



Machine Learning

Dimensionality Reduction

Motivation I:
Data Compression

Data Compression



Reduce data from
2D to 1D

Data Compression



Reduce data from
2D to 1D

$$x^{(1)} \in \mathbb{R}^2 \rightarrow z^{(1)} \in \mathbb{R}$$

$$x^{(2)} \in \mathbb{R}^2 \rightarrow z^{(2)} \in \mathbb{R}$$

\vdots

$$x^{(m)} \in \mathbb{R}^2 \rightarrow z^{(m)} \in \mathbb{R}$$

Data Compression

10000 \rightarrow 1000

Reduce data from 3D to 2D



$$x^{(i)} \in \mathbb{R}^3$$



$$z^{(i)} \in \mathbb{R}^2$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad z^{(i)} = \begin{bmatrix} z_1^{(i)} \\ z_2^{(i)} \end{bmatrix}$$



Machine Learning

Dimensionality Reduction

Motivation II:
Data Visualization

Data Visualization

$$x \in \mathbb{R}^{50}$$

$$x^{(i)} \in \mathbb{R}^{50}$$

Country	x_1 GDP (trillions of US\$)	x_2 Per capita GDP (thousands of intl. \$)	x_3 Human Development Index	x_4 Life expectancy	x_5 Poverty Index (Gini as percentage)	x_6 Mean household income (thousands of US\$)	...
→ Canada	1.577	39.17	0.908	80.7	32.6	67.293	...
China	5.878	7.54	0.687	73	46.9	10.22	...
India	1.632	3.41	0.547	64.7	36.8	0.735	...
Russia	1.48	19.84	0.755	65.5	39.9	0.72	...
Singapore	0.223	56.69	0.866	80	42.5	67.1	...
USA	14.527	46.86	0.91	78.3	40.8	84.3	...
...

Data Visualization

Country	z_1	z_2
Canada	1.6	1.2
China	1.7	0.3
India	1.6	0.2
Russia	1.4	0.5
Singapore	0.5	1.7
USA	2	1.5
...

$z^{(i)} \in \mathbb{R}^2$

Reduce data from 500 to 2D

Data Visualization



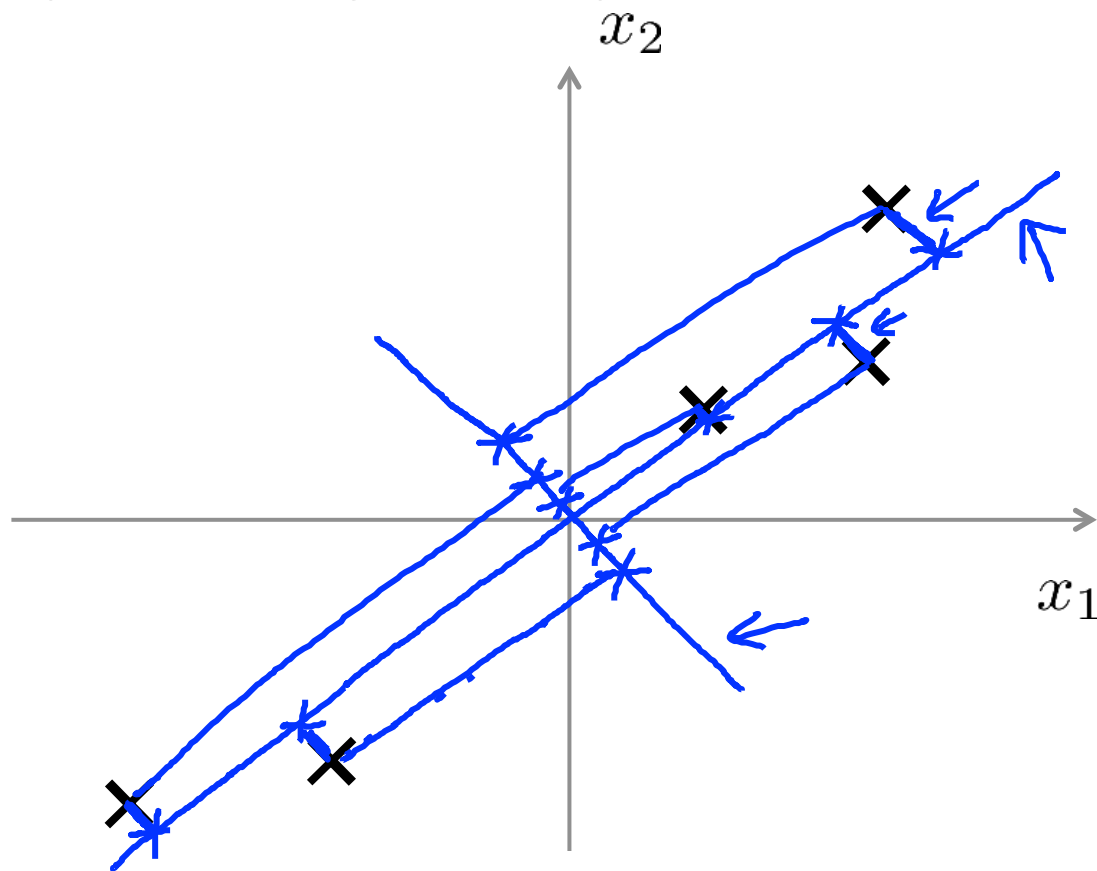


Machine Learning

Dimensionality Reduction

Principal Component
Analysis problem
formulation

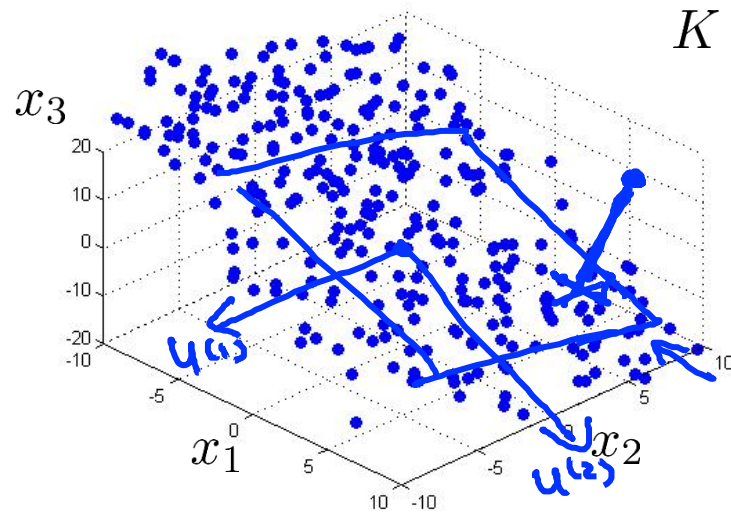
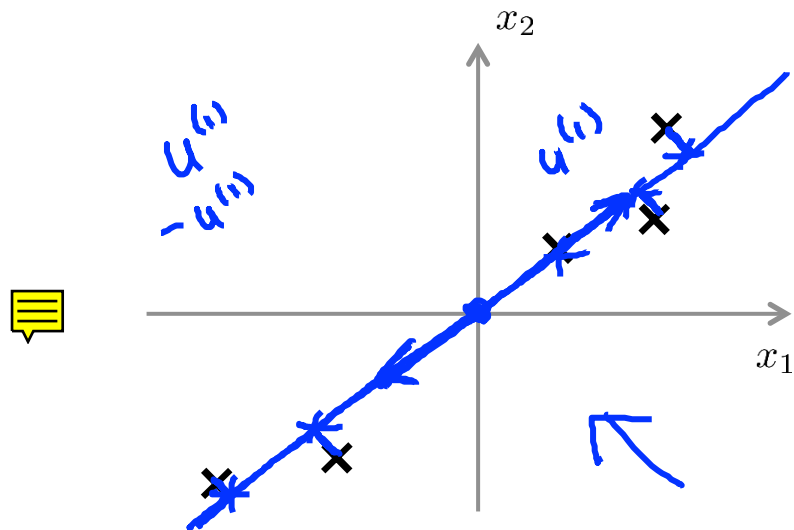
Principal Component Analysis (PCA) problem formulation



$$x \in \mathbb{R}^2$$

Principal Component Analysis (PCA) problem formulation

$$3D \rightarrow 2D$$
$$K = 2$$



Reduce from 2-dimension to 1-dimension: Find a direction (a vector $u^{(1)} \in \mathbb{R}^n$) onto which to project the data so as to minimize the projection error.

Reduce from n -dimension to k -dimension: Find k vectors $u^{(1)}, u^{(2)}, \dots, u^{(k)}$ onto which to project the data, so as to minimize the projection error.

PCA is not linear regression



PCA is not linear regression





Machine Learning

Dimensionality Reduction

Principal Component
Analysis algorithm

Data preprocessing

Training set: $x^{(1)}, x^{(2)}, \dots, x^{(m)}$ \leftarrow

Preprocessing (feature scaling/mean normalization):

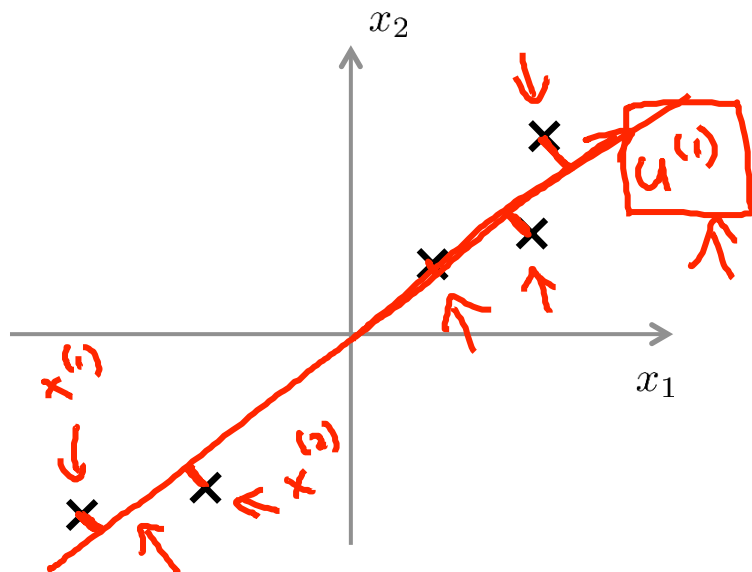
$$\mu_j = \frac{1}{m} \sum_{i=1}^m x_j^{(i)}$$

Replace each $x_j^{(i)}$ with $x_j - \mu_j$.

If different features on different scales (e.g., x_1 = size of house, x_2 = number of bedrooms), scale features to have comparable range of values.

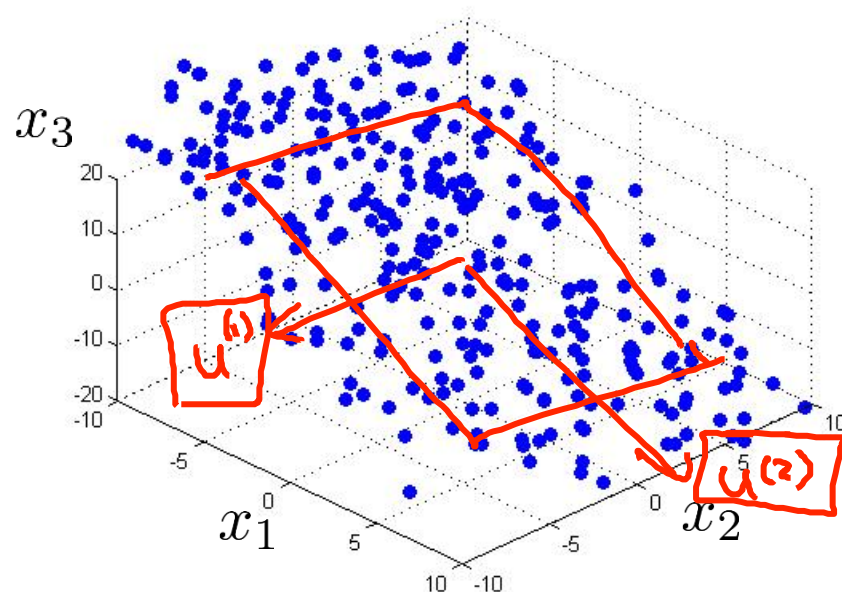
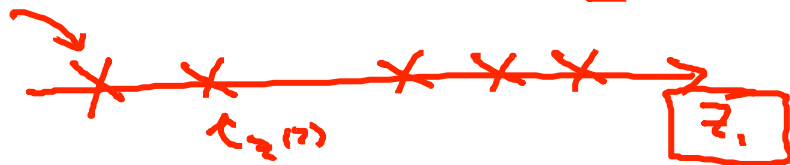
$$x_j^{(i)} \leftarrow \frac{x_j^{(i)} - \mu_j}{s_j}$$

Principal Component Analysis (PCA) algorithm



Reduce data from 2D to 1D

$$x^{(i)} \in \mathbb{R}^2 \rightarrow z^{(i)} \in \mathbb{R}$$



Reduce data from 3D to 2D

$$x^{(i)} \in \mathbb{R}^3 \rightarrow z^{(i)} \in \mathbb{R}^2$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Principal Component Analysis (PCA) algorithm

Reduce data from n -dimensions to k -dimensions

Compute "covariance matrix":

$$\Sigma = \frac{1}{m} \sum_{i=1}^n \underbrace{(x^{(i)})}_{n \times 1} \underbrace{(x^{(i)})^T}_{1 \times n}$$

Σ is labeled as $n \times n$ matrix. The term Σ is also labeled as Sigma.

Compute "eigenvectors" of matrix Σ :

$$\rightarrow [U, S, V] = \text{svd}(\text{Sigma});$$

U , S , and V are labeled as $n \times n$ matrix.

\rightarrow Singular value decomposition
 $\text{eig}(\text{Sigma})$

$$U = \begin{bmatrix} | & | & | & \dots & | \\ u^{(1)} & u^{(2)} & u^{(3)} & \dots & u^{(m)} \\ | & | & | & & | \end{bmatrix}$$

The first k columns are indicated by a bracket labeled k .

$$U \in \mathbb{R}^{n \times n}$$

$$u^{(1)}, \dots, u^{(k)}$$

Principal Component Analysis (PCA) algorithm

From $[U, S, V] = \text{svd}(\text{Sigma})$, we get:

$$\rightarrow U = \begin{bmatrix} | & | & & | \\ u^{(1)} & u^{(2)} & \dots & u^{(n)} \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$\underbrace{\hspace{10em}}_k$

$$x \in \mathbb{R}^n \rightarrow z \in \mathbb{R}^k$$

$$z^{(i)} = \underbrace{\begin{bmatrix} | & | & & | \\ u^{(1)} & u^{(2)} & \dots & u^{(k)} \\ | & | & & | \end{bmatrix}^T}_{\substack{n \times k \\ U_{\text{reduce}}}} x^{(i)} = \underbrace{\begin{bmatrix} \text{---} (u^{(1)})^T \text{---} \\ \vdots \\ \text{---} (u^{(k)})^T \text{---} \end{bmatrix}}_{\substack{k \times n \\ k \times 1}} \underbrace{x^{(i)}}_{\substack{n \times 1}}$$

$z \in \mathbb{R}^k$

Principal Component Analysis (PCA) algorithm summary

- After mean normalization (ensure every feature has zero mean) and optionally feature scaling:

$$\text{Sigma} = \frac{1}{m} \sum_{i=1}^m (x^{(i)})(x^{(i)})^T$$

→ $[U, S, V] = \text{svd}(\text{Sigma}) ;$

→ $\text{Ureduce} = U(:, 1:k) ;$

→ $z = \text{Ureduce}' * x ;$

↑

↑

$$x \in \mathbb{R}^n$$

~~$$x_0 = 1$$~~

$X = \begin{bmatrix} - & x^{(1)T} & - \\ & \vdots & \\ - & x^{(m)T} & - \end{bmatrix}$

→ $\boxed{\text{Sigma} = (1/m) * X' * X ;}$

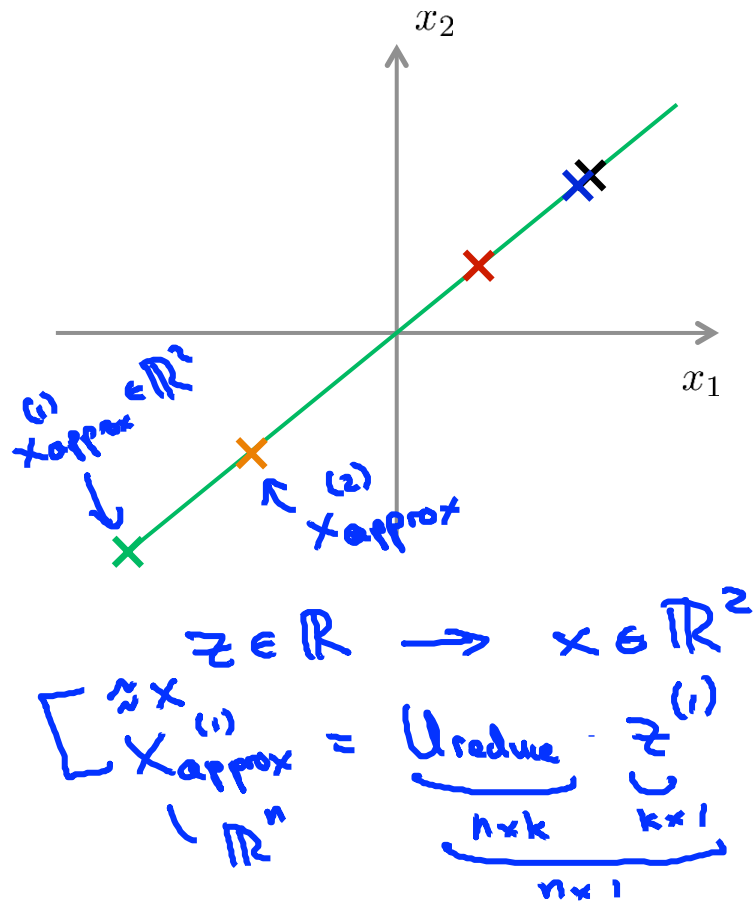
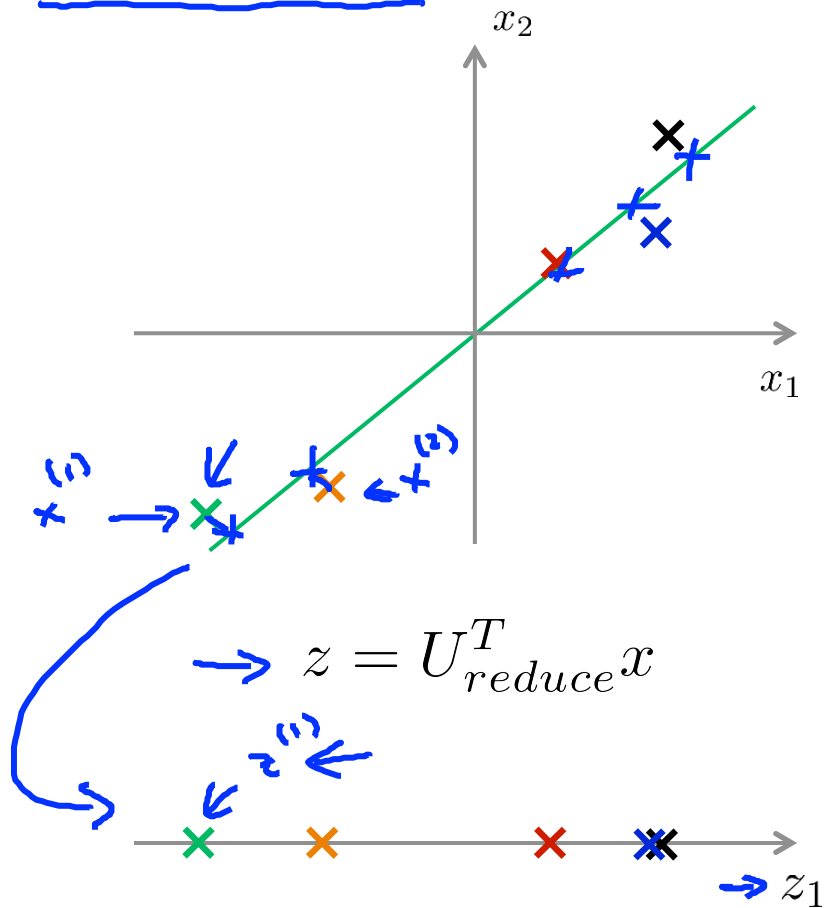


Machine Learning

Dimensionality Reduction

Reconstruction from
compressed
representation

Reconstruction from compressed representation





Machine Learning

Dimensionality Reduction

Choosing the number of principal components

Choosing k (number of principal components)

Average squared projection error: $\frac{1}{n} \sum_{i=1}^n \|x^{(i)} - x_{\text{approx}}^{(i)}\|^2$

Total variation in the data: $\frac{1}{n} \sum_{i=1}^n ||x^{(i)}||^2$

Typically, choose k to be smallest value so that

$$\begin{aligned} \rightarrow & \frac{\frac{1}{m} \sum_{i=1}^m \|x^{(i)} - x_{approx}\|^2}{\frac{1}{m} \sum_{i=1}^m \|x^{(i)}\|^2} \leq \frac{\cancel{0.01}}{\cancel{0.05} \quad 0.10} \quad \frac{\cancel{(1\%)}}{\cancel{5\%} \quad (10\%)} \end{aligned}$$

→ “~~99%~~ of variance is retained”
95 to 96%

Choosing k (number of principal components)

Algorithm:

Try PCA with $k=1$

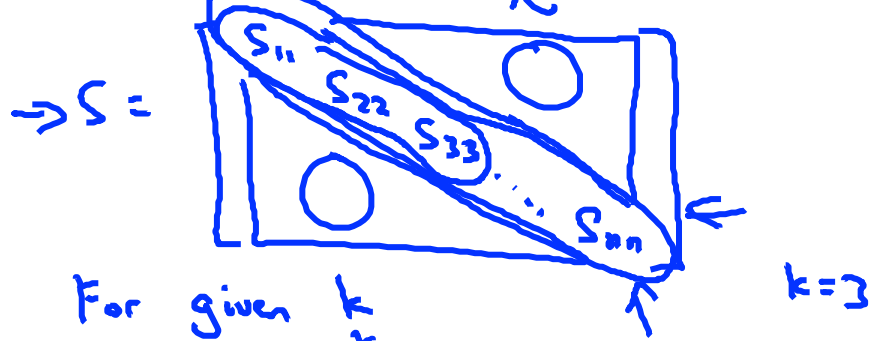
Compute $U_{reduce}, z^{(1)}, z^{(2)}, \dots, z^{(m)}, x_{approx}^{(1)}, \dots, x_{approx}^{(m)}$

Check if

$$\frac{\frac{1}{m} \sum_{i=1}^m \|x^{(i)} - x_{approx}\|^2}{\frac{1}{m} \sum_{i=1}^m \|x^{(i)}\|^2} \leq 0.01?$$

$k=17$

$$\rightarrow [U, S, V] = \text{svd}(\text{Sigma})$$



$$1 - \frac{\sum_{i=1}^k S_{ii}}{\sum_{i=1}^n S_{ii}} \leq 0.01$$

$$\frac{\sum_{i=1}^k S_{ii}}{\sum_{i=1}^n S_{ii}} \geq 0.99$$

Choosing k (number of principal components)

→ $[U, S, V] = \text{svd}(\text{Sigma})$

Pick smallest value of k for which

$$\frac{\sum_{i=1}^k S_{ii}}{\sum_{i=1}^m S_{ii}} \geq 0.99$$

$k=100$

(99% of variance retained)

↖



Machine Learning

Dimensionality Reduction

Advice for applying PCA

Supervised learning speedup

→ $(\underline{x^{(1)}}, y^{(1)}), (\underline{x^{(2)}}, y^{(2)}), \dots, (\underline{x^{(m)}}, y^{(m)})$

Extract inputs:

Unlabeled dataset: $\underline{x^{(1)}}, \underline{x^{(2)}}, \dots, \underline{x^{(m)}} \in \mathbb{R}^{10000}$

↓ PCA

$\underline{z^{(1)}}, \underline{z^{(2)}}, \dots, \underline{z^{(m)}} \in \mathbb{R}^{1000}$

New training set:

$(\underline{z^{(1)}}, y^{(1)}), (\underline{z^{(2)}}, y^{(2)}), \dots, (\underline{z^{(m)}}, y^{(m)})$

Note: Mapping $x^{(i)} \rightarrow z^{(i)}$ should be defined by running PCA only on the training set. This mapping can be applied as well to the examples $x_{cv}^{(i)}$ and $x_{test}^{(i)}$ in the cross validation and test sets

$$x^{(i)} \in \mathbb{R}^{10,000} \leftarrow \begin{matrix} 100 \\ 100 \end{matrix}$$

$x \rightarrow z$

$$h_{\theta}(z) = \frac{1}{1 + e^{-\theta^T z}}$$

$x \rightarrow z$

Application of PCA

- Compression

- Reduce memory/disk needed to store data
 - Speed up learning algorithm ←

Choose k by % of variance retain

- Visualization

$k=2$ or $k=3$

Bad use of PCA: To prevent overfitting

→ Use $z^{(i)}$ instead of $x^{(i)}$ to reduce the number of features to $k < n$. — 10000

Thus, fewer features, less likely to overfit.

Bad!

This might work OK, but isn't a good way to address overfitting. Use regularization instead.

$$\rightarrow \min_{\theta} \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \boxed{\frac{\lambda}{2m} \sum_{j=1}^n \theta_j^2} \leftarrow$$

PCA is sometimes used where it shouldn't be

Design of ML system:

- - Get training set $\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \dots, (x^{(m)}, y^{(m)})\}$
- - ~~Run PCA to reduce $x^{(i)}$ in dimension to get $z^{(i)}$~~
- - Train logistic regression on $\{(\cancel{z^{(1)}}), y^{(1)}), \dots, (\cancel{z^{(m)}}), y^{(m)})\}$
- - Test on test set: Map $x_{test}^{(i)}$ to $z_{test}^{(i)}$. Run $h_{\theta}(z)$ on $\{(z_{test}^{(1)}, y_{test}^{(1)}), \dots, (z_{test}^{(m)}, y_{test}^{(m)})\}$

→ How about doing the whole thing without using PCA?

→ Before implementing PCA, first try running whatever you want to do with the original/raw data $x^{(i)}$. Only if that doesn't do what you want, then implement PCA and consider using $z^{(i)}$.