Heap Sort

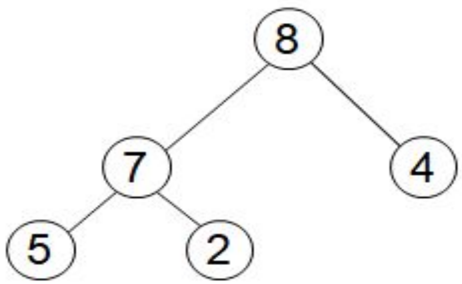
Heap

The Heap Data Structure

A heap is a nearly complete binary tree with the following two properties:

- Structural Property: all levels are full, except possibly the last one, which is filled from left to right
- Order (heap) property: for any node x

Parent (x)>=x



Heap

From the heap property, it follows that:
"The root is the maximum element of the heap!"

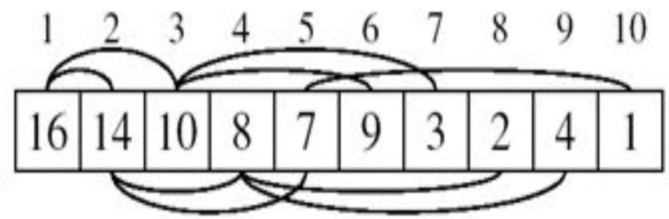
A heap is a binary tree that is filled in order

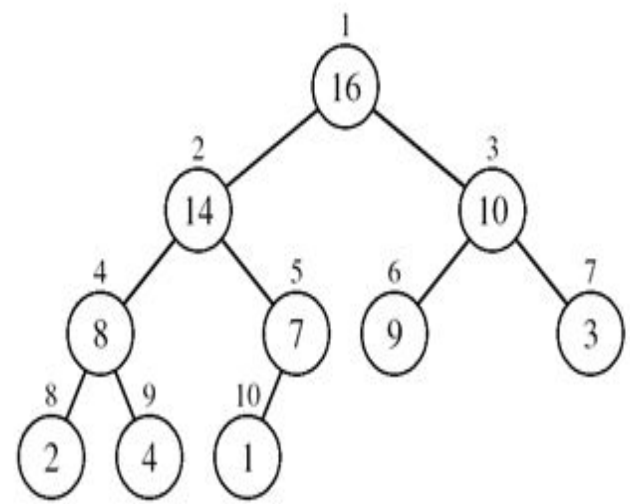
Array Representation of Heaps

- A heap can be stored as an array A
  - Root of tree is A[1]
  - Left child of A[i] = A[2i]
  - Right child of A[i] = A[2i + 1]
  - Parent of A[i] = A[i/2]
  - Heapsize[A] ≤ length[A]

- The elements in the subarray A[(n/2+1) .. n] are leaves

]





Heap Types

- **Max-heaps** (largest element at root), have the *max-heap property:*
  - for all nodes i, excluding the root:

    $A[PARENT(i)] \geq A[i]$

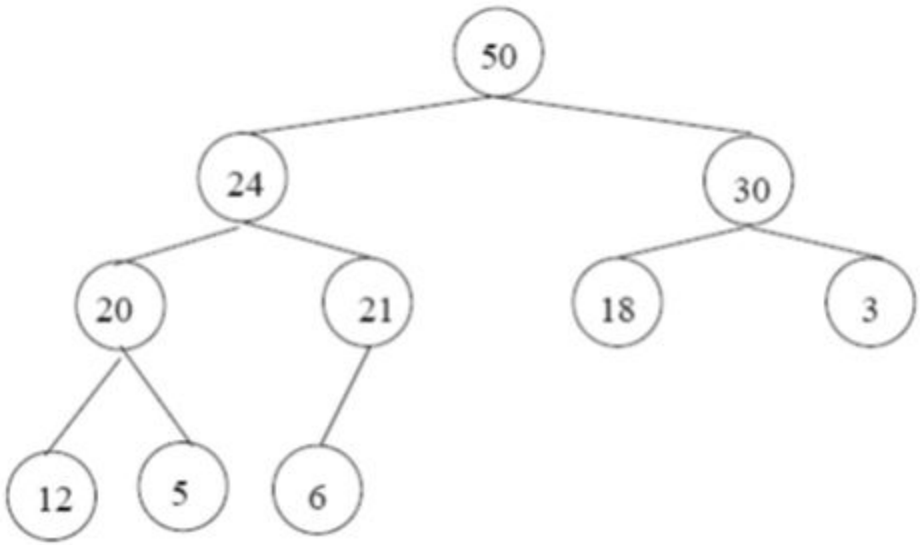
–

- **Min-heaps** (smallest element at root), have the *min-heap property:*
  - for all nodes i, excluding the root:

    $A[PARENT(i)] \leq A[i]$

Adding/ Deleting Nodes

- New nodes are always inserted at the bottom level (left to right)
- Nodes are removed from the bottom level (right to left)

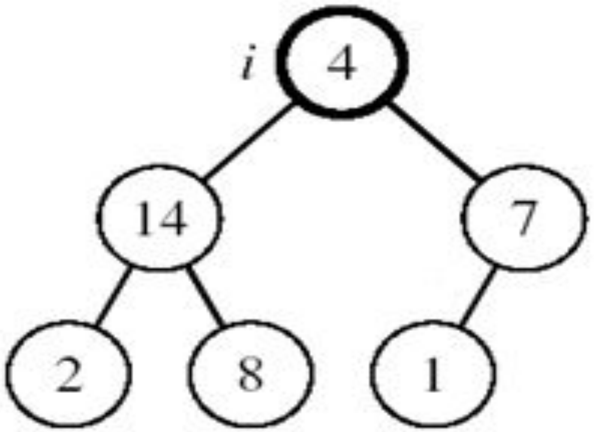


Operations on Heaps

- Maintain/ Restore the max-heap property
  - MAX-heapify
- Create a max-heap from an unordered array
  - Build-MAX-Heap
- Sort an array in Place
  - HEAPSORT
- Priority queues

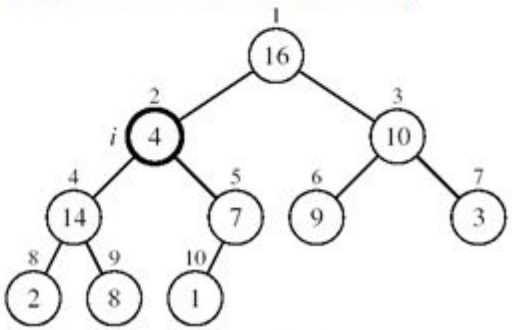
Maintaining the Heap Property

- Suppose a node is smaller than a child
  - Left and Right subtrees of i are max-heaps
- To eliminate the violation:
  - Exchange with larger child
  - Move down the tree
  - Continue until node is not smaller than children



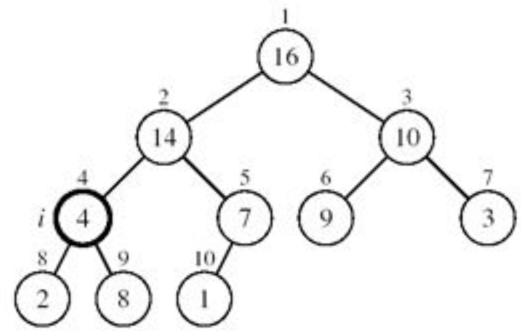
Example

MAX-HEAPIFY(A, 2, 10)



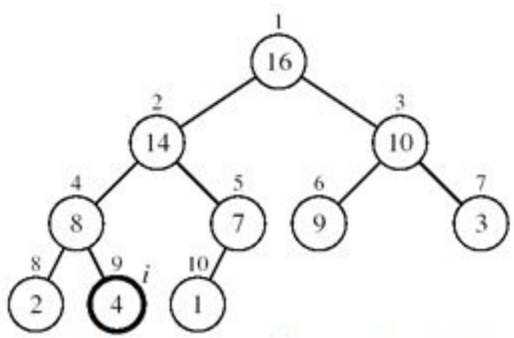
A[2] violates the heap property

A[2] ↔ A[4]



A[4] violates the heap property

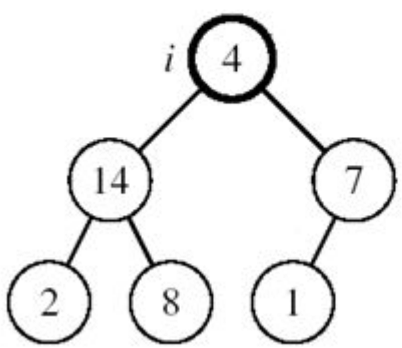
A[4] ↔ A[9]



Heap property restored

Maintaining the Heap Property

- Assumptions:
  - Left and Right subtrees of i are max-heaps
  - A[i] may be smaller than its children



Alg: MAX-HEAPIFY(A, i, n)

```
l ← LEFT(i)
r ← RIGHT(i)
if l ≤ n and A[l] > A[i]
   then largest ←l
   else largest ←i
if r ≤ n and A[r] > A[largest]
   then largest ←r
if largest ≠ i
   then exchange A[i] ↔ A[largest]
        MAX-HEAPIFY(A, largest, n)
```

MAX-HEAPIFY Running Time

- Intuitively:

  It traces a path from the root to a leaf (longest path length: d)
  At each level, it makes exactly 2 comparisons
  Total number of comparisons is $2d$
  Running time is $O(d)$ or $O(lgn)$

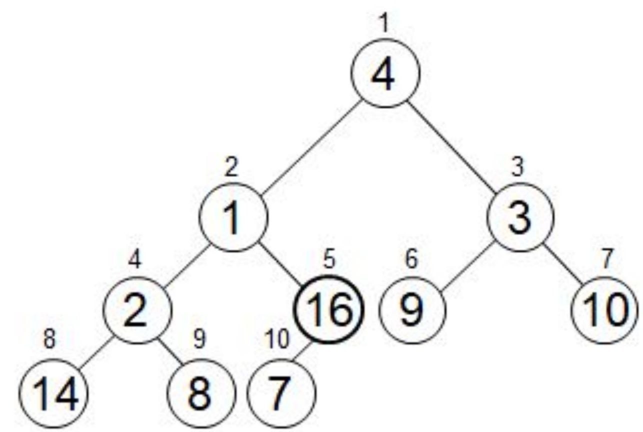
- Running time of MAX-HEAPIFY is O(lgn)
- Can be written in terms of the height of the heap, as being O(h)
  - Since the height of the heap is lgn

Building a Heap

- Convert an array A[1 … n] into a max-heap (n = length[A])
- The elements in the subarray A[(n/2+1) .. n] are leaves
- Apply MAX-HEAPIFY on elements between 1 and n/2

Alg: BUILD-MAX-HEAP(A)

1. n = length[A]
2. **for** i ← $\lfloor n/2 \rfloor$ **downto** 1
3. **do** MAX-HEAPIFY(A, i, n)



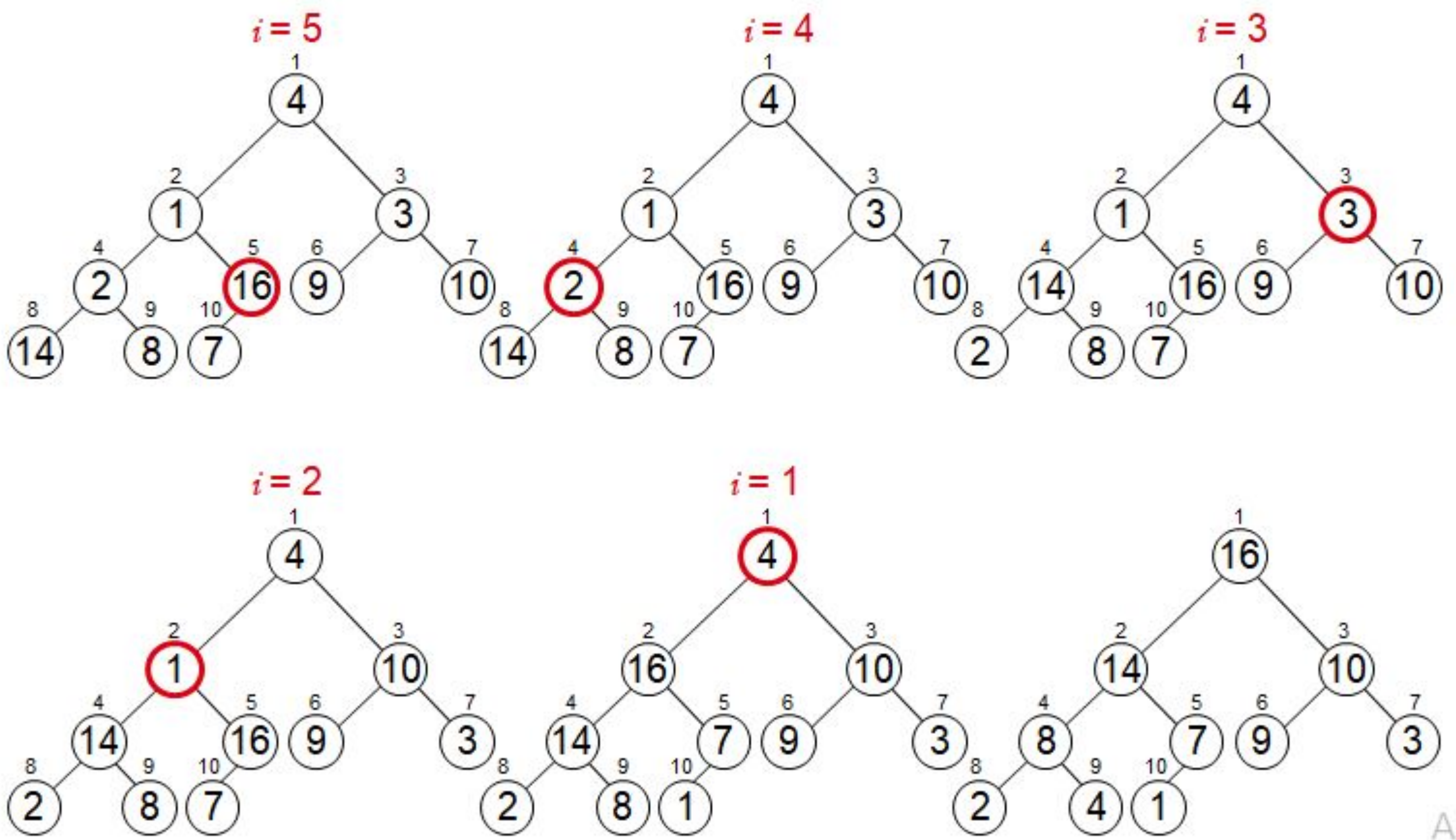
A:

4	1	3	2	16	9	10	14	8	7

Example

A:

4	1	3	2	16	9	10	14	8	7



Running Time of BUILD MAX HEAP

Alg: BUILD-MAX-HEAP(A)

1. n = length[A]
2. **for** i ← $\lfloor n/2 \rfloor$ **downto** 1
3. **do** MAX-HEAPIFY(A, i, n) $O(\lg n)$

$$\left.\begin{array}{l} \text{lines 2–3} \end{array}\right\} O(n)$$

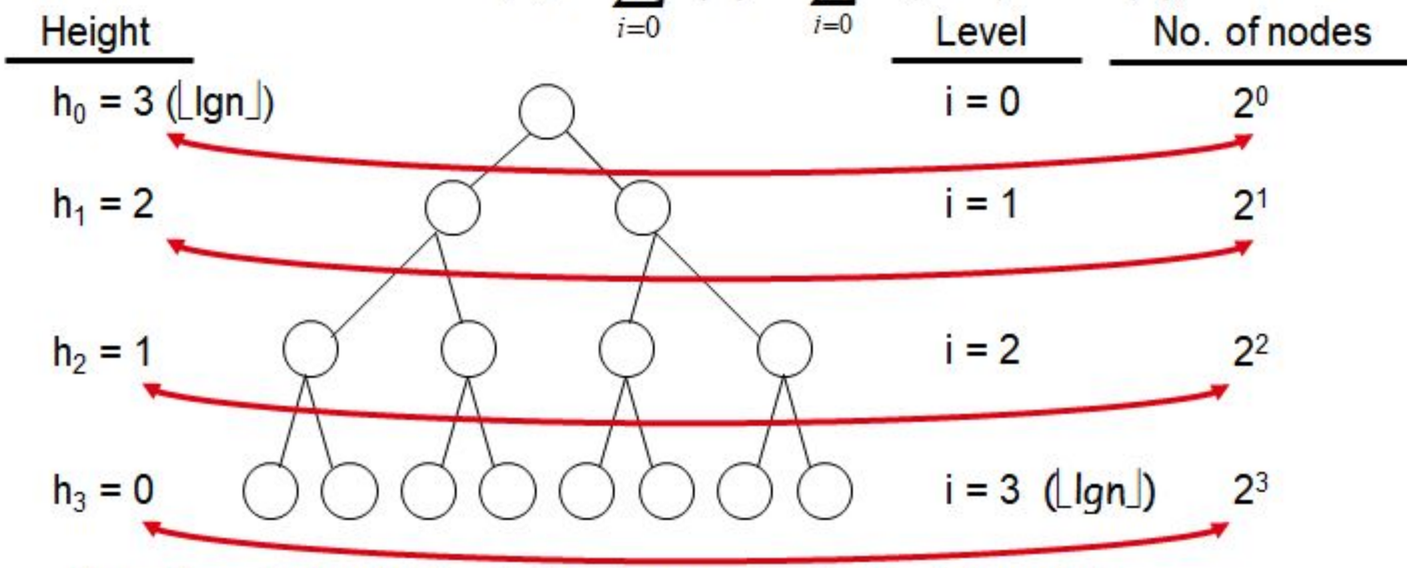
⇒ Running time: $O(n\lg n)$

- This is not an asymptotically tight upper bound

Running Time of BUILD MAX HEAP

HEAPIFY takes $O(h) \Rightarrow$ the cost of HEAPIFY on a node i is proportional to the height of the node i in the tree

$$\Rightarrow T(n) = \sum_{i=0}^{h} n_i h_i = \sum_{i=0}^{h} 2^i (h-i) = O(n)$$



$h_i = h - i$ height of the heap rooted at level i

$n_i = 2^i$ number of nodes at level i

Running Time of BUILD MAX HEAP

$$T(n) = \sum_{i=0}^{h} n_i h_i$$ Cost of HEAPIFY at level i * number of nodes at that level

$$= \sum_{i=0}^{h} 2^i (h-i)$$ Replace the values of n_i and h_i computed before

$$= \sum_{i=0}^{h} \frac{h-i}{2^{h-i}} 2^h$$ Multiply by 2^h both at the nominator and denominator and write 2^i as $\frac{1}{2^{-i}}$

$$= 2^h \sum_{k=0}^{h} \frac{k}{2^k}$$ Change variables: k = h - i

$$\leq n \sum_{k=0}^{\infty} \frac{k}{2^k}$$ The sum above is smaller than the sum of all elements to ∞ and h = lgn

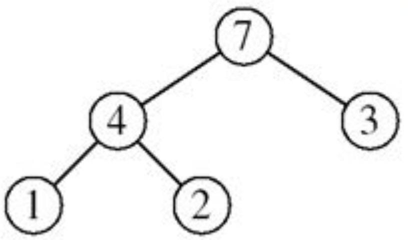
$$= O(n)$$ The sum above is smaller than 2

Running time of BUILD-MAX-HEAP: T(n) = O(n)

Heap Sort

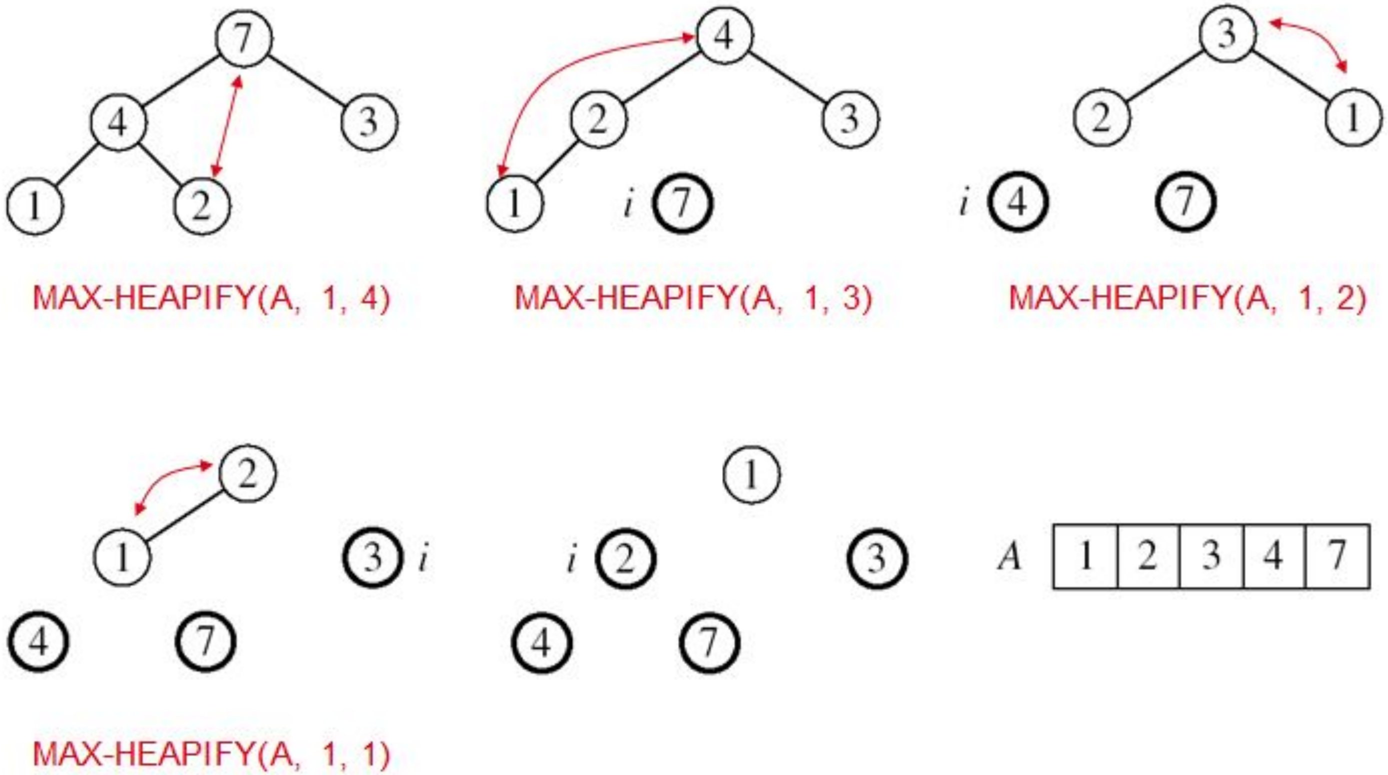
Heapsort

- Goal:
  - sort an array using heap representation



- Idea:
  - Build a max-heap from the array
  - Swap the root (the maximum element) with the last element in the array
  - “Discard” this last node by decreasing the heap size
  - Call MAX-HEAPIFY on the new root
  - Repeat this process until only one node remains

Example: A=[7, 4, 3, 1, 2]



Alg: HEAPSORT(A)

1. BUILD-MAX-HEAP(A) $O(n)$
2. **for** i ← length[A] **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. MAX-HEAPIFY(A, 1, i - 1) $O(\lg n)$

n-1 times

- Running time: $O(n\lg n)$ --- *Can be* shown to be $\Theta(n\lg n)$

Priority Queues

Properties

- Each element is associated with a value (priority)

- The key with the highest (or lowest) priority is extracted first

- Major operations

  Remove an element from the queue

  Insert an element in the queue

Operations on Priority Queues

- Max-priority queues support the following operations:
  - INSERT(S, x): inserts element x into set S
  - EXTRACT-MAX(S): removes and returns element of S with largest key
  - MAXIMUM(S): returns element of S with largest key
  - INCREASE-KEY(S, x, k): increases value of element x's key to k (Assume k $\geq$ x's current key value)

HEAP-MAXIMUM

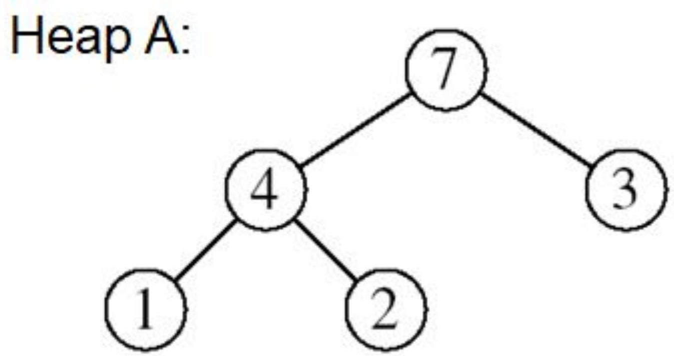
Goal:

- Return the largest element of the heap

Running time: $O(1)$

Alg: HEAP-MAXIMUM(A)

1. **return** A[1]



Heap-Maximum(A) returns 7

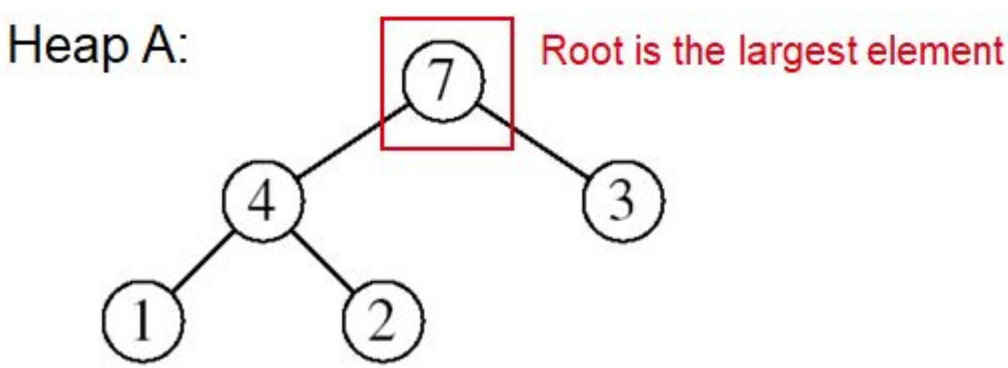
HEAP-EXTRACT-MAX

Goal:

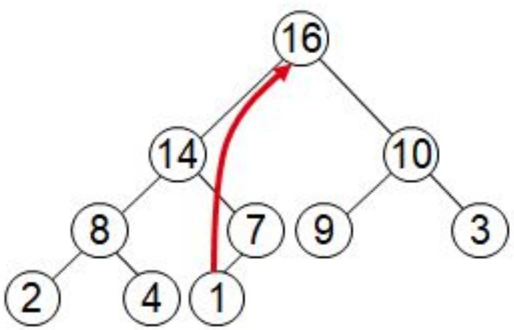
- Extract the largest element of the heap (i.e., return the max value and also remove that element from the heap

Idea:

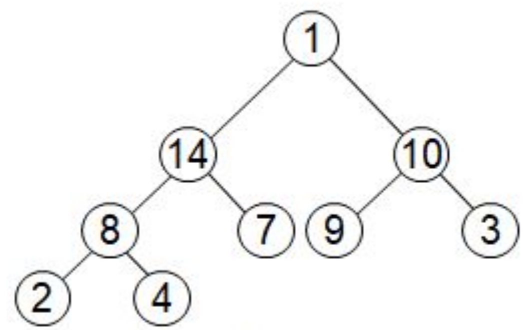
- Exchange the root element with the last
- Decrease the size of the heap by 1 element
- Call MAX-HEAPIFY on the new root, on a heap of size n-1



Example: HEAP-EXTRACT-MAX

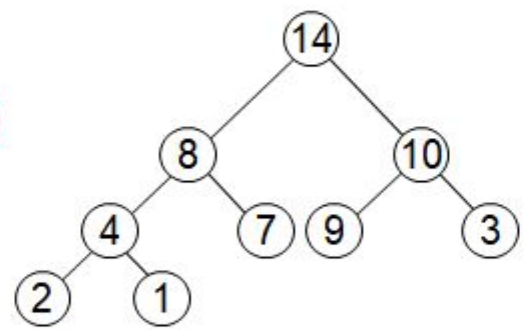


max = 16



Heap size decreased with 1

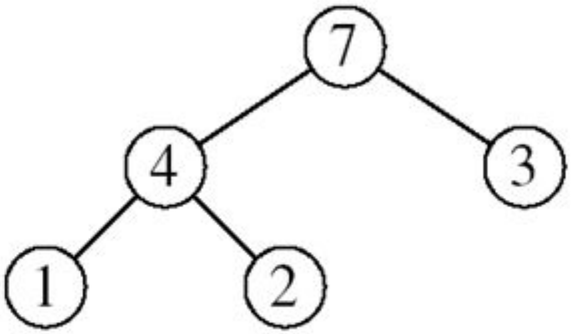
Call MAX-HEAPIFY(A, 1, n-1)



HEAP-EXTRACT-MAX

Alg: HEAP-EXTRACT-MAX(A, n)

1. **if** n < 1
2.   **then error** "heap underflow"
3. $max \leftarrow A[1]$
4. $A[1] \leftarrow A[n]$
5. MAX-HEAPIFY(A, 1, n-1) ▷ remakes heap
6. **return** max

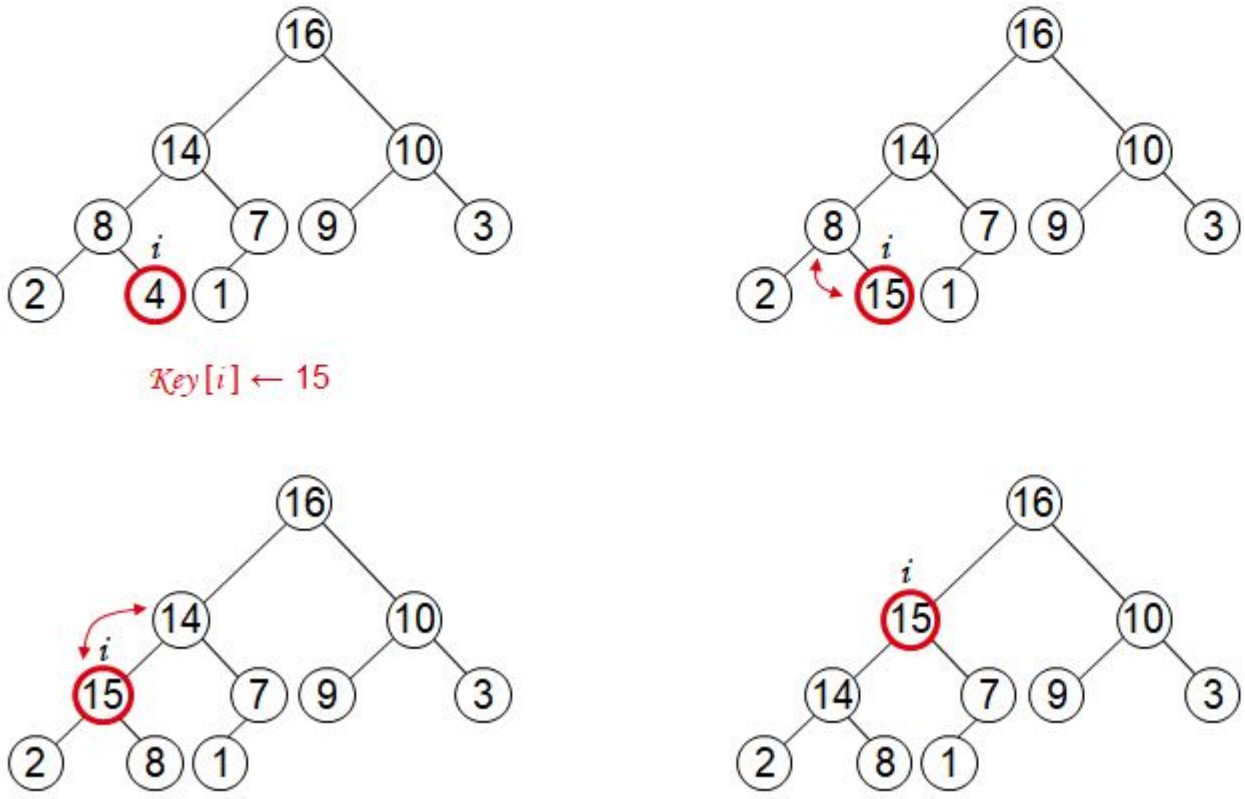


Running time: $O(\lg n)$

HEAP-INCREASE-KEY

- Goal:
  - Increases the key of an element i in the heap
- Idea:
  - Increment the key of A[i] to its new value
  - If the max-heap property does not hold anymore: traverse a path toward the root to find the proper place for the newly increased key

Example: HEAP-INCREASE-KEY

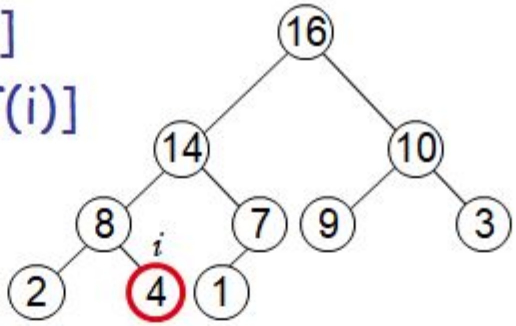


HEAP-INCREASE-KEY

Alg: HEAP-INCREASE-KEY(*A*, i, key)

```
if key < A[i]
   then error "new key is smaller than current key"
A[i] ← key
while i > 1 and A[PARENT(i)] < A[i]
    do exchange A[i] ↔ A[PARENT(i)]
        i ← PARENT(i)
```

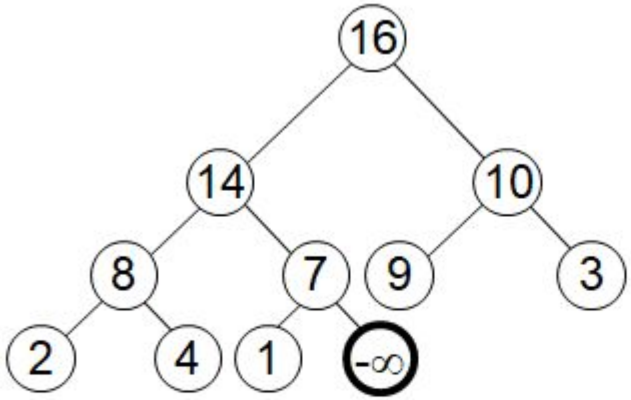
- Running time: $O(\lg n)$

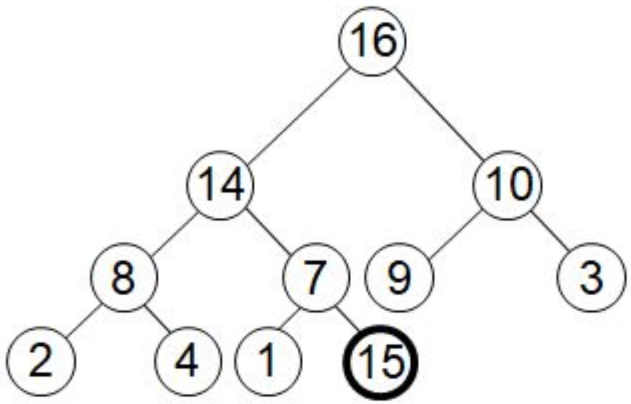


Key [i] ← 15

MAX-HEAP-INSERT

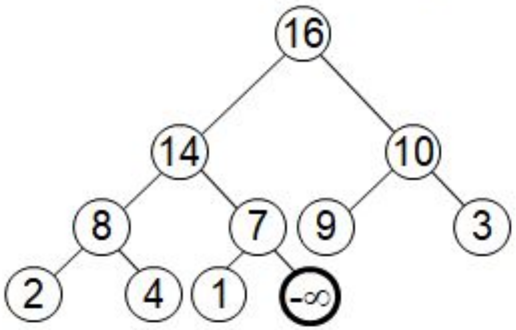
- Goal:
  - Inserts a new element into a max-heap
- Idea:
  - Expand the max-heap with a new element whose key is -∞
  - Calls HEAP-INCREASE-KEY to set the key of the new node to its correct value and maintain the max-heap property



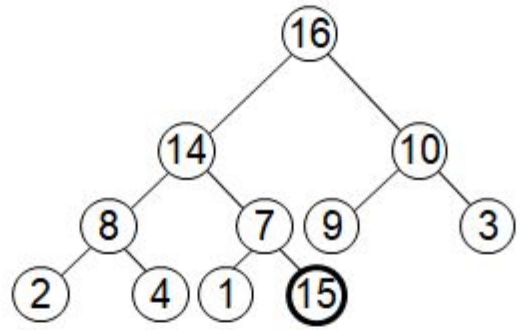


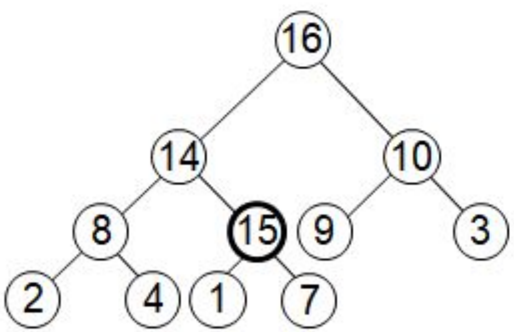
Example: MAX-HEAP-INSERT

Insert value 15:
- Start by inserting -∞

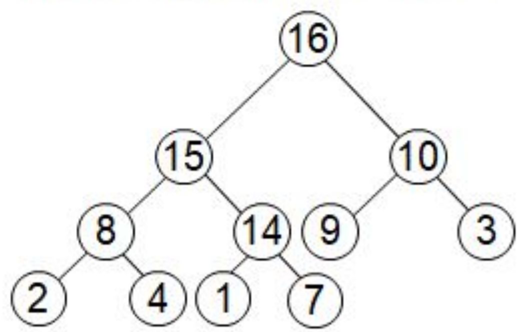


Increase the key to 15
Call HEAP-INCREASE-KEY on A[11] = 15



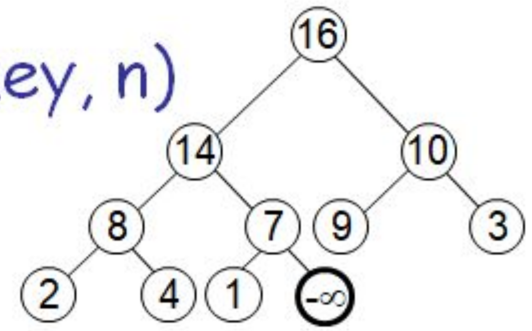


The restored heap containing
the newly added element



MAX-HEAP-INSERT

Alg: MAX-HEAP-INSERT(A, key, n)



1. heap-size[A] $\leftarrow$ n + 1
2. A[n + 1] $\leftarrow -\infty$
3. HEAP-INCREASE-KEY(A, n + 1, key)

Running time: $O(\lg n)$

Summary

- We can perform the following operations on heaps:
  - MAX-HEAPIFY $O(\lg n)$
  - BUILD-MAX-HEAP $O(n)$
  - HEAP-SORT $O(n\lg n)$
  - MAX-HEAP-INSERT $O(\lg n)$
  - HEAP-EXTRACT-MAX $O(\lg n)$
  - HEAP-INCREASE-KEY $O(\lg n)$
  - HEAP-MAXIMUM $O(1)$

Average $O(\lg n)$