# A Theory for the Economic Operation of a Smart Grid with Stochastic Renewables, Demand Response and Storage

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*Abstract*— **We are motivated by the problems faced by independent system operators in an era where renewables constitute a significant portion of generation and demand response is employed by a significant portion of loads. We address a key issue of designing architectures and algorithms which generate optimal demand response over a time window in a decentralized manner, for a smarter grid consisting of several stochastic renewables and dynamic loads. By *optimal demand response*, we refer to the demand response which maximizes the sum of the utilities of the agents, i.e., generators, loads, load serving entities, storage services, prosumers, etc., connected to the smart-grid. By *decentralized* we refer to the desirable case where neither the independent system operator (ISO) needs to know the dynamics/utilities/states of the agents, nor do the agents need to know the dynamics/utilities/states of each other. The communication between the ISO and agents is restricted to the ISO announcing prices, and the agents responding with their energy generation/consumption bids.**

**We begin with the deterministic case for which there is a complete solution. It features a price iteration scheme that results in optimality of social welfare. We also provide an optimal solution for the case where there is a common randomness affecting and observed by all agents. This solution can be computationally complex, though we provide approximations. For the more general partially observed randomness case, we exhibit a relaxation that significantly reduces complexity. We also provide an approximation strategy that leads to a model predictive control (MPC) approach. Simulation results illustrate the increase in social welfare utility compared to some alternative architectures.**

## I. INTRODUCTION

Traditionally, given demand (or a demand forecast), generation (or the planned generation) has been dispatched so as to balance demand in power systems. Since there are many generators capable of producing power at different cost curves, it is desirable to allocate the total power generation among the generators so that the total cost of generating the required power is minimized. This role of determining which generators are selected, and how much power they produce, has been played by the Independent System Operator (ISO).

This economic dispatch of generators is done by soliciting production bids (power vs. price curves) from each generator, and then, given the demand, choosing the generators such that the overall cost to purchase the power is minimized. Motivated by the problem of integrating renewable power generation from sources such as wind and solar [1] which vary over time, we consider the problem of demand response over a time interval, i.e., adjusting demand so that it can be part of the flexibility to match the variable generation [2]. The process of increasing or decreasing demand can be accomplished by decreasing or increasing the price of power. Thus the problem becomes one of choosing the level of demand as well as allocating the required generation among all the various producers of power including fossil fuel sources such as coal or gas as well as renewables such as solar or wind.

An additional and important consideration in this paper is that both power generators as well as loads are dynamic systems. Generators, for example, have ramping constraints as well as maximum power constraints. Similarly, loads also have certain dynamic constraints; an analysis of load data is conducted in [3]. Thus, all variables, including price as well as generator power outputs as well as loads, are *functions of time*.

A third consideration is that renewable power production is uncertain, which we model as a stochastic process. Thus, in addressing the dynamic evolution of the future demand as well as future non-renewable power generation over time, one needs to take future uncertainty of renewable power generation into account.

In this paper, we consider the resulting overall problem faced by the ISO: Given stochastic renewable generation which can only be known causally in time, how should an ISO choose the price causally as a function of time, and thereby the level of demand response elicited, and then allocate the net remaining generation among various conventional generators. A consideration which we will neglect in this paper is that of network constraints such as load flow constraints or robustness to contingencies. Some limited forms of constraints can be incorporated into the setup that we develop in this paper.

There are some complexities that are involved in the problem, and some constraints that a desirable solution has to satisfy. It is desirable that the communication from the ISO to the generators or loads contain only the price of power in each market interval. Also it would be desirable that the communication from each generator and load to the ISO, at each time instant only be the resulting power for desired production and demand response respectively. In particular, we would not like to require that each load provide a dynamic model of its behavior as well as its utility function to the ISO, and similarly for each generator. Not only is this a lot of information that the agents may not want disclose for reasons of privacy or business, but it also implies that the ISO solve a gigantic optimization problem encompassing every single

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agent in the system along with all their utility functions.

Thus we would like the system-wide optimization to be conducted by the agents themselves in a distributed manner, coupled only by the price announced by the ISO. Additional complexities are introduced by the fact that the agents are dynamic systems as well as the fact that the renewable power generation is stochastic. Our approach attempts to provide a comprehensive solution that takes into consideration all individual agents' utility functions. It has one feature that would be desirable to eliminate.

At each time our approach requires that the ISO interact with the agents in an extended transaction. The ISO announces a tentative price sequence over time, and the agents respond with power generation or consumption levels. Based on these responses, the ISO again announces a tentative price sequence, and so on. This process continues till the price sequence converges. This discovery process for price appears unavoidable if the agents are not to share thier dynamic models, utility functions and states with the ISO. This is similar to the Walrasian tatonnement process in general equilibrium theory [4].

We examine three models of increasing complexity, a deterministic model, a common completely observed randomness model, and a partially observed randomness model. For the deterministic model our solution is complete and leads to social welfare optimality under appropriate convexity conditons. Subsequently we consider the case where there is a common randomness affecting all agents which is observed by each of them. This includes the case where only certain portions of the overall randomness affect individual agents and in different ways, but all agents would need to know everyone's uncertainties. In this case we describe a complete solution, which leads to social welfare optimality of the total utility. We also propose an approximation scheme to reduce complexity that leads to a model predictive approach. In the most general case where the different agents experience differing randomness not observable by other agents, we propose a *free storage* relaxation that leads to a significant reduction in complexity. We also show that our MPC approach extends to this case. We present simulations comparing our algorithms with other schemes and illustrate the improvement in utility that can be realized.

Overall, our approach explores a theory for the operation of the ISO in an environment where it is needed to integrate stochastic renewables, demand response, and dynamic and other constraints as well as uncertainties in both generation and loads. For the system operator the challenge is to optimally balance the system when uncertainty arises both in supply and demand, without resorting to the expensive option of procuring large amounts of reserve and/or energy storage [5].The solution requires communication only of prices and energy production/consumption responses, as is desirable. Specifically it is not necessary for the ISO to be aware of the dynamics of producers or consumers or their individual utility functions. Similarly, the various agents need only know their own dynamic system models and states and utility function, and need not be aware of any attributes of other users, or even the existence of other users. The key issue in this case is of designing an architecture which yields optimal demand response in a decentralized manner while maintaining user privacy.

The current smart grid faces the challenge of renewable penetration and electricity price fluctuations. Our approach of modeling the users by a dynamical system in lieu of following a *static-optimization* is the key to generate demand response that plays the dual role of mitigating renewable penetration and price fluctuation via utility optimization. Thus our goal is to provide a framework in which Demand Response could participate in both energy and ancillary service markets [6], [7]. Of course, this demand response needs to be optimized in order to achieve our pre-set goals. This is done via utilization of the computational power and latent energy storage that is present in the *smart* users connected to the smart grid.

The paper is organized as follows: we begin with a survey of some related works in Section II and give a complete description of the problem in Section III. This is followed by a discussion of iterative bidding schemes and the ensuing optimal demand response in a deterministic setting (Section IV) and stochastic settings (Section V). Finally simulation results are presented (Section VI), to illustrate the approach.

## II. LITERATURE SURVEY

There have been many efforts since the deregulation of the electricity sector on a market-based framework to clear the system. Ilic et al. [8] proposed a two-layered approach that internalizes individual constraints of market participants while allowing the ISO to manage the spatial complexity. The approximated MPC algorithm is shown to work in many realistic cases.

In order to analyze the strategic interactions between the ISO and market participants, game theoretical approaches have been proposed in a number of papers. Zhu et al. [9] use a Stackelberg game framework for economic dispatch with demand response. The approach uses a two person game with ISO as leader and users aggregated into second player. The users change their demand based on a price signal so as to maximize their payoff function. The Economic Dispatch (ED) problem considered is a single time interval conventional dispatch without transmission line constraints. Jia and Tong [10] use a Stackelberg formulation to study the energy consumption scheduling problem for customers which are subjected to a time-varying price that is determined one day ahead of time. The trade-off between consumer surplus and retailer profit under different pricing schemes is investigated.

Wang et al. [11] formulates the trading of energy by storage units as a noncooperative game. Under certain assumptions on the strategy space and utility functions a Nash equilibrium is shown to exist. Mohsenian-Rad et al. [12] propose a distributed algorithm to obtain the optimal energy consumption schedule for each user. The problem of determining the user energy consumption schedule for the whole day is formulated as a deterministic linear program.

The classic paper [13] develops a theory of pricing in electrical networks, however the system model therein does not include the dynamics, and stochastic nature of the agents.

One of the major challenges in the above approaches is how to elicit optimal demand response without revealing the inherent dynamic nature of the loads to the ISO. In this paper, we model the users as stochastic dynamical systems and generate the optimal demand response in a decentralized and adaptive manner, thus maximizing the sum total of the utilities of the users, which in-turn allows for maximum renewable penetration while controlling price fluctuations.

## III. PROBLEM FORMULATION

We consider a smart-grid in which there are a total of $N$ agents. Each agent may be either a consumer or a producer of electricity. We model time as consisting of discrete periods. At each such discrete time $t$ each agent $i$ obtains or supplies $u_i(t)$ units of energy (equivalently power since it is proportional to it given the fixed period) to the grid, with $u_i(t) > 0$ signifying that user $i$ supplies energy to the grid at time $t$, while $u_i(t) < 0$ signifies an energy consumption. An important constraint is that there is net energy balance at each time over the whole grid: $\sum_{i=1}^{N} u_i(t) = 0$ for all $t$. This model does allow for storage too, since each storage device can be considered as an agent.

We model each agent as a dynamic system. The motivation in the case of an agent which is a generator is that it has ramp up constraints, thus necessitating a dynamic system model, or in the case of a load it may have similar ramp down constraints as well as delay in demand response. The state of user $i$ at time $t$, denoted $x_i(t) \in \mathcal{X}_i$, evolves as,

$$x_i(t+1) = f_i^t(x_i(t), u_i(t)), t = 1, 2, \ldots, T-1. \quad (1)$$

Thus the state of the grid resides on the space $\otimes \mathcal{X}_i$. We suppose that each agent $i$ has a stagewise utility function $F_i(\cdot) : \mathcal{X}_i \mapsto \mathbb{R}$, with the user preferring a state having higher utility. The total utility of user $i$ over the horizon $\{1, 2, \ldots, T\}$ is $\sum_{t=1}^{T} F_i(x_i(t))$. (The theory can be generalized in a straightforward way to utilities that are time-dependent.) The model (1) can incorporate constraints on inputs, for example reflecting bounds on ramp rates, such as $u_i(t) \in \mathcal{U}_i$. In that case, these constraint sets $\mathcal{U}_i$ are not dualized in the sequel, but simply carry over to the dual in (3). For simplicity of exposition we will not explicitly consider this case in the treatment here, but will consider such constraints in the numerical examples in Section VI.

With the above set-up, we are led to the following deterministic social welfare optimization problem (DSWOP):

$$\begin{aligned} &\max \sum_{i=1}^{N} \sum_{t=1}^{T} F_i(x_i(t)) \\ \text{subject to } &\sum_{i} u_i(t) = 0, \forall t = 1, 2, \ldots, T \\ &x_i(t+1) = f_i^t(x_i(t), u_i(t)), \text{ for} \\ &t = 1, 2, \ldots, T-1, i = 1, 2, \ldots, N. \quad \text{(DSWOP)} \end{aligned}$$

Subsequently we will consider the stochastic version of the problem caused by uncertainties due to weather, etc.

## IV. OPTIMAL DEMAND RESPONSE AND DECENTRALIZED SOLUTION VIA BIDDING

The above problem can be interpreted as giving the ISO the task of determining the $T$-dimensional vectors $\boldsymbol{u}_i := (u_i(1), u_i(2), \ldots, u_i(T))$, for $i = 1, 2, \ldots, N$, so as to maximize the social welfare $\sum_{i=1}^{N} \sum_{t=1}^{T} F_i(x_i(t))$. In this section, we will derive an easy-to-implement algorithm that does so, while satisfying information and action decentralization, with all communication between agents restricted simply to being either price announcements or purchasing or supply of energy decisions in response to prices. The ISO simply determines the appropriate prices causally, while each user optimizes its response causally.

The Lagrangian for the problem DSWOP is,

$$\begin{aligned} &\mathcal{L}(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N, \boldsymbol{\lambda}) \\ &:= \sum_{i=1}^{N} \sum_{t=1}^{T} F_i(x_i(t)) - \sum_{t=1}^{T} \lambda(t) \left( \sum_{i=1}^{N} u_i(t) \right), \quad (2) \end{aligned}$$

where $\lambda(t), t = 1, 2, \ldots, T$ are the Lagrangian multipliers associated with the constraints $\sum_i u_i(t) = 0, t = 1, 2, \ldots, T$ respectively. The Lagrange dual function is,

$$\begin{aligned} D(\boldsymbol{\lambda}) &:= \max_{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N} \mathcal{L}(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N, \boldsymbol{\lambda}) \\ &= \max_{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N} \sum_{i=1}^{N} \left( \sum_{t=1}^{T} F_i(x_i(t)) - \lambda(t) u_i(t) \right). \quad (3) \end{aligned}$$

The objective function (3) can be decomposed agent by agent since they are only coupled by price. Hence we consider the optimal problem for agent $i$ as one of maximizing the objective

$$\max_{\boldsymbol{u}_i} \sum_{t=1}^{T} F_i(x_i(t)) - \lambda(t) u_i(t), \quad (4)$$

for the dynamic system (1). The optimal cost is a function of the initial condition and the Lagrange multiplier sequence $\boldsymbol{\lambda} = (\lambda(1), \ldots, \lambda(T))$, and we denote it $V_i(x_i(0), \boldsymbol{\lambda})$. Therefore,

$$D(\boldsymbol{\lambda}) = \sum_{i=1}^{N} V_i(x_i(0), \boldsymbol{\lambda}).$$

We thus observe that the consideration of the dual problem has led us to a decentralized problem. Its solution involves the ISO first announcing the price vector $\boldsymbol{\lambda}$, and then each agent $i$ simply optimizing its own objective (4) by determining its vector $\boldsymbol{u}_i$. Thus neither the ISO, nor the other agents, need to know the utility function or dynamics or state of agent $i$. The dual problem is,

$$\begin{aligned} &\min D(\boldsymbol{\lambda}) \\ &\text{subject to} \quad \lambda(1), \ldots, \lambda(T) \geq 0. \quad (5) \end{aligned}$$

We will suppose that strong duality holds, i.e., the optimal values of DSWOP and (5) are equal. There are several sufficient conditions for strong duality. For example a sufficient condition is for the utility functions $\sum_{t=1}^{T} F_i(x_i(t))$ be convex and the feasibility region of the problem DSWOP nonempty. Denoting the optimal solution of the Dual problem by $\boldsymbol{\lambda}^\star$, we will suppose that,

$$
\begin{aligned}
D(\boldsymbol{\lambda}^\star) &= \sum_{i=1}^{N} V_i(x_i(0), \boldsymbol{\lambda}^\star) \\
&= \max_{\substack{\boldsymbol{u}_i, i=1,2,\ldots,N: \\ \sum_i u_i(t)=0, \\ \forall t=1,2,\ldots,T}} \sum_{i=1}^{N} \sum_{t=1}^{T} F_i(x_i(t)).
\end{aligned}
$$

The issue faced by the ISO is how to determine the optimal price vector $\boldsymbol{\lambda}^\star$. Since $D(\boldsymbol{\lambda})$ as well as $V_i(x_i(0), \boldsymbol{\lambda})$ are all concave functions of $\boldsymbol{\lambda}$, will consider the use of the sub-gradient $\frac{\partial D}{\partial \boldsymbol{\lambda}}$ for iterating on the price-vector $\boldsymbol{\lambda}$ so as to converge to the optimal price-vector $\boldsymbol{\lambda}^\star$:

$$
\begin{aligned}
\frac{\partial D}{\partial \boldsymbol{\lambda}} &= \sum_{i=1}^{N} \frac{\partial V_i}{\partial \boldsymbol{\lambda}} \\
&= \left( \sum_{i=1}^{N} u_i^{\lambda}(1), \sum_{i=1}^{N} u_i^{\lambda}(2), \ldots, \sum_{i=1}^{N} u_i^{\lambda}(T) \right),
\end{aligned}
$$

where $\boldsymbol{u}_i^{\boldsymbol{\lambda}} := \left( u_i^{\lambda}(1), u_i^{\lambda}(2), \ldots, u_i^{\lambda}(T) \right)$ is the vector that achieves the optimal utility for the $i$-th user for the price vector $\boldsymbol{\lambda}$ in (4).

We see that the iterations on the price vector $\boldsymbol{\lambda}$ generate the corresponding demand response,

$$
DR(\boldsymbol{\lambda}) = \frac{\partial \boldsymbol{u}^{\boldsymbol{\lambda}}}{\partial \boldsymbol{\lambda}}, \tag{6}
$$

where $\boldsymbol{u}^{\boldsymbol{\lambda}}$ contains the vectors $\boldsymbol{u}_i^{\boldsymbol{\lambda}}$ for agents $i$ which are consumers.

We thus obtain the following price iteration algorithm. Set $k$, the iteration index to 0. The ISO declares a price vector $\boldsymbol{\lambda}^0$ (chosen arbitrarily, but preferably close to the true price vector).

- The users $i = 1, 2, ..., N$ solve their individual optimal control problems and calculate their $\boldsymbol{u}_i^{\boldsymbol{\lambda}(0)}$. Then they separately submit their bids $\boldsymbol{u}_i^{\boldsymbol{\lambda}(0)}$.
- The ISO then updates the price vector as: $\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k - \alpha \left( \sum_i \boldsymbol{u}_i \right)$, where $\alpha > 0$ is a step size. Increment $k$ by one and go to the previous step.

There are several choices for the step size $\alpha$, and several convergence results for the resulting sub-gradient method [14].

## V. BIDDING WITH STOCHASTIC RENEWABLES AND DEMANDS

In the previous section, the dynamics of the users were assumed to be deterministic, i.e., the exact value of the system state at the next instant was completely determined by (1). This might be unrealistic keeping in mind the stochastic nature of renewable energy as well as user demands. We begin our treatment with a special case in which the theory can be fully developed. In this case, which we call the Common Completely Observed Case, the model of the stochastic uncertainties is known to all the agents and observed causally by all of them. This could include, for example, the cloud cover in Denver or wind speed and direction in Brazos County in Texas.

### A. *Common Observed Randomness Case*

Let $\omega = \omega(1), \omega(2), \ldots, \omega(T)$ be $T$ primitive random variables. For simplicity, let us suppose that each $\omega(t)$ assumes value in a finite set. The state of the $i$-th agent evolves as,

$$
X_i(t+1) = f_i^t(X_i(t), U_i(t), \omega(t)),
$$

and it is assumed that each agent observes $\omega$ causally in time, i.e., has access to $\omega(1), \omega(2), \ldots, \omega(t)$ at time $t$. Also everybody knows the probability law $P$ of $\omega$. The expression ‘Common in “Common Observed Randomness Case” refers to the fact that every subsystem knows the randomness that affects all subsystems.

The problem of interest is then to maximize

$$
\max \mathbb{E} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} F_i(X_i(t)) \right\}
$$

(Common Known Randomness Problem)

for the $N$ stochastic dynamic systems

$$
X_i(t+1) = f_i^t(X_i(t), U_i(t), \omega(t)). \tag{7}
$$

The inputs $U_i(t)$, for $i = 1, 2, ..., N$ at each $t$ have to satisfy the constraints

$$
\sum_i U_i(t) = 0, \text{ for each } t = 1, 2, \ldots, T. \tag{8}
$$

We consider the following ISO-based approach to solving this problem. Let $\omega^t = (\omega(1), \omega(2), \ldots, \omega(t))$ be the past of $\omega$ $(= \omega^T)$ until time $t$. The ISO announces a price random variable $\lambda(\omega) = \left( \lambda(1, \omega^1), \lambda(2, \omega^2), \ldots, \lambda(T, \omega^T) \right)$ for each $\omega$.

Note that the price announcement by the ISO is actually a *policy* announcement. (much like the Federal Reserve saying that interest rates will rise if there is a hurricane). The ISO thus announces that if the disturbances $\omega(1), \omega(2), \ldots, \omega(t)$ hit the system by time $t$, then the price will be $\lambda(t+1, \omega^t)$.

Based on this policy announcement, the individual agents also respond with a *policy*. Agent $i$ announces a policy $U_i(1, \omega^1), U_i(2, \omega^2), \ldots, U_i(T, \omega^T)$. The agents determine their policies individually simply by dynamic programming since each knows the probability law for the stochastic process $\omega$, their own dynamic system model, and the price random variable $\lambda(t, \omega^t)$.

Now we can see that this system is amenable to the same iteration for prices $\lambda(w)$ as before. The ISO first announces the future price policy as its first iterate, $\lambda^0(0, \omega^t), \lambda^0(1, \omega^{t+1}), \ldots, \lambda^0(T, \omega^T)$. Each agent $i$ then responds with future consumption/generation policy $U_i(0, \omega^t), U_i(1, \omega^{t+1}), \ldots, U_i(T, \omega^T)$. The ISO computes whether there is a net surplus or deficit of energy at each future time , $\sum_{i=1}^{N} U_i(0), \sum_{i=1}^{N} U_i(1), \ldots, \sum_{i=1}^{N} U_i(T)$. Based on this it iterates to produce a new iterate $\lambda^1(0, \omega^t), \lambda^1(1, \omega^{t+1}), \ldots, \lambda^1(T, \omega^T)$. This iteration can be based on a sub-gradient method where the increment is proportional to the energy surplus/deficit vector. Then the users again respond with the next iterate of the future consumption/generation policy $U_i(0, \omega^t), U_i(1, \omega^{t+1}), \ldots, U_i(T, \omega^T)$. This continues until there is convergence. This is a solution of the Common Known Randomness Problem which leads to optimal utility.

A major issue with the above solution is complexity, since $\omega$ lies in a large cardinality set $|\Omega|$. One could consider the following approximation algorithm.

**Approximation Algorithm with $k$-step Look-ahead** At each time $0 \leq s \leq T$, the ISO announces the prices $\lambda(s+1), \ldots, \lambda(s+k)$ for the next $k$ time periods, freezing the prices after $k$ periods. Iteration then takes place over the $k$-dimensional space, and at each step the iteration tries to achieve energy balance over the next $k$ time periods via bids. The idea is similar to the Model Predictive Control (MPC), so that optimization is performed only for $k$ horizon look-ahead instead of entire $T$ horizon, thus giving us a reduction in complexity. This policy will not approach the optimal policy even as $k \to \infty$ since it is what is called an "open loop feedback policy". At each time the future price sequence is assumed to be deterministic, not a fully uncertainty state-dependent policy.

### B. The Partially Observed Randomness Case

We now consider the more general case where each agent $i$ has a separate "private" stochastic process $\omega_i = \omega_i(1), \ldots, \omega_i(T)$ affecting only its system via the equation

$$X_i(t+1) = f_i^t(X_i(t), U_i(t), \omega_i(t)),$$

The stochastic process $\omega_i$ is not completely observed by the other agents, and only agent $i$ knows the law of process $\omega_i$. All the stochastic processes are assumed to be independent in the treatment below. The objective function and the constraints remain the same as in the Common Known Randomness Problem. However the assumption that an agent does not have access to the randomness of other agents makes it difficult to achieve co-operation amongst the agents.

If the goal is to optimize the utility over all decentralized policies, then the ISO needs to play a more active role so as to induce co operation amongst the agents. It needs to know much more about each individual agents' systems, their values of the states $X_i(t)$, utility functions $F_i(\cdot)$, their dynamics $f_i^t(X_i(t), U_i(t), \omega_i(t))$, and the probability distributions of their uncertainties $\omega_i$. Then the ISO could in principle decide the optimal allocations $\boldsymbol{U}(t)$ for each $t$, as a function of the state of the entire system via dynamic programming. This procedure of course suffers from the curse of dimensionality as the number of users is increased.

An optimal decentralized solution, where the solution is itself computed in an iterative decentralized manner to this is an interesting and open problem. We now present another approach, a relaxation, that provides some interesting approximation algorithms with much reduced complexity.

**Free Storage Relaxation for The Relaxed Partially Observed Randomness Problem**

Let us assume that the ISO has access to a subset of the randomness $\{\omega_i\}_{i=1}^{N}$, denoted by $\omega^{ISO}$, knows the law of $\omega^{ISO}$, and suppose that $\omega^{ISO}$ is a positive recurrent Markov process. This is the same as assuming that the ISO of a city has knowledge to the weather of the city, or has knowledge of events which might alter the electricity consumption in a big way, and knows the probability laws governing them. However, the ISO neither has knowledge of the utility functions $F_i$ of the agents, nor of their dynamics $f_i$, nor of the randomnesses vector $\omega_i$. Also it is assumed that $\omega^{ISO}$ and its law are known to each agent. The key idea for producing a tractable approximation is to relax the constraint of energy supply equal to energy consumption at each time $t$, and along each sample path of the stochastic uncertainty. We relax the above constraints to the following constraint almost sure long term average of conditional expectations being in balance:

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \sum_{t=1}^{T} \sum_{i=1}^{N} U_i(t) 1 \left( \omega^{ISO}(t) = w \right) \right) = 0,$$

where $w$ is any element of the state space of $\omega^{ISO}$. Intuitively it means that the power-balance constraint $\sum_{i=1}^{N} U_i(t)$ is allowed to be violated, however the fluctuation should balance out over time, conditioned on the ISO's observations. This can be regarded as corresponding to the availability of free storage that eliminates the energy balance requirement in each time period.

It can be shown that the optimal policy for this case is for the ISO to declare the price at time $t$ based on the value of $\omega^{ISO}(t)$, with users then choosing the quantities $U_i(t)$ based on the value of their state $X_i(t)$, and the process $\omega^{ISO}(t)$. That is, an agent $i$ need not know the values of the states $X_j(t), j \neq i$ of the other agents, nor their utility functions, in order to decide the quantity $U_i(t)$. The analysis and the proof of optimality in the case of large number of agents is analogous to the treatment of multi-armed bandits problems and activity allocation problems [15], [16], [17], [18] and uses the technique of large-deviations for Markov process. We note that the relaxation provides an upper bound on utility, which can be interpreted as the utility that can be realized in case there is free storage.

**The Limited Lookahead Approach for the Partially Observed Randomness Case** The Limited Lookahead approach can also be applied to the Partially Observed case to yield an approach that resembles the Model Predictive Control approach. The MPC approach discussed in Section V-A can be applied in order to develop an approximation algorithm for The Partially Known Randomness Case.

At each time $t$, the ISO fixes a $k$-step random price vector $\lambda^0$ for the next $k$ time instants. This vector will depend only upon the $\omega^{ISO}$ in a causal way. The agents respond to this vector via calculating optimal bids $U_i$ for the future time

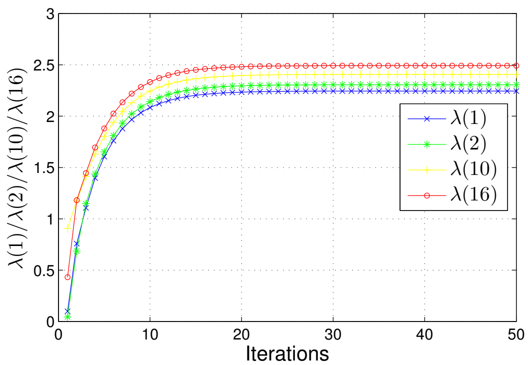


Fig. 1. Convergence of the price vector for deterministic case

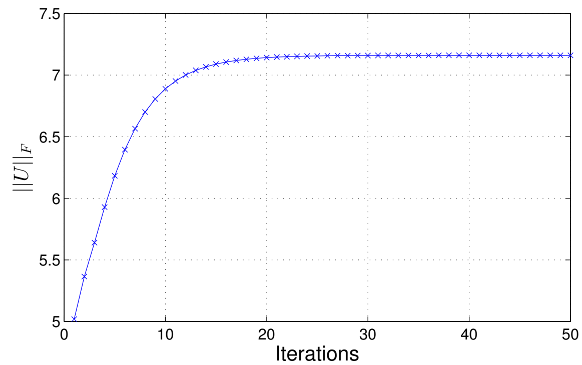


Fig. 2. Demand Response for deterministic case

periods. Then the ISO iterates the price upon receiving the agents' bids. The iterations continue till the changes in the iterates become small enough.

## VI. Numerical results

We illustrate the above algorithms by simple examples. We start with a deterministic case, followed by a stochastic case. We assume that the $N$ users are divided into two groups: users $i \in \{1, ..., M\}$ are residential consumers and users $i \in \{M+1, ..., N\}$ are power suppliers. We assume that the network constraints are not binding.

### A. *Deterministic case*

For a consumer $i$, $X_i(t)$ denotes the room temperature at time $t$ that evolves as,

$$X_i(t+1) = a_i X_i(t) + h_i - \beta_i U_i(t),\ i \in \{1, ..., M\}$$

where $h_i$ denotes ambient heating. For a supplier $i$, $X_i(t)$ denotes the power production level for the $i$-th user at time $t$, evolving as,

$$X_i(t+1) = a_i X_i(t) + U_i(t),\ i \in \{M+1, ..., N\}.$$

There are natural constraints associated with the state equations. For consumers, $U_i(t) \leq \frac{1}{\beta_i}(h_i + c_i)$, where $c_i$ is the maximal cooling rate. For suppliers, $U_i(t) \leq r_i$, where $r_i$ is the maximal ramp rate allowed.

For consumers, let

$$F_i(X_i(t)) = -\left(X_i(t) - \frac{\phi_{1i} + \phi_{2i}}{2}\right)^2 + m_i - \lambda(t)U_i(t)$$

where $[\phi_{1i}, \phi_{2i}]$ is $i$-th user's "comfort interval" for temperature, $m_i$'s are constants and $\lambda(t)$ is the price. For suppliers,

$$F_i(X_i(t)) = \lambda(t)X_i(t) - \left(C_{1i}X_i^2(t) + C_{2i}X_i(t) + C_{3i} + C_{4i}U_i(t)\right).$$

We can use $QCQP$ (Quadratically Constrained Quadratic Programming) to solve the problem. Suppose $M = 5$, $N = 10$, $h_i = \beta_i = 1$, $m_i = 2$, $a_i = 1$ and choose $\phi_{1i}$ uniformly from $[20, 21]$, $\phi_{2i}$ from $[24, 25]$, $r_i$ from $[0.5, 1.5]$, $C_{1i}$ from $[0.9, 1.1]$, $C_{2i}$ from $[0.1, 0.3]$, $C_{3i}$ from $[0.5, 1.0]$ and $C_{4i}$ from $[0.1, 0.5]$.

Fig. 1 plots the evolution of the price vector, where for legibility of display, we only plot 4 components of $\lambda$. It is easy to see that $\lambda$ converges rapidly, in less than 20 iterations.

Fig. 2 shows the demand response value as a function of iterations. For the deterministic case, this is simply the norm of matrix $U$, where $U$ is a $M \times T$ matrix with $u_{ij} = U_i(j)$. Here we use the Frobenius norm defined by $||U||_F = \sqrt{\sum_i \sum_j u_{ij}^2}$. We can see that demand response approaches a constant as iteration goes on.

### B. *Stochastic case*

We adopt the same notation as in the deterministic case, but modify the state equations by adding a random variable influencing the availability of renewables or stochasticity of demand. For consumers,

$$X_i(t+1) = a_i X_i(t) + h_i - \beta_i U_i(t) + W(t),\ i \in \{1, ..., M\}$$

where $W(t)$ is not necessarily i.i.d. because of geographical and temporal correlations. Similarly, for suppliers,

$$X_i(t+1) = a_i X_i(t) + U_i(t) + V(t),\ i \in \{M+1, ..., N\}$$

where $V(t)$ is not necessarily i.i.d. either.

In our simulation, for simplicity, we let $W(t)$ assume two values drawn uniformly from $[-0.5, 0.5]$, each with probability 0.5; and let $V(t)$ also take two values drawn uniformly from $[-0.2, 0.2]$, each with probability 0.5.

For each step in the model predictive control approach, the price vector $\lambda$ converges within 20 iterations, just as it does in the deterministic case. Fig. 3 plots the demand response value as a function of the iterations. Let $Q(t)$ be the vector containing only the $U_i$'s for $i \in \{1, ..., M\}$. For clarity of display, we only plot the first 4 steps and adopt the $L^2$ norm. As the optimization window moves, $||Q||$ converges faster; whether it converges from the above or below depends on the initial value.

### C. *Comparison with another candidate for ISO operation*

Consider an ISO that sets the price as follows: At time $t$, it assumes that the demand $D(t)$ is given, and based on the previous step's production level $X_i(t-1)$ and the marginal cost of each producer, it assigns the production level $X_i(t)$

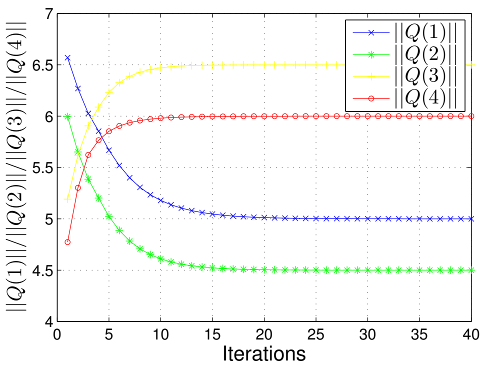


Fig. 3. Demand Response for stochastic case

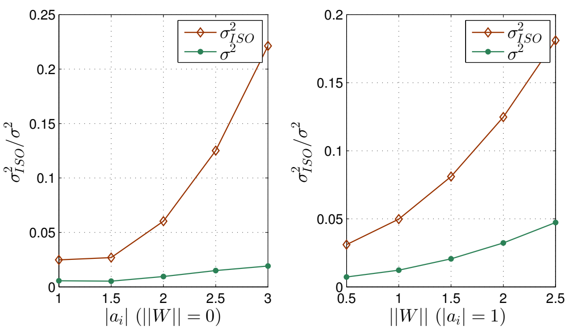


Fig. 4. Variance of price

for each producer so as to minimize the total production cost at time $t$. The associated Lagrange multiplier will then be the price at time $t$. We set $a_i = 3$ for $i \in \{1, ..., N\}$, and keep the same value for the other parameters. Each consumer tries to keep $X_i(t) = \frac{\phi_{1i}+\phi_{2i}}{2}$ for all $t$, and the resulting $U_i(t)$ will be used to calculate the demand input $D(t)$ to the ISO scheme, where $D(t) := \sum_i U_i(t)$ for $i \in \{1, ..., M\}$.

Fig. 4 summarizes the results. By fixing the uncertainty magnitude $||W|| = 0$, the figure on the left shows that $\lambda^{ISO}$, which is generated by the ISO scheme, has a bigger variance $\sigma^2_{ISO}$ than $\lambda$, which is obtained by our iterative approach. Moreover the difference in variance becomes even larger as $a_i$ increases. The figure on the right fixes $|a_i| = 1$ and plots changes in the variance of price as a function of $||W||$. Similarly in the left figure, the difference in variance increases as $||W||$ increases.

Next we compare the total utility of the entire system obtained by the two approaches. Notice that in our case the total utility is,

$$u = \sum_{t=1}^{T}\sum_{i=1}^{N} F_i(X_i(t)) =$$
$$\sum_{t=1}^{T}\left(\sum_{i\in\{1,...,M\}} -\left(X_i(t) - \frac{\phi_{1i}+\phi_{2i}}{2}\right)^2 + m_i\right.$$
$$\left. - \sum_{i\in\{M+1,...,N\}} C_{1i}X_i^2(t) + C_{2i}X_i(t) + C_{3i} + C_{4i}U_i(t)\right)$$

TABLE I

TOTAL UTILITY OBTAINED BY ISO SCHEME AND THE ITERATIVE APPROACH

| Name | The iterative approach | ISO scheme |
|---|---|---|
| Value | 427.1932 | 142.8451 |

TABLE II

TOTAL EXPECTED UTILITY OBTAINED BY ISO SCHEME AND THE ITERATIVE APPROACH

| Name | The MPC approach | ISO scheme |
|---|---|---|
| Value | 500.2578 | 159.2198 |

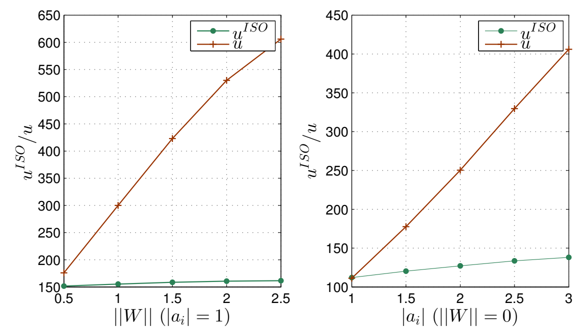


Fig. 5. Changes in total utility as $|a_i|$ or $||W||$ varies

as the $\lambda$ terms cancel out. We calculate the total utility incurred by the two schemes; the results are shown in Table I. It can be seen that the total utility obtained by our dynamic iterative approach is roughly three times the total utility obtained by the greedy ISO scheme. (Note that this coincides with the setting of $a_i = 3$.)

The above considers the case where demand is deterministic. When there is randomness on the demand side, the ISO scheme aims at minimizing the expected production cost at each time $t$. We assume for simplicity that the added noise term $W(t)$ is i.i.d. and takes values 0.5 and $-0.5$ each with probability 0.5. Similarly, as in the deterministic case, we calculate the total expected utility incurred by the two schemes and present the results in Table II. It can be seen that the total expected utility obtained by the MPC approach is more than three times the expected utility obtained by the ISO scheme. We conclude that in both deterministic and stochastic cases, our approach provides greater total utility than the alternative candidate ISO scheme.

We also compare the change in total utility obtained as a function of $a_i$ for the two schemes. We fix $|a_i| = 1$ and let $W(t)$ be i.i.d. taking values $||W||$ and $-||W||$ with probabilities 0.5 and 0.5 respectively. We observe the change in total utility while increasing the noise magnitude $||W||$. The result is shown in the left plot in Fig. 5. It can be seen that as $||W||$ increases, the total utility obtained by the MPC is not a strict linear function of the utility obtained by alternative candidate ISO scheme. The plot on the right fixes $||W|| = 0$, and shows that as $|a_i|$ increases, the difference in utility obtained increases as well.

## VII. Concluding Remarks

We have formulated the problem of allocating the power demands and generations over heterogeneous energy consuming or producing agents, connected to a smart-grid in a dynamic fashion, both under a deterministic setting and a stochastic setting when there are underlying uncertainties affecting both generation as well as consumption. We have proposed decentralized iterative algorithms to solve this problem. These algorithms work under the assumption of local knowledge, i.e., an agent needs to keep track of only its own randomness and its own system dynamics. We have shown that the ISO can play a central role in inducing co operation amongst the agents by declaring policies. We show that in the common completely observed randomness case, there is an ISO strategy that achieves social welfare optimality. It incorporates decentralized dynamics where there is no need for agents to be aware of each others' dynamics or states. The only communication from the ISO is price policy, and from the agents their energy consumption or generation in response to the price. We have also proposed more computationally tractable policies for this case. For the case of Partially Observed Randomness, we have indicated a relaxation that significantly reduces complexity, as well as an MPC approach that is tractable. Some comparative simulation results showing the increase in terms of the net social welfare are provided.

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