Linear Programming Problems

OPERATIONS RESEARCH

Prof. Aditya Suresh Kasar

Unit - 2

WHAT IS LINEAR PROGRAMMING?

A mathematical method of solving practical problems (such as the allocation of resources) by means of linear functions where the variables involved are subject to constraints.

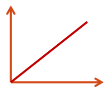
STRUCTURE OF LINEAR PROGRAMMING MODEL

The general structure of an LP model consists of following three basic components (or parts).

- **Decision variables (activities)**
- **The objective function**
- **The constraints**

ASSUMPTIONS OF AN LP MODEL

- **Linearity:** The amount of each resource used (or supplied) and its contribution to the profit (or cost) in objective function must be proportional to the value of each decision variable.



1 pen = 10 Rs.
2 pen = 20 Rs.
10 pen = 100 Rs.

- **Additivity:** The value of the objective function and the total amount of each resource used (or supplied), must be equal to the sum of the respective individual contribution (profit or cost) of the decision variables.

ASSUMPTIONS OF AN LP MODEL

- **Divisibility:** The solution values of decision variables are allowed to assume continuous values.

Ex:1 1 man = 5 days
2 men = 2.5 days
4 men = 1.25 days

Ex:2 10 units = 3men
1 unit = 3/10 men

- **Constant over time:** In LP models, it is assumed that all its parameters such as: availability of resources, profit (or cost) contribution per unit of decision variable and consumption of resources per unit of decision variable must be known and constant.

ADVANTAGES OF USING LINEAR PROGRAMMING

- Helps decision-makers to use their productive resources effectively.
- Improves the quality of decisions. The decision-making approach of the user of this technique becomes more objective and less subjective.
- Helps to arrive at optimal solution of a decision problem by taking into account constraints on the use of resources. For example, saying that so many units of any product may be produced does not mean that all units can be sold.
- LPP approach for solving decision problem highlight bottlenecks in the production processes.

LIMITATIONS OF LINEAR PROGRAMMING

- Linear programming assumes linear relationships among decision variables. However, in real-life problems, decision variables, neither in the objective function nor in the constraints are linearly related.
- While solving an LP model there is no guarantee that decision variables will get integer value. For example, how many men/machines would be required to perform a particular job, a non-integer valued solution will be meaningless. Rounding off the solution to the nearest integer will not yield an optimal solution.
- The linear programming model does not take into consideration the effect of time and uncertainty.
- Parameters in the model are assumed to be constant but in real-life situations, they are frequently neither known nor constant.
- Linear programming deals with only single objective, whereas in real-life situations a decision problem may have conflicting and multiple objectives.

APPLICATION AREAS OF LINEAR PROGRAMMING

- Applications in Agriculture
- Applications in Military
- Production Management
  - Product mix
  - Production planning
  - Assembly line balancing
  - Blending problems
  - Trim loss

APPLICATION AREAS OF LINEAR PROGRAMMING

- Financial Management
  - Portfolio selection
  - Profit Planning
- Marketing Management
  - Media selection
  - Travelling Salesman problem
  - Physical distribution
- Personnel Management

GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- Step 1: Identify the decision variables
- Step 2: Identify the problem data
- Step 3: Formulate the constraints
- Step 4: Formulate the objective function

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GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- **Step 1: Identify the decision variables**

(a) Express each constraint in words. For this you should first see whether the constraint is of the form ≥ (at least as large as), of the form ≤ (no larger than) or of the form = (exactly equal to).

(b) Express verbally the objective function.

(c) Verbally identify the decision variables with the help of Step (a) and (b). For this you need to ask yourself the question – What decisions must be made in order to optimize the objective function?

Having followed Step 1(a) to (c) decide the symbolic notation for the decision variables and specify their units of measurement. Such specification of units of measurement would help in interpreting the final solution of the LP problem.

GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- Step 1: Identify the decision variables
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GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- **Step 2: Identify the problem data**
  - To formulate an LP model, identify the problem data in terms of constants, and parameters associated with decision variables.
  - It may be noted that the decision-maker can control values of the variables but cannot control values in the data set.

GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- Step 1: Identify the decision variables
- Step 2: Identify the problem data
- **Step 3: Formulate the constraints**
- Step 4: Formulate the objective function

GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- **Step 3: Formulate the constraints**
  - Convert the verbal expression of the constraints in terms of resource requirement and availability of each resource.
  - Then express each of them as linear equality or inequality, in terms of the decision variables defined in Step 1.
  - Values of these decision variables in the optimal LP problem solution must satisfy these constraints in order to constitute an acceptable (feasible) solution.
  - Wrong formulation can either lead to a solution that is not feasible or to the exclusion of a solution that is actually feasible and possibly optimal.

GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- Step 1: Identify the decision variables
- Step 2: Identify the problem data
- Step 3: Formulate the constraints
- **Step 4: Formulate the objective function**

GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

- **Step 4: Formulate the objective function**
  - Identify whether the objective function is to be maximized or minimized.
  - Then express it in the form of linear mathematical expression in terms of decision variables along with profit (cost) contributions associated with them.

NUMERICAL: LP MODEL FORMULATION

1. A manufacturing company is engaged in producing three types of products: A, B and C. The production department produces, each day, components sufficient to make 50 units of A, 25 units of B and 30 units of C. The management is confronted with the problem of optimizing the daily production of the products in the assembly department, where only 100 man-hours are available daily for assembling the products. The following additional information is available:

Types of Product	Profit contribution per unit (Rs)	Assembly time per product (hrs)
A	12	0.8
B	20	1.7
C	45	2.5

The company has a daily order commitment for 20 units of products A and a total of 15 units of products B and C. Formulate this problem as an LP model so as to maximize the total profit.

NUMERICAL: LP MODEL FORMULATION

Solution (1).

Types of products: A, B and C.

Production each day: 50 units of A, 25 units of B and 30 units of C. Resource / Constraints

Assembly time per product (hrs.): A = 0.8 B = 1.7 C = 2.5 Constraints

Man-hours available daily: 100 hrs. Constraints

Daily order commitment: 20 units of products A and a total of 15 units of products B and C. Constraints

Profit contribution per unit (Rs): A = 12 B = 20 C = 45

Maximize the total profit Objective

NUMERICAL: LP MODEL FORMULATION

Solution (1). **LP model Formulation**

The data of the problem is summarized as follows:

Resources/Constraints	Product type			Total
	x_1 A	x_2 B	x_3 C	
Production Capacity (units)	50	25	30	
Man-hours per unit	0.8	1.7	2.5	100
Order commitment (units)	20	15		
Profit contribution (Rs/unit)	12	20	45	

Let x_1, x_2 and x_3 = number of units of products A, B and C to be produced, respectively.

NUMERICAL: LP MODEL FORMULATION

Solution (1).

Let x_1, x_2 and x_3 = number of units of products A, B and C to be produced, respectively.

Resources/Constraints	Product type			Total
	A (x1)	B (x2)	C (x3)	
Production Capacity (units)	50	25	30	
Man-hours per unit	0.8	1.7	2.5	100
Order commitment (units)	20	15		
(Z) Profit contribution (Rs/unit)	12	20	45	

The LP model

Maximize (total profit) $Z = 12\,x_1 + 20\,x_2 + 45\,x_3$

subject to the constraints

(i) Labor (assembly time) $0.8\,x_1 + 1.7\,x_2 + 2.5\,x_3 \le 100$,

and materials $x_1 \le 50$, $x_2 \le 25$, $x_3 \le 30$

(ii) Order commitment $x_1 \ge 20$; $x_2 + x_3 \ge 15$

and (non negativity) $x_1, x_2, x_3 \ge 0$.

NUMERICAL: LP MODEL FORMULATION

2. A company has two plants, each of which produces and supplies two products: A and B. The plants can each work up to 16 hours a day.

In plant 1, it takes three hours to prepare and pack 1,000 gallons of A and one hour to prepare and pack one quintal of B.

In plant 2, it takes two hours to prepare and pack 1,000 gallons of A and 1.5 hours to prepare and pack a quintal of B.

In plant 1, it costs Rs 15,000 to prepare and pack 1,000 gallons of A and Rs 28,000 to prepare and pack a quintal of B,

whereas in plant 2 these costs are Rs 18,000 and Rs 26,000, respectively.

The company is obliged to produce daily at least 10 thousand gallons of A and 8 quintals of B.

Formulate this problem as an LP model to find out as to how the company should organize its production so that the required amounts of the two products be obtained at the minimum cost.

NUMERICAL: LP MODEL FORMULATION

Solution (2). **LP model Formulation**
The data of the problem is summarized as follows:

Resources/Constraints		A	B	Total
Preparation Time (Hrs)	Plant 1	3 hrs/1000 gallons x_1	1 hr/quintal x_3	16
	Plant 2	2 hrs/1000 gallons x_2	1.5 hr/quintal x_4	16
Minimum Daily Production		10,000 gallons	8 quintals	
Cost of Production (Rs)	Plant 1	15000/1000 gallons	28000/quintals	
	Plant 2	18000/1000 gallons	26000/quintals	

x_1, x_2 = quantity of product A (in '000 gallons) to be produced in plant 1 and 2, respectively.
x_3, x_4 = quantity of product B (in quintals) to be produced in plant 1 and 2, respectively.

NUMERICAL: LP MODEL FORMULATION

Solution (2).

Decision variables Let

x1, x2 = quantity of product A (in '000 gallons) to be produced in plant 1 and 2, respectively.

x3, x4 = quantity of product B (in quintals) to be produced in plant 1 and 2, respectively.

Resources/Constraints		A	B	Total
Preparation Time (Hrs)	Plant 1	3 hrs/1000 gallons (x1)	1 hr/quintal (x3)	16
	Plant 2	2 hrs/1000 gallons (x2)	1.5 hr/quintal (x4)	16
Minimum Daily Production		10000 gallons	8 quintals	
Cost of Production (Rs)	Plant 1	15000/1000 gallons	28000/quintals	
	Plant 2	18000/1000 gallons	26000/quintals	

The LP model

Minimize (total cost) $Z = 15{,}000\,x_1 + 18{,}000\,x_2 + 28{,}000\,x_3 + 26{,}000\,x_4$

subject to the constraints

(i) Preparation time: $3\,x_1 + 2\,x_2 \le 16$,

$x_3 + 1.5\,x_4 \le 16$

(ii) Minimum daily production requirement: $x_1 + x_2 \ge 10$,

$x_3 + x_4 \ge 8$

and (non negativity): $x_1, x_2, x_3, x_4 \ge 0$.

NUMERICAL: LP MODEL FORMULATION

3. An electronic company is engaged in the production of two components C_1 and C_2 that are used in radio sets. Each unit of C_1 costs the company Rs 5 in wages and Rs 5 in material, while each of C_2 costs the company Rs 25 in wages and Rs 15 in material.

The company sells both products on one period credit terms, but the company's labor and material expenses must be paid in cash. The selling price of C_1 is Rs 30 per unit and of C_2 it is Rs 70 per unit.

Because of the company's strong monopoly in these components, it is assumed that the company can sell, at the prevailing prices, as many units as it produces. The company's production capacity is, however, limited by two considerations.

First, at the beginning of period 1, the company has an initial balance of Rs 4,000 (cash plus bank credit plus collections from past credit sales).

Second, the company has, in each period, 2,000 hours of machine time and 1,400 hours of assembly time.

The production of each C_1 requires 3 hours of machine time and 2 hours of assembly time, whereas the production of each C_2 requires 2 hours of machine time and 3 hours of assembly time.

Formulate this problem as an LP model so as to maximize the total profit to the company.

NUMERICAL: LP MODEL FORMULATION

Solution (3).

Resources/ Constraints	Components		Total Availability
	x_1 C1	x_2 C2	
Budget (₹)	10/unit	40/unit	₹ 4,000
Machine Time	3 hrs/unit	2 hrs/unit	2000 hours
Assembly Time	2 hrs/unit	3 hrs/unit	1400 hours
Selling Price	₹ 30	₹ 70	
Cost (wages + material) price	₹ 10	₹ 40	

Let x_1 and x_2 = number of units of components C_1 and C_2 to be produced, respectively.

NUMERICAL: LP MODEL FORMULATION

Solution (3).

Decision variables Let x_1, x_2 = number of units of components C_1 and C_2 to be produced, respectively.

The LP model

Maximize (total Profit) Z = Selling Price – Cost Price

$Z = (30 – 10)\ x_1 + (70 – 40)\ x_2$

$Z = 20\ x_1 + 30\ x_2$

Resources/ Constraints	Components		Total Availability
	C1	C2	
Budget (₹)	10/unit	40/unit	₹ 4,000
Machine Time	3 hrs/unit	2 hrs/unit	2000 hours
Assembly Time	2 hrs/unit	3 hrs/unit	1400 hours
Selling Price	₹ 30	₹ 70	
Cost (wages + material) price	₹ 10	₹ 40	

subject to the constraints

(i) Total budget available: $10\ x_1 + 40\ x_2 \leq 4000$,

(ii) Production Time: $3\ x_1 + 2\ x_2 \leq 2000$,

$2\ x_1 + 3\ x_2 \leq 1400$

and non negativity $x_1, x_2, \geq 0$.

NUMERICAL: LP MODEL FORMULATION

4. An advertising company wishes to plan an advertising campaign for three different media: television, radio and a magazine. The purpose of the advertising is to reach as many potential customers as possible. The following are the results of a market study:

	Television		Radio (₹)	Magazine (₹)
	Prime Day (₹)	Prime Time (₹)		
Cost of an advertising unit	40,000	75,000	30,000	15,000
Number of potential customers reached per unit	4,00,000	9,00,000	5,00,000	2,00,000
Number of women customers reached per unit	3,00,000	4,00,000	2,00,000	1,00,000

The company does not want to spend more than Rs 8,00,000 on advertising. It is further required that
(i) at least 2 million exposures take place amongst women,
(ii) the cost of advertising on television be limited to Rs 5,00,000,
(iii) at least 3 advertising units be bought on prime day and two units during prime time; and
(iv) the number of advertising units on the radio and the magazine should each be between 5 and 10.
Formulate this problem as an LP model to maximize potential customer reach

NUMERICAL: LP MODEL FORMULATION

Solution (4).

Decision variables Let x_1, x_2, x_3 and x_4 = number of advertising units bought in prime day and time on television, radio and magazine, respectively.

	Television		Radio (₹)	Magazine (₹)
	Prime Day (₹)	Prime Time (₹)		
Cost of an advertising unit	40,000	75,000	30,000	15,000
Number of potential customers reached per unit	4,00,000	9,00,000	5,00,000	2,00,000
Number of women customers reached per unit	3,00,000	4,00,000	2,00,000	1,00,000

The LP model

Maximize (total potential customer reach)

$Z = 4{,}00{,}000\ x_1 + 9{,}00{,}000\ x_2 + 5{,}00{,}000\ x_3 + 2{,}00{,}000\ x_4$

subject to the constraints

(i) Advertising budget: $40{,}000\ x_1 + 75{,}000\ x_2 + 30{,}000\ x_3 + 15{,}000\ x_4 \leq 8{,}00{,}000$

(ii) Number of women customers reached by the advertising campaign

$3{,}00{,}000x_1 + 4{,}00{,}000x_2 + 2{,}00{,}000x_3 + 1{,}00{,}000x_4 \geq 20{,}00{,}000$

(iii) Television advertising : (a) $40{,}000x_1 + 75{,}000x_2 \leq 5{,}00{,}000$; (b) $x_1 \geq 3$;

(c) x2 ≥ 2

(iv) Radio and magazine advertising : (a) $5 \leq x_3 \leq 10$; (b) $5 \leq x_4 \leq 10$

and non negativity $x_1, x_2, x_3, x_4 \geq 0$.

NUMERICAL: LP MODEL FORMULATION

5. A businessman is opening a new restaurant and has budgeted Rs 8,00,000 for advertisement, for the coming month. He is considering four types of advertising: (i) 30 second television commercials (ii) 30 second radio commercials (iii) Half-page advertisement in a newspaper (iv) Full-page advertisement in a weekly magazine which will appear four times during the coming month. The owner wishes to reach families (a) with income over Rs 50,000 and (b) with income under Rs 50,000. The amount of exposure of each media to families of type (a) and (b) and the cost of each media is shown below:

Media	Cost of Advertisement (Rs)	Exposure to Families with Annual Income Over Rs 50,000 (a)	Exposure to Families with Annual Income Rs 50,000 (b)
Television	40,000	2,00,000	3,00,000
Radio	20,000	5,00,000	7,00,000
Newspaper	15,000	3,00,000	1,50,000
Magazine	5,000	1,00,000	1,00,000

To have a balanced campaign, the owner has determined the following four restrictions:
(i) there should be no more than four television advertisements
(ii) there should be no more than four advertisements in the magazine
(iii) there should not be more than 60 per cent of all advertisements in newspaper and magazine put together
(iv) there must be at least 45,00,000 exposures to families with annual income of over Rs 50,000.
Formulate this problem as an LP model to determine the number of each type of advertisement to be given so as to maximize the total number of exposures.

NUMERICAL: LP MODEL FORMULATION

Solution (5).

Decision variables Let x_1, x_2, x_3 and x_4 = number of television, radio, newspaper, magazine advertisements to be pursued, respectively.

Media	Cost of Advertisement (Rs)	Exposure to Families with Annual Income Over Rs 50,000 (a)	Exposure to Families with Annual Income Rs 50,000 (b)
Television	40,000	2,00,000	3,00,000
Radio	20,000	5,00,000	7,00,000
Newspaper	15,000	3,00,000	1,50,000
Magazine	5,000	1,00,000	1,00,000

The LP model

Maximize (total number of exposures of both groups)

$Z = (2{,}00{,}000 + 3{,}00{,}000)\ x_1 + (5{,}00{,}000 + 7{,}00{,}000)\ x_2 + (3{,}00{,}000 + 1{,}50{,}000)\ x_3 + (1{,}00{,}000 + 1{,}00{,}000)\ x_4$

$Z = 5{,}00{,}000\ x_1 + 12{,}00{,}000\ x_2 + 4{,}50{,}000\ x_3 + 2{,}00{,}000\ x_4$

subject to the constraints

(i) Advertising budget: $40{,}000\ x_1 + 20{,}000\ x_2 + 15{,}000\ x_3 + 5{,}000\ x_4 \leq 8{,}00{,}000$

(ii) Maximum television advertisement: $x_1 \leq 4$

(iii) Maximum magazine advertisement: $x_4 \leq 4$ (because magazine will appear only four times in the next month)

(iv) Maximum newspaper and magazine advertisement: – 0.6x1 – 0.6x2 + 0.4x3 + 0.4x4 ≤ 0

(v) Exposure to families with income over Rs 50,000: $2{,}00{,}000\ x_1 + 5{,}00{,}000\ x_2 + 3{,}00{,}000\ x_3 + 1{,}00{,}000\ x_4 \geq 45{,}00{,}000$

and non negativity $x_1, x_2, x_3, x_4 \geq 0$.

NUMERICAL: LP MODEL FORMULATION

6. An electronic company produces three types of parts for automatic washing machines. It purchases casting of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines. The selling prices of parts A, B and C are Rs 8, Rs 10 and Rs 14 respectively. All parts made can be sold. Castings for parts A, B and C, respectively cost Rs 5, Rs 6 and Rs 10. The shop possesses only one of each type of casting machine. Costs per hour to run each of the three machines are Rs 20 for drilling, Rs 30 for shaping and Rs 30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the table:

Machine	Capacity per hour		
	Part A	Part B	Part C
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an LP model so as to maximize total profit to the company.

NUMERICAL: LP MODEL FORMULATION

Solution (6).

LP model formulation

Let x_1, x_2, x_3 = numbers of type A, B and C parts to be produced per hour, respectively.

Machine	Capacity per hour		
	Part A	Part B	Part C
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

Since 25 type A parts per hour can be run on the drilling machine at a cost of Rs 20, then Rs 20 / 25 = Re 0.80 is the drilling cost per type A part.

Similar reasoning for shaping and polishing gives

Profit per type A part = (8 – 5) – [20/25 + 30/25 + 30/40] = 0.25

Profit per type B part = (10 – 6) – [20/40 + 30/20 + 30/30] = 1

Profit per type C part = (14 – 10) – [20/25 + 30/20 + 30/40] = 0.95

On the drilling machine, one type A part consumes 1/25th of the available hour, a type B part consumes 1/40th, and a type C part consumes 1/25th of an hour. Thus, the drilling machine constraint is

$$x_1/25 + x_2/40 + x_3/25 \leq 1$$

Similarly, other constraints can be established.

NUMERICAL: LP MODEL FORMULATION

Solution (6).

Machine	Capacity per hour		
	Part A	Part B	Part C
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The LP model

Maximize (total profit) $Z = 0.25\ x_1 + 1.00\ x_2 + 0.95\ x_3$

subject to the constraints

(i) Drilling Machine: $x_1/25 + x_2/40 + x_3/25 \leq 1$

(ii) Shaping Machine: $x_1/25 + x_2/20 + x_3/20 \leq 1$

(iii) Polishing Machine: $x_1/40 + x_2/30 + x_3/40 \leq 1$

and non negativity $x_1, x_2, x_3 \geq 0$.

NUMERICAL: LP MODEL FORMULATION

7. A pharmaceutical company produces two pharmaceutical products: A and B. Production of both these products requires the same process – I and II. The production of B also results in a by-product C at no extra cost.

The product A can be sold at a profit of Rs 3 per unit and B at a profit of Rs 8 per unit. Some quantity of this by-product can be sold at a unit profit of Rs 2, the remainder has to be destroyed and the destruction cost is Re 1 per unit. Forecasts show that only up to 5 units of C can be sold. The company gets 3 units of C for each unit of B produced.

The manufacturing times are 3 hours per unit for A on process I and II, respectively, and 4 hours and 5 hours per unit for B on process I and II, respectively. Because the product C is a by product of B, no time is used in producing C. The available times are 18 and 21 hours of process I and II, respectively.

Formulate this problem as an LP model to determine the quantity of A and B which should be produced, keeping C in mind, to make the highest total profit to the company.

NUMERICAL: LP MODEL FORMULATION

Solution (7).

LP model formulation

The data of the problem is summarized as follows:

Constraints/Resources	Time(hrs) required by			Availability
	A	B	C	
Process I	3	4	-	18 hrs
Process II	3	5	-	21 hrs
By product ratio from B	-	1	3	5 units (max units that can be sold
Profit per unit (Rs)	3	8	2	

NUMERICAL: LP MODEL FORMULATION

Solution (7).

Decision variables: Let x_1, x_2 = units of product A and B to be produced, respectively

x_3, x_4 = units of product C to be produced and destroyed, respectively.

The LP model

Maximize (total profit) $Z = 3x_1 + 8x_2 + 2x_3 - x_4$

subject to the constraints

(i) Manufacturing constraints for product A and B: (a) $3x_1 + 4x_2 \le 18$, (b) $3x_1 + 5x_2 \le 21$

(ii) Manufacturing constraints for by-product C: (a) $x_3 \le 5$, (b) $-3x_2 + x_3 + x_4 = 0$

and non negativity $x_1, x_2, x_3 + x_4 \ge 0$.

Constraints/Resources	Time(hrs) required by			Availability
	A	B	C	
Process I	3	4	-	18 hrs
Process II	3	5	-	21 hrs
By product ratio from B	-	1	3	5 units (max units that can be sold
Profit per unit (Rs)	3	8	2	

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Unit - 2

INTRODUCTION

- An optimal as well as a feasible solution to an LP problem is obtained by choosing one set of values from several possible values of decision variables $x_1, x_2, \ldots, x_n$, that satisfies the given constraints simultaneously and also provides an optimal (maximum or minimum) value of the given objective function.
- For LP problems that have only two variables, it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints on a graph paper in order to locate the best (optimal) solution.
- The technique used to identify the optimal solution is called the graphical solution method (approach or technique) for an LP problem with two variables.
- Since most real-world problems have more than two decision variables, such problems cannot be solved graphically.
- However, graphical approach provides understanding of solving an LP problem algebraically, involving more than two variables.

IMPORTANT DEFINITIONS

- **Solution:** The set of values of decision variables xj (j = 1, 2, . . ., n) that satisfy the constraints of an LP problem is said to constitute the solution to that LP problem.
- **Feasible solution:** The set of values of decision variables xj (j = 1, 2, . . ., n) that satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the feasible solution to that LP problem.
- **Infeasible solution:** The set of values of decision variables xj (j = 1, 2, . . ., n) that do not satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the infeasible solution to that LP problem.

IMPORTANT DEFINITIONS

- **Basic solution:** For a set of m simultaneous equations in n variables (n > m) in an LP problem, a solution obtained by setting (n – m) variables equal to zero and solving for remaining m equations in m variables is called a basic solution of that LP problem. The (n – m) variables whose value did not appear in basic solution are called non-basic variables and the remaining m variables are called basic variables.
- **Basic feasible solution:** A feasible solution to an LP problem which is also the basic solution is called the basic feasible solution. That is, all basic variables assume non-negative values. Basic feasible solution is of two types:
  a) **Degenerate:** A basic feasible solution is called degenerate if the value of at least one basic variable is zero.
  b) **Non-degenerate**: A basic feasible solution is called non-degenerate if value of all m basic variables is non-zero and positive.

IMPORTANT DEFINITIONS

- **Optimum basic feasible solution:** A basic feasible solution that optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.
- **Unbounded solution:** A solution that can increase or decrease infinitely the value of the objective function of the LP problem is called an unbounded solution

EXTREME POINT SOLUTION METHOD

- Step 1 : Develop an LP model
- Step 2 : Plot constraints on graph paper and decide the feasible region
  a) Replace the inequality sign in each constraint by an equality sign.
  b) Draw these straight lines on the graph paper and decide each time the area of feasible solutions according to the inequality sign of the constraint. Shade the common portion of the graph that satisfies all the constraints simultaneously drawn so far.
  c) The final shaded area is called the feasible region (or solution space) of the given LP problem. Any point inside this region is called feasible solution and this provides values of x1 and x2 that satisfy all the constraints.

EXTREME POINT SOLUTION METHOD

- Step 3 : Examine extreme points of the feasible solution space to find an optimal solution
  a) Determine the coordinates of each extreme point of the feasible solution space.
  b) Compute and compare the value of the objective function at each extreme point.
  c) Identify the extreme point that gives optimal (max. or min.) value of the objective function.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Q.1. Use the graphical method to solve the following LP problem.

Maximize $Z = 15x_1 + 10x_2$

subject to the constraints

(i) $4x_1 + 6x_2 \leq 360$, (ii) $3x_1 + 0x_2 \leq 180$, (iii) $0x_1 + 5x_2 \leq 200$

and $x_1, x_2 \geq 0$.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 1.

Consider the first constraint $4x_1 + 6x_2 \leq 360$.

Treat this as the equation:

$4x_1 + 6x_2 = 360$ ---------------- 1

x_1	0	90
x_2	60	0

EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 1.

Consider the first constraint $4x_1 + 6x_2 \leq 360$.

Treat this as the equation:

$4x_1 + 6x_2 = 360$ ---------------- 1

x_1	0	90
x_2	60	0

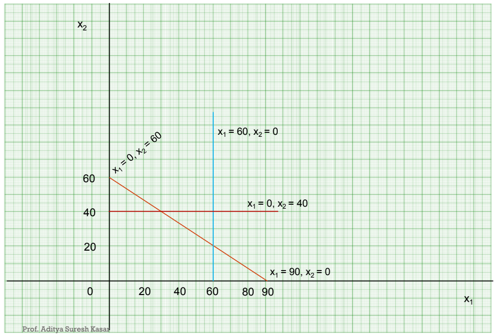
Similarly, the constraints

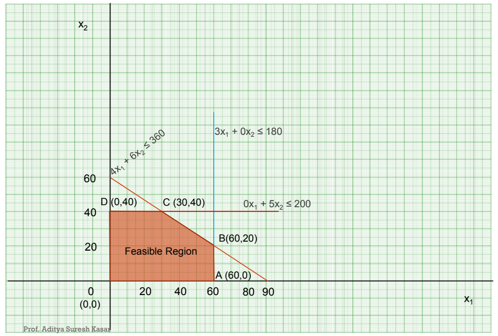
$3x_1 = 180$ -------------------------- 2

x_1	60
x_2	0

$5x_2 = 200$ -------------------------- 3

x_1	0
x_2	40





EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 1. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 15x_1 + 10x_2$
O	(0,0)	15 (0) + 10 (0) = 0
A	(60,0)	15(60) + 10(0) = 900
B	(60,20)	15(60) + 10(20) = 1,100
C	(30,40)	15(30 + 10(40) = 850
D	(0,40)	15(0) + 10(40) = 400

Since objective function Z is to be maximized, we conclude that maximum value of **Z = 1,100** is achieved at the point extreme **B (60, 20).**

Hence the optimal solution to the given LP problem is: $x_1 = 60$, $x_2 = 20$ and **Max Z = 1,100**.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Q.2. Use the graphical method to solve the following LP problem.

Maximize $Z = 2x_1 + x_2$

subject to the constraints

(i) $x_1 + 2x_2 \leq 10$, (ii) $x_1 + x_2 \leq 6$, (iii) $x_1 - x_2 \leq 2$, (iv) $x_1 - 2x_2 \leq 1$

and $x_1, x_2 \geq 0$.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 2.

$x_1 + 2x_2 = 10$ ------------- 1

x_1	0	10
x_2	5	0

$x_1 + x_2 = 6$ ----------------- 2

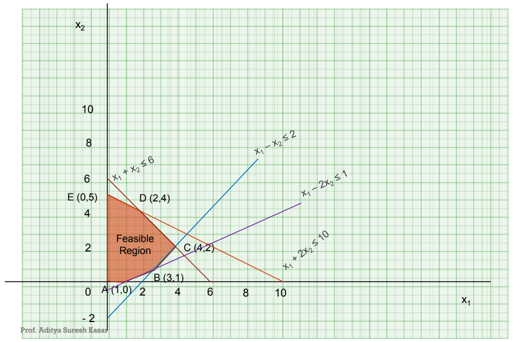
x_1	0	6
x_2	6	0

$x_1 - x_2 = 2$ ----------------- 3

x_1	0	2
x_2	- 2	0

$x_1 - 2x_2 = 1$ --------------- 4

x_1	0	1
x_2	- 0.5	0



EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 2. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 2x_1 + x_2$
O	(0,0)	2 (0) + 1 (0) = 0
A	(1,0)	2(1) + 1(0) = 2
B	(3,1)	2(3) + 1(1) = 7
C	(4,2)	2(4) + 1(2) = 10
D	(2,4)	2(2) + 1(4) = 8
E	(0,5)	2(0) + 1(5) = 5

Since objective function Z is to be maximized, we conclude that maximum value of **Z = 10** is achieved at the point extreme **C (4,2).**

Hence the optimal solution to the given LP problem is: $x_1 = 4$, $x_2 = 2$ and **Max Z = 10**.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Q.3. The ABC Company has been a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and marketing of AM and AM-FM radios. It has built a new plant that can operate 48 hours per week. Production of an AM radio in the new plant will require 2 hours and production of an AM-FM radio will require 3 hours. Each AM radio will contribute Rs 40 to profits while an AM-FM radio will contribute Rs 80 to profits. The marketing department, after extensive research has determined that a maximum of 15AM radios and 10 AM-FM radios can be sold each week.

(a) Formulate a linear programming model to determine the optimum production mix of AM and FM radios that will maximize profits.

(b) Solve this problem using the graphical method.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 3. Let us define the following decision variables;

x_1 and x_2 = number of units of AM radio and AM-FM radio to be produced, respectively.

Then LP model of the given problem is:

Maximize (total profit) $Z = 40x_1 + 80x_2$

subject to the constraints

(i) Plant : $2x_1 + 3x_2 \le 48$, (ii) AM radio : $x_1 \le 15$, (iii) AM-FM radio : $x_2 \le 10$

and $x_1, x_2 \ge 0$.

EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 3.

$2x_1 + 3x_2 \le 48$ ------------- 1

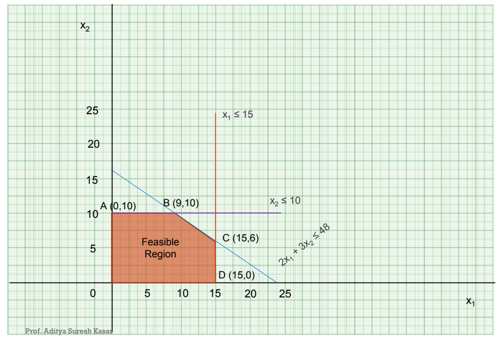
x_1	0	24
x_2	16	0

$x_1 \le 15$ ---------------------- 2

x_1	15
x_2	0

$x_2 \le 10$ ---------------------- 3

x_1	0
x_2	10



EXAMPLES ON MAXIMIZATION LP PROBLEM

Solution 3. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 40x_1 + 80x_2$
O	(0,0)	40 (0) + 80 (0) = 0
A	(0,10)	40(0) + 80(10) = 800
B	(9,10)	40(9) + 80(10) = 1,160
C	(15,6)	40(15) + 80(6) = 1,080
D	(15,0)	40(15) + 80(0) = 600

Since objective function Z is to be maximized, we conclude that maximum value of **Z = 1,160** is achieved at the point extreme **B (9,10).**

Hence the optimal solution to the given LP problem is: $x_1 = 9$, $x_2 = 10$ and **Max Z = 1,160**.

EXAMPLES ON MINIMIZATION LP PROBLEM

Q.4. Use the graphical method to solve the following LP problem.

Minimize $Z = 3x_1 + 2x_2$

subject to the constraints

(i) $5x_1 + x_2 \ge 10$, (ii) $x_1 + x_2 \ge 6$, (iii) $x_1 + 4x_2 \ge 12$

and $x_1, x_2 \ge 0$.

EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 4.

$5x_1 + x_2 = 10$ ------------- 1

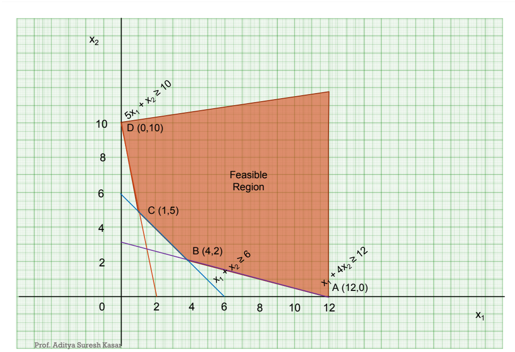
x_1	0	2
x_2	10	0

$x_1 + x_2 = 6$ ---------------------- 2

x_1	0	6
x_2	6	0

$x_1 + 4x_2 = 12$ ---------------------- 3

x_1	0	12
x_2	3	0



EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 4. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 3x_1 + 2x_2$
A	(12,0)	3(12) + 2(0) = 36
B	(4,2)	3(4) + 2(2) = 16
C	(1,5)	3(1) + 2(5) = 13
D	(0,10)	3(0) +2(10) = 20

Since objective function Z is to be minimized, we conclude that minimum value of **Z = 13** is achieved at the point extreme **C (1,5).**

Hence the optimal solution to the given LP problem is: $x_1 = 1$, $x_2 = 5$ and **Min Z = 13**.

EXAMPLES ON MINIMIZATION LP PROBLEM

Q.5. Use the graphical method to solve the following LP problem.

Minimize $Z = -x_1 + 2x_2$

subject to the constraints

(i) $-x_1 + 3x_2 \leq 10$, (ii) $x_1 + x_2 \leq 6$, (iii) $x_1 - x_2 \leq 2$

and $x_1, x_2 \geq 0$.

EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 5.

$-x_1 + 3x_2 = 10$ ---------------- 1

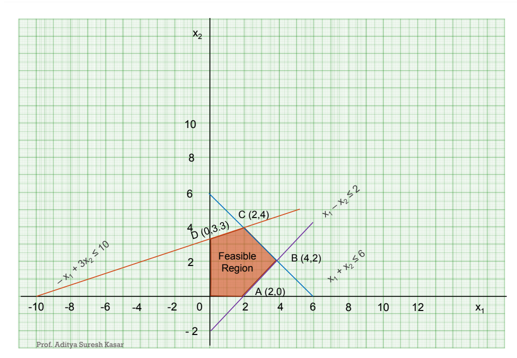
x_1	0	-10
x_2	3.3	0

$x_1 + x_2 = 6$ ---------------------- 2

x_1	0	6
x_2	6	0

$x_1 - x_2 = 2$ ---------------------- 3

x_1	0	2
x_2	-2	0



EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 5. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = -x_1 + 2x_2$
O	(0,0)	-1(0) + 2(0) = 0
A	(2,0)	-1(2) + 2(0) = -2
B	(4,2)	-1(4) + 2(2) = 0
C	(2,4)	-1(2) + 2(4) = 6
D	(0,3.3)	-1(0) +2(3.3) = 6.6

Since objective function Z is to be minimized, we conclude that minimum value of **Z = -2** is achieved at the point extreme **A (2,0).**

Hence the optimal solution to the given LP problem is: $x_1 = 2$, $x_2 = 0$ and **Min Z = - 2**.

EXAMPLES ON MINIMIZATION LP PROBLEM

Q.6. G.J. Breweries Ltd have two bottling plants, one located at 'G' and the other at 'J'. Each plant produces three drinks, whisky, beer and brandy named A, B and C respectively. The number of the bottles produced per day are shown in the table.

Drink	Plant at	
	G	J
Whisky	1,500	1,500
Beer	3,000	1,200
Brandy	2,000	5,000

A market survey indicates that during the month of July, there will be a demand of 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at G and J are 600 and 400 monetary units. For how many days each plant be run in July so as to minimize the production cost, while still meeting the market demand? Formulate this problem as an LP problem and solve that using graphical method.

EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 6. Let us define the following decision variables:

x_1 and x_2 = number of days of work at plant G and J, respectively.

Then LP model of the given problem is:

Minimize $Z = 600x_1 + 400x_2$

subject to the constraints

(i) Whisky : $1500x_1 + 1500x_2 = 20,000$

(ii) Beer : $3000x_1 + 1000x_2 = 40,000$

(iii) Brandy : $2000x_1 + 5000x_2 = 44,000$

and $x_1, x_2 \geq 0$.

EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 6.

$1500x_1 + 1500x_2 = 20,000$ ----- 1

x_1	0	13.33
x_2	13.33	0

$3000x_1 + 1000x_2 = 40,000$ ----- 2

x_1	0	13.33
x_2	40	0

$2000x_1 + 5000x_2 = 44,000$ ----- 3

x_1	0	22
x_2	8.8	0

EXAMPLES ON MINIMIZATION LP PROBLEM

Solution 6. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 600x_1 + 400x_2$
A	(22,0)	600(22) + 400(0) = 13,200
B	(12,4)	600(12) + 400(4) = 8,800
C	(0,40)	600(0) + 400(40) = 16,000

Since objective function Z is to be minimized, we conclude that minimum value of **Z = 8,800** is achieved at the point extreme **B (12,4).**

Hence the optimal solution to the given LP problem is: x_1 = 12, x_2 = 4 and **Min Z = 8,800**.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Q.7. A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintals of metal X and not more than 35 quintals of metal Y. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scrap supplied by A and B is given below.

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs 200 per quintal and that of B is Rs 400 per quintal. The firm wants to determine the quantities that it should buy from the two suppliers so that the total cost is minimized.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 7. Let us define the following decision variables:

x_1 and x_2 = quantity (in quintals) of scrap to be purchased from suppliers A and B, respectively.

Then LP model of the given problem is:

Minimize $Z = 200x_1 + 400x_2$

subject to the constraints

(i) Maximum purchase : $x_1 + x_2 \geq 200$

(ii) Scrap containing : $x_1/4 + 3x_2/4 \geq 100$ or $x_1 + 3x_2 \geq 400$ (Metal X)

$x_1/10 + x_2/5 \leq 35$ or $x_1 + 2x_2 \leq 350$ (Metal Y)

and $x_1, x_2 \geq 0$.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 7.

$x_1 + x_2 = 200$ -------------------- 1

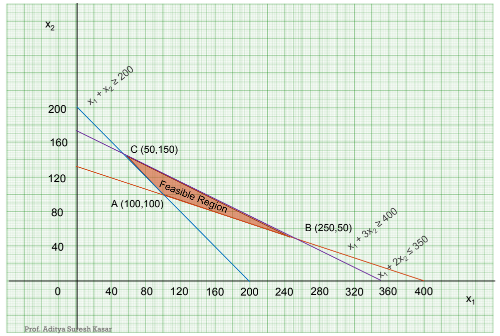
x_1	0	200
x_2	200	0

$x_1 + 3x_2 = 400$ -------------------- 2

x_1	0	400
x_2	133.33	0

$x_1 + 2x_2 = 350$ -------------------- 3

x_1	0	350
x_2	175	0



EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 7. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 200x_1 + 400x_2$
A	(100,100)	200(100) + 400(100) = 60,000
B	(250,50)	200(250) + 400(50) = 70,000
C	(50,150)	200(50) + 400(150) = 70,000

Since minimum (optimal) value of objective function, **Z = Rs 60,000** occurs at the extreme point **A (100, 100),** firm should buy $x_1 = x_2$ = 100 quintals of scrap each from suppliers A and B.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Q.8. Use the graphical method to solve the following LP problem.

Maximize $Z = 7x_1 + 3x_2$

subject to the constraints

(i) $x_1 + 2x_2 \geq 3$ (ii) $x_1 + x_2 \leq 4$ (iii) $0 \leq x_1 \leq 5/2$ (iv) $0 \leq x_2 \leq 3/2$

and $x_1, x_2 \geq 0$.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 8.

$x_1 + 2x_2 = 3$ -------------------- 1

x_1	0	3
x_2	3/2	0

$x_1 + x_2 = 4$ -------------------- 2

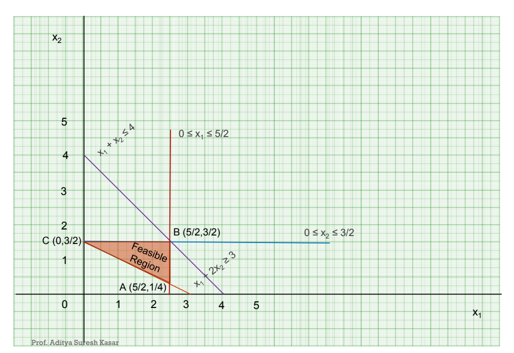
x_1	0	4
x_2	4	0

$x_1 = 5/2$ -------------------- 3

x_1	5/2
x_2	0

$x_2 = 3/2$ -------------------- 4

x_1	0
x_2	3/2



EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 8. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 7x_1 + 3x_2$
A	(5/2,1/4)	7(5/2) + 3(1/4) = 73/4
B	(5/2,3/2)	7(5/2) + 3(3/2) = 22
C	(0,3/2)	7(0) + 3(3/2) = 9/2

Since maximum (optimal) value of objective function, **Z = Rs 22** occurs at the extreme point **B (5/2, 3/2).**
Hence, the optimal solution to the given LP problem is: x1 = 5/2, x2 = 3/2 and Max Z = 22.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Q.9. Use the graphical method to solve the following LP problem.

Minimize $Z = 20x_1 + 10x_2$

subject to the constraints

(i) $x_1 + 2x_2 \le 40$, (ii) $3x_1 + x_2 \ge 30$, (iii) $4x_1 + 3x_2 \ge 60$

and $x_1, x_2 \ge 0$.

EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 9.

$x_1 + 2x_2 = 40$ -------------------- 1

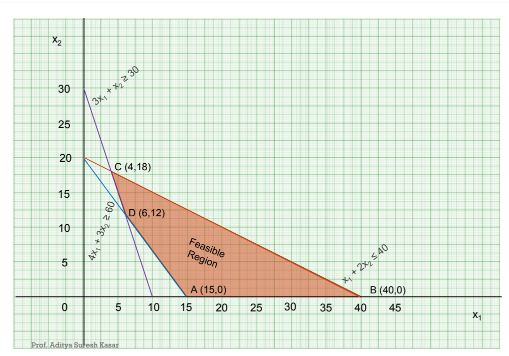
x_1	0	40
x_2	20	0

$3x_1 + x_2 = 30$ ------------------- 2

x_1	0	10
x_2	30	0

$4x_1 + 3x_2 = 60$ -------------------- 3

x_1	0	15
x_2	20	0



EXAMPLES ON MIXED CONSTRAINTS LP PROBLEMS

Solution 9. Evaluate objective function value at each extreme point of the feasible region as shown:

Extreme Points	Coordinates (x1, x2)	Objective Function Value $Z = 20x_1 + 10x_2$
A	(15,0)	20(15) + 10(0) = 300
B	(40,0)	20(40) + 10(0) = 800
C	(4,18)	20(4) + 10(18) = 260
D	(6,12)	20(6) + 10(12) = 240

Since minimum (optimal) value of objective function, **Z = Rs 240** occurs at the extreme point **D (6,12).**
Hence, the optimal solution to the given LP problem is: $x_1 = 6$, $x_2 = 12$ and Min Z = 240.

Linear Programming Problems

OPERATIONS RESEARCH

Prof. Aditya Suresh Kasar

Unit - 2

INTRODUCTION

- Most real-life problems when formulated as an LP model have more than two variables and therefore need a more efficient method to suggest an optimal solution for such problems.
- The concept of **Simplex Method** is similar to the graphical method. In the graphical method, extreme points of the feasible solution space are examined in order to search for the optimal solution that lies at one of these points.
- For LP problems with several variables, we may not be able to graph the feasible region, but the optimal solution will still lie at an extreme point of the many-sided, multidimensional figure (called an n-dimensional polyhedron) that represents the feasible solution space.
- The simplex method examines these extreme points in a systematic manner, repeating the same set of steps of the algorithm until an optimal solution is found. It is for this reason that it is also called the iterative method.

STANDARD FORM OF AN LP PROBLEM

The use of the simplex method to solve an LP problem requires that the problem be converted into its standard form. The standard form of the LP problem should have the following characteristics:

(i) All the constraints should be expressed as equations by adding slack or surplus and/or artificial variables.

(ii) The right-hand side of each constraint should be made non-negative (if not). This is done by multiplying both sides of the resulting constraint by – 1.

(iii) The objective function should be of the maximization type.

STANDARD FORM OF AN LP PROBLEM

The standard form of the LP problem is expressed as:

Optimize (Max or Min) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + 0s_1 + 0s_2 + \ldots + 0s_m$

subject to the linear constraints

$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + s_1 = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + s_2 = b_2$

. . .

. . .

. . .

$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + s_m = b_m$

and $x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0$

STANDARD FORM OF AN LP PROBLEM

This standard form of the LP problem can also be expressed in the compact form as follows:

Optimize (Max or Min) $Z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i$ (Objective function)

subject to the linear constraints

$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i ; \quad i = 1, 2, \ldots, m$ (Constraints)

and $x_j, s_i \geq 0, \quad$ for all i and j (Non-negativity conditions)

REMARKS

Any LP problem (maximization or minimization) may have

1. (a) no feasible solution, i.e. value of decision variables, xj (j = 1, 2, . . ., n) may not satisfy every constraint.

(b) a unique optimum feasible solution.

(c) more than one optimum feasible solution, i.e. alternative optimum feasible solutions.

(d) a feasible solution for which the objective function is unbounded (value of the objective function can be made as large as possible in a maximization problem or as small as possible in a minimization problem).

2. Any minimization LP problem can be converted into an equivalent maximization problem by changing the sign of cj's in the objective function.

$$\text{Minimize } \sum_{j=1}^{n} c_j x_j = \text{Maximize } \sum_{j=1}^{n} (-c_j) x_j$$

REMARKS

3. Any constraint expressed by equality (=) sign may be replaced by two weak inequalities.

For example,

$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$ is equivalent to following two simultaneous constraints,

$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1$ and $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \geq b_1$

4. Three types of additional variables, namely (i) slack variables (s) (ii) surplus variables (– s), and (iii) artificial variables (A) are added in the given LP problem to convert it into the standard form for the following reasons:

(a) These variables allow us to convert inequalities into equalities, thereby converting the given LP problem into its standard form. Such form will help in getting solution of the LP problem.

(b) These variables help decision-makers to draw economic interpretation from the final solution.

(c) These variables are used to get an initial feasible solution represented by the columns of the identity matrix.

REMARKS

- A **slack variable** represents an unused resource, either in the form of time on a machine, labour hours, money, warehouse space, etc. Since these variables don't yield any profit, therefore such variables are added to the objective function with zero coefficients.
- A **surplus variable** represents the amount by which solution values exceed a resource. These variables are also called negative slack variables. Surplus variables, like slack variables carry a zero coefficient in the objective function.

EXAMPLES

Q.1. Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

subject to the constraints

(i) $2x_1 + 3x_2 \leq 8$, (ii) $2x_2 + 5x_3 \leq 10$, (iii) $3x_1 + 2x_2 + 4x_3 \leq 15$

and $x_1, x_2, x_3 \geq 0$

EXAMPLES

Solution 1. **Step 1:** Introducing non-negative slack variables s_1, s_2 and s_3 to convert the given LP problem into its standard form:

Maximize $Z = 3X_1 + 5X_2 + 4X_3 + 0S_1 + 0S_2 + 0S_3$

subject to the constraints

(i) $2X_1 + 3X_2 + S_1 = 8$, (ii) $2X_2 + 5X_3 + S_2 = 10$, (iii) $3X_1 + 2X_2 + 4X_3 + S_3 = 15$

and $X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$

EXAMPLES

Solution 1. **Step 2:** Since all bi (RHS values) > 0, (i = 1, 2, 3), therefore choose the initial basic feasible solution as:
$X_1 = X_2 = X_3 = 0; S_1 = 8, S_2 = 10, S_3 = 15$ and Max Z = 0
This solution can also be read from the initial simplex Table below by equating row-wise values in the basis (B) column and solution values (x_B) column.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
0	S_1	8	2	3	0	1	0	0	
0	S_2	10	0	2	5	0	1	0	
0	S_3	15	3	2	4	0	0	1	
Z = 0		Z_j							
		$C_j - Z_j$							

EXAMPLES

Solution 1.
Step 3: To see whether the current solution given is optimal or not, calculate $C_j - Z_j = C_j - C_B B^{-1} a_j = C_j - C_B Y_j$ for non-basic variables x_1, x_2 and x_3 as follows.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
0	S_1	8	2	3	0	1	0	0	
0	S_2	10	0	2	5	0	1	0	
0	S_3	15	3	2	4	0	0	1	
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	3	5	4	0	0	0	

EXAMPLES

Solution 1. Since all $C_j - Z_j \geq 0$ (j = 1, 2, 3), the current solution is **not optimal**.
Variable X_2 is chosen to enter into the basis because $C_2 - Z_2 = 5$ is the largest positive number in the X_2-column, where all elements are positive. This means that for every unit of variable X_2, the objective function will increase in value by 5. The X_2- column is the **key column**.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
0	S_1	8	2	3	0	1	0	0	
0	S_2	10	0	2	5	0	1	0	
0	S_3	15	3	2	4	0	0	1	
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	3	5	4	0	0	0	

EXAMPLES

Solution 1. **Step 4:** The variable that is to leave the basis is determined by dividing the values in the X_B - column by the corresponding elements in the key column. Since the ratio, 8/3 is minimum in row 1, the basic variable S_1 is chosen to leave the solution (basis).

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio X_B/X_2
0	S_1	8	2	3	0	1	0	0	8/3
0	S_2	10	0	2	5	0	1	0	10/2
0	S_3	15	3	2	4	0	0	1	15/2
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	3	5	4	0	0	0	

EXAMPLES

Solution 1. **Step 4:** The variable that is to leave the basis is determined by dividing the values in the X_B - column by the corresponding elements in the key column. Since the ratio, 8/3 is minimum in row 1, the basic variable S_1 is chosen to leave the solution (basis).

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio X_B/X_2
0	S_1	8	2	3	0	1	0	0	**8/3** →
0	S_2	10	0	2	5	0	1	0	10/2
0	S_3	15	3	2	4	0	0	1	15/2
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	3	5	4	0	0	0	

EXAMPLES

Solution 1. **Step 5 (Iteration 1):** Since the key element enclosed in the circle is not 1, divide all elements of the key row by 3 in order to obtain new values of the elements in this row.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	
0	S_2								
0	S_3								
Z =		Z_j							
		$C_j - Z_j$							

EXAMPLES

Solution 1. **Step 5 (Iteration 1):** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R2 (new) → R2 (old) – 2 R1(new)

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	
0	S_2	14/3	-4/3	0	5	-2/3	1	0	
0	S_3								
Z =		Z_j							
		$C_j - Z_j$							

10 – 2 x 8/3 = 14/3   2 – 2 x 1 = 0   0 – 2 x 1/3 = -2/3   0 – 2 x 0 = 0
0 – 2 x 2/3 = -4/3   5 – 2 x 0/3 = 5   1 – 2 x 0/3 = 1

EXAMPLES

Solution 1. **Step 5 (Iteration 1):** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R3 (new) → R3 (old) – 2R1 (new)

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio X_B/X_2
5	X_2	8/3	2/3	1	0	1/3	0	0	
0	S_2	14/3	-4/3	0	5	-2/3	1	0	
0	S_3	29/3	5/3	0	4	-2/3	0	1	
Z = 40/3		Z_j							
		$C_j - Z_j$							

15 – 2 x 8/3 = 29/3   2 – 2 x 1 = 0   0 – 2 x 1/3 = -2/3   1 – 2 x 0 = 1
3 – 2 x 2/3 = 5/3   4 – 2 x 0/3 = 4   0 – 2 x 0/3 = 0

EXAMPLES

Solution 1.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	
0	S_2	14/3	-4/3	0	5	-2/3	1	0	
0	S_3	29/3	5/3	0	4	-2/3	0	1	
Z = 40/3		Z_j	10/3	5	0	5/3	0	0	
		$C_j - Z_j$	-1/3	0	4	-5/3	0	0	

EXAMPLES

Solution 1.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	0
0	S_2	14/3	-4/3	0	5	-2/3	1	0	14/3÷5
0	S_3	29/3	5/3	0	4	-2/3	0	1	29/3÷4
Z = 40/3		Z_j	10/3	5	0	5/3	0	0	
		$C_j - Z_j$	-1/3	0	4	-5/3	0	0	

EXAMPLES

Solution 1. **Step 4 (Iteration 1):** The variable that is to leave the basis is determined by dividing the values in the X_B - column by the corresponding elements in the key column. Since the ratio, 14/3 ÷ 5 is minimum in row 2, the basic variable S_2 is chosen to leave the solution (basis).

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	0
0	S_2	14/3	-4/3	0	5	-2/3	1	0	**14/3÷5** →
0	S_3	29/3	5/3	0	4	-2/3	0	1	29/3÷4
Z = 40/3		Z_j	10/3	5	0	5/3	0	0	
		$C_j - Z_j$	-1/3	0	4	-5/3	0	0	

EXAMPLES

Solution 1. **Step 5 (Iteration 2):** Since the key element enclosed in the circle is not 1, divide all elements of the key row by 5 in order to obtain new values of the elements in this row.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	0
0	S_2	14/3	-4/3	0	5	-2/3	1	0	14/3÷5
0	S_3	29/3	5/3	0	4	-2/3	0	1	29/3÷4
Z = 40/3		Z_j	10/3	5	0	5/3	0	0	
		$C_j - Z_j$	-1/3	0	4	-5/3	0	0	

EXAMPLES

Solution 1. **Step 5 (Iteration 2):** Since the key element enclosed in the circle is not 1, divide all elements of the key row by 5 in order to obtain new values of the elements in this row.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2								
4	X_3	14/15	-4/15	0	1	-2/15	1/5	0	
0	S_3								
Z =		Z_j							
		$C_j - Z_j$							

EXAMPLES

Solution 1. **Step 5 (Iteration 2):** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R3 (new) → R3 (old) – 4R2 (new)

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	
4	X_3	14/15	-4/15	0	1	-2/15	1/5	0	
0	S_3	89/15	41/15	0	0	-2/15	-4/5	1	
Z = 256/15		Z_j							
		$C_j - Z_j$							

29/3 – 4 x 14/15 = 89/15
5/3 – 4 x -4/15 = 41/15
0 – 4 x 0 = 0
4 – 4 x 1 = 0
-2/3 – 4 x -2/15 = -2/15
0 – 4 x 1/5 = -4/5
1 – 4 x 0 = 1

EXAMPLES

Solution 1.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio X_B/X_2
5	X_2	8/3	2/3	1	0	1/3	0	0	
4	X_3	14/15	-4/15	0	1	-2/15	1/5	0	
0	S_3	89/15	41/15	0	0	-2/15	-4/5	1	
Z = 256/15		Z_j	34/15	5	4	17/15	4/5	0	
		$C_j - Z_j$	11/15	0	0	-17/15	-4/5	0	

EXAMPLES

Solution 1.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	4
4	X_3	14/15	-4/15	0	1	-2/15	1/5	0	-
0	S_3	89/15	41/15	0	0	-2/15	-4/5	1	2.17
Z = 256/15		Z_j	34/15	5	4	17/15	4/5	0	
		$C_j - Z_j$	11/15	0	0	-17/15	-4/5	0	

EXAMPLES

Solution 1. **Step 4 (Iteration 2):** The variable that is to leave the basis is determined by dividing the values in the X_B - column by the corresponding elements in the key column. Since the ratio, 2.17 is minimum in row 3, the basic variable S_3 is chosen to leave the solution (basis).

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	4
4	X_3	14/15	-4/15	0	1	-2/15	1/5	0	-
0	S_3	89/15	41/15	0	0	-2/15	-4/5	1	2.17
Z = 256/15		Z_j	34/15	5	4	17/15	4/5	0	
		$C_j - Z_j$	11/15	0	0	-17/15	-4/5	0	

EXAMPLES

Solution 1. **Step 5 (Iteration 3):** Since the key element enclosed in the circle is not 1, divide all elements of the key row by 15/41 in order to obtain new values of the elements in this row.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	8/3	2/3	1	0	1/3	0	0	4
4	X_3	14/15	-4/15	0	1	-2/15	1/5	0	-
0	S_3	89/15	41/15	0	0	-2/15	-4/5	1	2.17
Z = 256/15		Z_j	34/15	5	4	17/15	4/5	0	
		$C_j - Z_j$	11/15	0	0	-17/15	-4/5	0	

EXAMPLES

Solution 1. **Step 5 (Iteration 3):** Since the key element enclosed in the circle is not 1, divide all elements of the key row by 15/41 in order to obtain new values of the elements in this row.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2								
4	X_3								
3	X_1	89/41	1	0	0	-2/41	-12/41	15/41	
Z =		Z_j							
		$C_j - Z_j$							

EXAMPLES

Solution 1. **Step 5 (Iteration 3):** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R1 (new) → R1 (old) – 2/3 R3 (new)

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	50/41	0	1	0	15/41	8/41	-10/41	
4	X_3								
3	X_1	89/41	1	0	0	-2/41	-12/41	15/41	
Z =		Z_j							
		$C_j - Z_j$							

EXAMPLES

Solution 1. **Step 5 (Iteration 3):** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R2 (new) → R2 (old) – 4/15 R3 (new)

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	50/41	0	1	0	15/41	8/41	-10/41	
4	X_3	62/41	0	0	1	-6/41	5/41	4/41	
3	X_1	89/41	1	0	0	-2/41	-12/41	15/41	
Z =		Z_j							
		$C_j – Z_j$							

EXAMPLES

Solution 1.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	50/41	0	1	0	15/41	8/41	-10/41	
4	X_3	62/41	0	0	1	-6/41	5/41	4/41	
3	X_1	89/41	1	0	0	-2/41	-12/41	15/41	
Z = 765/41		Z_j	3	5	4	45/41	24/41	11/41	
		$C_j – Z_j$	0	0	0	-45/41	-24/41	-11/41	

EXAMPLES

Solution 1. all $C_j – Z_j < 0$ for non-basic variables.
Therefore, the optimal solution is reached with, X_1 = 89/41, X_2 = 50/41, X_3 = 62/41 and the optimal value of Z = 765/41.

		C_j	3	5	4	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
5	X_2	50/41	0	1	0	15/41	8/41	-10/41	
4	X_3	62/41	0	0	1	-6/41	5/41	4/41	
3	X_1	89/41	1	0	0	-2/41	-12/41	15/41	
Z = 765/41		Z_j	3	5	4	45/41	24/41	11/41	
		$C_j – Z_j$	0	0	0	-45/41	-24/41	-11/41	

EXAMPLES

Q.2. A company makes two kinds of leather belts, belt A and belt B. Belt A is a high quality belt and belt B is of lower quality. The respective profits are Rs 4 and Rs 3 per belt. The production of each of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1,000 belts per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 of these are available per day. There are only 700 buckles a day available for belt B. What should be the daily production of each type of belt? Formulate this problem as an LP model and solve it using the simplex method.

EXAMPLES

Solution 2.

Let x_1 and x_2 be the number of belts of type A and B, respectively, manufactured each day.

Then the LP model would be as follows:

Maximize (total profit) $Z = 4x_1 + 3x_2$

subject to the constraints

(i) $2x_1 + x_2 \leq 1,000$ (Time availability),

(ii) $x_1 + x_2 \leq 800$ (Supply of leather)

(iii) $x_1 \leq 400$ (Buckles availability)

$x_2 \leq 700$ (Buckles availability)

and $x_1, x_2 \geq 0$

EXAMPLES

Solution 2. Standard form Introducing slack variables s_1, s_2, s_3 and s_4 to convert given LP model into its standard form as follows.

Maximize $Z = 4x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to the constraints

(i) $2x_1 + x_2 + s_1 = 1,000$

(ii) $x_1 + x_2 + s_2 = 800$

(iii) $x_1 + s_3 = 400$

(iv) $x_2 + s_4 = 700$

and $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

EXAMPLES

Solution 2. **Step 2:** An initial feasible solution is obtained by setting x1 = x2 = 0. Thus, the initial solution is:
s_1 = 1,000, s_2 = 800, s_3 = 400, s_4 = 700 and Max Z = 0.

		C_j	4	3	0	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	S_1	S_2	S_3	S_4	Minimum Ratio
0	S_1	1000	2	1	1	0	0	0	1000/2
0	S_2	800	1	1	0	1	0	0	800/1
0	S_3	400	1	0	0	0	1	0	400/1 →
0	S_4	700	0	1	0	0	0	1	-
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j – Z_j$	4 ↑	3	0	0	0	0	

EXAMPLES

Solution 2. **Iteration 1:**

		C_j	4	3	0	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	S_1	S_2	S_3	S_4	Minimum Ratio
0	S_1	200	0	1	1	0	-2	0	200/1 →
0	S_2	400	0	1	0	1	-1	0	400/1
4	X_1	400	1	0	0	0	1	0	-
0	S_4	700	0	1	0	0	0	1	700/1
Z = 1600		Z_j	4	0	0	0	4	0	
		$C_j – Z_j$	0	3 ↑	0	0	-4	0	

EXAMPLES

Solution 2. **Iteration 2:** Since all $c_j – z_j < 0$ correspond to non-basic variables columns, the current basic feasible solution is also the optimal solution. Thus, the company must manufacture, x_1 = 200 belts of type A and x_2 = 600 belts of type B in order to obtain the maximum profit of **Rs 2,600**.

		C_j	4	3	0	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	S_1	S_2	S_3	S_4	Minimum Ratio
3	X_2	600	0	1	-1	2	0	0	
0	S_2	200	0	0	-1	1	1	0	
4	X_1	200	1	0	1	-1	0	0	
0	S_4	100	0	0	1	-2	0	1	
Z = 2600		Z_j	4	3	1	2	0	0	
		$C_j – Z_j$	0	0	-1	-2	0	0	

EXAMPLES

Q.3. Maximize $Z = 4x_1 + 3x_2 + 6x_3$

Subject to constraints:

$2x_1 + 3x_2 + 2x_3 \le 440$

$4x_1 + 3x_3 \le 470$

$2x_1 + 5x_2 \le 430$

$x_1, x_2, x_3 \ge 0$.

Solve the following LPP by simplex method.

EXAMPLES

Solution 3. Standard form Introducing slack variables s_1, s_2, and s_3 to convert given LP model into its standard form as follows.

Maximize $Z = 4x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$2x_1 + 3x_2 + 2x_3 + s_1 \le 440$

$4x_1 + 3x_3 + s_2 \le 470$

$2x_1 + 5x_2 + s_3 \le 430$

and $x_1, x_2, x_3 s_1, s_2, s_3, \ge 0$

EXAMPLES

Solution 3. **Step 2:** An initial feasible solution is obtained by setting x1 = x2 = x3 = 0. Thus, the initial solution is: $s_1 = 440$, $s_2 = 470$, $s_3 = 430$, and Max Z = 0.

		C_j	4	3	6	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
0	S_1	440	2	3	2	1	0	0	440/2
0	S_2	470	4	0	3	0	1	0	470/3
0	S_3	430	2	5	0	0	0	1	-
Z = 0		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	4	3	6	0	0	0	

EXAMPLES

Solution 3. **Iteration 1:** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R1 (new) → R1 (old) – 2 R2(new)

		C_j	4	3	6	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
0	S_1	126.6	-0.66	3	0	1	-2/3	0	126.6/3
6	X_3	470/3	4/3	0	1	0	1/3	0	-
0	S_3	430	2	5	0	0	0	1	430/5
Z = 940		Z_j	8	0	6	0	6/3	0	
		$C_j - Z_j$	-4	3	0	0	-6/3	0	

440 – 2 x 470/3 = 126.6

2 – 2 x 4/3 = -0.66

3 – 2 x 0 = 3

2 – 2 x 1 = 0

1 – 2 x 0 = 1

0 – 2 x 1/3 = -2/3

0 – 2 x 0 = 0

EXAMPLES

Solution 3. **Iteration 2:** The new values of the elements in the remaining rows for the new Table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.
R3 (new) → R3 (old) – 5 R1(new)

		C_j	4	3	6	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
3	X_2	126.6/3	-0.66/3	1	0	1/3	-2/9	0	
6	X_3	470/3	4/3	0	1	0	1/3	0	
0	S_3	219	-2.78	0	0	-5/3	10/9	1	
Z = 1066.6		Z_j	7.34	3	6	1	4/3	0	
		$C_j - Z_j$	-3.34	0	0	-1	-4/3	0	

430 – 5 x 126.6/3 = 219

2 – 5 x -0.66/3 = -2.78

5 – 5 x 1 = 0

0 – 5 x 0 = 0

0 – 5 x 1/3 = -5/3

0 – 5 x -2/9 = 10/9

1 – 5 x 0 = 1

EXAMPLES

Solution 3. Since all $c_j - z_j < 0$ correspond to non-basic variables columns. Therefore, the optimal solution is reached with, $X_1 = 0$, $X_2 = 126.6/3$, $X_3 = 470/3$ and the optimal value of Z = 1066.6

		C_j	4	3	6	0	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
3	X_2	126.6/3	-0.66/3	1	0	1/3	-2/9	0	
6	X_3	470/3	4/3	0	1	0	1/3	0	
0	S_3	219	-2.78	0	0	-5/3	10/9	1	
Z = 1066.6		Z_j	7.34	3	6	1	4/3	0	
		$C_j - Z_j$	-3.34	0	0	-1	-4/3	0	

EXAMPLES

Q.4. Maximize $Z = 4x_1 + 2x_2 + 5x_3$

Subject to constraints:

$x_1 + 2x_2 + x_3 \le 430$

$3x_1 + 2x_3 \le 460$

$x_1 + 4x_2 \le 420$

$x_1, x_2, x_3 \ge 0$.

Solve the following LPP by simplex method.

EXAMPLES

Solution 4. Standard form Introducing slack variables s_1, s_2, and s_3 to convert given LP model into its standard form as follows.

Maximize $Z = 4x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$x_1 + 2x_2 + x_3 + s_1 \le 430$

$3x_1 + 0x_2 + 2x_3 + s_2 \le 460$

$x_1 + 4x_2 + 0s_3 \le 420$

and $x_1, x_2, x_3, s_1, s_2, s_3, \ge 0$

Linear Programming Problems

OPERATIONS RESEARCH

Prof. Aditya Suresh Kasar

Unit - 2

SIMPLEX ALGORITHM

In certain cases, it is difficult to obtain an initial basic feasible solution of the given LP problem. Such cases arise

(i) when the constraints are of the ≤ type,

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \quad x_j \ge 0$$

and value of few right-hand side constants is negative [i.e. bi < 0].

After adding the non-negative slack variable si (i = 1, 2, . . ., m), the initial solution so obtained will be

si = – bi for a particular resource, i.

This solution is not feasible because it **does not satisfy non-negativity conditions** of slack variables

(i.e. si ≥ 0).

Ex. $2x_1 + 4x_2 \le -12$,

$2x_1 + 4x_2 + s_1 = -12$

$s_1 = -12$ (initial solution)

SIMPLEX ALGORITHM

In certain cases, it is difficult to obtain an initial basic feasible solution of the given LP problem. Such cases arise

(ii) when the constraints are of the ≥ type,

$$\sum_{j=1}^{n} a_{ij}\, x_j \ge b_i, \quad x_j \ge 0$$

After adding surplus (negative slack) variable s_i, the initial solution so obtained will be, – s_i = bi or s_i = –bi

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i = b_i, \quad x_j \ge 0,\ s_j \ge 0$$

This solution is not feasible because it does not satisfy non-negativity conditions of surplus variables (i.e. $s_i \ge 0$). In such a case, artificial variables, A_i (i = 1, 2, . . ., m) are added to get an initial basic feasible solution.

The resulting system of equations then becomes:

$$\sum_{j=1}^{n} a_{ij}\, x_j - s_i + A_i = b_i$$

$$x_j,\ s_i,\ A_i \ge 0, \quad i = 1, 2, \ldots, m$$

Ex. $2x_1 + 4x_2 \ge 12$,

$2x_1 + 4x_2 - s_1 + A_1 = 12$

SIMPLEX ALGORITHM

These are m simultaneous equations with (n + m + m) variables (n decision variables, m artificial variables and m surplus variables).

An initial basic feasible solution of LP problem with such constraints can be obtained by equating (n + 2m – m) = (n + m) variables equal to zero.

Thus the new solution to the given LP problem is: $A_i = b_i$ (i = 1, 2, . . . , m), which is not the solution to the original system of equations because the two systems of equations are not equivalent.

Ex. $2x_1 + 4x_2 \ge 12$,

$2x_1 + 4x_2 - s_1 + A_1 = 12$

$A_1 = 12$ (initial solution)

Thus, to get back to the original problem, artificial variables must be removed from the optimal solution. There are two methods for removing artificial variables from the solution.

- Two-Phase Method
- **Big-M Method or Method of Penalties**

REMARKS

- Artificial variables have no meaning in a physical sense and are only used as a tool for generating an initial solution to an LP problem.
- Before the optimal solution is reached, all artificial variables must be dropped out from the solution mix.
- This is done by assigning appropriate coefficients to these variables in the objective function.
- These variables are added to those constraints with equality (=) and greater than or equal to (≥) sign.

BIG M METHOD

- Big-M method is a method of removing artificial variables from the basis.
- In this method, large undesirable (unacceptable penalty) coefficients to artificial variables are assigned from the point of view of the objective function.
- If the objective function Z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable.
- Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables.
- The penalty is supposed to be designated by – M, for a maximization problem, and + M, for a minimization problem, where M > 0.

STEPS

The Big-M method for solving an LP problem can be summarized in the following steps:

Step 1: Express the LP problem in the standard form by adding slack variables, surplus variables and/or artificial variables. Assign a zero coefficient to both slack and surplus variables. Then assign a very large coefficient + M (minimization case) and – M (maximization case) to artificial variable in the objective function.

Step 2: The initial basic feasible solution is obtained by assigning zero value to decision variables, x_1, x_2, ..., etc.

Step 3: Calculate the values of $c_j - z_j$ in last row of the simplex table and examine these values.

(i) If all $c_j - z_j \ge 0$, then the current basic feasible solution is optimal.

(ii) If for a column, k, $c_k - z_k$ is most negative and all entries in this column are negative, then the problem has an unbounded optimal solution.

STEPS

The Big-M method for solving an LP problem can be summarized in the following steps:

(iii) If one or more $c_j - z_j < 0$ (minimization case), then select the variable to enter into the basis (solution mix) with the largest negative $c_j - z_j$ value (largest per unit reduction in the objective function value). This value also represents the opportunity cost of not having one unit of the variable in the solution. That is,

$c_k - z_k = \text{Min}\,\{c_j - z_j : c_j - z_j < 0\}$

Step 4: Determine the key row and key element in the same manner as discussed in the simplex algorithm for the maximization case.

Step 5: Continue with the procedure to update solution at each iteration till optimal solution is obtained.

REMARKS

At any iteration of the simplex algorithm one of the following cases may arise:

1. If at least one artificial variable is a basic variable (i.e., variable that is present in the basis) with zero value and the coefficient M in each $c_j - z_j$ (j = 1, 2, . . ., n) values is non-negative, then the given LP problem has no solution. That is, the current basic feasible solution is degenerate.

2. If at least one artificial variable is present in the basis with a positive value and the coefficients M in each $c_j - z_j$ (j = 1, 2, . . ., n) values is non-negative, then the given LP problem has no optimum basic feasible solution. In this case, the given LP problem has a pseudo optimum basic feasible solution.

SUMMARY OF ADDITIONAL VARIABLES

Types of Constraint	*Extra Variable Needed*	*Coefficient of Additional Variables in the Objective Function*		*Presence of Additional Variables in the Initial Solution*
		Max Z	*Min Z*	
• Less than or equal to (≤)	A slack variable is added	0	0	Yes
• Greater than or equal to (≥)	A surplus variable is subtracted, and	0	0	No
	an artificial variable is added	$-M$	$+M$	Yes
• Equal to (=)	Only an artificial variable is added	$-M$	$+M$	Yes

EXAMPLES

Q.1. Use penalty (Big-M) method to solve the following LP problem.

Minimize $Z = 5x_1 + 3x_2$

subject to the constraints

(i) $2x1 + 4x_2 \leq 12$, (ii) $2x_1 + 2x_2 = 10$, (iii) $5x_1 + 2x_2 \geq 10$

and $x_1, x_2 \geq 0$.

EXAMPLES

Solution.1. Adding slack variable, s_1; surplus variable, s_2 and artificial variables, A_1 and A_2 in the constraints of the given LP problem, the standard form of the LP problem becomes.

Minimize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

(i) $2x_1 + 4x_2 + s_1 = 12$, (ii) $2x_1 + 2x_2 + A_1 = 10$ (iii) $5x_1 + 2x_2 - s_2 + A_2 = 10$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

EXAMPLES

Solution.1. An initial basic feasible solution: $s_1 = 12$, $A_1 = 10$, $A_2 = 10$ and Min $Z = 10M + 10M = 20M$ is obtained by putting $x_1 = x_2 = s_2 = 0$. It may be noted that the columns that correspond to the current basic variables and form the basis (identity matrix) are s_1 (slack variable), A_1 and A_2 (both artificial variables). The initial basic feasible solution is given in Table

		C_j	5	3	0	0	M	M	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	S_1	S_2	A_1	A_2	Minimum Ratio
0	S1	12	2	4	1	0	0	0	6
M	A1	10	2	2	0	0	1	0	5
M	A2	10	5	2	0	-1	0	1	2 →
Z = 20M		Z_j	7M	4M	0	-M	M	M	
		$C_j - Z_j$	5-7M	3-4M	0	M	0	0	

Since the value $c_1 - z_1 = 5 - 7M$ is the smallest value, therefore variable x_1 is chosen to enter into the basis (solution mix). To decide a current basic variable to leave the basis, calculate minimum ratio as shown

EXAMPLES

Solution.1. **Iteration 1:** Introduce variable x_1 into the basis and remove A_2 from the basis by applying the following row operations.

R_3 (new) → R_3 (old) ÷ 5 (key element); R_2 (new) → R_2 (old) – $2R_3$ (new). R_1 (new) → R_1 (old) – $2R_3$ (new).

		C_j	5	3	0	0	M	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	S_1	S_2	A_1	Minimum Ratio
0	S1	8	0	16/5	1	2/5	0	5/2 →
M	A1	6	0	6/5	0	2/5	1	5
5	x1	2	1	2/5	0	-1/5	0	5
Z = 10 + 6M		Z_j	5	(6M/5)+2	0	(2M/5)-1	M	
		$C_j - Z_j$	0	(-6M/5)+1	0	(-2M/5)+1	0	

Since the value of $c_2 - z_2$ in Table 4.27 is the largest negative value, variable x_2 is chosen to replace basic variable s_1 in the basis.

EXAMPLES

Solution.1. **Iteration 2:** To get an improved basic feasible solution, apply, the following row operations:
R1 (new) → R1 (old) × 5/16 (key element); R2 (new) → R2 (old) – (6/5) R1 (new). R3 (new) → R3 (old) – (2/5) R1 (new).

		C_j	5	3	0	0	M	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x1	x2	S1	S2	A1	Minimum Ratio
3	x2	5/2	0	1	5/16	1/8	0	20
M	A1	3	0	0	-3/8	1/4	1	12 →
5	x1	1	1	0	-1/8	-1/4	0	-
Z = 25/2 + 3M		Z_j	5	3	5/16-3M/8	M/4 -7/8	M	
		$C_j - Z_j$	0	0	3M/8-5/16	-M/4 + 7/8	0	

Since c4 – z4 < 0 (negative) in s2-column, the current solution is not optimal. Thus, non-basic variable s2 is chosen to replace artificial variable A1 in the basis.

EXAMPLES

Solution.1. **Iteration 3:** To get an improved basic feasible solution, apply the following row operations:
R2 (new) → R2 (old) × 4 (key element); R1 (new)→ R1 (old) – (1/8) R2 (new) R3 (new) → R3 (old) + (1/4) R2 (new).

		C_j	5	3	0	0	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X1	x2	S1	S2	Minimum Ratio
3	x2	1	0	1	1/2	0	
0	s2	12	0	0	-3/2	1	
5	x1	4	1	0	-1/2	0	
Z = 23		Z_j	5	3	-1	0	
		$C_j - Z_j$	0	0	1	0	

Since all cj – zj ≥ 0. Thus an optimal solution is arrived at with the value of variables as: x1 = 4, x2 = 1, s1 = 0, s2 = 12 and Min Z = 23.

EXAMPLES

Q.2. Use penalty (Big-M) method to solve the following LP problem.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 = 15$, (ii) $2x_1 + x_2 + 5x_3 = 20$, (iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4 \geq 0$

EXAMPLES

Solution.2. Since all constraints of the given LP problem are equations, therefore only artificial variables A1 and A2 are added in the constraints to convert given LP problem to its standard form. The standard form of the problem is stated as follows:

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

subject to the constraints

(i) $x_1 + 2x_2 + 3x_3 + A_1 = 15$, (ii) $2x_1 + x_2 + 5x_3 + A_2 = 20$

(iii) $x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$

EXAMPLES

Solution.2. An initial basic feasible solution is given in Table

		C_j	1	2	3	-1	-M	-M	
Basic variable coefficien t c_B	Basic variable B	Basic variables value B (= x_B)	X1	X2	X3	X4	A1	A2	Minimum Ratio
-M	A1	15	1	2	3	0	1	0	5
-M	A2	20	2	1	5	0	0	1	4 →
-1	X4	10	1	2	1	1	0	0	10
Z = -35M-10		Z_j	-3M-1	-3M-2	-8M-1	-1	-M	-M	
		$C_j - Z_j$	3M+2	3M+4	8M+4	0	0	0	

Since value of c3 – z3 is the largest positive value, the non-basic variable x3 is chosen to replace the artificial variable A2 in the basis.

EXAMPLES

Solution.2. Thus, to get an improved solution, apply the following row operations.

R2 (new) → R1 (old) ÷ 5 (key element) ; R1 (new) → R1 (old) – 3 R2 (new). R3 (new) → R3 (old) – R2 (new)

The improved basic feasible solution is shown in

		C_j	1	2	3	-1	-M	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X1	X2	X3	X4	A1	Minimum Ratio
-M	A1	3	-1/5	7/5	0	0	1	15/7 →
3	X3	4	2/5	1/5	1	0	0	20
-1	X4	6	3/5	9/5	0	1	0	30/9
Z = -3M+6		Z_j	M/5+3/5	-7M/5-6/5	3	-1	-M	
		$C_j – Z_j$	-M/5-2/5	7M/5+16/5 ↑	0	0	0	

Since value of c2 – z2 > 0 (positive) in Table, the non-basic variable x2 is chosen to replace the artificial variable A1 in the basis.

EXAMPLES

Solution.2. Thus, to get an improved solution, apply the following row operations.

R1 (new) → R1 (old) × (5/7) (key element); R2 (new) → R2 (old) – (1/5) R1 (new); R3 (new) → R3 (old) – (9/5) R1 (new).

The improved basic feasible solution is shown in

		C_j	1	2	3	-1	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X1	X2	X3	X4	Minimum Ratio
2	X2	15/7	-1/7	0	0	0	-
3	X3	25/7	3/7	0	1	0	25/3
-1	X4	15/7	6/7	0	0	1	15/6 →
Z = 90/7		Z_j	1/7	2	3	-1	
		$C_j – Z_j$	6/7 ↑	0	0	0	

Once again, the solution shown is not optimal as c1 – z1 > 0 in x1-column.

EXAMPLES

Solution.2. Thus, to get an improved solution, apply the following row operations.

R3 (new) → R3 (old) × (7/6) (key element) ; R1 (new) → R1 (old) + (1/7) R3 (new); R2 (new) → R2 (old) – (3/7) R3 (new).

The improved basic feasible solution is shown in

		C_j	1	2	3	-1	
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	X1	X2	X3	X4	Minimum Ratio
2	X2	15/6	0	1	0	1/6	
3	X3	15/6	0	0	1	-3/6	
1	X1	15/6	1	0	0	7/6	
Z = 15		Z_j	1	2	3	0	
		$C_j – Z_j$	0	0	0	-1	

since all cj – zj ≤ 0, therefore, an optimal solution is arrived at with value of variables as: x1 = 15/6, x2 = 15/6, x3 = 15/6 and Max Z = 15.

SOME COMPLICATIONS AND THEIR RESOLUTION

1. Unrestricted Variables:

- In actual practice, decision variables, xj (j = 1, 2, . . ., n) should have non-negative values.
- However, in many situations, one or more of these variables may have either positive, negative or zero value.
- Variables that can assume positive, negative or zero value are called **unrestricted variables**.
- Since use of the simplex method requires that all the decision variables must have non-negative value at each iteration, therefore in order to convert an LP problem involving unrestricted variables into an equivalent LP problem having only nonnegative variables, each of unrestricted variable is expressed as the difference of two non-negative variables.
- Let variable x_r be unrestricted in sign.
- We define two new variables say x_r' and x_r'' such that $x_r = x_r' – x_r''$; x_r' , $x_r'' \geq 0$

EXAMPLES

Q.3. Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 2x_2 + x_3$

subject to the constraints

(i) $2x_1 + 5x_2 + x_3 = 12$, (ii) $3x_1 + 4x_2 = 11$

and x_2, $x_3 \geq 0$, and x_1 unrestricted

EXAMPLES

Solution.3. Since x_1 is unrestricted in sign, introducing the nonnegative variables x1′ and x2″ so that

$x1 = x_1' – x_1''$, where x_1' , $x_1'' \geq 0$.

The standard form of the LP problem now becomes:

Maximize $Z = 3 (x_1' – x_1'') + 2x_2 + x_3 – MA_1$

subject to the constraints

(i) $2(x_1' – x_1'') + 5x_2 + x_3 = 12$, (ii) $3(x_1' – x_1'') + 4x_2 + A_1 = 11$

and x_1' , x_1'' , x_2, x_3 , $A_1 \geq 0$

EXAMPLES

Solution.3. The initial solution is shown in Table

		C_j	3	-3	2	1	-M	
Basic variable coefficient c_B	Basic variable B	Basic variable s value B (= x_B)	x_1'	x_1''	x_2	x_3	A_1	Minimum Ratio
1	x_3	12	2	-2	5	1	0	12/5 →
-M	A_1	11	3	-3	4	0	1	11/4
Z = -11M+12		Z_j	-3M+2	3M-2	-4M+5	1	-M	
		$C_j – Z_j$	3M+1	-3M-1	4M-3 ↑	0	0	

EXAMPLES

Solution.3. For this, apply the following row operations:

R1 (new) → R1 (old)/5 (key element) R2 (new) → R2 (old) – 4R1 (new).

		C_j	3	-3	2	1	-M	
Basic variable coefficient c_B	Basic variable B	Basic variable s value B (= x_B)	x_1'	x_1''	x_2	x_3	A_1	Minimum Ratio
1	x_3	12	2	-2	5	1	0	12/5 →
-M	A_1	11	3	-3	4	0	1	11/4
Z = -11M+12		Z_j	-3M+2	3M-2	-4M+5	1	-M	
		$C_j – Z_j$	3M+1	-3M-1	4M-3 ↑	0	0	

Duality in Linear Programming

OPERATIONS RESEARCH

Prof. Aditya Suresh Kasar

Unit - 2

INTRODUCTION

- In The term 'dual', in general, implies two or double.
- The concept of duality is very useful in mathematics, physics, statistics, engineering and managerial decision-making.
- For example, in a two-person game theory, one competitor's problem is the dual of the opponent's problem.
- In linear programming, duality implies that each linear programming problem can be analyzed in two different ways but would have equivalent solutions.
- Any LP problem (either maximization and minimization) can be stated in another equivalent form based on the same data.
- The new LP problem is called **dual linear programming problem** or in short dual.
- In general, it is immaterial which of the two problems is called **primal** or **dual**, since the dual of the dual is primal.

RULES FOR CONSTRUCTING THE DUAL FROM PRIMAL

The rules for constructing the dual from the primal and vice-versa using the symmetrical form of LP problem are:

1. A dual variable is defined corresponds to each constraint in the primal LP problem and vice versa. Thus, for a primal LP problem with m constraints and n variables, there exists a dual LP problem with m variables and n constraints and vice-versa.
2. The right-hand side constants $b_1, b_2, \ldots, b_m$ of the primal LP problem becomes the coefficients of the dual variables $y_1, y_2, \ldots, y_m$ in the dual objective function Z_y. Also the coefficients $c_1, c_2, \ldots, c_n$ of the primal variables $x_1, x_2, \ldots, x_n$ in the objective function become the right-hand side constants in the dual LP problem.
3. For a maximization primal LP problem with all ≤ (less than or equal to) type constraints, there exists a minimization dual LP problem with all ≥ (greater than or equal to) type constraints and vice versa. Thus, the inequality sign is reversed in all the constraints except the non-negativity conditions.

RULES FOR CONSTRUCTING THE DUAL FROM PRIMAL

The rules for constructing the dual from the primal and vice-versa using the symmetrical form of LP problem are:

4. The matrix of the coefficients of variables in the constraints of dual is the transpose of the matrix of coefficients of variables in the constraints of primal and vice versa.
5. If the objective function of a primal LP problem is to be maximized, the objective function of the dual is to be minimized and vice versa.
6. If the i^{th} primal constraint is = (equality) type, then the i^{th} dual variables is unrestricted in sign and vice versa.

EXAMPLES

Q.1. Write the dual to the following LP problem.

Maximize $Z = x_1 - x_2 + 3x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \le 10$, *(ii)* $2x_1 - x_2 - x_3 \le 2$, *(iii)* $2x_1 - 2x_2 - 3x_3 \le 6$

and $x_1, x_2, x_3 \ge 0$

EXAMPLES

Solution 1.

Maximize $Z = x_1 - x_2 + 3x_3$

subject to the constraints

(i) $x_1 + x_2 + x_3 \le 10$, *(ii)* $2x_1 - x_2 - x_3 \le 2$, *(iii)* $2x_1 - 2x_2 - 3x_3 \le 6$

and $x_1, x_2, x_3 \ge 0$

In the given LP problem there are m = 3 constraints and n = 3 variables. Thus, there must be m = 3 dual variables and n = 3 constraints.

Further, the coefficients of the primal variables, $c_1 = 1$, $c_2 = -1$, $c_3 = 3$ become right-hand side constants of the dual.

The right-hand side constants $b_1 = 10$, $b_2 = 2$, $b_3 = 6$ become the coefficients in the dual objective function.

Finally, the dual must have a **minimizing objective function** with all **≥ type constraints**.

EXAMPLES

Solution 1. Maximize $Z = x_1 - x_2 + 3x_3$

subject to the constraints (i) $x_1 + x_2 + x_3 \le 10$, (ii) $2x_1 - x_2 - x_3 \le 2$, (iii) $2x_1 - 2x_2 - 3x_3 \le 6$ and $x_1, x_2, x_3 \ge 0$

If y_1, y_2 and y_3 are dual variables corresponding to three primal constraints in the given order, the resultant dual is

Minimize $Z_y = 10y_1 + 2y_2 + 6y_3$

subject to the constraints

(i) $y_1 + 2y_2 + 2y_3 \ge 1$,

(iii) $y_1 - y_2 - 2y_3 \ge -1$,

(iii) $y_1 - y_2 - 3y_3 \ge 3$

and $y_1, y_2, y_3 \ge 0$

EXAMPLES

Q.2. Write the dual of the following LP problem.

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \ge 7$, (ii) $6x_1 + x_2 + 3x_3 \ge 4$, (iii) $7x_1 - 2x_2 - x_3 \le 10$ (iv) $x_1 - 2x_2 + 5x_3 \ge 3$,

(v) $4x_1 + 7x_2 - 2x_3 \ge 2$

and $x_1, x_2, x_3 \ge 0$

EXAMPLES

Solution 2.

Since the objective function of the given LP problem is of minimization, the direction of each inequality has to be changed to ≥ type by multiplying both sides by – 1. The standard primal LP problem so obtained is:

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \ge 7$, (ii) $6x_1 + x_2 + 3x_3 \ge 4$, (iii) $-7x_1 + 2x_2 + x_3 \ge -10$

(iv) $x_1 - 2x_2 + 5x_3 \ge 3$, (v) $4x_1 + 7x_2 - 2x_3 \ge 2$

and $x_1, x_2, x_3 \ge 0$

EXAMPLES

Solution 2. Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints (i) $3x_1 + 5x_2 + 4x_3 \ge 7$, (ii) $6x_1 + x_2 + 3x_3 \ge 4$, (iii) $7x_1 - 2x_2 - x_3 \le 10$

(iv) $x_1 - 2x_2 + 5x_3 \ge 3$, (v) $4x_1 + 7x_2 - 2x_3 \ge 2$ and $x_1, x_2, x_3 \ge 0$

If y_1, y_2, y_3, y_4 and y_5 are dual variables corresponding to the five primal constraints in the given order, the dual of this primal LP problem is stated as:

Maximize $Z_y = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to the constraints

(i) $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3$,

(ii) $5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \le -2$

(iii) $4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le 4$

and $y_1, y_2, y_3, y_4, y_5 \ge 0$

EXAMPLES

Q.3. Obtain the dual LP problem of the following primal LP problem:

Minimize Z = $x_1 + 2x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \le 160$, (ii) $x_1 - x_2 = 30$, (iii) $x_1 \ge 10$

and $x_1, x_2 \ge 0$

EXAMPLES

Solution 3.

Since the objective function of the primal LP problem is of minimization, change all ≤ type constraints to ≥ type constraints by multiplying the constraint on both sides by –1. Also write = type constraint equivalent to two constraints of the type ≥ and ≤. Then the given primal LP problem can be rewritten as:

Minimize $Z_x = x_1 + 2x_2$

subject to the constraint

(i) $-2x_1 - 4x_2 \ge -160$, (ii) $x_1 - x_2 \ge 30$ (iii) $x_1 - x_2 \le 30$ or $-x_1 + x_2 \ge -30$, (iv) $x_1 \ge 10$

and $x_1, x_2 \ge 0$

EXAMPLES

Solution 3 . Minimize $Z_x = x_1 + 2x_2$

subject to the constraint

(i) $-2x_1 - 4x_2 \ge -160$, (ii) $x_1 - x_2 \ge 30$ (iii) $x_1 - x_2 \le 30$ or $-x_1 + x_2 \ge -30$, (iv) $x_1 \ge 10$

and $x_1, x_2 \ge 0$

Let y_1, y_2, y_3 and y_4 be the dual variables corresponding to the four constraints in the given order.

The dual of the given primal LP problem can then be formulated as follows:

Maximize $Z_y = -160y_1 + 30y_2 - 30y_3 + 10y_4$

s.c (i) $-2y_1 + y_2 - y_3 + y_4 \le 1$, (ii) $-4y_1 - y_2 + y_3 \le 2$ and $y_1, y_2, y_3, y_4 \ge 0$

Remark Since second constraint in the primal LP problem is equality, therefore as per rule 6 corresponding second dual variable y (= y2 – y3) should be unrestricted in sign

Let $y = y_2 - y_3$ ($y_2, y_3 \ge 0$).

Maximize $Z_y = -160y_1 + 30y_2 - 30y_3 + 10y_4$

s.c (i) $-2y_1 + y + y_4 \le 1$, (ii) $-4y_1 - y \le 2$ and $y_1, y_4 \ge 0$ and y is unrestricted in sign.

EXAMPLES

Q.4. Obtain the dual LP problem of the following primal LP problem:

Minimize $Z_x = x_1 - 3x_2 - 2x_3$

subject to the constraints

(i) $3x_1 - x_2 + 2x_3 \le 7$, (ii) $2x_1 - 4x_2 \ge 12$, (iii) $-4x_1 + 3x_2 + 8x_3 = 10$

and $x_1, x_2 \ge 0$; x_3 unrestricted in sign.

EXAMPLES

Solution 4 . Minimize $Z_x = x_1 - 3x_2 - 2x_3$

s.c (i) $-3x_1 + x_2 - 2x_3 \ge -7$, (ii) $2x_1 - 4x_2 \ge 12$, (iii) $-4x_1 + 3x_2 + 8x_3 = 10$ and $x_1, x_2 \ge 0$ and x_3 is unrestricted in sign.

Let y_1, y_2 and y_3 be the dual variables corresponding to three primal constraints in the given order. Since the third constraint of the primal is an equation, the third dual variable y_3 will be unrestricted in sign. The dual of the given LP primal can be formulated as follows:

Maximize $Z_y = -7y_1 + 12y_2 + 10y_3$

subject to the constraints

(i) $-3y_1 + 2y_2 - 4y_3 \le 1$, (ii) $y_1 - 4y_2 + 3y_3 \le -3$, (iii) $-2y_1 + 8y_3 \le -2$

and $y_1, y_2 \ge 0$; y_3 unrestricted in sign.

EXAMPLES

Q.5. Obtain the dual of the following primal LP problem

Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints

(i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$

and $x_1, x_2, x_3 \ge 0$

EXAMPLES

Solution 5 . Maximize $Z_x = x_1 - 2x_2 + 3x_3$

subject to the constraints (i) $-2x_1 + x_2 + 3x_3 = 2$, (ii) $2x_1 + 3x_2 + 4x_3 = 1$ and $x_1, x_2, x_3 \ge 0$

Since both the primal constraints are of the equality type, the corresponding dual variables y_1 and y_2, will be unrestricted in sign. Following the rules of duality formulation, the dual of the given primal LP problem is

Minimize $Z_y = 2y_1 + y_2$

subject to the constraints

(i) $-2y_1 + 2y_2 \ge 1$, (ii) $y_1 + 3y_2 \ge -2$, (iii) $3y_1 + 4y_2 \ge 3$

and y_1, y_2 unrestricted in sign.

EXAMPLES

Q.6. Write the dual of the following primal LP problem

Maximize $Z = 3x_1 + x_2 + 2x_3 - x_4$

subject to the constraints

(i) $2x_1 - x_2 + 3x_3 + x_4 = 1$, (ii) $x_1 + x_2 - x_3 + x_4 = 3$

and $x_1, x_2 \ge 0$ and x_3, x_4 unrestricted in sign.

EXAMPLES

Solution 6 . Here we may apply the following rules of forming a dual of given primal LP problem.

(i) The x_3 and x_4 variables in the primal are unrestricted in sign, third and fourth constraints in the dual shall be equalities.

(ii) The given primal LP problem is of maximization; the first two constraints in the dual LP problem will therefore be ≥ type constraints.

(iii) Since both the constraints in the primal are equalities, the corresponding dual variables y_1 and y_2 will be unrestricted in sign. If y_1 and y_2 are dual variables corresponding to the two primal constraints in the given order, the dual of the given primal can be written as:

Minimize, $Z_y = y_1 + 3y_2$

subject to the constraints (i) $2y_1 + y_2 \ge 3$, (ii) $-y_1 + y_2 \ge 1$, (iii) $3y_1 - y_2 = 2$ (iv) $y_1 + y_2 = -1$

and y_1, y_2 unrestricted in sign.

Integer Linear Programming Gomory's Cut Method

OPERATIONS RESEARCH

Prof. Aditya Suresh Kasar

Unit - 3

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- Understand the limitations of simplex method in deriving integer solution to linear programming problems.
- Apply cutting plane methods to obtain optimal integer solution value of variables in an LP problem.
- Apply Branch and Bound method to solve integer LP problems.
- Appreciate application of integer LP problem in several areas of managerial decision-making.

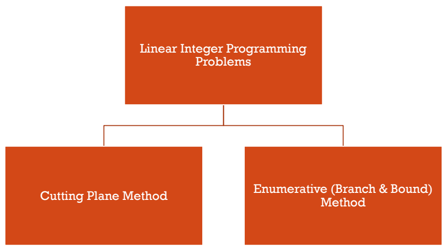
INTRODUCTION

- In linear programming, each decision variable, slack and/or surplus variable is allowed to take any discrete or fractional value.
- However, there are certain real-life problems in which the fractional value of the decision variables has no significance.
- The integer solution to a problem can, however, be obtained by rounding off the optimum value of the variables to the nearest integer value.
- This approach can be easy in terms of economy of effort, time, and the cost that might be required to derive an integer solution.

INTRODUCTION

- This solution, however, may not satisfy all the given constraints.
- Secondly, the value of the objective function so obtained may not be the optimal value.
- All such difficulties can be avoided if the given problem, where an integer solution is required, is solved by integer programming techniques.
- Integer LP problems are those in which some or all of the variables are restricted to integer (or discrete) values.
- An integer LP problem has important applications. Capital budgeting, construction scheduling, plant location and size, routing and shipping schedule, batch size, capacity expansion, fixed charge, etc., are few problems that demonstrate the areas of application of integer programming.

TYPES OF INTEGER LINEAR PROGRAMMING PROBLEMS



CUTTING PLANE SOLUTION CONCEPT

- The cutting-plane method to solve integer LP problems was developed by R.E. Gomory in 1956.
- This method is based on creating a sequence of linear inequalities called cuts.
- Such a cut reduces a part of the feasible region of the given LP problem, leaving out a feasible region of the integer LP problem.
- The hyperplane boundary of a cut is called the cutting plane.

CUTTING PLANE SOLUTION CONCEPT

Illustration Consider the following linear integer programming (LIP) problem

Maximize $Z = 14x_1 + 16x_2$

subject to the constraints

(i) $4x_1 + 3x_2 \leq 12$, (ii) $6x_1 + 8x_2 \leq 24$

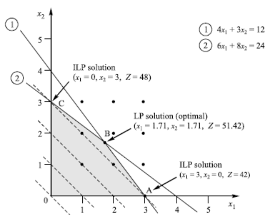
and $x_1, x_2 \geq 0$ and are integers.

Relaxing the integer requirement, the problem is solved graphically as shown in Fig.

The optimal solution to this LP problem is:

$x_1 = 1.71$, $x_2 = 1.71$ and Max $Z = 51.42$. This solution does not satisfy the integer requirement of variables x_1 and x_2.

Rounding off this solution to $x_1 = 2$, $x_2 = 2$ does not satisfy both the constraints and therefore, the solution is infeasible.



CUTTING PLANE SOLUTION CONCEPT

Illustration Consider the following linear integer programming (LIP) problem

Maximize $Z = 14x_1 + 16x_2$

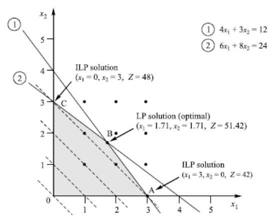
subject to the constraints

(i) $4x_1 + 3x_2 \leq 12$, (ii) $6x_1 + 8x_2 \leq 24$

and $x_1, x_2 \geq 0$ and are integers.

The dots in Fig. also referred to as lattice points, represent all of the integer solutions that lie within the feasible solution space of the LP problem.

However, it is difficult to evaluate every such point in order to determine the value of the objective function.



CUTTING PLANE SOLUTION CONCEPT

Illustration Consider the following linear integer programming (LIP) problem

Maximize $Z = 14x_1 + 16x_2$

subject to the constraints

(i) $4x_1 + 3x_2 \leq 12$, (ii) $6x_1 + 8x_2 \leq 24$

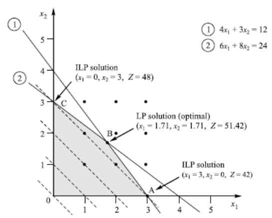
and $x_1, x_2 \geq 0$ and are integers.

In Fig. it may be noted that the optimal lattice point C, lies at the corner of the solution space OABC, obtained by cutting away the small portion above the dotted line.

This suggests a solution procedure that successively reduces the feasible solution space until an integer-valued corner is found.

The optimal integer solution is: $x_1 = 0$, $x_2 = 3$ and Max $Z = 48$.

The lattice point, C is not even adjacent to the most desirable LP problem solution corner, B.



STEPS OF GOMORY'S ALL INTEGER PROGRAMMING ALGORITHM

- **Step 1: Initialization**

  Formulate the standard integer LP problem. If there are any non-integer coefficients in the constraint equations, convert them into integer coefficients. Solve the problem by the simplex method, ignoring the integer value requirement of the variables.
- **Step 2: Test the optimality**

  (a) Examine the optimal solution. If all basic variables (i.e. $x_{Bi} = b_i \geq 0$) have integer values, then the integer optimal solution has been obtained and the procedure is terminated.

  (b) If one or more basic variables with integer value requirement have non-integer solution values, then go to Step 3.

STEPS OF GOMORY'S ALL INTEGER PROGRAMMING ALGORITHM

- **Step 3: Generate cutting plane**

  Choose a row r corresponding to a variable x_r that has the largest fractional value f_r and follow the procedure to develop a 'cut' (a Gomory constraint)

$$-f_r = s_g - \sum_{j \neq r} f_{rj}\, x_j, \quad \text{where} \quad 0 \leq f_{rj} < 1 \quad \text{and} \quad 0 < f_r < 1$$

  If there are more than one variables with the same largest fraction, then choose the one that has the smallest profit/unit coefficient in the objective function of maximization LP problem or the largest cost/unit coefficient in the objective function of minimization LP problem.

STEPS OF GOMORY'S ALL INTEGER PROGRAMMING ALGORITHM

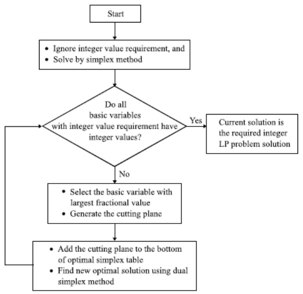
- **Step 4: Obtain the new solution**

  Add this additional constraint (cut) generated in **Step 3** to the bottom of the optimal simplex table.

  Find a new optimal solution by using the dual simplex method, i.e. choose a variable that is to be entered into the new solution having the smallest ratio: $\{(c_j - z_j)/y_{ij} ;\ y_{ij} < 0\}$ and return to **Step 2**.

  The process is repeated until all basic variables with integer value requirement assume non-negative integer values.

STEPS OF GOMORY'S ALL INTEGER PROGRAMMING ALGORITHM



EXAMPLES

Q.1. Solve the following Integer LP problem using Gomory's cutting plane method.

Maximize $Z = x_1 + x_2$

subject to the constraints

(i) $3x_1 + 2x_2 \leq 5$, (ii) $x_2 \leq 2$

and $x_1, x_2 \geq 0$ and are integers.

EXAMPLES

Solution 1.

Step-1: Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method.

		C_j	1	1	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	-2/3
1	x_2	2	0	1	0	1
Z * = 7/3		$C_j - Z_j$	0	0	-1/3	-1/3

Since all $C_j - Z_j \leq 0$, the optimal solution of LP problem is: $x_1 = 1/3$, $x_2 = 2$ and Max Z = 7/3.

EXAMPLES

Solution 1.

Step-2: In the current optimal solution, shown in Table all basic variables in the basis (x_B-column) did not assume integer value. Thus solution is not desirable. To obtain an optimal solution satisfying integer value requirement, go to step 3.

		C_j	1	1	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	-2/3
1	x_2	2	0	1	0	1
Z * = 7/3		$C_j - Z_j$	0	0	-1/3	-1/3

EXAMPLES

Solution 1.

Step-3: Since x1 is the only basic variable whose value is a non-negative fractional value, therefore consider first row (x_1-row) as source row in previous iteration to generate Gomory's cut as follows:

$$\frac{1}{3} = x_1 + 0.x_2 + \frac{1}{3} s_1 - \frac{2}{3}\ s_2 \text{ ----------- (x1-source row).}$$

The factoring of numbers (integer plus fractional) in the x1-source row gives ($X_{B1} = I_{B1} + F_{B1}$)

$$(0 + \tfrac{1}{3}) = (1 + 0)\ x_1 + (0 + \tfrac{1}{3})\ s_1\ (-1 + \tfrac{1}{3})\ s_2$$

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive. Rearranging all of the integer coefficients on the left-hand side, we get

$$\frac{1}{3} + (s_2 - x_1) = \frac{1}{3} s_1 + \frac{1}{3} s_2$$

Since value of variables x_1 and s_2 is assumed to be non-negative integer, left-hand side must satisfy

$$\frac{1}{3} \leq \frac{1}{3} s_1 + \frac{1}{3} s_2 \qquad -f_r = s_g - \sum_{j \neq r} f_{rj}\, x_j, \quad \text{where} \quad 0 \leq f_{rj} < 1 \quad \text{and} \quad 0 < f_r < 1$$

$$\frac{1}{3} + S_{g1} = \frac{1}{3} s_1 + \frac{1}{3} s_2 \text{ or } S_{g1} - \frac{1}{3} s_1 - \frac{1}{3} s_2 = -\frac{1}{3} \text{ --------(Cut 1)}$$

where s_{g1} is the new non-negative (integer) slack variable.

		C_j	1	1	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	s_1	s_2
1	x_1	1/3	1	0	1/3	-2/3
1	x_2	2	0	1	0	1
Z * = 7/3		$C_j - Z_j$	0	0	-1/3	-1/3

EXAMPLES

Solution 1.

Step-3: Adding this equation (also called Gomory's cut) at the bottom of last iteration, the new values so obtained is shown in Table below.

		C_j	1	1	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	s_1	s_2	S_{g1}
1	x_1	1/3	1	0	1/3	-2/3	0
1	x_2	2	0	1	0	1	0
0	S_{g1}	-1/3	0	0	-1/3	-1/3	1
Z * = 7/3		$C_j - Z_j$	0	0	-1/3	-1/3	0
Ratio: Min $C_j - Z_j / y_{3j}$ (<0)			-	-	1	1	-

EXAMPLES

Solution 1.

Step-4: Since the solution shown in Table is infeasible, apply the dual simplex method to find a feasible as well as an optimal solution. The key row and key column are marked in Table. The new solution is obtained by applying the following row operations. R3(new) → R3(old) × –3; R1(new) → R1(old) – (1/3) R3(new)

		C_j	1	1	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	s_1	s_2	S_{g1}
1	x_1	1/3	1	0	1/3	-2/3	0
1	x_2	2	0	1	0	1	0
0	S_{g1}	-1/3	0	0	-1/3	-1/3	1
Z * = 7/3		$C_j – Z_j$	0	0	-1/3	-1/3	0
Ratio: Min $C_j – Z_j / y_{3j}$ (<0)			-	-	1	1	-

EXAMPLES

Solution 1.

Step-4: The new solution is shown in Table

		C_j	1	1	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	s_1	s_2	S_{g1}
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	S_{g1}	1	0	0	1	1	-3
Z * = 2		$C_j – Z_j$	0	0	0	0	-1

Since all $c_j – z_j \leq 0$ and value of basic variables shown in x_B-column of Table is integer, the solution:
$x_1 = 0$, $x_2 = 2$, $s_{g1}= 1$ and Max Z = 2, is an optimal basic feasible solution of the given ILP problem.

EXAMPLES

Q.2. Solve the following Integer LP problem using the cutting plane method.

Maximize $Z = 2x_1 + 20x_2 – 10x_3$

subject to the constraints

(i) $2x_1 + 20x_2 + 4x_3 \leq 15$, (ii) $6x_1 + 20x_2 + 4x_3 = 20$

and $x_1, x_2, x_3 \geq 0$ and are integers.

EXAMPLES

Solution 2.

Step-1: Adding slack variable s_1 in the first constraint and artificial variable in the second constraint, the LP problem is stated in the standard form as:

Maximize $Z = 2x_1 + 20x_2 – 10x_3 + 0s_1 – MA_1$

subject to the constraints

(i) $2x_1 + 20x_2 + 4x_3 + s_1 = 15$, (ii) $6x_1 + 20x_2 + 4x_3 + A_1 = 20$

and $x_1, x_2, x_3, s_1, A_1 \geq 0$ and are integers.

EXAMPLES

Solution 2.

Step-1: The optimal solution of the LP problem, ignoring the integer value requirement using the simplex method is shown

		C_j	2	20	-10	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1
20	X_2	5/3	0	1	1/5	3/40
2	x_1	5/8	1	0	0	-1/4
Z = 15		$C_j – Z_j$	0	0	-14	-1

The non-integer optimal solution shown in Table: $x_1 = 5/4$, $x_2 = 5/8$, $x_3 = 0$ and Max Z = 15.

EXAMPLES

Solution 2.

Step-3: To obtain an optimal solution satisfying integer value requirement, we proceed to construct Gomory's constraint. In this solution, the value of both basic variables x1 and x2 are non-integer.
Since the fractional part of the value of basic variable x_2 = (0 + 5/8) is more than that of basic variable x_1 (= 1 + 1/4), the x_2-row is selected for constructing Gomory's cut as follows:

$$\frac{5}{8} = 0.\, x_1 + x_2 + \frac{1}{5} x_3 + \frac{3}{40}\ s_1 \text{ ----------- } (x_2\text{-source row}).$$

The factoring of the x_2-source row yields

$$\left(0 + \frac{5}{8}\right) = (1 + 0)\, x_2 + \left(0 + \frac{1}{5}\right) x_3 + \left(0 + \frac{3}{40}\right)\ s_1$$

$$\frac{5}{8} - x_2 = \frac{1}{5} x_3 + \frac{3}{40} s_1 \text{ or } \frac{5}{8} \leq \frac{1}{5} x_3 + \frac{3}{40} s_1$$

On adding a slack variable s_{g1}, the Gomory's fractional cut becomes:

$$\frac{5}{8} + S_{g1} = \frac{1}{5} x_3 + \frac{3}{40} s_1 \text{ or } S_{g1} - \frac{1}{5} x_3 - \frac{3}{40} s_1 = -\frac{5}{8} \text{ --------(Cut 1)}$$

EXAMPLES

Solution 2.

Step-3: Adding this additional constraint at the bottom of optimal simplex Table, the new values so obtained are shown in Table.

Iteration 1: Remove the variable s_{g1} from the basis and enter variable s_1 into the basis by applying the dual simplex method.

		C_j	2	20	-10	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1	S_{g1}
20	x_2	5/8	0	1	1/5	3/40	0
2	x_1	5/4	1	0	0	-1/4	0
0	S_{g1}	-5/8	0	0	-1/5	-3/40	1
Z = 15		$C_j – Z_j$	0	0	-14	-1	0
Ratio: Min $C_j – Z_j / y_{3j}$ (<0)			-	-	70	40/3	-

EXAMPLES

Solution 2.

Step-3: Adding this additional constraint at the bottom of optimal simplex Table, the new values so obtained are shown in Table.

Iteration 1: Remove the variable s_{g1} from the basis and enter variable s_1 into the basis by applying the dual simplex method.

		C_j	2	20	-10	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1	S_{g1}
20	x_2	0	0	1	0	0	-1
2	x_1	10/3	1	0	2/3	0	-10/3
0	S_1	25/3	0	0	8/3	1	-40/3
Z = 20/3		$C_j – Z_j$	0	0	-34/3	0	-40/3

The optimal solution shown in Table is still non-integer

EXAMPLES

Solution 2.

Step-3: Therefore, one more fractional cut needs to be generated. Since x1 is the only basic variable whose value is a non-negative fractional value, consider the x_1-row (because of largest fractional part) for constructing the cut:

$$\frac{10}{3} = x_1 + \frac{2}{3} x_3 + \frac{-10}{3}\ s_{g1} \text{ ----------- } (x_1\text{-source row}).$$

The factoring of the x_1-source row yields

$$\left(3 + \frac{1}{3}\right) = (1 + 0)\, x_1 + \left(0 + \frac{2}{3}\right) x_3 + \left(-4 + \frac{2}{3}\right)\ s_{g1}$$

$$\frac{1}{3} + (3 - x_1 + 4s_{g1}) = \frac{2}{3} x_3 + \frac{2}{3} s_{g1} \text{ or } \frac{1}{3} \leq \frac{2}{3} x_3 + \frac{2}{3} s_{g1}$$

On adding a slack variable s_{g2}, the Gomory's fractional cut becomes:

$$\frac{1}{3} + S_{g2} = \frac{2}{3} x_3 + \frac{2}{3} s_{g1} \text{ or } S_{g2} - \frac{2}{3} x_3 - \frac{2}{3} s_{g1} = -\frac{1}{3} \text{ --------(Cut 2)}$$

EXAMPLES

Solution 2.

Step-3: Adding this additional constraint at the bottom of optimal simplex Table, the new values so obtained are shown in Table.

		C_j	2	20	-10	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1	S_{g1}	S_{g2}
20	x_2	0	0	1	0	0	-1	0
2	x_1	10/3	1	0	2/3	0	-10/3	0
0	s_1	25/3	0	0	8/3	1	-40/3	0
0	S_{g2}	-1/3	0	0	-2/3	0	-2/3	1
Z = 20/3	$C_j - Z_j$	0	0	-34/3	0	40/3	0	
Ratio: Min $C_j - Z_j$ / y_{3j} (<0)		-	-	17	-	20	-	

EXAMPLES

Solution 2.

Step-3: Iteration 2: Enter non-basic variable x_3 into the basis to replace basic variable s_{g2} by applying the dual simplex method. The new solution is shown in Table.

		C_j	2	20	-10	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1	S_{g1}	S_{g2}
20	x_2	0	0	1	0	0	1	0
2	x_1	3	1	0	0	0	-4	0
0	s_1	7	0	0	0	1	-16	4
-10	x_3	1/2	0	0	1	0	1	-3/2
Z = 1	$C_j - Z_j$	0	0	0	0	-2	-17	

The optimal solution shown in Table is still non-integer because variable x_3 does not assume integer value.

EXAMPLES

Solution 2.

Step-3: Thus, a third fractional cut needs to be constructed with the help of the x_3-row:

$$\frac{1}{2} = x_3 + s_{g1} - \frac{3}{2} s_{g2} \text{ ----------- } (x_3 \text{ - source row}).$$

The factoring of the x_3 - source row yields

$$\left(0 + \frac{1}{2}\right) = (1 + 0)\, x_3 + (1 + 0)\, s_{g1} + \left(-2 + \frac{1}{2}\right) s_{g2}$$

$$\frac{1}{2} + (2 s_{g2} - x_3 - s_{g1}) = \frac{1}{2} s_{g2} \text{ or } \frac{1}{2} \le \frac{1}{2} s_{g2}$$

On adding a slack variable s_{g3}, the Gomory's fractional cut becomes:

$$\frac{1}{2} + S_{g3} = \frac{1}{2} s_{g2} \text{ or } S_{g3} - \frac{1}{2} s_{g2} = -\frac{1}{2} \text{ --------(Cut 3)}$$

EXAMPLES

Solution 2.

Step-3: Adding this additional constraint at the bottom of optimal simplex Table, the new values so obtained are shown in Table.

		C_j	2	20	-10	0	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1	S_{g1}	S_{g2}	S_{g3}
20	x_2	0	0	1	0	0	-1	0	0
2	x_1	3	1	0	0	0	-4	1	0
0	s_1	7	0	0	0	1	-16	4	0
-10	x_3	1/2	0	0	1	0	1	-3/2	0
0	S_{g3}	-1/2	0	0	0	0	0	-1/2	1
Z = 1	$C_j - Z_j$	0	0	0	0	-2	-17	0	
Ratio: Min $C_j - Z_j$ / y_{3j} (<0)		-	-	-	-	-	-34	-	

EXAMPLES

Solution 2.

Iteration 3: Remove the variable s_{g3} from the basis and enter variable s_{g2} into the basis by applying the dual simplex method. The new solution is shown in Table.

		C_j	2	20	-10	0	0	0	0
Basic variable coefficient c_B	Basic variable B	Basic variables value B (= x_B)	x_1	x_2	x_3	s_1	S_{g1}	S_{g2}	S_{g3}
20	x_2	0	0	1	0	0	-1	0	0
2	x_1	2	1	0	0	0	-4	0	2
0	s_1	3	0	0	0	1	-16	0	8
-10	x_3	2	0	0	1	0	1	0	-3
0	S_{g2}	1	0	0	0	0	0	1	-2
Z = -16	$C_j - Z_j$	0	0	0	0	-2	0	-34	

Since value of all basic variables is an integer value and all $C_j - Z_j \le 0$, the current solution is an integer optimal solution: $x_1 = 2$, $x_2 = 0$, $x_3 = 2$ and Max Z = – 16.

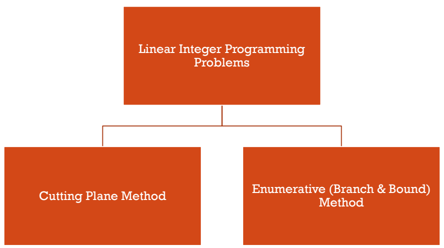
Integer Linear Programming Branch & Bound Method

OPERATIONS RESEARCH

Prof. Aditya Suresh Kasar

Unit - 3

TYPES OF INTEGER LINEAR PROGRAMMING PROBLEMS



BRANCH AND BOUND METHOD CONCEPT

- The Branch and Bound method developed first by A H Land and A G Doig is used to solve all-integer, mixed-integer and zero-one linear programming problems.
- The concept behind this method is to divide the feasible solution space of an LP problem into smaller parts called subproblems and then evaluate corner (extreme) points of each subproblem for an optimal solution.
- The branch and bound method starts by imposing bounds on the value of objective function that help to determine the subproblem to be eliminated from consideration when the optimal solution has been found.
- If the solution to a subproblem does not yield an optimal integer solution, a new subproblem is selected for branching.
- At a point where no more subproblem can be created, an optimal solution is arrived at.

BRANCH AND BOUND METHOD: PROCEDURE

Step 1: Initialization

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &= b_m \end{aligned}\right\} \quad \text{(LP-A)}$$

and $x_j \ge 0$ and non-negative integers.

Obtain the optimal solution of the given LP problem ignoring integer restriction on the variables.

(i) If the solution to this LP problem (say LP-A) is infeasible or unbounded, the solution to the given all-integer programming problem is also infeasible or unbounded, as the case may be.

(ii) If the solution satisfies the integer restrictions, the optimal integer solution has been obtained. If one or more basic variables do not satisfy integer requirement, then go to Step 2. Let the optimal value of objective function of LP-A be Z_1. This value provides an initial upper bound on objective function value and is denoted by Z_U.

(iii) Find a feasible solution by rounding off each variable value. The value of objective function so obtained is used as a lower bound and is denoted by Z_L.

BRANCH AND BOUND METHOD: PROCEDURE

Step 2: Branching step

(i) Let x_k be one basic variable which does not have an integer value and also has the largest fractional value.

(ii) Branch (or partition) the LP-A into two new LP subproblems (also called *nodes*) based on integer values of x_k that are immediately above and below its non-integer value. That is, it is partitioned by adding two mutually exclusive constraints:

$$x_k \leq [x_k] \quad \text{and} \quad x_k \geq [x_k] + 1$$

to the original LP problem. Here $[x_k]$ is the integer portion of the current non-integer value of the variable x_k. This is obviously is done to exclude the non-integer value of the variable x_k. The two new LP subproblems are as follows:

LP Subproblem B	*LP Subproblem C*
Max $Z = \sum_{j=1}^{n} c_j x_j$	Max $Z = \sum_{j=1}^{n} c_j x_j$
subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$	subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$
$x_k \leq [x_k]$	$x_k \geq [x_k] + 1$
and $x_j \geq 0$.	and $x_j \geq 0$.

BRANCH AND BOUND METHOD: PROCEDURE

Step 3: Bound step Obtain the optimal solution of subproblems B and C. Let the optimal value of the objective function of LP-B be Z_2 and that of LP-C be Z_3. The best integer solution value becomes the lower bound on the integer LP problem objective function value (Initially this is the rounded off value). Let the lower bound be denoted by Z_L.

Step 4: Fathoming step Examine the solution of both LP-B and LP-C

(i) If a subproblem yields an infeasible solution, then terminate the branch.

(ii) If a subproblem yields a feasible solution but not an integer solution, then return to Step 2.

(iii) If a subproblem yields a feasible integer solution, examine the value of the objective function. If this value is equal to the upper bound, an optimal solution has been reached. But if it is not equal to the upper bound but exceeds the lower bound, this value is considered as new upper bound and return to Step 2. Finally, if it is less than the lower bound, terminate this branch.

BRANCH AND BOUND METHOD: PROCEDURE

Step 5: Termination The procedure of branching and bounding continues until no further sub-problem remains to be examined. At this stage, the integer solution corresponding to the current lower bound is the optimal all-integer programming problem solution.

Remark The above algorithm can be represented by an enumeration tree. Each node in the tree represents a subproblem to be evaluated. Each branch of the tree creates a new constraint that is added to the original problem.

EXAMPLES

Q.1. Solve the following all integer programming problem using the branch and bound method.

Maximize $Z = 2x_1 + 3x_2$

subject to the constraints

(i) $6x_1 + 5x_2 \leq 25$, (ii) $x_1 + 3x_2 \leq 10$

and $x_1, x_2 \geq 0$ and integers.

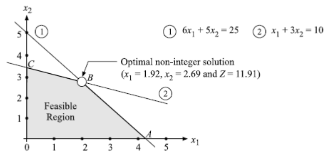
EXAMPLES

Solution 1.

LPP1: $x_1 = 1.92$, $x_2 = 2.69$, $Z = 11.91$

$x_2 \leq 2$ → LPP2

$x_2 \geq 3$ → LPP3



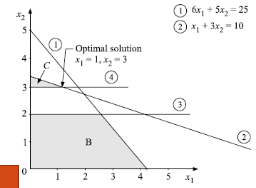
EXAMPLES

Solution 1.

LPP1: $x_1 = 1.92$, $x_2 = 2.69$, $Z = 11.91$

$x_2 \leq 2$ → LPP2: $x_1 = 2.5$, $x_2 = 2$, $Z = 11$

$x_2 \geq 3$ → LPP3: $x_1 = 1$, $x_2 = 3$, $Z = 11$



Optimal Solution

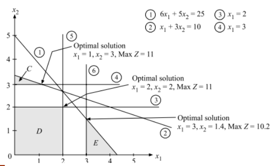
EXAMPLES

Solution 1.

LPP1: $x_1 = 1.92$, $x_2 = 2.69$, $Z = 11.91$

$x_2 \leq 2$ → LPP2: $x_1 = 2.5$, $x_2 = 2$, $Z = 11$

$x_2 \geq 3$ → LPP3: $x_1 = 1$, $x_2 = 3$, $Z = 11$



Optimal Solution

EXAMPLES

Solution 1.

LPP1: $x_1 = 1.92$, $x_2 = 2.69$, $Z = 11.91$

$x_2 \leq 2$ → LPP2: $x_1 = 2.5$, $x_2 = 2$, $Z = 11$

$x_2 \geq 3$ → LPP3: $x_1 = 1$, $x_2 = 3$, $Z = 11$

Optimal Solution

EXAMPLES

Solution 1.

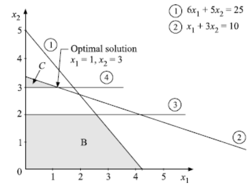
LPP1: $x_1 = 1.92$, $x_2 = 2.69$, $Z = 11.91$

$x_2 \leq 2$ → LPP2: $x_1 = 2.5$, $x_2 = 2$, $Z = 11$

Non Integer Solution
But further branching cannot improve Z
Fathomed Node

$x_2 \geq 3$ → LPP3: $x_1 = 1$, $x_2 = 3$, $Z = 11$

Integer Solution
Fathomed Node



Optimal Solution

EXAMPLES

Q.2. Solve the following all-integer programming problem using the branch and bound method

Minimize $Z = 3x_1 + 2.5x_2$

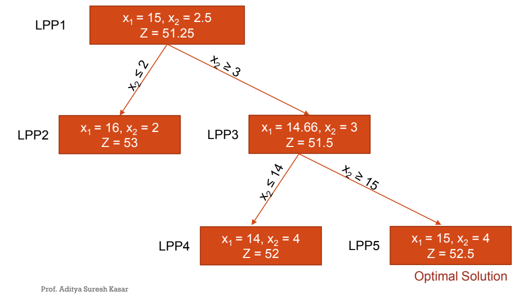
subject to the constraints

(i) $x_1 + 2x_2 \geq 20$, (ii) $3x_1 + 2x_2 \geq 50$

and $x_1, x_2 \geq 0$ and integers.

EXAMPLES

Solution 2.



Transportation Problem

Prof. Aditya Suresh Kasar

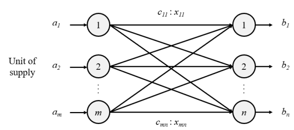
Unit - 4

TOPICS

INTRODUCTION TO TRANSPORTATION PROBLEM

- It is a special kind of ***Linear Programming Problem (LPP)***, in which goods are transported from a set of ***sources*** to a set of ***destination***, subject to the supply and demand of the source and destination respectively.

- Such that the total ***cost of transportation*** is **minimized**.

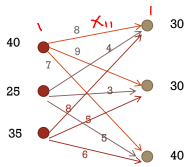


INTRODUCTION TO TRANSPORTATION PROBLEM

- The transportation model is an example of **Decision making under Certainty**.
- It is an example of decision making under certainty because everything is ***known*** or ***fixed***.
- As in the destinations of the goods are decided and known.
- The demand at each destination, the costs of each shipping route, and the supply at each source are understood to be known with a degree of certainty.

Back

FORMULATION OF TRANSPORTATION PROBLEM



Let X_{ij} be quantity transported from i to j

Minimize $\sum_i \sum_j C_{ij} X_{ij}$

$\sum_j X_{ij} \leq a_i$ *Supply*

$\sum_i X_{ij} \geq a_i$ Demand

$X_{ij} \geq 0$

Back

TYPES OF TRANSPORTATION PROBLEMS

- Balanced (Supply = Demand).

	1	2	3	Supply
A	2	3	4	10
B	5	7	9	30
C	10	2	5	60
Demand	20	30	50	100 |100

- Unbalanced (Supply ≠ Demand)

	1	2	3	Supply
A	2	3	4	10
B	5	7	9	30
C	10	2	5	60
Demand	20	25	45	90 | 100

Back

ASSUMPTIONS IN THE TRANSPORTATION MODEL

- Total quantity of item available at different sources is equal to total requirement at different destinations.
- Items can be transported conveniently from all the sources to the destinations.
- The unit transportation cost of the item from all the sources to the destination is certainly and precisely known.
- The transportation cost on a given route is directly proportional to the number of units shipped on that route.
- The objective is to minimize the total transportation cost for the organisation as the whole and not for the individual supply and distribution centres.

Back

DUAL OF TRANSPORTATION MODEL

- For a given basic feasible solution if we associate numbers (also called dual variables or multipliers) u_i and v_j with row i (i = 1, 2, . . ., m) and column j (j = 1, 2, . . ., n) of the transportation table, respectively, then u_i and v_j must satisfy the equation.
- ***$u_i + v_j = c_{ij}$, for each occupied cell (i, j)***
- These equations yield m + n – 1 equations in m + n unknown dual variables.
- The values of these variables can be determined by arbitrarily assigning a zero value to any one of these variables.
- The value of the remaining m + n – 2 variables can then be obtained algebraically by using the above equation for the occupied cells.

DUAL OF TRANSPORTATION MODEL (CONT..)

- The opportunity cost of each unoccupied cell (called non-basic variable or unused route) is calculated by using following equation that involves u_i and v_j values.:

  $d_{rs} = c_{rs} – (u_r + v_s)$, for each unoccupied cell (r, s)
- The relationship $(c_{ij} – u_i – v_j)\, x_{ij} = 0$ is known as ***complementary slackness*** for a transportation problem and indicates that
  a) if $x_{ij} > 0$ and solution is feasible, then $c_{ij} – u_i – v_j = 0$ or $c_{ij} = u_i + v_j$, for each occupied cell,
  b) if $x_{ij} = 0$ and $c_{ij} > u_i + v_j$, then it is not desirable to have $x_{ij} > 0$ in the solution mix because it would cost more to transport on a route (i, j),
  c) if $c_{ij} \le u_i + vj$ for some $x_{ij} = 0$, then x_{ij} can be brought into the solution mix.

DUAL OF TRANSPORTATION MODEL (CONT..)

- **Economic Interpretation of u_i's and v_j's:**
  - The u_i values measures the comparative advantage of additional unit of supply or shadow price (or value) of available supply at center i. This may also be termed as ***location rent***.
  - Similarly, the v_j values measures the comparative advantage of an additional unit of commodity demanded at demand center j. This may also be termed as ***market price.***

VARIATION IN TRANSPORTATION PROBLEM

- **Degeneracy and its Resolution**
- **Alternative Optimal Solutions**
- **Maximization Transportation Problem**
- **Unbalanced Supply and Demand**
- **Prohibited Transportation Routes**

DEGENERACY ITS RESOLUTION

A basic feasible solution for the general transportation problem must have exactly m + n – 1 (number of rows + number of columns – 1) positive allocations in the transportation table. If the number of occupied cells is less than the required number, m + n – 1, then such a solution is called degenerate solution.

* **Numerical**

ALTERNATIVE OPTIMAL SOLUTIONS

The existence of alternative optimal solutions can be determined by an inspection of the opportunity costs, d_{ij} (or P_{ij}) for the unoccupied cells. If **$d_{ij} = 0$**, for an unoccupied cell in an optimal solution, then an alternative optimal solution exists and can be obtained by bringing such an unoccupied cell in the solution mix without increasing the total transportation cost.

***Opportunity Cost is:* $d_{ij} = C_{ij} – (u_i + v_j)$ or $P_{ij} = (u_i + v_j) – C_{ij}$**

If $d_{ij} = 0$ or $P_{ij} = 0$, then an alternate optimum solution exist without increasing the total transportation cost.

* **Numerical**

UNBALANCED SUPPLY AND DEMAND

A situation may arise when the total available supply is not equal to the total demand. The following two cases may arise:

a) If the total supply exceeds the total demand, then an additional column (called a dummy demand center) can be added to the transportation table in order to absorb the excess supply. The unit transportation cost for the cells in this column is set equal to zero because these represent product items that are neither made nor sent.

b) If the total demand exceeds the total supply, a dummy row (called a dummy supply center) can be added to the transportation table to account for the excess demand quality. The unit transportation cost in such a case also, for the cells in the dummy row is set equal to zero.

* **Numerical-1**

MAXIMIZATION TRANSPORTATION PROBLEM

* **Numerical**

PROHIBITED TRANSPORTATION ROUTES

If situations like road hazards (snow, flood, etc.), traffic regulations, etc., arise, then it may not be possible to transport goods from certain sources to certain destinations. Such situations can be handled by assigning a very large cost, say M (or ∞) to such a route(s) (or cell).

- **Numerical-1**
- **Numerical-2**
- **Numerical-3**

STEPS TO SOLVE THE TRANSPORTATION PROBLEMS

- **Step A: Make a Transportation table.**

	1	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand	7	5	3	2	

Steps to Solve the Transportation Problems

- **Step B: Find a Basic Feasible Solution**

Methods available for finding IBFS are as mentioned below:

- North West - Corner Method
- Least - Cost Method (LCM)
- Vogel's Approximation Method (VAM)

Steps to Solve the Transportation Problems

- **Step C: Perform Optimality Test**
  - Make a optimality test to find whether the obtained feasible solution is optimal or not.
  - An optimality test can only be performed on that feasible solution in which;
    - Number of allocations is **m + n – 1** (where **m** is no. of rows, **n** is no. of columns)
    - The m+n-1 allocations should be in independent positions.
- **Methods used for Optimality Test**
  - Stepping Stone Method.
  - **The Modified Distribution (MODI) Method or u-v Methods.**

FINDING THE OPTIMUM SOLUTION BY MODI METHOD

- **Step 1:** Check whether m + n – 1 = no. allocated cells. (Degeneracy Test)
- If "**No**" Go to **Step 2**; if "**Yes**" Go to **Step 3**

FINDING THE OPTIMUM SOLUTION BY MODI METHOD

- **Step 2: Convert the necessary no. of unallocated cells into allocated cells to satisfy the above condition.**
  - Starting from the least value of the unallocated cell.
  - Check the ***loop formation*** one by one.
  - There should be No closed loop formation.
  - Select that cell as a new allocated cell and assign 'Ɛ'
- Go to **Step 3**

FINDING THE OPTIMUM SOLUTION BY MODI METHOD

- Closed loop.
  - Draw the lines horizontally and vertically towards allocated cells.
  - The turning point of the loop should be at the allocated cell.
  - Finally the loop should be complete at where we started.

X			X	
				X
		X	X	
	X			X

FINDING THE OPTIMUM SOLUTION BY MODI METHOD

- **Step 3**: Calculate the value of Ui and Vj for all the allocated cells by using the formula $\boxed{U_i + V_j = C_{ij}}$
- **Step 4:** Calculate Penalties Pij for all the unallocated cells by using the formula, $P_{ij} = \boxed{U_i + V_j - C_{ij}}$
- **Step 5:** Check the Optimality condition; all P_{ij} values should be ≤ (which means Zero or Negative values)
  - If Yes: Stop the Procedure >> "***The Optimality is Reached***"
  - Otherwise, Go to ***Step 6***

FINDING THE OPTIMUM SOLUTION BY MODI METHOD

- **Step 6:** Select the most (+ve) value of P_{ij} and consider the cell as the new allocated cell.
- **Step 7:** From the particular cell draw a closed loop passing through some allocated cells.
- **Step 8:** Starting from the new allocated cell alternatively assign (+) and (-) signs at the corners of the closed loop.
- **Step 9:** Select the minimum of the allocated value among (-) signed cells.
- **Step 10:** Frame the new iteration by applying the following steps:

FINDING THE OPTIMUM SOLUTION BY MODI METHOD

- **Step 10:** Frame the new iteration by applying the following steps:
  - Add and subtract that selected minimum value in all the (+) and (-) signed cells.
  - Copy the remaining cell value as it is.
  - Go to ***Step 1***.
- ****Numerical***

NUMERICAL: NORTH WEST CORNER METHOD

1.a. Find minimum Transportation Cost by **North West Method**.

	I	II	III	IV	Supply
A	9	8	3	2	**20**
B	3	4	2	1	**20**
C	5	1	3	9	**30**
Demand	**14**	**15**	**21**	**20**	

NUMERICAL: NORTH WEST CORNER METHOD

Solution:

	I	II	III	IV	Supply
A	9	8	3	2	**20**
B	3	4	2	1	**20**
C	5	1	3	9	**30**
Demand	**14**	**15**	**21**	**20**	70 / 70

NUMERICAL: NORTH WEST CORNER METHOD

Solution:

	I	II	III	IV	Supply
A	$\boxed{14}$ 9	$\boxed{6}$ 8	3	2	~~20~~ ~~6~~
B	3	$\boxed{9}$ 4	$\boxed{11}$ 2	1	~~20~~ ~~11~~
C	5	1	$\boxed{10}$ 3	$\boxed{20}$ 9	~~30~~ ~~20~~
Demand	~~14~~	~~15~~ ~~9~~	~~21~~ ~~10~~	~~20~~	70 / 70

Total Cost = (14x9)+(6x8)+(9x4)+(11x2)+(10x3)+(20x9) = 442

NUMERICAL: NORTH WEST CORNER METHOD

2. a. Find minimum Transportation Cost by North West Method.

	I	2	3	4	Supply
1	2	3	11	7	**6**
2	1	0	6	1	**1**
3	5	8	15	9	**10**
Demand	**7**	**5**	**3**	**2**	

NUMERICAL: NORTH WEST CORNER METHOD

Solution:

	I	2	3	4	Supply
1	2	3	11	7	**6**
2	1	0	6	1	**1**
3	5	8	15	9	**10**
Demand	**7**	**5**	**3**	**2**	17 / 17

NUMERICAL: NORTH WEST CORNER METHOD

Solution:

	I	2	3	4	Supply
1	$\boxed{6}$ 2	3	11	7	~~6~~
2	$\boxed{1}$ 1	0	6	1	~~1~~
3	5	$\boxed{5}$ 8	$\boxed{3}$ 15	$\boxed{2}$ 9	~~10~~ ~~5~~ ~~2~~
Demand	~~7~~ ~~1~~	~~5~~	~~3~~	~~2~~	17 / 17

Total Cost = (6x2)+(1x1)+(5x8)+(3x15)+(2x9) = 116

NUMERICAL: LEAST COST METHOD (LCM)

1.b. Find minimum Transportation Cost by **Least Cost Method**.

	I	II	III	IV	Supply
A	9	8	3	2	**20**
B	3	4	2	1	**20**
C	5	1	3	9	**30**
Demand	**14**	**15**	**21**	**20**	

NUMERICAL: LEAST COST METHOD (LCM)

Solution:

	I	II	III	IV	Supply
A	9	8	3	2	**20**
B	3	4	2	1	**20**
C	5	1	3	9	**30**
Demand	**14**	**15**	**21**	**20**	70 / 70

NUMERICAL: LEAST COST METHOD (LCM)

Solution:

	I	II	III	IV	Supply
A	9	8	$\boxed{20}$ 3	2	~~20~~
B	3	4	2	$\boxed{20}$ 1	~~20~~
C	$\boxed{14}$ 5	$\boxed{15}$ 1	$\boxed{1}$ 3	9	~~30~~ ~~15~~ ~~14~~
Demand	~~14~~	~~15~~	~~21~~ ~~1~~	~~20~~	70 / 70

Total Cost = (20x3)+(20x1)+(14x5)+(15x1)+(1x3) = 168

NUMERICAL: LEAST COST METHOD (LCM)

2.b. Find minimum Transportation Cost by Least Cost Method.

	I	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand	7	5	3	2	17 / 17

NUMERICAL: LEAST COST METHOD (LCM)

2.b. Find minimum Transportation Cost by Least Cost Method.

	I	2	3	4	Supply
1	6 2	3	11	7	~~6~~
2	1	1 0	6	1	~~1~~
3	1 5	4 8	3 15	2 9	~~10~~ ~~9~~ ~~5~~ ~~3~~
Demand	~~7~~ ~~1~~	~~5~~ ~~4~~	~~3~~	~~2~~	17 / 17

Total Cost = (6x2)+(1x0)+(1x5)+(4x8)+(3x15)+(2x9) = 112

NUMERICAL: VOGEL'S APPROXIMATION METHOD (VAM)

1.c. Find minimum Transportation Cost by **Vogel's Approximation Method.**

	I	II	III	IV	Supply
A	9	8	3	2	20
B	3	4	2	1	20
C	5	1	3	9	30
Demand	14	15	21	20	

NUMERICAL: VOGEL'S APPROXIMATION METHOD (VAM)

Solution:

	I	II	III	IV	Supply	Row Penalties			
A	9	8	3	20 2	~~20~~	1	1	1	1
B	14 3	4	6 2	1	~~20~~ ~~6~~	1	1	1	1
C	5	15 1	15 3	9	~~30~~ ~~15~~	2	2	x	x
Demand	~~14~~	~~15~~	~~21~~ ~~6~~	~~20~~	70 / 70				

Column Penalties				
	2	3	1	1
	2	x	1	1
	6	x	1	1
	x	x	1	1

Total Cost = (20x2)+(14x3)+(6x2)+(15x1)+(15x3) = **154**

NUMERICAL: VOGEL'S APPROXIMATION METHOD (VAM)

2.c. Find minimum Transportation Cost by Vogel's Approximation Method.

	I	2	3	4	Supply
1	2	3	11	7	6
2	1	0	6	1	1
3	5	8	15	9	10
Demand	7	5	3	2	

NUMERICAL: VOGEL'S APPROXIMATION METHOD (VAM)

Solution:

	I	2	3	4	Supply	Row Penalties			
1	2	3	11	7	6	1	1	5	x
2	1	0	6	1	1	1	x	x	x
3	5	8	15	9	10	3	3	4	4
Demand	7	5	3	2					

Column Penalties				
	1	3	5	6
	3	5	4	2
	3	x	4	2

Total Cost = (1x2)+(5x3)+(1x1)+(6x5)+(3x15)+(1x9) = **102**

NUMERICAL:

3. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCR, (b) LCM and (c) VAM.

Destination

Source

	DI	D2	D3	D4	Supply
S1	21	16	15	3	11
S2	17	18	14	23	13
S3	32	27	18	41	19
Demand	6	6	8	23	

NUMERICAL: NORTH WEST CORNER METHOD

Solution (3) (a) NWCR,

	DI	D2	D3	D4	Supply
S1	6 21	5 16	15	3	~~11~~ ~~5~~
S2	17	1 18	8 14	4 23	~~13~~ ~~12~~ ~~4~~
S3	32	27	18	19 41	~~19~~
Demand	~~6~~	~~6~~ ~~1~~	~~8~~	~~23~~ ~~19~~	43/43

Total Cost = (6x21)+(5x16)+(1x18)+(8x14)+(4x23)+(19x41)= 1,207

NUMERICAL: LEAST COST METHOD (LCM)

Solution (3) (b) LCM

	DI	D2	D3	D4	Supply
S1	21	16	15	11 3	~~11~~
S2	5 17	18	8 14	23	~~13~~ ~~5~~
S3	1 32	6 27	18	12 41	~~19~~ ~~18~~ ~~12~~
Demand	~~6~~ ~~1~~	~~6~~	~~8~~	~~23~~ ~~12~~	

Total Cost = (11x3)+(5x17)+(8x14)+(1x32)+(6x27)+(12x41) = 916

NUMERICAL: VOGEL'S APPROXIMATION METHOD (VAM)

Solution (3) (c) VAM.

	DI	D2	D3	D4	Supply
S1	21	16	15	11 3	~~11~~
S2	1 17	18	14	12 23	~~13~~ ~~1~~
S3	5 32	6 27	8 18	41	~~19~~ ~~11~~ ~~5~~
Demand	~~6~~ ~~5~~	~~6~~	~~8~~	~~23~~ ~~12~~	

Row Penalties			
12	x	x	x
3	3	3	x
9	9	9	9

Column Penalties			
4	2	1	20
15	11	4	18
15	11	4	x

Total Cost = (11x3)+(1x17)+(12x23)+(5x32)+(6x27)+(8x18) = 792

NUMERICAL

4. Determine an initial basic feasible solution to the following transportation problem by using
(a) NWCR, (b) LCM and (c) VAM.

Destination

Source

	DI	D2	D3	D4	Supply
S1	1	2	1	4	30
S2	3	3	2	1	30
S3	4	2	5	9	40
Demand	20	40	30	10	

NUMERICAL: NORTH WEST CORNER METHOD

Solution (4) (a) NWCR

	DI	D2	D3	D4	Supply
S1	20 1	2 10	1	4	~~30~~ ~~10~~
S2	3	30 3	2	1	~~30~~
S3	4	2	30 5	10 9	~~40~~
Demand	~~20~~	~~40~~ 30	~~30~~	~~10~~	

Total Cost = (20x1)+(10x2)+(30x3)+(30x5)+(10x9) = 370

NUMERICAL: LEAST COST METHOD (LCM)

Solution (4) (b) LCM

	DI	D2	D3	D4	Supply
S1	1	2	1	4	30
S2	3	3	2	1	30
S3	4	2	5	9	40
Demand	20	40	30	10	

Total Cost = 180

NUMERICAL: VOGEL'S APPROXIMATION METHOD (VAM)

Solution (4) (c) VAM.

	DI	D2	D3	D4	Supply
S1	20 1	2	10 1	4	~~10~~ ~~30~~
S2	3	3	20 2	10 1	20 ~~30~~
S3	4	40 2	5	9	~~40~~
Demand	~~20~~	~~40~~	30	~~10~~	

Row Penalties			
0	0	0	
1	1	1	
2	2	X	

Column Penalties			
2	0	1	3
2	0	1	X
2	X	1	X

Total Cost = (20x1)+(10x1)+(20x2)+(10x1)+(40x2)=160

NUMERICAL

5. Determine an initial basic feasible solution to the following transportation problem by using
(a) NWCR, (b) LCM and (c) VAM.

Destination

Source

	DI	D2	D3	D4	Supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

NUMERICAL

6. Determine an initial basic feasible solution to the following transportation problem by using
(a) NWCR, (b) LCM and (c) VAM.

Destination

Source

	DI	D2	D3	D4	Supply
O1	6	4	1	5	14
O2	8	9	2	7	16
O3	4	3	6	2	5
Demand	6	10	15	4	

NUMERICAL

7. Determine an initial basic feasible solution to the following transportation problem by using NWCR, and then optimize the solution using **MO**DIFIED **DI**STRIBUTION (**MODI**) method.

Destination

Source

	DI	D2	D3	D4	Supply
O1	3	1	7	4	250
O2	2	6	5	9	350
O3	8	3	3	2	400
Demand	200	300	350	150	

NUMERICAL (NWCM)

Solution 7:

Destination

Source

	DI	D2	D3	D4	Supply
O1	200 3	50 1	7	4	~~250~~ ~~50~~
O2	2	250 6	100 5	9	~~350~~ ~~100~~
O3	8	3	250 3	150 2	~~400~~ ~~150~~
Demand	~~200~~	~~300~~ ~~250~~	~~350~~ ~~250~~	~~150~~	

Total Cost = (200x3)+(50x1)+(250x6)+(100x5)+(250x3)+(150x2) = **3,700**

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 3 + 4 – 1 = 6; 6 = 6 (It is a Non Degeneracy Problem)

Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1= 3	V2= 1	V3= 0	V4=-1	
		DI	D2	D3	D4	Supply
U1= 0	O1	[200] 3	[50] 1	7	4	250
U2= 5	O2	2	[250] 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1= 3	V2= 1	V3= 0	V4= -1	
		DI	D2	D3	D4	Supply
U1= 0	O1	[200] 3	[50] 1	7	4	250
U2= 5	O2	2	[250] 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{13} = 0 + 0 - 7 = -7$ $P_{24} = 5 + (-1) - 9 = -5$

$P_{14} = 0 + (-1) - 4 = -5$ $P_{31} = 3 + 3 – 8 = -2$

$P_{21} = 5 + 3 – 2 = 6$ $P_{32} = 3 + 1 – 3 = 1$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

		V1= 3	V2= 1	V3= 0	V4= -1	
		DI	D2	D3	D4	Supply
U1= 0	O1	[200] 3	[50] 1	7	4	250
U2= 5	O2	2	[250] 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{13} = 0 + 0 - 7 = -7$ $P_{24} = 5 + (-1) - 9 = -5$

$P_{14} = 0 + (-1) - 4 = -5$ $P_{31} = 3 + 3 – 8 = -2$

$P_{21} = 5 + 3 – 2 = 6$ $P_{32} = 3 + 1 – 3 = 1$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1= 3	V2= 1	V3= 0	V4= -1	
		DI	D2	D3	D4	Supply
U1= 0	O1	[200] 3	[50] 1	7	4	250
U2= 5	O2	2	[250] 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{13} = 0 + 0 - 7 = -7$ $P_{24} = 5 + (-1) - 9 = -5$

$P_{14} = 0 + (-1) - 4 = -5$ $P_{31} = 3 + 3 – 8 = -2$

$\boxed{P_{21} = 5 + 3 – 2 = 6}$ $P_{32} = 3 + 1 – 3 = 1$

NUMERICAL (OPTIMUM SOLUTION)

Solution7. Step 7: From the particular cell draw a closed loop passing through some allocated cell

		V1= 3	V2= 1	V3= 0	V4= -1	
		DI	D2	D3	D4	Supply
U1= 0	O1	[200] 3	[50] 1	7	4	250
U2= 5	O2	2	[250] 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

P13 = 0 + 0 - 7 = -7 P24 = 5 + (-1) -9 = -5

P14 = 0 + (-1) - 4 = -5 P31 = 3 + 3 – 8 = -2

$\boxed{\text{P21 = 5+ 3 – 2 = 6}}$ P32 = 3 + 1 – 3 = 1

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		V1= 3	V2= 1	V3= 0	V4= -1	
		DI	D2	D3	D4	Supply
U1= 0	O1	(-) [200] 3	[50] (+) 1	7	4	250
U2= 5	O2	(+) 2	[250] (-) 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

P13 = 0 + 0 - 7 = -7 P24 = 5 + (-1) -9 = -5

P14 = 0 + (-1) - 4 = -5 P31 = 3 + 3 – 8 = -2

$\boxed{\text{P21 = 5+ 3 – 2 = 6}}$ P32 = 3 + 1 – 3 = 1

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 9: Select the minimum of the allocated value among the (-) signed cells.

		V1= 3	V2= 1	V3= 0	V4= -1	
		DI	D2	D3	D4	Supply
U1= 0	O1	(-) [200] 3	[50] (+) 1	7	4	250
U2= 5	O2	(+) 2	[250] (-) 6	[100] 5	9	350
U3= 3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

Minimum of the allocated value among (-) signed cells is **200**

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

	DI	D2	D3	D4	Supply
O1	[0] 3	[250] 1	7	4	250
O2	[200] 2	[50] 6	[100] 5	9	350
O3	8	3	[250] 3	[150] 2	400
Demand	200	300	350	150	

Minimum of the allocated value among (-) signed cells is **200**

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 3 + 4 – 1 = 6; 6 = 6 (It is a Non Degeneracy Problem)

Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1=-3	V2=1	V3=0	V4=-1	
		DI	D2	D3	D4	Supply
U1=0	O1	3	[250] 1	7	4	250
U2=5	O2	[200] 2	[50] 6	[100] 5	9	350
U3=3	O3	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 4: For unoccupied cells, calculate the penalties **P_{ij}** by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1= -3	V2= 1	V3= 0	V4= -1	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2= 5	**O2**	[200] 2	[50] 6	[100] 5	9	350
U3= 3	**O3**	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{11} = 0 + (-3) - 3 = -6$

$P_{13} = 0 + 0 - 7 = -7$

$P_{14} = 0 + (-1) - 4 = -5$

$P_{24} = 5 + (-1) - 9 = -5$

$P_{31} = 3 + (-3) - 8 = -8$

$P_{32} = 3 + 1 - 3 = 1$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 5: Since all P_{ij} values are not ≤ 0; the obtained solution is **not optimal**.

		V1= -3	V2= 1	V3= 0	V4= -1	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2= 5	**O2**	[200] 2	[50] 6	[100] 5	9	350
U3= 3	**O3**	8	3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{11} = 0 + (-3) - 3 = -6$

$P_{13} = 0 + 0 - 7 = -7$

$P_{14} = 0 + (-1) - 4 = -5$

$P_{24} = 5 + (-1) - 9 = -5$

$P_{31} = 3 + (-3) - 8 = -8$

$P_{32} = 3 + 1 - 3 = 1$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1= -3	V2= 1	V3= 0	V4= -1	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2= 5	**O2**	[200] 2	[50] 6	[100] 5	9	350
U3= 3	**O3**	8	● 3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{11} = 0 + (-3) - 3 = -6$

$P_{13} = 0 + 0 - 7 = -7$

$P_{14} = 0 + (-1) - 4 = -5$

$P_{24} = 5 + (-1) - 9 = -5$

$P_{31} = 3 + (-3) - 8 = -8$

$\boxed{P_{32} = 3 + 1 - 3 = 1}$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 7: From the particular cell draw a closed loop passing through some allocated cell

		V1= -3	V2= 1	V3= 0	V4= -1	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2= 5	**O2**	[200] 2	[50] 6	[100] 5	9	350
U3= 3	**O3**	8	● 3	[250] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{11} = 0 + (-3) - 3 = -6$

$P_{13} = 0 + 0 - 7 = -7$

$P_{14} = 0 + (-1) - 4 = -5$

$P_{24} = 5 + (-1) - 9 = -5$

$P_{31} = 3 + (-3) - 8 = -8$

$\boxed{P_{32} = 3 + 1 - 3 = 1}$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		V1= -3	V2= 1	V3= 0	V4= -1	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2= 5	**O2**	[200] 2	(-) [50] 6	[100] (+) 5	9	350
U3= 3	**O3**	8	(+) ● 3	[250] (-) 3	[150] 2	400
	Demand	200	300	350	150	

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 9: Select the minimum of the allocated value among the (-) signed cells.

		V1= -3	V2= 1	V3= 0	V4= -1	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2= 5	**O2**	[200] 2	(-) [50] 6	[100] (+) 5	9	350
U3= 3	**O3**	8	(+) ● 3	[250] (-) 3	[150] 2	400
	Demand	200	300	350	150	

Minimum of the allocated value among (-) signed cells is **50**

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 2: Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

	DI	**D2**	**D3**	**D4**	**Supply**
O1	3	[250] 1	7	4	250
O2	[200] 2	[0] 6	[150] 5	9	350
O3	8	[50] 3	[200] 3	[150] 2	400
Demand	200	300	350	150	

Minimum of the allocated value among (-) signed cells is **50**

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 3: Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 3 + 4 – 1 = 6; **6 = 6** (It is a Non Degeneracy Problem)

Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1=-2	V2=1	V3=1	V4=0	
		DI	**D2**	**D3**	**D4**	**Supply**
U1=0	**O1**	3	[250] 1	7	4	250
U2=4	**O2**	[200] 2	6	[150] 5	9	350
U3=2	**O3**	8	[50] 3	[200] 3	[150] 2	400
	Demand	200	300	350	150	

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 3: Step 4: For unoccupied cells, calculate the penalties **P_{ij}** by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1= -2	V2= 1	V3= 1	V4= 0	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	[250] 1	7	4	250
U2 = 4	**O2**	[200] 2	6	[150] 5	9	350
U3= 2	**O3**	8	[50] 3	[200] 3	[150] 2	400
	Demand	200	300	350	150	

$P_{11} = 0 + (-2) - 3 = -5$

$P_{13} = 0 + 1 - 7 = -6$

$P_{14} = 0 + 0 - 4 = -4$

$P_{22} = 4 + 1 - 6 = -1$

$P_{24} = 4 + 0 - 9 = -5$

$P_{31} = 2 + (-2) - 8 = -8$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 3: Step 5: Check the optimality condition; all Pij values should be <= 0 (which means zero or negative values) Since all Pij values are ≤ 0; the **obtained solution is Optimal**.

		V1= -2	V2= 1	V3= 1	V4= 0	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	$\boxed{250}$ 1	7	4	250
U2 = 4	**O2**	$\boxed{200}$ 2	6	$\boxed{150}$ 5	9	350
U3= 2	**O3**	8	$\boxed{50}$ 3	$\boxed{200}$ 3	$\boxed{150}$ 2	400
	Demand	200	300	350	150	

$P_{11} = 0 + (-2) - 3 = -5$

$P_{13} = 0 + 1 - 7 = -6$

$P_{14} = 0 + 0 - 4 = -4$

$P_{22} = 4 + 1 - 6 = -1$

$P_{24} = 4 + 0 - 9 = -5$

$P_{31} = 2 + (-2) - 8 = -8$

NUMERICAL (OPTIMUM SOLUTION)

Solution 7. Iteration 3: Step 5: Since all Pij values are ≤ 0; the **obtained solution is Optimal**.

		V1= -2	V2= 1	V3= 1	V4= 0	
		DI	**D2**	**D3**	**D4**	**Supply**
U1= 0	**O1**	3	$\boxed{250}$ 1	7	4	250
U2 = 4	**O2**	$\boxed{200}$ 2	6	$\boxed{150}$ 5	9	350
U3= 2	**O3**	8	$\boxed{50}$ 3	$\boxed{200}$ 3	$\boxed{150}$ 2	400
	Demand	200	300	350	150	

Total Cost = (250x1) + (200x2) + (150x5) + (50x3) + (200x3) + (150x2) = **2,450**

Total Cost = 3,700 (NWC)

NUMERICAL (DEGENERACY)

8. Determine an initial basic feasible solution to the following transportation problem by using **LCM**, and then optimize the solution using **MO**DIFIED **DI**STRIBUTION (**MODI**) method.

Destination

Source

	A	B	C	D	E	Supply
1	10	2	3	15	9	35
2	5	10	15	2	4	40
3	15	5	14	7	15	20
4	20	15	13	25	8	30
Demand	20	20	40	10	35	125/125

NUMERICAL (DEGENERACY)

Solution 8: Finding the Initial Basic Feasible Solution by LCM.

Destination

Source

	A	B	C	D	E	Supply
1	10	$\boxed{20}$ 2	$\boxed{15}$ 3	15	9	~~35~~ ~~15~~
2	5	10	15	$\boxed{10}$ 2	$\boxed{30}$ 4	~~40~~ ~~30~~
3	$\boxed{20}$ 15	5	14	7	15	20
4	20	15	$\boxed{25}$ 13	25	$\boxed{5}$ 8	~~30~~ ~~25~~
Demand	20	~~20~~	~~40~~ ~~25~~	~~10~~	~~35~~ ~~5~~	125/125

Total Cost = (20x2)+(15x3)+(10x2)+(30x4)+(20x15)+(25x13) +(5x8)= **890**

NUMERICAL (DEGENERACY)

Solution 8. Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 4 + 5 – 1 = 7; 8 ≠ 7 (It is a Degeneracy Problem)

Step 2: Convert the necessary no. of unallocated cells (1) into allocated cells to satisfy above condition.

Destination

Source

	A	B	C	D	E	Supply
1	10	$\boxed{20}$ 2	$\boxed{15}$ 3	15	9	35
2	5	10	15	$\boxed{10}$ 2	$\boxed{30}$ 4	40
3	$\boxed{20}$ 15	5	14	7	15	20
4	20	15	$\boxed{25}$ 13	25	$\boxed{5}$ 8	30
Demand	20	20	40	10	35	125/125

NUMERICAL (DEGENERACY)

Solution 8. Step 2: Starting from the least value of the unallocated cell. Check the loop formation one by one.

There should be **No Closed loop** formation. Select the cell as a new allocated cell & assign ' **Ɛ** '

Destination

Source

	A	B	C	D	E	Supply
1	10	$\boxed{20}$ 2	$\boxed{15}$ 3	15	9	35
2	5	10	15	$\boxed{10}$ 2	$\boxed{30}$ 4	40
3	$\boxed{20}$ 15	$\boxed{Ɛ}$ 5	14	7	15	20
4	20	15	$\boxed{25}$ 13	25	$\boxed{5}$ 8	30
Demand	20	20	40	10	35	125/125

NUMERICAL (DEGENERACY)

Solution 8. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 12	V2= 2	V3= 3	V4=-4	V5=-2	
		A	**B**	**C**	**D**	**E**	**Supply**
U1= 0	**1**	10	$\boxed{20}$ 2	$\boxed{15}$ 3	15	9	35
U2= 6	**2**	5	10	15	$\boxed{10}$ 2	$\boxed{30}$ 4	40
U3= 3	**3**	$\boxed{20}$ 15	$\boxed{Ɛ}$ 5	14	7	15	20 + Ɛ
U4= 10	**4**	20	15	$\boxed{25}$ 13	25	$\boxed{5}$ 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

NUMERICAL (DEGENERACY)

Solution 8. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ; $P_{ij} = U_i + V_j - C_{ij}$

		V1 = 12	V2= 2	V3= 3	V4=-4	V5=-2	
		A	**B**	**C**	**D**	**E**	**Supply**
U1= 0	**1**	10	$\boxed{20}$ 2	$\boxed{15}$ 3	15	9	35
U2= 6	**2**	5	10	15	$\boxed{10}$ 2	$\boxed{30}$ 4	40
U3= 3	**3**	$\boxed{20}$ 15	$\boxed{Ɛ}$ 5	14	7	15	20 + Ɛ
U4= 10	**4**	20	15	$\boxed{25}$ 13	25	$\boxed{5}$ 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

$P_{11} = 2$ $P_{15} = -11$ $P_{22} = -2$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $P_{21} = 13$ $P_{23} = -6$ $P_{34} = -8$ $P_{41} = 2$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

		V1 = 12	V2= 2	V3= 3	V4=-4	V4=-2	
		A	**B**	**C**	**D**	**E**	**Supply**
U1= 0	**1**	10	$\boxed{20}$ 2	$\boxed{15}$ 3	15	9	35
U2= 6	**2**	5	10	15	$\boxed{10}$ 2	$\boxed{30}$ 4	40
U3= 3	**3**	$\boxed{20}$ 15	$\boxed{Ɛ}$ 5	14	7	15	20 + Ɛ
U3= 10	**4**	20	15	$\boxed{25}$ 13	25	$\boxed{5}$ 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

$P_{11} = 2$ $P_{15} = -11$ $P_{22} = -2$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $P_{21} = 13$ $P_{23} = -6$ $P_{34} = -8$ $P_{41} = 2$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

V1 = 12   V2= 2   V3= 3   V4=-4   V4=-2

		A	B	C	D	E	Supply
U1= 0	1	10	[20] 2	[15] 3	15	9	35
U2= 6	2	● 5	10	15	[10] 2	[30] 4	40
U3= 3	3	[20] 15	[Ɛ] 5	14	7	15	20 + Ɛ
U3= 10	4	20	15	[25] 13	25	[5] 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

$P_{11} = 2$ $P_{15} = -11$ $P_{22} = -2$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $\boxed{P_{21} = 13}$ $P_{23} = -6$ $P_{34} = -8$ $P_{41} = 2$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Step 7: From the particular cell draw a closed loop passing through some allocated cell

V1 = 12   V2= 2   V3= 3   V4=-4   V4=-2

		A	B	C	D	E	Supply
U1= 0	1	10	[20] 2	[15] 3	15	9	35
U2= 6	2	● 5	10	15	[10] 2	[30] 4	40
U3= 3	3	[20] 15	[Ɛ] 5	14	7	15	20 + Ɛ
U3= 10	4	20	15	[25] 13	25	[5] 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

$P_{11} = 2$ $P_{15} = -11$ $P_{22} = -2$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $\boxed{P_{21} = 13}$ $P_{23} = -6$ $P_{34} = -8$ $P_{41} = 2$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

V1 = 12   V2= 2   V3= 3   V4=-4   V4=-2

		A	B	C	D	E	Supply
U1= 0	1	10	(-) [20] 2	[15] (+) 3	15	9	35
U2= 6	2	(+) ● 5	10	15	[10] 2	[30] (-) 4	40
U3= 3	3	(-) [20] 15	[Ɛ] (+) 5	14	7	15	20 + Ɛ
U3= 10	4	20	15	[25] (-) 13	25	[5] (+) 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

$P_{11} = 2$ $P_{15} = -11$ $P_{22} = -2$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $\boxed{P_{21} = 13}$ $P_{23} = -6$ $P_{34} = -8$ $P_{41} = 2$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Step 9: Select the minimum of the allocated value among the (-) signed cells.

V1 = 12   V2= 2   V3= 3   V4=-4   V4=-2

		A	B	C	D	E	Supply
U1= 0	1	10	(-) [20] 2	[15] (+) 3	15	9	35
U2= 6	2	(+) ● 5	10	15	[10] 2	[30] (-) 4	40
U3= 3	3	(-) [20] 15	[Ɛ] (+) 5	14	7	15	20 + Ɛ
U3= 10	4	20	15	[25] (-) 13	25	[5] (+) 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

Minimum of the allocated value among (-) signed cells is 20

NUMERICAL (DEGENERACY)

Solution 8. Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

V1 = 12   V2= 2   V3= 3   V4=-4   V4=-2

		A	B	C	D	E	Supply
U1= 0	1	10	[0] 2	[35] 3	15	9	35
U2= 6	2	[20] 5	10	15	[10] 2	[10] 4	40
U3= 3	3	15	[20+Ɛ] 5	14	7	15	20 + Ɛ
U3= 10	4	20	15	[5] 13	25	[25] 8	30
	Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

Minimum of the allocated value among (-) signed cells is 20

NUMERICAL (DEGENERACY)

Solution 8. Iteration 2: Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 4 + 5 – 1 = 7; 8 ≠ 7 (It is a Degeneracy Problem)

	A	B	C	D	E	Supply
1	10	2	[35] 3	15	9	35
2	[20] 5	10	15	[10] 2	[10] 4	40
3	15	[20+Ɛ] 5	14	7	15	20 + Ɛ
4	20	15	[5] 13	25	[25] 8	30
Demand	20	20 + Ɛ	40	10	35	125+ Ɛ

NUMERICAL (DEGENERACY)

Solution 8. Iteration 2: Step 2: Starting from the least value of the unallocated cell. Check the loop formation one by one.

There should be **No Closed loop** formation. Select the cell as a new allocated cell & assign ' Ɛ '

	A	B	C	D	E	Supply
1	10	[Ɛ] 2	[35] 3	15	9	35+Ɛ
2	[20] 5	10	15	[10] 2	[10] 4	40
3	15	[20+Ɛ] 5	14	7	15	20 + Ɛ
4	20	15	[5] 13	25	[25] 8	30
Demand	20	20 + 2Ɛ	40	10	35	125 + 2Ɛ

NUMERICAL (DEGENERACY)

Solution 8. Iteration 2: Step 3: **Calculate $U_i + V_j = C_{ij}$**

V1 = -1   V2= 2   V3= 3   V4=-4   V5=-2

		A	B	C	D	E	Supply
U1= 0	1	10	[Ɛ] 2	[35] 3	15	9	35+ Ɛ
U2= 6	2	[20] 5	10	15	[10] 2	[10] 4	40
U3= 3	3	15	[20+Ɛ] 5	14	7	15	20 + Ɛ
U4= 10	4	20	15	[5] 13	25	[25] 8	30
	Demand	20	20 + 2Ɛ	40	10	35	125 + 2Ɛ

NUMERICAL (DEGENERACY)

Solution 8. Iteration 2: Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship;

$P_{ij} = U_i + V_j - C_{ij}$

V1 = -1   V2= 2   V3= 3   V4=-4   V5=-2

		A	B	C	D	E	Supply
U1= 0	1	10	[Ɛ] 2	[35] 3	15	9	35+ Ɛ
U2= 6	2	[20] 5	10	15	[10] 2	[10] 4	40
U3= 3	3	15	[20+Ɛ] 5	14	7	15	20 + Ɛ
U4= 10	4	20	15	[5] 13	25	[25] 8	30
	Demand	20	20 + 2Ɛ	40	10	35	125 + 2Ɛ

$P_{11} = -11$ $P_{15} = -11$ $P_{23} = -6$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $P_{22} = -2$ $P_{31} = -13$ $P_{34} = -8$ $P_{41} = -11$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Iteration 2: Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

		V1 = -1	V2= 2	V3= 3	V4= -4	V5= -2	
		A	**B**	**C**	**D**	**E**	**Supply**
U1= 0	**1**	10	[ε] 2	[35] 3	15	9	**35+ ε**
U2= 6	**2**	[20] 5	10	15	[10] 2	[10] 4	**40**
U3= 3	**3**	15	[20+ε] 5	14	7	15	**20 + ε**
U4= 10	**4**	20	15	[5] 13	25	[25] 8	**30**
	Demand	**20**	**20 + 2ε**	**40**	**10**	**35**	**125 + 2ε**

$P_{11} = -11$ $P_{15} = -11$ $P_{23} = -6$ $P_{33} = -8$ $P_{35} = -14$ $P_{42} = -3$

$P_{14} = -19$ $P_{22} = -2$ $P_{31} = -13$ $P_{34} = -8$ $P_{41} = -11$ $P_{44} = -19$

NUMERICAL (DEGENERACY)

Solution 8. Iteration 2: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

		V1 = -1	V2= 2	V3= 3	V4= -4	V5= -2	
		A	**B**	**C**	**D**	**E**	**Supply**
U1= 0	**1**	10	[ε] 2	[35] 3	15	9	**35+ ε**
U2= 6	**2**	[20] 5	10	15	[10] 2	[10] 4	**40**
U3= 3	**3**	15	[20+ε] 5	14	7	15	**20 + ε**
U4= 10	**4**	20	15	[5] 13	25	[25] 8	**30**
	Demand	**20**	**20 + 2ε**	**40**	**10**	**35**	**125 + 2ε**

Total Cost = (35x3)+(20x5)+(10x2)+(10x4)+(20x5)+(5x13)+(25x8) = **630**

Total Cost = 890 (LCM)

Back

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

9. (a) Test the solution for optimality and find the Optimum Solution.

(b) Find one or more optimum solution.

Destination

Source

	X	**Y**	**Z**	**Supply**
A	[35] 6	[25] 3	5	**60**
B	[40] 5	2	[40] 2	**80**
C	12	[85] 7	8	**85**
Demand	**75**	**110**	**40**	**225/225**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. (a) Test the solution for optimality and find the Optimum Solution.

Step 1: Test for optimality. m + n - 1 = no. of allocations; 3+3-1=5; 5 = 5

	X	**Y**	**Z**	**Supply**
A	[35] 6	[25] 3	5	**60**
B	[40] 5	2	[40] 2	**80**
C	12	[85] 7	8	**85**
Demand	**75**	**110**	**40**	**225/225**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 6	V2= 3	V3= 3	
		X	**Y**	**Z**	**Supply**
U1= 0	**A**	[35] 6	[25] 3	5	**60**
U2= -1	**B**	[40] 5	2	[40] 2	**80**
U3= 4	**C**	12	[85] 7	8	**85**
	Demand	**75**	**110**	**40**	**225/225**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 6	V2= 3	V3= 3	
		X	**Y**	**Z**	**Supply**
U1= 0	**A**	[35] 6	[25] 3	5	**60**
U2= -1	**B**	[40] 5	2	[40] 2	**80**
U3= 4	**C**	12	[85] 7	8	**85**
	Demand	**75**	**110**	**40**	**225/225**

$P_{13} = 0+3-5 = -2$ $P_{31} = 4 + 3 - 12 = -5$

$P_{22} = -1 + 3 - 2 = 0$ $P_{33} = 4 + 3 - 8 = -1$

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

But since $P_{22} = 0$, an alternate optimal solution is possible.

		V1 = 6	V2= 3	V3= 3	
		X	**Y**	**Z**	**Supply**
U1= 0	**A**	[35] 6	[25] 3	5	**60**
U2= -1	**B**	[40] 5	2	[40] 2	**80**
U3= 4	**C**	12	[85] 7	8	**85**
	Demand	**75**	**110**	**40**	**225/225**

$P_{13} = 0+3-5 = -2$ $P_{31} = 4 + 6 - 12 = -2$

$\mathbf{P_{22} = -1 + 3 - 2 = 0}$ $P_{33} = 4 + 3 - 8 = -1$

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

But since $P_{22} = 0$, an alternate optimal solution is possible.

		V1 = 6	V2= 3	V3= 3	
		X	**Y**	**Z**	**Supply**
U1= 0	**A**	[35] 6	[25] 3	5	**60**
U2= -1	**B**	[40] 5	2	[40] 2	**80**
U3= 4	**C**	12	[85] 7	8	**85**
	Demand	**75**	**110**	**40**	**225/225**

Total Cost = (35x6)+(25x3)+(40x5)+(40x2)+(85x7) = **1,160**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1 = 6	V2= 3	V3= 3	
		X	**Y**	**Z**	**Supply**
U1= 0	**A**	[35] 6	[25] 3	5	**60**
U2= -1	**B**	[40] 5	2	[40] 2	**80**
U3= 4	**C**	12	[85] 7	8	**85**
	Demand	**75**	**110**	**40**	**225/225**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 7: From the particular cell draw a closed loop passing through some allocated cell

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	35 6	25 3	5	60
U2= -1	B	40 5	2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	(+) 35 6	25 (-) 3	5	60
U2= -1	B	(-) 40 5	(+) 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 9: Select the minimum of the allocated value among the (-) signed cells.

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	(+) 35 6	25 (-) 3	5	60
U2= -1	B	(-) 40 5	(+) 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

Minimum of the allocated value among (-) signed cells is **25**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	60 6	3	5	60
U2= -1	B	15 5	25 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 3 + 3 – 1 = 5; 5 = 5 (It is a Non-Degeneracy Problem)

		V1	V2=	V3=	
		X	Y	Z	Supply
U1=	A	60 6	3	5	60
U2=	B	15 5	25 2	40 2	80
U3=	C	12	85 7	8	85
	Demand	75	110	40	225/225

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	60 6	3	5	60
U2= -1	B	15 5	25 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	60 6	3	5	60
U2= -1	B	15 5	25 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

$P_{12} = 0+3-3 = 0$

$P_{13} = 0 + 3 – 5 = -2$

$P_{31} = 4 + 6 -12 = -2$

$P_{33} = 4 + 3 – 8 = -1$

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	60 6	3	5	60
U2= -1	B	15 5	25 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

$P_{12} = 0+3-3 = 0$

$P_{13} = 0 + 3 – 5 = -2$

$P_{31} = 4 + 6 -12 = -2$

$P_{33} = 4 + 3 – 8 = -1$

NUMERICAL (ALTERNATE OPTIMUM SOLUTION)

Solution 9. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

		V1= 6	V2= 3	V3= 3	
		X	Y	Z	Supply
U1= 0	A	60 6	3	5	60
U2= -1	B	15 5	25 2	40 2	80
U3= 4	C	12	85 7	8	85
	Demand	75	110	40	225/225

Total Cost = (35x3)+(25x3)+(40x5)+(40x2)+(85x7) = **1,160**

Back

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

10. Profit potential per units is given below along with demand and supply. Find the optimum transportation schedule.

Destination

Source

	M1	M2	M3	Supply
W1	25	22	23	2000
W2	15	20	18	1500
W3	18	17	16	1000
Demand	1200	1800	1000	

Back

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Since the Total Supply is not equal to Total Demand, the problem is ***unbalanced***.

Destination

Source

	M1	M2	M3	Supply
W1	25	22	23	2000
W2	15	20	18	1500
W3	18	17	16	1000
Demand	1200	1800	1000	4000/4500

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. To balance the Supply and Demand we add a Dummy column.

Destination

Source

	M1	M2	M3	M4	Supply
W1	25	22	23	0	2000
W2	15	20	18	0	1500
W3	18	17	16	0	1000
Demand	1200	1800	1000	500	4500/4500

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. To solve a maximization problem we convert it to a minimization problem by finding the Regret Matrix (i.e. Subtract maximum profit from all the rows and columns)

Destination

Source

	M1	M2	M3	M4	Supply
W1	25	22	23	0	2000
W2	15	20	18	0	1500
W3	18	17	16	0	1000
Demand	1200	1800	1000	500	4500/4500

Maximum Profit is 25

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. To solve a maximization problem we convert it to a minimization problem by finding the Regret Matrix (i.e. Subtract all the rows and columns from the maximum profit)

Destination

Source

	M1	M2	M3	M4	Supply
W1	0	3	2	25	2000
W2	10	5	7	25	1500
W3	7	8	9	25	1000
Demand	1200	1800	1000	500	4500/4500

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Find the Initial Basic Feasible Solution by Vogel's Approximation Method.

	M1	M2	M3	M4	Supply
W1	[1200] 0	3	[800] 2	25	2000
W2	10	[1500] 5	7	25	1500
W3	7	[300] 8	[200] 9	[500] 25	1000
Demand	1200	1800	1000	500	4500/4500

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Step 1: **Check of Degeneracy**: m + n - 1= no. of allocations; 3 + 4 – 1 = 6; 6 = 6 (It is a Non-Degeneracy Problem)

	M1	M2	M3	M4	Supply
W1	[1200] 0	3	[800] 2	25	2000
W2	10	[1500] 5	7	25	1500
W3	7	[300] 8	[200] 9	[500] 25	1000
Demand	1200	1800	1000	500	4500/4500

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 0 M1	V2= 1 M2	V3= 2 M3	V4= 18 M4	Supply
U1= 0	**W1**	[1200] 0	3	[800] 2	25	2000
U2= 4	**W2**	10	[1500] 5	7	25	1500
U3= 7	**W3**	7	[300] 8	[200] 9	[500] 25	1000
	Demand	1200	1800	1000	500	4500/4500

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 0 M1	V2= 1 M2	V3= 2 M3	V4= 18 M4	Supply
U1= 0	**W1**	[1200] 0	3	[800] 2	25	2000
U2= 4	**W2**	10	[1500] 5	7	25	1500
U3= 7	**W3**	7	[300] 8	[200] 9	[500] 25	1000
	Demand	1200	1800	1000	500	4500/4500

$P_{12} = 0+1-3 = -2$ $P_{21} = 4 + 0 - 10 = -6$ $P_{24} = 4+18-25 = -3$

$P_{14} = 0 + 18 - 25 = -7$ $P_{23} = 4 + 2 - 7 = -1$ $P_{31} = 7 + 0 - 7 = 0$

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

		V1 = 0	V2= 1	V3= 2	V4= 18	
		M1	**M2**	**M3**	**M4**	**Supply**
U1= 0	**W1**	(1200) 0	3	(800) 2	25	**2000**
U2= 4	**W2**	10	(1500) 5	7	25	**1500**
U3= 7	**W3**	7	(300) 8	(200) 9	(500) 25	**1000**
	Demand	**1200**	**1800**	**1000**	**500**	**4500/4500**

NUMERICAL (MAXIMIZATION PROBLEM & UNBALANCED)

Solution 10. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

		V1 = 0	V2= 1	V3= 2	V4= 18	
		M1	**M2**	**M3**	**M4**	**Supply**
U1= 0	**W1**	(1200) 25	22	(800) 23	0	**2000**
U2= 4	**W2**	15	(1500) 20	18	0	**1500**
U3= 7	**W3**	18	(300) 17	(200) 16	(500) 0	**1000**
	Demand	**1200**	**1800**	**1000**	**500**	**4500/4500**

Total maximum Profit = (1200x25)+(800x23)+(1500x20)+(300x17)+(200x16) + (500x0) = **86,700**

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NUMERICAL (UNBALANCED PROBLEM)

11. Consider the following unbalanced problem. Since there is not enough supplies, some of the demands at the destinations may not be satisfied. Suppose there are penalty cost for every unsatisfied demand unit which are given by 5, 3, 2 for destinations 1, 2, 3 respectively. Find the Optimal solution.

	Destination				
		1	**2**	**3**	**Supply**
Source	**1**	5	1	7	**10**
	2	6	4	6	**80**
	3	3	2	5	**15**
	Demand	**75**	**20**	**50**	**145/105**

Back

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Setup Transportation Problem Table

Since there are penalties mentioned for unsatisfied demand, we show them in the dummy row (instead of '0')

	1	**2**	**3**	**Supply**
1	5	1	7	**10**
2	6	4	6	**80**
3	3	2	5	**15**
Dummy	5	3	2	**40**
Demand	**75**	**20**	**50**	**145/145**

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Find Initial Basic Feasible Solution using VAM

	1	**2**	**3**	**Supply**
1	5	(10) 1	7	**10**
2	(60) 6	(10) 4	(10) 6	**80**
3	(15) 3	2	5	**15**
d	5	3	(40) 2	**40**
Demand	**75**	**20**	**50**	**145/145**

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Step 1: Optimality Test m + n - 1 = 6 , 4 + 3 – 1 = 6, 6 = 6

	1	**2**	**3**	**Supply**
1	5	(10) 1	7	**10**
2	(60) 6	(10) 4	(10) 6	**80**
3	(15) 3	2	5	**15**
d	5	3	(40) 2	**40**
Demand	**75**	**20**	**50**	**145/145**

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 3	V2= 1	V3= 3	
		1	**2**	**3**	**Supply**
U1= 0	**1**	5	(10) 1	7	**10**
U2= 3	**2**	(60) 6	(10) 4	(10) 6	**80**
U3= 0	**3**	(15) 3	2	5	**15**
U4= -1	**d**	5	3	(40) 2	**40**
	Demand	**75**	**20**	**50**	**145/145**

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 3	V2= 1	V3= 3	
		1	**2**	**3**	**Supply**
U1= 0	**1**	5	(10) 1	7	**10**
U2= 3	**2**	(60) 6	(10) 4	(10) 6	**80**
U3= 0	**3**	(15) 3	2	5	**15**
U4= -1	**d**	5	3	(40) 2	**40**
	Demand	**75**	**20**	**50**	**145/145**

$P_{11} = -2$ $P_{13} = -4$ $P_{32} = -1$ $P_{33} = -2$ $P_{41} = -3$ $P_{42} = -3$

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal.**

		V1 = 3	V2= 1	V3= 3	
		1	**2**	**3**	**Supply**
U1= 0	**1**	5	(10) 1	7	**10**
U2= 3	**2**	(60) 6	(10) 4	(10) 6	**80**
U3= 0	**3**	(15) 3	2	5	**15**
U4= -1	**d**	5	3	(40) 2	**40**
	Demand	**75**	**20**	**50**	**145/145**

$P_{11} = -2$ $P_{13} = -4$ $P_{32} = -1$ $P_{33} = -2$ $P_{41} = -3$ $P_{42} = -3$

NUMERICAL (UNBALANCED PROBLEM)

Solution 11. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

V1 = 3   V2= 1   V3= 3

		1	2	3	Supply
U1= 0	1	5	[10] 1	7	10
U2= 3	2	[60] 6	[10] 4	[10] 6	80
U3= 0	3	[15] 3	2	5	15
U4= -1	d	5	3	[40] 2	40
	Demand	75	20	50	145/145

Total Cost = (10x1)+(60x6)+(10x4)+(10x6)+(15x3)+(40x2) = **595**

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NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

12. ICT has 3 plants situated at Chennai, Nagpur and Udaipur having monthly manufacturing capacity of 1600 units, 1200 units and 1700 units respectively. These plants supply finished products to their 4 warehouses situated at Mumbai, Delhi, Hyderabad and Kolkata having their monthly demandof1000 units,1200 units, 800 units and 1200 units respectively. The transportation cost per unit (in Rupees) from plants to warehouses is as follows:

The shipment from Chennai to Hyderabad and Nagpur to Delhi are not possible due to certain operational problems. Determine the optimum transportation cost.

	Mumbai	Delhi	Hyderabad	Kolkata	Supply
Chennai	17	28	15	8	1600
Nagpur	68	14	38	58	1200
Udaipur	38	6	68	18	1700
Demand	1000	1200	800	1200	4200/4500

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NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Setup Transportation Problem Table

Assign infinite cost 'M' to the prohibited routes (Chennai to Hyderabad and Nagpur to Delhi) and since the problem is unbalanced, we balance it by adding a dummy column.

	Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
Chennai	17	28	M	8	0	1600
Nagpur	68	M	38	58	0	1200
Udaipur	38	6	68	18	0	1700
Demand	1000	1200	800	1200	300	4500

NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Finding the Initial Basic Feasible Solution by VAM.

	Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
Chennai	[1000] 17	28	M	[600] 8	0	1600
Nagpur	68	M	[800] 38	[100] 58	[300] 0	1200
Udaipur	38	[1200] 6	68	[500] 18	0	1700
Demand	1000	1200	800	1200	300	4500

NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Step 1: Optimality Test

m + n - 1 = 7 , 3 + 5 – 1 = 7, *7 = 7* (Non Degeneracy)

	Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
Chennai	[1000] 17	28	M	[600] 8	0	1600
Nagpur	68	M	[800] 38	[100] 58	[300] 0	1200
Udaipur	38	[1200] 6	68	[500] 18	0	1700
Demand	1000	1200	800	1200	300	4500

NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Step 3: **Calculate $U_i + V_j = C_{ij}$**

V1 = 17   V2= -4   V3= -12   V4= 8   V5= -50

		Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
U1= 0	Chennai	[1000] 17	28	M	[600] 8	0	1600
U2= 50	Nagpur	68	M	[800] 38	[100] 58	[300] 0	1200
U3= 10	Udaipur	38	[1200] 6	68	[500] 18	0	1700
	Demand	1000	1200	800	1200	300	4500

NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

V1 = 17   V2= -4   V3= -12   V4= 8   V5= -50

		Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
U1= 0	Chennai	[1000] 17	28	M	[600] 8	0	1600
U2= 50	Nagpur	68	M	[800] 38	[100] 58	[300] 0	1200
U3= 10	Udaipur	38	[1200] 6	68	[500] 18	0	1700
	Demand	1000	1200	800	1200	300	4500

P_{12} = 0+(-4)-28 = -32
P_{13} = 0-12-M = -M
P_{15} = 0-50-0 = -50
P_{21} = 50+17-68 = -1
P_{22} = 50-4-M = -M
P_{31} = 10+17-38 = -1
P_{33} = 10-12-68 = -70
P_{35} = 10-50-0 = -50

NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

V1 = 17   V2= -4   V3= -12   V4= 8   V5= -50

		Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
U1= 0	Chennai	[1000] 17	28	M	[600] 8	0	1600
U2= 50	Nagpur	68	M	[800] 38	[100] 58	[300] 0	1200
U3= 10	Udaipur	38	[1200] 6	68	[500] 18	0	1700
	Demand	1000	1200	800	1200	300	4500

P_{12} = 0+(-4)-28 = -32
P_{13} = 0-12-M = -M
P_{15} = 0-50-0 = -50
P_{21} = 50+17-68 = -1
P_{22} = 50-4-M = -M
P_{31} = 10+17-38 = -1
P_{33} = 10-12-68 = -70
P_{35} = 10-50-0 = -40

NUMERICAL (PROHIBITED ROUTES & UNBALANCED)

Solution 12. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

V1 = 17   V2= -4   V3= -12   V4= 8   V5= -50

		Mumbai	Delhi	Hyderabad	Kolkata	Dummy	Supply
U1= 0	Chennai	[1000] 17	28	M	[600] 8	0	1600
U2= 50	Nagpur	68	M	[800] 38	[100] 58	[300] 0	1200
U3= 10	Udaipur	38	[1200] 6	68	[500] 18	0	1700
	Demand	1000	1200	800	1200	300	4500

Total Cost = (1000x17)+(600x8)+(800x38)+(100x58)+(300x0)+(1200x6)+(500x18) = **74,200**

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NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

13. A company has factories at four different places, which supply warehouses A, B, C, D and E. Monthly factory capacities are 200, 175, 150, and 325 units respectively. Monthly warehouse requirements are 110, 90, 120, 230 and 160 units respectively. Unit shipping costs are given in the table. The costs are in rupees.

Shipment from 1 to B and 4 to D is not possible. Determine the optimum distribution to minimize the shipping cost.

To

From		A	B	C	D	E
	1	13	--	31	8	20
	2	14	9	17	6	10
	3	25	11	12	17	15
	4	10	21	13	--	17

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Setup Transportation Problem Table

To

From		A	B	C	D	E	Supply
	1	13	M	31	8	20	200
	2	14	9	17	6	10	175
	3	25	11	12	17	15	150
	4	10	21	13	M	17	325
	Demand	110	90	120	230	160	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Since Supply ≠ Demand we add a dummy column. For prohibited routes we assign a heavy penalty +M to these cells.

	A	B	C	D	E	d	Supply
1	13	M	31	8	20	0	200
2	14	9	17	6	10	0	175
3	25	11	12	17	15	0	150
4	10	21	13	M	17	0	325
Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Initial Basic Feasible Solution using VAM.

	A	B	C	D	E	d	Supply
1	13	M	31	[200] 8	20	0	200
2	14	9	17	[30] 6	[145] 10	0	175
3	25	[10] 11	12	17	15	[140] 0	150
4	[110] 10	[80] 21	[120] 13	M	[15] 17	0	325
Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 1: Optimality Test m + n - 1 = 9 , 4 + 6 – 1 = 9, 9 = 9

	A	B	C	D	E	d	Supply
1	13	M	31	[200] 8	20	0	200
2	14	9	17	[30] 6	[145] 10	0	175
3	25	[10] 11	12	17	15	[140] 0	150
4	[110] 10	[80] 21	[120] 13	M	[15] 17	0	325
Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 5	V2= 16	V3= 8	V4= 8	V5= 12	V6= 5	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	[10] 11	12	17	15	[140] 0	150
U4= 5	4	[110] 10	[80] 21	[120] 13	M	[15] 17	0	325
	Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$$P_{ij} = U_i + V_j - C_{ij}$$

		V1 = 5	V2= 16	V3= 8	V4= 8	V5= 12	V6= 5	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	[10] 11	12	17	15	[140] 0	150
U4= 5	4	[110] 10	[80] 21	[120] 13	M	[15] 17	0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -8$ $P_{12} = -M$ $P_{13} = -23$ $P_{15} = -8$ $P_{16} = 5$ $P_{21} = -11$ $P_{22} = 5$ $P_{23} = -11$ $P_{26} = 3$ $P_{31} = -25$ $P_{33} = -9$ $P_{34} = -14$ $P_{35} = -8$ $P_{44} = -M$ $P_{46} = 10$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

		V1 = 5	V2= 16	V3= 8	V4= 8	V5= 12	V6= 5	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	[10] 11	12	17	15	[140] 0	150
U4= 5	4	[110] 10	[80] 21	[120] 13	M	[15] 17	0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -8$ $P_{12} = -M$ $P_{13} = -23$ $P_{15} = -8$ $P_{16} = 5$ $P_{21} = -11$ $P_{22} = 5$ $P_{23} = -11$ $P_{26} = 3$ $P_{31} = -25$ $P_{33} = -9$ $P_{34} = -14$ $P_{35} = -8$ $P_{44} = -M$ $P_{46} = 10$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1 = 5	V2= 16	V3= 8	V4= 8	V5= 12	V6= 5	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	[10] 11	12	17	15	[140] 0	150
U4= 5	4	[110] 10	[80] 21	[120] 13	M	[15] 17	0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -8$ $P_{12} = -M$ $P_{13} = -23$ $P_{15} = -8$ $P_{16} = 5$ $P_{21} = -11$ $P_{22} = 5$ $P_{23} = -11$ $P_{26} = 3$ $P_{31} = -25$ $P_{33} = -9$ $P_{34} = -14$ $P_{35} = -8$ $P_{44} = -M$ **$P_{46} = 10$**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 7: From the particular cell draw a closed loop passing through some allocated cell

V1 = 5, V2= 16, V3= 8, V4= 8, V5= 12, V6= 5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	[10] 11	12	17	15	[140] 0	150
U4= 5	4	[110] 10	[80] 21	[120] 13	M	[15] 17	● 0	325
	Demand	110	90	120	230	160	140	710/850

P_{11} = -8, P_{12} = -M, P_{13} = -23, P_{15} = -8, P_{16} = 5, P_{21} = -11, P_{22} = 5, P_{23} = -11, P_{26} = 3, P_{31} = -25, P_{33} = -9, P_{34} = -14, P_{35} = -8, P_{44} = -M, **P_{46} = 10**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

V1 = 5, V2= 16, V3= 8, V4= 8, V5= 12, V6= 5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	(+) [10] 11	12	17	15	(-) [140] 0	150
U4= 5	4	[110] 10	(-) [80] 21	[120] 13	M	[15] 17	● (+) 0	325
	Demand	110	90	120	230	160	140	710/850

P_{11} = -8, P_{12} = -M, P_{13} = -23, P_{15} = -8, P_{16} = 5, P_{21} = -11, P_{22} = 5, P_{23} = -11, P_{26} = 3, P_{31} = -25, P_{33} = -9, P_{34} = -14, P_{35} = -8, P_{44} = -M, **P_{46} = 10**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 9: Select the minimum of the allocated value among the (-) signed cells.

V1 = 5, V2= 16, V3= 8, V4= 8, V5= 12, V6= 5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	(+) [10] 11	12	17	15	(-) [140] 0	150
U4= 5	4	[110] 10	(-) [80] 21	[120] 13	M	[15] 17	● (+) 0	325
	Demand	110	90	120	230	160	140	710/850

Minimum of the allocated value among (-) signed cells is **80**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

V1 = 5, V2= 16, V3= 8, V4= 8, V5= 12, V6= 5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= -5	3	25	[90] 11	12	17	15	[60] 0	150
U4= 5	4	[110] 10	21	[120] 13	M	[15] 17	[80] 0	325
	Demand	110	90	120	230	160	140	710/850

Minimum of the allocated value among (-) signed cells is **80**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 1: Optimality Test m + n - 1 = 9 , 4 + 6 – 1 = 9, 9 = 9

V1, V2=, V3=, V4=, V5=, V6=

		A	B	C	D	E	d	Supply
U1=	1	13	M	31	[200] 8	20	0	200
U2=	2	14	9	17	[30] 6	[145] 10	0	175
U3=	3	25	[90] 11	12	17	15	[60] 0	150
U4=	4	[110] 10	21	[120] 13	M	[15] 17	[80] 0	325
	Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 3: **Calculate $U_i + V_j = C_{ij}$**

V1 = 5, V2= 6, V3= 8, V4= 8, V5= 12, V6= -5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 5	3	25	[90] 11	12	17	15	[60] 0	150
U4= 5	4	[110] 10	21	[120] 13	M	[15] 17	[80] 0	325
	Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 4: For unoccupied cells, calculate the penalties **P_{ij}** by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

V1 = 5, V2= 6, V3= 8, V4= 8, V5= 12, V6= -5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 5	3	25	[90] 11	12	17	15	[60] 0	150
U4= 5	4	[110] 10	21	[120] 13	M	[15] 17	[80] 0	325
	Demand	110	90	120	230	160	140	710/850

P_{11} = -8, P_{12} = -M, P_{13} = -23, P_{15} = -8, P_{16} = -5, P_{21} = -11, P_{22} = -5, P_{23} = -11, P_{26} = -7, P_{31} = -15, P_{33} = 1, P_{34} = 4, P_{35} = 2, P_{42} = -10, P_{44} = -M

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

V1 = 5, V2= 6, V3= 8, V4= 8, V5= 12, V6= -5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 5	3	25	[90] 11	12	17	15	[60] 0	150
U4= 5	4	[110] 10	21	[120] 13	M	[15] 17	[80] 0	325
	Demand	110	90	120	230	160	140	710/850

P_{11} = -8, P_{12} = -M, P_{13} = -23, P_{15} = -8, P_{16} = -5, P_{21} = -11, P_{22} = -5, P_{23} = -11, P_{26} = -7, P_{31} = -15, P_{33} = 1, P_{34} = 4, P_{35} = 2, P_{42} = -10, P_{44} = -M

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

V1 = 5, V2= 6, V3= 8, V4= 8, V5= 12, V6= -5

		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 5	3	25	[90] 11	12	17	15	[60] 0	150
U4= 5	4	[110] 10	21	[120] 13	M	[15] 17	[80] 0	325
	Demand	110	90	120	230	160	140	710/850

P_{11} = -8, P_{12} = -M, P_{13} = -23, P_{15} = -8, P_{16} = -5, P_{21} = -11, P_{22} = -5, P_{23} = -11, P_{26} = -7, P_{31} = -15, P_{33} = 1, **P_{34} = 4**, P_{35} = 2, P_{42} = -10, P_{44} = -M

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1 = 5	V2= 6	V3= 8	V4= 8	V5= 12	V6= -5	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(30) 6	(145) 10	0	**175**
U3= 5	**3**	25	(90) 11	12	● 17	15	(60) 0	**150**
U4= 5	**4**	(110) 10	21	(120) 13	M	(15) 17	(80) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -8 P_{12} = -M P_{13} = -23 P_{15} = -8 P_{16} = -5 P_{21} = -11 P_{22} = -5 P_{23} = -11 P_{26} = -7 P_{31} = -15 P_{33} = 1 **P_{34} = 4** P_{35} = 2 P_{42} = -10 P_{44}= -M

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 7: From the particular cell draw a closed loop passing through some allocated cell.

		V1 = 5	V2= 6	V3= 8	V4= 8	V5= 12	V6= -5	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(30) 6	(145) 10	0	**175**
U3= 5	**3**	25	(90) 11	12	● 17	15	(60) 0	**150**
U4= 5	**4**	(110) 10	21	(120) 13	M	(15) 17	(80) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -8 P_{12} = -M P_{13} = -23 P_{15} = -8 P_{16} = -5 P_{21} = -11 P_{22} = -5 P_{23} = -11 P_{26} = -7 P_{31} = -15 P_{33} = 1 **P_{34} = 4** P_{35} = 2 P_{42} = -10 P_{44}= -M

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		V1 = 5	V2= 6	V3= 8	V4= 8	V5= 12	V6= -5	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(-) (30) 6	(145) (+) 10	0	**175**
U3= 5	**3**	25	(90) 11	12	(+) ● 17	15	(60) (-) 0	**150**
U4= 5	**4**	(110) 10	21	(120) 13	M	(15) (-) 17	(80) (+) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -8 P_{12} = -M P_{13} = -23 P_{15} = -8 P_{16} = -5 P_{21} = -11 P_{22} = -5 P_{23} = -11 P_{26} = -7 P_{31} = -15 P_{33} = 1 **P_{34} = 4** P_{35} = 2 P_{42} = -10 P_{44}= -M

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 9: Select the minimum of the allocated value among the (-) signed cells.

		V1 = 5	V2= 6	V3= 8	V4= 8	V5= 12	V6= -5	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(-) (30) 6	(145) (+) 10	0	**175**
U3= 5	**3**	25	(90) 11	12	(+) ● 17	15	(60) (-) 0	**150**
U4= 5	**4**	(110) 10	21	(120) 13	M	(15) (-) 17	(80) (+) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

Minimum of the allocated value among (-) signed cells is **15**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 2 Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

		V1 = 5	V2= 6	V3= 8	V4= 8	V5= 12	V6= -5	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(15) 6	(160) 10	0	**175**
U3= 5	**3**	25	(90) 11	12	(15) 17	15	(45) 0	**150**
U4= 5	**4**	(110) 10	21	(120) 13	M	17	(95) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

Minimum of the allocated value among (-) signed cells is **15**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 1: Optimality Test m + n - 1 = 9 , 4 + 6 – 1 = 9, 9 = 9

		V1	V2=	V3=	V4=	V5=	V6=	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1=	**1**	13	M	31	(200) 8	20	0	**200**
U2=	**2**	14	9	17	(15) 6	(160) 10	0	**175**
U3=	**3**	25	(90) 11	12	(15) 17	15	(45) 0	**150**
U4=	**4**	(110) 10	21	(120) 13	M	17	(95) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(15) 6	(160) 10	0	**175**
U3= 9	**3**	25	(90) 11	12	(15) 17	15	(45) 0	**150**
U4= 9	**4**	(110) 10	21	(120) 13	M	17	(95) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(15) 6	(160) 10	0	**175**
U3= 9	**3**	25	(90) 11	12	(15) 17	15	(45) 0	**150**
U4= 9	**4**	(110) 10	21	(120) 13	M	17	(95) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -12 P_{12} = -M P_{13} = -27 P_{15} = -8 P_{16} = -9 P_{21} = -15 P_{22} = -9 P_{23} = -15 P_{26} = -11 P_{31} = -15 P_{33} = 1 P_{35} = 6 P_{42} = -10 P_{44} = -M P_{45}= 4

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	(200) 8	20	0	**200**
U2= -2	**2**	14	9	17	(15) 6	(160) 10	0	**175**
U3= 9	**3**	25	(90) 11	12	(15) 17	15	(45) 0	**150**
U4= 9	**4**	(110) 10	21	(120) 13	M	17	(95) 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -12 P_{12} = -M P_{13} = -27 P_{15} = -8 P_{16} = -9 P_{21} = -15 P_{22} = -9 P_{23} = -15 P_{26} = -11 P_{31} = -15 P_{33} = 1 P_{35} = 6 P_{42} = -10 P_{44} = -M P_{45}= 4

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[15] 6	[160] 10	0	**175**
U3= 9	**3**	25	[90] 11	12	[15] 17	● 15	[45] 0	**150**
U4= 9	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -12 P_{12} = -M P_{13} = -27 P_{15} = -8 P_{16} = -9 P_{21} = -15 P_{22} = -9 P_{23} = -15 P_{26} = -11 P_{31} = -15 P_{33} = 1 **P_{35} = 6** P_{42} = -10 P_{44} = -M P_{45}= 4

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 7: From the particular cell draw a closed loop passing through some allocated cell.

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[15] 6	[160] 10	0	**175**
U3= 9	**3**	25	[90] 11	12	[15] 17	● 15	[45] 0	**150**
U4= 9	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -12 P_{12} = -M P_{13} = -27 P_{15} = -8 P_{16} = -9 P_{21} = -15 P_{22} = -9 P_{23} = -15 P_{26} = -11 P_{31} = -15 P_{33} = 1 **P_{35} = 6** P_{42} = -10 P_{44} = -M P_{45}= 4

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	(+) [15] 6	[160] (-) 10	0	**175**
U3= 9	**3**	25	[90] 11	12	[15] (-) 17	● (+) 15	[45] 0	**150**
U4= 9	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -12 P_{12} = -M P_{13} = -27 P_{15} = -8 P_{16} = -9 P_{21} = -15 P_{22} = -9 P_{23} = -15 P_{26} = -11 P_{31} = -15 P_{33} = 1 **P_{35} = 6** P_{42} = -10 P_{44} = -M P_{45}= 4

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 9: Select the minimum of the allocated value among the (-) signed cells.

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	(+) [15] 6	[160] (-) 10	0	**175**
U3= 9	**3**	25	[90] 11	12	[15] (-) 17	● (+) 15	[45] 0	**150**
U4= 9	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

Minimum of the allocated value among (-) signed cells is **15**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 3 Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

		V1 = 1	V2= 2	V3= 4	V4= 8	V5= 12	V6= -9	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[30] 6	[145] 10	0	**175**
U3= 9	**3**	25	[90] 11	12	17	[15] 15	[45] 0	**150**
U4= 9	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

Minimum of the allocated value among (-) signed cells is **15**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 1: Optimality Test m + n - 1 = 9 , 4 + 6 – 1 = 9, 9 = 9

		V1 =	V2=	V3=	V4=	V5=	V6=	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1=	**1**	13	M	31	[200] 8	20	0	**200**
U2=	**2**	14	9	17	[30] 6	[145] 10	0	**175**
U3=	**3**	25	[90] 11	12	17	[15] 15	[45] 0	**150**
U4=	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[30] 6	[145] 10	0	**175**
U3= 3	**3**	25	[90] 11	12	17	[15] 15	[45] 0	**150**
U4= 3	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[30] 6	[145] 10	0	**175**
U3= 3	**3**	25	[90] 11	12	17	[15] 15	[45] 0	**150**
U4= 3	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -6 P_{12} = -M P_{13} = -21 P_{15} = -8 P_{16} = -3 P_{21} = -9 P_{22} = -3 P_{23} = -9 P_{26} = -5 P_{31} = -15 P_{33} = 1 P_{34} = -6 P_{42} = -10 P_{44} = -M P_{45}= -2

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[30] 6	[145] 10	0	**175**
U3= 3	**3**	25	[90] 11	12	17	[15] 15	[45] 0	**150**
U4= 3	**4**	[110] 10	21	[120] 13	M	17	[95] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

P_{11} = -6 P_{12} = -M P_{13} = -21 P_{15} = -8 P_{16} = -3 P_{21} = -9 P_{22} = -3 P_{23} = -9 P_{26} = -5 P_{31} = -15 P_{33} = 1 P_{34} = -6 P_{42} = -10 P_{44} = -M P_{45}= -2

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	● 12	17	[15] 15	[45] 0	150
U4= 3	4	[110] 10	21	[120] 13	M	17	[95] 0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -6$ $P_{13} = -21$ $P_{16} = -3$ $P_{22} = -3$ $P_{26} = -5$ **$P_{33} = 1$** $P_{42} = -10$ $P_{45} = -2$

$P_{12} = -M$ $P_{15} = -8$ $P_{21} = -9$ $P_{23} = -9$ $P_{31} = -15$ $P_{34} = -6$ $P_{44} = -M$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 7: From the particular cell draw a closed loop passing through some allocated cell.

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	● 12	17	[15] 15	[45] 0	150
U4= 3	4	[110] 10	21	[120] 13	M	17	[95] 0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -6$ $P_{13} = -21$ $P_{16} = -3$ $P_{22} = -3$ $P_{26} = -5$ **$P_{33} = 1$** $P_{42} = -10$ $P_{45} = -2$

$P_{12} = -M$ $P_{15} = -8$ $P_{21} = -9$ $P_{23} = -9$ $P_{31} = -15$ $P_{34} = -6$ $P_{44} = -M$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	(+) ● 12	17	[15] 15	[45] (-) 0	150
U4= 3	4	[110] 10	21	[120] (-) 13	M	17	[95] (+) 0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -6$ $P_{13} = -21$ $P_{16} = -3$ $P_{22} = -3$ $P_{26} = -5$ **$P_{33} = 1$** $P_{42} = -10$ $P_{45} = -2$

$P_{12} = -M$ $P_{15} = -8$ $P_{21} = -9$ $P_{23} = -9$ $P_{31} = -15$ $P_{34} = -6$ $P_{44} = -M$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 9: Select the minimum of the allocated value among the (-) signed cells.

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	(+) ● 12	17	[15] 15	[45] (-) 0	150
U4= 3	4	[110] 10	21	[120] (-) 13	M	17	[95] (+) 0	325
	Demand	110	90	120	230	160	140	710/850

Minimum of the allocated value among (-) signed cells is **45**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 4 Step 10 : Frame the new iteration by applying the following steps;

- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

		V1 = 7	V2= 8	V3= 10	V4= 8	V5= 12	V6= -3	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	[45] 12	17	[15] 15	0	150
U4= 3	4	[110] 10	21	[75] 13	M	17	[140] 0	325
	Demand	110	90	120	230	160	140	710/850

Minimum of the allocated value among (-) signed cells is **45**

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 5 Step 1: Optimality Test m + n - 1 = 9 , 4 + 6 – 1 = 9, 9 = 9

		V1 =	V2=	V3=	V4=	V5=	V6=	
		A	B	C	D	E	d	Supply
U1=	1	13	M	31	[200] 8	20	0	200
U2=	2	14	9	17	[30] 6	[145] 10	0	175
U3=	3	25	[90] 11	[45] 12	17	[15] 15	0	150
U4=	4	[110] 10	21	[75] 13	M	17	[140] 0	325
	Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 5 Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 6	V2= 8	V3= 9	V4= 8	V5= 12	V6= -4	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	[45] 12	17	[15] 15	0	150
U4= 4	4	[110] 10	21	[75] 13	M	17	[140] 0	325
	Demand	110	90	120	230	160	140	710/850

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 5 Step 4: For unoccupied cells, calculate the penalties **P_{ij}** by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 6	V2= 8	V3= 9	V4= 8	V5= 12	V6= -4	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	[45] 12	17	[15] 15	0	150
U4= 4	4	[110] 10	21	[75] 13	M	17	[140] 0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -7$ $P_{13} = -22$ $P_{16} = -4$ $P_{22} = -3$ $P_{26} = -6$ $P_{34} = -6$ $P_{42} = -9$ $P_{45} = -1$

$P_{12} = -M$ $P_{15} = -8$ $P_{21} = -10$ $P_{23} = -10$ $P_{31} = -16$ $P_{36} = -1$ $P_{44} = -M$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 5 Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

		V1 = 6	V2= 8	V3= 9	V4= 8	V5= 12	V6= -4	
		A	B	C	D	E	d	Supply
U1= 0	1	13	M	31	[200] 8	20	0	200
U2= -2	2	14	9	17	[30] 6	[145] 10	0	175
U3= 3	3	25	[90] 11	[45] 12	17	[15] 15	0	150
U4= 4	4	[110] 10	21	[75] 13	M	17	[140] 0	325
	Demand	110	90	120	230	160	140	710/850

$P_{11} = -7$ $P_{13} = -22$ $P_{16} = -4$ $P_{22} = -3$ $P_{26} = -6$ $P_{34} = -6$ $P_{42} = -9$ $P_{45} = -1$

$P_{12} = -M$ $P_{15} = -8$ $P_{21} = -10$ $P_{23} = -10$ $P_{31} = -16$ $P_{36} = -1$ $P_{44} = -M$

NUMERICAL (PROHIBITED TRANSPORTATION ROUTES)

Solution 13. Iteration 5 Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

		V1 = 6	V2= 8	V3= 9	V4= 8	V5= 12	V6= -4	
		A	**B**	**C**	**D**	**E**	**d**	**Supply**
U1= 0	**1**	13	M	31	[200] 8	20	0	**200**
U2= -2	**2**	14	9	17	[30] 6	[145] 10	0	**175**
U3= 3	**3**	25	[90] 11	[45] 12	17	[15] 15	0	**150**
U4= 4	**4**	[110] 10	21	[75] 13	M	17	[140] 0	**325**
	Demand	**110**	**90**	**120**	**230**	**160**	**140**	**710/850**

Total Cost = (200x8)+(30x6)+(145x10)+(90x11)+(45x12)+(15x15)+(110x10)+(75x13)+(140x0) = **7,060**

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NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

14. XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities:

Warehouse Location (City) : A B C D

Capacity (Tonnes) : 90 50 80 60

The warehouses supply tobacco to cigarette companies in three cities that have the following demand:

The following railroad shipping costs per tonne (in hundred rupees) have been determined:

Because of railroad construction, shipments are temporarily prohibited from warehouse at city A to Bharat Cigarette company.

(a) Find the optimum distribution for XYZ tobacco company.

(b) Are there multiple optimum solutions? If yes, identify them.

Cigarette Company	Demand (Tonnes)
Bharat	120
Janata	100
Red Lamp	110

	Bharat	Janata	Red Lamp
A	7	10	5
B	12	9	4
C	7	3	11
D	9	5	7

Back

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14). Since the total demand of 330 units exceeds the total capacity of 280 units by 50 units of the product, a dummy company is created to handle the excess demand. The associated cost coefficients for the dummy warehouse location are taken as zero. Further, the cost element (i.e. 7) on the route city A – Bharat company is replaced by M, since the route is prohibited.

	Bharat	Janata	Red Lamp	Supply
A	M	10	5	90
B	12	9	4	50
C	7	3	11	80
D	9	5	7	60
Dummy	0	0	0	50
Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14). Since the total demand of 330 units exceeds the total capacity of 280 units by 50 units of the product, a dummy company is created to handle the excess demand. The associated cost coefficients for the dummy warehouse location are taken as zero. Further, the cost element (i.e. 7) on the route city A – Bharat company is replaced by M, since the route is prohibited.

	Bharat	Janata	Red Lamp	Supply
A	M	10	5	90
B	12	9	4	50
C	7	3	11	80
D	9	5	7	60
Dummy	0	0	0	50
Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.a). Initial Basic Feasible Solution using VAM.

	Bharat	Janata	Red Lamp	Supply
A	M	10	[90] 5	90
B	[30] 12	9	[20] 4	50
C	7	[80] 3	11	80
D	[40] 9	[20] 5	7	60
Dummy	[50] 0	0	0	50
Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.a). Step 1: Optimality Test m + n - 1 = 7 , 5 + 3 – 1 = 7, **7 = 7 (It is Non-Degeneracy Problem)**

	Bharat	Janata	Red Lamp	Supply
A	M	10	[90] 5	90
B	[30] 12	9	[20] 4	50
C	7	[80] 3	11	80
D	[40] 9	[20] 5	7	60
Dummy	[50] 0	0	0	50
Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.a). Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1 = 13	V2= 9	V3= 5	
		Bharat	Janata	Red Lamp	Supply
U1 = 0	A	M	10	[90] 5	90
U2 = -1	B	[30] 12	9	[20] 4	50
U3 = -6	C	7	[80] 3	11	80
U4 = -4	D	[40] 9	[20] 5	7	60
U5 = -13	Dummy	[50] 0	0	0	50
	Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.a). Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ;

$P_{ij} = U_i + V_j - C_{ij}$

		V1 = 13	V2= 9	V3= 5	
		Bharat	Janata	Red Lamp	Supply
U1 = 0	A	-M M	-1 10	[90] 5	90
U2 = -1	B	[30] 12	-1 9	[20] 4	50
U3 = -6	C	0 7	[80] 3	-12 11	80
U4 = -4	D	[40] 9	[20] 5	-6 7	60
U5 = -13	Dummy	[50] 0	-4 0	-8 0	50
	Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.a). Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

		V1 = 13	V2= 9	V3= 5	
		Bharat	Janata	Red Lamp	Supply
U1 = 0	A	-M M	-1 10	[90] 5	90
U2 = -1	B	[30] 12	-1 9	[20] 4	50
U3 = -6	C	0 7	[80] 3	-12 11	80
U4 = -4	D	[40] 9	[20] 5	-6 7	60
U5 = -13	Dummy	[50] 0	-4 0	-8 0	50
	Demand	120	100	110	330

Total Cost = (90x5)+(30x12)+(20x4)+(80x3)+(40x9)+(20x5)+(50x0) = **1,59,000**

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.b). Since P_{31} = 0; there exist an alternate optimum solution.

		V1 = 13	V2= 9	V3= 5	
		Bharat	Janata	Red Lamp	Supply
U1 = 0	A	-M / M	-1 / 10	90 / 5	90
U2 = -1	B	30 / 12	-1 / 9	20 / 4	50
U3 = -6	C	0 / 7	80 / 3	-12 / 11	80
U4 = -4	D	40 / 9	20 / 5	-6 / 7	60
U5 = -13	Dummy	50 / 0	-4 / 0	-8 / 0	50
	Demand	120	100	110	330

NUMERICAL (ALTERNATE OPTIMUM SOLUTION, PROHIBITED ROUTE & UNBALANCED)

Solution (14.b). The alternate optimum solution.

		V1 =	V2=	V3=	
		Bharat	Janata	Red Lamp	Supply
U1 =	A	M	10	90 / 5	90
U2 =	B	30 / 12	9	20 / 4	50
U3 =	C	40 / 7	40 / 3	11	80
U4 =	D	9	60 / 5	7	60
U5 =	Dummy	50 / 0	0	0	50
	Demand	120	100	110	330

Total Cost = (90x5)+(30x12)+(20x4)+(80x3)+(40x9)+(20x5)+(50x0) = **1,59,000**

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NUMERICAL

15. A departmental store wishes to purchase the following quantities of sarees:

Types of sarees : A B C D E

Quantity : 150 100 75 250 200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities mentioned below (all types of sarees combined);

Manufacturer : W X Y Z

Total quantity: 300 250 150 200

How should the orders be placed?

Sarees

		A	B	C	D	E
Manufacture	W	275	350	425	225	150
	X	300	325	450	175	100
	Y	250	350	475	200	125
	Z	325	275	400	250	175

NUMERICAL

Solution 15. Setup Transportation Problem Table. Since it is an unbalanced problem add a dummy column (125)

Sarees

		A	B	C	D	E	Dummy	Supply
Manufacture	W	275	350	425	225	150	0	300
	X	300	325	450	175	100	0	250
	Y	250	350	475	200	125	0	150
	Z	325	275	400	250	175	0	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Since it is a maximization problem (placement of order), we setup a opportunity cost matrix by subtracting maximum cost (475) from all the rows and columns.

Sarees

		A	B	C	D	E	Dummy	Supply
Manufacture	W	200	125	50	250	325	0	300
	X	175	150	25	300	375	0	250
	Y	225	125	0	275	350	0	150
	Z	150	200	75	225	300	0	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Finding the Initial Basic Feasible solution by VAM.

	A	B	C	D	E	Dummy	Supply
W	200	25 / 125	50	250 / 250	325	25 / 475	300
X	150 / 175	150	25	300	375	100 / 475	250
Y	225	75 / 125	75 / 0	275	350	475	150
Z	150	200	75	225	200 / 300	475	200
Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 1: Optimality Test m + n - 1 = 8 , 4 + 6 – 1 = 9, **9 ≠ 8 (It is a Degeneracy Problem)**

	A	B	C	D	E	Dummy	Supply
W	200	25 / 125	50	250 / 250	325	25 / 475	300
X	150 / 175	150	25	300	375	100 / 475	250
Y	225	75 / 125	75 / 0	275	350	475	150
Z	150	200	75	225	200 / 300	475	200
Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 2: Select the cell 3-2 (No closed lop formed) and add Ɛ.

	A	B	C	D	E	Dummy	Supply
W	200	25 / 125	50	250 / 250	325	25 / 475	300
X	150 / 175	150	25	300	375	100 / 475	250
Y	225	75 / 125	75 / 0	275	350	475	150
Z	150	200	Ɛ / 75	225	200 / 300	475	200
Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		V1=175	V2= 125	V3=0	V4=250	V5=225	V6=475	
		A	B	C	D	E	Dummy	Supply
U1 = 0	W	200	25 / 125	50	250 / 250	325	25 / 475	300
U2 = 0	X	150 / 175	150	25	300	375	100 / 475	250
U3 = 0	Y	225	75 / 125	75 / 0	275	350	475	150
U4 = 75	Z	150	200	Ɛ / 75	225	200 / 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 4: For unoccupied cells, calculate the penalties **P_{ij}** by using the relationship ; **$P_{ij} = U_i + V_j - C_{ij}$**

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	-25 200	25 125	-50 50	250 250	-100 325	25 475	300
U2 = 0	X	150 175	-25 150	-25 25	-50 300	-150 375	100 475	250
U3 = 0	Y	-50 225	75 125	75 0	-25 275	-125 350	0 475	150
U4 = 75	Z	100 150	0 200	Ɛ 75	100 225	200 300	75 475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 5: Since all Pij values are not ≤ 0; the obtained solution is **not optimal**.

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	-25 200	25 125	-50 50	250 250	-100 325	25 475	300
U2 = 0	X	150 175	-25 150	-25 25	-50 300	-150 375	100 475	250
U3 = 0	Y	-50 225	75 125	75 0	-25 275	-125 350	0 475	150
U4 = 75	Z	100 150	200	Ɛ 75	100 225	200 300	75 475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 6: Select the most (+) value of P_{ij} and consider the cell as the new allocated cell.

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	200	25 125	50	250 250	325	25 475	300
U2 = 0	X	150 175	150	25	300	375	100 475	250
U3 = 0	Y	225	75 125	75 0	275	350	475	150
U4 = 75	Z	150	200	Ɛ 75	225	200 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 7: From the particular cell draw a closed loop passing through some allocated cell.

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	200	25 125	50	250 250	325	25 475	300
U2 = 0	X	150 175	150	25	300	375	100 475	250
U3 = 0	Y	225	75 125	75 0	275	350	475	150
U4 = 75	Z	150	200	Ɛ 75	225	200 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 8: Starting from the new allocated cell, alternatively assign (+) and (-) signs at the corner of the closed loop.

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	200	25 (+) 125	50	250 (-) 250	325	25 475	300
U2 = 0	X	150 175	150	25	300	375	100 475	250
U3 = 0	Y	225	75 (-) 125	75 (+) 0	275	350	475	150
U4 = 75	Z	150	200	Ɛ (-) 75	(+) 225	200 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 9: Select the minimum of the allocated value among the (-) signed cells (Ɛ)

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	200	25 (+) 125	50	250 (-) 250	325	25 475	300
U2 = 0	X	150 175	150	25	300	375	100 475	250
U3 = 0	Y	225	75 (-) 125	75 (+) 0	275	350	475	150
U4 = 75	Z	150	200	Ɛ (-) 75	(+) 225	200 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 10 : Frame the new iteration by applying the following steps;
- Add and subtract that selected minimum value in all the (+) and (-) signed cells.
- Copy the remaining cell value as it is. And Go to Step 1

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=225)	Dummy (V6=475)	Supply
U1 = 0	W	200	25+Ɛ 125	50	250-Ɛ 250	325	25 475	300
U2 = 0	X	150 175	150	25	300	375	100 475	250
U3 = 0	Y	225	75-Ɛ 125	75+Ɛ 0	275	350	475	150
U4 = 75	Z	150	200	75	Ɛ 225	200 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 1: Optimality Test m + n - 1 = 8 , 4 + 6 – 1 = 9, **9 = 9 (It is non Degeneracy Problem)**

		A (V1=)	B (V2=)	C (V3=)	D (V4=)	E (V5=)	Dummy (V6=)	Supply
U1 =	W	200	25+Ɛ 125	50	250-Ɛ 250	325	25 475	300
U2 =	X	150 175	150	25	300	375	100 475	250
U3 =	Y	225	75-Ɛ 125	75+Ɛ 0	275	350	475	150
U4 =	Z	150	200	75	Ɛ 225	200 300	475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 3: **Calculate $U_i + V_j = C_{ij}$**

		A (V1=175)	B (V2= 125)	C (V3=0)	D (V4=250)	E (V5=325)	Dummy (V6=475)	Supply
U1 = 0	W	200	25+Ɛ 125	50	250-Ɛ 250	325	25 475	300
U2 = 0	X	150 175	150	25	300	375	100 475	250
U3 = 0	Y	225	75-Ɛ 125	75+Ɛ 0	275	350	475	150
U4 = -25	Z	150	200	75	Ɛ 225	200 300	475	200
	Demand	150	100	75	250	200	125	900

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NUMERICAL

Solution 15. Step 4: For unoccupied cells, calculate the penalties P_{ij} by using the relationship ; $P_{ij} = U_i + V_j - C_{ij}$

		A	B	C	D	E	Dummy	Supply
		V1=175	V2= 125	V3=0	V4=250	V5=325	V6=475	
U1 = 0	W	-25 200	25+ε 125	-50 50	250-ε 250	0 325	25 475	300
U2 = 0	X	150 175	-25 150	-25 25	-50 300	-50 375	100 475	250
U3 = 0	Y	-50 225	75-ε 125	75+ε 0	-25 275	-25 350	0 475	150
U4 = -25	Z	0 150	-100 200	-100 75	ε 225	200 300	-25 475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. Step 5: Since all Pij values are ≤ 0; the obtained solution is **optimal**.

		A	B	C	D	E	Dummy	Supply
		V1=175	V2= 125	V3=0	V4=250	V5=325	V6=475	
U1 = 0	W	-25 200	25+ε 125	-50 50	250-ε 250	0 325	25 475	300
U2 = 0	X	150 175	-25 150	-25 25	-50 300	-50 375	100 475	250
U3 = 0	Y	-50 225	75-ε 125	75+ε 0	-25 275	-25 350	0 475	150
U4 = -25	Z	0 150	-100 200	-100 75	ε 225	200 300	-25 475	200
	Demand	150	100	75	250	200	125	900

NUMERICAL

Solution 15. The Final allocations are as seen in the table.

	A	B	C	D	E	Dummy	Supply
W	275	25 350	425	250 225	150	25 0	300
X	150 300	325	450	175	100	100 0	250
Y	250	75 350	75 475	200	125	0	150
Z	325	275	400	250	200 175	0	200
Demand	150	100	75	250	200	125	900

W - B = 25, W - D = 250, X - A = 150, Y - B = 75, Y - C = 75, Z - E = 200

NUMERICAL

16. Find the optimum solution to the following transportation problem to minimize the cost.

Production Cost per Unit

	1	2	3	4	Supply
1	10	13	16	19	700
2	M	10	13	16	700
3	M	15	18	21	200
4	M	M	15	18	700
5	M	M	20	23	200
6	M	M	M	15	700
Demand	300	700	900	800	

Assignment Problem

Operations Research-I

Prof. Aditya Suresh Kasar

Module - 2

Topics

Numerical

Introduction to Assignment Problem

- It is a special type of ***Transportation Problem,*** in which each source should have the capacity to fulfil demand of any of the destination, such that the total ***Processing time*** is **minimized**.
- Some of the problems where the assignment technique may be useful are assignment of;
  a. workers to machines
  b. salesmen to different sales areas
  c. clerks to various checkout counters
  d. classes to rooms
  e. vehicles to routes
  f. contracts to bidders, etc.

Types of Assignment Problems

- Balanced (No. of Rows = No. of Columns)

	1	2	3	4
A	40	20	0	10
B	10	20	40	0
C	10	40	20	0
D	10	10	0	10

- Unbalanced (No. of Rows ≠ No. of Columns)

	1	2	3	4
A	40	20	0	10
B	10	20	40	0
C	10	40	20	0

Solution Methods of Assignment Problem

An Assignment problem can be solved by any of the following methods:

- Enumeration Method
- Simplex Method
- Transportation Method
- **Hungarian Method**

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Variation of the Assignment Problem

- Multiple Optimal Solution
- Maximization Case Problem
- Unbalanced Assignment Problem
- Restrictions on Assignments

Back

Multiple Optimal Solution

- While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number of zeros.
- Such a situation indicates that there are multiple optimal solutions with the same optimal value of objective function.
- Numerical-1

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Maximization Case Problem

- If instead of cost matrix, a profit (or revenue) matrix is given, then assignments are made in such a way that total profit is maximized.
- The profit maximization assignment problems are solved by converting them into a cost minimization problem in either of the following two ways:
  - a) Put a negative sign before each of the elements in the profit matrix in order to convert the profit values into cost values.
  - b) Locate the largest element in the profit matrix and then subtract all the elements of the matrix from the largest element including itself.
- The transformed assignment problem can be solved by using usual Hungarian method.
- Numerical-1

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Unbalanced Assignment Problem

- The Hungarian method for solving an assignment problem requires that the number of columns and rows in the assignment matrix should be equal.
- However, when the given cost matrix is not a square matrix, the assignment problem is called an unbalanced problem.
- In such cases before applying Hungarian method, dummy row(s) or column(s) are added in the matrix (with zeros as the cost elements) in order to make it a square matrix.
- Problem-1
- Problem-2

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Restrictions on Assignments

- Sometimes it may so happen that a particular resource (say a man or machine) cannot be assigned to a particular activity (say territory or job).
- In such cases, the cost of performing that particular activity by a particular resource is considered to be very large (written as M or ∞) so as to prohibit the entry of this pair of resource-activity into the final solution.
- Numerical-1

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Hungarian Method for Solving Assignment Problem

The Hungarian method (minimization case) can be summarized in the following steps:

- **Phase I: Row and Column Reduction.**

Step 1: Develop the cost matrix from the given problem

If the number of rows are not equal to the number of columns, then add required number of dummy rows or columns. The cost element in dummy rows/columns are always zero.

Step 2: Find the opportunity cost matrix

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row, and

(b) In the reduced matrix obtained from 2(a), identify the smallest element in each column and then subtract it from each element of that column. Each row and column now have at least one zero element.

Hungarian Method for Solving Assignment Problem

The Hungarian method (minimization case) can be summarized in the following steps:

- **Phase II: Optimum Solution.**

Step 3: Make assignments in the opportunity cost matrix

The procedure of making assignments is as follows:

- **Row Scanning:** Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making a square (□) around it. Draw a vertical line passing through that zero.
- After scanning the last row, check whether all the zeros are covered with lines. If Yes, go to Step 4, else do Column Scanning.

Hungarian Method for Solving Assignment Problem

Step 3: Make assignments in the opportunity cost matrix

The procedure of making assignments is as follows (cont.):

- **Column Scanning:** Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make assignment to this single zero by making a square (□) around it and draw a horizontal line passing through the zero.
- Check whether all the zeros are covered with lines, if Yes go to Step 4, else *Select the zeros diagonally opposite with each other*.

Hungarian Method for Solving Assignment Problem

Step 4: Check whether the no. of squares marked is equal to the no. of rows of the matrix.

If Yes, the solution obtained is optimal, else go to Step 5.

Step 5: Identify the minimum value of the undeleted cell value.

- Add the minimum undeleted cell value at the intersection points of the present matrix.
- Subtract the minimum undeleted cell value from all the undeleted cell values.
- All the other entries remain the same.

Step 6: Go to Step 3.

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Numerical:

1. A department of a company has five operators with five jobs to be performed. The time (in hours) that each man takes to perform each job is given in the effectiveness matrix. How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

Operators

Jobs	1	2	3	4	5
1	9	11	14	11	7
2	6	15	13	13	10
3	12	13	6	8	8
4	11	9	10	12	9
5	7	12	14	10	14

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Numerical:

Solution (1) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Operators

Jobs	1	2	3	4	5	Row Minimum
1	9	11	14	11	7	7
2	6	15	13	13	10	6
3	12	13	6	8	8	6
4	11	9	10	12	9	9
5	7	12	14	10	14	7

Numerical:

Solution (1) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Operators

Jobs	1	2	3	4	5	Row Minimum
1	2	4	7	4	0	7
2	0	9	7	7	4	6
3	6	7	0	2	2	6
4	2	0	1	3	0	9
5	0	5	7	3	7	7

Numerical:

Solution (1) ***Step 2: Find the opportunity cost matrix***

(b) Identify the smallest element in each column and then subtract it from each element of that column.

Operators

Jobs	1	2	3	4	5
1	2	4	7	4	0
2	0	9	7	7	4
3	6	7	0	2	2
4	2	0	1	3	0
5	0	5	7	3	7
Column Minimum	0	0	0	2	0

Numerical:

Solution (1) ***Step 2: Find the opportunity cost matrix***

(b) Identify the smallest element in each column and then subtract it from each element of that column.

Operators

Jobs	1	2	3	4	5
1	2	4	7	2	0
2	0	9	7	5	4
3	6	7	0	0	2
4	2	0	1	1	0
5	0	5	7	1	7
Column Minimum	0	0	0	2	0

Numerical:

Solution (1) ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Operators

Jobs	1	2	3	4	5
1	2	4	7	2	$\boxed{0}$
2	$\boxed{0}$	9	7	5	4
3	6	7	0	0	2
4	2	$\boxed{0}$	1	1	0
5	0	5	7	1	7

Numerical:

Solution (1) ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Operators

Jobs	1	2	3	4	5
1	2	4	7	2	$\boxed{0}$
2	$\boxed{0}$	9	7	5	4
3	6	7	$\boxed{0}$	0	2
4	2	$\boxed{0}$	1	1	0
5	0	5	7	1	7

Numerical:

Solution (1) ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Operators

Jobs	1	2	3	4	5
1	2	4	7	2	$\boxed{0}$
2	$\boxed{0}$	9	7	5	4
3	6	7	$\boxed{0}$	0	2
4	2	$\boxed{0}$	1	1	0
5	0	5	7	1	7

In our case no. of square marked is 4 ≠ 5 (no. of rows). Hence the solution is not optimal.

Numerical:

Solution (1) ***Step 5:*** Identify the minimum value of the undeleted cell value.

Operators

Jobs	1	2	3	4	5
1	2	4	7	2	0
2	0	9	7	5	4
3	6	7	0	0	2
4	2	0	1	1	0
5	0	5	7	1	7

Minimum value of undeleted cell = 1

Numerical:

Solution (1) ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Operators

Jobs	1	2	3	4	5
1	2	4	7	2	0
2	0	9	7	5	4
3	7	8	0	0	3
4	2	0	1	1	0
5	0	5	7	1	7

Minimum value of undeleted cell = 1

Numerical:

Solution (1) ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Operators

Jobs	1	2	3	4	5
1	2	4	6	1	0
2	0	9	6	4	4
3	7	8	0	0	3
4	2	0	0	0	0
5	0	5	6	0	7

Minimum value of undeleted cell = 1

Numerical:

Solution (1) ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Operators

Jobs	1	2	3	4	5
1	2	4	6		0
2	0	9	6	4	4
3	7	8	0	0	3
4	2	0	0	0	0
5	0	5	6	0	7

Numerical:

Solution (1) ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Operators

Jobs	1	2	3	4	5
1	2	4	6		0
2	0	9	6	4	4
3	7	8	0	0	3
4	2	0	0	0	0
5	0	5	6	0	7

Numerical:

Solution (1) ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Operators

Jobs	1	2	3	4	5
1	2	4	6		0
2	0	9	6	4	4
3	7	8	0	0	3
4	2	0	0	0	0
5	0	5	6	0	7

In our case no. of square marked is 5 = 5 (no. of rows). Hence the solution is optimal.

Numerical:

Solution (1)

Operators

Jobs	1	2	3	4	5
1	2	4	6	1	0
2	0	9	6	4	4
3	7	8	0	0	3
4	2	0	0	0	0
5	0	5	6	0	7

Job	Operator	Time
1	5	7
2	1	6
3	3	6
4	2	9
5	4	10
Total Time		**38 Hrs.**

Numerical:

2. A computer center has three expert programmers. The center wants three application programmes to be developed. The head of the computer center, after carefully studying the programmes to be developed, estimates the computer ***time in minutes*** required by the experts for the application programmes as follows:
Find the optimum assignment.

Programmers

Programmes	A	B	C
1	120	100	80
2	80	90	110
3	110	140	120

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Numerical:

Solution (2) ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Programmers

Programmes	A	B	C	Row Minimum
1	120	100	80	80
2	80	90	110	80
3	110	140	120	110

Numerical:

Solution (2) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Programmers

Programmes		A	B	C	Row Minimum
	1	40	20	0	80
	2	0	10	30	80
	3	0	30	10	110

Numerical:

Solution (2) ***Step 2: Find the opportunity cost matrix***

(b) Identify the smallest element in each column and then subtract it from each element of that column.

Programmers

Programmes		A	B	C
	1	40	20	0
	2	0	10	30
	3	0	30	10
	Column Minimum	0	10	0

Numerical:

Solution (2) ***Step 2: Find the opportunity cost matrix***

(b) Identify the smallest element in each column and then subtract it from each element of that column.

Programmers

Programmes		A	B	C
	1	40	10	0
	2	0	0	30
	3	0	20	10
	Column Minimum	0	10	0

Numerical:

Solution (2) ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Programmers

Programmes		A	B	C
	1	40	10	$\boxed{0}$
	2	0	0	30
	3	$\boxed{0}$	20	10

Numerical:

Solution (2) ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Programmers

Programmes		A	B	C
	1	40	10	$\boxed{0}$
	2	0	$\boxed{0}$	30
	3	$\boxed{0}$	20	10

Numerical:

Solution (2) ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Programmers

Programmes		A	B	C
	1	40	10	$\boxed{0}$
	2	0	$\boxed{0}$	30
	3	$\boxed{0}$	20	10

In our case no. of square marked is 3 = 3 (no. of rows). Hence the solution is optimal.

Numerical:

Solution (2):

Programmers

Programmes		A	B	C
	1	40	10	$\boxed{0}$
	2	0	$\boxed{0}$	30
	3	$\boxed{0}$	20	10

Programmes	Programmers	Time
1	C	80
2	B	90
3	A	110
Total Time		**280 Hrs.**

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Numerical:

3. A department of a company has five employees with five jobs to be performed. The time (in hours) that each man takes to perform each job is given in the effectiveness matrix. How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

Employees

Jobs		I	II	III	IV	V
	A	10	5	13	15	16
	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

Back

Numerical:

Solution (3) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Employees

Jobs		I	II	III	IV	V	Row Minimum
	A	10	5	13	15	16	5
	B	3	9	18	13	6	3
	C	10	7	2	2	2	2
	D	7	11	9	7	12	7
	E	7	9	10	4	12	4

Numerical:

Solution (3) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Employees

Jobs	I	II	III	IV	V	Row Minimum
A	5	0	8	10	11	5
B	0	6	15	10	3	3
C	8	5	0	0	0	2
D	0	4	2	0	5	7
E	0	5	6	0	8	4

Numerical:

Solution (3) ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning .

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	0	0	0
D	0	4	2	$\boxed{0}$	5
E	0	5	6	0	8

Numerical:

Solution (3) ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning .

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	0	0
D	0	4	2	$\boxed{0}$	5
E	0	5	6	0	8

Numerical:

Solution (3)***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	0	0
D	0	4	2	$\boxed{0}$	5
E	0	5	6	0	8

In our case no. of square marked is 4 ≠ 5 (no. of rows). Hence the solution is not optimal.

Numerical:

Solution (3) ***Step 5:*** Identify the minimum value of the undeleted cell value.

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	0	0
D	0	4	2	$\boxed{0}$	5
E	0	5	6	0	8

Minimum value of undeleted cell = 2

Numerical:

Solution (3) ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	10	7	$\boxed{0}$	2	0
D	0	4	2	$\boxed{0}$	5
E	0	5	6	0	8

Minimum value of undeleted cell = 2

Numerical:

Solution (3) ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	6	10	9
B	$\boxed{0}$	6	15	10	1
C	10	7	$\boxed{0}$	2	0
D	0	4	0	$\boxed{0}$	3
E	0	5	4	0	6

Minimum value of undeleted cell = 2

Numerical:

Solution (3) ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	6	10	9
B	$\boxed{0}$	6	15	10	1
C	10	7	0	2	0
D	0	4	0	0	3
E	0	5	4	$\boxed{0}$	6

Numerical:

Solution (3)***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Employees

Jobs	I	II	III	IV	V
A	5	$\boxed{0}$	6	10	9
B	$\boxed{0}$	6	15	10	1
C	10	7	0	2	$\boxed{0}$
D	0	4	$\boxed{0}$	0	3
E	0	5	4	$\boxed{0}$	6

Numerical:

Solution (3) ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Employees

Jobs	I	II	III	IV	V
A	5	0	6	10	9
B	0	6	15	10	1
C	10	7	0	2	0
D	0	4	0	0	3
E	0	5	4	0	6

In our case no. of square marked is 5 = 5 (no. of rows). Hence the solution is optimal.

Numerical:

Solution (3):

Employees

Jobs	I	II	III	IV	V
A	5	0	6	10	9
B	0	6	15	10	1
C	10	7	0	2	0
D	0	4	0	0	3
E	0	5	4	0	6

Job	Employees	Time
A	II	5
B	I	3
C	V	2
D	III	9
E	IV	4
Total Time		**23 Hrs.**

Back

Numerical: (Multiple Optional Solution/Diagonal Rule)

4. A department of a company has five operators with five jobs to be performed. The time (in hours) that each man takes to perform each job is given in the effectiveness matrix. How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

Operators

Jobs	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Back

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Operators

Jobs	A	B	C	D	Row Minimum
1	5	3	2	8	2
2	7	9	2	6	2
3	6	4	5	7	4
4	5	7	7	8	5

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Operators

Jobs	A	B	C	D	Row Minimum
1	3	1	0	6	2
2	5	7	0	4	2
3	2	0	1	3	4
4	0	2	2	3	5

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Operators

Jobs	A	B	C	D
1	3	1	0	6
2	5	7	0	4
3	2	0	1	3
4	0	2	2	3
Column Minimum	0	0	0	3

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Operators

Jobs	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0
Column Minimum	0	0	0	3

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Operators

Jobs	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Operators

Jobs	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Operators

Jobs	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

In our case no. of square marked is 3 ≠ 4 (no. of rows). Hence the solution is not optimal.

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 5:*** Identify the minimum value of the undeleted cell value.

Operators

Jobs	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Minimum value of undeleted cell = 1

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Operators

Jobs	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	2	0
4	0	2	3	0

Minimum value of undeleted cell = 1

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Operators

Jobs	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Minimum value of undeleted cell = 1

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Operators

Jobs	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Operators

Jobs	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Operators

Jobs	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4) ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Operators

Jobs	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

In our case no. of square marked is 4 = 4 (no. of rows). Hence the solution is optimal.

Numerical: (Multiple Optional Solution/Diagonal Rule)

Solution (4)

Operators

Jobs	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Job	Operators	Time
1	B	3
2	C	2
3	D	7
4	A	5
Total Time		17 Hrs.

Back

Numerical: (Maximization Problem)

5. A marketing manager has four salesmen and four sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that the sales per month (in hundred rupees) for each salesman in each district would be as follows:
Find the assignment of salesmen to districts that will result in maximum sales.

Sales Districts

Salesmen

	I	II	III	IV
A	45	38	30	22
B	35	29	20	14
C	35	29	20	14
D	27	20	15	10

Back

Numerical: (Maximization Problem)

Solution (5): Step 1: Convert it into a minimization problem by subtracting all the elements from the largest element.

Sales Districts

Salesmen

	I	II	III	IV
A	45	38	30	22
B	35	29	20	14
C	35	29	20	14
D	27	20	15	10

The Largest Element is 45

Numerical: (Maximization Problem)

Solution (5): Step 1: Convert it into a minimization problem by subtracting all the elements from the largest element.

Sales Districts

Salesmen

	I	II	III	IV
A	0	7	15	23
B	10	16	25	31
C	10	16	25	31
D	18	25	30	35

The Largest Element is 45

Numerical: (Maximization Problem)

Solution (5): ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Sales Districts

Salesmen

	I	II	III	IV	Row Minimum
A	0	7	15	23	0
B	10	16	25	31	10
C	10	16	25	31	10
D	18	25	30	35	18

Numerical: (Maximization Problem)

Solution (5): ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Sales Districts

Salesmen

	I	II	III	IV	Row Minimum
A	0	7	15	23	0
B	0	6	15	21	10
C	0	6	15	21	10
D	0	7	12	17	18

Numerical: (Maximization Problem)

Solution (5): ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Sales Districts

Salesmen

	I	II	III	IV
A	0	7	15	23
B	0	6	15	21
C	0	6	15	21
D	0	7	12	17
Column Minimum	0	6	12	17

Numerical: (Maximization Problem)

Solution (5): ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Sales Districts

Salesmen

	I	II	III	IV
A	0	1	3	6
B	0	0	3	4
C	0	0	3	4
D	0	1	0	0
Column Minimum	0	6	12	17

Numerical: (Maximization Problem)

Solution (5): ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Sales Districts

Salesmen

	I	II	III	IV
A	$\boxed{0}$		3	6
B	0	$\boxed{0}$	3	4
C	0	0	3	4
D	0		0	0

Numerical: (Maximization Problem)

Solution (5): ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Sales Districts

Salesmen

	I	II	III	IV
A	$\boxed{0}$		3	6
B	0	$\boxed{0}$	3	4
C	0	0	3	4
D	0		$\boxed{0}$	0

Numerical: (Maximization Problem)

Solution (5): ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Sales Districts

Salesmen		I	II	III	IV
	A	0		3	6
	B	0	0	3	4
	C	0	0	3	4
	D	0		0	0

In our case no. of square marked is 3 ≠ 4 (no. of rows). Hence the solution is not optimal.

Numerical: (Maximization Problem)

Solution (5): ***Step 5:*** Identify the minimum value of the undeleted cell value.

Sales Districts

Salesmen		I	II	III	IV
	A	0		3	6
	B	0	0	3	4
	C	0	0	3	4
	D	0		0	0

Minimum value of undeleted cell = 3

Numerical: (Maximization Problem)

Solution (5): ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	3	6
	B	0	0	3	4
	C	0	0	3	4
	D	3	4	0	0

Minimum value of undeleted cell = 3

Numerical: (Maximization Problem)

Solution (5): ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	0	3
	B	0	0	0	1
	C	0	0	0	1
	D	3	4	0	0

Minimum value of undeleted cell = 3

Numerical: (Maximization Problem)

Solution (5): ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	0	3
	B	0	0	0	1
	C	0	0	0	1
	D	3	4	0	0

Numerical: (Maximization Problem)

Solution (5): ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	0	3
	B	0	0	0	1
	C	0	0	0	1
	D	3	4	0	0

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical: (Maximization Problem)

Solution (5): ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	0	3
	B	0	0	0	1
	C	0	0	0	1
	D	3	4	0	0

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical: (Maximization Problem)

Solution (5): ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	0	3
	B	0	0	0	1
	C	0	0	0	1
	D	3	4	0	0

In our case no. of square marked is 4 = 4 (no. of rows). Hence the solution is optimal.

Numerical: (Maximization Problem)

Solution (5):

Sales Districts

Salesmen		I	II	III	IV
	A	0	1	0	3
	B	0	0	0	1
	C	0	0	0	1
	D	3	4	0	0

Salesmen	Territories	Time
A	I	45
B	II	29
C	III	20
D	IV	10
Total Time		**104 Hrs.**

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Numerical: (Unbalanced Problem)

6. A company has four machines that are to be used for three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table. What are the job-assignment pairs that shall minimize the cost?

Jobs

Machines

	A	B	C
1	9	26	15
2	13	27	6
3	35	20	15
4	18	30	20

Numerical: (Unbalanced Problem)

Solution (6). Step 1: Convert to square matrix by adding a dummy Column.

Jobs

Machines

	A	B	C	D
1	9	26	15	0
2	13	27	6	0
3	35	20	15	0
4	18	30	20	0

Numerical: (Unbalanced Problem)

Solution (6). *Step 2: Find the opportunity cost matrix*

(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Jobs

Machines

	A	B	C	D
1	9	26	15	0
2	13	27	6	0
3	35	20	15	0
4	18	30	20	0
Column Minimum	9	20	6	0

Numerical: (Unbalanced Problem)

Solution (6). *Step 2: Find the opportunity cost matrix*

(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Jobs

Machines

	A	B	C	D
1	0	6	9	0
2	4	7	0	0
3	26	0	9	0
4	9	10	14	0
Column Minimum	9	20	6	0

Numerical: (Unbalanced Problem)

Solution (6). *Step 3: Make assignments in the opportunity cost matrix*

Row Scanning

Jobs

Machines

	A	B	C	D
1	0	6	9	0
2	4	7	0	0
3	26	0	9	0
4	9	10	14	$\boxed{0}$

Numerical: (Unbalanced Problem)

Solution (6). *Step 3: Make assignments in the opportunity cost matrix*

Column Scanning

Jobs

Machines

	A	B	C	D
1	$\boxed{0}$	6	9	0
2	4	7	$\boxed{0}$	0
3	26	$\boxed{0}$	9	0
4	9	10	14	$\boxed{0}$

Numerical: (Unbalanced Problem)

Solution (6). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Jobs

Machines

	A	B	C	D
1	$\boxed{0}$	6	9	0
2	4	7	$\boxed{0}$	0
3	26	$\boxed{0}$	9	0
4	9	10	14	$\boxed{0}$

In our case no. of square marked is 4 = 4 (no. of rows). Hence the solution is optimal.

Numerical: (Unbalanced Problem)

Solution (6).

Jobs

Machines

	A	B	C	D
1	$\boxed{0}$	6	9	0
2	4	7	$\boxed{0}$	0
3	26	$\boxed{0}$	9	0
4	9	10	14	$\boxed{0}$

Jobs	Machines	Time
1	A	9
2	C	6
3	B	20
4	D	0
Total Time		**35 Hrs.**

Numerical: (Unbalanced Problem)

7. A company has four machines that are to be used for three jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table. What are the job-assignment pairs that shall minimize the cost?

Machines

Jobs

	A	B	C	D
1	18	24	28	32
2	8	13	17	19
3	10	15	19	22

Numerical: (Unbalanced Problem)

Solution (7). Step 1: Convert to square matrix by adding a dummy row

Machines

Jobs	A	B	C	D
1	18	24	28	32
2	8	13	17	19
3	10	15	19	22
4	0	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Machines

Jobs	A	B	C	D	Row Minimum
1	18	24	28	32	18
2	8	13	17	19	8
3	10	15	19	22	10
4	0	0	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Machines

Jobs	A	B	C	D	Row Minimum
1	0	6	10	14	18
2	0	5	9	11	8
3	0	5	9	12	10
4	0	0	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Machines

Jobs	A	B	C	D
1	0	6	10	14
2	0	5	9	11
3	0	5	9	12
4	0	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Machines

Jobs	A	B	C	D
1	0	6	10	14
2	0	5	9	11
3	0	5	9	12
4	0	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Machines

Jobs	A	B	C	D
1	0	6	10	14
2	0	5	9	11
3	0	5	9	12
4	0	0	0	0

In our case no. of square marked is 2 ≠ 4 (no. of rows). Hence the solution is not optimal.

Numerical: (Unbalanced Problem)

Solution (7). ***Step 5:*** Identify the minimum value of the undeleted cell value.

Machines

Jobs	A	B	C	D
1	0	6	10	14
2	0	5	9	11
3	0	5	9	12
4	0	0	0	0

Minimum value of undeleted cell = 5

Numerical: (Unbalanced Problem)

Solution (7). ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Machines

Jobs	A	B	C	D
1	0	6	10	14
2	0	5	9	11
3	0	5	9	12
4	5	0	0	0

Minimum value of undeleted cell = 5

Numerical: (Unbalanced Problem)

Solution (7). ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Machines

Jobs	A	B	C	D
1	0	1	5	9
2	0	0	4	6
3	0	0	4	7
4	5	0	0	0

Minimum value of undeleted cell = 5

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Machines

Jobs

	A	B	C	D
1	0	1	5	9
2	0	0	4	6
3	0	0	4	7
4	5	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Machines

Jobs

	A	B	C	D
1	0	1	5	9
2	0	0	4	6
3	0	0	4	7
4	5	0	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Machines

Jobs

	A	B	C	D
1	0	1	5	9
2	0	0	4	6
3	0	0	4	7
4	5	0	0	0

In our case no. of square marked is 3 ≠ 4 (no. of rows). Hence the solution is not optimal.

Numerical: (Unbalanced Problem)

Solution (7). ***Step 5:*** Identify the minimum value of the undeleted cell value.

Machines

Jobs

	A	B	C	D
1	0	1	5	9
2	0	0	4	6
3	0	0	4	7
4	5	0	0	0

Minimum value of undeleted cell = 4

Numerical: (Unbalanced Problem)

Solution (7). ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Machines

Jobs

	A	B	C	D
1	0	1	5	9
2	0	0	4	6
3	0	0	4	7
4	9	4	0	0

Minimum value of undeleted cell = 4

Numerical: (Unbalanced Problem)

Solution (7). ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Machines

Jobs

	A	B	C	D
1	0	1	1	5
2	0	0	0	2
3	0	0	0	3
4	9	4	0	0

Minimum value of undeleted cell = 4

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Machines

Jobs

	A	B	C	D
1	0	1	1	5
2	0	0	0	2
3	0	0	0	3
4	9	4	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Machines

Jobs

	A	B	C	D
1	0	1	1	5
2	0	0	0	2
3	0	0	0	3
4	9	4	0	0

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Machines

Jobs

	A	B	C	D
1	0	1	1	5
2	0	0	0	2
3	0	0	0	3
4	9	4	0	0

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical: (Unbalanced Problem)

Solution (7). ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Machines

Jobs	A	B	C	D
1	$\boxed{0}$	1	1	5
2	0	$\boxed{0}$	0	2
3	0	0	$\boxed{0}$	3
4	9	4	0	$\boxed{0}$

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical: (Unbalanced Problem)

Solution (7). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Machines

Jobs	A	B	C	D
1	$\boxed{0}$	1	1	5
2	0	$\boxed{0}$	0	2
3	0	0	$\boxed{0}$	3
4	9	4	0	$\boxed{0}$

In our case no. of square marked is 4 = 4 (no. of rows). Hence the solution is optimal.

Numerical: (Unbalanced Problem)

Solution (7).

Machines

Jobs	A	B	C	D
1	$\boxed{0}$	1	1	5
2	0	$\boxed{0}$	0	2
3	0	0	$\boxed{0}$	3
4	9	4	0	$\boxed{0}$

Jobs	Machines	Time
1	A	18
2	B	13
3	C	19
4	D	0
Total Time		**50 Hrs.**

Numerical: (Restrictions on Assignments)

8. In the modification of a plant layout of a factory four new machines M1, M2, M3, and M4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M2 cannot be placed at C and M3 cannot be placed at A. The cost of locating a machine at a place (in hundred rupees) is as follows.

Locations

Machines	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	-	10	9
M3	-	11	14	11	7
M4	14	8	12	7	8

Numerical: (Restrictions on Assignments)

Solution (8). Step 1: Convert to square matrix by adding a dummy column. Also assign a high cost, denoted by M, to the pair (M2, C) and (M3, A)

Locations

Machines	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	M	10	9
M3	M	11	14	11	7
M4	14	8	12	7	8
Dummy	0	0	0	0	0

Numerical: (Restrictions on Assignments)

Solution (8). ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Locations

Machines	A	B	C	D	E	Row Min
M1	9	11	15	10	11	**9**
M2	12	9	M	10	9	**9**
M3	M	11	14	11	7	**7**
M4	14	8	12	7	8	**7**
Dummy	0	0	0	0	0	**0**

Numerical: (Restrictions on Assignments)

Solution (8). ***Step 2: Find the opportunity cost matrix***
(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Locations

Machines	A	B	C	D	E	Row Min
M1	0	2	6	1	2	**9**
M2	3	0	M	1	0	**9**
M3	M	4	7	4	0	**7**
M4	7	1	5	0	1	**7**
Dummy	0	0	0	0	0	**0**

Numerical: (Restrictions on Assignments)

Solution (8). ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Locations

Machines	A	B	C	D	E
M1	$\boxed{0}$	2	6	1	2
M2	3	0	M	1	0
M3	M	4	7	4	$\boxed{0}$
M4	7	1	5	$\boxed{0}$	1
Dummy	0	0	0	0	0

Numerical: (Restrictions on Assignments)

Solution (8). ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Locations

Machines	A	B	C	D	E
M1	$\boxed{0}$	2	6	1	2
M2	3	$\boxed{0}$	M	1	0
M3	M	4	7	4	$\boxed{0}$
M4	7	1	5	$\boxed{0}$	1
Dummy	0	0	$\boxed{0}$	0	0

Numerical: (Restrictions on Assignments)

Solution (8). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Locations

Machines		A	B	C	D	E
	M1	$\boxed{0}$	2	6	1	2
	M2	3	$\boxed{0}$	M	1	0
	M3	M	4	7	4	$\boxed{0}$
	M4	7	1	5	$\boxed{0}$	1
	Dummy	0	0	$\boxed{0}$	0	0

In our case no. of square marked is 5 = 5 (no. of rows). Hence the solution is optimal.

Back

Numerical: (Restrictions on Assignments)

Solution (8).

Locations

Machines		A	B	C	D	E
	M1	$\boxed{0}$	2	6	1	2
	M2	3	$\boxed{0}$	M	1	0
	M3	M	4	7	4	$\boxed{0}$
	M4	7	1	5	$\boxed{0}$	1
	Dummy	0	0	$\boxed{0}$	0	0

Jobs	Machines	Time
M1	A	9
M2	B	9
M3	E	7
M4	D	7
dummy	C	0
Total Time		32 Hrs.

Back

Numerical:

9. A company operates in four territories, and four salesmen available for an assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory	I	II	III	IV
Annual Sales (Rs)	126000	105000	84000	63000

The four salesmen also differ in their ability. It is estimated that, working under the same condition, their yearly sales would be proportionately as follows:

Salesmen	A	B	C	D
Proportion	7	5	5	4

If the criterion is maximum expected total sales, the intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on; verify this answer by the assignment technique.

Back

Numerical:

Solution (9). Construct the Effectiveness Matrix: To avoid the fractional values of annual sales of each salesman in each territory, for convenience, consider their yearly sales as 21 (i.e. the sum of sales proportions), taking Rs 1,000 as one unit. Now divide the individual sales in each territory by 21 in order to obtain the required annual sales by each salesman.

Territory

Salesman		1	2	3	4	Sales Proportion
	A	42	35	28	21	7
	B	30	25	20	15	5
	C	30	25	20	15	5
	D	24	20	16	12	4
	Sales ('000 Rs)	6	5	4	3	

Eg. For territory 1 the Sales = 126 / 21 = 6, for territory 2 Sales = 105 / 21 = 5, for territory 3 Sales = 84 / 21 = 4 and for territory 4 Sales = 63 / 21 = 3

Back

Numerical:

Solution (9). ***Converting Maximization Problem into Minimization Problem:***

The given maximization assignment problem can be converted into a minimization assignment problem by subtracting from the highest element (i.e. 42) all the elements of the given table.

Territory

Salesman		1	2	3	4
	A	0	7	14	21
	B	12	17	22	27
	C	12	17	22	27
	D	18	22	26	30

Back

Numerical:

Solution (9). ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Territory

Salesman		1	2	3	4	Row Minimum
	A	0	7	14	21	0
	B	12	17	22	27	12
	C	12	17	22	27	12
	D	18	22	26	30	18

Back

Numerical:

Solution (9). ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row.

Territory

Salesman		1	2	3	4	Row Minimum
	A	0	7	14	21	0
	B	0	5	10	15	12
	C	0	5	10	15	12
	D	0	4	8	12	18

Back

Numerical:

Solution (9). ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Territory

Salesman		1	2	3	4
	A	0	7	14	21
	B	0	5	10	15
	C	0	5	10	15
	D	0	4	8	12
	Column Minimum	0	4	8	12

Back

Numerical:

Solution (9). ***Step 2: Find the opportunity cost matrix***

(a) Identify the smallest element in each column of cost matrix and then subtract it from each element of that column.

Territory

Salesman		1	2	3	4
	A	0	3	6	9
	B	0	1	2	3
	C	0	1	2	3
	D	0	0	0	0
	Column Minimum	0	4	8	12

Back

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Territory

Salesman

	1	2	3	4
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Territory

Salesman

	1	2	3	4
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Numerical:

Solution (9). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Territory

Salesman

	1	2	3	4
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

In our case no. of square marked is 2 ≠ 4 (no. of rows). Hence the solution is not optimal.

Numerical:

Solution (9). ***Step 5:*** Identify the minimum value of the undeleted cell value.

Territory

Salesman

	1	2	3	4
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Minimum value of undeleted cell = 1

Numerical:

Solution (9). ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Territory

Salesman

	1	2	3	4
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	1	0	0	0

Minimum value of undeleted cell = 1

Numerical:

Solution (9). ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Territory

Salesman

	1	2	3	4
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	1	0	0	0

Minimum value of undeleted cell = 1

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***

Row Scanning

Territory

Salesman

	1	2	3	4
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	1	0	0	0

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***

Column Scanning

Territory

Salesman

	1	2	3	4
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	1	0	0	0

Numerical:

Solution (9). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***

If **Yes** then the solution is optimal else Go to next Step.

Territory

Salesman

	1	2	3	4
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	1	0	0	0

In our case no. of square marked is 3 ≠ 4 (no. of rows). Hence the solution is not optimal.

Numerical:

Solution (9). ***Step 5:*** Identify the minimum value of the undeleted cell value.

Territory

Salesman	1	2	3	4
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	1	0	0	0

Minimum value of undeleted cell = 1

Numerical:

Solution (9). ***Step 5:*** Add the minimum undeleted cell value at the intersection points of the present matrix.

Territory

Salesman	1	2	3	4
A	0	2	5	8
B	0	0	1	2
C	0	0	1	2
D	2	1	0	0

Minimum value of undeleted cell = 1

Numerical:

Solution (9). ***Step 5:*** Subtract the minimum undeleted cell value from all the undeleted cell values.

Territory

Salesman	1	2	3	4
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

Minimum value of undeleted cell = 1

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***
Row Scanning

Territory

Salesman	1	2	3	4
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Territory

Salesman	1	2	3	4
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

Numerical:

Solution (9). ***Step 3: Make assignments in the opportunity cost matrix***
Column Scanning

Territory

Salesman	1	2	3	4
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

Since no further allocation is possible in row scanning and Column scanning, we select zeros diagonally opposite to each other.

Numerical:

Solution (9). ***Step 4: Check whether the no. square marked is equal to the no. of rows of the matrix.***
If **Yes** then the solution is optimal else Go to next Step.

Territory

Salesman	1	2	3	4
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

In our case no. of square marked is 4 = 4 (no. of rows). Hence the solution is optimal.

Numerical:

Solution (9).

Territory

Salesman	1	2	3	4
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

Salesman	Territory	Sales ('000 Rs)
A	1	42
B	2	25
C	3	20
D	4	12
Total Sales		99

Numerical:

10. Six salesmen are to be allocated to six sales regions. The earnings of each salesman at each region is given below. How can you find an allocation that earnings will be maximum?

Regions

Salesman	1	2	3	4	5	6
A	15	35	0	25	10	45
B	40	5	45	20	15	20
C	26	60	10	65	25	10
D	25	20	35	10	25	60
E	30	70	40	5	40	50
F	10	25	30	40	50	15

Numerical:

11. Consider a problem of assigning four clerks to four tasks. The time(hours) required to complete the tasks is given below :

Task

Clerk	A	B	C	D
1	4	7	5	6
2	-	8	7	4
3	3	-	5	3
4	6	6	4	-

Clerk 2 cannot be assigned to task A and clerk 3 cannot be assigned to task B and clerk 4 cannot be assigned to task D. Find all the optimum assignment schedules

Numerical:

12. A solicitor's firm employs typists on hourly piece rate basis for their daily work. There are five typists and their charges and speed are different. According to an earlier understanding only one job was given to one typist and the typist was paid for a full hour, even if he worked for a fraction of an hour. Find the least cost allocation for the following data:

Typist	Rate per hour (Rs.)	No. of Pages Typed/Hour
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of Pages
P	199
Q	175
R	145
S	298
T	178

Numerical:

13. A City Corporation has decided to carry out road repairs on four main arteries of the city. The government has agreed to make a special grant of Rs 50 lakh towards the cost with a condition that the repairs to be done at the lowest cost and quickest time. If the conditions warrant, a supplementary token grant will also be done considered favorably. The corporation has floated tenders and five contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

Contractors/Road	Cost of Repairs (Rs in lakh)			
	R_1	R_2	R_3	R_4
C_1	9	14	19	15
C_2	7	17	20	19
C_3	9	18	21	18
C_4	10	12	18	19
C_5	10	15	21	16

(a) Find the best way of assigning the repair work to the contractors and the costs.
(b) If it is necessary to seek supplementary grants, what should be amount sought?
(c) Which of the five contractors will be unsuccessful in his bid?

Game Theory

QUANTITATIVE TECHNIQUES

Prof. Aditya Suresh Kasar

LEARNING OBJECTIVES

After studying this chapter, you should be able to

- understand how optimal strategies are formulated in conflict and competitive environment.
- understand the principles of two-person zero-sum games.
- apply various methods to select and execute various optimal strategies to win the game.
- use dominance rules to reduce the size of a game payoff matrix and compute value of the game with mixed strategies.
- apply minimax and maximin principle to compute the value of the game, when there is a saddle point.
- make distinction between pure and mixed strategies.
- use linear programming approach to compute the value of the game when dominance rules do not apply.

INTRODUCTION

- In general, the term 'game' refers to a situation of conflict and competition in which two or more competitors (or participants) are involved in the decision-making process in anticipation of certain outcomes over a period of time.
- The competitors are referred to as *players*.
- A player may be an *individual, individuals*, or an *organization.*
- A few examples of competitive and conflicting decision environment, that involve the interaction between two or more competitors are:
  - Pricing of products, where sale of any product is determined not only by its price but also by the price set by competitors for a similar product
  - The success of any TV channel programme largely depends on what the competitors presence in the same time slot and the programme they are telecasting.
  - The success of a business strategy depends on the policy of internal revenue service regarding the expenses that may be disallowed,
  - The success of an advertising/marketing campaign depends on various types of services offered to the customers.

CLASSIFICATION

The models in the *theory of games* can be classified based on the following factors:

- ***Number of players***
- ***Sum of gains and losses***
- ***Strategy***

CLASSIFICATION

The models in the *theory of games* can be classified based on the following factors:

- ***Number of players*** If a game involves only two players (competitors), then it is called a *two-person game*. However, if the number of players are more, the game is referred to as *n-person game*.
- ***Sum of gains and losses***
- ***Strategy***

CLASSIFICATION

The models in the *theory of games* can be classified based on the following factors:

- ***Number of players***
- ***Sum of gains and losses*** If, in a game, the sum of the gains to one player is exactly equal to the sum of losses to another player, so that, the sum of the gains and losses equals zero, then the game is said to be a *zero-sum game*. Otherwise it is said to be *non-zero sum game*.
- ***Strategy***

CLASSIFICATION

The models in the *theory of games* can be classified based on the following factors:

- *Number of players*
- *Sum of gains and losses*
- ***Strategy*** The strategy for a player is the list of all possible actions (moves, decision alternatives or courses of action) that are likely to be adopted by him for every payoff (outcome). It is assumed that the players are aware of the rules of the game governing their decision alternatives (or strategies). The outcome resulting from a particular strategy is also known to the players in advance and is expressed in terms of numerical values (e.g. money, per cent of market share or utility).

TYPES OF STRATEGIES

The particular strategy that optimizes a player's gains or losses, without knowing the competitor's strategies, is called *optimal strategy.* The expected outcome, when players use their optimal strategy, is called *value of the game.*

- Generally, the following two types of strategies are followed by players in a game:
  - ***a) Pure Strategy*** A particular strategy that a player chooses to play again and again regardless of other player's strategy, is referred as *pure strategy*. The objective of the players is to maximize their gains or minimize their losses.
  - ***b) Mixed Strategy*** A set of strategies that a player chooses on a particular move of the game with some fixed probability are called *mixed strategies*. Thus, there is a probabilistic situation and objective of the each player is to maximize expected gain or to minimize expected loss by making the choice among pure strategies with fixed probabilities

TWO-PERSON ZERO-SUM GAMES

A game with only two players, say A and B, is called a two-person zero-sum game, only if one player's gain is equal to the loss of other player, so that total sum is zero.

- **Payoff matrix**
  - The payoffs (a quantitative measure of satisfaction that a player gets at the end of the play) in terms of gains or losses, when players select their particular strategies (courses of action), can be represented in the form of a matrix, called the payoff matrix.
  - Since the game is zero-sum, the gain of one player is equal to the loss of other and vice versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player, with the sign changed. Thus, it is sufficient to construct a payoff table only for one of the players.

TWO-PERSON ZERO-SUM GAMES

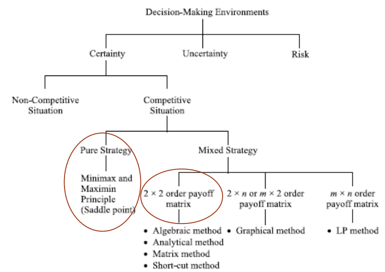
- **Payoff matrix**

- If player A has m strategies represented by the letters: A1, A2, . . ., Am and player B has n strategies represented by the letters: B1, B2, . . ., Bn. The numbers m and n need not be equal.
- The total number of possible outcomes is therefore m × n.
- It is assumed that each player not only knows his own list of possible strategies but also of his competitor.
- For convenience, it is assumed that player A is always a gainer whereas player B a loser.
- Let aij be the payoff that player A gains from player B if player A chooses strategy i and player B chooses strategy j.
- Then the payoff matrix is shown in the Table

Player A's Strategies	Player B's Strategies			
	B_1	B_2	. . .	B_n
A_1	a_{11}	a_{12}	. . .	a_{1n}
A_2	a_{21}	a_{22}	. . .	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	a_{m1}	a_{m2}	. . .	a_{mn}

- Since player *A* is assumed to be the gainer, therefore he wishes to gain as large a payoff *aij* as possible, player *B* on the other hand would do his best to reach as small a value of *aij* as possible. Of course, the gain to player *B* and loss to *A* must be – *aij*.

TWO-PERSON ZERO-SUM GAMES



ASSUMPTIONS OF THE GAME

1. Each player has available to him a finite number of possible strategies (courses of action). The list may not be the same for each player.
2. Players act rationally and intelligently.
3. List of strategies of each player and the amount of gain or loss on an individual's choice of strategy is known to each player in advance.
4. One player attempts to maximize gains and the other attempts to minimize losses.
5. Both players make their decisions individually, prior to the play, without direct communication between them.
6. Both players select and announce their strategies simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
7. The payoff is fixed and determined in advance.

PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES): GAMES WITH SADDLE POINT

- The objective of the study is to know how these players must select their respective strategies so that they are able to optimize their payoff.
- Such a decision-making criterion is referred to as the *minimax-maximin principle.*
- Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.
- ***Maximin principle*** For player *A* the minimum value in each row represents the least gain (payoff) to him, if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player *A* is called the *maximin principle,* and the corresponding gain is called the *maximin value of the game.*

PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES): GAMES WITH SADDLE POINT

- The objective of the study is to know how these players must select their respective strategies so that they are able to optimize their payoff.
- Such a decision-making criterion is referred to as the *minimax-maximin principle.*
- Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.
- ***Minimax principle*** For player *B*, who is assumed to be the looser, the maximum value in each column represents the maximum loss to him, if he chooses his particular strategy. These are written in the payoff matrix by column maxima. He will then select the strategy that gives the minimum loss among the column maximum values. This choice of player *B* is called the *minimax principle,* and the corresponding loss is the *minimax value of the game.*

PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES): GAMES WITH SADDLE POINT

- The objective of the study is to know how these players must select their respective strategies so that they are able to optimize their payoff.
- Such a decision-making criterion is referred to as the *minimax-maximin principle.*
- Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.
- ***Optimal strategy*** A course of action that puts any player in the most preferred position, irrespective of the course of action his competitor(s) adopt, is called as optimal strategy. In other words, if the maximin value equals the minimax value, then the game is said to have a *saddle* (*equilibrium*) *point* and the corresponding strategies are called *optimal strategies.*

PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES): GAMES WITH SADDLE POINT

- The objective of the study is to know how these players must select their respective strategies so that they are able to optimize their payoff.
- Such a decision-making criterion is referred to as the *minimax-maximin principle.*
- Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.
- ***Value of the game*** This is the expected payoff at the end of the game, when each player uses his optimal strategy, i.e. the amount of payoff, *V*, at an equilibrium point. A game may have more than one saddle points. A game with no saddle point is solved by choosing strategies with fixed probabilities.

PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES): GAMES WITH SADDLE POINT

Rules to Determine Saddle Point

The reader is advised to follow the following three steps, in this order, to determine the saddle point in the payoff matrix.

1. Select the minimum (lowest) element in each row of the payoff matrix and write them under 'row minima' heading. Then, select the largest element among these elements and enclose it in a rectangle,
2. Select the maximum (largest) element in each column of the payoff matrix and write them under 'column maxima' heading. Then select the lowest element among these elements and enclose it in a circle,
3. Find out the element(s) that is same in the circle as the well as rectangle and mark the position of such element(s) in the matrix. This element represents the value of the game and is called the saddle (or equilibrium) point.

NUMERICAL

Q 1. For the game with payoff matrix:

	Player B		
Player A	B_1	B_2	B_3
A_1	−1	2	−2
A_2	6	4	−6

determine the optimal strategies for players A and B. Also determine the value of game. Is this game (i) fair? (ii) strictly determinable?

NUMERICAL

Q 1. Solution:

In this example, gains to player A or losses to player B are represented by the positive quantities, whereas, losses to A and gains to B are represented by negative quantities. It is assumed that A wants to maximize his minimum gains from B.

Player A	Player B			
	B1	B2	B3 Saddle Point	Row Minimum
A1	-1	2	$\boxed{-2}$	-2 Maximin
A2	6	4	-6	-6
Column Maximum	6	4	-2	

Minimax

The payoff amount in the saddle-point position is also called *value of the game*. For this game, value of the game is, *V* = – 2, for player *A*. The value of game is always expressed from the point of view of the player whose strategies are listed in the rows. Also since the value of the game is not zero, the game is not fair.

NUMERICAL

Q 2. company management and the labour union are negotiating a new three year settlement.

Each of these has 4 strategies:

I : Hard and aggressive bargaining II : Reasoning and logical approach

III : Legalistic strategy IV : Conciliatory approach

The costs to the company are given for every pair of strategy choice.

	Company Strategies			
Union Strategies	*I*	*II*	*III*	*IV*
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	– 5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

NUMERICAL

Q 2. Solution:

Union Strategies	Company Startegies				
	I	II	III Saddle Point	IV	Row Minimum
I	20	15	$\boxed{12}$	35	12 Maximin
II	25	14	8	10	8
III	40	2	10	5	2
IV	-5	4	11	0	0
Column Maximum	40	15	12	35	

Minimax

Maximin = Minimax = Value of game = 12, therefore the company will always adopt strategy III – Legalistic strategy and union will always adopt strategy I – Hard and aggressive bargaining.

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

- In certain cases, no saddle point exists, i.e. maximin value ≠ minimax value.
- In all such cases, players must choose the mixture of strategies to find the value of game and an optimal strategy.
- The value of game obtained by the use of mixed strategies represents the least payoff, which player *A* can expect to win and the least which player *B* can expect to lose.
- The expected payoff to a player in a game with payoff matrix [*aij*] of order *m* × *n* is defined as:

$$E(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i \, a_{ij} \, q_j = \mathbf{P}^T \mathbf{A}\, \mathbf{Q} \text{ (in matrix notation)}$$

where P = (p1, p2, . . ., pm) and Q = (q1, q2, . . ., qn) denote probabilities (or relative frequency with which a strategy is chosen from the list of strategies) associated with m strategies of player A and n strategies of player, B respectively, where p1 + p2 + . . . + pm = 1 and q1 + q2 + . . . + qn = 1.

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Remark For solving a 2 × 2 game, without a saddle point, the following formula is also used. If payoff matrix for player *A* is given by:

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

then the following formulae are used to find the value of game and optimal strategies:

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}; \quad q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

where $p_2 = 1 - p_1; \quad q_2 = 1 - q_1$

and $$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

THE RULES (PRINCIPLES) OF DOMINANCE

The rules of dominance are especially used for the evaluation of two-person zero-sum games without a saddle (equilibrium) point. Certain dominance principles are stated as follows:

1. For player B, who is assumed to be the loser, if each element in a column, say Cr is greater than or equal to the corresponding element in another column, say Cs in the payoff matrix, then the column Cr is said to be dominated by column Cs and therefore, column Cr can be deleted from the payoff matrix. In other words, player B will never use the strategy that corresponds to column Cr because he will loose more by choosing such strategy.
2. For player A, who is assumed to be the gainer, if each element in a row, say Rr, is less than or equal to the corresponding element in another row, say Rs, in the payoff matrix, then the row Rr is said to be dominated by row Rs and therefore, row Rr can be deleted from the payoff matrix. In other words, player A will never use the strategy corresponding to row Rr, because he will gain less by choosing such a strategy.
3. A strategy say, k can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy k dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination would be decided as per rules 1 and 2 above.

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Example 1: A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in table represent wage increase while negative sign represents wage reduction. What are the optimal strategies for the company as well as the union? What is the game value?

Union Strategies	Company Strategies			
	C1	C2	C3	C4
U1	0.25	0.20	0.14	0.30
U2	0.27	0.16	0.12	0.14
U3	0.35	0.08	0.15	0.19
U4	-0.02	0.08	0.13	0.00

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: In this payoff matrix strategy *U*4 is dominated by strategy *U*1 as well as *U*3.

Union Strategies	Company Strategies				
	C1	C2	C3	C4	Sum
U1	0.25	0.20	0.14	0.30	0.89
U2	0.27	0.16	0.12	0.14	$\boxed{0.69}$
U3	0.35	0.08	0.15	0.19	0.77
~~U4~~	~~-0.02~~	~~0.08~~	~~0.13~~	~~0.00~~	$\boxed{0.19}$

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: After deleting U4 strategy, we get

Union Strategies	Company Strategies			
	C1	C2	C3	C4
U1	0.25	0.20	0.14	0.30
U2	0.27	0.16	0.12	0.14
U3	0.35	0.08	0.15	0.19
Sum	0.87	0.44	0.41	0.63

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: Company's point of view, strategy C1 is dominated by C2 and C3

Also C4 is dominated C2 and C3.

Union Strategies	Company Strategies			
	~~C1~~	C2	C3	~~C4~~
U1	~~0.25~~	0.20	0.14	~~0.30~~
U2	~~0.27~~	0.16	0.12	~~0.14~~
U3	~~0.35~~	0.08	0.15	~~0.19~~
Sum	$\boxed{0.87}$	0.44	0.41	$\boxed{0.63}$

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: Deleting strategies C1 and C4 we get

Union Strategies	Company Strategies	
	C2	C3
U1	0.20	0.14
U2	0.16	0.12
U3	0.08	0.15

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: Again strategy *U*2 is dominated by *U*1 and is, therefore, deleted to give

Union Strategies	Company Strategies		
	C2	C3	Sum
U1	0.20	0.14	0.34
~~U2~~	~~0.16~~	~~0.12~~	$\boxed{0.28}$
U3	0.08	0.15	0.23

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: Again strategy *U*2 is dominated by *U*1 and is, therefore, deleted to give

Union Strategies	Company Strategies	
	C2	C3
U1	0.20	0.14
U3	0.08	0.15

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution: Again strategy *U*2 is dominated by *U*1 and is, therefore, deleted to give

Union Strategies	Company Strategies			
	C2	C3	Oddments	Probability
U1	0.20	0.14	0.15 - 0.08 = 0.07	0.07/0.13 = 0.538
U3	0.08	0.15	0.2 - 0.14 = 0.06	0.06/0.13 = 0.461
Oddments	0.15 - 0.14 = 0.01	0.2 - 0.08 = 0.12		
Probability	0.01/0.21 = 0.076	0.12/0.21 = 0.923		

MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

Solution:

Value of the game, V : (0.538 × 0.20) + (0.461 × 0.08) = 0.1076 + 0.03688 = 0.1445

Optimal strategy for the company : (0, 0.076, 0.923, 0)

Optimal strategy for the union : (0.538, 0, 0.461, 0)

Union Strategies	Company Strategies		
	C2	C3	Probability
U1	0.20	0.14	0.07/0.13 = 0.538
U3	0.08	0.15	0.06/0.13 = 0.461
Probability	0.01/0.13 = 0.076	0.12/0.13 = 0.923	

Decision Theory

OPERATIONS RESEARCH-I

Prof. Aditya Suresh Kasar

Module - 3

TOPICS

INTRODUCTION TO DECISION THEORY

- The success or failure that an individual or organization experiences, depends to a large extent, on the ability of making ***acceptable decisions*** on time.
- To arrive at such a decision, a decision-maker needs to enumerate feasible and viable courses of action (alternatives or strategies), the projection of consequences associated with each course of action, and a measure of effectiveness (or an objective) to identify the best course of action.

INTRODUCTION TO DECISION THEORY

- The degree of knowledge is divided into four categories: complete knowledge (i.e. certainty), ignorance, risk and uncertainty as shown in Figure.

COMPONENTS OF DECISION THEORY

Consider a manufacturing company that is thinking of several alternatives to increase the production to meet the increasing market demands.

- **Decision alternatives (strategies):** There is a finite number of decision alternatives available to the decision-maker at each point in time when a decision is made. The number and type of such alternatives may depend on the previous decisions made and their outcomes.
  - Expand the present plant
  - Construct a new plant
  - Subcontract the production for extra demand

COMPONENTS OF DECISION THEORY

Consider a manufacturing company that is thinking of several alternatives to increase the production to meet the increasing market demands.

- **States of nature:** The state of nature is an event or scenario that is not under the control of decision makers. For instance, it may be the state of economy (e.g. inflation), a weather condition, a political development, etc.
  - High Demand
  - Moderate Demand
  - Low Demand
  - No Demand

COMPONENTS OF DECISION THEORY

Consider a manufacturing company that is thinking of several alternatives to increase the production to meet the increasing market demands.

- **Payoff:** It is a numerical value (outcome) obtained due to the application of each possible combination of decision alternatives and states of nature. The payoff values are always conditional values because of unknown states of nature.

Alternatives	*States of nature (product demand)*			
	High	*Moderate*	*Low*	*Nil*
Expand	₹ 50,000	₹ 25,000	– ₹ 25,000	– ₹ 45,000
Construct	₹ 70,000	₹ 30,000	– ₹ 40,000	– ₹ 80,000
Subcontract	₹ 30,000	₹ 15,000	– ₹ 1,000	– ₹ 10,000

STEPS OF DECISION-MAKING PROCESS

The decision-making process involves the following steps:

1. Identify and define the problem.
2. List all possible future events (not under the control of decision-maker) that are likely to occur.
3. Identify all the courses of action available to the decision-maker.
4. Express the payoffs (P_{ij}) resulting from each combination of course of action and state of nature.
5. Apply an appropriate decision theory model to select the best course of action from the given list on the basis of a criterion (measure of effectiveness) to get optimal (desired) payoff.

TYPES OF DECISION-MAKING ENVIRONMENTS

- Decision-Making under Certainty *(Transportation, Assignment, etc.)*
- Decision-Making under Uncertainty
- Decision-Making under Risk
- Posterior Probabilities and Bayesian Analysis

TYPES OF DECISION-MAKING ENVIRONMENTS

When probability of any outcome can not be quantified

- Equal Probabilities (Laplace) Criterion
- Optimism (Maximax or Minimin) Criterion
- Pessimism (Maximin or Minimax) Criterion
- Minimax Regret (Savage) Criterion
- Coefficient of Optimism (Hurwicz) Criterion

EQUAL PROBABILITIES (LAPLACE)/BAYES' CRITERION

- Since the probabilities of states of nature are not known, it is assumed that all states of nature will occur with equal probability, i.e. each state of nature is assigned an equal probability.
- Expected Payoff = 1/n [P_1+P_2+..........$+P_n$]

  where; n = number of events , P = Payoff values

Numerical-1

OPTIMISM (MAXIMAX OR MINIMIN) CRITERION

- In this criterion the decision-maker ensures that he should not miss the opportunity to achieve the largest possible profit (Maximax) or the lowest possible cost (Minimin). Thus, he selects the decision alternative that represents the maximum of the maxima (or minimum of the minima) payoffs (consequences or outcomes).
  - Maximax Criterion (Profit Matrix)
  - Minimin Criterion (Cost Matrix)

Numerical-1

PESSIMISM (MAXIMIN OR MINIMAX) CRITERION

- In this criterion the decision-maker ensures that he would earn no less (or pay no more) than some specified amount. Thus, he selects the decision alternative that represents the maximum of the minima (or minimum of the maxima in case of loss) payoff in case of profits.
  - Maximin Criterion (Profit matrix)
  - Minimax Criterion (Cost Matrix)

Numerical-1

MINIMAX REGRET (SAVAGE) CRITERION

- This criterion is also known as opportunity loss decision criterion or minimax regret decision criterion because decision-maker regrets for choosing wrong decision alternative resulting in an opportunity loss of payoff. The working method is summarized as follows:

  (a) From the given payoff matrix, develop an opportunity-loss (or regret) matrix as follows:

  (i) Find the best payoff corresponding to each state of nature

  (ii) Subtract all other payoff values in that column from this value.

  (b) For each decision alternative identify the worst (or maximum regret) payoff value. Record this value in the new row.

  (c) Select a decision alternative resulting in a smallest anticipated opportunity-loss value.

MINIMAX REGRET (SAVAGE) CRITERION

- Maximization Problem (Profit Matrix)
- Minimization Problem (Cost Matrix)

Numerical-1

COEFFICIENT OF OPTIMISM (HURWICZ) CRITERION

- This criterion suggests that a decision-maker should be neither completely optimistic nor pessimistic and, therefore, must display a mixture of both.
- Hurwicz, who suggested this criterion, introduced the idea of a coefficient of optimism (denoted by α) to measure the decision-maker's degree of optimism.
- This coefficient lies between 0 and 1, where 0 represents a complete pessimistic attitude about the future and 1 a complete optimistic attitude about the future.
  - Maximization Problem (Profit Matrix)
  - Minimization Problem (Cost Matrix)

Numerical-1

NUMERICAL (LAPLACE/BAYES' CRITERION)

1. The following matrix gives the payoff of different strategies (alternatives) S1, S2, S3 & S4 against conditions (Events) N1, N2, N3 & N4:

Determine the best strategy using Laplace Criterion.

Events

Strategies		N1	N2	N3	N4
	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000

NUMERICAL (LAPLACE/BAYES' CRITERION)

Solution (1). Calculate Expected Payoff (E.P)

Events

Strategies		N1	N2	N3	N4	E.P
	S1	1000	1500	750	0	812.50
	S2	250	2000	3750	3000	2250
	S3	-500	1250	3000	4750	2125
	S4	-1250	500	2250	4000	1375

S1= ¼ [1000+1500+750+0] = 812.50

S2= ¼ [250+2000+3750+3000] = 2250

S3= ¼ [-500+1250+3000+4750] = 2125

S4= ¼ [-1250+500+2250+4000] = 1375

NUMERICAL (LAPLACE/BAYES' CRITERION)

Solution (1). The maximum Payoff is for Strategy S2

Events

		N1	N2	N3	N4	E.P
Strategies	S1	1000	1500	750	0	812.50
	S2	250	2000	3750	3000	2250
	S3	-500	1250	3000	4750	2125
	S4	-1250	500	2250	4000	1375

Since Strategy S2 gives maximum payoff it is the best strategy among available strategies.

NUMERICAL (CRITERION OF OPTIMISM)

2. The following matrix gives the payoff of different strategies (alternatives) S1, S2, S3 & S4 against conditions (Events) N1, N2, N3 & N4:

Determine the best strategy using (a) Maximax Criterion (b) Minimin Criterion (consider the payoff matrix to be a cost matrix)

Events

		N1	N2	N3	N4
Strategies	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000

NUMERICAL (CRITERION OF OPTIMISM)

Solution (2). (a) **Maximax Criterion**

Identify the Maximum Payoff for every Strategy.

Events

		N1	N2	N3	N4	Maximum
Strategies	S1	1000	1500	750	0	1500
	S2	250	2000	3750	3000	3750
	S3	-500	1250	3000	4750	4750
	S4	-1250	500	2250	4000	4000

NUMERICAL (CRITERION OF OPTIMISM)

Solution (2). (a) **Maximax Criterion**

Identify the Maximum of Maximum Payoff of all the Strategies.

Events

		N1	N2	N3	N4	Maximum
Strategies	S1	1000	1500	750	0	1500
	S2	250	2000	3750	3000	3750
	S3	-500	1250	3000	4750	4750
	S4	-1250	500	2250	4000	4000

By Maximax Criterion Strategy S3 is the best selection

NUMERICAL (CRITERION OF OPTIMISM)

Solution (2). (b) **Minimin Criterion**

Identify the Minimum Payoff (cost) for each Strategy.

Events

		N1	N2	N3	N4	Minimum
Strategies	S1	1000	1500	750	0	0
	S2	250	2000	3750	3000	250
	S3	-500	1250	3000	4750	-500
	S4	-1250	500	2250	4000	-1250

NUMERICAL (CRITERION OF OPTIMISM)

Solution (2). (b) **Minimin Criterion**

Identify the Minimum of Minimum Payoff of all the Strategies.

	N1	N2	N3	N4	Minimum
S1	1000	1500	750	0	0
S2	250	2000	3750	3000	250
S3	-500	1250	3000	4750	-500
S4	-1250	500	2250	4000	-1250

By Minimin Criterion Strategy S4 is best selection.

NUMERICAL (CRITERION OF PESSIMISM)

3. The following matrix gives the payoff of different strategies (alternatives) S1, S2, S3 & S4 against conditions (Events) N1, N2, N3 & N4:

Determine the best strategy using (a) Maximin Criterion (b) Minimax Criterion (consider the payoff matrix to be a cost matrix)

Events

		N1	N2	N3	N4
Strategies	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000

NUMERICAL (CRITERION OF PESSIMISM)

Solution (3). (a) **Maximin Criterion**

Identify the Minimum Payoff for each Strategy.

Events

		N1	N2	N3	N4	Minimum
Strategies	S1	1000	1500	750	0	0
	S2	250	2000	3750	3000	250
	S3	-500	1250	3000	4750	-500
	S4	-1250	500	2250	4000	-1250

NUMERICAL (CRITERION OF PESSIMISM)

Solution (3). (a) **Maximin Criterion**

Identify the Maximum of Minimum Payoff of all the Strategies.

Events

		N1	N2	N3	N4	Minimum
Strategies	S1	1000	1500	750	0	0
	S2	250	2000	3750	3000	250
	S3	-500	1250	3000	4750	-500
	S4	-1250	500	2250	4000	-1250

By Maximin Criterion Strategy S2 is the best option

NUMERICAL (CRITERION OF PESSIMISM)

Solution (3). (b) **Minimax Criterion**

Identify the Maximum Payoff (cost) for each Strategy.

Events

Strategies		N1	N2	N3	N4	Maximum
	S1	1000	1500	750	0	1500
	S2	250	2000	3750	3000	3750
	S3	-500	1250	3000	4750	4750
	S4	-1250	500	2250	4000	4000

NUMERICAL (CRITERION OF PESSIMISM)

Solution (3). (b) **Minimax Criterion**

Identify the Minimum of Maximum Payoff of all the Strategies.

Events

Strategies		N1	N2	N3	N4	Maximum
	S1	1000	1500	750	0	1500
	S2	250	2000	3750	3000	3750
	S3	-500	1250	3000	4750	4750
	S4	-1250	500	2250	4000	4000

By Minimax Criterion Strategy S1 is the best option

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

4. The following matrix gives the payoff of different strategies (alternatives) S1, S2, S3 & S4 against conditions (Events) N1, N2, N3 & N4:
Determine the best strategy considering
(a) Maximization Problem (b) Minimization Problem (consider the payoff matrix to be a cost matrix)

Events

Strategies		N1	N2	N3	N4
	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (a) **Maximization Problem**

Draw the Regret Table (Maximum Payoff – Payoff) for every Event (Column)

Events

Strategies		N1	N2	N3	N4
	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000
	Column Maximum	**1000**	**2000**	**3750**	**4750**

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (a) **Maximization Problem**

Draw the Regret Table (Maximum Payoff – Payoff) for every Event (Column)

Events

Strategies		N1	N2	N3	N4
	S1	0	500	3000	4750
	S2	750	0	0	1750
	S3	1500	750	750	0
	S4	2250	1500	1500	750

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (a) **Maximization Problem**

Identify the Maximum Payoff for each Strategy.

Events

Strategies		N1	N2	N3	N4	Maximum
	S1	0	500	3000	4750	4750
	S2	750	0	0	1750	1750
	S3	1500	750	750	0	1500
	S4	2250	1500	1500	750	2250

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (a) **Maximization Problem**

Identify the Minimum of Maximum Payoff of all the Strategies.

Events

Strategies		N1	N2	N3	N4	Maximum
	S1	0	500	3000	4750	4750
	S2	750	0	0	1750	1750
	S3	1500	750	750	0	1500
	S4	2250	1500	1500	750	2250

For Maximization Problem by Minimax Regret Criterion the Strategy S3 is the best Option.

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (b) **Minimization Problem**

Draw the Regret Table (Cost – Minimum Cost) for every Event (Column)

Events

Strategies		N1	N2	N3	N4
	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000
	Column Minimum	**-1250**	**500**	**750**	**0**

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (b) **Minimization Problem**

Draw the Regret Table (Cost – Minimum Cost) for every Event (Column)

Events

Strategies		N1	N2	N3	N4
	S1	2250	1000	0	0
	S2	1500	1500	3000	3000
	S3	750	750	2250	4750
	S4	0	0	1500	4000

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (b) **Minimization Problem**

Identify the Maximum Payoff for each Strategy.

Events

Strategies		N1	N2	N3	N4	Maximum
	S1	2250	1000	0	0	2250
	S2	1500	1500	3000	3000	3000
	S3	750	750	2250	4750	4750
	S4	0	0	1500	4000	4000

NUMERICAL (MINIMAX REGRET / SAVAGE CRITERION)

Solution (4). (b) **Minimization Problem**

Identify the Minimum of Maximum Payoff of all the Strategies.

Events

Strategies		N1	N2	N3	N4	Maximum
	S1	2250	1000	0	0	2250
	S2	1500	1500	3000	3000	3000
	S3	750	750	2250	4750	4750
	S4	0	0	1500	4000	4000

For Minimization Problem by Minimax Regret Criterion the Strategy S1 is the best Option.

NUMERICAL (HURWICZ CRITERION)

5. The following matrix gives the payoff of different strategies (alternatives) S1, S2, S3 & S4 against conditions (Events) N1, N2, N3 & N4:

Determine the best strategy considering

(a) Maximization Problem (α = 0.6)

(b) Minimization Problem (α = 0.5)

Events

Strategies		N1	N2	N3	N4
	S1	1000	1500	750	0
	S2	250	2000	3750	3000
	S3	-500	1250	3000	4750
	S4	-1250	500	2250	4000

NUMERICAL (HURWICZ CRITERION)

Solution (5). (a) Maximization Problem (α = 0.6)

Wo = α x Max + (1 – α) x Min

Events

Strategies		N1	N2	N3	N4	Max	Min	Wo
	S1	1000	1500	750	0	1500	0	**900**
	S2	250	2000	3750	3000	3750	250	**2350**
	S3	-500	1250	3000	4750	4750	-500	**2650**
	S4	-1250	500	2250	4000	4000	-1250	**1900**

Wo (S1) = 0.6 x 1500 +(1- 0.6) x 0 = 900

Wo (S2) = 0.6 x 3750 +(1- 0.6) x 250 = 2350

Wo (S3) = 0.6 x 4750 +(1- 0.6) x -500 = 2650

Wo (S4) = 0.6 x 4000 +(1- 0.6) x -1250 = 1900

NUMERICAL (HURWICZ CRITERION)

Solution (5). (a) Maximization Problem (α = 0.6)

For a Maximization Problem consider the Maximum Wo value

Events

Strategies		N1	N2	N3	N4	Max	Min	Wo
	S1	1000	1500	750	0	1500	0	900
	S2	250	2000	3750	3000	3750	250	2350
	S3	-500	1250	3000	4750	4750	-500	**2650**
	S4	-1250	500	2250	4000	4000	-1250	1900

By Hurwicz Criterion the Strategy S3 is the best Option

NUMERICAL (HURWICZ CRITERION)

Solution (5). (b) Minimization Problem (α = 0.5)

Wo = α x Min + (1 – α) x Max

Events

Strategies		N1	N2	N3	N4	Min	Max	Wo
	S1	1000	1500	750	0	0	1500	750
	S2	250	2000	3750	3000	250	3750	2000
	S3	-500	1250	3000	4750	-500	4750	2125
	S4	-1250	500	2250	4000	-1250	4000	1375

Wo (S1) = 0.5 x 0 +(1- 0.5) x 1500 = 750

Wo (S2) = 0.5 x 250 +(1- 0.5) x 3750 = 2000

Wo (S3) = 0.5 x -500 +(1- 0.5) x 4750 = 2125

Wo (S4) = 0.5 x -1250 +(1- 0.5) x 4000 = 1375

NUMERICAL (HURWICZ CRITERION)

Solution (5). (b) Minimization Problem (α = 0.5)

For a Minimization problem consider the Minimum of Wo value.

Events

Strategies		N1	N2	N3	N4	Min	Max	Wo
	S1	1000	1500	750	0	0	1500	**750**
	S2	250	2000	3750	3000	250	3750	2000
	S3	-500	1250	3000	4750	-500	4750	2125
	S4	-1250	500	2250	4000	-1250	4000	1375

By Hurwicz Criterion the Strategy S1 is the best Option

NUMERICAL

6. A food products' company is contemplating the introduction of a revolutionary new product with new packaging or replacing the existing product at much higher price (S1). It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price (S2), or may mall a small change in the composition of the existing product, backing it with the word 'New' and a negligible increase in price (S3). The three possible states of nature or events are: (i) high increase in sales (N1), (ii) no change in sales (N2) and (iii) decrease in sales (N3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table: Which strategy should the concerned executive choose on the basis of

(a) **Maximin criterion** (b) **Maximax criterion** (c) **Minimax regret criterion** (d) **Laplace criterion**

States of Nature

Strategies		N1	N2	N3
	S1	7,00,000	3,00,000	1,50,000
	S2	5,00,000	4,50,000	0
	S3	3,00,000	3,00,000	3,00,000

NUMERICAL

Solution (6).a. **Maximin Criterion**

Identify the Maximum of Minimum Payoff of all the Strategies.

States of Nature

Strategies		N1	N2	N3	Minimum
	S1	7,00,000	3,00,000	1,50,000	1,50,000
	S2	5,00,000	4,50,000	0	0
	S3	3,00,000	3,00,000	3,00,000	3,00,000

NUMERICAL

Solution (6).a. **Maximin Criterion**

Identify the Maximum of Minimum Payoff of all the Strategies.

States of Nature

		N1	N2	N3	Minimum
Strategies	S1	7,00,000	3,00,000	1,50,000	1,50,000
	S2	5,00,000	4,50,000	0	0
	S3	3,00,000	3,00,000	3,00,000	**3,00,000**

By Maximin Criterion the company should adopt Strategy S3.

NUMERICAL

Solution (6).b. **Maximax Criterion**

Identify the Maximum of Maximum Payoff of all the Strategies.

States of Nature

		N1	N2	N3	Maximum
Strategies	S1	7,00,000	3,00,000	1,50,000	7,00,000
	S2	5,00,000	4,50,000	0	5,00,000
	S3	3,00,000	3,00,000	3,00,000	3,00,000

NUMERICAL

Solution (6).b. **Maximax Criterion**

Identify the Maximum of Maximum Payoff of all the Strategies.

States of Nature

		N1	N2	N3	Maximum
Strategies	S1	7,00,000	3,00,000	1,50,000	**7,00,000**
	S2	5,00,000	4,50,000	0	5,00,000
	S3	3,00,000	3,00,000	3,00,000	3,00,000

By Maximax Criterion the company should adopt Strategy S1.

NUMERICAL

Solution (6).c. **Minimax Regret Criterion**

Draw the Regret Table (Maximum Payoff – Payoff) for every Event (Column)

States of Nature

		N1	N2	N3
Strategies	S1	7,00,000 – 7,00,000 = 0	4,50,000 - 3,00,000 = 1,50,000	3,00,000 - 1,50,000 = 1,50,000
	S2	7,00,000 - 5,00,000 = 2,00,000	4,50,000 - 4,50,000 = 0	3,00,000 – 0 =3,00,000
	S3	7,00,000 - 3,00,000 = 4,00,000	4,50,000 - 3,00,000 = 1,50,000	3,00,000 - 3,00,000 = 0

NUMERICAL

Solution (6).c. **Minimax Regret Criterion**

Identify the Minimum of Maximum Payoff of all the Strategies.

States of Nature

		N1	N2	N3	Maximum
Strategies	S1	0	1,50,000	1,50,000	1,50,000
	S2	2,00,000	0	3,00,000	3,00,000
	S3	4,00,000	1,50,000	0	4,00,000

NUMERICAL

Solution (6).c. **Minimax Regret Criterion**

Identify the Minimum of Maximum Payoff of all the Strategies.

States of Nature

		N1	N2	N3	Maximum
Strategies	S1	0	1,50,000	1,50,000	**1,50,000**
	S2	2,00,000	0	3,00,000	3,00,000
	S3	4,00,000	1,50,000	0	4,00,000

By Minimax Regret Criterion the company should adopt Strategy S1.

NUMERICAL

Solution (6).d. **Laplace Criterion**

Assuming that each state of nature has a probability 1/3 of occurrence.

States of Nature

		N1	N2	N3	E.P
Strategies	S1	7,00,000	3,00,000	1,50,000	3,83,333.33
	S2	5,00,000	4,50,000	0	3,16,666.66
	S3	3,00,000	3,00,000	3,00,000	3,00,000

NUMERICAL

Solution (6).d. **Laplace Criterion**

Assuming that each state of nature has a probability 1/3 of occurrence.

States of Nature

		N1	N2	N3	E.P
Strategies	S1	7,00,000	3,00,000	1,50,000	**3,83,333.33**
	S2	5,00,000	4,50,000	0	3,16,666.66
	S3	3,00,000	3,00,000	3,00,000	3,00,000

By Laplace Criterion the company should adopt Strategy S1.

NUMERICAL

7. A manufacturer manufactures a product, of which the principal ingredient is a chemical X. At the moment, the manufacturer spends Rs 1,000 per year on supply of X, but there is a possibility that the price may soon increase to four times its present figure because of a worldwide shortage of the chemical. There is another chemical Y, which the manufacturer could use in conjunction with a third chemical Z, in order to give the same effect as chemical X. Chemicals Y and Z would together cost the manufacturer Rs 3,000 per year, but their prices are unlikely to rise. What action should the manufacturer take?

Apply the Maximin and Minimax Regret criteria for decision-making and give two sets of solutions. If the coefficient of optimism is 0.4, then find the course of action that minimizes the cost.

NUMERICAL

Solution (7). The data of the problem is summarized in the following table (negative figures in the table represents profit).

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)
	S1 (Use Y and Z)	-3000	-3000
	S2 (Use of X)	-4000	-1000

NUMERICAL

Solution (7). **Maximin Criterion**

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)	Minimum
	S1 (Use Y and Z)	-3000	-3000	-3000
	S2 (Use of X)	-4000	-1000	-4000

NUMERICAL

Solution (7). **Maximin Criterion**

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)	Minimum
	S1 (Use Y and Z)	-3000	-3000	**-3000**
	S2 (Use of X)	-4000	-1000	-4000

By Maximin Criterion the manufacturer should adopt Strategy S1.

NUMERICAL

Solution (7). **Minimax Regret Criterion**

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)
	S1 (Use Y and Z)	-3000 – (-3000) = 0	-1000 – (-3000) = 2000
	S2 (Use of X)	-3000- (-4000) = 1000	-1000 – (-1000) = 0

NUMERICAL

Solution (7). **Minimax Criterion**

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)	Maximum
	S1 (Use Y and Z)	0	2000	2000
	S2 (Use of X)	1000	0	1000

NUMERICAL

Solution (7). **Minimax Criterion**

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)	Maximum
	S1 (Use Y and Z)	0	2000	2000
	S2 (Use of X)	1000	0	**1000**

By Minimax Criterion the manufacturer should adopt Strategy S2.

NUMERICAL

Solution (7). **Hurwicz Criterion**

Given the coefficient of optimism equal to 0.4, the coefficient of pessimism will be 1 – 0.4 = 0.6.

Wo = α x Max + (1 – α) x Min

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)	Max	Min	Wo
	S1 (Use Y and Z)	-3000	-3000	-3000	-3000	-3000
	S2 (Use of X)	-4000	-1000	-1000	-4000	-2800

Wo (S1) = 0.4 x -3000 +(1- 0.4) x -3000 = -3000 Wo (S2) = 0.4 x -1000 +(1- 0.4) x -4000 = -2800

NUMERICAL

Solution (7). **Hurwicz Criterion**

Given the coefficient of optimism equal to 0.4, the coefficient of pessimism will be 1 – 0.4 = 0.6.

Wo = α x Max + (1 – α) x Min

States of Nature

Strategies		N1 (Price of X increases)	N2 (Price of X does not increase)	Max	Min	Wo
	S1 (Use Y and Z)	-3000	-3000	-3000	-3000	-3000
	S2 (Use of X)	-4000	-1000	-1000	-4000	**-2800**

By Hurwicz Criterion the manufacturer should adopt Strategy S2.

NUMERICAL

8. An investor is given the following investment alternatives and percentage rates of return.

States of Nature

Strategies		Low	Medium	High
	Regular Shares	7%	12%	15%
	Risky Shares	-10%	12%	25%
	Property	-12%	18%	30%

Over the past 300 days, 150 days have been medium market conditions and 60 days have had high market increases. On the basis of these data, state the optimum investment strategy for the investment.

NUMERICAL

Solution (8). According to the given information, the probabilities of low, medium and high market conditions would be 90/300 or 0.30, 150/300 or 0.50 and 60/300 or 0.20, respectively. The expected pay-offs for each of the alternatives are shown below:

States of Nature

Strategies		Low (0.3)	Medium (0.5)	High (0.2)
	Regular Shares	0.07 x 0.3	0.12 x 0.5	0.15 x 0.2
	Risky Shares	-0.10 x 0.3	0.12 x 0.5	0.25 x 0.2
	Property	-0.12 x 0.3	0.18 x 0.5	0.30 x 0.2

NUMERICAL

Solution (8). Calculate the Expected payoff for all the strategies.

States of Nature

Strategies		Low (0.3)	Medium (0.5)	High (0.2)	Expected Returns
	Regular Shares	0.07 x 0.3	0.12 x 0.5	0.15 x 0.2	0.136
	Risky Shares	-0.10 x 0.3	0.12 x 0.5	0.25 x 0.2	0.126
	Property	-0.12 x 0.3	0.18 x 0.5	0.30 x 0.2	0.230

E.R = 1/3 x [(0.07 x 0.3) + (0.12 x 0.5) + (0.15 x 0.2)] = 0.136

E.R = 1/3 x [(-0.10 x 0.3) + (0.12 x 0.5) + (0.25 x 0.2)] = 0.126

E.R = 1/3 x [(-0.12 x 0.3) + (0.18 x 0.5) + (0.30 x 0.2)] = 0.230

NUMERICAL

Solution (8).

States of Nature

Strategies		Low (0.3)	Medium (0.5)	High (0.2)	Expected Returns
	Regular Shares	0.07 x 0.3	0.12 x 0.5	0.15 x 0.2	0.136
	Risky Shares	-0.10 x 0.3	0.12 x 0.5	0.25 x 0.2	0.126
	Property	-0.12 x 0.3	0.18 x 0.5	0.30 x 0.2	**0.230**

Since the expected return of 23 per cent is the highest for property, the investor should invest in this alternative.

Decision Theory

OPERATIONS RESEARCH-I

Prof. Aditya Suresh Kasar

Module - 3

DECISION-MAKING UNDER RISK

In this decision-making environment, decision-maker has sufficient information to assign probability to the likely occurrence of each outcome (state of nature).

- Expected Monetary Value (EMV)
- Expected Opportunity Loss (EOL)
- Expected Value of Perfect Information (EVPI)

EXPECTED MONETARY VALUE (EMV)

- The Expected Monetary Value (EMV) for a given course of action is obtained by adding payoff values multiplied by the probabilities associated with each state of nature.
- Mathematically, EMV is stated as follows:

EMV (Course of action, S_j) = $\sum_i^m p_{ij} \, Pi$

Where,

m = number of possible states of nature

p_i = probability of occurrence of state of nature, Ni

p_{ij} = payoff associated with state of nature Ni and course of action, S_j

EXPECTED MONETARY VALUE (EMV)

The Procedure

1. Construct a payoff matrix listing all possible courses of action and states of nature. Enter the conditional payoff values associated with each possible combination of course of action and state of nature along with the probabilities of the occurrence of each state of nature.
2. Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and adding these weighted values for each course of action.
3. Select the course of action that yields the optimal EMV.

- Numerical-1

EXPECTED OPPORTUNITY LOSS (EOL)

- Expected opportunity loss (EOL), also called expected value of regret, is an alternative decision criterion for decision making under risk.
- The EOL is defined as the difference between the highest profit (or payoff) and the actual profit due to choosing a particular course of action in a particular state of nature.
- A course of action resulting to the minimum EOL is preferred.
- Mathematically, EOL is stated as follows:

EOL (State of Nature, N_i) = $\sum_i^m l_{ij} \, Pi$

Where, l_{ij} = opportunity loss due to state of nature, N_i and course of action, S_j

P_i = probability of occurrence of state of nature, Ni

EXPECTED OPPORTUNITY LOSS (EOL)

The Procedure

1. Prepare a conditional payoff values matrix for each combination of course of action and state of nature along with the associated probabilities.
2. For each ***state of nature*** calculate the ***conditional opportunity loss (COL)*** values by subtracting each payoff from the maximum payoff.
3. Calculate the ***EOL for each course of action*** by multiplying the probability of each state of nature with the COL value and then adding the values.
4. Select a course of action for which the EOL is minimum.

- Numerical-1

NUMERICAL (EMV)

9. Mr. X flies quite often from town A to town B. He can use the airport bus which costs ₹ 25 but if he takes it, there is a 0.08 chance that he will miss the flight. The stay in a hotel costs ₹ 270 with a 0.96 chance of being on time for the flight. For ₹ 350 he can use a taxi which will make 99 per cent chance of being on time for the flight.

If Mr. X catches the plane on time, he will conclude a business transaction that will produce a profit of ₹ 10,000, otherwise he will lose it.

Which mode of transport should Mr. X use? Answer on the basis of the EMV criterion.

NUMERICAL (EMV)

Solution (9).

State of Nature	Course of Action								
	Bus			Stay in Hotel			Taxi		
	Cost (a)	Prob. (b)	Expected Value (a x b)	Cost (a)	Prob. (b)	Expected Value (a x b)	Cost (a)	Prob. (b)	Expected Value (a x b)
Catches Flight	10000 - 25 = 9975	0.92	9177	10000 - 270 = 9730	0.96	9340.80	10000 - 350 = 9650	0.99	9553.50
Misses Flight	- 25	0.08	-2	- 270	0.04	-10.80	- 350	0.01	-3.50
		EMV:	9175		**EMV:**	9330		**EMV:**	9550

NUMERICAL (EMV)

Solution (9). Since EMV associated with course of action 'Taxi' is largest (= Rs 9,550), it is the logical alternative.

State of Nature	Course of Action								
	Bus			Stay in Hotel			Taxi		
	Cost	Prob.	Expected Value	Cost	Prob.	Expected Value	Cost	Prob.	Expected Value
Catches Flight	10000-25 = 9975	0.92	9177	10000-270 = 9730	0.96	9340.80	10000-350 = 9650	0.99	9553.50
Misses Flight	-25	0.08	-2	-270	0.04	-10.80	-350	0.01	-3.50
		EMV:	9175		**EMV:**	9330		**EMV:**	**9550**

NUMERICAL (EMV)

10. The manager of a flower shop promises its customers delivery within four hours on all flower orders. All flowers are purchased on the previous day and delivered to Parker by 8.00 am the next morning. The daily demand for roses is as follows.

Dozens of roses : 70 80 90 100

Probability : 0.1 0.2 0.4 0.3

The manager purchases roses for Rs 10 per dozen and sells them for Rs 30. All unsold roses are donated to a local hospital. How many dozens of roses should Parker order each evening to maximize its profits? What is the optimum expected profit?

NUMERICAL (EMV)

Solution (10). The quantity of roses to be purchased per day is considered as 'course of action' and the daily demand of the roses is considered as a 'state of nature' because demand is uncertain with known probability. From the data, it is clear that the flower shop must not purchase less than 70 or more than 100 dozen roses, per day. Also each dozen roses sold within a day yields a profit of Rs (30 – 10) = Rs 20 and otherwise it is a loss of Rs 10. Thus

Marginal profit (MP) = Selling price – Cost = 30 – 10 = Rs 20

Marginal loss (ML) = Loss on unsold roses = Rs 10

Conditional profit = MP × Roses sold – ML × Roses not sold

= 20 D if D ≥ S

= 20 D – 10(S – D) if D < S

where D = number of roses sold within a day and S = number of roses stocked.

NUMERICAL (EMV)

Solution (10). Conditional profit = MP × Roses sold – ML × Roses not sold
= 20 D if D ≥ S
= 20 D – 10(S – D) if D < S

State of Nature (Demand per Day)	Prob. (p)	Conditional Profit (Rs) due to Courses of Action (Purchase per Day) C.P				Expected Payoff (Rs) due to Courses of Action (Purchase per Day) E.P = p x C.P			
		70	80	90	100	70	80	90	100
70	0.1	1400	1300	1200	1100	140	130	120	110
80	0.2	1400	1600	1500	1400	280	320	300	280
90	0.4	1400	1600	1800	1700	560	640	720	680
100	0.3	1400	1600	1800	2000	420	480	540	600
					EMV:	1400	1570	1680	1670

NUMERICAL (EMV)

Solution (10). Since the highest EMV of Rs 1680 corresponds to the course of action 90, the flower shop should purchase ninety dozen roses everyday.

State of Nature (Demand per Day)	Prob. (p)	Conditional Profit (Rs) due to Courses of Action (Purchase per Day) C.P				Expected Payoff (Rs) due to Courses of Action (Purchase per Day) E.P = p x C.P			
		70	80	90	100	70	80	90	100
70	0.1	1400	1300	1200	1100	140	130	120	110
80	0.2	1400	1600	1500	1400	280	320	300	280
90	0.4	1400	1600	1800	1700	560	640	720	680
100	0.3	1400	1600	1800	2000	420	480	540	600
					EMV:	1400	1570	**1680**	1670

NUMERICAL (EMV)

11. A retailer purchases cherries every morning at Rs 50 a case and sells them for Rs 80 a case. Any case that remains unsold at the end of the day can be disposed of the next day at a salvage value of Rs 20 per case (thereafter they have no value). Past sales have ranged from 15 to 18 cases per day. The following is the record of sales for the past 120 days.

Cases sold : 15 16 17 18

Number of days : 12 24 48 36

Find out how many cases should the retailer purchase per day in order to maximize his profit.

NUMERICAL (EMV)

Solution (11).

Let Ni (i = 1, 2, 3, 4) be the possible states of nature (daily likely demand) and Sj (j = 1, 2, 3, 4) be all possible courses of action (number of cases of cherries to be purchased).

Marginal profit (MP) = Selling price – Cost = Rs (80 – 50) = Rs 30

Marginal loss (ML) = Loss on unsold cases = Rs (50 – 20) = Rs 30

Conditional profit = MP × Cases sold – ML × Cases unsold

= 30S if S ≥ N

= (80 – 50) S – 30(N – S) = 60S – 30N if S < N

NUMERICAL (EMV)

Solution (11). Conditional profit = MP × Cases sold – ML × Cases unsold
= 30S if S ≥ N
= (80 – 50) S – 30(N – S) = 60S – 30N if S < N

State of Nature (Demand per Day)	Prob. (p)	Conditional Profit (Rs) due to Courses of Action (Purchase per Day) C.P				Expected Payoff (Rs) due to Courses of Action (Purchase per Day) E.P = p x C.P			
		15	16	17	18	15	16	17	18
15	0.1	450	420	390	360	45	42	39	36
16	0.2	450	480	450	420	90	96	90	84
17	0.4	450	480	510	480	180	192	204	192
18	0.3	450	480	510	540	135	144	153	162
					EMV:	450	474	486	474

NUMERICAL (EMV)

Solution (11). Since the highest EMV of Rs 486 corresponds to the course of action 17, the retailer must purchase 17 cases of cherries every morning.

State of Nature (Demand per Day)	Prob. (p)	Conditional Profit (Rs) due to Courses of Action (Purchase per Day) C.P				Expected Payoff (Rs) due to Courses of Action (Purchase per Day) E.P = p x C.P			
		15	16	17	18	15	16	17	18
15	0.1	450	420	390	360	45	42	39	36
16	0.2	450	480	450	420	90	96	90	84
17	0.4	450	480	510	480	180	192	204	192
18	0.3	450	480	510	540	135	144	153	162
					EMV:	450	474	**486**	474

NUMERICAL (EMV)

12. The probability of demand for hiring cars on any day in a given city is as follows:

No. of cars demanded : 0 1 2 3 4

Probability : 0.1 0.2 0.3 0.2 0.2

Cars have a fixed cost of Rs 90 each day to keep the daily hire charges (variable costs of running) Rs 200. If the car-hire company owns **4 cars**, what is its daily expectation?

If the company is about to go into business and currently has no car, ***how many cars should it buy***?

NUMERICAL (EMV)

Solution (12).

Given that Rs 90 is the fixed cost and Rs 200 is variable cost.

The payoff values with 4 cars at the disposal of decision-maker are calculated as under:

Payoff = (No. of cars demanded x Variable Cost) – (No. of cars at disposal x Fixed Cost)

No. of cars demanded	0	1	2	3	4
Payoff (with 4 cars)	(0 x 200) – (90 x 4) = -360	(1 x 200) – (90 x 4) = -160	(2 x 200) – (90 x 4) = 40	(3 x 200) – (90 x 4) = 240	(4 x 200) – (90 x 4) = 240

NUMERICAL (EMV)

Solution (12). Thus, the daily expectation is obtained by multiplying the payoff values with the given corresponding probabilities of demand:

Daily Expectation = (– 360)(0.1) + (– 160)(0.2) + (40)(0.3) + (240)(0.2) + (440)(0.2) = Rs 80

No. of cars demanded	0	1	2	3	4
Probability	0.1	0.2	0.3	0.2	0.2
Payoff (with 4 cars)	-360	-160	40	240	440

NUMERICAL (EMV)

Solution (12).

State of Nature (Demand per Day)	Prob. (p)	Conditional Profit (Rs) due to Courses of Action (Purchase Car) C.P					Expected Payoff (Rs) due to Courses of Action (Purchase Car) E.P = p x C.P				
		0	1	2	3	4	0	1	2	3	4
0	0.1	0	-90	-180	-270	-360	0	-9	-18	-27	-36
1	0.2	0	110	20	-70	-160	0	22	4	-14	-32
2	0.3	0	110	220	130	40	0	33	66	39	12
3	0.2	0	110	220	330	240	0	22	44	66	48
4	0.2	0	110	220	330	440	0	22	40	66	88
						EMV:	0	90	140	130	80

NUMERICAL (EMV)

Solution (12). Since the EMV of Rs 140 for the course of action 2 is the highest, the company should buy 2 cars.

State of Nature (Demand per Day)	Prob.	Conditional Profit (Rs) due to Courses of Action (Purchase Car)					Expected Payoff (Rs) due to Courses of Action (Purchase Car)				
		0	1	2	3	4	0	1	2	3	4
0	0.1	0	-90	-180	-270	-360	0	-9	-18	-27	-36
1	0.2	0	110	20	-70	-160	0	22	4	-14	-32
2	0.3	0	110	220	130	40	0	33	66	39	12
3	0.2	0	110	220	330	240	0	22	44	66	48
4	0.2	0	110	200	330	440	0	22	40	66	88
						EMV:	0	90	**140**	130	80

NUMERICAL (EOL)

13. A company manufactures goods for a market in which the technology of the product is changing rapidly. The research and development department has produced a new product that appears to have potential for commercial exploitation. A further Rs 60,000 is required for development testing.

The company has 100 customers and each customer might purchase, at the most, one unit of the product. Market research suggests that a selling price of Rs 6,000 for each unit, with total variable costs of manufacturing and selling estimate as Rs 2,000 for each unit.

From previous experience, it has been possible to derive a probability distribution relating to the proportion of customers who will buy the product as follows:

Proportion of customers : 0.04 0.08 0.12 0.16 0.20

Probability : 0.10 0.10 0.20 0.40 0.20

Determine the ***Expected Opportunity Losses***, given no other information than that stated above, and state whether or not the company should develop the product.

NUMERICAL (EOL)

Solution (13). If p is the proportion of customers who purchase the new product, the company's conditional profit is:

(6,000 – 2,000) × 100 p – 60,000 = Rs (4,00,000 p – 60,000).

Let ***Ni (i = 1, 2, . . ., 5)*** be the possible ***states of nature***, i.e. proportion of the customers who will buy the new product and ***S1 (develop the product)*** and ***S2 (do not develop the product)*** be the two ***courses of action***.

NUMERICAL (EOL)

Solution (13). The conditional profit values (payoffs) for each pair of N_js and S_j s are shown

Proportion of Customers (p) (State of Nature)	Prob.	Conditional Profit = Rs (4,00,000 p – 60,000) Course of Action		Opportunity Loss (O.L) Max payoff - payoff		Prob. x O.L	
		S1 (Develop)	S2 (Do not Develop)	S1	S2	S1	S2
0.04	0.1	-44000	0	44000	0	4400	0
0.08	0.1	-28000	0	28000	0	2800	0
0.12	0.2	-12000	0	12000	0	2400	0
0.16	0.4	4000	0	0	4000	0	1600
0.20	0.2	20000	0	0	20000	0	4000
					EOL:	9600	5600

NUMERICAL (EOL)

Solution (13). Since the company seeks to minimize the expected opportunity loss, the company should select course of action S2 (do not develop the product) with minimum EOL.

Proportion of Customers (State of Nature)	Prob.	Conditional Profit = Rs (4,00,000 p – 60,000) Course of Action		Opportunity Loss (O.L) Max payoff - payoff		Prob. x O.L	
		S1 (Develop)	S2 (Do not Develop)	S1	S2	S1	S2
0.04	0.1	-44000	0	44000	0	4400	0
0.08	0.1	-28000	0	28000	0	2800	0
0.12	0.2	-12000	0	12000	0	2400	0
0.16	0.4	4000	0	0	4000	0	1600
0.20	0.2	20000	0	0	20000	0	4000
					EOL:	9600	**5600**

NUMERICAL (EVPI)

14. XYZ Inc. is a small mining company that purchased tract of land, larger mining company thought not worth their time and money. A consulting geologist has determined that there is 20% chance the tracts contain a profitable gold deposit.

Mining the location will require an investment of ₹ 80,000. If there is no gold, all the ₹ 80,000 would be lost. However if the gold is present, the geologist estimates enough to generate ₹ 600,000 in profit.

Finally a larger mining company has learned of the report and is willing to buy the land outright; generating an immediate ₹ 75,000 profit and eliminating the risk of finding no gold.

What should XYZ Inc. do?

NUMERICAL (EVPI)

Alternatives (Course of Action):

- The options available to the decision maker. Oftentimes DO NOTHING is a viable option. (Mine Land, Sell Land)

States of Nature:

- Random factors outside the control of the decision maker that nonetheless influence the outcome of the decision. (Gold, No Gold)

Prior Probabilities:

- The relative likelihood of each possible state of nature. Most often expressed as estimates based on secondary information. Not always available.(0.2, 0.8)

Payoffs:

- A quantitative result of each alternative / state of nature combination.

NUMERICAL (EVPI)

Solution (14). Setup a Decision Table

States of Nature	Prior Probabilities	Alternatives/Course of Action	
		Mine Land	Sell Land
Gold	0.2	₹ 600,000	₹ 75,000
No Gold	0.8	₹ -80,000	₹ 75,000

NUMERICAL (EVPI)

Maximax Criterion

The decision for hopeless optimistic. What is the best that can happen?

<u>**No probabilities are involved**</u>

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold		₹ 600,000	₹ 75,000
No Gold		₹ -80,000	₹ 75,000

NUMERICAL (EVPI)

Maximax Criterion

The decision for hopeless optimistic. What is the best that can happen?

<u>**No probabilities are involved**</u>

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold		**₹ 600,000**	₹ 75,000
No Gold		₹ -80,000	₹ 75,000

Mine the Land!!!

NUMERICAL (EVPI)

Maximin Criterion

The decision for the hardened pessimist or the very cautious.

What is the best of the worst that can happen?!?!

<u>**No probabilities are involved**</u>

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold		₹ 600,000	₹ 75,000
No Gold		₹ -80,000	₹ 75,000
Minimum		₹ -80,000	₹ 75,000

NUMERICAL (EVPI)

Maximin Criterion

The decision for the hardened pessimist or the very cautious.

What is the best of the worst that can happen?!?!

<u>**No probabilities are involved**</u>

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold		₹ 600,000	₹ 75,000
No Gold		₹ -80,000	₹ 75,000
Minimum		₹ -80,000	**₹ 75,000**

Sell the Land!!!

NUMERICAL (EVPI)

Equally Likely Criterion

Given even state of probabilities, and using the ***weighted average***, what is the ***best payoff***?

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold	0.5	0.5 x ₹ 600,000	0.5 x ₹ 75,000
No Gold	0.5	0.5 x ₹ -80,000	0.5 x ₹ 75,000
	Expected Payoff:	₹ 260,000	₹ 75,000

NUMERICAL (EVPI)

Equally Likely Criterion

Given even state of probabilities, and using the ***weighted average***, what is the ***best payoff***?

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold	0.5	0.5 x ₹ 600,000	0.5 x ₹ 75,000
No Gold	0.5	0.5 x ₹ -80,000	0.5 x ₹ 75,000
	Expected Payoff:	**₹ 260,000**	₹ 75,000

DIG FOR GOLD!!!!

NUMERICAL (EVPI)

Expected Monetary Value (EMV)

Given the state probabilities, and using a ***weighted average***, what is the ***best payoff***?

We do not have perfect information, just our best analysis.

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold	0.2	0.2 x ₹ 600,000	0.2 x ₹ 75,000
No Gold	0.8	0.8 x ₹ -80,000	0.8 x ₹ 75,000
	Expected Payoff (EMV):	₹ 56,000	₹ 75,000

NUMERICAL (EVPI)

Expected Monetary Value (EMV)

Given the state probabilities, and using a ***weighted average***, what is the ***best payoff***?

We do not have perfect information, just our best analysis.

States of Nature	Prior Probabilities	Course of Action	
		Mine Land	Sell Land
Gold	0.2	0.2 x ₹ 600,000	0.2 x ₹ 75,000
No Gold	0.8	0.8 x ₹ -80,000	0.8 x ₹ 75,000
	Expected Payoff (EMV):	₹ 56,000	**₹ 75,000**

SELL THE LAND!

NUMERICAL (EVPI)

Expected Value of Perfect Information (EVPI)

- Solid, quality, reliable information can be hard to come by. Firms often hire a specialist, consultants, academics etc. to use their expertise in order to greatly improve the quality of fundamental data.
  - Geologist, Economist, Assessment specialist etc.
- The Goal is to reduce RISK and increase CERTAINTY

What is the Value of Perfect Information? What should a company be wiling to pay for such a information?"

This is called the EVPI

(Expected Value of Perfect Information)

NUMERICAL (EVPI)

Expected Value of Perfect Information (EVPI)

EVPI = **Expected Profit with Perfect Information (EPPI)** – Maximum EMV

States of Nature	Probabilities	Course of Action	
		Mine Land	Sell Land
Gold	0.2	0.2 x ₹ 600,000	~~₹ 75,000~~
No Gold	0.8	~~₹ -80,000~~	0.8 x ₹ 75,000
	Expected Payoff (EPPI):	**₹ 120,000**	**₹ 60,000**

1. Choose Best alternative under each state of nature.
2. Multiply its payoff by the probability for that state of nature.
3. Repeat for each state of nature.

EPPI = 120,000 + 60,000 = $ 180,000

NUMERICAL (EVPI)

Expected Value of Perfect Information (EVPI)

EVPI = **Expected Profit with Perfect Information (EPPI)** – **Maximum EMV**

States of Nature	Probabilities	Course of Action	
		Mine Land	Sell Land
Gold	0.2	0.2 x ₹ 600,000	0.2 x ₹ 75,000
No Gold	0.8	0.8 x ₹ -80,000	0.8 x ₹ 75,000
	Expected Payoff (EMV):	₹ 56,000	**₹ 75,000**

EVPI = ₹ 180,000 – ₹ 75,000 = ₹ 105,000

NUMERICAL (EVPI)

EVPI = ₹ 180,000 – ₹ 75,000 = ₹ 105,000

Therefore if XYZ Inc. hired a geologist who was able to provide the Perfect Information with respect to the probabilities of finding or not finding Gold, the MOST they should be willing to pay is $ 105,000.

Of course this assumes the payoff amounts for each alternative/ state combination is correct and of course that the geologist calculated perfect probabilities

NUMERICAL

15. A certain piece of equipment has to be purchased for a construction project at a remote location. This equipment contains an expensive part that is subject to random failure. Spares of this part can be purchased at the same time the equipment is purchased. Their unit cost is ₹ 1,500 and they have no scrap value. If the part fails on the job and no spare is available, the part will have to be manufactured on a special order basis. If this is required, the total cost including down time of the equipment, is estimated at ₹ 9,000 for each such occurrence. Based on previous experience with similar parts, the following probability estimates of the number of failures expected over the duration of the project are provided below:

Failure	: 0	1	2
Probability	: 0.80	0.15	0.05

a. Determine optimal EMV* and optimal number of spares to purchase initially.
b. Based on opportunity losses, determine the optimal course of action and optimal value of EOL.
c. Determine the expected profit with perfect information and Expected Value of Perfect Information.

NUMERICAL

Solution (15).a. Using the conditional costs and the probabilities of states of nature, the expected monetary value can be calculated for each of three states of nature as shown

State of Nature (Spares Required)	Prob.	Conditional Cost Due to Course of Action (No. of Spares purchased)			Weighted Cost Due to Course of Action		
		0	1	2	0	1	2
0	0.8	0	1500	3000	0	1200	2400
1	0.15	9000	1500	3000	1350	225	450
2	0.05	18000	10500	3000	900	525	150
				EMV:	2250	1950	3000

NUMERICAL

Solution (15).a. Since weighted cost = ₹ 1,950 is lowest due to course of action, S2, it should be chosen. If the EMV is expressed in terms of profit, then **EMV* = EMV(S2) = – ₹ 1,950**.

Hence, the optimal number of spares to be purchased initially should be one.

State of Nature (Spares Required)	Prob.	Conditional Cost Due to Course of Action (No. of Spares purchased)			Weighted Cost Due to Course of Action		
		0	1	2	0	1	2
0	0.8	0	1500	3000	0	1200	2400
1	0.15	9000	1500	3000	1350	225	450
2	0.05	18000	10500	3000	900	525	150
				EMV:	2250	**1950**	3000

NUMERICAL

Solution (15).b. Since we are dealing with conditional costs rather than conditional profits, the lower value for each state of nature shall be considered for calculating opportunity losses.

State of Nature (Spares Required)	Prob.	Conditional Cost Due to Course of Action (No. of Spares purchased)			Conditional Opportunity Loss Due to Course of Action (Cost - Min Cost)			Weighted Opportunity Loss Due to Course of Action		
		0	1	2	0	1	2	0	1	2
0	0.8	0	1500	3000	0	1500	3000	0	1200	2400
1	0.15	9000	1500	3000	7500	0	1500	1125	0	225
2	0.05	18000	10500	3000	15000	7500	0	750	375	0
							EOL:	1875	1575	2625

NUMERICAL

Solution (15).b. Since minimum, EOL* = EOL(S2) = ₹ 1,575, therefore adopt course of action S2 and purchase one spare.

State of Nature (Spares Required)	Prob.	Conditional Cost Due to Course of Action (No. of Spares purchased)			Conditional Opportunity Loss Due to Course of Action			Weighted Opportunity Loss Due to Course of Action		
		0	1	2	0	1	2	0	1	2
0	0.8	0	1500	3000	0	1500	3000	0	1200	2400
1	0.15	9000	1500	3000	7500	0	1500	1125	0	225
2	0.05	18000	10500	3000	15000	7500	0	750	375	0
							EOL:	1875	1575	2625

NUMERICAL

Solution (15).c. The expected profit with perfect information (EPPI) can be determined by selecting the optimal course of action for each state of nature, multiplying its conditional values by the corresponding probability and then adding these products.

State of Nature (Spares Required)	Prob.	Optimal Course of Action	Cost of optimal Course of Action	
			Conditional Cost (Minimum Cost)	Weighted Opportunity Loss
0	0.8	0	0	0.80 x 0 = 0
1	0.15	1	1500	0.15 x 1500 = 225
2	0.05	2	3000	0.05 x 3000 =150
			Total (EPPI) :	**375**

NUMERICAL

Solution (15).c. Thus expected profit with perfect information is, EPPI = Rs 375.

Expected value of perfect information then is: EVPI = EPPI – EMV* = – 375 – (– 1,950) = Rs 1,575.

It may be observed that, EVPI = EOL* = Rs 1,575

State of Nature (Spares Required)	Prob.	Optimal Course of Action	Cost of optimal Course of Action	
			Conditional Cost (Minimum Cost)	Weighted Opportunity Loss
0	0.8	0	0	0.80 x 0 = 0
1	0.15	1	1500	0.15 x 1500 = 225
2	0.05	2	3000	0.05 x 3000 =150
			Total (EPPI):	**375**

NUMERICAL

16. XYZ Company manufactures parts for passenger cars and sells them in lots of ***10,000 parts each***. The company has a policy of inspecting each lot before it is actually shipped to the retailer. Five inspection categories, established for quality control, represent the percentage of defective items contained in each lot. These are given in the following table. The daily inspection chart for past ***100 inspections*** shows the following rating or breakdown inspection: Due to this the management is considering two possible courses of action:

S1: Shut down the entire plant operations and thoroughly **inspect** each machine.

S2 : Continue production as it now exists but offer the customer a **refund** for defective items that are discovered and subsequently returned.

The first alternative will **cost ₹ 600** while the second alternative will cost the company **₹ 1 for each defective item** that is returned. What is the optimum decision for the company? Find the EVPI.

Rating	Proportion of defective item	Frequency
Excellent (A)	0.02	25
Good (B)	0.05	30
Acceptable (C)	0.10	20
Fair (D)	0.15	20
Poor (E)	0.20	5
	Total =	100

NUMERICAL

Solution (16). **Points of interest:**

- Sells them in lots of = ***10,000 parts each***
- No of. Inspection= ***100 inspections***
- Alternatives (Course of Action) = Inspect (cost ₹ 600) or

Refund (₹ 1 for each defective item)

The cost of Refund is calculated as follows: No. of parts x Proportion of defective items x Refund Cost

For lot A : 10,000 × 0.02 × 1.00 = **Rs 200**

For lot B: 10000 x 0.05 x 1.00 = **500**

For lot C: 10000 x 0.10 x 1.00 =**1000**

For lot D: 10000 x 0.15 x 1.00 = **1500**

For lot E: 10000 x 0.20 x 1.00 = **2000**

NUMERICAL

Solution (16). The cost of refund is calculated as follows:

For lot A : 10,000 × 0.02 × 1.00 = **Rs 200** | For lot B: 10000 x 0.05 x 1.00 = **500** | For lot C: 10000 x 0.10 x 1.00 =**1000**

For lot D: 10000 x 0.15 x 1.00 = **1500** | For lot E: 10000 x 0.20 x 1.00 = **2000**

State of Nature (Defective Rate)	Prob.	Conditional Cost Due to Course of Action		Conditional Opportunity Loss Due to Course of Action (Cost - Min Cost)		Weighted Opportunity Loss Due to Course of Action	
		Inspect	Refund	Inspect	Refund	Inspect	Refund
(A) 0.02	0.25	600	200	400	0	100	0
(B) 0.05	0.30	600	500	100	0	30	0
(C) 0.10	0.20	600	1000	0	400	0	80
(D) 0.15	0.20	600	1500	0	900	0	180
(E) 0.20	0.05	600	2000	0	1400	0	70
					EOL:	130	330

NUMERICAL

Solution (16). Since the cost of refund is more than the cost of inspection, the plant should be shut down for inspection. Also, **EVPI = EOL of inspection = ₹ 130**

State of Nature (Defective Rate)	Prob.	Conditional Cost Due to Course of Action		Conditional Opportunity Loss Due to Course of Action (Cost - Min Cost)		Weighted Opportunity Loss Due to Course of Action	
		Inspect	Refund	Inspect	Refund	Inspect	Refund
(A) 0.02	0.25	600	200	400	0	100	0
(B) 0.05	0.30	600	500	100	0	30	0
(C) 0.10	0.20	600	1000	0	400	0	80
(D) 0.15	0.20	600	1500	0	900	0	180
(E) 0.20	0.05	600	2000	0	1400	0	70
					EOL:	**130**	330

NUMERICAL

17. A toy manufacturer is considering a project of manufacturing a dancing doll with three different movement designs. The doll will be sold at an average of ₹ 10. The first movement design using 'gears and levels' will provide the lowest tooling and set up cost of ₹ 1,00,000 and ₹ 5 per unit of variable cost. A second design with spring action will have a fixed cost of ₹ 1,60,000 and variable cost of ₹ 4 per unit. Yet another design with weights and pulleys will have a fixed cost of ₹ 3,00,000 and variable cost ₹ 3 per unit. The demand events that can occur for the doll and the probability of their occurrence is given below:

(a) Construct a payoff table for the above project.

(b) Which is the optimum design?

(c) How much can the decision-maker afford to pay in order to obtain perfect information about the demand?

	Demand (units)	Probability
Light Demand	25,000	0.10
Moderate Demand	1,00,000	0.70
Heavy Demand	1,50,000	0.20

NUMERICAL

Solution (17). The calculations for EMV: Payoff = (Demand × Selling price) – (Fixed cost + Demand × Variable cost) = Revenue – Total variable cost – Fixed cost

State of Nature (Demand)	Prob.	Conditional Payoff(₹) Due to Course of Action (Choice of Movement)			Expected Payoff (₹) Due to Course of Action		
		Gears & Levels	Spring Action	Wt. & Pulleys	Gears & Levels	Spring Action	Wt. & Pulleys
Light	0.10	25,000	-10,000	-1,25,000	2,500	-1,000	-12,500
Moderate	0.70	4,00,000	4,40,000	4,00,000	2,80,000	3,08,000	2,80,000
Heavy	0.20	6,50,000	7,40,000	7,50,000	1,30,000	1,48,000	1,50,000
				EMV:	4,12,500	4,55,000	4,17,000

NUMERICAL

Solution (17). The calculations for EMV: Payoff = (Demand × Selling price) – (Fixed cost + Demand × Variable cost) = Revenue – Total variable cost – Fixed cost

State of Nature (Demand)	Prob.	Conditional Payoff(₹) Due to Course of Action (Choice of Movement)			Expected Payoff (₹) Due to Course of Action		
		Gears & Levels	Spring Action	Wt. & Pulleys	Gears & Levels	Spring Action	Wt. & Pulleys
Light	0.10	25,000	-10,000	-1,25,000	2,500	-1,000	-12,500
Moderate	0.70	4,00,000	4,40,000	4,00,000	2,80,000	3,08,000	2,80,000
Heavy	0.20	6,50,000	7,40,000	7,50,000	1,30,000	1,48,000	1,50,000
				EMV:	4,12,500	**4,55,000**	4,17,000

Optimum Design: Since EMV is largest for spring action, it is the one that must be selected.

NUMERICAL

Solution (17). The maximum amount of money that the decision-maker would be willing to pay in order to obtain perfect information regarding demand for the doll will be:

State of Nature (Demand)	Prob.	Conditional Payoff(₹) Due to Course of Action (Choice of Movement)			Maximum Payoff	Maximum Payoff x Prob.
		Gears & Levels	Spring Action	Wt. & Pulleys		
Light	0.10	25,000	-10,000	-1,25,000	25,000	2,500
Moderate	0.70	4,00,000	4,40,000	4,00,000	4,40,000	3,08,000
Heavy	0.20	6,50,000	7,40,000	7,50,000	7,50,000	1,50,000
					EPPI:	**4,60,500**

EVPI = Expected payoff with perfect information – Expected payoff under uncertainty (EMV)

EVPI = 4,60,500 – 4,55,000 = ₹ 5,500

NUMERICAL

18. A TV dealer finds that the cost of holding a TV in stock for a week is ₹ 50. Customers who cannot obtain new TV sets immediately tend to go to other dealers and he estimates that for every customer who cannot get immediate delivery he loses an average of Rs. 200. For one particular model of TV the probabilities of demand of 0, 1, 2, 3, 4 and 5 TV sets in a week are 0.05, 0.10, 0.20, 0.30, 0.20 and 0.15, respectively.

(a) How many televisions per week should the dealer order? Assume that there is no time lag between ordering and delivery.

(b) Compute EVPI.

(c) The dealer is thinking of spending on a small market survey to obtain additional information regarding the demand levels. How much should he be willing to spend on such a survey.

NUMERICAL

Solution (18). If D denotes the demand and S the number of televisions stored (ordered), then the conditional cost values are computed and are:

Cost function = 50 S + 200 (D – S), when D ≥ S

= 50 D + 50 (S – D), when D < S

NUMERICAL

Solution (18). Cost function = 50 S + 200 (D – S), when D ≥ S

= 50 D + 50 (S – D), when D < S

State of Nature (Demand)	Prob.	Conditional Cost(₹) Course of Action (Stock)					
		0	1	2	3	4	5
0	0.05	0	50	100	150	200	250
1	0.10	200	50	100	150	200	250
2	0.20	400	250	100	150	200	250
3	0.30	600	450	300	150	200	250
4	0.20	800	650	500	350	200	250
5	0.15	1000	850	700	550	400	250
Expected Cost (EMV):		590	450	330	250	230	250

NUMERICAL

Solution (18). The dealer should order 4 televisions per week (since optimal EMV* is 230)

State of Nature (Demand)	Prob.	Conditional Cost(₹) Course of Action (Stock)					
		0	1	2	3	4	5
0	0.05	0	50	100	150	200	250
1	0.10	200	50	100	150	200	250
2	0.20	400	250	100	150	200	250
3	0.30	600	450	300	150	200	250
4	0.20	800	650	500	350	200	250
5	0.15	1000	850	700	550	400	250
Expected Cost (EMV):		590	450	330	250	**230**	250

NUMERICAL

Solution (18). Expected Value for Perfect Information (EVPI)

State of Nature (Demand)	Probability	Minimum Cost for Perfect Information	Expected Cost for Perfect Information
0	0.05	0	0
1	0.10	50	5
2	0.20	100	20
3	0.30	150	45
4	0.20	200	40
5	0.15	250	37.5
		ECPI:	**147.5**

EVPI = Conditional Cost (EMV) – ECPI = 230 – 147.5 = Rs. 82.5.

NUMERICAL

Solution (18).

On the basis of the given data, the dealer should not be willing to spend more than

₹ 82.5 for the market survey.

NUMERICAL

19. A producer of boats has estimated the following distribution of demand for a particular kind of boat:

No. demanded: 0 1 2 3 4 5 6

Probability : 0.14 0.27 0.27 0.18 0.09 0.04 0.01

Each boat cost him Rs.7000 and he sells them for Rs.10000 each. Any boats that are left unsold at the end of the season must be disposed off for Rs. 6000 each. How many boats should be in stock so as to maximize his expected profit?

NUMERICAL

Solution (19). Marginal Profit = 10,000 – 7000 = 3000

Marginal Loss = 6000 – 7000 = -1000

State of Nature (Demand)	Prob. (p)	Course of Action (Stock)						
		0	1	2	3	4	5	6
0	0.14	0	-1000	-2000	-3000	-4000	-5000	-6000
1	0.27	0	3000	2000	1000	0	-1000	-2000
2	0.27	0	3000	6000	5000	4000	3000	2000
3	0.18	0	3000	6000	9000	8000	7000	6000
4	0.09	0	3000	6000	9000	12000	11000	10000
5	0.04	0	3000	6000	9000	12000	15000	14000
6	0.01	0	3000	6000	9000	12000	15000	18000
	EMV:	0	2440	3800	4080	3640	2840	1880

NUMERICAL

Solution (19). He should stock 3 boats since the optimal EMV = 4080 corresponding to 3 boats (Stock)

State of Nature (Demand)	Prob. (p)	Course of Action (Stock)						
		0	1	2	3	4	5	6
0	0.14	0	-1000	-2000	-3000	-4000	-5000	-6000
1	0.27	0	3000	2000	1000	0	-1000	-2000
2	0.27	0	3000	6000	5000	4000	3000	2000
3	0.18	0	3000	6000	9000	8000	7000	6000
4	0.09	0	3000	6000	9000	12000	11000	10000
5	0.04	0	3000	6000	9000	12000	15000	14000
6	0.01	0	3000	6000	9000	12000	15000	18000
	EMV:	0	2440	3800	4080	3640	2840	1880

NUMERICAL

20. A bookstore sells a particular book of tax laws for Rs. 100. It purchases the book for Rs. 80 per copy. Since some of the tax laws change every year, the copies unsold at the end of a year become outdated and can be disposed off for Rs.30 each. According to past experience, the annual demand for this book is between 18 and 23 copies.

Assuming that the order for this book can be placed only once during the year, the problem before the store's manager is to decide how many copies of the book should be purchased for the next year. For this problem, since the annual demand varies between18 and 23 copies, there are six possible events: E1, E2, E3, E4, E5 and E6, 18, 19, 20, 21, 22 and 23 copies respectively. Also, there are six possible strategies, or course of action. They are: A1, A2, A3, A4, A5 and A6, buy 18, 19, 20, 21, 22 and 23 copies respectively.

(I)Create a pay – off table and hence find best course of action using

(I) (i) Maximax (ii) Maximin (iii) Laplace (iv) Minimax regret and (v) Hurwicz (α=0.6) criteria

(II) If the probabilities of the occurrence of the various events are respectively 0.05, 0.1, 0.3, 0.4, 0.1 and 0.05 then find best course of action using: (i) EMV (ii) EOL Methods and also find EVPI.

Decision Theory

OPERATIONS RESEARCH-I

Prof. Aditya Suresh Kasar

Module - 3

POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS

- An initial probability statement to evaluate expected payoff is called a prior probability distribution, but if the probability statement has been revised due to additional information, then such a probability statement is called a posterior probability distribution.
- The analysis of problems using posterior probabilities with new expected payoffs and additional information, is called prior-posterior analysis.

POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS

Conditional Probability

A conditional probability is the probability of one event B, conditional on the occurrence of another event A. It is denoted by

P(B|A)

Read as "probability of B given A". P(B|A) can be computed by,

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

The conditional probability is based on that A actually occurs, therefore

$P(A) > 0$

POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS

Bayes' Theorem Statement

Let A1, A2, . . ., An be mutually exclusive and collectively exhaustive outcomes. Their probabilities P(A1), P(A2), . . ., P(An) are known.

There is an experimental outcome B for which the conditional probabilities P(B | A1), P(B | A2), . . ., P(B | An) are also known.

Given the information that outcome B has occurred, the revised conditional probabilities of outcomes Ai, i.e. P(Ai | B), i = 1, 2, . . ., n are determined by using the following relationship:

$$P(A_i \mid B) = \frac{P(A_i \text{ and } B)}{P(B)} = \frac{P(A_i)\, P(B \mid A_i)}{P(B)}$$

Numerical

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Example 1

Consider the stock price per share of Google. The probability of the stock price going up is 90%. This probability will vary with general market conditions.
The probability of the good market and the stock price going up is 67.5%. Find the probability of the good market given that the stock price went up.

Solution:

$$P(\text{good} \mid \text{up}) = \frac{P(\text{up and good})}{P(\text{up})} = \frac{0.675}{0.9} = 0.75$$

The probability of a good market given that the stock price went up is 75%.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

A: up
B: good

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Example 2

Extending from example 1, suppose the market is good 73.5% of the time. Find the probability of the stock price per share of Google going up given a good market.

Solution:

$$P(\text{up} \mid \text{good}) = \frac{P(\text{good and up})}{P(\text{good})} = \frac{0.675}{0.735} = 0.918$$

The probability of stock price going up given a good market is 91.8%.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

A: good
B: up

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Chain Rule

The Chain Rule of probability can be used to compute the joint probabilities of both events A and B occurring.

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Rearranging the chain rule gives the conditional probability formula.

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

If event B actually occurs that is, $P(B) > 0$
Then we can compute the probability of A given B by,

$$P(A|B) = \frac{P(B \text{ and } A)}{P(B)} = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(A) \times P(B|A)}{P(B)}$$

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Posterior Probabilities

Now, we have a formula to compute posterior probabilities

$$P(A|B) = \frac{P(A) \times P(B|A)}{P(B)}$$

Given the prior probabilities P(A) and P(B)

And the conditional probabilities P(B|A)

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Example 3

The stock price per share of Google is considered. The probability of the stock price going up is 90%.
Given the stock price went up, the market is good 75% of the times, fair 20% of the time and bad 5% of the time. When the stock price went down, those numbers were 60%, 30% and 10% respectively.
Use this information to find the probability of the stock price going up given a fair market.

Solution: P(market condition | stock price)

Stock Prices	Prior Probabilities	Market Condition		
		Good	Fair	Bad
Up	0.9	0.75	0.2	0.05
Down	0.1	0.6	0.3	0.1

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Example 3

Solution: P(stock price) X P(market condition | stock price) = P(stock price and market condition)

Stock Prices	Prior Probabilities	Market Condition Conditional Probabilities			Market Condition Joint Probabilities		
		Good	Fair	Bad	Good	Fair	Bad
Up	0.9	0.75	0.2	0.05	0.9 x 0.75 = 0.675	0.9 x 0.20 = 0.180	0.9 x 0.05 = 0.045
Down	0.1	0.6	0.3	0.1	0.1 x 0.60 = 0.06	0.1 x 0.30 = 0.03	0.1 x 0.10 = 0.01
		Marginal Probabilities :			0.735	0.21	0.055

$$P(\text{up} \mid \text{fair}) = \frac{P(\text{up}) \times P(\text{fair} \mid \text{up})}{P(\text{fair})} = \frac{0.9 \times 0.2}{0.21} = \frac{0.18}{0.21} = 0.857$$

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Exercise

The stock price per share of Google is considered. The probability of the stock price going up is 90%.
Given the stock price went up, the market is good 75% of the times, fair 20% of the time and bad 5% of the time. When the stock price went down, those numbers were 60%, 30% and 10% respectively.
Use this information to find the probability of.

a. The stock price going up given a bad market = P(up | bad) = 0.045 / 0.055 = **0.818**
b. The stock price going down given a good market = P(down | good) = 0.06 / 0.735 = **0.082**
c. The stock price going down given a fair market = P(down | fair) = 0.03 / 0.21 = **0.143**
d. The stock price going down given a bad market = P(down | bad) = 0.01 / 0.055 = **0.182**

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

19. A company receives shipments of certain items. It should decide whether to accept or reject the shipment, on the basis of inspection of a sample selected from the shipment. From the past experience, it is known that the percentage of defective items in a batch of shipment is either 0, 2% or 5%, the probabilities for which are 0.5, 0.3 and 0.2 respectively.
The company can accept only those batches which have no defectives. The cost of rejecting a good batch i.e. batch with no defectives is ₹ 200. The cost of accepting a defective batch is ₹ 600.
A sample of 10 items has been selected from the shipment and two items are found to be defective.
The conditional probabilities of getting 2 defectives in a sample of 10 items from a batch of 0 , 2% and 5% defectives are calculated as 0.083, 0.185 and 0.265 respectively.
Determine whether the shipment should be accepted.

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (19). The following table summarizes the given problem.

The **cost of rejecting** a good batch i.e. batch with no defectives is **₹ 200**.

The **cost of accepting** a defective batch is **₹ 600**.

State of Nature (% of Defective Items)	Prior Prob.	Conditional Prob.	Course of Action	
			Accept	Reject
0	0.5	0.083	0	200
2	0.3	0.185	600	0
5	0.2	0.265	600	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (19). E.V (accept) = (0 x 0.277) + (600 x 0.370) + (600 x 0.353) = 433.8

E.V (reject) = (200 x 0.277) + (0 x 0.370) + (0 x 0.353) = 55.40

State of Nature (% of Defective Items)	Prior Prob. (1)	Conditional Prob. (2)	Joint Prob. (3) = 1 x 2	Posterior Prob. (4) = 3/M.P	Course of Action	
					Accept	Reject
0	0.5	0.083	0.0415	0.277	0	200
2	0.3	0.185	0.0555	0.370	600	0
5	0.2	0.265	0.530	0.353	600	0
	Marginal Prob. (M.P):		0.150	EMV:	433.8	55.40

Since EMV (reject) is lower, the company should Reject the shipment.

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

20. A manufacturing company is considering the introduction of a new product to its existing product range. The company is to make decision on the market survey of the demand for the product which is as follows:

Event (sales)	Probability	Action (Profit in '000)	
		Manufacturer	Do not Manufacture
High	0.25	180	0
Low	0.75	-50	0

The company wants to have further information on which to base the decision and contacts a consultant cell. The consultant cell provides the following additional information:

Event (sales)	Action (Profit in '000)		
	High	Indecisive	Low
High	0.5	0.3	0.2
Low	0.1	0.5	0.4

If the consultant cell charges ₹ 10,000 for providing the information.

(a) Determine the decision company should take with prior information as well as with the additional information.

(b) Perform a complete posterior analysis and state whether it will be economical to engage the consultants.

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (a) EMV calculations with prior information.

State of Nature (Sales)	Prior Prob.	Course of Action Profit in ('000)	
		Manufacture	Do not Manufacture
High	0.25	180	0
Low	0.75	-50	0
	EMV (prior):	7.5	0

Based on EMV criterion the company should **(Manufacture)** the product to earn profit of 7,500.

EVPI = EPPI – EMV = [(180 x 0.25) + (0 x0.75) – 7.5] x 1000 = 37,500

EVPI = 37,500

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) The probability information supplied by the consultant cell is in the form of conditional probabilities often 'likelihoods'

State of Nature (Sales)	Prior Prob.	Conditional Prob.			Joint Prob.		
		(H)	(I)	(L)	(H)	(I)	(L)
High	0.25	0.5	0.3	0.2	0.125	0.075	0.050
Low	0.75	0.1	0.5	0.4	0.075	0.375	0.300
	Marginal Prob. (M.P):				0.200	0.450	0.350

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) Compute the Posterior probabilities.

State of Nature (Sales)	Posterior Prob.		
	(H)	(I)	(L)
High	0.125/0.2 = 0.625	0.075/0.45 = 0.167	0.05/0.35 = 0.143
Low	0.075/0.2 = 0.375	0.375/0.45 = 0.833	0.3/0.35 = 0.857

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) Use posterior probabilities to compute the revised EMV & EVPI

Sales Forecast (High)

State of Nature (Sales)	Posterior Prob. (H)	Course of Action	
		Manufacture	Do not Manufacture
High	0.625	180	0
Low	0.375	-50	0
	EMV:	93.75	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) Use posterior probabilities to compute the revised EMV & EVPI

Sales Forecast (Indecisive)

State of Nature (Sales)	Posterior Prob. (I)	Course of Action	
		Manufacture	Do not Manufacture
High	0.167	180	0
Low	0.833	-50	0
	EMV:	-11.59	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) Use posterior probabilities to compute the revised EMV & EVPI

Sales Forecast (Low)

State of Nature (Sales)	Posterior Prob. (L)	Course of Action	
		Manufacture	Do not Manufacture
High	0.143	180	0
Low	0.857	-50	0
	EMV:	-17.11	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) Posterior EMV

Likelihood	Marginal Prob.	Course of Action	Optimal EMV	Expected EMV
H	0.200	A1	93.75	18.75
I	0.450	A2	0	0
L	0.350	A2	0	0

Posterior EMV = 18,750

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (20). (b) Thus from the information provided by the consultant, the company has EMV increased from ₹ 7500 to ₹ 18,750.

The maximum that this company shall be willing to pay the consultant cell is (18,750 – 7500) = **₹ 11,250**

Thus EVPI for additional information = ₹ 11,250

Since the consultant charges are 10,000, the company should go in for their advice and get the additional information.

By doing so, the company gains **₹ 1,250**

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

21. A company is considering to introduce a new product to its existing product range. It has defined two levels of sales as 'high' and 'low' on which it wants to base its decision and has estimated the changes that each market level will occur, together with their costs and consequent profits or losses. This information is summarized below:

State of Nature	Probability	Course of Action	
		Market the product ('000)	Do not Market the product ('000)
High	0.3	150	0
Low	0.7	-40	0

The company's marketing manager suggests that a market research survey may be undertaken to provide further information on which the company should base its decision. Based on the company's past experience with a certain market research organization, the marketing manager assesses its ability to give good information in the light of subsequent actual sales achievements. This information is given below:

Actual sales	Market Research (Survey outcome)		
	'High' sales forecast	Indecisive survey report	'Low' sales forecast
Market 'High'	0.5	0.3	0.2
Market 'Low'	0.1	0.4	0.5

The market research survey costs Rs 20,000, state whether or not there is a case for employing the market research organization.

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (a) EMV calculations with prior information.

State of Nature (Sales)	Prior Prob.	Course of Action Profit in ('000)	
		Market the product ('000)	Do not Market the product ('000)
High	0.3	150	0
Low	0.7	-40	0
	EMV (prior):	**17**	**0**

Based on EMV criterion the company should **(Market)** the product to earn profit of 17,000.

EVPI = EPPI – EMV = [(150 x 0.3) + (0 x0.7)] - 17 x 1000 = 28,000

EVPI = 28,000

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) The probability information supplied by the consultant cell is in the form of conditional probabilities often 'likelihoods'

State of Nature (Sales)	Prior Prob.	Conditional Prob.			Joint Prob.		
		(H)	(I)	(L)	(H)	(I)	(L)
High	0.3	0.5	0.3	0.2	0.15	0.09	0.06
Low	0.7	0.1	0.4	0.5	0.07	0.28	0.35
	Marginal Prob. (M.P):				0.22	0.37	0.41

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) Compute the Posterior probabilities.

State of Nature (Sales)	Posterior Prob.		
	(H)	(I)	(L)
High	0.15/0.22 = 0.681	0.09/0.37 = 0.243	0.06/0.41 = 0.146
Low	0.07/0.22 = 0.318	0.28/0.37 = 0.756	0.35/0.41 = 0.853

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) Use posterior probabilities to compute the revised EMV & EVPI

Sales Forecast (High)

State of Nature (Sales)	Posterior Prob. (H)	Course of Action	
		Market	Do not Market
High	0.681	150	0
Low	0.318	-40	0
	EMV:	89.43	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) Use posterior probabilities to compute the revised EMV & EVPI

Sales Forecast (Indecisive)

State of Nature (Sales)	Posterior Prob. (I)	Course of Action	
		Market	Do not Market
High	0.243	150	0
Low	0.756	-40	0
	EMV:	6.21	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) Use posterior probabilities to compute the revised EMV & EVPI

Sales Forecast (Low)

State of Nature (Sales)	Posterior Prob. (L)	Course of Action	
		Market	Do not Market
High	0.146	150	0
Low	0.853	-40	0
	EMV:	-12.22	0

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) Posterior EMV

Likelihood	Marginal Prob.	Course of Action	Optimal EMV	Expected EMV
H	0.22	A1	89.43	19.67
I	0.37	A1	6.21	2.29
L	0.41	A2	0	0

Posterior EMV = 21,967

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (21). (b) Thus from the information provided by the consultant, the company has EMV increased from ₹ 17,000 to ₹ 21,967.

The maximum that this company shall be willing to pay the consultant cell is (21,967 – 17,000) = **₹ 4,967**

Thus EVPI for additional information = ₹ 4,967

Since the market survey cost 20,000, the company should **not go for the market survey**.

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

22. A firm is currently considering the purchase of new equipment to manufacture parts required by the north sea oil development programme. There are three types of equipment which could be purchased:

(1) Conventional manual operation (CMO) (2) Numerical controlled (NC) (3) Computer numerically controlled (CNC)

The capital cost of the equipment rises from (1) through (3). The profit resulting from whichever course of action is finally taken, depends on the size of the market to be supplied; at present this is uncertain. The market has been classified into three broad categories: poor, fair, and good. The profit (losses are given a negative signs) are shown for each market size/machine tool type in the table below.

Equipment	Profit (million) Market		
	Poor	Fair	Good
CMO	0.5	1.0	1.5
NC	0	1.5	2.5
CNC	-1.5	0.5	3.5

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

the chances of the market being poor, fair or good are assessed at 30%, 50% and 20% respectively by the firm's management. A market research group could be employed to provide information on which the market will be realized. Past experience with work from this group shows that its information cannot be relied on to be absolutely accurate. The management assesses chances of it indicating a poor (I), fair (II) and good (III) market when these are the actual states off the market to be as follows:

Actual State	Indicated state		
	I	II	III
Poor	0.7	0.2	0.1
Fair	0.2	0.7	0.1
Good	0	0.2	0.8

You are required to

(a) Prepare the conditional/expected opportunity loss table based on the market assessment made by the firm.
(b) Calculate the expected value of perfect information.
(c) Prepare the conditional expected opportunity loss table on the information which might be available from the market research; and
(d) State the maximum amount it would be worthwhile for the firm to pay for the market research.

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) a. The conditional profit table:

Market	Prob.	Indicated state		
		CMO	NC	CNC
Poor	0.3	0.5	1.0	1.5
Fair	0.5	0	1.5	2.5
Good	0.2	-1.5	0.5	3.5

Opportunity Loss Table:

Market	Prob.	Indicated state		
		CMO	NC	CNC
Poor	0.3	0.5	1.0	1.5
Fair	0.5	0	1.5	2.5
Good	0.2	-1.5	0.5	3.5
	EOL:	0.65	0.35	1.1

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) b. EVPI is the EOL for the optimal act i.e.. ₹ 0.35 million corresponding to buying of NC equipment.

c. The joint probabilities and their marginal probabilities are then computed.

Market	Prior Prob.	Conditional probabilities			Joint probabilities		
		(I)	(II)	(III)	(I)	(II)	(III)
Poor	0.3	0.7	0.2	0.1	0.21	0.06	0.03
Fair	0.5	0.2	0.7	0.1	0.10	0.35	0.05
Good	0.2	0	0.2	0..8	0	0.04	0.16
				Marginal probabilities:	0.31	0.45	0.24

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) c. The required posterior probabilities are now obtained by dividing each joint probability by the corresponding marginal probability.

Market	Posterior probabilities		
	(I)	(II)	(III)
Poor	0.21/0.31 = 0.68	0.13	0.12
Fair	0.10/0.31 = 0.32	0.78	0.21
Good	0/0.31 = 0	0.09	0.67

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) c. Posterior analysis is now performed.

For State (I)

Market	Posterior probability	Buying decision		
		CMO	NC	CNC
Poor	0.68	0	0.5	2.0
Fair	0.32	0.5	0	1.0
Good	0	2.0	1.0	0
	EOL:	0.16	0.34	1.68

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) c. Posterior analysis is now performed.

For State (II)

Market	Posterior probability	Buying decision		
		CMO	NC	CNC
Poor	0.13	0	0.5	2.0
Fair	0.78	0.5	0	1.0
Good	0.09	2.0	1.0	0
	EOL:	0.57	0.155	1.04

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) c. Posterior analysis is now performed.

For State (III)

Market	Posterior probability	Buying decision		
		CMO	NC	CNC
Poor	0.12	0	0.5	2.0
Fair	0.21	0.5	0	1.0
Good	0.67	2.0	1..0	0
	EOL:	1.445	0.73	**0.45**

NUMERICAL (POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS)

Solution (22) c. It is observed from the posterior analysis that when the given information is state (I), the firm will buy CMO, when the given information is state (II), the firm will buy NC, and when it is state (III), the firm will buy CNC.

d. We now calculate the revised (posterior) EOL and then the expected value of market research.

State	Marginal probability	Buying decision	Optimal EOL	Expected EOL
I	0.31	CMO	0.16	0.05
II	0.45	NC	0.155	0.07
III	0.24	CNC	0.45	0.11
			Posterior EOL:	**0.23**

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Solution (22) d. We now calculate the revised (posterior) EOL and then the expected value of market research.

Thus the EOL can be reduced from **₹ 0.35 million** to **₹ 0.23 million** by conducting the market research. The net gain from market research is **₹ 0.12 million**, which is the **expected value** for market research, i.e. the maximum amount the firm shall be willing to pay to the market research group as the price for providing additional information.