# Asymptotic Analysis

# Analysis of Algorithms

- An *algorithm* is a finite set of precise instructions for performing a computation or for solving a problem.
- What is the goal of analysis of algorithms?
  - To compare algorithms mainly in terms of running time but also in terms of other factors (e.g., memory requirements, programmer's effort etc.)
- What do we mean by running time analysis?
  - **Determine how running time increases as the size of the problem increases.**

# Input Size

- Input size (number of elements in the input)
  - size of an array
  - polynomial degree
  - # of elements in a matrix
  - # of bits in the binary representation of the input
  - vertices and edges in a graph

# Types of Analysis

- Worst case
  - Provides an upper bound on running time
  - An absolute guarantee that the algorithm would not run longer, no matter what the inputs are
- Best case
  - Provides a lower bound on running time
  - Input is the one for which the algorithm runs the fastest

$$\boxed{Lower\ Bound \le Running\ Time \le Upper\ Bound}$$

- Average case
  - Provides a prediction about the running time
  - Assumes that the input is random

# How do we compare algorithms?

- We need to define a number of <u>objective measures</u>.

(1) Compare execution times?

***Not good***: times are specific to a particular computer !!

(2) Count the number of statements executed?

***Not good***: number of statements vary with the programming language as well as the style of the individual programmer.

# Ideal Solution

- Express running time as a function of the input size *n* (i.e., *f(n)*).
- Compare different functions corresponding to running times.
- Such an analysis is independent of machine time, programming style, etc.

# Example

- Associate a "cost" with each statement.
- Find the "total cost" by finding the total number of times each statement is executed.

***Algorithm 1***

| | Cost |
|---|---|
| arr[0] = 0; | $c_1$ |
| arr[1] = 0; | $c_1$ |
| arr[2] = 0; | $c_1$ |
| ... | ... |
| arr[N-1] = 0; | $c_1$ |

$c_1+c_1+\ldots+c_1 = c_1 \times N$

***Algorithm 2***

| | Cost |
|---|---|
| for(i=0; i<N; i++) | $c_2$ |
| arr[i] = 0; | $c_1$ |

$(N+1) \times c_2 + N \times c_1 =$

$(c_2 + c_1) \times N + c_2$

# Another Example

- ***Algorithm 3*** *Cost*

| | Cost |
|---|---|
| `sum = 0;` | $c_1$ |
| `for(i=0; i<N; i++)` | $c_2$ |
| `  for(j=0; j<N; j++)` | $c_2$ |
| `    sum += arr[i][j];` | $c_3$ |
| | ------------ |

$$c_1 + c_2 \times (N+1) + c_2 \times N \times (N+1) + c_3 \times N^2$$

# Asymptotic Analysis

- To compare two algorithms with running times *f(n)* and *g(n)*, we need a **rough measure** that characterizes **how fast each function grows.**
- <u>*Hint:*</u> use *rate of growth*
- Compare functions in the limit, that is, **asymptotically!**

  (i.e., for large values of *n*)

# Rate of Growth

- Consider the example of buying *elephants* and *goldfish*:

  **Cost**: cost_of_elephants + cost_of_goldfish

  **Cost** ~ cost_of_elephants (approximation)

- The low order terms in a function are relatively insignificant for **large** $n$

$$n^4 + 100n^2 + 10n + 50 \quad \sim \quad n^4$$

*i.e.*, we say that $n^4 + 100n^2 + 10n + 50$ and $n^4$ have the same **rate of growth**

# Asymptotic Notation

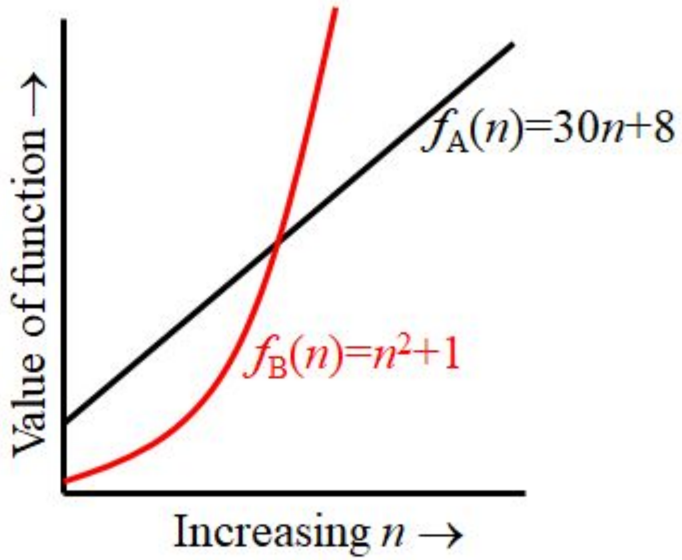
- O notation: asymptotic "less than":
  - $f(n)=O(g(n))$ implies: $f(n)$ "$\leq$" $g(n)$
- $\Omega$ notation: asymptotic "greater than":
  - $f(n)=\Omega(g(n))$ implies: $f(n)$ "$\geq$" $g(n)$
- $\Theta$ notation: asymptotic "equality":
  - $f(n)=\Theta(g(n))$ implies: $f(n)$ "$=$" $g(n)$

# Big-O Notation

- We say $f_A(n)=30n+8$ is *order n*, or O (n)
  It is, at most, roughly *proportional* to $n$.
- $f_B(n)=n^2+1$ is *order* $n^2$, or $O(n^2)$. It is, at most, roughly proportional to $n^2$.
- In general, any $O(n^2)$ function is faster-growing than any $O(n)$ function.

# Visualizing Orders of Growth

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



# More Examples …

- $n^4 + 100n^2 + 10n + 50$ is $O(n^4)$
- $10n^3 + 2n^2$ is $O(n^3)$
- $n^3 - n^2$ is $O(n^3)$
- constants
  - 10 is $O(1)$
  - 1273 is $O(1)$

# Back to Our Example

| *Algorithm 1* | Cost | *Algorithm 2* | Cost |
|---|---|---|---|
| arr[0] = 0; | $c_1$ | for(i=0; i<N; i++) | $c_2$ |
| arr[1] = 0; | $c_1$ | arr[i] = 0; | $c_1$ |
| arr[2] = 0; | $c_1$ | | |
| ... | | | |
| arr[N-1] = 0; | $c_1$ | | |
| | ---------- | | ------------ |
| $c_1+c_1+\ldots+c_1 = c_1 \times N$ | | $(N+1) \times c_2 + N \times c_1 = (c_2 + c_1) \times N + c_2$ | |

- Both algorithms are of the same order: *O(N)*

# Example (cont'd)

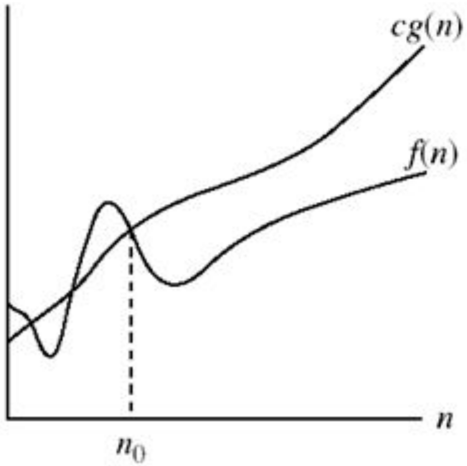
| ***Algorithm 3*** | ***Cost*** |
| --- | --- |
| `sum = 0;` | $c_1$ |
| `for(i=0; i<N; i++)` | $c_2$ |
| `  for(j=0; j<N; j++)` | $c_2$ |
| `  sum += arr[i][j];` | $c_3$ |
| | ------------ |

$c_1 + c_2 \times (N+1) + c_2 \times N \times (N+1) + c_3 \times N^2 = O(N^2)$

# Asymptotic notations

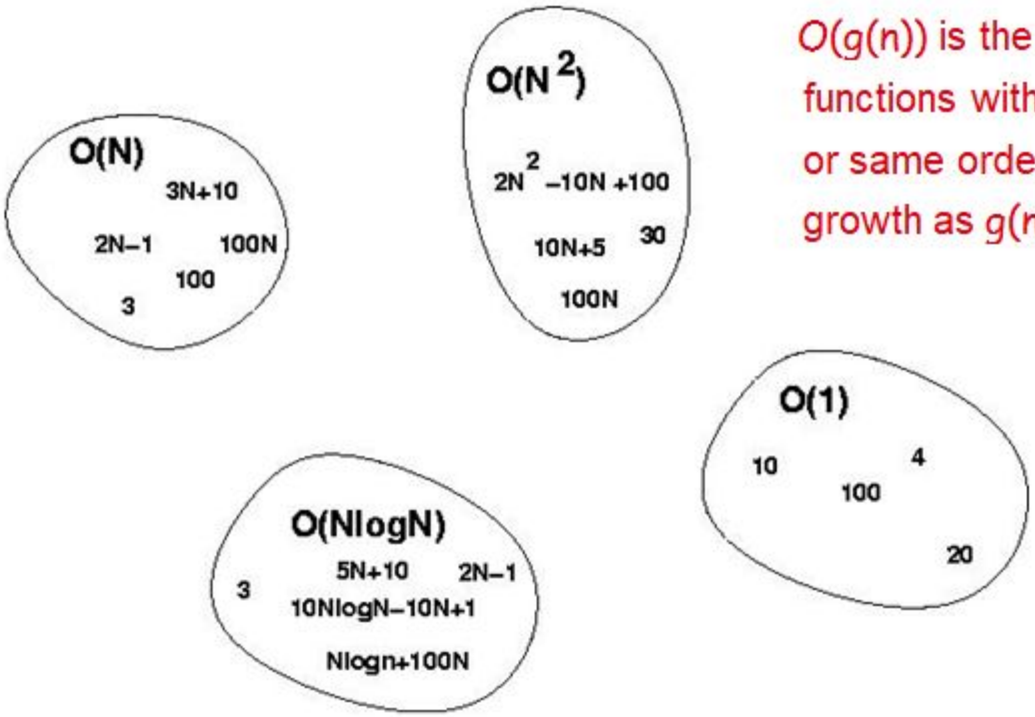
- *O-notation*

$O(g(n)) = \{f(n) :$ there exist positive constants $c$ and $n_0$ such that
$0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$ .




$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

# Big-O Visualization




*O*(*g*(n)) is the set of functions with smaller or same order of growth as *g*(n)

# Examples

- $2n^2 = O(n^3)$: $2n^2 \le cn^3 \Rightarrow 2 \le cn \Rightarrow c = 1$ and $n_0 = 2$

- $n^2 = O(n^2)$: $n^2 \le cn^2 \Rightarrow c \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

- $1000n^2 + 1000n = O(n^2)$:

$1000n^2 + 1000n \le 1000n^2 + n^2 = 1001n^2 \Rightarrow c = 1001$ and $n_0 = 1000$

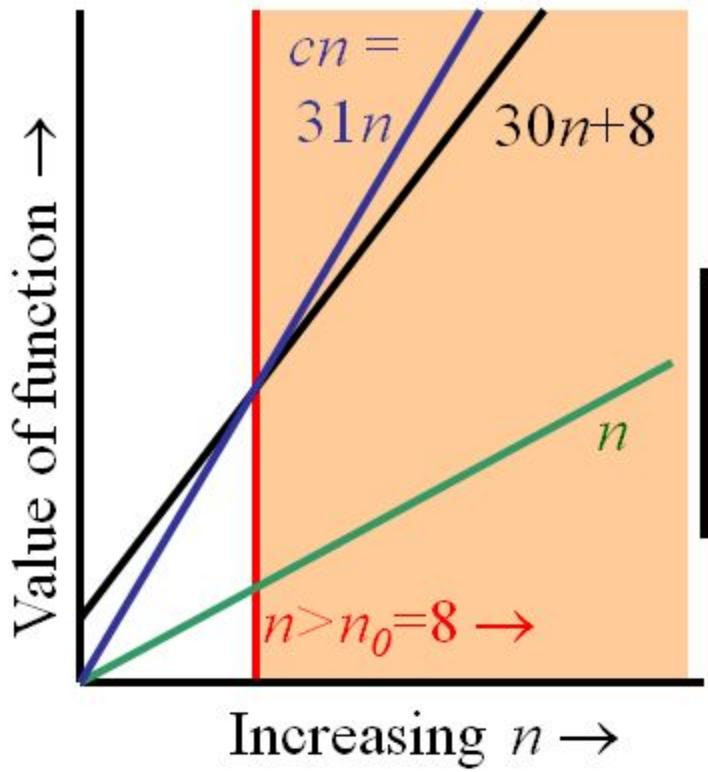
- $n = O(n^2)$: $n \le cn^2 \Rightarrow cn \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

# More Examples

- Show that $30n+8$ is $O(n)$.
  - Show $\exists c, n_0$: $30n+8 \leq cn$, $\forall n > n_0$ .
    - Let $c=31$, $n_0=8$. Assume $n>n_0=8$. Then $cn = 31n = 30n + n > 30n+8$, so $30n+8 < cn$.

# Big-O example, graphically

- Note $30n+8$ isn't less than $n$ *anywhere* ($n>0$).
- It isn't even less than $31n$ *everywhere*.
- But it *is* less than $31n$ everywhere to the right of $n=8$.




$30n+8 \in O(n)$

# No Uniqueness

- There is no unique set of values for $n_0$ and $c$ in proving the asymptotic bounds
- Prove that $100n + 5 = O(n^2)$
  - $100n + 5 \leq 100n + n = 101n \leq 101n^2$

    for all $n \geq 5$

    $n_0 = 5$ and $c = 101$ is a solution
  - $100n + 5 \leq 100n + 5n = 105n \leq 105n^2$

    for all $n \geq 1$

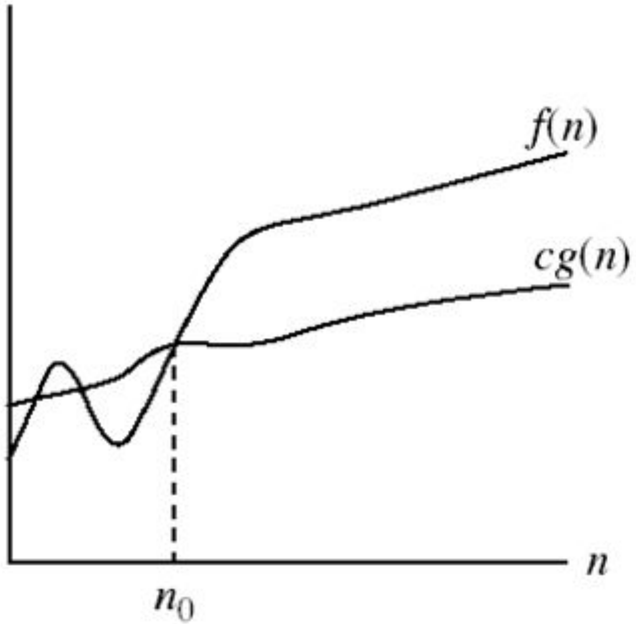
    $n_0 = 1$ and $c = 105$ is also a solution

Must find **SOME** constants c and $n_0$ that satisfy the asymptotic notation relation

# Asymptotic notations (cont.)

- *$\Omega$ - notation*

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$




$\Omega$(g(n)) is the set of functions with larger or same order of growth as g(n)

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

# Examples

- $5n^2 = \Omega(n)$

  $\exists\ c, n_0$ such that: $0 \leq cn \leq 5n^2 \Rightarrow cn \leq 5n^2 \Rightarrow c = 1$ and $n_0 = 1$

- $100n + 5 \neq \Omega(n^2)$

  $\exists\ c, n_0$ such that: $0 \leq cn^2 \leq 100n + 5$

  $100n + 5 \leq 100n + 5n\ (\forall\ n \geq 1) = 105n$

  $cn^2 \leq 105n \Rightarrow n(cn - 105) \leq 0$

  Since n is positive $\Rightarrow cn - 105 \leq 0 \Rightarrow n \leq 105/c$

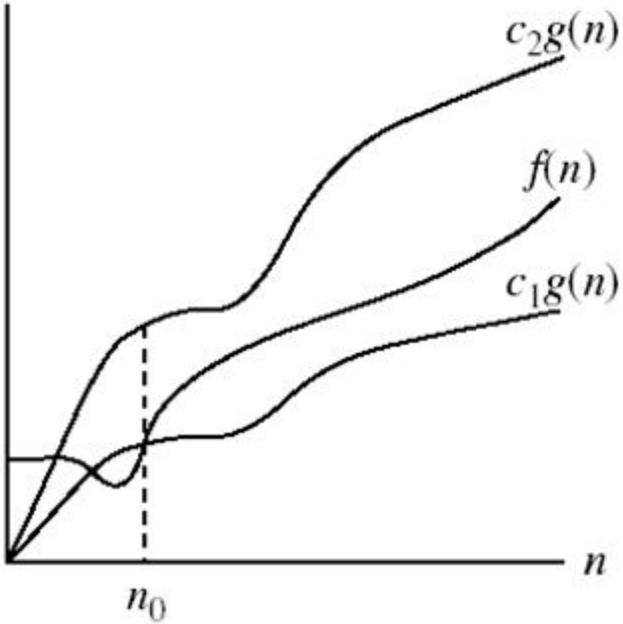
  $\Rightarrow$ contradiction: $n$ cannot be smaller than a constant

- $n = \Omega(2n)$, $n^3 = \Omega(n^2)$, $n = \Omega(\log n)$

# Asymptotic notations (cont.)

- $\Theta$*-notation*

$\Theta(g(n)) = \{f(n) :$ there exist positive constants $c_1$, $c_2$, and $n_0$ such that
$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$ .




$\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

# Examples

- $n^2/2 - n/2 = \Theta(n^2)$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \leq \frac{1}{2} n^2 \; \forall n \geq 0 \quad \Rightarrow \quad c_2 = \frac{1}{2}$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \geq \frac{1}{2} n^2 - \frac{1}{2} n * \frac{1}{2} n \; ( \forall n \geq 2 ) = \frac{1}{4} n^2$

    $\Rightarrow \quad c_1 = \frac{1}{4}$

- $n \neq \Theta(n^2)$: $c_1 n^2 \leq n \leq c_2 n^2$

  $\Rightarrow$ only holds for: $n \leq 1/c_1$

# Examples

- $6n^3 \neq \Theta(n^2)$: $c_1 n^2 \leq 6n^3 \leq c_2 n^2$

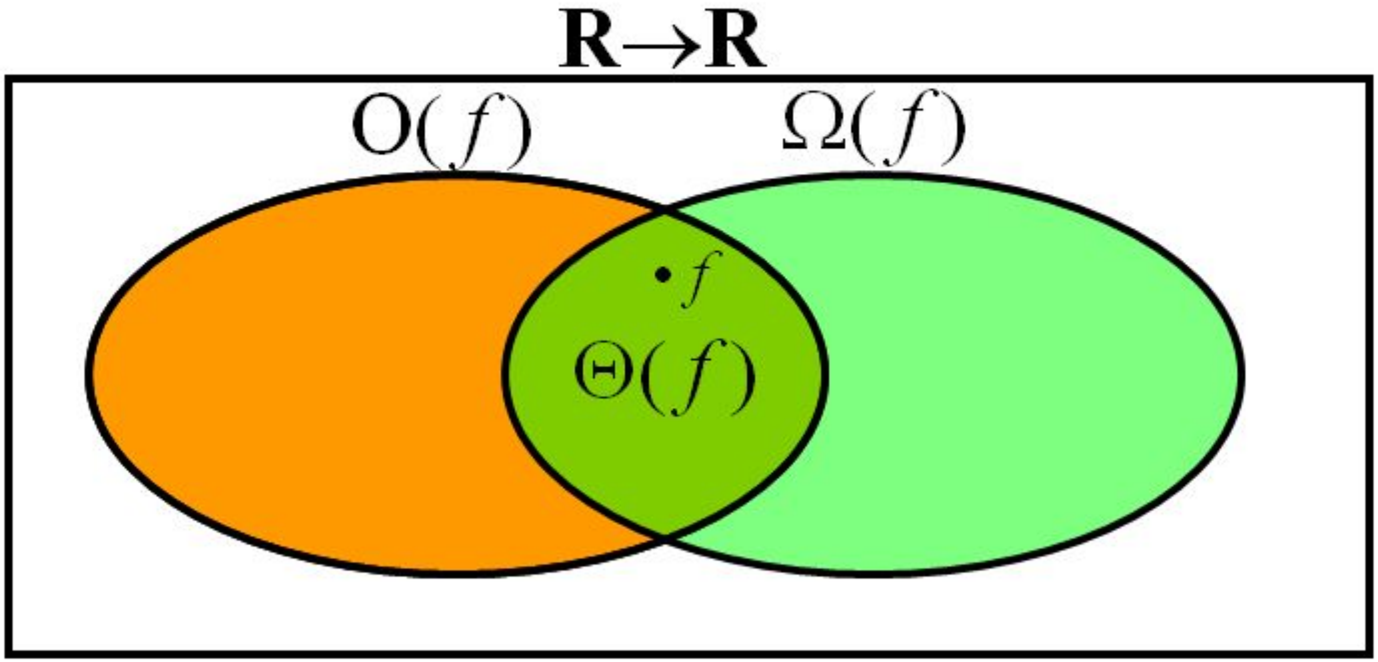
  $\Rightarrow$ only holds for: $n \leq c_2 / 6$

- $n \neq \Theta(\log n)$: $c_1 \log n \leq n \leq c_2 \log n$

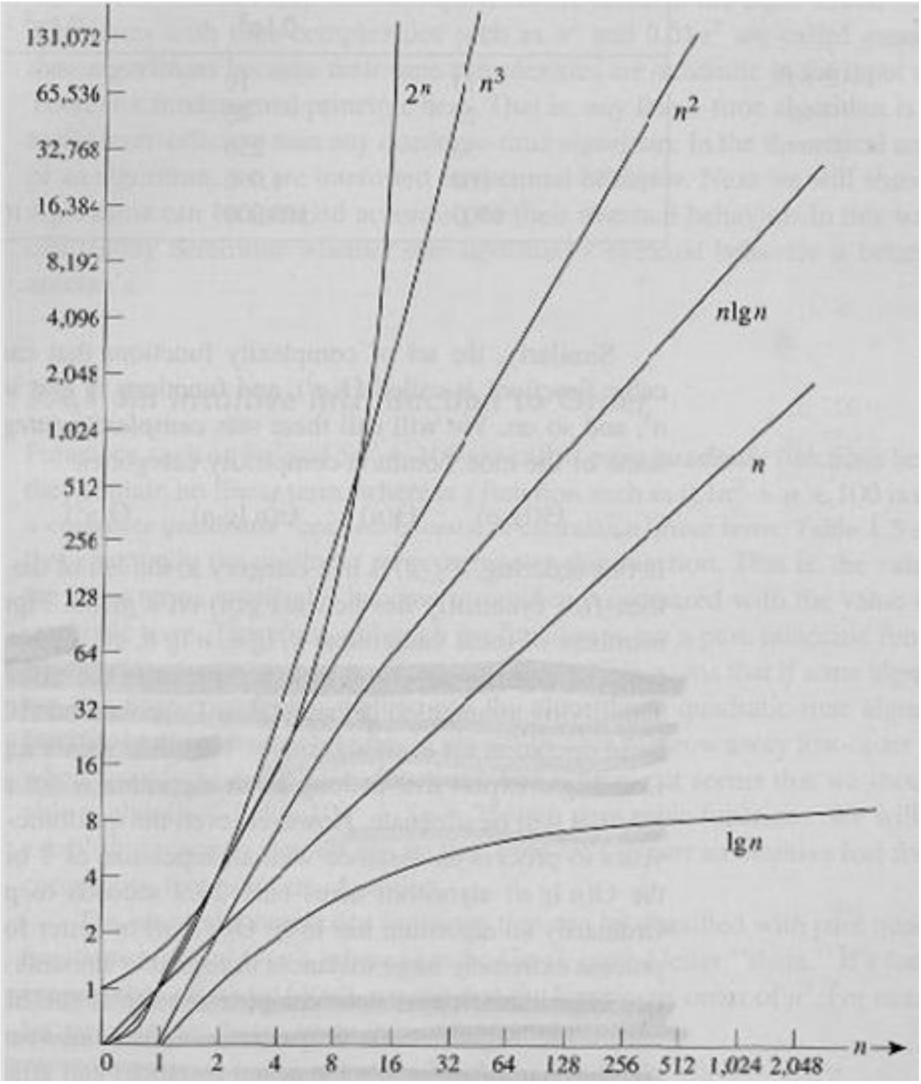
  $\Rightarrow c_2 \geq n/\log n, \forall\, n \geq n_0$ - impossible

# Relations Between Different Sets

- Subset relations between order-of-growth sets.



# Common orders of magnitude



# Common orders of magnitude

**Table 1.4** Execution times for algorithms with the given time complexities

| $n$ | $f(n) = \lg n$ | $f(n) = n$ | $f(n) = n \lg n$ | $f(n) = n^2$ | $f(n) = n^3$ | $f(n) = 2^n$ |
|---|---|---|---|---|---|---|
| 10 | 0.003 μs* | 0.01 μs | 0.033 μs | 0.1 μs | 1 μs | 1 μs |
| 20 | 0.004 μs | 0.02 μs | 0.086 μs | 0.4 μs | 8 μs | 1 ms† |
| 30 | 0.005 μs | 0.03 μs | 0.147 μs | 0.9 μs | 27 μs | 1 s |
| 40 | 0.005 μs | 0.04 μs | 0.213 μs | 1.6 μs | 64 μs | 18.3 min |
| 50 | 0.006 μs | 0.05 μs | 0.282 μs | 2.5 μs | 125 μs | 13 days |
| $10^2$ | 0.007 μs | 0.10 μs | 0.664 μs | 10 μs | 1 ms | $4 \times 10^{13}$ years |
| $10^3$ | 0.010 μs | 1.00 μs | 9.966 μs | 1 ms | 1 s | |
| $10^4$ | 0.013 μs | 10 μs | 130 μs | 100 ms | 16.7 min | |
| $10^5$ | 0.017 μs | 0.10 ms | 1.67 ms | 10 s | 11.6 days | |
| $10^6$ | 0.020 μs | 1 ms | 19.93 ms | 16.7 min | 31.7 years | |
| $10^7$ | 0.023 μs | 0.01 s | 0.23 s | 1.16 days | 31,709 years | |
| $10^8$ | 0.027 μs | 0.10 s | 2.66 s | 115.7 days | $3.17 \times 10^7$ years | |
| $10^9$ | 0.030 μs | 1 s | 29.90 s | 31.7 years | | |

*1 μs = $10^{-6}$ second.
†1 ms = $10^{-3}$ second.

# Logarithms and properties

- In algorithm analysis we often use the notation "log n" without specifying the base

Binary logarithm $\lg n = \log_2 n$

Natural logarithm $\ln n = \log_e n$

$$\lg^k n = (\lg n)^k$$

$$\lg \lg n = \lg(\lg n)$$

$$\log x^y = y \log x$$

$$\log xy = \log x + \log y$$

$$\log \frac{x}{y} = \log x - \log y$$

$$a^{\log_b x} = x^{\log_b a}$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

# More Examples

- For each of the following pairs of functions, either f(n) is O(g(n)), f(n) is Ω(g(n)), or f(n) = Θ(g(n)). Determine which relationship is correct.
  - $f(n) = \log n^2$; $g(n) = \log n + 5$ $\quad f(n) = \Theta(g(n))$
  - $f(n) = n$; $g(n) = \log n^2$ $\quad f(n) = \Omega(g(n))$
  - $f(n) = \log \log n$; $g(n) = \log n$ $\quad f(n) = O(g(n))$
  - $f(n) = n$; $g(n) = \log^2 n$ $\quad f(n) = \Omega(g(n))$
  - $f(n) = n \log n + n$; $g(n) = \log n$ $\quad f(n) = \Omega(g(n))$
  - $f(n) = 10$; $g(n) = \log 10$ $\quad f(n) = \Theta(g(n))$
  - $f(n) = 2^n$; $g(n) = 10n^2$ $\quad f(n) = \Omega(g(n))$
  - $f(n) = 2^n$; $g(n) = 3^n$ $\quad f(n) = O(g(n))$

# Properties

- *Theorem:*

$$f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n)) \text{ and } f = \Omega(g(n))$$

- Transitivity:
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for $O$ and $\Omega$
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for $O$ and $\Omega$
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

# Asymptotic Notations in Equations

- On the right-hand side
  - $\Theta(n^2)$ stands for some anonymous function in $\Theta(n^2)$

  $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means:

  There exists a function $f(n) \in \Theta(n)$ such that

  $$2n^2 + 3n + 1 = 2n^2 + f(n)$$

- On the left-hand side

  $2n^2 + \Theta(n) = \Theta(n^2)$

  No matter how the anonymous function is chosen on the left-hand side, there is a way to choose the anonymous function on the right-hand side to make the equation valid.

# Common Summations

- Arithmetic series: $$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$
- Geometric series: $$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$
  - Special case: $|x| < 1$: $$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$
- Harmonic series: $$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \ln n$$
- Other important formulas: $$\sum_{k=1}^{n} \lg k \approx n \lg n$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \ldots + n^p \approx \frac{1}{p+1} n^{p+1}$$