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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

Duffing oscillator (or Duffing Equation) is one of the most significant and classical nonlinear ordinary differential equations in view of its diverse applications in science and engineering. This paper attempts to study some applications of Duffing oscillator and also develop an alternative computational method that may be used to simulate it. In developing the computational method for simulating the Duffing oscillator, power series was adopted as the basis function with the integration carried out within a one-step interval. The computational method developed was applied on some modeled Duffing oscillators and from the results obtained; it is evident that the method developed is computationally reliable.

Keywords: Chaos; damping; Duffing oscillator; nonlinear; simulations.

2010 AMS subject classification: 65L05, 65L06, 65D30.

1 Introduction

Over the recent decades, many physical phenomena have been modeled using nonlinear ordinary differential equations. One of these equations, called the Duffing oscillator has received remarkable attention due to its classical applications in sciences, engineering and biology. It is named after a German electrical engineer

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Georg Duffing in 1981. Given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation since it replicates similar dynamics in our natural world. Duffing oscillator occurs as a result of the motion of a body subjected to a nonlinear spring power, linear sticky damping and periodic powering. Oscillations of mechanical systems under the action of a periodic external force can be revealed using Duffing oscillator, [1].

This paper presents the applications and computational simulations of Duffing oscillators given by the form;

$$y''(t)+\eta y'(t)+\mu y(t)+\gamma y^3(t)= f(t) \tag{1}$$

with initial conditions,

$$y(0)=\alpha,\ y'(0)=\beta \tag{2}$$

where $\eta,\ \mu, \gamma,\ \alpha\, and\ \ \beta$ are real constants and $f(t)$ is a real-valued function.

Numerous works have focused on the development of efficient methods for simulating Duffing oscillators. These methods include; Laplace decomposition method [2], restarted Adomian decomposition method [3], memetic computing [4], differential transform method [5], modified differential transform method [6], improved Taylor matrix method [7], variational iteration method [8,9], modified variational iteration method [10], trigonometrically fitted Obrechkoff method [11], among others.

In this paper, we shall study some applications of Duffing oscillators most especially in damping and chaos theory and also develop an alternative computational method for simulating the Duffing equations. It is expected that this method will be more efficient and computationally reliable than the existing ones.

2 Applications of Duffing Oscillator

It is important to note that the Duffing oscillator is a simple model that shows different types of oscillations such as chaos and limit cycles. The terms associated with the system in equation (1) represents;

$y'(t)$: small damping
η: ratio (coefficient) of viscous damping (it controls the size of damping)
$\mu y(t)+\gamma y^3(t)$: nonlinear restoring force acting like a hard spring (with μ controlling the size of stiffness and γ controlling the size of nonlinearity)
$f(t)$: small periodic force

2.1 Damping

Duffing oscillators are routinely associated with damping in physical systems. Damping is an influence within or upon oscillatory system that has the effect of reducing, restricting or preventing its oscillation. Damping is produced by processes that dissipate the energy stored in the oscillation. Examples include viscous drag in mechanical systems, resistance in electronic oscillators, and absorption and scattering of light in optical oscillators. Damping not based on energy loss can be important in other oscillating systems such as those that occur in biological systems. The damping of a system can be described as being one of the following;

- *Overdamped:* the system returns (exponentially decays) to equilibrium without oscillating
- *Critically damped*: the system returns to equilibrium as quickly as possible without oscillating
- *Underdamped:* the system oscillates (at reduced frequency compared to the undamped case) with the amplitude gradually decreasing to zero
- *Undamped:* the system oscillates at its natural resonant frequency

As a practical example, consider a door that uses a spring to close the door once open. This can lead to any of the above types of damping. If the door is undamped, it will swing back and forth forever at a particular resonant frequency. If it is underdamped, it will swing back and forth with decreasing size of the swing until it comes to a stop. If it is critically damped, then it will return to closed as quickly as possible without oscillating. Finally, if it is overdamped, it will return to 'closed' without oscillating but more slowly depending on how overdamped it is.

2.2 Chaos theory

Chaos theory is one of the most significant achievements of nonlinear science, [12]. The Duffing oscillator is also routinely associated with mathematical chaotic behavior. These chaotic behaviors exist in many natural systems such as weather and climate, [13]. It also occurs spontaneously in some systems with artificial components, such as road traffic. Chaos theory has applications in several disciplines including meteorology, sociology, environmental sciences, etc. According to [14], chaos is "is a periodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial condition(s)". Although there is no set definition of chaos, mathematicians agree that there are three properties that must exist in a dynamical system in order to be classified as chaotic:

- It must have periodic long-term behavior meaning that the solution of the system settles into an irregular patter as $t \to \infty$. The solution does not repeat or oscillate in a periodic manner.
- It is sensitive to initial conditions. This means that any small change in the initial condition can change the trajectory, which may give a significantly different long-term behavior.
- It must be "deterministic" which means that the irregular behavior of the system is due to the nonlinearity of the system, rather than outside forces.

Thus, Duffing oscillators find applications in Chaos theory, which is the field of study in mathematics that studies the behavior of dynamical systems that are highly sensitive to initial condition(s) - a response popularly referred to as the butterfly effect. Small difference in initial conditions (such as those of rounding errors in numerical computation) yields widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general [15]. This happens even though these systems are deterministic, meaning that their future behavior if fully determined by their initial conditions, with no random elements involved, [15].

The theory was summarized by [16] as "*Chaos: when the present determines the future, but the approximate present does not approximately determine the future*".

In general, the challenge with chaotic systems, as described by [17], is that computation errors are progressively increased without bounds.

3 Derivation of the Method

In this section, a discrete block method of the form,

$$A^{(0)}\mathbf{Y}_m^{(i)} = \sum_{i=0}^{1} h^i e_i y_n^{(i)} + h^2 d_i f(y_n) + h^2 b_i f(\mathbf{Y}_m),\ i = 0,1 \tag{3}$$

is derived for the global solution of problem of the form (3) on the interval $\left[x_n, x_{n+1}\right]$. The initial assumption is that the solution on the interval $\left[x_n, x_{n+1}\right]$ is locally approximated by the basis function (approximate solution),

$$y(x) = \sum_{j=0}^{r+s-1} \tau_j x^j \tag{4}$$

where τ_j are the real coefficients to be determined, r is the number of interpolation points, s is the number of collocation points and $h = x_n - x_{n-1}$ is a constant step-size of the partition of the interval $[\alpha, \beta]$ which is given by $\alpha = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = \beta$.

In this research, we assume that the polynomial (4) must pass through the interpolation points $(x_{n+s}, y_{n+s}), s = \frac{1}{2}, \frac{3}{4}$ and the interpolation points $(x_{n+r}, f_{n+r}), r = 0\left(\frac{1}{4}\right)1$ and we require that the following $(r+s)$ equations must be satisfied:

$$\sum_{j=0}^{r+s-1} \tau_j x^j = y_{n+s},\ s = \frac{1}{2}, \frac{3}{4} \tag{5}$$

$$\sum_{j=0}^{r+s-1} j(j-1)\tau_j x^{j-2} = f_{n+r},\ r = 0\left(\frac{1}{4}\right)1 \tag{6}$$

The $(r+s)$ undetermined coefficients τ_j are obtained by solving the system of nonlinear equations (5) and (6) using Gauss elimination method. This gives a continuous hybrid linear multistep method of the form;

$$y(x) = \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(t) y_{n+\frac{3}{4}} + h^2 \left(\sum_{j=0}^{1} \beta_j(t) f_{n+j} + \beta_k(t) f_{n+k} \right),\ k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \tag{7}$$

The coefficients $\alpha_{\frac{1}{2}}, \alpha_{\frac{3}{4}}, \beta_0, \beta_{\frac{1}{4}}, \beta_{\frac{1}{2}}, \beta_{\frac{3}{4}}, \beta_1$ are given by;

$$\left.\begin{aligned}
&\alpha_{\frac{1}{2}}(t) = 3 - 4t \\
&\alpha_{\frac{3}{4}}(t) = 4t - 2 \\
&\beta_0(t) = \frac{1}{11520}(4096t^6 - 15360t^5 + 22400t^4 - 16000t^3 + 5760t^2 - 950t + 51) \\
&\beta_{\frac{1}{4}}(t) = -\frac{1}{2880}(4096t^6 - 13824t^5 + 16640t^4 - 7680t^3 + 954t - 189) \\
&\beta_{\frac{1}{2}}(t) = \frac{1}{1920}(4096t^6 - 12288t^5 + 12160t^4 - 3840t^3 - 322t + 201) \\
&\beta_{\frac{3}{4}}(t) = -\frac{1}{2880}(4096t^6 - 10752t^5 + 8960t^4 - 2560t^3 + 142t - 39) \\
&\beta_1(t) = \frac{1}{11520}(4096t^6 - 9216t^5 + 7040t^4 - 1920t^3 + 66t - 9)
\end{aligned}\right\} \tag{8}$$

where $t = \dfrac{x - x_n}{h}$, $y_{n+j} = y(t_n + jh)$ is the numerical approximation to the analytic solution $y(t_{n+j})$ and $y_{n+j} = f_{n+j} = f\left((t_n + jh), y(t_n + jh), y'(t_n + jh)\right)$ is the approximation to $y'(t_{n+j})$.

The continuous method (7) is then solved for the independent solution at the grid points to give the continuous block method:

$$y(t) = \sum_{j=0}^{1} \frac{(jh)^{(m)}}{m!} y_n^{(m)} + h^2 \left(\sum_{j=0}^{1} \sigma_j(t) f_{n+j} + \sigma_k f_{n+k} \right),\ k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \tag{9}$$

The coefficients $\sigma_i,\ i = 0\left(\frac{1}{4}\right)1$ gives,

$$\left.\begin{aligned}
\sigma_0(t) &= \frac{1}{90}\left(32t^6 - 120t^5 + 175t^4 - 125t^3 + 45t^2\right) \\
\sigma_{\frac{1}{4}}(t) &= -\frac{1}{45}\left(64t^6 - 216t^5 + 260t^4 - 120t^3\right) \\
\sigma_{\frac{1}{2}}(t) &= \frac{1}{15}\left(32t^6 - 96t^5 + 95t^4 - 30t^3\right) \\
\sigma_{\frac{3}{4}}(t) &= -\frac{1}{45}\left(64t^6 - 168t^5 + 140t^4 - 40t^3\right) \\
\sigma_1(t) &= \frac{1}{90}\left(32t^6 - 72t^5 + 55t^4 - 15t^3\right)
\end{aligned}\right\} \tag{10}$$

We then evaluate (9) at $t = \frac{1}{4}\left(\frac{1}{4}\right)1$ to give the method of the form (3) where,

$$\mathbf{Y}_m = \begin{bmatrix} y_{n+\frac{1}{4}} & y_{n+\frac{1}{2}} & y_{n+\frac{3}{4}} & y_{n+1} \end{bmatrix}^T,\quad \mathbf{y}_n^{(i)} = \begin{bmatrix} y_{n-\frac{1}{4}}^{(i)} & y_{n-\frac{1}{2}}^{(i)} & y_{n-\frac{3}{4}}^{(i)} & y_n^{(i)} \end{bmatrix}^T$$

$$\mathbf{F}(\mathbf{Y}_m) = \begin{bmatrix} f_{n+\frac{1}{4}} & f_{n+\frac{1}{2}} & f_{n+\frac{3}{4}} & f_{n+1} \end{bmatrix}^T,\quad \mathbf{f}(\mathbf{y}_n) = \begin{bmatrix} f_{n-\frac{1}{4}} & f_{n-\frac{1}{2}} & f_{n-\frac{3}{4}} & f_n \end{bmatrix}^T$$

$A^{(0)}$ is a 4×4 identity matrix given by $A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

When $i = 0$:

$$e_0=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\quad e_1=\begin{bmatrix}0&0&0&\frac{1}{4}\\0&0&0&\frac{1}{2}\\0&0&0&\frac{3}{4}\\0&0&0&1\end{bmatrix}\quad d_0=\begin{bmatrix}0&0&0&\frac{367}{23040}\\0&0&0&\frac{53}{1440}\\0&0&0&\frac{147}{2560}\\0&0&0&\frac{7}{90}\end{bmatrix}$$

$$b_0=\begin{bmatrix}\frac{3}{128}&\frac{-47}{3840}&\frac{29}{5760}&\frac{-7}{7680}\\\frac{1}{10}&\frac{-1}{48}&\frac{1}{90}&\frac{-1}{480}\\\frac{117}{640}&\frac{27}{1280}&\frac{3}{128}&\frac{-9}{2560}\\\frac{4}{15}&\frac{1}{15}&\frac{4}{45}&0\end{bmatrix}$$

When $i=1$:

$$e_1=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\quad d_1=\begin{bmatrix}0&0&0&\frac{251}{2880}\\0&0&0&\frac{29}{360}\\0&0&0&\frac{27}{320}\\0&0&0&\frac{7}{90}\end{bmatrix}\quad b_1=\begin{bmatrix}\frac{323}{1440}&\frac{-11}{120}&\frac{53}{1440}&\frac{-19}{2880}\\\frac{31}{90}&\frac{1}{15}&\frac{1}{90}&\frac{-1}{360}\\\frac{51}{160}&\frac{9}{40}&\frac{21}{160}&\frac{-3}{320}\\\frac{16}{45}&\frac{2}{15}&\frac{16}{45}&\frac{7}{90}\end{bmatrix}$$

4 Analysis of the Method

In this section, we shall analyze the basic properties of the method.

4.1 Order of accuracy and error constant

According to [18], the linear operator associated with the discrete block method (3) is defined as;

$$L\{y(t):h\}=\mathbf{A}^{(0)}\mathbf{Y}_m^{(i)}-\sum_{i-0}^{1}h^i e_i y_n^{(i)}-h^2\left(d_0 f(y_n)+b_0\mathbf{F}(\mathbf{Y}_m)\right)\tag{11}$$

Assuming that $y(t)$ is sufficiently differentiable, we write the terms in (11) as a Taylor series expansion about the point t to obtain the expression;

$$L\{y(t):h\}=c_0y(t)+c_1hy'(t)+c_2h^2y''(t)+...+c_ph^py^p(t)+c_{p+1}h^{p+1}y^{p+1}(t)+c_{p+1}h^{p+2}y^{p+2}(t) \quad (12)$$

where the constant coefficients $c_p,\ p=0,1,2,...$ are given by;

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k}\alpha_j \\ c_1 &= \sum_{j=0}^{k}\left(j\alpha_j-\beta_j\right) \\ &. \\ &. \\ &. \\ c_p &= \sum_{j=0}^{k}\left[\frac{1}{q!}j^q\alpha_j-\frac{1}{(q-1)!}j^{q-1}\beta_j\right],\ q=2,3,... \end{aligned}\right\} \quad (13)$$

The block method (3) is said to be of uniform accurate order p, if p is the largest positive integer for which $\bar{c}_0=\bar{c}_1=\bar{c}_2=...=\bar{c}_p=\bar{c}_{p+1}=0,\ \bar{c}_{p+2}\neq\bar{0}$. $\bar{c}_{p+2}$ is called the error constant and the local truncation error of the method is given by;

$$\bar{t}_{n+k}=\bar{c}_{p+2}h^{(p+2)}y^{(p+2)}(t)+O\left(h^{(p+3)}\right) \quad (14)$$

It has therefore been established from our computations that the block method (3) has coefficients of h given by $\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{c}_6=\bar{0}$, implying that the order $p=\begin{bmatrix}5 & 5 & 5 & 5\end{bmatrix}^T$ and the error constant is give by $\bar{c}_7=\begin{bmatrix}6.4789\times10^{-7} & 1.5501\times10^{-6} & 2.4523\times10^{-6} & 3.1002\times10^{-6}\end{bmatrix}^T$.

4.2 Consistency

The hybrid block method (3) is consistent since it has order $p=5\geq1$. According to [19], consistency controls the magnitude of the local truncation error committed at each stage of the computation.

4.3 Zero-stability

Definition 4.1 [19]: The block method (3) is said to be zero-stable, if the roots $z_s,\ s=1,2,...,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z)=\det(zA^{(0)}-e_0)$ satisfies $|z_s|\leq1$ and every root satisfying $|z_s|=1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h\rightarrow0,\rho(z)=z^{r-\mu}(z-1)^{\mu}$, where μ is the order of the matrices $A^{(0)}$ *and* e_0.

For our method,

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \tag{15}$$

Therefore, $\rho(z) = j \to \infty \Rightarrow z_1 = z_2 = z_3 = 0,\ z_4 = 1$. Hence, the hybrid block method is zero-stable. It is important to note that the main consequence of zero-stability is to control the propagation of the error as the integration proceeds.

4.4 Convergence

The hybrid block method is convergent since it is consistent and zero-stable.

Theorem 4.1 [20]

A linear multistep method is convergent if and only if it is stable and consistent.

4.5 Region of absolute stability

Definition 4.2 [21]

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y'' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

In determining the stability polynomial of our method, the boundary locus method will be adopted. This gives;

$$\begin{aligned} \bar{h}(w) = &-h^8\left(\frac{7}{3686400}w^3 - \frac{1}{4915200}w^4\right) + h^6\left(\frac{1}{1474560}w^4 - \frac{1123}{2211840}w^3\right) \\ &- h^4\left(\frac{307}{9216}w^3 - \frac{31}{92160}w^4\right) - h^2\left(\frac{5}{192}w^4 + \frac{59}{96}w^3\right) + w^4 - 2w^3 \end{aligned} \tag{16}$$

The stability region is shown in Fig. 1.

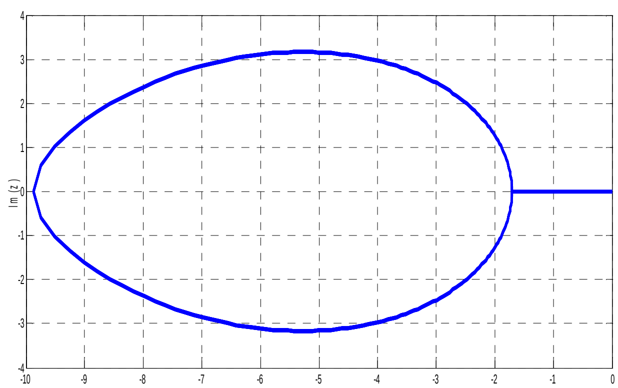

Fig. 1. Stability region of the method

The stability region in the Fig. 1 is A-stable.

5 Implementation, Numerical Experiments and Discussion of Results

5.1 Implementation

The hybrid block method (3) was derived using the Scientific Work Place 5.5 was used to derive the hybrid block method and also to compute the stability polynomial of the method. The method was implemented on Duffing oscillators with the aid of MATLAB 2015a programming language. The stability region was also plotted using MATLAB.

5.2 Numerical experiments

Three important Duffing equations that find applications in science and engineering in terms of modeling damp and chaotic systems shall be considered. The method developed in this research shall be implemented on these problems and the results obtained shall be compared with those of the existing methods. The following notations shall be used in the Tables below:

ESS-End point absolute errors obtained in [11].

EBM-Absolute error in [7].

ETG-Absolute error in [5].

Problem 5.1:

Consider the following undamped Duffing oscillator of the form;

$$y''(t)+y(t)+y^3(t)=B\cos\Omega t \tag{17}$$

with initial conditions,

$$y(0)=\alpha,\ y'(0)=0 \tag{18}$$

where,

$$\alpha=0.200426728067, B=0.002,\ \Omega=1.01$$

The exact solution to the problem is

$$y(t)=\sum_{i=0}^{3}A_{2i+1}Cos\left(\left(2i+1\right)\Omega t\right) \tag{19}$$

where,

$$\left\{\begin{matrix}A_1, A_3, A_5,\\ A_7, A_9\end{matrix}\right\}=\left\{\begin{matrix}0.200179477536, 0.0024946143, 0.000000304014,\\ 0.000000000374, 0.000000000000\end{matrix}\right\}$$

Source: [11].

Problem 5.2:

Consider the undamped Duffing oscillator,

$$y''(t)+3y(t)+2y^3(t)=\cos(t)\sin(2t) \tag{20}$$

with the initial conditions,

$$y(0)=0,\ y'(0)=1 \tag{21}$$

The exact solution is given by,

$$y(t)=\sin(t) \tag{22}$$

Source: [7].

Problem 5.3:

Consider the damped Duffing oscillator,

$$y''(t)+y'(t)+y(t)+y^3(t)=\cos^3(t)-\sin(t) \tag{23}$$

whose initial conditions are,

$$y(0)=1,\ y'(0)=0 \tag{24}$$

The exact solution is given by,

$$y(t)=\cos(t) \tag{25}$$

Source: [5].

Table 1. Comparison of the end-point absolute errors in [11] with that of the new method developed for Problem 5.1

h	**Error**	**ESS**
$\frac{M}{500}$	8.813783e-013	1.81e-010
$\frac{M}{1000}$	1.114692e-012	8.02e-012
$\frac{M}{2000}$	2.953554e-012	5.52e-012
$\frac{M}{3000}$	2.339406e-012	7.28e-012
$\frac{M}{4000}$	1.859929e-012	6.99e-012
$\frac{M}{5000}$	1.328992e-012	6.65e-012

Note: $M=10$ *in Table 1 above*

Table 2. Showing the results for problem 5.2 in comparison with the absolute errors in [7]

t	Exact solution	Computed solution	Error	EBM	Time/s
0.1000	0.0998334166468282	0.0998334166471306	3.024248e-013	3.603424e-07	0.0917
0.2000	0.1986693307950612	0.1986693307955197	4.584944e-013	1.020596e-05	0.0926
0.3000	0.2955202066613396	0.2955202066612664	7.316370e-014	2.357701e-05	0.0930
0.4000	0.3894183423086505	0.3894183423069583	1.692257e-012	9.788940e-07	0.0935
0.5000	0.4794255386042030	0.4794255385996061	4.596878e-012	1.601644e-05	0.0940
0.6000	0.5646424733950354	0.5646424733862804	8.754997e-012	3.106965e-05	0.0945
0.7000	0.6442176872376910	0.6442176872237844	1.390665e-011	8.5059594-06	0.0950
0.8000	0.7173560908995227	0.7173560908799302	1.959244e-011	2.193132e-05	0.0955
0.9000	0.7833269096274833	0.7833269096022861	2.519718e-011	3.183986e-05	0.0960
1.0000	0.8414709848078964	0.8414709847778973	2.999911e-011	3.225774e-05	0.0965

Table 3. Showing the results for problem 5.3 in comparison with the absolute errors in [5]

t	Exact solution	Computed solution	Error	ETG	Time/s
0.1000	0.9950041652780257	0.9950041652770839	9.418022e-013	1×10^{-06}	0.1275
0.2000	0.9800665778412416	0.9800665778319209	9.320766e-012	6×10^{-10}	0.1284
0.3000	0.9553364891256060	0.9553364891018900	2.371603e-011	7×10^{-10}	0.1290
0.4000	0.9210609940028851	0.9210609939604013	4.248379e-011	4×10^{-10}	0.1296
0.5000	0.8775825618903728	0.8775825618264685	6.390422e-011	5×10^{-10}	0.1301
0.6000	0.8253356149096783	0.8253356148233559	8.632239e-011	1.7×10^{-09}	0.1307
0.7000	0.7648421872844885	0.7648421871762232	1.082653e-010	1.06×10^{-08}	0.1312
0.8000	0.6967067093471655	0.6967067092186436	1.285219e-010	3.99×10^{-08}	0.1318
0.9000	0.6216099682706645	0.6216099681244809	1.461836e-010	1.273×10^{-07}	0.1324
1.0000	0.5403023058681398	0.5403023057074929	1.606468e-010	3.599×10^{-07}	0.1330

5.3 Discussion of results

We implemented the hybrid block method developed on three modeled Duffing oscillators and from the results obtained in Tables 1, 2 and 3, it is obvious that the new method developed is more efficient than the existing ones with which we compared our results.

6 Conclusion

We have studied some applications of Duffing oscillators and also developed a computational method for solving such problems using the power series approximate solution. The method developed was consistent, convergent, zero-stable and A-stable. This paper therefore recommends the use of this method for solving not only Duffing oscillators but second order nonlinear (and linear) differential equations of the form (1).

Competing Interests

Author has declared that no competing interests exist.

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