→ Random Process

- It is also known as stochastic process.
- Consider a random experiment with sample space, S, if a time function $X(t, s)$ is assigned to each outcome $s \in S$ and where $t \in T$, then the family of all such functions denoted $X(t, s)$ is called Random process,

$$x(t), \; t \in [0, \infty]$$

- For a fixed $t = \tau$ OR $t \in \tau$ where τ is any moment,

$$X(t, s) = \underbrace{X(\tau, s)}_{\text{random variable}}$$

As s varies over the sample space S

- On the other hand for fixed $s = s_i$ is a single function of time t, called as 'Sample function' OR '~~as~~ ensemble member' OR 'realisation of the random process'

- If both t and s are fixed, then $X(t, s)$ is a real number.

→ Classification of Random Process.

- Let E be the state space of the random process
- Let T be the index set OR parameter set dependent on time.

- If the index set T is discrete, then the process is called discrete parameter or discrete time process. It is also called a random sequence and denoted by

$$\{X_n : n = 1, 2, 3, \ldots\ldots\}$$

- If T and E are both discrete, the random process is called as discrete random process or discrete random sequence.

eg 1 → If X_n represents the number of heads obtained in the n^{th} toss of two 2 fair coins then,

→ $X_n, n \geq 1$ is a discrete random process

→ $E = \{0, 1, 2\}$

→ $T = \{1^{st}, 2^{nd}, 3^{rd}, \ldots\ldots, n^{th} \ldots\}$

eg 2 → If X_n represents the temperature at the end of the n^{th} hour of a day, then the state

→ $X_n = \{1, 2, \ldots\ldots n^{th}\}$ can take any value in an ~~inter~~ interval is continuous

→ Continuous Random Process

- If T is discrete and E is continuous then random process is called continuous random sequence

eg 3 → Let $N(t)$ be the number of customers who have visited the bank from $t = 9$ until t, the random variable $N(t_1)$ is a discrete random variable thus $N(t)$ is a discrete valued random process however since t and take any value between 9 to 16, $N(t)$ is a continuous time random process.

- A continuous time random process is a random process where J is an interval on the real line such as $[-1, 1]$, $[0, \infty)$, $(-\infty, \infty)$ where

$$\{ X(t) : t \in J \}$$

- A discrete time random process is a random process X_n where J is a countable set such as N or Z

→ Deterministic and Non-Deterministic Random Process

- A random process is called a deterministic random process if future values of any sample function can be predicted exactly from past observation or past value.

eg → $x(t) = (\cos wt)(A)$

→ $x(1) = A\cos w$

→ $x(2) = A\cos 2w$

- Otherwise called as non-deterministic.

eg → share market, exchange rate, etc.

→ Description of Random Process

- Consider a random process $X(t) : t \in T$ for a fixed t_1, $X(t_1) = X_1$ is a random variable and it's cumulative distribution function CDF denoted by ~~$F_X(t_1, x_1)$~~ $F_X(x_1, t_1)$ and defined as ~~$F_X($~~

$$F_X(x_1, t_1) = P(X(t_1) \le x_1)$$

and called as first order distribution of random process $X(t)$.

- For given t_1 and t_2 if your $x(t_1) = x_1$ and $x(t_2) = x_2$

~~$F_x(x_1, t_1)$~~

$$F_x(x_1, t_1, t_2) = P(x(t_1) \le x_1, x(t_2) \le x_2)$$

is called second order distribution of random process $x(t)$

eg $\xrightarrow{Q_1}$ compound interest

You have \$1000 to put in an account with interest rate R compounded annually i.e if X_n is the value of the account at year n, then $X_n = 1000(1+R)^n$ for $n = 0, 1, 2, \ldots\ldots$

Then the value of R is a random variable when you put the money in the bank but it does not change after a while. Then in particular assume that R follows uniform distribution where limits are $R \sim U(0.04, 0.05)$

$U(a, b)$ $\quad f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$

$$E(x) = \text{mean} = \frac{a+b}{2}$$

$$V(x) = \frac{(b-a)^2}{12}$$

a) Find all possible sample function for the random process $\{X_n, n = 0, 1, 2, 3, \ldots\}$

b) Find the expected value of your account at year 3 is $E[X_3]$.

ANS → here the randomness in X_n comes from the random variable R as soon as we know R, the entire sequence will be known i.e. $X_n, n = 0, 1, 2, 3$

In particular if $R = r$, then this
$X_n = 1000(1+r)^n, \; n = 0, 1, 2, \ldots$

Thus here sample function are of the form
$f(n) = 1000(1+r)^n, \; n = 0, 1, 2, 3 \ldots$
where $r \in [0.04, 0.05]$ ~~for an~~
we obtain a sample function for the random X_n.

→ $X_n = 1000(1+R)^n, \; n = 0, 1, 2 \ldots$

$X_3 = 1000(1+R)^3$
put $Y = 1+R$

here $R \sim U[0.04, 0.05]$
$\therefore 1+R \sim U[1.04, 1.05]$

$$F_Y(y) = \begin{cases} \frac{1}{1.05-1.04} = \frac{1}{0.01} = 100, & 1.04 < y < 1.05 \\ 0 & , \text{otherwise} \end{cases}$$

$$E(X_3) = E[1000(1+R^3)]$$

$$= E[1000\,Y^3]$$

$$= 1000\,E[Y^3]$$

$$= 1000 \int_{1.04}^{1.05} 100y^3\,dy$$

$$= 1000 \times \overset{25}{\cancel{100}} \left[\frac{y^4}{\cancel{4}}\right]_{1.04}^{1.05}$$

$$= \cancel{10}\ 25000\,[y^4]_{1.04}^{1.05}$$

$$= 1141.19225 \approx 1141.2$$

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→ Discrete and Continuous Random Process

- If the state space E of a random process is discrete, then the random process is called as a discrete random process also called as "chain". In this the state space E is assumed to be {0, 1, 2, 3}

eg → If $x(t)$ represent the number of SMS received in a cell in the interval $(0, t)$, then $\{x(t)\}$ is a discrete random process

- If the state space E of a random process is continuous, then the random process is called as a continuous random process OR continuous state space.

eg → If $x(t)$ is the minimum temperature recorded in a city in a interval $(0, t)$ then $\{x(t)\}$ is a continuous random process.

Q2 Let $\{x(t), t \in [0, \infty]\}$ be defined as

$x(t) = A + Bt, \forall t \in [0, \infty]$ where A and B are independent normal random variables.

$N(1,1) \longrightarrow N(\mu, \sigma^2)$ OR $N(m, \sigma^2)$

a) Find all possible sample function for this random process

b) Define the random variable ~~by~~ $Y = x(1)$ find the PDF of Y

c) ~~Also find~~ Let also. $Z = x(2)$, find $E[YZ]$

a)

ANS → Here we note that the randomness in $\{x(t)\}$ comes from the two random variables A and B. The random variable A can take any real value $A \in R$ and also B can take any real value $B \in R$. As soon as we know the values of A and B the entire process $\{x(t)\}$ is known

In particular if $A = a$ and $B = b$ where $a, b \in R$ then, we obtain a sample function for $\{x(t)\}$

$$x(t) = a + bt, \forall t \in [0, \infty]$$

b) If normal distribution is given $N(\mu, \sigma^2)$, then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Given

$$A, B \sim N(1, 1)$$

$\therefore$ mean $= 1$
variance $= 1$

$Y = x(1)$

$$\therefore Y = x(1) = A + B(1)$$
$$= A + B$$

here $x(t)$ follows normal distribution, therefore Y also follows normal distribution

$$A + B \sim N(\mu, \sigma^2)$$
$$\cancel{\forall C}\ Y = x(1) = A + B$$

ie $Y = A + B$

$$\therefore Y \sim N(\mu, \sigma^2)$$

$$\text{mean} = E[Y] = E[A+B]$$
$$= E[A] + E[B]$$
$$= 1 + 1$$
$$= 2$$

$$\therefore E[Y] = 2$$

$$\text{var}(Y) = \text{var}(A+B)$$
$$= \text{var}(A) + \text{var}(B)$$
$$= 1 + 1$$
$$= 2$$

$$\text{var}(Y) = 2$$

This means $Y = X(1) = A + B \sim N(2, 2)$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{(\sqrt{2})\sqrt{2\pi}} e^{\frac{(y-2)^2}{4}}$$

$$= \frac{1}{2\sqrt{\pi}} e^{\frac{(y-2)^2}{4}}$$

c) $Z = X(2)$
$Z = A + B(2)$
$Z = A + 2B$

$$E[YZ] = E[X(1) \cdot X(2)]$$
$$= E[(A+B)(A+2B)]$$
$$= E[A^2 + 3AB + 2B^2]$$

$$= E[A^2] + 3E[A]E[B] + 2[B^2]$$

$$\because \ var(A) = E[A^2] - \{E[A]\}^2$$

$$1 = E[A^2] - 1$$

$$E[A^2] = 2$$

Similarly

$$var(B) = E[B^2] - \{E[B]\}^2$$

$$E[B^2] = 2$$

$\cancel{E[A^2]}$

$$E[YZ] = 2 + 3(2) + 2(2)$$

$$= 2 + 3 + 4$$

$$= \cancel{12}\ 9$$

$$\cancel{\text{mean} = E[Y] = E[A+B] = E[A] + E[B] = 1 + 1 = 2}$$

→ Mean, Autocorrelation and Auto-covariance

- For a fixed value of t, $x(t)$ is a random variable thus mean of $x(t)$ denoted by

$$\mu_x(t) = E[x(t)]$$
$$\cancel{= \Sigma x p(x)}$$

for discrete $E(x) = \Sigma x p(x)$

Here $\mu_x(t)$ is a function of time and it is often called as 'ensemble mean' or 'ensemble average' of $x(t)$

- For two time intervals t_1 and t_2, $x(t_1)$ and $x(t_2)$ are random variables. The correlation between them is defined as

$$R_{xx}(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$$

$\Rightarrow \rho = E[XY]$

and called as autocorrelation or correlation of the process $\{x(t)\}$.

Autocorrelation serves as a measures of the dependence among the random variables of $x(t)$

- The autocovariance of or covariance of the process $x(t)$ is defined as

$$C_{xx}(t_1, t_2) = E[\{x(t_1) - \mu_x(t_1)\}\{x(t_2) - \mu_x(t_2)\}]$$

$$\therefore C_{xx}(t_1, t_2) = E[x(t_1)x(t_2) - x(t_1)\mu_x(t_2) - x(t_2)\mu_x(t_1) + \mu_x(t_1)\mu_x(t_2)]$$

$$= E[x(t_1)x(t_2)] - \mu_x(t_2)E[x(t_1)] - \mu_x(t_1)E[x(t_2)] + \mu_x(t_1)\mu_x(t_2)$$

$$= E[x(t_1)x(t_2)] - \mu_x(t_2)\mu_x(t_1) - \cancel{\mu_x(t_1)\mu_x(t_2)} + \cancel{\mu_x(t_1)\mu_x(t_2)}$$

$$= E[x(t_1)x(t_2)] - \mu_x(t_2)\mu_x(t_1)$$

Q Let $\{x(t) : t \in [0,1]\}$ be defined as $x(t) = T + (1-t)$ where T is a uniform random variable in interval $(0,1)$

a) Find the CDF of $x(t)$.
b) Find $E[x(t)]$ and $C_{xx}(t_1 t_2)$

→ For uniform distribution

$$X \sim U(a,b)$$

pdf → $f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$

cdf → $F_x(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x > 1 \end{cases}$

Given that,

$x(t) = T + (1-t)$, as T follows uniform distribution. Therefore the CDF of $x(t)$

$F_x(t) \ (T \leq t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{}, & 0 < t < 1 \\ 1, & t > 1 \end{cases}$

~~$x(t) \leq t$~~

$$x(t) \le t = \begin{cases} 0, & T + (1-t) < 0 \\ T-(1-t), & 0 < T+(1-t) < 1 \\ 1, & T(1-t) > 0 \end{cases}$$

$$= \begin{cases} 0, & T < t-1 \\ T-(1-t), & 0 < T < t) \\ 1 & T > t \end{cases}$$

$$(x(t) \le x) = P[T+(1-t) \le x]$$

$$= P[T \le (x-1+t)]$$

$$= P[T \le t']$$

where $t' = x-1+t$

$$= \begin{cases} 0, & x-1+t = t' < 0 \\ t', & 0 < t' = (x-1+t) < 1 \\ 1, & t' = x-1+t > 1 \end{cases}$$

$$= \begin{cases} 0, & x < \cancel{t=} 1-t \\ t', & 0 < x < 2-t \\ 1, & x < 2-t \end{cases}$$

which is the required CDF.

and pdf $x(t) = \begin{cases} \frac{1}{1-0} = 1, & 0 < x < 2- \\ 0, & \text{otherwise} \end{cases}$

~~$E(x(t)) =$~~

Since $x(t) \sim U(1-t, 2-t)$

here

$a = 1-t$

$b = 2-t$

$x(t) = T + (1-t)$

$\therefore$ Mean, $E(x(t)) = \frac{a+b}{2}$

$= \frac{1-t+2-t}{2}$

$= \frac{3-2t}{2}$

Correlation

$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$

$= E[(T+(1-t_1))(T+(1-t_2))]$

$= E[T^2 + T - Tt_2 + T + 1 - t_2 - t_1T - t_1 + t_1t_2]$

$= E\left[T^2 + 2T - T(t_1+t_2) + t_1t_2 + 1 - t_1 - t_2\right]$

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YOUVA

$$= E[T^2] + [2-t_1-t_2]E[T] + (1-t_1)(1-t_2)$$

$T \sim U(0,1)$

$\therefore$ Mean $E[T] = \frac{1}{2}$

$$Var(T) = E[T^2] - E[T]^2$$

$\hookrightarrow \frac{(b-a)^2}{12}$

$$\therefore \frac{1}{12} = E[T^2] - (E[T])^2$$

$$E[T^2] = \frac{1}{12} + \frac{1}{4}$$

$$= \frac{4+12}{48} = \frac{16}{48} = \frac{8}{24} = \frac{1}{3}$$

$$\therefore R_{xx}(t_1,t_2) = \frac{1}{3} + (2-t_1-t_2)\frac{1}{2} + (1-t_1)(1-$$

$\therefore$ Covariance

$$C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - \mu_x(t_1)\mu_x(t_2)$$

$$= \frac{1}{3} + \frac{(2-t_1-t_2)}{2} + (1-t_1)(1-t_1)$$

$$-\left(\frac{3}{2}-t_1\right)\left(\frac{3}{2}-t_2\right)$$

$$= \frac{1}{3} + 1 - \frac{t_1}{2} - \frac{t_2}{2} + 1 - t_1 - t_2 + \cancel{t_1 t_2} - \frac{9}{4}$$

$$+ \frac{3}{2}t_1 + \frac{3}{2}t_2 - \cancel{t_1 t_2}$$

$$= \left(\frac{1}{3} + 2\right) + t_1 + t_2 - t_1 - t_2 - \frac{9}{4}$$

$$= \frac{7}{3} - \frac{9}{4}$$

$$= \cancel{\frac{24 - 27}{12}} \qquad = \frac{28 - 27}{12}$$

$$= \cancel{\frac{-3}{12}} \qquad = \cancel{\frac{-1}{4}} \qquad = \frac{1}{12}$$

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Q. Consider a random process $x(t)$ where $x(t) = A\cos(\omega t + \Theta)$ where A and Θ are independent and uniform random variable from $(-K, K)$ and $(-\pi, \pi)$. Find the mean, auto-correlation, ~~and~~ auto-variance and

$\rightarrow$ $A \sim U(-K, K)$

$$f(A) = \begin{cases} \frac{1}{2K}, & -K < A < K \\ 0, & \text{otherwise} \end{cases}$$

$$f(\Theta) = \begin{cases} \frac{1}{2\pi}, & -\pi < \Theta < \pi \\ 0, & \text{otherwise} \end{cases}$$

mean $E[A] = \frac{a+b}{2} = \frac{K+(-K)}{2} = 0$

$var[A] = \frac{(b-a)^2}{12} \quad \frac{4K^2}{12} = \frac{K^2}{3}$

similarly

$E[\Theta] = 0$

$var(\Theta) = \frac{\pi^2}{3}$

mean of $x(t)$, $\mu_x(t) = E[x(t)]$

$= E[A\cos(\omega t + \theta)]$

$= E[A] E[\cos(\omega t + \theta)]$

$= 0 \qquad (E[A] = 0)$

$R_{xx}(t_1, t_2) = E[x(t_1) x(t_2)]$

$= E[A\cos(\omega t_1 + \theta) A\cos(\omega t_2 + \theta)]$

$= E[A^2 \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)]$

$= E[A^2] \; E[(\cos\omega t_1 + \theta)(\cos\omega t_2 + \theta)]$

But

$var(A) = E[A^2] - E[A]^2$

$\frac{k^2}{3} = E[A^2] - 0$

$E[A^2] = \frac{k^2}{3}$

$\therefore R_{xx}(t_1, t_2) = \frac{k^2}{3} E[(\cos\omega t_1 + \theta)(\cos\omega t_2 + \theta)]$

$\rightarrow ①$

for

$E[(\cos\omega t_1 + \theta)(\cos\omega t_2 + \theta)]$

$E[x] = \int_a^b x f(x)\, dx \rightarrow$ for continuous distribution

$= \int_{-\pi}^{+\pi} \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta) \frac{1}{2\pi}\, d\theta$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2}{2} \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta)\, d\theta$$

$[\because 2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos(\omega t_1 + \theta + \omega t_2 + \theta) + \cos(\omega(t_1 - t_2))$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2))\, d\theta$$

$$= \frac{1}{4\pi} \left[\frac{\sin(\omega(t_1 + t_2) + 2\theta)}{2} + \theta \cos(\omega(t_1 - t_2)) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi} \left[\frac{\sin(\omega(t_1 + t_2) + 2\pi)}{2} + \pi\cos(\omega(t_1 - t_2)) - \frac{\sin(\omega(t_1 + t_2) - 2\pi)}{2} + \pi\cos(\omega(t_1 - t_2)) \right]$$

$$= \frac{1}{4\pi} \left[2\pi\cos(\omega(t_1 - t_2)) + \sin(\omega(t_1 + t_2))/2 - \sin(\omega(t_1 + t_2))/2 \right]$$

$$\left[\begin{array}{l} \because \sin(2\pi + \theta) = \sin\theta \\ \sin(2\pi - \theta) = -\sin\theta \end{array} \right]$$

$$= \frac{1}{4\pi} (2\pi \cos(\omega(t_1 - t_2))$$

$$= \frac{\cos(\omega(t_1 \mp t_2))}{2}$$

$$R_{xx}(t_1, t_2) = \frac{k^2}{6} (\cos(\omega(t_1 - t_2))$$

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$$

$$= \frac{k^2}{6} (\cos(\omega(t_1 - t_2)))$$

$\Rightarrow$ when mean $= 0$, ie $x(t) = 0$

$$R_{xx}(t_1, t_2) = C_{xx}(t_1, t_2)$$

$$C_{xx}(t, t) = \frac{k^2}{6} (\cos(\omega(t - t)))$$

$$C_{xx}(t^2) = \frac{k^2}{6} = \sigma^2 x(t)$$

Q $x(t) = B\cos(50t + \phi)$ where B and ϕ are independent random variables, where B is a random variable with $B = 0$ and variance $= 1$ and ϕ is $U \sim [-\pi, \pi]$. Find the mean and auto-correlation of the process.

$\rightarrow$ $E[B] = 0$

$var[B] = 1$

$$\phi \sim U(-\pi, \pi]$$

$$f(\phi) = \begin{cases} \frac{1}{2\pi}, & -\pi < \phi < \pi \\ 0, & \text{otherwise} \end{cases}$$

$E[\phi] = \frac{a+b}{2} = \frac{\pi - \pi}{2} = 0$

$var(\phi) = \frac{2\pi^2}{12} = \frac{\pi^2}{6}$

mean of $x(t) = E[x(t)]$
$= E[B\cos(50t + \phi)]$
$= E[B]\cos(50t + \phi)$
$= 0$

$R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$
$= E[B^2 \cos(50t_1 + \phi)\cos(50t_2 + \phi)]$

$= E[B^2]\, E[\cos(50t_1 + \phi)\cos(50t_2 + \phi)]$
$\longrightarrow (1)$

$var(B) = E[B^2] - [E[B]]^2$

$E[B^2] = 1 + 0$
$= 1$

$E[\cos(50t_1 + \phi)\cos(50t_2 + \phi)]$

$E(x) = \int_{-\pi}^{\pi} \cos(50t_1 + \phi)\cos(50t_2 + \phi)\, d\theta\, f(\theta$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cancel{2}\cos(50(t_1 + t_2) + 2\phi) + \cos(50(t_1 - t_2)}{2}$

$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos(50(t_1 + t_2) + 2\phi) + \cos(50(t_1 - t_2$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[\frac{-\sin}{\cos} \frac{(50(t_1+t_2)+2\phi)}{2} + \phi\cos(50(t_1-t\right.$$

$$= \frac{1}{4\pi} \left[\frac{\sin}{\cos} \frac{(50(t_1+t_2)+2\pi)}{2} + \pi\cos(50(t_1-t_2 \right.$$

$$\left. + \sin \frac{(2\pi - 50(t_1+t_2))}{2} + \pi\cos(50(t_1-t_2 \right]$$

$$= \frac{1}{4\pi} \left(2\pi\cos\frac{(50(t_1-t_2))}{2} \right)$$

$$= \cos\left(\frac{50}{2}(t_1-t_2)\right)$$

$$C_{xx}(t_1, t_2) = \cos\left(\frac{50}{2}(t_1-t_2)\right)$$

Q Suppose $x(t)$ is a random process with mean $=3$ and auto-correlation $R(t_1, t_2) = 9 + 4e^{-0.2|t_1-t_2|}$. Determine the mean, variance and covariance of the random variable $z = x(\underset{t_1}{5})$ and $w = x(\underset{t_2}{8})$

Given

$\mu_x(t) = E(x(t)) = 3 \quad \forall t \in T$

$\therefore$ mean of z & $w = 3$

$\therefore$ $E[Z] = E(x(5)) = 3$
$E[W] = E(x(8)) = 3$

$R(Z, W) = E[x(t_1)x(t_2)] = 9 + 4e^{-0.2|t_1 - t_2|}$
$= E[ZW]$

$C_{xx}(Z, W) = E[x(t_1)x(t_2) - \mu_x(t_1)\mu_x(t_2)]$
$= E[ZW] - E[Z]E[W]$
$= \cancel{9} + 4e^{-0.2|t_1 - t_2|} - \cancel{9}$
$= 4e^{-0.2|(5-8)|}$

$C_{xx}(Z, W) = 4e^{-0.2|3|}$
$= 4e^{-0.6}$

when $Z = W$

is coverted
$C_{xx}(Z, W) = \text{variance}(Z, W)$

Now
$\text{var}(Z) = C_{xx}(Z, Z)$

$= R_{ZZ}(5, 5) - E(Z)E(Z)$

$= 9 - 9 + 4e^{-0.2|5-5|}$

$= 4$

similarly,
$\text{var}(W) = 4$

SSR

→ Stationary Random Process (~~Strict Sense~~

- A random process is stationary if it's statistical properties (PDF, CDF) do not change by time i.e. for stationary process $x(t)$ and $x(t+\Delta t)$ have the same probability distribution have the same probability distribution

$$f_x(t) = f_x(t+\Delta) \ \forall t, t+\Delta$$

$$F_x(t) = F_x(t+\Delta) \ \forall t, t+\Delta$$

- ∴ A random process is stationary if the time shift does not change it's statistical properties

- $\{x(t) : t \in T\}$ is strict sense stationary or simply stationary if for all $t_1, t_2, \dots t_r$ belong to T and all Δ belong to Real Number

→ $F_x(t_1)\,x(t_2)\,x(t_3) \dots x(t_r)(x_1, x_2, \dots x_r)$
$= F_x(t_1+\Delta)\,x(t_2+\Delta) \dots x(t_r+\Delta)$
$(x_1, x_2, \dots x_r)$

OR

→ $f\cancel{(x_1(t_1)\,x(t_2))}\, f(x_1, x_2, \dots x_r, t_1, t_2, \dots t_r)$

$$= f_x(x_1, x_2, \dots x_r, t_1+\Delta, t_2+\Delta, \dots t_r+\Delta)$$

Q The mean and variance of a first order stationary process are constant.

→ Let $\{x(t)\}$ be a first order stationary process.

~~$f_x(t) = f$~~

$f_x(x;t) = f_x(x;t+\Delta t)$

Our target is to show $E[x(t+\Delta)] = E[x(t)]$ and
$var[x(t+\Delta)] = var(x(t))$

$$E(x(t+\Delta)) = \int_{-\infty}^{\infty} x f_x(x;t+\Delta)\,dx$$

$$= \int_{-\infty}^{\infty} x f_x(x;t)\,dx$$

$$= E[x(t)]$$

∴ mean is constant

Let $E(x(t)) = \mu_x$ = constant

$var(t+\Delta) = E[(x(t+\Delta)-\mu_x)^2]$

$\Rightarrow \boxed{E[x] = E[(x-\mu)^2]}$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x;t+\Delta)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x;t)\,dx$$

$= var(x^{\cdot}, t)$

$\therefore$ variance is constant

~~$\therefore$ let var(x(t)) =~~

Q If $\{x(t) = A\cos\lambda t + B\sin\lambda t \,;\, t > 0\}$ is a random process, where A and B are independent random variables, each of which assumes the value -2 and 1 ~~with~~ with probabilities $\frac{1}{3}$ and $\frac{2}{3}$

respectively. Show that $\{x(t)\}$ is not strick sense stationary

→ For not strict sense we need to show one of ~~m~~ moments is not constant

~~E~~

A/B	-2	1
$p(x)$	$\frac{1}{3}$	$\frac{2}{3}$

$mean = \Sigma x\, p(x)$

$E[A] = -\frac{2}{3} + \frac{2}{3}$ $\quad (\because \Sigma A\, P(A))$

$= 0$

$\therefore E[B] = 0$ $\quad (\because \Sigma B\, P(B))$

$$E[A^2] = \sum A^2 P(A)$$
$$= \frac{4}{3} + \frac{2}{3}$$
$$= \frac{6}{3} = 2$$

$$E[B^2] = 2$$

$\therefore E[AB] = E[A]E[B] = 0$ (because A & B are independent)

$$E(x(t)) = \cancel{f x f}\ E[A\cos\lambda t + B\sin\lambda t]$$
$$= E[A]E[\cos\lambda t] + E[B]E[\sin\lambda t]$$
$$= 0$$
$$= \text{constant}$$

$$E[x^2(t)] = E[(A\cos\lambda t + B\sin\lambda t)^2]$$
$$= E[A^2\cos^2\lambda t + B^2\sin^2\lambda t + 2AB\sin\lambda t\cos\lambda t]$$
$$= E[A^2]E[\cos^2\lambda t] + E[B^2]E[\sin^2\lambda t]$$
$$+ 2E[A]E[B]E[\sin\lambda t\cos\lambda t]$$

$$= 2\cos^2\lambda t + 2\sin^2\lambda t + 0$$
$$= 2$$
$$= \text{constant}$$

Q. Consider a random process $x(t) = \cos(t+\phi)$ where ϕ is a random variable with density function $f(\phi) = \frac{1}{\pi}$, $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$. Check whether or not the process is stationary.

$$E[x(t)] = E[\cos(t+\phi)]$$

$$= \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, f(\phi)\, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \cos(t+\phi)\, \frac{1}{\pi}\, d\phi$$

$$= \frac{1}{\pi}\left[\sin(t+\phi)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\sin\left(t+\frac{\pi}{2}\right) - \sin\left(t-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\sin\left(t+\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}-t\right)\right]$$

$$= \frac{1}{\pi}\left[\cos t + \cos t\right]$$

$$= \frac{2\cos t}{\pi} \longrightarrow \text{which depends on } t$$

$\therefore$ The process is not stationary.

Q Examine whether the Poisson process pf $p[x(t) = k] = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$, $k = 0, 1, 2, 3, \ldots$

→ For poisson process,

mean of $x(t) = \lambda t$ → depends on t, hence not constant

∴ The given process is not ~~constant~~ stationary.

Q For the sine wave process $x(t) = Y \cos \omega_0 t$, $-\infty < t$ ω_0 is constant, the amplitude Y is a random variable with uniform distribution in the interval 0 to 1. Check whether the process is stationary or not.

$$Y \sim U[0, 1]$$

$$f(y) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$E[Y] = \frac{1}{2}$ $\quad$ $var[Y] = \frac{1}{12}$

$$\begin{aligned} E[x(t)] &= E[Y \cos \omega_0 t] \\ &= E[Y]\, E[\cos \omega_0 t] \\ &= \frac{E}{2}[\cos \omega_0 t] \end{aligned}$$

→ depends on t, hence not constant

∴ The given process is not stationary

Weak Sense Stationary Process
OR
Wide Sense Stationary Process (WSS)

- A random process $x(t)$ where $t \in T$ is called wide ~~sta~~ sense stationary if it's mean function and correlation function do not change by shift in time, ie $\{x(t) : t \in T\}$ is said to be WSS if for all t_1, t_2 belongs to T ~~which~~ we have,

i) $E[x(t)] = E[x(t+\Delta)]$

ii) $R_{xx}(t_1, t_2) = R_{xx}(t_1+\Delta, t_2+\Delta)$

$$= R_{xx}(t_1 - t_2)$$

Q Consider the random process $\{x(t) : t \in T\}$ $x(t) = \cos(t + A)$ where $A \sim U[0, 2\pi]$. Show that $x(t)$ is WSS.

→ $f(A) = \begin{cases} \dfrac{1}{2\pi}, & 0 \leq A < 2\pi \\ 0, & \text{otherwise} \end{cases}$

mean of ~~A~~ $E[A] = \pi$

$var[A] = \dfrac{\pi^2}{3}$

$$E[x(t)] = E[\cos(t+A)]$$

$$= \int_0^{2\pi} \cos(t+A) f(A)\, dA$$

$$= \int_0^{2\pi} \frac{\cos(t+A)}{2\pi}\, dA$$

$$= \frac{1}{2\pi} \left[\sin(t+A)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[\sin(2\pi + t) - \sin t\right]$$

$$= \frac{1}{2\pi} [\sin t - \sin t]$$

$$= \frac{1}{2\pi} (0)$$

$$= 0, \forall t$$

$\therefore E[x(t)]$ is constant

$\therefore E[x(t)] = E[x(t+\Delta)]$

$$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$$

$$= E[\cos(t_1 + A)\cos(t_2 + A)]$$

$$= \int_0^{2\pi} \cos(t_1 + A)\cos(t_2 + A) f(A)\, dA$$

$$= \int_0^{2\pi} \frac{\cos(t_1 + A + t_2 + A)\cos(t_1 + A - t_2 + A)}{4\pi}\, dA$$

$$= \int_0^{2\pi} \frac{\cos(t_1+t_2+2A) + \cos(t_1-t_2)}{4\pi}\, dA$$

$$= \frac{1}{4\pi}\left[\frac{\sin(t_1+t_2+2A)}{2} + [\cos(t_1-t_2)]A\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{\sin(t_1+t_2+4\pi)}{2} + 2\pi\cos(t_1-t_2) - \frac{\sin(t_1+t_2)}{2} - 0\right]$$

$$= \frac{1}{4\pi}\left[\cancel{\frac{\sin(t_1+t_2)}{2}} - \cancel{\frac{\sin(t_1+t_2)}{2}} + 2\pi\cos(t_1-t_2)\right]$$

$$= \frac{\cos(t_1-t_2)}{2}$$ — depends on $t_1 - t_2$

$\therefore R_{xx}(t_1, t_2) = R_{xx}(t_1 - t_2)$

Hence, the given process is WSS

Q Given a random variable Y with characteristic function $\phi(\omega) = E[e^{i\omega Y}]$ and a random process defined by $x(t) = \cos(\lambda t + Y)$. ~~S.T.~~ Show that $x(t)$ is WSS if $\phi(1) = \phi(2) = 0$.

$\rightarrow$ $\phi(1) = E[e^{iY}]$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore \phi(1) = E[\cos Y + i\sin Y]$

$0 + i0 = E[\cos Y + i\sin Y]$

$0 + i0 = E[\cos Y] + iE[\sin Y]$

$\Rightarrow \cancel{\cos Y}\ E[\cos Y] = 0 \ \& \ E[\sin Y] = 0$

similarly

$E[\cos 2Y] = 0 \ \& \ E[\sin 2Y] = 0$

$E[x(t)] = E[\cos(\lambda t + Y)]$

$= E[\cos\lambda t \cos Y - \sin\lambda t \sin Y]$

$= E[\cos\lambda t]E[\cos Y] - E[\sin\lambda t]E[\sin Y]$

$= 0, \forall Y$

$\therefore E[x(t)]$ is constant $\forall t$

$$R_{xx}(t_1, t_2) = E[x(t_1)\,x(t_2)]$$

$$= E[\cos(\lambda t_1 + Y)\cos(\lambda t_2 + Y)]$$

$$= E\left[\frac{\cos(\lambda t_1 + \lambda t_2 + 2Y) + \cos(\lambda t_1 - \lambda t_2)}{2}\right]$$

$$= E\left[\frac{\cos(\lambda t_1 + \lambda t_2)\cos 2Y - \sin(\lambda t_1 + \lambda t_2)\sin 2Y + \cos(\lambda t_1 - \lambda t_2)}{2}\right]$$

$$= \frac{\cos(\lambda t_1 \pm \lambda t_2)}{2} E[\cos 2Y] - \frac{\sin(\lambda t_1 \pm \lambda t_2)}{2} E[\sin 2Y] + \frac{\cos(\lambda t_1 - \lambda t_2)}{2}$$

$$= \frac{\cos(\lambda t_1 - \lambda t_2)}{2} \rightarrow \text{depends on } t_1 - t_2$$

$$\therefore R_{xx}(t_1, t_2) = R_{xx}(t_1 - t_2)$$