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VECTOR CALCULUS

Vector field:

Def: Let R be a region in the xy plane. A vector field F assigns to every point (x, y) in R a vector $F(x,y)$ with two components

$$F(x,y) = M(x,y)\,i + N(x,y)\,j$$

$$F(x,y,z) = M(x,y,z)\,i + N(x,y,z)\,j + P(x,y,z)\,k$$

Ex: Position vector at (x,y) is

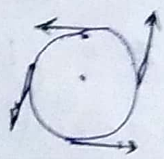
$$R = xi + yj$$

$$|R| = r = \sqrt{x^2+y^2}$$

$$\frac{|R|}{r} = \frac{x}{r}i + \frac{y}{r}j$$

Ex: Spin field or rotation field or turning field

$$S = -yi + xj \qquad |S| = r$$



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Directional Derivatives

$f(x,y)$

$\frac{\partial f}{\partial x}$ → keeps y constant and provides slope in x direction.

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$\frac{\partial f}{\partial y} \rightarrow x$ constant & gives slope in y direction

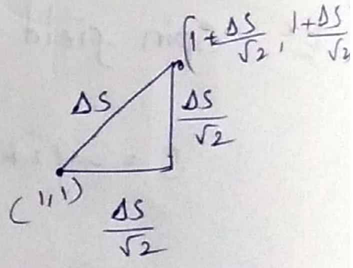
Many other directions are possible, such as 45° line

Given $f(x,y)$ around $P(x_0, y_0)$ and a direction $\vec{u}$ (unit vector) Find the derivative of f in the direction of $\vec{u}$

$$45^\circ \rightarrow \vec{u} = \frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}$$

Ex1 $z = xy$ when (x,y) moves a ~~direction~~ distance Δs in the 45° direction from $(1,1)$ what is $\frac{\Delta z}{\Delta s}$

Soln: The step is Δs times unit vector $\vec{u}$ starting from $x = y = 1$, the step ends at $x = y = 1 + \frac{\Delta s}{\sqrt{2}}$



$z = xy$ is then

$$z = \left(1 + \frac{\Delta s}{\sqrt{2}}\right)^2$$

$$= 1 + \sqrt{2}\,\Delta s + \frac{1}{2}(\Delta s)^2$$

$$\Delta z = \sqrt{2}\,\Delta s + \frac{1}{2}(\Delta s)^2$$

$\frac{\Delta z}{\Delta s} \rightarrow \sqrt{2}$ as $\Delta s \rightarrow 0$ (slope in 45° direction)

$\underline{\underline{Def^n}}$: The derivative of f in the direction $\vec{u}$ at the point p is

$$D_u f(P) = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \lim_{\Delta s \to 0} \frac{f(P + \vec{u}\,\Delta s) - f(P)}{\Delta s}$$

$\downarrow$
Directional derivative in the direction u of the function f at point P

$$\vec{u} = u_1 i + u_2 j$$
$\downarrow$
unit vector

$P(x_0, y_0)$ step $\Delta s \rightarrow Q(x_0 + u_1 \Delta s, \; y_0 + u_2 \Delta s)$

$u = (1, 0)$ $\quad Q \rightarrow (x_0 + \Delta s, y_0)$

$$D_{(1,0)} f = \frac{\partial f}{\partial x}$$

$$\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} \rightarrow \frac{\partial f}{\partial x}$$

The directional derivative $D_u f$ in the direction $\vec{u} = (u_1, u_2)$ $(u_1 i + u_2 j)$ equals $D_u f = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$

unit $\leftarrow$ vector

Slopes in all directions are known from slopes in two directions

$\underline{\underline{Ex}}$: $f = xy$ and $P = (1, 1)$ and $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Find $D_u f(P)$

$\underline{\underline{sol}}$ $f_x = y$, $f_y = x$ Both equal to 1 at P.

$D_u f(P) = f_x u_1 + f_y u_2$

$= 1\left(\frac{1}{\sqrt{2}}\right) + 1\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$ as ~~stoy~~ before

* Directional derivative of a scalar function gives scalar values.

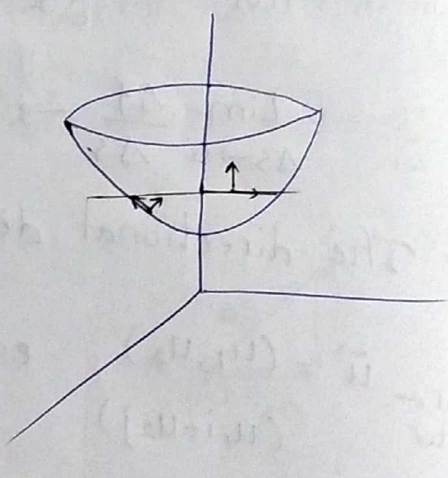
Ex 3 $\quad f = 3x + y + 1$

Slope in (u_1, u_2) direction

$D_u f = 3u_1 + u_2$

Slope in x direction $\rightarrow$ $D_{(1,0)} f = 3$

In which direction slope is maximum?



Gradeent:

The direction of the gradeent vector is the direction of maximum slope

$\Delta f = f_x u_1 + f_y u_2$

$\underset{\text{direction}}{(u_1, u_2)} \cdot \underset{\text{Gradient vector}}{(f_x, f_y)}$

Gradient vector

$\boxed{f_x i + f_y j}$

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$\underline{Def}^n$ The gradient $f(x,y)$ is the vector whose components are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j \quad \left(\text{Add } \frac{\partial f}{\partial z} k \text{ in } 3D\right)$$

Nyabla/ Gradient

Gradient of a function is a vector quantity

But function is scalar.

For plane $3x+y+1$, the gradient is constant vector $(3,1)$

If the way to climb the plane

Consider $f = x^2 + xy$

$$\nabla f = (2x + y, x)$$

Gradientis in the xyplane and tells us which way ~~om~~ on the surface is up (steepest ascent)

The directional derivative is $D_u f = (\text{grad} f) \cdot \vec{u}$

The ~~slod~~ slope $D_u f$ is largest when $\vec{u}$ is parallel to grad f.

The maximum slope (steepness) is the length :

$$|\text{grad } f| = \sqrt{f_x^2 + f_y^2} \quad \text{Cste}$$

$$u = \frac{\nabla f}{|\nabla f|}, \text{ the slope is } \nabla f \cdot \vec{u} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$

Ex 3 $\quad f(x,y) = 3x + y + 1$

$$\nabla f = (3, 1)$$

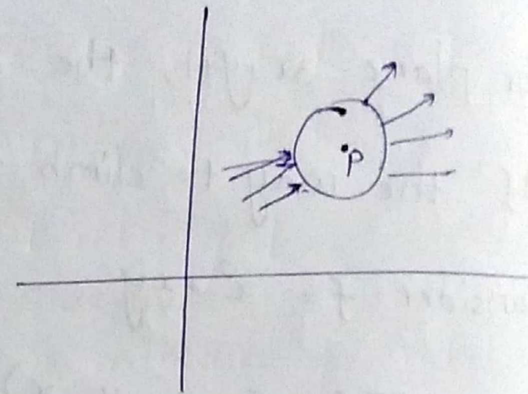
Its the steepest slope is in the direction $\vec{u} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

The maximum slope is $|\nabla f| = \sqrt{10}$

Divergence:

Defined on vector field

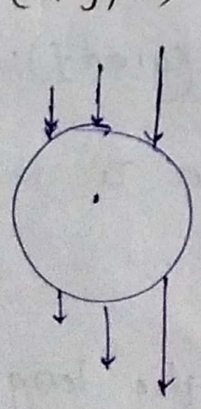
$$F(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$$



$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{Div } F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \rightarrow \text{scalar}$$

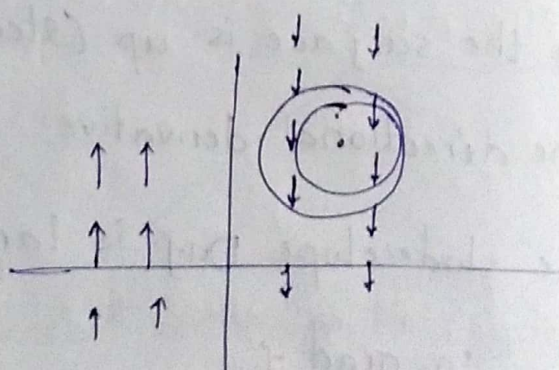
Ex: $\quad F(x,y,z) = -x\hat{j}$



$P = 0$

$Q = -x$

$R = 0$



$$\text{Div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 0 + 0$$

$$= 0$$

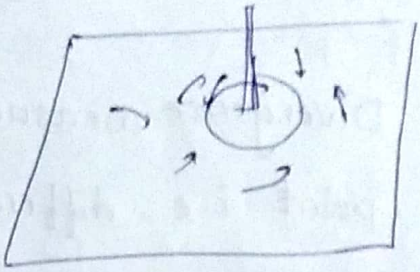
If inflow = outflow   divergence = 0

inflow > outflow   divergence < 0

inflow < outflow   divergence > 0

Curl:

The measurement of rotation of the vector $f^{\wedge}$ in the neighbourhood of P.



curl(f) at P, then the paddle would rotate counter clockwise

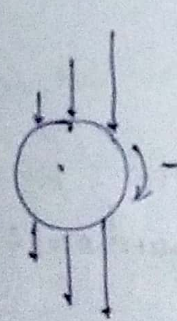
curl < 0 → ~~anti~~clockwise

curl > 0 → anticlockwise

curl = 0 → no rotation.

$$\text{curl } F = \nabla \times F$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$



→ spins clockwise

since vector increasing as we go right

curl < 0

$F = -xj$, $P = 0$, $Q = -x$ $R = 0$

$$F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & 0 \end{vmatrix} = -\hat{k}$$

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Operator	Symbol	Field taken as input	Output Field
Gradient	∇	Scalar	Vector
Divergence	$\nabla\cdot$ (div)	Vector	Scalar
Curl	$\nabla\times$ (curl)	Vector	Vector

Divergence measures "spreading" of $\vec{F}$ around a point i.e., difference between the amount of $\vec{F}$ existing from an infinitesimally small ball around the point and the amount entering it.

The curl is in some way a measure of "rotation" of the field. If we imagine to place a microscopic paddle wheel at a point in fluid moving with velocity $\vec{F}$, will it rotate or not?

Ex: $$F(x,y,z) = \frac{x}{\sqrt{x^2+y^2}}\, i + \frac{y}{\sqrt{x^2+y^2}}\, j$$

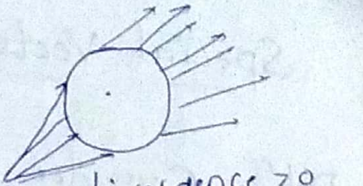
Find divergence and curl.

* Curl is only defined on 3 dimensional coordinates.

$$\text{Div } \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{\sqrt{x^2+y^2}}\right] + \frac{\partial}{\partial z}[0]$$

$$= \frac{1}{\sqrt{x^2+y^2}} > 0$$

$$\operatorname{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$$



divergence > 0

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= (0-0)\,i - (0-0)\,j + \left[-\frac{1}{2}(2x)\,y\,(x^2+y^2)^{-3/2} - \left(-\frac{1}{2}\,2y\right) x\,(x^2+y^2)^{-3/2} \right]$$

$$= 0$$

Laplacian Operator

The Laplacian Δf $(\nabla^2 f)$ of a scalar field is obtained by $\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

[Trace of Hessian matrix]

A scalar field whose laplacian vanishes everywhere is called harmonic function.

A vector laplacian $\vec{\Delta}$ is the laplacian operator applied componentwise to vector-fields $\vec{\Delta} \vec{F}$

$$\vec{\Delta}\vec{F} = (\Delta F_1)\,i + (\Delta F_2)\,j + (\Delta F_3)\,k$$

Ex: $f = x^2 - y^2$ Compute the Laplacian

$\Delta f = 2 - 2 = 0$

Special Vector field and potentials:

Def^n: Consider a vector field $\vec{F}$ defined on $D \subset R^3$

If $\vec{\nabla} \times \vec{F} = \vec{0}$, then $\vec{F}$ is called irrotational (or curlfree)

If $\vec{\nabla} \cdot \vec{F} = 0$, then $\vec{F}$ is called solenoidal (or divergencefree or incompressible).

dot

If $\vec{F} = \vec{\nabla}\phi$ for some field ϕ, then $\vec{F}$ is called conservative and ϕ is called scalar potential of $\vec{F}$

If $\vec{F} = \vec{\nabla} \times \vec{A}$ for some vector field $\vec{A}$ then $\vec{A}$ is called vector potential of $\vec{F}$.

Ex: 1 $\vec{F} = 2x\,i - 2y\,j$

Is $\vec{F}$ irrotational or solenoidal ?

Is $\vec{F}$ conservative ?

$\vec{\nabla} = 2i - 2j$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2 - 2 + 0 = 0 \quad [\text{Solenoidal}]$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2y & 0 \end{vmatrix} = 0 \quad [\text{irrotational}]$$

$$\vec{F} = \vec{\nabla}\underbrace{(x^2 - y^2)}_{\phi = x^2 - y^2} = \vec{\nabla} \times (2xy\,\hat{k})$$

$\vec{F}$ admit both scalar and vector potentials

so is conservative.

Ex 2 $\vec{G} = yz\,i + xz\,j + xy\,k$

$P = yz \quad Q = xz \quad R = xy$

$$\nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\nabla \cdot F = 0$ [Solenoidal]

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= i\,(x-x) - j\,(y-y) + k\,(z-z)$$

$= 0$ [irrotational]

$$\vec{G} = \vec{\nabla}\,(xyz\,i + xyz\,j + xyz\,k)$$

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Identities (Vector-differential)

Let f be a scalar field and $\vec{F}$ be a vector field both defined in a domain $D \subset R^3$ Then the following identities hold true

i) $\vec{\nabla} \cdot (\vec{\nabla} f) = \Delta f$ (Divergence of grad f is laplacian)

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial (\vec{\nabla} f)_1}{\partial x} + \frac{\partial (\vec{\nabla} f)_2}{\partial y} + \frac{\partial (\vec{\nabla} f)_3}{\partial z}$$

$$\left[\vec{\nabla} f = (\nabla f)_1 i + (\nabla f)_2 j + (\nabla f)_3 k\right]$$

$$= \frac{\partial}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial}{\partial y} \frac{\partial f}{\partial y} + \frac{\partial}{\partial z} \frac{\partial f}{\partial z}$$

$$\vec{\nabla} \cdot (\vec{\nabla} f) = \Delta f$$

ii) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ (Divergence of curl f is zero)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \frac{\partial (\nabla \times F)_1}{\partial x} + \frac{\partial ((\nabla \times F)_2)}{\partial y} + \frac{\partial ((\nabla \times F)_3)}{\partial z}$$

$$= \frac{\partial \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)}{\partial x} + \frac{\partial \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)}{\partial y} + \frac{\partial \left(\frac{\partial F_2}{\partial x} - \frac{\partial F}{\partial y}\right)}{\partial z}$$

$$= \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y} + \cdots$$

$$= 0$$

iii) $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$

Curl of grad f is zero vector

$$(\vec{\nabla} \times (\vec{\nabla} f))_1 = \frac{\partial((\nabla f)_3)}{\partial y} - \frac{\partial((\nabla f)_2)}{\partial z}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)$$

$$= 0$$

similar to other components too.

iv) $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \vec{\Delta} \vec{F}$

Vector differential identities for two fields,

product rules for differential operators

Let f & g be scalar fields, $\vec{F}$ & $\vec{G}$ be vector fields, all defined in $D \subset R^3$. Then the following identities hold true.

$$\vec{\nabla}(fg) = f\,\vec{\nabla} g + g\,\vec{\nabla} f$$

$$\vec{\nabla} \cdot (\vec{F} \times \vec{G}) = (\vec{\nabla} \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\vec{\nabla} \times \vec{G})$$

$$\Delta(fg) = (\Delta f) g + 2\,\vec{\nabla} f \cdot \vec{\nabla} g + f(\Delta g)$$

Note:

1. From the definition of the curl operator, we note that $\vec{F}$ is irrotational precisely when

$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3 k$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \text{ and } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

2. If $\vec{F}$ admits a scalar or a vector potential, then the potential is not unique for all constant scalars $\lambda \in R$ and for all scalar fields g

if $\vec{F} = \vec{\nabla}\phi$ then $\vec{F} = \vec{\nabla}(\phi + \lambda)$

if $\vec{G} = \vec{\nabla} \times \vec{A}$ then $\vec{G} = \vec{\nabla} \times (\vec{A} + \vec{\nabla} g)$

Ex: Consider the field $\vec{F} = z\hat{i} + x\hat{k}$

It is irrotational and solenoidal

If ϕ is a scalar potential of $\vec{F}$

$$\vec{F} = \vec{\nabla}\phi \Rightarrow \frac{\partial \phi}{\partial x} = z \Rightarrow \phi = xz + f(y, z)$$

$$\frac{\partial \phi}{\partial y} = 0 \Rightarrow \frac{\partial (xz + f(yz))}{\partial y} = \frac{\partial f(yz)}{\partial y} = 0$$

$$\Rightarrow \phi = xz + g(z)$$

$$\frac{\partial \phi}{\partial z} = x \Rightarrow \frac{\partial (xz + g(z))}{\partial z} = x + \frac{\partial g(z)}{\partial z} = x$$

$$\Rightarrow \phi = xz + \lambda$$

So for every real constant λ, the fields $xz + \lambda$ are scalar potentials of $\vec{F}$.

$$\vec{F} = z\hat{i} + x\hat{k}$$

$$\vec{F} = \vec{\nabla} \times \vec{A}$$

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Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = z$$

$$\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = 0$$

$$\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = x$$

Many choices for $\vec{A}$

$$\vec{A_1} = 0 = \vec{A_3}$$

$$\frac{\partial A_2}{\partial z} = -z \rightarrow ①$$

$$\frac{\partial A_2}{\partial x} = x \rightarrow ② \Rightarrow A_2 = \frac{x^2}{2} + b(z)$$

$$① \Rightarrow \frac{\partial\left(\frac{x^2}{2} + b(z)\right)}{\partial z} = -\partial$$

$$\frac{\partial b(z)}{\partial z} = -z$$

$$b(z) = -\frac{z^2}{2}$$

$$A_2 = \frac{x^2}{2} - \frac{z^2}{2}$$

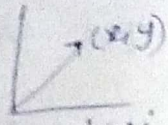
$\vec{\nabla} \times (\vec{\nabla}\phi) = \vec{0}$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

All conservative fields are irrotational.

All fields admitting a vector potential are solenoidal.

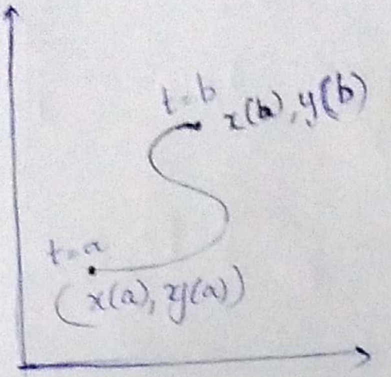
Position Vector:

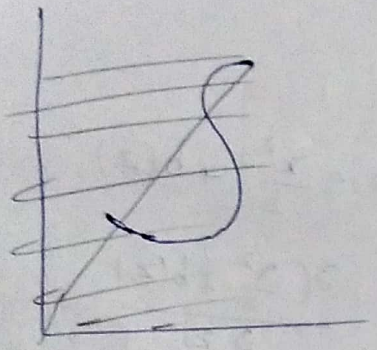


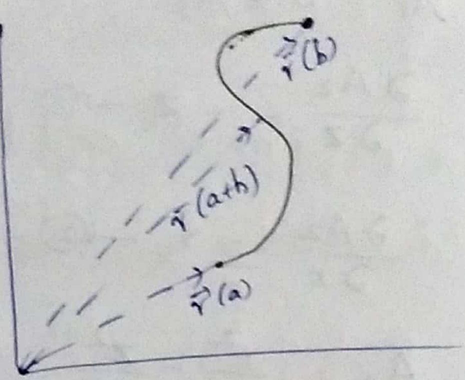
$\vec{r} = xi + yj$

Curve C: $x = x(t)$

$y = y(t)$ $\quad a \leq t \leq b$



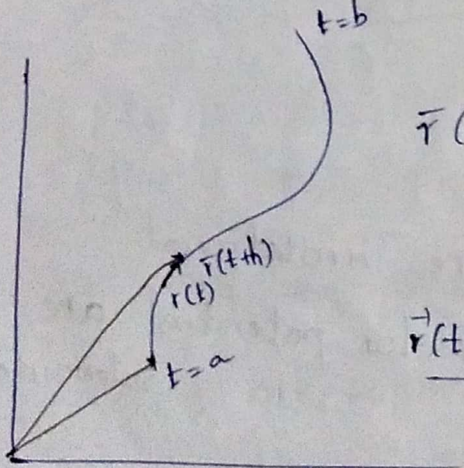




$$\bar{r}(t) = x(t)i + y(t)j \quad a \leq t \leq b$$

$$\bar{r}(a) = x(a)i + y(a)j$$

$$\bar{r}(a+h) = x(a+h)i + y(a+h)$$



$$\bar{r}(t+h) - \vec{r}(t)$$

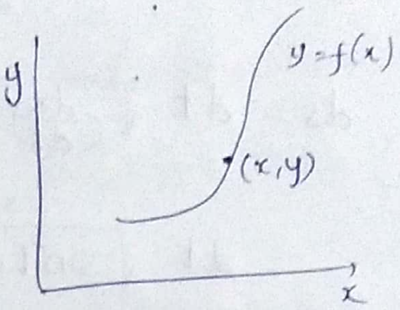
$$= x(t+h)i + y(t+h)i - (x(t)i + y(t)j)$$

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \frac{x(t+h) - x(t)}{h}i + \frac{y(t+h) - y(t)}{h}j$$

$h \Leftrightarrow \Delta t$

$$\lim_{h\to 0} \rightarrow \vec{r}'(t) \text{ (or) } \frac{d\vec{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j$$

Displacement vector along the curve



$$d\vec{s} = dx\, i + dy\, j$$

$$ds = |ds| = \sqrt{(dx)^2 + (dy)^2}$$

$$y = f(x) \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Parametric form $(x(t), y(t))$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$a \le t \le b \quad \int_c ds = \int_a^b \frac{ds}{dt} \cdot dt$$

<u>Ex</u>: Find the length of the curve described by

$$y(x) = \frac{1}{2}x^2 + x - 3, \quad 1 \le x \le 3$$

Distance between two infinitesimally nearly points on the curve is

$$ds = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx \sqrt{1 + (x+1)^2}$$

$$ds = dx \sqrt{x^2 + 2x + 2}$$

$$\int ds = \int^3 \sqrt{x^2 + 2x + 2}\, dx = 6.34$$

Ex 2 The curves $x = \cos t$

$y = \sin t \qquad 0 \le t \le 6\pi$

$z = t$

$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t \qquad \frac{dz}{dt} = 1$$

$$ds = dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= dt\sqrt{\sin^2 t + \cos^2 t + 1}$$

$$ds = dt\sqrt{2}$$

$$\int_C ds = \int_0^{6\pi} \sqrt{2}\, dt = 6\pi\sqrt{2}$$

Line integral (scalar)

A line integral in two dimensions may be written as

$$\int_C f(x,y)\, dw$$

$f(x,y) \rightarrow$ scalar function to be integrated

Ex: $f(x,y) = x^2 + 4y^2$

$c \rightarrow$ This is the curve along which integration takes place

Ex: $y = x^2$ or $x = \sin y$

or $x = t-1 \,;\, y = t^2$

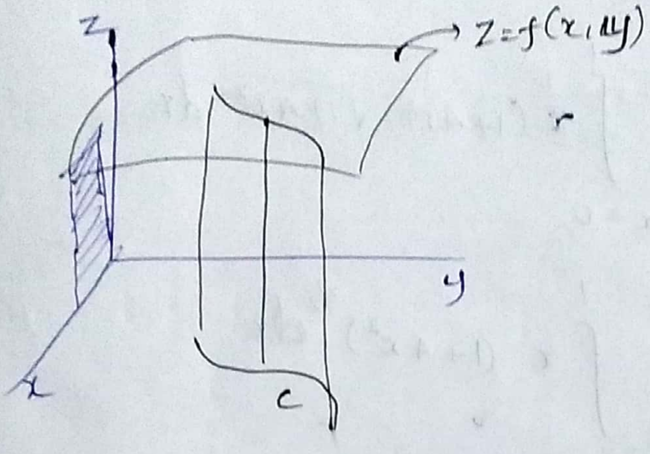
$\vec{r}(t) = xi + yi = (t-1)\,i + t^2\,j$

$dw \to$ variable of integration

may be dx, dy or ds.

$ds = \sqrt{(dx)^2+(dy)^2}$

The integral $\int_C f(x,y)\, ds$ represents the area beneath the surface $z=f(x,y)$ but above the curve c.



The curve can be given in x,y form or parametric form with limits.

For calculating line integrals, express all quantities in terms of a single variable.

Ex 3 Find $\int_C x(1+4y)\, dx$ where c is the curve $y=x^2$, starting from $x=0, y=0$ and ending $x=1, y=1$

soln.

$$\int_C x(1+4y)\, dx = \int_{x=0}^{1} x(1+4x^2)\, dx = \int_0^1 (x+4x^3)\, dx$$

$$= \left(\frac{x^2}{2} + x^4\right)_0^1$$

$$= \frac{3}{2}$$

$\int_C x(1+4y)\,ds$ $C: y = x^2$

$x=0, y=0 \rightarrow x=1, y=1$

$$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= dx\sqrt{1+(2x)^2}$$

$$ds = dx\sqrt{1+4x^2}$$

$$\int_C x(1+4y)\,ds = \int_{x=0}^{1} x(1+4x^2)\sqrt{1+4x^2}\,dx$$

$$= \int_0^1 x\,(1+4x^2)^{3/2}\,dx$$

$$= \frac{1}{20}\left(5^{5/2}-1\right)$$

<u>Ex: 4</u> Find $\int_C xy\,ds$ $C: x = 3t^2,\ y = t^3-1$ for t varying from 0 to 1

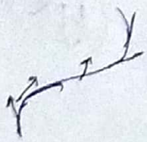
$$\int_C xy\,ds = \int_{t=0}^{1} (3t^2)(t^3-1)\,ds$$

$$ds = dt\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = dt\sqrt{(6t)^2+(3t^2)^2}$$

$$\int_C xy\,ds = \int_{t=0}^{1} (3t^2)(t^3-1)\sqrt{36t^2+9t^4}\,dt$$

Line integral (vector)

Work done $= \vec{F} \cdot d\vec{r}$



$\vec{F} = F_x i + F_y j + F_z k$

$d\vec{r} = dx\, i + dy\, j + dz\, k$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (F_x\, dx + F_y\, dy + F_z\, dz)$$

Ex4: $\vec{F} = 2xy\, i - 5x\, j$ & $C: y = x^3,\ 0 \le x \le 1$

$dy = 3x^2 dx$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (2xy\, dx - 5x\, dy)$$

$$= \int_0^1 (2x\, x^3\, dx - 5x \cdot 3x^2\, dx)$$

$$= \int_0^1 (2x^4 - 15x^3)\, dx$$

$$= \frac{-67}{20}$$

Ex5: Consider the vector field $\vec{F} = y^2 z^3\, \hat{i} + 2xy\, z^3\, \hat{j} + 3xy^2 z^2\, \hat{k}$

$C_1: x = t,\ y = t,\ z = t \quad (0 \le t \le 1)$

$C_2: x = t^2,\ y = t,\ z = t^2 \quad (0 \le t \le 1)$

$O(0,0,0)$ to $A(1,1,1)$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} (y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz)$$

$x = t$
$dx = dt$
$y = t$
$dy = dt$
$z = t$
$dz = dt$

$$= \int_0^1 (t^5\,dt + 2t^5\,dt + 3t^5\,dt)$$

$$= \int_0^1 (t^5 + \cancel{2}5t^5)\,dt$$

$$= \frac{1}{6} + \frac{5}{6}$$

$$= 1$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$$

$x = t^2$
$dx = 2t\,dt$
$y = t$
$dy = dt$
$z = t^2$
$dz = 2t\,dt$

$$= \int_0^1 t^2(t^2)^3\,2t\,dt + 2t^2t(t^2)^3\,dt + 3t^2t^2t^4\,2t\,dt$$

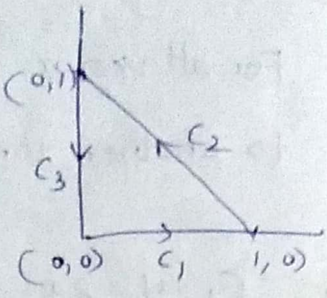
$$= \int_0^1 (2t^9 + 2t^9 + 6t^9)\,dt$$

$$= \int_0^1 10t^9\,dt$$

$$= 1$$

Work done by both paths is same

Ex: Evaluate $\int_C (x\hat{i} + y\hat{j})\, d\vec{r}$



piecewise smooth curve/loop

Over C_1 $\quad 0 \le x \le 1,\ y = 0$

$\vec{r}(t) = t\hat{i} \quad , \quad 0 \le t \le 1$

$\frac{d\vec{r}}{dt} = \hat{i}$

$I_{C_1} = \int_0^1 (t\hat{i}) \cdot \hat{i}\, dt = \int_0^1 t\, dt = \frac{1}{2}$

Over C_2 $\quad x + y = 1$ in 1^{st} quadrant

$\bar{r}(t) = t\hat{i} + (1-t)\hat{j} \quad ; \quad 0 \le t \le 1$

$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j}$

$I_{C2} = \int_0^1 (t\hat{i} + (1-j)\hat{j}) \cdot (\hat{i} - \hat{j})\, dt$

$= \int_0^1 t - (1-t)\, dt = \int_0^1 (2t-1)\, dt$

$= [t^2 - t]_0^1 = 0$

Over C_3 $\quad y$ is from 1 to 0

$0 \le y \le 1 \quad ; \quad x = 0$

$\vec{r}(t) = (1-t)\hat{j} \qquad 0 \le t \le 1$

$\frac{d\,\vec{r}(t)}{dt} = -\hat{j}$

$$I_{C_3} = \int_0^1 (1-t)\hat{j} \cdot (-\hat{j})\, dt$$

$$= -\int_0^1 (1-t)\, dt = \left(-t + \frac{t^2}{2}\right)_0^1$$

$$= -1 + \frac{1}{2} = -\frac{1}{2}$$

$$I_C = I_{C_1} + I_{C_2} + I_{C_3} = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

<u>Work done in case of conservative vector field $\vec{F}$</u>

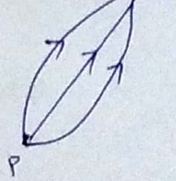
Let $\vec{F}$ be a conservative vector field. Hence, there exists a scalar function $\phi(x,y,z)$ such that $\vec{F} = \text{grad}(\phi)$. Now, $\vec{\nabla} = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

$$\vec{F} = F_1 i + F_2 j + F_3 k$$

$$W = \int_P^Q \vec{F} \cdot d\vec{r} = \int_P^Q (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_P^Q \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right)$$

$$= \int_P^Q d\phi$$



$$= \left[\phi(x,y,z)\right]_P^Q$$

$$= \phi(Q) - \phi(P)$$

Work done is independent of path in case of conservative

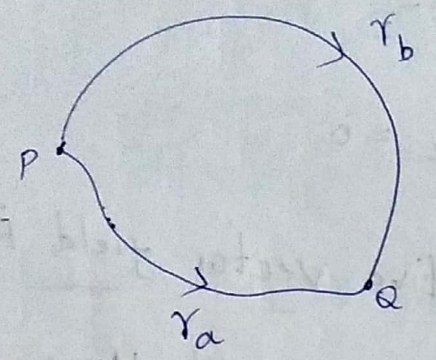
$\vec{F} = \vec{\nabla}\phi \iff \int_P^Q \vec{F}\cdot d\vec{r} \iff \oint \vec{F}\cdot d\vec{r} = 0$

conservative — path independent — $\forall$ loops

$\Downarrow$

$\vec{\nabla} \times \vec{F} = \vec{0}$

(irrotational)



$r = r_a - r_b$

$\downarrow$

loop path

Show that $\vec{F} = (yz-1)\hat{i} + (z+xz+z^2)\hat{j} + (y+xy+2yz)\hat{k}$ is conservative. Also find the work done by $\vec{F}$ in moving a particle from $(1,2,2)$ to $(2,3,4)$

sol^n

$$\text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz-1 & z+xz+z^2 & y+xy+2yz \end{vmatrix} = \hat{i}\,(1+x+2z-1-x-2z) + \hat{j}\,(y-y) + \hat{k}\,(z-z)$$

$$= 0$$

$\vec{F} = \vec{\nabla}\phi$

$$\phi = xyz - x + yz + yz^2 + c$$

$$\text{Work done} = \int_{(1,2,2)}^{(2,3,4)} \vec{F}\cdot d\vec{r} = \phi(2,3,4) - \phi(1,2,2)$$

$$= 67$$

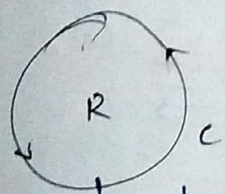
Green's Theorem:

It connects line integral to double integral

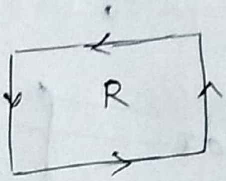
Let c be a piecewise smooth simple ~~curve~~ closed curve bounding a region R. If $f, g, \frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continous on R then

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

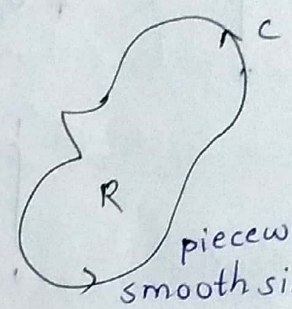
Integration being carried out in the positive direction on C counter clockwise.



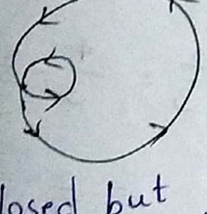
simple closed smooth curve



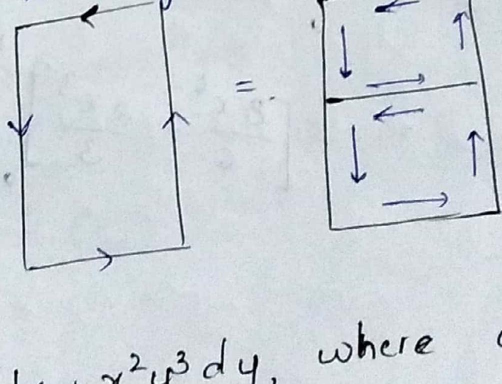
piecewise smooth simple closed curve.



piecewise smooth simple closed.

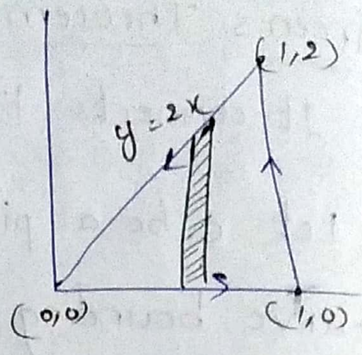

closed but not simple

Line integral = Line integral.



Ex: Evaluate $\oint_C xy\,dx + x^2y^3\,dy$, where C is the curve that is the boundary of the triangle having vertices $(0,0), (1,0), (1,2)$

$f = xy \qquad \frac{\partial f}{\partial y} = x$



$g = x^2y^3 \qquad \frac{\partial g}{\partial x} = 2xy^3$

$$\iint_R (2xy^3 - x)\, dx\, dy$$

$$\int_0^1 \int_0^{2x} (2xy^3 - x)\, dy\, dx$$

$$\int_0^1 \left(2x\frac{y^4}{4} - xy\right)_0^{2x} dx = \int_0^1 \left(\frac{x}{2} \cdot 16x^4 \cancel{dx} - 2x^2\right) dx$$

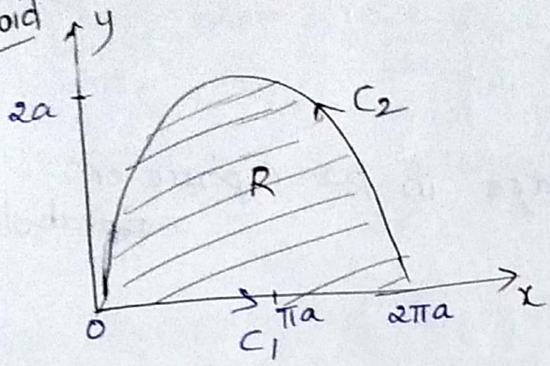
$$= \cancel{\int_0^1 8x^5\, dx = \int_0^1 8 \cdot \frac{1}{6} = \frac{4}{3}}$$

$$= \int_0^1 (8x^5 - 2x^2)\, dx$$

$$= \left[\frac{8x^6}{6} - \frac{2x^3}{3}\right]_0^1 = \frac{8}{6} - \frac{2}{3} = \frac{4}{3} - \frac{2}{3}$$

$$= \frac{2}{3}$$

cycloid



C_2 $x = a(\theta - \sin\theta)$ θ runs

$y = a(1-\cos\theta)$ from 2π to 0

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1$$

$$\vec{F} = -y\,i$$

C_1 $y = 0,\quad x = t \quad 0 \le t \le 2\pi a$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = 0$$

$dx = a(1-\cos\theta)\,d\theta$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int (-y)\,dx$$

$$= \int_{2\pi}^{0} -a^2(1-\cos\theta)^2\,d\theta$$

$$= a^2 \int_{0}^{2\pi} (1-\cos\theta)^2\,d\theta$$

$$= 3\pi a^2$$

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17/11/21

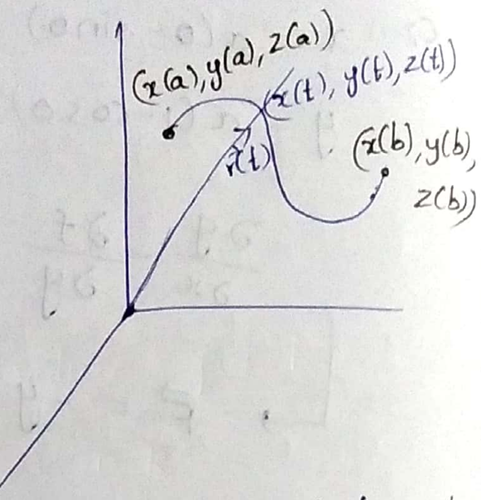
Surface integral:

Integration over a surface in R^3 (sphere or parabolic)

Points (x, y, z) on curve

$x = x(t)$, $y = y(t)$, $z = z(t)$

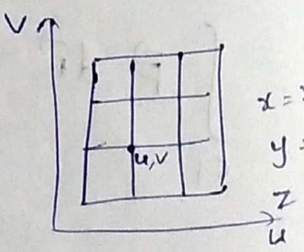
$a \le t \le b$



Parameterization is transformation a subset of R into a curve

Parameterization of surfaces to define a surface integral

we will use 2 variables u & v to parameterize a surface Σ in R^3

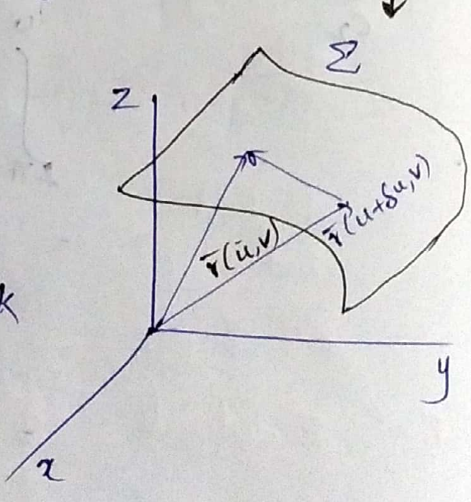


$x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$ for (u,v) in some region R in R^2

Position vector of a point on Σ



$\vec{r}(u,v) = x(u,v)i + y(u,v)j + z(u,k)k$

for (u,v) in R

$$\frac{\partial \vec{r}}{\partial u}(u,v) = \frac{\partial x}{\partial u}(u,v)\, i + \frac{\partial y}{\partial u}(u,v) + \frac{\partial z}{\partial u}(u,v)\, k$$

$$\frac{\partial \vec{r}}{\partial v}(u,v) = \frac{\partial x}{\partial v}(u,v) i + \frac{\partial y}{\partial u}$$

Parameterization of $\Sigma \equiv$ transforming a region of R^2 (in u,v plane) into a 2 dimensional surface in R^3.

Vertical ~~tin~~ gridlines in R (u is constant) → mapped to curves on Σ and the variable u is constant along $\vec{r_0}(u,v)$ So the tangent vector to those curves at a point (u,v) is $\frac{\partial \vec{r}}{\partial v}$

Similarly horizontal lines (v is constant) → mapped to curves on Σ and the variable v is constant along $\vec{r}(u,v)$. So the tangent vector to those curves at a point (u,v) is $\frac{\partial \vec{r}}{\partial u}$

Consider the pt (u,v) in R. Suppose the rectangle drawn has a small width and a heigh of, Δu and Δv respectively.

Corner points (u,v), $(u+\Delta u, v)$, $(u+\Delta u, v+\Delta v)$, and $(u, v+\Delta v)$

Area of the rectangle. $A = \Delta u \Delta v$.

Mapped to surface area $d\sigma$ which is close to the area of the parallelogram which has adjacent sides

$\vec{r}(u+\Delta u, v) - \vec{r}(u,v)$ and $\vec{r}(u, v+\Delta v) - \vec{r}(u,v)$

$$\frac{\partial \vec{r}}{\partial u} = \frac{\vec{r}(u+\Delta u) - \vec{r}(u,v)}{\Delta u} \quad \text{and} \quad \frac{\partial \vec{r}}{\partial v} = \frac{\vec{r}(u, v+\Delta v) - \vec{r}(u,v)}{\Delta v}$$

$$d\sigma = \left\| \left(\Delta u \frac{\partial \vec{r}}{\partial u}\right) \times \left(\Delta v \frac{\partial \vec{r}}{\partial v}\right) \right\| = \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| \Delta u \, \Delta v$$

Total surface area S of Σ

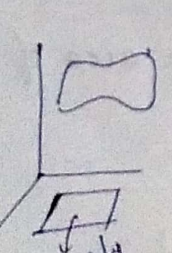
$$S = \iint_R \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

$$\Downarrow$$

$$S = \iint_\Sigma 1 \cdot d\sigma$$

Replace 1 by a general real valued function $f(x,y,z)$ in R^3.

Let Σ be a surface in R^3 parameterized by $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ for (u,v) in some region R in R^2. Let $\vec{r}(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$ be the position vector for any point in Σ and Let $f(x,y,z)$ be a real valued f^n defined on some subset of R^3 that contains Σ. The surface integral of $f(x,y,z)$ over Σ is

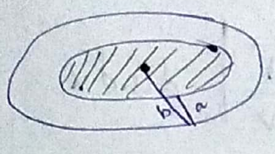


$$\iint_\Sigma f(x,y,z)\, d\sigma = \iint_R f(x(u,v), y(u,v), z(u,v)) \cdot \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

Not only flat can be any surface
If flat can directly done by double integral.

Particular case, for surface area S of Σ is $S = \iint_\Sigma 1 \cdot d\sigma$

Ex:



Torus T obtained by revolving a circle of radius a in the yz plane around the z-axis, circle's centre is at a distance b from the z-axis $(0<a<b)$. Find the surface area of T.

Sol Torus can be parameterized as

$$x = (b+a\cos u)\cos v$$

$$y = (b+a\cos u)\sin v \qquad \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le 2\pi \end{array}$$

$$z = a\sin u$$

Position vector

$$\vec{r}(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

$$= (b+a\cos u)\cos v\, i + (b+a\cos u)\sin v\, j + a\sin u\, k$$

$$\frac{\partial \vec{r}}{\partial u} = -a\sin u\cos v\, i - a\sin u\sin v\, j + a\cos u\, k$$

$$\frac{\partial \vec{r}}{\partial v} = -(b+a\cos u)\sin v\, i + (b+a\cos u)\cos v\, j + 0k$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = -a(b+a\cos u)\cos v\cos u\, i - a(b+a\cos u)\cos u\sin v\, j - a(b+a\cos u)\sin v\, k$$

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = a(b+a\cos u)$$

Surface area of T is

$$S = \iint_{\Sigma} 1\, d\sigma = \int_0^{2\pi}\int_0^{2\pi} \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\, du\, dv$$

$$= \int_0^{2\pi} (abu + a^2 \sin u)\Big|_0^{2\pi} dv$$

$$= \int_0^{2\pi} 2\pi ab\, dv$$

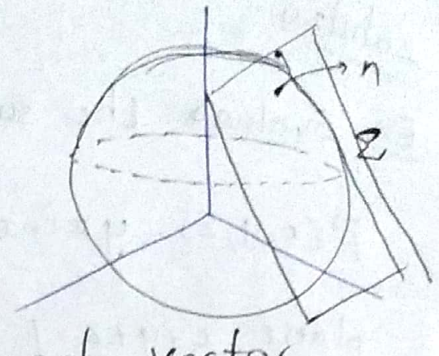
$$= 4\pi^2 ab.$$

Since $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are tangent to the surface Σ (i.e., lie in the tangent plane to Σ at each point on Σ), then $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is perpendicular to the tangent plane to the surface at each point of Σ

$$\iint_{\Sigma} f(x,y,z)\, d\sigma = \iint_{R} f(x(u,v), y(u,v), z(u,v))\, \|\vec{n}\|\, d\sigma$$

$$\left(\vec{n} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)\right)$$

$\vec{n}$ is normal vector to Σ



Surface integral of a 3-dimensional vector field over a surface:

Let Σ be a surface in R^3 and let

$\vec{F}(x,y,z) = f_1(x,y,z)\, i + f_2(x,y,z)\, j + f_3(x,y,z)\, k$ be

a vector field defined on some subset of R^3 that

contains Σ. The surface integral of $\vec{F}$ over Σ is

$$\iint_{\Sigma} \vec{F} \cdot d\vec{\sigma} = \iint_{\Sigma} \underbrace{\vec{F} \cdot \hat{n}}_{\text{real valued } f^n}\, d\sigma$$

where at any point on Σ, $\hat{n}$ is the outward

unit normal vector to Σ

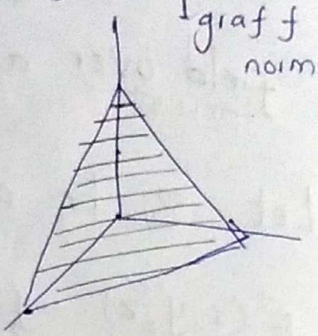
20/11/20

Ex: Evaluate the surface integral $\iint_{\Sigma} \vec{F} \cdot d\vec{\sigma}$ where

$\vec{F}(x,y,z) = yzi + xzj + xyk$ and Σ is the part of the plane $x+y+z=1$ with $x \geq 0$, $y \geq 0$ and $z \geq 0$.

$f(x,y,z) = c$

$\perp$ graf f is normal vector



Sol^n unit normal vector

$$\hat{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$(1,1,1)$
$\downarrow$
Normal vector

Parameterize Σ over R

$$\frac{1}{\hat{n} \cdot x} = \sqrt{3}$$

$$x = u, \quad y = v, \quad z = 1-(u+v) \quad \text{for} \quad \begin{matrix} 0 \leq u \leq 1 \\ 0 \leq v \leq 1-u \end{matrix}$$

$$\vec{F} \cdot \hat{n} = (yz, xz, xy) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}(yz + xz + xy)$$

$$= \frac{1}{\sqrt{3}}((x+y)z + xy)$$

$$= \frac{1}{\sqrt{3}}((u+v)(1-(u+v)) + uv)$$

$$= \frac{1}{\sqrt{3}}((u+v) - (u+v)^2 + uv)$$

for (u,v) in R and $\vec{r}(u,v) = ui + vj + (1-(u+v))k$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (1,0,-1) \times (0,1,-1) = (1,1,1)$$

Thus integrating over R gives

$$\iint_{\Sigma} \vec{F} \cdot \hat{n}\, d\sigma$$

$$= \iint_{R} \left(\vec{F}\left(x(u,v), y(u,v), z(u,v)\right) \cdot \hat{n}\right) \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| du\, dv$$

$$= \int_0^1 \int_0^{1-u} \frac{1}{\sqrt{3}} \left(u+v-(u+v)^2+uv\right) \sqrt{3}\, dv\, du$$

$$= \int_0^1 \left[\frac{(u+v)^2}{2} - \frac{(u+v)^3}{3} + \frac{uv^2}{2}\right]_{v=0}^{v=1-u} du$$

$$= \int_0^1 \left(\frac{1}{6} + \frac{u}{2} - \frac{3u^2}{2} + \frac{5u^3}{6}\right) du$$

$$= \left[\frac{u}{6} + \frac{u^2}{4} - \frac{u^3}{2} + \frac{5u^4}{24}\right]_0^1$$

$$= \frac{1}{8}$$

If the eq^n of Σ is given as $z = f(x,y)$

$$x = u, \quad y = v, \quad z = f(u,v)$$

$$\vec{F}(u,v) = ui + vj + f(u,v)\, k$$

$$\frac{\partial \vec{r}}{\partial u} = i + f_u k, \quad \frac{\partial \vec{r}}{\partial v} = j + f_v k$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$$

$$= -f_u i - f_v j + k$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{1 + f_u^2 + f_v^2}$$

$$= \sqrt{1 + f_x^2 + f_y^2}$$

Surface area of Σ $\quad A = \iint_{R^*} \sqrt{1+f_x^2+f_y^2}\, dx\, dy$

R^* is the region in the xy plane obtained by the projection of the surface Σ.

Surface integral of $g(x,y,z)$ over Σ is

$$I = \iint_{\Sigma} g(x,y,z)\, d\sigma$$

$$= \iint_{R^*} g(x, y, f(x,y))\, d\sigma$$

$$= \iint_{R^*} \sqrt{1+f_x^2+f_y^2}\, dx\, dy.$$

If $x = f_1(y,z)$ $\quad = \iint_{R^*} \sqrt{1+f_y^2+f_z^2}\, dy\, dz$

$y = m(x,z)$ $\quad = \iint \sqrt{1+m_x^2+m_z^2}\, dx\, dz$

Ex: Find the surface area of hemisphere

$$x^2+y^2+z^2=a^2, \quad z \geq 0$$

sol^n $$z = \sqrt{a^2-x^2-y^2} = f(x,y)$$

$$f_x = z_x = \frac{1}{2}(a^2-x^2-y^2)^{-1/2}(-2x)$$

$$z_x = \frac{-x}{(a^2-x^2-y^2)^{1/2}}$$

$$f_y = z_y = \frac{1}{2}(a^2-x^2-y^2)^{-1/2}(-2y)$$

$$z_y = \frac{-y}{(a^2-x^2-y^2)^{1/2}}$$

$$1+z_x^2+z_y^2 = 1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}$$

$$= \frac{a^2}{a^2-x^2-y^2}$$

$$A = \iint_{R^*} \sqrt{1+z_x^2+z_y^2}\, dx\, dy \qquad R^*: x^2+y^2 \leq a^2$$

$$= a \iint_{R^*} \left(\sqrt{a^2-x^2-y^2}\right)^{-1} dx\, dy$$

$$= a \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{1}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$$

$$= 2\pi a^3$$

If S is piecewise smooth having smooth surfaces $S_1, S_2, \ldots S_k$

$$\iint_S g(x,y,z)\, dA = \left(\iint_{S_1} + \iint_{S_2} + \cdots + \iint_{S_k} \right) g(x,y,z)\, dA$$

<u>Vector field (surface integral)</u>

$\Sigma : x = f(y,z)$

$$\iint_\Sigma \vec{v}\cdot\hat{n}\, d\sigma = \iint_R \vec{v}\cdot\hat{n}\, \frac{dy\,dz}{\hat{n}\cdot\hat{i}}$$

R is the projection of Σ onto yz plane.

Computing surface integrals can often be tedious. The following theorem provides an easier way in the case when Σ is a closed surface, that is when Σ encloses a bounded solid in R^3.

For example, spheres, cubes and ellipsoids are closed surfaces but planes and paraboloids are not.

<u>Gauss's Divergence theorem</u>: Let Σ be a closed surface in R^3 which bounds a solid D and let $\vec{F}(x,y,z) = F_1(x,y,z)\,i + F_2(x,y,z)\,j + F_3(x,y,z)\,k$ be a vector field defined on some subset of R^3

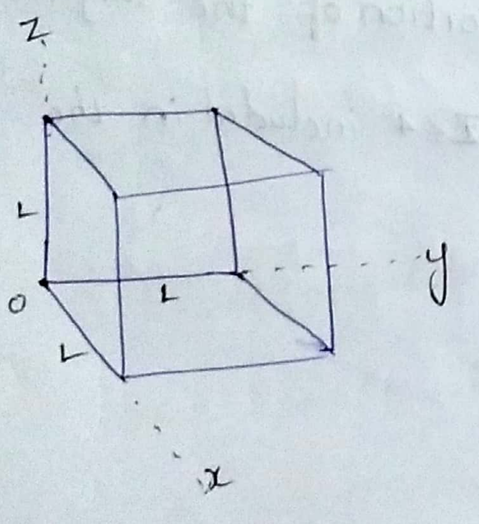
that contains Σ

Then $$\iint_{\Sigma} \vec{F} \cdot d\vec{\sigma} = \iiint_{D} \operatorname{div} \vec{F} \, dv$$

$\operatorname{div} \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$ is called the

$\vec{\nabla} \cdot \vec{F}$ divergence of $\vec{F}$

<u>Test</u>: $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$ defined over a cube of side L lying in the first octant with a vertex at the origin



$\vec{\nabla} \cdot \vec{F} = y + z + x$

$$\iiint_{D} \vec{\nabla} \cdot \vec{F} \, dv = \int_0^L \int_0^L \int_0^L (x+y+z) \, dx \, dy \, dz$$

$$= \frac{L^4}{2} + \frac{L^4}{2} + \frac{L^4}{2} = \frac{3L^4}{2}$$

The flux integral only has non zero contributions from the three sides located $x = L$, $y = L$, $z = L$
corresponding unit normal vectors: $\hat{n} = i$, $\hat{n} = j$, $\hat{n} = k$

$$\iint_{\Sigma} \vec{F}\cdot\hat{n}\,d\sigma = \int_0^L\int_0^L Ly\,dy\,dz + \int_0^L\int_0^L Lz\,dx\,dz$$

$$+ \int_0^L\int_0^L Lx\,dx\,dy$$

$$= \frac{L^4}{2} + \frac{L^4}{2} + \frac{L^4}{2}$$

$$= \frac{3L^4}{2}$$

Evaluate $\iint_S \vec{v}\cdot\hat{n}\,d\sigma$ where $\vec{v} = z^2\hat{i} + xy\,\hat{j} + y^2\hat{k}$

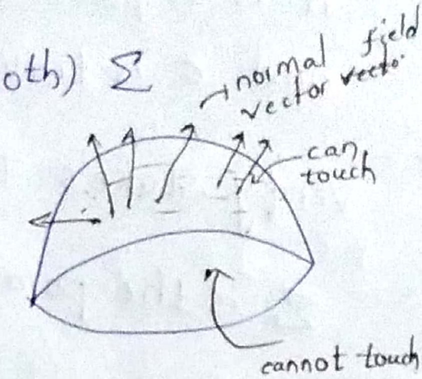
and surface S is the portion of the surface of cylinder $x^2+y^2=36$, $0 \le z \le 4$ included in the first octant.

Stoke's theorem: (line & surface integral)

i) orientable surface (piecewise smooth) Σ

bounded by piecewise

smooth simple closed curve C.



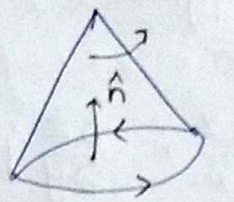
but u know there is space inside

Walk along C with your

head pointing is direction

of $\hat{n}$, then the surface

would be on your left



Then we say $\hat{n}$ is

positive vector and

C is traversed $\hat{n}$-positively

Nonorientable

Ex : Mobious strip

Let Σ be an orientable piecewise smooth surface

in R^3 whose boundary is a simple closed curve C,

and let $\vec{F}(x,y,z) = P(x,y,z)\, i + Q(x,y,z)\, j + R(x,y,z)\, k$

be smooth vector field defined on some subset of R^3

that contains Σ then

$$\oint_C \vec{F} \cdot d\vec{x} = \iint_\Sigma (\text{curl } \vec{F}) \cdot \hat{n} \, d\sigma$$

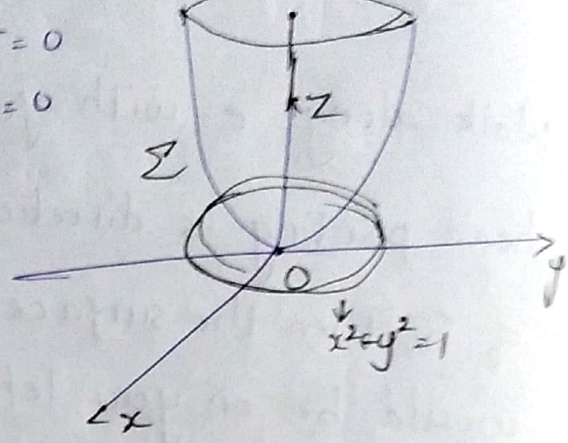
where $\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + (\quad) j + (\quad) k$

$\hat{n}$ is a positive unit normal vector over Σ and c is traversed $\hat{A}$-positively

Verify Stoke's theorem for $\vec{F} = zi + xj + yk$ when Σ is the paraboloid $z = x^2 + y^2$ such that $z \le 1$

$z^2 - x^2 - y^2 = 0$

$f(x, y, z) = 0$



Sol:

$$\hat{n} = \frac{\text{grad } F}{|\text{grad } F|}$$

$$\hat{n} = \frac{-\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

$$= \frac{-2xi - 2yj + k}{\sqrt{1 + 4x^2 + 4y^2}}$$

$\Sigma : \vec{r}(x, y) = xi + yj + (x^2 + y^2)k$

for (x, y) in $D: \{(x,y) : x^2 + y^2 \le$

$x = u$

$y = v$

$z = u^2 + v^2$

$$\text{Curl } \vec{F} = (1-0)i + (1-0)j + (1-0)k = i + j + k$$

$$\text{Curl } \vec{F} \cdot \hat{n} = \frac{-2x - 2y + 1}{\sqrt{1 + 4x^2 + 4y^2}}$$

$$\iint_\Sigma (\text{curl } \vec{F}) \cdot \hat{n} \, d\sigma = \iint_D \frac{-2x - 2y + 1}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2} \, dA$$

$$= \iint_D (-2x - 2y + 1) \, dA$$

$$= \int_0^{2\pi}\int_0^1 (-2r\cos\theta - 2r\sin\theta + 1)\, r\, dr\, d\theta$$

$$= \pi.$$

The boundary curve C $\quad d\vec{r} = \left(\frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k\right) dt$

$x^2 + y^2 = 1$, in $z = 1$

$x = \cos t$, $y = \sin t$, $z = 1$, $0 \le t \le 2\pi$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [1(-\sin t) + (\cos t)(\cos t) + \sin t)(0)]\, dt$$

$$= \int_0^{2\pi} \left(-\sin t + \frac{1 + \cos 2t}{2}\right) dt$$

$$= \pi$$

Volume Integrals

① V bounded by S $\qquad \phi$ is a scalar f^n

$\delta V_1, \delta V_2, \dots \delta V_n$

$\delta V_i \to P_i(x_i, y_i, z_i)$

$\Sigma\, \phi(P_i) \cdot \delta V_i$ where $\phi(P_i) = \phi(x_i, y_i, z_i)$

Limit of sum as $n \to \infty$: Volume integral of ϕ over V.

$$\iiint_V \phi\, dv \qquad dv = dx\, dy\, dz$$

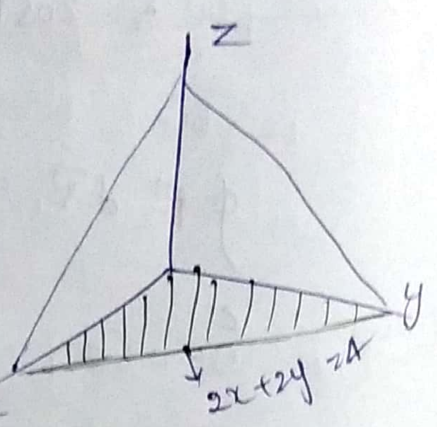
② Vector volume integral: $\iiint\limits_V \vec{F}\, dv$ ↓ vector.

Ex: If $\vec{F} = (2x^2-3x)\,\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$

Find $\iiint\limits_V (\vec{\nabla} \times \vec{F})\, dv$, V is the region R

bounded by the coordinate planes and the

plane $2x+2y+z=4$.



Sol

$$\iiint\limits_V \vec{\nabla} \times \vec{F}\, dv$$

$$\vec{\nabla} \times \vec{F} = \hat{j} - 2y\,\hat{k}$$

R: $2x \le 4$, $2x+2y \le 4$, $2x+2y+z \le$

$x \le 2$, $y \le 2-x$, $z \le 4-2x-$

$$\iiint\limits_V \vec{\nabla} \times \vec{F}\, dv = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\hat{j} - 2y\,\hat{k})\, dz\, dy\, dx$$

Ex: Evaluate $\iiint\limits_V \phi \, dV$, where $\phi(x, y, z) = 45x^2y$

and V is the volume /closed region bounded by

the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.

$$\iiint\limits_V \phi \, dV = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2y \, dz \, dy \, dx$$