DEPARTMENT OF PHYSICS
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PH2140 Mathematics on the Computer Assignment 8 (II) 28 Sep 2015

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1. Establish the following identity:

$$U = \exp\left(\frac{i\theta\boldsymbol{\sigma}\cdot\boldsymbol{n}}{2}\right) = I_{2\times 2}\cos(\theta/2) + i(\boldsymbol{\sigma}\cdot\boldsymbol{n})\sin(\theta/2)$$

where $\boldsymbol{\sigma}$ represents a "vector" whose components are the three Pauli matrices; i.e., $\boldsymbol{\sigma} \equiv [\sigma_x, \sigma_y, \sigma_z]$. You might want to remember to following points:

1. $\exp(M)$ for a matrix M is defined via the series: $\exp(M) = I + \sum_{k=1}^{\infty} M^k/k!$, where M^k is M multiplied k times with itself.
2. The vector $\boldsymbol{n}$ above is by definition a unit vector in 3 dimensions; i.e., $\boldsymbol{n}\cdot\boldsymbol{n} = 1$. Make sure you take this into account properly.

Comment: The above identity provides an alternate way of describing rotations (by the angle θ about the axis $\boldsymbol{n}$) in 3 dimensions, the usual description being in terms of the 3×3 rotation matrices $R_{\boldsymbol{n}}(\theta)$.

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2. Establish the above comment through following steps:

1. Construct a 2×2 matrix $H = \boldsymbol{r}\cdot\boldsymbol{\sigma} = x\sigma_x + y\sigma_y + z\sigma_z$.
2. Define H' as the transform of H under U:

$$H' = UHU^\dagger$$

3. Compare the components of H' and H and hence make the connection with rotations in 3 dimensions.