RANDOM DISTANCES WITHIN A RECTANGLE AND BETWEEN TWO RECTANGLES

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Introduction

The distributions of random distances within a figure, and between two figures in a plane, have important applications in statistics, specially in problems of *topographic variation* (*Vide*: B. Matérn, 1947, p. 123; B. Ghosh, 1949, p. 20; M. N. Ghosh, 1949, p. 85; Garwood, 1947; Armitage, 1949). Such distributions have been worked out within a rectangle*, and between two rectangles with similar orientations, and the results for some particular cases have been briefly recorded in earlier notes (B. Ghosh, 1943*a*, 1943*b*). As some authors have required the results for other cases not covered by the earlier notes, the general method of evaluating such distributions will be described here, following which any required particular case can be tackled.

A rigorous statement of the problems may be given first. Let us consider a rectangle (which is usually a *sample-unit* in *area sampling*) with sides equal to a and b along x and y axes ($x = 0$ to a; $y = 0$ to b). The rectangle, as part of the statistical field, is composed of "points", each "point" being the centre of a very small square called the *basic cell* (*vide* B. Ghosh, 1949, pp. 13-14); all the square cells are of the same size and have their sides parallel to x and y axes. A cell being extremely small compared with the rectangle, for all practical purposes the co-ordinates, x and y, of the "points" may be regarded as varying continuously with their joint *probability density function* (p. d. f.) given by $f(x, y) = 1/(ab)$ for $x = 0$ to a, and $y = 0$ to b. Two "points" P_1 and P_2 are located randomly and independently (in the stochastic sense) within the rectangle, with their co-ordinates denoted by (x_1, y_1) and (x_2, y_2) respectively. Consider the interval-vector, I, connecting the two points, P_1 and P_2, with its components along x and y given by (x_2-x_1) and (y_2-y_1). It is required to find out the p. d. f., $f(R)$, of R, the length of I, defined by $R = +\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}$. This problem may be called briefly the "one rectangle" problem.

Next, we consider two rectangles with similar orientations defined by say, $x = 0$ to a and $y = 0$ to b for the first rectangle, and $x = (a+e)$ to $(a+c+e)$ and $y = (b+f)$ to $(b+d+f)$ for the second. Two points P_1 and P_2 are located randomly in the two rectangles, P_1 in the first rectangle and P_2 in the second. The p. d. f. of (x_1, y_1), the

* Some particular results covered by (Ghosh 1943a) have been given later by other authors; see also Santalo (1947).

co-ordinates of P_1, is given by $f(x_1, y_1) = 1/(ab)$, for $x_1 = 0$ to a and $y_1 = 0$ to b; similarly for P_2 we have $f(x_2, y_2) = 1/(cd)$, for $x_2 = (a+e)$ to $(a+c+e)$ and $y_2 = (b+f)$ to $(b+d+f)$. In this problem also we require $f(R)$, R being the length of the interval I connecting P_1 and P_2. This may be called the "two rectangles" problem.

Problem of One Rectangle

Let us define $X = |x_1 - x_2|$ and $Y = |y_1 - y_2|$, so that $R = +\sqrt{(X^2+Y^2)}$. Since P_1 and P_2 are independent, and for any point, x, y are independent, obviously X, Y also will be independently distributed. Theorem 1 regarding the p. d. f. of X is stated below, the proof of which, being quite simple, is omitted here.

Theorem 1. If x_1 and x_2 are two stochastically independent and random values from the same distribution whose p. d. f. is given by $f(x) = 1/a$ for $x = 0$ to a, then the p. d. f. of $X = |x_1 - x_2|$ is given by $f(X) = 2(a-X)/a^2$, for $X = 0$ to a.

Here X can also be regarded as the distance between two random points in a straight line of length a. Similarly we get the p. d. f. of Y. Since X, Y are independent their joint p. d. f. is given by

$$f(X, Y) = \frac{4(a-X)(b-Y)}{a^2b^2}, \text{ for } X = 0 \text{ to } a \text{ and } Y = 0 \text{ to } b. \tag{1}$$

We now transform the variables (X, Y) to (R, θ), where $\tan\theta = Y/X$, (R already defined). Within the effective ranges $X = 0$ to a and $Y = 0$ to b, $f(X, Y)dXdY$ is therefore transformed to

$$f(R, \theta)dRd\theta = \frac{4R}{a^2b^2}(a - R\cos\theta)(b - R\sin\theta)dRd\theta. \tag{2}$$

Integrating (2) over θ, between the appropriate limits, θ_1 (lower) and θ_2 (upper), we get the p. d. f. of R, $f(R) = (4R/a^2b^2)\phi(R)$, where

$$\phi(R) = ab(\theta_2 - \theta_1) + aR(\cos\theta_2 - \cos\theta_1) - bR(\sin\theta_2 - \sin\theta_1) - \frac{R^2}{4}(\cos 2\theta_2 - \cos 2\theta_1). \tag{3}$$

Care is necessary in finding the values θ_1 and θ_2. Consider the "effective rectangle," $X = 0$ to a, $Y = 0$ to b, in the (X, Y)-plane. (In this case we can assume $a \geq b$, without any loss of generality, since it is merely a matter of choosing the x and y axes suitably). From inspection it will be seen that the "effective rectangle" can be divided in terms of R into three convenient ranges, (i) $R = 0$ to b, (ii) $R = b$ to a, and (iii) $R = a$ to $\sqrt{(a^2+b^2)}$. Further, in range (i) $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$; in range (ii) $\theta_1 = 0$, $\theta_2 = \sin^{-1}(b/R)$; and in range (iii) $\theta_1 = \cos^{-1}(a/R)$, $\theta_2 = \sin^{-1}(b/R)$. Putting these values in equation (3) and simplifying, we get the following theorem.

Theorem 2. The p. d. f. of R, the distance between two independent random points in a rectangle with sides a and b, ($a \geq b$), is given by $f(R) = (4R/a^2b^2)\phi(R)$, where

$$\phi(R) = \tfrac{1}{2}\pi ab - aR - bR + \tfrac{1}{2}R^2, \text{ for } R = 0 \text{ to } b\,;$$

$$\phi(R) = ab\sin^{-1}(b/R) + a\sqrt{(R^2-b^2)} - aR - \tfrac{1}{2}b^2, \text{ for } R = b \text{ to } a\,;$$

and $\quad \phi(R) = ab\{\sin^{-1}(b/R) - \cos^{-1}(a/R)\} + a\sqrt{(R^2-b^2)}$

$$+ b\sqrt{(R^2-a^2)} - \tfrac{1}{2}(R^2+a^2+b^2), \quad \text{for } R = a \text{ to } \sqrt{(a^2+b^2)}.$$

The first four moments of this distribution are given in the appendix (eqn. 15). At any transitional value of R, $f(R)$ and df/dR have the same values in both the adjacent ranges, while the values of d^2f/dR^2 are not so. When $a = b$ (square), the second range in Theorem 2, $R = b$ to a, is non-existent.

Problem of Two Rectangles

Here also we define $X = |x_1 - x_2|$, $Y = |y_1 - y_2|$. The joint p. d. f. of x_1 and x_2 is given by $f(x_1, x_2) = 1/(ac)$, for $x_1 = 0$ to a and $x_2 = (a+e)$ to $(a+c+e)$. Transforming (x_1, x_2) to (u, v), with $u = (x_2 - x_1)$, $v = x_1$, we have $f(u, v) = 1/(ac)$ in the effective ranges for x_1 and x_2. Integrating $f(u, v)$ over v, between the appropriate limits, v_1 (lower) and v_2 (upper), we get,

$$f(u) = \frac{v_2 - v_1}{ac}, \text{ in the effective ranges of } x_1, x_2. \tag{4}$$

From inspection of the "effective rectangle," $x_1 = 0$ to a, $x_2 = (a+e)$ to $(a+c+e)$, in the (x_1, x_2)-plane we have three convenient ranges for u, (i) $u = e$ to $(e+g)$, (ii) $u = (e+g)$ to $(e+h)$, and (iii) $u = (e+h)$ to $(e+g+h)$, where g stands for the smaller of the two values, a and c, and h for the greater. (If $a = c$, then $g = h = a$). If $a \geq c$, it can be shown from inspection of (x_1, x_2)-diagram that in range (i) $v_1 = (e+h-u)$, $v_2 = h$, so that $(v_2 - v_1) = (u-e)$; in range (ii) $v_1 = (e+h-u)$, $v_2 = (e+g+h-u)$, and $(v_2 - v_1) = g$; and in range (iii) $v_1 = 0$, $v_2 = (e+g+h-u)$, and $(v_2 - v_1) = (e+g+h-u)$. If alternatively $a < c$, the values of $(v_2 - v_1)$ in the three ranges will remain unaltered in terms of g and h. Since $X = u$, we have the following result from eqn. (4):—

Theorem 3. If x_1, x_2 are independently distributed with p. d. f.'s, $f(x_1) = 1/a$, for $x_1 = 0$ to a, and $f(x_2) = 1/c$, for $x_2 = (a+e)$ to $(a+c+e)$, the p. d. f. of $X = |x_1 - x_2|$ is given by $f(X) = \phi(X)/(ac)$, where $\phi(X) = (X-e)$ for $X = e$ to $(e+g)$; $\phi(X) = g$, for $X = (e+g)$ to $(e+h)$; and $\phi(X) = (g+h+e-X)$, for $X = (e+h)$ to $(e+g+h)$. (The symbols, g, h, have been explained before).

Here X may also be regarded as the distance between two random points P_1, P_2, selected respectively from two straight lines of lengths a and c, one line lying on the extension of the other, with their nearest points separated by a distance, e. Or, the two straight lines of lengths a and c, parallel to the x-axis, may have their nearest points separated by a distance e along x, and f along y; here X will represent the absolute value of the x-component of I, the interval connecting P_1 and P_2, while the y-component of I will be a constant equal to f.

Corollary I to theorem 3. If $a = c$, $f(X) = \phi(X)/a^2$, where $\phi(X) = (X-e)$, for $X = e$ to $(a+e)$; and $\phi(X) = (2a+e-X)$, for $X = (a+e)$ to $(2a+e)$.

In case the effective ranges of x_1 and x_2 are overlapping, we have to formally take e as negative. Further X will not always be $(+u)$, but sometimes $(-u)$ as well, and so some care has to be taken in changing over from u to X. A case of complete overlapping with $a = c$, and $e = -a$ (formally), is of some practical interest, and can be derived from corollary 1.

Corollary 2 to theorem 3. In case of "complete overlapping" ($a = c$, $e = -a$). $f(X) = 2(a-X)/a^2$. for $X = 0$ to a. (This result may be compared with theorem 1).

Returning to the general problem of "two rectangles," $f(Y)$ can be derived from theorem 3, by replacing a, c, e, g, h with b, d, f, p, q respectively, (p, q) being defined with respect to (b, d) in the same manner as (g, h) with respect to (a, c). Further $f(X, Y) = f(X)f(Y)$. Defining R, θ as in the problem of "one rectangle," we transform $f(X, Y)dXdY$ to $f(R, \theta)dRd\theta$ and then integrate out θ between appropriate limits θ_1 and θ_2, to get $f(R)$. The problem here is, however, more complicated than the "one rectangle" problem. The "effective rectangle" $X = e$ to $(e+g+h)$, $Y = f$ to $(f+p+q)$ in the (X, Y)-diagram is divided into 9 "compartments" formed by the combinations of three effective ranges of X and three ranges of Y, whereas in the "one-rectangle problem" there is only one such "compartment". Here as we change θ, for a given value of R, we may have to pass through several compartments, and the function $f(R, \theta)$ and the lower and upper limits of θ will be different in different compartments. These points will be clear from the study of the (X, Y)-diagram. In the most general case, there will be 16 transitional points in the "effective rectangle" given by the combinations of $X = e$, $(e+g)$, $(e+h)$, $(e+g+h)$ and $Y = f$, $(f+p)$, $(f+q)$, $(f+p+q)$. By considering the *iso-R* lines in the (X, Y)-diagram it will be seen that there will be 16 transitional values of R, passing through the 16 transitional points, and so there will be 15 effective ranges of R instead of only 3 such ranges of the "one-rectangle" problem. Though any particular case with known values of a, b, c, d, e, f can be always solved, it is no use attempting a general solution, which will be too much involved. Some special cases are discussed below to illustrate the method.

Equal Squares with Common Diagonal Line

Here $a = b = c = d$, $e = f$. The joint p. d. f. $f(X, Y) = f(X)f(Y)$ can be written down from corollary 1 to theorem 3. The "effective rectangle" in the (X, Y) diagram is divided into four compartments: the first $C(I)$ is defined as $X = e$ to $(e+a)$, $Y = e$ to $(e+a)$; $C(II)$ is $X = (e+a)$ to $(e+2a)$, $Y = e$ to $(e+a)$; $C(III)$ is $X = e$ to $(e+a)$, $Y = (e+a)$ to $(e+2a)$; and $C(IV)$ is $X = (e+a)$ to $(e+2a)$, $Y = (e+a)$ to $(e+2a)$. Though there are 9 transitional points, $X = e$, $(e+a)$, $(e+2a)$ with $Y = e$, $(e+a)$, $(e+2a)$, because of symmetry in X and Y, we have only 6 transitional values of R, given by $R_1 = \sqrt{2}e$, $R_2 = \sqrt{\{e^2+(e+a)^2\}}$, $R_3 = \sqrt{2}(e+a)$, $R_4 = \sqrt{\{e^2+(e+2a)^2\}}$, $R_5 = \sqrt{\{(e+a)^2+(e+2a)^2\}}$, and $R_6 = \sqrt{2}(e+2a)$. Thus there are only 5 effective ranges of R, (R_1 to R_2, R_2 to R_3, . . ., R_5 to R_6) in this case.

First Range ($R = R_1$ to R_2). In this range the point in the (X, Y)-diagram is solely confined to $C(I)$, in which $f(X, Y) = (X-e)(Y-e)/a^4$, and so $f(R, \theta) = R(R\cos\theta - e)(R\sin\theta - e)/a^4$. The limits of θ are given by $\theta_1 = \sin^{-1}(e/R)$ and $\theta_2 = \cos^{-1}(e/R)$, as will be clear from inspection of the (X, Y)-diagram. Integrating $f(R, \theta)$ over θ, from θ_1 to θ_2, and simplifying, we have

$$f(R) = \frac{R}{a^4}\left\{c^2\left(\cos^{-1}\frac{c}{R} - \sin^{-1}\frac{c}{R}\right) - 2c\sqrt{(R^2-c^2)} + \tfrac{1}{2}R^2 + c^2\right\}. \tag{5}$$

Second Range ($R = R_2$ *to* R_3). This range represents the strip of the "effective rectangle" in the (X, Y)-diagram between the two circular lines $R = R_2$, $R = R_3$. It will be seen that for a given R as one increases θ, one has to pass through $C(II)$, $C(I)$ and $C(III)$ successively.

In $C(II)$, we have $f(R, \theta) = R(2a + c - R\cos\theta)(R\sin\theta - c)/a^4$; further the limits of θ are $\theta_1 = \sin^{-1}(c/R)$, $\theta_2 = \cos^{-1}(c+a)/R$. Integrating $f(R, \theta)$ over θ, from θ_1 to θ_2, we have,

$$R[-\tfrac{3}{2}a^2 - c^2 - 2ac + c\sqrt{\{R^2-(a+c)^2\}} + (2a+c)\sqrt{(R^2-c^2)}$$
$$- c(2a+c)\{\cos^{-1}(a+c)/R - \sin^{-1}(c/R)\}]/a^4 = \alpha, \text{ say.} \tag{6}$$

Next, in $C(I)$, $f(R, \theta) = R(R\cos\theta - c)(R\sin\theta - c)/a^4$, and the limits $\theta_1 = \cos^{-1}(a+c)/R$, $\theta_2 = \sin^{-1}(a+c)/R$. (These limits θ_1, θ_2 are different from the limits for the same compartment, $C(I)$, in the first range, R_1 to R_2). Integrating $f(R, \theta)$ over θ, between θ_1 and θ_2, we have,

$$R[-\tfrac{1}{2}R^2 + a^2 - c^2 + 2c\sqrt{\{R^2-(a+c)^2\}}$$
$$+ c^2\left\{\sin^{-1}\left(\frac{a+c}{R}\right) - \cos^{-1}\left(\frac{a+c}{R}\right)\right\}\Big]\Big/ a^4 = \beta, \text{ say.} \tag{7}$$

Finally, in $C(III)$, integrating $f(R, \theta) = R(R\cos\theta - c)(2a + c - R\sin\theta)/a^4$ over θ between $\theta_1 = \sin^{-1}(a+c)/R$ and $\theta_2 = \cos^{-1}(c/R)$, we have,

$$R\left[-c^2 - \tfrac{3}{2}a^2 - 2ac - \frac{R^2}{2} + (2a+c)\sqrt{(R^2-c^2)} + c\sqrt{\{R^2-(a+c)^2\}}\right.$$
$$\left. - c(2a+c)\left\{\cos^{-1}\left(\frac{c}{R}\right) - \sin^{-1}\left(\frac{a+c}{R}\right)\right\}\right]\Big/ a^4 = \gamma, \text{ say.} \tag{8}$$

Now adding together the contributions from $C(II)$, $C(I)$ and $C(III)$ represented by α, β and γ (equns. 6, 7, 8), we have, in the range $R = R_2$ to R_3,

$$f(R) = R\left[2c(a+c)\left\{\sin^{-1}\left(\frac{a+c}{R}\right) - \cos^{-1}\left(\frac{a+c}{R}\right)\right\}\right.$$
$$+ c(2a+c)\left\{\sin^{-1}\left(\frac{c}{R}\right) - \cos^{-1}\left(\frac{c}{R}\right)\right\} + 4c\sqrt{\{R^2-(a+c)^2\}}$$
$$\left. + (4a+2c)\sqrt{(R^2-c^2)} - \tfrac{3}{2}R^2 - 2a^2 - 4ac - 3c^2\right]\Big/ a^4. \tag{9}$$

Proceeding in this manner we can work out the expressions for $f(R)$ in the other three ranges of R also. Now we shall state without proof the results for $c = 0$, as that case is of some practical importance.

Equal Squares with Corner-point Contact

Here $a = b = c = d$, $e = f = 0$. The p. d. f. $f(R)$ is given in the following five ranges, as $f(R) = R\phi(R)/a^4$, where

$$\phi(R) = \tfrac{1}{2}R^2, \text{ for } R = 0 \text{ to } a;$$

$$\phi(R) = -\tfrac{3}{2}R^2 + 4aR - 2a^2, \text{ for } R = a \text{ to } \sqrt{2}a;$$

$$\phi(R) = 4a^2\{\cos^{-1}(a/R) - \sin^{-1}(a/R)\} - 8a\sqrt{(R^2-a^2)} + \tfrac{1}{2}R^2 + 4aR + 2a^2, \text{ for } R = \sqrt{2}a \text{ to } 2a;$$

$$\phi(R) = 4a^2\{\cos^{-1}(a/R) - \sin^{-1}(a/R)\} - 8a\sqrt{(R^2-a^2)} + \tfrac{3}{2}R^2 + 6a^2, \text{ for } R = 2a \text{ to } \sqrt{5}a;$$

and

$$\phi(R) = 4a^2\{\sin^{-1}(2a/R) - \cos^{-1}(2a/R)\} + 4a\sqrt{(R^2-4a^2)} - \tfrac{1}{2}R^2 - 4a^2, \text{ for } R = \sqrt{5}a \text{ to } 2\sqrt{2}a. \qquad (10)$$

It can be shown that at any transitional value of R, $f(R)$ has got the same values in both the adjacent ranges. The mean value of R comes out as $1.473a$ approximately (*vide* appendix).

Equal Adjacent Squares

This case is also of practical importance, and will be briefly discussed here, without details of proof. Suppose the squares are adjacent in the x-direction, so that we may put $a = b = c = d$, $e = 0$, $f = -a$. The p. d. f. of X, $f(X)$ is given by corollary 1 to theorem 3, and $f(Y)$ by corollary 2. In the (X, Y)-diagram we have only two compartments $C(I)$ and $C(II)$, $C(I)$ being given by $X = 0$ to a, $Y = 0$ to a, and $C(II)$ by $X = a$ to $2a$, $Y = 0$ to a; there are six transitional points $X = 0, a, 2a$ combined with $Y = 0, a$, and only five transitional values of $R = 0, a, \sqrt{2}a, 2a, \sqrt{5}a$. The p. d. f. of R, is given in four ranges, as $f(R) = 2R\phi(R)/a^4$, where

$$\phi(R) = aR - \tfrac{1}{2}R^2, \text{ for } R = 0 \text{ to } a;$$

$$\phi(R) = 2a^2\cos^{-1}(a/R) - 2a\sqrt{(R^2-a^2)} - 2aR + R^2 + \tfrac{3}{2}a^2, \text{ for } R = a \text{ to } \sqrt{2}a;$$

$$\phi(R) = 2a^2\sin^{-1}(a/R) + 2a\sqrt{(R^2-a^2)} - 2aR - \tfrac{1}{2}a^2, \text{ for } R = \sqrt{2}a \text{ to } 2a;$$

and

$$\phi(R) = 2a^2\{\sin^{-1}(a/R) - \cos^{-1}(2a/R)\} + 2a\sqrt{(R^2-a^2)} + a\sqrt{(R^2-4a^2)} - \tfrac{1}{2}R^2 - \tfrac{5}{2}a^2, \text{ for } R = 2a \text{ to } \sqrt{5}a. \qquad (11)$$

For any transitional value of R, $f(R)$ has got the same value in both the adjacent ranges. The mean R comes out as $1.088a$ approximately (*vide* appendix).

Indirect Methods

A method has been developed for evaluating $f(R)$ indirectly for some new cases, with the help of already known expressions of $f(R)$ for some other cases; a simple illustration of this method is given below. Consider a rectangle with adjacent sides equal to a and $2a$, in which two independent random points P_1 and P_2 are located, and let the p. d. f. $f(R)$, be denoted by f_1. Now considering the rectangle as made up of two equal adjacent squares of side a, it will be seen that the probability of P_1 and P_2 belonging to the same square is equal to the probability of P_1 and P_2 belonging to different squares, each probability being $\frac{1}{2}$. So, if f_2 denotes the p. d. f., $f(R)$, within a square of side a, and f_3 denotes the p. d. f., $f(R)$, between two equal adjacent squares of side a, it will be clear with a little thought that f_1 will be equal to $\frac{1}{2}(f_2+f_3)$. So of these three functions, f_1, f_2, f_3, if any two are already known, the third can be easily found out in this indirect manner. For this particular example of course, we can easily find out f_1 and f_2 from theorem 2, and f_3 is given by eqn. (11), and so we can verify the relation $f_1 = \frac{1}{2}(f_2+f_3)$. Denoting the mean R for the three distributions f_1, f_2, f_3 by M_1, M_2, M_3, we also have $M_1 = \frac{1}{2}(M_2+M_3)$, which relation can also be easily verified, as from the appendix we find the approximate values of M_1, M_2, M_3 as $0.804a$, $0.521a$ and $1.088a$.

For plane figures of other shapes (non rectangular) it will not usually be possible to derive the expressions for $f(R)$ theoretically. But, if necessary, approximate nature of the distribution $f(R)$ can be ascertained by empirical methods, *e.g.* experimental sampling.

Appendix

The k-th moment about the origin, α_k, is the integral of $\{R^k f(R)\}$ over the whole range of R. For evaluating these moments for both the problems of "one rectangle" and "two rectangles" the following integrals will be required:

$$\int R^n \sin^{-1}\left(\frac{m}{R}\right)dR, \int R^n \cos^{-1}\left(\frac{m}{R}\right)dR, \text{ and } \int R^n \sqrt{(R^2-m^2)}\,dR,$$

with positive integral values of n. If we require α_k for $k = 0, 1, 2, 3, 4$ only, the necessary values of n are 1, 2, 3, 4, 5. By integration by parts, we have

$$\int \sin^{-1}\left(\frac{m}{R}\right) R^n dR = \frac{1}{n+1} R^{n+1} \sin^{-1}\left(\frac{m}{R}\right) + \frac{m}{n+1}\int \frac{R^n dR}{\sqrt{(R^2-m^2)}},$$

$$\int \cos^{-1}\left(\frac{m}{R}\right) R^n dR = \frac{1}{n+1} R^{n+1} \cos^{-1}\left(\frac{m}{R}\right) - \frac{m}{n+1}\int \frac{R^n dR}{\sqrt{(R^2-m^2)}}. \tag{12}$$

So ultimately we require integrals of the form

$$I_n = \int \frac{R^n dR}{\sqrt{(R^2-m^2)}}, \quad \text{and} \quad J_n = \int R^n \sqrt{(R^2-m^2)}\,dR,$$

for $n = 1, 2, 3, 4, 5$.

By successive integration by parts, and putting $P = \sqrt{(R^2-m^2)}$, we have, for odd values of n,

$$I_1 = P;\ I_3 = \tfrac{1}{3}P^3 + m^2P;\ I_5 = \tfrac{1}{5}P^5 + \tfrac{2}{3}m^2P^3 + m^4P;$$
$$J_1 = \tfrac{1}{3}P^3;\ J_3 = \tfrac{1}{5}P^5 + \tfrac{1}{3}m^2P^3;\ J_5 = \tfrac{1}{7}P^7 + \tfrac{2}{5}m^2P^5 + \tfrac{1}{3}m^4P^3. \tag{13}$$

We further put $Q = \cosh^{-1}(R/m)$: of the two roots of $\cosh^{-1}(R/m)$, the principal value, $[\log_e\{R + \surd(R^2 - m^2)\} - \log_e m]$, is to be taken. Now by successive integration by parts, we have, for even values of n,

$$I_2 = \tfrac{1}{2}RP + \tfrac{1}{2}m^2Q\,; \quad I_4 = \tfrac{1}{8}RP(2R^2 + 3m^2) + \tfrac{3}{8}m^4Q\,;$$

$$J_2 = \tfrac{1}{8}RP(2R^2 - m^2) - \tfrac{1}{8}m^4Q\,; \quad J_4 = \tfrac{1}{48}RP(8R^4 - 2m^2R^2 - 3m^4) - \tfrac{1}{16}m^6Q. \tag{14}$$

Using these relations (12), (13), (14), we can work out the values of α_k for different cases.

"One-Rectangle" Case. For $f(R)$ given in theorem 2, we have, putting $M = \surd(a^2 + b^2)$, for even values of k, $\alpha_0 = 1$ (as it should be),

$$\alpha_2 = \tfrac{1}{6}M^2\,; \quad \alpha_4 = \tfrac{1}{15}a^4 + \tfrac{1}{18}a^2b^2 + \tfrac{1}{15}b^4\,;$$

and for odd values of k,

$$\alpha_1 = \frac{1}{6}\left\{\frac{b^2}{a}\cosh^{-1}\left(\frac{M}{b}\right) + \frac{a^2}{b}\cosh^{-1}\left(\frac{M}{a}\right)\right\} + \frac{1}{15}\left(\frac{a^3}{b^2} + \frac{b^3}{a^2}\right) - \frac{1}{15}M\left(\frac{a^2}{b^2} + \frac{b^2}{a^2} - 3\right);$$

$$\alpha_3 = \frac{1}{20}\left\{\frac{a^4}{b}\cosh^{-1}\left(\frac{M}{a}\right) + \frac{b^4}{a}\cosh^{-1}\left(\frac{M}{b}\right)\right\} + \frac{2}{105}\left(\frac{a^5}{b^2} + \frac{b^5}{a^2}\right)$$

$$- M\left\{\frac{2}{105}\left(\frac{a^4}{b^2} + \frac{b^4}{a^2}\right) - \frac{5}{84}M^2\right\}. \tag{15}$$

With known values of a and b, the values of mean, variance, skewness (γ_1) and kurtosis (γ_2) can be worked out from eqns. (15). For example, the approximate values of α_1 (mean) for $a = b$ and $a = 2b$ are given by $0.521a$ and $0.402a$ respectively. It may be noted here that some of the numerical values for these measures have been wrongly printed in (B. Ghosh 1943a).

"Two Rectangles" Case. For the special cases of "equal squares with corner-point contact" (eqn. 10), and "equal adjacent squares" (eqn. 11), it has been verified that $\alpha_0 = 1$; the approximate values of the mean distance (α_1) are $1.473a$ and $1.088a$ respectively in the two cases.

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