DEPARTMENT OF PHYSICS
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PH2140 Mathematics on the Computer   Assignment 9   5 Oct 2015

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1. Gamma function

Check out the syntax and definition of the Gamma function $\Gamma[z]$ in `Mathematica`.

1. Verify that, for positive integers N, $\Gamma[N] = (N-1)!$.
2. The Gamma function can be extended to real numbers through it's integral representation. Plot $\Gamma[x]$ and note it's singular points.
3. Using the series expansion of $\Gamma[x]$, establish the Sterling's approximation:

$$N! \approx \sqrt{2\pi N}\, e^{-N} N^N \qquad (N \gg 1) \tag{1}$$

   Note: This approximation is very widely used in statistical physics. Check how good the above approximation is by plotting $\log[N!]$ vs. log of the RHS above. You can use `ListPlot` for this.

4. (a) A N sphere of radius R is a sphere in N dimensions, defined as a locus of points whose distances from a given point are less than or equal to R. Show that the volume of a N sphere of radius R is given by

$$V_N(R) = \frac{\pi^{N/2}}{\Gamma(N/2+1)} R^N \tag{2}$$

   (b) Check the above expression gives the desired result for $N = 3$.

   (c) Plot $V_N(1)$ as a function of N (you may assume N take all real values, or try a `ListPlot`).

   (d) While you are at it, use the above result to calculate the surface area $S_N(R)$ of a N sphere. For $N = 3$, you should get $4\pi R^2$.

2. Legendre Polynomials

Check out the syntax and definition of Legendre polynomials $P_\ell(x)$ in `Mathematica`.

1. Show that these are solutions of the differential equation:

$$\begin{aligned} \frac{d}{d\theta}\left(\sin\theta \frac{dy}{d\theta}\right) &= -\ell\,(\ell+1)\,(\sin\theta) y & (3) \\ y &= P_\ell(\cos\theta) & (4) \end{aligned}$$

2. Verify Rodrigues formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell \left(x^2-1\right)^\ell \tag{5}$$

3. Establish the orthogonality of these polynomials:

$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)dx = \left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'} \tag{6}$$

4. Given that the defining differential is second order, you must expect two linearly independent solution. Find the second solution, and show that it is generically unphysical. *Hint*: Check out the $\ell = 0$ case first.

5. Given two points with position vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$, calculate the multipole expansion of

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

in terms of $P_\ell(\cos\Omega)$, for the case when $|\boldsymbol{x}| \gg |\boldsymbol{x}'|$. Here,

$$\cos\Omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

6. Find the electrostatic potential outside a sphere of radius a, with surface charge density $\sigma(\theta') =$ (constant) $\cos\theta'$ pasted on it.

3. Spherical harmonics

Check out the syntax and definition of spherical harmonics Y_ℓ^m in `Mathematica`.

1. Verify that

$$Y_\ell^m(\theta,\phi) \quad = \quad (-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m\left(\cos\theta\right)\, e^{im\phi} \tag{7}$$

where $P_\ell^m(x)$ are the associated Legendre polynomials (check in `Mathematica`).

2. Make a polar plot of $|Y_\ell^m(\theta,0)|$. That is, plot the curve $r = |Y_\ell^m(\theta,0)|$ in the $x-z$ plane.

3. Show that

$$\int Y_\ell^m(\theta,\phi)Y_{\ell'}^{m'}(\theta,\phi)\,d\Omega = \delta_{\ell\ell'}\delta_{mm'} \tag{8}$$

4. Show that

$$\sum_{m=-\ell}^{\ell} \left(Y_\ell^m(\theta,\phi)\right)^* \, Y_\ell^m(\theta,\phi) = \frac{2\ell+1}{4\pi} \tag{9}$$

5. Show that

$$\sum_{m=-\ell}^{\ell} \left(Y_\ell^m(\theta',\phi')\right)^* \, Y_\ell^m(\theta,\phi) = \frac{2\ell+1}{4\pi} P_\ell(\cos\Omega) \tag{10}$$

   where $\cos\Omega$ was defined in previous exercise.