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IMPRORER INTEGRALS

Proper integral:

$\int_a^b f(x)\,dx$ is proper if the range of integration is finite and the integrand is bounded.

[Values of a & b are finite]

Improper Integral:

The integral $\int_a^b f(x)\,dx$ is improper if

i) a and/or b $= -\infty$ or ∞ and $f(x)$ is bounded (1^{st} kind)

ii) $f(x)$ is unbounded at one or more points of $a \leq x \leq b$ (2^{nd} kind)

iii) Both (i) and (ii) (3^{rd} kind)

$\int_0^1 \frac{1}{x}\,dx \rightarrow$ improper of 2^{nd} kind

$\int_0^2 \sqrt{x^2+1}\,dx \rightarrow$ proper integral

$\int_{-\infty}^5 \frac{1}{x}\,dx \rightarrow$ improper of 3^{rd} kind

$\int_0^1 \frac{\sin x}{x}\,dx \rightarrow$ proper integral since $\underset{x\to 0}{Lt} \frac{\sin x}{x} = 1$

$\int_0^\infty \cos x\,dx \rightarrow$ improper of 1^{st} kind.

$\int_{-\infty}^{\infty} x^2 \, dx \longrightarrow$ improper of 3^{rd} kind

$\int_{0}^{\infty} \frac{1}{(1-x)^2} dx \longrightarrow$ improper integral of 3^{rd} kind.

Evaluation of improper integrals of 1^{st} kind:

$$\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty}\left[\int_a^R f(x)\,dx\right]$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{R\to\infty}\left[\int_{-R}^{b} f(x)\,dx\right]$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1\to\infty}\int_{-R_1}^{c} f(x)\,dx + \lim_{R_2\to\infty}\int_{c}^{R_2} f(x)\,dx$$

$$= \lim_{\substack{R_1\to\infty \\ R_2\to\infty}} \int_{-R_1}^{R_2} f(x)\,dx.$$

① $\int_2^{\infty} \frac{2x^2}{x^4-1} dx$

$$\frac{2x^2}{x^4-1} = \frac{2x^2}{(x^2+1)(x^2-1)}$$

$$= \frac{1}{x^2-1} + \frac{1}{x^2+1}$$

$$= \lim_{R\to\infty}\int_2^R \left(\frac{1}{x^2-1} + \frac{1}{x^2+1}\right) dx$$

$$= \lim_{R\to\infty}\int_2^R \frac{1}{x^2-1}\,dx + \lim_{R\to\infty}\int_2^R \frac{1}{x^2+1}\,dx$$

$$= \lim_{R\to\infty} \int_2^R \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \lim_{R\to\infty} \left(\tan^{-1}x\right)_2^R$$

$$= \lim_{R\to\infty}\left(\frac{1}{2}\int_2^R \frac{1}{x-1}\,dx - \frac{1}{2}\int_2^R \frac{1}{x+1}\,dx\right) + \lim_{R\to\infty}\left[\tan^{-1}R - \tan^{-1}(2)\right]$$

$$= \lim_{R\to\infty}\left(\frac{1}{2}\ln\frac{x-1}{x+1}\right)_2^R + \frac{\pi}{2} - \tan^{-1}(2)$$

$$= \cancel{\tfrac{1}{2}} \lim_{R\to\infty}\left(\frac{1}{2}\ln\frac{R-1}{R+1}\right) + \frac{1}{2}\ln 3 + \frac{\pi}{2} - \tan^{-1}(2)$$

$$= \lim_{R\to\infty} \frac{1}{2}\ln\left(\frac{1-\frac{1}{R}}{1+\frac{1}{R}}\right) + \frac{1}{2}\ln 3 + \frac{\pi}{2} - \tan^{-1}(2)$$

$$\int_2^\infty \frac{2x}{x^4-1}\,dx = \frac{1}{2}\ln 3 + \frac{\pi}{2} - \tan^{-1}(2)$$

(2) $\int_0^\infty \sin x\,dx \longrightarrow$ improper integral of 1st kind.

$$\lim_{R\to\infty}\int_0^R \sin x\,dx = \lim_{R\to\infty}\left[-\cos x\right]_0^R$$

$$= \lim_{R\to\infty}\left[1 - \cos R\right]$$

$$= 1 - \lim_{R\to\infty}\cos R$$

The integration value is not converging.

It does not converge to a finite value.

Evaluation of improper integrals of 2nd kind:

$\int_a^b f(x)\,dx$ $f(x)$ is unbounded in $a \le x \le b$

i) If $f(x) \to \infty$ as $x \to b$ then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx$$

ii) If $f(x) \to \infty$ as $x \to a$ then

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon^+}^{b} f(x)\,dx$$

iii) If $f(x) \to \infty$ as $x \to c$ where $a < c < b$

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon \to 0^+} \int_{c+\epsilon^*} f(x)\,dx$$

iv) If $f(x) \to \infty$ as $x \to a$ & $x \to b$

$$\int_a^b f(x)\,dx = \lim_{\substack{\epsilon_1 \to 0^+ \\ \epsilon_2 \to 0^+}} \int_{a+\epsilon_1}^{b-\epsilon_2} f(x)\,dx.$$

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$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$ [2nd kind]

$$= \lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{\epsilon \to 0^+} \left[- \frac{(1-x)^{1/2}}{1/2} \right]_0^{1-\epsilon}$$

$$= \lim_{\epsilon \to 0^+} \left[-2\sqrt{1-x} \right]_0^{1-\epsilon}$$

$$= \lim_{\epsilon \to 0^+} \left[-2\sqrt{\epsilon} + 2 \right]$$

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = 2$$

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$$\int_0^2 \frac{dx}{2x-x^2} \quad [2^{nd} \text{ kind}]$$

$$\frac{1}{2x-x^2} = \frac{1}{2}\left[\frac{1}{2-x}+\frac{1}{x}\right]$$

$$= \lim_{\epsilon_1 \to 0^+} \int_{\epsilon_1}^{1} \frac{dx}{(2x-x^2)} + \lim_{\epsilon_2 \to 0^+} \int_{1}^{2-\epsilon_2} \frac{dx}{(2x-x^2)}$$

$$= \lim_{\epsilon_1 \to 0^+} \left[\frac{1}{2}\ln\left(\frac{x}{2-x}\right)\right]_{\epsilon_1}^{1} + \lim_{\epsilon_2 \to 0^+} \left[\frac{1}{2}\ln\left(\frac{x}{2-x}\right)\right]_{1}^{2-\epsilon_2}$$

Divergent.

[Type 1]

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx$$

$$= \lim_{a\to\infty} \left[\ln x\right]_1^a$$

$$= \lim_{a\to\infty} \ln a$$

Diverges

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx$$

$$= \lim_{a\to\infty} \left[-\frac{1}{x}\right]_1^a$$

$$= \lim_{a\to\infty} \left(1-\frac{1}{a}\right)$$

$$= 1$$

$\int_1^\infty \frac{1}{x}\,dx \longrightarrow$ diverges

$\int_1^\infty \frac{1}{x^2}\,dx \longrightarrow$ converges to 1

$\int_a^\infty \frac{1}{x^P}\,dx$ Find conditions for convergence & divergence.

(a is +ve).

$$\int_a^\infty \frac{1}{x^P}\,dx = \lim_{R\to\infty} \left[\int_a^R f(x)\,dx\right]$$

$$= \lim_{R\to\infty} \left[\int_a^R \frac{1}{x^P}\,dx\right]$$

$$= \lim_{R\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^R$$

$$= \lim_{R\to\infty} \frac{1}{(1-p)} \cdot \left(\frac{1}{x^{p-1}}\right)_a^R$$

$$= \frac{1}{(1-p)} \lim_{R\to\infty} \left(\frac{1}{R^{(p-1)}} - \frac{1}{a^{p-1}}\right)$$

$$= \frac{1}{(1-p)} \quad \frac{-1}{a^{p-1}}$$

$$\int_a^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \cdot \frac{1}{a^{p-1}} \qquad p \neq 1$$

If $p = 1$ $\quad \lim_{R\to\infty} \int_a^R \frac{1}{x}\,dx = \ln\left(\frac{R}{a}\right)$

$$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \ln\left(\frac{R}{a}\right) & p = 1 \\ \frac{1}{1-p}\left[\frac{1}{R^{(p-1)}} - \frac{1}{a^{p-1}}\right] & p \neq 1 \end{cases}$$

$p \leq 1 \longrightarrow$ divergent

$p > 1 \longrightarrow$ converging to $\frac{1}{p-1} \cdot \frac{1}{a^{(p-1)}}$

$\int_a^b \frac{1}{(x-a)^p}\,dx$ Find conditions for convergence & divergence.

$$\int_a^b \frac{1}{(x-a)^p}\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b \frac{1}{(x-a)^p}\,dx$$

$$= \lim_{\epsilon \to 0^+} \left[\frac{(x-a)^{-p+1}}{-p+1} \right]_{a+\epsilon}^{b}$$

$$= \lim_{\epsilon \to 0^+} \left[\frac{(b-a)^{-p+1}}{(-p+1)} - \frac{\epsilon^{(-p+1)}}{(-p+1)} \right]$$

$$= \frac{1}{1-p} \lim_{\epsilon \to 0^+} \left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}} \right]$$

If $p = 1$

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} \frac{1}{(x-a)^p} dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} \frac{1}{(x-a)} dx$$

$$= \lim_{\epsilon \to 0^+} \left[\ln(x-a) \right]_{a+\epsilon}^{b}$$

$$= \lim_{\epsilon \to 0^+} \left[\ln(b-a) - \ln \epsilon \right]$$

$$\int_a^b \frac{1}{(x-a)^p} dx = \begin{cases} \frac{1}{(1-p)} \lim_{\epsilon \to 0^+} \left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}} \right] & p \neq 1 \\ \lim_{\epsilon \to 0^+} \left[\ln(b-a) - \ln \epsilon \right] & p = 1 \end{cases}$$

$$\int_a^b \frac{1}{(x-a)^p} dx \to \begin{cases} p \geq 1 & \text{diverging} \\ p < 1 & \text{converges to } \frac{1}{1-p} \cdot \frac{1}{(b-a)^{p-1}} \end{cases}$$

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Ex: $\int_0^1 \frac{1}{x^p}\,dx$

divergent if $p \geq 1$

and convergent if $p < 1$

Rule:

Suppose f and g are defined on J

i) If $0 \leq f(x) \leq g(x) \quad \forall x \in J$ and

$\int_J g(x)\,dx$ exists then $\int_J f(x)\,dx$ exists.

ii) If $\int_J |f(x)|\,dx$ exists then $\int_J f(x)\,dx$ exists.

Ex: $\int_0^1 \frac{\cos x}{x^p}\,dx$ Find condition for convergence & divergence.

$$\left|\frac{\cos x}{x^p}\right| \leq \frac{1}{x^p}$$

$\int_0^1 \left|\frac{\cos x}{x^p}\right| dx$ converges for $p < 1$

$\int_0^1 \frac{\cos x}{x^p}\,dx$ converges for $p < 1$

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Beta Function:

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx} \qquad m>0, n>0$$

$m, n \in R$

Put $x = 1-y$

$$\beta(m,n) = -\int_1^0 (1-y)^{m-1} y^{n-1}\, dy$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1}\, dy$$

$$\boxed{\beta(m,n) = \beta(n,m)} \qquad \beta(1,3) = \beta(3,1)$$

Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} (2\sin\theta\cos\theta)\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}$$

Gamma Function:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad (n>0)$$

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1 \qquad \boxed{\Gamma(1) = 1}$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\, dx = \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$\Gamma(1) = 0\cdot\Gamma(0)$, so $\Gamma(0)$ is undefined so are gamma f^n for all $-ve$ integers

But $\Gamma(1) = 1$

$\Gamma(n)$ is undefined for $n = 0, -1, -2,$

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(2) = 1\cdot\Gamma(1) = 1!$

$\Gamma(3) = 2\,\Gamma(2) = 2!$

$\Gamma(4) = 3\cdot\Gamma(3) = 3!$

$\Gamma(n)$ is defined $\forall\, n$ $0<n\le 1$ $\quad 1<n\le 2 \quad 2<n\le 3, \ldots$

$-1<n<0 \quad -2<n<1, \ldots$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x} x^{-1/2}\, dx \qquad \text{put } x = y^2, \quad dx = 2y\,dy$$

$$= \int_0^\infty e^{-y^2} \frac{1}{y}\cdot 2y\, dy$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\,dy$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \quad \text{(Normal distribution)}$$

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Normal pdf

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\mu = 0$$
$$2\sigma^2 = 1$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$= 4\cdot\frac{\pi}{2}\cdot\int_0^\infty e^{-r^2}\, r\,dr$$

$$= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty$$

$$= \pi$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi\cdot z)}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(-\tfrac{1}{2}\right)\Gamma\left(\tfrac{3}{2}\right) = \frac{\pi}{-1} \qquad \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$$

Relation between β and Γ:

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$t = x^2$

$$\Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt = 2\int_0^\infty e^{-x^2}\, x^{2m-1}\, dx$$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}\, r^{2(m+n)-1} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta\, dr$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta \times 2\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr$$

$$= \beta(m,n)\,\Gamma(m+n)$$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \sqrt{\pi}\cdot\sqrt{\pi} = \pi$$

$$\beta(1,1) = \frac{\Gamma(1)\,\Gamma(1)}{\Gamma(2)} = \frac{1}{2}$$

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{\gamma\left(\frac{p+1}{2}\right) \gamma\left(\frac{q+1}{2}\right)}{2\gamma\left(\frac{p+1+q+1}{2}\right)}$$

$q = 0, \quad p = n$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{\gamma\left(\frac{n+1}{2}\right) . \sqrt{\pi}}{\gamma\left(\frac{n+2}{2}\right) . 2}$$

$n = 3$ $$\int_0^{\pi/2} \sin^3 x \, dx = \frac{\gamma(2)}{\gamma(5/2)} \cdot \frac{\sqrt{\pi}}{2}$$

$$\gamma\left(\frac{5}{2}\right) = \frac{3}{2}\gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\gamma\left(\frac{1}{2}\right)$$

$$\int_0^{\pi/2} \sin^3 x \, dx = \frac{\cancel{2}}{\frac{3}{4} \cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$$= \frac{4}{3}$$

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① Evaluate $I = \int_0^{\infty} e^{-h^2x^2}\,dx$

$$\boxed{\int_0^{\infty} e^{-ax^n}\,dx \quad \text{Put } ax^n = t}$$

$h^2x^2 = t$

$hx = \sqrt{t}$

$h\,dx = \frac{1}{2\sqrt{t}}\,dt$

$dx = \frac{1}{2h\sqrt{t}}\,dt$

$$I = \int_0^{\infty} e^{-t}\frac{1}{2h\sqrt{t}}\,dt = \frac{1}{2h}\int_0^{\infty} e^{-t}\,t^{-1/2}\,dt$$

$$= \frac{1}{2h}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2h}$$

② $I = \int_0^{1/2} x^3(1-4x^2)^{1/2}\,dx$

$4x^2 = t$

$x^2 = \frac{t}{4}$

$x = \frac{\sqrt{t}}{2}$

$dx = \frac{1}{2}\cdot\frac{1}{2\sqrt{t}}\,dt$

$dx = \frac{1}{4\sqrt{t}}\,dt$

$$I = \int_0^1 \left(\frac{\sqrt{t}}{2}\right)^3 \cdot (1-t)^{1/2} \cdot \frac{1}{4\sqrt{t}}\, dt$$

$$= \frac{1}{32} \int_0^1 t \cdot (1-t)^{1/2}\, dt$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$= \frac{1}{32}\, \beta\left(2, \frac{3}{2}\right)$$

$$= \frac{1}{32} \cdot \frac{\Gamma(2) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{1}{32} \cdot \frac{2 \times \frac{\sqrt{\pi}}{2}}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{\sqrt{\pi}}{2}}$$

$$I = \frac{1}{120}$$

(3) $$I = \int_0^1 (x \log x)^4\, dx$$

$$\log x = -t$$

$$x = e^{-t}$$

$$dx = -e^{-t}\, dt$$

$$I = \int_{\infty}^{0} \left(e^{-t}\,(-t)\right)^4 \left(-e^{-t}\, dt\right)$$

$$= \int_0^{\infty} e^{-5t}\, t^4\, dt$$

$5t = y$

$t = \frac{y}{5}$

$dt = \frac{1}{5}\,dy$

$$I = \int_0^\infty e^{-y} \left(\frac{y}{5}\right)^4 \frac{dy}{5}$$

$$= \frac{1}{5^5} \int_0^\infty e^{-y} \cdot y^4\,dy$$

$$= \frac{1}{3125} \Gamma(5)$$

$$= \frac{4!}{3125}$$

$$I = \frac{24}{3125}$$

(4)

$$I = \int_0^{2a} x^2 \sqrt{2ax - x^2}\,dx$$

$$\boxed{\begin{gathered}\int x^m (b - x^n)^p\,dx \\ x^n = bt\end{gathered}}$$

$$= \int_0^{2a} x^{5/2} \sqrt{2a - x}\,dx$$

$x = 2at$

$dx = 2a\,dt$

$$I = \int_0^1 (2at)^{5/2} (2a - 2at)^{1/2}\, 2a\,dt$$

$$= \int_0^1 (2a)^{5/2}\, t^{5/2}\, (1-t)^{1/2}\, (2a)^{3/2}\,dt$$

$$= 16a^4 \int_0^1 t^{5/2} (1-t)^{1/2} dt$$

$$= 16a^4 \, \beta\left(\frac{7}{2}, \frac{3}{2}\right)$$

$$= 16a^4 \frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma(5)}$$

$$= 16a^4 \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2}}{24}$$

$$= \frac{15 \pi a^4}{24}$$

$$I = \frac{5}{8} \pi a^4$$

(5) $$\int_5^9 \sqrt[4]{(9-x)(x-5)} \, dx$$

$$x - 5 = 4t$$

$$\boxed{\begin{array}{c} \int_a^b (x-a)^m (b-x)^n \, dt \\ (x-a) = (b-a)\, t \end{array}}$$

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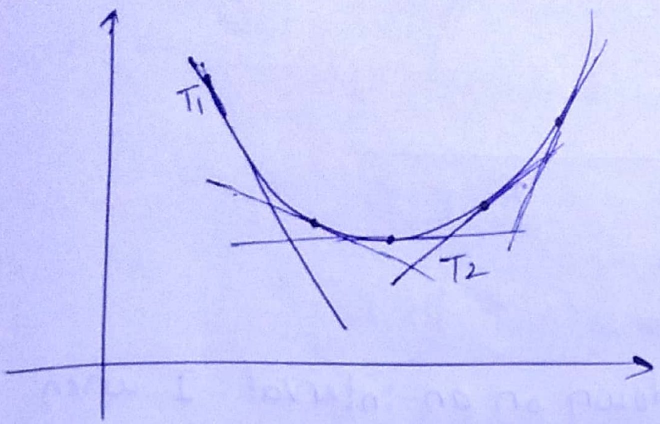
CONCAVE / CONVEX



- $f'(x) > 0$ increasing

$f'(x) < 0$ decreasing

slope of T_2 > slope of T_1

$f''(x) > 0$ (since slope fun is increasing)

$f''(x)>0$ slope is increasing

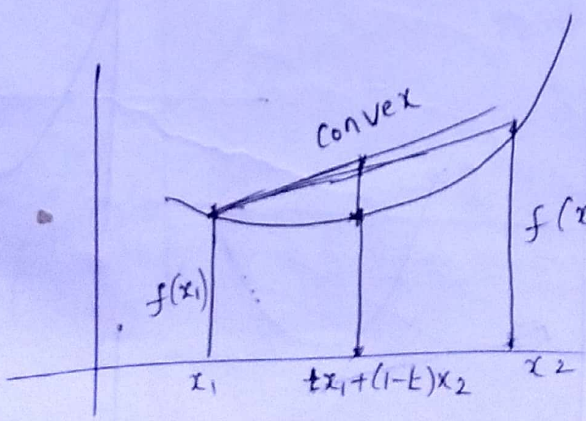
$f''(x)<0$ slope is decreasing

For concave-up (convex)

Convex:

A curve is concave up on an internal I when

$f''(x) > 0 \quad \forall x$ in I



$0 \le t \le 1$

$$t f(x_1) + (1-t) f(x_2) \ge f(t x_1 + (1-t) x_2)$$

Any tangent lies below the curve

Any secant lies above the curve.

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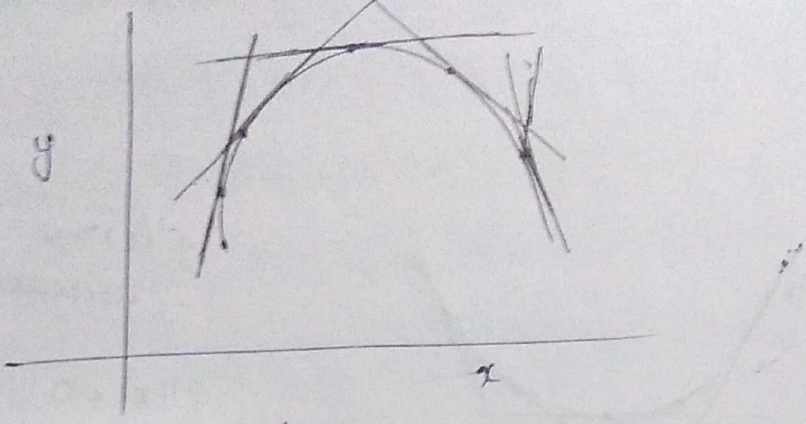
Concave down:

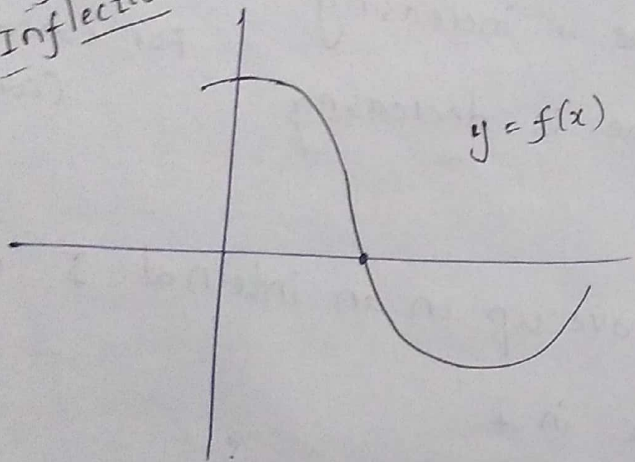


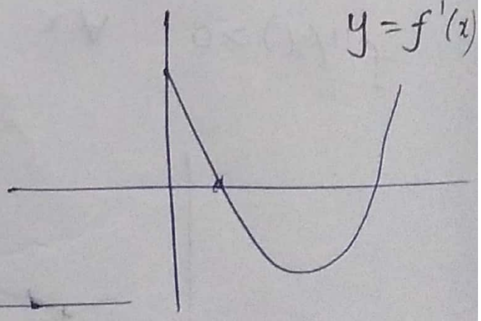
$f''(x) < 0$

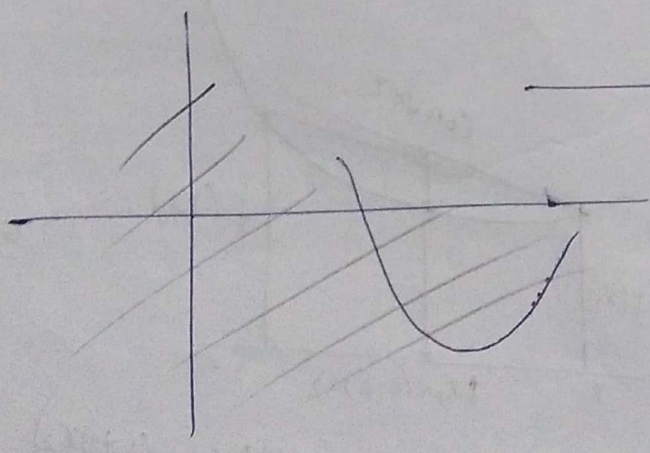
A curve is concave down on an interval I when

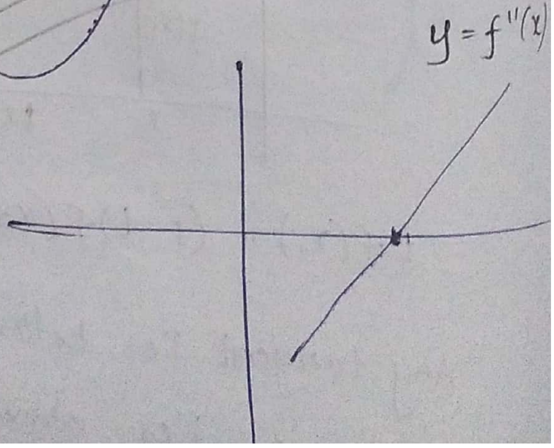
$f''(x) < 0 \quad \forall x \text{ in } I$

Point of Inflection:



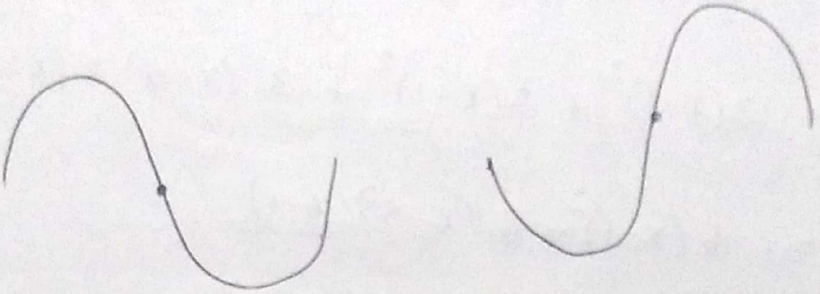






Points of inflection:

A point of inflection occurs at a point where $f''(x)=0$ and there is a change in concavity of the curve at the ~~o~~point



Ex: $y = x^3 + x$

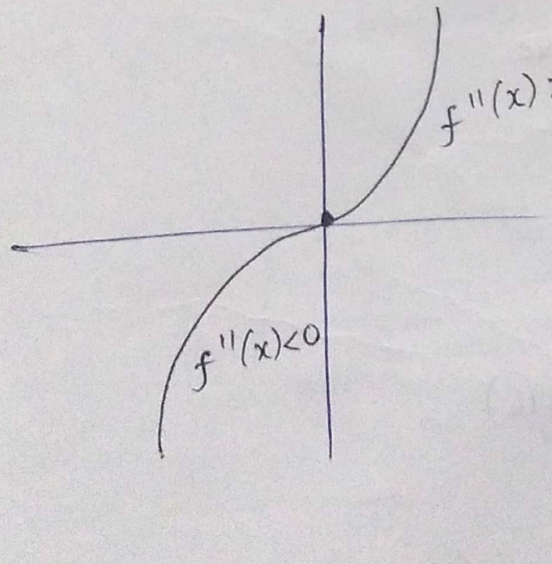
$f'(x) = 3x^2 + 1 > 0$

$f''(x) = 6x = 0$

$x = 0$

x	<0	0	>0
$f''(x)$	$-ve$	0	$+ve$

Concavity is changing at the point o.



Point of inflextion (0,0)

Find the point of inflection of the function

$$f(x) = (x-1)^3 (x-5)$$

$f(x) = (x-1)^3 (x-5)$

$f'(x) = (x-1)^3 + 3(x-1)^2 (x-5)$

$f''(x) = 3(x-1)^2 + 3(x-1)^2 + 3\cdot(x-5)\,2(x-1)$

$= 6(x-1)^2 + 6(x-5)(x-1)$

$= 6\left[(x-1)^2 + (x-5)(x-1)\right]$

$= 6\left[(x-1)\left[x-1+x-5\right]\right]$

$= 6(x-1)(2x-6)$

$= 12(x-1)(x-3)$

x	<1	1	$(1,3]$	$(3, \dots$
$f''(x)$	$-ve$	0	$-ve$	$+ve$

x	<3		3	>3
	<1	$[1\ 3]$		
$f''(x)$	$+ve$	$\mp ve$	0	$+ve$

Points of inflection

$x=1 \quad f(x)=0 \quad (1,0)$

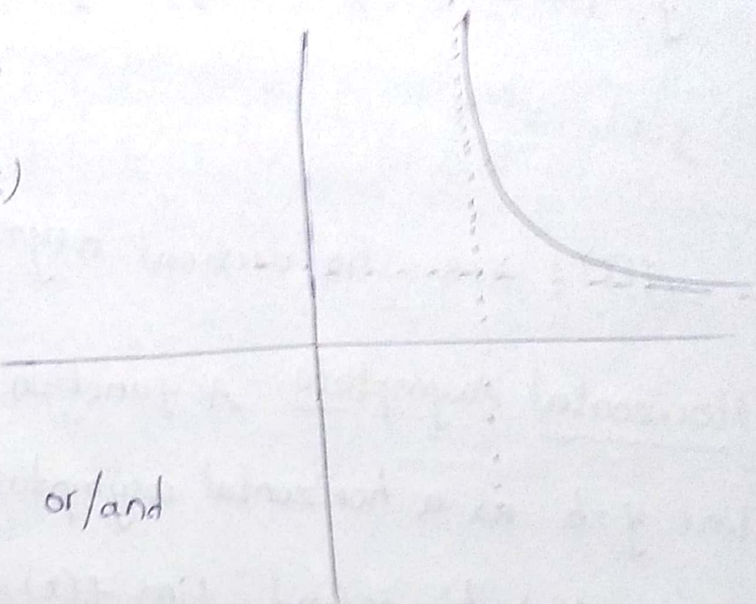
$x=3 \quad f(x)=8(-2) \quad (3,-16)$

	$(-\infty,1)$	1	$(1,3)$	3	$(3,\infty)$
$f''(x)$	>0	0	<0	0	>0

Asymptotes:

Vertical Asymptotes:

A function $y = f(x)$ has the line $x = a$ as a vertical asymptote if



$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or/and}$$

$$\lim_{x \to a^-} f(x) = \pm\infty$$

Ex 1 $\quad f(x) = \frac{5x-1}{x-3}$

$$\lim_{x \to 3^-} f(x) = -\infty$$

$x = 3$ is a Vertical asymptote

$$\lim_{x \to 3^+} f(x) = +\infty$$

Ex 2 $\quad f(x) = \cancel{(x-5)(x-1)} \quad \frac{x(x-1)}{(x-5)(x+2)}$

$$\left.\begin{array}{l} x = 5 \\ x = -2 \end{array}\right\} \text{Asymptotes}$$

Ex 3 $\quad f(x) = \frac{x-1}{x^2+1}$ has no vertical asymptote

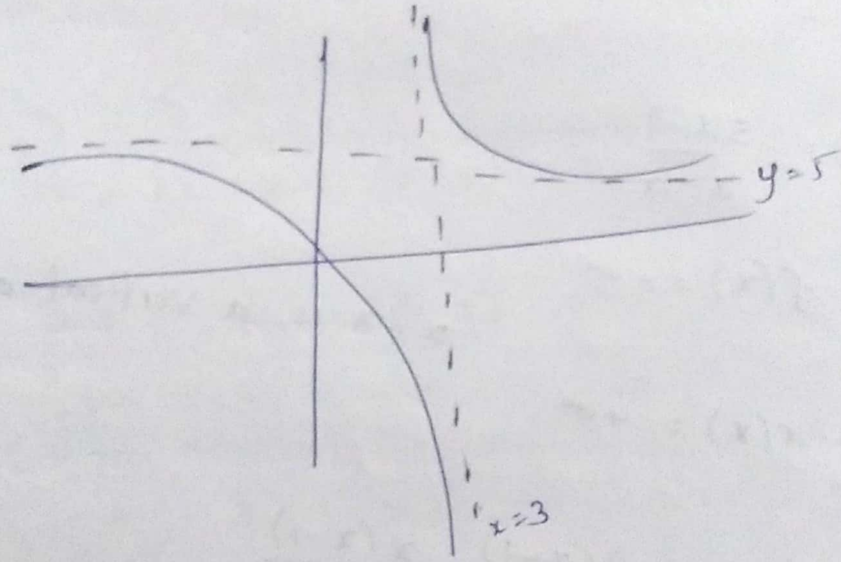
$y = \tan x$, $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$

$f(x) = \frac{x^2-9}{x-3}$

$f(x) = x+3$ no vertical asymptote

Horizontal Asymptote: A function $y=f(x)$ has the line $y=b$ as a horizontal asymptote if

$\lim_{x \to +\infty} f(x) = b$ or/and $\lim_{x \to -\infty} f(x) = b$



$y = f(x) = \frac{5x-1}{x-3} = 5 + \frac{14}{x-3}$

$\lim_{x \to \infty} f(x) = 5$

Line $y=5$ is horizontal asymptote

Rules: A rational function of the form

$$f(x) = \frac{\text{Pol}^n \text{ of degree } n}{\text{Pol}^n \text{ of degree } m}$$

has a horizontal asymptote if the degree of denominator is higher than equal to the degree of numerator $m \geq n$

$$f(x) = \frac{5x^2-7}{9x^2+3}$$

$$\lim_{x\to\infty} f(x) = \frac{5}{9}$$

$$y = \frac{5}{9}$$

* If $m>n$ then the line $y=0$ is horizontal asymptote
* If $m=n$ then the eq^n of horizontal asymptote is

  y = ratio of leading coefficients.

* If $m<n$ there is no horizontal asymptote.

Ex: $f(x) = \dfrac{5x^3-3}{12x^4+x^2+7}$

$y=0$ is horizontal asymptote.

Ex: $f(x) = \dfrac{5x^3+3x+1}{12x^2+3}$

$\lim_{x\to\infty} f(x)$ diverges   No horizontal asymptote.

Oblique Asymptotes (or Slant Asymptotes)

Defn A function $y=f(x)$ has the line $y=mx+b$ as an oblique (or slant) asymptote if $f(x)=mx+b+g(x)$ and $\lim_{x\to\infty} g(x)=0$

Ex: $f(x)=\dfrac{x^2+1}{x-1}$

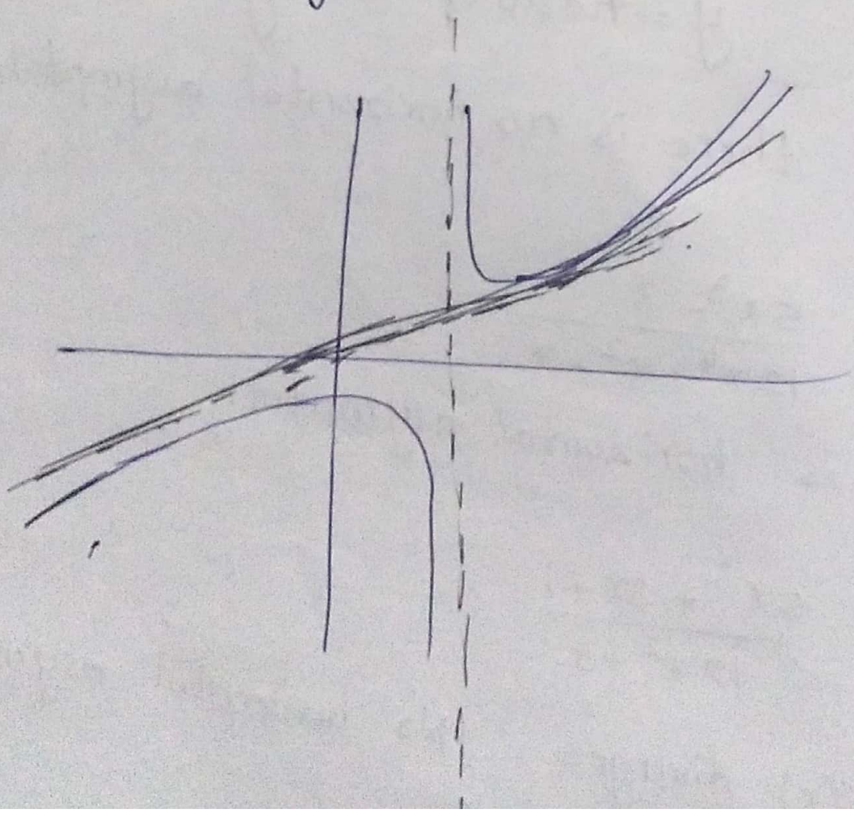
$$f(x) = (x+1)+\frac{3}{(x-1)}$$

$$\lim_{x\to\infty} \frac{3}{x-1} = 0$$

$y=(x+1)$ is oblique asymptote of $f(x)$

* Rational f^n $\dfrac{\text{pol } n}{\text{pol } m}$ $\boxed{n=m+1}$

oblique asymptotes exists iff $n=m+1$



Note:

pol^n $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 x$

$f(x) = \sin x$

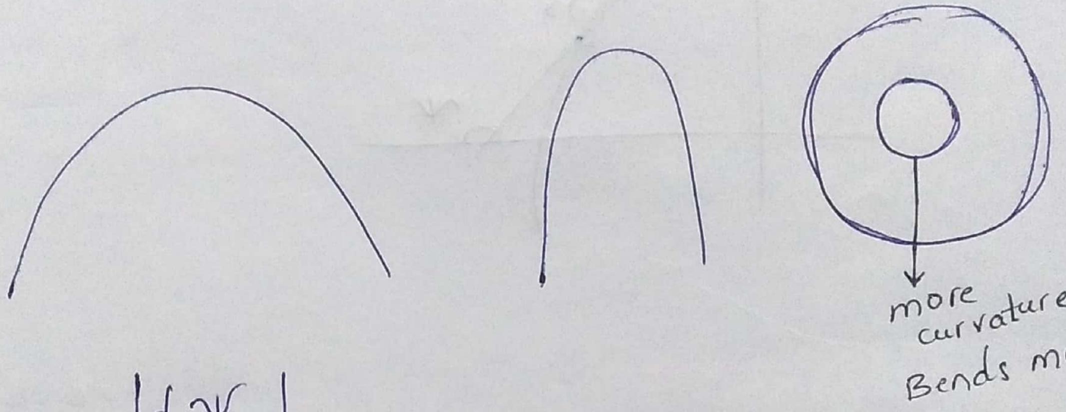
$f(x) = \cos x$

have no asymptotes.

$f(x) = e^{kx}$ $y = 0$ horizontal asymptote

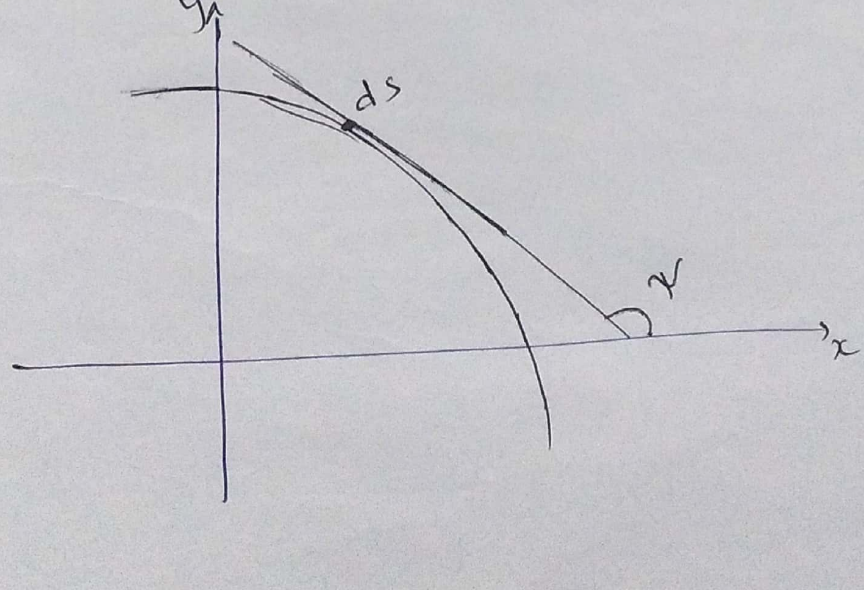
No vertical asymptote.

Curvature:



$$k = \left| \frac{d\psi}{ds} \right|$$

s is the measure of arc length along the curve

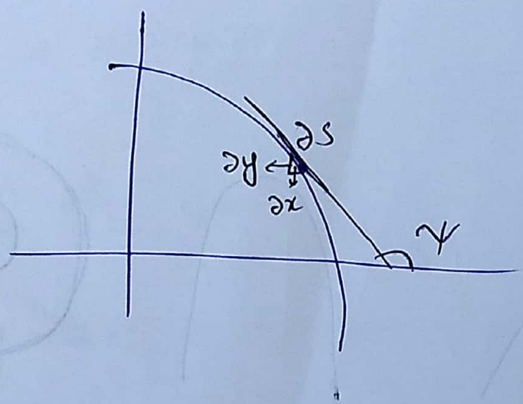


Curvature is the magnitude of the rate of change of ψ w.r.t the distance moved along the curve

$$K = \left|\frac{d\psi}{ds}\right|$$

(Kappa)

$$\frac{d\psi}{ds} = \frac{d\psi}{dx} \Big/ \frac{ds}{dx}$$



$$\partial s^2 = \partial x^2 + \partial y^2$$

$$\left(\frac{\partial s}{\partial x}\right)^2 = 1 + \left(\frac{\partial y}{\partial x}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{ds}{dx} = \left[1 + \{f'(x)\}^2\right]^{1/2}$$

$$\frac{df}{dx} = \tan\psi$$

$$\frac{d^2 f}{dx^2} = \sec^2\psi \frac{d\psi}{dx} = (1 + \tan^2\psi)\frac{d\psi}{dx}$$

$$\frac{d\psi}{dx} = \frac{f''(x)}{1 + (f'(x))^2}$$

$$K = \left|\frac{d\psi}{ds}\right| = \left|\frac{d\psi}{dx} \Big/ \frac{ds}{dx}\right| = \left|\frac{f''(x)}{\left[1 + (f'(x))^2\right]^{3/2}}\right|$$

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At each point on a curve with equation $y = f(x)$, the tangent line turns at a certain rate. A measure of this rate is curvature. K is defined by

$$K = \left| \frac{f''(x)}{[1+(f'(x))^2]^{3/2}} \right|$$

Ex: $f(x) = x^2$

$$\frac{df}{dx} = 2x \qquad \frac{d^2f}{dx^2} = 2$$

$$K = \frac{2}{[1+(2x)^2]^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$$

Circle: $f(x) = (a^2-x^2)^{1/2}$.

$$\frac{df}{dx} = \frac{1}{2}(a^2-x^2)^{-1/2}(-2x)$$

$$\frac{df}{dx} = -\frac{x}{\cancel{2}}(a^2-x^2)^{-1/2}$$

$$\frac{df}{dx} = \frac{-x}{(a^2-x^2)^{\frac{1}{2}}}$$

$$\frac{d^2f}{dx^2} = \frac{1}{2}x(a^2-x^2)^{-3/2}(-2x) - (a^2-x^2)^{-1/2}$$

$$= \frac{-x^2}{(a^2-x^2)^{3/2}} - \frac{1}{(a^2-x^2)^{1/2}}$$

$$\frac{d^2f}{dx^2} = \frac{-a^2}{(a^2-x^2)^{3/2}}$$

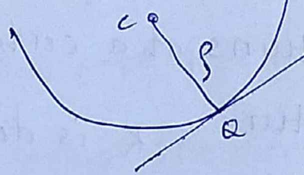
$$1+[f'(x)]^2 = 1+\frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$$

$$K = \left| \frac{\frac{-a^2}{(a^2-x^2)^{3/2}}}{\left(\frac{a^2}{a^2-x^2}\right)^{3/2}} \right| = \frac{1}{a}.$$

Unit of K: radian/m.

Radius of curvature: The reciprocal of the curvature of a curve at any point P is called the radius of curvature at P denoted by ρ

$$\rho = \frac{ds}{d\psi} = \frac{(1+y_1^2)^{3/2}}{y_2}$$



Parametric eq^{n}s

$x = f(t)$, $y = \phi(t)$

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = y'/x'$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}\left[\frac{y'}{x'}\right] \cdot \frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{(x')^2} \cdot \frac{1}{x'}$$

$$\rho = \left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2} \Big/ \frac{x'y''-y'x''}{(x')^3}$$

Ex: Find the radius of curvature at point (i) $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ if the f^n is $x^3+y^3 = 3axy$.

$x^3 + y^3 = 3axy$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

$$(y^2 - ax)\frac{dy}{dx} = 3a - y^2 \rightarrow ①$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -1$$

Differentiating ①

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -\frac{32}{3a}$$

$$\rho_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{[1+(-1)^2]^{3/2}}{(-32/3a)} = \frac{3a}{8\sqrt{2}}$$

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Function of several variables

Limits:

$y = f(x)$ (one variable)

$\lim_{x \to a} f(x) = L$ $\lim_{x\to a^-} f(x) = \lim_{x \to a^+} f(x)$

x can apprach a ~~from~~ in 2 ways (left & right).

$Z = f(x,y)$ Two variable extension.

domain $\subseteq R^2$

$(x,y) \to (a,b)$

(x,y) can approach (a,b) in several different paths (straigh line, parabolic etc).

The function $f(x,y)$ is said to tend to the limit l as $(x,y) \to (a,b)$ if the limit is independent of the path followed by the pointy (x,y) as $(x,y) \to (a,b)$ and

We write $\lim_{\substack{x \to a \\ y \to b}} f(x,y) = l$

The function $f(x,y)$ defined in a region R is said to ~~be~~ tend to the limit l as $x \to a$ and $y \to b$ iff corresponding to a +ve number ϵ, $\exists$ another +ve numbker δ s.t

$|f(x,y) - l| < \epsilon$ for $0 < (x-a)^2 + (y-b)^2 < \delta^2$

For every point (x,y) in R.

Notation:

i) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

ii) $\lim\limits_{\substack{x\to a \\ y\to b}} f(x,y) = L$

iii) $f(x,y)$ approaches L as (x,y) approaches (a,b)

Finding Limits using Numerical Methods:

Ex:

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1.$$

x \ y	-1	-0.5	-0.2	0	0.2	0.5	1
-1.0	0.455	0.759		0.841			
-0.5				0.989			
-0.2				0.999			
0	0.841	0.989	0.999				
0.2							
0.5							
1.0							

Finding Limits Using Analytical Method:

Let L, M and R are real numbers and

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \&$$

$$\lim_{(x,y)\to(a,b)} g(x,y) = M \quad \text{then}$$

The following hold

1) $\lim_{(x,y)\to(a,b)} x = a, \quad \lim_{(x,y)\to(a,b)} y = b$

If c is a const. $\lim_{(x,y)\to(a,b)} c = c$

2) $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$

3) $\lim_{(x,y)\to(a,b)} K f(x,y) = KL$ [K is constant]

4) $\lim_{(x,y)\to(a,b)} [f(x,y) \cdot g(x,y)] = LM$

5) $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad (M \neq 0)$

6) If r and s are integers with no common factors and $s \neq 0$ then

$$\lim_{(x,y)\to(a,b)} [f(x,y)]^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number.

Limits of polynomials and rational f^n's $\dfrac{P(x,y)}{Q(x,y)}$

1) To find the limit of a polynomial we simply plug in the point.

2) To find the limit of a rational function, we plug in the point as long as denominator is not 0.

Ex: $\displaystyle\lim_{(x,y)\to(1,2)} x^6 y + 2xy.$ (polynomial)

$$= 1^6 \cdot 2 + 2(1)(2)$$
$$= 2 + 4$$
$$= 6$$

Ex $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 y}{x^4+y^2}$ (Rational f^n)

$$= \frac{1^2 \cdot 1}{1^4 + 1^2}$$
$$= \frac{1}{2}$$

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x-y)(x^2+xy+y^2)}{(x-y)}$$

$$= \lim_{(x,y)\to(0,0)} x^2+xy+y^2$$

$$= 0$$

Limit along a path

One way to prove that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist is to prove that this limit $(x,y)\to(a,b)$ has different values along two different paths.

Ex: $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ $(x,y)\to(0,0)$

when $x=0$ or $y=0$ $f(x,y)$ is 0.

so the limit of $f(x,y)$ approaching the origin along either the x or y axis is 0.

Along $y=mx$ $f(x,y) = \dfrac{m^2x^3}{x^2+m^4x^4}$

As $x\to 0$ $f(x,y)\to 0$

So along every line through origin $f(x,y)$ approches 0.

Along $x=y^2$ $f(x,y) = \dfrac{y^2\cdot y^2}{y^4+y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}$

Additional Techniques:

Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$

Let $\epsilon > 0$

then $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| = \dfrac{x^2}{x^2+y^2} \, 3|y|$

$\dfrac{x^2}{x^2+y^2} \leq 1$, $|y| \leq \sqrt{x^2+y^2} < \delta$

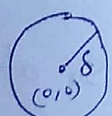
So $\dfrac{x^2}{x^2+y^2} \cdot 3|y| < 3 \cdot 1 \cdot \delta$

We want to force this to be less than ϵ by picking δ "small enough". ~~chose~~

choose $\delta = \epsilon/3$

then $\left|\dfrac{3x^2y}{x^2+y^2}\right| < \epsilon$

$|f(x,y) - l| < \epsilon$ for every $\sqrt{(x-a)^2+(y-b)^2} < \delta$.



Continuity : $f(x)$ is continous at $x=a$ if

$$\lim_{x \to a} f(x) = f(a).$$

$f(x,y)$ is continous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

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Ex: $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ $\quad (x,y) \neq (0,0)$

Prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Let $\epsilon > 0$ be an arbitrary small number

$0 < \sqrt{x^2+y^2} < \delta$

$0 < |x| = \sqrt{x^2} < \sqrt{x^2+y^2} < \delta$

$0 < |x| < \delta$

$0 < |y| < \delta$

$$|f(x,y) - 0|$$

$$= \left| \frac{xy(x^2-y^2)}{x^2+y^2} - 0 \right|$$

$$= |xy| \left| \frac{x^2-y^2}{x^2+y^2} \right| \qquad \left| \frac{x^2-y^2}{x^2+y^2} \right| \leq 1$$

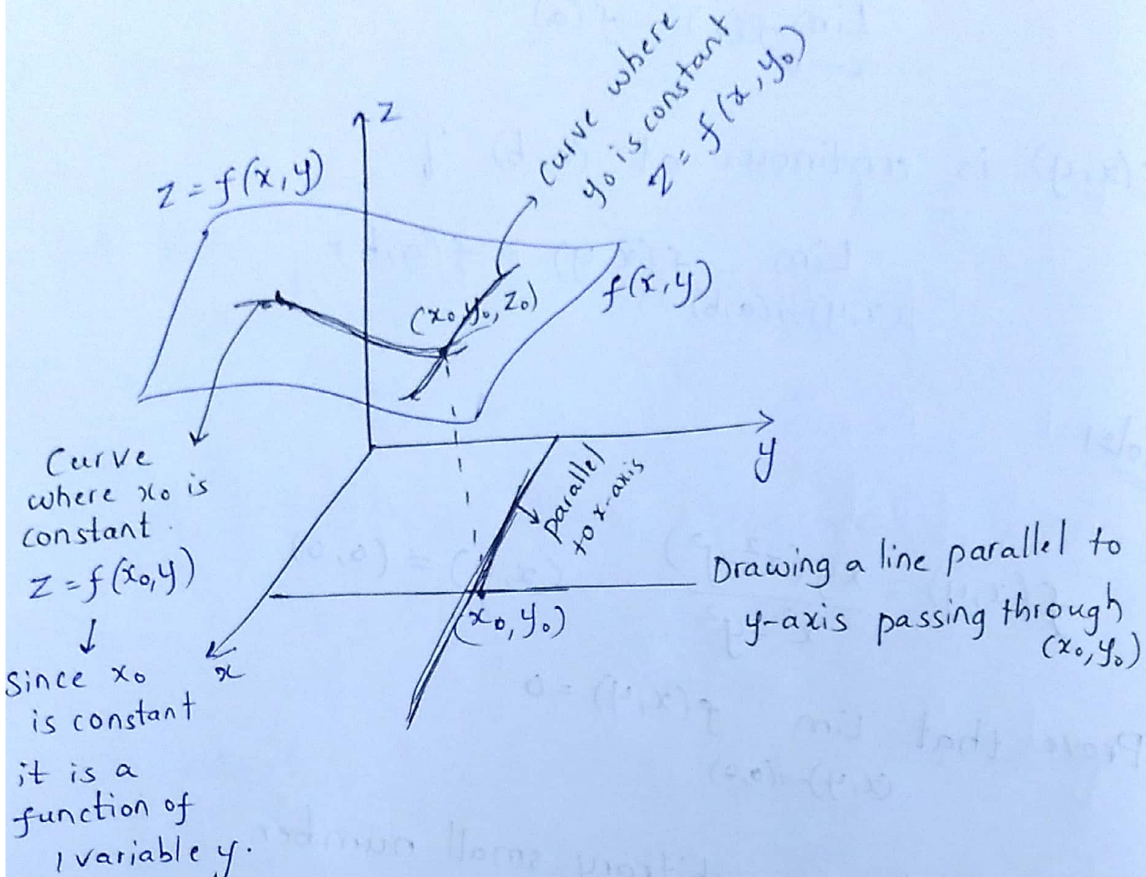
$$\leq |xy| = |x||y|$$

$$\leq \delta^2$$

choose $\delta = \sqrt{\epsilon} \implies |f(x,y) - 0| < \epsilon$

So $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Partial Derivatives



First order partial derivative:

Let $z = f(x,y)$ be a f^n of 2 variables x & y

If we keep y as const and vary x alone, z is a f^n of x only. The derivative of z with respect to x, treating y as const, is called the partial derivative of z w.r.t x and is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, $f_x(x,y)$, $D_x f$

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}.$$

f_x & f_y ~~ar~~ can be differentiated further partially w.r.t x and y.

Thus $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or f_{xx}

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or $\cancel{f_{xy}}$ f_{yx}

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ $\qquad$ $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}$

$$f_{xy} = f_{yx}$$

⇒ which variable to be treated as const

$x = r\cos\theta, \quad y = r\sin\theta$

$r = x\sec\theta \quad , \quad r^2 = x^2 + y^2$

$\left(\frac{\partial r}{\partial x}\right)_\theta = \sec\theta \qquad \frac{\partial r}{\partial x}$

Convention: $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x}\right)_y \qquad \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y}\right)_x$

Ex: Find the first and second order partial derivatives of $z = x^3 + y^3 - 3axy$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2 - 3ay\right)$

$= 6x$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(3y^2 - 3ax\right)$$

$$= 6y$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 - 3ay\right)$$

$$= -3a$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(3y^2 - 3ax\right)$$

$$= -3a$$

Total Derivative

$$u = f(x, y)$$

$$x = \phi(t), \quad y = \psi(t) \qquad u = f(\phi(t), \psi(t))$$

Chain Rule: $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

If $u = f(x, y, z)$

x, y, z are f^n's of t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

Ex: $u = \sin(x/y)$

$x = e^t$ and $y = t^2$

Find $\frac{du}{dt} = ?$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{y}\cos(x/y)\cdot e^t + \left(\frac{-x}{y^2}\right)\cos(x/y)(2t)$$

$$\frac{du}{dt} = \left(\frac{e^t}{y} - \frac{2xt}{y^2}\right)\cos(x/y) = \left(\frac{e^t}{t^2} - \frac{2e^t \cdot t}{t^4}\right)\cos\left(\frac{e^t}{t^2}\right)$$

$$= \frac{(t-2)e^t}{t^3}\cos\left(\frac{e^t}{t^2}\right)$$

$$u = \sin\left(\frac{e^t}{t^2}\right)$$

$$\frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right)\cdot\left(\frac{-e^t(2t) + t^2 e^t}{t^4}\right)$$

$$= \cos\left(\frac{e^t}{t^2}\right)\left(\frac{-2+t}{t^3}\right)e^t$$

$$= \cos\left(\frac{e^t}{t^2}\right)\cdot\left(\frac{t-2}{t^3}\right)e^t$$

Extension of the chain Rule:

$$z = f(x,y) \qquad \phi = \phi(s,t) \qquad y = \psi(s,t)$$

$$z = f(\phi(s,t), \psi(s,t))$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex. $z = x^2 + y^2$

$x = r\cos\theta$ Find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$

$y = r\sin\theta$.

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= 2x\cos\theta + 2y\sin\theta = 2\frac{x^2}{r} + 2\frac{y^2}{r}$$

$$= \frac{2}{r}(x^2+y^2) = 2r$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= 2x\,\cancel{\cos\theta}(-r\sin\theta) + 2y\,r\cos\theta$$

$$= -2rx\frac{y}{r} + 2yr\cdot\frac{x}{r} = 0$$

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Find Limit of

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})}$$

$$\lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y})$$

$$= 0$$

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$= 0 \qquad (x,y) = (0,0)$$

$x = 0$
$y = 0$
$x = y$
$x = y^2$

Is f continuous at $(0,0)$?

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=0}} \frac{x^2-y^2}{x^2+y^2} = -1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^2-y^2}{x^2+y^2} = 1$$

$\therefore$ Limit does not exist at $(0,0)$

$\therefore$ It is not continuoust at $(0,0)$

A f^n $f(x,y)$ is continuous at a pt (a,b) if the following is true:

i) (a,b) is in the domain of f.

ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Find $\lim\limits_{(x,y)\to(1,0)} f(x,y)$

$$f(x,y) = \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}$$

Along $x=1 \longrightarrow 0$

Along $y=0 \longrightarrow 0$

Along $y=(x-1) \longrightarrow 0$

Squeeze theorem : Additional Technique

$$g(x,y) \le f(x,y) \le h(x,y)$$

if $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = L$

then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$

$g(x,y) = 0 \qquad h(x,y) = \ln x$

$$\lim_{(x,y)\to(1,0)} \ln x = 0$$

$$\therefore \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$$

$$z = xe^{xy}, \quad x = t^2, \; y = t^{-1}$$

Find $\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = xye^{xy} + e^{xy} = t^2 \cdot \frac{1}{t} \cdot e^t + e^t$$

$$= te^t + e^t$$

$$\frac{\partial z}{\partial y} = x^2 \cdot e^{xy} = t^4 e^t$$

$$\frac{dz}{dt} = (te^t + e^t) \cdot (2t) + (t^4 e^t) \cdot \left(\frac{-1}{t^2}\right)$$

$$= 2t^2 e^t + 2te^t - t^2 e^t$$

$$= t^2 e^t + 2te^t$$

$$= te^t (t+2)$$

Special case: $z = f(x, y) \qquad y = g(x)$

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$Z = x\log(xy) + y^3 \qquad y = \cos(x^2+1)$$

$$\frac{\partial z}{\partial x} = x \frac{1}{xy} \cdot y + \log(xy)$$

$$= 1 + \log(xy)$$

$$\frac{\partial z}{\partial y} = x \frac{1}{xy} \cdot x + 3y^2 = \frac{x}{y} + 3y^2$$

$$\frac{dy}{dx} = -\sin(x^2+1)\,(2x)$$

$$= -2x\sin(x^2+1)$$

$$\frac{dz}{dx} = 1+\log(xy) + \left(\frac{x}{y}+3y^2\right)(-2x\sin(x^2+1))$$

$$= 1+\log(x\cos(x^2+1)) + \left(\frac{x}{\cos(x^2+1)} + 3\cos(x^2+1)\right)(-2x\sin(x^2+1))$$

$$= 1+\log(x\cos(x^2+1)) - 2x^2\tan(x^2+1) - 6x\cos(x^2+1)\sin(x^2+1)$$

Implicit differentiation:

$$f(x,y) = 0 \qquad y = f(x)$$

$$0 = f_x + f_y \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$x\cos(3y) + x^3y^5 = 3x - e^{xy}$$

$$f(x,y) = x\cos(3y) + x^3y^5 - 3x + e^{xy}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\cos 3y + 3x^2y^5 - 3 + ye^{xy}}{-3x\sin(3y) + 5x^3y^4 + xe^{xy}}$$

Implicit f^n of 3 variables

$$f(x,y,z) = 0$$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ for $x^2\sin(2y-5z) = 1 + y\cos(6zx)$

$$f(x,y,z) = x^2\sin(2y-5z) - 1 - y\cos(6zx)$$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = -\frac{2x\sin(2y-5z) + 6zy\sin(6zx)}{-5x^2\cos(2y-5z) + 6xy\sin(6zx)}$$

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Jacobians

If u and v are f^ns of two independent variables x and y, then the determinant

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called

the Jacobian of u,v w.r.t x,y.

u, v, w w.r.t x, y, z

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

If u,v are f^ns of r,s and r,s are f^ns of x, y

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

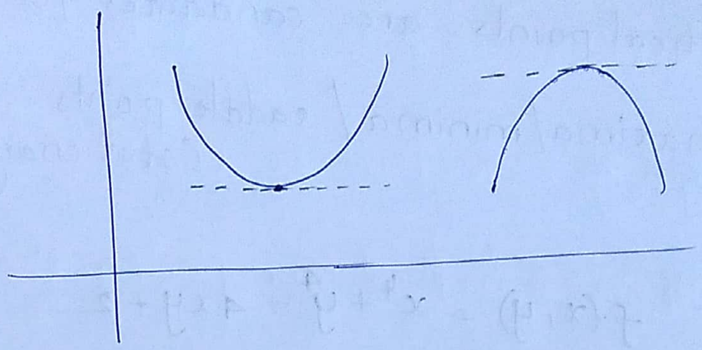
Ex: $\frac{\partial(x,y)}{\partial(r,\theta)} = ?$ $\quad x = r\cos\theta$, $y = r\sin\theta$

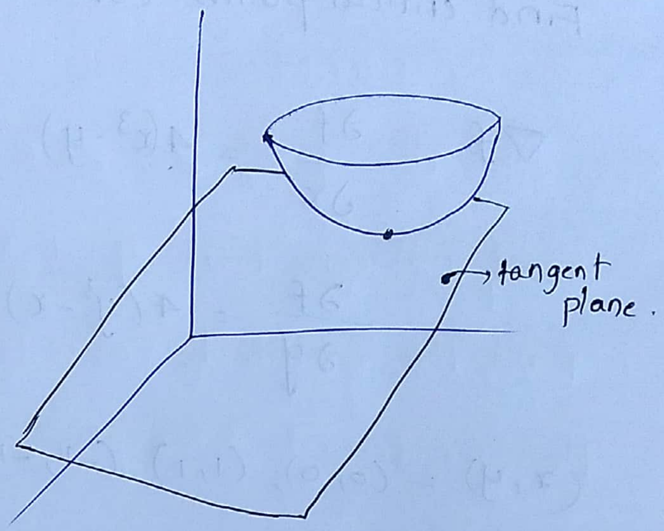
$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$y = f(x)$

$$\frac{dy}{dx} = 0$$

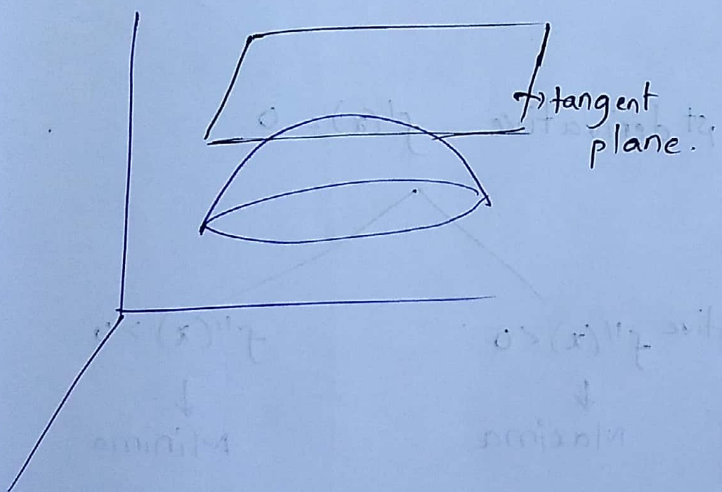
(For maxima or minima)





$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$



Maxima

Minima

Saddle point.

$z = f(x, y)$

A point (a,b) in the plane is called a critical point of a $f^n(x, y)$ if $\nabla f(a,b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ $\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

Critical points are candidates for extrema

Local maxima/minima/saddle points
(stationary points)

Ex1 $f(x,y) = x^4 + y^4 - 4xy + 2$

Find critical points (stationary points)

$\nabla f \quad \dfrac{\partial f}{\partial x} = 4(x^3 - y)$

$\dfrac{\partial f}{\partial y} = 4(y^3 - x)$

$(x,y) = (0,0), (1,1), (-1,-1)$ are critical points

$y = f(x)$ 1st derivative $f'(x) = 0$

2nd derivative $f''(x) < 0$ $\downarrow$ Maxima

$f''(x) > 0$ $\downarrow$ Minima

2nd Derivative Test:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \rightarrow \text{Hessian matrix.}$$

$$D = f_{xx} f_{yy} - f_{xy}^2 \quad \text{(discriminant)}$$

$$H \begin{cases} \rightarrow \text{positive definite} \\ \rightarrow \text{Negative definite} \end{cases}$$

Second Derivative test: Assume (a, b) is a critical point for $f(x, y)$

i) If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a minima

ii) If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a maxima

iii) If $D < 0$ then (a, b) is a saddle point

$D = 0$ undetermined, needs further study.

Ex 2 $f(x, y) = \frac{x^3}{3} - x - \left(\frac{y^3}{3} - y\right)$

Find critical points

classify the critical points

∇f $\quad \frac{\partial f}{\partial x} = x^2 - 1 \Rightarrow x = \pm 1$

$\frac{\partial f}{\partial y} = -(y^2 - 1) \Rightarrow y = \pm 1$

Critical points

$(1, 1)$

$(1, -1)$

$(-1, 1)$

$(-1, -1)$

$f_{xx} = 2x \qquad f_{yy} = -2y$

$f_{xy} = 0 \qquad f_{yx} = 0$

$$H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$$

$D = -4xy$

At $(1, 1) \rightarrow$ saddle point

$(1, -1) \rightarrow$ Minima

$(-1, 1) \rightarrow$ Maxima

$(-1, -1) \rightarrow$ saddle point.

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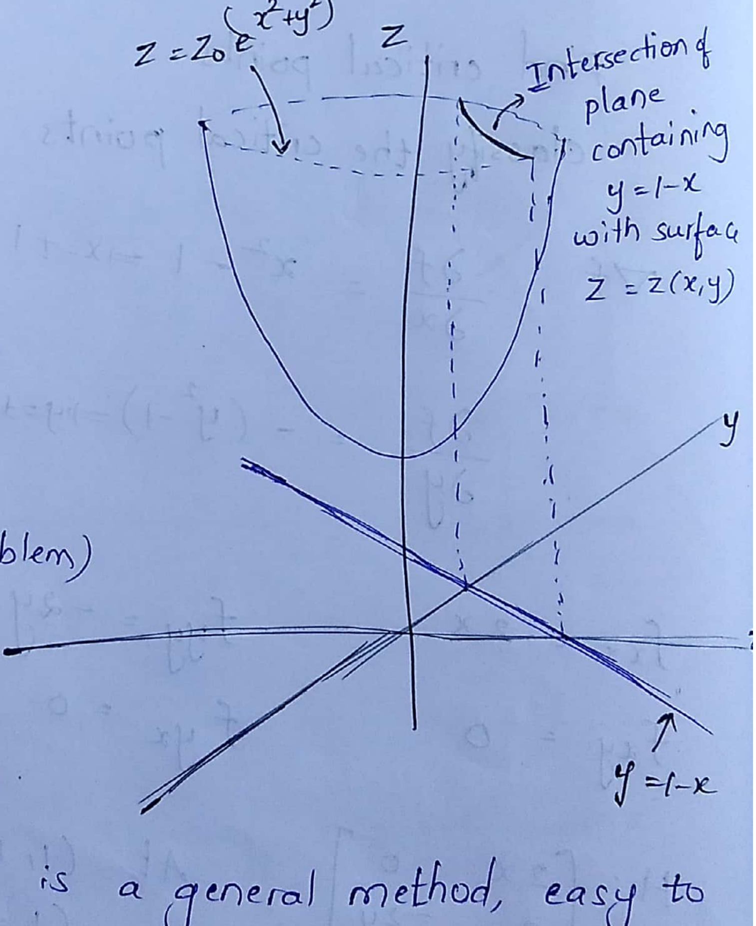
Lagrange's method of Undetermined Multipliers

It is a method for finding the minimum or maximum value of a function subject to one or more constraints.

→ Consider the f^n : $Z = Z_0 e^{(x^2+y^2)}$

Additional condition: $y = 1-x$.

Find the smallest value of z consistent with this constraint. This plane cuts the surface in a curve.



Straight forward sol^n

$$Z = Z_0 e^{x^2+y^2}$$
$$= Z_0 e^{x^2+(1-x)^2}$$

Find minimum
(one variable problem)

this may become difficult sometimes.

Lagrange's method is a general method, easy to apply and which is readily extended to multiple constraint cases.

More general problem

Find (x,y) that minimize (or maximize) a f^n $h=h(x,y)$

subject to a constraint of the form $g(x,y)=c$

If we parameterize this problem, we have

$$h = h(x,y) = h\left(x(t), y(t)\right) = h(t)$$

$$c = g(x,y) = g(t)$$

$$\frac{dc}{dt} = \frac{dg}{dt} = 0$$

The solution we seek is the point at which 'h' is

an extremum $\frac{dh}{dt} = 0$.

$$\frac{dh}{dt} = \left(\frac{\partial h}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial h}{\partial y}\right)_x \frac{dy}{dt} = 0 \longrightarrow ①$$

$$\frac{dg}{dt} = \left(\frac{\partial g}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial g}{\partial y}\right)_x \frac{dy}{dt} = 0 \longrightarrow ②$$

$$0 = \frac{dh}{dt} - \lambda\frac{dg}{dt} = \left(\frac{\partial h}{\partial x} - \lambda\frac{\partial g}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial h}{\partial y} - \lambda\frac{\partial g}{\partial y}\right)_x \frac{dy}{dt}$$

$$\frac{\partial h}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0$$

$$\frac{\partial h}{\partial y} - \lambda\frac{\partial g}{\partial y} = 0$$

$$g(x,y) = c$$

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Ex 1 Find the minimum and maximum value of

$f(x,y) = x+y$, given $x^2+y^2=1$

$f(x,y) = x+y$

$g(x,y) = x^2+y^2-1$

$f_x = 1 \qquad g_x = 2x$

$f_y = 1 \qquad g_y = 2y$

$1 = \lambda 2x \rightarrow (1)$

$1 = \lambda 2y \rightarrow (2)$

$x^2+y^2 = 1 \rightarrow (3)$

$\frac{(1)}{(2)} \Rightarrow \frac{x}{y} = 1$

$x = y$

$2x^2 = 1$

$x = \pm \frac{1}{\sqrt{2}}$

$y = \pm \frac{1}{\sqrt{2}}$

sol^{n}s $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

So f is maximum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

minimum at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Extension to $f(x, y, z)$ will have 4 eqn s to solve

$$f_x - \lambda g_x = 0$$
$$f_y - \lambda g_y = 0$$
$$f_z - \lambda g_z = 0$$
$$g(x, y, z) = 0$$

$h = f(x, y, z)$

$(x_1, y_1) \to f(x_1, y_1)$
$(x_2, y_2) \to f(x_2, y_2)$
$(x_3, y_3) \to f(x_3, y_3)$

Ex2 $f(x, y, z) = x^m y^n z^p$

Find maximum value of f given that $x+y+z=a$

$f(x,y,z) = x^m y^n z^p$
$g(x,y,z) = x+y+z-a$

$$f_x = m x^{m-1} y^n z^p$$
$$g_x = 1$$
$$f_y = n x^m y^{n-1} z^p$$
$$g_y = 1$$
$$f_z = p x^m y^n z^{p-1}$$
$$g_z = 1$$

$m x^{m-1} y^n z^p = \lambda \to (1)$

$n x^m y^{n-1} z^p = \lambda \to (2)$

$p x^m y^n z^{p-1} = \lambda \to (3)$

$\frac{(1)}{(3)} \Rightarrow \frac{m}{p} \frac{z}{x} = 1$

$mz = px$

$\frac{m}{x} = \frac{z}{p}$

$\frac{(1)}{(2)} \Rightarrow \frac{m}{n} \cdot \frac{y}{x} = 1$

$my = nx$

$\frac{m}{x} = \frac{n}{y}$

$\frac{(2)}{(3)} \Rightarrow \frac{n}{p} \cdot \frac{z}{y} = 1$

$nz = py$

$\frac{n}{y} = \frac{z}{p}$

$$-\lambda = m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z}$$

$$\frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$$

sol

$$x = am(m+n+p)$$

$$y = an(m+n+p)$$

$$z = ap(m+n+p)$$

Maximum value of $f(x,y,z) = \dfrac{a^{m+n+p} \cdot m^m n^n p^p}{(m+n+p)^{m+n+p}}$

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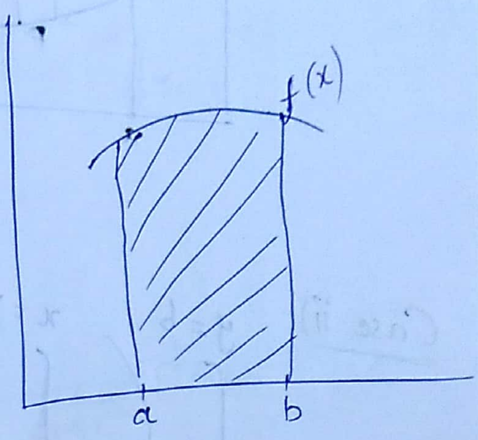
Multiple Integral

Double Integral:

$$y = f(x) \Rightarrow \int_a^b f(x)\,dx = I$$

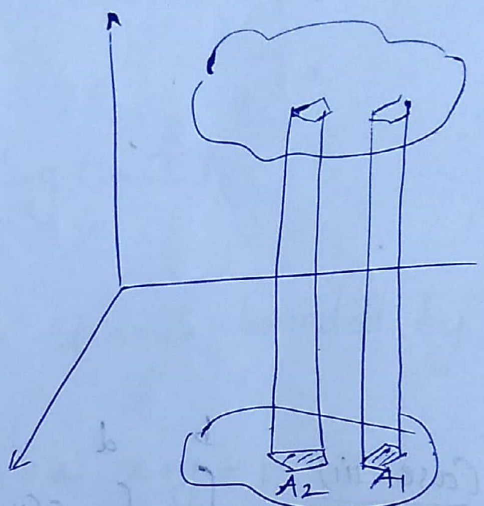


$$S = f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \dots + f(x_n)\,\delta x_n$$

as $n \to \infty$ $S \to I$

$$I = \iint_A f(x,y)\,dx\,dy$$

$$= f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \dots + f(x_n,y_n)\,\delta A_n$$

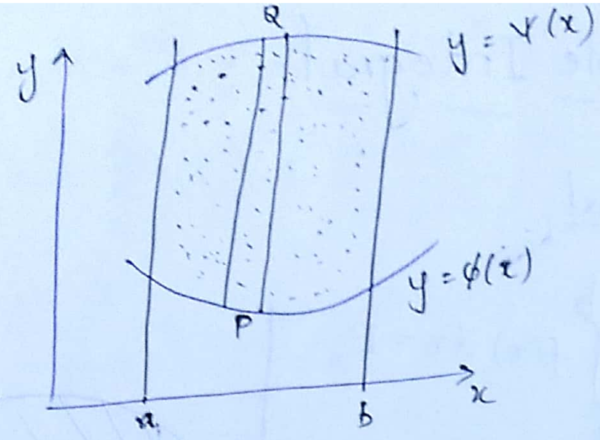


as $n \to \infty$ $\delta A_n \to 0$

$$\iint_A f(x,y)\,dA = \underset{\substack{n\to\infty \\ \delta A_n \to 0}}{Lt} \sum_{r=1}^{\infty} f(x_r, y_r)\,\delta A_r$$

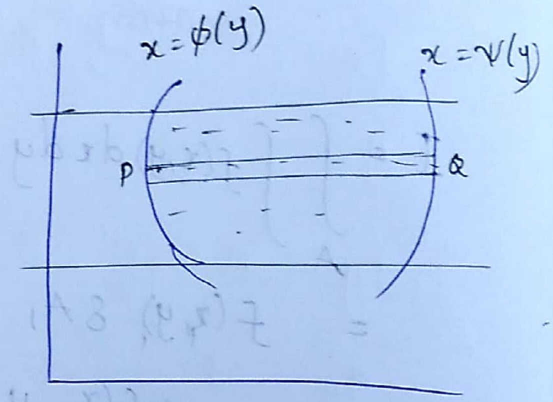
Case i:

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left(\int_{y=\phi(x)}^{y=\psi(x)} f(x,y)\,dy \right) dx.$$



<u>Case ii)</u>

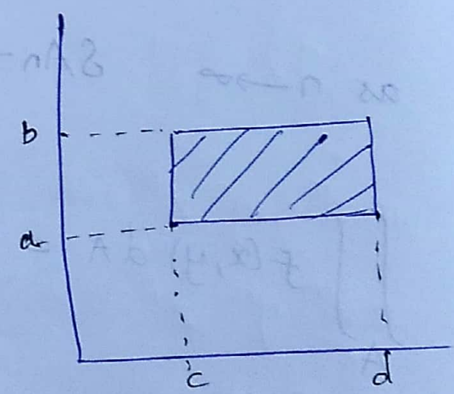
$$\int_{y=a}^{y=b} \left(\int_{x=\phi(y)}^{x=\psi(y)} f(x,y)\,dx \right) dy$$



<u>Case iii)</u>

$$\int_a^b \int_c^d f(x,y)\,dx\,dy$$

independent of order of integration.



Ex:
$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2+y^2)}\, dy\, dx$$

$$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

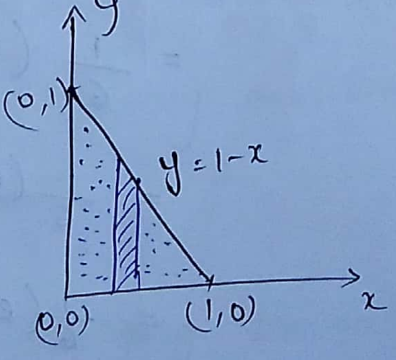
$$\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4}\, dx = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4} \left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4} \log(1+\sqrt{2})$$

Ex 2 $\iint e^{2x+3y}\, dx\, dy$ over the triangle bounded by the lines $x=0$, $y=0$ & $x+y=1$

$0 \le x \le 1$, $0 \le y \le 1-x$



$$\int_0^1 \int_0^{1-x} e^{2x+3y}\, dy\, dx$$

$$\int_0^1 \frac{1}{3}\left[e^{2x+3y}\right]_0^{1-x} dx$$

$$\frac{1}{3}\int^1 \left(e^{2x+3-3x} - e^{2x}\right) dx$$

$$= \frac{1}{3}\int_0^1 \left(e^{-x+3} - e^{2x}\right) dx$$

$$= \frac{1}{3}\left[-e^{-x+3} - \frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{1}{3}\left[-e^{2} + e^{3} - \frac{e^{2}}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{3}\left[\frac{-3e^{2}}{2} + e^{3} + \frac{1}{2}\right]$$

$$= \frac{1}{6}\left[-3e^{2} + 2e^{3} + 1\right]$$

$$= \frac{1}{6}\left(2e^{3} - 3e^{2} + 1\right)$$

$$= \frac{1}{6}\left(2e^{3} - 2e^{2} - e^{2} + 1\right)$$

$$= \frac{1}{6}\left(2e^{2}(e-1) - (e^{2}-1)\right)$$

$$= \frac{1}{6}\left(2e^{2}(e-1) - (e+1)(e-1)\right)$$

$$= \frac{1}{6}(e-1)\left(2e^{2} - e - 1\right)$$

$$= \frac{1}{6}(e-1)\left(2e^{2} - 2e + e - 1\right)$$

$$= \frac{1}{6}(e-1)\left(2e(e-1) + 1(e-1)\right)$$

$$= \frac{1}{6}(e-1)^{2}(2e+1)$$

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Change of Order of Integration:

The concept of change of order of integration evolved to help in handling typical integrals occuring in evaluation of double integrals.

When the limits of given integral

$$\int_a^b \int_{y=\phi(x)}^{y=\psi(x)} f(x,y)\,dy\,dx$$ are clearly drawn and

demarcated then ~~the~~ the region of integration is we can change the order of integration by performing integration first w.r.t x as a f^n of y and then w.r.t y from c to d.

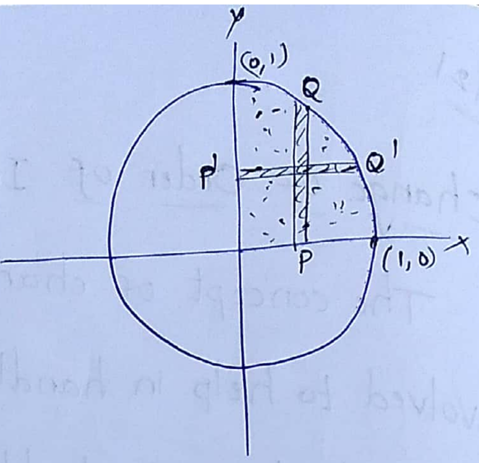
$$I = \int_c^d \int_{x=\phi(y)}^{x=\psi(y)} f(x,y)\,dx\,dy$$

Sometimes the demarcated region may have to be split into two-to-three parts for defining new limits for each region in the changed order.

Ex1 Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$ by changing the order of integration.

sol

$$I = \int_0^1 y^2 \left(\int_0^{\sqrt{1-y^2}} dx \right) dy$$



$$= \int_0^1 y^2 \, [x]_0^{\sqrt{1-y^2}} \, dy$$

$$= \int_0^1 y^2 (1-y^2)^{1/2} \, dy$$

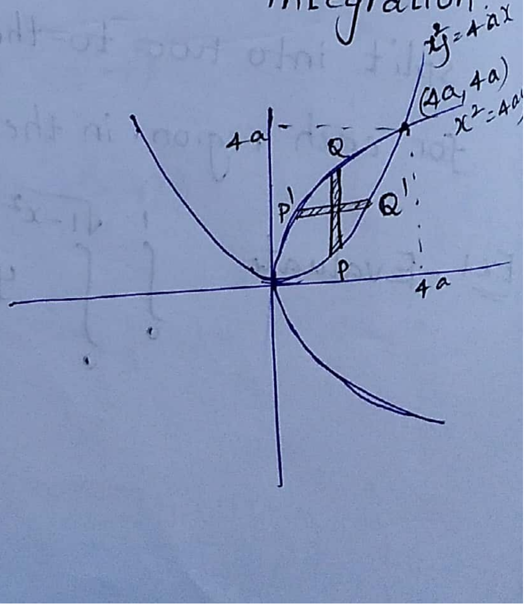
$$= \int_0 \quad \text{Substitute } y = \sin\theta, \quad dy = \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= \frac{(2-1)\cdot(2-1)}{4\cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{16}$$

Ex 2 Find $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \, dx$ by changing the order of integration.



$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy \, dx.$$

$$\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx \, dy$$

$$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}} \, dy$$

$$= \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left[2\sqrt{a} \, \frac{y^{3/2}}{3/2} - \frac{y^3}{4a} \right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,\sqrt{(4a)^3} - \frac{(4a)^3}{4a}$$

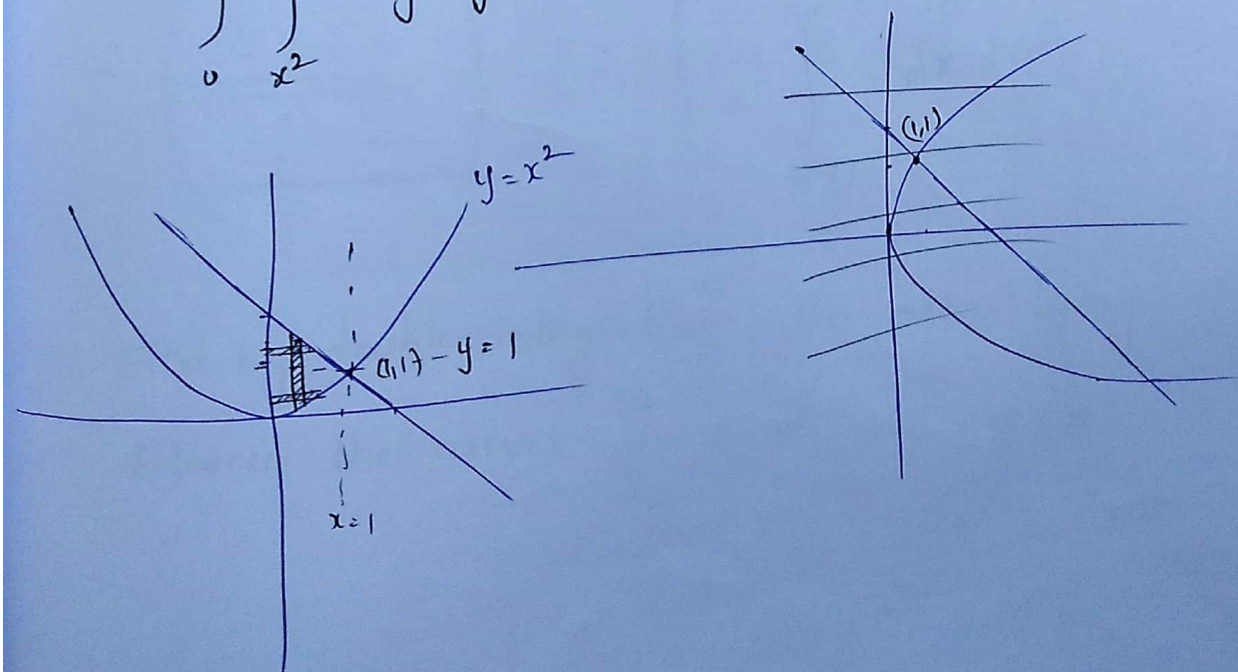
$$= \frac{4}{3} \cdot 2 \cdot (4a^2) - 16a^2$$

$$= \frac{32}{3}a^2 - 16a^2$$

$$= \frac{16}{3}a^2$$

Ex3 change the order of integration of

$$\int_0^1 \int_{x^2}^{2-x} xy \, dy\, dx \quad \text{and Evaluate.}$$



$$\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy + \int_1^2 \int_0^{2-y} xy \, dx \, dy$$

$$= \int_0^1 y \cdot \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 y \cdot \left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_0^1 \frac{y^2}{2} dy + \int_1^2 \frac{y(2-y)^2}{2} dy$$

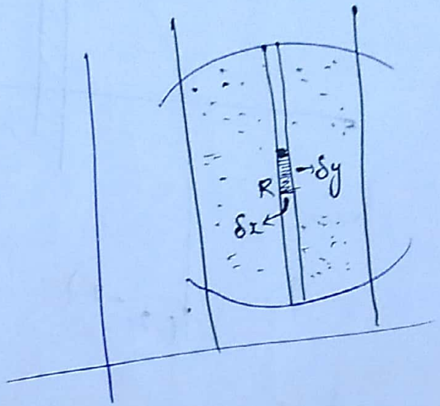
$$= \left[\frac{y^3}{3 \cdot 2}\right]_0^1 + \int_1^2 \frac{y(4+y^2-4y^2)}{2} dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{3}{8}.$$

Area enclosed by plane curve:

Consider the area bounded by the two const curves $y=\phi(x)$ & $y=\psi(x)$ & the two ordinates $x=a$, $x=b$



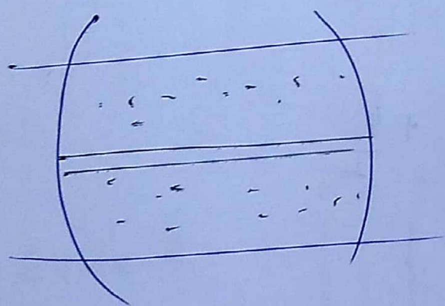
$R(x,y)$ $S(x+\delta x, y+\delta y)$

Area in shade $= \delta x\, \delta y$.

All such small rectangles on PQ are of width δx and y changes from $\phi(x)$ to $\psi(x)$

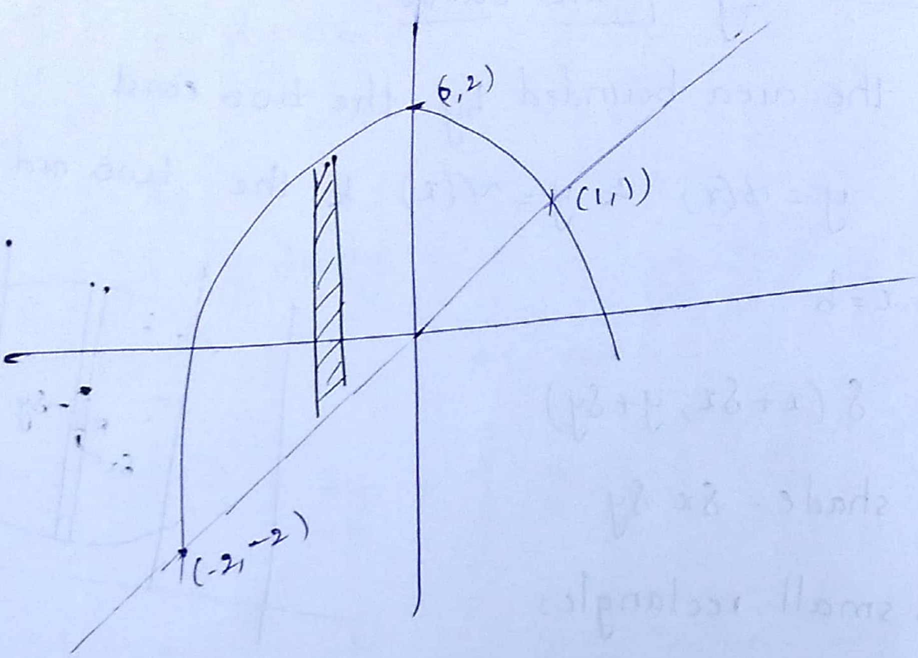
$$\text{Area of PQ} = \lim_{\delta y \to 0} \sum \delta x\, \delta y = \delta x \lim_{\delta y \to 0} \sum_{\phi(x)}^{\psi(x)} \delta y$$

$$= \delta x \int_{\phi(x)}^{\psi(x)} dy.$$



$$\int_a^b \int_{\phi(y)}^{\psi(y)} dx\, dy.$$

Find by double integration, the area lying between the curves $y=2-x^2$ and $y=x$



$$A = \int_{-2}^{1} \left(\int_{x}^{2-x^2} dy \right) dx$$

$$= \int_{-2}^{1} (2 - x^2 - x)\, dx$$

$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^{1}$$

$$= 2 - \frac{1}{3} - \frac{1}{2} + 4 \cancel{+} - \frac{8}{3} + \frac{4}{2}$$

$$= \cancel{10+1}\ 4 \cancel{-\frac{7}{3}} = 5 - \frac{1}{2}$$

$$= \frac{9}{2}$$

Triple Integral

$$\iiint f(x,y,z)\, dV = \sum_{r=1}^{n} F(x_r, y_r, z_r)\, \delta V_r$$

if exists as $x \to \infty$, $\delta V_r \to 0$

For evaluation purpose

$\iiint_V f(x,y,z)\, dV$ is expressed as repeated

integrals $\displaystyle\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dz\, dy\, dx$.

$$I = \int_{x=a}^{x=b}\int_{y=\phi_1(x)}^{y=\phi_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

5/11/21

Ex: Evaluate $\displaystyle\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$\int_0^a\int_0^x e^{x+y}\left(\int_0^{x+y} e^z\, dz\right) dy\, dx$$

$$\int_0^a\int_0^x e^{x+y}\cdot\left[e^{x+y}-1\right] dy\, dx$$

$$= \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy\, dx$$

$$= \int_0^a e^{2x} \int_0^x \left(e^{2y}\, dy\right) dx - \int_0^a e^{x} \int_0^x \left(e^{y}\, dy\right) dx$$

$$= \int_0^a e^{2x} \frac{\left(e^{2x} - 1\right)}{2} dx - \int_0^a e^{x} \left(e^{x} - 1\right) dx$$

$$= \int_0^a \frac{\left(e^{4x} - e^{2x}\right)}{2} dx - \int_0^a \left(e^{2x} - e^{x}\right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x}\right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x}\right]_0^a$$

$$= \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a}\right) - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \left(\frac{1}{4} + \frac{1}{8}\right)$$

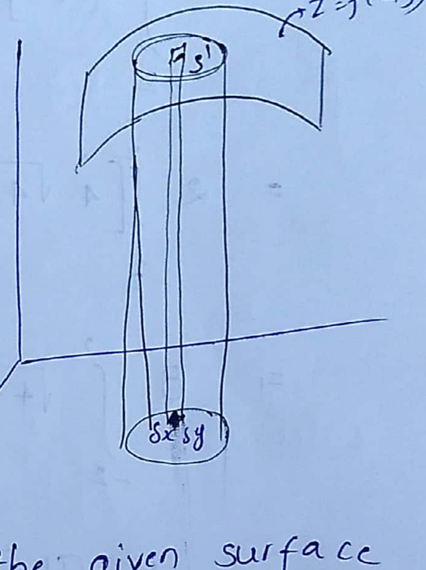
$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$$

Volumes as double integrals

Surface $z = f(x,y)$

A portion $s' \rightarrow$ orthogonal projection on XY plane is S.

Divide S into elementary rectangles of area $\delta x \, \delta y$ by drawing lines parallel to X and Y axes with each of these rectangles as base, draw prism having its length ∥ to Z-axes.



∴ Volume of this prism between S & the given surface $z = f(x,y)$ is $z \, \delta x \, \delta y$.

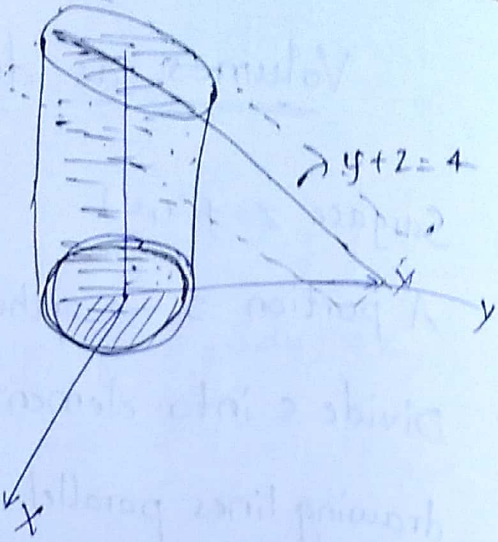
Hence the volume of the solid on S as base, bounded by the given surface with generators ∥ to Z-axis

$$= \iint_S z \, dx \, dy$$

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum \sum z \, \delta x \, \delta y = \iint_S f(x,y) \, dx \, dy$$

Ex1 Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ & $z = 0$

$z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy plane



$$V = 2 \int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} (4-y)\, dx\, dy$$

$$= 2 \int_{-2}^{2} (4-y) \sqrt{4-y^2}\, dy$$

$$= 2 \int_{-2}^{2} \Big[4\sqrt{4-y^2} - \underbrace{y\sqrt{4-y^2}}_{\text{odd fun}^c} \Big] dy$$

$$= 8 \int_{-2}^{2} \sqrt{4-y^2}\, dy - 0$$

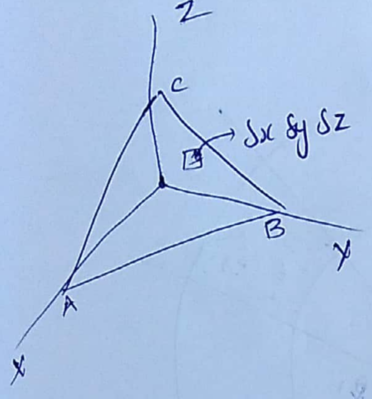
$$= 8 \left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2} \sin^{-1} \frac{y}{2} \right]_{-2}^{2}$$

$$= 16\pi$$

Volume as triple integral:

Divide the given solid by planes ∥ to the coordinate planes into rectangular parallelopiped of volume $\delta x\, \delta y\, \delta z$.

$\therefore$ The total volume $= \lim\limits_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \sum\sum\sum \delta x\, \delta y\, \delta z$



$$= \iiint 1 \cdot dx\, dy\, dz$$

with appropriate limits of integration.

Ex: Calculate the volume of the solid bounded by $x=0$, $y=0$, $x+y+z=a$ and $z=0$.

$$V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$$

$$= \int_0^a \int_0^{a-x} (a-x-y)\, dy\, dx$$

$$= \int_0^a \left[(a-x)y - \frac{y^2}{2}\right]_0^{a-x} dx = \int_0^a \left((a-x)^2 - \frac{(a-x)^2}{2}\right) dx$$

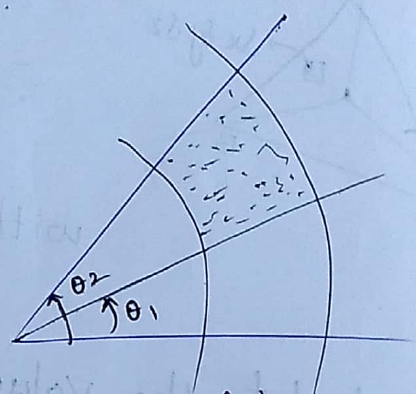
$$= \frac{1}{2} \int_0^a (a-x)^2\, dx$$

$$= \frac{1}{2}\left[-\frac{(a-x)^3}{3}\right]_0^a$$

$$= \frac{a^3}{6}.$$

Double integral in Polar Coordinates:

$$\int_{\theta_1}^{\theta_2} \int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} f(r,\theta)\, dr\, d\theta.$$



$r_1 = f_1(\theta)$ $\quad r_2 = f_2(\theta)$

Calculate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ & $r = 4\sin\theta$.

Change of Variables

Appropriate choice of coordinates often eases evaluation of a double or a triple integral.

Double integral $x, y \to u, v$

$$x = \phi(u, v)$$
$$y = \psi(u, v)$$

Then $$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{uv}} f(\phi(u,v), \psi(u,v))\,|J|\,du\,dv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Triple Integrals

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R_{uvw}} f(\phi_1(u,v,w), \phi_2(u,v,w), \phi_3(u,v,w))\,|J|\,du\,dv\,dw$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$x = \phi_1(u,v,w)$$
$$y = \phi_2(u,v,w)$$
$$z = \phi_3(u,v,w)$$

Particular case: Cartesian to polar

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r \qquad x = r\cos\theta \qquad y = r\sin\theta$$

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

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Ex:1 Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

Hence show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\frac{\pi}{2}}$

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta.$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^\infty e^{-r^2}(-2r)\,dr\right\}d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[e^{-r^2}\right]_0^\infty d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy = \frac{\pi}{4}$$

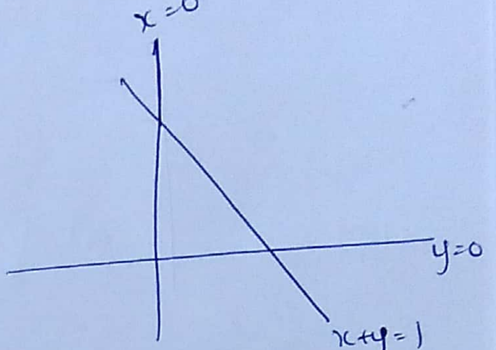
$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Ex 2: Evaluate

$$\iint_D xy\left(\sqrt{(1-x-y)}\right)\,dx\,dy$$

where D is the region bounded by $x=0$, $y=0$, $x+y=1$ using the transformation

$$x+y=u, \quad y=uv$$



$x = u - uv = u(1-v)$ $\qquad$ $u = 0$ to 1

$y = uv$ $\qquad$ $y = 0$ to 1

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

$$\int_0^1\int_0^1 xy\sqrt{1-x-y}\,dx\,dy$$

$$= \int_0^1\int_0^1 u(1-v)\,uv\,\sqrt{1-u(1-v)-uv}\;.\;u\,du\,dv$$

$$= \int_0^1 u^3(1-u)^{1/2}\,du \times \int_0^1 v(1-v)\,dv$$

(BSG Book)

Put $u = \sin^2\theta$

$$= \frac{2}{945}$$