NUMERICAL METHODS

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CHAPTER 5

Finite Differences

5.1 INTRODUCTION

The calculus of finite differences plays an important role in Numerical methods. It deals with the variations in a function when the independent variable changes by finite jumps which may be equal or unequal. In contrast, the infinitesimal calculus deals with the relationships that exist between the values assumed by the function, whenever the independent variable changes continuously in a given interval. In this chapter, we shall study the variations in a function due to the changes in the independent variable by equal intervals.

5.2 FINITE DIFFERENCES

Let $y=f(x)$ be a discrete function. If $x_0, x_0+h, x_0+2h, \ldots, x_0+nh$ are the successive values of x, where two consecutive values differ by a quantity h, then the corresponding values of y are $y_0, y_1, y_2, \ldots, y_n$. The value of the independent variable x is usually called the argument and the corresponding functional value is known as the entry. The arguments and entries can be shown in a tabular form as follows:

Argument x	x_0	$x_1 = x_0+h$	$x_2 = x_0+2h$	$\ldots$	$x_n = x_0+nh$
Entry $y=f(x)$	$y_0 = f(x_0)$	$y_1 = f(x_0+h)$	$y_2 = f(x_0+2h)$	$\ldots$	$y_n = f(x_0+nh)$

To determine the values of $f(x)$ or $f'(x)$ etc., for some intermediate arguments, the following three types of differences are found useful:

(i) Forward differences
(ii) Backward differences and
(iii) Central differences

5.3 FORWARD DIFFERENCES

If we subtract from each value of y (except y_0) the preceding value of y, we get $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ respectively, known as the first differences of y. These results which may be denoted $\Delta y_0, \Delta y_1, \ldots, \Delta y_n$

i.e. $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$

where Δ is a symbol representing an operation of forward difference, are called *first forward differences*. Thus, the first forward differences are given by

$$\Delta y_i = y_{i+1} - y_i, \quad i = 0, 1, 2, \ldots, n.$$

Now, the second forward differences are defined as the differences of the first differences, that is,

$$\begin{aligned}
\Delta^2 y_0 &= \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \\
&= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \\
\Delta^2 y_1 &= \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1 \\
&\ldots \qquad \ldots \qquad \ldots \\
\Delta^2 y_n &= \Delta y_{n+1} - \Delta y_n = y_{n+2} - 2y_{n+1} + y_n
\end{aligned}$$

Here, Δ^2 is called *second forward difference operator*.
Similarly, the third forward differences are :

$$\begin{aligned}
\Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta^2 y_1 - \Delta^2 y_0 = \Delta(\Delta y_1) - \Delta(\Delta y_0) \\
&= \Delta(y_2 - y_1) - \Delta(y_1 - y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0 \\
&= (y_3 - y_2) - 2(y_2 - y_1) + y_1 - y_0 \\
&= y_3 - 3y_2 + 3y_1 - y_0 \\
\Delta^3 y_1 &= \Delta^2 y_2 - \Delta^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\Delta^3 y_n &= \Delta^2 y_{n+1} - \Delta^2 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n
\end{aligned}$$

In general, the nth differences are defined as

$$\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k \tag{5.1}$$

In function notation, the forward differences are as written below:

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta^2 f(x) &= f(x+2h) - 2f(x+h) + f(x) \\
\Delta^3 f(x) &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)
\end{aligned}$$

and so on, where h is the interval of differencing.

The forward differences are usually arranged in a tabular form in the following manner.

x argument	$y=f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	$y_0=f(x_0)$					
		Δy_0				
$x_1=x_0+h$	$y_1=f(x_1)$		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2=x_0+2h$	$y_2=f(x_2)$		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3=x_0+3h$	$y_3=f(x_3)$		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4=x_0+4h$	$y_4=f(x_4)$		$\Delta^2 y_3$			
		Δy_4				
$x_5=x_0+5h$	$y_5=f(x_5)$					

The first term in the table y_0 is called the *leading term* and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called *leading differences*. It can be seen that the differences $\Delta^k y_i$ with a subscript 'i' lie along the diagonal sloping downwards; that is, forward with respect to the direction of x. The above difference table is known as *Forward difference table* or *Diagonal difference table*.

Properties of Δ The operator 'Δ' satisfies the following properties:

(i) $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$, i.e. Δ is linear

(ii) $\Delta[\alpha f(x)] = \alpha \Delta f(x)$, α being a constant

(iii) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x)$, where m and n are positive integers

(iv) $\Delta[f(x).\, g(x)] \neq f(x).\, \Delta g(x)$

Observation 1 We can express any higher order forward difference of y_0 in terms of the entries $y_0, y_1, y_2, \ldots, y_n$.

From
$$\Delta y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$
and so on, we can see that the coefficients of the entries on the RHS are binomial coefficients. Therefore, in general,

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-)^n y_0 \tag{5.2}$$

Observation 2 We can express any value of y in terms of leading entry y_0.

We know that $y_1 - y_0 = \Delta y_0$

$$\therefore \qquad y_1 = y_0 + \Delta y_0 = (1 + \Delta) y_0 \tag{5.3}$$

Now, $y_2 = y_1 + \Delta y_1 = (1 + \Delta) y_1 = (1 + \Delta)^2 y_0$ [using Eqn (5.3)]

Similarly, $y_3 = (1 + \Delta)^3 y_0$ and so on. In general,

$$y_n = (1 + \Delta)^n y_0 = y_0 - {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + \cdots + \Delta^n y_0 \tag{5.4}$$

5.4 BACKWARD DIFFERENCES

The differences $y_1 - y_0$, $y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ respectively, are called the *first backward differences*, where ∇ is the backward difference operator called *nabla operator*.

$$\therefore \qquad \nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1, \ldots, \nabla y_n = y_n - y_{n-1}$$

Now the second backward differences are defined as the differences of the first backward differences, i.e.

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1$$
$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 = y_3 - 2y_2 + y_1 \text{ and so on.}$$

In general,

$$\nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1} \tag{5.5}$$

In function notation, these are written as

$$\nabla f(x) = f(x) - f(x - h)$$
$$\nabla f(x + h) = f(x + h) - f(x)$$
$$\nabla^2 f(x + 2h) = f(x + 2h) - 2f(x + h) + f(x)$$
$$\nabla^3 f(x + 3h) = f(x + 3h) - 3f(x + 2h) + 3f(x + h) - f(x)$$

and so on, where h is the interval of differencing.

These backward differences are arranged in a tabular form in the following manner. In this table, the difference $\nabla^k y_i$ with a fixed subscript 'i' lies along the diagonal sloping upwards; that is, backwards with respect to the direction of increasing argument x.

x argument	$y = f(x)$ entry	1st difference	2nd difference	3rd difference	4th difference	5th difference
x_0	y_0					
		∇y_1				
$x_1 = x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\Delta^3 y_3$		
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		Δy_3		$\Delta^3 y_4$		$\Delta^5 y_5$
$x_3 = x_0 + 3h$	y_3		$\nabla^2 y_4$		$\Delta^4 y_5$	
		Δy_4		$\Delta^3 y_5$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_5$			
		Δy_5				
$x_5 = x_0 + 5h$	y_5					

Properties of ∇

(i) $\nabla[f(x) \pm g(x)] = \nabla f(x) \pm \nabla g(x)$, i.e ∇ is a linear operator.

(ii) $\nabla[\alpha f(x)] = \alpha \nabla f(x)$, α being a constant.

(iii) $\nabla^m \nabla^n f(x) = \nabla^{m+n} f(x)$, m and n being positive integers.

(iv) $\nabla[f(x)\, g(x)] \neq [\nabla f(x)].\, g(x)$.

Observation We can express any value of y in terms of y_n and the backward differences ∇y_n, $\nabla^2 y_n$, etc.

By definition, $$y_n - y_{n-1} = \nabla y_n$$

or $$y_{n-1} = y_n - \nabla y_n = (1 - \nabla) y_n \tag{5.6}$$

Now, $y_{n-2} = y_{n-1} - \nabla y_{n-1} = (1 - \nabla) y_{n-1} = (1 - \nabla)^2 y_n$ [using Eqn (5.6)]

Similarly, $y_{n-3} = (1 - \nabla)^3 y_n$ and so on.

In general, $$y_{n-k} = (1 - \nabla)^k y_n \tag{5.7}$$

$$\therefore \quad y_{n-k} = y_n - {}^kC_1 \nabla y_n + {}^kC_2 \nabla^2 y_n - \cdots + (-1)^k \nabla^k y_n \tag{5.8}$$

5.5 CENTRAL DIFFERENCES

Sometimes, it is more convenient to employ another system of differences known as *central differences*. In this system the symbol δ is used instead of Δ and is known as *central difference operator*. The subscript of δy for any difference is the average of the subscripts of the two members of the difference.

$$\therefore \quad y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2},\ y_3 - y_2 = \delta y_{5/2},\ \cdots$$

respectively, polynomials of nth degree, $(n-1)$th degree, $(n-2)$th degree etc. we can express $f(x)$ as

$$f(x) = A_0 + A_1 x^{(1)} + A_2 x^{(2)} + A_3 x^{(3)} + \cdots + A_n x^{(n)} \tag{5.9}$$

where $A_0, A_1, A_2, A_3, \ldots, A_n$, are to be determined.

Now,

$$\begin{aligned}
\Delta f(x) &= A_1 + 2A_2 x^{(1)} + 3A_3 x^{(2)} + \cdots + nA_n x^{(n-1)} \\
\Delta^2 f(x) &= 2A_2 + 6A_3 x^{(1)} + \cdots + n(n-1)A_n x^{(n-2)} \\
\Delta^3 f(x) &= 6A_3 + \cdots + n(n-1)(n-2)A_n x^{(n-3)} \\
&\ldots \qquad \ldots \qquad \ldots \\
\Delta^n f(x) &= n(n-1)(n-2)\ldots 2.1.A_n x^{(0)} = A_n(n!)
\end{aligned}$$

Putting $x = 0$ in the above equations, we get

$$f(0) = A_0;\ \Delta f(0) = A_1;\ \Delta^2 f(0) = 2A_2 \Rightarrow A_2 = \frac{1}{2!}\Delta^2 f(0)$$

$$\Delta^3 f(0) = 6A_3 \Rightarrow A_3 = \frac{1}{3!}\Delta^3 f(0);$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\Delta^n f(0) = A_n(n!);\ \Rightarrow A_n = 1/n!\ \Delta^n f(0)$$

Substituting the above values in Eqn (5.9), we get

$$f(x) = f(0) + \Delta f(0)\, x^{(1)} + \frac{1}{2!}\Delta^2 f(0)\, x^{(2)} + \frac{1}{3!}\Delta^3 f(0)\, x^{(3)}$$

$$+ \cdots + \frac{1}{n!}\Delta^n f(0)\, x^{(n)} \tag{5.10}$$

Thus, any polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice versa.

Note: We have some other methods to express a polynomial in factorial notation. They are best illustrated through examples.

Example 5.1 Construct a forward difference table from the following data:

x	0	1	2	3	4
y_x	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 y_1$, y_x and y_5.

Solution The forward difference table is as given below:

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 1$				
		$\Delta y_0 = 0.5$			
$x_1 = 1$	$y_1 = 1.5$		$\Delta^2 y_0 = 0.2$		
		$\Delta y_1 = 0.7$		$\Delta^3 y_0 = 0$	
$x_2 = 2$	$y_2 = 2.2$		$\Delta^2 y_1 = 0.2$		$\Delta^4 y_0 = 0.4$
		$\Delta y_2 = 0.9$		$\Delta^3 y_1 = 0.4$	
$x_3 = 3$	$y_3 = 3.1$		$\Delta^2 y_2 = 0.6$		
		$\Delta y_3 = 1.5$			
$x_4 = 4$	$y_4 = 4.6$				

Now, $\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1$

$$= 4.6 - 3(3.1) + 3(2.2) - 1.5 = 0.4$$

Again, from Observation 2 of Section 5.3, we have

$$y_x = y_0 + {}^xC_1 \Delta y_0 + {}^xC_2 \Delta^2 y_0 + {}^xC_3 \Delta^3 y_0 + {}^xC_4 \Delta^4 y_0$$

$$= 1 + x(0.5) + \frac{1}{2} x(x-1)(0.2) + \frac{1}{3!} x(x-1)(x-2)(0)$$

$$+ \frac{1}{4!} x(x-1)(x-2)(x-3)(0.4)$$

$$= 1 + \frac{1}{2} x + \frac{1}{10}(x^2 - x) + \frac{1}{60}(x^4 - 6x^3 + 11x^2 - 6x)$$

$$\therefore \quad y_x = \frac{1}{60}(x^4 - 6x^3 + 17x^2 + 18x + 60)$$

$$\therefore \quad y_5 = \frac{1}{60}[5^4 - 6(5)^3 + 17(5)^2 + 18(5) + 60] = 7.5$$

Example 5.2 Evaluate (i) $\Delta \cos x$, (ii) $\Delta \log f(x)$, (iii) $\Delta^2 \sin(px + q)$, (iv) $\Delta \tan^{-1} x$, and (v) $\Delta^n e^{ax+b}$

Solution Let h be the interval of differencing.

(i) $\Delta \cos x = \cos(x + h) - \cos x$

$$= -2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}$$

(ii) $\Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(iii) $\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\therefore \Delta^2 \sin(px+q) = 2\sin\frac{ph}{2}\,\Delta\left[\sin\left(px+q+\frac{1}{2}(\pi+ph)\right)\right]$$

$$= \left(2\sin\frac{ph}{2}\right)^2 \sin\left(px+q+2.\frac{1}{2}(\pi+ph)\right)$$

(iv) $\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+x(x+h)}\right] = \tan^{-1}\frac{h}{1+x(x+h)}$$

(v) $\Delta e^{(ax+b)} = e^{a(x+h)+b} - e^{ax+b}$

$$= e^{(ax+b)}(e^{ah}-1)$$

$$\Delta^2 e^{(ax+b)} = \Delta[\Delta(e^{ax+b})] = \Delta[(e^{ab}-1)e^{ax+b}]$$

$$= (e^{ah}-1)^2\Delta(e^{ax+h})$$

$$= (e^{ah}-1)^2 e^{ax+h} \qquad [\because e^{ah}-1 \text{ is a constant}]$$

Proceeding on, we get, $\Delta^n(e^{ax+b}) = (e^{ah}-1)^n e^{ax+b}$

Example 5.3 Evaluate (i) $\Delta[f(x)\,g(x)]$ and (ii) $\Delta\left[\frac{f(x)}{g(x)}\right]$

Solution Let h be the interval of differencing.

(i) $\Delta[f(x)\,g(x)] = f(x+h)\,g(x+h) - f(x)\,g(x)$

$$= f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)$$

$$= f(x+h)[g(x+h)-g(x)] + g(x)[f(x+h)-f(x)]$$

$$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$$

(ii) $$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - g(x+h)f(x)}{g(x+h)g(x)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)}$$

$$= \frac{g(x)\Delta f(x) - f(x)\Delta g(x)]}{g(x+h)g(x)}$$

Example 5.4 If $y = (3x+1)(3x+4)\ldots(3x+22)$, prove that
$\Delta^4 y = 136080\,(3x+13)(3x+16)(3x+19)(3x+22)$

Solution The given equation $y = (3x+1)(3x+4)\ldots(3x+22)$ contains eight factors:

$$\therefore \quad y = 3^8\left(x+\frac{1}{3}\right)\left(x+\frac{4}{3}\right)\ldots\left(x+\frac{22}{3}\right)$$

$$= 3^8\left(x+\frac{22}{3}\right)^{(8)}$$

$$\therefore \quad \Delta y = 3^8.8\left(x+\frac{22}{3}\right)^{(7)}$$

$$\Delta^2 y = 3^8.8.7\left(x+\frac{22}{3}\right)^{(6)}$$

$$\Delta^3 y = 3^8.8.7.6\left(x+\frac{22}{3}\right)^{(5)}$$

and

$$\Delta^4 y = 3^8.8.7.6.5\left(x+\frac{22}{3}\right)^{(4)}.$$

$$\therefore \Delta^4 y = 11022480\left(x+\frac{22}{3}\right)\left(x+\frac{22}{3}-1\right)\left(x+\frac{22}{3}-2\right)\left(x+\frac{22}{3}-3\right)$$

$$= 136080\,(3x+22)(3x+19)(3x+16)(3x+13)$$

Solution

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$2 = y_0$				
		1			
4	$3 = y_1$		1		
		2		0	
6	$5 = y_2$		1		–3
		3		–3	
8	$8 = y_3$		–2		–3
		1		$\Delta^3 y_2 = -6$	
10	$9 = y_4$		$\Delta^2 y_3 = -8$		–3
		$\Delta y_4 = -7$		$\Delta^3 y_3 = -9$	
12	$y_5 = 2$		$\Delta^2 y_4 = -11$		
		$\Delta y_5 = -24$			
14	$y_6 = -22$				

Since the values of y belongs to a polynomial of degree four, the fourth differences must be constant. Then from the table,

$$\begin{aligned}
\Delta^3 y_2 - (-3) &= -3 & \therefore \Delta^3 y_2 &= -6 \\
\Delta^3 y_3 - \Delta^3 y_2 &= -3 & \Rightarrow \Delta^3 y_3 &= -3 + \Delta^3 y_2 = -9 \\
\Delta^2 y_3 - (-2) &= \Delta^3 y_2 & \Rightarrow \Delta^2 y_3 &= \Delta^3 y_2 - 2 = -8 \\
\Delta^2 y_4 - \Delta^2 y_3 &= \Delta^3 y_3 & \Rightarrow \Delta^2 y_4 &= \Delta^3 y_3 + \Delta^2 y_3 = -9 - 8 = -17 \\
\Delta y_4 - 1 &= \Delta^2 y_3 & \Rightarrow \Delta y_4 &= 1 + \Delta^2 y_3 = 1 - 8 = -7 \\
\Delta y_5 - \Delta y_4 &= \Delta^2 y_4 & \Rightarrow \Delta y_5 &= \Delta y_4 + \Delta^2 y_4 = -7 - 17 = -24 \\
y_5 - y_4 &= \Delta y_4 & \Rightarrow y_5 &= y_4 + \Delta y_4 = 9 - 7 = 2 \\
\text{and} \quad y_6 - y_5 &= \Delta y_5 & \Rightarrow y_6 &= y_5 + \Delta y_5 = 2 - 24 = -22
\end{aligned}$$

Hence the missing entries corresponding to $x = 12, 14$ are 2 and – 22.

5.11 OTHER DIFFERENCE OPERATORS

So far we have studied the operators Δ, ∇ and δ. Now we shall introduce other operators like E, μ, D etc. which also play a vital role in numerical methods.

Shift Operator E

If h is the interval of differencing in the argument x then the operator E is defined as

$$Ef(x) = f(x + h).$$

It is also called *translation operator* due to the reason that it results the next value of the function.

$$E^2 f(x) = E[E f(x)] = E f(x+h) = f(x+2h)$$

Similarly, $E^3 f(x) = f(x+3h)$, $E^4 f(x) = f(x+4h)$ and $E^n f(x) = f(x+nh)$.

If y_x is the function $f(x)$ then $E y_x = y_{x+h} \ldots$ and $E^n y_x = y_{x+nh}$ the inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$

Similarly,

$$E^{-n} f(x) = f(x-nh).$$

In general,

$$E^n f(x) = f(x+nh) \text{ for any real } n. \tag{5.11}$$

Note: If $y_0, y_1, \ldots, y_n$ are the consecutive values of y_x then

$$E y_0 = y_1, E^2 y_3 = y_5, \ldots, E^k y_n = y_{n+k}$$

Averaging Operator μ

The averaging operator μ is defined by

$$\mu f(x) = \tfrac{1}{2}[f(x+h/2) + f(x-h/2)]$$

i.e.

$$\mu y_x = \tfrac{1}{2}[y_{(x+h/2)} + y_{(x-h/2)}] \tag{5.12}$$

Differential Operator D

The differential operator D is defined as $D f(x) = \dfrac{d}{dx} f(x)$

In general,

$$D^n f(x) = \frac{d^n}{dx^n} f(x) \tag{5.13}$$

Unit Operator 1

The unit operator 1 is defined as $1 \cdot f(x) = f(x)$

Note: All the above operators are linear and obey index laws.

Relation between Δ and E

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x) = (E-1) f(x)$$
$$\Delta = E - 1 \text{ or } E = 1 + \Delta$$

Relation between E and ∇

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x) = (1 - E^{-1}) f(x)$$

$\therefore \nabla = 1 - E^{-1}$ or $E^{-1} = 1 - \nabla$

$\therefore E = (1-\nabla)^{-1} \qquad [\because (E^{-1})^{-1} = E]$

Relation between E and δ

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$
$$= E^{1/2} f(x) - E^{-1/2} f(x)$$
$$= (E^{1/2} - E^{-1/2}) f(x)$$

$\therefore \delta = E^{1/2} - E^{1/2}$

Again, $\delta = E^{1/2}(1 - E^{-1}) = E^{1/2}\nabla$

and $\delta = E^{-1/2}(E - 1) = E^{-1/2}\Delta$

$\therefore \delta = E^{1/2}\nabla = E^{-1/2}\nabla$

Relation between E and μ

$$\mu f(x) = \tfrac{1}{2}[f(x + h/2) + f(x - h/2)]$$
$$= \tfrac{1}{2}[E^{1/2} f(x) + E^{-1/2} f(x)]$$
$$= \tfrac{1}{2}(E^{1/2} + E^{-1/2}) f(x)$$

$\therefore \mu = \tfrac{1}{2}(E^{1/2} + E^{-1/2})$

Relation of D with other Operators

We know that $Df(x) = \dfrac{d}{dx} f(x) = f'(x)$ etc.

By Taylor's series

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

or $$Ef(x) = f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \ldots$$

$$= \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots\right] f(x)$$
$$= e^{hD} f(x)$$

$\therefore E = e^{hD}$

Taking logarithms on both sides, we get

$$hD = \log E = \log(1+\Delta)$$

$$\therefore \quad D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} - \frac{\Delta^4}{4!} + \ldots\right] \tag{5.16}$$

Also, $\nabla = 1 - E^{-1}, \ \therefore E^{-1} = 1 - \nabla$

i.e. $e^{-hD} = 1 - \nabla$

Taking logarithm on both sides,

$$-hD = \log(1-\nabla)$$

$$\therefore \quad D = \frac{1}{h}\left[\nabla - \frac{\nabla^2}{2!} + \frac{\nabla^3}{3!} - \frac{\nabla^4}{4!} + \ldots\right] \tag{5.17}$$

Again, $\sin(hD) = \dfrac{e^{hD} - e^{-hD}}{2} = \dfrac{E - E^{-1}}{2}$

$$= \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]\left[E^{1/2} + e^{-1/2}\right] = \mu\delta$$

$$\therefore \quad hD = \sin^{-1}(\mu\delta)$$

or

$$D = \frac{1}{h}\sin^{-1}(\mu\delta) \tag{5.18}$$

Example 5.11 Prove the following results:

(i) $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$ *(M.U, B.E., 1996)*

(ii) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$ *(B.R, B.E., 1996)*

(iii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ *(Madurai B.E., 1998)*

(iv) $1 + \mu^2\delta^2 = (1 + \frac{1}{2}\delta^2)^2$

(v) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{(1 + \delta^2/4)}$ *(M.U, B.E., 1997, Madurai B.E., 1988)*

(vi) $\mu^{-1} = 1 - \dfrac{1}{8}\delta^2 + \dfrac{3}{128}\delta^4 - \dfrac{5}{1024}\delta^6 + \ldots$

Solution (i) We have,

$$\begin{aligned}
\Delta\nabla f(x) &= \Delta[\nabla f(x)] = \Delta[f(x) - f(x-h)] \\
&= \Delta f(x) - \Delta f(x-h) \\
&= [f(x+h) - f(x)] - [f(x) - f(x-h)] \\
&= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)
\end{aligned}$$

$$\therefore \quad \Delta\nabla = \Delta - \nabla$$

Similarly, $\nabla\Delta f(x) = \nabla[\Delta f(x)] = \nabla[f(x+h) - f(x)]$

$$= \nabla f(x+h) - \nabla f(x)$$
$$= [f(x+h) - f(x)] - [f(x) - f(x-h)]$$
$$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)$$

$$\therefore \quad \nabla\Delta = \Delta - \nabla$$

Again, $\delta^2 f(x) = [E^{1/2} - E^{-1/2}]^2 f(x)$

$$= (E + E^{-1} - 2) f(x)$$
$$= f(x+h) + f(x-h) - 2f(x)$$
$$= f(x+h) - f(x)] - [f(x) - f(x-h)]$$
$$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla) f(x)$$

$$\therefore \quad \delta^2 = \Delta - \nabla$$

Hence, $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$

(ii) $$\text{RHS} = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\nabla\Delta}$$

$$= \frac{(\Delta + \nabla)(\Delta - \nabla)}{(\Delta - \nabla)} = \Delta + \nabla = \text{LHS}$$

(iii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2}) E^{1/2}$

$$= E + 1 = 1 + \Delta + 1 = 2 + \Delta$$

(iv) $$1 + \mu^2\delta^2 = 1 + \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]^2 \left[E^{1/2} - E^{-1/2}\right]^2$$

$$= 1 + \left[\frac{E - E^{-1}}{2}\right]^2$$

$$= \frac{4 + (E - E^{-1})^2}{4} = \left[\frac{E + E^{-1}}{2}\right]^2 \qquad \text{(i)}$$

Now, $$\left[1 + \frac{1}{2}\delta^2\right]^2 = \left[1 + \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]^2\right]^2$$

$$= \left[1 + \frac{1}{2}\left[E + E^{-1} - 2\right]\right]^2 = \left[\frac{E + E^{-1}}{2}\right]^2 \qquad \text{(ii)}$$

Hence, from Eqns (i) and (ii), we have

$$1+\mu^2\delta^2 = \left[1+\frac{1}{2}\delta^2\right]^2$$

(v) $$\text{RHS} = \frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$$

$$= \frac{1}{2}\delta+\left[\delta+\sqrt{4+\delta^2}\right]$$

$$= \frac{1}{2}\delta+\left[(E^{1/2}-E^{-1/2})+\sqrt{4+(E^{1/2}-E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta+\left[(E^{1/2}-E^{-1/2})+(E^{1/2}+E^{-1/2})\right]$$

$$= \frac{1}{2}(E^{1/2}-E^{-1/2})(2E^{1/2})$$

$$= E-1 = \Delta = \text{LHS}$$

(vi) By definition, we have

$$\mu^2 = \left[\frac{1}{2}(E^{1/2}+E^{-1/2})\right]^2$$

$$= \frac{1}{4}\left[(E^{1/2}-E^{-1/2})^2+4\right]$$

$$= \frac{1}{4}(\delta^2+4) = \frac{\delta^2}{4}+1$$

$$\therefore \mu = \left[1+\frac{\delta^2}{4}\right]^{1/2} \text{ or } \mu^{-1} = \left[1+\frac{\delta^2}{4}\right]^{-1/2}$$

$$\therefore \mu^{-1} = 1-\frac{1}{2}\frac{\delta^2}{4}+\frac{1}{2!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left[\frac{\delta^2}{4}\right]^2-\frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left[\frac{\delta^2}{4}\right]^3+\cdots$$

$$= 1-\frac{1}{8}\delta^2+\frac{3}{128}\delta^4-\frac{5}{1024}\delta^6+\cdots$$

Example 5.12 Prove the following :

(i) $\left(\frac{\Delta^2}{E}\right)u_x \neq \frac{\Delta^2 u_x}{Eu_x}$ (M.U, B.E., 1996)

(ii) $\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E(e^x)}{\Delta^2 e^x} = e^x$ (Kerala, B.Tech, 1990)

Solution

(i)
$$\text{LHS} = E^{-1}(\Delta^2 u_x) = E^{-1}(E-1)^2 u_x$$
$$= (E - 2 + E^{-1})u_x$$
$$= u_{x+h} - 2u_x + u_{x-h}$$

$$\text{RHS} = \frac{(E-1)^2 u_x}{U_{x+h}} = \frac{(E^2 - 2E + 1)u_x}{u_{x+h}}$$

$$= \frac{u_{x+h} - 2u_{x+h} + u_x}{u_{x+h}}$$

$\therefore$ LHS $\neq$ RHS

(ii)
$$\text{LHS} = E^{-1}(\Delta^2 e^x)\frac{e^{x+h}}{e^x(e^h - 1)^2}$$

$$= E^{-1}[e^x(e^h-1)^2]\frac{e^h}{(e^h-1)^2} = E^{-1}e^x.e^h$$

$$= e^{x-h}e^h = e^x = \text{RHS}$$

Example 5.13 Using the method of seperation of symbols, prove that

(i) $u_1x + u_2x^2 + u_3x^3 + \cdots = \frac{x}{1-x}u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 u_1 + \cdots$

(Kerala, B.Tech., 1985)

(ii) $u_0 + u_1 + u_2 + \cdots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \cdots + \Delta^n u_0$.

(Rourkela, B.Tech., 1985)

Solution

(i) We know that $u_{x+h} = E^h u_x$

$\therefore$ LHS $= xu_1 + x^2Eu_1 + x^3E^2u_1 + \cdots$

$$= x[1 + xE + x^2E^2 + \cdots]u_1$$

$$= x\left(\frac{1}{1-xE}\right)u_1 = x\left(\frac{1}{1-x(1+\Delta)}\right)u_1$$

$$= x\left(\frac{1}{1-x-x\Delta}\right)u_1 = \frac{x}{1-x}\left[1-\frac{x\Delta}{1-x}\right]^{-1} u_1$$

$$= \frac{x}{1-x}\left[1+\frac{x\Delta}{1-x}+\frac{x^2\Delta^2}{(1-x)^2}+\cdots\right]u_1$$

$$= \frac{x}{(1-x)}u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \cdots = \text{RHS}$$

(ii) $\text{LHS} = u_0 + u_1 + u_2 + \cdots + u_n$

$$= u_0 + Eu_0 + E^2u_0 + \cdots + E^n u_0$$

$$= (1 + E + E^2 + \cdots + E^n)u_0$$

$$= \left[\frac{E^{n+1}-1}{E-1}\right]u_0 = \left[\frac{(1+\Delta)^{n+1}-1}{\Delta}\right]u_0$$

$$= \frac{1}{\Delta}[(1 + {}^{n+1}C_1\Delta + {}^{n+1}C_2\Delta^2 + {}^{n+1}C_3\Delta^3 + \cdots + \Delta^{n+1} - 1]\, u_0$$

$$= \frac{1}{\Delta}[{}^{n+1}C_1\Delta + {}^{n+1}C_2\Delta^2 + {}^{n+1}C_3\Delta^3 + \cdots + \Delta^{n+1}]\, u_0$$

$$= {}^{n+1}C_1 u_0 + {}^{n+1}C_2\Delta^2 u_0 + {}^{n+1}C_3\Delta^2 u_0 + \cdots + \Delta^n u_0 = \text{RHS}$$

5.12 SUMMATION OF SERIES

One of the very important applications of the calculus of finite differences is to obtain a formula for summing a series of upto n terms. The method is best illustrated by the following examples.

Example 5.14 Find the sum to n terms of the series $u_0 + u_1 + u_2 + \cdots + u_n$, where the general term u_x is the first difference of another function. Using this concept, find the sum to n terms of the following series.

(i) $1.3.5 + 2.4.6 + 3.5.7 + \cdots$

(ii) $\frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} + \cdots$

Solution It is given that the general term u_x of the given series

$$u_1 + u_2 + u_3 + \cdots + u_n$$

is the first difference of another function.

Hence, from Eqn (i), sum of the series is

$$= e^x [u_0 + x\,\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0]$$

$$= e^x [1 + 7x + \frac{12x^2}{2!} + \frac{6x^3}{3!}]$$

$$= e^x (1 + 7x + 6x^2 + x^3)$$

EXERCISE 5.1

1. Tabulate the forward differences for the given data:

x	1	2	3	4	5	6	7	8	9
y	1	8	27	64	125	216	343	512	729

2. Form a table of backward differences of the function

$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$

3. Form the difference table of $f_x = x^4 - 5x^3 + 6x^2 + x - 2$ for the values of $x = -3, -2, -1, 0, 1, 2, 3$. Extend the table in both directions to give f_{-4}, f_{-5}, f_4, f_5.

4. Show that

(i) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$

(ii) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$ (iii) $\delta^2 y_5 = y_6 - 2y_5 + y_4$

5. If $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 2000$, $y_4 = 100$ show that $\Delta^4 y_0 = -7459$.

6. If the interval of differencing is unity, prove that

(i) $\Delta \sin x = 2\sin\frac{1}{2} \cos (x + \frac{1}{2})$

(ii) $\Delta f(x) = \dfrac{-\Delta f(x)}{f(x) f(x+1)}$

(iii) $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right) = \tan^{-1} \dfrac{1}{2n^2}$

(iv) $\Delta \dfrac{2^x}{x!} = \dfrac{2^x (1-x)}{(x+1)!}$

(v) $\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$

(vi) $\Delta^2\left[\dfrac{5x+12}{x^2+5x+6}\right] = \dfrac{10x+32}{(x+2)(x+3)(x+4)(x+5)}$

(vii) $\Delta^n e^x = (e-1)^n e^x$

(viii) $\Delta^n(1/x) = \dfrac{(-1)^n n!}{x(x+1)x+2)\ldots(x+n)}$

7. If h is the interval of differencing, prove that

(i) $\Delta^2\cos 2x = -4\sin^2 h\cos 2(x+h)$ (*Kerala B.E., 1989, M.U, B.E., 1996*)

(ii) $\Delta^3 a^{cx+d} = (a^{ch}-1)^3 a^{cx+d}$

(iii) $\Delta^n\sin(ax+b) = 2\sin(ah/2)^n\sin\left(ax+b+\dfrac{nah+n\pi}{2}\right)$

8. Show that

(i) $\Delta^3[(1-x)(1-2x)(1-3x)] = -36$ if $h = 1$.

(ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)] = 24\times 2^{10}\times 10!$ if $h = 2$.

9. Find the seventh term of the sequence 2, 9, 28, 65, 126, . . . and also find the general term.

10. Evaluate $\Delta^2 f(x)$ if $f(x)$ is

(i) $\dfrac{1}{x(x+4)x+8)}$ (ii) $\dfrac{1}{(3x+1)(3x+4)(3x+7)}$

11. Find $\Delta^3 f(x)$ if $f(x)$ is $(3x+1)(3x+4)(3x+7)\ldots(3x+19)$

12. Express the following in factorial notation.

(i) $f(x) = 2x^3 - 3x^2 + 3x - 10$

(ii) $f(x) = x^3 - 2x^2 + x - 1$

(iii) $f(x) = 3x^4 - 4x^3 + 6x^2 + 2x + 1$

(iv) $f(x) = x^4 - 3x^3 - 5x^2 + 6x - 7$ and get their successive forward differences.

13. Obtain the function whose first difference is $x^3 + 3x^2 + 5x + 12$.

14. Express the following in factorial notation taking $h = 2$ and find their differences of second order.

(i) $f(x) = 7x^4 + 12x^3 - 6x^2 + 5x - 3$

(ii) $f(x) = x^3 - 3x^2 + 5x + 7$

27) $u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \cdots = e^x \left(u_0 + x\,\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \cdots\right)$

28) $u_0 + {}^xC_1\,\Delta u_1 + {}^xC_2\,\Delta^2 u_2 + \cdots = u_x + {}^xC_1\,\Delta^2 u_{x-1} + {}^xC_2\,\Delta^4 u_{x-2} + \cdots$

29) Sum the series to n terms:

(i) $1.2.3 + 2.3.4 + 3.4.5 + \cdots$

(ii) $4.5.6. + 5.6.7 + 6.7.8 + \cdots$

(iii) $2.5 + 5.8 + 8.11 + \cdots$

30) Using the method of finite differences, find the sum to n terms of the series whose nth term is $n(n-1)(n-2)$.

31) Using the method of finite differences, find the sum of the first

(i) n squares and (ii) n cubes.

32) Sum the series using the identity of Example 5.15.

(i) $5 + \frac{4x}{1!} + \frac{5x^2}{2!} + \frac{14x^3}{3!} + \frac{37x^4}{4!} + \cdots$

(ii) $1 + \frac{4x}{1!} + \frac{10x^2}{2!} + \frac{20x^3}{3!} + \frac{35x^4}{4!} + \cdots$

33) Using Montmort's theorem, sum the series

$1.3 + 3.5x + 5.7x^2 + 7.9x^3 + \cdots$

ANSWERS

9) $344, (n+1)^3 + 1$

10) (i) $\Delta^2 f(x) = \frac{192}{x(x+4)(x+8)(x+12)(x+16)}$

(ii) $\Delta^2 f(x) = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$

11) $\Delta^3 f(x) = 459270\,(3x+19)\,(3x+16)\,(3x+13)\,(3x+10)$

12) (i) $f(x) = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$

(ii) $f(x) = x^{(3)} + x^{(2)} - 1;$

(iii) $f(x) = 3x^{(4)} + 14x^{(3)} + 15x^{(2)} + 7x^{(1)} + 1;$

(iv) $f(x) = x^{(4)} + 9x^{(3)} + 11x^{(2)} + 5x^{(1)} - 7$

13) $f(x) = \frac{1}{4}x^{(4)} + 2x^{(3)} + \frac{9}{2}x^{(2)} + 12x^{(1)} + \text{constant}$

14) (i) $f(x) = 7x^{(4)} + 96x^{(3)} + 262x^{(2)} + 97x^{(1)} - 3$

$\Delta f(x) = 56x^{(3)} + 576x^{(2)} + 1048x^{(1)} + 194$

$\Delta^2 f(x) = 336x^{(2)} + 2304x^{(1)} + 2096$

(ii) $f(x) = x^{(3)} + 3x^{(2)} + 3x^{(1)} + 7$

$\Delta f(x) = 6x^{(2)} + 12x^{(1)} + 6$

$\Delta^2 f(x) = 24x^{(1)} + 24$

15) $f(x) = 244$, error $= -10$

18) $y_6 = 138$

19) 31, 129, 351

20) 16.1

21) Error is at $x = 2$; $y(2) = 0$

29) (i) $\frac{1}{4}n(n+1)(n+2)(n+4)$

(ii) $\frac{1}{4}[(n+6)(n+5)(n+4)(n+3) - 360]$

(iii) $n(3n^2 + 6n + 1)$

30) $\frac{1}{4}(n+1)(n)(n-1)(n-2)$

31) (i) $\frac{n(n+1)(2n+1)}{6}$ (ii) $\frac{n^2(n+1)^2}{4}$

32) (i) $e^x(x^3 + x^2 - x + 5)$ (ii) $e^x\left(1 + 3x + \frac{3x^2}{2} + \frac{x^3}{6}\right)$

33) $\frac{3 + 6x - x^2}{(1-x)^3}$.

06.11.23

EMEP

Shifting operator

$$Ef(x) = f(x+h)$$

$$E^2 f(x) = E(Ef(x))$$

$$= E[f(x+h)]$$

$$= f(x+2h)$$

$$E^3 f(x) = f(x+3h)$$

$$\therefore E^n f(x) = f(x+nh)$$

Relation betw Δ & E.

$$\Delta f(x) = f(x+h) - f(x)$$

$$= Ef(x) - f(x)$$

$$\Delta f(x) = f(x)(E-1)$$

$$\therefore \Delta = E-1$$

Relation betw ∇ & E^{-1}

$$\nabla f(x) = f(x) - f(x-h) \rightarrow E^{-1}$$

$$= f(x) - E^{-1} f(x)$$

$$\nabla f(x) = (1-E^{-1}) f(x)$$

$$\nabla = 1 - E^{-1}$$

① Evaluate $\Delta^5 e^x$

$$\Delta^5 = (E-1)^5$$

$$= E^5 - 5E^4 + \frac{5\times4}{2!}E^3 - \frac{5\times4\times3}{3!}E^2 + \frac{5\times4\times3\times2}{4!}E - 1$$

$(a+b)^n$

$$\left[\because (a+b)^n = a^n + {}^nc_1\, a^{n-1}b + {}^nc_2\, a^{n-b}b + \cdots + {}^nc_n\, b^n\right]$$

$$\Delta^5 e^x = \left(E^5 - 5E^4 + \frac{5\times4}{2}E^3 - \frac{5\times4\times3}{3!}E^2 + \frac{5\times4\times3\times2}{4!}E - 1\right)e^x$$

$$= e^{x+5h} - 5e^{x+4h} + 10e^{x+3h} - 10e^{x+2h} + 5e^{x+h} - e^x$$

exm-2

Prove that $\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^x$

$$L.H.S = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$$

$$= \left(\frac{E-1}{E}\right)^2 e^x \cdot \frac{Ee^x}{(E-1)^2 e^x}$$

$$= \frac{E^2-2E+1}{E}e^x \cdot \frac{Ee^x}{(E^2-2E+1)e^x}$$

$$= (E - 2 + E^{-1})e^x \cdot \frac{e^{x+h}}{E^2 e^x - 2Ee^x + e^x}$$

$$= (Ee^x - 2e^x + E^{-1}e^x) \frac{e^{x+h}}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= (e^{x+h} - 2e^x + e^{x-h}) \left(\frac{e^h}{e^{2h} - 2e^h + 1}\right)$$

$$= e^x \cdot \frac{(e^{2h} - 2e^h + 1)}{e^{2h} - 2e^h + 1}$$

$$= e^x (e^h - 2 + e^{-h})$$

Prove that,

exm আসবে।

$$\left(\frac{\Delta^2}{E}\right) x^3 = 6x$$

Fundamental theorem of difference colums.

Statement: The nth difference of a polynomial of degree n is constant and the higher under difference are zero. i.e if $f(x)$ is a polynomial of degree n is x and then the nth diff of $f(x)$ is constant and $(n+1)$ th diff is zero.

Proof:

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

where n is a positive integer and a, b, a_1 $\ldots a_n$ are constants.

We have, $\Delta f(x) = f(x+h) - f(x) = [a_0 + a_1(x+h) + a_2(x+h)^2 + \cdots + a_n(x+h)^n] - [a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n]$

$= a_1 h + a_2 [(x+h)^2 - x^2] + a_2 [(x+h)^3 - x^3] + \cdots + a_n [(x+h)^4 - x^4]$

$= a_1 h + a_2 [{}^2C_1\, xh + h^2] + a_3 [{}^3C_1\, x^2 h + {}^3C_2\, xh^2 + h^3] + \cdots + a_n [{}^nC_1\, x^{n-1} h + {}^nC_2\, x^{n-2} h^2 + \cdots + {}^nC_n\, h^n]$

$= b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-2} x^{n-2} + n a_n h\, x^{n-1}$ ——— (1)

where, $b_0, b_1, b_2 \ldots b_{n-2}$ are constant coefficients. So, $\Delta f(x)$ is a polynomial of degree $(n-1)$ in which coefficient of x^{n-1} is $n.h.$ the coefficient of x^n in $f(x)$.

Let, $\Delta f(x) = \phi(x)$

$\therefore \Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta \phi(x)$ $x^0 = 1$

By ①, $\Delta\phi(x)$ is of degree $(n-2)$

$= (n-1)h.n.h.a_n = h(n-1)h^2 a_n$

$\therefore$ nth diff of polynomial

$\Delta^h f(x) = n(n-1)(n-2) \cdots 3.2.2\, h^n a_n x^{n-h}$

$= h!\, h^n a_n x^0$

$= h!\, h^n a_n \longrightarrow$ Constant

Math:

Proof of operator

Prove that,

① $\Delta\nabla = \nabla\Delta = \Delta - \nabla$

$\Delta\nabla f(x) = \Delta[f(x) - f(x-h)]$

$= \Delta f(x) - \Delta f(x-h)$

$= f(x+h) - f(x) - [f(x) - f(x-h)]$

$= \Delta f(x) - \nabla f(x)$

$\Delta\nabla = \Delta - \nabla$

(2) $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$$= \frac{\Delta^2 - \nabla^2}{\Delta\nabla} = \frac{(\Delta+\nabla)(\Delta-\nabla)}{\Delta-\nabla} = \Delta+\nabla$$

(Ans).

exm আসবে

(3) Find missing term,

x	0	1	2	3	4	5	6
$f(x)$	5	11	22	40	=	140	–

~~x f(x) Δ~~

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	5	6				
1	11	6	5			
2	22	11	7			
3	40	18	m-40-18 = 28			
4	m	m-40	140-m-40			
5	140	140-m				
6	n	n-140				