DIJKSTRA'S ALGORITHM

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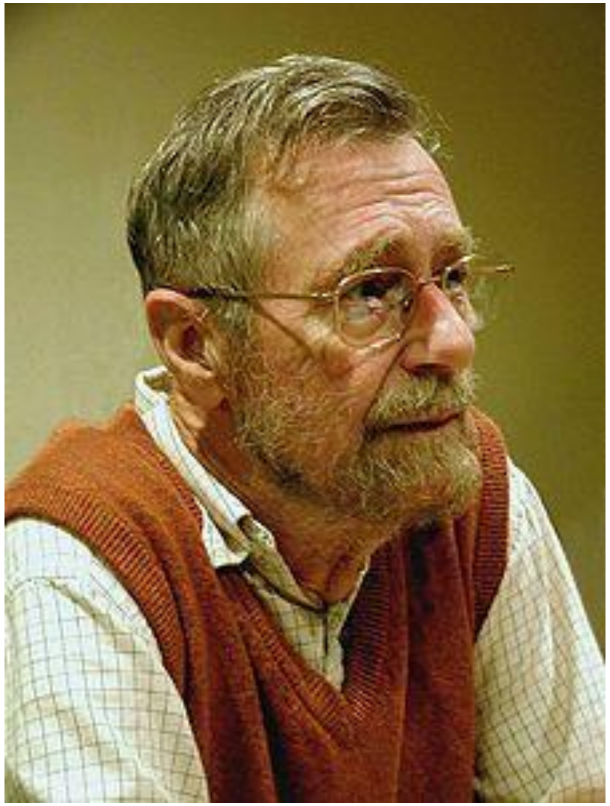
THE AUTHOR: EDSGER WYBE DIJKSTRA



"Computer Science is no more about computers than astronomy is about telescopes."

http://www.cs.utexas.edu/~EWD/

EDSGER WYBE DIJKSTRA

- May 11, 1930 – August 6, 2002

- Received the 1972 A. M. Turing Award, widely considered the most prestigious award in computer science.

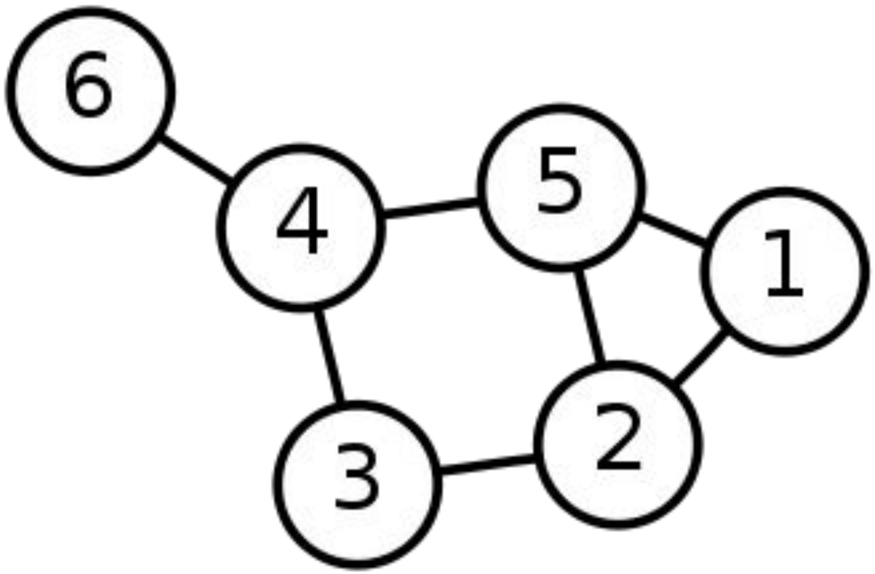
- The Schlumberger Centennial Chair of Computer Sciences at The University of Texas at Austin from 1984 until 2000

- Made a strong case against use of the GOTO statement in programming languages and helped lead to its deprecation.

- Known for his many essays on programming.

SINGLE-SOURCE SHORTEST PATH PROBLEM

Single-Source Shortest Path Problem - The problem of finding shortest paths from a source vertex *v* to all other vertices in the graph.



DIJKSTRA'S ALGORITHM

Dijkstra's algorithm - is a solution to the single-source shortest path problem in graph theory.

Works on both directed and undirected graphs. However, all edges must have nonnegative weights.

Approach: Greedy

Input: Weighted graph $G=\{E,V\}$ and source vertex $v \in V$, such that all edge weights are nonnegative

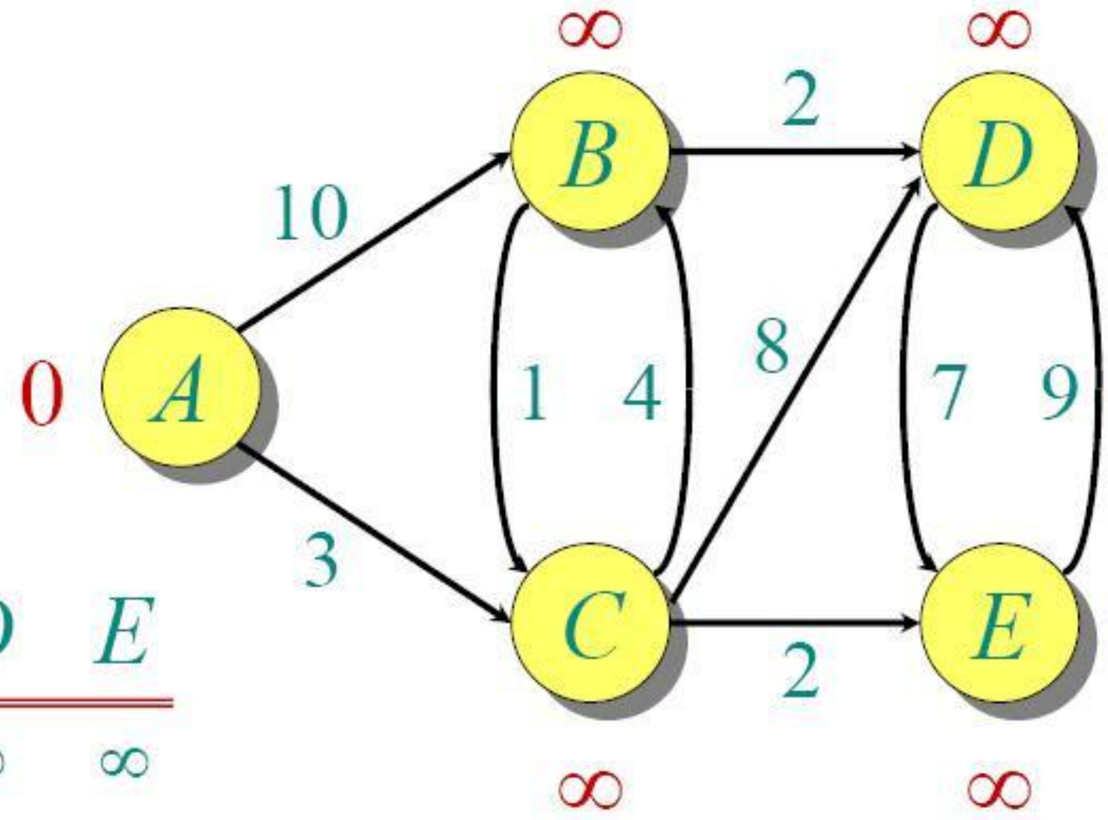
Output: Lengths of shortest paths (or the shortest paths themselves) from a given source vertex $v \in V$ to all other vertices

DIJKSTRA'S ALGORITHM - PSEUDOCODE

```
dist[s] ←0                          (distance to source vertex is zero)
for  all v ∈ V–{s}
       do  dist[v] ←∞               (set all other distances to infinity)
S←∅                                 (S, the set of visited vertices is initially empty)
Q←V                                 (Q, the queue initially contains all vertices)
while Q ≠∅                          (while the queue is not empty)
do   u ← mindistance(Q,dist)(select the element of Q with the min. distance)
     S←S∪{u}                        (add u to list of visited vertices)
      for all v ∈ neighbors[u]
           do  if   dist[v] > dist[u] + w(u, v)        (if new shortest path found)
                    then      d[v] ←d[u] + w(u, v)     (set new value of shortest path)
           (if desired, add traceback code)
return dist
```

DIJKSTRA ANIMATED EXAMPLE

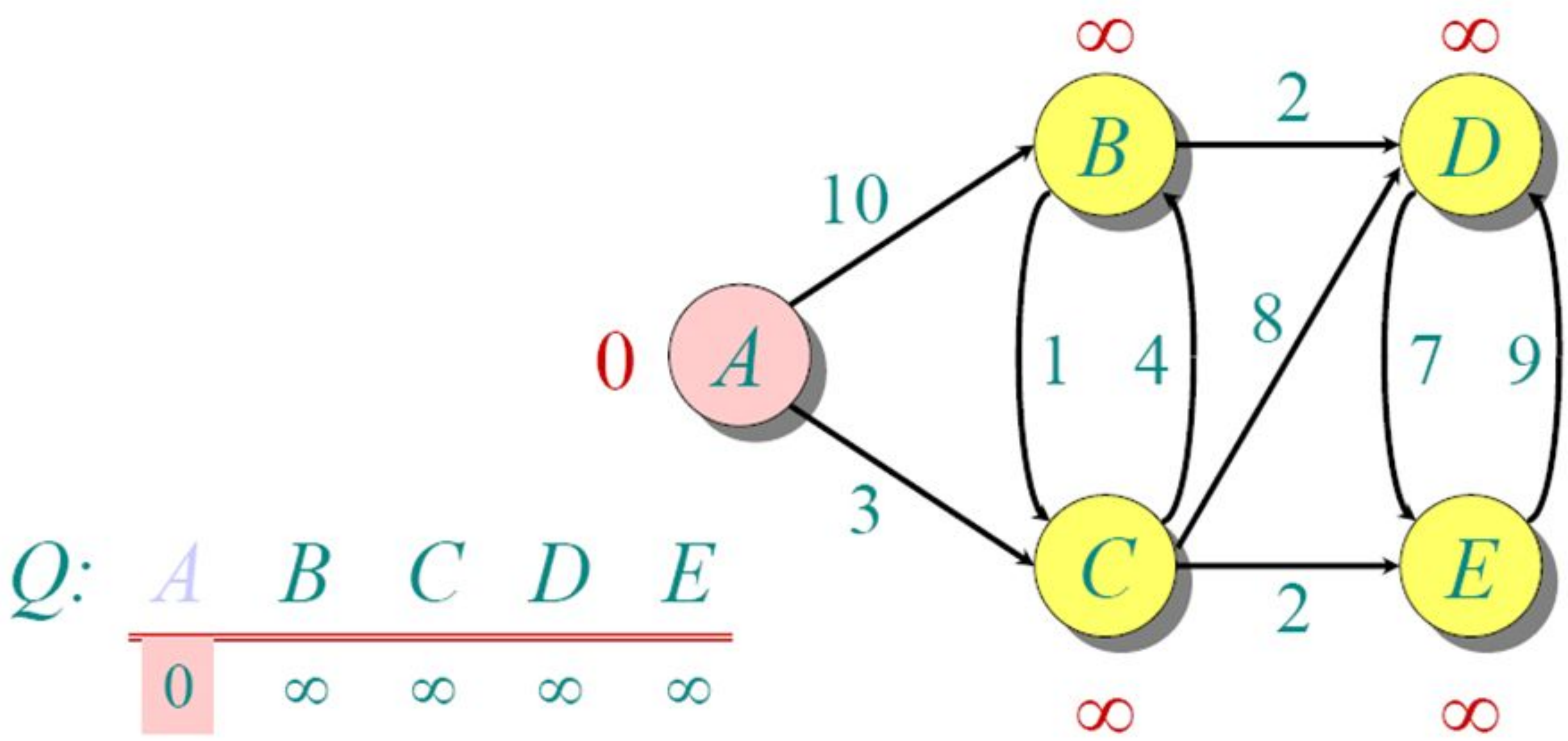
Initialize:



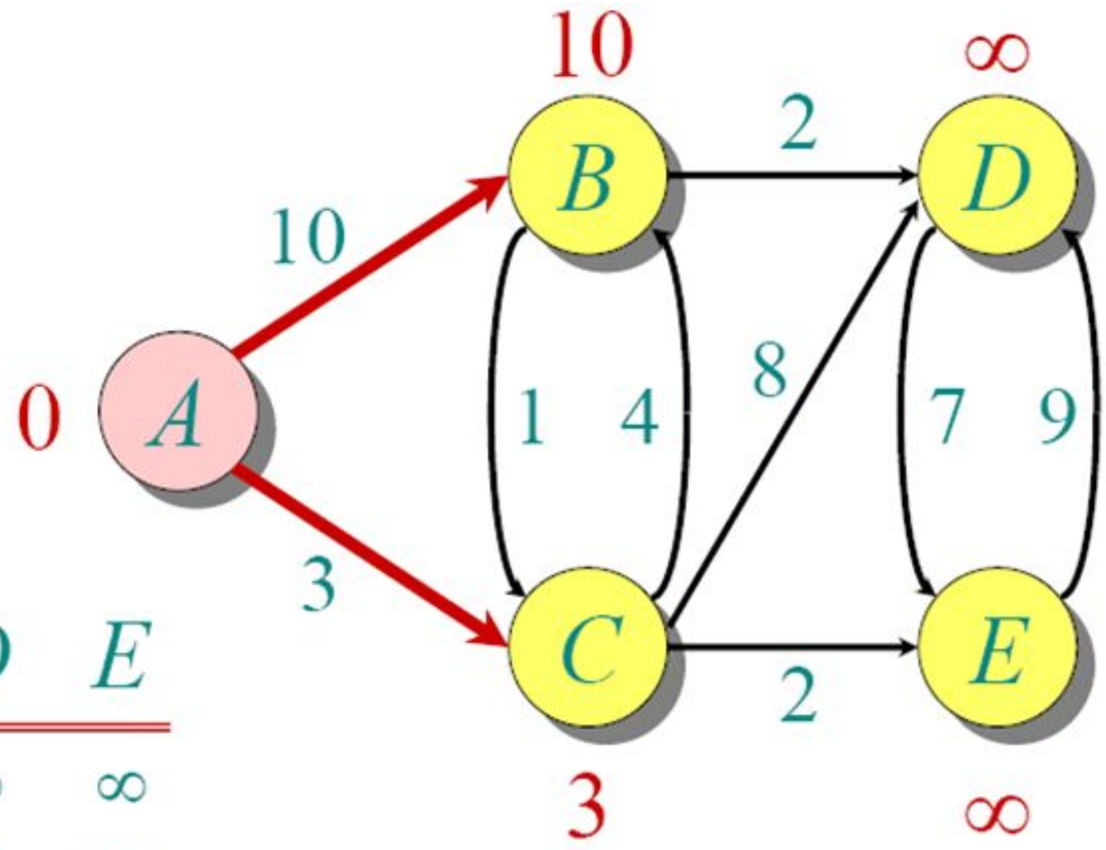
Q:	A	B	C	D	E
	0	∞	∞	∞	∞

$S: \{\}$

DIJKSTRA ANIMATED EXAMPLE



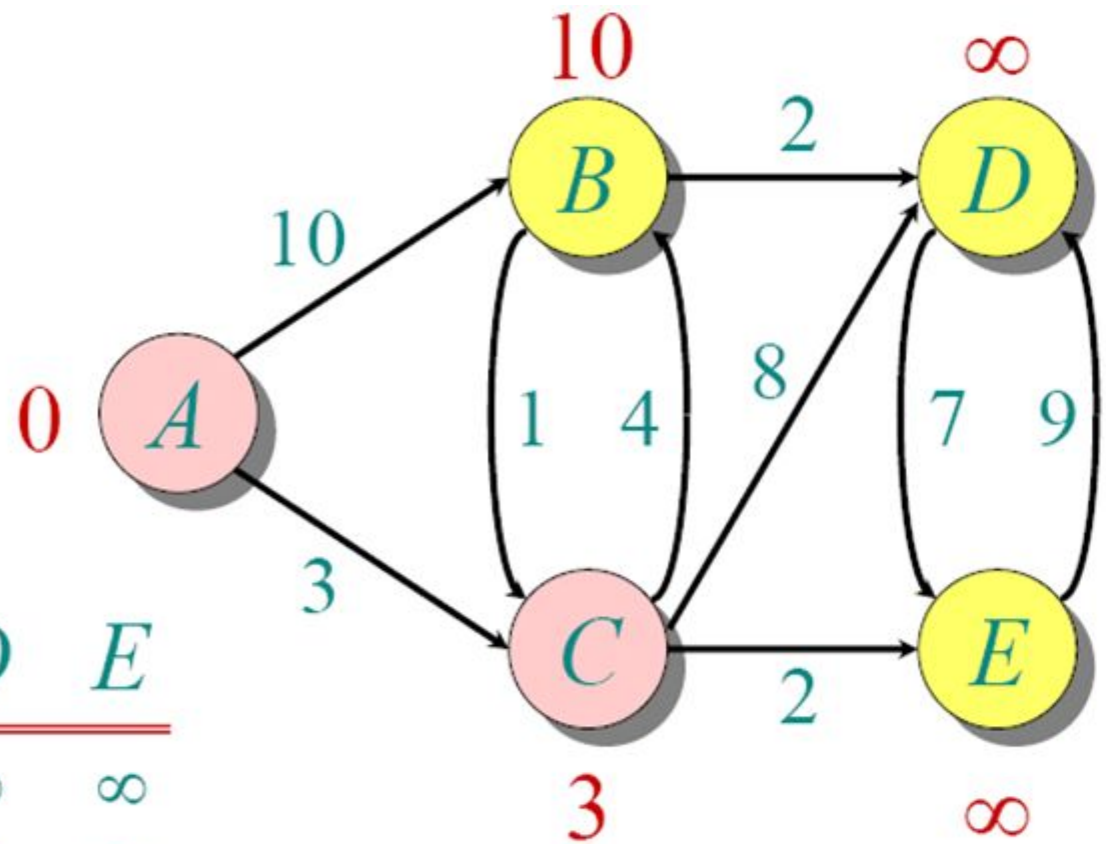
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

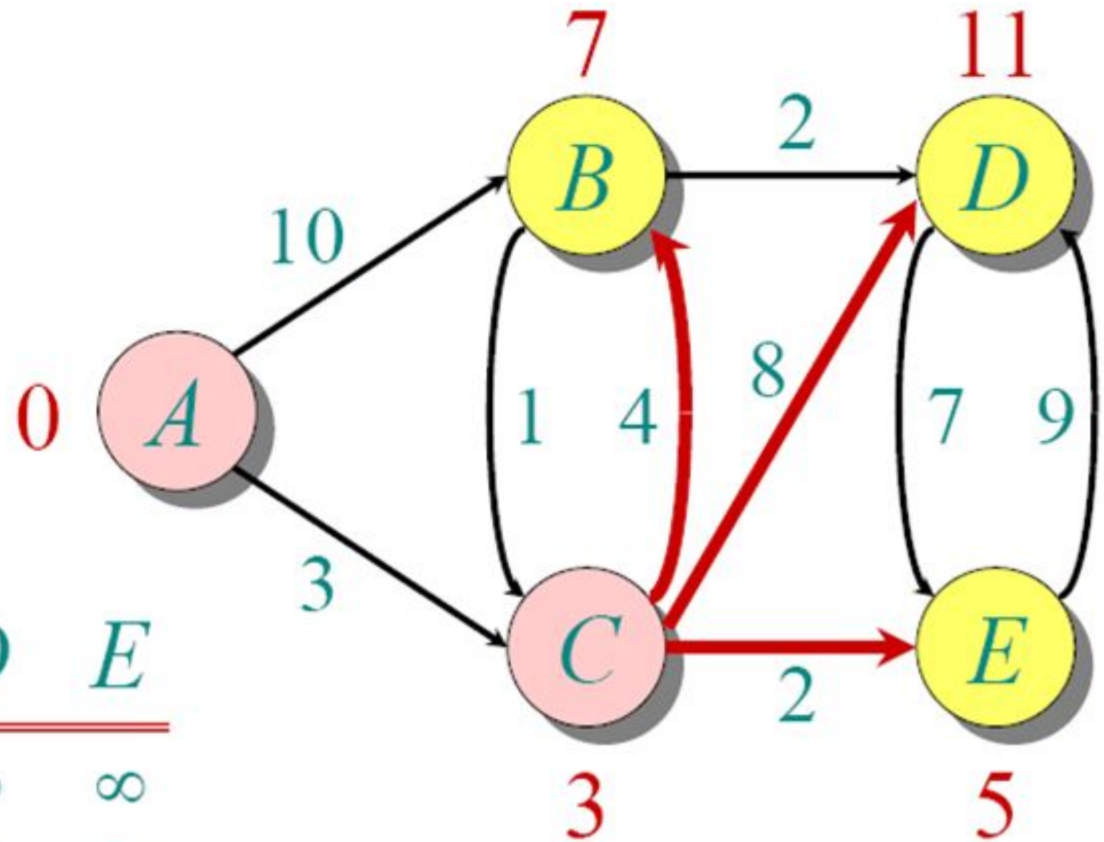
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A, C }

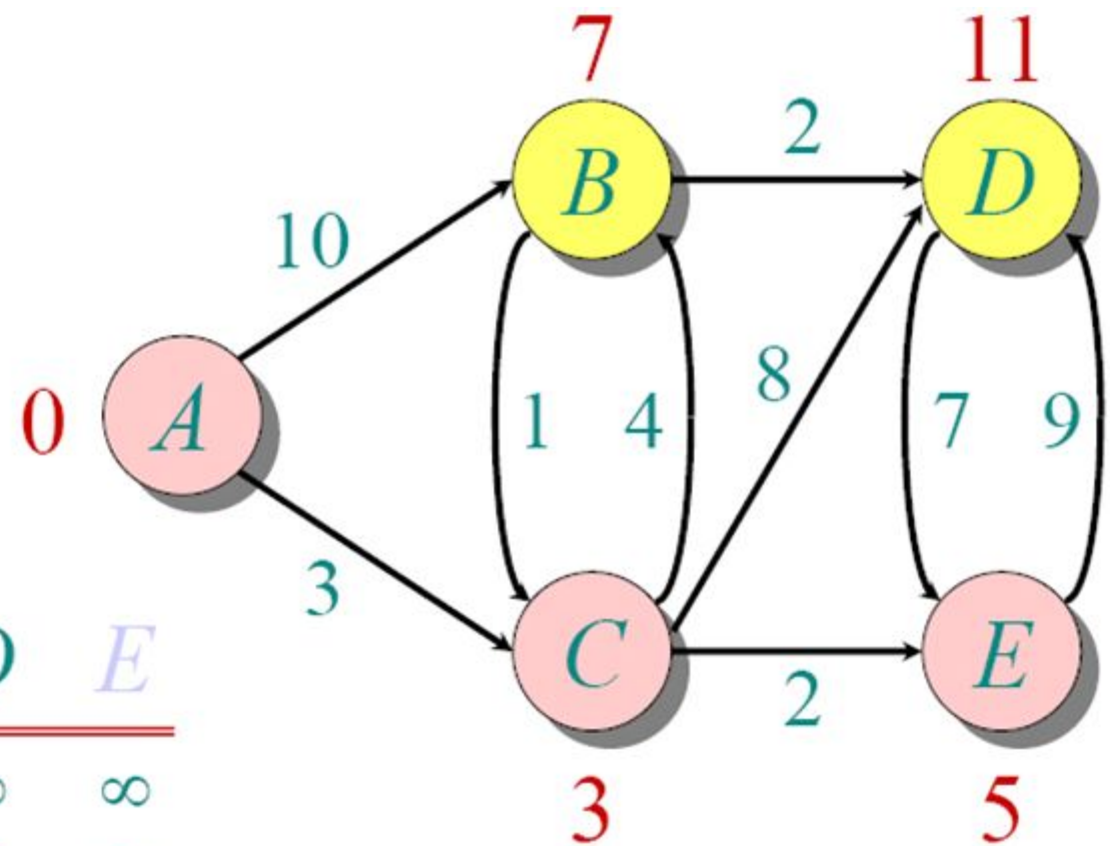
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C }

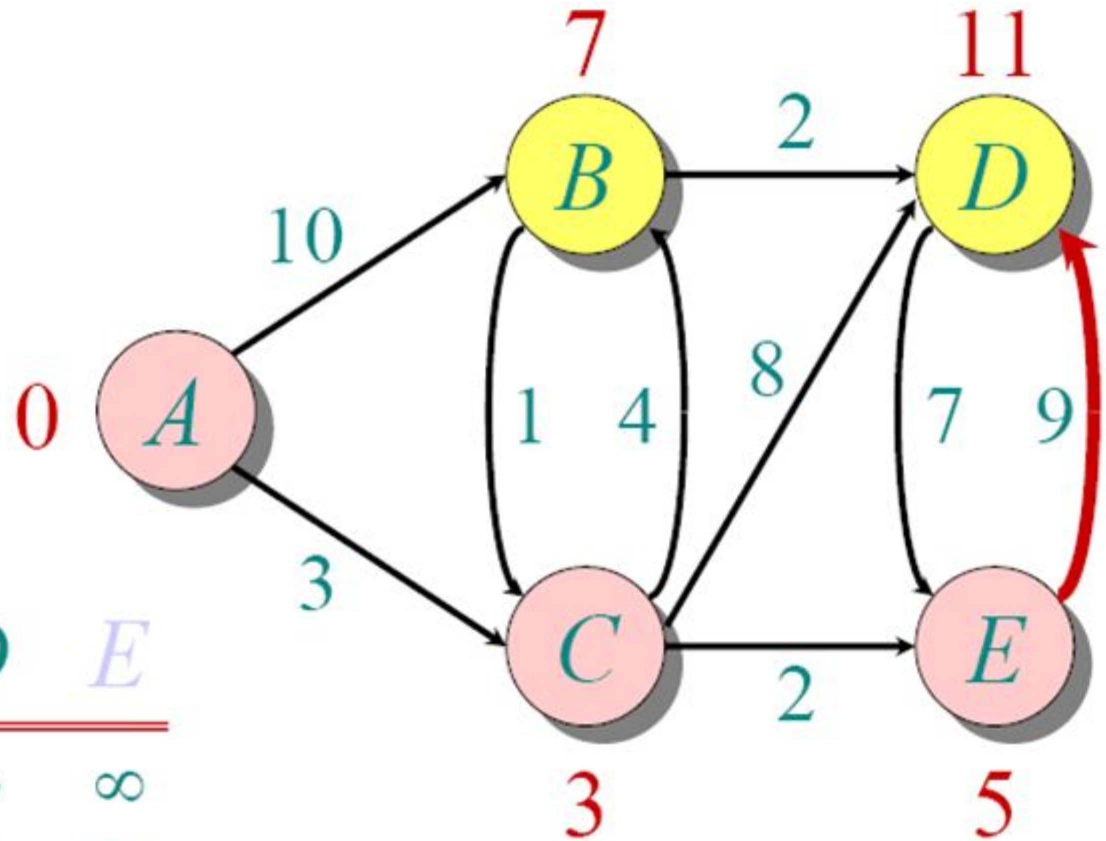
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }

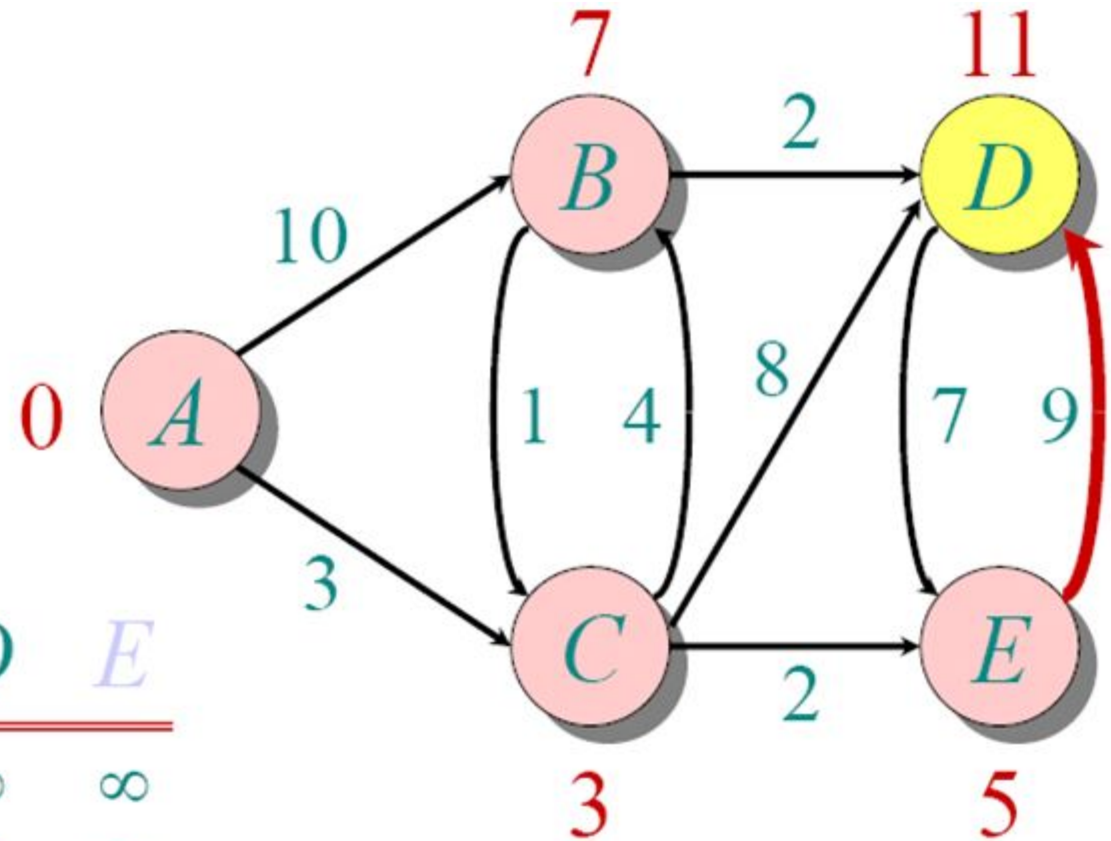
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

$S: \{ A, C, E \}$

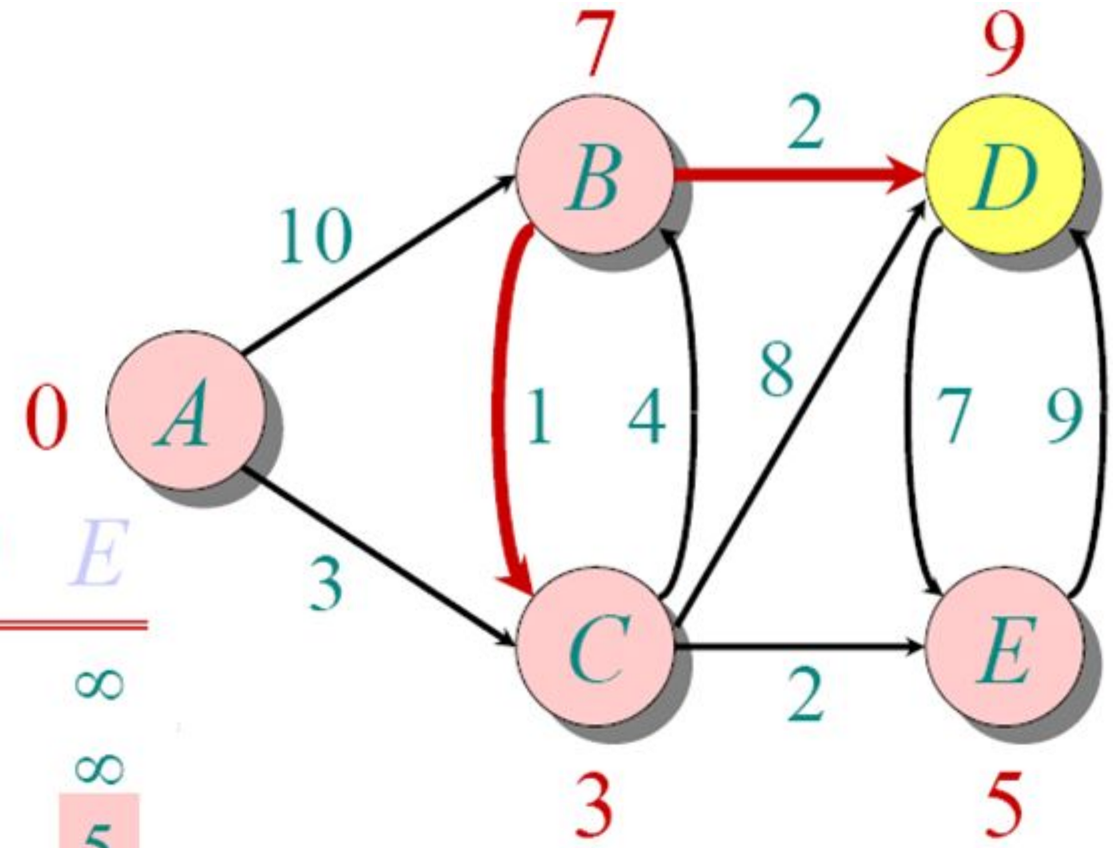
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

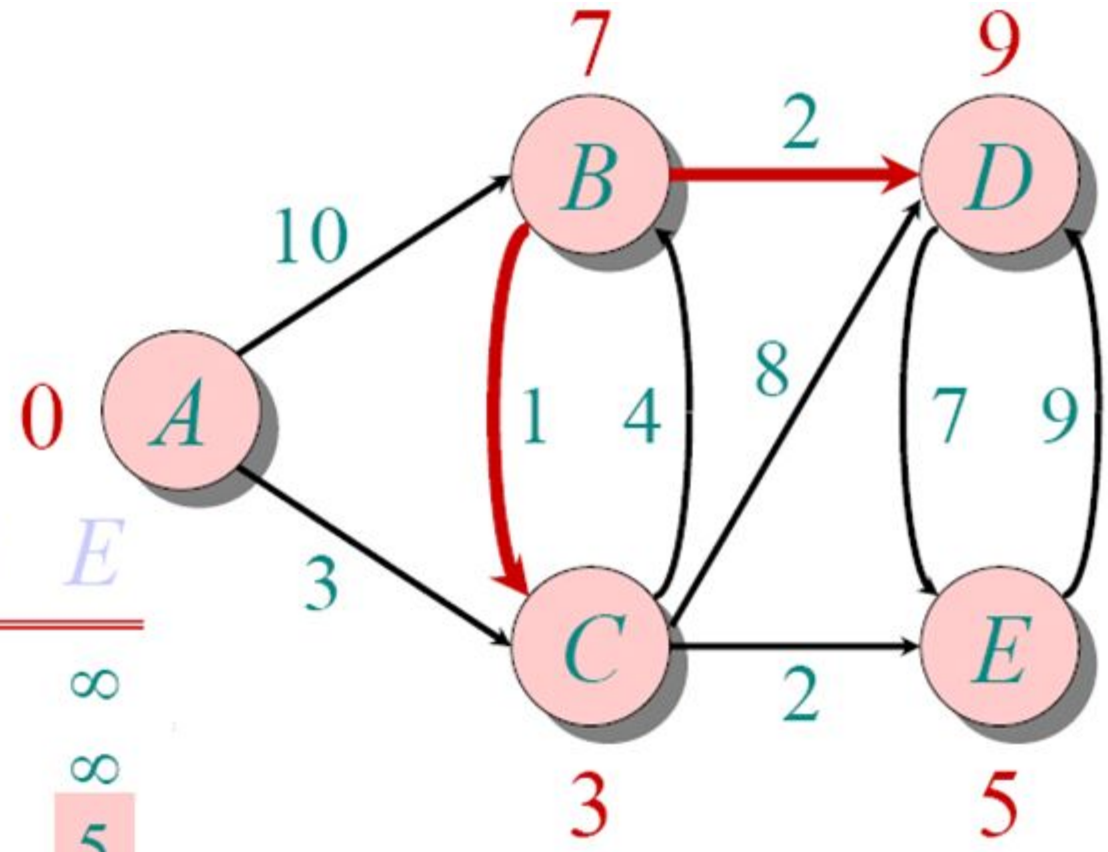
DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B }

DIJKSTRA ANIMATED EXAMPLE



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B*, *D* }

IMPLEMENTATIONS AND RUNNING TIMES

The simplest implementation is to store vertices in an array or linked list. This will produce a running time of

$O(|V|^2 + |E|)$

For sparse graphs, or graphs with very few edges and many nodes, it can be implemented more efficiently storing the graph in an adjacency list using a binary heap or priority queue. This will produce a running time of

$O((|E|+|V|) \log |V|)$

DIJKSTRA'S ALGORITHM - WHY IT WORKS

- As with all greedy algorithms, we need to make sure that it is a correct algorithm (e.g., it *always* returns the right solution if it is given correct input).

- A formal proof would take longer than this presentation, but we can understand how the argument works intuitively.

- If you can’t sleep unless you see a proof, see the second reference or ask us where you can find it.

DIJKSTRA'S ALGORITHM - WHY IT WORKS

- To understand how it works, we'll go over the previous example again. However, we need two mathematical results first:

- **Lemma 1**: Triangle inequality
  If δ(u,v) is the shortest path length between u and v,
  δ(u,v) ≤ δ(u,x) + δ(x,v)
- **Lemma 2**:
  The subpath of any shortest path is itself a shortest path.

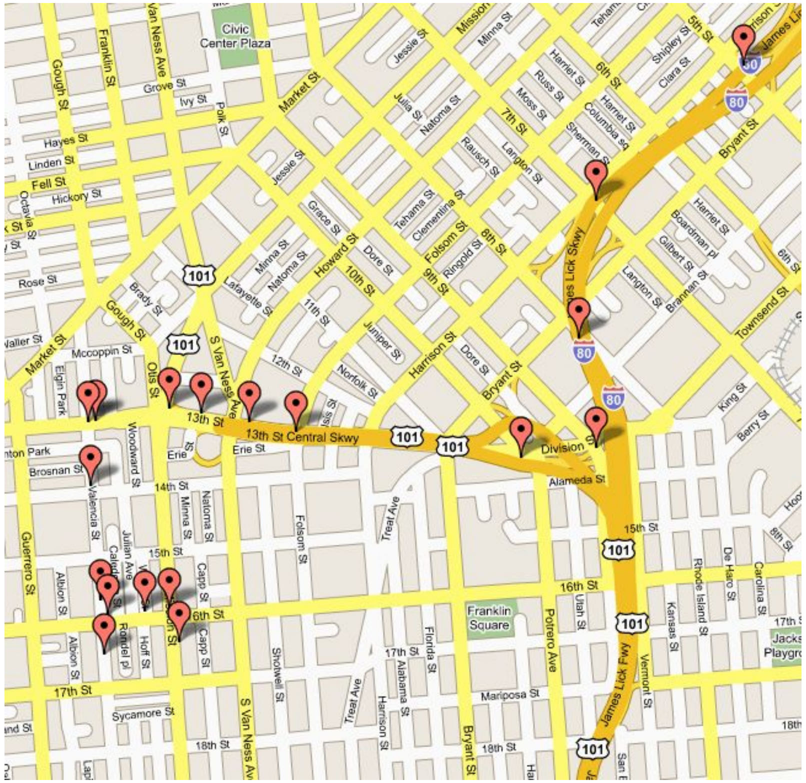
- The key is to understand why we can claim that anytime we put a new vertex in S, we can say that we already know the shortest path to it.
- Now, back to the example…

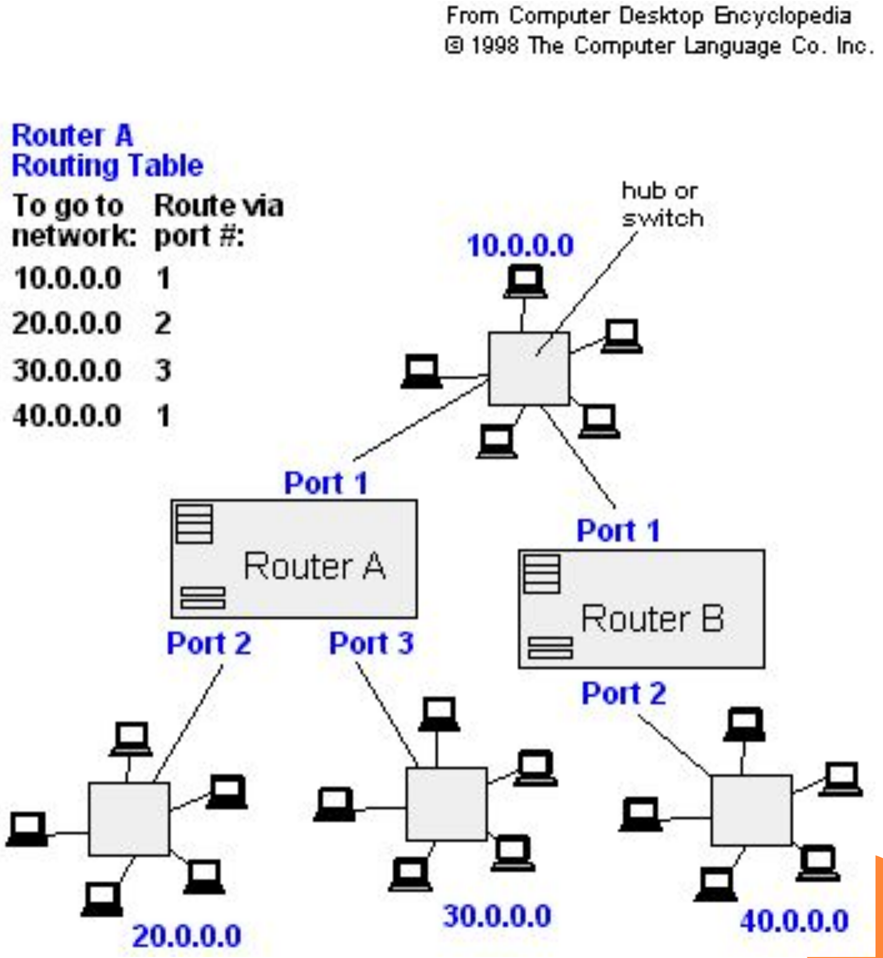
DIJKSTRA'S ALGORITHM - WHY USE IT?

- As mentioned, Dijkstra’s algorithm calculates the shortest path to every vertex.
- However, it is about as computationally expensive to calculate the shortest path from vertex u to every vertex using Dijkstra’s as it is to calculate the shortest path to some particular vertex v.
- Therefore, anytime we want to know the optimal path to some other vertex from a determined origin, we can use Dijkstra’s algorithm.

APPLICATIONS OF DIJKSTRA'S ALGORITHM

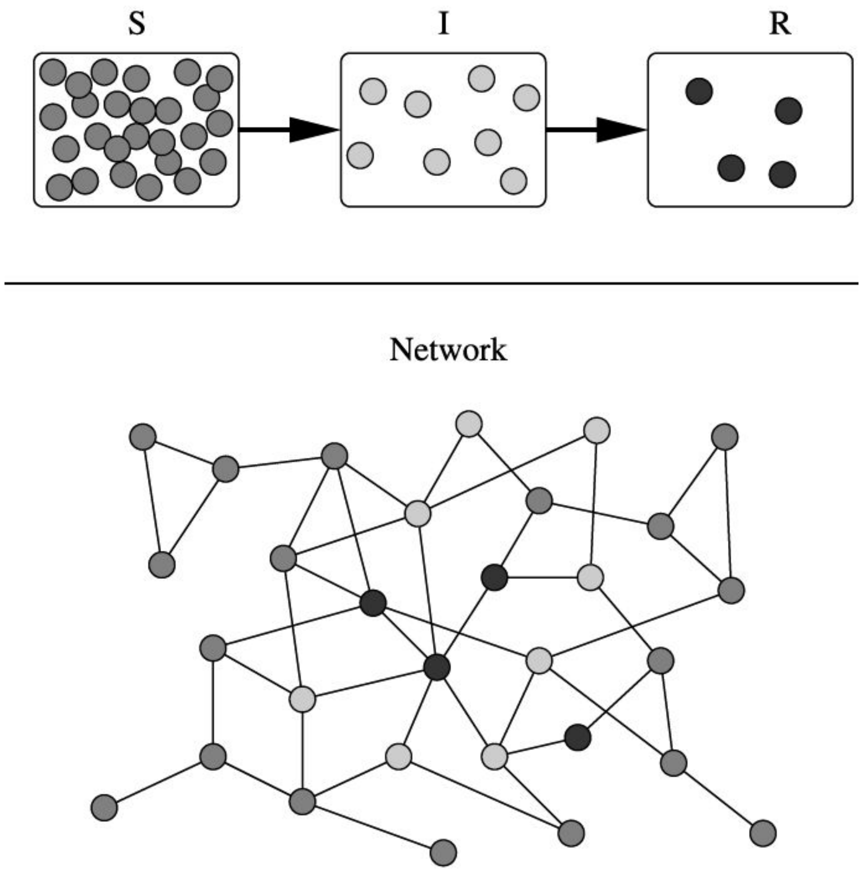
- Traffic Information Systems are most prominent use
- Mapping (Map Quest, Google Maps)
- Routing Systems





APPLICATIONS OF DIJKSTRA'S ALGORITHM

- One particularly relevant this week: epidemiology
- Prof. Lauren Meyers (Biology Dept.) uses networks to model the spread of infectious diseases and design prevention and response strategies.
- Vertices represent individuals, and edges their possible contacts. It is useful to calculate how a particular individual is connected to others.
- Knowing the shortest path lengths to other individuals can be a relevant indicator of the potential of a particular individual to infect others.



SHORTEST PATH PROBLEM

- Weighted path length (cost): The sum of the weights of all links on the path.

- The single-source shortest path problem: Given a weighted graph G and a source vertex s, find the shortest (minimum cost) path from s to every other vertex in G.

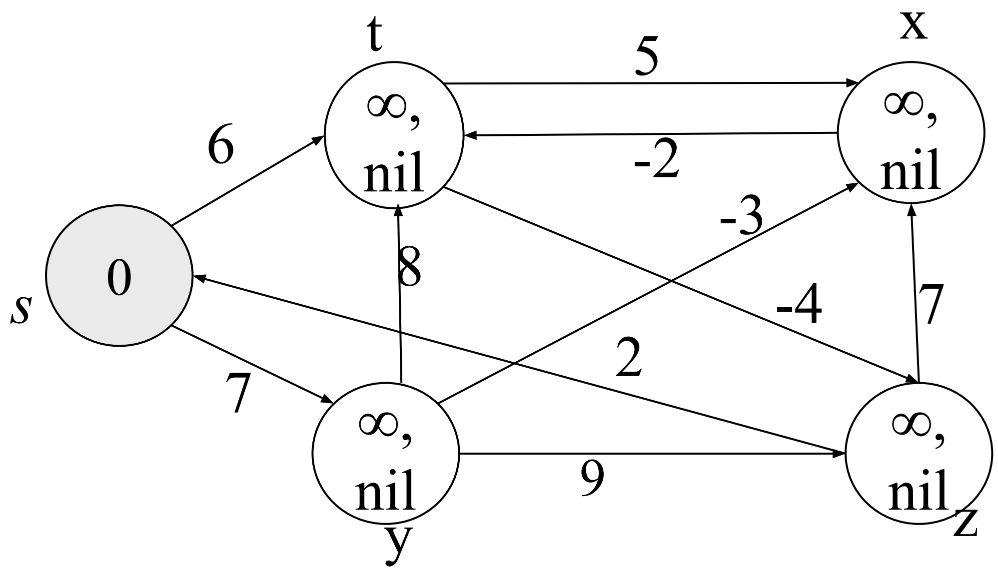
DIFFERENCES

- Negative link weight: The Bellman-Ford algorithm works; Dijkstra's algorithm doesn't.

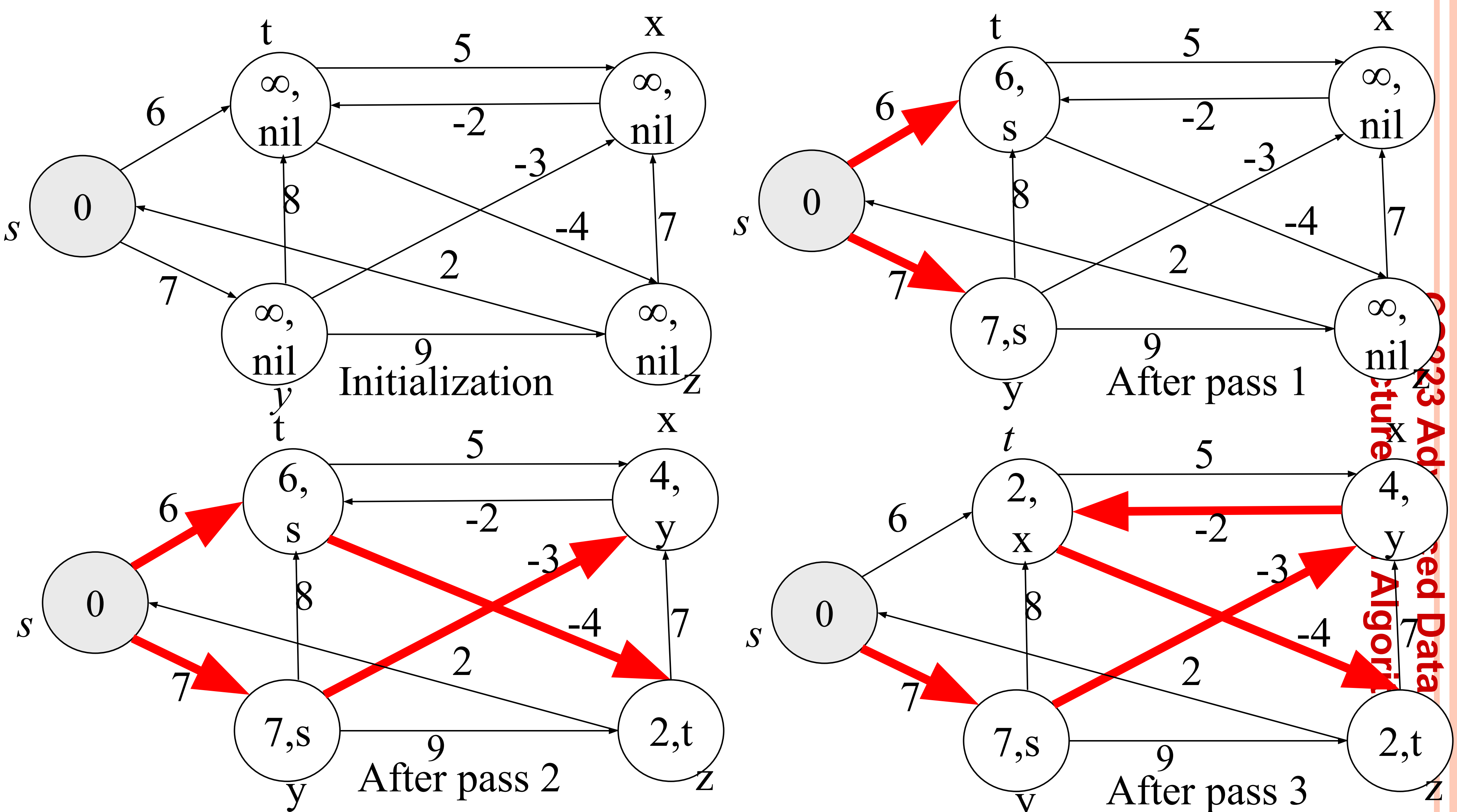
- Distributed implementation: The Bellman-Ford algorithm can be easily implemented in a distributed way. Dijkstra's algorithm cannot.

- Time complexity: The Bellman-Ford algorithm is higher than Dijkstra's algorithm.

THE BELLMAN-FORD ALGORITHM



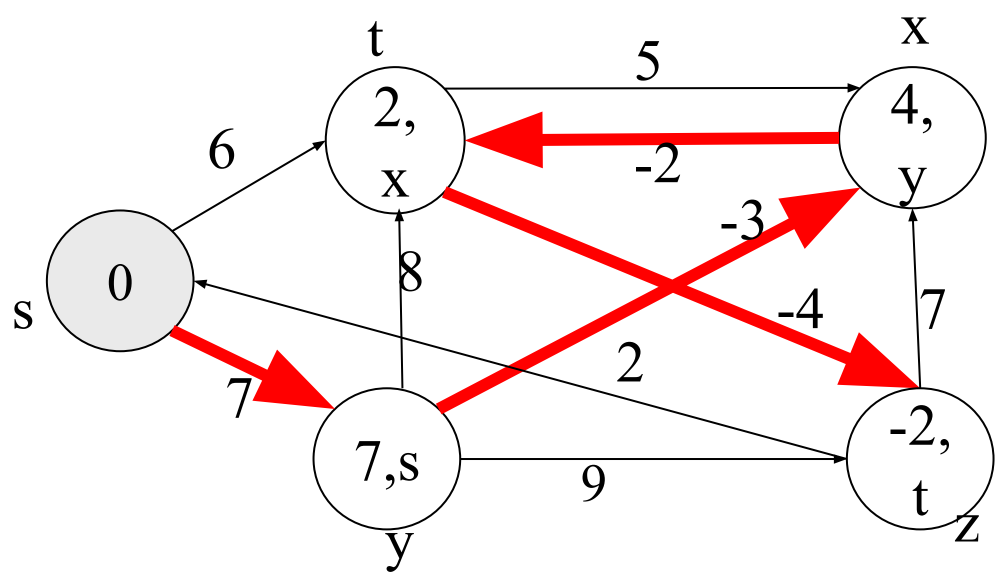
THE BELLMAN-FORD ALGORITHM



The order of edges examined in each pass:

(t, x), (t, z), (x, t), (y, x), (y, t), (y, z), (z, x), (z, s), (s, t), (s, y)

THE BELLMAN-FORD ALGORITHM



After pass 4

The order of edges examined in each pass:

(t, x), (t, z), (x, t), (y, x), (y, t), (y, z), (z, x), (z, s), (s, t), (s, y)

THE BELLMAN-FORD ALGORITHM

```
Bellman-Ford(G, w, s)
    Initialize-Single-Source(G, s)
    for i := 1 to |V| - 1 do
        for each edge (u, v) ∈ E do
            Relax(u, v, w)
    for each vertex v ∈ u.adj do
        if d[v] > d[u] + w(u, v)
            then return False      // there is a negative cycle
    return True
```

```
Relax(u, v, w)
if d[v] > d[u] + w(u, v)
    then  d[v] := d[u] + w(u, v)
          parent[v] := u
```

TIME COMPLEXITY

```
Bellman-Ford(G, w, s)
    Initialize-Single-Source(G, s)                  O(|V|)
    for i := 1 to |V| - 1 do
        for each edge (u, v) ∈ E do
            Relax(u, v, w)                          O(|V||E|)
    for each vertex v ∈ u.adj do                    O(|E|)
        if d[v] > d[u] + w(u, v)
            then return False    // there is a negative cycle
    return True
```

Time complexity: $O(|V||E|)$

Algorithm	Negative Edge Weights	Positive Edge Weights > 1	Undirected Cycles	Runtime
DFS	✔	✔	✘	$O(n + e)$
BFS	✘	✘	✔	$O(n + e)$ or $O(g^d)$
Dijkstra	✘	✔	✔	$O(e + n \log(n))$
Bellman-Ford	✔	✔	✔	$O(n * e)$

REFERENCES

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