transform of $f(x) = \sin kx, \ 0 < x < \pi$, if k is not an integer.

   Ans. $F_s(n) = -\frac{1}{2}\left[\frac{\sin(k+n)\pi}{k+n} - \frac{\sin(k-n)\pi}{k-n}\right]$
7. Find the finite Fourier sine and cosine transform of $f(x) = 1, \ 0 < x < l$.

   Ans. $F_c(n) = 0, F_s(n) = \frac{l\left[1-(-1)^n\right]}{n\pi}$
8. If the Fourier sine transform of $f(x)$ is given by $\frac{[1-\cos n\pi]}{n^2\pi^2}$ $(0 < x < \pi)$, find $f(x)$.

   Ans. $f(x) = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-\cos n\pi]}{n^2}\sin nx$

1)The finite Fourier sine transform of $f(x) = 2x, \ 0 < x < 4$

ans $\frac{-32}{s\pi}\cos s\pi$

2) If Fourier finite cosine transform of $f(x)$ is

$$F_c(s) = \frac{\left(\sin\left(\frac{s\pi}{2}\right) - \cos s\pi\right)}{(2s+1)\pi} \quad if \ \ s = 1, 2, 3, \ldots$$
$$= \frac{2}{\pi}, \ \ s = 0 \ \ where \ 0 < x < 6$$

then $f(x)$ is

ans- $\frac{1}{3\pi}+\frac{1}{3}\sum_{s=1}^{\infty}\frac{\left(\sin\left(\frac{s\pi}{2}\right)-\cos s\pi\right)}{(2s+1)\pi}\cos\left(\frac{s\pi x}{6}\right)$

Fourier Transforms

Tutorial

1. Using Fourier Integral representation, show that $\int_0^\infty \frac{\cos \varpi x}{1+\varpi^2} d\varpi = \frac{\pi}{2} e^{-x} \left(x \ge 0 \right)$.
2. Express the function $f(x) = \begin{cases} e^{kx}, & for\ x<0 \\ e^{-kx}, & for\ x>0 \end{cases}$ as Fourier Integral and hence show that $\int_0^\infty \frac{\cos \varpi x}{\varpi^2 + k^2} d\varpi = \frac{\pi}{2k} e^{-kx}, x>0, k>0$.
3. Express $e^{-x} \sin x$ as a Fourier sine integral and show that $e^{-x} \sin x = \frac{4}{\pi} \int_0^\infty \frac{\varpi \sin \varpi x}{\varpi^4 + 4} d\varpi, \left(x \ge 0 \right)$
4. Find the Fourier transform of $f(x) = \begin{cases} x^2, & for\ |x|<a \\ 0, & for\ |x|>a \end{cases}$.

   Ans. $F(s) = \frac{2}{s^3}\left[\left(a^2 s^2 - 2 \right) \sin sa + 2as \cos sa \right]$
5. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & for\ |x| \le 1 \\ 0, & for\ |x|>1 \end{cases}$.

   Ans. $F(s) = \frac{4}{s^3}\left[\sin s - s \cos s \right]$
6. Find the Fourier cosine transform of $f(x) = e^{-ax},\ a>0$. **Ans.** $F_c(s) = \frac{a}{a^2 + s^2}$
7. Find $f(x)$ if its cosine transform is $F_c(s) = e^{-s}$. **Ans.** $f(x) = \frac{2}{\pi \left(1+s^2 \right)}$
8. Find the finite Fourier sine transform of $f(x) = 2x,\ 0<x<4$. **Ans.** $F_c(n) = \frac{-32(-1)^n}{n\pi}$
9. Find the finite Fourier sine and cosine transform of $f(x) = x^2,\ 0<x<\pi$.

   Ans. $F_c(n) = \frac{2\pi \left(-1 \right)^n}{n^2}, F_s(n) = \frac{2\left[\left(-1 \right)^n - 1 \right]}{n^3} - \frac{\pi^2 \left(-1 \right)^n}{n}$
10. Find the finite Fourier cosine transform of $f(x) = \sin kx,\ 0<x<\pi$, if k is not an integer.

    Ans. $F_s(n) = \frac{1}{2}\left[\frac{-\cos\left(k+n \right)\pi}{k+n} - \frac{\cos\left(k-n \right)\pi}{k-n} + \frac{1}{k+n} + \frac{1}{k-n} \right]$

1. Motivation: Laplace transform a very powerful technique is that it replaces operations of calculus by operations of algebra. Laplace transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It finds very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing. Laplace transforms help in solving complex problems with a very simple approach.

2. Prerequisite:

Function, the concept of limit, continuity, ordinary derivative of function, rules and formulae of differentiation and integration of function of one independent variable.

3. Objective: The Laplace transform method solves differential equations and corresponding initial and boundary value problems. The Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. It is also useful in problems where the mechanical or electrical driving force has discontinuities, is impulsive or is a complicated periodic function.
The Laplace transform also has the advantage that it solves problems directly, initial and boundary value problems without determining a general solution.

4. Key Notations:

(1) $L\{f(t)\}$: Laplace transform of a function

(2) $L^{-1}\{f(t)\}$: Inverse Laplace transform of a function

5. Key Definitions:

(1) LAPLACE TRANSFORM: Let $f(t)$ be a function defined for all positive values of t , then $\phi(s)=\int_0^\infty e^{-st}\, f(t)\,dt$ provided the integral exists, is called the Laplace Transform of $f(t)$.

It is denoted as

$$L\{f(t)\}=\phi(s)=\int_0^\infty e^{-st}\, f(t)\,dt$$

(2) INVERSE LAPLACE TRANSFORM: If $L\{f(t)\}=\phi(s)=\int_0^\infty e^{-st}\, f(t)\,dt$ then $f(t)$ is called the Inverse Laplace transform of $\phi(s)$.

It is denoted as $L^{-1}[\phi(s)]=f(t)$.

6. Important Formulae/ Theorems / Properties:

LAPLACE TRANSFORM:

STANDARD FORMULAE:

1) $L(e^{at})=\dfrac{1}{s-a}$ $\quad (s>a)$

2) $L(1)=\dfrac{1}{s}$ $\quad (s>0)$

3) $L(\sin at)=\dfrac{a}{s^2+a^2}$

4) $L(\cos at) = \frac{s}{s^2+a^2}$

5) $L(\sinh at) = \frac{a}{s^2-a^2} \qquad (s > |a|)$

6) $L(\cosh at) = \frac{s}{s^2-a^2} \qquad (s > |a|)$

7) $L(t^n) = \frac{\overline{|n+1}}{s^{n+1}}$

7. SAMPLE PROBLEMS:

I. Exercise can be solved based on following sample problem.

LAPLACE TRANSFORM BY DEFINITION:

Ex. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & for\ 0 < t < \pi \\ \sin t & for\ t > \pi \end{cases}$$

Solution: By the definition of Laplace transform we have,

$$L\left[f(t)\right] = \int_0^\infty e^{-st}\ f(t)\,dt = \int_0^\pi e^{-st}\cos t\,dt = +\int_\pi^\infty e^{-st}\sin t\,dt$$

$$But \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{\left(a^2+b^2\right)}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{\left(a^2+b^2\right)}\left[a\sin bx - b\cos bx\right]$$

$$L\left[f(t)\right] = \frac{1}{s^2+1}\left[e^{-st}\left(-s\cos t + \sin t\right)\right]_0^\pi + \frac{1}{s^2+1}\left[e^{-st}\left(-s\sin t - \cos t\right)\right]_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}(s) - (-s)\right] + \frac{1}{s^2+1}\left[-e^{-s\pi}\right]$$

$$= \frac{1}{s^2+1}\left[s + (s-1)e^{-s\pi}\right]$$

Unsolved Problem

1) $f(t) = (t-2)^2 \quad t > 2\ , \quad f(t) = 0 \quad 0 < t < 2$ $\qquad Ans: \frac{2}{s^3}e^{-2s}$

2) $f(t) = t \quad 0 < t < a$

$= b \quad t > a$ $\qquad Ans: \frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$

3) $f(t) = t\ ,\ 0 < t < 3$

$= 6\ ,\ t > 3$ $\qquad Ans: \frac{1}{s^2} + \left[\frac{3}{s} - \frac{1}{s^2}\right]e^{-3s}$

II Exercise can be solved based on following sample problem.

LAPLACE TRANSFORM BY LINEARITY PROPERTY:

$L\{k_1 f(t)+k_2 g(t)\}=k_1 L\{f(t)\}+k_2 L\{g(t)\}$

Ex. Find the Laplace transform of $\sin(\omega t+\alpha)$

Solution: By linearity property, we have

$$L[\sin(\omega t+\alpha)]=L[\sin\omega t\cos\alpha+\cos\omega t\sin\alpha]$$
$$=\cos\alpha\, L(\sin\omega t)+\sin\alpha\, L(\cos\omega t)$$
$$=\cos\alpha\frac{\omega}{s^2+\omega^2}+\sin\alpha\frac{s}{s^2+\omega^2}$$

Unsolved Problem

Evaluate

1) $L[t^2-e^{-2t}+\cosh^2 3t]$ $\quad Ans: \frac{2}{s^3}-\frac{1}{(s+2)}+\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-6^2}\right]$

2) $L[(\sin 2t-\cos 2t)^2]$ $\quad Ans: \frac{1}{s}-\frac{4}{s^2+4^2}$

3) $L[\cos(wt+b)]$ $\quad Ans: \frac{s}{s^2+w^2}\cos b-\frac{w}{s^2+w^2}\sin b$

4) $L[\sin(5t+3)]$ $\quad Ans: \frac{5}{s^2+5^2}\cos 3+\frac{s}{s^2+5^2}\sin 3$

5) $L[\cos t\cos 2t\cos 3t]$ $\quad Ans: \frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+2^2}+\frac{s}{s^2+4^2}+\frac{s}{s^2+6^2}\right]$

6) $L[\sin^5 t]$ $\quad Ans: \frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$

CHANGE OF SCALE PROPERTY:

If $L[f(t)]=\phi(s)$ then $L[f(at)]=\frac{1}{a}\phi\left(\frac{s}{a}\right)$

Ex. If $L[f(t)]=\log\left(\frac{s+3}{s+1}\right)$**,** find $L[f(2t)]$.

Solution: By change of scale property, we have

$$L[f(2t)]=\frac{1}{2}\phi\left(\frac{s}{2}\right)$$

$$L[f(2t)]=\frac{1}{2}\log\left[\frac{\left(\frac{s}{2}\right)+3}{\left(\frac{s}{2}\right)+1}\right]$$
$$=\frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

Unsolved Problem

If $L[f(t)] = \phi(s)$ **then** $L[f(at)] = \frac{1}{a}\phi(\frac{s}{a})$

1) Find $L[f(2t)]$ if $L[f(t)] = \log\left(\frac{s+3}{s+1}\right)$ $\quad Ans: \frac{1}{2}\log\frac{(s+6)}{(s+2)}$

2) Find $L[(erf\, 2\sqrt{t})]$, If $L\left\{\text{ert}\sqrt{\text{t}} = \frac{1}{s\sqrt{s+1}}\right\}$ $\quad Ans: \frac{2}{s\sqrt{s+4}}$

3) Find $L\{\cos 4t\}$, If $L\{\cos t\} = \frac{s}{s^2+1}$

FIRST SHIFTING THEOREM:

If $L[f(t)] = \phi(s)\, then\ L[e^{-at} f(t)] = \phi(s+a)\ \& L[e^{at} f(t)] = \phi(s-a)$

Ex. Find the Laplace transform of $\sin 2t \cos t \cosh 2t$.

Solution: We know that

$$\sin 2t \cos t = \frac{1}{2} 2\sin 2t \cos t = \frac{1}{2}\left[\left(\sin 3t + \sin t\right)\right]$$

$$\cosh 2t = \frac{e^{2t} + e^{-2t}}{2}$$

$$\therefore \sin 2t \cos t \cosh 2t = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)\left(\sin 3t + \sin t\right)$$

$$\therefore \sin 3t = \frac{3}{s^2+9}$$

$$\therefore L\left[e^{2t}\sin 3t\right] = \frac{3}{\left(s-2\right)^2+9}, \quad L\left[e^{-2t}\sin 3t\right] = \frac{3}{\left(s+2\right)^2+9}$$

$$\therefore L\left(e^{2t}\sin 3t\right) + L\left(e^{-2t}\sin 3t\right) = 3\left[\frac{1}{\left(s-2\right)^2+9} + \frac{1}{\left(s+2\right)^2+9}\right]$$

$$= \frac{3.2\left(s^2+13\right)}{s^4+10s^2+13^2}$$

$$Now\ \sin t = \frac{1}{s^2+1}$$

$$\therefore L\left(e^{2t}\sin t\right) = \frac{1}{\left(s-2\right)^2+1}\ ,\quad L\left(e^{-2t}\sin t\right) = \frac{1}{\left(s+2\right)^2+1}$$

$$L\left(e^{2t}\sin t\right) + L\left(e^{-2t}\sin t\right) = \frac{2\left(s^2+5\right)}{s^4-6s^2+5^2}$$

Form (1), (2) and (3), we get

$$L\left[\sin 2t \cos t \cosh 2t\right] = \frac{3\left(s^2+13\right)}{s^4+10s^2+13^2} + \frac{\left(s^2+5\right)}{s^4-6s^2+5^2}$$

Unsolved Problem

If $L[f(t)]=\phi(s)\, then \;\; L[e^{-at}f(t)]=\phi(s+a) \;\; \& L[e^{at}f(t)]=\phi(s-a)$

Evaluate

1) $L[e^{-3t}t^4]$

$Ans:\dfrac{4!}{(s+3)^5}$

2) $L[\sinh(\frac{t}{2})\sin(\frac{\sqrt{3}}{2}t)]$ $\qquad Ans:\dfrac{\sqrt{3}\,s}{2(s^4+s^2+1)}$

3) $L[\dfrac{\cos 2t\sin t}{e^t}]$ $\qquad Ans:\dfrac{(s^2+2s-2)}{(s^2+2s+10)(s^2+2s+2)}$

4) $L[e^{-4t}\sinh t\sin t]$ $\qquad Ans:\dfrac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$

5) $L[e^t\sin 2t\sin 3t]$ $\qquad Ans:\dfrac{12s}{(s^2-2s+2)(s^2-2s+26)}$

6) $L[e^{-3t}\cosh 5t\sin 4t]$ $\qquad Ans:\dfrac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+80)}$

7) $L[\sin 2t\cos t\cosh 2t]$ $\qquad Ans:\dfrac{3(s^2+13)}{(s^4+10s^2+13^2)}+\dfrac{(s^2+5)}{(s^4-6s^2+5^2)}$

8) $L[e^{-4t}\cosh t\sin t]$ $\qquad Ans:\dfrac{(s^2+8s+18)}{(s^2+6s+10)(s^2+10s+26)}$

EFFECT OF MULTIPLICATION BY t^n :

$If\;\; L[f(t)]=\phi(s)\;\; then\;\; L[t^n f(t)]=(-1)^n\dfrac{d^n}{ds^n}\phi(s)$

Ex. Find the Laplace transform of $t\sqrt{1+\sin t}$.

Solution:

$$\sqrt{1+\sin t}=\sqrt{\left[\sin^2\left(\frac{t}{2}\right)+\cos^2\left(\frac{t}{2}\right)+2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\right]}$$

$$=\sqrt{\left[\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\right]^2}=\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)$$

$$\therefore\;\; L\sqrt{1+\sin t}=L\left[\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{2}\right)\right]$$

$$=\frac{\left(1/2\right)}{s^2+\left(1/2\right)^2}+\frac{s}{s^2+\left(1/2\right)^2}$$

$$=\frac{1}{2}\frac{4}{\left(4s^2+1\right)}+\frac{4s}{\left(4s^2+1\right)}$$

$$=\frac{4s+2}{\left(4s^2+1\right)}=\frac{2\left(2s+1\right)}{\left(4s^2+1\right)}$$

$$L\left[\sqrt{1+\sin t}\right]=-\frac{d}{ds}\left[\frac{2\left(2s+1\right)}{4s^2+1}\right]$$

$$=-2\left[\frac{\left(4s^2+1\right)2-\left(2s+1\right)8s}{\left(4s^2+1\right)^2}\right]$$

$$=-2\left[\frac{-8s^2-8s+2}{\left(4s^2+1\right)^2}\right]$$

$$=4\frac{\left(4s^2+4s-1\right)}{\left(4s^2+1\right)^2}$$

Unsolved Problem

$If\ L[f(t)]=\phi(s)\ \ then\ \ L[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}\phi(s)$

Evaluate

1) $L[t\sin^3 t]$ $\qquad Ans:\frac{24s(s+5)}{(s^2+1)^2(s^2+9)^2}$

2) $L[t\sin 2t\cosh t]$ $\qquad Ans:2\left[\frac{(s-1)}{(s^2-2s+5)^2}+\frac{(s+1)}{(s^2+2s+5)^2}\right]$

3) $L[t\cos^2 t]$ $\qquad Ans:-\frac{1}{2s^2}+\frac{1}{2}\frac{s^2-2^2}{(s^2+2^2)^2}$

4) $L[te^{3t}\sin 4t]$ $\qquad Ans:\frac{8(s-3)}{(s^2-6s+2s)^2}$

5) $L[te^{3t}\sin t]$ $\qquad Ans:\frac{(2s-6)}{(s^2-6s+10)^2}$

6) $L[t\sqrt{1+\sin t}]$ $\qquad Ans:\frac{4(4s^2+4s-1)}{(4s^2+1)^2}$

7) $L[te^{3t}erf\sqrt{t}]$

$Ans:\frac{(3s-7)}{2(s-3)^2(s-2)^{\frac{3}{2}}}$

8) $L[te^{3t}\sin 2t]$ $\qquad Ans:\frac{4(s-3)}{(s^2-6s+13)^2}$

9) $L\left[t^2\sin 3t\right]$ $\qquad Ans:-\quad 18\frac{(s^2-3)}{(s^2+9)^3}$

EFFECT OF DIVISION BY t

If $L[f(t)] = \phi(s)$ *then* $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \phi(s)\, ds$

Ex. Find $L\left[\frac{\sin^2 t}{t^2}\right]$

Solution: We know that

$$L\left(\sin^2 t\right) = L\left[\frac{1-\cos t}{2}\right]$$

$$= \frac{1}{2}\left[L(1) - L(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

By effect of division, we have

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log\left(s^2+4\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right)$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right) ds$$

Integrating by parts

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)s - \int s\,\frac{s^2}{s^2+4}\left(\frac{s^2\,2s - \left(s^2+4\right)2s}{s^4}\right) ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 8\int \frac{ds}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 2\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

Unsolved Problem

If $L[f(t)]=\phi(s)\ then\ L\left[\frac{1}{t}f(t)\right]=\int_s^\infty \phi(s)\,ds$

Evaluate

1) $L\left[\frac{1}{t}(1-\cos t)\right]$ $Ans: \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

2) $L\left[\frac{1}{t}(e^{-at}-e^{-bt})\right]$ $Ans: \log\left(\frac{s+b}{s+a}\right)$

3) $L\left[\frac{\sin^2 2t}{t}\right]$ $Ans: \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

4) $L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right]$ $Ans: \frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)$

5) $L\left[\frac{\sin^2 t}{t^2}\right]$ $Ans: 2\cot^{-1}\left(\frac{s}{2}\right)+s\log\frac{\sqrt{s^2+a^2}}{s}$

6) $L\left[\frac{1-\cos t}{t^2}\right]$ $Ans: - \quad \frac{\pi}{2}-\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right)-\tan^{-1}s$

LAPLACE TRANSFORM OF DERIVATIVE

$If \quad L[f(t)]=\phi(s)\ then$

$$L[f'(t)]=s\phi(s)-f(0)$$

$$L[f''(t)]=s^2\phi(s)-sf(0)-f'(0)$$

$$L[F'''(t)]=s^3\phi(s)-s^2f(0)-sf'(0)-f''(0)$$

$$L\left\{f^n(t)\right\}=s^n\phi(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)\cdots\cdots f^{(n-1)}(0)$$

Ex. Find $L\left[f(t)\right]$ and $L\left[f'(t)\right]$, where $f(t)=\frac{\sin t}{t}$

Solution: $\because \quad L\left[f(t)\right]=\overline{f}(s)$

$$L\left[\frac{\sin t}{t}\right]=\int_s^\infty L\left[\sin t\right]ds$$

$$=\int_s^\infty \frac{1}{s^2+1}ds$$

$$=\left[\tan^{-1}s\right]_s^\infty$$

$$=\cot^{-1}s$$

$\therefore \quad \overline{f}(s)=\cot^{-1}s$

$$L\left[f'(t)\right]=s\,\overline{f}(s)-f(0)$$

$$=s\cot^{-1}s-\lim_{t\to0}\frac{\sin t}{t}$$

But $f(0)=\lim_{t\to0}\frac{\sin t}{t}$ is an indeterminate form, which can be solved by L'Hospital's Rule

$$\therefore\ \lim_{t\to0}\frac{\sin t}{t}=\lim_{t\to0}\frac{\cos t}{1}\quad\left(\textit{By differentiating numerator and deonimnatorseparately}\right)$$

$$=1$$

$$\therefore\quad L\left[f'(t)\right]=L\left[f(t)\right]-f(0)$$

$$=s\cot^{-1}s-1$$

Unsolved Problem

1) Find $L\left\{f(t)\right\}$ *and* $L\left\{f'(t)\right\}$

i) If $f(t)=\dfrac{\sin t}{t}$ — $Ans: s\cot^{-1}s-1$

ii) $f(t)=3\ ,\ 0\le t<5$
$\quad =0\ ,\ t>5$ — $Ans: \dfrac{3}{s}\left(1-e^{-5s}\right)$

iii) $f(t)=t\ ,\ 0\le t<3$
$\quad =6\ ,\ t>3$ — $Ans: \dfrac{1}{s^2}+e^{-3s}\left(\dfrac{3}{s}-\dfrac{1}{s^2}\right)\ ,\ \dfrac{1}{s}+e^{-3s}\left(3-\dfrac{1}{s}\right)$

2) If $L\left\{2\sqrt{\frac{t}{\pi}}\right\}=\frac{1}{s^{3/2}}$, Show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\}=\frac{1}{\sqrt{s}}$

3) If $L\left\{t\sin\omega t\right\}=\dfrac{2\omega s}{\left(s^2+\omega^2\right)^2}$,evaluate i) $L\left\{\omega t\cos\omega t+\sin\omega t\right\}$ ii) $L\left\{2\cos\omega t-\omega t\sin\omega t\right\}$

Ans :i) $\dfrac{2\omega s}{\left(s^2+\omega^2\right)^2}$, ii) $\dfrac{2s^3}{\left(s^2+\omega^2\right)^2}$

LAPLACE TRANSFORM OF INTEGRAL

If $L\left[f(t)\right]=\phi(s)$ *then* $L\left[\int_0^t f(u)\,du\right]=\dfrac{\phi(s)}{s}$

Ex. Find the Laplace transform of $\int_0^t t\,e^{-4t}\sin 3t\,dt$

Solution:

$$L\left[t\sin 3t\right]=-\frac{d}{ds}L\left[\sin 3t\right]$$

$$=-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$=\frac{6s}{\left(s^2+9\right)^2}$$

$$L\left[t\,e^{-4t}\sin 3t\right]=\frac{6(s+4)}{\left(s^2+8s+25\right)^2}$$

$$L\left[\int_0^t t\,e^{-4t}\sin 3t\right]=\frac{1}{s}L\left\{t\,e^{-4t}\sin 3t\right\}$$

$$=\frac{6(s+4)}{s\left(s^2+8s+25\right)^2}$$

Unsolved Problem

Evaluate

1) $L\left[\int_0^t \mathrm{t\cosh t\,dt}\right]$ $\qquad Ans:\dfrac{-(s^2+a^2)}{s(s^2-a^2)^2}$

2) $L\left[e^{3t}erf\sqrt{t}\right]$ $\qquad Ans:\dfrac{1}{(s-3)\sqrt{(s-2)}}$

3) $L\left[t\,erf\,3\sqrt{t}\right]$

4) $L\left[\int_0^t t\cos^2 t\,dt\right]$ $\qquad Ans:\dfrac{1}{2s^3}+\dfrac{1}{2}\dfrac{s^2-2^2}{s(s^2+2^2)^2}$

5) $L\left[\int_0^t \frac{1-e^{-t}}{t}dt\right]$ $\qquad Ans:\dfrac{1}{s}\log\left(\dfrac{s+1}{s}\right)$

6) $L\left[\int_0^t \frac{\sin t}{t}dt\right]$ $\qquad Ans:\dfrac{1}{s}\cot^{-1}s$

7) $L\left[\int_0^t\int_0^t\int_0^t t\sin t\,dt\,dt\,dt\right]$ $\qquad Ans:\dfrac{2}{s^2\left(s^2+1\right)^2}$

EVALUATION OF INTEGRAL USING LAPLACE TRANSFORMS

Ex. Evaluate $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}du\,dt$

Solution: By comparing the given integral $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}du\,dt$ with the Definition of Laplace transform

L [f (t)] = $\int_0^\infty e^{-st}\quad f(t)\quad dt$ we get,

s=1 and $f(t)=\int_0^t \frac{\sin u}{u}du$

$$L\left[\sin u\right]=\frac{1}{s^2+1}$$

$$L\left[\frac{\sin u}{u}\right]=\int_s^\infty L\left[\sin u\right]ds$$

Now, $$=\int_s^\infty \frac{1}{s^2+1}ds$$

$$=\left[\tan^{-1}s\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}s$$

$$=\cot^{-1}s$$

$$L\left[\int_0^t \frac{\sin u}{u}du\right]=\frac{1}{s}L\left\{\frac{\sin u}{u}\right\}$$

$$=\frac{1}{s}\cot^{-1}s$$

Now, $\int_0^\infty e^{-st}\int_0^t \frac{\sin u}{u}du\,dt=\frac{1}{s}\cot^{-1}s$

Putting s=1, we get

$$\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}du\,dt=\cot^{-1}1=\frac{\pi}{4}$$

Unsolved Problem

1) Show that $\int_0^\infty e^{-2t}\sin^3 t\,dt=\frac{6}{65}$

2) Show that $\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t\sinh t}{t}dt=\frac{\pi}{8}$

4) Show that $\int_0^\infty\left(\frac{\sin 2t+\sin 3t}{te^t}\right)dt=\frac{3\pi}{4}$

5) Show that $\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}dt=\frac{\pi}{3}$

6) Show that $\int_0^\infty t^3e^{-t}\sin t\,dt=0$

7) Show that $\int_0^\infty\int_0^t e^{-t}\frac{\sin u}{u}\,du\,dt$

8) If $\int_0^\infty e^{-2t}\cos(t-\alpha)\sin(t+\alpha)dt=\frac{3}{8}$ find α $\qquad Ans:\alpha=\frac{\pi}{4}$

INVERSE LAPLACE TRANSFORM:

STANDARD FORMULAE:

1) $L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

2) $L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}$

3) $L^{-1}\left[\frac{1}{s}\right]=1$

4) $L^{-1}\left[\frac{1}{s^n}\right]=\frac{t^{n-1}}{n}$

5) $L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}$

6) $L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

7) $L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$

8) $L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$

III Exercise can be solved based on following sample problem.

INVERSE BY DIRECT FORMULAE

Ex. Find the inverse Laplace transform of $\frac{3s+4}{s^2+16}$

Solution:

$$L^{-1}\left(\frac{3s+4}{s^2+16}\right)=3L^{-1}\left(\frac{s}{s^2+16}\right)+L^{-1}\left(\frac{4}{s^2+16}\right)$$
$$=3L^{-1}\left(\frac{s}{s^2+4^2}\right)+L^{-1}\left(\frac{4}{s^2+4^2}\right)$$
$$=3\cos 4t+\sin 4t$$

Unsolved Problem

Evaluate

1) $L^{-1}\left[\frac{1}{s^2+9}\right]$ $\qquad Ans:\frac{\sin 3t}{3}$

2) $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$ $\qquad Ans:2t^2-3t+1$

3) $L^{-1}\left[\frac{3s+4\sqrt{3}}{s^2+7}\right]$ $\qquad Ans:\cos\sqrt{7}t+\frac{4\sqrt{3}}{\sqrt{7}}\sin\sqrt{7}t$

INVERSE BY FIRST SHIFTING THEOREM

$L^{-1}\left[\phi(s+a)\right]=e^{-at}\,L^{-1}\left[\phi(s)\right]$

Ex. Find the inverse Laplace transform of $\frac{4s+12}{s^2+8s+12}$

Solution:

$$L^{-1}\left[\frac{4s+12}{s^2+8s+12}\right]=L^{-1}\left[\frac{4(s+4)-2^2}{(s+4)^2-2^2}\right]$$

$$=L^{-1}\left[\frac{4(s+4)}{(s+4)^2-2^2}\right]+L^{-1}\left[\frac{2^2}{(s+4)^2-2^2}\right]$$

By First shifting theorem, we have

$$=4e^{-4t}L^{-1}\left[\frac{s}{s^2-2^2}\right]-4e^{-4t}L^{-1}\left[\frac{2^2}{s^2-2^2}\right]$$

$$=4e^{-4t}\cosh 2t-4e^{-4t}\frac{1}{4}\sinh 2t$$

$$=e^{-4t}\left(4\cosh 2t-\sinh 2t\right)$$

Unsolved Problem

Evaluate

1) $L^{-1}\left[\frac{2s+2}{s^2+2s+10}\right]$ $\qquad$ $Ans: 2e^{-t}\cos 3t$

2) $L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]$ $\qquad$ $Ans: e^{-2t}\cos\sqrt{3}t$

3) $L^{-1}\left[\frac{2s+3}{s^2+2s+2}\right]$ $\qquad$ $Ans: 2e^{-t}\cos t+e^{-t}\sin t$

INVERSE BY PARTIAL FRACTION

Ex. Find the inverse Laplace transform of $\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$

Solution: $$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]=L^{-1}\left[\frac{(s+1)^2+2}{\left[(s+1)^2+2^2\right]\left[(s+1)^2+1^2\right]}\right]$$

$$=e^{-t}L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]$$

Let

$$s^2=x$$

And hence

$$\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]=\left[\frac{x+2}{(x+4)(x+1)}\right]$$

$$\left[\frac{x+2}{(x+4)(x+1)}\right]=\frac{a}{x+4}+\frac{b}{x+1}$$

$$\therefore \quad x+2=a(x+1)+b(x+4)$$

When x=-1, 1=3b;   when x=-4, -2=-3a

$$\left[\frac{s^2+2}{\left(s^2+4\right)\left(s^2+1\right)}\right]=\frac{2}{3}\frac{1}{s^2+4}+\frac{1}{3}\frac{1}{s^2+1}$$

$$L^{-1}\left[\frac{s^2+2}{\left(s^2+4\right)\left(s^2+1\right)}\right]=\frac{2}{3}L^{-1}\left[\frac{1}{s^2+4}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=\frac{2}{3}\frac{1}{2}\sin 2t+\frac{1}{3}\sin t$$

$$\therefore\quad L^{-1}\left[\frac{s^2+2s+3}{\left(s^2+2s+5\right)\left(s^2+2s+2\right)}\right]=\frac{e^{-t}}{3}\left(\sin 2t+\sin t\right)$$

Unsolved Problem

Evaluate

1) $L^{-1}\left[\frac{3s+1}{(s+1)\left(s^2+2\right)}\right]$ $\quad Ans: -\frac{2}{3}e^{-t}+\frac{2}{3}\cos\sqrt{2}t+\frac{7}{3\sqrt{2}}\sin\sqrt{2}t$

2) $L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]$ $\quad Ans: \frac{1}{a^2-b^2}(a\sin at-b\sin bt)$

4) $L^{-1}\left[\frac{s+2}{s^2(s+3)}\right]$ $\quad Ans: \frac{1}{9}\left(1+6t-e^{-3t}\right)$

5) $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]$ $\quad Ans: \frac{e^{-t}}{3}\left(\sin 2t+\sin t\right)$

$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$ $\quad Ans: \frac{1}{3}\left(\cos t-\cos 2t\right).$

7) $L^{-1}\left\{\frac{s}{1+s^2+s^4}\right\}$ $\quad Ans: \frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\sinh\frac{t}{2}$

8) $L^{-1}\left\{\frac{21s-33}{(s+1)(s-2)^3}\right\}$ $\quad Ans: 2e^{-t}-2e^{2t}+6te^{2t}+\frac{3}{2}t^2e^{2t}$

11) $L^{-1}\left\{\frac{s^2}{(s+a)^3}\right\}$ $\quad e^{-at}\left[1-2at+\frac{a^2t^2}{2}\right]$

12) $L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$ $\quad Ans: -\quad -1+t+e^{-t}$

13) $L^{-1}\left[\frac{s}{s^4+4a^4}\right]$ $\quad Ans: -\quad \frac{1}{2a^2}\sin at\sinh at.$

INVERSE BY CONVOLUTION THEOREM

Let $L\left[f_1(t)\right]=\phi_1(s)\ and\ L[f_2(t)]=\phi_2(s)\ then$

$$L^{-1}[\phi_1(s).\phi_2(s)]=\int_0^t f_1(u).f_2(t-u)\,du$$

$$where\ \ f_1(t)=L^{-1}[\phi_1(s)]\ \&\ f_2(t)=L^{-1}[\phi_2(s)]$$

Ex. Find the inverse Laplace transform of $\dfrac{(s+2)^2}{(s+2)^2+2^2}$

Solution: By convolution theorem, we have

$$L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right]=L^{-1}\left[\frac{(s+2)^2}{\left((s+2)^2+2^2\right)^2}\right]$$

$$=e^{-2t}\,L^{-1}\left[\frac{s^2}{\left(s^2+2^2\right)^2}\right]$$

$$L^{-1}\frac{s}{s^2+2^2}=\cos 2t$$

$$\therefore\ L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right]=\int_0^t \cos 2u\ \cos 2(t-u)\,du$$

$$=\frac{1}{2}\int_0^t \cos 2t+\cos(4u-2t)\,du$$

$$=\frac{1}{2}\left[u\cos 2t+\frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$=\frac{1}{2}\left[t\cos 2t+\frac{1}{4}\sin 2t+\frac{1}{4}\sin 2t\right]$$

$$=\frac{1}{2}\left[t\cos 2t+\frac{1}{2}\sin 2t\right]$$

$$\therefore\quad L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right]=\frac{e^{-2t}}{2}\left[t\cos 2t+\frac{1}{2}\sin 2t\right]$$

$$=\frac{e^{-2t}}{4}\left[2t\cos 2t+\sin 2t\right]$$

Unsolved Problem

Evaluate

1) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right]$ $\qquad Ans\,\dfrac{1}{2}\left[\sinh at+at\cosh at\right]$

2) $L^{-1}\left[\dfrac{1}{(s-2)^4(s+3)}\right]$ $\qquad Ans:\dfrac{e^{-3t}}{625}-e^{2t}\left[\dfrac{1}{625}-\dfrac{t}{125}+\dfrac{t^2}{50}-\dfrac{t^3}{30}\right]$

3) $L^{-1}\left[\dfrac{(s+2)^2}{(s^2+4s+8)^2}\right]$ $\qquad Ans:\dfrac{e^{-2t}}{4}[\sin 2t+2t\cos 2t]$

4) $L^{-1}\left[\dfrac{1}{(s-2)(s+2)^2}\right]$ $\qquad Ans:\dfrac{1}{16}[e^{2t}-e^{-2t}-4te^{-2t}]$

5) $L^{-1}\left\{\dfrac{s^2}{(s^2+1)(s^2+4)}\right\}$ $\qquad Ans:\dfrac{1}{3}(2\sin 2t-\sin t)$

6) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ $\qquad$ $Ans:\frac{t\sin at}{2a}$

7) $L^{-1}\left\{\frac{1}{\left(s^2+1\right)^3}\right\}$ $\qquad$ $Ans:\frac{1}{8}\left[\left(3-t^2\right)\sin t-3t\cos t\right]$

8) $L^{-1}\left\{\frac{s^2+s}{\left(s^2+1\right)\left(s^2+2s+2\right)}\right\}$ $\qquad$ $Ans:\frac{1}{5}\left[3\cos t+\sin t-3e^{-t}\cos t+e^{-t}\sin t\right]$

HEAVISIDE UNIT STEP FUNCTION

LAPLACE TRANSFORM OF HEAVISIDE UNIT STEP FUNCTION:

$$L\left[H(t-a)\right]=\frac{1}{s}e^{-as}$$

$$L\left[H(t)\right]=\frac{1}{s}$$

$$L\left[f(t)\,H(t-a)\right]=e^{-as}\,L\left[f(t+a)\right]$$

$$L\left[f(t)\,H(t)\right]=L\left[f(t)\right]$$

Ex 1. Express the function $f(t)=\begin{cases}\cos t & 0<t<\pi\\ \cos 2t & \pi<t<2\pi\\ \cos 3t & t>2\pi\end{cases}$ as Heaviside's unit step functions and find their Laplace transform.

Solution: By the formulae of Heaviside's unit step function, we have

$$\begin{aligned}f(t) &= \cos t\left[H(t)-H(t-\pi)\right]+\cos 2t\left[H(t-\pi)-H(t-2\pi)\right]+\cos 3t\,H(t-2\pi)\\ &=\cos t\,H(t)+\left(\cos 2t-\cos t\right)H(t-\pi)+\left(\cos 3t-\cos 2t\right)H(t-2\pi)\end{aligned}$$

$$\begin{aligned}L\left[f(t)\right]&=L\left[\cos t\right]+e^{-\pi s}\,L\left[\cos 2\left(t+\pi\right)-\cos\left(t+\pi\right)\right]+e^{-2\pi s}\,L\left[\cos 3\left(t+2\pi\right)-\cos 2\left(t+2\pi\right)\right]\\ &=L\left[\cos t\right]+e^{-\pi s}\,L\left[\cos 2t+\cos t\right]+e^{-2\pi s}\,L\left[\cos 3t-\cos 2t\right]\\ &=\frac{s}{s^2+1}+e^{-\pi s}\left[\frac{s}{s^2+4}+\frac{s}{s^2+1}\right]+e^{-2\pi s}\left[\frac{s}{s^2+9}+\frac{s}{s^2+4}\right]\end{aligned}$$

Ex 2. Find the Laplace transform of $(1+2t-3t^2+4t^3)H(t-2)$.

Solution: Here

$$f(t)=1+2t-3t^2+4t^3 \ \ and \ \ a=2$$
$$\therefore \ f(t+2)=1+2(t+2)-3(t+2)^2+4(t+2)^3$$
$$=4t^3+21t^2+38t+25$$
$$L\left[f(t+2)\right]=L\left[4t^3+21t^2+38t+25\right]$$
$$=4\frac{3!}{s^4}+21\frac{2!}{s^3}+38\frac{1}{s^2}+25\frac{1}{s}$$
$$L\left[f(t)H(t-2)\right]=e^{-2s}\left[\frac{24}{s^4}+\frac{42}{s^3}+\frac{38}{s^2}+\frac{25}{s}\right]$$

Unsolved Problem

1) Prove that $L\left[H(t-a)\right]=\dfrac{e^{-as}}{s}$

2) Prove that $L\left[H(t-a)\,f(t-a)\right]=e^{-as}\phi(s)$

3) Evaluate $L\left[\sin t\,H\left(t-\dfrac{\pi}{2}\right)-H\left(t-\dfrac{3\pi}{2}\right)\right]$ $\qquad Ans: \dfrac{se^{-\frac{\pi s}{2}}}{s^2+1}-\dfrac{e^{-\frac{3\pi s}{2}}}{s}$

4) Evaluate

$L\left[(1+2t-3t^2+4t^3)H(t-2)\right]$ *and hence evaluate* $\int_0^{\infty} e^{-t}(1+2t-3t^2+4t^3)H(t-2)\,dt$

$$Ans: e^{-2s}\left[\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right],\ \frac{129}{e^2}$$

6) $f(t)=t-1,\ \ 1<t<2$ $\qquad Ans: \dfrac{e^{-s}}{s^2}-\dfrac{2e^{-2s}}{s^2}+\dfrac{e^{-3s}}{s^2}$

$\quad =3-t,\ \ 2<t<3$

7) $f(t)=\cos t,\ \ 0<t<\pi$ $\qquad Ans: \dfrac{1}{s^2+1}\left\{s+e^{-\pi s}(s-1)-e^{-2\pi s}\right\}$

$\quad =\sin t,\ \ \pi<t<2\pi$

LAPLACE TRANSFORM OF DIRAC-DELTA (UNIT IMPULSE) FUNCTIONS

$$L\left[\delta(t-a)\right]=e^{-as}$$

$$L\left[\delta(t)\right]=1$$

Ex. Find the Laplace transform of $\sin 2t\,\delta(t-2)$.

Solution: By taking $f(t)=\sin at \ \ and \ \ a=2$, we have

$$L\left[\sin 2t\,\delta(t-2)\right]=L\left[f(t)\right]\delta(t-2)$$
$$=e^{-as}\ f(a)$$
$$=e^{-2s}\sin 4$$

Unsolved Problem

1) Prove that $L\left[\delta(t-a)\right]=e^{-as}$

2) Prove that $L\left[f(t)\delta(t-a)\right]=e^{-as}f(a)$

3) Find $L\left\{t^4 4\left(t-2\right)+t^2\delta\left(t-2\right)\right\}$ $\qquad Ans: e^{-2s}\left\{4+\frac{16}{s}+\frac{32}{s^2}+\frac{48}{s^3}+\frac{48}{s^4}+\frac{24}{s^5}\right\}$

4) Prove that $L\left\{t^2H\left(t-2\right)-\cos t\,\delta\left(t-4\right)\right\}=\frac{2e^{-2s}}{s^3}\left[1+2s+2s^2\right]-e^{-4s}\cosh 4$

5) Prove that $\int_0^\infty t^2e^{-t}\sin t\,\delta\left(t-2\right)dt=4e^{-2}\sin 2$

LAPLACE TRANSFORM OF PERIODIC FUNCTION

If f (t) is a periodic function of period T then $L[f(t)]=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$

Ex. Find the Laplace transform of $f(t)=\begin{cases}1 & 0\le t<a\\ -1 & a<t<2a\end{cases}$ and f (t) is periodic with period 2a.

Solution: Since f (t) is periodic with period 2a.

$$L\left[f(t)\right]=\frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}\,f(t)\,dt$$

$$=\frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(1)\,dt+\int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$=\frac{1}{1-e^{-2as}}\left[\left\{\frac{-e^{-st}}{s}\right\}_0^a+\left\{\frac{e^{-st}}{s}\right\}_a^{2a}\right]$$

$$=\frac{1}{s}\frac{1}{1-e^{-2as}}\left(1-e^{-as}\right)^2=\frac{1}{s}\frac{1-e^{-as}}{1+e^{-as}}$$

$$=\frac{1}{s}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right]$$

$$=\frac{1}{s}\tanh\left\{\frac{as}{2}\right\}$$

Unsolved Problem

If $f(t)$ is a periodic function of period T then $L[f(t)]=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$

1) Find Laplace Transform of $f\left(t\right)=kt,\ \ 0<t<1$ $\qquad Ans: \frac{k}{s^2}-\frac{ke^{-s}}{s\left(1-e^{-s}\right)}$

2) Find Laplace Transform of $\begin{aligned} f(t)&=t\ \ ,0<t<1\\ &=0\ ,1<t<2\end{aligned}$

3) Find Laplace Transform of

$$f(t) = a\sin pt \quad for\ 0 < t < \frac{\pi}{p}$$

$$= 0 \qquad for\ \frac{\pi}{p} < t < \frac{2\pi}{p} \qquad\qquad Ans: \frac{ap}{1-e^{-\frac{s\pi}{p}}}\frac{1}{s^2+p^2}$$

$$and\ f(t)\ is\ periodic\ with\ the\ period\ \frac{2\pi}{p}$$

INVERSE LAPLACE TRANSFORM BY HEAVISIDE UNIT STEP FUNCTION:

$$L^{-1}\left[\frac{1}{s}\right] = H(t)$$

$$L^{-1}\left[\frac{1}{s}e^{-as}\right] = H(t-a)$$

$$L^{-1}\left[e^{-as}\,\phi(s)\right] = f(t-a)\,H(t-a)$$

Ex. Find the inverse Laplace transform of $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$.

Solution: Here $\phi(s) = \dfrac{1}{(s+4)^{5/2}}$, We know that

$$f(t) = L^{-1}\,\phi(s) = L^{-1}\frac{1}{(s+4)^{5/2}}$$

$$= e^{-4t}\,L^{-1}\frac{1}{s^{5/2}} \quad (By\ first\ shifting\ theorem)$$

$$= e^{-4t}\ \frac{t^{3/2}}{\Gamma 5/2} \qquad ,$$

$$= \frac{e^{-4t}t^{3/2}}{\left(3/2\right)\left(1/2\right)\Gamma 1/2}$$

$$= \frac{4e^{-4t}t^{3/2}}{3\sqrt{\pi}}$$

$$\therefore \quad L^{-1}\frac{e^{4-3s}}{(s+4)^{5/2}} = \frac{4e^4}{3\sqrt{\pi}}e^{-4t}\,(t-3)^{3/2}\ H(t-3)$$

Unsolved Problem

1) $L^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}\right]$ $\qquad$ $Ans: \dfrac{4}{3\sqrt{\pi}}e^{-4(t-4)}(t-3)^{\frac{3}{2}}H(t-3)$

2) $L^{-1}\left[\dfrac{(s+1)e^{-s}}{s^2+s+1}\right]$ $\qquad$ $Ans: e^{-\frac{(t-1)}{2}}\left[\cos\sqrt{3}\dfrac{(t-1)}{2}+\dfrac{1}{\sqrt{3}}\sin(\sqrt{3}\dfrac{(t-1)}{2})\right]H(t-1)$

3) $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2(s^2+1)}\right]$ $\qquad$ $Ans: (t-\pi)+\sin(t-\pi).H(t-\pi)$

4) $L^{-1}\left[\dfrac{se^{-\frac{s}{2}}+\pi e^{-s}}{(s^2+\pi^2)}\right]$ $\qquad$ $Ans: \sin\pi t\left[H\left(t-\dfrac{1}{2}\right)+H(t-1)\right]$

IV Exercise can be solved based on following sample problem.

APPLICATIONS OF LAPLACE TRANSFORM

Ex. Solve the following equation by using Laplace transform

$$\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t, \quad given\ \ that \quad y(0)=1 \ .$$

Solution: Let $L(y)=\overline{y}$ **.** Taking Laplace transform on both the sides, we get

$$L(y')+2L(y)+L\left[\int_0^t y\,dt\right]=L(\sin t)$$

But

$$L(y')=sL(y)-y(0)=s\overline{y}-1$$

$$L\left[\int_0^t y\,dt\right]=\frac{1}{s}L(y)=\frac{1}{s}\overline{y}$$

$$L(\sin t)=\frac{1}{s^2+1}$$

$\therefore$ The equation becomes

$$\therefore \quad \left(\frac{s^2+2s+1}{s}\right)\overline{y}=\frac{s^2+2}{s^2+1}$$

$$\therefore \qquad \overline{y}=\frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $\dfrac{s(s^2+2)}{(s+1)^2(s^2+1)}=\dfrac{a}{s+1}+\dfrac{b}{(s+1)^2}+\dfrac{cs+d}{s^2+1}$

$$\therefore \quad s(s^2+2)=a(s+1)(s^2+1)+b(s^2+1)+(cs+d)(s+1)^2$$

Putting $s=-1,$ $\qquad -3=$ 2b $\qquad \therefore \qquad b=-\dfrac{3}{2}$

Putting $s=0,$ $\qquad 0=a+b+d$

Equating the coefficients of s^2 and s^3, $\qquad$ we get

$0 = a + b + 2c + d \quad and \;\; 1 = a = c$

$\because \;\; b = -\frac{3}{2} \;\; , \;\; a + d = \frac{3}{2}$

$and \quad a + 2c + d = \frac{3}{2}$

$But \qquad a + d = \frac{3}{2} \qquad \therefore 2c = 0 \qquad \therefore c = 0$

$\because \; 1 = a + c \; and \;\; c = 0 \qquad\qquad \therefore a = 1$

$a + d = \frac{3}{2} \qquad\qquad and \quad a = 1 \quad \therefore d = \frac{1}{2}$

$\therefore \qquad a = 1, \;\; b = -\frac{3}{2}, \;\; c = 0, \quad d = \frac{1}{2}$

$$\therefore \qquad \bar{y} = \frac{1}{s+1} - \frac{3}{2}\frac{1}{(s+1)^2} + \frac{1}{2}\frac{1}{(s^2+1)}$$

$$\therefore \qquad y = L^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}e^{-t} L^{-1}\frac{1}{s^2} + \frac{1}{2}L^{-1}\frac{1}{(s^2+1)}$$

$$\therefore \qquad y = e^{-t} - \frac{3}{2}te^{-t} + \frac{1}{2}\sin t$$

Ex. Solve by using Laplace transform

$(D^2 + 2D + 5)Y = e^{-t}\sin t, \quad when\, y(0) = 0, y'(0) = 1$

Solution: Let $L(y) = \bar{y}$ **.** Taking Laplace transform on both the sides, we get

$L(y'') + 2L(y') + 5L(y) = L\left(e^{-t}\sin t\right)$

But

$L(y') = sL(y) - y(0) = s\bar{y}$

$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$

$L\left(e^{-t}\sin t\right) = \frac{1}{(s+1)^2 + 1}$

$\therefore$ The equation becomes

$$\therefore \quad (s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2 + 1}$$

$$(s^2 + 2s + 5)\bar{y} = 1 + \frac{1}{s^2 + 2s + 2} = \frac{s^2 + 2s + 3}{s^2 + 2s + 2}$$

$$\therefore \qquad \bar{y} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)}$$

Let $\bar{y} = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)} = \dfrac{as + b}{(s^2 + 2s + 5)} + \dfrac{cs + d}{(s^2 + 2s + 2)}$

After simplication, we get

$$\therefore \quad \bar{y}=\frac{2}{3}\frac{1}{(s^2+2s+5)}+\frac{1}{3}\frac{1}{(s^2+2s+2)}=\frac{2}{3}\frac{1}{(s+1)^2+2^2}+\frac{1}{3}\frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplace transform

$$\therefore \quad y=\frac{2}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)+\frac{1}{3}e^{-t}L^{-1}\left(\frac{1}{s^2+1^2}\right)$$

$$y=\frac{2}{3}e^{-t}.\frac{1}{2}\sin 2t+\frac{1}{3}e^{-t}\sin t=\frac{e^{-t}}{3}\left(\sin 2t+\sin t\right)$$

Unsolved Problem

1) Solve $3\frac{dy}{dt}+2y=e^{3t}, \quad y=1 at\ t=0$ $\qquad Ans:\frac{10}{11}e^{-\frac{2}{3}t}+\frac{1}{11}e^{3t}$

2) Solve $\frac{dy}{dt}+3y=2+e^{-t}, \quad y=1\ at\ t=0$ $\qquad Ans:\frac{2}{3}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}$

3) Solve $\frac{dx}{dt}+x=\sin wt,\ x(0)=2$ $\qquad Ans:\frac{1}{1+w^2}[(2w^2+w+2)e^{-t}-w\cos wt+\sin wt]$

4) Solve $(D^2-3D+2)y=4e^{2t}, with\ y(0)=-3, y'(0)=5$ $\qquad Ans:-\quad y=-7e^t+4e^{2t}+4te^{2t}$

5) Solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+8y=1, where\ y(0)=0, y'(0)=1$ $\qquad Ans:\frac{1}{8}-\frac{1}{8}e^{-2t}\cos 2t+\frac{3}{8}e^{-2t}\sin 2t$

6) Solve $\frac{d^2y}{dt^2}+y=t, \quad where\ y(0)=1, y'(0)=0$ $\qquad Ans:t+\cos t-\sin t$

7) Solve $\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t,$ given that $y(0)=1.$ $\qquad Ans:-\quad y=e^{-t}-\frac{3}{2}e^{-t}\cdot t+\frac{1}{2}\sin t$

8) Solve $\frac{d^2y}{dt^2}+9y=\cos 2t$ with $y(0)=1\ \&\ y\left(\frac{\pi}{2}\right)=-1$ $\qquad y=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\frac{4}{5}\sin 3t$

9) Solve $\frac{d^2y}{dt^2}-4y=3e^t$ where $y(0)=0\ \&\ y'(0)=3$ $\qquad Ans:-e^t+\frac{3}{2}e^{2t}-\frac{1}{2}e^{-2t}$

TUTORIAL-I

(TUTORIAL ON LAPLACE TRANSFORM BY DEFINITION, LAPLACE TRANSFORM BY LINEARITY PROPERTY, CHANGE OF SCALE PROPERTY, FIRST SHIFTING THEOREM)

Q.1.

$$f(t)=1 \ , 0\le t<1$$
$$=e^t \ , 1\le t\le 4$$
$$=0 \ , t>4$$

$$Ans: \frac{1-e^{-s}}{s}+\frac{e^{4(1-s)}-e^{(1-s)}}{1-s}$$

Q.2.

$$f(t)=\cos(t-a)\ , t>a$$
$$=0 \qquad , t<a$$

$$Ans: \frac{se^{-sa}}{s^2+1}$$

Q.1. $L[\sin\sqrt{t}]$ $\qquad Ans: \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{4s}}$

Q.2. $L[\cosh^5 t]$ $\qquad Ans: \frac{s(s^4-30s^2+149)}{(s^2-1)(s^2-9)(s^2-25)}$

Q.3. $\left(\sqrt{t}-1\right)^2$ $\qquad Ans: \frac{1}{s^2}-\frac{1}{s^2}\sqrt{\frac{\pi}{s}}+\frac{1}{s}$

Q.4. Evaluate $\left(\sqrt{t}\pm\frac{1}{\sqrt{t}}\right)^3$ $\qquad \frac{\boxed{5/2}}{s^{5/2}}\pm\frac{3\boxed{3/2}}{s^{3/2}}+\frac{3\boxed{1/2}}{s^{1/2}}\pm\frac{\boxed{-1/2}}{s^{-1/2}}$

Q.5. Evaluate $\sqrt{1+\sin t}$ $\qquad \frac{s}{s^2+(1/2)^2}+\frac{1/2}{s^2+(1/2)^2}$

Q.6. Evaluate $L\{\sinh at\sin at\}$ $\qquad \frac{2a^2s}{s^4+4a^4}$

Q.7. Evaluate $L\left[\left(t^2\sinh t\right)^2\right]$ $\qquad 6\left[\frac{1}{(s-2)^5}-\frac{2}{s^5}+\frac{1}{(s+2)^2}\right]$

Q.8. Evaluate $L[t^5\cosh t]$ $\qquad Ans: \frac{1}{60}\left[\frac{1}{(s-1)^6}+\frac{1}{(s+1)^6}\right]$

Q.9. $L[e^{-3t}\cosh 4t\sin 3t]$ $\qquad Ans: \frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$

Q.10. Evaluate $L\left\{\frac{\sqrt{1+\sin 4t}}{e^{2t}}\right\}$

Q.11. Evaluate $L\left[\frac{\cos 2t\sin t}{e^t}\right]$ $\qquad Ans: -\frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$

Q.12. Evaluate $t^n e^{-at}+\sin^3 t$ $\qquad Ans: -\frac{\boxed{n+1}}{(s+a)^{n+1}}-\frac{6}{(s^2+1)(s^2+9)}$

Q.13. Evaluate $L\left[e^{2t}\sin^4 t\right]$ $\qquad Ans:-\frac{1}{8}\left[\frac{3}{s-2}-\frac{4(s-2)}{s^2-4s+8}+\frac{s-4}{s^2-4s+20}\right]$

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TUTORIAL-II

(TUTORIAL ON EFFECT OF MULTIPLICATION BY t^n, EFFECT OF DIVISION BY t,)

1) $L[te^{-2t}\sinh 4t]$ $\qquad Ans:\frac{8(s+2)}{(s^2+4s-12)^2}$

2) $L\left\{t\cos\left(\omega t-\alpha\right)\right\}$, where ω & α are constants. $\qquad Ans:-\ \frac{(s^2-w^2)\cos\alpha+2ws\sin\alpha}{(s^2+w^2)^2}$

3) $L\left\{\left(t\ \sinh 2t\right)^2\right\}$ $\qquad Ans:-\ \frac{1}{2}\left[\frac{1}{(s-4)^3}-\frac{2}{s^3}+\frac{1}{(s+4)^3}\right]$

4) $L\left\{\left(t+\sin 2t\right)^2\right\}$ $\qquad Ans:-\ \frac{2}{s^3}+\frac{8s}{(s^2+2^2)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}$

5) $L\left\{t\sin 2t\cosh t\right\}$ $\qquad Ans:-2\left[\frac{s-1}{(s^2-2s+5)^2}+\frac{s+1}{(s^2+2s+5)^2}\right]$

6) $L\left\{t\,e^{-3t}\cos 2t\right\}$ $\qquad Ans:\frac{s^2+6s+5}{\left(s^2+6s+13\right)^2}$

7) $L\left[t\sqrt{1-\sin t}\right]$ $\qquad Ans:\frac{4(4s^2-4s-1)}{(4s^2+1)^2}$ *or* $\frac{-4(4s^2-4s-1)}{(4s^2+1)^2}$

8) $L\left[t\cos(wt-\alpha)\right]$ $\qquad Ans:-\frac{(s^2-w^2)\cos\alpha+2ws\sin\alpha}{(s^2+w^2)^2}$

9) $L\left[t^3\sin t\right]$ $\qquad Ans:-\ 24s\frac{(s^2-1)}{(s^2+1)^4}$

10) $L\left[\frac{\sinh 2t}{t}\right]$ $\qquad Ans:\frac{1}{2}\log\frac{s+2}{s-2}$

11) $L\left[\frac{(1-\cos t)}{t^2}\right]$ $\qquad Ans:\frac{\pi}{2}-\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right)-\tan^{-1}s$

12) $L\left[\frac{\cosh 2t\sin 2t}{t}\right]$ $\qquad Ans:\frac{1}{2}\left\{\cot^{-1}\left(\frac{s-2}{2}\right)+\cot^{-1}\left(\frac{s+2}{2}\right)\right\}$

13) $L\left\{\dfrac{\sin t}{t\,e^{t}}\right\}$ $\quad$ $Ans:-\quad \cot^{-1}(s+1)$

14) $L\left\{\dfrac{\sin t\sin 5t}{t}\right\}$ $\quad$ $Ans:-\quad \dfrac{1}{2}\log\left(\dfrac{s^2+36}{s^2+16}\right)$

15) $L\left\{\dfrac{2\sin t\sin 2t}{t}\right\}$ $\quad$ $Ans:-\quad \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+1}\right)$

16) $L\left[\dfrac{1}{t}(e^{-t}\sin at)\right]$ $\quad$ $Ans:-\quad \cot^{-1}\left(\dfrac{s-1}{a}\right)$

17) $L\left[\dfrac{e^{2t}\sin t}{t}\right]$ $\quad$ $Ans:-\quad \cot^{-1}(s-2)$

18) $L\left[\dfrac{\cosh 3t\sin^2 2t}{t}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{8}\left[\log\dfrac{(s+3)^2+4}{s+3}+\log\dfrac{(s-3)^2+4}{s-3}\right]$

19) $L\left[\dfrac{\sinh at}{t}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{2}\log\left(\dfrac{s-a}{s+a}\right)$

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TUTORIAL-III

(TUTORIAL LAPLACE TRANSFORM OF DERIVATIVE, LAPLACE TRANSFORM OF INTEGRAL, EVALUATION OF INTEGRAL USING LAPLACE TRANSFORMS)

1) Find $L\{f(t)\}$ and $L\{f'(t)\}$

$i)\ f(t)=e^{-5t}\sin t$ $\qquad Ans: \dfrac{s}{s^2+10s+36}$

$ii)\ f(t)=t+1,\quad 0<t<2$
$\quad =3\ ,\quad t>2$

2) Evaluate $L\left[\dfrac{d}{dt}\left(\dfrac{1-\cos 2t}{t}\right)\right]$ $\qquad Ans:-\quad s\log\left(\dfrac{\sqrt{s^2+2^2}}{s}\right)$

3) $L\left\{\int_0^t e^t t^3\,dt\right\}$ $\qquad Ans: \dfrac{6}{s(s+1)^4}$

4) $L\left\{e^{-3t}\int_0^t t\sin 3t\,dt\right\}$ $\qquad Ans: \dfrac{6}{\left(s^2+6s+18\right)^2}$

5) $L\left\{t\int_0^t e^{-4t}\sin 3t\,dt\right\}$ $\qquad Ans: \dfrac{3\left(3s^2+16s+25\right)}{s^2\left(s^2+8s+25\right)^2}$

6) $\int_0^t u\cos^2 u\,du.$ $\qquad Ans:-\quad \dfrac{1}{2s^3}+\dfrac{1}{2}\cdot\dfrac{s^2-2^2}{s(s^2+2^2)^2}$

7) $\int_0^t ue^{-3u}\sin 4u\,du$ $\qquad Ans:-\quad \dfrac{1}{s}\cdot\dfrac{4}{s^2+6s+25}$

8) $\int_0^t \dfrac{e^{-u}\sin u}{u}du$ $\qquad Ans:-\quad \dfrac{1}{s}\cot^{-1}(s+1)$

9) Show that $\int_0^\infty e^{-t}\dfrac{\sin^2 t}{t}dt=\dfrac{1}{4}\log 5$

10) Show that $\int_0^\infty \dfrac{\sin^2 t}{t^2}dt=\dfrac{\pi}{2}$

11) Show that $\int_0^\infty e^{-3t}t^2\sinh 2t\,dt=\dfrac{124}{125}$

12) Show that $\int_0^\infty e^{-t}\left(\int_0^t u^4\sinh u\cosh u\ du\right)dt=-24$

13) Show that $\int_0^\infty e^{-4t}\cosh^3 t\,dt=\dfrac{12}{35}$

14) Show that $\int_0^\infty e^{-2t}t\ \sin^2 t\,dt=\dfrac{1}{8}$

15) Show that $\int_0^\infty e^{-2t}\left[\int_0^t\left(\dfrac{1-e^{-u}}{u}\right)du\right]dt=\dfrac{1}{2}\log\left(\dfrac{3}{2}\right)$

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TUTORIAL-IV

INVERSE BY DIRECT FORMULAE, INVERSE BY FIRST SHIFTING THEOREM,INVERSE BY PARTIAL FRACTION,

1) $L^{-1}\left(\dfrac{1}{s^2-25}\right)$ $\quad$ $Ans:-\quad \dfrac{1}{5}\sin 5t$

2) $L^{-1}\left[\dfrac{7s}{s^2+4}\right]$ $\quad$ $Ans: 7\cos 2t$

3) $L^{-1}\left[\dfrac{3+2s+s^2}{s^3}\right]$ $\quad$ $Ans:-\quad 3\cdot\dfrac{t^2}{2}+2\cdot t+1$

4) $L^{-1}\left[\dfrac{1}{(s-2)^{3/2}}\right]$ $\quad$ $Ans:-\quad 2e^{2t}\sqrt{t/\pi}$

5) $L^{-1}\left[\dfrac{6s-4}{s^2-4s+20}\right]$ $\quad$ $Ans:-\quad 6e^{2t}\cos 4t+2e^{2t}\sin 4t$

6) $L^{-1}\left[\dfrac{1}{(s+1)^2}+\dfrac{s-2}{s^2-4s+5}+\dfrac{s-2}{s^3-4s+3}\right]$ $\quad$ $Ans:-\quad e^{-t}t+e^{2t}\cos t+e^{2t}\cosh t$

7) $L^{-1}\left\{\dfrac{1}{s}\cos\dfrac{1}{s}\right\}$ $\quad$ $Ans: 1-\dfrac{t^2}{(2!)^2}+\dfrac{t^4}{(4!)^2}$

8) $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]$ $\quad$ $Ans: e^t\left(\cosh 2t+5\sinh 2t\right)$

9) $L^{-1}\left[\dfrac{s^2+16s-24}{s^4+20s^2+64}\right]$ $\quad$ $Ans:-\quad \dfrac{4}{3}\cos 2t-\dfrac{7}{3}\cdot\dfrac{1}{2}\sin 2t-\dfrac{4}{3}\cdot\cos 4t+\dfrac{10}{3}\cdot\dfrac{1}{4}\sin 4t$

10) $L^{-1}\left[\dfrac{14s+10}{49s^2+28s+13}\right]$ $\quad$ $Ans: \dfrac{2}{7}e^{-\frac{2}{7}t}\left(\cos\dfrac{3}{7}t+5\sin\dfrac{3}{7}t\right)$

10) $L^{-1}\left[\dfrac{2s}{s^4+4}\right]$ $\quad$ $Ans: \sin t\sinh t$

11) $L^{-1}\left\{\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$ $\quad$ $Ans: 2e^{-3t}+3\cos t-\sin t$

12) $L^{-1}\left(\dfrac{1}{s^3-a^3}\right)$ $\quad$ $Ans: \dfrac{1}{3a^2}\left[e^{at}-e^{-\frac{at}{2}}\left\{\cos\dfrac{\sqrt{3}}{2}at+\sqrt{3}\sin\dfrac{\sqrt{3}}{2}at\right\}\right]$

13) $L^{-1}\left[\dfrac{s^2+10s+13}{(s-1)(s^2-5s+6)}\right]$ $\quad$ $Ans:-\quad 12e^{t}-37e^{2t}+26e^{3t}$

14) $L^{-1}\left[\dfrac{s}{(s+1)^2(s^2+1)}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{2}\left[\sin t - te^{-t}\right]$

15) $L^{-1}\left[\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right]$ $\quad$ $Ans:-\quad e^{t} - e^{-t}\cos 2t + \dfrac{3}{2}e^{-t}\sin 2t$

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TUTORIAL-V

INVERSE BY CONVOLUTION THEOREM, INVERSE BY DIFFERENTIATION OF ϕ (s).

1) $L^{-1}\left[\dfrac{1}{s^2(s+1)^2}\right]$ $\quad$ $Ans: te^{-t} + 2e^{-t} + t - 2$

2) $L^{-1}\left[\dfrac{1}{s(s+a)^2}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{a^2}\left[1-(1+at)e^{-at}\right]$

3) $L^{-1}\left[\dfrac{1}{s(s^2-9)}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{9}\left[\cosh 3t - 1\right]$

4) $L^{-1}\left[\dfrac{1}{(s-3)(s+3)^2}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{36}\left[e^{3t} - 6te^{-3t} - e^{-3t}\right]$

5) $L^{-1}\left[\dfrac{s^2}{(s^2+9)(s^2+4)}\right]$ $\quad$ $Ans:-\quad \dfrac{1}{5}(3\sin 3t - 2\sin 2t)$

6) $L^{-1}\left[\dfrac{1}{s^2-s-6}\right]$ $\quad$ $Ans:-\quad \dfrac{e^{3t}-e^{-2t}}{5}$

7) $L^{-1}\left[\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$ $\quad$ $Ans:-\quad \dfrac{e^{-t}}{3}\left[\sin 2t + \sin t\right]$

8) $L^{-1}\left[\dfrac{(s+2)^2}{(s^2+4s+8)^2}\right]$ $\quad$ $Ans:-\quad \dfrac{e^{-2t}}{4}\left[2t\cos 2t + \sin 2t\right]$

9) $L^{-1}\left[\dfrac{(s+3)^2}{(s^2+6s+5)^2}\right]$ $\quad$ $Ans: \dfrac{e^{-3t}}{4}[2t\cosh 2t + \sinh 2t]$

10) $L^{-1}\left[\dfrac{1}{(s^2+4s+13)^2}\right]$ $\quad$ $Ans: \dfrac{e^{-2t}}{18}\left[\dfrac{\sin 3t}{3} - t\cos 3t\right]$

11) $L^{-1}\left[\dfrac{s+2}{(s^2+4s+5)^2}\right]$ $\quad$ $Ans:-\quad e^{-2t}\cdot\dfrac{1}{2}t\sin t$

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TUTORIAL-VI

(TUTORIAL ON, HEAVISIDE UNIT STEP FUNCTION, LAPLACE TRANSFORM OF DIRAC-DELTA (UNIT IMPULSE) FUNCTIONS, LAPLACE TRANSFORM OF PERIODIC FUNCTION INVERSE LAPLACE TRANSFORM BY HEAVISIDE UNIT STEP FUNCTION)

Q.3.Evaluate

$L\left[t^2 H(t-2)\right]$ $\quad Ans: e^{-2s}\left[\frac{4}{s}+\frac{4}{s^2}+\frac{2}{s^3}\right]$

Q.4.Evaluate

$f(t)=e^{-3t}H(t-2)$ $\quad Ans: \frac{e^{-2(s+3)}}{s+3}$

Q.5.Evaluate $f(t)=t^2, \ 0<t<1$

$=4t, \ t>1$

Q.6. Express in terms of unit step function & find LT

$$f(t)=E, \ a<t<b$$
$$=0, \ t>b$$

$Ans: E\left[H(t-a)-H(t-b)\right], E\left[\frac{e^{-as}}{s}-\frac{e^{-bs}}{s}\right]$

Q.7. Find $L\left\{(\sin t)\delta\left(t-\frac{\pi}{2}\right)-t^2\delta(t-2)\right\}$ $\quad Ans: e\frac{-\pi s}{2}-4e^{-2s}$

Q.8. Prove that $\int_0^\infty e^{-t}\sin t\,\delta'(t-a)\,dt=(\sin a-\cos a)e^{-a}$

Q.9. Find Laplace Transform of

$$f(t)= 1 \ for\, 0\le t<a$$
$$=-1 \ for \ a<t<2a, \ and \ f(t) \ is \ periodic \ function \ with \ the \ period \ 2a.$$

Q.10. Find $L\left[E\left|\sin\omega t\right|\right]$ if $f(t+2)=f(t)$ $\quad Ans: \frac{E\omega\cot\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2}$

1) Evaluate $L^{-1}\left[\frac{8e^{-3s}}{s^2+4}\right]$ $\quad Ans: 4\sin 2(t-3)H(t-3)$

2) Evaluate $L^{-1}\left[\frac{e^{-5s}}{(s-2)^4}\right]$ $\quad Ans: \frac{1}{3!}e^{2(t-5)}(t-5)^3H(t-5)$

3) Find the inverse Laplace transform of $L^{-1}\left[\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}\right]$ $\quad \frac{4}{3\sqrt{\pi}}\cdot e^{-4(t-4)}\cdot(t-3)^{3/2}H(t-3)$

TUTORIAL-VII

APPLICATION OF LAPLACE TRANSFORM.

4) Solve $(D^2+2D+5)y=e^{-t}\sin t, with\ y(0)=0, y'(0)=1$ $Ans: \frac{1}{3}e^{-t}(\sin 2t+\sin t)$

5) Solve $(D^2+9)y=18t, with\ y(0)=0, y(\frac{\pi}{2})=0$ $Ans: 2t+\pi\sin 3t$

6) Solve $\frac{dy}{dt}+2y+\int_0^t y\,dt=\sin t\ with\ y(0)=1$ $Ans: e^{-t}-\frac{3}{2}e^{-t}.t+\frac{1}{2}\sin t$

7) $Solve\ L\frac{dI}{dt}+RI=Ee^{-at}$, where $I(0)=0$. $Ans:-\ \frac{E}{R-La}\left[e^{-at}-e^{-(R/L)t}\right]$

8) Solve $\frac{d^2y}{dt^2}+4y=f(t)$ with condition $f(t)=H(t-2), y(0)=0, y'(0)=1$

$Ans:-\ \frac{1}{2}\sin 2t+\frac{1}{4}\cdot 1\cdot H(t-1)-\frac{1}{4}\cos 2(t-2)H(t-2)$

9) Solve $(D^2+2D+5)y=e^{-t}\sin t, with\ y(0)=0, y'(0)=1$ $Ans:-\ \frac{e^{-t}}{3}(\sin 2t+\sin t)$

10) Solve $(D+1)^2 y$, given $y(0)=2, y'(0)=5$ $Ans:-\ y=e^{-t}(t^3+7t+2)$.

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