Euler and Hamiltonian Paths

Certain graph problems deal with finding a path between two vertices such that each edge is traversed exactly once, or finding a path between two vertices while visiting **each vertex** exactly once. These paths are better known as **Euler path** and **Hamiltonian path** respectively.

Euler paths and circuits :

- An Euler path is a path that uses every edge of a graph **exactly once**.
- An Euler circuit is a circuit that uses every edge of a graph exactly once.
- An Euler path starts and ends at different vertices.
- An Euler circuit starts and ends at the same vertex.

There are simple criteria for determining whether a multigraph has a Euler path or a Euler circuit. For any multigraph to have a Euler circuit, all the degrees of the vertices must be even.

Theorem – "A connected multigraph (and simple graph) with at least two vertices has a Euler circuit if and only if **each** of its vertices has an **even degree**."

Proof of the above statement is that every time a circuit passes through a vertex, it adds twice to its degree. Since it is a circuit, it starts and ends at the same vertex, which makes it contribute one degree when the circuit starts and one when it ends. In this way, every vertex has an even degree.
Since the Koningsberg graph has vertices having odd degrees, a Euler circuit does not exist in the graph.

Theorem – "A connected multigraph (and simple graph) has an Euler path but not an Euler circuit if and only if it has **exactly two vertices** of odd degree."

The proof is an extension of the proof given above. Since a path may start and end at different vertices, the vertices where the path starts and ends are allowed to have odd degrees.

Hamiltonian paths and circuits :

Hamiltonian Path – A simple path in a graph Gthat passes through **every vertex exactly once** is called a Hamiltonian path.

Hamiltonian Circuit – A simple circuit in a graph Gthat passes through every vertex exactly once is called a Hamiltonian circuit.

Unlike Euler paths and circuits, there is no simple necessary and sufficient criteria to determine if there are any Hamiltonian paths or circuits in a graph. But there are certain criteria which rule out the existence of a Hamiltonian circuit in a graph, such as- if there is a vertex of degree one in a graph then it is impossible for it to have a Hamiltonian circuit.

There are certain theorems which give sufficient but not necessary conditions for the existence of Hamiltonian graphs.