DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH2140 Mathematics on the Computer   Assignment 6   8 Sep 2015

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1. Integral curves of a vector field

As mentioned in the lecture, given a vector field $\boldsymbol{v}(\boldsymbol{x})$, it's integral curves are defined as parametrized curves $\boldsymbol{x}(\lambda)$ satisfying

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda} = \boldsymbol{v}(\boldsymbol{x}(\lambda))$$

Aside: Convince yourself that if you change the normalization of the vector field, you can always absorb this change by changing the curve parameter λ. This can be helpful in certain situations because it allows you to choose a normalized vector field (that is, $\hat{\boldsymbol{v}} = \boldsymbol{v}/\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$), and then find the integral curves.

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1. Use `NIntegrate` to find integral curves of the Electric field $\boldsymbol{E}(\boldsymbol{x})$ corresponding to a configuration of three charges, $Q_1 = +1, Q_2 = +1, Q_3 = -2$ (in suitable units), arranged at the vertices of an equilateral triangle.
2. Use `ContourPlot` to plot the equipotential lines for the above configuration. What is the relation between these lines and the integral curves of $\boldsymbol{E}(\boldsymbol{x})$?

Aside: Can you say anything about electrostatic equilibrium in such a configuration by looking at one (or both) of the curves above? (Look up *Earnshaw's theorem*!)

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2. Non-linear PDE

Consider the partial differential equation for $\phi(t, x)$

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = m^2\phi - \lambda\phi^3$$

where m and λ are positive constants. The above equation is non-linear because of the ϕ^3 term on the RHS.

1. Verify (of course, using `Mathematica`) that

$$u(t,x) = \frac{m}{\sqrt{\lambda}} \tanh\left[\frac{m}{\sqrt{2}}\left(\frac{x - x_0 - Vt}{\sqrt{1-V^2}}\right)\right]$$

   is a solution of the given equation.

   Here, $-1 < V < 1$, and x_0 is a parameter (you will see it's significance if you plot the above solution at different times.

2. Generate the above solution using `NDSolve`. (Recall our discussion regarding initial and boundary conditions from the lecture.) You may choose $m = 1, \lambda = 0.5, x_0 = 0$ and give the results for three values of V.

Aside: The above equation arises as a solution of an important class of toy models obtained by adding a potential proportional to $\lambda\phi^4$ to the standard lagrangian for the wave equation. It describes localized "lumps" of energy density of typical width $\sim 1/m$. You may verify this by plotting the energy density $\rho(x)$ for the above solution (with $V = 0$), which is given by

$$\rho(x) = \frac{m^4}{2\lambda}\mathrm{sech}^4\left[\frac{m\,(x - x_0)}{\sqrt{2}}\right]$$
