# Properties of Auto-correlation

→ $R_{xx}(t_1, t_2)$

here

$$t_2 = t_1 + \tau$$

$\therefore R_{xx}(t_1, t_1 + \tau)$

here

$$t_2 - t_1 = t_1 - t_1 + \tau$$
$$= \tau$$

$\therefore R_{xx}(\tau) = R_{xx}(t_1, t_2)$

If $t_1 = t_2$

$$R_{xx}(t_1 - t_2) = R_{xx}(0)$$

I $|R_{xx}(\tau)| \leq R_{xx}(0)$,
means $R_{xx}(\tau)$ is bounded by it's value at the origin (or the largest value of $R_{xx}(\tau)$ occurs at $\tau = 0$).

II $R_{xx}(\tau) = R_{xx}(-\tau)$
means the $R_{xx}(\tau)$ is an even function (symmetric)

III $R_{xx}(0) = E[x^2(t)]$
means that the largest value of the auto carelation function, $R_{xx}(0)$ is equal to the second moment of the random process.

$E[x^2(t)]$ is usually referred as mean square value.

IV $R_{xx}(\tau)$ cannot have an arbitrary shape. This means that any arbitary function cannot be an autocorelation function.

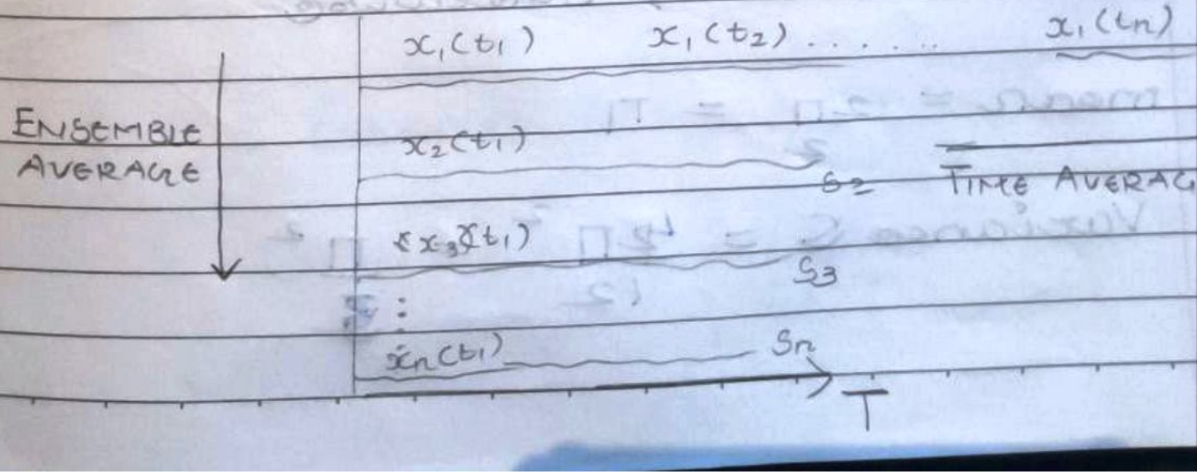
V. If $x(t)$ has no periodic components and is ergodic and expectation is not equal to zero
ie $E[x(t)] = \mu_x(t) \neq 0$
Then, $\lim_{|\tau| \to \infty} R_{xx}(\tau) = \mu_x^2$

## → Ergodic Process

- A random process is said to be ergodic if it statistical properties can be deduced from a single sufficiently long random sample of the process.

Clearly for a process to be ergodic it has to necessarily be stationary but not all stationary process are ergodic.

- Condition for the process to be ergodic →
A random process is said to be ergodic if the time average of the processes tend to the appropriate ensemble average.



Time Average, Mean $\bar{x} = \frac{1}{2\pi}\int_{-T}^{T} x(t)\, dt$

We have to show that
ensemble function = mean of time average ~~mean~~
$E[x(t)] = \bar{x}$

Q A random process has sample function of the form $x(t) = A\cos(\omega t + \Theta)$ where $\omega$ is a constant and A is a random variable that has a magnitude of +1 and −1 with equal probabilities and $\Theta$ is a random variable that is uniformly distributed between 0 and $2\pi$. Assume that the random variable A and $\Theta$ are independent.

i) Is x(t) a WSS?

ii) Is x(t) a mean ergodic process?

| A | 1 | −1 |
| --- | --- | --- |
| P(x) | $(\frac{1}{2})$ | $(\frac{1}{2})$ |

$\Theta \sim U[0, 2\pi]$

$$p(\Theta) = \begin{cases} \frac{1}{2\pi}, & 0 < \Theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

mean $= \frac{2\pi}{2} = \pi$

Variance $= \frac{4\pi^2}{12} = \frac{\pi^2}{3}$

$$E[A] = \Sigma x\, p(x)$$
$$= \frac{1}{2} - \frac{1}{2}$$
$$= 0$$

$$E(x(t)) = E[A\cos(\omega t + \theta)]$$
$$= E[A]\, E[\cos(\omega t + \theta)]$$
$$= 0$$

$$R_{xx}(t_1, t_2) = E(x(t_1)\, x(t_2))$$
$$= E[A\cos(\omega t_1 + \theta)\, A\cos(\omega t_2 + \theta)]$$

$$= E[A^2]\, E[\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)] \longrightarrow (1)$$

$$E[A^2] = \Sigma x^2\, p(x)$$
$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1$$

$$\therefore R_{xx}(t_1, t_2) = 1\; E[\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)]$$

$$= \frac{1}{2}\cos\omega(t_1 - t_2)$$

$$= \frac{1}{2}\cos(\omega\tau)$$

$\therefore \{x(t)\}$ is WSS

b) ~~time average~~

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} x(t)\,dt$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2\pi}\int_{0}^{2\pi} A\cos(\omega t+\theta)\,dt$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2\pi}\int_{0}^{2\pi} A\cos(\omega t+\theta)\,dt$$

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2\pi}\left[\frac{(A\,\cancel{\cos}\,\sin(\omega t+\theta)}{\omega}\right]_{0}^{2\pi}$$

$$\bar{x} = \lim_{T\to\infty} \frac{1\cdot A}{2\pi\omega}\left[\sin(2\pi\omega+\theta)-\sin\theta\right]$$

$$\bar{x} = \lim_{T\to\infty} \frac{A}{2\pi\omega}\left[\sin(2\pi\omega+\theta)-\sin\theta\right]$$

$$\bar{x} = \lim_{T\to\infty} \frac{A}{2\pi\omega}\left[\sin\theta-\sin\theta\right]$$

$$\bar{x} = \lim_{T\to\infty} 0$$

$$\bar{x} = 0$$

$$\therefore \bar{x} = E[x(t)]$$

$\therefore$ given process is ergodic

# POWER SPECTRAL DENSITY

## UNIT – II

→ Fourier Transform

- It is a mathematical formula that relates a signal sample in time or space to the same signal sampled in

In signal processing, the FT can reveal important characteristic of a signal namely it's frequency component. FT is a mathematical technique that transforms a function of time $x(t)$ to a function of frequency $X(\omega)$.

If a function $x(t)$ is defined on an interval $(-\infty, \infty)$ is a peacewise continuous in each finite interval and is absolutely integrable in $(-\infty, \infty)$ then the integral,

Q] $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$ is called the fourier

transform of $f$ $F[\{f(x)\}] = F(s)$.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

→ Power Spectral Density (PSD)

- The goal of spectral analysis is to start with a signal and identify the strength of a sinusoidal components that make up the signal. The strength of magnitude of the sinusoides are displays as a function of frequency.

- PSD → If $x(t)$ is a stationary process, either it is (SSS or WSS) with auto-correlation function $R_{xx}(\tau)$, then the fourier transform of $R_{xx}(\tau)$ the PSD function of $\{x(t)\}$ and denoted as $S_{xx}(\omega)$ OR $S_x(\omega)$ (OR $S(\omega)$).

It is given by,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} \, d\tau \quad \text{—— (1)}$$

- Sometimes $\omega$ is replaced by $2\pi f$
$\omega \to 2\pi f$, where $f$ is the frequency variable in which case the PSD function will be a function of $f$, $S(f) =$

$$\int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} \, d\tau \quad \text{—— (2)}$$

- Given the PSD function $S(\omega)$ the auto correlation function $R(\tau)$ given by Fourier Inverse Transform $S(\omega)$

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} \, d\omega$$

OR

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i2\pi f\tau} df \longrightarrow (4)$$

- Properties of PSD

I The value of PSD function at 0 frequency ($\omega$ is 0 in eq (1) & (2)) equals the total area under the graph of the autocorelation function.

II The mean square value (avg power) of WSS equals to the total area under the graph of the spectral density. (putting $f = 0$ in eq (4)) we get $R[0] = E[x^2(t)]$

III $S(-\omega) = S(\omega)$ which means $S(\omega)$ is even function for real random process.

IV The spectral density of a process (real or complex) is a real function of $\omega$ which is non negative.

V The spectral density and the auto corelation function of a real WSS process form a Fourier cosine Transform pair.

VI If $\int_{-\infty}^{\infty} |R(\tau)| d\tau < \infty$ then $S(\omega)$ is a continuous function of $\omega$

Mathematical Expression of Property

1) $S(0) = \int_{-\infty}^{\infty} R(\tau)\,d\tau$

5) $R(\tau) = \frac{1}{\pi} \int_{0}^{\infty} S(\omega) \omega \tau \, d\omega$

= fourier inverse cosine transform of $\frac{1}{2} S(\omega)$

→ White Noise

- A random process $wt$ whose PSD, $S(\omega)$ is constant at all frequencies, ie

$S_\omega(w) = \frac{N_0}{2}$ where $N_0$ is a real positive constant is called White Noise

Q1 Find the PSD of the WSS random process whose auto correlation function is ~~$R(\tau)$~~ =
$R(\tau) = e^{-a|\tau|} \cos b\tau$

→ $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

$= \int_{-\infty}^{\infty} e^{-a|\tau|} \cos b\tau \, e^{-i\omega\tau} d\tau$

$\cancel{= \int_{-\infty}^{\infty} e^{-a|\tau|} \cos b t}$

$= \int_{-\infty}^{0} e^{a(\tau)} \cos b\tau \, e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-a\tau} \cos b\tau \, e^{-i\omega\tau} d\tau$

$$= \int_{-\infty}^{0} e^{+(a\tau + i\omega\tau)} \cos b\tau \, d\tau + \int_{0}^{\infty} e^{-(a\tau + i\omega\tau)} \cos b\tau \, d\tau$$

$$= \left[ \frac{e^{a\tau - i\omega\tau}}{(a\tau - i\omega\tau)^2 + b^2} \left[ b\sin b\tau + (a\tau - i\omega\tau)\cos bx \right] \right]_{-\infty}^{0}$$

$$+ \left[ \frac{e^{-(a\tau + i\omega\tau)}}{(a\tau + i\omega\tau)^2 + b^2} \left[ b\sin bx - (a\tau + i\omega\tau)\cos bx \right] \right]_{0}^{\infty}$$

$$= \frac{e^{(a - i\omega)0}}{(a - i\omega)^2 + b^2} \left[ b\sin b\tau + (a - i\omega)\cos b\tau \right]$$

$$- \frac{e^{(a - i\omega) - \infty}}{(a - i\omega)^2 + b^2} \left[ b\sin bx + (a - i\omega)\cos bx \right.$$

$$= \left[ \frac{e^{(a - i\omega)\tau}}{(a - i\omega)^2 + b^2} \left[ b\sin b\tau + a - i\omega \cos b\tau \right] \right]_{-\infty}^{0}$$

$$+ \left[ \frac{e^{-(a + i\omega)\tau}}{(a + i\omega)^2 + b^2} \left[ b\sin b\tau - a + i\omega \cos b\tau \right] \right]_{0}^{\infty}$$

$$= \frac{1}{(a - i\omega)^2 + b^2} \left[ (a - i\omega) - 0 \right] + \frac{1}{(a + i\omega)^2 + b^2} \left[ (a + i\omega) + 0 \right]$$

$$= \frac{a-i\omega}{(a-i\omega)^2+b^2} + \frac{a+i\omega}{(a+i\omega)^2+b^2}$$

$$= \frac{a-i\omega}{a^2-2ai\omega+\omega^2+b^2} + \frac{a+i\omega}{a^2+2ai\omega+\omega^2+b^2}$$

$$= \frac{a-i\omega}{a^2+\omega^2+b^2-2ai\omega} \times \frac{a^2+\omega^2+b^2+2ai\omega}{a^2+\omega^2+b^2+2ai\omega}$$

$$+ \frac{a+i\omega}{a^2+\omega^2+b^2+2ai\omega} \times \frac{a^2+\omega^2+b^2-2ai\omega}{a^2+\omega^2+b^2-2ai\omega}$$

$$= \frac{(a-i\omega)(a^2+\omega^2+b^2+2ai\omega)}{(a^2+\omega^2+b^2)^2-(2ai\omega)^2}$$

$=$

Q For a random process $x(t)$, auto relation is given

$$R(\tau) = 1 + e^{-\alpha|\tau|}$$

$\rightarrow$ $$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} (1 + e^{-\alpha|\tau|}) e^{-i\omega\tau} d\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} d\tau + \int_{-\infty}^{\infty} e^{-\alpha|\tau|} e^{-i\omega\tau} d\tau \longrightarrow (1)$$

$\Rightarrow$ NOTE $\Rightarrow$ $$\int_{-\infty}^{\infty} e^{-i\omega\tau} d\tau = 2\pi\delta(\omega)$$

Now $$\int_{-\infty}^{\infty} e^{-\alpha|\tau|} e^{-\omega\tau} d\tau = \int_{-\infty}^{0} e^{\alpha\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-i\omega\tau} d\tau$$

$$\therefore S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} d\tau + \int_{-\infty}^{0} e^{\alpha\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{\alpha\tau} e^{-i\omega\tau} d\tau \longrightarrow (2)$$

$= \cancel{e^{\alpha}}$

$$\int_{-\infty}^{0} e^{(\alpha-i\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(\alpha+i\omega)\tau}\, d\tau$$

$$\left[\frac{e^{(\alpha-i\omega)\tau}}{\alpha-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+i\omega)\tau}}{-(\alpha+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{\alpha-i\omega} + \frac{1}{\alpha+i\omega}$$

$$= \frac{\alpha+i\omega+\alpha-i\omega}{\alpha^2-i^2\omega^2}$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} \longrightarrow (3)$$

eq (2) becomes

$$S(\omega) = 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2+\omega^2}$$

Q The autocorrelation function of the random telegraph signal process is given by $R(\tau) = a^2 e^{-2y|\tau|}$ Find the PSD

$$\rightarrow S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau} a^2 e^{-2y|\tau|}\, d\tau$$

$$= \int_{-\infty}^{0} e^{-i\omega\tau} a^2 e^{2y\tau}\, d\tau + \int_{0}^{\infty} e^{-i\omega\tau} a^2 e^{-2y\tau}\, d\tau$$

$$= a^2 \left[ \frac{e^{(2y-i\omega)\tau}}{2y-i\omega} \right]_{-\infty}^{0} + a^2 \left[ \frac{e^{-(2y+i\omega)\tau}}{-(2y+i\omega)} \right]_{0}^{\infty}$$

$$= a^2 \frac{1}{2y-i\omega} + \frac{a^2}{2y+i\omega}$$

$$= a^2 \frac{2y+i\omega + 2y - i\omega}{4y^2+\omega^2}$$

$$= \frac{4y a^2}{4y^2+\omega^2}$$

HW

Q Find the PSD of random process $x(t)$ where $x(t)$ is given as

$x(t) = a\cos(bt + Y)$

Find $x(t)$'s PSD where $Y$ is random variable $Y \sim U[0, 2\pi]$ and $a, b$ are constants.

Q1 method-2

$R(\tau) = e^{-a|\tau|}\cos b\tau$

$$S(\omega) = \int_{-\infty}^{\infty} e^{-a|\tau|}\cos b\tau\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a|\tau|}\cos\omega t(\cos\omega\tau - i\sin\omega\tau)$$

$$= \int_{-\infty}^{\infty} e^{-a|\tau|}\cos b\tau\cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-a|\tau|}\cos b\tau\sin b\tau\, d\tau$$

---

$\rightarrow$ $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even

$= 0$ if $f(x)$ is odd.

---

$$S(\omega) = 2\int_{0}^{\infty} e^{-a|\tau|}\cos b\tau\cos\omega\tau\, d\tau - i\times(0$$

$$= 2\int_{0}^{\infty} e^{-a\tau}\cos b\tau\cos\omega\tau\, d\tau$$

$$= \int_{0}^{\infty} e^{-a\tau}\cos(\tau(b+\omega)) + \cos\tau(b-\omega)\, d\tau$$

$$= \int_{0}^{\infty} e^{-a\tau}\cos\tau(b+\omega) + \int_{0}^{\infty} e^{-a\tau}\cos\tau(b-\omega)\, d\tau$$

---

$\rightarrow$ $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \left[\frac{e^{-a\tau}}{a^2+(b+\omega)^2}\left[-a\cos((b+\omega)\tau)+(b+\omega)\sin((b+\omega)\tau)\right]\right]_0^\infty$$

$$+ \left[\frac{e^{-a\tau}}{a^2+(b-\omega)^2}\left[-a\cos\tau(b-\omega)+(b-\omega)\sin\tau(b-\omega)\right]\right]_0^\infty$$

$$= 0 + \frac{a}{a^2+(b+\omega)^2} + 0 + \frac{a}{a^2+(b-\omega)^2}$$

$$= \frac{a(a^2+(b-\omega)^2) + a(a^2+(b+\omega)^2)}{a^4+a^2(b-\omega)^2+a^2(b+\omega)^2+(b+\omega)^2(b-\omega)^2}$$

$$= \frac{a^3+ab^2-\cancel{2abw}+aw^2+a^3+ab^2+aw^2+\cancel{2abw}}{a^4+a^2b^2-\cancel{2a^2b\omega}+a^2\omega^2+a^2b^2+\cancel{2a^2b\omega}+a\omega^2 \; b^4-2b^3\omega+b^2\omega^2+2b^3\omega-4b^2\omega^2+2b\omega^3+b^2\omega^2+2b\omega^3+\omega^4}$$

Q A random process has the autocorrelation function $R_{xx}(\tau) = \frac{4\tau^2+6}{\tau^2+1}$ find $\mu_x^2$ value, the mean value and the variance of the process

$$\rightarrow R_{xx}(\tau) = \frac{4\tau^2+4}{\tau^2+1} + \frac{2}{\tau^2+1}$$

$$= 4 + \frac{2}{\tau^2+1}$$

$$\mu_x^2 = \lim_{|\tau|\to\infty} R_{xx}(\tau)$$

$$= \lim_{|\tau|\to\infty} 4 + \frac{2}{\tau^2+1}$$

$$= 4$$

$$\therefore \mu_x = \sqrt{\mu_x^2} = \sqrt{4} = 2$$

~~var = x~~

$E[x^2(\tau)] = 6$ $\quad [\because E(x^2(\tau)) = R_{xx}(0)]$

$$\therefore \text{variance} = 6 - 4$$
$$= 2$$

Q Compute the variance of random process $x(t)$, whose auto-correlation function is given by

$$R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

$$\mu_x^2 = \lim_{|\tau| \to \infty} 25 + \frac{4}{1+6\tau^2}$$

$$\mu_x^2 = 25$$

$$\mu_x = \sqrt{25} = 5$$

$$E[x^2(\tau)] = 25 + \frac{4}{1+6(0)^2} = 29$$

$$\therefore \text{variance} = 29 - 25$$
$$= 4$$

→ Overview of Linear System With Deterministic Inputs

- Deterministic Signal
  A signal is said to deterministic; if there is no uncertainity with respect to its value at any instant of time.
  OR
  Signal which can be defined exactly by a mathematical formula are known as deterministic signals.

- Consider a system with deterministic input signal $x(t)$ and deterministic response $y(t)$

→ $\underset{\text{signal}}{\xrightarrow{x(t)}}$ $\underset{\text{process}}{\boxed{h(t)}}$ $\underset{\text{output}}{\xrightarrow{y(t)}}$

Time-Domain

→ $\xrightarrow{x(t)}$ $\boxed{H(W)}$ $\xrightarrow{y(t)}$

Frequency-Domain    Fourier Transform

- For linear time invariant system, the response of the system, to an input is the ~~convolu~~ convolution of $x(t)$ and $h(t)$

convolution → $x(t) * h(t)$

ie $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) * h(t-\tau) d\tau$

Also

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) * x(t-\tau)\,d\tau$$

In the frequency domain we can compute, the fourier transform of $y(t)$ as follows,

$$Y(w) = F[y(t)] = \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt$$

$$\cancel{= \int_{-\infty}^{\infty} x(\tau)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau\, e^{-i\omega t} dt$$

Interchanging the order of integration and since $x(\tau)$ does not depend ~~to~~ on $t$

$$\cancel{= \int_{-\infty}^{\infty} h(t-\tau}$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t-\tau)\, e^{-i\omega t}\, dt\right) d\tau$$

But the inner integral is the fourier transform of $h(t-\tau)$ which is $e^{-i\omega\tau} H(w)$ Therefore

$$= \int_{-\infty}^{\infty} x(\tau)\, H(w)\, e^{-i\omega\tau}\, d\tau$$

$$= H(w) \int_{-\infty}^{\infty} x(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$Y(w) = H(w)\,X(w)$$

Thus The fourier transform of The output signal is The product of The system response and The fourier transform of The input signal.

→ Linear System with continuous time random inputs.

Problem Statement → Given that $x(t)$ is the input to a linear time invariant system, with impulse response $h(t)$ and $Y(t)$ is the corresponding output of the system. Can we determine The mean and autocorrelation function of $Y(t)$ if those of $x(t)$ as Known.

Solution

→ The output process is given by

$$Y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Thus The mean of $Y(t)$ is given by,

$$\mu_Y(t) = E[Y(\tau)] = E\left[\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau\right]$$

$$= \int_{-\infty}^{\infty} E[x(\tau)]\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \mu_x(\tau)\,h(t-\tau)\,d\tau$$

$$\cancel{= \mu_x(\tau) * h(t-\tau)}$$

$$= \mu_x(t) * h(t)$$

We can also write,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$\cancel{\mu_y(t) = h(t) * x(t)}$$

Thus The mean of the output process is The convolution of The mean of the input process.

$$\mu_y(t) = h(t) * \mu_x(t)$$

- The cross-correlation function between The input process $x(t)$ and the output process $y(t)$ is given by

$$R_{xy}(t) = \int_{-\infty}^{\infty} R_{xx}(\tau - \mu)\, h(\mu)\, d\mu$$

$$= R_{xx}(\tau) * h(\tau)$$

If we take the fourier transform of both sides we obtain the cross power spectral density between $x(t)$ and $y(t)$.

$$S_{xy}(\omega) = H(\omega) S_{xx}(\omega)$$

Thus the transformed function of The system is given by, $H(\omega) = \dfrac{S_{xy}(\omega)}{S_{xx}(\omega)}$

In a similar way, it can be shown as a cross correlation function between the output process $y(t)$ and WSS. The input process is given by

$$S_{YX}(\omega) = H^*(\omega) S_{XX}(\omega)$$

where this $H^*(\omega)$ is the complex conjugate of $H(\omega)$ and it is given by

$$H^*(\omega) = \frac{S_{YX}(\omega)}{S_{XX}(\omega)}$$

Finally the autocorrelation function of the output of a linear time invariant system (LTI) with a WSS process input is given by

$$R_{YY}(\tau) = E[\underset{t_1}{Y(\tau)}\, Y(\underbrace{t+\tau}_{t_2})]$$

$$= E[Y(t) \int_{-\infty}^{\infty} h(u) X(t+\tau-u)\,du$$

$$= R_{YX}(\tau) * h(u)$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$$

$$= h(-\tau) * h(\tau) * R_{XX}(\tau)$$

## → Cross Power Spectral Density

The cross power spectrum, $S_{xy}(\omega)$ of two continuous random process $x(t)$ and $y(t)$ is defined as the fourier transform of $R_{xy}(\tau)$

ie $S_{xy}(\omega) = F[R_{xy}(\tau)]$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} \, d\tau$$

Hence the inverse fourier transform of $S_{xy}(\omega)$ is given by,

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega t} \, d\omega$$

- Properties

→ If $S_{xy}(\omega)$ is CPSD of two random process $x(t)$ and $y(t)$ then the following properties hold.

$$S_{xy}(\omega) = S_{yx}(-\omega)$$

→ $S_{xy}(-\omega) = S^{*}_{xy}(\omega)$

→ If $x(t)$ and $y(t)$ are orthogonal, then $S_{xy}(\omega) = 0$

Q The CPSD of real random process x(t) and y(t) is given by

$$S_{xy}(\omega) = \begin{cases} a+jb\omega & |\omega| < 1 \\ 0 & , \text{otherwise} \end{cases}$$

Find the cross corelation function.

→ CPSD is given by

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-1}^{1} a+jb\omega(e^{i\omega\tau}) d\omega$$

$$\cancel{= \frac{1}{2\pi} \int_{-1}^{1} a+jb\omega(\cos\omega t t}$$

$$\cancel{= \frac{1}{2\pi}\left[e^{i\omega t}\left(a\omega + \frac{jb\omega^2}{2}\right) - \int\right.}$$

$$= \frac{1}{2\pi}\left[\frac{(a+jb\omega)e^{i\omega t}}{it} - \int jb \frac{e^{i\omega t}}{it}\right]_{-1}^{1}$$

$$= \frac{1}{2\pi}\left[\frac{(a+jb\omega)e^{i\omega t}}{it} - \frac{jbe^{i\omega t}}{(it)^2}\right]_{-1}^{1}$$

$$= \frac{1}{2\pi}\left[\frac{(a+jb)e^{i\tau}}{i\tau} - \frac{jbe^{i\tau}}{(i\tau)^2} - \frac{(a-jb)e^{-i\tau}}{i\tau} + \frac{jbe^{-i\tau}}{(i\tau)^2}\right]$$

$$\cancel{= \frac{1}{2\pi}\left[\frac{(a+jb}{i\tau}\right.}$$

$$\cancel{= \frac{1}{2\pi}\left[\right.}$$

take $i$ as $j$

$$= \frac{1}{2\pi}\left[\frac{(a+jb)e^{j\tau}}{j\tau} - \frac{jbe^{j\tau}}{j^2\tau^2} - \frac{(a-jb)e^{-j\tau}}{j\tau} + \frac{jbe^{-j\tau}}{j\tau^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{a}{j\tau}e^{j\tau} + \frac{b}{\tau}e^{j\tau} - \frac{be^{j\tau}}{j\tau^2} - \frac{ae^{-j\tau}}{j\tau} + \frac{be^{-j\tau}}{\tau} + \frac{be^{-j\tau}}{j\tau^2}\right]$$

$$= \frac{1}{2\pi}\left[e^{j\tau}\left(\frac{a}{j\tau} + \frac{b}{\tau} - \frac{b}{j\tau^2}\right) + e^{-j\tau}\left(-\frac{a}{j\tau} + \frac{b}{\tau} + \frac{b}{j\tau^2}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{a}{j\tau}\left(e^{j\tau} - e^{-j\tau}\right) + \frac{b}{\tau}\left(e^{j\tau} + e^{-j\tau}\right) + \frac{b}{j\tau^2}\left(-e^{j\tau} + e^{-j\tau}\right)\right]$$

$$\cancel{\frac{1}{2\pi}\left(\frac{a}{j\tau}(\cos\theta + j\sin\theta - \cos\theta\right.}$$

$$= \frac{1}{2\pi}\left[\frac{a}{j\tau}(2j\sin\theta\tau) + \frac{b}{\tau}(2\cos\tau)\right.$$

$$\left. - \frac{b}{j\tau^2}(2j\sin\tau)\right]$$

$$R_{xy}(\tau) = \frac{a}{\pi\tau}\sin\tau + \frac{b}{\pi\tau}\cos\tau - \frac{b}{\pi\tau^2}\sin\tau$$

$$\boxed{\begin{aligned} \because\ e^{i\theta} - e^{-i\theta} &= \cos\theta + j\sin\theta - (\cos\theta \mp j\sin\theta) \\ &= 2j\sin\theta \\ e^{i\theta} + e^{-i\theta} &= \cos\theta + j\sin\theta + (\cos\theta - j\sin\theta) \\ &= 2\cos\theta \end{aligned}}$$

$$\text{RHS} = \frac{1}{\pi}\left[\frac{p}{\tau}\sin B\tau + \frac{q}{\tau}\cos B\tau - \frac{q}{B\tau^2}\sin B\tau\right]$$

Q. Find the cross corelation function of a process whose cross power spectral density is

$$S_{xy}(\omega) = \begin{cases} p + \dfrac{jq\omega}{B} & |\omega| < B \\ 0 & \text{, otherwise} \end{cases}$$

$$\rightarrow R_{xy}(\tau) = \frac{1}{2\pi}\int_{-B}^{B}\left(p + \frac{jq\omega}{B}\right)e^{j\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\left[\left(p + \frac{jq\omega}{B}\right)\frac{e^{j\omega\tau}}{j\tau} - \int \frac{jq\omega}{B^2}\frac{e^{j\omega\tau}}{j\tau}\,d$$

$$= \frac{1}{2\pi}\left[\left(p + \frac{jq\omega}{B}\right)\frac{e^{j\omega\tau}}{j\tau} + \frac{jq\omega}{B^2}\frac{e^{j\omega\tau}}{j^2\tau^2}\right]_{-\tau}$$

$$\cancel{= \frac{1}{2\pi}\left[p + \frac{jpB_0}{B}\left(\frac{e^{j\omega\tau}}{j\tau}\right) + \frac{jq\omega}{B^2}\frac{e^{j\omega\tau}}{j^2\tau^2} - \left(p + \frac{jp\omega}{-B}\right)\left(\frac{e^{j\omega\tau}}{j\tau}\right) + \frac{jq\omega}{B^2}\frac{e^{j\omega\tau}}{j^2\tau^2}\right.}$$

$$= \frac{1}{2\pi}\left[\left(p + \frac{jqB}{B}\right)\frac{e^{jB\tau}}{j\tau} + \frac{jqB}{B^2}\frac{e^{jB\tau}}{j^2\tau^2} - \left[\left(p + \frac{jp(-B)}{B}\right)\frac{e^{j(-B\tau)}}{j\tau} + \frac{jq(-B)}{B^2}\frac{e^{-j}}{j^2}\right.\right.$$

$=$

$$= \frac{1}{2\pi}\left[\frac{p e^{jB\tau}}{j\tau} + \cancel{j}\frac{q e^{jB\tau}}{\cancel{j}\tau} + \frac{jq}{B}\frac{e^{jB\tau}}{j^2\tau^2}\right.$$

$$-\left(\frac{p e^{-jB\tau}}{j\tau} - \frac{\cancel{j}q}{\cancel{j}\tau} e^{-jB\tau} + \frac{jq}{B}\frac{e^{-jB\tau}}{j^2\tau^2}\right)$$

$$= \frac{1}{2\pi}\left[\frac{p}{j\tau}\left(e^{jB\tau} - e^{-jB\tau}\right) + \frac{q}{\tau}\left(e^{jB\tau} - e^{jB\tau}\right) + \frac{q e^{jB\tau}}{Bj\tau^2}\left(e^{+jB\tau} + e^{-jB\tau}\right)\right]$$

Q The PSD of a random process $X(t)$ is given b[y]

$$S_{xx}(\omega) = \begin{cases} \pi & \text{if } |\omega| < 1 \\ 0 & \text{, otherwise} \end{cases}$$

Find the autocorrelation function.

$$\rightarrow R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{+1} \pi\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-1}^{1}$$

$$= \frac{1}{2j\tau}\left[e^{j\tau} - e^{-j\tau}\right]$$

$$= \frac{1}{\cancel{2j}\tau}\, \cancel{2j}\sin\tau$$

$$= \frac{\sin\tau}{\tau}$$

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Q. If $Y(t) = x(t+a) - x(t-a)$
PT
$$R_{yy}(\tau) = 2R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a)$$

Also PT
$$S_{yy}(\omega) = 4\sin^2 a\omega \; S_{xx}(\omega)$$

$\rightarrow$ $y(t) = x(t+a) - x(t-a)$

$$R_{yy}(\tau) = E[y(t_1) \cdot y(t_2)]$$

$$= E[x(t_1+a) - x(t_1-a) \times (x(t_2+a) - x(t_2-a)]$$

$$= E\left[\cancel{xt_1} + xa - \cancel{xt_1} + xa \times \cancel{xt_2} + xa - \cancel{xt_2} + xa\right]$$

$$= E[\cancel{2}\, 2xa \times 2xa]$$

$$= E[4(xa)^2]$$

$$R_{yy}(\tau) = E[x(t_1+a)x(t_2+a)] - E[x(t_1+a)x(t_2-a)] - E[x(t_1-a)x(t_2-a)]$$

$$+E[x(t_1-a)x(t_2-a)]$$

$$\therefore R_{YY}(\tau) = R_{xx}(t_1+a, t_2+a)$$

$$- R_{xx}(t_1+a, t_2 \mp a)$$

$$- R_{xx}(t_1-a, t_2+a)$$

$$+ R_{xx}(t_1-a, t_2-a)$$

$$= R_{xx}(\tau) - R_{xx}(\tau+2a)$$

$$- R_{xx}(\tau-2a) + R_{xx}(\tau)$$

$$\Rightarrow R_{YY}(\tau) = 2R_{xx}(\tau) - R_{xx}(\tau+2a) - R_{xx}(\tau-2a)$$

$\cancel{S_{xx}(\omega) = \int_{-\infty}^{\infty}}$ apply fourier transform

$$S_{YY}(\omega) = 2F[R_{xx}(\tau)] - F[R_{xx}(\tau+2a)]$$

$$- F[R_{xx}(\tau-2a)]$$

$$= 2S_{xx}(\omega) - e^{i2a\omega} S_{xx}(\omega)$$

$$- e^{i-2a\omega} S_{xx}(\omega)$$

$$= 2S_{xx}(\omega) - S_{xx}(\omega)[e^{i2a\omega} + e^{i-2a\omega}$$

$$= 2S_{xx}(\omega) - S_{xx}(\omega)[2\cos 2a\omega]$$

$$= 2S_{xx}\omega\ (1 - \cos 2a\omega)$$

$$= 4\sin^2 a\omega\ S_{xx}(\omega)$$