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Analysis of the fully discrete fat boundary method

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Abstract The Fat Boundary Method is a method of the Fictitious Domain class, which was proposed to solve elliptic problems in complex geometries with non-conforming meshes. It has been designed to recover optimal convergence at any order, despite of the non-conformity of the mesh, and without any change in the discrete Laplace operator on the simple shape domain. We propose here a detailed proof of this high-order convergence, and propose some numerical tests to illustrate the actual behaviour of the method.

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1 Introduction

The term *Fictitious Domain Method*, in its most general sense, refers to a class of solution strategies aimed at solving elliptic problems in complex geometries. The typical situation, which is met for example in the context of composite material [26] or in the

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modelling of fluid particle flows [25], is the one of a perforated domain, say a cube in $\mathbb{R}^n$ minus a collection B of (possibly many) balls.

Mesh generation for domains of this kind can be costly and delicate (in particular when the balls move, like in fluid particle simulations). The fictitious domain approach simplifies this task by extending the computational domain to a simpler shape (the cubic box itself) and reformulating the problem in such a way that only computations on the extended domain are performed. Simple cartesian meshes can then be used, for which, in addition, fast solvers are available.

Among the most popular of such strategies, let us mention here as an example the *Fictitious Domain Methods* itself, introduced in the 1960s [31], and the *Immersed Boundary Method* introduced in the 1970s [27]. Since their introduction, these methods have not only been widely used for different applications (see for example [25] and the overview [28]), but have inspired, in the last decade, a variety of methods based on similar ideas. Let us mention for instance the so called *Physalis* [29], a Cartesian Grid method for moving 2D objects in viscous flows [30], the *Immersed Interface Method* [19], a new Cartesian Grid Finite Volume Method [24], a fixed-mesh ALE Method for moving domains [11], a direct forcing method in the simulations of particulate flows called *Proteus* [12] and more recently the *Smoothed profile method for particulate flows* [20]).

As far as we know, error estimates for most of these methods have never been precisely studied in the literature, especially when using high order discretization. In general, the price to pay for sacrificing the geometrical conformity is a degradation of the accuracy when compared with boundary fitted method. To make this point clear, let us consider the simple example of a Poisson problem in a domain obtained by subtracting an interior ball B from the unit cube of $\mathbb{R}^n$. Let then u be the solution to $-\Delta u = 1$ on $\Omega\backslash\bar{B}$, with homogeneous Dirichlet conditions on ∂B and $\partial\Omega$. First of all we remark that extending such a problem to a problem on Ω in such a way that the solution is globally smooth is far from being easy. It is quite easy to realize that the straightforward C^∞ extension to $f = 1$ in B yields a solution which presents a jump in the normal derivative across ∂B. In fact it is not sufficient to smoothly extend the right hand side f, but it is necessary that the extension is compatible with the Dirichlet condition on the curve ∂B, and, at present, there is no direct way of obtaining such a compatible extension. Therefore the solution to the extended problem will not be smooth. A direct calculation yields the well known result that in such case the best approximation error can not even be of order one, as one realizes by extending, for simplicity, u by 0 within B (such an extension will not behave worse than any other extension presenting a jump in the normal derivative) and by considering a cartesian mesh over Ω, and the associated space V_h of Q_1 finite elements, that is of all continuous functions which, restricted to any element of the mesh, are first order polynomials in each variable. Though the approximation of u by a function $v_h \in V_h$ can be expected to behave nicely (at the first order in h) away from ∂B, if we consider ω_h the set of cubes which intersect ∂B (see Fig. 1), in each cube C of ω_h, the gradient of u is either 0 (inside B) or has norm of order 1 (outside B). Then for any Q_1 functions, the integral over C of $|\nabla(u - u_h)|^2$ is of the order of h^n. As the number of such cubes behaves like $1/h^{n-1}$, the contribution of the interfacial zone in the global error is expected to behave at best like $(h^n/h^{n-1})^{1/2} = \sqrt{h}$. As far as the global error (over

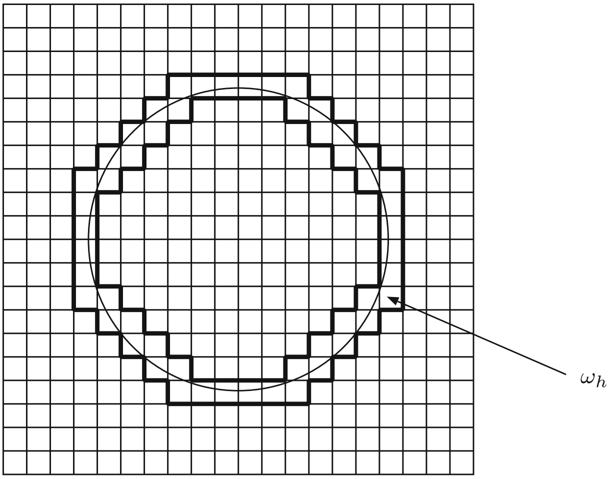

Fig. 1 Non-conforming mesh

the whole domain) is concerned, a better estimate cannot be expected. This order of convergence has been also confirmed for the approximation provided by several fictitious domain methods, see for instance [13,2] for an error estimate for the Lagrange multiplier approach, or [22] for the full error estimate for the penalty method on a cartesian mesh.

Yet, the actual domain of interest is $\Omega\backslash\overline{B}$, and, still considering for simplicity the example mentioned above, it is clear that when restricted to such a domain, the solution u to our problem is smooth. One could therefore hope for a better order of approximation, at least outside the set $\bar{B} \cup \omega_h$. Unfortunately the gain that one obtains in considering such restricted region is in general disappointingly small. For the fictitious domain method with Lagrange multiplier, the error on the restricted region increases only to an order $1 - \varepsilon$, independently of the order of approximation of the space V_h [4]. If this is a satisfactory result when considering P_1 or Q_1 finite elements, it shows that the method is far from being optimal when considered in the framework of high order approximation methods, which are then, in general, not well suited to be used in the fictitious domain approach. Nevertheless, a variant of the *Immersed Boundary Method* has been developed recently [14] and it was observed that for some particular cases it converges at a second order rate. In the same way, *Physalis* has been extended in order to get a second-order convergence for three-dimensional particle simulation [34]. Although it may be surprising, at our best knowledge, these observed rates of convergence have never been proved except in the framework of first order finite elements for the *Fictitious Domain methods with Lagrange Multipliers* [13,2], the *Penalty method* [1,3,22], and the *Fat Boundary Method* [21,7].

In this paper we consider precisely the Fat Boundary Method, introduced in [21] and especially designed to recover optimal convergence at any order, despite of the non-conformity of the mesh, and without any change in the discrete Laplace operator on the simple shape domain. The optimal behaviour of the method has been already supported by some numerical tests (see [7] for first order elements in dimension three, and [33], comparing the FBM with the Fictitious domain method with Lagrange multiplier, both with high order elements in dimension one) as well as by a first theoretical analysis for a simplified semi-discrete approach [7].

We will prove here that, for any dimension and any order of approximation, the FBM has an optimal behaviour in terms of accuracy. More precisely we will show that, provided the solution of the problem is smooth in the original computational domain $\Omega\backslash\overline{B}$, the approximation error achieved by the method in $\Omega\backslash\overline{B}$ is of the best possible order allowed by the approximation space considered. Moreover we propose numerical experiments with high order elements in both dimensions one and two, which corroborate this behaviour.

The paper is structured as follows:

Section 2 presents the principles of FBM method as introduced in [21] : reformulation of the initial problem (in $\Omega\backslash\overline{B}$) in the form of two coupled problems, a local one set in a neighbourhood ω of $\gamma = \partial B$ and a global one set in the domain Ω. We introduce the two main ingredients of the space discretization: an approximation space U_h over Ω, and a discrete normal derivative operator ∂_ν^h to account for the way the local solution is reinjected at the global level. We consider in fact a general class of such operators, we simply assume it verifies some properties (an example of such a construction in given in Sect. 6).

In Sect. 3 we prove some technical lemmas on which the analysis will be based. In particular, Lemma 1 asserts equivalence between the initial fixed point problem (FBM coupled formulation) and an auxiliary fixed point problem, on which forthcoming analysis will be focused.

In Sect. 4 we prove the convergence of the iterative algorithm (in its new formulation). The convergence at the continuous level was based on some monotonicity property of the operator involved ([21]). We show that this monotonicity is preserved at the discrete level, up to a first order term in the discretization parameters (Lemma 2). This quasi-monotonicity is sufficient to establish convergence of the iterative scheme, under some restriction in the relaxation parameter (Theorem 2). We finally establish that the discrete solution to the initial problem enjoys some stability property with respect to the data (Corollary 2).

Section 5 is dedicated to error estimates : Corollary 5 for the error away from obstacle B, and Subsect. 5.3 for the local error (i.e. in ω).

In Sect. 6 we propose an explicit way to define the discrete normal derivative (which controls the way the local step influences the global one), and we present in the last section some numerical tests to verify the actual behaviour of the algorithm in terms of convergence.

2 The fat boundary method

Let Ω be a domain in $\mathbb{R}^n$, which for simplicity we will assume to be of class C^∞, with boundary Γ and let B denote a collection of smooth subsets of positive measure, whose boundary will be denoted by γ (see Fig. 2 for the case of a single inclusion).

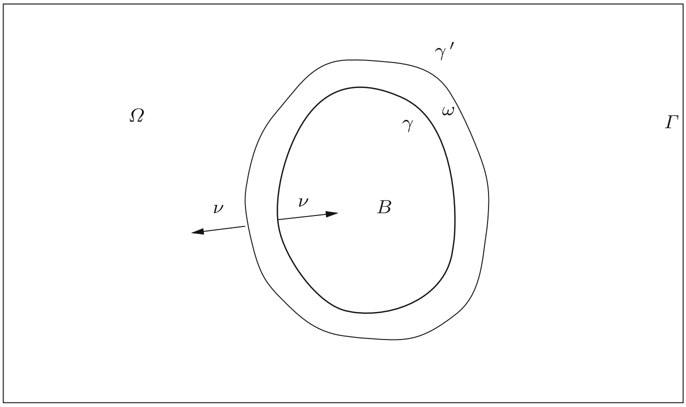

Fig. 2 Notations

We aim at solving the Poisson problem in $\Omega\backslash\bar{B}$

$$-\triangle u = f \text{ in } \Omega\backslash\bar{B}, \qquad u = 0 \text{ on } \Gamma \cup \gamma. \tag{1}$$

The idea underlying the *Fat Boundary Method* is to replace the above problem by two coupled new problems, one of which is set in the whole domain Ω, the other one being set in a suitable (narrow) domain $\omega \subset \Omega\backslash\bar{B}$ delimited by γ and by a smooth artificial boundary $\gamma' \subset \Omega\backslash\bar{B}$ (see Fig. 2). To this aim we introduce an extension to Ω of the right hand side f, originally defined in $\Omega\backslash\bar{B}$, obtained by setting $f = 0$ in B. By abuse of notation we also call such an extension f, and we will assume $f \in L^2(\Omega)$. We then replace problem (1) by the following two coupled problems:

$$-\triangle u = f + \delta_\gamma \partial_\nu v \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

$$-\triangle v = f \quad \text{in } \omega, \quad v = 0 \quad \text{on } \gamma, \quad v = u \quad \text{on } \gamma', \tag{3}$$

with unknowns $u \in H_0^1(\Omega)$ and $v \in H_{0,\gamma}^1(\omega)$, where $H_{0,\gamma}^1(\omega)$ denotes the subset of $H^1(\omega)$ of functions with trace vanishing on γ

$$H_{0,\gamma}^1(\omega) = \{u \in H^1(\omega) : \ u|_\gamma = 0\},$$

and where, for $\eta \in H^{-1/2}(\gamma)$, we denote by $\eta\delta_\gamma$ the $H^{-1}(\Omega)$ element defined by

$$\int\limits_\Omega (\eta\delta_\gamma) w = \int\limits_\gamma \eta w, \quad \forall w \in H_0^1(\Omega).$$

The idea is to find the solution of the two coupled problems (2) and (3) by a suitable fixed point procedure, namely, given u^n, solve

$$-\triangle v^n = f \quad \text{in } \omega, \quad v^n = 0 \quad \text{on } \gamma, \quad v^n = u^n \quad \text{on } \gamma'; \tag{a}$$

let next $\tilde{u}^{n+1}$ be defined as the solution of

$$-\triangle\tilde{u}^{n+1} = f + \delta_\gamma \partial_\nu v^n \qquad \text{in } \Omega, \quad \tilde{u}^{n+1} = 0 \quad \text{on } \partial\Omega; \tag{b}$$

finally apply a relaxation step with a parameter θ and set

$$u^{n+1} = \theta u^n + (1-\theta)\tilde{u}^{n+1}.$$

It has been proven in [21] that there exists a $\theta_0 \in]0, 1[$ such that for all $\theta \in]\theta_0, 1[$ the resulting sequence u^n converges to a limit $u \in H^1_0(\Omega)$ whose restriction to $\Omega\backslash\bar{B}$ is the solution of Problem (1).

In practice the two problems are solved numerically. Problem (**b**) is treated by a Galerkin method. To this aim, we will denote by $U_h \subset H^1_0(\Omega)$ a family of discretization spaces depending on a mesh-size parameter $h = h_\Omega$ and satisfying suitable properties which will be specified in Sect. 3.

As far as problem (**a**) is concerned a Galerkin method is also possible, but it is not necessarily the best option. Since the only feedback given by problem (**a**) to problem (**b**) is in terms of the trace on γ of the outer normal derivative of the solution v, we will assume that given any $f \in L^2(\omega)$ and $\eta \in H^{1/2}(\gamma')$ we have a way of numerically computing (with suitable accuracy) an approximation $\partial^h_\nu v$ to the outer normal derivative $\partial_\nu v$ on γ of the solution v to the problem

$$-\triangle v = f \quad \text{in } \omega, \quad v = 0 \quad \text{on } \gamma, \quad v = \eta \quad \text{on } \gamma',$$

without specifically choosing a solution method (see Sect. 6 for a possible solution method satisfying all the assumptions that we will need in the following).

The fully discrete FBM iteration can be defined as follows. Let $u^n_h \in U_h$ be given. Let $\tilde{v}^n \in H^1_{0,\gamma}(\omega)$ be the solution of

$$-\triangle\tilde{v}^n = f \quad \text{in } \omega, \quad \tilde{v}^n = 0 \quad \text{on } \gamma, \quad \tilde{v}^n = u^n_h \quad \text{on } \gamma'.$$

We define $\tilde{u}^{n+1}_h \in U_h$ as the unique element satisfying, for all $w_h \in U_h$

$$\int_\Omega \nabla u^{n+1}_h \cdot \nabla w_h = \int_\Omega f w_h + \int_\gamma \partial^h_\nu \tilde{v}^n w_h.$$

As in the continuous framework we define u^{n+1}_h as

$$u^{n+1}_h = \theta u^n_h + (1-\theta)\tilde{u}^{n+1}_h. \tag{\textbf{h}}$$

Remark 1 The FBM iteration can be viewed as a relaxed Dirichlet–Neumann type overlapping Schwartz iteration, in which one of the two subdomains, namely ω is entirely embedded in the other, which, in its turn, is a fictitious domain, that is, it is constructed as an extension of $\Omega\backslash\bar{B}$, the original domain of definition of the problem considered. The main difference with respect to classical Dirichlet–Neumann iterations is that the Neumann data is not imposed at the boundary of the domain Ω but at an interior curve γ, coinciding with a portion of the boundary of the original domain $\Omega\backslash\bar{B}$. In this framework the analysis of the method requires not only to cope with the interaction of non matching grids (the use of which is inherent to the fictitious domain approach) but also to to deal with the artificial singularity across γ, by showing that it does not affect the convergence rate of the method.

3 Analysis of the fully discrete FBM

Let $\Pi_h : H_0^1(\Omega) \to U_h$ denote the Galerkin projection, that is, for $u \in H_0^1(\Omega)$, $\Pi_h u$ is the unique element in U_h such that for all $w_h \in U_h$

$$\int_\Omega \nabla \Pi_h u \cdot \nabla w_h = \int_\Omega \nabla u \cdot \nabla w_h.$$

Let $\mathcal{T}_f^h : H_0^1(\Omega) \to U_h$ be defined as follows:

$$\int_\Omega \nabla(\mathcal{T}_f^h u) \cdot \nabla w_h = \int_\Omega f w_h + \int_\gamma \partial_\nu^h v w_h, \quad \forall w_h \in U_h$$

v being the unique solution of

$$-\triangle v = f, \text{ in } \omega, \qquad \text{with boundary conditions} \qquad v_{|\gamma} = 0, \text{ and } v_{|\gamma'} = (\Pi_h u)_{|\gamma'}.$$

It is easy to check that, provided that the iterative procedure (**h**) described in the previous section converges, the limit function u_h is a fixed point of the operator $\mathcal{T}_f^h$:

$$u_h = \mathcal{T}_f^h(u_h).$$

We aim here at proving that $\mathcal{T}_f^h$ has indeed a unique fixed point which is a good approximation to the solution u of Eq. (1), for which we want to provide an error estimate. Moreover we want to prove that the iterative procedure (**h**) actually converges to the fixed point u_h, provided the relaxation parameter θ is in a suitable range.

Clearly the behavior of (**h**) will depend on the choice of the approximation space U_h and on the discrete outer normal derivative operator ∂_ν^h. The forthcoming analysis will be carried out under the following assumptions.

3.1 Assumptions on U_h

Given any fixed concentric spheres G_0 and G with $G_0 \subset\subset G \subset\subset \Omega$ there exists an h_0 such that for all $h \leq h_0$ we have for some $R \geq 1$, $M > 1$

A1 For each $u \in H^1(G)$ there exists $\eta \in U_h$ such that for any $0 \le s \le R, s \le \ell \le M$

$$\|u - \eta\|_{s,G} \lesssim h_\Omega^{\ell-s} \|u\|_{\ell,G}.$$

Moreover if $u \in H_0^1(G_0)$ then η can be chosen to satisfy $\eta \in H_0^1(G)$.

A2 Let $\varphi \in C_0^\infty(G_0)$ and $u_h \in U_h$. Then there exists $\eta \in U_h \cap H_0^1(G)$ such that

$$\|\varphi u_h - \eta\|_{1,G} \le C(\varphi, G, G_0) h_\Omega \|u_h\|_{1,G}.$$

A3 For each $h \le h_0$ there exists a domain G_h with $G_0 \subset\subset G_h \subset\subset G$ such that if $0 \le t \le s \le R$ then for all $u_h \in U_h$ we have that

$$\|u_h\|_{s,G_h} \lesssim h^{t-s} \|u_h\|_{t,G_h}.$$

Assumptions (**A1**) and (**A3**) are quite standard and they are satisfied by a wide variety of approximation spaces, among which we can count all finite element spaces defined on quasiuniform meshes. The parameters R and M play respectively the role of the regularity and order of approximation of the approximation space U_h (for example for Pq finite elements we have that $R = 3/2 - \varepsilon$ and $M = q + 1$). Assumption (**A2**) is less common but it is also satisfied by a wide class of approximation spaces (including finite elements) (see for instance [23,5]).

3.2 Assumptions on ∂_ν^h

As already mentioned we do not want to choose a particular method for solving problem (**a**). We only assume that, given the Laplacian in ω and the trace on γ' of any function v in $H_{0,\gamma}^1(\omega)$ with $\triangle v \in L^2(\omega)$, we are able to compute an approximation ∂_ν^h of $\partial_\nu v$ on γ. In order to prove convergence of the discrete FBM we need the discrete outer normal derivative operator ∂_ν^h to be linear, stable and to provide a sufficiently good approximation for the outer normal derivative in the natural $H^{-1/2}(\gamma)$-norm. More precisely, we make the following assumptions, for some parameter $S \ge 2$.

B1 If $v \in H_{0,\gamma}^1(\omega)$ is such that $\triangle v \in L^2(\omega)$ then

$$\|\partial_\nu^h v\|_{-1/2,\gamma} \lesssim \|\triangle v\|_{0,\omega} + \|v\|_{1/2,\gamma'}. \tag{4}$$

B2 If $v \in H^s(\omega)$, $2 \le s \le S$ then

$$\|\partial_\nu v - \partial_\nu^h v\|_{-1/2,\gamma} \lesssim h_\omega^{s-1} \|v\|_{s,\omega}$$

Remark 2 In Sect. 6 we will show how to construct one particular operator ∂_ν^h that satisfies assumptions (**B1**) and (**B2**).

3.3 A continuous counterpart of the method

In order to prove convergence of the iterative procedure we need to introduce an auxiliary iterative procedure which is (in the sense specified by Lemma 1 in the following) equivalent to the fully discrete FBM iterative scheme (**h**).

Given $u^n \in H_0^1(\Omega)$ let v^n be the solution of

$$-\triangle v^n = f, \quad \text{in } \omega, \qquad \text{with boundary conditions} \qquad v^n_{|\gamma} = 0, \ \text{and } v^n_{|\gamma'} = (\Pi_h u^n)_{|\gamma'}.$$

Define $\tilde{u}^{n+1}$ as the solution of

$$-\triangle \tilde{u}^{n+1} = f + \partial_\nu^h v^n \delta_\gamma, \qquad \text{with boundary conditions} \qquad \tilde{u}^{n+1}_{|\Gamma} = 0.$$

Finally define u^{n+1} as

$$u^{n+1} = \theta u^n + (1-\theta)\tilde{u}^{n+1}. \tag{$\star$}$$

Let us also define the corresponding operator $\mathcal{T}_f^\star : H_0^1(\Omega) \to H_0^1(\Omega)$ as

$$-\triangle(\mathcal{T}_f^\star u) = f + \partial_\nu^h v^\star \delta_\gamma,$$

$v^\star$ being the unique solution of

$$-\triangle v^\star = f, \ \text{ in } \omega, \qquad \text{with boundary conditions} \qquad v^\star_{|\gamma} = 0, \ \text{and } v^\star_{|\gamma'} = (\Pi_h u)_{|\gamma'}.$$

Lemma 1 *Let $u^\star$ be a fixed point for the operator $\mathcal{T}_f^\star$. Then $u_h = \Pi_h u^\star$ is a fixed point of $\mathcal{T}_f^h$. Conversely let u_h be a fixed point of $\mathcal{T}_f^h$ then, letting $u^\star \in H_0^1(\Omega)$ be defined as $u^\star = \mathcal{T}_f^\star u_h$, we have that $u^\star$ is a fixed point for $\mathcal{T}_f^\star$. Moreover, provided $u_h^0 = \Pi_h u^0$, if the iterative procedure ($\star$) converges then so does the iterative procedure (**h**).*

Proof We start by observing that the following identities hold

$$\mathcal{T}_f^\star \circ \Pi_h = \mathcal{T}_f^\star, \qquad \mathcal{T}_f^h = \Pi_h \circ \mathcal{T}_f^\star$$

Let now $u^\star$ be a fixed point for $\mathcal{T}_f^\star$: $\mathcal{T}_f^\star u^\star = u^\star$. We have

$$\mathcal{T}_f^h \Pi_h u^\star = \Pi_h \mathcal{T}_f^\star \Pi_h u^\star = \Pi_h \mathcal{T}_f^\star u^\star = \Pi_h u^\star,$$

whence $u_h = \Pi_h u^\star$ is a fixed point for $\mathcal{T}_f^h$.

Let us now assume that u_h is a fixed point for $\mathcal{T}_f^h$: $\mathcal{T}_f^h u_h = u_h$. We have

$$\mathcal{T}_f^\star \mathcal{T}_f^\star u_h = \mathcal{T}_f^\star \Pi_h \mathcal{T}_f^\star u_h = \mathcal{T}_f^\star \mathcal{T}_f^h u_h = \mathcal{T}_f^\star u_h,$$

whence $u^\star = \mathcal{T}_f^\star u_h$ is a fixed point for $\mathcal{T}_f^\star$.

Let us now compare the two iterative procedures (**h**) and ($\star$). Let $u_h^0 = \Pi_h u^0$. By induction we can prove that $u_h^n = \Pi_h u^n$ for all n. In fact, assuming that $u_h^{n-1} = \Pi_h u^{n-1}$, we have

$$\begin{aligned} u_h^n &= \theta u_h^{n-1} + (1-\theta)\mathcal{T}_f^h u_h^{n-1} = \theta \Pi_h u^{n-1} + (1-\theta)\Pi_h \mathcal{T}_f^\star \Pi_h u^{n-1} \\ &= \Pi_h(\theta u^{n-1} + (1-\theta)\mathcal{T}_f^\star u^{n-1}) = \Pi_h u^n. \end{aligned}$$

By the continuity of Π_h we immediately deduce that if u^n converges to a fixed point $u^\star$ of $\mathcal{T}_f^\star$, then u_h^n converges to the fixed point $u_h = \Pi_h u^\star$ of $\mathcal{T}_f^h$. □

3.4 Preliminary bounds

In the analysis of the fully discrete FBM we will make use of several known results which, for the sake of clarity, we collect in this section. The first bound that we will need is a kind of inverse inequality which holds for harmonic functions.

Proposition 1 *Let $\Omega' \subset\subset \Omega\backslash\bar{B}$ and let u satisfy $-\triangle u = 0$ in $\Omega\backslash\bar{B}$, $u = 0$ on $\partial\Omega$. Then we have*

$$\|u\|_{2,\Omega_0} \lesssim \|u\|_{1,\Omega\backslash\bar{B}}. \tag{5}$$

Proof Let $\rho \in C^\infty(\Omega)$ with $\rho = 1$ in Ω_0 and $\rho = 0$ in B. Clearly it holds that

$$\|u\|_{2,\Omega'} = \|\rho u\|_{2,\Omega'} \leq \|\rho u\|_{2,\Omega\backslash\bar{B}}.$$

We now observe that $\rho u = 0$ on $\partial(\Omega\backslash\bar{B})$. By standard regularity results on the solution of the Poisson equation with homogeneous Dirichlet boundary conditions ([8]) we can then write

$$\|u\|_{2,\Omega'} \lesssim \|\triangle(\rho u)\|_{0,\Omega\backslash\bar{B}}. \tag{6}$$

Leibnitz derivation rule gives us (recall that $\triangle u = 0$ in $\Omega\backslash\bar{B}$)

$$\triangle(\rho u) = u\triangle\rho + \rho\triangle u + 2\nabla\rho\cdot\nabla u = u\triangle\rho + 2\nabla\rho\cdot\nabla u$$

whence the thesis. Remark that the constant in the inequality depends, through the function ρ, on the geometry of Ω, Ω' and B. □

A key tool in the forthcoming analysis is the following theorem by Nitsche and Schatz [23].

Theorem 1 *Let $\Omega_0 \subset\subset \Omega_1 \subseteq \Omega$ and let U_h satisfy assumptions (**A1–3**). Let $u \in H^\ell(\Omega_1) \cap H_0^1(\Omega)$, with $1 \leq \ell \leq M$, $u_h \in U_h$ and let p be a non negative* *integer, arbitrary but fixed. Suppose that $u - u_h$ satisfies*

$$\int_{\Omega} \nabla(u - u_h) \cdot \nabla w_h = 0, \forall w_h \in U_h \cap H_0^1(\Omega_1).$$

Then there exists h_1 such that if $h_\Omega < h_1$ we have, for $s = 0, 1$,

$$\|u - u_h\|_{s,\Omega_0} \lesssim h_\Omega^{\ell - s} \|u\|_{\ell,\Omega_1} + \|u - u_h\|_{-p,\Omega_1}.$$

Remark 3 In the following we will need to apply the above theorem with $\Omega_0 = \Omega \backslash (\bar{\omega} \cup \bar{B})$ and $\Omega_1 = \Omega \backslash \bar{B}'$, with $B \subset\subset B' \subset\subset \omega \cup \bar{B}$. Since $\partial\Omega_0 \cap \partial\Omega_1 = \partial\Omega \neq \emptyset$, we do not have $\Omega_0 \subset\subset \Omega_1$. It is however possible (see [6] for a similar result) to verify that Theorem 1 still holds in this case.

In view of the above remark, by combining Theorem 1 with a standard duality estimate [10] we obtain the following Corollary:

Corollary 1 *Under the assumptions of Theorem 1 let $u \in H_0^1(\Omega)$ such that $\triangle u = 0$ in $\Omega \backslash \bar{B}$. Then if $h_\Omega < h_1$ we have*

$$\|u - \Pi_h u\|_{1,\Omega \backslash (\bar{\omega} \cup \bar{B})} \lesssim h_\Omega |u|_{1,\Omega}.$$

Proof Let Ω' be such that $\Omega \backslash (\bar{\omega} \cup \bar{B}) \subset\subset \Omega' \subset\subset \Omega \backslash \bar{B}$. By applying Theorem 1 with $\Omega_0 = \Omega \backslash (\bar{\omega} \cup \bar{B})$ and $\Omega_1 = \Omega'$, we get

$$\|u - \Pi_h u\|_{s,\Omega \backslash (\bar{\omega} \cup \bar{B})} \lesssim h_\Omega \|u\|_{2,\Omega'} + \|u - \Pi_h u\|_{0,\Omega'}.$$

Using (5) we can bound the first term of the sum on the right hand side, while a standard duality argument yield

$$\|u - \Pi_h u\|_{0,\Omega'} \lesssim h_\Omega \|u - \Pi_h u\|_{1,\Omega} \lesssim h_\Omega \|u\|_{1,\Omega}. \tag{7}$$

Applying Poincaré's inequality we obtain the thesis. □

In the following we will also make use of the following identity, which already played a key role in the analysis of both continuous and semi-discrete FBM (see [21,7]):

Proposition 2 *Let $\hat{\Omega}$ be a bounded domain with $\partial\hat{\Omega} = \hat{\gamma} \cup \gamma'$ and let $u \in H^1_{0,\hat{\gamma}}(\hat{\Omega})$ such that $\triangle u = 0$ in $\hat{\Omega}$. Then we have*

$$\int_{\gamma'} u \partial_{\hat{\nu}} u = |u|^2_{1,\hat{\Omega}},$$

where we denote by $\hat{\nu}$ the outer normal on γ'.

4 Convergence of the iterative procedure

Let us now come to the analysis of the discrete FBM. We want to start by proving existence and uniqueness of the fixed point u_h of the operator $\mathcal{T}_f^h$ as well as convergence of the iterative procedure (**h**). Thanks to Lemma 1 we rather study the properties of $\mathcal{T}_f^\star$ and we do so by analysing the iterative procedure ($\star$). We start by considering the case $f \equiv 0$. We denote by $\mathcal{T}^\star : H_0^1(\Omega) \to H_0^1(\Omega)$ the operator defined as follows.

$$-\triangle(\mathcal{T}^\star u) = \partial_\nu^h v \delta_\gamma,$$

v being the unique solution of

$$-\triangle v = 0, \quad \text{in } \omega, \qquad \text{with boundary conditions} \qquad v_{|\gamma} = 0, \text{ and } v_{|\gamma'} = (\Pi_h u)_{|\gamma'}.$$

($\mathcal{T}^\star$ coincides with $\mathcal{T}_f^\star$ for $f \equiv 0$). The following lemma holds.

Lemma 2 *Under assumptions* (**A1–3**) *and* (**B1–2**)*, provided $h_\Omega \le h_1$ (h_1 given by Theorem 1), if u satisfies $\triangle u = 0$ in $\Omega \backslash \bar{B}$ and in B, then*

$$\int_\Omega \nabla \mathcal{T}^\star u \cdot \nabla u \lesssim (h_\Omega + h_\omega)|u|_{1,\Omega}^2.$$

Proof Using the definition of $\mathcal{T}^\star$ we can write:

$$\int_\Omega \nabla \mathcal{T}^\star u \cdot \nabla u = \int_\gamma \partial_\nu^h v u = \int_\gamma \partial_\nu v u + \int_\gamma (\partial_\nu^h - \partial_\nu) v u.$$

On the other hand, since $v_{|\gamma} = 0$ and both $\triangle u = 0$ and $\triangle v = 0$ in ω we can write

$$0 = -\int_\omega \triangle u v = \int_\omega \nabla u \cdot \nabla v - \int_{\gamma'} \partial_\nu u v = \int_{\gamma'} \partial_\nu v u + \int_\gamma \partial_\nu v u - \int_{\gamma'} \partial_\nu u v$$

whence

$$\int_\gamma \partial_\nu v u = \int_{\gamma'} \partial_\nu u v - \int_{\gamma'} \partial_\nu v u.$$

Now, using $-\triangle u = 0$ in $\Omega\backslash(\bar{B}\cup\bar{\omega})$ and $u = 0$ on $\partial\Omega$ we have that (recall that $\partial(\Omega\backslash(\bar{B}\cup\bar{\omega})) = \gamma'\cup\partial\Omega$)

$$\int_{\gamma'}\partial_\nu u v = \int_{\gamma'}\partial_\nu u \Pi_h u = \int_{\gamma'} u\partial_\nu u + \int_{\gamma'}(\Pi_h u - u)\partial_\nu u$$
$$= -|u|^2_{1,\Omega\backslash(\bar{B}\cup\bar{\omega})} + \int_{\gamma'}(\Pi_h u - u)\partial_\nu u.$$

Moreover, we can write

$$\int_{\gamma'}\partial_\nu v u = \int_{\gamma'}\partial_\nu v(\Pi_h u) + \int_{\gamma'}\partial_\nu v(u - \Pi_h u)$$
$$= \int_{\gamma'}\partial_\nu v v + \int_{\gamma'}\partial_\nu v(u - \Pi_h u)$$
$$= |v|^2_{1,\omega} + \int_{\gamma'}(u - \Pi_h u)\partial_\nu v$$

which gives us,

$$\int_\Omega \nabla\mathcal{T}^\star u\cdot\nabla u = -|u|^2_{1,\Omega\backslash(\bar{B}\cup\bar{\omega})} - |v|^2_{1,\omega} + \int_\gamma(\partial^h_\nu - \partial_\nu)vu$$
$$+\int_{\gamma'}(\Pi_h u - u)\partial_\nu u - \int_{\gamma'}(u - \Pi_h u)\partial_\nu v$$
$$= -|u|^2_{1,\Omega\backslash(\bar{B}\cup\bar{\omega})} - |v|^2_{1,\omega} + I + II + III.$$

The terms II and III are easily bound by assumption (**B1**) and Corollary 1: if $h_\Omega < h_0$

$$\begin{aligned} II &\lesssim \|\Pi_h u - u\|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})}(\|\triangle u\|_{0,\Omega\backslash(\bar{\omega}\cup\bar{B})} + |u|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})}) \\ &\lesssim h_\Omega|u|_{1,\Omega}|u|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})} \end{aligned} \tag{8}$$

and

$$III \lesssim \|\Pi_h u - u\|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})}(\|\triangle v\|_{0,\omega} + |v|_{1,\omega}) \lesssim h_\Omega|u|_{1,\Omega}|v|_{1,\omega}. \tag{9}$$

We then need to bound I. Remark that, since in general $U_h|_{\gamma'} \not\subset H^{3/2}(\gamma')$, $v^\star$ does not verify $v^\star \in H^2(\omega)$, so we cannot use assumption (**B2**) to bound I directly. In order to estimate I we then introduce the solution $\bar{v}$ of

$$-\triangle\bar{v} = 0,\ \text{ in } \omega,\ \text{ with boundary conditions } \bar{v}_{|\gamma} = 0,\quad \bar{v}_{\gamma'} = u.$$

We remark that $\triangle u = 0$ in $\Omega\backslash\bar{B}$ and $\gamma' \subset \Omega\backslash\bar{B}$ imply $u|_{\gamma'} \subset H^{3/2}(\gamma')$, whence $\bar{v} \in H^2(\omega)$. We now add and subtract $\partial_\nu^h \bar{v}$ and $\partial_\nu \bar{v}$ in I, obtaining

$$
\begin{aligned}
I &= \int_\gamma (\partial_\nu^h v - \partial_\nu v) u = \int_\gamma u \partial_\nu^h (v - \bar{v}) + \int_\gamma u(\partial_\nu^h - \partial_\nu)\bar{v} + \int_\gamma u \partial_\nu (\bar{v} - v) \\
&= IV + V + VI.
\end{aligned}
$$

Let us bound the three terms separately. As far as V is concerned, using (**B2**) we have

$$
V \leq \|(\partial_\nu^h - \partial_\nu)\bar{v}\|_{-1/2,\gamma} \|u\|_{1/2,\gamma} \lesssim h_\omega \|\bar{v}\|_{2,\omega} \|u\|_{1/2,\gamma}.
$$

Now we remark that, since $\triangle \bar{v} = 0$ we can bound

$$
\|\bar{v}\|_{2,\omega} \lesssim \|\bar{v}\|_{3/2,\partial\omega} = \|\bar{v}\|_{3/2,\gamma'} = \|u\|_{3/2,\gamma'} \lesssim \|u\|_{2,\Omega\backslash(\bar{\omega}\cup\bar{B})}.
$$

Using (5) and bounding the $H^1(\Omega\backslash(\bar{B}\cup\bar{\omega}))$ norm with the corresponding semi-norm (which we can do by Poincaré's inequality, since $u = 0$ on $\partial\Omega$) we can bound

$$
V \lesssim h_\omega |u|_{1,\Omega}^2.
$$

Let us now bound the term IV (the term VI is bounded by an analogous argument). Since $\triangle(\bar{v} - v) = 0$, applying (4) and Corollary (1) gives (recall that on γ' it holds that $v = \Pi_h u$)

$$
\begin{aligned}
IV &\lesssim \|\partial_\nu^h (\bar{v} - v)\|_{-1/2,\gamma} \|u\|_{1/2,\gamma} \lesssim \|\bar{v} - v\|_{1/2,\gamma'} \|u\|_{1/2,\gamma} \\
&= \|u - \Pi_h u\|_{1/2,\gamma'} \|u\|_{1/2,\gamma} \lesssim \|u\|_{1/2,\gamma} \|u - \Pi_h u\|_{1,\Omega\backslash(\bar{B}\cup\bar{\omega})} \lesssim h_\Omega |u|_{1,\Omega}^2.
\end{aligned}
$$

Analogously, as far as VI is concerned we get the bound

$$
VI \lesssim h_\Omega |u|_{1,\Omega}^2.
$$

Collecting the bounds on IV, V, and VI, we obtain

$$
I \lesssim (h_\omega + h_\Omega)|u|_{1,\Omega}^2.
$$

We now collect the bounds on I, II and III and obtain that for some suitable constants c_1, c_2 and c_3

$$
\begin{aligned}
\int_\Omega \nabla \mathcal{T}^\star u \cdot \nabla u &\leq -|u|_{1,\Omega\backslash(\bar{B}\cup\bar{\omega})}^2 - |v|_{1,\omega}^2 + c_1 h_\Omega \|u\|_{1,\Omega}^2 \\
&\quad + c_2 h_\Omega |u|_{1,\Omega} |v|_{1,\omega} + c_3 h_\omega |u|_{1,\Omega}^2.
\end{aligned}
$$

Applying a Schwartz–Hölder inequality of the form $ab = 2a(b/2) \leq a^2 + b^2/4$ we obtain

$$\int\limits_{\Omega} \nabla \mathcal{T}^{\star} u \cdot \nabla u \leq -|v|_{1,\omega}^2 + c_1 h_{\Omega} \|u\|_{1,\Omega}^2 + |v|_{1,\omega}^2 + c_2^2 h_{\Omega}^2 \frac{1}{4} |u|_{1,\Omega}^2 + c_3 h_{\omega} |u|_{1,\Omega}^2,$$

which, since $h_{\Omega}^2 \lesssim h_{\Omega}$, immediately yields the thesis. □

We can then prove the main result of this section.

Theorem 2 *Let* (**A1–3**) *and* (**B1–2**) *hold. There exist* $h_0 \in]0, h_1[$ *and* $\theta_0 \in]0, 1[$ *such that if* θ, h_{Ω} *and* h_{ω} *satisfy respectively* $\theta_0 < \theta < 1$ *and* $h_{\Omega} + h_{\omega} \leq h_0$ *then the iterative procedure* $(\star)$ *converges to a limit* $u^{\star}$*, provided that the initial guess* u_0 *verifies* $-\triangle u_0 = f$ *both in* B *and in* $\Omega \backslash \bar{B}$.

Proof We will prove that the iterative procedure verifies a contractivity property. With u_0 and u_1 given, with $-\triangle(u_i) = f$ in B and in $\Omega \backslash B$, let U_0 and U_1 be defined as

$$U_i = \theta u_i + (1-\theta) \mathcal{T}_f^{\star} u_i, \qquad i = 0, 1.$$

We start by observing that we have

$$\begin{aligned} U_0 - U_1 = \theta(u_0 - u_1) + (1-\theta)\mathcal{T}^{\star}(u_0 - u_1) &= (u_0 - u_1) \\ &+ (1-\theta)(\mathcal{T}^{\star}(u_0 - u_1) - (u_0 - u_1)). \end{aligned}$$

Set $d = u_0 - u_1$. Since $h_{\Omega} < h_{\Omega} + h_{\omega} \leq h_0 \leq h_1$, using Lemma 2 and observing that the operator $\mathcal{T}^{\star}$ from $H_0^1(\Omega)$ to itself is bounded, we can write

$$\begin{aligned} |U_0 - U_1|_{1,\Omega}^2 &= |d|_{1,\Omega}^2 + (1-\theta)^2 |\mathcal{T}^{\star} d - d|_{1,\Omega}^2 + 2(1-\theta) \int\limits_{\Omega} \nabla(\mathcal{T}^{\star} d - d) \cdot \nabla d \\ &= |d|_{1,\Omega}^2 + (1-\theta)^2 |\mathcal{T}^{\star} d - d|_{1,\Omega}^2 - 2(1-\theta)|d|_{1,\Omega}^2 \\ &\quad + 2(1-\theta) \int\limits_{\Omega} \nabla \mathcal{T}^{\star} d \cdot \nabla d \\ &\leq (1 - 2(1-\theta)(1 - C(h_{\Omega} + h_{\omega})) + (1-\theta)^2 C_{\Omega}) |d|_{1,\Omega}^2. \end{aligned}$$

It is not difficult to verify that, provided $1 - C(h_{\omega} + h_{\Omega}) > 0$, if

$$1 - \theta < 2(1 - C(h_{\omega} + h_{\Omega}))/C_{\Omega},$$

then $(1 - 2(1-\theta)(1 - C(h_{\Omega} + h_{\omega})) + (1-\theta)^2 C_{\Omega}) < 1$.

We cannot at this point apply the Banach fixed point Theorem directly since contractivity only holds for the subset of elements in $H_0^1(\Omega)$ with Laplacian equal to (the same) f. It is however easy to verify that the proof of the Banach fixed point theorem can be repeated without any modification in the present case. As a result we obtain that the operator $\theta \mathbf{1} + (1-\theta)\mathcal{T}_f^{\star}$ has a unique fixed point $u^{\star}$. □

Consequently both $\mathcal{T}_f^\star$ and $\mathcal{T}_f^h$ have a unique fixed point and, provided u_h^0 is chosen as $\Pi_h u^0$ with $-\Delta u^0 = f$ in B and in $\Omega\backslash\bar{B}$, the iterative procedure (**h**) converges to the fixed point of the latter.

As far as stability is concerned we have the following Theorem.

Theorem 3 *Under the assumptions of Theorem 2, the fixed point $u^\star$ of $\mathcal{T}_f^\star$ verifies*

$$\|u^\star\|_{1,\Omega} \lesssim \|f\|_{-1,\Omega} + \|f\|_{L^2(\omega)}.$$

Proof We recall that, being a fixed point for $\mathcal{T}_f^\star$, by definition $u^\star$ verifies

$$-\triangle u^\star = f + \partial_\nu^h v^\star \delta_\gamma,$$

where $v^\star \in H^1_{0,\gamma}(\omega)$ is the unique solution of

$$-\triangle v^\star = f \quad \text{in } \omega \text{ with boundary conditions } v^\star_{|\gamma'} = \Pi_h u^\star_{|\gamma'}.$$

Letting $u^f \in H_0^1(\Omega)$ be the unique solution of $-\triangle u^f = f \quad \text{in } \Omega$, we can split $v^\star$ as $v^\star = v_1^\star + v_2^\star + v_3^\star$ with $v_1^\star$, $v_2^\star$ and $v_3^\star$ verifying respectively:

$$-\triangle v_1^\star = f, \;\; v^\star_{1|\partial\omega} = 0, \qquad -\triangle v_2^\star = 0, \;\; v^\star_{2|\gamma} = 0, \;\; v^\star_{2|\gamma'} = \Pi_h u^f_{|\gamma'},$$

while

$$-\triangle v_3^\star = 0, \;\; v^\star_{3|\gamma} = 0, \;\; v^\star_{3|\gamma'} = \Pi_h(u^\star - u^f)_{|\gamma'}.$$

Clearly we have $u^\star = u^f + u_1^\star + u_2^\star + u_3^\star$ with $u_i^\star \in H_0^1(\Omega)$ verifying

$$-\triangle u_i^\star = \partial_\nu^h v_i^\star \delta_\gamma, \quad i = 1, 2, 3.$$

We can easily bound u^f, $u_1^\star$ and $u_2^\star$ in terms of f as

$$\|u^f\|_{1,\Omega} \le \|f\|_{-1,\Omega}, \quad \|u_1^\star\|_{1,\Omega} \le \|f\|_{L^2(\omega)}, \quad \|u_2^\star\|_{1,\Omega} \le \|f\|_{-1,\Omega}.$$

Letting $u_0^\star = u^\star - u^f$, we observe that $u_3^\star = \mathcal{T}^\star u_0^\star$ so that we can write

$$\begin{aligned}\|u^\star\|^2_{1,\Omega} &= (u^\star, u^f) + (u^\star, u_1^\star) + (u^\star, u_2^\star) + (u^f, \mathcal{T}^\star u_0^\star) + (u_0^\star, \mathcal{T}^\star u_0^\star)\\ &\le C\|u^\star\|_{1,\Omega}\|u^f\|_{1,\Omega} + \|u^\star\|_{1,\Omega}\|f\|_{L^2(\omega)} + \|u^\star\|_{1,\Omega}\|f\|_{-1,\Omega}\\ &\quad + \|f\|_{-1,\Omega}\|u_0\|_{1,\Omega} + C(h_\Omega + h_\omega)\|u_0^\star\|^2_{1,\Omega}.\end{aligned}$$

Observing that $\|u_0^\star\|_{1,\Omega} \lesssim \|u^\star\|_{1,\Omega}$ we easily obtain

$$(1 - C(h_\Omega + h_\omega))\|u^\star\|_{1,\Omega} \lesssim \|f\|_{-1,\Omega} + \|f\|_{L^2(\omega)}, \tag{10}$$

which, provided h_Ω and h_ω are small enough, yields the thesis. □

Corollary 2 *Under the assumptions of Theorem 2, provided h_Ω and h_ω are sufficiently small, there exists a θ_0 such that if $u_h^0 = \Pi_h u^0$ with u^0 verifying $-\triangle u_0 = f$ and $\theta_0 < \theta < 1$ then the sequence $\{u_h^n\}$ converges to a limit u_h which verifies*

$$\|u_h\|_{1,\Omega} \lesssim \|f\|_{-1,\Omega} + \|f\|_{L^2(\omega)}. \tag{11}$$

Remark 4 The stability estimates (10) and (11) are suboptimal, since we find on the righthand side the $L^2(\omega)$ norm of the data f. This suboptimality is a consequence of the suboptimality of the assumption on the operator ∂_ν^h. It is not difficult to see that if assumption (**B1**) is replaced by a stronger assumption, then (10) and (11) could be improved. Just to make an example, for the operator ∂_ν^h proposed in Sect. 6 the following stability estimate holds:

$$\|\partial_\nu^h v\|_{-1/2,\gamma} \lesssim \|\triangle v\|_{(H^1(\omega))'} + \|v\|_{1/2,\gamma'}.$$

In such a case it is not difficult to check that (10) and (11) could be improved by replacing the $L^2(\omega)$ norm of f on the righthand side with the weaker $(H^1(\omega))'$ norm.

5 Error estimate

5.1 Error in $H_0^1(\Omega)$

We start by estimating the error between u_h and u globally over Ω. Clearly, we cannot expect such an error to be better than the best approximation error, which, since u has no reason to be regular over γ, is, in general, of order $h_\Omega^{1/2-\varepsilon}$. We can write:

$$\begin{aligned}\|u - u_h\|_{1,\Omega} = \|u - \Pi_h u^\star\|_{1,\Omega} &\leq \|u - \Pi_h u\|_{1,\Omega} \\ &+\|\Pi_h(u - u^\star)\|_{1,\Omega} \lesssim \|u - \Pi_h u\|_{1,\Omega} + \|u - u^\star\|_{1,\Omega}.\end{aligned}$$

By Cea's Lemma [10] the first contribution can be bound by the best approximation error in $H_0^1(\Omega)$ as

$$\|u - \Pi_h u\|_{1,\Omega} \lesssim \inf_{v_h \in U_h} \|u - v_h\|_{1,\Omega}.$$

Let us then concentrate on the second contribution, which is a sort of consistency error. We have the following Lemma.

Lemma 3 *Let $u^\star$ be the fixed point of the iterative procedure ($\star$). Then, under the assumptions of Lemma 2 if $h_\Omega + h_\omega$ is sufficiently small, we have that*

$$\|u - u^\star\|_{1,\Omega} \lesssim \|(\partial_\nu - \partial_\nu^h)u\|_{-1/2,\gamma} + \|u - \Pi_h u\|_{1/2,\gamma'}. \tag{12}$$

Proof We introduce the auxiliary function $\tilde{v} \in H^1_{0,\gamma}(\omega)$ solution of

$$-\triangle\tilde{v} = f, \quad \text{in } \omega, \qquad \tilde{v}_{|\gamma'} = \Pi_h u_{|\gamma'}.$$

If we define $\tilde{u} \in H^1_0(\Omega)$ as the unique solution of

$$-\triangle\tilde{u} = f + \partial_\nu^h \tilde{v}\delta_\gamma, \tag{13}$$

one can easily check that

$$\tilde{u} - u^\star = \mathcal{T}^\star(u - u^\star).$$

Let now $w \in H^1_0(\Omega)$ be any given function. We can write:

$$\begin{aligned}
\int_\Omega \nabla(u - u^\star)\cdot\nabla w &= \int_\gamma (\partial_\nu u - \partial_\nu^h v^\star)w \\
&= \int_\gamma (\partial_\nu u - \partial_\nu^h u)w + \int_\gamma (\partial_\nu^h u - \partial_\nu^h \tilde{v})w + \int_\gamma (\partial_\nu^h \tilde{v} - \partial_\nu^h v^\star)w
\end{aligned}$$

Since $\partial_\nu^h v^\star\delta_\gamma = -(f + \Delta u^\star)$ and $\partial_\nu^h \tilde{v}\delta_\gamma = -(f + \Delta\tilde{u})$ we obtain

$$\begin{aligned}
\int_\Omega \nabla(u - u^\star)\cdot\nabla w &= \int_\gamma (\partial_\nu u - \partial_\nu^h u)w + \int_\gamma \partial_\nu^h(u - \tilde{v})w + \int_\Omega \nabla(\tilde{u} - u^\star)\cdot\nabla w \\
&= I + II + III.
\end{aligned}$$

Let us bound the three contributions after taking $w = u - u^\star$. We have

$$I \lesssim \|(\partial_\nu - \partial_\nu^h)u\|_{-1/2,\gamma}\|u - u^\star\|_{1,\Omega},$$

while, by (**B1**),

$$II \lesssim \|\partial_\nu^h(u - \tilde{v})\|_{-1/2,\gamma}\|u - u^\star\|_{1,\Omega} \le c_3\|u - \Pi_h u\|_{1/2,\gamma'}\|u - u^\star\|_{1,\Omega}.$$

We can bound the last contribution by applying Lemma 2, which gives us:

$$\begin{aligned}
III &= \int_\Omega \nabla(\tilde{u} - u^\star)\cdot\nabla(u - u^\star) = \int_\Omega \nabla\mathcal{T}^\star(u - u^\star)\cdot\nabla(u - u^\star) \\
&\le C(h_\Omega + h_\omega)|u - u^\star|^2_{1,\Omega}.
\end{aligned}$$

Putting everything together we obtain

$$(1 - C(h_\Omega + h_\omega))|u - u^\star|_{1,\Omega} \lesssim \|(\partial_\nu - \partial_\nu^h)u\|_{-1/2,\gamma} + \|u - \Pi_h u\|_{1/2,\gamma'}.$$

which, provided $h_\Omega + h_\omega \leq 1/(2C)$, implies (12). □

We now observe that if the solution to the original problem verifies $u \in H^s(\Omega\backslash\bar{B})$ with $s < 3/2$, then the extended solution stemming from the Fat Boundary formulation also verifies $u \in H^s(\Omega)$. Bounding $\|u - \Pi_h u\|_{1/2,\gamma'}$ with $\|u - \Pi_h u\|_{1,\Omega}$ and applying Cea's Lemma we then easily obtain the following corollary.

Corollary 3 *Under the assumptions of Lemma 3, if the solution u of (1) verifies $u \in H^s(\Omega\backslash\bar{B}) \cap H^2(\omega)$, $s < 3/2$, then the following error estimate holds*

$$\|u - u_h\|_{1,\Omega} \lesssim \|(\partial_\nu - \partial_\nu^h)u\|_{-1/2,\gamma} + \|u - \Pi_h u\|_{1,\Omega} \lesssim h_\omega\|u\|_{2,\omega} + h_\Omega^{s-1}\|u\|_{s,\Omega}.$$

If the solution to the original problem verifies $u \in H^s(\Omega\backslash\bar{B})$ for $s \geq 3/2$ then, in general, the extended solution only verifies $u \in H^r(\Omega)$ for all $r < 3/2$. This yields the following corollary

Corollary 4 *Under the assumptions of Lemma 3, if the solution u of (1) verifies $u|_{\Omega\backslash\bar{B}} \in H^s(\Omega\backslash\bar{B})$ with $s \geq 3/2$ and $u|_\omega \in H^2(\omega)$ then the following error estimate holds*

$$\|u - u_h\|_{1,\Omega} \lesssim h_\omega\|u\|_{2,\omega} + h_\Omega^{1/2}\log h_\Omega\|u\|_{3/2,\Omega\backslash\bar{B}}.$$

5.2 Error in $H^1(\Omega\backslash(\bar{\omega} \cup \bar{B}))$

Let us assume that the continuous solution u is regular in $\Omega\backslash\bar{B}$. Then, we can hope to get a better estimate if we measure the error in a subset of this domain. More precisely we have the following lemma.

Proposition 3 *Under the assumptions of Theorem 2 if $u|_{\Omega\backslash\bar{B}} \in H^s(\Omega\backslash\bar{B})$ for $s \leq M$ we have*

$$\|u - u_h\|_{1,\Omega\backslash(\bar{B}\cup\bar{\omega})} \lesssim h_\Omega^{s-1}(|u|_{s,\Omega\backslash\bar{B}} + \|u\|_{1,\Omega}) + \|(\partial_\nu - \partial_\nu^h)u\|_{-1/2,\gamma}$$

Proof Since $u_h = \Pi_h u^\star$ we have

$$\begin{aligned}\|u - u_h\|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})} &\leq \|u - \Pi_h u\|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})} + \|\Pi_h(u - u^\star)\|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})} \\ &\leq \|u - \Pi_h u\|_{1,\Omega\backslash(\bar{\omega}\cup\bar{B})} + \|\Pi_h(u - u^\star)\|_{1,\Omega} \\ &= I + II.\end{aligned}$$

Using Lemma 3 we can bound

$$II \lesssim \|(\partial_\nu - \partial_\nu^h)u\|_{-1/2,\gamma} + I$$

It only remains to bound I. We can apply Theorem 1, by which, setting $p = \max\{0, s-2\}$, we can bound

$$\|u - \Pi_h u\|_{1,\Omega\setminus(\bar{\omega}\cup\bar{B})} \lesssim h_\Omega^{s-1}|u|_{s,\Omega\setminus\bar{B}} + \|u - \Pi_h u\|_{-p,\Omega}.$$

A standard duality argument gives us

$$\|u - \Pi_h u\|_{2-s,\Omega} \lesssim h^{p+1}\|u\|_{1,\Omega} \lesssim h^{s-1}\|u\|_{1,\Omega}, \tag{14}$$

which implies the thesis. □

Corollary 5 *Under the assumptions of Proposition 3, if $u \in H^s(\Omega\setminus\bar{B})$, $1 < s \leq \min\{M, S\}$*

$$\|u - u_h\|_{1,\Omega\setminus(\bar{B}\cup\bar{\omega})} \lesssim (h_\Omega^{s-1} + h_\omega^{s-1})|u|_{s,\Omega\setminus\bar{B}}$$

Remark 5 For the sake of simplicity we assumed throughout the paper that the domain Ω is of class C^∞. The role played by this assumption is to ensure that for regular data the solution of the problem considered is regular. More precisely we implicitly used several times the bound

$$\begin{aligned} -\Delta u = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \text{ with } f \in H^s(\Omega) \Longrightarrow u \in H^{s+2}(\Omega) \text{ and} \\ \|u\|_{s+2,\Omega} \lesssim \|f\|_{s,\Omega}. \end{aligned} \tag{15}$$

For $s = 0$ such a property was used in the proof of Proposition 1 and, implicity, in the duality bound (7). For $s \leq M - 2$ such property is needed in the proof of Proposition 3 in order to obtain (14). The results of the paper (and in particular Theorem 2, Theorem 3 and Corollary 5) hold for all domains such that (15) holds for $s = 0$ (Theorem 2, Theorem 3) and for $s \leq M - 2$ (Corollary 5). Recall that for $s = 0$, property (15) holds for convex polygonal domains ([15]). For the particular case of Ω being a square or cube, by using a reflection argument (see [32], [23]) it is possible to prove that (15) holds for all $s \geq 0$ provided $f \in C_0^\infty(\bar{\Omega})$ which is sufficient for proving Corollary 5.

5.3 Error in $H^1(\omega)$

For the reasons that we already explained, we cannot hope to obtain an optimal approximation of u by means of u_h in the vicinity of γ.

However, once u_h is computed, we can obtain a better approximation to u in ω by finding numerically, by our favorite method, the solution $v^\star$ to the equation

$$-\triangle v^\star = f \quad \text{in } \omega, \qquad v^\star = u_h \quad \text{on } \gamma', \qquad v = 0 \quad \text{on } \gamma.$$

We observe that

$$\|u - v^\star\| \lesssim \|u - u_h\|_{1/2,\gamma'} \lesssim \|u - u_h\|_{1,\Omega\setminus(\bar{B}\cup\bar{\omega})}.$$

We do not have any regularity result on $v^\star$ so we cannot bound the resulting approximation error directly. Nevertheless, if we choose an H^1 stable and linear solution method we can still obtain an optimal error estimate. In fact, denoting by $\hat{\Pi}_h$ the chosen discrete solution operator we can write

$$\begin{aligned}\|u - \hat{\Pi}_h v^\star\|_{1,\omega} &\leq \|u - \hat{\Pi}_h u\|_{1,\omega} + \|\hat{\Pi}_h (u - v^\star)\|_{1,\omega} \\ &\lesssim \|u - \hat{\Pi}_h u\|_{1,\omega} + \|u - v^\star\|_{1,\omega}.\end{aligned}$$

This bound clearly allows to get the best possible approximation error permitted by the chosen solution method $\hat{\Pi}_h$ when applied to the continuous solution u.

6 Computation of $\partial_\nu^h v$

In the previous sections we assumed that the approximate outer normal derivative operator ∂_ν^h satisfied assumptions (**B1–2**). In order to be as general as possible we avoided to explicitly choose such an operator. The aim of this section is to propose one possible way of constructing an operator satisfying the required properties. We recall that the operator ∂_ν^h must give an approximation of $\partial_\nu v$ for $v \in H^1_{0,\gamma}(\omega)$ with, as input data, $\triangle v$ and $v|_{\gamma'}$. Let then $v \in H^1_{0,\gamma}(\omega)$, with $\triangle v \in L^2(\omega)$ and let $f = -\triangle v$ and $g = v|_{\gamma'}$. Let now $V_h \subset H^1_{0,\gamma}(\omega)$ denote a family of approximation spaces depending on a mesh size parameter h_ω and satisfying the following standard assumptions

C1 For each $u \in H^1(\omega)$ there exists $\eta \in V_h$ such that for any $0 \leq s \leq R, s \leq \ell \leq M$

$$\|u - \eta\|_{s,\omega} \lesssim h_\omega^{\ell - s} \|u\|_{\ell,\omega}.$$

C2 If $0 \leq t \leq s \leq R$ then for all $w_h \in V_h$ we have that

$$\|w_h\|_{s,\omega} \lesssim h_\omega^{t-s} \|w_h\|_{t,\omega}.$$

Let $V_h^0 = V_h \cap H^1_0(\omega)$. Given f and g let $v_h \in V_h$ denote the unique element satisfying

$$\begin{gathered} v_h = 0 \quad \text{on } \gamma, \qquad v_h = \pi_h g \quad \text{on } \gamma' \\ \int_\omega \nabla v_h \cdot \nabla w_h = \int_\omega f w_h \quad \text{for all } w_h \in V_h^0, \end{gathered}$$

where we denote by $\pi_h : L^2(\gamma') \to V_h|_{\gamma'}$ the L^2 orthogonal projection. Let now $\mathcal{L}^H : V_h|_{\gamma'} \to V_h \cap H^1_{0,\gamma}(\omega)$ denote the discrete harmonic lifting operator: $\mathcal{L}^H \eta_h \in V_h$ is the unique function such that

$$\begin{gathered} \mathcal{L}^H \eta_h = 0 \quad \text{on } \gamma, \qquad \mathcal{L}^H \eta_h = \eta_h \quad \text{on } \gamma' \\ \int_\omega \nabla \left(\mathcal{L}^H \eta_h\right) \cdot \nabla w_h = 0 \quad \text{for all } w_h \in V_h^0. \end{gathered}$$

We define $\partial_\nu^h v \in V_h|_\gamma$ as the unique element such that for all $\eta_h \in V_h|_\gamma$

$$\int\limits_\gamma \partial_\nu^h v \eta_h = \int\limits_\omega \nabla v_h \cdot \nabla \mathcal{L}^H \eta_h + \int\limits_\omega f \mathcal{L}^H \eta_h.$$

Proposition 4 *If $v \in H^1_{0,\gamma}(\omega)$ verifies $\triangle v \in L^2(\omega)$, then we have*

$$\|\partial_\nu^h v\|_{-1/2,\gamma} \lesssim \|\triangle v\|_{(H^1(\omega))'} + \|v\|_{1/2,\gamma'} \tag{16}$$

Moreover, if $v \in H^s(\omega)$, $2 \le s \le M$ then we have

$$\|\partial_\nu v - \partial_\nu^h v\|_{-1/2,\gamma} \lesssim h_\omega^{s-1} \|v\|_{s,\omega}.$$

Proof It is not difficult to realize that $\partial_\nu^h v$ coincide with the restriction to γ of the discrete multiplier $\lambda_h \in V_h|_{\partial\omega}$ for the solution of the above Dirichlet problem by the Lagrange multiplier method by Babuska: find $u_h \in V_h$, $\lambda_h \in V_h|_{\partial\omega}$ such that $\forall w_h \in V_h$, $\mu_h \in V_h|_{\partial\omega}$ we have

$$\int\limits_\omega \nabla v_h \cdot \nabla w_h + \int\limits_{\partial\omega} \lambda_h w_h = \int\limits_\omega f w_h$$
$$\int\limits_{\partial\omega} v_h \mu_h = \int\limits_{\gamma'} g \mu_h$$

The thesis follows then by applying the standard estimates for such a method ([3,9]). □

7 Numerical tests

The aim of this section is to validate numerically the previous theoretical results, especially those from Sects. 5.2 (corollary 5) and 5.3. In the latter we showed that if we use high order finite elements (P_q for instance), the FBM is also of order q in every sub-domain of $\Omega\backslash\bar{B}$. To do so, we consider two kinds of errors, a global one, computed in the sub-domain $\Omega\backslash(\bar{B} \cup \bar{\omega})$ and a local one, computed in ω. For the first one, we use the global solution u_h and, as explained in Sect. 5.3, we use the local solution v_h for the second error.

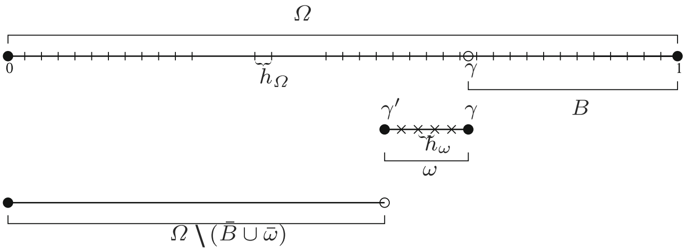

Fig. 3 Global and local domains

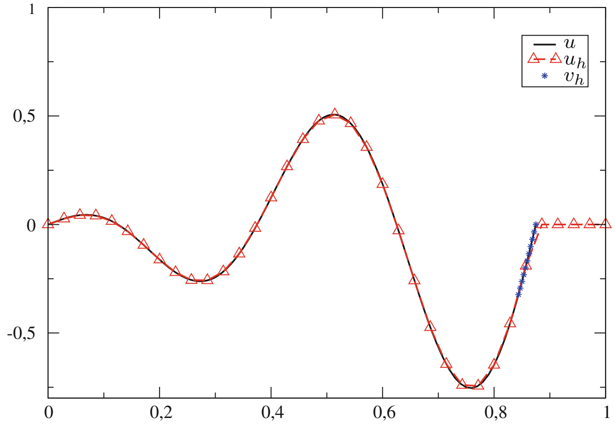

Fig. 4 Global, Local and Exact Solutions

7.1 One-dimensional case

For this numerical test, the domain Ω is the interval $]0, 1[$ and B its subset $]\gamma, 1[$. So that the initial problem is given by:

$$-\frac{d^2u}{dx^2} = f \quad \text{on }]0, \gamma[, \tag{17}$$

$$u(0) = 0, \quad u(\gamma) = 0. \tag{18}$$

A typical example of the different sub-domains is given by Fig. 3.

For this example the data are chosen in such a way that the exact solution of the initial problem (17)–(18) is given by this expression:

$$u(x) = x\cos(4\pi x), \quad \forall x \in]0, \gamma[. \tag{19}$$

The curve of this function is presented in Fig. 4. A typical couple of functions (u_h, v_h) that approximates the solutions of problems (2) and (3) is also presented in this figure. One can note the role of the local function v_h to better approximate the solution in the vicinity of γ. We can see easily that v_h is much closer to the solution than u_h.

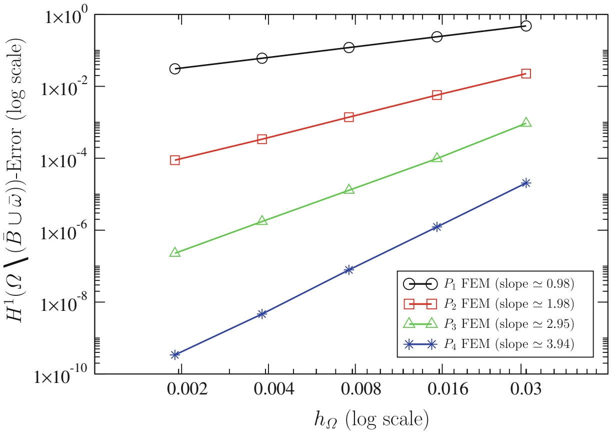

Fig. 5 One dimensional example. Global $H^1(\Omega\backslash(\bar{B} \cup \bar{\omega}))$ errors

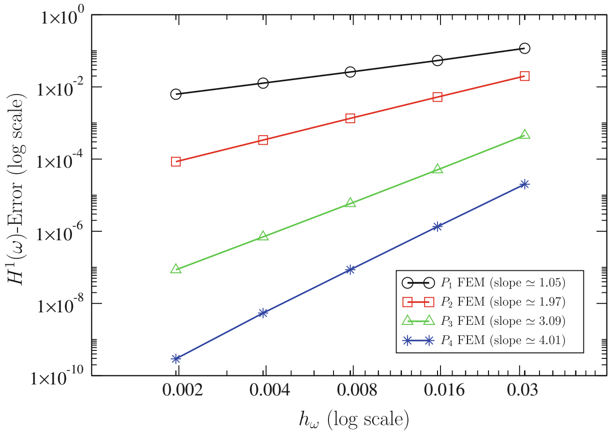

Fig. 6 One dimensional example. Local $H^1(\omega)$ errors

In order to validate numerically the error estimates of Sect. 5, we compute the global error in $\Omega \setminus (\bar{B} \cup \bar{\omega})$ and the local error in ω (see Fig. 3). In our case $\gamma = \frac{7}{8}$ and $\gamma' = \frac{6.5}{8}$. The mesh sizes h_Ω and h_ω are equal and chosen such that the two grids do not match (in particular γ doesn't have to belong to the global mesh. Otherwise we retrieve conforming finite elements case).

Figure 5 shows the dependence of the global errors in $H^1(\Omega \setminus (\bar{B} \cup \bar{\omega}))$-norm upon the mesh step size h_Ω for different finite elements degrees (P_1, P_2, P_3 and P_4). The local $H^1(\omega)$ errors are shown in Fig. 6. In both cases we retrieve the optimal order of the used finite elements.

7.1.1 Convergence of the fix point algorithm

Previous results (Theorem 2) suggest that the range of θ (relaxation parameter) could be small. But as illustrated by the following example, the choice of this parameter is

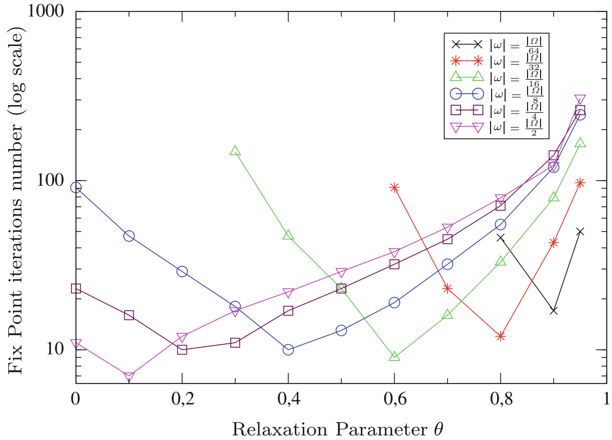

Fig. 7 Convergence of the Fix Point algorithm versus the relaxation parameter θ for different sizes of ω ($|\omega|$ decreases from left to right)

rather less restrictive in practice. Usually an number of iteration between 10 and 20 is sufficient.

We return to the previous numerical test and we fix the mesh sizes h_ω and h_Ω (and keep them of the same order) while varying the relaxation parameter $\theta \in [0, 1[$. For each value of θ, we note the number of iterations needed for the convergence of the fix point algorithm. We repeat the same experience for different sizes of the local domain ω:

$$|\omega| = |\gamma - \gamma'| \in \left\{ \frac{|\Omega|}{64}, \frac{|\Omega|}{32}, \frac{|\Omega|}{16}, \frac{|\Omega|}{8}, \frac{|\Omega|}{4}, \frac{|\Omega|}{2} \right\},$$

$|\Omega|$ being the size of the global domain (it's equal to one in our case).

The results of these P_1 calculations are presented in Fig. 7. They allow us to make the following remarks:

- Each of these curves gives an optimal value of θ which corresponds to the minimal number of iterations. These numerical values are smaller than the theoretical ones given in [21] which can be considered as upper bounds.
- The range of convergence decreases with the size of ω. For example, for $|\omega| = \dfrac{|\Omega|}{64}$ the fix point algorithm didn't converge for values of θ smaller than 0.8.
- Provided that we are in the range of convergence, changing the mesh sizes does not affect the iteration's number.
- The iteration number is independent of the finite element order.

7.2 Two-dimensional case

As in Sect. 7.1, we try in this one to retrieve the same numerical validations using two dimensional example. The numerical test concerns the resolution of a Poisson's

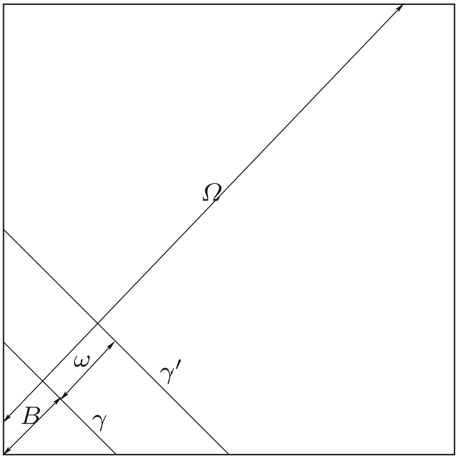

Fig. 8 Two-dimensional geometry

problem and the computation of the global and the local H^1 errors for different values of mesh sizes and using different finite elements degrees (P_1, P_2, P_3 and P_4) finite elements.

Figure 8 shows our geometry in two dimensions. Typically, Ω is the whole square $]0,1[^2$, B is the triangle given by the left bottom corner of the square and the line segment $\gamma = \{(x, y) \in [0, 1]^2,\ s.t.\, x + y = \frac{1}{4}\}$. The artificial interface γ' is given by $\gamma' = \{(x, y) \in [0, 1]^2,\ s.t.\, x + y = \frac{1}{2}\}$, so that the local domain ω is defined in such a way that $\omega \subset \Omega$ and $\partial\omega = \gamma \cup \gamma'$. We chose this geometry to avoid the loss of optimal order due to the bad approximation of curved boundaries by straight high order finite elements. An alternative way to deal with curved boundaries is to use the procedure described in [18] which permits the construction of isoparametric finite elements that preserve the optimal order.

As in the one dimensional example, the meshes (the local one and the global one) don't match and the corresponding mesh sizes h_Ω and h_ω are chosen of the same order. An example of these meshes is given by Fig. 9. We use a cartesian mesh for the global domain Ω which makes it easy the use of fast solvers and/or efficient standard preconditioners.

Recall that our initial problem is given by the following equations:

$$-\Delta u = f \quad \text{in } \Omega \backslash \bar{B}, \tag{20}$$

$$u = g \quad \text{on } \partial\Omega \tag{21}$$

$$u = 0 \quad \text{on } \gamma \tag{22}$$

As in the one-dimensional case, we compute two types of errors, a local one computed in ω and a global one computed in $\Omega \setminus (\bar{B} \cup \bar{\omega})$. All These two-dimensional computations are done using the FreeFem++[1] software ([17]).

[1] http://www.freefem.org/ff++.

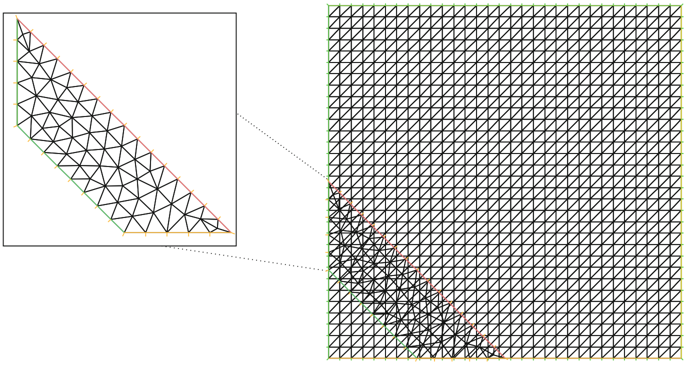

Fig. 9 Example of local and global meshes

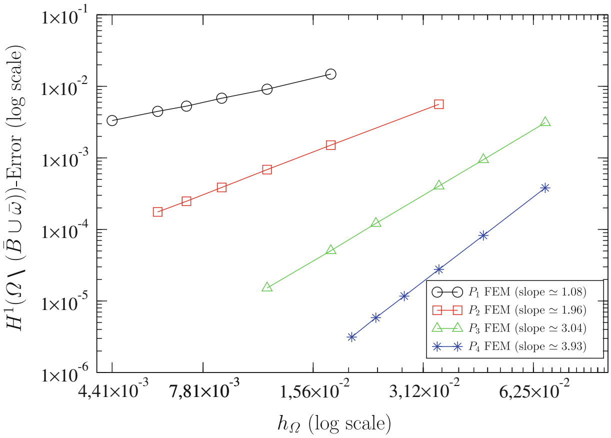

Fig. 10 Two dimensional example. Global $H^1(\Omega\backslash(\bar{B} \cup \bar{\omega}))$ errors

The right-hand side f and the boundary condition g are chosen in such a way that the exact solution u is given by this expression :

$$u(x, y) = (x + y - 1)(x + y)cos(6\pi(x + y)) + 3\pi(x + y)(x + y) - 3\frac{\pi}{16}$$

Figure 10 presents the dependence of the global $H^1(\Omega \setminus (\bar{B} \cup \bar{\omega}))$ errors upon the mesh size h_Ω when using P_1, P_2, P_3 and P_4 finite elements. The local $H^1(\omega)$ errors are presented in the Fig. 11. In both local and global resolutions we retrieve the optimal finite element order. Note that for P_4 finite elements we didn't consider very fine meshes to prevent round-off errors that could pollute the numerical ones (this could lead to a saturation of the error).

Remark 6 We have similar results of the convergence of the Fix Point algorithm as in the case of the previous one dimensional example. In all our 2D calculations, a number of iterations around twenty was sufficient for convergence.

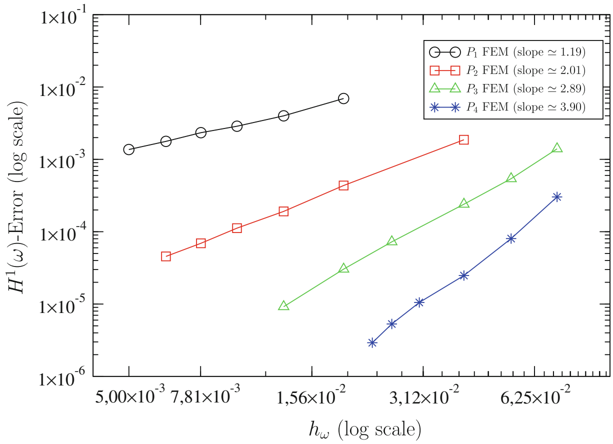

Fig. 11 Two dimensional example. Local $H^1(\omega)$ errors

8 Concluding remarks

In this work, we proved the high-order convergence of the fully discrete Fat Boundary Method. Some numerical tests are also proposed to illustrate such behaviors in one and two dimensions.

Unlike other fictitious domain-like methods where the convergence rate is of order $\sqrt{h}$ even when using high-order Galerkin methods (see [13,16,33]), our method preserves the optimal finite elements convergence rate under some reasonable assumptions.

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