DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH2140 Mathematics on the Computer   Assignment 10   19 Oct 2015

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1. Probability Distributions

1. Find the mean and variance of the Poisson distribution.
2. For the normal distribution $\mathcal{N}(x)$, explicitly calculate (i.e., without using in-built commands) $\langle x^k \rangle$ for $k = 1, 2, 3$.

   (You will encounter many such calculations in *quantum mechanics*.)
3. For the Binomial distribution $\mathcal{B}(x)$, explicitly calculate (i.e., without using in-built commands) $\langle n^k \rangle$ for $k = 1, 2, 3$.

   (You will encounter these repeatedly in *statistical physics*.)

2. Central limit theorem

The central limit theorem is an extremely important result in probability and statistics. It states that, given certain mathematical conditions, the distribution of the *arithmetic mean* of a sufficiently large number of identical and independently distributed (iid) random variables X_k, tends to a *normal distribution*, regardless of the underlying distribution. (More general variants exist; those interested are encouraged to explore further on web.)

1. Verify the above theorem using `Mathematica`, using the tools discussed in the lecture, for *two* probability distributions $f(x)$ (you can choose any two).
2. Using appropriate plots, show that the distribution of

   $$S_n = \frac{X_1 + X_2 + \ldots X_n}{n}$$

   is very well fitted, for large n, by a normal distribution of mean μ and variance σ^2/n, where μ and σ^2 represent mean and variance of X_k.
3. Show that $\sqrt{n}\,(S_n - \mu)$ tends to a normal distribution with mean 0 and variance σ^2.

3. Random numbers

1. **Estimating the value of π:** Consider a square of size $2L$, and a circle of radius L inscribed within it. The ratio α of area of square to that of the circle is then given by $\pi/4$. Therefore, you can estimate the value of $\pi = 4\alpha$ as follows:

   Fill the square with N points whose (x, y) coordinates are independent uniformly distributed random variables. Count how many of these lie inside the circle, call this number m. For large N, N and m would be proportional to the area of the square and the circle (respectively). Use this to estimate α, and hence π. Repeat the process 10 times and `ListPlot` your estimate of π.

   This problem is a simple demonstration of the so called *Monte Carlo* methods, which are simulation methods that use sequences of random numbers to perform a simulation. A slightly more non-trivial way of estimating π using such methods is known as the *Buffon's needle* problem. Those interested can browse the web for more information.

2. **Wigner semi-circle law:** Let M be a real, symmetric $N \times N$ matrix, whose off-diagonal entries are chosen from a given probability distribution, and diagonal entries from the same or different probability distribution.

   (a) Write a `Mathematica` routine to generate such matrices.

   (b) Calculate the eigenvalues of such a matrix for $N = 3000$.

   (c) Make a `Histogram` of the above eigenvalues.

   (d) Fit the appropriately normalized histogram with the distribution

   $$\begin{aligned} f(x) &= \frac{2}{\pi R^2}\sqrt{R^2 - x^2} \qquad && (-R < x < R) \\ &= 0 && (|x| > R) \end{aligned}$$

   and determine R.

   (e) Find the mean and variance of the above distribution.

   The above result concerning distribution of eigenvalues of (large) real, symmetric matrices is known as the *Wigner semicircle law.*